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Stochastic processes induced by Dirichlet (B-) splines: modelling multivariate asset price dynamics

by

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Abstract

We consider a new class of processes, called LG processes, defined as linear combinations of independent gamma processes. Their distributional and path-wise properties are explored by following their relation to polynomial and Dirichlet (B-) splines. In particular, it is shown that the density of an LG process can be expressed in terms of Dirichlet (B-) splines, introduced independently by Ignatov and Kaishev (1987, 1988, and 1989) and Karlin et al. (1986). We further show that the well known variance-gamma (VG) process, introduced by Madan and Seneta (1990), and the Bilateral Gamma (BG) process, recently considered by Küchler and Tappe (2008) are special cases of an LG process. Following this LG interpretation, we derive new (alternative) expressions for the VG and BG densities and consider their numerical properties. The LG process has two sets of parameters, the B-spline knots and their multiplicities, and offers further flexibility in controlling the shape of the Levy density, compared to the VG and the BG processes. Such flexibility is often desirable in practice, which makes LG processes interesting for financial and insurance applications.

Multivariate LG processes are also introduced and their relation to multivariate Dirichlet and simplex splines is established. Expressions for their joint density, the underlying LG-copula, the characteristic, moment and cumulant generating functions are given. A method for simulating LG sample paths is also proposed, based on the Dirichlet bridge sampling of Gamma processes, due to Kaishev and Dimitrova (2009). A method of moments for estimation of the LG parameters is also developed. Multivariate LG processes are shown to provide a competitive alternative in modelling dependence, compared to the multivariate asymmetric VG process considered by Cont and Tankov (2004) and Luciano and Schoutens (2006), and to its generalization by Luciano and Semeraro (2007) and Semeraro (2008). Application of multivariate LG processes in modelling the joint dynamics of multiple exchange rates is also considered.

Keywords: LG process; (multivariate) variance gamma process; bilateral gamma process; Dirichlet spline; B-spline; simplex spline; Dirichlet bridge sampling; cumulants; FX modelling.

1 Introduction

An important strand of literature on financial modelling in recent years is devoted to developing more realistic stochastic models incorporating appropriate Lévy processes as drivers of the price dynamics of financial assets. Examples of such processes are the Variance Gamma process introduced by Madan and Seneta (1990) (see also Madan et al. 1998) and the so called Bilateral Gamma (BG) process considered recently by Küchler and Tappe (2008). The three parameter VG process of Madan et al. (1998) is constructed by randomly changing the time in a Brownian motion with certain drift and volatility parameters, following a Gamma process with unit mean rate and certain variance rate parameter. The BG process is a generalization of the VG process and its increments have a four parameter Bilateral Gamma distribution, which represents two Gamma distributions, one for the positive and one for the negative half-lines, adjoined together at the origin. Both VG and BG processes are pure jump, infinite activity, finite-variation, Lévy processes, that inherit these properties from the Gamma processes underlying their construction. For an excellent account on properties of Gamma processes which play an important role throughout this paper, we refer to Yor (2007).

The exponential VG process has proved a successful alternative to Geometric Brownian motion in a number of applications, for example in option pricing (see Kaishev and Dimitrova 2009 and the references therein) and in credit risk modelling (see Schoutens and Cariboni 2009). The ability of the VG process to capture both upward and downward jumps as well as very small movements (jitters) in stock prices have been highlighted by Stein et al. (2007) who give an extensive list of further references on the VG model and its applications.

Many real life financial applications require modelling the joint dynamics of multiple, possibly dependent asset price processes. A typical example would be the necessity to model the joint movement of foreign currencies exchange rates. In such cases, developing models involving appropriate multivariate Lévy processes, capable of capturing different dependence patterns is of utmost importance. In order to meet such demands, recently, attempts to extend the VG model to more than one dimension have been undertaken in several directions. For example, Luciano and Schoutens (2006) considered a multivariate VG model, in which dependence is achieved by applying a random time change according to a common Gamma process, in the corresponding, differently parameterized, univariate Brownian motions. The level of dependence in this construction is controlled only through the Gamma variance rate parameter which imposes some limitations on its flexibility (see the numerical illustration in Section 4). Further generalizations of this construction, due to Luciano and Semeraro (2007) and Semeraro (2008), allow for a decomposition of the time change in a common and idiosyncratic parts.

The univariate BG process with its four parameters offers somewhat extended flexibility, compared to the univariate VG. However, to the best of our knowledge, no multivariate versions of the BG process have been considered in the literature.

In this paper we propose a new class of Lévy processes defined as linear combinations of independent Gamma processes. In what follows, it will be convenient to refer to such linear combinations as *LG processes*. It is directly verified (see Section 2) that both the Variance Gamma (VG) process and the Bilateral Gamma process are special cases of an LG process represented as particular linear combinations of two Gamma processes.

Our aim in this paper is to introduce univariate and multivariate LG processes, explore their properties and illustrate how they can be applied in modelling the joint behavior of empirical asset price processes. As the VG and the BG, LG processes also preserve some of the nice features of the Gamma processes used for their construction. They are pure jump Lévy processes of finite variation which may jump infinitely many times on a finite time interval. We show that LG processes are intrinsically related to the so called Dirichlet splines and polynomial B-splines, and posses some of their interesting geometric properties. In particular, we give explicit expressions, in terms of multivariate Dirichlet (B-) splines, of the joint density of the LG distribution, generating multivariate LG processes. Dirichlet splines, which have been independently introduced by Karlin et al. (1986) and by Ignatov and Kaishev (1987, 1988, 1989) who call them *generalized B-splines*, are densities of linear transformations of Dirichlet random variables. When the shape parameters of the underlying Gamma processes are integer, the corresponding LG density is expressed in terms of multivariate simplex splines, introduced by De Boor (1976). We give also some new expressions, in terms of univariate Dirichlet (B-) splines, for the densities of the VG and BG distributions. The proposed approach allows for the uniform treatment of the wide class of LG processes in terms of multivariate Dirichlet (B-) splines for which methods of their efficient numerical evaluation exist (see Section 3).

The structure of the paper is as follows. In section 2, we introduce univariate LG processes, note their relation to the Variance Gamma and Bilateral Gamma processes, explore their distributional properties and give the Lévy triplet and martingale conditions, which characterize them. In section 3, we introduce the multivariate version of an LG process, establish expressions in terms of multivariate Dirichlet (B-) splines for the joint density of its underlying joint LG distribution, give its underlying LG copula, its characteristic, moment and cumulant generating functions. We also provide a method of moments, based on expressing them in terms of cumulants, for estimating the LG parameter. In Section 4 we illustrate how the multivariate LG processes are applied in modelling the dynamics of the joint movement of the exchange rates of a set of currencies. Section 5 provides conclusions and some further comments.

2 Linear combinations of Gamma (LG-) processes

Our aim here will be to consider a new class of stochastic processes, defined as linear combinations of independent Gamma processes and explore their distributional and pathwise properties.

For the purpose, denote by $G_i(t;\alpha_i,\lambda)$, $i=0,\ \dots,\ n$ a collection of $n+1$ independent *Gamma* processes, defined on a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$, with mean rate $\alpha_i/\lambda>0$ and variance rate $\alpha_i/\lambda^2>0$, where $\alpha_i>0$ and $\lambda>0$, $i=0,\ \dots,\ n$. For a fixed t, $t>0$, the density of $G_i(t;\alpha_i,\lambda)$ is

$$f_{G_i}(x;\alpha_i,\lambda,t)=\frac{\lambda^{\alpha_i t}}{\Gamma(\alpha_i t)}\,x^{\alpha_i t-1}\,e^{-\lambda x},$$

where $x>0$. Let us recall that the Gamma process, $G_i(t;\alpha_i,\lambda)$, is a pure jump, finite variation process which jumps infinitely many times up to time t and has independent, gamma distributed increments. It plays a central role in contemporary financial modelling. For a detailed account on the properties of Gamma processes and their application in finance and insurance, we refer to Yor (2007), Fu (2007), Dufresne et al. (1991), Dickson and Waters (1993), Madan et al. (1998). We will use the gamma processes, $G_i(t;\alpha_i,\lambda)$, $i=0,\ \dots,\ n$, as building blocks and define the process of interest in this paper, as follows

Definition 1. Given a set of real valued distinct parameters $\delta=\{\delta_0,\ \dots,\ \delta_n\}$, define the process $LG(t;\delta,\alpha,\lambda,n)$ as a linear combination of the independent gamma processes, $G_i(t;\alpha_i,\lambda)$, $i=0,\ \dots,\ n$, i.e.,

$$LG(t;\delta,\alpha,\lambda,n)=\delta_0\,G_0(t;\alpha_0,\lambda)+\dots+\delta_n\,G_n(t;\alpha_n,\lambda), \tag{1}$$

where $\alpha=\{\alpha_0,\ \dots,\ \alpha_n\}$. For the sake of brevity we call such linear combinations, *LG processes*.

In what follows we will sometimes abbreviate $LG(t;\delta,\alpha,\lambda,n)$, to $LG(t)$ and the two notations will be used interchangeably.

Let us note that the three parameter Variance Gamma process, introduced by Madan et al. (1998), is a special case of an LG process. To see this recall that the VG process, $VG(t;\theta,\sigma,\nu)$ is defined as

$$VG(t;\theta,\sigma,\nu)=B\left(G\left(t;\ \frac{1}{\nu},\ \frac{1}{\nu}\right);\theta,\sigma\right),$$

where $B(t;\theta,\sigma)$ is a Brownian motion with drift $\theta\in\mathbb{R}$ and volatility $\sigma>0$, and $G\left(t;\ \frac{1}{\nu},\ \frac{1}{\nu}\right)$ is a Gamma process with mean rate 1 and variance rate $\nu>0$. It is not difficult to see that the VG process admits the alternative, LG representation

$$VG(t;\theta,\sigma,\nu)=\delta_0\,G_0(t;\alpha_0,1)+\delta_1\,G_1(t;\alpha_1,1), \tag{2}$$

where $\delta_0=-\frac{\sqrt{\theta^2+2\sigma^2/\nu}-\theta}{2}\,\nu$, $\delta_1=\frac{\sqrt{\theta^2+2\sigma^2/\nu}+\theta}{2}\,\nu$; $\alpha_0=\alpha_1=\frac{1}{\nu}$, which is a special case of an $LG(t;\delta,\alpha,\lambda,n)$ process with $\lambda=1$ and $n=1$.

Equality (2) follows from the fact that the characteristic function of the VG process (see Madan et al. 1998), can be expressed as

$$\phi_{VG(t)}(u) = \left(\frac{1}{1 - i\,\theta\,\nu\,u + \frac{1}{2}\,\sigma^2\,\nu\,u^2} \right)^{\frac{1}{\nu}\,t}$$

$$= \left(\frac{1}{1 + i\,u\,|\delta_0|} \right)^{\alpha_0\,t} \left(\frac{1}{1 - i\,u\,\delta_1} \right)^{\alpha_1\,t} = \mathbb{E}\left[e^{i\,u\,\{\delta_0\,G_0(t;\alpha_0,1) + \delta_1\,G_1(t;\alpha_1,1)\}} \right],$$

where $|\delta_0|$ is the absolute value of δ_0. Furthermore, a linear combination of say, p, VG processes is also a LG process, i.e.,

$$\mathrm{VG}_1(t;\,\theta_0,\,\sigma_1,\,\nu_1) + \ldots + \mathrm{VG}_{p-1}(t;\,\theta_{p-1},\,\sigma_{p-1},\,\nu_{p-1}) = \mathrm{LG}(t;\,\delta,\,\alpha,\,1,\,2\,p)$$

where $\delta = \{\delta_0,\,\ldots,\,\delta_{2\,p-1}\}$, $\alpha = \{\alpha_0,\,\ldots,\,\alpha_{2\,p-1}\}$ and $\delta_j = -\frac{\sqrt{\theta_j^2 + 2\,\sigma_j^2/\nu_j}\, - \theta_j}{2}\,\nu_j$ $j = 0,\,\ldots,\,p-1$, $\delta_j = \frac{\sqrt{\theta_{j-p}^2 + 2\,\sigma_{j-p}^2/\nu_{j-p}}\, + \theta_{j-p}}{2}\,\nu_{j-p}$, $j = p,\,\ldots,\,2\,p-1$ and $\alpha_j = \alpha_{p+j} = 1/\nu_j,\ j = 0,\,\ldots,\,p-1$.

It can be shown that the *Bilateral Gamma* (BG) process, recently considered by Küchler and Tappe (2008), is also a special case of an LG process. The BG processes are associated with the bilateral gamma distribution, $\Gamma(\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^-)$, with parameters $\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^- > 0$, defined as the convolution

$$\Gamma(\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^-) := \Gamma(\alpha^+,\,\lambda^+) * \Gamma(\alpha^-,\,-\lambda^-),$$

where $\Gamma(\alpha\,,\,\lambda)$ is a generalized Gamma distribution with parameters $\alpha > 0,\ \lambda \in \mathbb{R}\setminus\{0\}$. The density of $\Gamma(\alpha\,,\,\lambda)$ is given by

$$f_{BG}(x;\,\alpha,\,\lambda) = \frac{|\lambda|^\alpha}{\Gamma(\alpha)}\,|x|^{\alpha-1}\,e^{-|\lambda|\,|x|}(\mathbb{1}_{\{\lambda>0\}}\,\mathbb{1}_{\{x>0\}} + \mathbb{1}_{\{\lambda<0\}}\,\mathbb{1}_{\{x<0\}}), \tag{3}$$

where $x \in \mathbb{R}$ and $\mathbb{1}_{\{\cdot\}}$ is the indicator function. As can be seen from (3), when $\lambda > 0$, this is the well-known Gamma distribution, concentrating mass on $\mathbb{R}_+$, whereas, for $\lambda < 0$, the generalized Gamma distribution is simply a Gamma distribution on the negative half axis, $\mathbb{R}_-$. The corresponding bilateral gamma process, $\mathrm{BG}(t;\,\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^-)$ is a pure jump Lévy process, whose increments have bilateral gamma distribution and in particular, for fixed t, $t > 0$,

$$\mathrm{BG}(t;\,\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^-) \sim \Gamma(\alpha^+\,t,\,\lambda^+,\,\alpha^-\,t,\,\lambda^-).$$

For further properties of the BG distribution and processes, and some applications in finance, we refer to Küchler and Tappe (2008).

It is directly verified that, the BG process is a four parameter generalization of the VG process and admits the following representation as an LG process

$$\mathrm{BG}(t;\, \alpha^+,\, \lambda^+,\, \alpha^-,\, \lambda^-) = \delta_0\, G_0(t;\, \alpha_0,\, 1) + \delta_1\, G_1(t;\, \alpha_1,\, 1),$$

where $\delta_0 = -1/\lambda^-$; $\delta_1 = 1/\lambda^+$; $\alpha_0 = \alpha^-$, $\alpha_1 = \alpha^+$, $\lambda = 1$ and $n = 1$. As in the case of VG, linear combinations of BG processes are also LG processes, i.e.,

$$\mathrm{BG}_1(t;\, \alpha_0^+,\, \lambda_0^+,\, \alpha_0^-,\, \lambda_0^-) + \ldots + \mathrm{BG}_p\big(t;\, \alpha_{p-1}^+,\, \lambda_{p-1}^+,\, \alpha_{p-1}^-,\, \lambda_{p-1}^-\big) = \mathrm{LG}(t;\, \delta,\, \alpha,\, 1,\, 2\,p),$$

where $\delta = \left\{-1/\lambda_0^-,\, \ldots,\, -1\big/\lambda_{p-1}^-,\, 1\big/\lambda_0^+,\, \ldots,\, 1\big/\lambda_{p-1}^+\right\}$ and $\alpha = \left\{\alpha_0^-,\, \ldots,\, \alpha_{p-1}^-,\, \alpha_0^+,\, \ldots,\, \alpha_{p-1}^+\right\}$.

2.1 Distributional properties

From Definition 1, for fixed t, say $t = 1$, it is directly seen that the characteristic function, $\phi_{\mathrm{LG}}(z) = \mathbb{E}\left[e^{i\,z\,\mathrm{LG}(t)}\right]$, of a LG process is given by

$$\phi_{\mathrm{LG}}(z) = \prod_{j=0}^{n}\left(\frac{\lambda}{\lambda - i\,\delta_j\,z}\right)^{\alpha_j},\ \ z \in \mathbb{R}.$$

The cumulant generating function, $\Psi(u) = \ln \mathbb{E}\left[e^{u\,\mathrm{LG}(t)}\right]$, $u \in \mathbb{R}$ is

$$\Psi(u) = \sum_{j=0}^{n} \alpha_j \ln \frac{\lambda}{\lambda - \delta_j\,u},$$

where

$$\frac{\lambda}{\max_{j\in D_-}\{\delta_j\}} < u < \frac{\lambda}{\max_{j\in D_+}\{\delta_j\}},$$

$$D_- = \ \{i \in I : \mathrm{sgn}(\delta_i) = -1\},\ \ D_+ = \{i \in I : \mathrm{sgn}(\delta_i) = +1\},\ I = \{1,\ \ldots,\ n\}.$$

The cumulants $\kappa_w = \Psi^{(w)}(0)$, where

$$\Psi^{(w)}(u) = (w-1)!\sum_{j=0}^{n} \alpha_j\,\delta_j^w\,(\lambda - \delta_j\,u)^{-w},\ \ w = 1,\ 2,\ \ldots$$

are then obtained as

$$\kappa_w = (w-1)!\sum_{j=0}^{n} \frac{\alpha_j}{\lambda^w}\,\delta_j^w,\ \ w = 1,\ 2,\ \ldots. \tag{4}$$

We can now use (4) and specify the mean, μ_{LG}, the variance, ν_{LG}, the Charliers skewness, χ_{LG}, and the kurtosis, τ_{LG}, of LG (t) as

$$\mathbb{E}[\mathrm{LG}(t)] = \mu_{LG} = \kappa_1 = \sum_{i\in D_+} \frac{\alpha_i\,\delta_i}{\lambda} - \sum_{i\in D_-} \frac{\alpha_i\,|\delta_i|}{\lambda},$$

$$\mathrm{Var}[\mathrm{LG}(t)] = \nu_{LG} = \kappa_2 = \sum_{i\in D_+} \frac{\alpha_i\,\delta_i^2}{\lambda^2} + \sum_{i\in D_-} \frac{\alpha_i\,|\delta_i|^2}{\lambda^2}$$

$$\chi_{LG} = \kappa_3 \Big/ (\kappa_2)^{3/2} = \sum_{j=0}^{n} 2\,\alpha_j\,\delta_j^3\,\lambda^{-3} \Bigg/ \left(\sum_{j=0}^{n} \alpha_j\,\delta_j^2\,\lambda^{-2}\right)^{3/2}$$

$$\tau_{LG} = 3 + \kappa_4 \Big/ (\kappa_2)^2 = 3 + \sum_{j=0}^{n} 6\,\alpha_j\,\delta_j^4\,\lambda^{-4} \Bigg/ \left(\sum_{j=0}^{n} \alpha_j\,\delta_j^2\,\lambda^{-2}\right)^{2}.$$

Let us now give an expression for the density of $LG\,(t)$. For the purpose, we will need some notation and background results. Denote by

$$S_n = \left\{\boldsymbol{x} = (x_1,\ \ldots,\ x_n) \in \mathbb{R}^n : x_i \geq 0, \text{ for all } i,\ \sum_{i=1}^{n} x_i \leq 1\right\},$$

the standard n-simplex and recall that the random vector $(\theta_0,\ ...,\ \theta_n)$, has Dirichlet distribution $\mathfrak{D}(\alpha_0,\ ...,\ \alpha_n)$ on S_n, with (real) parameters $\alpha_0 > 0,\ ...,\ \alpha_n > 0$, i.e., $(\theta_0,\ ...,\ \theta_n) \in \mathfrak{D}(\alpha_0,\ ...,\ \alpha_n)$, if $\theta_0 = 1 - \theta_1 - ... - \theta_n$ and the joint probability density of $\theta_1,\ ...,\ \theta_n$ with respect to the Lebesgue measure is

$$f_{\theta_1,\ldots,\theta_n}(\boldsymbol{x}) = \frac{\Gamma(\alpha_0 + ... + \alpha_n)}{\prod_{i=0}^{n} \Gamma(\alpha_i)} \prod_{j=0}^{n} x_j^{\alpha_j - 1}\, \mathbb{1}_{\{\boldsymbol{x}\in S_n\}},$$

where $x_0 = 1 - x_1 - \ldots - x_n$. We will use the shorter notation $(\theta_0,\ ...,\ \theta_n) \in \mathfrak{D}(1)$ if $\alpha_j = 1,\ j = 0,\ \ldots,\ n$. We will now establish the following property of a LG process, which will be used in the sequel.

Lemma 1. For a fixed t, $t > 0$, the process $LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$, defined in (1), admits the representation

$$LG(t;\,\delta,\,\alpha,\,\lambda,\,n) = B(t)\,\Gamma(t), \tag{5}$$

where $\Gamma(t) = \sum_{i=0}^{n} G_i(t;\,\alpha_i,\,\lambda)$, $B(t) = \delta_0\,\theta_0 + ... + \delta_n\,\theta_n$ and the random variables $\theta_0,\ ...,\ \theta_n$, have a Dirichlet distribution $\mathfrak{D}(\alpha_0\,t,\ ...,\ \alpha_n\,t)$ with (real) parameters $\alpha_0\,t > 0$, ..., $\alpha_n\,t > 0$, i.e., $(\theta_0,\ ...,\ \theta_n) \in \mathfrak{D}(\alpha_0\,t,\ ...,\ \alpha_n\,t)$ and $B(t)$ is independent of $\Gamma(t)$.

Proof. Representation (5) follows from the fact that, for fixed t, the r.v.s $\theta_0, \ldots, \theta_n$, coincide in distribution with the random variables $G_i(t; \alpha_i, \lambda) / \Gamma(t)$, $i = 0, \ldots, n$ (see e.g. Wilks 1962), and by the theorem of Sukhatme (1937), the latter are independent of $\Gamma(t)$ which yields the independence of $B(t)$ and $\Gamma(t)$.□

Lemma 1 is fundamental in the study of LG processes since it links their underlying LG distribution to the classical polynomial splines and in general to the so called, generalized B-splines (known also as Dirichlet splines). This link, as will be demonstrated, provides a different, spline-approximation insight into the distributional properties of LG processes. It is interesting, both from the theoretical and numerical point of view, since the theory of polynomial spline functions is well developed (see e.g. Schumaker 1981) and offers also numerically efficient recurrence formulas for the evaluation of (B-)splines (see De Boor 2001) which, as we will see, can be useful in dealing with LG distributions.

In order to follow the link of the distribution of $LG(t; \delta, \alpha, \lambda, n)$ to splines, provided by Lemma 1, let us first note that, for integer values of the parameters $\alpha_0 t > 0, \ldots, \alpha_n t > 0$, the density, $f_{B(t)}(x)$, of the random variable, $B(t)$, coincides with a polynomial B-spline. This is an important probabilistic interpretation of B-splines, established independently by Ignatov and Kaishev (1985, 1989) and Karlin et al. (1986). In order to give a more precise formulation of this result, which will be used in the sequel, let us recall some background properties of polynomial B-splines. Let $\delta = \{\delta_0, \ldots, \delta_n\}$ denote a set of distinct real values, called knots of the spline and denote by $\alpha = \{\alpha_0, \ldots, \alpha_n\}$ the set of their corresponding integer-valued multiplicities. The multiplicity $\alpha_i = 1, 2, \ldots$, equals the number of repetitions of the knot δ_i in the set of possibly coincident knots of the spline. Let us recall that the polynomial B-spline $M\left(x; \begin{matrix}\delta_0, & \ldots, & \delta_n \\ \alpha_0, & \ldots, & \alpha_n\end{matrix}\right)$ of order $r = \alpha_0 + \ldots + \alpha_n - 1$ (degree $r-1$) with knots $\delta = \{\delta_0, \ldots, \delta_n\}$ of multiplicities $\alpha = \{\alpha_0, \ldots, \alpha_n\}$ coincides with a polynomial of degree $r-1$ between its adjacent (distinct) knots and is defined as the r-th order divided difference of the function $f(y) = r(y-x)_+^{r-1}$, i.e.,

$$M\left(x; \begin{matrix}\delta_0, & \ldots, & \delta_n \\ \alpha_0, & \ldots, & \alpha_n\end{matrix}\right) = [\delta_0, \ldots, \delta_n]\, f(y).$$

The B-spline $M\left(x; \begin{matrix}\delta_0, & \ldots, & \delta_n \\ \alpha_0, & \ldots, & \alpha_n\end{matrix}\right)$ has the following explicit representations. If knots are pairwise distinct, i.e., their multiplicities $\alpha_0 = 1, \ldots, \alpha_n = 1$, then

$$M\left(x; \begin{matrix}\delta_0, & \ldots, & \delta_n \\ 1, & \ldots, & 1\end{matrix}\right) = n \sum_{i=0}^{n} (\delta_i - x)_+^{n-1} \Big/ \prod_{j=0, j\neq i}^{n} (\delta_i - \delta_j)^{n-1}$$

If some of the knots coincide, i.e., $\alpha_0 \geq 1, \ ..., \ \alpha_n \geq 1$, then

$$M\left(x; \begin{matrix}\delta_0, & ..., & \delta_n \\ \alpha_0, & ..., & \alpha_n\end{matrix}\right) = \sum_{i=0}^{n} D^{\alpha_i - 1}\, \xi_i(\delta_i)\,/\,(\alpha_i - 1)\,!,$$

where $\xi_i(y) = r(y-x)_+^{d-1} \Big/ \prod_{\substack{j=0 \\ j\neq i}}^{n} (y-\delta_j)^{n-1}$ and $D^{\alpha_i - 1}$ denotes the $(\alpha_i - 1)$-th derivative.

The following theorem, due to Ignatov and Kaishev (1989) establishes an important probabilistic interpretation of polynomial B-splines which we will use to study the distributional properties of LG processes.

Theorem 1. (Ignatov and Kaishev 1989). The polynomial B-spline $M\left(x; \begin{matrix}\delta_0, & ..., & \delta_n \\ \alpha_0, & ..., & \alpha_n\end{matrix}\right)$ of degree $\alpha_0 + ... + \alpha_n$ coincides with the density $f_B(x)$, with respect to the Lebesgue measure of the random variable

$$B = \delta_0\, \theta_0 + ... + \delta_n\, \theta_n,$$

where the random variables $\theta_0, \ ..., \ \theta_n$ have joint Dirichlet distribution with parameters, $\alpha_0, \ ..., \ \alpha_n$, i.e., $(\theta_0, \ ..., \ \theta_n) \in \mathfrak{D}(\alpha_0, \ ... \ \alpha_n)$.

Let us note that the Dirichlet parameters $\alpha_0, \ ..., \ \alpha_n$, may in general take real values. In this case the density, $f_B(x)$, has been viewed by Ignatov and Kaishev (1987, 1988) as a generalized B-spline. Independently, Karlin et al. (1986) have also considered similar generalization of B-splines. Later, such densities have been named Dirichlet splines (see Neuman 1994 and zu Castell 2002). Here and thereafter, we will use the two terms, generalized B-splines and Dirichlet splines interchangeably. For consistency with the polynomial B-spline notation, we will alternatively denote, $f_B(x)$ as $M_g\left(x; \begin{matrix}\delta_0, & ..., & \delta_n \\ \alpha_0, & ..., & \alpha_n\end{matrix}\right)$, to stress its interpretation as a generalized B-spline i.e. a Dirichlet spline. We will make use of the following properties of generalized B-splines.

Denote by $\delta = \{\delta_0, \ ..., \ \delta_n\}$, the set of distinct knots, $\delta_i \in \mathbb{R}$, and by $\alpha = \{\alpha_0, \ ..., \ \alpha_n\}$ the set of (positive real) multiplicities of $\delta = \{\delta_0, \ ..., \ \delta_n\}$. Denote also by $\hat{\alpha}_i$, the integer part of α_i, and by $\overline{\alpha}_i = \alpha_i - \hat{\alpha}_i$, its fractional part. Without loss of generality, assume that, $\overline{\alpha}_i > 0$, $i = 0, \ \ldots, \ r$ and that, and $\overline{\alpha}_i = 0$, $i = r+1, \ \ldots, \ r+m$, $(n = r+m)$.

The generalized B-spline can be expressed as the following divided difference (see Ignatov and Kaishev 1988)

$$M_g\left(x; \begin{matrix}\delta_0, & ..., & \delta_n \\ \alpha_0, & ..., & \alpha_n\end{matrix}\right) = \begin{cases} \left[\begin{matrix}\delta_0, & ..., & \delta_r, & \delta_{r+1},, & ..., & \delta_{r+m} \\ \hat{\alpha}_0, & ..., & \hat{\alpha}_r, & \alpha_{r+1}, & & \alpha_{r+m}\end{matrix}\right] H(u)\,, & \text{if } x \in [\![\mathcal{D}]\!] \\ 0 & , \quad \text{otherwise} \end{cases}$$

where

$$H(u) = \frac{\Gamma(\alpha_0 + \ldots + \alpha_{r+m})}{\Gamma(l-1)\,\Gamma(\overline{\alpha}_0)\ \ldots\ \Gamma(\overline{\alpha}_r)} \int_{S_r} \left(u - x + \sum_{i=0}^{r} (\delta_i - u)\, y_i\right)_+^{l-2} y_0^{\overline{\alpha}_0 - 1} \ldots y_r^{\overline{\alpha}_r - 1}\, d y_0 \ldots d y_r,$$

$l = \sum_{i=0}^{r+m} \hat{\alpha}_i$, $(l \geq 2)$, $S_r = \{(y_0,\ \ldots,\ y_r) : 0 \leq y_i,\ i = 0,\ \ldots,\ r,\ y_0 + \ldots y_r \leq 1\}$ and $\mathcal{D}$ is the set of all δ_i's for which $\hat{\alpha}_i \geq 1$, $[\![\mathcal{D}]\!]$ denotes the convex hull of $\mathcal{D}$.

The numerical evaluation of generalized B-splines is facilitated by their representation in terms of classical polynomial B-splines, due to Kaishev (1991). For further properties of generalized B-splines (i.e. Dirichlet splines) we refer to Neuman (1994) and zu Castell (2002).

We can now formulate and prove the following proposition which expresses the density of $LG(t)$ in terms of Dirichlet splines.

Proposition 1. For fixed t, the density, $f_{LG(t)}(x)$, of $LG(t;\, \delta,\, \alpha,\, \lambda,\, n)$ is given by

$$f_{LG(t)}(x) = \int_0^{+\infty} \frac{\lambda^{(\alpha_0 + \ldots + \alpha_n)\, t}}{\Gamma((\alpha_0 + \ldots + \alpha_n)\, t)}\, y^{(\alpha_0 + \ldots + \alpha_n)\, t - 2}\, e^{-\lambda y}\, M_g\left(\begin{array}{c} \frac{x}{y};\ \delta_0,\ \ldots,\ \delta_n \\ \alpha_0\, t,\ \ldots,\ \alpha_n\, t \end{array}\right) d y, \tag{6}$$

where $M_g\left(\begin{array}{c} \frac{x}{y};\ \delta_0,\ \ldots,\ \delta_n \\ \alpha_0\, t,\ \ldots,\ \alpha_n\, t \end{array}\right)$, is a Dirichlet spline with knots, $\delta_0,\ \ldots,\ \delta_n$, of (real) multiplicities, $\alpha_0\, t,\ \ldots,\ \alpha_n\, t$.

Proof. By Lemma 1, we have that $LG(t;\, \delta,\, \alpha,\, \lambda,\, n)$ is expressed as a product of two independent random variables with known densities. More precisely, the random variable, $\Gamma(t) = \sum_{i=0}^{n} G_i(t;\, \alpha_i,\, \lambda)$, is gamma distributed with parameters $(\alpha_0 + \ldots + \alpha_n)\, t$ and λ, i.e., $\Gamma(t) \sim \text{Gamma}((\alpha_0 + \ldots + \alpha_n)\, t,\ \lambda)$, whereas, by Theorem 1, the density $f_{B(t)}(x)$, of the random variable, $B(t)$, coincides with a generalized B-spline. We will denote the density of $\Gamma(t)$, as $f_{\Gamma(t)}(x)$.

Thus, we have

$$\begin{aligned} f_{LG(t)}(x) &= \frac{d}{d x}\, P(B(t)\, \Gamma(t) \leq x) \\ &= \frac{d}{d x}\, P(B(t) \leq x / \Gamma(t)) \\ &= \frac{d}{d x} \int_0^{+\infty} P(B(t) \leq x / y)\, f_{\Gamma(t)}(y)\, d y \\ &= \int_0^{+\infty} f_{B(t)}(x / y)\, f_{\Gamma(t)}(y)\, \frac{1}{y}\, d y\, . \end{aligned}$$

The result now follows, noting that

$$f_{\Gamma(t)}(y) = \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\,t}}{\Gamma((\alpha_0+\ldots+\alpha_n)\,t)}\; y^{(\alpha_0+\ldots+\alpha_n)\,t-1}\, e^{-\lambda\, y}, \tag{7}$$

and that $f_{B(t)}(x/y)$ coincides with a generalized B-spline, $M_g\left(x/y;\; \underset{\alpha_0 t,}{\delta_0,}\; \underset{\ldots,}{\ldots,}\; \underset{\alpha_n t}{\delta_n}\right)$. □

Several properties of the process $LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$ easily follow from Proposition 1.

Corollary 1. If $\alpha_i\, t$ are integer valued, the density, $f_{LG(t)}(x)$, of $LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$ is given by

$$f_{LG(t)}(x) = \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\,t}}{\Gamma((\alpha_0+\ldots+\alpha_n)\,t)}\; y^{(\alpha_0+\ldots+\alpha_n)\,t-2}\, e^{-\lambda\, y}\, M\left(\frac{x}{y};\; \underset{\alpha_0 t,}{\delta_0,}\; \underset{\ldots,}{\ldots,}\; \underset{\alpha_n t}{\delta_n}\right) d\,y, \tag{8}$$

where $M\left(\frac{x}{y};\; \underset{\alpha_0 t,}{\delta_0,}\; \underset{\ldots,}{\ldots,}\; \underset{\alpha_n t}{\delta_n}\right)$, is a polynomial B-spline with knots, $\delta_0,\; \ldots,\; \delta_n$, of multiplicities, $\alpha_0\, t,\; \ldots,\; \alpha_n\, t$.

Corollary 2. The density of the increments, $LG(t+h;\,\delta,\,\alpha,\,\lambda,\,n) - LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$, $h > 0$ is given by

$$\int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\,h}}{\Gamma((\alpha_0+\ldots+\alpha_n)\,h)}\; y^{(\alpha_0+\ldots+\alpha_n)\,h-2}\, e^{-\lambda\, y}\, M_g\left(\frac{x}{y};\; \underset{\alpha_0 h,}{\delta_0,}\; \underset{\ldots,}{\ldots,}\; \underset{\alpha_n h}{\delta_n}\right) d\,y.$$

Proof. We have

$$\begin{gathered} LG(t+h;\,\delta,\,\alpha,\,\lambda,\,n) - LG(t;\,\delta,\,\alpha,\,\lambda,\,n) = \\ \delta_0[G_0(t+h;\,\alpha_0,\,\lambda) - G_0(t;\,\alpha_0,\,\lambda)] + \ldots + \delta_n[G_n(t+h;\,\alpha_n,\,\lambda) - G_n(t;\,\alpha_n,\,\lambda)], \end{gathered}$$

which, for fixed t and $h > 0$, is a linear combination of gamma variates $g_i = [G_i(t+h;\,\alpha_i,\,\lambda) - G_i(t;\,\alpha_i,\,\lambda)]$ with density

$$f_{g_i}(x;\,\alpha_i,\,\lambda,\,h) = \frac{\lambda^{\alpha_i\, h}}{\Gamma(\alpha_i\, h)}\; x^{\alpha_i\, h-1}\, e^{-\lambda\, x}.$$

Obviously, for fixed t and $h > 0$ we can write

$$LG(t+h;\,\delta,\,\alpha,\,\lambda,\,n) - LG(t;\,\delta,\,\alpha,\,\lambda,\,n) = \left[(\delta_0\, g_0 + \ldots + \delta_n\, g_n) \Big/ \sum_{i=0}^{n} g_i\right] \left[\sum_{i=0}^{n} g_i\right],$$

Hence, the Corollary follows in view of the independence of $\left[\sum_{i=0}^{n} g_i\right]$ from $\left[(\delta_0\, g_0 + \ldots + \delta_n\, g_n) \big/ \sum_{i=0}^{n} g_i\right]$, by the theorem of Sukhatme (1937). □

We conclude this section by noting that the following proposition which is a direct consequence of the scaling property of the gamma distribution provides an alternative way of expressing the underlying LG distribution, as a linear combination of $n+1$ gamma variates with different shape and scale parameters.

Proposition 2. The process $LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$ admits the representation

$$LG(t;\,\delta,\,\alpha,\,\lambda,\,n) = \operatorname{sgn}(\delta_0)\,G_0(t;\,\alpha_0,\,\lambda/\,|\,\delta_0\,|) + \ldots + \operatorname{sgn}(\delta_n)\,G_n(t;\,\alpha_n,\,\lambda/\,|\,\delta_n\,|). \tag{9}$$

It has to be noted that extensive literature exists which deals with the distribution underlying (11), in the special case when $\operatorname{sgn}(\delta_j) = +1$, $j = 0, \ldots, n$. In the latter case, an explicit formula for the density of $LG(t;\,\delta,\,\alpha,\,\lambda,\,n)$ when t is fixed, $t > 0$, is given by Moschopoulos (1985).

2.2 The Variance Gamma and the Bilateral Gamma special cases

New expressions for the density of the Variance Gamma, $VG(t;\,\theta,\,\sigma,\,\nu)$ and the Bilateral Gamma processes directly follow from their LG representation, Proposition 1 and Corollary 1. We have

Corollary 3. For fixed t, the density, $f_{VG(t)}(x;\,\theta,\,\sigma,\,\nu)$, of the Variance Gamma process, $VG(t;\,\theta,\,\sigma,\,\nu)$ is given by

$$f_{VG(t)}(x;\,\theta,\,\sigma,\,\nu) = \int_0^{+\infty} \frac{1}{\Gamma(2\,t/\nu)}\, y^{2\,t/\nu-2}\, e^{-y}$$
$$M_g\left(\frac{x}{y};\, -\underset{t/\nu}{\frac{\sqrt{\theta^2 + 2\,\sigma^2/\nu} - \theta}{2}\,\nu},\, \underset{t/\nu}{\frac{\sqrt{\theta^2 + 2\,\sigma^2/\nu} + \theta}{2}\,\nu}\right) d\,y, \tag{10}$$

where $M_g\left(\frac{x}{y};\,\underset{t/\nu}{\cdot},\,\underset{t/\nu}{\cdot}\right)$ coincides with a classical polynomial B-spline of degree $\frac{2\,t}{\nu} - 2$ if $\frac{t}{\nu}$ is integer. Recall that a different expression for the density $f_{VG(t)}(x;\,\theta,\,\sigma,\,\nu)$, has been given by Madan et al. (1998) as follows

$$f_{VG(t)}(x;\,\theta,\,\sigma,\,\nu) = \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\,\pi\,y}}\, e^{-\frac{(x-\theta\,y)^2}{2\,\sigma^2\,y}}\, \frac{1}{\nu^{\frac{t}{\nu}}\,\Gamma(t/\nu)}\, y^{t/\nu-1}\, e^{-\frac{y}{\nu}}\, d\,y.$$

For the density of the Bilateral Gamma process we have ,

Corollary 4. For fixed, $t > 0$, the density, $f_{BG(t)}(x)$, of the Bilateral Gamma process, $\mathrm{BG}(t;\,\alpha^+,\,\lambda^+,\,\alpha^-,\,\lambda^-)$, is given by

$$f_{BG(t)}(x) = \int_0^{+\infty} \frac{1}{\Gamma((\alpha^- + \alpha^+)\,t)}\, y^{(\alpha^-+\alpha^+)\,t-2}\, e^{-y}\, M_g\left(\frac{x}{y};\, \underset{\alpha^-\,t}{-(\lambda^-)^{-1}},\, \underset{\alpha^+\,t}{(\lambda^+)^{-1}}\right) d\,y. \tag{11}$$

where $M_g\left(\frac{x}{y}; \underset{\alpha^- t}{\cdot}, \underset{\alpha^+ t}{\cdot}\right)$ coincides with a polynomial B-spline of degree $\alpha^- t + \alpha^+ t - 2$ if the parameters, $\alpha^- t$, $\alpha^+ t$, are integer. For comparison with (11), for $t = 1$, the density, $f_{BG(t)}(x)$ given by Küchler and Tappe (2008) is

$$f_{BG}(x) = \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{(\alpha^+ + \alpha^-)/2}\, \Gamma(\alpha^+)}\, x^{(\alpha_0+\alpha_1)/2-1}\, e^{-(x(\lambda^+ - \lambda^-))/2}\, W_{(\alpha^+ - \alpha^-)/2, (\alpha^+ + \alpha^- - 1)/2}(x(\lambda^+ + \lambda^-)),$$

where $W_{\omega,\mu}(z)$ is the Whittaker function defined as

$$W_{\omega,\mu}(z) = \frac{z^{\omega}\, e^{-z/2}}{\Gamma(\mu - \omega + 1/2)} \int_0^{+\infty} t^{\mu-\omega-1/2}\, e^{-t} \left(1 + \frac{t}{z}\right)^{\mu+\omega-1/2} dt$$

$$\text{for } \mu - \omega > -\frac{1}{2}.$$

In conclusion, let us note that expressions (6), (8) (10) and (11), involving generalized or polynomial B-splines, are numerically appealing, due to the recurrent computation of polynomial B-splines (see De Boor 1976) and the cubature formula for generalized B-splines (i.e. Dirichlet splines) in terms of polynomial B-splines, due to Kaishev (1991).

2.3 The Lévy triplet and related properties

As known, (see e.g. Cont and Tankov 2004, Section 3.4), the characteristic triplet, (γ, A, κ), i.e., the Lévy triplet of a (multivariate) Lévy process, comprised by, a (real) vector γ, a positive definite (covariance) matrix A and a positive measure κ, related to its Lévy-Itô decomposition, uniquely determines its distribution. Following the Lévy-Khinchin representation formula, it is possible to express the characteristic function, $\phi_{\mathrm{LG}}(z) = \mathbb{E}\left[e^{i z \mathrm{LG}(t)}\right]$, of a LG process, in terms of its corresponding Lévy triplet (γ, A, κ) and deduce some path-wise properties. The following Proposition gives the Lévy triplet of an LG process.

Proposition 3. *LG*$(t; \delta, \alpha, \lambda, n)$ is a Lévy process with characteristic triplet $(\gamma, 0, \kappa_{LG})$, where the Lévy measure $\kappa_{LG}(dx)$ is given by

$$\kappa_{LG}(dx) = \left(\sum_{i \in D_-} \frac{\alpha_i\, e^{-\lambda \frac{|x|}{|\delta_i|}}}{|x|} \mathbb{1}_{x<0} + \sum_{i \in D_+} \frac{\alpha_i\, e^{-\lambda \frac{x}{\delta_i}}}{x} \mathbb{1}_{x>0} \right) dx \tag{12}$$

with $D_- = \{i \in I : \mathrm{sign}(\delta_i) = -1\}$, $D_+ = \{i \in I : \mathrm{sign}(\delta_i) = +1\}$, $I = \{0, \ ..., n\}$ and

$$\gamma = \frac{1}{\lambda}\left(\sum_{i\in D_+}\alpha_i\,\delta_i\left(1-e^{-\frac{\lambda}{\delta_i}}\right)-\sum_{i\in D_-}\alpha_i\,|\delta_i|\left(1-e^{-\frac{\lambda}{|\delta_i|}}\right)\right)<\infty\,. \tag{13}$$

Proof. Since $LG(t;\delta,\alpha,\lambda,n)$ is defined as a linear combination of the gamma processes, $G_i(t;\alpha_i,\lambda)$, $i=0,\ \ldots,n$, which are Lévy processes, $LG(t;\delta,\alpha,\lambda,n)$ is also a Lévy process (see, e.g. Theorem 4.1 of Cont and Tankov 2004). Expression (12) for the Lévy measure $\kappa_{LG}\,dx$ follows from the additivity property of the Lévy measure (see e.g. Proposition 5.3, Theorem 4.1 and Example 4.1 of Cont and Tankov 2004) and representation (11), noting that the Lévy measure of the process $\beta_i(t)=\mathrm{sgn}(\delta_i)\,G_i(t;\alpha_i,\lambda/|\,\delta_i\,|)$ is

$$\kappa_{\beta_i}(dx)=\left(\frac{\alpha_i\exp(-\lambda\,|\,x\,|\,/\,|\,\delta_i\,|)}{|\,x\,|}\,\mathbb{1}_{\{x<0,\,\delta_i<0\}}+\frac{\alpha_i\exp(-\lambda\,x\,/\,|\,\delta_i\,|)}{x}\,\mathbb{1}_{x>0,\,\delta_i>0}\right)dx.$$

Clearly, there is no Brownian motion component in the definition of LG $(t;\ \delta,\ \alpha,\ \lambda,\ n)$, hence the second parameter of the characteristic triplet is 0.

Due to the fact that, the drift parameter of the gamma process, $G_i(t;\alpha_i,\lambda)$, is 0, from Corollary 3.1 of Cont and Tankov (2004), we have that

$$\gamma=\int_{|x|\le 1}x\,\kappa_{LG}(dx), \tag{14}$$

and by substituting (12) in (14) we obtain (13). □

From the analytical properties of its characteristic triplet, $(\gamma,0,\kappa_{LG})$, it is straightforward to deduce that the LG process has piece-wise constant trajectories, is a process of finite variation and infinite activity (i.e., may have infinitely many small jumps). These path-wise properties are in fact inherited from the gamma processes, underlying the definition of an LG process (see Definition 1). Let us also note that, the LG process offers extended flexibility in controlling its Lévy measure, $\kappa_{LG}(dx)$, compared to the VG and BG processes. In the case of an LG process, one can manipulate its parameters and alter its Lévy measure, $\kappa_{LG}(dx)$ so that the distribution of the size of only the positive jumps, or only the negative jumps changes, (see Fig. 1, right panel). This may often be desirable in practical applications, but is not possible for the VG process. Changing the VG parameters, θ, σ and ν, affects both the positive and the negative parts of its Lévy measure, which is illustrated in Fig. 1, left panel.

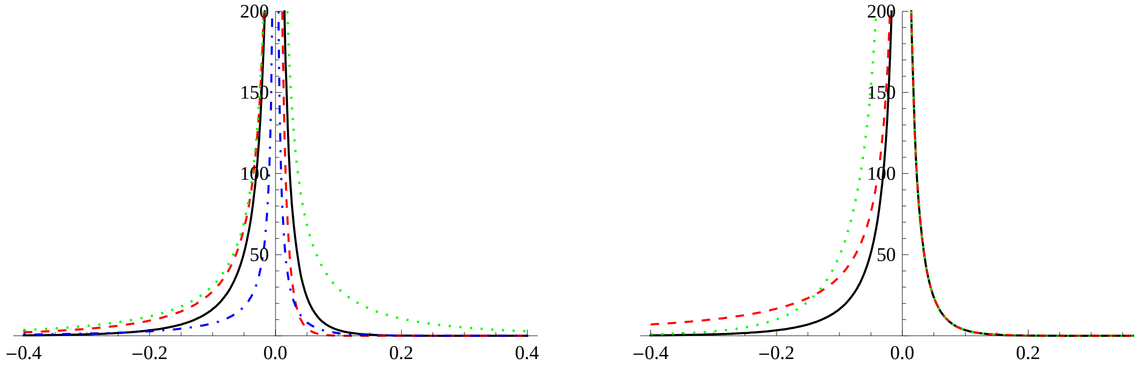

Fig. 1 left panel: Lévy measure of a VG process for the following four sets of parameters $\theta = -0.29$, $\sigma = 0.19$, $\nu = 0.25$ (solid line); $\theta = -0.99$, $\sigma = 0.19$, $\nu = 0.25$, (dashed); $\theta = -0.29$, $\sigma = 0.99$, $\nu = 0.25$ (dotted) and $\theta = -0.29$, $\sigma = 0.19$, $\nu = 0.95$ (dot-dashed); right panel: The LG Lévy measure for $\lambda = 1$**,** and the following three sets of parameters $\delta_0 = -0.11$, $\delta_1 = 0.04$, $\alpha_0 = \alpha_1 = 3.99$ (solid line); $\delta_0 = -1.13$, $\delta_1 = 0.04$, $\alpha_0 = \alpha_1 = 3.99$ (dashed); and $\delta_0 = -0.11$, $\delta_1 = 0.04$, $\alpha_0 = 11.98$, $\alpha_1 = 3.99$ (dotted);

As known, (see Proposition 3.18 of Cont and Tankov 2004) the exponent of a (univariate) Lévy process with characteristic triplet, (γ, A, κ) is a martingale if and only if, $\int_{|x|\geq 1} e^x \kappa(dx) < \infty$ and

$$\frac{A}{2} + \gamma + \int_{-\infty}^{+\infty} (e^x - 1 - x\, \mathbb{1}_{|x|\leq 1})\, \kappa_{LG}(dx) = 0. \tag{15}$$

Based on this result, Propositions 4 and 5 establish the conditions for the exponent of an LG process to be a martingale a property which is important in financial applications.

Proposition 4. Given $n \geq 1$, $\alpha_i > 0$, $\delta_i \neq 0$, $i = 0, \ldots, n$, $\int_{|x|\geq 1} e^x \kappa_{LG}(dx) < \infty$ if $\lambda > \max_{i \in D_+} \{\delta_i\}$.

Proof. It can be directly verified, substituting $\kappa_{LG}(dx)$ from (12) that, for $x > 0$,

$$\int_{|x|\geq 1} e^x \kappa_{LG}(dx) = \sum_{i \in D_+} \int_1^{+\infty} \frac{\alpha_i \exp[-x(\lambda/\delta_i - 1)]}{x}\, dx. \tag{16}$$

We have that,

$$\int_1^{+\infty} \frac{\alpha_i \exp[-x(\lambda/\delta_i - 1)]}{x}\, dx = \begin{cases} \alpha_i E_1(\lambda/\delta_i - 1) < \infty, & \text{if } \lambda > \delta_i \\ \text{diverges}, & \text{otherwise} \end{cases},$$

where $E_1(\lambda/\delta_i - 1)$ denotes the *Exponential Integral* (defined in section 5.1.4 of Abramowitz and Stegun 1972), evaluated at $\lambda/\delta_i - 1 > 0$, from where it can be seen that, in order for the sum in (16) to converge, the condition $\lambda > \max_{i \in D_+} \{\delta_i\}$ needs to be imposed.

Similarly, it can be verified that, for $x < 0$, we have that

$$\int_{|x|\geq 1} e^x \,\kappa_{LG}(dx) = \sum_{i\in D_-}\int_1^{+\infty}\frac{\alpha_i \exp[-|x|(\lambda/\delta_i| + 1)]}{|x|}\,dx = -\sum_{i\in D_-}\alpha_i\,\mathrm{Ei}(\lambda/\delta_i - 1) < \infty,$$

where $\mathrm{Ei}(\lambda/\delta_i - 1)$ denotes the Exponential Integral function (defined in section 5.1.2 of Abramowitz and Stegun 1972), evaluated at $\lambda/\delta_i - 1 < 0$, from where it can be seen that, in order for the sum in (16) to converge, no additional conditions on the parameters λ and δ_i need to be imposed. □

Proposition 5. There exist $n \geq 1$, $\alpha_i > 0$, $\delta_i \neq 0$, $i = 0, \ldots, n$ and $\lambda > \max_{i\in D_+}\{\delta_i\}$ such that, the $\exp(LG(t;\, \delta,\, \alpha,\, \lambda,\, n))$ is a martingale, i.e.,

$$\int_{-\infty}^{+\infty}(e^x - 1)\,\kappa_{LG}(dx) = 0. \tag{17}$$

Proof. Clearly, the necessary condition, $\lambda > \max_{i\in D_+}\{\delta_i\}$, established by Lemma 4, can be met for arbitrary positive real parameters $\{\delta_i\}$, $i \in D_+$. The necessary and sufficient condition (17), for $\exp(LG(t;\, \delta,\, \alpha,\, \lambda,\, n))$ to be a martingale, directly follows from (15) and (14), noting that, for a LG process, $A = 0$. For the integral in (17), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(e^x - 1)\,\kappa_{LG}(dx) = \\
&\int_{-\infty}^{0} x\sum_{j=0}^{\infty}\frac{x^j}{(j+1)!}\sum_{i\in D_-}\alpha_i\,\frac{e^{-\lambda\frac{|x|}{|\delta_i|}}}{|x|}\,dx + \int_{0}^{+\infty} x\sum_{j=0}^{\infty}\frac{x^j}{(j+1)!}\sum_{i\in D_+}\alpha_i\,\frac{e^{-\lambda\frac{x}{\delta_i}}}{x}\,dx = \\
&\sum_{i\in D_+}\alpha_i\sum_{j=0}^{\infty}\int_0^{+\infty}\frac{x^j}{(j+1)!}\,e^{-\lambda\frac{x}{\delta_i}}\,dx - \sum_{i\in D_-}\alpha_i\sum_{j=0}^{\infty}\int_{-\infty}^{0}\frac{x^j}{(j+1)!}\,e^{\lambda\frac{x}{|\delta_i|}}\,dx = \\
&\sum_{i\in D_+}\alpha_i\sum_{j=0}^{\infty}\left(\frac{\delta_i}{\lambda}\right)^{j+1}\frac{1}{j+1} - \sum_{i\in D_-}\alpha_i\sum_{j=0}^{\infty}(-1)^j\left(\frac{|\delta_i|}{\lambda}\right)^{j+1}\frac{1}{j+1} = \\
&-\sum_{i\in D_+}\alpha_i\ln\left(1-\frac{\delta_i}{\lambda}\right) - \sum_{i\in D_-}\alpha_i\ln\left(1+\frac{|\delta_i|}{\lambda}\right),
\end{aligned} \tag{18}$$

where $\ln\left(1 - \frac{\delta_i}{\lambda}\right)$ is well defined, given that, $\lambda > \max_{i\in D_+}\{\delta_i\}$, as required by Proposition 4. It is not difficult to see that the right-hand side of (18) vanishes if $n \geq 1$, the sets D^- and D^+ have equal cardinality and if

$$-\alpha_{i^-}\ln\left(1+\frac{|\delta_{i^-}|}{\lambda}\right) = \alpha_{i^+}\ln\left(1-\frac{\delta_{i^+}}{\lambda}\right),$$

where $i^- \in D^-$ and $i^+ \in D^+$, which holds true if

$$-\alpha_{i^-} = \alpha_{i^+},\ \ |\delta_{i^-}| = -\frac{\delta_{i^+}\,\lambda}{\delta_{i^+} - \lambda} \text{ and } \lambda > \max_{i\in D_+}\{\delta_i\}. \tag{19}$$

Hence, for a fixed $n \geq 1$, one can always chose a set $\{\delta_i\}$, $i \in D_+$ and select values, λ, $|\delta_{i^-}|$, $i^- \in D_-$, and α_i, $i \in I$, according to (19), so that (18) vanishes, which completes the proof of the asserted existence. □

Remark 1. Propositions 4 and 5 state that, it is possible to select the parameters n, δ, α and λ of a LG process in such a way that the exponent, $\exp(LG(t;\, \delta,\, \alpha,\, \lambda,\, n))$, is a martingale. However, it is not difficult to see from the LG representation, (3), of a VG process that, there does not exist a set of VG parameters, $(\theta,\, \sigma,\, \nu)$ for which $\exp(VG(t;\, \theta,\, \sigma,\, \nu))$ is a martingale.

We conclude this section by briefly indicating that the (univariate) LG process can be used for modelling asset price dynamics. Define the (risk-neutral) asset price process, $S(t)$ as

$$S(t) = S(0)\exp((r - q + \omega)\, t + LG(t;\, \delta,\, \alpha,\, \lambda,\, n)) \tag{20}$$

where r - the (constant) risk-free rate, q - the dividend yield, and the constant ω is chosen so that $\mathbb{E}(S(t)) = S(0)\exp((r - q)\, t)$, i.e.

$$\omega = \sum_{i\in D_+} \alpha_i \log\left(1 - \frac{\delta_i}{\lambda}\right) + \sum_{i\in D_-} \alpha_i \log\left(1 + \frac{|\delta_i|}{\lambda}\right) \tag{21}$$

which follows from Proposition 5. We therefore require $\lambda > \max_{i\in D_+}\{\delta_i\}$. Note that, in the special case of the $VG(t;\, \theta,\, \sigma,\, \nu)$ process (21) yields

$$\omega = \frac{1}{\nu}\log\left(1 - \theta\,\nu - \frac{\sigma^2\,\nu}{2}\right),$$

where $1 > \frac{\sqrt{\theta^2 + 2\,\sigma^2 / \nu} + \theta}{2}\,\nu$ (which implies $1 > \left(\theta + \sigma^2 \big/ 2\right)\nu$).

The model given by (20) can be used in (exotic) option pricing and pricing participating life insurance contracts. Due to volume limitations, details of how this is done are outside the scope of this paper and will appear elsewhere.

3 Multivariate LG processes

In what follows we will consider the multivariate generalization of univariate LG processes, defined in Section 2, which, as we will illustrate in Section 4, can be very useful in modelling the joint dynamics of possibly dependent prices of multiple assets. We start with the following definition.

Definition 2. Define the multivariate LG process, $\mathbf{LG}(t) = (\mathrm{LG}_1(t), \ldots, \mathrm{LG}_s(t))'$, $(s \geq 1)$ as

$$\begin{aligned} \mathrm{LG}_1(t) &= \delta_{1,0}\, G_0(t;\alpha_0,\lambda) + \ldots + \delta_{1,n}\, G_n(t;\alpha_n,\lambda) \\ &\vdots \\ \mathrm{LG}_s(t) &= \delta_{s,0}\, G_0(t;\alpha_0,\lambda) + \ldots + \delta_{s,n}\, G_n(t;\alpha_n,\lambda), \end{aligned} \tag{22}$$

where $\boldsymbol{\delta_j} = (\delta_{1,j}, \ldots, \delta_{s,j})'$, $\boldsymbol{\delta_j} \in \mathbb{R}^s$, $j = 0, \ldots, n$, are pairwise distinct, $n \geq s$, $\lambda > 0$, $\alpha = \{\alpha_0, \ldots, \alpha_n\}$, $\alpha_j > 0$, $j = 0, \ldots, n$ and $G_j(t;\alpha_j,\lambda)$, $j = 0, \ldots, n$ are independent *Gamma* processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Multivariate LG processes are illustrated graphically in Fig. 2 where we have simulated sample paths from two and three dimensional LG processes with coordinates

$$\begin{aligned} LG_1(t) &= -5\,G_0(t;1,20) - 2\,G_1(t;10,20) + 2\,G_2(t;30,20), \\ LG_2(t) &= -2\,G_0(t;1,20) + 1\,G_1(t;10,20) + 2\,G_2(t;30,20), \end{aligned}$$

with $\mu_{LG_1} = 1.75$, $\nu_{LG_1} = 0.46$, $\mu_{LG_2} = 3.40$, $\nu_{LG_2} = 0.34$ and

$$\begin{aligned} LG_1(t) &= -5\,G_0(t;1,20) - 2\,G_1(t;10,20) + 2\,G_2(t;30,20), \\ LG_2(t) &= -6\,G_0(t;1,20) - 3\,G_1(t;10,20) + 1\,G_2(t;30,20), \\ LG_3(t) &= -3\,G_0(t;1,20) - 1\,G_1(t;10,20) + 1\,G_2(t;30,20), \end{aligned}$$

with $\mu_{LG_1} = 1.75$, $\nu_{LG_1} = 0.46$, $\mu_{LG_2} = -0.3$, $\nu_{LG_2} = 0.39$, and $\mu_{LG_3} = 0.85$, $\nu_{LG_3} = 0.12$ respectively, where μ_{LG_i}, ν_{LG_i} $i = 1, 2, 3$ are the corresponding (marginal) mean and variance rates. As can be seen from Fig. 2 all coordinates jump together, which is a consequence of the fact that for a fixed t, a multivariate LG process represents a linear transformation of a set of Gamma processes, in this case these are $G_0(t;1,20)$, $G_1(t;10,20)$ and $G_2(t;30,20)$. It should also be noted that simulation of a multivariate LG is straightforward since it requires simulating Gamma sample paths which is done very efficiently, applying the Dirichlet bridge sampling method, recently proposed by Kaishev and Dimitrova (2009).

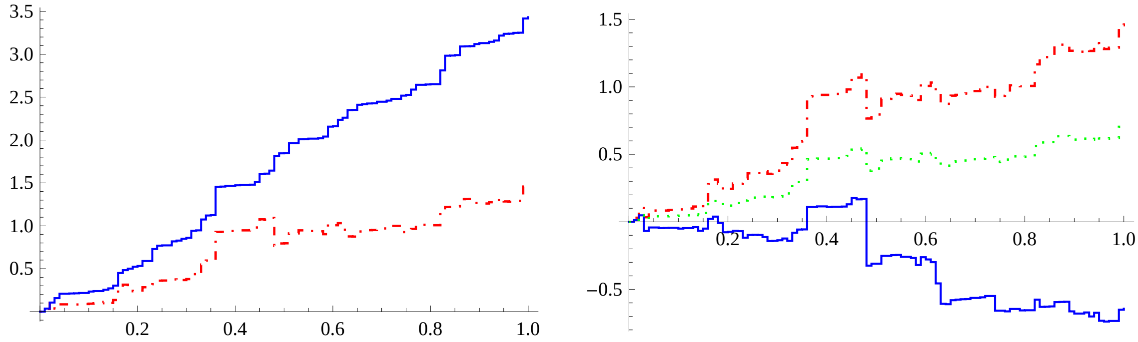

Fig. 2 Sample paths of a two and three-dimensional LG processes.

Before proceeding further, we will need to introduce the following notation. For a given set $A \subset \mathbb{R}^s$, $\mathbb{1}_A(x)$, $[\![A]\!]$, $\mathrm{vol}_s(A)$, $\dim(A)$ denotes the indicator function, the closed convex hull, the s-dimensional Lebesgue measure and the dimension respectively. By $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, … we denote elements (vectors) in the Euclidean space $\mathbb{R}^s$ $(s \geq 1)$, i.e., $\boldsymbol{x} = (x_1, \ldots, x_s)'$ where, ', means transposition and we use subscripts to index vectors, i.e., $\boldsymbol{x}_j = (x_{1,j}, \ldots, x_{s,j})'$, $j = 0, 1, \ldots$. We denote by $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{s} x_i y_i$ the inner product of $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^s$.

3.1 Distributional properties

In what follows, we study distributional properties of multivariate LG processes and establish their relation to multivariate splines. For the purpose we will need to introduce multivariate B-splines, known also as simplex splines. A simplex spline is a multivariate version of the univariate polynomial B-spline defined in Section 2.1. Simplex splines, were first introduced by De Boor (1976) as follows.

Definition 3. (De Boor 1976)**.** Let $\mathfrak{S} = [\![\boldsymbol{y}_0, \ldots, \boldsymbol{y}_r]\!]$ be any r-simplex in $\mathbb{R}^r$, $\mathbb{R}^r = \mathbb{R}^s \times \mathbb{R}^{r-s}$, such that $\boldsymbol{y}_j\,|_{\mathbb{R}^s} = \boldsymbol{\delta}_j$, $j = 0, \ldots, r$, i.e., the first s coordinates of $\boldsymbol{y}_j$ agree with the vector $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $s \geq 1$. The multivariate (simplex) spline $M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r)$ is defined as

$$M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r) = \mathrm{vol}_{r-s}(\{\boldsymbol{u} \in \mathfrak{S} : \boldsymbol{u}\,|_{\mathbb{R}^s} = \boldsymbol{x}\}) / \mathrm{vol}_r(\mathfrak{S}).$$

Note that Definition 3 allows for coalescent knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r$ of which say, $n + 1 < r + 1$ knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ may be distinct with corresponding multiplicities $\alpha_0, \ldots, \alpha_n$. If there are $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ pairwise distinct knots with multiplicities $\alpha_0, \ldots, \alpha_n$, $M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r)$, will be alternatively denoted as $M\left(\boldsymbol{x}; \underset{\alpha_0,}{\boldsymbol{\delta}_0}, \ldots, \underset{\alpha_n}{\boldsymbol{\delta}_n}\right)$, $v \in \mathbb{R}^s$ and also as $M\left(x_1, \ldots, x_s; \underset{\alpha_0,}{\boldsymbol{\delta}_0}, \ldots, \underset{\alpha_n}{\boldsymbol{\delta}_n}\right)$.

The simplex spline, $M(\boldsymbol{x}; \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r)$ is a piecewise polynomial of total degree not exceed ing $r - s$ with $r - s - 1$ continuous derivatives when the knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r$ are in general position. The knots, $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_r$, are said to be in general position if for $j = 1, \ldots, s$ and for arbitrary, different indexes $0 \leq i_1, \ldots, i_{j+1} \leq r$, we have

$$\det\begin{pmatrix} 1 & \delta_{1,i_1} & \ldots & \delta_{j,i_1} \\ 1 & \delta_{1,i_2} & \ldots & \delta_{j,i_2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \delta_{1,i_{j+1}} & \ldots & \delta_{j,i_{j+1}} \end{pmatrix} \neq 0.$$

The numerical evaluation of multivariate simplex splines is facilitated by the following recurrence relation, due to Micchelli (1980)

$$M(\boldsymbol{x};\,\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_r) = \frac{r}{r-s}\sum_{j=0}^{r}\lambda_j\, M(\boldsymbol{x};\,\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_{j-1},\ \boldsymbol{\delta}_{j+1},\ \dots,\ \boldsymbol{\delta}_r), \tag{23}$$

whenever $r > s$ and the numbers, $\lambda_j \in \mathbb{R}$, are such that, $\boldsymbol{x} = \sum_{j=0}^{r} \lambda_j\, \boldsymbol{\delta}_j$, $\sum_{j=0}^{r} \lambda_j = 1$.

For further properties of simplex splines see e.g., Neamtu (2001), Cohen et al. (2001) and Prautzsch et al. (2002).

We will now recall that simplex splines have a nice probabilistic interpretation established independently by Karlin et al. (1986) and Ignatov and Kaishev (1985, 1989) which we will exploit in studying the properties of multivariate LG processes. Given the set of knots $\Delta = \{\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_r\}$, $\boldsymbol{\delta}_j = (\delta_{1,j},\ \dots,\ \delta_{s,j})'$, $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $j = 0,\ \dots,\ r$, consider the random vector $\boldsymbol{B} = (B_1,\ \dots,\ B_s)'$, defined by

$$\boldsymbol{B} = \boldsymbol{\delta}_0\, \theta_0 + \dots + \boldsymbol{\delta}_r\, \theta_r, \tag{24}$$

with coordinates $B_i = \delta_{i,0}\, \theta_0 + \dots + \delta_{i,r}\, \theta_r$, $i = 1,\ \dots,\ s$, where the random vector $\boldsymbol{\theta} = (\theta_0,\ \dots,\ \theta_r)'$, is Dirichlet distributed with parameters $\alpha = \{1,\ \dots,\ 1\}$, i.e. $(\theta_0,\ \dots,\ \theta_r) \in \mathfrak{D}(1)$.

It will be convenient to view the vectors $\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_r$ as points in $\mathbb{R}^s$, $s \geq 1$. Note that in (24), we allow some of the points $\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_n$ to coalesce. Let us assume that only $n+1$ of them are pairwise distinct, say $\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_n$, each repeated with multiplicity $\alpha_0,\ \dots,\ \alpha_n$, $\alpha_0 + \dots + \alpha_n = r+1$. Then, given the set of distinct knot parameters, $\Delta = \{\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_r\}$, following a well known property of the Dirichlet distribution (see e.g., Wilks 1962), the random vector $\boldsymbol{B} = (B_1,\ \dots,\ B_s)'$, defined by (24), can be rewritten as

$$\boldsymbol{B} = \boldsymbol{\delta}_0\, \theta_0 + \dots + \boldsymbol{\delta}_n\, \theta_n, \tag{25}$$

with coordinates $B_i = \delta_{i,0}\, \theta_0 + \dots + \delta_{i,n}\, \theta_n$, $i = 1,\ \dots,\ s$, where the random vector $\boldsymbol{\theta} = (\theta_0,\ \dots,\ \theta_n)'$, is Dirichlet distributed with parameters $\alpha = \{\alpha_0,\ \dots,\ \alpha_n\}$, i.e., $(\theta_0,\ \dots,\ \theta_n) \in \mathfrak{D}(\alpha_0,\ \dots,\ \alpha_n)$.

Assume also that the parameters α, Δ, r, and n, are such that the distribution of the linear transformation $\boldsymbol{B}$ and its marginal distributions exist and are non-degenerate. Denote by $f_{\boldsymbol{B}}(\boldsymbol{x})$ the density of $\boldsymbol{B}$. The following result establishes the probabilistic interpretation of simplex splines.

Theorem 2. (Ignatov and Kaishev 1985, 1989). Let $\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_n$ be fixed pairwise distinct vectors in $\mathbb{R}^s$, $n \geq s$, with dimension $\dim([\![\{\boldsymbol{\delta}_0,\ \dots,\ \boldsymbol{\delta}_n\}]\!]) = s$, then the density $f_{\boldsymbol{B}}(\boldsymbol{x})$ with respect to the s-dimensional Lebesgue measure of the random vector $\boldsymbol{B}$, defined as in (25), coincides with the simplex spline

$$M\left(\boldsymbol{x};\, \underset{\alpha_0,}{\boldsymbol{\delta}_0},\, \underset{\ldots,}{\ldots},\, \underset{\alpha_n}{\boldsymbol{\delta}_n}\right)$$

with knots $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$ having (integer) multiplicities, $\alpha_0, \ldots, \alpha_n$, $\alpha_0 + \ldots + \alpha_n = r + 1$.

As in the univariate case, the Dirichlet parameters $\alpha_0, \ldots, \alpha_n$, may in general take real values. In this case the density, $f_{\boldsymbol{B}}(\boldsymbol{x})$, has been viewed by Ignatov and Kaishev (1987, 1988) as a multivariate generalized B-spline i.e., as multivariate Dirichlet spline. Independently, Karlin et al. (1986) have also considered similar generalization of multivariate simplex splines. For some further properties of multivariate Dirichlet splines see Karlin et al. (1986), Ignatov and Kaishev (1987, 1988) and Neuman (1994).

The following proposition gives for fixed $t > 0$ an expression for the joint density of the multivariate LG process in terms of multivariate Dirichlet splines.

Proposition 6. Let $\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n$, $\boldsymbol{\delta}_j \in \mathbb{R}^s$, $n \geq s$, be pairwise distinct and let $dim\, [\![\{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}]\!] = s$, then the density of LG is

$$f_{LG}(x_1, \ldots, x_s) =$$
$$\int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\,t}}{\Gamma((\alpha_0+\ldots+\alpha_n)\,t)}\, y^{(\alpha_0+\ldots+\alpha_n)\,t-(s+1)}\, e^{-\lambda y}\, M_g\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y};\, \underset{\alpha_0 t,}{\boldsymbol{\delta}_0},\, \underset{\ldots,}{\ldots},\, \underset{\alpha_n t}{\boldsymbol{\delta}_n}\right) d\,y, \tag{26}$$

where $\Gamma(\cdot)$ is the gamma function and $M_g\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y};\, \underset{\alpha_0 t,}{\boldsymbol{\delta}_0},\, \underset{\ldots,}{\ldots},\, \underset{\alpha_n t}{\boldsymbol{\delta}_n}\right)$ is a multivariate Dirichlet spline with knots $\Delta = \{\boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_n\}$, of multiplicities $\{\alpha_0\, t, \ldots \alpha_n\, t\}$.

Proof. We have that the multivariate LG process can be represented as

$$\boldsymbol{LG}(t; \Delta, \alpha, \lambda, n) = \boldsymbol{B}(t) \times \Gamma(t) \tag{27}$$

where $\boldsymbol{B}(t)$ is defined as in (25) and has a joint density $f_{\boldsymbol{B}(t)}(\boldsymbol{x})$, which, by Theorem 2, coincides with a generalized B-spline, and where the random variable, $\Gamma(t) = \sum_{i=0}^{n} G_i(t; \alpha_i, \lambda)$, independent of $\boldsymbol{B}(t)$, is gamma distributed with parameters $(\alpha_0 + \ldots + \alpha_n)\, t$ and λ. Thus, we have

$$f_{LG}(x_1, \ldots, x_s) = \frac{\partial}{\partial x_1 \ldots \partial x_s} P(B_1(t) \times \Gamma(t) \leq x_1, \ldots, B_s(t) \times \Gamma(t) \leq x_s)$$
$$= \frac{\partial}{\partial x_1 \ldots \partial x_s} P\left(B_1(t) \leq \frac{x_1}{\Gamma(t)}, \ldots, B_s(t) \leq \frac{x_s}{\Gamma(t)}\right)$$
$$= \int_0^{+\infty} \frac{\partial}{\partial x_1 \ldots \partial x_s} P\left(B_1(t) \leq \frac{x_1}{y}, \ldots, B_s(t) \leq \frac{x_s}{y}\right)\right) f_{\Gamma(t)}(y)\, d\,y$$

$$= \int_0^{+\infty} f_{\boldsymbol{B}(t)}\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}\right) f_{\Gamma(t)}(y) \frac{1}{y^s} \, dy \, .$$

The result now follows, in view of (7) and noting that, by Theorem 2, $f_{\boldsymbol{B}(t)}\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}\right)$ coincides with a multivariate Dirichlet spline, $M_g\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,}{\boldsymbol{\delta}_0}, \underset{\ldots,}{\ldots,} \underset{\alpha_n t}{\boldsymbol{\delta}_n}\right)$. □

In case $\alpha_i t$, $i = 0, \ldots, n$, are integers then $M_g(\cdot)$ is a classical multivariate polynomial simplex spline, given by Definition 3 and its evaluation can be successfully performed using e.g. Michelli's recurrence (23). When the multiplicities $\alpha_i t$, $i = 0, \ldots, n$ are non-integer, to the best of our knowledge, the evaluation of multivariate Dirichlet splines has not been sufficiently explored. Recurrence formulas for the moments of multivariate Dirichlet splines and simplex splines have been established by Neuman (1994).

In order to provide some insight into the dependence properties of multivariate LG processes, next we give its underlying copula.

Proposition 7. The copula $C_{LG}(u_1, \ldots, u_s)$, is given as

$$C_{LG}(u_1, \ldots, u_s) =$$
$$\int_{-\infty}^{F^{-1}_{\mathrm{LG}_1}(u_1)} \cdots \int_{-\infty}^{F^{-1}_{\mathrm{LG}_s}(u_s)} \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\, t}}{\Gamma((\alpha_0+\ldots+\alpha_n)\, t)} \, y^{(\alpha_0+\ldots+\alpha_n)\, t-(s+1)} \, e^{-\lambda y} \tag{28}$$
$$M_g\left(\frac{x_1}{y}, \ldots, \frac{x_s}{y}; \underset{\alpha_0 t,}{\boldsymbol{\delta}_0}, \underset{\ldots,}{\ldots,} \underset{\alpha_n t}{\boldsymbol{\delta}_n}\right) dy \, dx_s \ldots dx_1,$$

where $u_i \in [0, 1]$, and

$$F_{\mathrm{LG}_i}(x) = \int_{-\infty}^{x} \int_0^{+\infty} \frac{\lambda^{(\alpha_0+\ldots+\alpha_n)\, t}}{\Gamma((\alpha_0+\ldots+\alpha_n)\, t)} \, y^{(\alpha_0+\ldots+\alpha_n)\, t-2} \, e^{-\lambda y} \, M_g\left(\frac{x}{y}; \underset{\alpha_0 t,}{\delta_{j,0}}, \underset{\ldots,}{\ldots,} \underset{\alpha_n t}{\delta_{j,n}}\right) dy \, dz,$$
$$i = 1, \ldots s.$$

Proof. Expression (28) follows from the Sklar's Theorem and expressions (26) and (6).□

Let us note that the LG copula $C_{LG}(u_1, \ldots, u_s)$ is related to the (new) class of the so-called Dirichlet (B-) spline copulas, introduced by Kaishev (2006 b). Both B-spline copulas and LG copulas are quite flexible, and by controlling the knots, Δ, of the Dirichlet spline, and their multiplicities, α, one can model and reproduce a wide range of dependence structures arising in financial applications. This is illustrated in section 4, on the example of multivariate FX modelling. For further results and applications of B-spline copulas see Kaishev (2006 b).

The next proposition gives the characteristic function of a multivariate LG process, which will be needed in order to develop a method of moments for estimating the LG parameters.

Proposition 8. The characteristic function, $\phi(z)$, of the multivariate LG process, $\mathbf{LG}(t) = (\mathrm{LG}_1(t), \ldots, \mathrm{LG}_s(t))'$ given by Definition 2 is

$$\phi_{\mathrm{LG}}(\boldsymbol{z}) = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - i\,(\boldsymbol{\delta}_j \cdot \boldsymbol{z})} \right)^{\alpha_j}, \tag{29}$$

where $\boldsymbol{\delta}_j = (\delta_{1,j}, \ldots, \delta_{s,j})' \in \mathbb{R}^s$, $j = 0, \ldots, n$, $\boldsymbol{z} = (z_1, \ldots, z_s)' \in \mathbb{R}^s$ and $\lambda > 0$.

Proof. From Definition 2, for fixed t, say $t = 1$ and $\boldsymbol{z} = (z_1, \ldots, z_s)' \in \mathbb{R}^s$, it is directly seen that the characteristic function

$$\begin{aligned}\phi_{\mathrm{LG}}(\boldsymbol{z}) &= \mathbb{E}\left[e^{i\,(\boldsymbol{z} \cdot \mathbf{LG}(t))}\right] = \mathbb{E}\left[e^{i\left(\sum_{j=0}^{n} \boldsymbol{\delta}_j \cdot \boldsymbol{z}\, G_j(t;\alpha_j,\lambda)\right)}\right] \\ &= \prod_{j=0}^{n} \mathbb{E}\left[e^{i\,\boldsymbol{\delta}_j \cdot \boldsymbol{z}\, G_j(t;\alpha_j,\lambda)}\right] = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - i\,(\boldsymbol{\delta}_j \cdot \boldsymbol{z})} \right)^{\alpha_j},\end{aligned}$$

which completes the proof of the asserted expression for $\phi_{\mathrm{LG}}(\boldsymbol{z})$. □

In order to develop a method of moments for estimating the LG parameters, we will give here the moment generating function (mgf)

$$M_{\mathrm{LG}}(\boldsymbol{z}) = \mathbb{E}\left[e^{\boldsymbol{z} \cdot \mathbf{LG}(t)}\right] = \prod_{j=0}^{n} \left(\frac{\lambda}{\lambda - \boldsymbol{\delta}_j \cdot \boldsymbol{z}} \right)^{\alpha_j},$$

and the cumulant generating function (cgf)

$$K_{\mathrm{LG}}(\boldsymbol{z}) = \log M_{\mathrm{LG}}(\boldsymbol{z}) = \sum_{j=0}^{n} -\alpha_j \log\left(1 - \frac{1}{\lambda}\, \boldsymbol{\delta}_j \cdot \boldsymbol{z}\right) \tag{30}$$

of the LG random vector.

3.2 LG parameter estimation: method of moments

There are two sources of difficulty related to estimating the parameters of a multivariate LG process, given an appropriate data set. Firstly, it is the curse of dimensionality, i.e., the dimension s may be very high which is typically the case in some credit risk modelling applications. Secondly, the underlying dependence pattern may be rather complex, requiring significant number, $n+1$ of knot parameters in each coordinate, and hence a large number of parameters overall.

Due to the latter difficulties, the calibration of a multivariate LG process, based on maximum likelihood estimation utilizing expression (26), is not so straightforward and may require, developing a special purpose optimization algorithm, using exhaustive numerical optimization methods such as, adapted simulated annealing. Development of such methods is outside our scope and will be a subject of another paper. Here we will develop a method of moments for the estimation of the LG parameters, which is simpler to implement and as will be illustrated in section 4, serves well the purpose of calibrating an FX model driven by a multivariate LG process.

In order to develop a method of moments for the estimation of LG parameters, we will need the following piece of general multivariate cumulant theory, provided by McCullagh (2008). In what follows we shall somewhat depart from the notation used so far and use the notationally convenient, Einstein's summation convention in order to denote scalar products. Thus, $z_r X_r$ denotes the linear combination $z_1 X_1 + \ldots + z_s X_s$, where X_i, $i = 1, \ldots, s$ are the coordinates of a random vector $\boldsymbol{X} = (X_1, \ldots, X_s)$. The square of a linear combination $(z_r X_r)^2 = (z_{r_1} X_{r_1})(z_{r_2} X_{r_2}) = z_{r_1} z_{r_2} X_{r_1} X_{r_2}$ is a sum of s^2 terms and for higher powers, $(z_r X_r)^l = z_{r_1} \ldots z_{r_2} X_{r_1} \ldots X_{r_l}$ is the sum of s^l terms. Following McCullagh (2008), we denote $\kappa_r = \mathbb{E}(X_r)$ the components of the mean vector, $\kappa_{r_1 r_2} = \mathbb{E}(X_{r_1} X_{r_2})$, $r_1, r_2 = 1, \ldots, s$ the components of the matrix of second moments, $\kappa_{r_1 r_2 r_3} = \mathbb{E}\left(X_{r_1} X_{r_2} X_{r_3}\right)$, $r_1, r_2, r_3 = 1, \ldots, s$, the elements of the third moment matrix and so on, for the elements of the matrices of higher order moments. The Taylor expansions of the moment generating function, $M_{\boldsymbol{X}}(\boldsymbol{z}) = \mathbb{E}\left[e^{z_r X_r}\right]$, and the cumulant generating function, $K_{\boldsymbol{X}}(\boldsymbol{z}) = \log M_{\boldsymbol{X}}(\boldsymbol{z})$, are then given as

$$M_{\boldsymbol{X}}(\boldsymbol{z}) = 1 + z_{r_1} \kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2} \kappa_{r_1 r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} \kappa_{r_1 r_2 r_3} + \ldots$$

and

$$K_{\boldsymbol{X}}(\boldsymbol{z}) = z_{r_1} \kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2} \kappa_{r_1, r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} \kappa_{r_1, r_2, r_3} + \ldots, \tag{31}$$

where κ_{r_1} denotes simultaneously first order moments and first order cumulants. The coefficients κ_{r_1, r_2}, κ_{r_1, r_2, r_3}, … in the expansion of $K_{\boldsymbol{X}}(\boldsymbol{z})$ are the corresponding second third and higher order cumulants. Note that, the latter are distinguished notationally from the corresponding moments, $\kappa_{r_1 r_2}$, $\kappa_{r_1 r_2 r_3}$, … by the commas separating subscripts. Equating the coefficients in the expansion of

$$K_{\boldsymbol{X}}(\boldsymbol{z}) = \log\left(1 + z_{r_1} \kappa_{r_1} + \frac{1}{2!} z_{r_1} z_{r_2} \kappa_{r_1 r_2} + \frac{1}{3!} z_{r_1} z_{r_2} z_{r_3} \kappa_{r_1 r_2 r_3} + \ldots\right)$$

to the corresponding coefficients in the expansion (31), it can be seen that each of the moments $\kappa_{r_1 r_2}$, $\kappa_{r_1 r_2 r_3}$, … can be expressed, as a sum over partitions of the subscripts, where each term in the sum is a product of cumulants, as follows

$$\kappa_{r_1 r_2} = \kappa_{r_1,r_2} + \kappa_{r_1} \kappa_{r_2} \tag{32}$$

$$\begin{gathered}\kappa_{r_1 r_2 r_3} = \\ \kappa_{r_1,r_2,r_3} + \kappa_{r_1,r_2} \kappa_{r_3} + \kappa_{r_1,r_3} \kappa_{r_2} + \kappa_{r_2,r_3} \kappa_{r_1} + \kappa_{r_1} \kappa_{r_2} \kappa_{r_3} = \kappa_{r_1,r_2,r_3} + \kappa_{r_1,r_2} \kappa_{r_3}[3] + \kappa_{r_1} \kappa_{r_2} \kappa_{r_3}\end{gathered} \tag{33}$$

$$\kappa_{r_1 r_2 r_3 r_4} = \kappa_{r_1,r_2,r_3,r_4} + \kappa_{r_1,r_2,r_3} \kappa_{r_4}[4] + \kappa_{r_1,r_2} \kappa_{r_3,r_4}[3] + \kappa_{r_1,r_2} \kappa_{r_3} \kappa_{r_4}[6] + \kappa_{r_1} \kappa_{r_2} \kappa_{r_3} \kappa_{r_4}, \tag{34}$$

where the numbers in the square brackets indicate a sum over distinct partitions of the subscripts, having the same block sizes. Note that there are s equations $\kappa_{r_1} = \kappa_{r_1}$ relating the first order moments to the first order cumulants. In general, there are s^k equations for the moments of order $k = 1, 2, \ldots$ however, there are only $\binom{s+k-1}{k}$ distinct equations which coincides with the number of distinct moments of order k. Equations, (32) - (34), have been given by McCullagh (2008). Here we further give the sets of equations, relating the fifth and the sixth order moments with the corresponding cumulants

$$\begin{gathered}\kappa_{r_1 r_2 r_3 r_4 r_5} = \kappa_{r_1,r_2,r_3,r_4,r_5} + \kappa_{r_1,r_2,r_3,r_4} \kappa_{r_5}[5] + \kappa_{r_1,r_2,r_3} \kappa_{r_4,r_5}[10] + \\ \kappa_{r_1,r_2,r_3} \kappa_{r_4} \kappa_{r_5}[5] + \kappa_{r_1,r_2} \kappa_{r_3,r_4} \kappa_{r_5}[15] + \kappa_{r_1,r_2} \kappa_{r_3} \kappa_{r_4} \kappa_{r_5}[10] + \kappa_{r_1} \kappa_{r_2} \kappa_{r_3} \kappa_{r_4} \kappa_{r_5},\end{gathered} \tag{35}$$

$$\begin{gathered}\kappa_{r_1 r_2 r_3 r_4 r_5 r_6} = \kappa_{r_1,r_2,r_3,r_4,r_5,r_6} + \kappa_{r_1,r_2,r_3,r_4,r_5} \kappa_{r_6}[6] + \\ \kappa_{r_1,r_2,r_3,r_4} \kappa_{r_5,r_6}[15] + \kappa_{r_1,r_2,r_3,r_4} \kappa_{r_5} \kappa_{r_6}[15] + \kappa_{r_1,r_2,r_3} \kappa_{r_4,r_5,r_6}[10] + \\ \kappa_{r_1,r_2,r_3} \kappa_{r_4,r_5} \kappa_{r_6}[60] + \kappa_{r_1,r_2,r_3} \kappa_{r_4} \kappa_{r_5} \kappa_{r_6}[20] + \kappa_{r_1,r_2} \kappa_{r_3,r_4} \kappa_{r_5,r_6}[15] + \\ \kappa_{r_1,r_2} \kappa_{r_3,r_4} \kappa_{r_5} \kappa_{r_6}[45] + \kappa_{r_1,r_2} \kappa_{r_3} \kappa_{r_4} \kappa_{r_5} \kappa_{r_6}[15] + \kappa_{r_1} \kappa_{r_2} \kappa_{r_3} \kappa_{r_4} \kappa_{r_5} \kappa_{r_6}.\end{gathered} \tag{36}$$

In what follows we will derive expressions for the cumulants of the random vector $\mathbf{LG}(t)$, in terms of the unknown parameters, Δ, α, λ and n. By substituting these expressions in the right-hand side of equations (32)-(36) and equating the theoretical moments, κ_{r_1}, $\kappa_{r_1 r_2}$, $\kappa_{r_1 r_2 r_3}$, … to their corresponding empirical counterparts, one can solve the appropriate set of equations and obtain estimates of the unknown parameters. In what follows we will elaborate further on the details related to this method. The following proposition gives an expression for the cumulants of $\mathbf{LG}(t)$ in terms of the unknown parameters, Δ, α, λ and n.

Proposition 9. The cumulant, $\kappa_{r_1,\ldots,r_w}$ of the random vector $\mathbf{LG}(t)$ is

$$\kappa_{r_1,\ldots,r_w} = (w-1)! \sum_{j=0}^{n} \frac{\alpha_j}{\lambda^w} \delta_{r_1,j} \delta_{r_2,j} \ldots \delta_{r_w,j}, \tag{37}$$

where $w = 1, 2, \ldots, r_i = 1,\ldots,s, i = 1, \ldots, w$.

Proof. The cgf of the random vector, $\mathbf{LG}(t)$ can be expressed as in (31). On the other hand, from (30), we have

$$
\begin{aligned}
K_{\mathrm{LG}}(\boldsymbol{z}) &= \sum_{j=0}^{n} \alpha_j \left(-\log\left(1 - \frac{1}{\lambda}\, \boldsymbol{\delta}_j \cdot \boldsymbol{z} \right) \right) \\
&= \sum_{j=0}^{n} \alpha_j \left(\frac{1}{\lambda}\, \boldsymbol{\delta}_j \cdot \boldsymbol{z} + \frac{1}{2} \left(\frac{1}{\lambda}\, \boldsymbol{\delta}_j \cdot \boldsymbol{z} \right)^2 + \frac{1}{3} \left(\frac{1}{\lambda}\, \boldsymbol{\delta}_j \cdot \boldsymbol{z} \right)^3 + \ldots \right) \\
&= \sum_{j=0}^{n} \left(\left(\frac{\alpha_j}{\lambda}\, \boldsymbol{\delta}_j \right) \cdot \boldsymbol{z} + \frac{1}{2} \left(\left(\frac{\alpha_j^{1/2}}{\lambda}\, \boldsymbol{\delta}_j \right) \cdot \boldsymbol{z} \right)^2 + \frac{1}{3} \left(\left(\frac{\alpha_j^{1/3}}{\lambda}\, \boldsymbol{\delta}_j \right) \cdot \boldsymbol{z} \right)^3 + \ldots \right) \\
&= \sum_{j=0}^{n} \left(\left(\frac{\alpha_j}{\lambda} \right) \delta_{r_1,j}\, z_{r_1} + \frac{1}{2} \left(\frac{\alpha_j^{1/2}}{\lambda} \right) \delta_{r_1,j}\, z_{r_1} \left(\frac{\alpha_j^{1/2}}{\lambda} \right) \delta_{r_2,j}\, z_{r_2} + \right. \\
&\qquad \left. \frac{1}{3} \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_1,j}\, z_{r_1} \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_2,j}\, z_2 \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_3,j}\, z_{r_3} + \ldots \right) \\
&= z_{r_1} \sum_{j=0}^{n} \left(\frac{\alpha_j}{\lambda} \right) \delta_{r_1,j} + \frac{1}{2!}\, z_{r_1} z_{r_2} \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/2}}{\lambda} \right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/2}}{\lambda} \right) \delta_{r_2,j} + \\
&\qquad \frac{1}{3!}\, z_{r_1} z_{r_2} z_{r_3}\, 2! \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_2,j} \left(\frac{\alpha_j^{1/3}}{\lambda} \right) \delta_{r_3,j} + \ldots
\end{aligned} \tag{38}
$$

Hence, comparing the coefficients of the corresponding terms z_{r_1}, $z_{r_1} z_{r_2}$, $z_{r_1} z_{r_2} z_{r_3}, \ldots$ in (31) and (38) we have

$$
\begin{aligned}
\kappa_{r_1,\ldots,r_w} &= (w-1)! \sum_{j=0}^{n} \left(\frac{\alpha_j^{1/w}}{\lambda} \right) \delta_{r_1,j} \left(\frac{\alpha_j^{1/w}}{\lambda} \right) \delta_{r_2,j} \ldots \left(\frac{\alpha_j^{1/w}}{\lambda} \right) \delta_{r_w,j} \\
&= (w-1)! \sum_{j=0}^{n} \frac{\alpha_j}{\lambda^w}\, \delta_{r_1,j}\, \delta_{r_2,j} \ldots \delta_{r_3,j},
\end{aligned}
$$

which coincides with the asserted expression (37). □

We can now use (37) in order to express the cumulants on the right hand side of equations (32)-(36) and therefore, express the theoretical moments, $\kappa^{r_1 \ldots r_w}$ in terms of the unknown parameters, Δ, α, λ and n. Then, equate the theoretical moments, $\kappa_{r_1 \ldots r_w}$ to their empirical counterparts and solve with respect to Δ, α, λ, assuming n is appropriately chosen. It will be instructive to make the definition of the moments, $\kappa_{r_1 \ldots r_w}$ a bit more precise.

Definition 4. For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)' \in \mathbb{N}^s$, define the moment, $\mathbb{E}\left(X_1^{\beta_1} X_2^{\beta_2} \ldots X_s^{\beta_s}\right)$, of order $|\boldsymbol{\beta}|$, ($|\boldsymbol{\beta}| = (\beta_1 + \ldots + \beta_s)$) of the random vector $\mathbf{LG}(t)$ as

$$\mathbb{E}\left(X_1^{\beta_1} X_2^{\beta_2} \ldots X_s^{\beta_s}\right) = \int_{\mathbb{R}^s} \boldsymbol{x}^{\boldsymbol{\beta}} f_{LG}(\boldsymbol{x})\, d\boldsymbol{x} = \kappa_{r_1 \ldots r_{\beta_1} r_{\beta_1+1} \ldots r_{\beta_1+\beta_2} \ldots r_{\beta_1+\ldots+\beta_{s-1}+1} \ldots r_{|\boldsymbol{\beta}|}},$$

where $\boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \ldots x_s^{\beta_s}$ and $r_1 = \ldots = r_{\beta_1} = 1$, $r_{\beta_1+1} = \ldots = r_{\beta_1+\beta_2} = 2$, $\ldots$, $r_{\beta_1+\ldots+\beta_{s-1}+1} = \ldots = r_{|\boldsymbol{\beta}|} = s$.

The empirical moments of $\mathbf{LG}$ (t) can now be defined as follows.

Definition 5. For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)' \in \mathbb{N}^s$, define the empirical moment $\hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}}$ of order $|\boldsymbol{\beta}|$, as

$$\hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}} = \frac{1}{N} \sum_{l=1}^{N} x_{1,l}^{\beta_1} \ldots, x_{s,l}^{\beta_s},$$

where $\{x_{1,l} \ldots, x_{s,l}\}_{l=1}^{N}$ is a sample of N i.i.d observations on $\mathbf{LG}$ (t).

In order to estimate the parameters $(\Delta, \alpha, \lambda)$, based on $\{x_{1,l} \ldots, x_{s,l}\}_{l=1}^{N}$, we apply the method of moments and solve

$$\left\{ \kappa_{r_1 \ldots r_{|\boldsymbol{\beta}|}}(\Delta, \alpha, \lambda) = \hat{\kappa}_{r_1 \ldots r_{|\boldsymbol{\beta}|}}, \quad |\boldsymbol{\beta}| = 1, 2, 3, \ldots \right. \tag{39}$$

with respect to $(\Delta, \alpha, \lambda)$.

Remark 2. Note that there are $\binom{s+k-1}{k}$ distinct moments of order $|\boldsymbol{\beta}| = k = 1, 2, 3, \ldots$. In an application, one would need to select the number of equations, p, in (39), to be equal to the number, $(s+1)\times(n+1)+1$, of unknown parameters, $(\Delta, \alpha, \lambda)$, starting from moments of order 1 and increasing up to a maximum order k^*, where $k^* = \inf\left\{k : p \le \sum_{j=1}^{k} \binom{s+j-1}{j}\right\}$.

The method of moments described here is illustrated in section 4 on the example of FX modelling.

4 Modelling the joint dynamics of exchange rates

Let $S_1(t), S_2(t), \ldots, S_s(t)$, $t \ge 0$, be the exchange rates of a set of s currencies against a common reference currency. We are interested in modelling the joint dynamics of $S_1(t), S_2(t), \ldots, S_s(t)$, over a finite time interval $[0, T]$. We view $S_j(t)$, as the price of a risky asset with dynamics

$$S_j(t) = S_j(0) \exp\{X_j(t)\}, \; j = 1, \ldots, s,$$

where $X_j(t)$, $j = 1, \ldots, s$ are the coordinates of an appropriate s-variate stochastic process driving the joint FX dynamics. In what follows we will compare and contrast the modelling results we obtain, under two alternative choices for the processes $X_j(t)$, $j = 1, \ldots, s$, a multivariate VG model proposed by Luciano and Schoutens (2006) and a multivariate LG process, defined as in Definition 2.

In Fig. 3, we give the (historic) joint co-movement of the exchange rates of three currencies ($s = 3$), the Euro (EUR), the GB Pound (GBP) and the Japanese Yen (JPY) to the US Dollar as the reference (domestic) currency for the period 30.06.2008-30.06.2009.

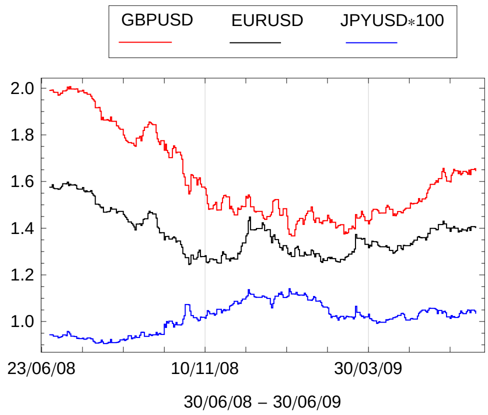

Fig. 3 Joint co-movement of the exchange rates of GBP/USD, EUR/USD, and JPY/USD, viewed from top to bottom.

As can be seen, examining Fig. 3 visually, there are different degrees of inter-dependence in the three FX trajectories. The exchange rates GBP/USD and EUR/USD exhibit stronger mutual correlation while at the same time, each of them is less correlated with the JPY/USD exchange rate. This is confirmed also if one analyses Fig. 4-5 which provide scatter plots and histograms of the corresponding log returns at unit time intervals, $\ln(S_j(t)/S_j(t-1))$, $t = 1, 2, \ldots$ given by

$$\ln(S_j(t)/S_j(t-1)) = X_j(t) - X_j(t-1) \overset{d}{\equiv} X_j.$$

Examining Fig. 5 one can see that the (marginal) distributions of the corresponding (historic) daily log returns seem to exhibit heavier tails than in the normal case, which is a bit more expressed for the EUR/USD and JPY/USD. As has been noted by Daal and Madan (2005), a univariateVG density is an appropriate choice for fitting empirical FX data.

The two dimensional scatter plots of the three pairs of log returns given in Fig. 4, show that the pair GBP/USD versus EUR/USD exhibits positive dependence with stronger upper tail dependence, the pair JPY/USD versus EUR/USD looks evenly scattered around the origin, while the pair JPY/USD versus GBP/USD looks somewhat negatively correlated.

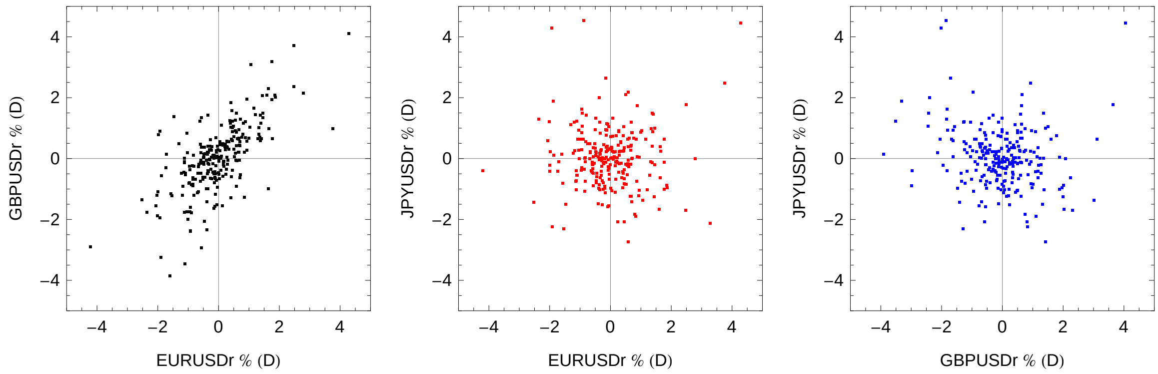

Fig. 4 Bilateral scatter plots of the empirical log returns EUR/USD, GBP/USD and JPY/USD.

First, we model the co-movement of the three FX rates, EUR/USD, GBP/USD and JPY/USD, indexed by $j = 1, 2, 3$ respectively, applying the multivariate VG process, proposed by Luciano and Schoutens (2006), as follows

$$S_j(t) = S_j(0)\exp\left\{m_j\, t + B\left(G\left(t; \frac{1}{\nu}, \frac{1}{\nu}\right); \theta_j, \sigma_j\right)\right\} = S_j(0)\exp\{m_j\, t + \mathrm{VG}_j(t; \theta_j, \sigma_j, \nu)\},$$

where $B\left(G\left(t; \frac{1}{\nu}, \frac{1}{\nu}\right); \theta_j, \sigma_j\right)$ is a Brownian motion with drift θ_j and volatility σ_j, for the j-th currency, $j = 1, 2, 3$, and $G\left(t; \frac{1}{\nu}, \frac{1}{\nu}\right)$ is a common Gamma process with mean rate 1 and variance rate ν, and m_j is a drift parameter. Obviously, there are 10 unknown parameters in total in this model and this is the maximum possible number of parameters for a three dimensional application ($s = 3$). Denote by $Z_j = \ln(S_j(t)/S_j(t-1))$ the corresponding log returns. Then, it is not difficult to see that the joint density of the log returns for the three currencies is given as

$$f_{Z_1,Z_2,Z_3}(z_1, z_2, z_3) = \int_0^\infty \frac{\nu^{-1/\nu}}{\Gamma(1/\nu)}\, y^{(1/\nu-1)}\, e^{-\frac{y}{\nu}} \prod_{j=1}^{3} \frac{1}{\sigma_j\sqrt{y}}\, \varphi\left(\frac{z_j - m_j - \theta_j\, y}{\sigma_j\sqrt{y}}\right) dy. \tag{40}$$

where $\varphi()$ is the standard normal pdf. In order to calibrate the three dimensional Luciano and Schoutens (2006) VG model, we have fixed $\nu = 1$ and have used the method of maximum likelihood in order to estimate the unknown parameters, m_j, θ_j, σ_j $j = 1, 2, 3$. In Fig. 5, we give the histograms of (historic) daily log returns and fitted marginal $\mathrm{VG}_j(t; \theta_j, \sigma_j, \nu)$ densities and, as can be seen, they fit reasonably well the data.

In Fig. 6 we give the two dimensional scatter plots simulated from the corresponding fitted three dimensional VG distributions. As can be seen from Fig 6 if one compares it with the corresponding empirical scatter plots of Fig 4, the 10 parameter, Luciano and Schoutens (2006) model fails to capture the underlying dependence in the data, especially for the EUR/USD, GBP/USD pair of currencies.

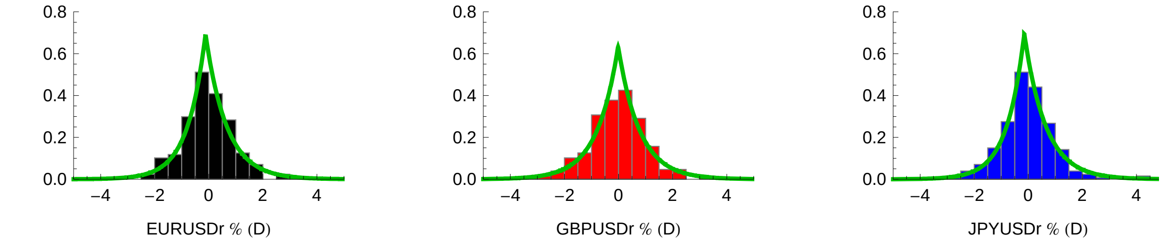

Fig. 5 Marginal VG densities fitted to (histograms of) historic daily log returns of the exchange rates of EUR/USD, GBP/USD and JPY/USD.

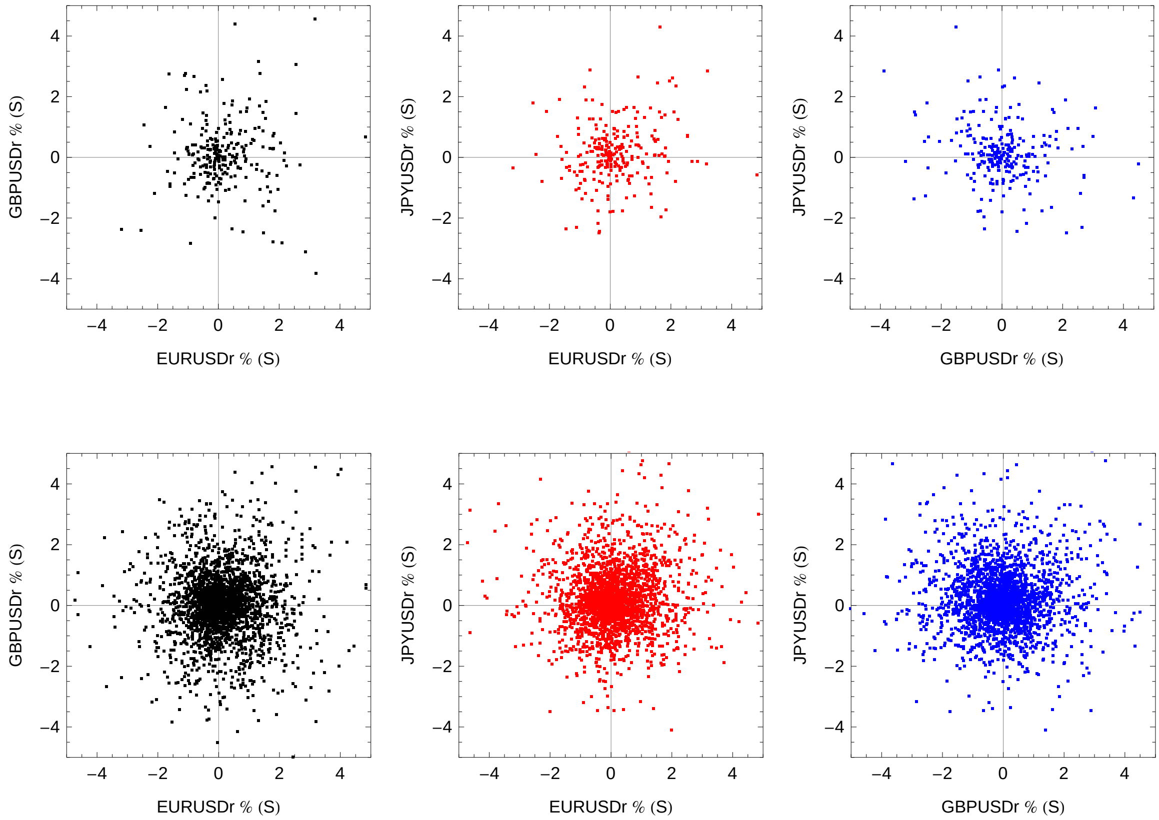

Fig. 6 Bilateral scatter plots of the simulated three dimensional VG log returns EUR/USD, GBP/USD and JPY/USD, upper panel: 254 simulated data points; lower panel: 2540 simulated data points.

Secondly, we model the co-movement of the three FX rates, EUR/USD, GBP/USD and JPY/USD, applying the multivariate LG process, proposed in this paper, as follows

$$S_j(t) = S_j(0)\exp(\mathrm{LG}_j(t;\, \delta_{j,0},\, \ldots,\, \delta_{j,n},\, \alpha,\, \lambda,\, n)), \tag{41}$$

where $j = 1, 2, 3$, $n = 3$, $\lambda = 1$, $\alpha = \{1, 1, 1, 1\}$ and $\delta_{j,0}, \ldots, \delta_{j,n}$, are the 12 knot parameters. Note that 12 is the minimum possible number of knot parameters, since four is the minimum number of knots which span a volume in $\mathbb{R}^3$ ($s = 3$). For the purpose of estimating, $\delta_{j,0}, \ldots, \delta_{j,3}$, $j = 1, 2, 3$, we consider the joint distribution of the corresponding log returns, Z_j , $j = 1, 2, 3$, which is a three dimensional LG distribution.

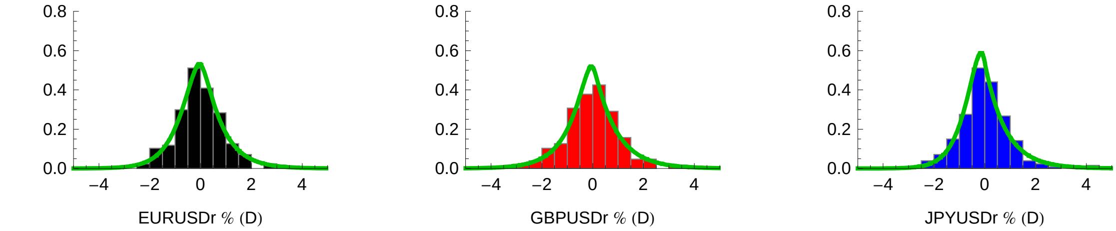

Fig. 7 Marginal LG densities fitted to (histograms of) historic daily log returns of the exchange rates of EUR/USD, GBP/USD and JPY/USD.

We have used the method of moments, developed in section 3.2, in order to estimate $\delta_{j,i}$, $j = 1, 2, 3,$ $i = 0, 1, 2, 3,$ by equating the first, second and third order theoretical moments, κ_{r_1}, $r_1 = 1, 2, 3,$ $\kappa_{r_1 r_2}$, $r_1 = 1, 2, 3,$ $r_2 = 1, 2, 3,$ $r_1 < r_2$, $\kappa_{r_1 r_2 r_3}$, $r_1 = r_2 = r_3 = 1, 2, 3$, of the random vector (Z_1, Z_2, Z_3), given by (32) and (33), to their empirical counterparts, following (39).

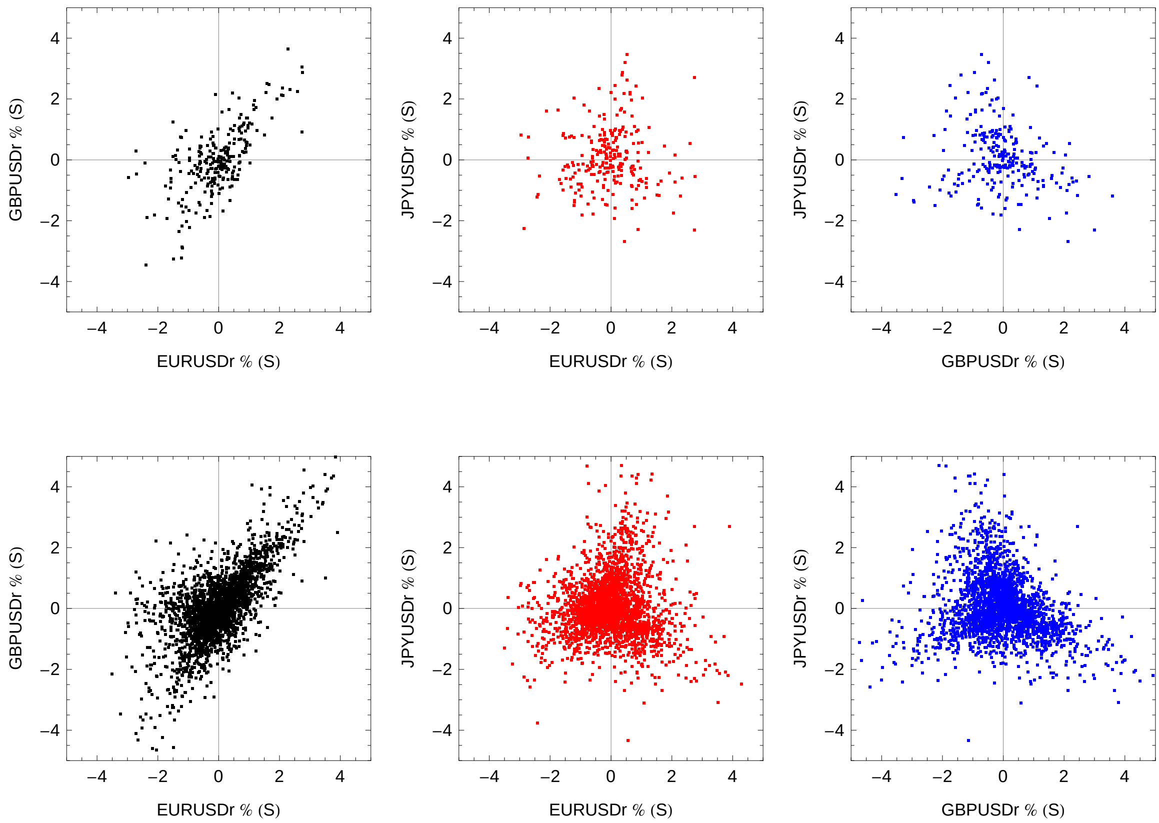

Fig. 8 Bilateral scatter plots of the simulated LG log returns EUR/USD, GBP/USD and JPY/USD, upper panel: 254 simulated data points, lower panel: 2540 simulated data points.

In Fig. 8, we give the two dimensional scatter plots simulated from the corresponding fitted three dimensional LG distributions.

Comparing the scatter plots from Fig. 8 with the corresponding empirical scatter plots of Fig. 4, the 12 parameter, LG model captures the underlying dependence in the data, both for GBP/USD, EUR/USD and GBP/USD, JPY/USD pairs of exchange rates. As can be seen from Fig. 8, the multivariate LG vector can take any value in $\mathbb{R}^3$ but the scatter plots reveal a triangular shape inherited from the domain of the three dimensional Dirichlet spline, namely the pyramid configuration defined by its four knots, δ_j, $j = 0, 1, 2, 3$ in $\mathbb{R}^3$. In contrast to the Luciano and Schoutens (2006) VG model, for which the number of parameters is limited to a max of 10, it is possible to increase the number of LG parameters, say to six knots δ_j $j = 0, 1, 2, 3, 4, 5$ and use the method of moments in order to get a better estimate of the underlying dependence structure.

In Fig. 9, we give sample paths simulated from the three dimensional LG model (41) which illustrates the higher correlation between the EUR/USD and GBP/USD which has also been empirically observed (see Fig. 3).

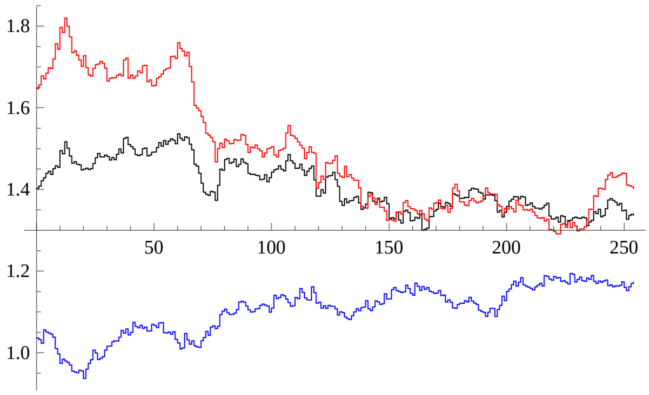

Fig. 9 Joint co-movement of the exchange rates of EUR/USD, GBP/USD and JPY/USD simulated from the LG model.

5 Comments and conclusions

The proposed (univariate) LG process is a more flexible generalization of the well-known VG process and the BG process since it allows to use any number of parameters according to the requirements of a particular application and control both the positive and the negative parts of the corresponding Lévy measure.

An enlightening link between the LG distribution, and (univariate) B-splines and Dirichlet splines is established and alternative formulas for the density of the VG and BG are given. The use of a LG process, as the driver of a stock price dynamics, in pricing exotic options and participating life insurance contracts is briefly indicated.

The proposed multivariate generalization of the LG process is very flexible, since it allows to incorporate any required number of parameters and to model complex dependence patterns between asset price processes. It is a competitive alternative to multivariate Lévy copulas and other multivariate generalizations of the VG process, based for instance, on a common random time change in a multivariate Brownian motion.

We have also explored some of the properties of multivariate LG processes in terms of multivariate simplex B-splines and Dirichlet splines. In particular, we have given explicit expressions of the joint LG density and the underlying LG copula function in terms of Dirichlet splines, and also the LG characteristic, moment and cumulant generating functions. The latter have been used in section 3.2 to develop, a reasonably simple method of moments, based on their relation to cumulants, for the purpose of calibrating the LG model parameters.

We have also illustrated the modelling power of a multivariate LG process on the example of FX modelling of the exchange rates of three currencies, the EUR the GBP and the JPY to the US Dollar. Results demonstrate that in contrast to the three dimensional, 10 parameter VG model, proposed by Luciano and Schoutens (2006), the three dimensional, 12 parameter LG model, captures better the different (bilateral) patterns of dependence between the FX rates of the three currencies although there is still scope to improve its performance by introducing additional parameters.

Ongoing research is related to exploring market consistent LG parameter calibration and properties of the LG copula, which is a new promising member of the relatively limited family of multivariate copulas, richly enough parametrized so as to capture complex dependence patterns in truly multivariate financial and insurance applications. It is worth mentioning that yet another new class of copulas, related to the LG copulas, called Dirichlet (B-) spline copulas have been proposed and explored by Kaishev (2006b).

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