ARBITRARILY LARGE MORITA FROBENIUS NUMBERS

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Abstract. We construct blocks of finite groups with arbitrarily large Morita Frobenius numbers, an invariant which determines the size of the minimal field of definition of the associated basic algebra. This answers a question of Benson and Kessar. This also improves upon a result of the second author where arbitrarily large $\mathcal{O}$-Morita Frobenius numbers are constructed.

1. Introduction

Let ℓ be a prime and k an algebraically closed field of characteristic ℓ. For a finite-dimensional k-algebra A we define the n^{th} Frobenius twist of A, denoted $A^{(\ell^n)}$, as follows: as a set, and indeed as a ring, $A^{(\ell^n)}$ is equal to A, but for $\lambda \in k$ and $a \in A^{(\ell^n)}$ we set $\lambda \cdot a = \lambda^{\ell^{-n}} a$ (the multiplication on the right hand side being that of A). That is, if we think of a k-algebra as a ring with a distinguished embedding $k \hookrightarrow Z(A)$, then that embedding is precomposed with the n^{th} power of the Frobenius automorphism to obtain the k-algebra structure of $A^{(\ell^n)}$. The result of this construction is clearly isomorphic to A as a ring, but not necessarily as a k-algebra. This leads to the following notion, first defined by Kessar [9].

Definition 1.1. *The* Morita Frobenius number *of A, denoted* $\mathrm{mf}(A)$*, is the smallest $n \in \mathbb{N}$ such that A is Morita equivalent to $A^{(\ell^n)}$ as a k-algebra.*

As an alternative characterisation, Kessar [9] showed that, for a basic algebra A, $\mathrm{mf}(A)$ is the smallest $n \in \mathbb{N}$ such that $A \cong k \otimes_{\mathbb{F}_{\ell^n}} A_0$ for some $\mathbb{F}_{\ell^n}$-algebra A_0. For fixed $n \in \mathbb{N}$ there are only finitely many possibilities for A_0 in any given dimension, which is why Morita Frobenius numbers are being used to approach Donovan's famous finiteness conjecture (more on this further below).

In the present paper we are interested in the Morita Frobenius numbers of blocks of finite groups G. For a block B of kG, the Frobenius twist $B^{(\ell^n)}$ is isomorphic as a k-algebra to $\sigma^n(B)$, where σ is the ring automorphism

$$\sigma : \ kG \longrightarrow kG, \ \sum_{g\in G} \alpha_g g \mapsto \sum_{g\in G} \alpha_g^{\ell} g.$$

We can therefore think of a Frobenius twist of a block simply as another block of the same group algebra, Galois conjugate to the original one. And while there is no bound on the number of Galois conjugates of a block, Benson and Kessar [1] observed that Morita Frobenius numbers of blocks tend to be very small, with no known example exceeding Morita Frobenius number two. This prompted them to ask the following.

Question (Benson-Kessar, [1, Question 6.2])**.** *Is there a universal bound on the Morita Frobenius numbers of ℓ-blocks of finite groups?*

This question, to which we give a negative answer in the present article, has gained much interest in recent years. In [1, Examples 5.1, 5.2] Benson and Kessar constructed blocks with

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Morita Frobenius number two, the first discovered to be greater than one. The relevant blocks all have a normal, abelian defect group and abelian ℓ' inertial quotient with a unique isomorphism class of simple modules. It was also proved that amongst such blocks the Morita Frobenius numbers cannot exceed two [1, Remark 3.3]. In work of Benson, Kessar and Linckelmann [2, Theorem 1.1] the bound of two was extended to blocks that don't necessarily have a unique isomorphism class of simple modules. One can also define Morita Frobenius numbers over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero with residue field k, and in [2] it was also shown that the aforementioned bound of two applies equally to the Morita Frobenius numbers of the corresponding blocks defined over $\mathcal{O}$. Finally, Farrell [6, Theorem 1.1] and Farrell and Kessar [7, Theorem 1.1] proved that the Morita Frobenius number of any block of a finite quasi-simple group is at most four (both over k and over $\mathcal{O}$).

Our main result (see Theorem 6.5) is the k-analogue of [10, Theorem 3.6], where the corresponding result is proved for blocks defined over $\mathcal{O}$. Note that the result in the current paper takes significantly longer to prove as Weiss' criterion to detect ℓ-permutation modules does not hold over k.

Theorem. *For any $n \in \mathbb{N}$ there exists an ℓ-block B of kG, for some finite group G, such that* $\mathrm{mf}(B) = n$.

Hence the questions [1, Questions 6.2, 6.3] (the second of which is the one mentioned above) both have negative answers. The blocks realising arbitrarily large Frobenius numbers have elementary abelian defect groups and metabelian ℓ' inertial quotients, and turn out to be Morita equivalent to a *twisted tensor product* of the algebra $k[D \rtimes P]$ with itself, where D is an elementary abelian ℓ-group and P is a cyclic ℓ'-group. It should be mentioned that while the blocks realising Morita Frobenius number two in [1] are described as *quantum complete intersections*, such algebras can also be realised as iterated twisted tensor products of group algebras of cyclic groups.

To close, let us quickly explain how the initial motivation to consider Morita Frobenius numbers came from their link with Donovan's conjecture.

Conjecture 1.2 (Donovan)**.** *Let D be a finite ℓ-group. Then, amongst all finite groups G and blocks B of kG with defect group isomorphic to D, there are only finitely many Morita equivalence classes.*

If true, Donovan's conjecture would imply that Morita Frobenius numbers are bounded in terms of a function of the isomorphism class of the defect group.

Conjecture 1.3 (see [1, Question 6.1])**.** *Let D be a finite ℓ-group. Then, amongst all finite groups G and blocks B of kG with defect group isomorphic to D, the Morita Frobenius number* $\mathrm{mf}(B)$ *is bounded.*

While we cannot contribute much to this, we should point out that the defect of our blocks realising Morita Frobenius number $n \in \mathbb{N}$ grows exponentially in n. Hence our result does not contradict Conjecture 1.3 and it would in fact be consistent with a logarithmic bound of Morita Frobenius numbers in terms of the rank of D. In [9, Theorem 1.4] Kessar proved that Donovan's conjecture is equivalent to Conjecture 1.3 together with the so-called Weak Donovan conjecture, which further highlights the importance of understanding and bounding Morita Frobenius numbers.

Conjecture 1.4 (Weak Donovan)**.** *Let D be a finite ℓ-group. Then there exists a constant $c(D) \in \mathbb{N}$ such that if G is a finite group and B is a block of kG with defect group isomorphic to D, then the entries of the Cartan matrix of B are bounded by $c(D)$.*

The article is organised as follows. In §2 we introduce the block $B(\theta)$, which becomes the focus of study for the remainder of the paper. We describe the simple $B(\theta)$-modules in §3 and in §4 we study a certain subalgebra of $B(\theta)$ more closely. We introduce $B(\theta)_0$, a k-algebra Morita equivalent to $B(\theta)$, in §5 and prove our main theorem in §6.

Notation. For an ℓ'-group G, $e_\chi \in kG$ will denote the primitive central idempotent corresponding to $\chi \in \operatorname{Irr}(G)$ and $1_G \in \operatorname{Irr}(G)$ will signify the trivial character. We will often use the fact that $\operatorname{IBr}(G) = \operatorname{Irr}(G)$ for such a group G.

For an arbitrary group G, a normal subgroup $N \lhd G$ and $\chi \in \operatorname{Irr}(N)$, we set $\operatorname{Irr}(G|\chi)$ to be the set of characters of G appearing as a non-zero constituent in $\chi \uparrow_N^G$. For $x \in kN$ and $g \in G$, we denote by $x^g = g^{-1}xg \in kN$. Similarly, for $\chi \in \operatorname{Irr}(N)$, we signify by χ^g the character of N given by $\chi^g(h) = \chi(h^{g^{-1}})$, for all $h \in N$. Note this definition ensures that $e_\chi^g = e_{\chi^g}$. If $\chi, \chi' \in \operatorname{Irr}(N)$ such that $\chi' = \chi^g$, for some $g \in G$, we write $\chi \sim_G \chi'$.

2. Setup

In this section we will define the groups and blocks which we later show realise arbitrarily large Morita Frobenius numbers over k. All notation introduced in this section will be used throughout the paper. We start by setting, for $i \in \{1, 2\}$,

$$P_i = (\mathbb{F}_p, +) = C_p$$

for some prime $p \neq \ell$, and

$$D_i = \prod_{P_i} C_\ell \Big/ \langle (x, \dots, x) \mid x \in C_\ell \rangle \cong C_\ell^{p-1}.$$

We set $d_i^x = (1, \dots, 1, d, 1, \dots, 1) \in D_i$, where d is a fixed generated of C_ℓ, and the position of d is the direct factor of $\prod C_\ell$ labeled by $x \in P_i$. In particular, all d_i^x taken together generate D_i. The group P_i acts on D_i by permuting the direct factors, i.e. by setting $(d_i^x)^y = d_i^{xy}$ for $x, y \in P_i$. Hence we can form the algebra

$$A_i = k[D_i \rtimes P_i].$$

Let $L = \langle g_0 \rangle \subseteq \mathbb{F}_p^\times \cong C_{p-1}$ be an ℓ'-subgroup of order $r > 1$. Set

$$H = \langle g_1, g_2, g_z : g_1^r = g_2^r = g_z^r = 1, [g_1, g_z] = [g_2, g_z] = 1, [g_1, g_2] = g_z \rangle$$

where we adopt the convention that $[g, h] = ghg^{-1}h^{-1}$, and define the subgroups

$$L_1 = \langle g_1 \rangle,\ L_2 = \langle g_2 \rangle \text{ and } Z = \langle g_z \rangle.$$

We have $L_1 \cong L_2 \cong Z \cong L \cong C_r$. Note we have an action of $\mathbb{F}_p^\times$ (and hence L) on each P_i given by multiplication. We can now define an action of H on $(D_1 \rtimes P_1) \times (D_2 \rtimes P_2)$, with kernel Z, in the following way. If $\{i, j\} = \{1, 2\}$, then L_i acts on $D_i \rtimes P_i$ by setting

$$(d_i^x y)^w = d_i^{(x^w)} y^w \quad \text{for } x, y \in P_i \text{ and } w \in L_i,$$

and setting the action of L_i on $D_j \rtimes P_j$ to be trivial.

Definition 2.1 (The group G, and the block $B(\theta)$)**.** *Define*

$$G = ((D_1 \rtimes P_1) \times (D_2 \rtimes P_2)) \rtimes H.$$

and the following subgroups of G

$$D = D_1 \times D_2, \quad \textit{and} \quad E = (P_1 \times P_2) \rtimes H.$$

Let $\theta \in \mathrm{Irr}(Z)$ be a faithful character, and let e_θ be the associated central-primitive idempotent. Define a block $B(\theta) = kGe_\theta$.

Definition 2.2. *Let $\theta \in \mathrm{Irr}(Z)$ be faithful, as before, and let $\{i, j\} = \{1, 2\}$. For each $\chi \in \mathrm{Irr}(L_i)$ define an element $h^\theta_{\chi,i} \in L_j$ such that*

$$\chi(-) = \theta([h^\theta_{\chi,i}, -]) \tag{1}$$

and

$$h^\theta_{\chi,i} h^\theta_{\eta,i} = h^\theta_{\chi\eta,i} \quad \text{for all } \chi, \eta \in \mathrm{Irr}(L_i). \tag{2}$$

We will often refer to $h^\theta_{\chi,i}$ as $h_{\chi,i}$ where the choice of θ is clear from the context.

Note that in the foregoing definition, the existence of an $h^\theta_{\chi,i}$ satisfying (1) is guaranteed by [8, Lemma 4.1] and the uniqueness of such an $h^\theta_{\chi,i}$ in L_j follows by the fact that $C_H(L_i) = Z \times L_i$. In order to see that (2) holds we note that, since $[H, H] \subseteq Z \subseteq Z(H)$, we have

$$[h^\theta_{\chi,i}, g][h^\theta_{\eta,i}, g] = h^\theta_{\chi,i} g (h^\theta_{\chi,i})^{-1} g^{-1} [h^\theta_{\eta,i}, g] = h^\theta_{\chi,i} [h^\theta_{\eta,i}, g] g (h^\theta_{\chi,i})^{-1} g^{-1} = [h^\theta_{\chi,i} h^\theta_{\eta,i}, g]$$

for all $g \in L_i$. Effectively, the above just fixes an isomorphism between $\mathrm{Irr}(L_i) \cong \mathrm{Hom}(C_r, k^\times) \cong C_r$ and $L_j \cong C_r$.

3. Simple modules and Brauer characters

From now on, unless we are explicitly considering $B(\theta)$ and $B(\theta')$ for two $\theta, \theta' \in \mathrm{Irr}(Z)$, we denote $B(\theta)$ simply by B. Since D acts trivially on every simple B-module, we can and do identify $\mathrm{IBr}(B)$ with $\mathrm{Irr}(E|\theta)$. In what follows, by an abuse of notation, we often use 1 to denote 1_{P_i}, for $i = 1, 2$. We define the following elements of $\mathrm{Irr}(E|\theta)$,

$$\begin{aligned}
(1, 1) &= (\theta \otimes 1_{P_1 \times (P_2 \rtimes L_2)}) \uparrow^E_{Z \times P_1 \times (P_2 \rtimes L_2)} = (\theta \otimes 1_{(P_1 \rtimes L_1) \times P_2}) \uparrow^E_{Z \times (P_1 \rtimes L_1) \times P_2}, \\
(\phi, 1) &= (\theta \otimes \phi \otimes 1_{P_2 \rtimes L_2}) \uparrow^E_{Z \times P_1 \times (P_2 \rtimes L_2)}, \\
(1, \psi) &= (\theta \otimes 1_{P_1 \rtimes L_1} \otimes \psi) \uparrow^E_{Z \times (P_1 \rtimes L_1) \times P_2}, \\
(\phi, \psi) &= (\theta \otimes \phi \otimes \psi) \uparrow^E_{Z \times P_1 \times P_2},
\end{aligned} \tag{3}$$

for all $\phi \in \mathrm{Irr}(P_1) \setminus \{1\}$ and $\psi \in \mathrm{Irr}(P_2) \setminus \{1\}$. Note that, since $C_{L_1}(L_2) = C_{L_2}(L_1) = \{1\}$ and θ is faithful, $\mathrm{Stab}_H(\theta \otimes 1_{L_i}) = Z \times L_i$, for all $i = 1, 2$. Also, as any non-trivial $\phi \in \mathrm{Irr}(P_i)$ is faithful, $\mathrm{Stab}_{L_i}(\phi) = \{1\}$, for all $i = 1, 2$. It follows that all the characters in (3) are indeed irreducible.

Lemma 3.1. $\mathrm{Irr}(E|\theta) = \{(\phi, \psi) | \phi \in \mathrm{Irr}(P_1),\ \psi \in \mathrm{Irr}(P_2)\}$ *and $(\phi, \psi) = (\phi', \psi')$ if and only if either $\phi = \phi'$ and $\psi = \psi'$, or $\phi, \psi, \phi', \psi' \neq 1$ and $\phi \sim_{L_1} \phi'$, $\psi \sim_{L_2} \psi'$. Moreover,*

$$\deg(1, 1) = \deg(\phi, 1) = \deg(1, \psi) = r, \quad \deg(\phi, \psi) = r^2,$$

for all $\phi \in \mathrm{Irr}(P_1) \setminus \{1\}$ and $\psi \in \mathrm{Irr}(P_2) \setminus \{1\}$.

Proof. We claim that

$$\begin{aligned}\{(1,1)\} &= \mathrm{Irr}(E|\theta\otimes 1_{P_1}\otimes 1_{P_2}),\\ \{(\phi,1)\}_{\phi\in\mathrm{Irr}(P_1)\setminus\{1\}} &= \bigcup_{\mu\in\mathrm{Irr}(P_1)\setminus\{1\}} \mathrm{Irr}(E|\theta\otimes\mu\otimes 1_{P_2}),\\ \{(1,\psi)\}_{\psi\in\mathrm{Irr}(P_2)\setminus\{1\}} &= \bigcup_{\nu\in\mathrm{Irr}(P_2)\setminus\{1\}} \mathrm{Irr}(E|\theta\otimes 1_{P_1}\otimes\nu),\\ \{(\phi,\psi)\}_{\phi\in\mathrm{Irr}(P_1)\setminus\{1\},\psi\in\mathrm{Irr}(P_2)\setminus\{1\}} &= \bigcup_{\substack{\mu\in\mathrm{Irr}(P_1)\setminus\{1\}\\ \nu\in\mathrm{Irr}(P_2)\setminus\{1\}}} \mathrm{Irr}(E|\theta\otimes\mu\otimes\nu).\end{aligned}$$

These equalities can all be readily checked. The main point is that, by the comments preceding the lemma, L_1 acts regularly on $\mathrm{Irr}(Z\times L_2|\theta)$ and L_2 on $\mathrm{Irr}(Z\times L_1|\theta)$. These facts are needed to prove the first three equalities. The fourth is more straightforward. The fact that there are no duplicates, other than the desired ones, is again a consequence of the regularity of these actions. It is a simple task to verify the degrees. □

Proposition 3.2. (1) *Let $g_i\in\mathrm{Aut}(P_i)\cong\mathbb{F}_p^\times$ for $i\in\{1,2\}$. The following automorphism of G induces an automorphism of $B(\theta)$,*

$$\begin{aligned} d_i^x y &\mapsto d_i^{(x^{g_i})}y^{g_i}, && \text{for all } i\in\{1,2\} \text{ and } x,y\in P_i,\\ h &\mapsto h, && \text{for all } h\in H.\end{aligned}$$

Furthermore, the corresponding permutation of $\mathrm{IBr}(B(\theta))$ is given by $(\phi,\psi)\mapsto(\phi^{g_1},\psi^{g_2})$, for all $\phi\in\mathrm{Irr}(P_1),\psi\in\mathrm{Irr}(P_2)$.

(2) *The following automorphism of G induces an isomorphism $B(\theta)\xrightarrow{\sim}B(\theta^{-1})$,*

$$\begin{aligned} (x_1,x_2) &\mapsto (x_2,x_1), && \text{for all } (x_1,x_2)\in(D_1\rtimes P_1)\times(D_2\rtimes P_2),\\ z &\mapsto z^{-1}, && \text{for all } z\in Z,\\ g_i &\mapsto g_j, && (\text{for } \{i,j\}=\{1,2\}),\end{aligned}$$

where g_1 and g_2 are the generators for L_1 and L_2 defined in §2. Also, for the topmost assignment recall that P_1 and P_2, as well as D_1 and D_2, are defined as two copies of the same group, i.e. we may identify them. Furthermore, the corresponding bijection $\mathrm{IBr}(B(\theta))\longrightarrow\mathrm{IBr}(B(\theta^{-1}))$ is given by $(\phi,\psi)\mapsto(\psi,\phi)$, where we identify $\mathrm{Irr}(P_1)$ and $\mathrm{Irr}(P_2)$.

Proof. This is all straightforward to check. □

4. Generators and relations for $A_i = k[D_i\rtimes P_i]$

Let us now give a description of the (isomorphic) algebras A_i for $i\in\{1,2\}$ in terms of quiver and relations. This description will be used implicitly throughout the remainder of the paper. For the sake of readability, we will use the same notation for the generators of A_1 and A_2.

Definition 4.1. *Set*

$$s_\phi=\sum_{g\in P_i}\phi(g^{-1})d_i^g\in k[D_i]\subset A_i \quad \textit{for } \phi\in\mathrm{Irr}(P_i)\setminus\{1\},$$

as well as

$$s_{\psi,\phi}=e_\psi s_\phi\in A_i \quad \textit{for } \psi\in\mathrm{Irr}(P_i) \textit{ and } \phi\in\mathrm{Irr}(P_i)\setminus\{1\}.$$

Note that $\mathrm{Irr}(P_i)\setminus\{1\}$ equals the set of constituents of the (multiplicity-free) $k[P_i]$-module $k\otimes_{\mathbb{F}_\ell}D_i$ and hence, by [5, Proposition 5.2], of $J(k[D_i])/J^2(k[D_i])$.

Proposition 4.2 (see [5, Proposition 5.3]). (1) *The e_ψ for $\psi \in \mathrm{Irr}(P_i)$ form a full set of primitive idempotents in A_i.*

(2) *The $s_{\psi,\phi}$ map to a basis of $J(A_i)/J^2(A_i)$ and $e_\psi s_{\psi,\phi} = s_{\psi,\phi} e_{\psi\phi}$. That is, the $s_{\psi,\phi}$ correspond to arrows in the quiver of A_i.*

(3) *The relations between the arrows are generated by*

$$s_{\psi,\phi} s_{\psi\phi,\zeta} = s_{\psi,\zeta} s_{\psi\zeta,\phi} \quad \text{for } \psi \in \mathrm{Irr}(P_i) \text{ and } \phi, \zeta \in \mathrm{Irr}(P_i) \setminus \{1\}$$

and

$$s_{\psi,\phi} s_{\psi\phi,\phi} \cdots s_{\psi\phi^{\ell-1},\phi} = 0 \quad \text{for } \psi \in \mathrm{Irr}(P_i) \text{ and } \phi \in \mathrm{Irr}(P_i) \setminus \{1\}.$$

A basis of $J(A_i)$ is given by elements of the form

$$s_{\psi,\boldsymbol{\phi}} = s_{\psi,\phi_1} s_{\psi\phi_1,\phi_2} \cdots s_{\psi\phi_1\cdots\phi_{m-1},\phi_m} \text{ where } \psi \in \mathrm{Irr}(P_i),\ \boldsymbol{\phi} = (\phi_1, \dots, \phi_m) \in \bigcup_{m=1}^{\infty} (\mathrm{Irr}(P_i) \setminus \{1\})^m.$$

To be more precise, we get a basis when we let $\boldsymbol{\phi}$ range over a transversal of

$$\bigcup_{m=1}^{\infty} (\mathrm{Irr}(P_i) \setminus \{1\})^m / \operatorname{Sym}_m,$$

where, in addition, $(\phi_1, \dots, \phi_m)$ must not involve any element of $\mathrm{Irr}(P_i) \setminus \{1\}$ more than $\ell - 1$ times. Let $\mathcal{I}$ denote the set of all possible values for $\boldsymbol{\phi}$ which give rise to non-zero $s_{\psi,\boldsymbol{\phi}}$'s, and let $\mathcal{I}/\sim$ denote equivalence classes of $\boldsymbol{\phi}$'s that give rise to the same $s_{\psi,\boldsymbol{\phi}}$ (all of this is independent of ψ). We have $|\mathcal{I}/\sim| = \ell^{p-1} - 1$, as $\mathcal{I}/\sim$ is naturally in bijection with maps $\mathrm{Irr}(P_1) \setminus \{1\} \longrightarrow \{0, 1, \dots, \ell - 1\}$ which are not identically zero.

5. The algebra $B(\theta)_0$

We now need to distinguish between the two sets of generators for A_1 and A_2 introduced earlier. We will always do this implicitly though, and keep the notation from the previous section. Also, $\theta \in \mathrm{Irr}(Z)$ will denote a fixed faithful character in this section.

Definition 5.1. *We define the k-algebra $B(\theta)_0 = C_{B(\theta)}(kHe_\theta)$.*

As with $B(\theta)$, we will usually denote $B(\theta)_0$ simply by B_0. Note that, by Lemma 3.1,

$$|\mathrm{Irr}(H|\theta)| = |\mathrm{Irr}(E|\theta \otimes 1_{P_1} \otimes 1_{P_2})| = |\{(1, 1)\}| = 1$$

and so $kHe_\theta \cong M_r(k)$. In particular,

$$B \cong B_0 \otimes_k kHe_\theta \cong B_0 \otimes_k M_r(k),$$

where the first isomorphism is given by multiplication. Naturally this shows that B and B_0 are Morita equivalent. Moreover, the dimensions of the simple B_0-modules are equal to the dimensions of the corresponding simple B-modules divided by r. Therefore, since by Lemma 3.1 $\deg(\phi, \psi) = r^2$, for any $\phi \neq 1$ and $\psi \neq 1$, B_0 is not basic, but it is sufficiently small for our purposes. The structure of the algebra B_0 described in Definition 5.2 and Proposition 5.4 (1)–(4) below is also known as a *twisted tensor product* of A_1 and A_2, a notion originally introduced in [4] (see also [3] for the special type of twisted tensor product that appears in our context).

Definition 5.2. (1) *Define a linear map*

$$\pi = \pi_\theta : B_0 \longrightarrow A_1 \otimes_k A_2$$

as the restriction of the linear map $kGe_\theta \longrightarrow k[(D_1 \rtimes P_1) \times (D_2 \rtimes P_2)] \cong A_1 \otimes_k A_2$ *which sends* nhe_θ *to* n *for any* $n \in (D_1 \rtimes P_1) \times (D_2 \rtimes P_2)$ *and* $h \in L_1 \cdot L_2 \subset H$ *(note that* $L_1 \cdot L_2$ *is not a group).*

(2) *For* $i \in \{1, 2\}$ *let*

$$A_i = \bigoplus_{\chi \in \mathrm{Irr}(L_i)} A_i^\chi$$

be the decomposition of A_i *as an* L_i*-module into isotypical components, i.e.* $a^g = \chi(g)a$ *whenever* $a \in A_i^\chi$ *and* $g \in L_i$*. We refer to the elements of any one of the spaces* A_i^χ *as* homogeneous.

(3) *For* $i \in \{1, 2\}$ *define the linear map* $\iota_i = \iota_{i,\theta}$ *as follows:*

$$\iota_i : A_i \longrightarrow B_0,\ a \mapsto a h_{\chi,i}^{-1} e_\theta \quad \text{for all } a \in A_i^\chi \text{ and } \chi \in \mathrm{Irr}(L_i).$$

Remark 5.3. *We will often use without further mention that* $e_1 \in A_i^1$ *for* $i \in \{1, 2\}$*. Analogous statements are not true for the other idempotents.*

The next proposition summarises the properties of the maps π, ι_1 and ι_2, which relate the structure of B_0 to that of $A_1 \otimes_k A_2$, which we understand completely by §4. The ι_i turn out to be actual algebra homomorphisms. The map π induces a bijection between B_0 and $A_1 \otimes_k A_2$. And while π is not an algebra isomorphism, it nevertheless shares some of the properties of an algebra isomorphism (e.g. point (5) of the proposition below would be obvious if π were a isomorphism).

Proposition 5.4. (1) *The map* $\iota_i : A_i \hookrightarrow B_0$ *is a* k*-algebra homomorphism for* $i \in \{1, 2\}$*.*

(2) *For* $\chi \in \mathrm{Irr}(L_1)$ *and* $\eta \in \mathrm{Irr}(L_2)$ *we have*

$$\iota_1(a)\iota_2(b) = \theta([h_{\eta,2}, h_{\chi,1}])\iota_2(b)\iota_1(a) \quad \text{for all } a \in A_1^\chi, b \in A_2^\eta. \tag{4}$$

(3) *The map* $\pi : B_0 \longrightarrow A_1 \otimes_k A_2$ *is bijective.*

(4) *For all* $a \in A_1$ *and* $b \in A_2$ *we have*

$$\pi(\iota_1(a)\iota_2(b)) = a \otimes b. \tag{5}$$

(5) *For all* $i \geqslant 1$ *we have*

$$\pi(J^i(B_0)) = J^i(A_1 \otimes_k A_2) = \sum_{j=1}^{i} J^j(A_1) \otimes_k J^{i-j}(A_2).$$

(6) *Let* $a_1, a_2 \in A_1$ *and* $b_1, b_2 \in A_2$ *be homogeneous elements. Then*

$$\pi(\iota_1(a_1)\iota_2(b_1)B_0\iota_1(a_2)\iota_2(b_2)) = (a_1 \otimes b_1)(A_1 \otimes_k A_2)(a_2 \otimes b_2).$$

Proof. (1) For $a \in A_i^\chi$ and $b \in A_i^\eta$, we have

$$\iota_i(a)\iota_i(b) = a h_{\chi,i}^{-1} e_\theta b h_{\eta,i}^{-1} e_\theta = ab h_{\chi,i}^{-1} h_{\eta,i}^{-1} e_\theta = ab h_{\chi\eta,i}^{-1} e_\theta = \iota_i(ab),$$

using that $h_{\chi,i} h_{\eta,i} = h_{\chi\eta,i}$, as we saw earlier. The above shows that ι_i is a k-algebra homomorphism, since the various A_i^χ span A_i.

(2) We have

$$\begin{aligned} \iota_1(a)\iota_2(b) &= a h_{\chi,1}^{-1} e_\theta b h_{\eta,2}^{-1} e_\theta = a b^{h_{\chi,1}} h_{\chi,1}^{-1} h_{\eta,2}^{-1} e_\theta \\ &= \eta(h_{\chi,1}) b a h_{\eta,2}^{-1} h_{\chi,1}^{-1} \theta([h_{\eta,2}, h_{\chi,1}^{-1}]) e_\theta = b a h_{\eta,2}^{-1} h_{\chi,1}^{-1} e_\theta \\ &= \chi(h_{\eta,2}^{-1}) b h_{\eta,2}^{-1} a h_{\chi,1}^{-1} e_\theta = \theta([h_{\eta,2}, h_{\chi,1}])\iota_2(b)\iota_1(a). \end{aligned}$$

(3) Surjectivity of π will follow immediately from point (4) below. For injectivity we compare dimensions. The image of π has dimension $\dim(A_1)\dim(A_2) = |D_1|^2|P_1|^2$. On the other hand $\dim(C_{B(\theta)}(H))\dim(kHe_\theta) = \dim(C_{B(\theta)}(H))r^2 = \dim(B) = |D_1|^2|P_1|^2r^2$, which shows that $\dim(B_0) = \dim(C_{B(\theta)}(H)) = |D_1|^2|P_1|^2$, which is the same as the dimension of the image of π.

(4) It suffices to check formula (5) for a and b homogeneous. So assume $a \in A_1^\chi$ and $b \in A_2^\eta$. As in the proof of (2) we have $\iota_1(a)\iota_2(b) = abh_{\eta,2}^{-1}h_{\chi,1}^{-1}e_\theta$, Since, slightly counter-intuitively, $h_{\chi,1} \in L_2$ and $h_{\eta,2} \in L_1$, we have that $h_{\eta,2}^{-1}h_{\chi,1}^{-1} \in L_1L_2$. Therefore, by definition, π maps the above element to $a \otimes b$.

(5) Recall that π was defined as the restriction of a linear map $\hat{\pi} : kGe_\theta \longrightarrow k[(D_1 \rtimes P_1) \times (D_2 \rtimes P_2)] \cong A_1 \otimes_k A_2$ which is a homomorphism of left $k[D_1 \times D_2]$-modules. In particular, $\hat{\pi}$ will map $J^i(kGe_\theta)$ onto $J^i(A_1 \otimes A_2)$ for all $i \geqslant 0$. This uses that $D_1 \times D_2$ is a normal Sylow ℓ-subgroup of G, and therefore $J(kG) = J(k[D_1 \times D_2])kG$ and an analogous expression for $J(A_1 \otimes A_2)$. Now $J^i(kGe_\theta) = J^i(B_0)kHe_\theta$, and by the definition of $\hat{\pi}$ we have $\hat{\pi}(J^i(B_0)kHe_\theta) = \pi(J^i(B_0))$, which proves the claim.

(6) It suffices to check that
$$\pi(\iota_1(a_1)\iota_2(b_1)(\iota_1(A_1^\chi)\iota_2(A_2^\eta))\iota_1(a_2)\iota_2(b_2)) = (a_1 \otimes b_1)(A_1^\chi \otimes_k A_2^\eta)(a_2 \otimes b_2)$$
for all $\chi \in \mathrm{Irr}(L_1)$ and $\eta \in \mathrm{Irr}(L_2)$. However, by formula (4), the relevant factors in the argument of π above commute up to a non-zero scalar (which does not affect the image). Hence the left hand side of the above is equal to $\pi(\iota_1(a_1A_1^\chi a_2)\iota_2(b_1A_2^\eta b_2))$, which equals the right hand side of the above by point (4). □

Proposition 5.5 (Explicit formula for ι_i). *For $\{i, j\} = \{1, 2\}$ we have*
$$\iota_i(a) = \sum_{g\in L_i}(ae_{1_{L_j}}e_\theta)^g \quad \text{for all } a \in A_i = k[D_i \rtimes P_i]. \tag{6}$$

Proof. Assume without loss of generality that $i = 1$ and $j = 2$. It suffices to prove this for $a \in A_1^\chi$ for a fixed $\chi \in \mathrm{Irr}(L_1)$. Note the idempotent $e_{1_{L_2}}$ can also be written as $r^{-1}\sum_{\eta\in\mathrm{Irr}(L_1)} h_{\eta,1}$. Now by character orthogonality (also using $h_{\eta,1} = [h_{\eta,1}, g^{-1}]h_{\eta,1}^g$),
$$\pi\left(\sum_{g\in L_1} a^g h_{\eta,1}^g e_\theta\right) = \pi\left(\sum_{g\in L_1}\chi(g)\eta(g)ah_{\eta,1}e_\theta\right) = \begin{cases} 0 & \text{if } \chi \neq \eta^{-1} \\ ra \otimes 1 & \text{if } \chi = \eta^{-1} \end{cases},$$
for all $\eta \in \mathrm{Irr}(L_1)$. As $\pi(\iota_1(a)) = a \otimes 1$, summing over all such η gives that π applied to both sides of (6) holds true. Since π is bijective, the result follows. □

By the above, the algebra $B_0 = B(\theta)_0$ can be thought of as being graded by $\mathrm{Irr}(P_1)\times\mathrm{Irr}(P_2)$, and the character θ determines how the homogeneous components commute through equation (4). We will ultimately recover θ from B_0 by showing that certain subspaces of the homogeneous components are preserved under isomorphisms modulo $J^3(B_0)$. However, we will proceed in a more elementary way, and for that we will need idempotents and certain arrows explicitly.

Definition 5.6. (1) *For $\phi \in \mathrm{Irr}(P_1), \psi \in \mathrm{Irr}(P_2)$ set*
$$\varepsilon_{(\phi,1)} = \iota_1(e_\phi)\iota_2(e_1) \quad \textit{and} \quad \varepsilon_{(1,\psi)} = \iota_1(e_1)\iota_2(e_\psi).$$
For $\phi \in \mathrm{Irr}(P_1) \setminus \{1\}, \psi \in \mathrm{Irr}(P_2) \setminus \{1\}$ set
$$\varepsilon_{(\phi,\psi)} = \iota_1(e_{[\phi]})\iota_2(e_{[\psi]}), \quad \textit{where } e_{[\psi]} = \sum_{g\in L_i} e_{\psi^g} \in A_i^1.$$

(2) *For $\psi, \phi \in \mathrm{Irr}(P_1)$, $\xi, \zeta \in \mathrm{Irr}(P_2)$ such that $\phi \neq 1$, $\zeta \neq 1$ set*

$$S_{\psi,\phi} = \iota_1(s_{\psi,\phi})\iota_2(e_1) \quad \text{and} \quad T_{\xi,\zeta} = \iota_1(e_1)\iota_2(s_{\xi,\zeta}).$$

(3) *For $1 \neq \phi \in \mathrm{Irr}(P_1)$, $1 \neq \zeta \in \mathrm{Irr}(P_2)$, $\chi \in \mathrm{Irr}(L_1)$ and $\eta \in \mathrm{Irr}(L_2)$ set*

$$\tilde{S}_\phi^\chi = \sum_{g\in L_1} \chi(g^{-1})S_{1,\phi^g}S_{\phi^g,(\phi^{-1})^g} \quad \text{and} \quad \tilde{T}_\zeta^\eta = \sum_{g\in L_2} \eta(g^{-1})T_{1,\zeta^g}T_{\zeta^g,(\zeta^{-1})^g}.$$

Remark 5.7. *Note that by Proposition 5.4* (2)*, for $\{i, j\} = \{1, 2\}$, the element $\iota_j(e_1)$, and more generally every element of $\iota_j(A_j^1)$, commutes with every element of $\iota_i(A_i)$. It follows that the $\varepsilon_{(\psi,\phi)}$ are idempotents, and*

$$S_{\psi,\phi} = \varepsilon_{(\psi,1)}S_{\psi,\phi}\,\varepsilon_{(\psi\phi,1)} \quad \text{and} \quad T_{\xi,\zeta} = \varepsilon_{(\xi,1)}T_{\xi,\zeta}\,\varepsilon_{(\xi\zeta,1)}. \tag{7}$$

Moreover, it follows that

$$S_{\psi_1,\phi_1}\cdots S_{\psi_m,\phi_m} = \iota_1(s_{\psi_1,\phi_1}\cdots s_{\psi_m,\phi_m})\iota_2(e_1) \quad \text{for any } m > 0,$$

$$S_{\psi_1,\phi_1}\cdots S_{\psi_m,\phi_m}T_{\xi_1,\zeta_1}\cdots T_{\xi_n,\zeta_n} = \iota_1(s_{\psi_1,\phi_1}\cdots s_{\psi_m,\phi_m}e_1)\iota_2(e_1 s_{\xi_1,\zeta_1}\cdots s_{\xi_n,\zeta_n}) \quad \text{for any } m,n > 0,$$

for any admissible choice of ψ_i, ϕ_i, ξ_i and ζ_i. This will be useful since the image of the right hand side under π can readily be determined using Proposition 5.4 (4).

Proposition 5.8. *If (ψ, ϕ) and (ψ', ϕ') label two different Brauer characters of B (using the notation of §3), then $\varepsilon_{(\phi,\psi)}$ and $\varepsilon_{(\phi',\psi')}$ are distinct orthogonal idempotents. The idempotent $\varepsilon_{(\phi,\psi)}$ annihilates all simple B_0-modules except for the one corresponding to the character (ϕ, ψ). In particular, the idempotents $\varepsilon_{(\psi,1)}$ and $\varepsilon_{(1,\phi)}$ are primitive in B_0 for any $\psi \in \mathrm{Irr}(P_1)$, $\phi \in \mathrm{Irr}(P_2)$.*

Proof. Let $\phi \in \mathrm{Irr}(P_1)$ and $\psi \in \mathrm{Irr}(P_2)$. Using Proposition 5.5 and the fact that $\iota_2(e_1) = e_{1_{P_2}}$ (immediately from the definition of ι_2) we get

$$\varepsilon_{(\phi,1)} = \iota_1(e_\phi)\iota_2(e_1) = \sum_{g\in L_1}(e_\phi e_{1_{L_2}} e_\theta)^g e_{1_{P_2}} = \sum_{g\in L_1}(e_\phi e_{1_{P_2\rtimes L_2}} e_\theta)^g.$$

The idempotent on the right hand side is clearly the central-primitive idempotent in kE which belongs to the induced module given in equation (3). That is, $\varepsilon_{(\phi,1)}$ is the idempotent attached to the character $(\phi, 1)$. By the same argument $\varepsilon_{(1,\psi)}$ belongs to the character $(1, \psi)$. If $\phi \neq 1$, $\psi \neq 1$ then, using only the definition of ι_i this time,

$$\varepsilon_{(\phi,\psi)} = \iota_1(e_{[\phi]})\iota_2(e_{[\psi]}) = \sum_{g\in L_1} e_{\phi^g}\, e_\theta \sum_{h\in L_2} e_{\psi^h}\, e_\theta = \sum_{g\in L_1}\sum_{h\in L_2}(e_\phi e_\psi e_\theta)^{gh},$$

which is clearly the central-primitive idempotent in kE which belongs to the character (ϕ, ψ). The distinctness and orthogonality of the various $\varepsilon_{(\phi,\psi)}$'s follows immediately from the fact that these are central-primitive idempotents belonging to distinct simple modules (albeit in the algebra kE). As each simple kE-module gives rise to a simple kG-module, whose restriction to B_0 is a direct sum of copies of a single simple module (since B_0 and B are naturally Morita equivalent), the claim regarding the action of these idempotents on the simple B_0-modules follows as well. It is also clear that the $\varepsilon_{(\phi,1)}$ and $\varepsilon_{(1,\psi)}$ are primitive, as the corresponding simple B_0-modules are one-dimensional. □

Lemma 5.9. (1) *For all $\psi, \phi \in \mathrm{Irr}(P_1)$ and $\xi, \zeta \in \mathrm{Irr}(P_2)$ with $\phi \neq 1$ and $\zeta \neq 1$,*

$$\begin{aligned}\varepsilon_{(\psi,1)}(J(B_0)/J^2(B_0))\varepsilon_{(\psi\phi,1)} &= \langle S_{\psi,\phi}\rangle_k + J^2(B_0),\\ \varepsilon_{(1,\xi)}(J(B_0)/J^2(B_0))\varepsilon_{(1,\xi\zeta)} &= \langle T_{\xi,\zeta}\rangle_k + J^2(B_0),\end{aligned}$$

and all of these spaces are one-dimensional. That is, the $S_{\psi,\phi}$ *and* $T_{\xi,\zeta}$ *correspond to arrows in the quiver of* B_0*. Moreover,*

$$\varepsilon_{(\psi,1)}(J(B_0)/J^2(B_0))\varepsilon_{(\psi,1)} = 0 \quad \textit{and} \quad \varepsilon_{(1,\xi)}(J(B_0)/J^2(B_0))\varepsilon_{(1,\xi)} = 0.$$

(2) *Let* $\phi_1, \ldots, \phi_\ell \in \mathrm{Irr}(P_1) \setminus \{1\}$ *such that* $\phi_1 \ldots \phi_\ell = 1$*. Then*

$$S_{1,\phi_1} S_{\phi_1,\phi_2} \cdots S_{\prod_{i=1}^{\ell-1} \phi_i, \phi_\ell} \in J^{\ell+1}(B_0)$$

if and only if $\phi_1 = \cdots = \phi_\ell$*. An analogous statement holds for the* $T_{\xi,\zeta}$*'s.*

(3) *The sets*

$$\left\{S_{1,\phi} S_{\phi,\phi^{-1}}\right\}_{\phi \in \mathrm{Irr}(P_1)\setminus\{1\}} \quad \textit{and} \quad \left\{S_{1,\phi} S_{\phi,\phi^{-1}} T_{1,\zeta} T_{\zeta,\zeta^{-1}}\right\}_{\phi \in \mathrm{Irr}(P_1)\setminus\{1\}, \zeta \in \mathrm{Irr}(P_2)\setminus\{1\}}$$

are linearly independent modulo $J^3(B_0)$ *and* $J^5(B_0)$*, respectively. An analogous statement to the first one holds for the* $T_{\xi,\zeta}$*'s.*

(4) *For any* $1 \neq \phi \in \mathrm{Irr}(P_1)$*,* $1 \neq \zeta \in \mathrm{Irr}(P_2)$*,* $\chi \in \mathrm{Irr}(L_1)$ *and* $\eta \in \mathrm{Irr}(L_2)$

$$\tilde{S}^{\chi}_{\phi} \tilde{T}^{\eta}_{\zeta} = \theta([h_{\eta,2}, h_{\chi,1}]) \tilde{T}^{\eta}_{\zeta} \tilde{S}^{\chi}_{\phi}.$$

Proof. (1) First of all note that "$\supseteq$" is clear by equation (7). By Proposition 5.4 (4) it follows that $\pi(S_{\psi,\phi}) = s_{\psi,\phi} \otimes e_1$, and this element is not contained in $J^2(A_1 \otimes_k A_2)$. Hence $S_{\psi,\phi} \notin J^2(B_0)$ by Proposition 5.4 (5). That is, the spaces on the right hand side are all one-dimensional.

By definition we have

$$\sum_{\psi \in \mathrm{Irr}(P_1)} \varepsilon_{(\psi,1)} = \iota_2(e_1).$$

Since the other inclusion is already known, to prove "$\subseteq$" it will suffice to show that $\iota_2(e_1)(J(B_0)/J^2(B_0))\iota_2(e_1)$ (which contains all of the $\varepsilon_{(\psi,1)}(J(B_0)/J^2(B_0))\varepsilon_{(\psi\phi,1)})$) is spanned by elements of the form $S_{\psi,\phi} + J^2(B_0)$. Since π is bijective we may as well consider the images under π. Using Proposition 5.4 (5) and (6), as well as $\pi(\iota_2(e_1)) = 1 \otimes e_1$, we have

$$\begin{aligned} & \pi(\iota_2(e_1)(J(B_0)/J^2(B_0))\iota_2(e_1)) \\ = \ & \left[J(A_1)/J^2(A_1) \otimes_k e_1(A_2/J(A_2))e_1\right] \oplus \left[A_1/J(A_1) \otimes_k e_1(J(A_2)/J^2(A_2))e_1\right], \end{aligned}$$

which is spanned by elements of the form $s_{\psi,\phi} \otimes e_1 + J^2(A_1 \otimes_k A_2) = \pi(S_{\psi,\phi} + J^2(B_0))$, since the second bracket is zero. This proves the first claim.

The second claim follows from the fact that, just like the other spaces we considered, $\varepsilon_{(\phi,1)}(J(B_0)/J^2(B_0))\varepsilon_{(\phi,1)}$ is contained in $\iota_2(e_1)(J(B_0)/J^2(B_0))\iota_2(e_1)$. We saw that the latter is spanned by the $S_{\mu,\nu} + J^2(B_0)$. But $\varepsilon_{(\phi,1)} S_{\mu,\nu} \varepsilon_{(\phi,1)} = 0$ for all choices of μ, ν.

(2) By Proposition 5.4 (4) (and Remark 5.7) we have

$$\pi(S_{1,\phi_1} S_{\phi_1,\phi_2} \cdots S_{\prod_{i=1}^{\ell-1} \phi_i, \phi_\ell}) = s_{1,\phi_1} s_{\phi_1,\phi_2} \cdots s_{\prod_{i=1}^{\ell-1} \phi_i, \phi_\ell} \otimes e_1.$$

By our knowledge of the basis of A_1 the right hand side is contained in $J^{\ell+1}(A_1 \otimes_k A_2)$ if and only if $\phi_1 = \ldots = \phi_\ell$. Now the assertion follows by Proposition 5.4 (5).

(3) By Proposition 5.4 (4) (and Remark 5.7) we have

$$\pi(S_{1,\phi} S_{\phi,\phi^{-1}}) = s_{1,\phi} s_{\phi,\phi^{-1}} \otimes e_1, \quad \pi(S_{1,\phi} S_{\phi,\phi^{-1}} T_{1,\zeta} T_{\zeta,\zeta^{-1}}) = s_{1,\phi} s_{\phi,\phi^{-1}} \otimes s_{1,\zeta} s_{\psi,\zeta^{-1}},$$

and by our knowledge of the bases of A_1 and A_2 these elements are linearly independent modulo $J^3(A_1 \otimes_k A_2)$ and $J^5(A_1 \otimes_k A_2)$, respectively. Our claim follows using Proposition 5.4 (5).

(4) By Remark 5.7 we see that

$$\tilde{S}^\chi_\phi = \iota_1 \left(\sum_{g \in L_1} \chi(g^{-1}) s_{1,\phi^g} s_{\phi^g,(\phi^{-1})^g} \right) \iota_2(e_1) \in \iota_1(e_1 A_1^\chi e_1)\iota_2(e_1),$$

that is, $\tilde{S}^\chi_\phi = \iota_1(\tilde{s}^\chi_\phi)\iota_2(e_1)$ for some $\tilde{s}^\chi_\phi \in e_1 A_1^\chi e_1$. Analogously we have $\tilde{T}^\eta_\zeta = \iota_1(e_1)\iota_2(\tilde{t}^\eta_\zeta)$ for some $\tilde{t}^\eta_\zeta \in e_1 A_2^\eta e_1$. Hence

$$\tilde{S}^\chi_\phi \tilde{T}^\eta_\zeta = \iota_1(\tilde{s}^\chi_\phi)\iota_2(\tilde{t}^\eta_\zeta) \quad \text{and} \quad \tilde{T}^\eta_\zeta \tilde{S}^\chi_\phi = \iota_2(\tilde{t}^\eta_\zeta)\iota_1(\tilde{s}^\chi_\phi).$$

The statement now follows directly from Proposition 5.4 (2). □

6. Morita equivalences

From now on let $\theta, \theta' \in \mathrm{Irr}(Z)$ be two faithful characters. We keep all other notation from the previous section. The equivalent in $B_0' = B(\theta')_0$ of the various elements given in Definition 5.6 will be denoted with a prime, e.g. $\varepsilon'_{(\phi,\psi)}$, $S'_{\psi,\phi}$, $(\tilde{S}^\chi_\phi)'$, and so on. We will show under which conditions B_0 and B_0' are Morita equivalent. This will immediately enable us to prove the main theorem of this paper.

Proposition 6.1. (1) *A Morita equivalence between B_0 and B_0' preserves the dimensions of simple modules. In particular, any such Morita equivalence is afforded by an isomorphism.*

(2) *An isomorphism $\tau : B_0 \longrightarrow B_0'$ can be modified by an inner automorphism such that*

$$\left\{ \sum_{\phi \in \mathrm{Irr}(P_1)} \tau(\varepsilon_{(\phi,1)}), \sum_{\psi \in \mathrm{Irr}(P_2)} \tau(\varepsilon_{(1,\psi)}) \right\} = \left\{ \sum_{\phi \in \mathrm{Irr}(P_1)} \varepsilon'_{(\phi,1)}, \sum_{\psi \in \mathrm{Irr}(P_2)} \varepsilon'_{(1,\psi)} \right\}.$$

Proof. We will prove this by finding distinguishing Morita invariant properties of the simple modules in B_0 and the attached idempotents. By definition we have $\pi(\varepsilon_{(\phi,\psi)}) = e_{[\phi]} \otimes e_{[\psi]}$ if $\phi \neq 1$ and $\psi \neq 1$. Using Proposition 5.4 (5) and (6), we get

$$\begin{aligned} \pi(\varepsilon_{(\phi,\psi)} J(B_0)/J^2(B_0)\varepsilon_{(\phi,\psi)}) \quad = \quad & e_{[\phi]}(J(A_1)/J^2(A_1))e_{[\phi]} \otimes_k e_{[\psi]}(A_2/J(A_2))e_{[\psi]} \\ \oplus \quad & e_{[\phi]}(A_1/J(A_1))e_{[\phi]} \otimes_k e_{[\psi]}(J(A_2)/J^2(A_2))e_{[\psi]}, \end{aligned}$$

which is clearly non-zero (e.g. $s_{\phi,\phi^{-1}\phi^g} \otimes e_{[\psi]}$ gives a non-trivial on the right hand side for any $1 \neq g \in L_1$). In particular the simple modules belonging to the characters of the form (ϕ, ψ) all have non-trivial self-extensions. This implies the first assertion since by Lemma 5.9 (1) the other simples do not have non-trivial self-extensions.

From Lemma 5.9 (1) we already know that the Ext^1 between two simple modules labeled by $(\phi, 1)$ and $(\phi', 1)$ is one-dimensional if $\phi \neq \phi' \in \mathrm{Irr}(P_1)$. The analogous statement holds for the simples labeled by $(1, \zeta)$ and $(1, \zeta')$. It therefore suffices to show that there are no non-trivial extensions between the simples labeled $(\phi, 1)$ and $(1, \zeta)$, where $1 \neq \phi \in \mathrm{Irr}(P_1)$ and $1 \neq \zeta \in \mathrm{Irr}(P_2)$. The sum of the $\varepsilon_{(\phi,1)}$ for $\phi \neq 1$ is equal to $f_1 = \iota_1(1 - e_1)\iota_2(e_1)$, and, analogously, the sum of the $\varepsilon_{(1,\zeta)}$ for $\zeta \neq 1$ is equal to $f_2 = \iota_1(e_1)\iota_2(1 - e_1)$. Note that f_1 and f_2 are suitable for application of Proposition 5.4 (6), and we have $\pi(f_1) = (1 - e_1) \otimes e_1$ and $\pi(f_2) = e_1 \otimes (1 - e_1)$. Hence

$$\begin{aligned} \pi(\varepsilon_{(\phi,1)} J(B_0)\varepsilon_{(1,\zeta)}) &\subseteq ((1 - e_1) \otimes e_1)J(A_1 \otimes_k A_2)(e_1 \otimes (1 - e_1)) \\ &= (1 - e_1)J(A_1)e_1 \otimes_k e_1 J(A_2)(1 - e_1) \subseteq J^2(A_1 \otimes_k A_2). \end{aligned}$$

It follows that $\varepsilon_{(\phi,1)}(J(B_0)/J^2(B_0))\varepsilon_{(1,\zeta)} = 0$, that is, there are no non-trivial extensions between the corresponding simple modules. □

Remark 6.2. *From Proposition 5.8 and Proposition 6.1 (2) it follows that a τ as in Proposition 6.1 (2) will, up to an inner automorphism, satisfy either*

$$\tau(\varepsilon_{(\phi,1)}) = \varepsilon'_{(\sigma(\phi),1)}, \quad \textit{or} \quad \tau(\varepsilon_{(\phi,1)}) = \varepsilon'_{(1,\sigma(\phi))},$$

for a bijective map σ from $\mathrm{Irr}(P_1)$ *to either* $\mathrm{Irr}(P_1)$ *or* $\mathrm{Irr}(P_2)$*. The analogous statement holds for the $\varepsilon_{(1,\zeta)}$, and thus, in particular, $\tau(\varepsilon_{(1,1)}) = \varepsilon'_{(1,1)}$.*

Proposition 6.3. *Let $\tau : B_0 \longrightarrow B'_0$ be an isomorphism such that*

$$\tau(\varepsilon_{(\phi,1)}) = \varepsilon'_{(\sigma(\phi),1)} \quad \textit{for a bijective map } \sigma : \mathrm{Irr}(P_1) \longrightarrow \mathrm{Irr}(P_1),$$

and $\sigma(1) = 1$. Then σ is a group automorphism.

Proof. All we need to show is that $\sigma(\zeta^i) = \sigma(\zeta)^i$ for all $i \geqslant 0$ for some arbitrary generator $\zeta \in \mathrm{Irr}(P_1)$. For $i < 2$ this is clear. By way of induction we may assume that $\sigma(\zeta^j) = \sigma(\zeta)^j$ for all $j < i$, and $i \geqslant 2$. Now for any $\psi \in \mathrm{Irr}(P_1)$ and $1 \neq \phi \in \mathrm{Irr}(P_1)$ we can write (using Lemma 5.9 (1))

$$\tau(S_{\psi,\phi}) + J^2(B'_0) = c_{\psi,\phi} S'_{\sigma(\psi),\sigma(\psi)^{-1}\sigma(\psi\phi)} + J^2(B'_0) \quad \text{for some } c_{\psi,\phi} \in k^{\times}.$$

By applying τ to the element from Lemma 5.9 (2) it follows that

$$S'_{\sigma(\psi),\sigma(\psi)^{-1}\sigma(\psi\phi)} S'_{\sigma(\psi\phi),\sigma(\psi\phi)^{-1}\sigma(\psi\phi^2)} \cdots S'_{\sigma(\psi\phi^{\ell-1}),\sigma(\psi\phi^{\ell-1})^{-1}\sigma(\psi\phi^{\ell})} \in J^{\ell+1}(B'_0), \tag{10}$$

again for any $\psi \in \mathrm{Irr}(P_1)$ and $1 \neq \phi \in \mathrm{Irr}(P_1)$. By Lemma 5.9 (2) all second indices occurring in (10), that is, all $\sigma(\psi\phi^q)^{-1}\sigma(\psi\phi^{q+1})$ for $0 \leqslant q < \ell$, must be equal for the element to be contained in $J^{\ell+1}(B'_0)$. In particular, if we specialise $\psi = \zeta^{i-2}$, $\phi = \zeta$ and look at $q = 0$ and $q = 1$, we get

$$\sigma(\zeta^{i-2})^{-1}\sigma(\zeta^{i-1}) = \sigma(\zeta^{i-1})^{-1}\sigma(\zeta^i).$$

The left hand side is equal to $\sigma(\zeta)$ by the induction hypothesis, which implies that $\sigma(\zeta^i) = \sigma(\zeta^{i-1})\sigma(\zeta) = \sigma(\zeta)^i$, which completes the induction step. □

Of course the analogue of the above statement with $(\phi, 1)$ swapped for $(1, \zeta)$ holds as well.

Proposition 6.4. *The block $B(\theta)$ is Morita equivalent to $B(\theta')$ if and only if $\theta' = \theta^{\pm 1}$.*

Proof. We first note that if $\theta' = \theta^{\pm 1}$, then, by Lemma 3.2 (2), B is Morita equivalent to B'. Conversely, suppose B is Morita equivalent to B'. Of course this implies that B_0 and B'_0 are Morita equivalent. Moreover, by Proposition 6.1 (1), any such Morita equivalence must preserve the dimensions of the simple modules and so we may assume that it is induced by a k-algebra isomorphism $\tau : B_0 \to B'_0$.

By Remark 6.2 we may assume, after pre-composing with an inner automorphism and the isomorphism from Lemma 3.2 (2) (in which case we replace θ by θ^{-1}), that $\tau(\varepsilon_{(\phi,1)}) = \varepsilon'_{(\sigma_1(\phi),1)}$ and $\tau(\varepsilon_{(1,\zeta)}) = \varepsilon'_{(1,\sigma_2(\zeta))}$ for maps $\sigma_i : \mathrm{Irr}(P_i) \longrightarrow \mathrm{Irr}(P_i)$. Furthermore, by Proposition 6.3 we may assume that the σ_i are group automorphisms of $\mathrm{Irr}(P_i)$. Certainly every group automorphism of $\mathrm{Irr}(P_i)$ is induced by one of P_i and so, possibly after pre-composing τ with an automorphism as in Lemma 3.2 (1), we may assume that both σ_i are the identity. That is, we may assume

$$\tau(\varepsilon_{(\phi,1)}) = \varepsilon'_{(\phi,1)}, \quad \tau(\varepsilon_{(1,\zeta)}) = \varepsilon'_{(1,\zeta)} \quad \text{for any } \phi \in \mathrm{Irr}(P_1) \text{ and } \zeta \in \mathrm{Irr}(P_2). \tag{11}$$

From now on we fix a $\phi \in \mathrm{Irr}(P_1) \setminus \{1\}$ and $\zeta \in \mathrm{Irr}(P_2) \setminus \{1\}$ and define the spaces

$$\boldsymbol{S}_\phi = \langle \tilde{S}^\chi_\phi | \chi \in \mathrm{Irr}(L_1) \rangle_k = \langle S_{1,\phi^h} S_{\phi^h,(\phi^h)^{-1}} | h \in L_1 \rangle_k,$$
$$\boldsymbol{T}_\zeta = \langle \tilde{T}^\eta_\zeta | \eta \in \mathrm{Irr}(L_2) \rangle_k = \langle T_{1,\zeta^h} T_{\zeta^h,(\zeta^h)^{-1}} | h \in L_2 \rangle_k.$$

The first assertion of Lemma 5.9 (3) gives that $\boldsymbol{S}_\phi$ and $\boldsymbol{T}_\zeta$ both have dimension r modulo $J^3(B_0)$, with bases $(\tilde{S}^\chi_\phi + J^3(B_0))_{\chi \in \mathrm{Irr}(L_1)}$ and $(\tilde{T}^\eta_\zeta + J^3(B_0))_{\eta \in \mathrm{Irr}(L_2)}$ respectively. From the second assertion of Lemma 5.9 (3) and Lemma 5.9 (4), we get that

$$\begin{aligned} \{x \in \boldsymbol{S}_\phi + J^3(B_0) | \quad & xy - yx \in J^5(B_0) \text{ for all } y \in \boldsymbol{T}_\zeta + J^3(B_0)\} \\ = \quad & \langle \tilde{S}^1_\phi \rangle_k + J^3(B_0) \end{aligned} \tag{12}$$

and similarly

$$\begin{aligned} \{x \in \boldsymbol{T}_\zeta + J^3(B_0) | \quad & xy - yx \in J^5(B_0) \text{ for all } y \in \boldsymbol{S}_\phi + J^3(B_0)\} \\ = \quad & \langle \tilde{T}^1_\zeta \rangle_k + J^3(B_0). \end{aligned} \tag{13}$$

Of course, the analogous assertions for the algebra B_0' hold as well.

Equation (11) combined with Lemma 5.9 (1) implies that

$$\tau(\langle S_{\mu,\nu} \rangle_k) + J^2(B_0) = \langle S'_{\mu,\nu} \rangle_k + J^2(B_0'),$$
$$\tau(\langle T_{\gamma,\delta} \rangle_k) + J^2(B_0) = \langle T'_{\gamma,\delta} \rangle_k + J^2(B_0'),$$

for all $\mu, \nu \in \mathrm{Irr}(P_1)$ and $\gamma, \delta \in \mathrm{Irr}(P_2)$, $\nu \neq 1$ and $\delta \neq 1$. Therefore, by Lemma 5.9 (3),

$$\tau(S_{1,\phi^g} S_{\phi^g,(\phi^g)^{-1}}) + J^3(B_0) = u_g S'_{1,\phi^g} S'_{\phi^g,(\phi^g)^{-1}} + J^3(B_0'),$$
$$\tau(T_{1,\zeta^h} T_{\zeta^h,(\zeta^h)^{-1}}) + J^3(B_0) = v_h T'_{1,\zeta^h} T'_{\zeta^h,(\zeta^h)^{-1}} + J^3(B_0'),$$

for all $g \in L_1$, $h \in L_2$ and uniquely determined $u_g, v_h \in k^\times$. Furthermore, (12) and (13) give

$$\tau(\langle \tilde{S}^1_\phi \rangle_k) + J^3(B_0) = \langle (\tilde{S}^1_\phi)' \rangle_k + J^3(B_0'),$$
$$\tau(\langle \tilde{T}^1_\zeta \rangle_k) + J^3(B_0) = \langle (\tilde{T}^1_\zeta)' \rangle_k + J^3(B_0').$$

Therefore, all the u_g's are equal, say u. Similarly we set v to be the common value of the v_h's. In particular,

$$\tau(\tilde{S}^\chi_\phi) + J^3(B_0) = u(\tilde{S}^\chi_\phi)' + J^3(B_0'),$$
$$\tau(\tilde{T}^\eta_\zeta) + J^3(B_0) = v(\tilde{T}^\eta_\zeta)' + J^3(B_0'),$$

for all $\chi \in \mathrm{Irr}(L_1)$ and $\eta \in \mathrm{Irr}(L_2)$. From Lemma 5.9 (4) we get the identities

$$\tilde{S}^\chi_\phi \tilde{T}^\eta_\zeta = \theta([h^\theta_{\eta,2}, h^\theta_{\chi,1}]) \tilde{T}^\eta_\zeta \tilde{S}^\chi_\phi \quad \text{and} \quad (\tilde{S}^\chi_\phi)'(\tilde{T}^\eta_\zeta)' = \theta'([h^{\theta'}_{\eta,2}, h^{\theta'}_{\chi,1}])(\tilde{T}^\eta_\zeta)'(\tilde{S}^\chi_\phi)'$$

Using the second assertion of Lemma 5.9 (3) we can apply τ to the first and compare to the second modulo $J^5(B_0')$. That gives

$$\eta(h^\theta_{\chi,1}) = \theta([h^\theta_{\eta,2}, h^\theta_{\chi,1}]) = \theta'([h^{\theta'}_{\eta,2} h^{\theta'}_{\chi,1}]) = \eta(h^{\theta'}_{\chi,1}),$$

for all $\chi \in \mathrm{Irr}(L_1)$ and $\eta \in \mathrm{Irr}(L_2)$. Therefore, $h^\theta_{\chi,1} = h^{\theta'}_{\chi,1}$, for all $\chi \in \mathrm{Irr}(L_1)$. Finally, since $[L_1, L_2] = Z$, it follows from Definition 2.2 that $\theta = \theta'$. □

Theorem 6.5. *Let ℓ be a prime and $n \in \mathbb{N}$. Then there exists an ℓ-block B of kG, for a finite group G, such that* $\mathrm{mf}(B) = n$.

Proof. Of course, we are setting to G and B to be as in the rest of the article. We just need an appropriate choice of p and r and θ.

Set $r = \ell^n + 1$. By the Dirichlet prime number theorem, we can set p to be a prime congruent to 1 modulo ℓ and modulo r. Set θ to be any faithful, linear character of Z. Note that $B(\theta)^{(\ell^m)} = B(\theta^{\ell^m})$, for all $m \in \mathbb{N}$, and so, by Proposition 6.4, $\mathrm{mf}(B)$ is the smallest $m \in \mathbb{N}$, such that $\theta^{\ell^m} = \theta^{\pm 1}$ or equivalently that $r|(\ell^m \pm 1)$. It is now clear that $\mathrm{mf}(B) = n$. □

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