Spectrally equivalent time-dependent double wells and unstable anharmonic oscillators

Andreas Fring and Rebecca Tenney

Department of Mathematics, City University London,
Northampton Square, London EC1V 0HB, UK
E-mail: a.fring@city.ac.uk, rebecca.tenney@city.ac.uk

ABSTRACT: We construct a time-dependent double well potential as an exact spectral equivalent to the explicitly time-dependent negative quartic oscillator with a time-dependent mass term. Defining the unstable anharmonic oscillator Hamiltonian on a contour in the lower-half complex plane, the resulting time-dependent non-Hermitian Hamiltonian is first mapped by an exact solution of the time-dependent Dyson equation to a time-dependent Hermitian Hamiltonian defined on the real axis. When unitary transformed, scaled and Fourier transformed we obtain a time-dependent double well potential bounded from below. All tranformations are carried out npn-perturbatively so that all Hamiltonians in this process are spectrally exactly equivalent in the sense that they have identical instantaneous energy eigenvalue spectra.

1. Introduction

Anharmonic oscillators have a wide range of applications in quantum mechanics as they describe for instance delocalization and decoherence of quantum states, e.g. [1]. They also occur naturally in relativistic models, e.g. [2]. From a mathematical point of view their nonlinear nature make them ideal testing grounds for various approximation methods, such as perturbative approaches [3]. Based on a perturbative expansion of the energy eigenvalues it was shown in [4] that the quartic anharmonic oscillator with mass term is spectrally equivalent to a double well potential with linear symmetry breaking. The first hint about the fact that even the unstable quartic anharmonic oscillator posses a well defined bounded real spectrum, despite being unbounded from below on the real axis, was proved in [5, 6], where it was proven that its energy eigenvalues series is Borel summable. The spectral equivalence between an unstable anharmonic oscillator and a complex double well potential was then proven directly by Buslaev and Grecchi in [7].

Subsequently the unstable quartic anharmonic oscillator without mass term was treated in [8] as part of the general series of $\mathcal{PT}$-symmetric potentials $V(x) = x^2(ix)^\varepsilon$, i.e. $\varepsilon = 2$, where it was shown numerically that the Hamiltonians in this series have real and positive spectra for $\varepsilon \geq 2$. Applying the techniques developed in this area of non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics [9, 10] Jones and Mateo [11] showed that the two Hamiltonians

$$H = p^2 - gx^4, \qquad \text{and} \qquad h = \frac{p^4}{64g} - \frac{1}{2}p + 16gx^2, \tag{1.1}$$

are spectrally equivalent. This was established by first defining H on a suitable contour in the complex plane, $x \to -2i\sqrt{1+ix}$, within the Stoke wedges where the corresponding wavefunctions decay asymptotically. Subsequently the resulting complex Hamiltonian was similarity transformed to a Hermitian Hamiltonian h that is well defined on the real axis.

Here our central aim is to extend the analysis by making the Hamiltonian explicitly time-dependent $H \to H(t)$ through the inclusion of an explicit time-dependence into the coefficients. The similarity transformation acquires then the form

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\partial_t\eta(t)\eta^{-1}(t), \tag{1.2}$$

often referred to as the time-dependent Dyson equation [12, 13, 14, 15, 16, 17, 18, 19, 20], in which $H \neq H^\dagger$ is a non-Hermitian explicitly time-dependent Hamiltonian, $h = h^\dagger$ a Hermitian explicitly time-dependent Hamiltonian and $\eta(t)$ the time-dependent Dyson map. The latter can be used to define a time-dependent metric $\rho(t)$ via the relation $\rho(t) = \eta^\dagger(t)\eta(t)$. Spectral equivalence is then understood on the level of the instantaneous energy eigenvalues for the operators $h(t)$ and the corresponding operator for the non-Hermitian system

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\eta^{-1}(t)\partial_t\eta(t). \tag{1.3}$$

Note while $\tilde{H}$ is observable it is not a Hamiltonian governing the time-evolution and satisfying the time-dependent Schrödinger equation. On the other hand the Hamiltonian $H(t)$ is not observable. Besides the aforementioned interest in the unstable anharmonic oscillator itself, there are not many known exact solutions [15, 17, 21, 18, 22, 19, 23, 24, 25, 26, 27, 28, 29, 30] to the time-dependent Dyson equation (1.2), so that any new exact solution provides valuable insights.

2. The time-dependent unstable harmonic oscillator

The Hamiltonian we investigate here is similar to the one in equation (1.1), but with time-dependent coefficient functions and an additional mass term

$$H(z,t) = p^2 + \frac{m(t)}{4}z^2 - \frac{g(t)}{16}z^4, \qquad m \in \mathbb{R}, g \in \mathbb{R}^+. \tag{2.1}$$

Defining $H(z,t)$ now on the same contour in the lower-half complex plane $z = -2i\sqrt{1+ix}$ as suggested by Jones and Mateo [11], it is mapped into the non-Hermitian Hamiltonian

$$H(x,t) = p^2 - \frac{1}{2}p + \frac{i}{2}\{x, p^2\} - m(t)(1+ix) + g(t)(x-i)^2, \tag{2.2}$$

with $\{\cdot,\cdot\}$ denoting the anti-commutator. Next we attempt to solve the time-dependent Dyson equation (1.2) to find a Hermitian counterpart h. Making the following general Ansatz for the Dyson map

$$\eta(t) = e^{\alpha(t)x} e^{\beta(t)p^3 + i\gamma(t)p^2 + i\delta(t)p}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \tag{2.3}$$

we use the Baker-Campbell-Hausdorff formula to compute the adjoint action of $\eta(t)$ on all terms appearing in $H(x,t)$

$$\begin{aligned}
\eta x \eta^{-1} &= x + \delta + 6\alpha\beta p + 2\gamma p + 3i\alpha^2\beta + 2i\alpha\gamma - 3i\beta p^2, && (2.4)\\
\eta p \eta^{-1} &= p + i\alpha, && (2.5)\\
\eta x^2 \eta^{-1} &= x^2 - 9\beta^2 p^4 - 12i\beta\left(3\alpha\beta + \gamma\right)p^3 + \left(54\alpha^2\beta^2 + 36\alpha\beta\gamma + 4\gamma^2 - 6i\beta\delta\right)p^2 && (2.6)\\
&\quad + 4(3\alpha\beta + \gamma)\left[\delta + i\alpha(3\alpha\beta + 2\gamma)\right]p + 2\left(\delta + 3i\alpha^2\beta + 2i\alpha\gamma\right)x \\
&\quad + (6\alpha\beta + 2\gamma)\{x,p\} - 3i\beta\left\{x,p^2\right\} - (3\alpha^2\beta + 2\alpha\gamma - i\delta)^2, \\
\eta p^2 \eta^{-1} &= p^2 - \alpha^2 + 2i\alpha p, && (2.7)\\
\eta\{x,p^2\}\eta^{-1} &= \{x,p^2\} - 6i\beta p^4 + (24\alpha\beta + 4\gamma)p^3 + \left(36i\alpha^2\beta + 12i\alpha\gamma + 2\delta\right)p^2 - 2\alpha^2 x && (2.8)\\
&\quad + 4\left(i\alpha\delta - 6\alpha^3\beta - 3\alpha^2\gamma\right)p - 2i\alpha^2\left(3\alpha^2\beta + 2\alpha\gamma - i\delta\right) + 4i\alpha\{x,p\}.
\end{aligned}$$

The gauge like terms in (1.2) and (1.3) are calculated to

$$\begin{aligned}
i\dot{\eta}\eta^{-1} &= ix\dot{\alpha} + i\dot{\beta}p^3 - \left(3\dot{\beta}\alpha + \dot{\gamma}\right)p^2 - \left(3i\dot{\beta}\alpha^2 + 2i\dot{\gamma}\alpha + \dot{\delta}\right)p + \dot{\beta}\alpha^3 + \dot{\gamma}\alpha^2 - i\dot{\delta}\alpha, && (2.9)\\
i\eta^{-1}\dot{\eta} &= ix\dot{\alpha} + i\dot{\beta}p^3 - (3\dot{\alpha}\beta + \dot{\gamma})p^2 - (2i\gamma\dot{\alpha} + \dot{\delta})p - i\delta\dot{\alpha}, && (2.10)
\end{aligned}$$

where as commonly used we abbreviate partial derivatives with respect to t by an overdot. Using the expressions in (2.4)-(2.9) for the evaluation of (1.2) and demanding the right hand side to be Hermitian yields the following constraints for the coefficient functions in the Dyson map

$$\alpha = \frac{\dot{g}}{6g}, \quad \beta = \frac{1}{6g}, \quad \gamma = \frac{12g^3 + 6mg^2 + \dot{g}^2 - g\ddot{g}}{4\dot{g}g^2}, \quad \delta = c_1\frac{g}{\dot{g}} - \frac{g\ln g}{2\dot{g}}, \tag{2.11}$$

with $c_1 \in \mathbb{R}$ being an integration constant. Moreover, the time-dependent coefficient functions in the Hamiltonian (2.1) must be related by the third order differential equation

$$9g^2\left(\dddot{g} - 6g\dot{m}\right) + 36g\dot{g}\left(gm - \ddot{g}\right) + 28\dot{g}^3 = 0. \tag{2.12}$$

Integrating once and introducing a new parameterization function $\sigma(t)$, we solve this equation by

$$g = \frac{1}{4\sigma^3}, \qquad \text{and} \qquad m = \frac{4c_2 + \dot{\sigma}^2 - 2\sigma\ddot{\sigma}}{4\sigma^2}, \tag{2.13}$$

with $c_2 \in \mathbb{R}$ denoting the integration constant corresponding to the only integration we have carried out. The time-dependent Hermitian Hamiltonian in equation (1.2) then results to

$$h(x,t) = \sigma^3 p^4 + f_{pp}(t)p^2 + f_x(t)x + f_p(t)p + f_{xp}(t)\{x,p\} + f_{xx}(t)x^2 + C(t). \tag{2.14}$$

with

$$f_{pp} = \frac{\sigma\left\{\sigma\left[2\left(\sigma\left(\dot{\sigma}^2-4c_2\right)-2\right)\ddot{\sigma}+16c_2^2+\dot{\sigma}^4\right]+16c_2\right\}+4}{4\sigma\dot{\sigma}^2}, \quad f_{xp} = \frac{\left(\sigma\left(\dot{\sigma}^2-4c_2\right)-2\right)}{4\sigma^2\dot{\sigma}},$$

$$f_p = \frac{2c_1\left[\sigma\left(4c_2+\dot{\sigma}^2-2\sigma\ddot{\sigma}\right)+2\right]+\ln\left(4\sigma^3\right)}{12\sigma\dot{\sigma}^2}, \quad f_x = -\frac{2c_1+\ln\left(4\sigma^3\right)}{12\sigma^2\dot{\sigma}}, \quad f_{xx} = \frac{1}{4\sigma^3},$$

$$C = \frac{\left(2c_1+\ln\left(4\sigma^3\right)\right)^2+36\dot{\sigma}^2\left(4c_2^2+\ddot{\sigma}\right)}{144\sigma\dot{\sigma}^2} + \frac{1}{8}\left(\dot{\sigma}^2-4c_2\right)\ddot{\sigma} - \frac{\dot{\sigma}^2}{4\sigma^2}$$

We may choose to set $c_1 = c_2 = 0$ and reintroduce the original time-dependent coefficient functions $g(t)$, $m(t)$ so that the Hamiltonian simplifies to

$$h(x,t) = \frac{p^4}{4g} + \left(\frac{18g^2(2g+m)}{\dot{g}^2} + \frac{\dot{g}^2}{72g^3} - \frac{2g+m}{4g}\right)p^2 - \frac{3\left(g^2m+g^3\right)\ln g}{\dot{g}^2}p + \frac{g^2\ln(g)}{\dot{g}}x$$
$$+\left(\frac{\dot{g}}{12g} - \frac{6g^2}{\dot{g}}\right)\{x,p\} + gx^2 + \frac{1296g^8\ln^2 g+\dot{g}^6-36\dot{g}^4g^2(2g+m)}{5184g^5\dot{g}^2} - \frac{m}{2}. \quad (2.15)$$

Notice that $\sigma(t)$ can be any function, but the coefficient functions $g(t)$ and $m(t)$ must be related by (2.12) that is (2.13).

The massless case for $m(t) = 0$ is more restrictive and leads to $\sigma(t)$ being a second order polynomial $\sigma(t) = \kappa_0 + \kappa_1 t + \kappa_2 t^2$ with real constants κ_i. This case is consistently recovered from (2.13) with the choice $c_2 = \kappa_1\kappa_3 - \kappa_2^2/4$. The solution found for the time-independent case in [11], would be obtained from (2.3) in the limits $\alpha \to 0$, $\beta \to 1/6g$, $\gamma \to 0$, $\delta \to i$ and $m \to 0$. While this limit obviously exists for α and β, the constraints for γ and δ are different from those reported in (2.11). In fact, setting $\delta(t) \to i\delta(t)$ enforces g to be time-independent and there is no time-dependent solution corresponding to that choice. The energy operator $\tilde{H}$ defined in (1.3) is obtained directly by adding $H(x,t)$ in (2.2) and the gauge-like term in (2.10).

Let us now eliminate the terms in $h(x,t)$ proportionate to x and $\{x,p\}$ by means of a unitary transformation

$$U = e^{-i\frac{f_{xp}}{2f_{xx}}p^2 - i\frac{f_x}{2f_{xx}}p}, \quad (2.16)$$

which leads to the unitary transformed Hamiltonian

$$\hat{h}(x,t) = \sigma^3p^4 + \left(f_{pp} - \frac{f_{xp}^2}{f_{xx}}\right)p^2 + \left(f_p - \frac{f_xf_{xp}}{f_{xx}}\right)p + f_{xx}x^2 + C - \frac{f_x^2}{4f_{xx}}. \quad (2.17)$$

Similarly as in the time-independent case [11], we may scale this Hamiltonian, albeit now with a time-dependent function, $x \to (f_{xx})^{-1/2}x$. Subsequently we Fourier transform $\hat{h}(x,t)$ so that it is viewed in momentum space. In this way we obtain a spectrally equivalent Hamiltonian with a time-dependent potential

$$\tilde{h}(y,t) = p_y^2 + \sigma^3f_{xx}^2y^4 + \left(f_{xx}f_{pp} - f_{xp}^2\right)y^2 + \left(\sqrt{f_{xx}}f_p - \frac{f_xf_{xp}}{\sqrt{f_{xx}}}\right)y + C - \frac{f_x^2}{4f_{xx}}, \quad (2.18)$$

$$= \frac{g}{4}y^2\left(y^2 + \frac{\dot{g}^2}{36g^3} + \frac{72g^2m}{\dot{g}^2} - \frac{m}{g} + 2\right) + \frac{\left(36g^2m+\dot{g}^2\right)\sqrt{g}\ln g}{12\dot{g}^2}y \quad (2.19)$$
$$+\frac{\dot{g}^4}{5184g^5} - \frac{\dot{g}^2m}{144g^3} - \frac{\dot{g}^2}{72g^2} - \frac{m}{2},$$

where for simplicity we have set $c_1 = c_2 = 0$ in (2.19). The potential in $\tilde{h}(y,t)$ is a double well that is bounded from below. We illustrate this for a specific choice of $\sigma(t)$, that is $g(t)$ and $m(t)$, in figure 1.

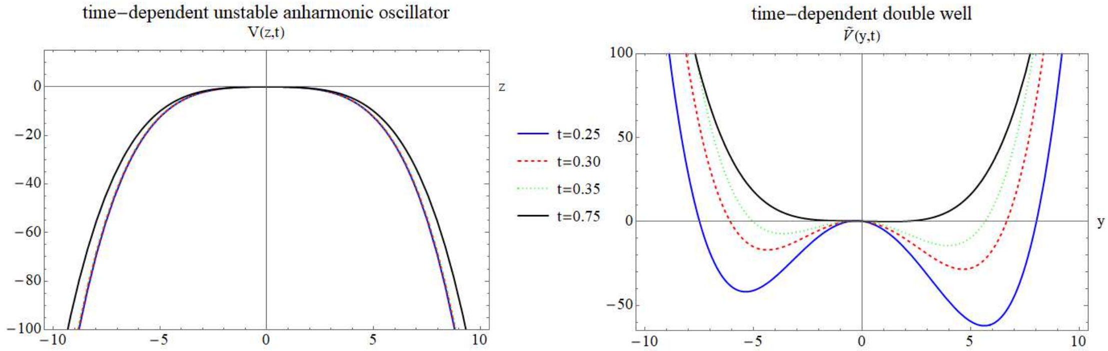

Figure 1: Spectrally equivalent time-dependent anharmonic oscillator potential $V(z,t)$ in (2.1) and time-dependent double well potential $\tilde{V}(y,t)$ in (2.19) for $\sigma(t) = \cosh t$, $g(t) = 1/4\cosh^3 t$, $m(t) = (\tanh^2 t - 2)/4$ at different values of time.

3. Conclusions

We have proven the remarkable fact that the time-dependent unstable anharmonic oscillator is spectrally equivalent to a time-dependent double well potential that is bounded from below. The transformations we carried out are summarized as follows:

$$H(z,t) \overset{z\to x}{\to} H(x,t) \overset{\text{Dyson}}{\to} h(x,t) \overset{\text{unitary transform}}{\to} \hat{h}(x,t) \overset{\text{Fourier}}{\to} \tilde{h}(y,t).$$

We have first transformed the time-dependent anharmonic oscillator $H(z,t)$ from a complex contour in a Stokes wedge to the real axis $H(x,t)$. The resulting non-Hermitian Hamiltonian $H(x,t)$ was then mapped by mean of a time-dependent Dyson map $\eta(t)$ to a time-dependent Hermitian Hamiltonian $h(x,t)$. It turned out that the Dyson map can not be obtained by simply introducing time-dependence into the known solution for the time-independent case [11], but it required to complexify one of the constants and the inclusion of two additional factors. In order to obtain a potential Hamiltonian we have unitary transformed $h(x,t)$ into a spectrally equivalent Hamiltonian $\hat{h}(x,t)$, which when Fourier transformed leads to $\tilde{h}(y,t)$ that involved a time-dependent double well potential.

A detailed analysis of the spectra and eigenfunctions using approximation methods for time-dependent potential [31] is left for future investigations. Moreover, it is well known that Dyson maps are not unique, in the time-dependent as well as time-independent case, and it might therefore be interesting to explore whether additional spectrally equivalent Hamiltonians to $H(z,t)$ can be found in the same fashion for new type of maps.

Acknowledgments: RT is supported by a City, University of London Research Fellowship.

References

[1] G. J. Milburn and C. A. Holmes, Dissipative quantum and classical Liouville mechanics of the anharmonic oscillator, Phys. Rev. Lett. **56**(21), 2237 (1986).

[2] G. Gabrielse, H. Dehmelt, and W. Kells, Observation of a relativistic, bistable hysteresis in the cyclotron motion of a single electron, Phys. Rev. Lett. **54**(6), 537 (1985).

[3] R. Seznec and J. Zinn-Justin, Summation of divergent series by order dependent mappings: Application to the anharmonic oscillator and critical exponents in field theory, J. Math. Phys. **20**(7), 1398–1408 (1979).

[4] A. A. Andrianov, The large N expansion as a local perturbation theory, Annals of Physics **140**(1), 82–100 (1982).

[5] S. Graffi and V. Grecchi, The Borel sum of the double-well perturbation series and the Zinn-Justin conjecture, Phys. Lett. B **121**(6), 410–414 (1983).

[6] E. Caliceti, V. Grecchi, and M. Maioli, Double wells: perturbation series summable to the eigenvalues and directly computable approximations, Comm. Math. Phys. **113**(4), 625–648 (1988).

[7] V. Buslaev and V. Grecchi, Equivalence of unstable anharmonic oscillators and double wells, J. of Phys. A: Math. and Gen. **26**(20), 5541 (1993).

[8] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. **80**, 5243–5246 (1998).

[9] A. Mostafazadeh, Pseudo-Hermitian Representation of Quantum Mechanics, Int. J. Geom. Meth. Mod. Phys. **7**, 1191–1306 (2010).

[10] C. M. Bender, P. E. Dorey, C. Dunning, A. Fring, D. W. Hook, H. F. Jones, S. Kuzhel, G. Levai, and R. Tateo, PT Symmetry: In Quantum and Classical Physics, (World Scientific, Singapore) (2019).

[11] H. F. Jones and J. Mateo, An Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ Potential, Phys. Rev. **D73**, 085002 (2006).

[12] C. Figueira de Morisson Faria and A. Fring, Time evolution of non-Hermitian Hamiltonian systems, J. Phys. **A39**, 9269–9289 (2006).

[13] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator, Physics Letters B **650**(2), 208–212 (2007).

[14] M. Znojil, Time-dependent version of crypto-Hermitian quantum theory, Physical Review D **78**(8), 085003 (2008).

[15] H. Bíla, Adiabatic time-dependent metrics in PT-symmetric quantum theories, arXiv preprint arXiv:0902.0474 (2009).

[16] J. Gong and Q.-H. Wang, Time-dependent PT-symmetric quantum mechanics, J. Phys. A: Math. and Theor. **46**(48), 485302 (2013).

[17] A. Fring and M. H. Y. Moussa, Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians, Phys. Rev. A **93**(4), 042114 (2016).

[18] A. Fring and T. Frith, Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians, Phys. Rev. A **95**, 010102(R) (2017).

[19] A. Fring and T. Frith, Mending the broken PT-regime via an explicit time-dependent Dyson map, Phys. Lett. A , 2318 (2017).

[20] A. Mostafazadeh, Energy observable for a quantum system with a dynamical Hilbert space and a global geometric extension of quantum theory, Phys. Rev. D **98**(4), 046022 (2018).

[21] A. Fring and M. H. Y. Moussa, Non-Hermitian Swanson model with a time-dependent metric, Phys. Rev. A **94**(4), 042128 (2016).

[22] A. Fring and T. Frith, Metric versus observable operator representation, higher spin models, Eur. Phys. J. Plus , 133: 57 (2018).

[23] B. Khantoul, A. Bounames, and M. Maamache, On the invariant method for the time-dependent non-Hermitian Hamiltonians, The European Physical Journal Plus **132**(6), 258 (2017).

[24] M. Maamache, O. K. Djeghiour, N. Mana, and W. Koussa, Pseudo-invariants theory and real phases for systems with non-Hermitian time-dependent Hamiltonians, The European Physical Journal Plus **132**(9), 383 (2017).

[25] J. Cen, A. Fring, and T. Frith, Time-dependent Darboux (supersymmetric) transformations for non-Hermitian quantum systems, J. of Phys. A: Math. and Theor. **52**(11), 115302 (2019).

[26] A. Fring and T. Frith, Solvable two-dimensional time-dependent non-Hermitian quantum systems with infinite dimensional Hilbert space in the broken PT-regime, J. of Phys. A: Math. and Theor. **51**(26), 265301 (2018).

[27] A. Fring and T. Frith, Quasi-exactly solvable quantum systems with explicitly time-dependent Hamiltonians, Phys. Lett. A **383**(2-3), 158–163 (2019).

[28] A. Fring and T. Frith, Eternal life of entropy in non-Hermitian quantum systems, Phys. Rev. A **100**(1), 010102 (2019).

[29] A. Fring and T. Frith, Time-dependent metric for the two-dimensional, non-Hermitian coupled oscillator, Mod. Phys. Lett. A **35**(08), 2050041 (2020).

[30] W. Koussa and M. Maamache, Pseudo-invariant approach for a particle in a complex time-dependent linear potential, Int. J. of Theor. Phys. , 1–14 (2020).

[31] A. Fring and R. Tenney, Time-independent approximations for time-dependent optical potentials, The European Physical Journal Plus **135**(2), 163 (2020).