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Pollution, mortality and time consistent abatement taxes[☆]

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ABSTRACT

We study dynamically consistent policy in a neoclassical overlapping generations growth model where pollution externalities undermine health but are mitigated via tax-financed abatement. With arbitrarily constant taxation, two steady states arise: an unstable 'poverty trap' and a 'neoclassical' steady state near which the dynamics might either be monotonically convergent or oscillating. When the planner chooses a time consistent abatement path that maximizes a weighted intergenerational sum of expected utility, the optimal tax is zero at low levels of capital and then a weakly increasing function of the capital stock. The non-homogeneity of the tax function along with its feedback effect on savings induces additional steady states, stability reversals and oscillations.

1. Introduction

Pollution, especially particulate matter (*PM*), nitrogen dioxide, sulphur dioxide, and ozone, leads to increase of cardiovascular and respiratory diseases resulting in premature mortality.[1] There is a three-way link between pollution, mortality and economic growth: while economic growth reduces mortality rates through the effect of higher income (and nutrition) and better health outcomes,[2] it also generates pollution which in turn increases mortality. Changes in mortality in turn affect savings decisions and thus, growth and pollution. This paper studies time consistent taxes in an overlapping generations model that incorporates this three-way effect and pollution is treated as an externality.

Recent economic literature has recognized the possibility that multiple steady states, poverty traps and cycles can arise from the interplay between the three factors and proposed various policy options, via either golden-rule, steady state analysis or Pigouvian taxes, to offset these outcomes.[3] It has, however, not studied optimal policy. This viewpoint is important as it addresses the issues of dynamic consistency and implementability which are both problematic in overlapping generations. There is a well-known

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1 Water pollution, carcinogens of both gaseous and soil contaminant types, heavy metals (such as mercury), persistent organic pollutants (POPs such as DDT, dioxin), etc. are other types of pollution that lead to premature mortality. There is robust micro evidence that exposure to particulate matter PM_{10} and $PM_{2.5}$, leads to increased cardiovascular disease, chronic obstructive pulmonary disease (COPD) and, controlling for other factors, an increase in mortality (see Ayres, 2006; Huang et al., 2012; Evans et al., 2013; Miller et al., 2007; Pope et al., 2004; HEI, 2010; Viegi et al., 2006). A 10 μg per cubic meter increase in PM_{10} leads to an increase in mortality by 0.51% and if other gases such as ozone, nitrogen dioxide, sulfur dioxide and carbon monoxide which are correlated with PM_{10} are taken into account the distribution of mortality shifts to the right (Samet et al., 2000). These effects are present in both developed and developing countries.

2 Preston (1975) was one of the earliest papers to document the positive effect of income on life expectancy. The recent survey by Cutler et al. (2006) documents this effect across countries and within countries. In their interpretation, income alone is not sufficient but it is correlated with willingness for effective public health delivery.

3 See Jouvet et al. (2010), Mariani et al. (2010), Varvarigos (2008, 2014), Palivos and Varvarigos (2017) and Raffin and Seegmuller (2014). Also see Stokey (1998) who studies the first best problem in a long-lived agent model with environmental externalities but no mortality effects. Wang et al. (2015) study a complementary model where pollution affects morbidity but not mortality. For earlier studies of taxes relying on steady state analysis to correct environmental externalities in overlapping generations models see Bovenberg and Heijdra (1998), John and Pecchenino (1994), and John et al. (1995).

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commitment problem in imposing taxes on future generations (see Ghiglino and Tvede, 2000, and the survey by Erosa and Gervais, 2001). John et al. (1995) highlight the problem of using pre-committed Pigouvian taxes in such an overlapping generations set-up with externalities. We characterize the optimal tax function, and in addition show that the state contingent optimal taxes can alter the transition dynamics in the model.

We study a two-period overlapping generations model where the probability of survival into old age is determined endogenously.[4] Production of a single consumption-capital good creates pollution as a by-product. Increased pollution increases the probability of premature death but increased income has a prophylactic effect on mortality. The positive effect of income on mortality follows the literature which has pointed out that increased income can counteract some of the adverse effects of pollution via better nutrition and greater access to health care.

The two contrary forces that affect mortality can result in a non-convexity that gives rise to multiple interior steady states: a low capital, locally unstable steady state with lower per capita consumption and life expectancy and a high capital "neoclassical" steady state with higher per capita consumption and life expectancy.[5] The latter steady state can be either locally stable or induce oscillations around it; in most of our analysis, we assume the former possibility. Moreover, an increase in the tax rate can increase the steady state capital in the neoclassical steady state while simultaneously widening the basin of attraction of the trivial steady state.[6]

The main contribution of the paper is to characterize optimal abatement policy in a second best context. The government imposes a wage tax on the young in each period. This tax affects their savings behavior and next period's capital stock, imposing contradictory externalities on the next generation: a higher capital stock means higher incomes which reduce the next generation's mortality but also means higher emissions which increase it. It is not possible in our model to offset the externality entirely by means of the wage tax.[7] Thus, the government has only a second-best instrument to maximize a weighted sum of life-cycle utilities of all generations, subject to each generation's incentive constraints regarding savings behavior.

We establish the existence of an optimal tax function and derive its main characteristics. First, below a threshold level of capital, the optimal tax is zero and there is no pollution abatement, as at low levels of pollution, the marginal effects of additional pollution are negligible. [8] Second, in the region of positive taxation, the optimal tax is weakly increasing in the capital stock. Third, we show via numerical simulations that the optimal tax at any given capital stock increases with the size of the inter-generational discount factor of the planner.

While the characteristics are unsurprising and plausible, the first of these, *i.e.* the non-homogeneity of the optimal tax, creates further intriguing dynamic possibilities. In particular, it can induce multiple versions of each type of steady state, whereas under arbitrary taxation there would be at most one of each. In other words, whenever no abatement can be optimal in a steady state of either type, the possibility arises that there is another steady state of the same type with positive abatement.[9] Moreover such multiplicity of steady states can lead to a reversal in stability properties relative to the arbitrary tax case: a poverty trap can become a sink while a neoclassical steady state can become a source. Finally, optimal taxes may introduce endogenous fluctuations in the neighborhood of either type of steady state. Such dynamics arise even when the government places relatively high weight on the utility of future generations.

Such results are also relevant to a broader literature that has addressed the interaction of economic policies and endogenous fluctuations in dynamic general equilibrium (see Woodford, 1994a). One strand of this literature (see Goenka and Liu, 2012; Grandmont, 1985) shows that state-dependent economic policies can be used to stabilize endogenous economic fluctuations. Another strand shows that simple, non-state dependent feedback policies can themselves be a source of endogenous economic fluctuations (see Goenka, 1994a,b; Grandmont, 1986; Smith, 1994; Woodford, 1994b), while state-dependent feedback policies may eliminate these. In this paper we present a different type of difficulty: when the private response to optimal policy shows potential non-convexities and the policy-maker is restricted to second-best instruments, state-dependent policies can generate non-linear dynamics in the evolution of state variables.

Our results have both a mathematical and a policy interpretation. From the mathematical standpoint, arbitrarily constant taxes mean that the path of capital is traced out by iterating a single first-order difference equation that links next period's capital stock to the current one. As is well known, this can be graphically depicted by a single one-dimensional phase diagram that maps each period's capital stock into next period's. In the presence of pollution externalities this map can generate two interior steady states: a poverty trap and a neoclassical steady state, differentiated not just by levels of capital but also by the respective slopes of the phase diagram as it crosses each steady state.

Under optimal taxation, a stable tax *function* replaces a constant tax and a shifting family of phase diagrams replace the single one. This adds an additional dimension to the dynamics and, together with the non-homogeneity of the tax function, drives the reversal in dynamic stability. For example, if it is possible to have a steady state with zero optimal taxation, then there can be one or more additional steady states with positive optimal taxation. If a zero-tax steady state is a poverty trap then the additional steady state(s) with positive taxation can be either poverty trap(s) or neoclassical. If the former, the positive-tax poverty trap is locally stable. If zero taxation is also optimal at a neoclassical steady state then, under the added contingency that higher taxes lead to higher capital in a neoclassical steady state, two additional positive-tax, neoclassical steady states arise, of which one will be locally unstable.

The emergence of fluctuations induced entirely by optimal policy is also related to the shifts in the phase diagram as the tax rate evolves over time. If a small change in the abatement tax is likely to produce a large shift in the phase diagram then the economy can jump from a low capital, low tax equilibrium in

4 See Chakraborty (2004) and Bhattacharya and Qiao (2007).

5 In addition, most overlapping generations models yield a *trivial* steady state with zero economic activity and ours is no exception.

6 The former possibility has been shown in the literature to arise in a variety of contexts: environmental degradation imposes costs that are external to each decision-maker so any policy that offsets this externality helps reduce these costs and if the balance is right, actually promotes growth (see Economides and Philippopoulos, 2008; John et al., 1995; Mariani et al., 2010; Palivos and Varvarigos, 2017 for an analysis of such effects in a variety of settings).

7 In Section 3, we discuss why the wage tax is the only reasonable tax instrument in our framework.

8 Palivos and Varvarigos, 2017 derive a similar result for a policy that maximizes survival probability. Note that life-cycle welfare per generation does not increase unambiguously with survival probability as the direct positive effect of enhanced survival and the indirect positive effect due to greater incentives to accumulate capital can be offset by the indirect negative effect that higher survival rates have on the net return to savings. Our welfare criterion captures all three effects.

9 The classification of a steady state as poverty trap versus neoclassical is determined by the slope of the capital evolution map in the neighborhood of that steady state.

one period to a high capital, high tax equilibrium in the next. This happens even if the same economy evolves monotonically under an arbitrarily constant tax rate.

In Section 2, the benchmark model with arbitrary tax policy is developed, its equilibrium and dynamic properties are characterized and the effect of higher abatement taxes on steady states is analyzed. Section 3 studies the second-best optimal tax. In this section we first characterize properties of the optimal tax function, and then study the dynamics of the equilibrium trajectories. The final section concludes.

2. Model

We study a discrete time, $t = 0, 1, \ldots$, overlapping generations model. In each period a generation consisting of a continuum, normalized to measure one, of identical agents is born. An agent born in period t lives for at most two periods: the period of birth and can survive to old age with probability π_t. A young agent supplies one unit of labor inelastically receiving the wage w_t which is used to finance current consumption, c_t^y and savings for old age, s_t. Old agents have no labor endowment and live entirely off the proceeds of their savings. Following the literature on uncertain lifetimes, we assume that there is a perfect annuity market in which young agents buy annuities from perfectly competitive intermediaries who lend out the proceeds to firms for investment in productive capital. Each unit of time t investment results in one unit of time $t+1$ capital, k_{t+1} which becomes immediately available for production and fully depreciates in that period. Thus,

$$k_{t+1} = s_t \tag{1}$$

At time $t = 0$, k_0 is exogenously given.

The production function is constant returns to scale Cobb–Douglas and can be expressed in intensive form:

$$y_t = Ak_t^{\alpha};$$

where y is output per worker and k is capital per worker.

The gross returns to capital and labor r_t and w_t respectively, are equal to their marginal products:

$$w_t = (1-\alpha)Ak_t^{\alpha}; \tag{2}$$

$$r_t = \frac{\alpha A}{k_t^{1-\alpha}}. \tag{3}$$

As a positive fraction of savers do not live into old age, the return on period t savings for those who survive is r_{t+1}/π_t.

The production of final output creates a flow of pollutants proportional to gross output. However, because population is normalized to unity, per-capita and gross quantities are numerically identical so for notational consistency we use lower case z to denote pollution flows and relate it to y, and write it as $z_t = \gamma y_t$. The pollution we are modeling here consists of PM_{10} and similar particulate matter and pollutants such as NO_x which have been linked to health effects. Evidence shows that these pollutants are short-lived, except in certain areas characterized by their geography and the nature of economic activity, so that they can be treated as a flow (Varotsos et al., 2005; Windsor and Tuomi, 2001; Zeka et al., 2005).[10]

Environmental policy consists of a planner imposing an environmental tax, τ_t on the wage incomes of the contemporaneous young,[11] the proceeds of which are spent on operating a clean-up technology that reduces the flow of pollutants. We assume that this technology can only be operated by a central authority so that individual agents do not have the means to abate privately.[12] The efficiency of this technology is denoted by $\chi > 0$, and the reduction in pollution flows, is assumed to be a linear function of tax-financed expenditures. Thus, the net flow of pollutants is:

$$z_t = \gamma y_t - \chi \tau_t w_t;$$

which, after substituting for w_t and redefining terms, simplifies to

$$z_t = \gamma(1-\psi\tau_t)Ak_t^{\alpha}. \tag{4}$$

where $\psi = \chi(1-\alpha)/\gamma$. We assume $\psi \in [0, 1]$ to avoid the possibility that as a result of abatement, the flow of pollution is negative.

The probability of survival into old age is identical for all agents and is represented by a twice continuously differentiable function of y_t and z_t. Longevity is increasing in per-capita income and decreasing in pollution. If per-capita income is zero, the survival probability is set to zero regardless of the stock of pollution and as the stock of pollution approaches infinity, survival probability tends to zero regardless of the level of income.

Assumption 1.

$$\pi_t = \pi(k_t) = \pi(y(k_t), z(k_t)); \tag{5}$$

$$\pi \in [0, 1], \quad \forall y \geq 0 \; \& \; \forall z \geq 0; \tag{6}$$

$$\frac{\partial \pi}{\partial y} \equiv \pi_y(y, z) \geq 0, \quad \forall y > 0; \tag{7}$$

$$\frac{\partial \pi}{\partial z} \equiv \pi_z(y, z) \leq 0, \quad \forall z > 0; \tag{8}$$

$$\pi(0, z) = 0, \;\; \forall z \geq 0; \tag{9}$$

$$\pi(y, \infty) = 0, \;\; \forall y \geq 0. \tag{10}$$

The only consequence of pollution in this model is that it creates a negative external effect on expected lifetimes. Given the overlapping generations framework this externality affects the young generation alone by affecting their expected lifetime utility. As only the young work, the output is not affected by pollution directly. Thus, there is a potential for welfare improvement by means of a tax on the young, the proceeds of which are spent on abating pollution. Future generations are affected indirectly through the effects of pollution on savings of the current generation, i.e. the next period's capital stock.

In order to derive closed form solutions, we assume log-linear utility[13]:

$$U_t = \ln c_t^y + \pi_t \ln c_{t+1}^o;$$

which the agent maximizes subject to the life-cycle budget constraints:

$$c_t^y \leq (1-\tau_t)w_t - s_t; \tag{11}$$

$$c_{t+1}^o \leq \frac{r_{t+1}}{\pi_t} s_t; \tag{12}$$

where c_t^y is consumption when young, s_t is the young agent's savings and c_{t+1}^o is *ex post* consumption for an agent who survives into old-age.

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[10] This is different from the issue of greenhouse gas build-up that arises in the global warming literature. In earlier versions of the paper, Goenka et al. (2012) we show that allowing for persistence of pollution does not affect results under some conditions.

[11] The reason for restricting the incidence of environmental taxes to the young generation is explained in Section 3 where the optimal tax policy is derived.

[12] Some papers have considered private abatement in contexts in which the benefits of pollution abatement a re unambiguously positive, see John and Pecchenino (1994), John et al. (1995), and Mariani et al. (2010). In our model this is not the case, see Sections 4.1 and 4.2.

[13] The qualitative results hold under more general specifications of *CRRA* utility functions.

Taking the first-order condition with respect to savings,

$$-\frac{1}{c_t^y}+\frac{\pi_t}{c_{t+1}^o}\frac{r_{t+1}}{\pi_t}=0;$$

and combining with Eqs. (11), (12) and (3), results in the savings function:

$$s_t=\frac{\pi_t}{1+\pi_t}A\cdot(1-\tau_t)(1-\alpha)k_t^{\alpha}.$$

2.1. Equilibrium

Using the market clearing condition, *i.e.* substituting into Eq. (1) we have:

$$k_{t+1}=\frac{\pi_t}{1+\pi_t}A\cdot(1-\tau_t)(1-\alpha)k_t^{\alpha}. \tag{13}$$

The path of the capital stock is traced out by recursive application of Eq. (13) from a given k_0 while the accompanying evolution of the flow of pollution follows from recursively applying Eq. (4). The other variables are updated similarly.

2.2. Dynamics

We first characterize dynamics for a fixed, exogenous tax rate, τ. This will aid understanding the state-contingent tax policy.

A steady state is described by the following equations:

$$\pi=\pi(k)=\pi(y(k),z(k)); \tag{14}$$

$$k=\frac{\pi(k)}{1+\pi(k)}A\cdot(1-\tau)(1-\alpha)k^{\alpha}; \tag{15}$$

$$z=\gamma(1-\psi\tau)Ak^{\alpha}; \tag{16}$$

$$y=Ak^{\alpha}; \tag{17}$$

where π, k, z and y denote steady state values of the respective variables. Eq. (15) can be written as

$$k=\mathtt{G}(k);$$

where

$$\mathtt{G}(k)=\frac{\pi(k)}{1+\pi(k)}\Gamma k^{\alpha};$$

and $\Gamma=A\cdot(1-\tau)(1-\alpha)$ is a constant. Evaluating $\mathtt{G}(\cdot)$ at $k=0$,

$$\mathtt{G}(0)=\frac{0}{1+0}\Gamma(0)^{\alpha}=0;$$

implies a trivial steady state exists at $k=0$.

If π, the survival probability, were constant, then $\mathtt{G}(k)$ would represent a standard concave neoclassical growth mapping, with $\mathtt{G}'(0)=\infty$, $\mathtt{G}''(k)<0\ \forall k$, so that a unique interior steady state exists and the dynamics would be globally stable. However, with endogenous survival probability, other possibilities exist.

Lemma 1. $\lim_{k\downarrow 0}\pi'(k)<\infty$ *is a sufficient condition for* $\mathtt{G}'(0)=0$.

Proof. See Appendix A. ■

Lemma 1 establishes the possibility of multiple steady states. While it is stated in terms of the reduced-form version of the survival probability, it is instructive to take into account the chain of dependence of π on y and z and through these variables, on k. Given the Cobb–Douglas production function assumed throughout the paper, we can express $\pi'(k)$ as:

$$\pi'(k)=\pi_y\frac{y}{k}+\pi_z\frac{z}{k}.$$

In order for the sufficient condition to hold, π_y and π_z should have exponents in k which are large enough to offset the denominator. The following specialization of Assumption 1 is sufficient to ensure this outcome, and we impose it from hereon:

Assumption 2.

$$\pi_t=\pi((y_t)^{\vartheta},(z_t)^{\delta});$$

$$\min\{\vartheta,\delta\}\geq\frac{1}{\alpha}.$$

To establish existence of multiple interior steady states, the steady state equation can be rearranged as follows:

$$\Gamma=\frac{1+\pi(k)}{\pi(k)}k^{1-\alpha}.$$

Given the function $\pi(k)$ and any finite and positive value of k, the right-hand side will be positive and finite. Since Γ is exogenous and positively related to A for $\tau<1$ and $\alpha<1$, there always exists A large enough that

$$\Gamma>\frac{1+\pi(k)}{\pi(k)}k^{1-\alpha}.$$

This leads to the following result, stated without proof:

Lemma 2. *For any* $\alpha\in(0,1)$ *and* $\tau\in(0,1)$ *there exists an* $\hat{A}<\infty$ *and a* $\hat{k}<\infty$ *and associated* $\hat{\Gamma}$: $\hat{\Gamma}=(1+\pi(\hat{k}))/(\pi(\hat{k}))\hat{k}^{1-\alpha}$, *such that* $\Gamma>\hat{\Gamma}$, $\mathtt{G}(\Gamma,\hat{k})>\hat{k}$.

Lemma 2 implies that so long as total factor productivity (TFP) is high enough (given a function $\pi(k)$), $\mathtt{G}(k)$ will exceed k for a non-empty interval of values of k. Along with the results on the slope and level of $\mathtt{G}(k)$ derived earlier, this leads to the following proposition

Proposition 1. *If TFP, A, is large enough, and Assumption 2 holds, then there are two interior steady states,* k_ℓ^* *and* k_h^*, *such that* $k_\ell^*<\hat{k}<k_h^*$.

The higher steady state, k_h^* has more capital and therefore more consumption as well as a higher flow of pollution. Despite this, it offers a greater survival probability. In the steady state, the survival probability is

$$\pi(k)=\frac{k^{1-\alpha}}{\Gamma-k^{1-\alpha}};$$

which is increasing in k.

Fig. 1 represents the phase diagram for this one-dimensional dynamic system, depicting k_{t+1} as an S-shaped function of k_t for a given tax rate, τ.

$\mathtt{G}(k)$ is S-shaped upwards, sharing its origin with the 45° line and intersecting it at two other points k_ℓ^*, k_h^*. Since, for points which lie between the origin and k_ℓ^*, $\mathtt{G}(k)$ lies below the 45° line, any path starting off with $k_0\in(0,k_\ell^*)$ will converge to the trivial steady state, while for points between k_ℓ^* and k_h^*, $\mathtt{G}(k)$ lies above the 45° line, any path starting off at $k_0>k_\ell^*$ will grow towards k_h^*. k_ℓ^* represents a poverty trap not just in the sense that it is the steady state with lower levels of economic activity and pollution flows, but also in the sense that it represents a threshold below which the equilibrium path of the economy converges asymptotically towards zero and is thus called the "poverty trap". k_h^* resembles (locally) the steady state of a neoclassical growth model in the sense that $\mathtt{G}(k)$ cuts the 45° line from above and, for this reason, we label it as a "neoclassical" steady state.

The intuition behind Proposition 1 is as follows: as an economy starts off from a very low level of capital, the negative effects of pollution on life expectancy are dominated by the positive effects of income. This is what makes the transformation map $\mathtt{G}(k)$ slope upward. Assumption A2 ensures that at low levels of capital, both effects are, however, small so that longevity increases only gradually at first. Thus, changes in savings are also dampened, making $\mathtt{G}(k)$ slope upward relatively slowly at low levels of capital. Along with Lemma 2 and the associated restriction on the

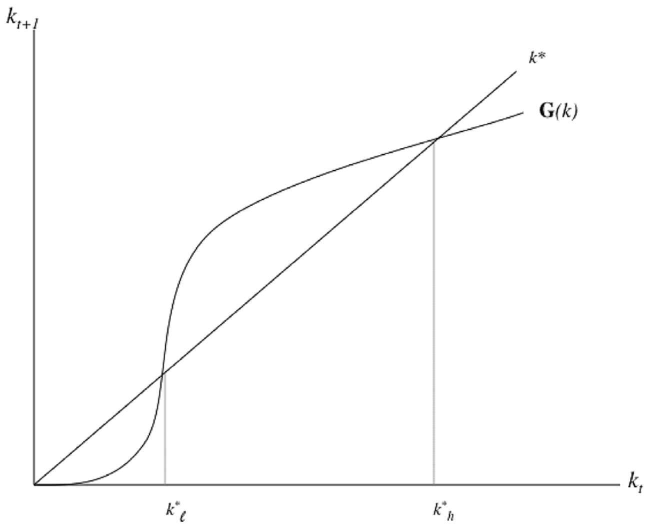

Fig. 1. Multiple steady states.

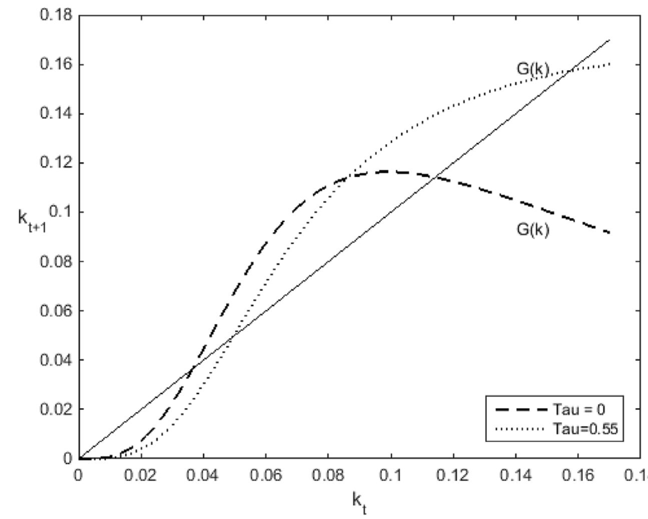

Fig. 2. A uniform increase in an exogenous tax rate.

magnitude of TFP, this is what generates multiple steady states. If the marginal effect of higher capital on longevity was infinitely large at zero capital the poverty trap would disappear, but a neoclassical steady state could still arise.

If G(k) slopes upward as it crosses the 45° line, then k_h^* will be locally stable. At the same time, the concavity of G(k) can lead to it to slope downward as it crosses the 45° line. A necessary condition for this to happen is $\pi'(k) < 0$, which can happen at high enough values of k. This can lead to oscillations and limit cycles in the stock of capital and the flow of pollution around the upper steady state.[14] This result is consistent with Palivos and Varvarigos (2017) who show that the negative effects of pollution on longevity can lead to endogenous fluctuations.[15] In our paper we extend this insight by showing that optimal policy can generate cycles even when the underlying economy with arbitrary abatement policy exhibits monotone convergence to the neoclassical steady state.

2.3. The effect of arbitrary tax policy

To understand the effect of an arbitrary abatement policy on growth, we differentiate G(k), with respect to τ:

$$\left.\frac{\partial \mathrm{G}(k)}{\partial \tau}\right|_k = \left[-\frac{\pi}{1+\pi} - \frac{\pi_z \gamma \psi (1-\tau) A k^\alpha}{(1+\pi)^2}\right](1-\alpha)Ak^\alpha; \tag{18}$$

where π_z is the partial of π with respect to z alone (the effect of k on z is accounted for by the rest of the numerator in the second term). The above derivative is ambiguous in sign because $\pi_z < 0$. An increase in τ lowers net wage incomes, which at constant π shifts G(k) downwards. On the other hand, a higher τ raises π via the abatement effect on z. This tends to work against the downward shift in G(k). But the latter effect is weighted by k^α and is likely to be dominated by the direct effect of τ on wage income at low values of k. Thus G(k) is likely to shift down at low levels of k but it might shift up at higher levels.

Fig. 2 maps the shift in G(k) in the context of a specific numerical example, details of which are provided in Appendix C.

[14] Note that G(k) cannot slope downwards at the low steady state, even if $\pi'(k) < 0$.

[15] Seegmuller and Verchère (2004) and Cao et al. (2011) propose an alternate mechanism through which pollution can lead to endogenous fluctuations, namely that the marginal disutility of pollution increases with the stock of pollution.

An increase in the exogenous abatement tax moves both types of steady state to the right. The shift of the poverty trap widens the basin of attraction for the trivial steady state, while the neoclassical steady state moves towards higher capital. Note that in our example, the phase diagram associated with the original tax rate is downward sloping at the neoclassical steady state indicating the presence of oscillations.

3. Optimal taxes

In this section we characterize the sequence of state-contingent, second-best taxes that are imposed to fund pollution abatement. In choosing the sequence of optimal abatement taxes to maximize the social planner maximizes a weighted sum of lifetime utilities of each generation born at time $t+i$, $i \geq 0$, with $0 \leq \beta < 1$ representing the inter-generational discount factor. The welfare function is

$$W_t = \pi_{t-1} c_t^o + \sum_{i=0}^{\infty} \beta^i U_{t+i};$$

where

$$U_{t+i} = \ln c_{t+i}^y + \pi_t \ln c_{t+i+1}^o; \quad i \geq 0.$$

The planner imposes a sequence of wage taxes rates $\{\tau_{t+i}\}_{i=0}^{\infty}$ to maximize the above.

A wage tax is the natural policy instrument in the model. Ours is a one-sector model which can be viewed as having only one choice variable for private agents, namely savings for old age, because of the life-cycle budget constraint given by Eqs. (11) and (12). The only possible instruments are taxes on output, capital and wages.[16] An output tax, because of constant returns to scale, amounts to a uniform tax on wage and capital incomes.

[16] We do not have a 'dirty' sector which could be taxed to fund transfers to a 'green' sector; neither do our agents have access to technologies that might offset pollution. So the type of Pigouvian taxes that can tilt incentives towards green activities are not available in our model. Some of the related papers in the literature, *e.g.* John and Pecchenino (1994) and Mariani et al. (2010) consider private abatement activity. This is not applicable in our model since the pollution externality arising from agents' savings decisions is passed on to agents not alive at the time the decisions are made. It should also be noted that in their papers, it is always welfare-improving to tax polluting activities and encourage abatement but in our model, reducing pollution may not improve welfare, given the dual external determinants of mortality.

Taxing capital incomes is problematic as it makes the old in the initial period worse off and introduces an inter-generational conflict where there is none. Hence, only wage taxes have the potential to be weakly welfare-increasing, albeit in a second-best way because of their effects on savings. Likewise, the planner is constrained to non-negative tax rates as any subsidy to the current young can only come at the expense of the current old.

Since the planner's policies are, by construction, welfare-neutral with respect to the surviving old at time t, we confine our attention to a truncated welfare function $\tilde{W}_t$ that excludes time t old. It is well known that in the absence of viable commitment strategies, the path of optimal taxes in an overlapping-generations economy may be time-inconsistent (Erosa and Gervais, 2001). To avoid this, we use dynamic programming to formulate each period's policy choice as a function of the state of the economy.

$$\tilde{W}_t = V(k_t) = \max_{\tau_t}\left[U_t + \beta V(k_{t+1})\right].$$

Plugging in private decisions regarding c_t^y, c_{t+1}^o and k_{t+1} from Eqs. (11)–(13) respectively into the objective function, we have

$$V(k_t) = \max_{\tau_t}\left[\ln\left(\frac{(1-\tau_t)(1-\alpha)Ak_t^{\alpha}}{1+\pi(k_t)}\right) + \pi(k_t)\ln\left(\frac{\hat{A}(1-\tau_t)^{\alpha}k_t^{2\alpha}}{\pi(k_t)^{1-\alpha}(1+\pi(k_t))^{\alpha}}\right) + \beta V(k_{t+1})\right]; \quad (19)$$

where $\hat{A} \equiv \alpha(1-\alpha)^{\alpha}A^{1+\alpha}$ is a constant. Taking the first-order condition:

$$\frac{\partial V_t}{\partial \tau_t} = \Omega_t\frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} + \beta\frac{\partial V_{t+1}}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial \tau_t} \leq 0; \quad (20)$$

where

$$\Omega_t = \ln c_{t+1}^o - \frac{2-\alpha+\pi_t}{1+\pi_t};$$

when Eq. (20) is < 0 implies a zero tax. Next, taking the derivative $\partial V_t/\partial k_t$ of the value function at time t and updating it by one period, we get

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = \frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1}\frac{\partial \pi_{t+1}}{\partial k_{t+1}}.$$

Finally taking into account the dependence of k_{t+1} on τ_t via Eq. (13),

$$\frac{\partial k_{t+1}}{\partial \tau_t} = \frac{A(1-\alpha)k_t^{\alpha}}{1+\pi_t}\left[\left(\frac{1-\tau_t}{1+\pi_t}\right)\frac{\partial \pi_t}{\partial \tau_t} - \pi_t\right].$$

Putting everything together we can express the first-order condition as

$$\frac{\partial V_t}{\partial \tau_t} = \Omega_t\frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} + \beta\left[\frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1}\frac{\partial \pi_{t+1}}{\partial k_{t+1}}\right] \times\left[\frac{A(1-\alpha)k_t^{\alpha}}{1+\pi_t}\left\{\left(\frac{1-\tau_t}{1+\pi_t}\right)\frac{\partial \pi_t}{\partial \tau_t} - \pi_t\right\}\right]. \quad (21)$$

The terms in Eq. (21) represent the following effects: (i) the direct effects of a tax on the wage income of the current young, (ii) the indirect effects working through induced changes in survival probability and (iii) the intergenerational spillover induced by the effect of current taxes on the capital stock available to the next generation's young workers. The direct effects reduce both consumption and savings by the young, and are negative. These are captured by the second term in the optimality condition.

The indirect effects are captured by the first term, Ω_t. An environmental tax raises survival probability, leading to higher expected utility in old age. At the same time the higher survival probability reduces actual consumption at both young and old age, the first because savings are increasing in survival probability; the second because although individuals save more the return to their annuities yields less because of the higher survival rate of the population. This effect can be confirmed from Eq. (19) in which the term capturing the optimal old-age consumption decreases in π. The intuition is that while per-capita old-age capital increases by a factor of $[\pi/(1+\pi)]^{\alpha}$, the market return on a unit annuity decreases by a factor $1/\pi$.

Finally, the intergenerational effect depends on a combination of three factors: the effect of a current abatement tax on capital stock in the next period; the effect of a higher capital stock next period on the lifetime utility of the next generation and the magnitude of the intergenerational discount factor. The first two of these effects are both ambiguous, consisting themselves of further sub-effects, but whatever their sign, their magnitude is proportional to the intergenerational discount factor β.

Before proceeding to further disentangle these effects we shall first consider the case of $\beta = 0$: this is the case of a myopic government concerned only with the welfare of a single contemporaneous generation. This is a benchmark case which yields tractable results that are extended to the general case.

3.1. Myopic social planner

When $\beta = 0$, first-order condition, Eq. (20), reduces to

$$\frac{dV_t}{d\tau_t} \equiv \mathtt{H}(k_t,\tau_t) = \left[\ln c_{t+1}^o - \frac{2-\alpha+\pi_t}{1+\pi_t}\right]\cdot\frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} \leq 0; \quad (22)$$

when Eq. (22) is < 0 this implies $\tau_t = 0$.

With some further manipulations to be described below, the above condition will underly a policy function, $\tau_t = h(k_t)$. Substituting the solution into Eq. (13) for capital accumulation yields $k_{t+1} = \mathtt{G}(k_t, h(k_t))$. The dynamic path of the economy is traced out by repeated iteration of the above. A steady state of the economy with optimal taxes is given by a pair k and $\tau = h(k)$ such that $k = \mathtt{G}(k, h(k))$.

Proposition 2. *If k_0 is below some threshold level $\underline{k}$, then the optimal environmental tax, $\tau^* = 0$.*

Proof. From (22) we see that a necessary condition for $\tau^* > 0$ is

$$\Omega_t = \left[\ln c_{t+1}^o - \frac{2-\alpha+\pi_t}{1+\pi_t}\right] > 0.$$

At low levels of initial capital, k_0, this is not going to hold. This is because the negative term in Ω_t is always non-zero while the positive term approaches minus infinity, given the logarithmic specification, as the capital stock approaches zero. Thus there exists some threshold level $\underline{k}$; such that for any $k_0 < \underline{k}$, $\Omega < 0$. ■

An optimal tax-financed abatement policy trades off less consumption for agents when they are young in exchange for higher life expectancy. But as we noted in the discussion following Eq. (21), an increase in life expectancy does not necessarily increase old-age utility: while the higher survival probability increases expected utility at given levels of old-age consumption, it also lowers the return on per-capita savings and this leads, blue all else equal, to a decline in per-capita old-age consumption. At low levels of capital, these indirect effects dominate as old consumption is low and the fall in return of per-capita savings outweighs the increase in expected utility due to the change in probability of survival for the given level of savings. For higher levels of capital, when old age consumption is higher, the increase in expected utility through increase in probability of survival will outweigh the decrease due to decrease in savings.

Pursuing this intuition further, we expect that once a critical level of capital has been reached such that the optimal tax becomes positive, any further increase in the capital stock will lead to higher taxes. This is because the benefits from higher survival probability are likely to rise faster than costs associated with it. Thus, starting from an initial situation in which the first-order condition for taxes holds with equality, a small increase in k_t will tend to increase the benefit from higher taxes relative to the costs, necessitating an increase in the tax rate.

We begin by studying the behavior of Ω. In principle, there will be an arbitrarily high level of k_t such that $\Omega_t > 0$. This is because the first term in Ω_t has the potential to increase monotonically with k_t, at least after some threshold, while the second term is always bounded in the interval $[(3-\alpha)/2, (2-\alpha)]$ and within this interval, it falls as π_t increases. c^o_{t+1} rises monotonically with k_t even when π_t rises as well. If along the dynamic path, the detrimental effects of pollution make π_t start declining in k_t, then c^o_{t+1} rises even faster with k_t. At some level of development, Ω_t will be positive and increasing in capital. The other negative term in the first-order condition is similarly bounded above at $(1+\alpha)$, when evaluated at a zero tax rate. Thus, at a second critical level of development, an interior solution will arise for a positive optimal tax. The question is what level of development has to be reached before it arises and to what extent this level coincides with potential steady states of the economy.

To pursue these conjectures more rigorously, we first establish some general conditions for the applicability of a positive environmental tax at some threshold level of income. Let the right-hand side of Eq. (22) be denoted by:

$$\mathrm{H}(k_t, \tau_t) = \Omega_t \cdot \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t}.$$

The first condition needed for a well-behaved tax function is

$$\left.\frac{\partial \mathrm{H}}{\partial \tau_t}\right|_{\mathrm{H}=0} < 0.$$

In other words, the second-order condition is satisfied whenever the first-order condition holds as an equality.

The second condition ensuring a well-behaved tax function is:

$$\left.\frac{\partial \mathrm{H}}{\partial k_t}\right|_{\tau=0,\mathrm{H}=0} > 0.$$

Thus, evaluated at the point where the first-order condition first holds with equality at a zero tax, it is upward sloping in k_t. Note that at very low levels of the capital stock this may not be true, but what is required is that it holds in the neighborhood of the threshold where an optimal tax first arises.

To explore the above conditions further, differentiate H with respect to its arguments (time scripts will be suppressed as all variables are contemporaneous). After some manipulation, these derivatives can be written as

$$\frac{\partial \mathrm{H}}{\partial \tau} = \Omega \frac{\partial^2 \pi}{\partial \tau^2} - \frac{2\alpha}{1-\tau}\frac{\partial \pi}{\partial \tau} - \frac{1+\alpha\pi}{(1-\tau)^2} - \frac{\pi(1+\pi)+(1-\alpha)}{\pi(1+\pi)^2}\left(\frac{\partial \pi}{\partial \tau}\right)^2; \quad (23)$$

$$\frac{\partial \mathrm{H}}{\partial k} = \frac{\partial \Omega}{\partial k}\frac{\partial \pi}{\partial \tau} + \Omega\frac{\partial^2 \pi}{\partial \tau \partial k} - \frac{\alpha}{1-\tau}\frac{\partial \pi}{\partial k}; \quad (24)$$

where

$$\frac{\partial \Omega}{\partial k} = \frac{2\alpha}{k} - \frac{(1+\pi)^2 - \pi - \alpha}{(1+\pi)^2}\nu_{\pi k};$$

where $\nu_{\pi k}$ is the elasticity of survival probability with respect to capital. This is eventually decreasing in k due to the positive and eventually diminishing effects of greater income and the negative and eventually increasing effects of higher pollution. It can turn negative at some point; however, we shall restrict our analysis to cases where it remains strictly positive.

None of the above terms can be signed unambiguously but two comments are in order. First, as noted before, a positive effect of k on Ω is necessary for the first-order condition to eventually hold. What this in turn requires is that along the infra-marginal path of capital, *i.e.* before the first-order condition kicks in, there is some range of values of k where the elasticity of survival probability with respect to the capital stock (taking into account both the beneficial and detrimental effects) is sufficiently small. As noted above, this elasticity will eventually diminish with growth in the capital stock, implying the existence of a threshold value of capital after which $\partial\Omega/\partial k > 0$.

Proposition 3. *There exists $\tilde{k} > 0$, such that for all $k > \tilde{k}$, $\frac{\partial \Omega}{\partial k} > 0$.*

From hereon we neglect consideration of values of k below this threshold, as for the purposes of deriving an environmental tax, such values of k cannot admit positive solutions of τ. Second, a sufficient condition for the second-order condition for τ to be negative is that π is concave in τ. However, this is likely to be too restrictive, given the following relationship between the second-order derivatives of π with respect to τ and z:

$$\frac{\partial^2 \pi}{\partial \tau^2} = (\psi\gamma A k^\alpha)^2 \frac{\partial^2 \pi}{\partial z^2}.$$

Thus, π will be concave in τ if and only if it is downwards concave in z. But given the likely impact of pollution levels on survival probability, this portion of the $\pi - z$ relationship applies at lower levels of pollution, when it is less likely that the first-order condition for an optimal tax will hold as an equality. At higher levels, it is unlikely that π is concave in τ. This rules out imposing concavity on the $\pi - \tau$ relationship as a sufficient condition for ensuring the validity of the second-order condition.

To proceed further, we turn to the specific example of the survival probability assumed earlier.

$$\pi = \pi^A \pi^B = \left[\frac{y^\vartheta}{1+y^\vartheta}\right]\left[\frac{1}{1+z^\delta}\right].$$

In the following subsections we first analyze the sign of $\partial^2\pi/\partial\tau^2$ and then the sign of $\partial^2\pi/(\partial\tau\partial k)$

3.1.1. The second-order condition, $\partial\mathrm{H}/\partial\tau$

The following expressions are derived for the specific functional form for π (time scripts are again suppressed).

$$\frac{\partial \pi}{\partial \tau} = \pi^A \frac{\psi\delta\gamma A k^\alpha z^{\delta-1}}{(1+z^\delta)^2} > 0; \quad (25)$$

$$\frac{\partial^2 \pi}{\partial \tau^2} = \pi^A \frac{(\psi\gamma A k^\alpha)^2 \delta z^{\delta-2}}{(1+z^\delta)^3}\left[(\delta+1)z^\delta - (\delta-1)\right].$$

By comparing the two expressions, the latter can be written as

$$\frac{\partial^2 \pi}{\partial \tau^2} = \pi^A \left(\frac{\psi\gamma A k^\alpha \delta}{z(1+z^\delta)} \cdot \frac{\partial \pi}{\partial \tau}\right)\left[(\delta+1)z^\delta - (\delta-1)\right] \times \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \text{ as } z^\delta \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \frac{\delta-1}{\delta+1}; \quad (26)$$

confirming the dependence of the sign of $\partial^2\pi/\partial\tau^2$ on that of $\partial^2\pi/\partial z^2$. To proceed further with an analysis of the second-order condition, i.e. Eq. (23), note from Eq. (4) that:

$$\gamma A k_t^\alpha = \frac{z_t}{1-\psi\tau_t}.$$

Suppressing time subscripts, let us write this as

$$\gamma A k^{\alpha} = \frac{z}{1-\psi\tau}.$$

Then (26) can be further modified:

$$\frac{\partial^2 \pi}{\partial \tau^2} = \pi^A \left(\frac{\psi \delta z}{z(1-\psi\tau)(1+z^{\delta})} \cdot \frac{\partial \pi}{\partial \tau} \right) \left[(\delta+1) z^{\delta} - (\delta - 1) \right].$$

Now, from Eq. (22),

$$\Omega \leq \frac{1+\alpha\pi}{1-\tau} \frac{1}{\partial\pi/\partial\tau}, \; \forall \, \tau.$$

Thus, taking the term involving $\partial^2\pi/\partial\tau^2$ in Eq. (23),

$$\Omega \frac{\partial^2 \pi}{\partial \tau^2} \leq \left(\frac{1+\alpha\pi}{1-\tau} \frac{\psi \delta z}{z(1-\psi\tau)(1+z^{\delta})} \right) \left[(\delta+1) z^{\delta} - (\delta - 1) \right].$$

Combining with one of the other terms in Eq. (23)

$$\Omega \frac{\partial^2 \pi}{\partial \tau^2} - \frac{1+\alpha\pi}{(1-\tau)^2} \leq \left[\frac{1+\alpha\pi}{1-\tau} \right] \times \left[\frac{\psi\delta z - \left[(\delta+1) z^{\delta} - (\delta-1) \right]}{z(1-\psi\tau)(1+z^{\delta})} - \frac{1}{1-\tau} \right]. \quad (27)$$

The sign of the above term will depend on the sign of the term inside square brackets. After some manipulation, the sign of the latter can be shown to be negative if the following holds:

$$-\frac{[1-\psi\{1+\delta(1-\tau)\}]z^{\delta}}{(1-\psi\tau)(1+z^{\delta})(1-\tau)} < 0.$$

A sufficient condition for the above term to be negative for all values of endogenous variables is $\psi < 1/(1+\delta)$.[17]

We have therefore established:

Lemma 3. *A sufficient condition for $\partial \mathtt{H}/\partial\tau$ to be negative at all values of endogenous variables and along the entire dynamic path is $\psi < 1/(1+\delta)$.*

Recall that $\psi = \frac{\chi(1-\alpha)}{\gamma}$, where χ is the effectiveness of the abatement technology and γ is how polluting the productive activity. As we would expect, if the first is low enough and/or the second high enough, then the second order condition holds, or in other words there is an interior solution.

3.1.2. The sign of $\partial \mathtt{H}/\partial k$

Note the following derivatives for the assumed functional form (time indices continue to be suppressed):

$$\frac{\partial \pi^A}{\partial k} = \frac{\alpha}{k} \frac{\vartheta y^{\vartheta}}{(1+y^{\vartheta})^2};$$

$$\frac{\partial \pi^B}{\partial k} = -\frac{\alpha\gamma(1-\psi\tau)Ak^{\alpha}}{k} \frac{\delta z^{\delta-1}}{(1+z^{\delta})^2}.$$

Using the definitions of π^A, π^B, and π, and rearranging, we can combine the above derivatives

$$\frac{\partial \pi}{\partial k} = \frac{\alpha\pi}{k} \left[\frac{\vartheta y^{\vartheta}}{(1+y^{\vartheta})y^{\vartheta}} - \frac{\delta z^{\delta}}{(1+z^{\delta})} \right]; \quad (28)$$

which implies that

$$\nu_{\pi k} = \alpha \left[\frac{\vartheta y^{\vartheta}}{(1+y^{\vartheta})y^{\vartheta}} - \frac{\delta z^{\delta}}{(1+z^{\delta})} \right]$$

where $\nu_{\pi k}$ has been defined as the *elasticity* of π with respect to k.[18]

Now, to derive the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$, we proceed in two steps. We first derive an expression for $\partial^2\pi/(\partial\tau\partial k)$ and then use it to evaluate the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$.

The first step is accomplished by taking the total derivative of $\partial\pi/\partial\tau$, Eq. (25), with respect to k. After imposing some definitions and equalities, and rearranging terms, it can be shown that:

$$\frac{k}{\partial\pi/\partial\tau} \frac{\partial^2\pi}{\partial\tau\partial k} = \nu_{\pi k} + \alpha\delta \frac{z}{z(1+z^{\delta})} > 0.$$

The full derivation is outlined in Appendix C. From here it is easy to establish the following:

Lemma 4. $\mathtt{H}(k,\tau) = 0 \Longrightarrow \partial \mathtt{H}/\partial k \geq 0.$

Proof. First, the expression for $\partial^2\pi/\partial\tau\partial k$ implies that

$$\frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{\partial\pi}{\partial\tau} \frac{1}{k} \nu_{\pi k}.$$

Second, $\mathtt{H} = 0$ implies that

$$\Omega = \frac{1+\alpha\pi}{1-\tau} \frac{1}{\partial\pi/\partial\tau}.$$

Therefore, referring to Eq. (24),

$$\Omega \frac{\partial^2\pi}{\partial\tau\partial k} = \frac{1+\alpha\pi}{1-\tau} \frac{1}{\partial\pi/\partial\tau} \frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{1+\alpha\pi}{1-\tau} \frac{1}{k} \nu_{\pi k}.$$

Now, referring to the negative term in Eq. (24),

$$\frac{\alpha}{(1-\tau)} \frac{\partial\pi}{\partial k} = \frac{\alpha\pi}{(1-\tau)k} \nu_{\tau k}.$$

Combining the two terms in Eq. (24),

$$\Omega \frac{\partial^2\pi}{\partial\tau\partial k} - \frac{\alpha\pi}{(1-\tau)k}\nu_{\pi k} \geq \frac{1+\alpha\pi}{1-\tau} \frac{1}{k} \nu_{\pi k} - \frac{\alpha\pi}{(1-\tau)k} \nu_{\pi k} \geq \frac{1}{(1-\tau)k} \nu_{\pi k} \geq 0. \quad \blacksquare$$

Note that we have derived the above result for all values of τ. Thus, as an economy's capital stock increases, the slack in $\mathtt{H}$ diminishes until finally an interior solution is reached.

3.1.3. Positive taxes

We can now establish:

Proposition 4. *If $\psi < \frac{1}{1+\delta}$ and $k \geq \tilde{k}$ then, there (i) exists an optimal policy function, $\tau = h(k)$, $h : [\tilde{k}, \infty) \longrightarrow [0, 1]$; (ii) $h(k)$ is (weakly) increasing in k.*

Proof. The first part follows from the strict monotonicity of $\mathtt{H}$ in both τ and k. Since $\mathtt{H}$ is strictly decreasing in τ for all k under the assumed conditions, then for any k in the relevant interval, either (i) $\mathtt{H}(k, 0) \leq 0$, or (ii) $\mathtt{H}(k, 1) > 0$ or (iii) $\mathtt{H}(k, \tau) = 0$ for some $\tau \in [0, 1]$. Moreover, τ uniquely solves the relevant case for $\mathtt{H}$ at given k, because for any $\tau' > \tau$, in case (i) $\tau = 0$ and $\tau' > 0$ worsens the slack in $\mathtt{H}$; in case (ii) if $\tau = 1$ then τ' lies outside the unit interval and in case (iii) since $\mathtt{H}(\tau, k) = 0$ for $\tau \in [0, 1]$, then $\mathtt{H}(\tau', k) < 0$. Similar argument rules out the possibility that $\tau' < \tau$ also solves $\mathtt{H}$ for a given k.

17 By extending the comparison with the sign of $\Omega \cdot \partial^2\pi/\partial\tau^2$ to other terms in the expression for $\partial^2 \mathtt{H}/\partial\tau^2$ even weaker conditions can be derived. But as with the above, to ensure negativity of the second-order condition for all admissible values of endogenous variables, the above condition still applies.

18 Throughout the analysis, we assume that $\nu_{\pi k}$ remains positive, although as we have noted before, a negative value is entirely possible under some conditions, and if it happens there can be oscillations around the high steady state.

The second part follows from

$$\left.\frac{\partial h(k)}{\partial k}\right|_{\mathrm{H}=0} = -\frac{\mathrm{H}_k}{\mathrm{H}_\tau} \geq 0;$$

while $\forall k \in [\tilde{k}, \infty)$, $\mathrm{H}(0, k) < 0 \Rightarrow h(k) = 0$ and $\mathrm{H}(1, k) > 0 \Rightarrow h(k) = 1$. ■

To understand why $h(k)$ is (weakly) increasing after a threshold has been passed, recall the first-order condition, Eq. (22), which we have denoted as $H(k_t, \tau_t)$. Lemma 3 established that the second order condition is satisfied, *i.e.* $\partial \mathrm{H}_t/\partial \tau_t < 0$, so long as the abatement technology is not too powerful in abating pollution (low χ). In Lemma 4, we established that, starting from any point at which $\mathrm{H}_t = 0$, *i.e.* the first-order condition holds as an equality, $\partial \mathrm{H}_t/\partial k_t > 0$, implying that a small increase in k starting from an initial optimum will induce some slack in the first order condition.

The intuition for this is that once the capital stock is high enough that the first-order condition binds, a further increment to the capital stock (at the initial tax rate) will raise the expected utility from given old-age consumption because of higher survival probability, by at least as much as it decreases utility from lower young-age consumption due to higher savings and from old-age consumption due to lower *per-capita* returns on savings. Thus higher capital increases the net benefit from taxing pollution, and this induces an increase in the optimal tax rate via the second-order condition.

3.2. Long-lived social planner

We look at the continuation utility of future generations in the first-order condition (20) for determining the optimal tax on the current generation:

$$\beta\left[\frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1}\frac{\partial \pi_{t+1}}{\partial k_{t+1}}\right] \times \left[\frac{A(1-\alpha)k_t^\alpha}{1+\pi_t}\left\{\left(\frac{1-\tau_t}{1+\pi_t}\right)\frac{\partial \pi_t}{\partial \tau_t} - \pi_t\right\}\right].$$

The term in second square brackets represents the effect of a higher current tax on next period's capital stock. A necessary condition for this to be positive is that the tax-financed increase in abatement activity increases the survival probability for the current young by enough to offset the negative income effect of the higher tax. The term in the first square brackets represents the effect of a higher stock of capital next period on the welfare of the next generation. That in turn depends in part on the effect of the higher capital stock on the survival probability of next period's young. Even if that is positive, the overall effect on their welfare might not be because of the term Ω_{t+1} which could be negative at low initial values of capital, for similar reasons as were identified in the case of Ω_t: higher survival probability raises the utility from given old-age consumption but lowers both young-age and old-age consumption levels; thus if the initial level of old-age consumption is low this contributes a negative effect. This discussion indicates that it will be difficult to assign a sign to the inter-generational effect on current optimal taxation on an *a priori* basis.

Since we have already derived using analytical methods a well-behaved tax policy function without incorporating the inter-generational effect and our main aim is to verify the intuition outlined above for how incorporating such effects might modify the policy function we proceed by way of numerical examples which map the policy function at varying levels of the steady state capital stock.

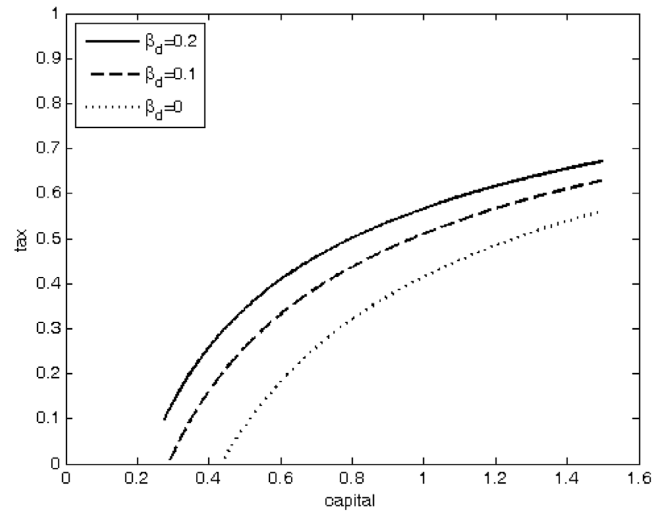

Fig. 3. The policy function.

We start by defining the steady state version of the optimal tax equation

$$\left[\Omega\frac{\partial \pi}{\partial \tau} - \frac{1+\alpha\pi}{1-\tau}\right] + \beta\left[\frac{\alpha(1+2\pi)}{k} + \Omega\frac{\partial \pi}{\partial \tau}\right] \times \left[\frac{A(1-\alpha)k^\alpha}{1+\pi}\left\{\left(\frac{1-\tau}{1+\pi}\right)\frac{\partial \pi}{\partial k} - \pi\right\}\right] \leq 0; \tag{29}$$

(when Eq. (29) is < 0 implies $\tau = 0$) where $\partial\pi/\partial\tau$ is given by Eq. (25) and $\partial\pi/\partial k$ is given by Eq. (28).

MATLAB was used to trace out the policy function. Taking an interval of values of steady state capital, k, Eq. (29) was recursively solved for the optimal value of the steady state abatement tax, τ at varying levels of the inter-generational discount factor, β. The results are in Fig. 3.[19] Other parameter values were set as in Fig. 2.

We can see that the qualitative properties of the policy function are as hypothesized: at any value of β, the optimal tax is zero at sufficiently low levels of k. As k rises, an upward sloping and concave tax emerges. The main effect of higher β is to shift the policy function upwards so that at any level of k the planner is more likely to undertake active abatement and to set a higher tax if positive. This is line with conventional wisdom regarding the effect of far-sighted environmental policy.

At the same time, in our model, the reaction of the capital stock to taxes can reflect the underlying non-convexities of this economy. We have seen in Section 3.1 that, at any arbitrary tax, there can be multiple steady states and that an increase in the abatement tax rate can have ambiguous effects on the steady state capital stock. It is the interaction of state-contingent environmental policy with the behavior of the capital stock that can introduce non-linearities and change the dynamics.

3.3. Dynamics of the optimal tax

A steady state with optimal taxation is characterized by two equations.

$$k = \frac{\pi(k,\tau)}{1+\pi(k,\tau)}A \cdot (1-\tau)(1-\alpha)k^\alpha; \tag{30}$$

[19] For $\beta = 0.2$, MATLAB could not solve for the optimal tax in certain regions of parameter space because of the highly nonlinear nature of Eq. (29). Hence for $\beta = 0.2$, the policy function does not intersect the capital axis at $\tau = 0$ but this was not an issue for $\beta = 0.1$ or 0.

$$\tau = h(k); \tag{31}$$

where Eq. (30) represents the steady state reaction function of private agents and Eq. (31) represents the steady state policy function of the social planner. We assume that $h(k)$ satisfies Proposition 4 for both a myopic and a long-lived social planner. A solution to the above equations is represented by a pair (k^*, τ^*).

The dynamics of the economy with optimal taxes are traced out by recursive application of the tax policy function and the phase diagram for the capital stock. For any capital $k_t > \tilde{k}$, $\tau_t = h(k_t)$. Then, next period's capital stock follows:

$$k_{t+1} = \frac{\pi(k_t, \tau_t)}{1 + \pi(k_t, \tau_t)} A(1-\tau_t)(1-\alpha)k_t^{\alpha} = \mathtt{G}(k_t, \tau_t);$$

and so on.

This represents a first-order difference equation in k_t for any arbitrary k_0. Linearizing around a steady state, the local dynamics are determined by the sign and magnitude of the expression

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k^*, \tau^*) + \mathtt{G}_2(k^*, \tau^*)h'(k^*); \tag{32}$$

where $\mathtt{G}_1(k, \tau) = \mathtt{G}'(k)$, as given by Eq. (34) and $\mathtt{G}_2(k, \tau)$ is given by Eq. (18); see Appendix B.

It is instructive to compare Eq. (32) with the case of exogenous abatement, in which

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}'(k^*).$$

In this case, the dynamics of the capital stock are driven by a non-time varying $\mathtt{G}(k)$ function for a given τ. In the case of optimal abatement, the $\mathtt{G}$ function shifts (in (k_{t+1}, k_t) space) each period as the tax varies along the optimal path. This generates the possibility of additional dynamic complexity arising from a dynamic tax policy. To rule out any further complexity in the exogenous-tax case, we assume that $\mathtt{G}_1(k, \tau) > 0$ throughout this section.

Define $k^* = g(\tau)$, as the value of k^* which solves Eq. (30) for any admissible τ. Then $\tau^* = h(k^*)$ solves the optimal tax at this steady state.

It is easy to show that

$$g'(\tau) = \frac{\mathtt{G}_2(k^*, \tau)}{1 - \mathtt{G}_1(k^*, \tau)}.$$

Using the above, Eq. (32) can be expressed as:

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k^*, \tau^*) + g'(\tau^*)(1 - \mathtt{G}_1(k^*, \tau^*))h'(k^*); \tag{33}$$

where the sign of $g'(\tau^*)$ is the same as (*resp.* the opposite of) the sign of $\mathtt{G}_2(k^*, \tau_2)$, when $1 - \mathtt{G}_1(k^*, \tau^*) > 0$ (*resp.* < 0), as in the neoclassical steady state (*resp.* as in the poverty trap).

We now examine the local dynamics, first at a neoclassical steady state and then at a poverty trap. In the next sub-section, we shall use a graphical approach to study local dynamics but since our diagrams will involve multiple steady states we shall use * to denote a steady state only if the diagram shows a single steady state; otherwise other indices will be used.

3.3.1. Local dynamics around a neoclassical steady state

In this case, $\mathtt{G}_1(k, \tau) < 1$, $1 - \mathtt{G}_1(k, \tau) > 0$. Then $g'(\tau) > 0$ (*resp.* < 0) as $\mathtt{G}_2(k, \tau) > 0$ (*resp.* < 0). By suitable rearrangement of Eq. (33), it can be shown that

$$\frac{dk_{t+1}}{dk_t} \left\{ \begin{array}{c} > 1 \\ \in [0, 1] \\ < 0 \end{array} \right\} \text{as } g'(\tau)h'(k) \left\{ \begin{array}{l} > 1 \\ \in \left[-\frac{\mathtt{G}_1(k,\tau)}{1-\mathtt{G}_1(k,\tau)}, 1 \right] \\ < -\frac{\mathtt{G}_1(k,\tau)}{1-\mathtt{G}_1(k,\tau)} \end{array} \right\}.$$

We can see that the local dynamics around a neoclassical steady state are no longer necessarily convergent, as was the case in the exogenous tax economy. They will depend on two factors: (i) whether $g'(\tau)$ is positive or negative, *i.e.* whether an increase in the tax rate shifts the neoclassical steady state up or down; (ii) the slope of $g(\tau)$ relative to the slopes of the other two main steady state relationships: $h(k^*)$ and $\mathtt{G}_1(k^*, \tau^*)$.

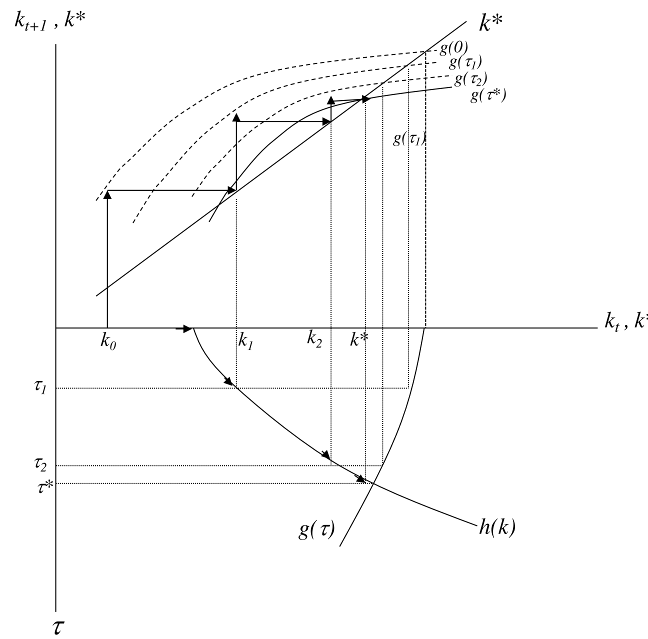

Fig. 4. A well-behaved neoclassical steady state.

Whether $g'(\tau)$ is negative or positive, but its magnitude is not too large, the dynamic path converges monotonically. When $g'(\tau)$ is positive and relatively large, the steady state becomes a source. When $g'(\tau)$ is negative and relatively large, fluctuations can arise near the steady state.

Fig. 4 represents the dynamics near a "well-behaved" neoclassical steady state, *i.e.* one in which an increase in the tax rate shifts the phase diagram $\mathtt{G}.(k)$ downwards at the steady state. This case is considered first as it helps outline the graphical approach that will be used in what follows. The top panel of Fig. 4 shows a family of phase diagrams relating k_{t+1} to k_t. These maps are drawn to display local, rather than global, dynamics so we do not depict them starting from the origin. Each map is underpinned by a specific value of the tax, τ_t. The lower panel depicts the functions $g(\tau)$ and $h(k)$ in $(\tau - k)$ space. $h(k)$ is always upward sloping in this space while, in keeping with the assumed nature of this steady state, $g(\tau)$ is downward sloping. Their intersection gives the combination of steady state capital and steady state taxes, (τ^*, k^*). This is the unique long-run steady state.

Starting at $k_0 < \underline{k}$, the latter defined in Proposition 2 as the minimum level of capital associated with active environmental policy, the optimal tax at $t = 0$ is $\tau_0 = 0$. The steady state associated with this tax is the highest dashed phase diagram on the top panel, which is labeled $\mathtt{G}(0)$. If the tax rate was held constant at this level, the capital stock would evolve monotonically towards $k = g(0)$ through iterative application of this map. At $t = 0$, next period's capital, k_1, is given by the vertical projection to this map from k_0. But when the economy reaches k_1, the optimal tax for that period no longer equals zero. Indeed, as drawn, the threshold level of capital is crossed and optimal $\tau_1 > 0$, as given by the projection down from k_1 to $h(k)$. At τ_1, the horizontal projection to $g(\tau)$ gives the new steady state level of capital associated with a tax rate, τ_1. This means that the phase diagram in the upper panel shifts downwards so it intersects the 45° line at $g(\tau_1)$. The vertical projection from k_1 to the new phase diagram gives k_2 and

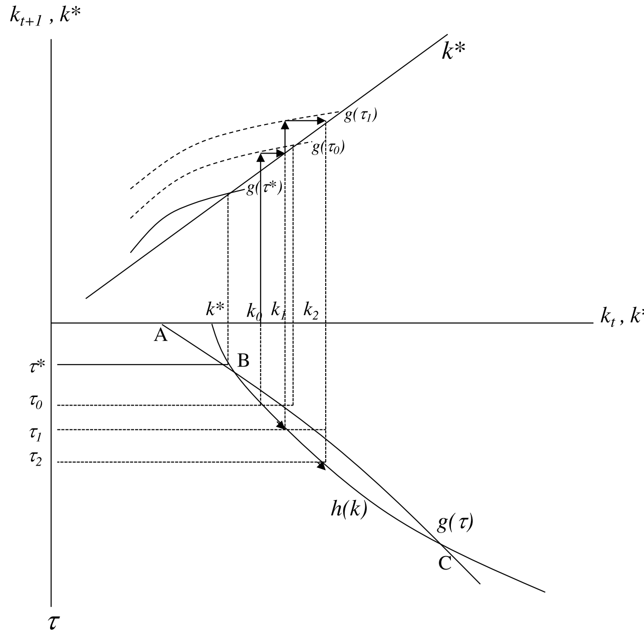

Fig. 5. Multiple neoclassical steady states.

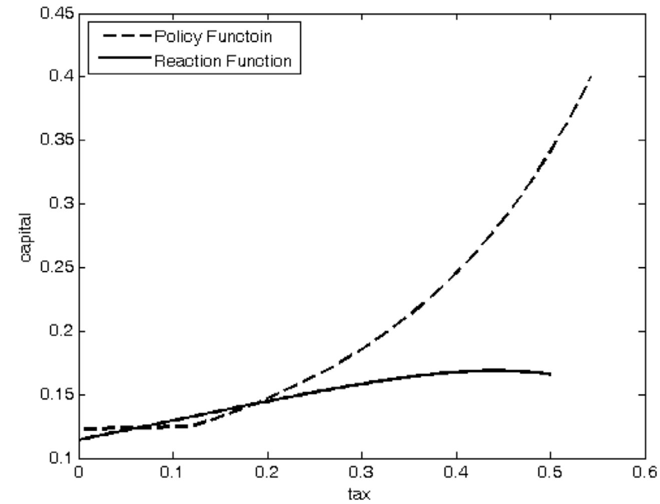

Fig. 6. Multiple neoclassical steady states, $\beta = 0.9$.

so on. The dynamics are monotonically convergent with both k_t and τ_t rising in ever shorter steps towards the steady state.

Fig. 5 shows the more interesting case of multiple neoclassical steady states.[20] Since the policy function $h(k)$ is always upward sloping in $k - \tau$ space, for multiple neoclassical steady states to arise, $g(\tau)$ also has to be positively sloped at some point, *i.e.* the steady state capital stock increases with greater abatement taxes. In addition, as drawn, $g'(\tau)h'(k) > 1$ so that the combined effect of an optimal tax that increases in the capital stock and the feedback from a higher tax to a higher steady state capital stock is relatively strong. We see that three steady states arise in this case, one with zero optimal abatement (A) and two with positive abatement (B and C). Since $G'(k)$ cuts the 45° line from above in each of the steady states, all three are of the type that we have designated as neoclassical. This contrasts with the case of an exogenous abatement tax where there can be at most one neoclassical steady state.

We consider dynamics near the middle steady state, B. Graphically, both $g(\tau)$ and $h(k)$ slope upwards in $(\tau - k)$ space *and* $h(k)$ cuts $g(\tau)$ from above at this steady state; hence, for any initial $k_0 > k_B$ (as shown in the diagram), $g(\tau_0) > k_0$. And since each potential steady state for a given tax rate is locally stable, $k_1 > k_0$ so that the economy moves away from k_B, *i.e.* B is unstable.

To understand better the possibility of multiple neoclassical states under optimal abatement, note that steady state A arises because the k-axis intercept of $g(\tau)$ lies to the left of that of $h(k)$. Given the relative slopes of $g(\tau)$ and $h(k)$, it can be seen that if $g(\tau)$ intersected the k-axis to the right of $h(k)$ then both A and B would disappear and only the stable steady state C would be possible. In this sense, the existence of a neoclassical steady state with optimally zero abatement appears to be a pre-condition for multiple neoclassical steady states with optimally positive abatement to emerge . We shall observe a similar feature in the context of multiple poverty traps in Fig. 8.

Fig. 6 illustrates a parametrized example of the multiplicity depicted above. It was computed using MATLAB and the same parameter values as the ones that generated Fig. 2, with the addition of the intergenerational discount factor β which is set at 0.9. In Fig. 6, there is a no-abatement steady state at $k_{n0} = 0.118$ and two positive abatement steady states with ($\tau_1 = 0.065$, $k_{n1} = 0.124$) and ($\tau_2 = 0.185$, $k_{n2} = 0.1422$) respectively. Given the respective slopes of the tax policy and capital reaction functions, the steady state K_{n1} is unstable and the other two are stable.

Fig. 7 shows the case when $g'(\tau) < 0$ and its magnitude is relatively large. Note that a large magnitude of $g'(\tau)$ implies that its diagrammatic representation in the bottom panel of Fig. 7, makes it appear relatively flat. As drawn, Fig. 7 shows the dynamic path starting at k_0 cycling between the pair (τ_0, k_0) and (τ_1, k_1). This happens a large magnitude of $g'(\tau)$ implies one or both of: (i) steady state capital is quite sensitive to changes in the tax rate, *i.e.* a small increase in the tax rate lowers the capital stock by a large amount, (ii) starting from an initial capital stock in the neighborhood of the steady state the rate of convergence towards that steady state is high, *i.e.* there is a large jump in the next period's capital stock in the direction of the steady state.[21]

These two effects reinforce each other to produce a cycle. In Fig. 7, there is a unique steady state (k^*, τ^*) at the point where $g(\tau)$ and $h(k)$ intersect. The economy starts at a capital stock, k_0 that lies below k^*. Given k_0, the optimal tax rate is τ_0 and in the upper panel of Fig. 7, the dynamics of the capital stock are governed by the phase diagram, $\mathtt{G}(\tau_0)$, associated with τ_0. As we can see, the capital stock jumps to k_1 which is greater than k^*. At k_1, the tax rate increases from τ_0 to τ_1 leading to a new phase diagram, $\mathtt{G}(\tau_1)$. Since the steady state associated with $\mathtt{G}(\tau_1)$ lies below k_1, next period's capital moves towards that steady state in a large step, bringing the economy back to k_0. As drawn, the cycle is locally stable but this is not necessarily going to be the case. The point is that oscillations can arise if these features are present.

We summarize these results under the following Proposition, stated without further proof.

Proposition 5. *Suppose there exists a neoclassical steady state* (k, τ) *in an economy with optimal taxation. Then, given* $\mathtt{G}_1(k, \tau) < 1$ *and* $(1 - \mathtt{G}_1(k, \tau_1)) > 0$,

[20] In depicting this case graphically we continue making the assumption that under arbitrarily constant taxation, there would be only a single steady state of the neoclassical type.

[21] In technical terms, a large magnitude for $g'(\tau)$ can arise either because $\mathtt{G}_2(k, \tau)$ is large in magnitude or because $\mathtt{G}_1(k, \tau)$ is large, or both. The first suggests possibility (i) and the second suggests possibility (ii).

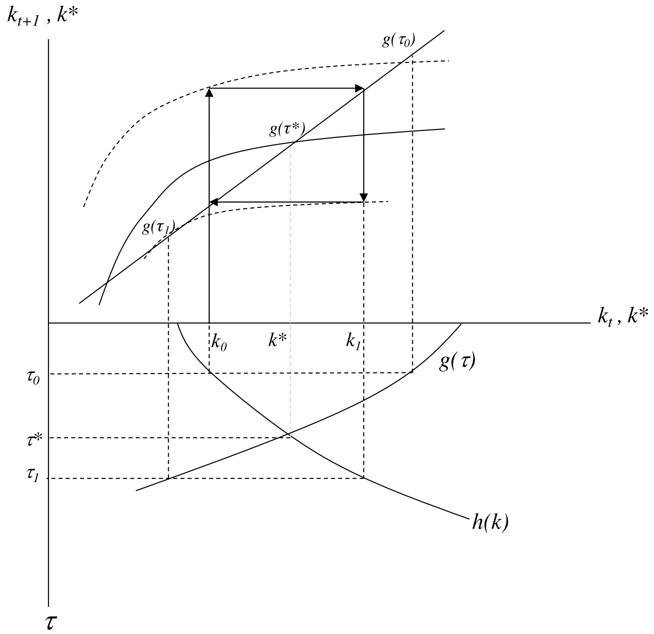

Fig. 7. Oscillations around a neoclassical steady state.

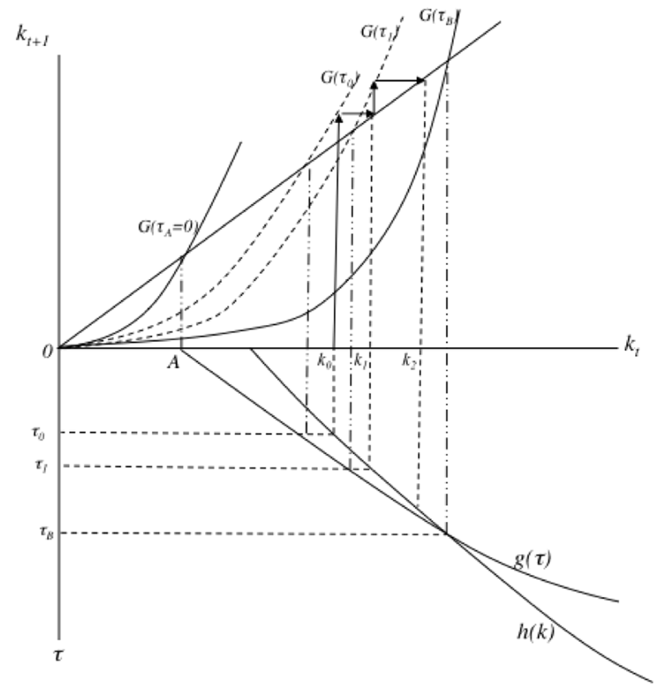

Fig. 8. A locally stable poverty trap.

(i) the steady state will be locally unstable if $g'(\tau)h'(k) > 1$*;*
(ii) there will be local fluctuations around the steady state if

$$g'(\tau)h'(k) < -[\mathtt{G}_1(k,\tau)/(1-\mathtt{G}_1(k,\tau))] < 0$$

(iii) the dynamics will be monotonically convergent if

$$-[\mathtt{G}_1(k,\tau)/(1-\mathtt{G}_1(k,\tau))] < g'(\tau)h'(k) < 0.$$

3.3.2. Local dynamics around a poverty trap

In this case of a poverty trap, the phase diagram cuts the 45° line from below; therefore $\mathtt{G}_1(k,\tau) > 1$ and $1 - \mathtt{G}_1(k,\tau) < 0$. Thus $g'(\tau) > 0$ (*resp.* < 0) as $\mathtt{G}_2(k,\tau) < 0$ (*resp.* > 0). To remain consistent with the discussion following Eq. (18) in Section 3, we shall exclude the case $g'(\tau) < 0$ from further consideration. Thus, Eq. (33) can be written more clearly as

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k,\tau) - g'(\tau)(\mathtt{G}_1(k,\tau) - 1)h'(k).$$

It can now be established by suitable rearrangement that[22]

$$\frac{dk_{t+1}}{dk_t}\left\{\begin{array}{c} >1 \\ \in[0,1] \\ <0 \end{array}\right\} \text{ as } g'(\tau)h'(k)\left\{\begin{array}{c} <1 \\ \in\left[1, \frac{\mathtt{G}_1(k,\tau)}{\mathtt{G}_1(k,\tau)-1}\right] \\ > \frac{\mathtt{G}_1(k^*,\tau^*)}{\mathtt{G}_1(k^*,\tau^*)-1} \end{array}\right\}.$$

Whereas a poverty trap was monotonically a source in the case of exogenous taxes, it can now be a sink. There can also be fluctuations around the poverty trap, depending on how strong the interaction of optimal policy with private sector capital accumulation decisions is.

Fig. 8 shows the case of two poverty traps one of which (steady state A) has zero taxes and is unstable while the other (steady state *B*) has positive taxes and is stable.[23] In this case, $h'(k)g'(\tau) > 1$, so that as k increases $h(k)$ cuts $g(\tau)$ from above in the lower panel. As in the neoclassical case, $h(k)$ is increasing and there is a steady state with no abatement, *A*, which in this case is locally unstable and a poverty trap as $G' > 1$. Also note that if $g(\tau)$ cut the k-axis to the right of $h(k)$, then neither *A* nor *B* would arise. As in the case of multiple neoclassical steady states, this again reflects the necessity of there being a no-abatement poverty trap steady state for there to be another poverty trap with positive abatement.

Consider now the dynamics near *B*. Starting at an initial capital, $k_0 < k_B$ and tax rate $\tau_0 < \tau_B$, the phase diagram associated with τ_0 would result in a steady state $g(\tau_0)$ which lies below k_0. Because $\mathtt{G}(\tau_0)$ is, for constant τ, unstable, this means that $k_1 > k_0$. Then $\tau_1 > \tau_0$ and $g(\tau_1)$ lies above $g(\tau_0)$ but below k_1. Thus $k_2 > k_1$, $\tau_2 > \tau_1$ and the economy is on a path that converges to (τ_B, k_B).

Intuitively Fig. 8 depicts a case in which abatement taxes become optimal only at a relatively high level of capital but are subsequently fairly sensitive to increases in capital. This results in $h(k)$ cutting $g(\tau)$ from above. When the initial capital stock is below the steady state, the optimal tax rate associated with that capital stock maps into an associated (transitory) steady state which lies below the initial capital stock. This results in next period's capital stock being higher than the initial one and closer to the long-run steady state.

Finally, the possibility of cycles around a poverty trap is illustrated in Fig. 9. As drawn, the phase diagram $\mathtt{G}(\tau)$ shown in the upper panel is "steep", *i.e.* meaning that k_{t+1} is quite sensitive to changes in k_t. This in turn implies that relatively small changes in steady state tax rates can induce large changes in steady state capital. When the economy starts at k_0, the tax rate is τ_0 and the dynamics follows $\mathtt{G}(\tau_0)$, along this map, capital increases by a large amount to $k_1 > k^*$. This causes the tax rate at $t = 1$ to increase to τ_1 causing a large shift in the phase diagram to $\mathtt{G}(\tau_1)$ which now lies on the other side of $\mathtt{G}(\tau_B)$. Given k_1, there is a large drop in capital to $k_2 < k^*$. As drawn, the cycle is convergent but the cycle could equally be stable or convergent.[24]

22 It is implicit in the above that for any variable $x > 1$, $x/(x-1) \to 1$ from above as $x \to \infty$.

23 Again, we depict only local dynamics so the neoclassical steady state associated with this economy is not stated.

24 It is worth noting the difference with Palivos and Varvarigos [2010]; while they argue that environmental taxation can be used to eliminate cycles

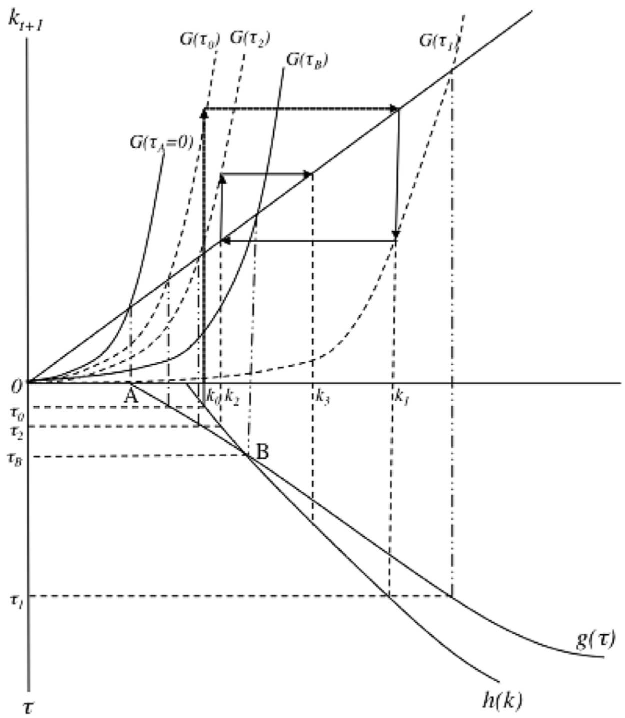

Fig. 9. Cycles around a poverty trap.

Proposition 6. *Suppose there exists a poverty trap* (k, τ) *in an economy with optimal taxation. Then, given that* $\mathtt{G}_1(k,\tau) > 1$, $\mathtt{G}_1(k,\tau)/[\mathtt{G}_1(k,\tau)-1] > 1$,
(i) the steady state will be locally unstable if $g'(\tau)h'(k) < 1$*;*
(ii) the steady state will be convergent if

$$\mathtt{G}_1(k,\tau)/[\mathtt{G}_1(k,\tau)-1] > g'(\tau)h'(k) > 1;$$

(iii) there will be local oscillations if

$$g'(\tau)h'(k) > \mathtt{G}_1(k,\tau)/[\mathtt{G}_1(k,\tau)-1] > 1.$$

4. Conclusions

This paper has shown that the combined effect of income and pollution on life expectancy can lead to multiple interior steady states, with an unstable poverty trap and a stable, neoclassical steady state. We examined the comparative static effects of exogenous tax abatement policy and showed that this will widen the basin of the poverty trap and can stimulate higher capital accumulation at the neoclassical steady state.

The main contribution of the paper has been the characterization of the optimal environmental taxation where a forward-looking planner sets taxes taking as given the optimal saving decisions of each generation. We show that the optimal tax to abate pollution is zero below a threshold capital and above this threshold, weakly increasing in the capital stock. From a policy point of view, this suggests that economies that are close to or just emerging from a poverty trap might impose zero or low levels of environmental protection but eventually this will rise along the growth path.

More importantly, we have shown that optimal policy might itself contribute to complex dynamics in several ways: first, when a steady state exists at which zero abatement is optimal, optimal policy can create additional steady states with higher capital stocks and positive levels of optimal abatement; second, by reversing the stability properties around a given steady state type, *i.e.* a poverty trap can be locally stable while a neoclassical steady state might be unstable; third, by inducing oscillations and cycles around steady states which would otherwise generate monotonic dynamics.

With respect to the last finding, we offer a word of caution. Although there is evidence that short term fluctuations in air quality can lead to fluctuations in mortality rates (see Evans and Smith, 2005; Huang et al., 2012), it is not clear that these phenomena are in turn part of a general business cycle or driven by seasonality. The main lesson that we would like to emphasize through these findings is that in cases such as the one we have studied, where the impact of state variables on economic outcomes is not uniformly monotonic, optimal policy itself can contribute to economic fluctuations and multiplicity of steady states, rather than reduce them. Thus, models that impose steady state conditions to derive optimal policy can be misleading about both the transitional dynamics and the asymptotic outcomes.

Appendix A. Proof of Lemma 1

Proof. Note that π is continuous and differentiable in its arguments which in turn are continuous and differentiable in k. Therefore, π is continuous and differentiable in k and $\mathtt{G}(k)$ is continuous and differentiable in k. Taking derivatives of both terms in $\mathtt{G}(k)$ and rearranging:

$$\mathtt{G}'(k) = \left[\frac{\Gamma k^{\alpha}}{1+\pi(k)}\right]\left[\alpha\frac{\pi(k)}{k} + \frac{\pi'(k)}{1+\pi(k)}\right]; \quad (34)$$

it can be seen that the shape of $\mathtt{G}(k)$ can be quite different from the standard neoclassical mapping, depending on how $\pi'(k)$ varies with k. Taking the right-hand limit of the two terms inside square brackets, i.e. as $k \downarrow 0$, the first term clearly goes to zero and the limit of the second term inside square brackets can be expressed as

$$\alpha \cdot \left\{\lim_{k\downarrow 0}\frac{\pi(k)}{k}\right\} + \left\{\lim_{k\downarrow 0}\frac{\pi'(k)}{1+\pi(k)}\right\};$$

where the limit of the first term inside curly brackets is given by L'Hopital's Rule for right-hand limits[25] as:

$$\lim_{k\downarrow 0}\frac{\pi(k)}{k} = \lim_{k\downarrow 0}\pi'(k).$$

It can be seen that $\lim_{k\downarrow 0}\pi'(k) < \infty$ is a sufficient condition for both the terms inside curly brackets to remain finite so that $G'(k)$ approaches zero as $k \downarrow 0$. ■

Appendix B. Derivation of $\partial^2\pi/\partial\tau\partial k$

Recall that

$$\frac{\partial\pi}{\partial\tau} = \frac{\pi^A\psi\gamma Ak^{\alpha}\delta z^{\delta-1}}{(1+z^{\delta})^2}$$

Note that we can also write this as

$$\frac{\partial\pi}{\partial\tau} = \frac{\pi\psi\gamma Ak^{\alpha}\delta z^{\delta-1}}{1+z^{\delta}}$$

associated with the impact of pollution on uncertain lifetimes, our results show that second-best welfare-maximizing environmental taxes can in themselves be a source of oscillations.

25 See Proposition 4.37 and Remark 4.38 in Gharpade and Limaye (2006). For this proposition to apply here, we must have $\lim_{k\downarrow 0} k = 0$ and $\lim_{k\downarrow 0}\pi(k) = 0$. The former right-hand limit follows and the latter follows by Assumption 1 Eq. (9).

Taking the derivative of the above with respect to k (after some straightforward rearrangement):

$$\frac{\partial^2\pi}{\partial\tau\partial k}=\frac{\alpha}{k}\frac{\partial\pi}{\partial\tau}+\frac{1}{\pi^A}\frac{\partial\pi}{\partial\tau}\frac{\partial\pi^A}{\partial k}+\frac{1}{z(1+z^\delta)}\frac{\partial\pi}{\partial\tau}[(\delta-1)-(\delta+1)z^\delta]\frac{\partial z}{\partial k}$$

where

$$\frac{\partial\pi^A}{\partial k}=\frac{\alpha}{k}\frac{\vartheta y^\vartheta}{(1+y^\vartheta)^2}=\frac{\alpha}{k}\frac{\vartheta y^\vartheta}{(1+y^\vartheta)}\frac{\pi(1+z^\delta)}{(\underline{\pi}+y^\vartheta)}$$

and

$$\frac{\partial z}{\partial k}=\frac{\alpha\gamma(1-\psi\tau)Ak^\alpha}{k}=\frac{\alpha z}{k}$$

The right hand side of the main derivative can be written as

$$\frac{\partial\pi}{\partial\tau}\left[\frac{\alpha}{k}+\frac{(1+z^\delta)}{\pi^A}\frac{\alpha\pi}{k}\frac{\vartheta y^\vartheta}{(1+y^\vartheta)y^\vartheta}+\frac{\alpha}{k}\frac{z-\phi z'}{z(1+z^\delta)}[(\delta-1)-(\delta+1)z^\delta]\right]$$

Finally, expanding the term in square brackets involving z^δ and noting the definition of π, we get

$$\frac{\partial\pi}{\partial\tau}\left[\frac{\alpha}{k}+\frac{1}{\pi}\left\{\frac{\alpha\pi}{k}\left(\frac{\vartheta y^\vartheta}{(1+y^\vartheta)y^\vartheta}-\frac{z\delta z^{\delta-1}}{(1+z^\delta)}\right)+\frac{\alpha\pi\delta}{k}\frac{z}{(1+z^\delta)z}-\frac{\alpha\pi}{k}\frac{z}{z}\right\}\right];$$

from which, noting the definition of $\partial\pi/\partial k$, it follows that

$$\frac{\partial^2\pi}{\partial\tau\partial k}=\frac{\partial\pi}{\partial\tau}\frac{1}{k}\left[\alpha+\frac{k}{\pi}\frac{\partial\pi}{\partial k}+\frac{\alpha\delta z}{(1+z^\delta)z}-\alpha\right];$$

leading to the desired result.

Appendix C. An example of the survival probability function

Assuming the functional form:

$$\pi=\pi^A\pi^B$$

where

$$\pi^A=\frac{y^\vartheta}{1+y^\vartheta}$$

then it can be shown that $\pi^A_y>0$ and that $\pi^A_{yy}\le 0$ if and only if $y\le[(\vartheta-1)/(1+\vartheta)]^{1/\vartheta}$ so that for any $\vartheta>1$, $\pi^A(y)$ is S-shaped upwards.

If similarly,

$$\pi^B=\frac{1}{1+z^\delta}$$

then it can be shown that $\pi^B_z<0$ and that $\pi^B_{zz}\le 0$ if and only if $z\le[(\delta-1)/(1+\delta)]^{1/\delta}$ so that for any $\delta>1$, $\pi^B(z)$ is reverse S-shaped downwards.

Thus, the above function satisfies the sufficient conditions for multiple steady states, and after imposing the steady state relationship between y, z and k and totally differentiating, that a sufficient condition for $\pi'(k)$ to satisfy the conditions of Lemma 1 as k approaches zero is $\vartheta>1/\alpha$. For this case, it can be shown that a weaker condition

$$\vartheta>\frac{1-\alpha}{\alpha}$$

suffices to generate $\mathtt{G}'(0)=0$.

This is because the combination of the terms

$$\mathtt{G}'(k)=\left[\frac{\Gamma k^\alpha}{1+\pi(k)}\right]\left[\alpha\frac{\pi(k)}{k}-\frac{\pi'(k)}{1+\pi(k)}\right].$$

can converge to zero even if each term inside the square brackets does not.

We now assign the following parameter values

$\alpha=1/3, A=2.4, \gamma=1.11, \vartheta=9, \delta=5, \psi=0.8;$

and using MATLAB, we trace out the phase diagram of the capital stock for $\tau=0$ and $\tau=0.55$. The results have been depicted in Fig. 2.

The original steady states, at $\tau=0$, are $k_\ell=0.035$ and $k_h=0.114$ respectively. An increase in the abatement tax to $\tau=0.55$ causes a downward shift in $\mathtt{G}(k)$ at low levels of capital stock but upwards at the high capital stock. There are two new steady states, $k^{*\prime}_\ell=0.050$ and $k^{*\prime}_h=0.158$ respectively. Compared with their respective predecessors, *both* steady states have higher levels of capital stock. The dynamic implication is that the basin of attraction of a trivial steady state has now increased, while economies that start off to the right of $k^{*\prime}_\ell$ can now converge to a higher steady state than before. Thus, with an increase in the exogenous tax, it is possible that long-run cross-country inequality will increase.

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