Simple Explicit Formula for Near-Optimal Stochastic Lifestyling

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Abstract

In life-cycle economics, the Samuelson paradigm (Samuelson, 1969) states that the optimal investment is in constant proportions out of lifetime wealth composed of current savings and the present value of future income. It is well known that in the presence of credit constraints this paradigm no longer applies. Instead, optimal life-cycle investment gives rise to so-called stochastic lifestyling (Cairns et al., 2006), whereby for low levels of accumulated capital it is optimal to invest fully in stocks and then gradually switch to safer assets as the level of savings increases. In stochastic lifestyling not only does the ratio between risky and safe assets change but also the mix of risky assets varies over time. While the existing literature relies on complex numerical algorithms to quantify optimal lifestyling, the present paper provides a simple formula that captures the main essence of the lifestyling effect with remarkable accuracy.

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1. Introduction

Operations research has analysed pension finance from two angles. The first looks at practical methodology for asset–liability management of a pension scheme as a whole (Sodhi, 2005; Mulvey et al., 2008). The second seeks to characterize the optimal mix of risky and risk-free investments for individual members of a pension scheme as they progress from early working life to retirement (Cairns

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et al., 2006; Zhang and Ewald, 2010). This second stream is informed by and linked to a wider literature on optimal investment and consumption with constraints (Zariphopoulou, 1994; Vila and Zariphopoulou, 1997; Xia, 2011; Nutz, 2012; Kilianová and Ševčovič, 2013).

In contrast to the considerable mathematical and numerical sophistication needed to arrive at optimal pension portfolios, there is notable absence of portfolio rules that are simple to implement and yet do not compromise welfare of investors. The practical need for such rules is significant but this demand has not been met by academia, despite five decades of research. In an isolated contribution, Ayres and Nalebuff (2013) propose simple heuristic rules for life-cycle portfolio allocation and evaluate their welfare, without analyzing their optimality. This paper offers an insight how one may bridge the gap between optimality and ease of implementation.[1]

Consider a model with d risky assets whose dynamics are given by the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \tag{1.1}$$

where B are d uncorrelated Brownian motions, $\mu \in \mathbb{R}^d$, and $\Sigma = \sigma\sigma^\top \in \mathbb{R}^{d\times d}$ is regular. Assume further that there is a risk-free asset with value $S^0 = e^{rt}$. An individual who starts working at time 0 and retires at time T makes pension contributions at the deterministic rate y_t per unit of time. The task of the pension fund manager is to invest these contributions on behalf of the individual so as to maximize the expected utility of the terminal value of the pension plan. To aid tractability, it is customary to consider utility functions of the form

$$U_\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.$$

The analysis can be extended to $\gamma = 1$ with $U_1(x) = \ln x$ and we will do so in due course.

We seek the optimal investment plan π^* that solves

$$\pi^* = \underset{\pi\geq 0,\ \pi\mathbf{1}\leq 1}{\arg\max}\ E\left[U_\gamma(W_T)\right] \text{ subject to} \tag{1.2a}$$

$$dW_t = (rW_t + y_t)\,dt + \pi_t W_t\left(\frac{dS_t}{S_t} - r\mathbf{1}dt\right). \tag{1.2b}$$

Here W_t denotes accumulated savings and π represents the proportions invested in the risky assets.[2] Parameter γ captures the risk-aversion of the individual account

[1]Thanks to their tractability our results have been adopted by Allianz in a spreadsheet modeller available to individual pension account clients in Slovakia.

[2]By convention, π is a row vector while S, μ, and $\mathbf{1}$ are column vectors.

holder. The restrictions imposed on π reflect typical institutional constraints faced by pension funds. In addition to shortsale constraints on risky assets, $\pi \geq 0$, there is a credit constraint that prevents the fund manager from borrowing against the value of future contributions, $\pi\mathbf{1} \leq 1$.

It is well known that without constraints on π and without contributions ($y_t = 0$) the optimal investment strategy is given by

$$\pi^* = \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}}{\gamma} = \arg\max_{\pi \in \mathbb{R}^d} \pi(\mu - r\mathbf{1}) - \frac{\gamma}{2}\pi\Sigma\pi^\top. \tag{1.3}$$

In the context of the optimization problem (1.2), one is thus lead to consider a heuristic fixed proportions strategy

$$\pi^{(1)} = \arg\max_{\pi \geq 0, \pi\mathbf{1} \leq 1} \pi(\mu - r\mathbf{1}) - \frac{\gamma}{2}\pi\Sigma\pi^\top. \tag{1.4}$$

Suppose the weights in (1.3) are strictly positive. Taken as a function of risk aversion γ, the optimal weights $\pi^{(1)}$ are no longer equal to the risky mix from (1.3) adjusted for the leverage constraint $\pi\mathbf{1} \leq 1$, as given by the formula

$$\pi^{(0)} = \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}}{\max\left((\mu - r\mathbf{1})^\top \Sigma^{-1}\mathbf{1}, \gamma\right)}. \tag{1.5}$$

Instead, for low levels of the risk aversion parameter γ the *relative weights* in $\pi^{(1)}$ change in a way that entails substitution towards the riskier assets as γ decreases.

One might reasonably expect that strategy (1.4) would provide satisfactory heuristic approximation of the fully optimal investment strategy. However, numerical experiments reveal that the character of the optimal investment changes more dramatically than suggested by equation (1.4). Simulations capture a phenomenon known in pension finance as stochastic lifestyling, a term coined by Cairns et al. (2006), whereby it is optimal early on to invest the accumulated savings in stocks and then gradually switch the investment into bonds and safe deposits as the retirement approaches and the total amount of savings increases. Thus the optimal strategy behaves *as if* the risk-aversion coefficient were lower for low levels of accumulated funds.

Because the fully optimal strategy π^* in (1.2) has to be computed numerically by *dynamic* programming and because it is a non-linear function of both time t and the accumulated savings W_t, at first sight it is difficult to see how one can characterize the lifestyling effect explicitly. In this paper we point out that there is an excellent heuristic approximation of the lifestyling effect, given by a formula that is no less explicit than equation (1.4).

To arrive at the correct lifestyling formula, one must adopt Samuelson's view of the investment weights (1.3). When the individual savings plan can borrow as well as invest at the risk-free rate r Samuelson (1969), and more explicitly Hakansson (1970), have pointed out that the presence of contributions does not affect the constant proportions strategy (1.3) provided that the risky investment is made out of lifetime pension wealth

$$\overline{W}_t = W_t + \mathrm{PV}_t,$$

where PV_t is the present value of all future pension contributions as of time t.

If we denote by $\overline{\pi}_t$ the proportions of risky investment out of lifetime pension wealth $\overline{W}_t$, the credit constraint $\pi_t \mathbf{1} \leq 1$ is transformed to $\overline{\pi}_t \mathbf{1} \leq \alpha_t$, where

$$\alpha_t = \frac{W_t}{\overline{W}_t} \tag{1.6}$$

is the ratio of the already accumulated savings to the entire lifetime pension capital. Observe that in the Samuelson world the heuristic strategy $\pi^{(1)}$ corresponds to

$$\overline{\pi}^{(1)}(\alpha_t) = \alpha_t \pi^{(1)}.$$

Observe also that if the sum of weights $\pi^{(1)}\mathbf{1}$ is strictly less than 1 then the sum of weights in $\overline{\pi}^{(1)}(\alpha_t)$ will be strictly less than α_t for all $\alpha_t \in (0,1)$ which is unlikely to be optimal. We therefore also consider a modified heuristic

$$\overline{\pi}^{(2)}(\alpha_t) = \min\left(\frac{\alpha_t}{\pi^{(1)}\mathbf{1}}, 1\right)\pi^{(1)},$$

that corresponds to cash-in-hand investment proportions

$$\pi^{(2)}(\alpha_t) = \frac{\pi^{(1)}}{\max\left(\pi^{(1)}\mathbf{1}, \alpha_t\right)}. \tag{1.7}$$

However, the key breakthrough of this paper is achieved by formulating a heuristic strategy directly in the Samuelson world, in the form

$$\overline{\pi}^{(3)}(\alpha_t) = \underset{\pi \geq 0, \pi\mathbf{1} \leq \alpha_t}{\arg\max}\ \pi(\mu - r\mathbf{1}) - \frac{\gamma}{2}\pi\Sigma\pi^\top,$$

which, when expressed as proportions out of accumulated savings W_t, yields

$$\pi^{(3)}(\alpha_t) = \frac{\overline{\pi}^{(3)}(\alpha_t)}{\alpha_t} = \underset{\pi \geq 0, \pi\mathbf{1} \leq 1}{\arg\max}\ \pi(\mu - r\mathbf{1}) - \frac{\alpha_t\gamma}{2}\pi\Sigma\pi^\top. \tag{1.8}$$

We show that, unlike $\pi^{(1)}$ and $\pi^{(2)}(\alpha_t)$, the strategy $\pi^{(3)}(\alpha_t)$ is an excellent approximation to the fully optimal strategy and can therefore serve as a simple rule of thumb for pension plan providers who wish to offer a choice of lifestyling strategies to their clients, while also specifying the sense in which such lifestyling is optimal. To reduce the barriers to application further, we analyze the explicit dependence of $\pi^{(3)}$ on α_t for a given set of binding constraints. For example, assuming that the constraints $\pi \geq 0$ are not binding, the near-optimal strategy $\pi^{(3)}$ is of the form

$$\pi^{(3)}(\alpha_t) = \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}}{\gamma \alpha_t} + \frac{\mathbf{1}^\top \Sigma^{-1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \min\left(1 - \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1} \mathbf{1}}{\gamma \alpha_t}, 0\right). \quad (1.9)$$

Note that the non-negativity constraint will become binding for α_t small enough, at which point, for typical parameter values, the formula directs all accumulated savings to be invested in stocks. Interestingly, $\mathbf{1}^\top \Sigma^{-1} / \mathbf{1}^\top \Sigma^{-1} \mathbf{1}$ is the classical Markowitz minimum variance portfolio.

Formula (1.9) captures the main essence of the lifestyling effect, representing in a nutshell the main conceptual contribution of our paper. It not only shows the change in portfolio composition as a function of α_t for fixed risk aversion, but it also neatly demonstrates that the portfolio composition will change with decreasing γ when there are no future contributions to consider ($\alpha_t = 1$). According to the formula, the near-optimal investment proportions do behave as if the risk aversion were lower for low levels of accumulated funds, with effective risk aversion equal to $\alpha_t \gamma$.

The article is organized as follows. Section 2 introduces what we call the 'Samuelson transform', linking a model with gradual contributions to an equivalent model where all capital is paid up-front but there are additional constraints on how the capital can be invested. We review the mathematical theory guaranteeing existence of an optimal strategy in the world with contributions and via the Samuelson link also in the world without contributions but with investment constraints. In Section 3 we provide economic analysis of the competing strategies, both in terms of welfare impact and portfolio weights. We close this section with a thorough robustness analysis. Section 4 concludes.

2. Theory

2.1. Samuelson transform

We denote by $Y_t = \int_0^t y(u)du$ the cumulative pension contribution up to and including time t. Function y is assumed to be deterministic, non-negative, and integrable on $[0, T]$. The price process of all assets, including the risk-free asset, is denoted by $S = (S^0, S^{1:d})$. We assume $S^{1:d}$ is a geometric Brownian motion with drift as described in equation (1.1), while $S_t^0 = e^{rt}$ represents a bank account with risk-free deposit rate r. Risk-free borrowing is excluded.

The process

$$\mathrm{PV}_t = \int_t^T e^{-r(u-t)} dY_u,$$

is the present value at time t of all contributions in the period $(t, T]$.

Definition 2.1. *We say that* φ *is a self-financing strategy for price process* S *and cumulative contributions* Y*, writing* $\varphi \in \Theta(S,Y)$*, if* φ *is predictable,* S*–integrable, and*

$$\varphi_0 S_0 + \int_0^t \varphi_u dS_u + Y_t = \varphi_t S_t.$$

We denote by $\Theta_x(S,Y)$ *the set of all self-financing strategies with initial capital* x*,*

$$\Theta_x(S,Y) = \{\varphi \in \Theta(S,Y) : \varphi_0 S_0 = x\}.$$

Consider the following transformation of trading strategies $\varphi \mapsto \overline{\varphi}$:

$$\overline{\varphi}_t^{1:d} = \varphi_t^{1:d}, \tag{2.1}$$

$$\overline{\varphi}_t^0 = \varphi_t^0 + e^{-rt}\mathrm{PV}_t. \tag{2.2}$$

We call (2.1–2.2) the *Samuelson transform.* Using the numeraire change technique of Geman et al. (1995) it is readily seen that the Samuelson transform is a one-to-one mapping between $\Theta_x(S,Y)$ and $\Theta_{x+\mathrm{PV}_0}(S,0)$.

We can now turn our attention to a situation where borrowing against future contributions is no longer possible.

Definition 2.2. *Consider an arbitrary self-financing strategy* $\varphi \in \Theta_x(S,Y)$ *with an arbitrary contribution process Y. Assume that* $\varphi \geq 0$ *and* $S \geq 0$*. We define the vector of proportions,* $\pi(\varphi)$*, invested in available risky assets by*

$$\pi_i(\varphi) = \frac{\varphi^i S^i}{\varphi S}, \qquad i \in \{1,\ldots,d\},$$

using the convention $0/0 = 0$*.*

Proposition 2.3. *Suppose* $S \geq 0$*. The Samuelson transform is a one-to-one mapping between* $\mathcal{A}_x = \{\varphi \in \Theta_x(S,Y) : \pi(\varphi) \geq 0, \pi(\varphi)\mathbf{1} \leq 1\}$*, and*

$$\overline{\mathcal{A}}_{x+\mathrm{PV}_0} = \{\overline{\varphi} \in \Theta_{x+\mathrm{PV}_0}(S,0) : \pi(\overline{\varphi}) \geq 0, \pi(\overline{\varphi})\mathbf{1} \leq 1 - \mathrm{PV}/\overline{\varphi}S\}. \tag{2.3}$$

Proof. $\pi(\varphi) \geq 0 \wedge \pi(\varphi)\mathbf{1} \leq 1 \iff \varphi^0 S^0 \geq 0 \wedge \varphi^{1:d} \geq 0 \iff \overline{\varphi}^0 S^0 \geq \mathrm{PV} \wedge \overline{\varphi}^{1:d} \geq 0 \iff \pi(\overline{\varphi}) \geq 0 \wedge \pi(\overline{\varphi})\mathbf{1} \leq 1 - \mathrm{PV}/\overline{\varphi}S.$ □

Proposition 2.3 clarifies the link between the classical Samuelson paradigm and the situation where the risk-free borrowing against future contributions is precluded. While in the classical case the sum of risky proportions is unconstrained, there is now in (2.3) a stochastic constraint on the total proportion invested in the risky assets. The risky proportion must not exceed $1 - \mathrm{PV}/\overline{\varphi}S$ in Samuelson's world without contributions. In economic terms, risky investment can only be financed from past contributions and from past capital gains. Below, we investigate how this constraint influences the leverage and the relative proportions invested in risky assets.

2.2. Hamilton–Jacobi–Bellman equations

In this subsection we relate the optimal investment strategy to the solutions of two Hamilton–Jacobi–Bellman (HJB) equations. The twin representation turns out to be important in the proof of existence and uniqueness (Subsection 2.3) and in the proof of optimality (Subsection 2.4) but most importantly it provides economic motivation for the near-optimal strategy (Subsection 3.3).

For the sake of brevity, hereafter we consider a constant contribution rate y. We begin by writing out formally the partial differential equation (PDE) in the world with contributions,

$$0 = \sup_{\pi \geq 0, \pi \mathbf{1} \leq 1} v_t + v_x(y + (r + \pi(\mu - r\mathbf{1}))\,x) + \frac{x^2}{2} v_{xx} \pi \Sigma \pi^\top, \tag{2.4a}$$

$$v(T, x) = \frac{x^{1-\gamma}}{1-\gamma}. \tag{2.4b}$$

The terms standing by v_x and v_{xx} originate from the dynamics of accumulated savings W in (1.2b). In Samuelson's world without contributions the corresponding HJB equation reads

$$0 = \sup_{\overline{\pi} \geq 0, \overline{\pi} \mathbf{1} \leq 1 - \mathrm{PV}_t/\overline{x}} \overline{v}_t + \overline{x}\overline{v}_{\overline{x}}(r + \overline{\pi}(\mu - r\mathbf{1})) + \frac{\overline{x}^2}{2} \overline{v}_{\overline{x}\overline{x}} \overline{\pi} \Sigma \overline{\pi}^\top, \tag{2.5a}$$

$$\overline{v}(T, \overline{x}) = \frac{\overline{x}^{1-\gamma}}{1-\gamma}, \tag{2.5b}$$

corresponding to lifetime pension wealth dynamics

$$d\overline{W}_t = r\overline{W}_t dt + \overline{\pi}_t \overline{W}_t \left(\frac{dS_t}{S_t} - r\mathbf{1}dt \right). \tag{2.6}$$

Similarly, the value function corresponding to the heuristic strategy $\pi^{(i)}$ for $i \in \{0, 1, 2, 3\}$ in the world with contributions is formally given as a solution of

$$0 = v_t^{(i)} + v_x^{(i)} \left(y + \left(r + \pi^{(i)}(\mu - r\mathbf{1})x \right) \right) + \frac{x^2}{2} v_{xx} \pi^{(i)} \Sigma \pi^{(i)\top}, \tag{2.7a}$$

$$v^{(i)}(T, x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{2.7b}$$

where $\pi^{(i)}$ is taken to be a fixed function of (t, x) as indicated in the introduction. In the Samuelson world, one obtains an analogous PDE for the strategies $\overline{\pi}^{(i)}$,

$$0 = \overline{v}_t^{(i)} + \overline{x}\overline{v}_{\overline{x}}^{(i)}(r + \overline{\pi}^{(i)}(\mu - r\mathbf{1})) + \frac{\overline{x}^2}{2} \overline{v}_{\overline{x}\overline{x}}^{(i)} \overline{\pi}^{(i)} \Sigma \overline{\pi}^{(i)\top}, \tag{2.8a}$$

$$\overline{v}^{(i)}(T, \overline{x}) = \frac{\overline{x}^{1-\gamma}}{1-\gamma}. \tag{2.8b}$$

The two sets of equations are equivalent in the sense that every $\mathcal{C}^{1,2}$ solution of the initial value problem (2.4) generates a $\mathcal{C}^{1,2}$ solution of (2.5) via transformation $\overline{v}(t,\overline{x}) = v(t,\overline{x}-\mathrm{PV}_t)$. Conversely, any $\mathcal{C}^{1,2}$ solution of (2.5) gives rise to a $\mathcal{C}^{1,2}$ solution of (2.4) through $v(t,x) = \overline{v}(t,x+\mathrm{PV}_t)$. The same correspondence holds between (2.7) and (2.8).

If, for the time being, we accept as given that (2.4), resp. (2.5), admit optimal controls π^*, resp. $\overline{\pi}^*$, then there is also a relationship between (2.4) and (2.7) to the extent that if one substitutes π^* for $\pi^{(i)}$ in (2.7) one obtains a solution of (2.4). The same correspondence holds between (2.5) and (2.8) on replacing $\overline{\pi}^{(i)}$ with $\overline{\pi}^*$.

Before we examine the optimal controls it is helpful to associate a coefficient of risk aversion to each indirect utility,

$$R(t,x) = -\frac{x v_{xx}(t,x)}{v_x(t,x)}, \tag{2.9}$$

$$\overline{R}(t,\overline{x}) = -\frac{\overline{x}\,\overline{v}_{\overline{x}\overline{x}}(t,\overline{x})}{\overline{v}_{\overline{x}}(t,\overline{x})}. \tag{2.10}$$

The optimal portfolio strategy is related to the following deterministic mean-variance utility $f : [0,\infty)\times(0,\infty)\to\mathbb{R}$, with risky investment constraint α and risk aversion ρ,

$$f(\alpha,\rho) = \sup_{\pi\geq 0,\pi\mathbf{1}\leq\alpha} \pi(\mu - r\mathbf{1}) - \frac{\rho}{2}\pi\Sigma\pi^\top. \tag{2.11}$$

Due to strict convexity in π and compactness of the optimization region there is a unique optimizer in the deterministic problem (2.11) which we denote $\hat{\pi}(\alpha,\rho)$,

$$\hat{\pi}(\alpha,\rho) = \operatorname*{arg\,max}_{\pi\geq 0,\pi\mathbf{1}\leq\alpha} \pi(\mu - r\mathbf{1}) - \frac{\rho}{2}\pi\Sigma\pi^\top. \tag{2.12}$$

We note for future use that $\hat{\pi}(\alpha,\rho)$ is self-similar, that is, for $\alpha > 0$ one has

$$\hat{\pi}(\alpha,\rho) = \alpha\hat{\pi}(1,\alpha\rho), \tag{2.13}$$

with the convention $0\times\infty = 0$.

Using the newly established notation the formal optimal controls in (2.4) and (2.5) can be written as

$$\pi^*(t,x) = \hat{\pi}(1,R(t,x)), \tag{2.14a}$$

$$\overline{\pi}^*(t,\overline{x}) = \hat{\pi}(1-\mathrm{PV}_t/\overline{x},\overline{R}(t,\overline{x})). \tag{2.14b}$$

Furthermore, the self-similarity of $\hat{\pi}(\alpha,\rho)$ yields

$$\pi^*(t,x) = (1+\mathrm{PV}_t/x)\,\overline{\pi}^*(t,x+\mathrm{PV}_t),$$
$$\overline{\pi}^*(t,\overline{x}) = (1-\mathrm{PV}_t/\overline{x})\,\pi^*(t,\overline{x}-\mathrm{PV}_t).$$

Economically this is no surprise in the light of our analysis in Subsection 2.1.

2.3. Existence and uniqueness

The advantage of the world with contributions is that it measures investment in natural units – out of accumulated funds. In addition, it is mathematically better behaved in that it can be transformed to a strictly parabolic quasilinear PDE whose properties, albeit mathematically involved, are well understood in specialist literature.[3]

Theorem 2.4. *Under the assumption*

$$\mu_i > r, \qquad \text{for some } i \in \{1, \ldots, d\}, \tag{2.15}$$

the inital value problems (2.4–2.8) have a unique classical solution belonging to $\mathcal{C}^{1,2}([0,T] \times (0,\infty))$. *The corresponding maximizers* $\pi^*(t,x)$ *and* $\overline{\pi}^*(t,x)$ *from* (2.14) *have the property that* $x\pi^*(t,x)$, *resp.* $x\overline{\pi}^*(t,x)$, *is locally Lipschitz–continuous in* x, *uniformly in* t, *on* $[0,T] \times [0,\infty)$.

Proof. 1) The difficult part is to reformulate the problem into a form where strict parabolicity can be established. We follow the strategy of Kilianová and Ševčovič (2013) whose key result is summarized in Proposition A.2. One begins with equation (A.6) formally obtained from (2.4a) by a logarithmic transformation $x \to e^z, v(t,x) \to u(t,z)$. Momentarily granting the assumptions of Proposition A.2 one establishes the existence and properties of an auxiliary function $\rho(t,z)$ from (A.3). Subsequently, from ρ one constructs via (A.5) u as a solution of (A.6) with a further property $1 - u_{zz}/u_z = \rho$. Therefore, the indirect risk aversion coefficient $R(t,x) = -xv_{xx}/v_x = \rho(t, \ln x)$ belongs to $\mathcal{C}^{1,2}([0,T] \times (0,\infty))$.

2) It is now readily seen that $v(t,x) = u(t, \ln x)$ is a unique classical solution of the HJB equation (2.4a) and likewise $\overline{v}(t,\overline{x}) = v(t, \overline{x} - \mathrm{PV}_t)$ is a unique classical solution of the HJB equation (2.5a).

3) To invoke Proposition A.2, it remains to prove that under the assumptions of Theorem 2.4 function g,

$$g(\rho) = f(1,\rho) = \sup_{\pi \geq 0, \pi \mathbf{1} \leq 1} \pi(\mu - r\mathbf{1}) - \frac{\rho}{2}\pi\Sigma\pi^\top,$$

possesses locally Lipschitz-continuous derivative with the property

$$0 < \inf_{\rho \in (0,\gamma]} -g'(\rho) \leq \sup_{\rho \in (0,\gamma]} -g'(\rho) < \infty. \tag{2.16}$$

[3] A related constrained optimization problem is studied in Vila and Zariphopoulou (1997). Their proofs make it clear that a rigorous mathematical treatment of the problem is technically demanding. We follow an alternative line of attack proposed in Kilianová and Ševčovič (2013) that allows us to condense the technical arguments considerably.

Since the region $A = \{\pi \in \mathbb{R}^d : \pi \geq 0, \pi\mathbf{1} \leq 1\}$ is compact, one has

$$\sup_{\pi \in A} \frac{1}{2}\pi\Sigma\pi^\top < \infty, \tag{2.17}$$

and by Milgrom and Segal (2002) g is differentiable everywhere on $(0,\infty)$ with

$$g'(\rho) = -\frac{1}{2}\hat{\pi}(1,\rho)\Sigma\hat{\pi}(1,\rho)^\top. \tag{2.18}$$

Combination of (2.17) and (2.18) proves the right-hand side inequality in (2.16). By Klatte (1985, Theorem 2), $\hat{\pi}(1,\rho)$ is a locally Lipschitz-continuous function of ρ and therefore g' is also Lipschitz-continuous by (2.18). It remains to show that

$$\inf_{\rho \in (0,\gamma)} -g'(\rho) > 0, \tag{2.19}$$

which is where the assumption '$\mu_i > r$ for some $i \in \{1,\ldots,d\}$' is required. Inequality (2.19) holds through delicate estimates in Lemma A.1.

4) To establish the local Lipschitz property of $x\pi^*(t,x)$ note that

$$x\pi^*(t,x) = x\hat{\pi}(1, R(t,x)). \tag{2.20}$$

We have shown in step 3) that $\hat{\pi}(1,.)$ is locally Lipschitz-continuous and since $R(t,x) \in \mathcal{C}^{1,2}([0,T]\times(0,\infty))$ the claim follows. Similar argument applies to $x\overline{\pi}^*(t,x)$.

5) For the heuristic strategies $\pi^{(i)} = \pi^{(i)}(t,x)$, $i \in \{0,1,2,3\}$, the situation is easier because $\pi^{(i)}$ are explicit functions of (t,x) and the resulting PDE is linear. Logarithmic transformation $z = \ln x$ with $u(t,z) = v(t,e^z)$ transforms the initial value problem (2.7) to

$$0 = u_t^{(i)} + u_z^{(i)}\left(ye^{-z} + r + \pi^{(i)}(\mu - r\mathbf{1}) - \frac{1}{2}\pi^{(i)}\Sigma\pi^{(i)\top}\right) + \frac{1}{2}u_{zz}^{(i)}\pi^{(i)}\Sigma\pi^{(i)\top}, \tag{2.21a}$$

$$u^{(i)}(T,z) = \frac{e^{z(1-\gamma)}}{1-\gamma}. \tag{2.21b}$$

By Lemma A.1, equation (2.21a) is strictly parabolic for $i \in \{0,1,2,3\}$. Existence of classical $\mathcal{C}^{1,2}$ solution follows from standard linear PDE theory (Ladyzhenskaya et al., 1968, Theorem III.12.1, Lieberman, 1996, Theorem 5.14).

6) In the case $\gamma = 1$ we take $U_1(x) = \lim_{\gamma\to 1}\frac{x^{1-\gamma}-1}{1-\gamma} = \ln x$ and the arguments in steps 1)–5) go through with $u^{(i)}(T,z) = u(T,z) = z$. □

2.4. Optimality

We say $\overline{\pi}(t,\omega)$ is an admissible control if it is progressively measurable (Fleming and Soner, 2006, Definition IV.2.1) and $0 \le \overline{\pi}\mathbf{1} \le 1 - \mathrm{PV}/\overline{W}$ for $\overline{W}$ from (2.6),

$$\frac{d\overline{W}_t}{\overline{W}_t} = (r + \overline{\pi}(\mu - r\mathbf{1}))dt + \overline{\pi}\sigma dB_t. \tag{2.22}$$

Observe that SDE (2.22) has a unique strong solution for any progressively measurable $\overline{\pi}$ with values in the compact set $0 \le \overline{\pi}1 \le 1$ (Fleming and Soner, 2006, paragraph after equation IV.2.4).

Comparison principle yields the estimate $|\overline{v}(t,x)| \le e^{C(T-t)}x^{1-\gamma}/\,|1-\gamma|$ for $\gamma > 0, \gamma \ne 1$ and a suitably chosen $C > 0$ dependent on γ. For $\gamma \in (0,1)$ the verification theorem (Fleming and Soner, 2006, Corollary IV.3.1) yields directly that $\overline{\pi}^*(t,\overline{W}_t)$ is the optimal Markov control policy. Because Theorem IV.3.1 in Fleming and Soner (2006) requires the value function to be dominated by a positive power of the endogenous state variable, for $\gamma > 1$ we pass to $\overline{W}^{-1}$ whose SDE reads

$$\begin{aligned}\overline{W}_t d\overline{W}_t^{-1} &= -\overline{W}_t^{-1} d\overline{W}_t + \overline{W}_t^{-2} d\big[\,\overline{W},\overline{W}\,\big]_t\\ &= \Big(\overline{\pi}\Sigma\overline{\pi}^\top - r - \overline{\pi}(\mu - r\mathbf{1})\Big)\,dt - \overline{\pi}\sigma dB_t.\end{aligned}$$

Hence, by Appendix D in Fleming and Soner (2006) $\overline{W}^{-1}$ satisfies for any $m > 0$

$$E\left[\left(\sup_{0\le t\le T} \overline{W}_t^{-1}\right)^m\right] < \infty.$$

This means $\overline{v}(t,\overline{W}_t)$ is a process of class (D) (Jacod and Shiryaev, 2003, Definition I.1.46) and a local supermartingale for any admissible strategy $\overline{\pi}$, hence a supermartingale (Karatzas and Kardaras, 2007, Appendix 3). It is furthermore a local martingale and therefore a true martingale (Jacod and Shiryaev, 2003, Proposition I.1.47) for the optimal strategy $\overline{\pi}^*(t,\overline{W}_t)$ which therefore remains an optimal Markov policy also for $\gamma > 1$.

Finally, for $\gamma = 1$ one has $U_1(x) = \ln x$. By comparison principle, the solution $\overline{v}(t,x)$ satisfies the estimate $\ln x \le \overline{v}(t,x) \le \ln x + C(T-t)$ for a suitably chosen $C > 0$. By the Itô formula

$$d\ln\overline{W}_t = \left(r + \overline{\pi}(\mu - r\mathbf{1}) - \frac{1}{2}\overline{\pi}\Sigma\overline{\pi}^\top\right)dt + \overline{\pi}\sigma dB_t$$

and therefore $\overline{v}(t,\overline{W}_t)$ is a process of class (D). Once again, this implies $\overline{\pi}^*(t,\overline{W}_t)$ is an optimal Markov policy.

The optimality results are summarized in the following theorem.

Theorem 2.5. *Recall the formal value function v in (2.4), the corresponding risk aversion function R in (2.9), and the optimal strategy π^* in (2.14a). The following statements hold.*

1) *The solution v of (2.4) is the value function of the corresponding optimal control problem, that is, it satisfies*

$$v(0,x) = \sup_{\pi(\varphi)\in\mathcal{A}_x} E_t\left[\frac{1}{1-\gamma}\left(\varphi_T S_T\right)^{1-\gamma}\right]. \tag{2.23}$$

2) *For any $x \geq 0$ there is a unique process W satisfying*

$$\begin{aligned} dW_t &= (y + rW_t)dt + W_t\pi^*(t,W_t)\left(\frac{dS_t}{S_t} - r\mathbf{1}dt\right), \\ W_0 &= x. \end{aligned}$$

3) *The optimal strategy φ in (2.23) satisfies*

$$\begin{aligned} \varphi_t^i &= \pi_i^*(t,W_t)\frac{W_t}{S_t^i}, \qquad i \in \{1,\dots,d\} \\ \varphi_t^0 &= e^{-rt}W_t(1-\pi^*(t,W_t)\mathbf{1}), \end{aligned}$$

and $\varphi S = W$.

3. Economic analysis and numerical robustness

3.1. Illustrative example

Consider the log-normal model of asset returns described in the introduction. Below we present, for illustration, a stylized model using figures broadly consistent with equity and corporate bond markets of developed economies. Numerically, we will take risk-free return of $r = 1\%$ and two risky assets with drifts $\mu_1 = 2\%$ (representing bond returns), $\mu_2 = 10\%$ (representing stock returns), volatilities 5%, 25% respectively and correlation -0.05, yielding the covariance matrix

$$\Sigma = \begin{bmatrix} 0.0025 & -0.000\,625 \\ -0.000\,625 & 0.0625 \end{bmatrix}.$$

The investment horizon has been set to $T = 40$ years. We have used the cumulative contribution process $Y_t = t/T$ so that the cumulative contribution is normalized to 1. The present framework provides methodology capable of analyzing and comparing results for various non-linear contribution profiles, but in the interest of brevity we do not consider them here.

Table 1: Certainty equivalents and internal rates of return for the heuristic strategies $\pi^{(i)}$, $i = 0, 1, 2$, and the optimal strategy, π^*.

γ	$\mathrm{CE}^{(0)}$	$\mathrm{IRR}^{(0)}$	$\mathrm{CE}^{(1)}$	$\mathrm{IRR}^{(1)}$	$\mathrm{CE}^{(2)}$	$\mathrm{IRR}^{(2)}$	CE^*	IRR^*
2	2.2584	3.64%	3.3353	5.16%	3.3353	5.16%	3.6501	5.50%
5	1.9720	3.08%	2.0153	3.17%	2.0153	3.17%	2.1782	3.49%
8	1.6872	2.42%	1.6872	2.42%	1.7510	2.58%	1.8164	2.74%

We examine three levels of relative risk aversion; low with $\gamma = 2$, moderate ($\gamma = 5$), and high ($\gamma = 8$). We report the utility of competing strategies both in terms of certainty equivalent wealth and in terms of certainty equivalent internal rate of return.[4]

To obtain the function $R(t, x)$ in (2.9), we solve the quasilinear second-order Cauchy problem (A.3) using the methodology of Kilianová and Ševčovič (2013). The solution $\rho(t, z) = R(t, e^z)$ is computed on a Cartesian grid $[0, 40] \times [-12, 6]$ with temporal step of 0.01 and spatial step of 0.001, with left boundary condition of Robin type and right boundary condition of Neumann type using the built-in Matlab function `pdepe`. The initial value problem (2.4) is then solved numerically by applying the method of characteristics to the linear PDE (A.5) with starting values $x \in \{e^{-10}, e^{-9}, \ldots, e^{-5}\}$. These results are subsequently extrapolated to $x = 0$ by linear regression in x. The optimal control π^* is obtained via (2.12) and (2.20).

3.2. Heuristic strategies $\pi^{(1)}$ and $\pi^{(2)}$

Let us begin by comparing the performance of the optimal strategy π^*, computed numerically as described above, with the rescaled Samuelson strategy $\pi^{(0)}$, computed explicitly from equation (1.5). Table 1 shows that π^* significantly outperforms the naive strategy for low and medium levels of risk aversion, while with high risk aversion the outperformance is relatively modest.

To gain better understanding where the outperformance originates from, we first analyze the case $\gamma = 8$ where the welfare loss is relatively small. We report in Table 2 the optimal portfolio weights $\pi^*(t, W_t)$ out of accumulated savings (cash in hand) W_t. The naive weights $\pi^{(0)}$ in this case coincide with $\pi^{(1)}$ and are equal to

$$\frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}}{\gamma} = (54.6\%, 18.6\%).$$

[4]The certainty equivalent is computed from the formula $\mathrm{CE} = \left(E((\varphi_T S_T)^{1-\gamma})\right)^{1/(1-\gamma)}$. The certainty equivalent internal rate of return is given as the interest rate ρ satisfying $\mathrm{CE} = \int_0^T e^{\rho(T-t)} y(t) dt$.

Table 2: Optimal strategy $\pi^*(t, W_t)$ as a function of t and W_t with $\gamma = 8$.

W_t	$t = 0$		$t = 10$		$t = 20$		$t = 30$		$t = 39.975$	
10^{-5}	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
0.01	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.581	0.197
0.05	0.000	1.000	0.000	1.000	0.000	1.000	0.107	0.893	0.553	0.188
0.1	0.000	1.000	0.000	1.000	0.158	0.842	0.451	0.549	0.550	0.187
0.2	0.244	0.756	0.349	0.651	0.475	0.525	0.625	0.375	0.548	0.186
0.3	0.423	0.577	0.496	0.504	0.582	0.419	0.683	0.317	0.548	0.186
0.5	0.569	0.431	0.614	0.386	0.668	0.332	0.730	0.270	0.547	0.186
1	0.681	0.319	0.706	0.294	0.734	0.266	0.676	0.230	0.547	0.186
2	0.740	0.260	0.723	0.246	0.670	0.228	0.611	0.208	0.547	0.186
20	0.569	0.193	0.564	0.192	0.559	0.190	0.553	0.188	0.546	0.186

Table 3: Heuristic strategy $\pi^{(2)}(\alpha_t)$ as a function of t and W_t with $\gamma = 8$.

W_t	$t = 0$		$t = 10$		$t = 20$		$t = 30$		$t = 39.975$	
10^{-5}	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253
0.01	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.581	0.197
0.05	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.553	0.188
0.1	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.550	0.187
0.2	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.548	0.186
0.3	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.548	0.186
0.5	0.747	0.253	0.747	0.253	0.747	0.253	0.747	0.253	0.547	0.186
1	0.747	0.253	0.747	0.253	0.747	0.253	0.676	0.230	0.547	0.186
2	0.747	0.253	0.723	0.246	0.670	0.228	0.611	0.208	0.547	0.186
20	0.569	0.193	0.564	0.192	0.559	0.190	0.553	0.188	0.546	0.186

We observe that for high levels of cash in hand there is good agreement between the optimal and the naive strategy, with the optimal weights tending towards $\pi^{(0)} = \pi^{(1)}$ as $W_t \to \infty$. For low level of accumulated savings the difference is substantial, however, with the optimal portfolio being invested fully in stocks while portfolios $\pi^{(0)} = \pi^{(1)}$ are not fully invested between stocks and bonds.

Staying with the case $\gamma = 8$, let us now turn to strategy $\pi^{(2)}$ which coincides with $\pi^{(1)}$ for high level of accumulated funds by construction (see eqs. 1.6 and 1.7). Its numerical values, obtained from the explicit formula (1.7), are displayed in Table 3. We observe that $\pi^{(2)}$ is better behaved for low levels of accumulated funds where it becomes fully invested in bonds and stocks, $\pi^{(2)}(\alpha) = \pi^{(1)}/(\pi^{(1)}\mathbf{1}) = (74.7\%, 25.3\%)$ for $\alpha \le \pi^{(1)}\mathbf{1} \approx 73\%$, although the split is such that the funds are far from being fully invested in stocks. We conclude that the welfare difference between the optimal strategy π^* on the one hand, and the heuristic strategies $\pi^{(0)} = \pi^{(1)}$ and $\pi^{(2)}$ on the other hand, reflects the economic value of correct lifestyling strategy at *low levels* of accumulated capital.

Let us now examine the case $\gamma = 2$ whose optimal strategy is displayed in

Table 4: Optimal strategy $\pi^*(t, W_t)$ as a function of t and W_t with $\gamma = 2$.

W_t	$t = 0$		$t = 10$		$t = 20$		$t = 30$		$t = 39.975$	
10^{-5}	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
0.01	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.311	0.689
0.05	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.342	0.659
0.1	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.345	0.655
0.2	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.347	0.653
0.3	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.348	0.652
0.5	0.000	1.000	0.000	1.000	0.000	1.000	0.077	0.923	0.348	0.652
1	0.000	1.000	0.029	0.971	0.104	0.897	0.212	0.788	0.349	0.651
2	0.147	0.853	0.180	0.820	0.225	0.775	0.281	0.720	0.349	0.651
20	0.328	0.672	0.332	0.668	0.337	0.664	0.342	0.658	0.349	0.651

Table 4. We show later in Subsection 3.3 that for high values of cash in hand W_t the optimal weights $\pi^*(t, W_t)$ tend to the expression

$$\pi^{(1)} = \frac{\hat{\pi}}{\gamma} + \zeta \min\left(1 - \frac{\hat{\pi}\mathbf{1}}{\gamma}, 0\right), \tag{3.1}$$

where

$$\zeta = \frac{\mathbf{1}^\top \Sigma^{-1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \tag{3.2}$$

is known as the minimum variance portfolio (Ingersoll, 1987, eq. 4.8). In the present example we have $\hat{\pi} = (437\%, 148\%)$, $\hat{\pi}\mathbf{1} = 5.85$, and $\zeta = (95.3\%, 4.7\%)$. Thus, as the risk aversion falls below 5.85 there is a strong substitution away from bonds towards stocks. The substitution continues until the risk aversion reaches the level of $1.27 = \hat{\pi}\mathbf{1} - \hat{\pi}_1/\zeta_1$ below which all accumulated savings are to be invested in stocks only.

For $\gamma = 2$ the portfolio weights $\pi^{(1)} = \pi^{(2)}$ are fully invested in proportions

$$\frac{\hat{\pi}}{2} - \zeta\left(\frac{5.85}{2} - 1\right) = (34.9\%, 65.1\%)$$

while the naive strategy $\pi^{(0)}$ uses almost the opposite ratio

$$\pi^{(0)} = \frac{\hat{\pi}}{\hat{\pi}\mathbf{1}} = (74.7\%, 25.3\%).$$

Therefore, in addition to the discrepancy between π^* and $\pi^{(0)}$ for low values of W_t which was present already for $\gamma = 8$, $\pi^{(0)}$ faces additional discrepancy of the portfolio weights for high level of accumulated savings. The combined effect makes the strategy $\pi^{(0)}$ substantially suboptimal for low levels of risk aversion.

Table 5: Welfare performance of strategies π^* and $\pi^{(3)}$ for different levels of risk aversion.

γ	CE*	IRR*	CE$^{(3)}$	IRR$^{(3)}$
2	3.6501	5.50%	3.6496	5.50%
5	2.1782	3.49%	2.1774	3.49%
8	1.8164	2.74%	1.8161	2.74%

3.3. Near-optimal strategy $\pi^{(3)}$

Previous subsection has highlighted that the optimal trading strategy π^* substantially outperforms the strategy $\pi^{(0)}$ based on mechanical rescaling of fixed Samuelson's portfolio weights $\hat{\pi}$ and, to a lesser extent, also the heuristic strategies $\pi^{(1)}$ and $\pi^{(2)}$. This happens for two reasons: firstly, the relative mix of stocks and bonds in the optimal portfolio varies with the value of the accumulated savings, moving progressively from stocks to bonds as the value of the savings increases over time. Secondly, for high savings levels the relative weights in stocks and bonds do depend on the risk aversion when risk aversion falls below the sum of credit-unconstrained weights $\hat{\pi}\mathbf{1}$. In this subsection we will examine the 'lifestyling' phenomenon in more detail, with the view to providing an analytic approximation of the switching formula.

On inspection of the HJB PDE (2.5a), one notes that the optimal portfolio is given by

$$\overline{\pi}^*(t,\overline{W}_t) = \underset{\overline{\pi}\geq 0, \overline{\pi}\mathbf{1}\leq\alpha_t}{\arg\max}\ \overline{\pi}(\mu - r\mathbf{1}) - \frac{1}{2}\overline{R}(t,\overline{W}_t)\overline{\pi}\Sigma\overline{\pi}^\top,$$

where $\overline{R}(t,\overline{W}_t)$ from equation (2.10) is the state-dependent coefficient of relative risk aversion of the indirect utility function and $\alpha_t = 1 - \mathrm{PV}_t/\overline{W}_t$. From a purely engineering point of view it makes sense to examine the suboptimal strategy where we replace state-dependent value $\overline{R}(t,\overline{W}_t)$ with the constant $\gamma = \overline{R}(T,\overline{W}_T)$,

$$\overline{\pi}^{(3)}(\alpha_t) = \arg \max_{\overline{\pi}\geq 0, \overline{\pi}\mathbf{1}\leq\alpha_t} \overline{\pi}(\mu - r\mathbf{1}) - \frac{\gamma}{2}\overline{\pi}\Sigma\overline{\pi}^\top = \hat{\pi}(\alpha_t,\gamma) = \alpha_t\hat{\pi}(1,\alpha_t\gamma). \quad (3.3)$$

In the world with contributions this strategy reads (see Eqs. 1.8 and 2.13)

$$\pi^{(3)}(\alpha_t) = \overline{\pi}^{(3)}(\alpha_t)/\alpha_t = \hat{\pi}(1,\alpha_t\gamma). \quad (3.4)$$

The strategies $\pi^{(i)}, \overline{\pi}^{(i)}$, $i \in \{0,1,2,3\}$ dispense with the need to solve a dynamic programming problem and leave us with a much simpler task of constrained quadratic programming. Whether $\overline{\pi}^{(3)}$ is a good approximation to the optimal strategy $\overline{\pi}^*$ now depends on how close the actual indirect risk aversion $\overline{R}(t,\overline{W}_t)$ is to the fixed value γ.

Table 6: Near-optimal strategy $\pi^{(3)}(\alpha_t)$ as a function of t and W_t with $\gamma = 8$.

W_t	$t=0$		$t=10$		$t=20$		$t=30$		$t=39.975$	
10^{-5}	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
0.01	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.581	0.197
0.05	0.000	1.000	0.000	1.000	0.000	1.000	0.084	0.916	0.553	0.188
0.1	0.000	1.000	0.000	1.000	0.118	0.882	0.443	0.557	0.550	0.187
0.2	0.180	0.820	0.313	0.687	0.460	0.540	0.622	0.378	0.548	0.186
0.3	0.387	0.613	0.476	0.524	0.574	0.426	0.682	0.318	0.548	0.186
0.5	0.553	0.447	0.606	0.394	0.665	0.335	0.730	0.270	0.547	0.186
1	0.678	0.323	0.704	0.296	0.734	0.267	0.676	0.230	0.547	0.186
2	0.740	0.260	0.723	0.246	0.670	0.228	0.611	0.208	0.547	0.186
20	0.569	0.193	0.564	0.192	0.559	0.190	0.553	0.188	0.546	0.186

In Table 5 one observes that the investment strategy $\pi^{(3)}$ is for all practical purposes indistinguishable from the fully optimal investment π^* in terms of welfare. On inspection of the portfolio weights in Tables 2 and 6, we note the largest discrepancy between the two strategies occurs for $t = 0$ at the savings level of $W = 0.2$ (recall that $\mathrm{PV}_0 = 0.82$) and it amounts to about 6 percentage points shift towards stocks for the $\pi^{(3)}$ strategy. Thus the near-optimal weights $\pi^{(3)}$ tend to be slightly riskier than the fully optimal investment for middling savings levels.

Generally speaking, the agreement between π^* and $\pi^{(3)}$ is guaranteed to be excellent for very low and very high savings levels, since in the former case both strategies invest the entire cash in hand in stocks, while in the latter case we have already seen the optimal weights of both strategies tend to the value $\pi^{(3)}(1) = \pi^{(2)}(1) = \pi^{(1)}$ given in (3.1).

Recall that the heuristic strategies $\pi^{(1)}$, resp. $\pi^{(3)}$, are based on replacing $R(t, W_t)$, resp. $\overline{R}(t, \overline{W}_t)$, with γ. We observe numerically in Table 7 that $\overline{R}$ can deviate quite substantially from the constant value γ. Hence the superior performance of strategy $\pi^{(3)}$ over $\pi^{(1)}$ does not stem from $\overline{R}$ being closer to γ than R is. Instead, $\pi^{(3)}$ does so well because the largest discrepancy between $\overline{R}$ and γ occurs at low levels of α_t and here both strategies invest everything in stocks. From theory we know $R(t, W_t) \leq \gamma$ (see Eq. A.4), which translates to

$$\overline{R}(t, \overline{W}_t) \leq \frac{\gamma}{\alpha_t}.$$

On the other hand, the numerical results in Table 7 suggest $\gamma \leq \overline{R}(t, \overline{W}_t)$ for which no theoretical proof is available as yet.

Let us now take a closer look at formula (3.3). By completing the square we have

$$\overline{\pi}^{(3)}(\alpha) = \underset{\pi \geq 0, \pi^\top \mathbf{1} \leq \alpha}{\arg\min} \left\| \pi\sigma - \gamma^{-1}(\mu - r\mathbf{1})^\top \sigma^{-1} \right\|^2. \tag{3.5}$$

Table 7: Values of $\overline{R}$ as a function of t and W_t for $\gamma \in \{2, 8\}$.

$W_t \setminus t$	0	10	20	30	30.9
10^{-5}	5.55	4.72	3.84	2.92	2.00
0.01	5.28	4.49	3.66	2.79	2.00
0.05	4.51	3.86	3.19	2.50	2.00
0.1	3.94	3.41	2.86	2.31	2.00
0.2	3.32	2.92	2.52	2.15	2.00
0.3	2.98	2.67	2.35	2.08	2.00
0.5	2.62	2.40	2.19	2.03	2.00
1	2.27	2.15	2.07	2.02	2.00
2	2.11	2.07	2.03	2.01	2.00
20	2.01	2.01	2.00	2.00	2.00

(a) $\gamma = 2$

$W_t \setminus t$	0	10	20	30	30.9
10^{-5}	13.10	11.97	10.75	9.42	8.01
0.01	12.42	11.36	10.22	8.99	8.00
0.05	10.68	9.85	8.98	8.22	8.00
0.1	9.52	8.90	8.41	8.14	8.00
0.2	8.72	8.48	8.25	8.06	8.00
0.3	8.55	8.35	8.16	8.03	8.00
0.5	8.33	8.19	8.07	8.01	8.00
1	8.11	8.05	8.01	8.00	8.00
2	8.01	8.00	8.00	8.00	8.00
20	8.00	8.00	8.00	8.00	8.00

(b) $\gamma = 8$

Since the expression on the right-hand side of (3.5) is strictly convex in π, those constraints in (3.5) that are not binding can be safely removed and the binding constraints applied with equality. Therefore, if some constraints in (3.5) are binding, (3.5) is equivalent to

$$\overline{\pi}^{(3)}(\alpha) = \underset{A_2\pi^\top = b_2}{\arg\max} \|A_1\pi^\top - b_1\|^2, \tag{3.6}$$

where $A_1 = \sigma^\top$, $b_1 = \sigma^{-1}(\mu - r\mathbf{1})/\gamma$ and A_2, b_2 represent the binding constraints. Assuming that at least one constraint is binding, the solution of (3.6) is given in Černý (2009, Corollary 4.2) as

$$\overline{\pi}^{(3)}(\alpha)^\top = A_1^{-1}b_1 + (A_1^\top A_1)^{-1}A_2^\top(A_2(A_1^\top A_1)^{-1}A_2^\top)^{-1}(b_2 - A_2A_1^{-1}b_1)\,. \tag{3.7}$$

Suppose that the only binding constraint in (3.4) is $\pi\mathbf{1} = \alpha$. In this case $A_2 = \mathbf{1}^\top = (1, 1, \ldots, 1) \in \mathbb{R}^d$, $b_2 = \alpha$ and (3.7) takes the form

$$\overline{\pi}^{(3)}(\alpha) = \frac{\hat{\pi}}{\gamma} + \zeta\left(\alpha - \frac{\hat{\pi}\mathbf{1}}{\gamma}\right), \tag{3.8}$$

where $\hat{\pi}$ from equation (1.3) represents the optimal unit risk-aversion weights without credit constraint and ζ from equation (3.2) is the minimum variance portfolio.

Recall that in our numerical illustration the lifestyling correction vector takes the value $\zeta = (95.3\%, 4.7\%)$. For high level of risk aversion $\gamma = 8$ the constraint $\pi\mathbf{1} \leq \alpha$ becomes binding below $\hat{\alpha} = 5.85/8 \approx 73\%$. The optimal investment switches 100% to stocks below $\alpha = 15.7\%$. For low level of risk aversion $\gamma = 2$ the constraint $\pi\mathbf{1} \leq \alpha$ binds for *all* values of $\alpha \in [0, 1]$ and the investment switches fully into stocks for all α below 63.4%. For γ below $1.27 = \hat{\pi}\mathbf{1} - \hat{\pi}_1/\zeta_1$ it is optimal to invest the entire cash in hand in stocks *at all times*.

Table 8: Summary of welfare performance of the optimal strategy π^* relative to heuristic strategies $\pi^{(i)}$, $i \in \{0,1,2,3\}$ over 324 model parametrizations specified in equations (3.9a–e).

γ	$\frac{\mathrm{CE}^*-\mathrm{CE}^{(0)}}{\mathrm{CE}^*}$		$\frac{\mathrm{CE}^*-\mathrm{CE}^{(1)}}{\mathrm{CE}^*}$		$\frac{\mathrm{CE}^*-\mathrm{CE}^{(2)}}{\mathrm{CE}^*}$		$\frac{\mathrm{CE}^*-\mathrm{CE}^{(3)}}{\mathrm{CE}^*}$	
	avg	max	avg	max	avg	max	avg	max
1	52.55%	87.78%	1.45%	6.55%	1.45%	6.55%	0.004%	0.083%
2	34.40%	80.07%	5.73%	12.57%	5.52%	12.31%	0.03%	0.19%
5	13.42%	49.17%	7.71%	14.50%	5.81%	13.69%	0.06%	0.39%
8	8.49%	31.57%	6.67%	14.52%	3.60%	12.98%	0.05%	0.37%

3.4. Robustness analysis

In this subsection we provide compelling evidence that the illustrative example of Subsections 3.1–3.3 is representative of general results for plausible parameter values. For this purpose, we consider 324 different parametrizations obtained as a $3 \times 3 \times 3 \times 3 \times 4$ Cartesian product of the following parameter values,

$$\mu_1 \in \{1.5\%, 2\%, 3\%\}, \tag{3.9a}$$
$$\mu_2 \in \{7\%, 10\%, 13\%\}, \tag{3.9b}$$
$$\sigma_1 \in \{3\%, 5\%, 7\%\}, \tag{3.9c}$$
$$\sigma_2 \in \{20\%, 25\%, 30\%\}, \tag{3.9d}$$
$$\rho \in \{-20\%, -5\%, 5\%, 20\%\}. \tag{3.9e}$$

The full set of results is available online in Černý and Melicherčík (2019). An aggregate summary is reported in Table 8.

We note that strategy $\pi^{(3)}$ offers and excellent approximation of π^* across the board. Looking at the detailed results over the 324 individual parametrizations, we observe the largest discrepancies occur for $\rho = -0.2$ and high expected bond return $\mu_1 = 0.03$ in combination with low bond return volatility $\sigma_1 = 0.03$.

4. Conclusions

We have analyzed optimal investment for an individual pension savings plan. As a result of the plan's inability to borrow against future contributions the Samuelson paradigm of investment in constant proportions out of total wealth including current savings and present value of future contributions changes in two important respects. Firstly, for high levels of accumulated savings the relative investment in risky bonds and stocks becomes a function of investor's risk aversion, with strong substitution from bonds towards stocks for lower values of risk aversion. Secondly, for low levels of accumulated savings it becomes optimal to switch entirely to stocks, in an investment pattern known as stochastic lifestyling (Cairns et al., 2006).

Since the computation of the fully optimal strategy is prohibitively technical for practitioners, we have put forward a near-optimal strategy involving only a static constrained quadratic programme (CQP), easily implementable in a spreadsheet. This CQP strategy is shown to be practically indistinguishable from the optimal investment in terms of its welfare implications. We have provided an explicit formula (3.8) which helps visualize the lifestyling effect and further lowers the technical barrier towards its implementation.

Three aspects of this research merit further investigation, in our view. As with any suboptimal strategy, it is desirable to have explicit bounds on the degree of suboptimality. The information relaxation approach of Brown and Smith (2014) is able to estimate the efficiency loss of suboptimal strategies when the optimal strategy is prohibitively expensive to compute. In our setting the optimal strategy is computationally feasible but perhaps the same approach can produce explicit error bounds.

Secondly, we have observed in our numerical simulations that the indirect relative risk-aversion coefficient $\overline{R}$ for the optimal strategy in the Samuelson world (2.10) satisfies $\overline{R} \geq \gamma$, implying that the near-optimal strategy $\pi^{(3)}$ is more aggresive than the optimal strategy $\overline{\pi}^*$. It is known from the comparison principle for parabolic equations that in the world with contributions the corresponding indirect relative risk-aversion coefficient (2.9) obeys $R \leq \gamma$, yielding $\overline{R} \leq \gamma/\alpha_t$. A mathematical proof of $\overline{R} \geq \gamma$ seems rather more elusive at present, cf. Xia (2011).

Last but not least, the near-optimality result has repercussions for the wider life-cycle portfolio allocation literature (Ayres and Nalebuff, 2013) and deserves to be explored further in that context.

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Appendix A. Proofs

Lemma A.1. *Let Σ be a positive definite matrix in $\mathbb{R}^{d\times d}$. Under the assumption* (2.15) *function $\hat{\pi}$ from equation* (2.12) *satisfies*

$$0 < \inf_{\rho\in(0,\gamma]} \hat{\pi}(1,\rho)\Sigma\hat{\pi}(1,\rho)^\top. \tag{A.1}$$

Moreover, for the investment strategies $\pi^{(i)}, i = 0,1,2,3$ one has

$$0 < \inf_{(t,x)\in[0,T)\times\mathbb{R}^+} \pi^{(i)}(t,x)\Sigma\pi^{(i)}(t,x)^\top. \tag{A.2}$$

Proof. Let i be the index for which $\mu_i > r$. Let c_i denote the i–th diagonal term of the matrix Σ and define

$$q_\rho(\pi) = \pi(\mu - r\mathbf{1}) - \frac{\rho}{2}\pi\Sigma\pi^\top.$$

Consider $\tilde{\pi} = (0,0,\ldots,\tilde{\pi}_i,0,\ldots,0)$ with

$$\tilde{\pi}_i = \min\left(\frac{\mu_i - r}{\gamma c_i}, 1\right) > 0.$$

For $\frac{\mu_i - r}{\gamma c_i} \leq 1$ we obtain

$$q_\rho(\tilde{\pi}) = \frac{(\mu_i - r)^2}{\gamma c_i} - \frac{\rho}{2\gamma}\frac{(\mu_i - r)^2}{\gamma c_i} \geq \frac{1}{2}\frac{(\mu_i - r)^2}{\gamma c_i}.$$

For $(\mu_i - r)/(\gamma c_i) > 1$ we have $\tilde{\pi}_i = 1$ and therefore

$$q_\rho(\tilde{\pi}) = (\mu_i - r) - \frac{1}{2}\rho c_i \geq (\mu_i - r) - \frac{1}{2}\gamma c_i \geq \frac{1}{2}(\mu_i - r).$$

From the above estimates one obtains

$$\begin{aligned}\inf_{0<\rho\leq\gamma}\left(\sup_{\tilde{\pi}\mathbf{1}\leq\pi\mathbf{1}\leq 1,\pi\geq 0} q_\rho(\pi)\right) &\geq \inf_{0<\rho\leq\gamma} q_\rho(\tilde{\pi}) \\ &\geq \min\left(\frac{1}{2}\frac{(\mu_i - r)^2}{\gamma c_i}, \frac{1}{2}(\mu_i - r)\right) = \delta > 0.\end{aligned}$$

On the other hand, setting $\varepsilon = \frac{1}{2}\frac{\delta}{\mathbf{1}^\top|\mu - r\mathbf{1}|} > 0$ one obtains for all $\rho > 0$

$$\sup_{\pi\mathbf{1}\leq\varepsilon,\pi\geq 0} q_\rho(\pi) \leq \pi(\mu - r\mathbf{1}) \leq \delta/2 < \delta.$$

Therefore, arguing by contradiction, the optimal strategy verifies

$$\inf_{0<\rho\leq\gamma} \hat{\pi}(1,\rho)\mathbf{1} > \varepsilon,$$

which in view of the assumed regularity of σ guarantees (A.1).

It remains to prove (A.2). Recall $\pi^{(0)}$ and $\pi^{(1)}$ are constant and different from the zero vector therefore the result follows by positive definiteness of Σ. We have

$$\pi^{(2)} = \frac{\pi^{(1)}}{\max(\pi^{(1)}\mathbf{1}, \alpha_t)}$$

and therefore in view of $\alpha(t,x) = x/(\mathrm{PV}_t + x) \leq 1$

$$\begin{aligned}
0 < \pi^{(1)}\Sigma\pi^{(1)\top} &\leq \pi^{(1)}\Sigma\pi^{(1)\top} \inf_{(t,x)\in[0,T)\times\mathbb{R}_+} \frac{1}{\max(\pi^{(1)}\mathbf{1}, \alpha(t,x))} \\
&\leq \inf_{(t,x)\in[0,T)\times\mathbb{R}^+} \pi^{(2)}(t,x)\Sigma\pi^{(2)}(t,x)^\top.
\end{aligned}$$

Finally, from (3.4) recall $\pi^{(3)}(t,x) = \hat{\pi}(1, \alpha(t,x)\gamma)$. Inequality (A.2) now follows from (A.1) because $0 \leq \alpha(t,x) \leq 1$. □

Proposition A.2 (Kilianová and Ševčovič 2013). *Assume $g : \mathbb{R}_+ \to \mathbb{R}$ is differentiable, its derivative is Lipschitz-continuous and satisfies inequality* (2.16). *Then the following statements hold.*

1) *The Cauchy problem*

$$\begin{aligned}
\partial_t\rho - \partial_z^2 g(\rho) + \partial_z[(y(t)e^{-z} + r)\rho - (1-\rho)g(\rho)] &= 0, \\
\rho(T,z) &= \gamma,
\end{aligned} \tag{A.3}$$

has a unique solution $\rho(t,z)$ in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$. This solution satisfies

$$0 < \rho(t,z) \leq \gamma \text{ on } [0,T)\times\mathbb{R}, \tag{A.4}$$

and it is Hölder-continuous of degree $H^{1+\lambda/2, 2+\lambda}$ for any $0 < \lambda < \frac{1}{2}$.

2) *For ρ from part* 1) *the linear PDE*

$$\begin{aligned}
u_t + u_z\left(ye^{-z} + r + g(\rho)\right) &= 0, \\
u(T,z) &= \frac{e^{(1-\gamma)z}}{1-\gamma}.
\end{aligned} \tag{A.5}$$

has a unique classical solution u.

3) *Function $u(t,z)$ from part* 2) *is the unique $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$ solution of the Cauchy problem*

$$\begin{aligned}
u_t + u_z\left(ye^{-z} + r + g\left(1 - \frac{u_{zz}}{u_z}\right)\right) &= 0, \\
u(T,z) &= \frac{e^{(1-\gamma)z}}{1-\gamma}.
\end{aligned} \tag{A.6}$$

4) *Conversely, if* u *denotes the unique classical solution from item* 3) *then* $\rho = 1 - u_{zz}/u_z$ *is the unique classical solution of* (A.3).

Proof. Combine Theorems 3.3 and 5.2 and Proposition 3.4 in Kilianová and Ševčovič (2013). □