Cass Business School
CITY UNIVERSITY LONDON

Faculty of Actuarial Science and Insurance

Actuarial Research Paper
No. 193

Surplus Analysis for Variable Annuities with a GMDB Option

Steven Haberman
Gabriella Piscopo

January 2010

ISBN 978-1-905752-25-6

Cass Business School
106 Bunhill Row
London EC1Y 8TZ
Tel +44 (0)20 7040 8470
www.cass.city.ac.uk

Surplus Analysis for Variable Annuities with a GMDB option

Haberman S.†, Piscopo G.††

Abstract

In this paper, we analyze the insurance surplus for a Variable Annuity contract with a Guaranteed Minimum Death Benefit (GMDB) option. Initially, we derive the first two moments of the distribution of the surplus; and subsequently, we develop the whole distribution using a stochastic model which involves an integrated analysis of financial and mortality risk for a portfolio of annuities with GMDB embedded options. We offer a model according which the premium can be modified as per the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the study enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency. Numerical examples illustrate the results.

JEL classification: G22

Keywords: Guaranteed Minimum Death Benefit option, financial risk, mortality risk, surplus, Variable Annuity.

1. Introduction

The Variable Annuity market has increased considerably in the past decade, when bullish financial markets and low interest rates have tempted investors to look for higher returns. Variable Annuities are very attractive to consumers, because they provide participation in the stock market. They are unit-linked annuity contracts, usually with a single premium payment up-front which is invested in one of several funds and they normally are designed with some embedded guarantees. "*As VAs are essentially a new product class in the U.K., an industry standard definition does not yet exists…they are similar to unit linked retirement saving vehicle such as unit linked annuities… however the availability of guarantees distinguish them*" (Ledlie et al. 2008). One of these features is the Guaranteed Minimum Death Benefit, an increasing-strike put option with a stochastic maturity date. In the basic form of product, when the insured dies, the beneficiary obtains a death benefit, which is equal to the maximum of the invested premium and the account value linked to the

† Professor of Actuarial Science and Deputy Dean of Cass Business School, City University, 106 Bunhill Row London EC1Y 8TZ, UK. Internet address: s.haberman@city.ac.uk

†† Dipartimento di Matematica e Statistica, Facoltà di Economia Università degli Studi di Napoli "FedericoII", via Chintia, Complesso Monte S.Angelo 80126 Napoli- Italy. Internet address: Gabriella.Piscopo.1@city.ac.uk

fund. This guarantee is paid for by the VA holder in the form of a perpetual fee that is deducted from the account value linked to the underlying asset.

The purpose of this paper is to study the insurance surplus over time for a portfolio of Variable Annuities with GMDB options. There are 2 theoretical foundations for this work: on the one hand, we take into account the actuarial literature concerning the valuation of the Variable Annuity and GMDB option (Bauer, Kling and Russ (2006); Coleman, Yuying and Patron (2006); Milevsky M. and Posner (2001), Milevsky M. and Salisbury(2002), Milevsky M.A. and Promislow (2001)); on the other hand, we look at the actuarial research literature on insurance surplus and insolvency probability (Coppola et al. (2003), Dahl (2004), Hoedemakers et al. (2005), Lysenko and Parker (2007), Marceau and Gaillardetz (1999) Parker (1996) and Parker (1994)). The abovementioned papers deal with the stochastically discounted value of future cash flows in respect of life insurance and life annuity contracts. The innovative contribution of our work is to apply this methodology to a new product like a Variable Annuity with a GMDB option, extending the models appearing in the literature in order to study a product with a payments linked to a fund account. In the manner of Lysenko and Parker (2007), we adopt a definition of surplus as the difference between the retrospective gain and prospective loss: if we fix a valuation date r, the accumulated value to time r of the insurance cash flows that occurred between times 0 and r represents the retrospective gain and the present value at time r of the cash flows that occur after r is the prospective loss. We modify the model proposed by Lysenko and Parker (2007) in order to capture the uncertainty of a death benefit linked to a fund account. Further, we do not approximate the true probability function of surplus by its limiting distribution as in Lysenko and Parker, which takes into account the investment risk but treats the cash flows as given and equal to their expected value. Instead, in order to explore the longevity risk, we simulate the impact of both the financial and mortality factors on the retrospective gains and prospective losses. We adopt the same financial assumptions as in the Black and Scholes framework. The mortality hypothesis is based on the stochastic model suggested by Cox and Lin (2005) and developed by Ballotta, Esposito and Haberman (2006).

The paper is organized as follows: in section 2 we describe the model; in section 3 we define the surplus as the difference between the retrospective gain and prospective loss and derive the first two moments of its distribution; in section 4 we develop the financial model. Numerical results are shown in section 5 under a deterministic approach. In section 6, we develop the simulations and construct the surplus distribution following a stochastic approach and, in particular, we identify three components, relating respectively to interest, fund and mortality risks. Concluding remarks are offered in section 7.

2. The model

In this work, we consider a portfolio of identical Variable Annuities with a GMDB option, which are issued to a group of m policyholders who are aged x with the same risk characteristics, and whose survival probability distribution are independent and identical; the final age is n. The product is composed of an annuity, with annual payment R, and a GMDB option; there is a single premium, paid at time 0 and invested in a fund. Let V_t be the value of the account at time t, which is linked to a unit fund. Following the standard assumptions in the literature, we model the evolution of the account value as:

$$dV_t = (\mu - \eta)V_t dt + \sigma V_t dW_t$$

(1)

where W_t is a standard Brownian motion under the real probability space, η is the drift rate, δ is the charge paid for the GMDB option. The risk neutral process for V_t is:

$$dV_t = (r - \eta)V_t dt + \sigma V_t dW_t^Q \qquad (2)$$

where r is the risk free rate and W_t^Q is a Brownian motion under a new Girsanov transformed measure Q.

The payoff of the GMDB option at the time t=τ is:

$$G_\tau = Max\left[e^{g\tau}V_o, V_\tau\right], 0 \leq \tau \leq n \qquad (3)$$

where τ is the stochastic time of death and g is the guaranteed rate.

The premium is calculated according to the equivalence principle:

$$P = Ra_{n,i} + D_0 \qquad (4)$$

where $a_{n,i}$ is the actuarial value of an annuity, i is the technical rate used to price the annuity and D_0 is the value of the GMDB option at t=0 (see Haberman and Piscopo (2008)). Appendix A provides details on how to describe the GMDB payoff and calculate its expected value. VAs, like unit linked contracts, can be structured in different ways: both of the constituent living and death benefits or just one of them can be linked to a fund account. In our case, only the death benefit is invested in a fund and so the premium can be ideally decomposed into a sterling part and a unit part.:

$$P = P' + P'' \qquad (5)$$

where *P'* is the sterling part, relating to the annuity, and *P''* is the unit part, relating to the GMDB option and which is invested in a fund.

Let r be a valuation date at which we estimate the surplus linked to this contract.

Let $RC_j^{(r)}$ be the net cash flow at time j for $0 \leq j \leq r$; it is called retrospective cash inflow at time r. It is given by:

$$
\begin{aligned}
RC_j^{(r)} &= \sum_{i=1}^{m} \left[P 1_{\{j=0\}} - R\alpha_{i,j} 1_{\{j \rangle 0\}} - G_j \delta_{i,j} 1_{\{j \rangle 0\}} \right] = \\
&= mP1_{\{j=0\}} - R\left(\sum_{i=1}^{m} \alpha_{i,j} \right) 1_{\{j \rangle 0\}} - G_j \left(\sum_{i=1}^{m} \delta_{i,j} \right) 1_{\{j \rangle 0\}} = \\
&= mP1_{\{j=0\}} - R\alpha_j 1_{\{j \rangle 0\}} - G_j \delta_j 1_{\{j \rangle 0\}}
\end{aligned} \tag{6}
$$

where

$$
\alpha_{i,j} = \begin{cases} 1 \text{ if policyholder i is alive at time j} \\ 0 \text{ otherwise} \end{cases} ;
$$

$$
\delta_{i,j} = \begin{cases} 1 \text{ if policyholder i is dies in year j} \\ 0 \text{ otherwise} \end{cases}
$$

α_j is the number of people from the initial group of m policyholder who survive to time j and δ_j is the number of deaths in year j. Let m_r be the size of the portfolio at time r; for $0 < j \leq$ n-r we have:

$$
\begin{aligned}
\left\{ \alpha_j \middle| \alpha_r = m_r \right\} &\approx BIN\left(m_r, {}_j p_{x+r} \right) \\
\left\{ \delta_j \middle| \alpha_r = m_r \right\} &\approx BIN\left(m_r, {}_{j-1} q_{x+r} \right)
\end{aligned}
$$

We consider r =0, since we study all cash flows as viewed from time 0. We have for k<j:

$$
\begin{aligned}
&E_0\left[\alpha_{i,j}\right] = m\ {}_j p_{x_i} \\
&E_0\left[\delta_{i,j}\right] = m\ {}_{j-1|} q_{x_i} \\
&Var_0\left[\alpha_{i,j}\right] = m\ {}_j p_{x_i} (1 - {}_j p_{x_i}) \\
&Var_0\left[\delta_{i,j}\right] = m\ {}_{j-1|} q_{x_i} (1 - {}_{j-1|} q_{x_i}) \\
&Cov_0\left[\alpha_{i,k}, \alpha_{i,j}\right] = m\ {}_j p_{x_i} (1 - {}_k p_{x_i}) \\
&Cov_0\left[\delta_{i,k}, \delta_{i,j}\right] = -m\ {}_{j-1|} q_{x_i} \times {}_{k-1|} q_{x_i} \\
&Cov_0\left[\delta_{i,j}, \alpha_{i,j}\right] = -m\ {}_{j-1|} q_{x_i} \times {}_j p_{x_i} \\
&Cov_0\left[\delta_{i,k}, \alpha_{i,j}\right] = -m\ {}_{k-1|} q_{x_i} \times {}_j p_{x_i} \\
&Cov_0\left[\alpha_{i,k}, \delta_{i,j}\right] = m(1 - {}_k p_{x_i})\ {}_{j-1|} q_{x_i}
\end{aligned}
$$

Calculation of the cash flow moments is straightforward. Under the reasonable assumption of independence between G_j and δ_j or α_j we have:

$$
E\left[RC_j^{(r)}\right] = mP1_{\{j=0\}} - RE\left[\alpha_j\right] 1_{\{j \rangle 0\}} - E\left[G_j\right] E\left[\delta_j\right] 1_{\{j \rangle 0\}} \tag{7}
$$

where $E\left[G_j\right]=E\left[Max\left(e^{gj}V_o,V_j\right)\right]$ can be calculated by simulating the evolution of the fund value and the corresponding GMDB value.

Moreover, we can calculate the variance of the retrospective cash flow:

$$Var\left[RC_j^r\right]=R^2Var\left[\alpha_j\right]1_{\{j\rangle 0\}}+Var\left[G_j\delta_j\right]1_{\{j\rangle 0\}}+2RCov\left[\alpha_j,G_j\delta_j\right]1_{\{j\rangle 0\}} \tag{8}$$

and the covariance of the retrospective cash flows:

$$Cov\left[RC_k^{(r)},RC_j^{(r)}\right]=$$

$$=R^2Cov\left[\alpha_k,\alpha_j\right]+Cov\left[G_k\delta_k,G_j\delta_j\right]+RCov\left[\alpha_k,G_j\delta_j\right]+RCov\left[\alpha_j,G_k\delta_k\right] \tag{9}$$

Now we fix our attention on the time period after r. Let $PC_j^{(r)}$ be the net cash flow plus the value of the shares invested in the fund that occurs j time units after r for $0 \leq j \leq$ n-r, where n is the final age underlying the life table; this is called the prospective cash outflow at time r. It is given by:

$$\begin{aligned}PC_j^{(r)} &= \sum_{i=1}^{m}\left[R\alpha_{i,(r+j)}1_{\{j\rangle 0\}}+G_{r+j}\delta_{i,(r+j)}1_{\{j\rangle 0\}}\right]=\\ &=R\left(\sum_{i=1}^{m}\alpha_{i,(r+j)}\right)1_{\{j\rangle 0\}}+G_{r+j}\left(\sum_{i=1}^{m}\delta_{i,(r+j)}\right)1_{\{j\rangle 0\}}=\\ &=R\alpha_{r+j}1_{\{j\rangle 0\}}+G_{r+j}\delta_{r+j}1_{\{j\rangle 0\}}\end{aligned} \tag{10}$$

We can derive formulae for the moments of the cash flow in the same manner as before.

$$E\left[PC_j^{(r)}\right]=RE\left[\alpha_{r+j}\right]1_{\{j\rangle 0\}}+E\left[G_{r+j}\right]E\left[\delta_{r+j}\right]1_{\{j\rangle 0\}} \tag{11}$$

$$Var\left[PC_j^r\right]=R^2Var\left[\alpha_{r+j}\right]1_{\{j\rangle 0\}}+Var\left[G_{r+j}\delta_{r+j}\right]1_{\{j\rangle 0\}}+2RCov\left[\alpha_{r+j},G_{r+j}\delta_{r+j}\right]1_{\{j\rangle 0\}} \tag{12}$$

$$\begin{aligned}&Cov\left[PC_j^r,PC_k^r\right]=R^2Cov\left[\alpha_{r+k},\alpha_{r+j}\right]+Cov\left[G_{r+k}\delta_{r+k},G_{r+j}\delta_{r+j}\right]+\\ &+RCov\left[\alpha_{r+k},G_{r+j}\delta_{r+j}\right]+RCov\left[\alpha_{r+j},G_{r+k}\delta_{r+k}\right]\end{aligned} \tag{13}$$

Next, we introduce two random variables, the retrospective gain and the prospective loss, which will be used to define the surplus.

3. Retrospective gain, prospective loss and surplus

The <u>retrospective gain</u> at time r is the difference between the accumulated value to time r of past premiums collected and benefits paid. It can be expressed in terms of $RC_j^{(r)}$ as follows:

$$RG_r=\sum_{j=0}^{r}RC_j^r e^{I(j,r)} \tag{14}$$

where I(s,r) denotes the force of interest accumulation function between times s and r if $0 \leq s \leq r$ and the force of interest actualization function if $r \leq s \leq$ n-r ; it is given by:

$$
I(s,r) = \begin{cases} \sum_{j=s+1}^{r} \lambda(j) & \text{if s < r} \\ 0 & \text{if s = r} \\ -\sum_{j=s}^{r+1} \lambda(j) & \text{if s > r} \end{cases}
$$

and λ(j) is the force of interest in period (j-1,j].

It is reasonable to assume independence between the fund value and interest rate. Thus, we obtain:

$$
E[RG_r] = \sum_{j=0}^{r} E\left[RC_j^r\right] E\left[e^{I(j,r)}\right] \tag{15}
$$

$$
\begin{aligned}
Var[RG_r] &= E\left[RG_r^2\right] - \left[E[RG_r]\right]^2 = \\
&= \sum_{k=0}^{r} \sum_{j=0}^{r} E\left[RC_k^r RC_j^r\right] E\left[e^{I(k,r)+I(j,r)}\right] - \left\{ \sum_{j=0}^{r} E\left[RC_j^r\right] E\left[e^{I(j,r)}\right] \right\}^2 = \\
&= \sum_{k=0}^{r} \sum_{j=0}^{r} \left\{ Cov\left[RC_k^r, RC_j^r\right] + E\left[RC_k^r\right] E\left[RC_j^r\right] \right\} E\left[e^{I(k,r)+I(j,r)}\right] - \left\{ \sum_{j=0}^{r} E\left[RC_j^r\right] E\left[e^{I(j,r)}\right] \right\}^2
\end{aligned} \tag{16}
$$

The <u>prospective loss</u> at time r is the difference between the discounted values to time r of future benefits to be paid and premiums to be collected (although, in this case, there are no future premiums since the contract has a single premium at time 0). The prospective loss can be expressed in terms of $PC_j^{(r)}$ as follows:

$$
PL_r = \sum_{j=0}^{n-r} PC_j^r e^{I(r,r+j)} \tag{17}
$$

The moments of PL_r can be calculated in a similar way to the moments of RG_r.

At this point of the analysis, we define the net stochastic <u>surplus</u> as the difference between the retrospective gain and the prospective loss:

$$
S_r = RG_r - PL_r = \sum_{j=0}^{n} FC_j^r e^{I(j,r)} \tag{18}
$$

where FC_j^r is the generic cash flow (outflow or inflow) at time j.

Thanks to our previous results, we can calculate the expected value and variance of surplus per policy:

$$
E[S_r / m] = E[RG_r / m] - E[PL_r / m] = \frac{1}{m} \sum_{j=0}^{n} E\left[FC_j^r\right] E\left[e^{I(j,r)}\right] \tag{19}
$$

$$Var[S_r / m] = Var\left[\sum_{j=0}^{n} \frac{FC_j^r e^{I(j,r)}}{m}\right] =$$

$$= \frac{1}{m^2}\left\{\sum_{j=0}^{n} Var\left[FC_j^r e^{I(j,r)}\right] + \sum_{j=0}^{n}\sum_{\substack{k=0 \\ k\neq j}}^{n} Cov\left(FC_j^r e^{I(j,r)}, FC_k^r e^{I(k,r)}\right)\right\} \tag{20}$$

In Appendix B, we develop the above formulae.

4. Financial hypothesis

In accordance with the Black & Scholes' framework, we model the evolution of the unit fund as in (1):

$$dV_t = (\mu - \eta)V_t dt + \sigma V_t dW_t \tag{21}$$

Since W_t is a standard Brownian motion, it follows that:

$$E_0[V_j] = V_0 \exp\{(\mu-\eta)j\}$$
$$E_0[V_j^2] = V_0^2 \exp\{2(\mu-\eta)j + \sigma^2 j\}$$
$$E_0[G_j] = E[Max(e^{gj}V_0, V_j] = Max(e^{gj}V_0, E_0[V_j])$$
$$Var[G_j] = Var[V_j]$$
$$Cov[G_j, G_k] = Cov[V_j, V_k] = 0$$

Moreover, we model the force of interest by a conditional autoregressive process AR(1), given the force of interest at time zero. This model is considered by Bellhouse and Panjer (1981) and Marceau and Gaillardetz (1999). Let δ(t) the force of interest in the period (t-1, t]:

$$\delta(t) - \delta = \varphi[\delta(t-1) - \delta] + \gamma\varepsilon(t)$$

Where $\{\varepsilon(k)\}$ is a sequence of independent and identically distributed standard normal variables and δ is the long term mean of the process. We assume $|\phi| < 1$ to ensure the process is stationary in covariance. The moment of the accumulation function are derived in Cairns and Parker (1997).

We assume the independence between the fund and the forces of interest.

5.Numerical Results: the first two moments of the Surples

In this section, we apply the model and show numerical results for a portfolio of identical Variable Annuities with a GMDB option. We consider a group of 1000 policyholders aged 50 with the same risk characteristics, whose survival probability distributions are independent and identically. The mortality table used in our calculation is the SIM2002 based on the Italian male population, with the maximum age n=110. We set R=1 and i=0.04; under this hypothesis, the premium calculated according to the equivalence principle is equal to 17; where, the sterling part $P^{'} = a_{(110-50),i}$ is equal to 16 and the unit part $P^{''} = D_0 = V_0$ is equal to 1. We set

Δ	0.06
δ_0	0.05
Φ	0.8
Γ	0.01

σ^2	0.03
μ-η	0.06
G	0.04

We have carried out 100000 simulations. We evaluate the first 2 moments of the surplus at different dates r and show the results in Figures 1:

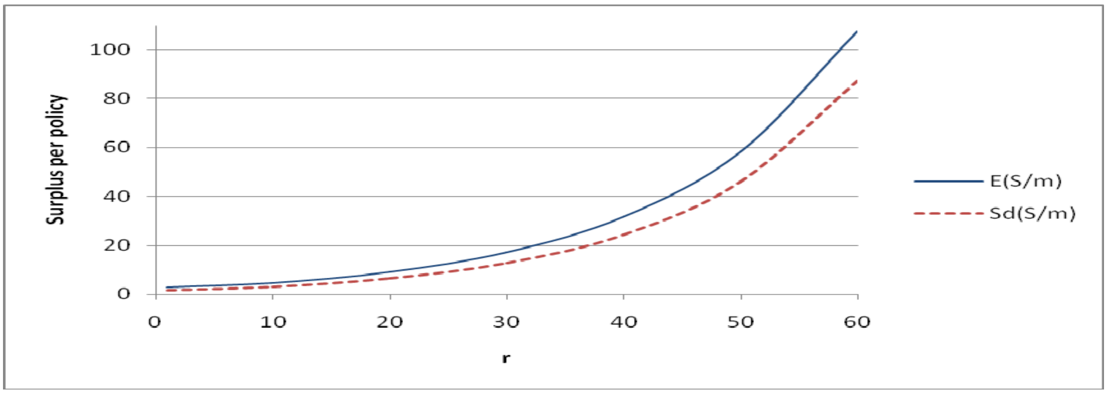

Figure 1: Expected Value and variance of the Surplus per policy

We note that, as the valuation date increases, the standard deviation of the surplus increases. In order to understand this, we have to consider that the standard deviation of the surplus is affected by the uncertainty about the cash flows following the premium and by the variance of the interest rate. When r increases, we have to accumulate a greater number of retrospective cash flows for a longer time and discount a smaller number of prospective cash flows for a shorter period. Consequently, the variance of the capitalized cash gains increases and that of the discounted losses decreases. Numerical investigation shows that the first effect prevails over the second one.

6. Distribution Function of Surplus: a stochastic approach

In the previous section, we have studied the first two moments of the stochastic surplus for a homogeneous portfolio of Variable Annuity contracts with GMDB options. Although the analysis of moments is useful, it is only the first step towards exploring the random behaviour of the surplus. We note that the standard deviation as a risk measure is inappropriate when dealing with asymmetric distributions and it is necessary to study the whole probability function of surplus.

Lysenko and Parker (2007) suggest a recursive method to construct this distribution; the complexity of the product we are considering makes necessary a simulation approach.

One of the objectives of this study is to assess the probability of insolvency, i.e. the probability that it will fall below zero. In order to achieve this purpose, we simulate the evolution of surplus under a mortality and financial stochastic model. Unlike the approach of Lysenko and Parker (2007), we do not approximate the true probability function of surplus by its limiting distribution, which takes into account the investment risk but treats cash flows as given and equal to their expected value. Instead, in order to take account also the longevity risk, we simulate the impact of both financial and mortality factors on retrospective gains and prospective losses.

The financial assumptions are the same as described previously. Also, we need a mortality assumption in order to avoid underestimation or overestimation of the surplus. In this respect, we consider the stochastic model suggested by Cox and Lin (2005) and developed by Ballotta, Esposito and Haberman (2006) and which is described below.

Our calculation is based on the actuarial table used before; however, we estimate the expected value of the number of survivors at age x+t, E[l(x+t)], in a stochastic framework. It is possible to prove that l(x+t) is approximately distributed as a normal random variable with mean equal to l(x) ${}_tp_x$ and variance equal to l(x) ${}_tp_x(1-{}_tp_x)$. However, the recent actuarial literature highlights the fact that the empirical data show perturbations in survival probabilities due to random shocks. Accordingly, we simulate the survival probabilities adjusted for shocks as follows:

$$p'_{x+t} = p_{x+t}{}^{(1-\varepsilon_t)} \tag{22}$$

where ε_t is the shock in the expected probability at time t. Ballotta, Esposito and Haberman (2006) assume that ε_t follows a beta distribution with parameter a and b and the sign of the shocks depends on the random number k(t) simulated from the uniform distribution U(0,1). In particular, we set:

$$\varepsilon(t) \quad \text{if} \quad k(t) < c$$

$$-\varepsilon(t) \quad \text{if} \quad k(t) \geq c$$

where c is a parameter which depends on the expectation of future mortality trend.

The importance of assigning a random sign to ε_t is that, in this way, the model captures not only the long period variations in mortality rates, but also the short period fluctuations due to exceptional circumstances.

We carry out 100000 simulations under different financial and mortality hypotheses. The results are shown in three sections concerning interest rate risk, fund risk and mortality risk.

a. Interest Rate Risk: Numerical Results

We construct the distribution of the surplus per policy at different valuation dates under the hypothesis of the previous section. Results are shown in Figure 2 and summarized in Table 1:

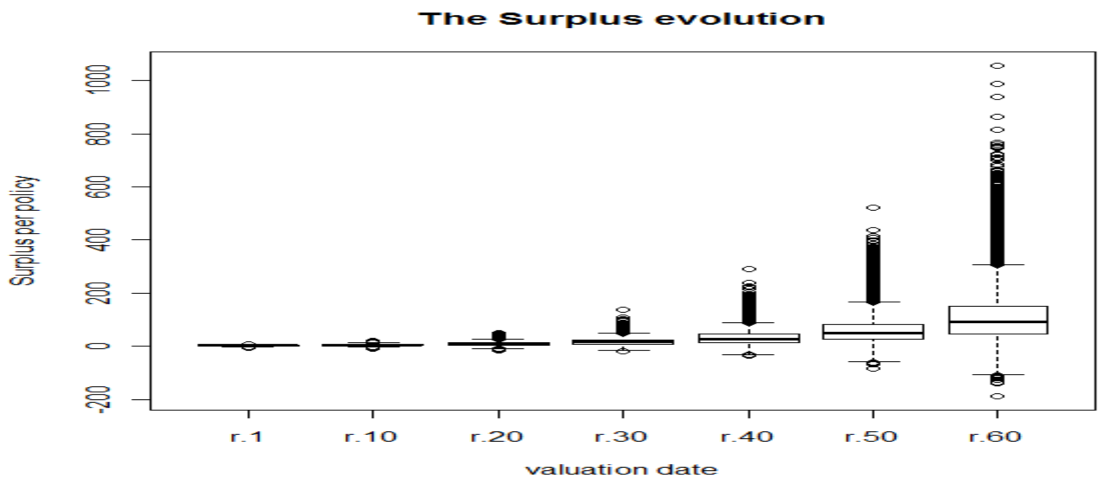

Figure 2: Boxplot of surplus per policy at different valuation dates

S1	S10	S20	S30
Min. :-7.116	Min. :-8.306	Min. :-14.194	Min. :-19.904
1st Qu.: 1.722	1st Qu.: 2.615	1st Qu.: 4.637	1st Qu.: 8.167
Median : 2.833	Median : 4.698	Median : 8.586	Median : 15.590
Mean : 2.712	Mean : 4.767	Mean : 9.181	Mean : 17.209
3rd Qu.: 3.832	3rd Qu.: 6.828	3rd Qu.: 13.063	3rd Qu.: 24.427
Max. : 7.807	Max. :20.882	Max. : 50.022	Max. :137.021

S40	S50	S60
Min. :-36.40	Min. :-83.98	Min. :-188.80
1st Qu.: 14.65	1st Qu.: 26.26	1st Qu.: 47.16
Median : 28.19	Median : 50.89	Median : 92.25
Mean : 31.80	Mean : 58.50	Mean : 107.72
3rd Qu.: 44.90	3rd Qu.: 82.29	3rd Qu.: 150.80
Max. :289.38	Max. :523.09	Max. :1057.44

Table 1 : Summary indices of the surplus distribution per policy at different valuation dates

We want to verify what happens under a different scenario for the force of interest. In particular, we investigate the effect of a reduction in the long mean of the force of interest. We compare the distribution of the Surplus per policy at the valuation date r=1 under the following scenarios:

	ScenarioI	ScenarioII
δ	0.06	0.04
δ_0	0.05	0.05
φ	0.8	0.8
Γ	0.01	0.01

The distributions of surplus are shown and compared in the next figure and table:

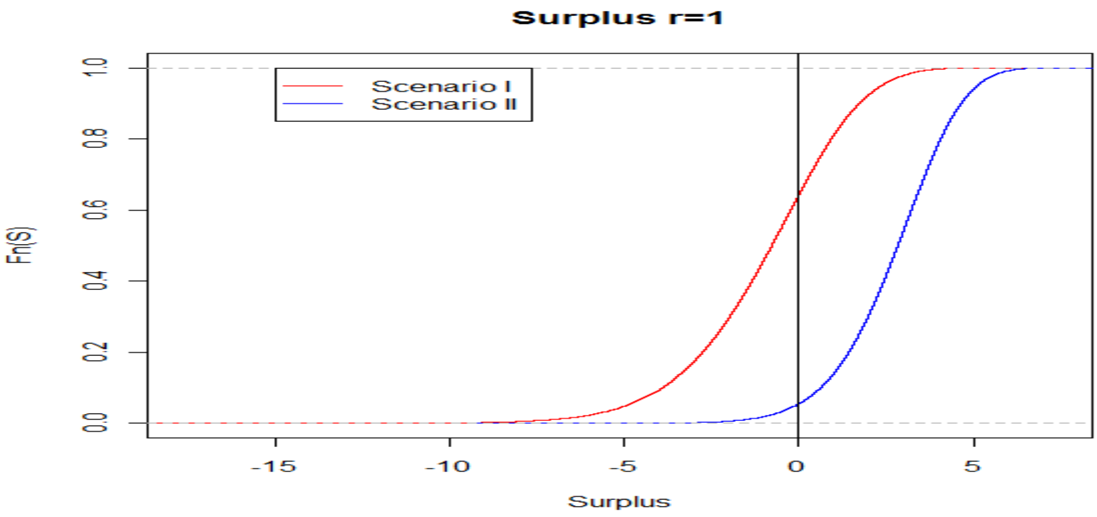

Figure 3: The Surplus per policy at r=1 under different scenarios for the forces of interes

As the long rate of return of the assets in which the insurer invests premium decrases, the cumulative distribution of the surplus moves to the left and, consequently, the probability of insolvency increases.

		ScenarioI	ScenarioII
Prob(S(1)/1000)≤0		5.48%	64.35%
Quantile	**1%**	-1.5826	-7.1447
	5%	-0.0938	-4.9480
	10%	0.6194	-3.9147
	90%	4.6504	1.7327
	95%	5.1123	2.3662
	99%	5.9288	3.4636

Table 2: Probability of Insolvency and significant percentile of the surplus distribution

This comparison highlights the importance of a correct investment strategy in order to avoid the insolvency. In this case, the insurer has to invest the collected premiums into assets with a long mean of the rate of return equal to 0.06 in order to have a positive surplus since the first year and not ask other money to shareholders. We note that if the insurer invests the premiums into assets that in mean yield a return equal to the guaranteed rate on GMDB the probability of insolvency at r=1 is 64.35%.

b. Fund Risk: Numerical Results

In this section, we study the effect of shifts in the distributions of fund value. As we wish to produce a sensitivity analysis, we fix the hypothesis concerning the interest rate distribution according the parameters used in section 5 and change those concerning the fund. In particular, we compare the surplus distributions under the following four scenarios:

Scenario	I	II	III	IV
μ-η	0.06	0.06	0.06	0.05
σ^2	0.03	0.04	0.03	0.03
g	0.04	0.04	0.05	0.04

The results are summarized in the next table:

Scenario I	Scenario II	Scenario III	Scenario IV
Min. : 19.904	Min. :-40.943	Min. :-20.354	Min. : -14.547
1st Qu. : 8.167	1st Qu.: 8.013	1st Qu.: 7.611	1st Qu.: 9.485
Median : 15.590	Median : 15.434	Median : 15.019	Median : 16.901
Mean : 17.209	Mean : 17.022	Mean : 16.642	Mean : 18.532
3rd Qu. : 24.427	3rd Qu.: 24.251	3rd Qu.: 23.849	3rd Qu.: 25.726
Max. :137.021	Max.: 136.487	Max. : 136.568	Max. : 138.754

Table 3 : Summary indices of the surplus distribution per policy under different scenarios

As expected, as the volatility of the fund increases the variance of the surplus increases and as the guaranteed rate increases the mean of the surplus distribution decreases. Moreover, as the drift of the fund process decreases, the distribution of the surplus moves on the right, as shown in Figure 4, because the amounts of death benefits paid decrease.

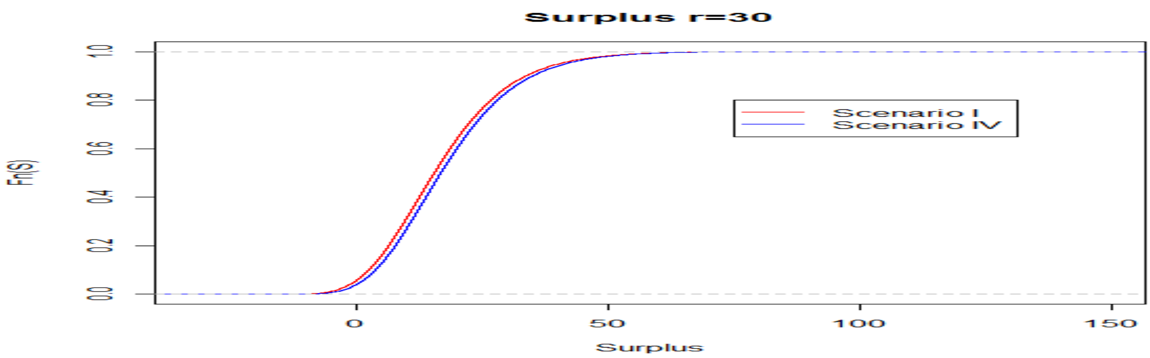

Figure 4: the distribution of surplus per policy under two different hypothesis for the fund process.

c.Mortality Risk: Numerical Results

In this section, we study the effect of shifts in the parameters of stochastic mortality model. In the same manner as the previous sections, we aim to produce a sensitivity analysis, and so we fix the hypothesis concerning the interest rate and the fund evolution as in section 5 and change the mortality table. In particular, we use the mortality model described and set a=0.5 and b=4.5. we evaluate the surplus per policy at r=30. We consider three cases for the value of c: c = 0, 0.5, 1. In the first case, there will be improvements in life expectancy at every date; in other words, all of the shocks are expected to be positive. Conversely, in the second case further improvements of an already high expectancy of life are impossible and all shocks are expected to be negative.

Under the hypothesis c=1, the outflows linked to annuities increase and, under the hypothesis c=0, they decrease. In Figure 16, we show the cdf of the number of deaths that occur in each year under the hypotheses c=0, c=0,5 and c=1. Under the hypothesis c=0, the pdf of the number of deaths is translated to the left and it has a fatter left tail and a less fat right tail than the other two curves. On the contrary, if c=1 the pdf of the number of deaths is translated to the right and it has a less fat left tail and a fatter right tail than the other two curves

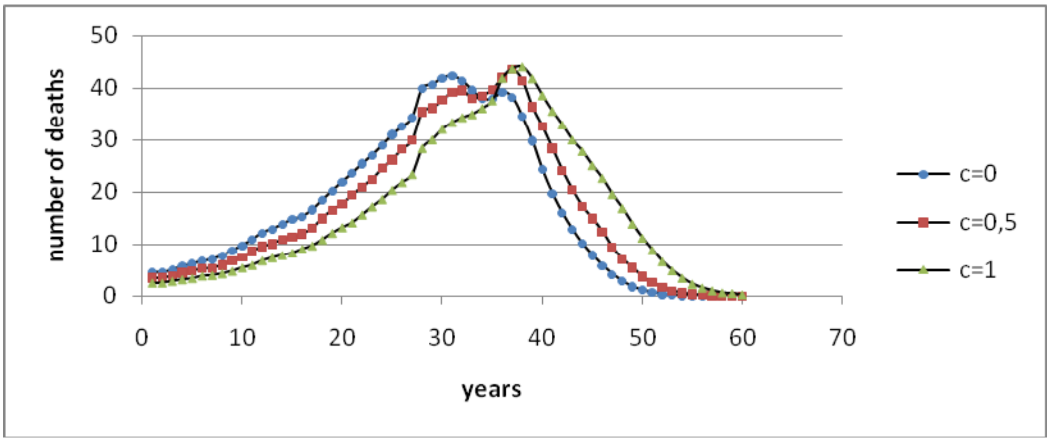

Figure 5: Pdf of the number of deaths under the hypotheses c=0, c=0,5 and c=1[2].

As c increases, the outflows linked to the annuity increase; moreover, the payments related to the GMDB option increase too, because they are rolled over and they are linked to a fund value that increases with time. Consequently, the cumulative distribution of the surplus moves on the left and the probability of insolvency increases. These results are illustrated in Figure 6 and Table 4.

[2] Figure 5 shows an unusual feature: each curve has two modes. The 2-modal feature can be found also in a recent Belgian males table (see Pitacco et al. (2008)).

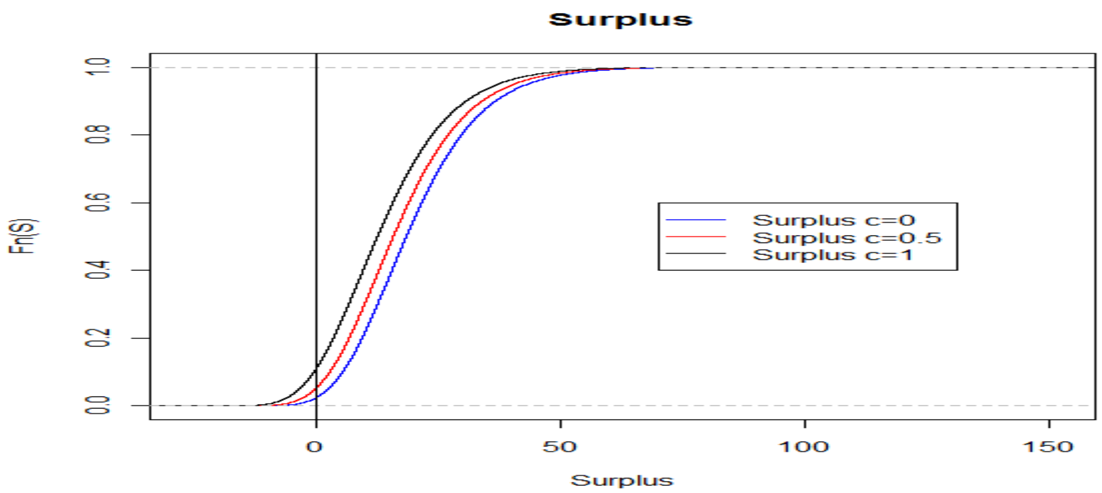

Figure 6: the distiubution of the Surplus per policy under different mortality hypothesis

		c=0	c=0.5	c=1
Prob(Sr/1000)≤0		2,36%	5,17%	10,85%
Quantile	**1%**	-2.4539	-5.1057	-8.2820
	5%	2.5213	-0.1077	-3.3526
	10%	5.4728	2.8347	-0.3719
	90%	36.7409	34.009	30.7354
	95%	43.3471	40.6048	37.3256
	99%	57.6183	54.8743	51.6409

Table 4: Probability of Insolvency and significant percentile of the surplus distribution under different mortality hypothesis.

7. Conclusions

The surplus is an important indicator of an insurance company's financial position and there exists a considerable actuarial literature on the topic (see, for example, Coppola et al. (2003), Hoedemakers et al. (2005), Lysenko and Parker (2007), Marceau and Gaillardetz (1999) and Parker (1996)). The contribution of this paper has been to analyze the behaviour of the insurance surplus for a portfolio of Variable Annuities with GMDB options. In order to achieve this purpose, we have simulated the evolution of the surplus under a mortality and financial stochastic model. The results are presented on the basis of a simplified model; thus, the cash flows arise from the benefit and premium payment streams and are assumed to be dependent only on the mortality experience of the portfolio so that expenses and lapse options are not considered; and further random lifetimes and rates of return are assumed to be independent. These are limitations of our framework and in subsequent work the model will be improved by introducing a more complex and realistic model structure and assumptions.

Despite the above limitations, we believe that the paper is useful in enhancing an insurer's understanding of the stochastic behaviour underlying a Variable Annuity product with a GMDB option and that it provides the first study of the surplus in respect of this recently developed insurance product. Indeed, up to this time, the literature has offered only pricing models for GMDB, but has not studied the evolution of cash flows. We deem this consideration is important in the perspective of the liquidity and insolvency risk management. We have considered both financial and mortality risk and have outlined a comprehensive description of the interaction of different risk factors on the GMDB value.As general rule, if a death benefit is added to an annuity, there is a sort of "mortality natural hedging effect", i.e. the impact of longevity risk is reduced because the annuity is paid for a longer period but the actual value of the death benefit decreases. The GMDB options can represent an exception to this rule; in section 6.3 we have shown that as the estimated life extends the outflows linked to the annuity increase and at the same time the payments related to the GMDB option increase too, because they are rolled over and they are linked to a fund value that increases with time. Therefore, under the hypothesis of a growing fund, the effect of "natural hedging" is nullified. Hence, it is not sufficient to study the impact of each risk factor on the GMDB value, but it is necessary to examine their interaction.

In the paper, numerical examples show a significant impact of the interest, fund and mortality risks on the surplus distribution, insomuch as the insolvency probability increases considerably in many cases. With regard to this point, an advantage of the model used is that it allows an ex ante assessment of the insurer's solvency throughout the duration of contract. Consequently, a change to the design of the product can be made, and, in particular, the premium can be modified according to the forecasts of mortality probabilities, interest rate and fund evolution. Moreover, the model enables us to determine the premium that leads to a required probability of insolvency, and so it can be used for an evaluation of the adequacy of solvency, which is consistent with recent regulatory changes.

References

Ballotta L., Esposito G., Haberman S. (2006). Modelling the Fair Value of Annuities Contracts: The Impact of Interest Rate Risk and Mortality Risk. Actuarial Research Paper No.176, Cass Business School.

Ballotta L., Haberman S. (2006), The fair valuation problem of guaranteed annuity options: The stochastic mortality environment case, Insurance: Mathematics and Economics 38, 195-214

Bauer D., Kling A., Russ J. (2006). A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities. Working Paper ULM University

Bellhouse D.R., Panjer H.H. (1981). Stochastic modeling of interest rates with applications to life contingiencies. Journal of Risk and Insurance, 48(4), 628-637

Cairns A.J.G., Parker G. (1997). Stochastic pension fund modeling. Insurance: Mathematics and Economics, 21, 43-79

Coleman T.F., Li Y., Patron M. (2006). Hedging guarantees in variable annuities under both equity and interest rate risks. Insurance: Mathematics and Economics 38, 215-228

Coppola M., Di Lorenzo E., Sibillo M. (2003). Stochastic analysis in life office management: applications to large annuity portfolios. Applied Stochastic Models in Business and Industry 19, 31-42.

Cox S.H., Lin Y. (2005). Securitization of mortality risk in life annuities. The Journal of Risk and Insurance 72, 227-252

Dahl M. (2004). Stochastic mortality in life insurance: market reserves and mortality- linked insurance contracts. Insurance: Mathematics and Economics 35, 113-136

Haberman S., Piscopo G. (2008). Mortality Risk and the Valuation of Annuities with Guaranteed Minimum Death Benefit Options: Application to the Italian Population. Actuarial Research Paper No. 187, Cass Business School.

Hoedemakers T., Darkiewicz G., Goovaerts M. (2005). Approximations for life annuity contracts in a stochastic financial environment. Insurance: Mathematics and Economics 37, 239-269.

Ledlie M.C., Corry D.P., Finkelstein G.S., Ritchie A.J., Su K., Wilson D.C.E (2008). Variable annuities. Presented to the Institute of Actuaries.

Lysenko N., Parker G. (2007). Stochastic Analysis of Life Insurance Surplus. Presented to the 2007 AFIR Colloquium.

Marceau E., Gaillardetz P. (1999). On life insurance reserves in a stochastic mortality and interest rates environment. Insurance: Mathematics and Economics, 25, 261-280.

Milevsky M., Posner S.E. (2001). The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds. The Journal of Risk and Insurance 68, 1, 91-126.

Milevsky M., Salisbury T.S. (2002), The Real Option to Lapse and the Valuation of Death-Protected Investments. Working Paper York University and the Fields Institute, Toronto.

Milevsky M.A., Promislow S.D. (2001), Mortality derivatives and the option to annuities. Insurance: Mathematics and Economics 29, 299-318

Parker G. (1994). Limiting distribution of the present value of a portfolio. ASTIN Bulletin, 24, 47-60.

Parker G. (1996). A portfolio of endowment policies and its limiting distribution. ASTIN Bulletin, 26, 25-33.

Pitacco E., Denuit M., Haberman S., Olivieri A. (2008). Modelling Longevity Dynamics for Pensions and Annuity Business. Oxford University Press, Oxford.

APPENDIX A

Let T be the future lifetime random variable expressed in continuous time, $F_x(t)$ be its *cdf* and $f_x(t)$ be its *pdf*; therefore, for an individual aged x the probability of death before time t is

$$F_x(t) = P\,(\,T \le t\,) = 1- {}_tp_x = 1- \exp\left\{-\int_0^t \varsigma(x+s)ds\right\}$$

where ζ denotes the force of mortality.

Let V_t be the account value at time t linked to fund value. Following standard assumptions in the literature, we model the evolution of the account as:

$$dV_t = (\mu - \eta)V_t dt + \sigma V_t dW_t$$

where W_t is a standard Brownian motion, μ is the drift rate, η is the charge paid for the GMDB option.

The risk neutral process for V_t is:

$$dV_t = (r - \eta)V_t dt + \sigma V_t dW_t^Q$$

where r is the risk free rate and W_t^Q is a Brownian motion under a new Girsanov transformed measure Q. The solution of the SDE is:

$$V_t = V_0 e^{(r-\eta-\sigma^2/2)t+\sigma W_t^Q}\,.$$

Now we describe the GMDB payoff. At the random date of death τ the beneficiary will receive

$$D_\tau^p = \max(e^{g\tau}V_0, V_\tau) = e^{g\tau}\,\max(V_0 - e^{-g\tau}V_\tau, 0) + V_\tau$$

where g is the guaranteed rate.

The value of the GMDB option at τ is the sum of the fund value and a put option whose strike price is the initial value V_0, with an underlying asset V_τ discounted by the guaranteed growth rate g. Since the maturity is stochastic and τ and V_τ are independent, the present value of GMDB is given by the expectations under τ and V_τ:

$$D_0^p = E_t\left\{E^Q\left\{e^{-r\tau}D_\tau^p \middle| \tau = t\right\}\right\}$$

If we fixed the date τ, we have at τ an European Option, whose actual value can be calculated with Black and Scholes formula. Therefore, the previous formula can be interpreted as a decomposition of the actual value of GMDB in the actual value of a continuous sequence of European put option. Substituting the expression of D_τ^p

$$D_0^p = E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-r\tau}V_\tau \middle| \tau = t\right\}\right\}$$

We can observe that

$$E^Q\left\{e^{-r\tau}V_\tau\right\} = e^{-\eta\tau}V_0$$

since we have assumed that V_t is a geometric Brownian motion with drift equal to r-δ and so its expected value is:

$$E_{(0)}^Q\left\{V_\tau\right\} = e^{(r-\eta)\tau}V_0$$

Consequently:

$$D_0^p = E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-r\tau}V_\tau \middle| \tau = t\right\}\right\} =$$
$$= E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-\eta\tau} \middle| \tau = t\right\}\right\}$$

We observe that for a fixed date T

$$E^Q\left\{e^{-(r-g)T}\max(V_0 - e^{-gT}V_T, 0) + e^{-\eta T}\right\} \equiv$$
$$\equiv V_0\left(\left[e^{-\tilde{r}T}N(-d_2) - e^{-\eta T}N(-d_1) + e^{-\eta T}\right]\right) \equiv$$
$$\equiv V_0\left[BS\left(\tilde{r}, \eta, \sigma, T\right) + e^{-\eta T}\right)\right]$$

where $d_1 = \dfrac{(\tilde{r} - \eta + \sigma^2/2)\sqrt{T}}{\sigma}$; $d_2 = d_1 - \sigma\sqrt{T}$; $\tilde{r} = r - g$; N(.) is the cumulative probability function for a random variable normally distributed.

If we consider both the expectations:

$$D_0^p = \int_0^{\omega - x} f_x(t)V_0\left[BS\left(\tilde{r}, \eta, \sigma, t\right) + e^{-\delta t}\right)\right]dt$$

In the discrete case we have:

$$D_0^p = \sum_{t=1}^{\omega - x} {}_t p_x q_{x+t} V_0\left[BS(\tilde{r}, \eta, \sigma.t) + e^{-\delta t}\right]$$

for a policyholder aged x at inception of the contract.

The value of GMDB is a weighted average of the values of ω-x European put options, where the weights are the postponed probability of death in t, i.e. the probability of survival until t and death between t and t+1.

APPENDIX B

The variance of the cash flows (both retrospective or prospective) is given by the following formula:

$$Var\left[FC_j^r\right] = R^2 Var\left[\alpha_j\right] + Var\left[G_j\delta_j\right] + 2RCov\left[\alpha_j, G_j\delta_j\right]$$

where

- $Var\left[G_j\delta_j\right] = E\left[G_j^2\right]E\left[\delta_j^2\right] - \left(E\left[G_j\right]\right)^2\left(E\left[\delta_j\right]\right)^2$;
- $Cov\left[\alpha_j, G_j\delta_j\right] = E\left[G_j\right]Cov\left[\alpha_j, \delta_j\right]$

  in fact

$$\begin{aligned} Cov\left[\alpha_j, G_j\delta_j\right] &= E\left[\alpha_j G_j\delta_j\right] - E\left[\alpha_j\right]E\left[G_j\delta_j\right] = E\left[G_j\right]E\left[\alpha_j\delta_j\right] - E\left[\alpha_j\right]E\left[G_j\right]E\left[\delta_j\right] = \\ &= E\left[G_j\right]\left\{E\left[\alpha_j\delta_j\right] - E\left[\alpha_j\right]E\left[\delta_j\right]\right\} = E\left[G_j\right]Cov\left[\alpha_j, \delta_j\right] \end{aligned};$$

The covariance of the cash flows is:

$$Cov\left[FC_k^{(r)}, FC_j^{(r)}\right] =$$

$$= R^2 Cov\left[\alpha_k, \alpha_j\right] + Cov\left[G_k\delta_k, G_j\delta_j\right] + RCov\left[\alpha_k, G_j\delta_j\right] + RCov\left[\alpha_j, G_k\delta_k\right]$$

where

- $$\begin{aligned} Cov\left[G_k\delta_k, G_j\delta_j\right] &= E\left[G_k\delta_k G_j\delta_j\right] - E\left[G_k\delta_k\right]E\left[G_j\delta_j\right] = \\ &= E\left[G_k G_j\right]E\left[\delta_k\delta_j\right] - E\left[G_k\right]E\left[\delta_k\right]E\left[G_j\right]E\left[\delta_j\right] = \\ &= E\left[G_k\right]E\left[G_j\right]E\left[\delta_k\delta_j\right] - E\left[G_k\right]E\left[\delta_k\right]E\left[G_j\right]E\left[\delta_j\right] = E\left[G_k\right]E\left[G_j\right]\left\{E\left[\delta_k\delta_j\right] - E\left[\delta_k\right]E\left[\delta_j\right]\right\} = \\ &= E\left[G_k\right]E\left[G_j\right]Cov\left[\delta_k, \delta_j\right] \end{aligned}$$
- $Cov\left[\alpha_k, G_j\delta_j\right] = E\left[G_j\right]Cov\left[\alpha_k, \delta_j\right]$
- $Cov\left[\alpha_j, G_k\delta_k\right] = E\left[G_k\right]Cov\left[\alpha_j, \delta_k\right]$

Finally, the variance of the surplus can be calculated:

$$Var[S_r / m] = Var\left[\sum_{j=0}^{n} \frac{FC_j^r e^{I(j,r)}}{m}\right] = \frac{1}{m^2}\left\{\sum_{j=0}^{n} Var\left[FC_j^r e^{I(j,r)}\right] + \sum_{j=0}^{n}\sum_{\substack{k=0\\k\neq j}}^{n} Cov\left(FC_j^r e^{I(j,r)}, FC_k^r e^{I(k,r)}\right)\right\}$$ where

- $Var\left[FC_j^r e^{I(j,r)}\right] =$

$$= E\left[FC_j^2 e^{I(j,r)2}\right] - \left\{E\left[FC_j^r e^{I(j,r)}\right]\right\}^2 = E\left[FC_j^2\right]E\left[e^{I(j,r)2}\right] - E\left[FC_j\right]^2 E\left[e^{I(j,r)}\right]^2 =$$

- $Cov\left[FC_j^r e^{I(j,r)}, FC_k^r e^{I(k,r)}\right] =$

$$= E\left[FC_j^r FC_k^r e^{I(j,r)+I(k,r)}\right] - E\left[FC_j^r\right]E\left[e^{I(j,r)}\right]E\left[FC_k^r\right]E\left[e^{I(k,r)}\right] =$$
$$= E\left[FC_j^r FC_k^r\right]E\left[e^{I(j,r)+I(k,r)}\right] - E\left[FC_j^r\right]E\left[e^{I(j,r)}\right]E\left[FC_k^r\right]E\left[e^{I(k,r)}\right] =$$
$$= E\left[FC_j^r FC_k^r\right]\left\{Cov\left[e^{I(j,r)}, e^{I(j,r)}\right] - E\left[e^{I(j,r)}\right]E\left[e^{I(k,r)}\right]\right\} - E\left[FC_j^r\right]E\left[e^{I(j,r)}\right]E\left[FC_k^r\right]E\left[e^{I(k,r)}\right]$$

Cass Business School
CITY UNIVERSITY LONDON

FACULTY OF ACTUARIAL SCIENCE AND INSURANCE

Actuarial Research Papers since 2001

Report Number	Date	Publication Title	Author
160.	January 2005	Mortality Reduction Factors Incorporating Cohort Effects. ISBN 1 90161584 7	Arthur E. Renshaw Steven Haberman
161.	February 2005	The Management of De-Cumulation Risks in a Defined Contribution Environment. ISBN 1 901615 85 5.	Russell J. Gerrard Steven Haberman Elena Vigna
162.	May 2005	The IASB Insurance Project for Life Insurance Contracts: Impart on Reserving Methods and Solvency Requirements. ISBN 1-901615 86 3.	Laura Ballotta Giorgia Esposito Steven Haberman
163.	September 2005	Asymptotic and Numerical Analysis of the Optimal Investment Strategy for an Insurer. ISBN 1-901615-88-X	Paul Emms Steven Haberman
164.	October 2005.	Modelling the Joint Distribution of Competing Risks Survival Times using Copula Functions. I SBN 1-901615-89-8	Vladimir Kaishev Dimitrina S, Dimitrova Steven Haberman
165.	November 2005.	Excess of Loss Reinsurance Under Joint Survival Optimality. ISBN1-901615-90-1	Vladimir K. Kaishev Dimitrina S. Dimitrova
166.	November 2005.	Lee-Carter Goes Risk-Neutral. An Application to the Italian Annuity Market. ISBN 1-901615-91-X	Enrico Biffis Michel Denuit
167.	November 2005	Lee-Carter Mortality Forecasting: Application to the Italian Population. ISBN 1-901615-93-6	Steven Haberman Maria Russolillo
168.	February 2006	The Probationary Period as a Screening Device: Competitive Markets. ISBN 1-901615-95-2	Jaap Spreeuw Martin Karlsson
169.	February 2006	Types of Dependence and Time-dependent Association between Two Lifetimes in Single Parameter Copula Models. ISBN 1-901615-96-0	Jaap Spreeuw
170.	April 2006	Modelling Stochastic Bivariate Mortality ISBN 1-901615-97-9	Elisa Luciano Jaap Spreeuw Elena Vigna.
171.	February 2006	Optimal Strategies for Pricing General Insurance. ISBN 1901615-98-7	Paul Emms Steve Haberman Irene Savoulli
172.	February 2006	Dynamic Pricing of General Insurance in a Competitive Market. ISBN1-901615-99-5	Paul Emms
173.	February 2006	Pricing General Insurance with Constraints. ISBN 1-905752-00-8	Paul Emms
174.	May 2006	Investigating the Market Potential for Customised Long Term Care Insurance Products. ISBN 1-905752-01-6	Martin Karlsson Les Mayhew Ben Rickayzen

Report Number	Date	Publication Title	Author
175.	December 2006	Pricing and Capital Requirements for With Profit Contracts: Modelling Considerations. ISBN 1-905752-04-0	Laura Ballotta
176.	December 2006	Modelling the Fair Value of Annuities Contracts: The Impact of Interest Rate Risk and Mortality Risk. ISBN 1-905752-05-9	Laura Ballotta Giorgia Esposito Steven Haberman
177.	December 2006	Using Queuing Theory to Analyse Completion Times in Accident and Emergency Departments in the Light of the Government 4-hour Target. ISBN 978-1-905752-06-5	Les Mayhew David Smith
178.	April 2007	In Sickness and in Health? Dynamics of Health and Cohabitation in the United Kingdom. ISBN 978-1-905752-07-2	Martin Karlsson Les Mayhew Ben Rickayzen
179.	April 2007	GeD Spline Estimation of Multivariate Archimedean Copulas. ISBN 978-1-905752-08-9	Dimitrina Dimitrova Vladimir Kaishev Spiridon Penev
180.	May 2007	An Analysis of Disability-linked Annuities. ISBN 978-1-905752-09-6	Ben Rickayzen
181.	May 2007	On Simulation-based Approaches to Risk Measurement in Mortality with Specific Reference to Poisson lee-Carter Modelling. ISBN 978-1-905752-10-2	Arthur Renshaw Steven Haberman
182.	July 2007	High Dimensional Modelling and Simulation with Asymmetric Normal Mixtures. ISBN 978-1-905752-11-9	Andreas Tsanakas Andrew Smith
183.	August 2007	Intertemporal Dynamic Asset Allocation for Defined Contribution Pension Schemes. ISBN 978-1-905752-12-6	David Blake Douglas Wright Yumeng Zhang
184.	October 2007	To split or not to split: Capital allocation with convex risk measures. ISBN 978-1-905752-13-3	Andreas Tsanakas
185.	April 2008	On Some Mixture Distribution and Their Extreme Behaviour. ISBN 978-1-905752-14-0	Vladimir Kaishev Jae Hoon Jho
186.	October 2008	Optimal Funding and Investment Strategies in Defined Contribution Pension Plans under Epstein-Zin Utility. ISBN 978-1-905752-15-7	David Blake Douglas Wright Yumeng Zhang
187.	May 2008	Mortality Risk and the Valuation of Annuities with Guaranteed Minimum Death Benefit Options: Application to the Italian Population. ISBN 978-1-905752-16-4	Steven Haberman Gabriella Piscopo
188.	January 2009	The Market Potential for Privately Financed Long Term Care Products in the UK. ISBN 978-1-905752-19-5	Leslie Mayhew
189.	June 2009	Whither Human Survival and Longevity or the Shape of things to Come. ISBN 978-1-905752-21-8	Leslie Mayhew David Smith
190	October 2009	ilc: A Collection of R Functions for Fitting a Class of Lee Carter Mortality Models using Iterative fitting Algorithms* ISBN 978-1-905752-22-5	Zoltan Butt Steven Haberman
191.	October 2009	Decomposition of Disease and Disability Life Expectancies in England, 1992-2004. ISBN 978-1-905752-23-2	Domenica Rasulo Leslie Mayhew Ben Rickayzen

Report Number	Date	Publication Title	Author
192.	October 2009	Exploration of a Novel Bootstrap Technique for Estimating the Distribution of Outstanding Claims Reserves in General Insurance. ISBN 978-1-905752-24-9	Robert Cowell
193.	January 2010	Surplus Analysis for Variable Annuities with a GMDB Option. ISBN 978-1-905752-25-6	Steven Haberman Gabriella Piscopo

Statistical Research Papers

1.	December 1995.	Some Results on the Derivatives of Matrix Functions. ISBN 1 874 770 83 2	P. Sebastiani
2.	March 1996	Coherent Criteria for Optimal Experimental Design. ISBN 1 874 770 86 7	A.P. Dawid P. Sebastiani
3.	March 1996	Maximum Entropy Sampling and Optimal Bayesian Experimental Design. ISBN 1 874 770 87 5	P. Sebastiani H.P. Wynn
4.	May 1996	A Note on D-optimal Designs for a Logistic Regression Model. ISBN 1 874 770 92 1	P. Sebastiani R. Settimi
5.	August 1996	First-order Optimal Designs for Non Linear Models. ISBN 1 874 770 95 6	P. Sebastiani R. Settimi
6.	September 1996	A Business Process Approach to Maintenance: Measurement, Decision and Control. ISBN 1 874 770 96 4	Martin J. Newby
7.	September 1996.	Moments and Generating Functions for the Absorption Distribution and its Negative Binomial Analogue. ISBN 1 874 770 97 2	Martin J. Newby
8.	November 1996.	Mixture Reduction via Predictive Scores. ISBN 1 874 770 98 0	Robert G. Cowell.
9.	March 1997.	Robust Parameter Learning in Bayesian Networks with Missing Data. ISBN 1 901615 00 6	P.Sebastiani M. Ramoni
10.	March 1997.	Guidelines for Corrective Replacement Based on Low Stochastic Structure Assumptions. ISBN 1 901615 01 4.	M.J. Newby F.P.A. Coolen
11.	March 1997	Approximations for the Absorption Distribution and its Negative Binomial Analogue. ISBN 1 901615 02 2	Martin J. Newby
12.	June 1997	The Use of Exogenous Knowledge to Learn Bayesian Networks from Incomplete Databases. ISBN 1 901615 10 3	M. Ramoni P. Sebastiani
13.	June 1997	Learning Bayesian Networks from Incomplete Databases. ISBN 1 901615 11 1	M. Ramoni P.Sebastiani
14.	June 1997	Risk Based Optimal Designs. ISBN 1 901615 13 8	P.Sebastiani H.P. Wynn
15.	June 1997.	Sampling without Replacement in Junction Trees. ISBN 1 901615 14 6	Robert G. Cowell
16.	July 1997	Optimal Overhaul Intervals with Imperfect Inspection and Repair. ISBN 1 901615 15 4	Richard A. Dagg Martin J. Newby
17.	October 1997	Bayesian Experimental Design and Shannon Information. ISBN 1 901615 17 0	P. Sebastiani. H.P. Wynn

Report Number	Date	Publication Title	Author
18.	November 1997.	A Characterisation of Phase Type Distributions. ISBN 1 901615 18 9	Linda C. Wolstenholme
19.	December 1997	A Comparison of Models for Probability of Detection (POD) Curves. ISBN 1 901615 21 9	Wolstenholme L.C
20.	February 1999.	Parameter Learning from Incomplete Data Using Maximum Entropy I: Principles. ISBN 1 901615 37 5	Robert G. Cowell
21.	November 1999	Parameter Learning from Incomplete Data Using Maximum Entropy II: Application to Bayesian Networks. ISBN 1 901615 40 5	Robert G. Cowell
22.	March 2001	FINEX : Forensic Identification by Network Expert Systems. ISBN 1 901615 60X	Robert G.Cowell
23.	March 2001.	Wren Learning Bayesian Networks from Data, using Conditional Independence Tests is Equivalant to a Scoring Metric ISBN 1 901615 61 8	Robert G Cowell
24.	August 2004	Automatic, Computer Aided Geometric Design of Free-Knot, Regression Splines. ISBN 1-901615-81-2	Vladimir K Kaishev, Dimitrina S.Dimitrova, Steven Haberman Richard J. Verrall
25.	December 2004	Identification and Separation of DNA Mixtures Using Peak Area Information. ISBN 1-901615-82-0	R.G.Cowell S.L.Lauritzen J Mortera,
26.	November 2005.	The Quest for a Donor : Probability Based Methods Offer Help. ISBN 1-90161592-8	P.F.Mostad T. Egeland., R.G. Cowell V. Bosnes Ø. Braaten
27.	February 2006	Identification and Separation of DNA Mixtures Using Peak Area Information. (Updated Version of Research Report Number 25). ISBN 1-901615-94-4	R.G.Cowell S.L.Lauritzen J Mortera,
28.	October 2006	Geometrically Designed, Variable Knot Regression Splines : Asymptotics and Inference. ISBN 1-905752-02-4	Vladimir K Kaishev Dimitrina S.Dimitrova Steven Haberman Richard J. Verrall
29.	October 2006	Geometrically Designed, Variable Knot Regression Splines : Variation Diminishing Optimality of Knots. ISBN 1-905752-03-2	Vladimir K Kaishev Dimitrina S.Dimitrova Steven Haberman Richard J. Verrall
30.	November 2008	Scheduling Reentrant Jobs on Parallel Machines with a Remote Server. ISBN 978-1-905752-18-8	Konstantin Chakhlevitch Celia Glass

Papers can be downloaded from

http://www.cass.city.ac.uk/facact/research/publications.html

Faculty of Actuarial Science and Insurance

Cass Business School

Cass Business School
106 Bunhill Row,
London EC1Y 8TZ.

ISBN 978-1-905752-25-6