PATH COMBINATORICS AND LIGHT LEAVES FOR QUIVER HECKE ALGEBRAS

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Abstract. We recast the classical notion of "standard tableaux" in an alcove-geometric setting and extend these classical ideas to all "reduced paths" in our geometry. This broader path-perspective is essential for implementing the higher categorical ideas of Elias–Williamson in the setting of quiver Hecke algebras. Our first main result is the construction of light leaves bases of quiver Hecke algebras. These bases are richer and encode more structural information than their classical counterparts, even in the case of the symmetric groups. Our second main result provides path-theoretic generators for the "Bott–Samelson truncation" of the quiver Hecke algebra.

The symmetric group lies at the intersection of two great categorical theories: Khovanov–Lauda and Rouquier's categorification of quantum groups and their knot invariants [KL09, Rou] and Elias–Williamson's diagrammatic categorification in terms of endomorphisms of Bott–Samelson bimodules. The purpose of this paper and its companion [BCH] is to construct an explicit isomorphism between these two diagrammatic worlds. The backbone of this isomorphism is provided by the "light-leaves" bases of these algebras.

The light leaves bases of diagrammatic Bott–Samelson endomorphism algebras were crucial in the calculation of counterexamples to the expected bounds of Lusztig's and James' conjectures [Wil17]. These bases are structurally far richer than any known basis of the quiver Hecke algebra — they vary with respect to each possible choice of reduced word/path-vector in the alcove geometry — this richer structure is necessary in order to construct a basis in terms of the "Soergel 2-generators" of these algebras. In this paper we show that the (quasi-hereditary quotients of) quiver Hecke algebras, $\mathcal{H}_n^\sigma$ for $\sigma \in \mathbb{Z}^\ell$, have light leaves bases indexed by paths in an alcove geometry of type

$$(A_{h-1} \times \ldots \times A_{h-1}) \backslash \widehat{A}_{h\ell-1}$$

where each point in this geometry corresponds to an ℓ-multipartition with at most h columns in each component (we denote the set of such ℓ-multipartitions by $\mathscr{P}_{h,\ell}(n)$) under the restriction that $\sigma \in \mathbb{Z}^\ell$ satisfies a higher level analogue of the condition $p > h + 1$.

Theorem A (Light leaves bases for quiver Hecke algebras). *For each $\lambda \in \mathscr{P}_{h,\ell}(n)$ we fix a reduced path $\mathsf{Q}_\lambda \in \mathrm{Path}_{h,\ell}(\lambda)$ and for each $\mathsf{S} \in \mathrm{Path}_{h,\ell}(\lambda)$, we fix an associated reduced path vector $\underline{\mathsf{P}}_{\mathsf{S}}$ terminating with Q_λ (this notation is defined in Sections 1 and 2). We have that*

$$\{\psi^{\mathsf{S}}_{\underline{\mathsf{P}}_{\mathsf{S}}} \psi^{\underline{\mathsf{P}}_{\mathsf{T}}}_{\mathsf{T}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda), \lambda \in \mathscr{P}_{h,\ell}(n)\}$$

is a cellular basis of $\mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$ where we quotient by the ideal generated by the idempotent

$$\mathsf{e}_h = \sum_{\substack{i_{k+1}=i_k+1 \\ 1 \leqslant k \leqslant h}} e_{(i_1,\ldots,i_n)},$$

which kills all simples indexed by ℓ–partitions with more than h columns in any given component.

We then consider the so-called "Bott–Samelson truncations" to the principal block

$$\mathsf{f}_{n,\sigma}(\mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma)\mathsf{f}_{n,\sigma} \qquad \text{for} \qquad \mathsf{f}_{n,\sigma} = \sum_{\substack{\mathsf{S} \in \mathrm{Std}_{n,\sigma}(\lambda) \\ \lambda \in \mathscr{P}_{h,\ell}(n)}} e_{\mathsf{S}}$$

which we will show (in the companion paper [BCH]) are isomorphic to the (breadth enhanced) diagrammatic Bott–Samelson endomorphism algebras of Elias–Williamson [EW16]. The charm of this isomorphism is that it allows one to view the current state-of-the-art regarding p-Kazhdan–Lusztig theory (in type A) entirely within the context of the group algebra of the symmetric group, without the need for calculating intersection cohomology groups, or working with parity sheaves, or appealing to the deepest results of 2-categorical Lie theory. In this paper, we specialise Theorem A by making certain path-theoretic choices which allow us to reconstruct Elias–Williamson's generators entirely within the quiver Hecke algebra itself, using our language of paths.

Theorem B. *The Bott–Samelson truncation of the Hecke algebra* $\mathsf{f}_{n,\sigma}(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma)\mathsf{f}_{n,\sigma}$ *is generated by horizontal and vertical concatenation of the elements*

$$e_{\mathsf{P}_{\boldsymbol{\alpha}}}, \quad \mathsf{fork}^{\boldsymbol{\alpha}\varnothing}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}, \quad \mathsf{spot}^{\varnothing}_{\boldsymbol{\alpha}}, \quad \mathsf{hex}^{\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}}_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}, \quad \mathsf{com}^{\boldsymbol{\gamma}\boldsymbol{\beta}}_{\boldsymbol{\beta}\boldsymbol{\gamma}}, \quad e_{\mathsf{P}_\emptyset}, \ \textit{and} \ \mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset}$$

(this notation is defined in Section 3) for $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \Pi$ *such that* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *label an arbitrary pair of non-commuting reflections and* $\boldsymbol{\beta}$ *and* $\boldsymbol{\gamma}$ *label an arbitrary pair of commuting reflections.*

The paper is structured as follows. In Section 1 we construct a "classical-type" cellular basis of $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$ in terms of tableaux but using a slightly exotic dominance ordering — the proofs in this section are a little dry and can be skipped on the first reading. In Section 2, we upgrade this basis to a "light leaf type" construction and prove Theorem A. Finally, in Section 3 of the paper we illustrate how Theorem allows us to reconstruct the precise analogue of the light leaves basis for the Bott–Samelson endomorphism algebras for regular blocks of quiver Hecke algebras (as a special case of Theorem A, written in terms of the generators of Theorem B). We do this in the exact language used by Elias, Libedinsky, and Williamson in order to make the construction clear for a reader whose background lies in either field.

This paper is a companion to [BCH], but the reader should note that the results here are entirely self-contained (although we refer to [BCH] for further development of ideas, examples, etc).

1. A tableaux basis of the quiver Hecke algebra

We let $\mathfrak{S}_n$ denote the symmetric group on n letters and let $\ell : \mathfrak{S}_n \to \mathbb{Z}$ denotes its length function. We let $\leqslant$ denote the (strong) Bruhat order on $\mathfrak{S}_n$. Given $\underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ and $s_r = (r, r+1) \in \mathfrak{S}_n$ we set $s_r(\underline{i}) = (i_1, \dots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \dots, i_n)$.

Definition 1.1 ([BK09, KL09, Rou])**.** *Fix $e > 2$. The* quiver Hecke algebra *(or KLR algebra), $\mathcal{H}_n$, is defined to be the unital, associative $\mathbb{Z}$-algebra with generators*

$$\{e_{\underline{i}} \mid \underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\},$$

subject to the relations

$$e_{\underline{i}} e_{\underline{j}} = \delta_{\underline{i},\underline{j}} e_{\underline{i}} \qquad \textstyle\sum_{\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n} e_{\underline{i}} = 1_{\mathcal{H}_n}; \qquad y_r e_{\underline{i}} = e_{\underline{i}} y_r \qquad \psi_r e_{\underline{i}} = e_{s_r \underline{i}} \psi_r \qquad y_r y_s = y_s y_r \tag{R1}$$

for all $r, s, \underline{i}, \underline{j}$ and

$$\psi_r y_s = y_s \psi_r \quad \text{for } s \neq r, r+1 \qquad\qquad \psi_r \psi_s = \psi_s \psi_r \quad \text{for } |r-s| > 1 \tag{R2}$$

$$y_r \psi_r e_{\underline{i}} = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e_{\underline{i}} \qquad\qquad y_{r+1} \psi_r e_{\underline{i}} = (\psi_r y_r + \delta_{i_r, i_{r+1}}) e_{\underline{i}} \tag{R3}$$

$$\psi_r \psi_r e_{\underline{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e_{\underline{i}} & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r) e_{\underline{i}} & \text{if } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1}) e_{\underline{i}} & \text{if } i_{r+1} = i_r - 1 \end{cases} \tag{R4}$$

$$\psi_r \psi_{r+1} \psi_r = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} - 1) e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} + 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} + 1) e_{\underline{i}} & \text{if } i_r = i_{r+2} = i_{r+1} - 1 \\ \psi_{r+1} \psi_r \psi_{r+1} e_{\underline{i}} & \text{otherwise} \end{cases} \tag{R5}$$

for all permitted r, s, i, j. We identify such elements with decorated permutations and the multiplication with vertical concatenation, $\circ$, of these diagrams in the standard fashion of [BK09, Section 1] *and as illustrated in Figure 1. We let $*$ denote the anti-involution which fixes the generators.*

Definition 1.2. *Fix $e > 2$ and $\sigma \in \mathbb{Z}^\ell$. The* cyclotomic quiver Hecke algebra*, $\mathcal{H}_n^\sigma$, is defined to be the quotient of $\mathcal{H}_n$ by the relation*

$$y_1^{\sharp\{\sigma_m \mid \sigma_m = i_1, 1 \leqslant m \leqslant \ell\}} e_{\underline{i}} = 0 \qquad \text{for } \underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n. \tag{1.1}$$

A long-standing belief in modular Lie theory is that we should (first) restrict our attention to fields whose characteristic, p, is greater than the Coxeter number, h, of the algebraic group we are studying. This allows one to consider a "regular" or "principal block" of the algebraic group in question. What does this mean on the other side of the Schur–Weyl duality relating $\mathrm{GL}_h(\Bbbk)$ and $\Bbbk\mathfrak{S}_n$? By the second fundamental theorem of invariant theory, the kernel of the group algebra of the symmetric group acting on n-fold h-dimensional tensor space is the element

$$\textstyle\sum_{g\in\mathfrak{S}_{h+1}\leqslant\mathfrak{S}_n} \mathrm{sgn}(g)g \in \Bbbk\mathfrak{S}_n$$

which can be rescaled to be an idempotent providing $p > h+1$ and, indeed, in terms of the quiver Hecke algebra presentation, this idempotent can be written in the form

$$\mathsf{e}_h = \sum_{\substack{i_{k+1}=i_k+1\\ 1\leqslant k\leqslant h}} e_{(i_1,\dots,i_n)}. \tag{1.2}$$

For $\ell > 1$ this element generates the ideal if and only if $\in \mathbb{N}$ and $\sigma \in \mathbb{Z}^\ell$ satisfy:

Standing assumption: Fix integers $h, \ell \in \mathbb{Z}_{>0}$ and $e \geqslant h\ell$ and the charge $\sigma \in \mathbb{Z}^\ell$ so that $h < |\sigma_i - \sigma_j| < e - h$ for $1 \leqslant i \neq j \leqslant \ell$. We say that such a charge is (h, e)-admissible.

This brings us to the algebras of interest in this paper and its companion [BCH].

Definition 1.3. *For (h, e)-admissible $\sigma \in \mathbb{Z}^\ell$, we define $\mathscr{H}_n^\sigma := \mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$.*

As we see in Figure 1, the y_k elements are visualised as dots on strands; we hence refer to them as KLR dots. Given $p < q$ we set

$$w_q^p = s_p s_{p+1}\dots s_{q-1} \quad w_p^q = s_{q-1}\dots s_{p+1}s_p \qquad \psi_q^p = \psi_p\psi_{p+1}\dots\psi_{q-1} \qquad \psi_p^q = \psi_{q-1}\dots\psi_{p+1}\psi_p.$$

and given an expression $\underline{w} = s_{i_1}\dots s_{i_p} \in \mathfrak{S}_n$ we set $\psi_{\underline{w}} = \psi_{i_1}\dots\psi_{i_p} \in \mathcal{H}_n$. We let $\boxtimes$ denote the horizontal concatenation of both KLR diagrams and residue sequences. Finally, we define the degree as follows,

$$\deg(e_{\underline{i}}) = 0 \quad \deg(y_r) = 2 \quad \deg(\psi_r e_{\underline{i}}) = \begin{cases} -2 & \text{if } i_r = i_{r+1} \\ 1 & \text{if } i_r = i_{r+1} \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

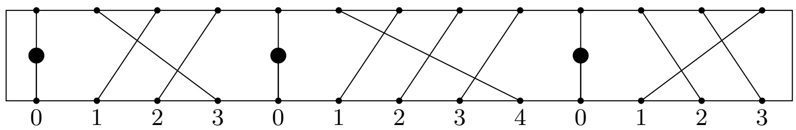


FIGURE 1. The element $y_1\psi_4^2 e_{(0,1,2,3)} \boxtimes y_1\psi_5^2 e_{(0,1,2,3,4)} \boxtimes y_1\psi_2^4 e_{(0,1,2,3)}$.

1.1. **Box configurations, partitions, residues and tableaux.** We define a box-configuration to be a subset of

$$\{[i,j,m] \mid 0 \leqslant m < \ell, 1 \leqslant i,j \leqslant n\}$$

and we let $\mathcal{B}_\ell(n)$ denote the set of all box-configurations with n boxes. We refer to a box $[i,j,m] \in \lambda \in \mathcal{B}_\ell(n)$ as being in the ith row and jth column of the mth component of λ. Given a box, $[i,j,m]$, we define the content of this box to be $\mathsf{ct}[i,j,m] = \sigma_m + j - i$ and we define its residue to be $\mathrm{res}[i,j,m] = \mathsf{ct}[i,j,m] \pmod e$. We refer to a box of residue $r \in \mathbb{Z}/e\mathbb{Z}$ as an r-box.

We define a composition, λ, of n to be a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \dots$, equals n. We say that λ is a partition if, in addition, this sequence is weakly decreasing. An ℓ-multicomposition (respectively ℓ-multipartition) $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ of n is an ℓ-tuple of compositions (respectively partitions) such that $|\lambda^{(0)}| + \dots + |\lambda^{(\ell-1)}| = n$. Given $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell-1)}) \in \mathscr{P}_\ell(n)$, the Young diagram of λ is defined to be the box configuration,

$$\{[i,j,m] \mid 1 \leqslant j \leqslant \lambda_i^{(m)}, 0 \leqslant m < \ell\}.$$

We do not distinguish between the multipartition and its Young diagram. We let $\mathscr{P}_{h,\ell}(n)$ (respectively $\mathcal{C}_{h,\ell}(n)$) denote the subsets of ℓ-multipartitions (respectively ℓ-multicompositions) with at most h

columns in each component. We let $\varnothing$ denote the empty multipartition. Given $\lambda \in \mathscr{P}_{h,\ell}(n)$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable (respectively addable) boxes of the Young diagram of λ so that the resulting diagram is the Young diagram of an ℓ-multipartition. We let $\mathrm{Rem}_r(\lambda) \subseteq \mathrm{Rem}(\lambda)$ and $\mathrm{Add}_r(\lambda) \subseteq \mathrm{Add}(\lambda)$ denote the subsets of boxes of residue $r \in \mathbb{Z}/e\mathbb{Z}$.

Definition 1.4. *We define the reverse lexicographic ordering on boxes as follows. Let $1 \leqslant i, i', j, j' \leqslant n$ and $1 \leqslant m, m' \leqslant \ell$. We write $[i,j,m] \succ [i',j',m']$ if $i < i'$, or $i = i'$ and $m < m'$, or $i = i'$ and $m = m'$ and $j < j'$.*

Definition 1.5. *We define the* reverse lexicographic ordering *on $\mathcal{B}_\ell(n)$ as follows. Given $\lambda, \mu \in \mathcal{B}_\ell(n)$, we write $\lambda \succ \mu$ if the lexicographically minimal box $\square \in (\lambda \cup \mu) \setminus (\lambda \cap \mu)$ belongs to μ.*

Example 1.6. *For the symmetric group, the reverse lexicographic ordering is equal to the transpose of the usual lexicographic ordering. In other words $\lambda \succ \mu$ if there exists some $t \geqslant 1$ such that*

$$\sum_{1 \leqslant i \leqslant t} \lambda_i^T < \sum_{1 \leqslant i \leqslant t} \mu_i^T \quad \text{and} \quad \sum_{1 \leqslant i \leqslant k} \lambda_i^T = \sum_{1 \leqslant i \leqslant k} \mu_i^T$$

for all $1 \leqslant k \leqslant t$ where T denotes the transpose partition. More generally, $\succ$ is a total refinement of the so-called "FLOTW" dominance order on $\mathscr{P}_{h,\ell}(n)$ in [Bow, BC18, LP].

Given $\lambda \in \mathcal{B}_\ell(n)$, we define a λ-tableau to be a filling of the boxes of $[\lambda]$ with the numbers $\{1, \dots, n\}$. For $\lambda \in \mathcal{B}_\ell(n)$, we say that a λ-tableau is row-standard, column-standard, or simply standard if the entries in each component increase along the rows, increase along the columns, or increase along both rows and columns, respectively. We say that a λ-tableau S has shape λ and write $\mathsf{Shape}(\mathsf{S}) = \lambda$. Given $\lambda \in \mathcal{B}_\ell(n)$, we let $\mathrm{Tab}(\lambda)$ denote the set of all tableaux of shape $\lambda \in \mathcal{B}_\ell(n)$. Given $\mathsf{T} \in \mathrm{Tab}(\lambda)$ and $1 \leqslant k \leqslant n$, we let $\mathsf{T}^{-1}(k)$ denote the box $\square \in \lambda$ such that $\mathsf{T}(\square) = k$. We let $\mathrm{RStd}(\lambda), \mathrm{CStd}(\lambda), \mathrm{Std}(\lambda) \subseteq \mathrm{Tab}(\lambda)$ denote the subsets of all row-standard, column-standard, and standard tableaux, respectively. We let $\mathrm{Std}(n) = \cup_{\lambda \in \mathscr{P}_{h,\ell}(n)} \mathrm{Std}(\lambda)$ for $n \in \mathbb{N}$.

We extend the reverse lexicographic ordering to tableaux in the following manner. Given S, we write $\mathsf{S}{\downarrow_{\leqslant k}}$ (respectively $\mathsf{S}{\downarrow_{\geqslant k}}$) for the subtableau of S consisting solely of the entries 1 through k (respectively of the entries k through n). Given $\mathsf{S}, \mathsf{T} \in \mathrm{Std}(\lambda)$ we write $\mathsf{S} \succ \mathsf{T}$ if there exists some $t \geqslant 1$ such that $\mathsf{Shape}(\mathsf{S}{\downarrow_{\leqslant k}}) = \mathsf{Shape}(\mathsf{S}{\downarrow_{\leqslant k}})$ for all $t \leqslant k \leqslant n$ and $\mathsf{Shape}(\mathsf{S}{\downarrow_{\leqslant t}}) \succ \mathsf{Shape}(\mathsf{S}{\downarrow_{\leqslant t}})$. Given $\lambda \in \mathcal{B}_\ell(n)$, we let T_λ denote the maximal tableau in the $\succ$-ordering. Namely, the λ-tableau in which we place the largest entry in the minimal $\succ$-node of λ, then continue in this fashion inductively. Finally, given S, T two λ-tableaux, we let $w^{\mathsf{S}}_{\mathsf{T}} \in \mathfrak{S}_n$ be the permutation such that $w^{\mathsf{S}}_{\mathsf{T}}(\mathsf{S}) = \mathsf{T}$.

Example 1.7. *For $\lambda = ((2,1^2),(2^2,1),(1^3))$, we have standard λ-tableaux*

$$\mathsf{T}_\lambda = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 6 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array}\ , \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 7 & 8 \\ \hline 11 \\ \cline{1-1} \end{array}\ , \begin{array}{|c|}\hline 5 \\ \hline 9 \\ \hline 12 \\ \hline \end{array} \right) \qquad \mathsf{S} = \left(\begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array}\ , \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 7 & 8 \\ \hline 11 \\ \cline{1-1} \end{array}\ , \begin{array}{|c|}\hline 4 \\ \hline 9 \\ \hline 12 \\ \hline \end{array} \right)$$

and we have that $w^{\mathsf{S}}_{\mathsf{T}_\lambda} = (4,5)(2,6)$.

Given any $[r,c,m] \in \lambda$ with $r \neq 1$, we define the associated Garnir belt to be the collection of boxes

$$\{[r,j,k] \mid 1 \leqslant j \leqslant c, k \leqslant m\} \cup \{[r-1,j,k] \mid c \leqslant j \leqslant \lambda_r^{(m)}, m \leqslant k\}$$

and we define the associated Garnir tableau to be the $\succ$-minimal tableau, $\mathsf{G}_{[r,c,m]} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)$ (or simply G when the context is clear), which coincides with T_λ outside of the Garnir belt.

Example 1.8. *For $\lambda = ((3^2,2^2),(4^2,3,2),(4^2,3,1))$ and the node $[3,3,1]$, the associated Garnir tableau is given by*

$$\mathsf{G}_{[3,3,1]} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 12 & 13 & 14 \\ \hline 17 & 18 \\ \cline{1-2} 31 & 32 \\ \cline{1-2} \end{array}\ , \begin{array}{|c|c|c|c|}\hline 4 & 5 & 6 & 7 \\ \hline 15 & 16 & 22 & 23 \\ \hline 19 & 20 & 21 \\ \cline{1-3} 33 & 34 \\ \cline{1-2} \end{array}\ , \begin{array}{|c|c|c|c|}\hline 8 & 9 & 10 & 11 \\ \hline 24 & 25 & 26 & 27 \\ \hline 28 & 29 & 30 \\ \cline{1-3} 35 \\ \cline{1-1} \end{array} \right)$$

where here we have coloured the Garnir belt in yellow. We note that this is of a different combinatorial flavour to the Garnir belts of [Mat99] *as we are working with a different weighting on our Hecke algebra, or equivalently a "twisted" Fock-Uglov-space ordering, or equivalently a Cherednik algebra which is not Morita equivalent to the cyclotomic q-Schur algebra. See* [BC18, LP, LPRH] *for more details.*

Let $\lambda \in \mathscr{P}_{h,\ell}(n)$. Given $1 \leqslant k \leqslant n$, we let $\mathcal{A}_{\mathsf{T}}(k)$, (respectively $\mathcal{R}_{\mathsf{T}}(k)$) denote the set of all addable $\mathrm{res}(\mathsf{T}^{-1}(k))$-boxes (respectively all removable $\mathrm{res}(\mathsf{T}^{-1}(k))$-boxes) of the ℓ-partition $\mathrm{Shape}(\mathsf{T}{\downarrow}_{\{1,\dots,k\}})$ which are less than $\mathsf{T}^{-1}(k)$ in the $\succeq$-order. We define the degree of $\mathsf{T} \in \mathrm{Std}(\lambda)$ as follows,

$$\deg(\mathsf{T}) = \sum_{k=1}^{n} \left(|\mathcal{A}_{\mathsf{T}}(k)| - |\mathcal{R}_{\mathsf{T}}(k)|\right).$$

1.2. **Generator/partition combinatorics.** Our cellular basis will provide a stratification of $\mathscr{H}_n^\sigma$ in which each layer is generated by an idempotent correspond to some multipartition. Whence we wish to understand the effect of multiplying a generator of a given layer in the cell-stratification by a KLR "dot generator". This leads us to define combinatorial analogues of the dot generators as maps on the set of box configurations.

Definition 1.9. *Let $\lambda \in \mathcal{B}_\ell(n)$ and let $[i,j,m] \in \lambda$ be an r-box for some $r \in \mathbb{Z}/e\mathbb{Z}$. We say that $[i,j,m]$ is left-justified if either (i) there exists some $[i,j-p,m] \in \lambda$ for $1 \leqslant p \leqslant e$ or (ii) $j=1$ and $[i-1,j,m] \in \lambda$.*

Definition 1.10. *Let $\lambda \in \mathcal{B}_\ell(n)$, $r \in \mathbb{Z}/e\mathbb{Z}$. For $\alpha \in \lambda$ an r-box, we define*

$$Y_\alpha(\lambda) = \lambda - \alpha + \beta$$

where $\beta \notin \lambda$ is the minimal r-box in the lexicographic ordering such that $\beta \succ \alpha$ and such that β is left-justified. We write $\lambda \ggcurly \mu$ if $\lambda = Y_\alpha(\mu)$ for some $\alpha \in \mu$ and we then extend $\ggcurly$ to a partial ordering on $\mathcal{B}_\ell(n)$ by taking the transitive closure. Suppose that $\{[i_k,j_k,m_k] \mid 0 < k \leqslant n\}$ is a set of r-boxes and that $Y_{[i_k,j_k,m_k]}(\lambda + [i_k,j_k,m_k]) = \lambda + [i_{k+1},j_{k+1},m_{k+1}]$ for $k \geqslant 1$. We define

$$Y^n_{[i_1,j_1,m_1]}(\lambda + [i_1,j_1,m_1]) = (\lambda + [i_n,j_n,m_n]).$$

We remark that $\lambda \ggcurly \mu$ implies that $\lambda \succ \mu$ and so $\ggcurly$ is a coarsening of $\succ$.

Example 1.11. *Let $h=3$ and $e=5$ and $\ell=1$. For $\lambda = (3,2^2,1^6) \in \mathcal{B}_1(13)$, we have that $Y_{[3,2,0]}(\lambda) = (3,2,1^6) \cup [2,6,0]$. One can stagger this process by thinking of the node $[3,2,0]$ as moving to the next lexicographically smallest position (in the array $\{[x,y,0] \mid 1 \leqslant x,y \leqslant 13\}$) namely $[2,11,0]$, but then noticing that $[2,11,0]$ is not left-justified, and so then moving the node to $[2,6,0]$, which is left-justified. We have that $Y^2_{[3,2,0]}(\lambda) = Y_{[2,6,0]}((3,2,1^6)+[2,6,0]) = (3,2,1^6)+[1,5,0]$.*

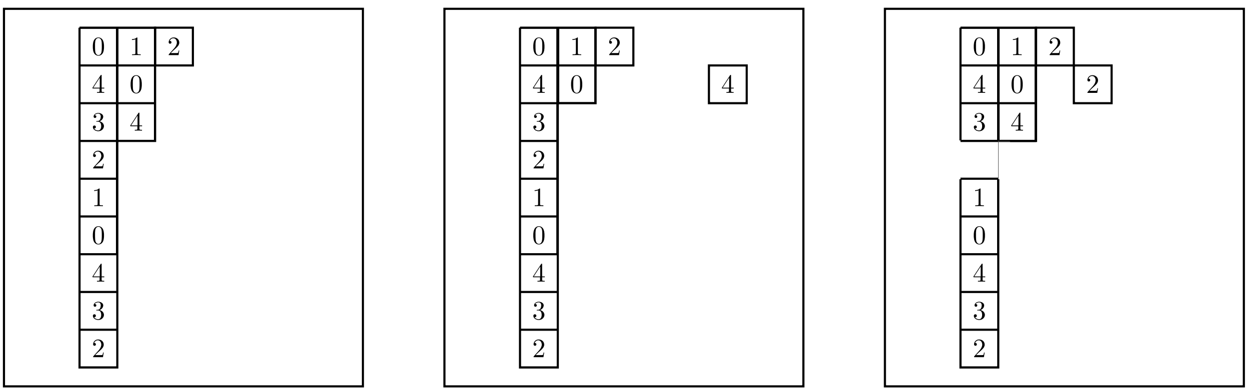


FIGURE 2. The box configurations, λ, $Y_{[3,2,0]}(\lambda)$, and and $Y^2_{[4,1,0]}(\lambda)$

Example 1.12. *Let $h=3$ and $e=5$ and $\ell=1$ and $r=2$. For $\lambda = (3,2^2,1^6) \in \mathcal{B}_1(13)$, we have that $Y^2_{[4,1,0]}(\lambda) = (3,2^2,1^6) \cup [2,4,0] - [4,1,0]$ (as depicted in Figure 2). We notice that $Y^1_{[4,1,0]}(\lambda)$ does not involve moving the 2-box through any nodes (as $[4,1,0]$ is in the first column). In fact, it takes $\ell+1$ (equal to 2 in this case) applications of the Y_α-operator in order to move an r-box $\alpha = [i,1,m] \in \lambda \in \mathscr{P}_{h,\ell}(n)$ past a node of* adjacent residue *in λ (and that this is the $(r+1)$-box equal to $[i-1,j,m]$). One should compare this with the case $\alpha = [i,j,m]$ for $m > 1$ where α must pass through the $(r-1)$-node $[i,j-1,m] \in \lambda$. This disparity appears leads to a ± 1 in Proposition 1.17.*

Given an idempotent generator, $e_{\underline{j}}$ for $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$, of the KLR algebra, we wish to identify to which layer of our stratification our idempotent belongs. To this end we make the following definition.

Definition 1.13. *Associated to any $\underline{j} = (j_1, \dots, j_n) \in (\mathbb{Z}/e\mathbb{Z})^n$, we have an element $\mathsf{J} \in \mathrm{Std}(n) \cup \{0\}$ given by placing the entry $k = 1, 2, \dots, n$ in the lexicographically least addable j_k-box of the partition* $\mathrm{Shape}(\mathsf{J}_{\leqslant k-1})$*, and formally setting $\mathsf{J} = 0$ if no such box exists for some $1 \leqslant k \leqslant n$.*

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 4 & 0 & \\ \cline{1-2} 3 & 4 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} 0 & & \\ \cline{1-1} 4 & & \\ \cline{1-1} 3 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 13 \\ \hline 3 & 4 & \\ \cline{1-2} 5 & 6 & \\ \cline{1-2} 7 & & \\ \cline{1-1} 8 & & \\ \cline{1-1} 9 & & \\ \cline{1-1} 10 & & \\ \cline{1-1} 11 & & \\ \cline{1-1} 12 & & \\ \cline{1-1} \end{array}$$

FIGURE 3. Let $e = 5$ and $\ell = 1$. On the left, we have the residues for the partition $\lambda = (3, 2^2, 1^6)$. On the right, we have the tableau $\mathsf{J} \in \mathrm{Std}(3, 2^2, 1^6)$ for $\underline{i} = (0, 1, 4, 0, 3, 4, 2, 1, 0, 4, 3, 2, 2)$.

Developing the above idea further, we wish to provide a "standardisation process" to inductively show that $e(\underline{j}) = \psi^{\mathsf{J}}_{\mathsf{T}_\lambda} \psi^{\mathsf{T}_\lambda}_{\mathsf{J}} + \cdots$ where the omitted terms are higher in the $\succ$-ordering.

Definition 1.14. *For $\lambda \in \mathscr{P}_{h,\ell}(n)$, $1 \leqslant k \leqslant n$, and $r \in \mathbb{Z}/e\mathbb{Z}$, we let $\square \notin \lambda$ denote the maximal r-box in the lexicographic ordering such that $\mathsf{T}^{-1}(k-1) \succ \square$. We define $\mathsf{T}_\lambda \cup \boxed{k}$ to be the tableau such that*

$$\left(\mathsf{T}_\lambda \cup \boxed{k}\,\right)[i,j,m] = \begin{cases} \mathsf{T}_\lambda[i,j,m] & \textit{if } [i,j,m] \succ \square \\ k & \textit{if } [i,j,m] = \square \\ 1 + \mathsf{T}_\lambda[i,j,m] & \textit{if } [i,j,m] \prec \square \end{cases}$$

Example 1.15. *Let $\lambda = (3, 2^2, 1^6)$ and $e = 5$ and $r = 2 \in \mathbb{Z}/e\mathbb{Z}$. The tableaux $\mathsf{T}_\lambda \cup \boxed{k}$ for $k = 13, 12, 11, 10, 9, 8, 7, 5, 3$ are recorded in Figure 4. Examples of how these tableaux lift to label our "standardisation process" in $\mathscr{H}_n^\sigma$ are given in Figures 6 to 8.*

1.3. **A tableaux theoretic basis.** We are now ready to construct our first basis of $\mathscr{H}_n^\sigma := \mathcal{H}_n^\sigma / \mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$. This basis will serve as the starting point for our light-leaves bases of Theorem A. The combinatorics of this basis will be familiar to anyone who has studied symmetric groups and cyclotomic Hecke algebras (but with respect to the, less familiar, $\succ$-ordering). We require the following observation, which says that residues form into diagonals which are "spaced far apart" within the tableau T_λ.

Lemma 1.16. *Let $\lambda \in \mathscr{P}_{h,\ell}(n)$ and $\sigma \in (\mathbb{Z}/e\mathbb{Z})^\ell$ be (h,e)-admissible. Let $[i,j,m]$ and $[i',j',m']$ be two nodes belonging to some Garnir belt of λ and such that $|\mathrm{res}[i,j,m] - \mathrm{res}[i',j',m']| \leqslant 1$. Then either $[i,j,m] = [i' \pm 1, j', m']$ or $[i,j,m] = [i', j' \pm 1, m']$.*

Proof. This follows immediately from the definitions, since $\sigma \in (\mathbb{Z}/e\mathbb{Z})^\ell$ is (h,e)-admissible. $\square$

We define $\mathscr{H}^{\geqslant \lambda} = \mathscr{H}_n^\sigma \langle e_{\mathsf{T}_\nu} \mid \nu \geqslant \lambda \rangle \mathscr{H}_n^\sigma$ for $\geqslant$ any ordering on $\mathscr{P}_{h,\ell}(n)$; we formally set $\mathscr{H}^{\geqslant 0} = 0$ to be the zero ideal. The following proof provides an algorithm for writing $e_{\mathsf{J}} \in \mathscr{H}^{\succeq \mathrm{Shape}(\mathsf{J})}$ for any $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$. We provide a running example of this procedure for the tableau of Figure 3. For this example, the eight steps (indexed by 9 tableaux) of this procedure are illustrated in Figure 4. More generally, the following proposition gives an inductive proof of the fact that $e_{\underline{j}} = 0$ if $\mathrm{Shape}(\underline{j}) = 0$.

Proposition 1.17. *For $\lambda \in \mathscr{P}_{h,\ell}(n)$, we have that:*

(1) for $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$, the element $e_{\underline{j}}$ belongs to $\mathscr{H}^{\succeq \mathrm{Shape}(\mathsf{J})}$.
(2) for $\lambda \in \mathscr{P}_{h,\ell}(n)$, the element $y_a e_{\mathsf{T}^\lambda}$ belongs to $\mathscr{H}^{\succ \lambda}$ for any $1 \leqslant a \leqslant n$.

Proof. We assume, for $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^{n-1}$ with $\mathrm{Shape}(\mathsf{J}) = \lambda \in \mathscr{P}_{h,\ell}(n-1)$, two stronger statements (which clearly imply (1) and (2) of the proposition). We refine the induction by the ordering $\succ$ on $\lambda \in \mathscr{P}_{h,\ell}(n-1)$. We will show, by induction, that

$$\begin{cases} e_{\underline{j}} \in \pm\psi^{\mathsf{J}}_{\mathsf{T}_{\lambda'}}\psi^{\mathsf{T}_{\lambda'}}_{\mathsf{J}} + \mathscr{H}^{\not\succ\lambda'} & \text{if } \mathrm{Shape}(\mathsf{J}) = \lambda' \in \mathscr{P}_{h,\ell}(n) \\ e_{\underline{j}} = 0 & \text{if } \mathrm{Shape}(\mathsf{J}) = 0 \end{cases} \tag{1.3}$$

for $\underline{j} \in (\mathbb{Z}/e\mathbb{Z})^n$, and coefficients $a_{\mathsf{TU}} \in \Bbbk$.

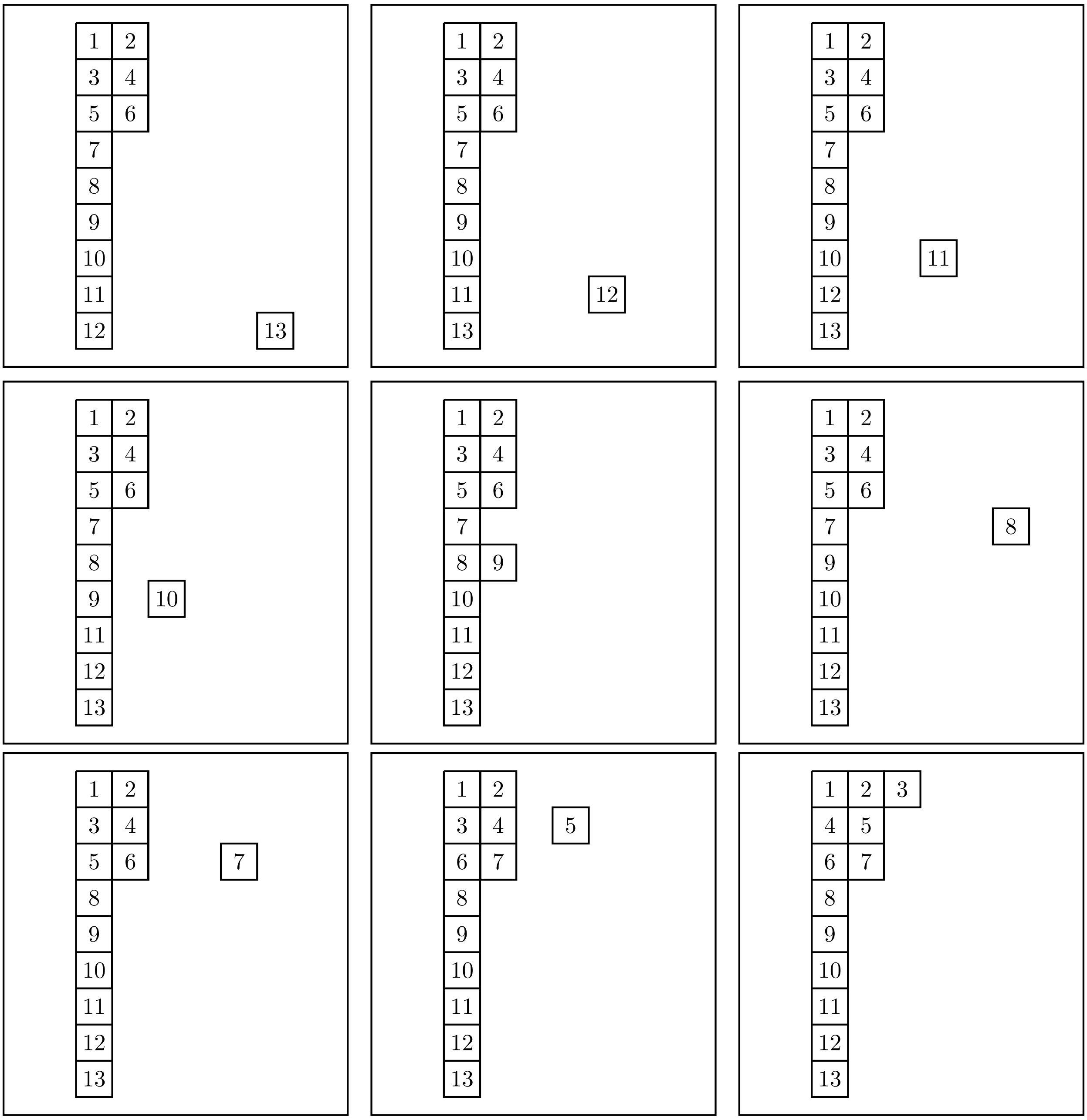


FIGURE 4. Let $e = 5$ and $\ell = 1$ and $\lambda = (2^3, 1^6)$ and $r = 2 \in \mathbb{Z}/e\mathbb{Z}$. We record the tableaux $\mathsf{T}_\lambda \cup \square$ with the 2-box $\square$ determined by $k = 13, 12, 11, 10, 9, 8, 7, 5, 3$.

For any $\lambda \in \mathscr{P}_{h,\ell}(n-1)$ and $\alpha = \mathsf{T}_\lambda^{-1}(a)$ for any $1 \leqslant a \leqslant n-1$, we set

$$\beta = \begin{cases} Y_\alpha(\lambda) = \lambda \cup \beta - \alpha & \text{if } \alpha = [i,j,m] \text{ for } j > 1 \\ Y_\alpha^{\ell+1}(\lambda) = \lambda \cup \beta - \alpha & \text{if } \alpha = [i,1,m] \end{cases},$$

and we set $1 \leqslant b < a$ equal to $\beta = \mathsf{T}^{-1}_{\lambda\cup\beta-\alpha}(b)$. (The two cases above are explained via Example 1.12.) Our second claim is that

$$y_a e_{\mathsf{T}_\lambda} \in \pm\psi^a_b e_{\mathsf{T}_{\lambda\cup\beta-\alpha}}\psi^b_a + \sum_{\{\mu \mid \mu \succcurlyeq \lambda-\alpha, \beta\notin\mu\}} \mathscr{H}^{\succeq(\mu\cup\beta)}. \tag{1.4}$$

We will equation (1.3) and (1.4) both hold for $\lambda \in \mathscr{P}_{h,\ell}(n)$. The proof is algorithmic and we keep track of the signs at each stage in the algorithm; the interested reader can calculate the signs 1.3 and 1.4 by keeping track of which cases occur at each point in the algorithm (this amounts to keeping track of the residues of the boxes $\mathsf{J}^{-1}(n) \succ \square \in \lambda$ such that $\square$ is the the final column of each row).

Claim (1)**.** Recall our inductive assumption 1.3; namely that for any $(j_1,\dots,j_{n-1}) \in (\mathbb{Z}/e\mathbb{Z})^{n-1}$ with $\mathrm{Shape}(\mathsf{J}_{<\mathsf{n}}) = \mu \succ \lambda \in \mathscr{P}_{h,\ell}(n-1)$, we have that

$$e_{j_1,\dots,j_{n-1}} = \pm\psi^{\mathsf{J}_{<\mathsf{n}}}_{\mathsf{T}_\mu}\psi^{\mathsf{T}_\mu}_{\mathsf{J}_{<\mathsf{n}}} + \mathscr{H}^{\succcurlyeq\mu}, \tag{1.5}$$

we will now prove this for $\mu = \lambda$. We have that

$$(\pm\psi^{\mathsf{J}_{<\mathsf{n}}}_{\mathsf{T}_\lambda}\psi^{\mathsf{T}_\lambda}_{\mathsf{J}_{<\mathsf{n}}}) \boxtimes e_{j_n} = \pm(\psi^{\mathsf{J}_{<\mathsf{n}}}_{\mathsf{T}_\lambda} \boxtimes e_{j_n})(e_{\mathsf{T}_\lambda} \boxtimes e_{j_n})(\psi^{\mathsf{T}_\lambda}_{\mathsf{J}_{<\mathsf{n}}} \boxtimes e_{j_n}) \tag{1.6}$$

and so (since $e_{\mathsf{T}_\lambda} \boxtimes e_{j_n} = e_{\mathsf{T}_\lambda\cup\boxed{n}}$, by the definition of this tableau) it suffices to show that

$$\begin{cases} e_{\mathsf{T}_\lambda\cup\boxed{n}} \in \pm\psi^{\mathsf{T}_\lambda\cup\boxed{n}}_{\mathsf{T}_{\lambda'}}\psi^{\mathsf{T}_{\lambda'}}_{\mathsf{T}_\lambda\cup\boxed{n}} + \mathscr{H}^{\succcurlyeq\mathrm{Shape}(\mathsf{J})} & \text{if } \mathrm{Shape}(\mathsf{J}) = \lambda' \in \mathscr{P}_{h,\ell}(n) \\ e_{\mathsf{T}_\lambda\cup\boxed{n}} = 0 & \text{if } \mathrm{Shape}(\mathsf{J}) = 0 \end{cases} \tag{1.7}$$

The proof of equation (1.7) is algorithmic and proceeds on the ordering $\succcurlyeq$. In order to do this, we set $\alpha = [x,y,z] \notin \lambda$ to be a j_n-node such that $\mathsf{J}^{-1}(n) \succ [x,y,z]$. We set $\beta = [x',y',z'] \notin \lambda$ to be the j_n-node determined by

$$\lambda \cup [x',y',z'] = \begin{cases} Y_{[x,y,z]}(\lambda\cup[x,y,z]) & \text{if } \mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1) \neq \mathrm{res}([x,y,z]); \\ Y^{\ell+1}_{[x,y,z]}(\lambda\cup[x,y,z]) & \text{if } \mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1) = \mathrm{res}([x,y,z]). \end{cases}$$

We set $a = \mathsf{T}^{-1}_{\lambda\cup\alpha}(\alpha)$ and $b = \mathsf{T}^{-1}_{\lambda\cup\beta}(\beta)$. We will prove 1.7 (which says that we can move $\boxed{n}$ through T_λ to the point $\mathsf{J}^{-1}(n)$) by factorising this to a single statement about moving a node α through a single row to the point β. Assuming only equation (1.5), we will show that

$$e_{\mathsf{T}_{\lambda\cup\alpha}} \in \psi^{\mathsf{T}_{\lambda\cup\alpha}}_{\mathsf{T}_{\lambda\cup\beta}} e_{\mathsf{T}_{\lambda\cup\beta}} \psi^{\mathsf{T}_{\lambda\cup\beta}}_{\mathsf{T}_{\lambda\cup\alpha}} + \mathscr{H}^{(\succcurlyeq\lambda)\cup\beta} \quad \text{where} \quad \mathscr{H}^{(\succcurlyeq\lambda)\cup\beta} = \sum_{\substack{\mu\succcurlyeq\lambda \\ \beta\notin\mu}} \mathscr{H}^{\succeq(\mu\cup\beta)} \tag{1.8}$$

This can be thought of, diagrammatically, as moving a box through each row, one at a time as illustrated in Figures 4 and 5. We remark that any term in $\mathscr{H}^{(\succcurlyeq\lambda)\cup\beta}$ can be ignored by induction. We include a running example of our algorithm for $e = 5$ and $\ell = 1$ and $\lambda = (2^3, 1^6)$. There are four cases to consider, depending on the residue of the final node in the column (i.e. the residue of the strand labelled by $\mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1)$). We reiterate that the $\mathrm{res}(\alpha) = \mathrm{res}(\beta) = r \in \mathbb{Z}/e\mathbb{Z}$.

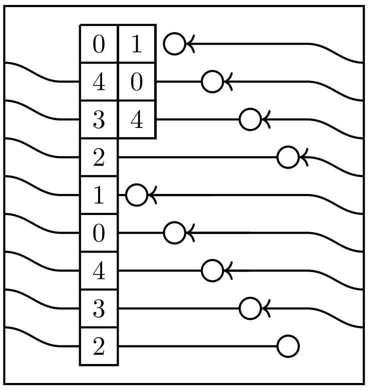


FIGURE 5. For $n = 13$ and $\lambda = (2^3, 1^6)$, we illustrate how the idempotent labelled by J in Figure 3 is rewritten in the form 1.7. The box moves through each row until it comes to rest at the point $\mathsf{J}^{-1}(13) = [1,3,0]$. This involves 8 applications of 1.8 to deduce 1.7.

(*i*) Suppose $\mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1)$ has residue $r \in \mathbb{Z}/e\mathbb{Z}$. By application of relations R3 and R4, we have that

$$e_{\mathsf{T}_{\lambda\cup\alpha}} = \psi^a_{a-1} e_{\mathsf{T}_{\lambda\cup\alpha}} y_{a-1} \psi^{a-1}_a y_{a-1} - y_k \psi_{a-1} e_{\mathsf{T}_{\lambda\cup\alpha}} y_{a-1} \psi_{a-1}. \tag{1.9}$$

An example of the visualisation of the idempotents on the righthand-side of equation (1.9) is given in the first step of Figure 4; the corresponding righthand-side of equation (1.9) is depicted in Figure 6. Now, by induction and equation (1.4) we have that

$$\begin{aligned} e_{\mathsf{T}_{\lambda\cup\alpha}} &= \psi^{a-1}_a \psi^{a-1}_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_a y_{a-1} - y_a \psi^a_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_a + \mathscr{H}^{(\not\succ\lambda)\cup\beta} & (1.10)\\ &= \psi^a_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_a y_{a-1} - y_a \psi^a_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_a + \mathscr{H}^{(\not\succ\lambda)\cup\beta} & (1.11)\end{aligned}$$

as required.

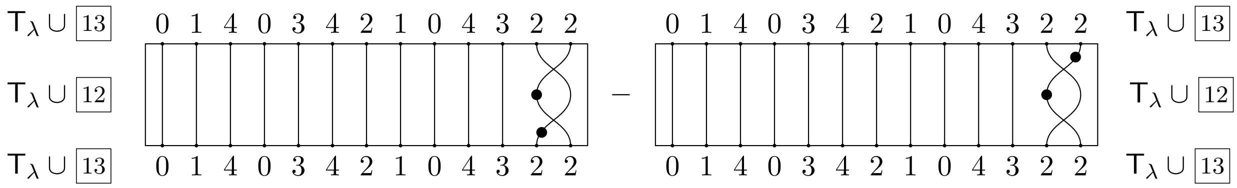


FIGURE 6. We continue with the example in Figures 4 and 5 for $\lambda = (2^6, 1^3)$. The righthand-side of equation (1.9) for $a = 13$ (and hence $b = 12$).

(*ii*) Now suppose $\mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1) = [i,j,m]$ has residue $r+1 \in \mathbb{Z}/e\mathbb{Z}$. If $[i, j-1, m] \in \lambda$ then the $(a-2)$th, $(a-1)$th and ath strands have residues r, $r+1$, and r respectively. By relation R5 we have that

$$\begin{aligned} e_{\mathsf{T}_{\lambda\cup\alpha}} &= e_{\mathsf{T}_{\lambda\cup\alpha}} \psi_{a-2}\psi_{a-1}\psi_{a-2} e_{\mathsf{T}_{\lambda\cup\alpha}} - e_{\mathsf{T}_{\lambda\cup\alpha}} \psi_{a-1}\psi_{a-2}\psi_{a-1} e_{\mathsf{T}_{\lambda\cup\alpha}} \\ &= -e_{\mathsf{T}_{\lambda\cup\alpha}} \psi_{a-2}\psi_{a-1} y_{a-1} \psi_{a-1}\psi_{a-2} e_{\mathsf{T}_{\lambda\cup\alpha}} + e_{\mathsf{T}_{\lambda\cup\alpha}} \psi_{a-1}\psi_{a-2} y_{a-2} \psi_{a-2}\psi_{a-1} e_{\mathsf{T}_{\lambda\cup\alpha}}. \end{aligned} \tag{1.12}$$

We set $\mu = \mathrm{Shape}(\mathsf{T}_\lambda{\downarrow}_{<a-2})$. The two terms in 1.12 factor through the elements

$$\underbrace{e_{\mathsf{T}_\lambda{\downarrow}_{<a-2}} \boxtimes e_{r+1}}_{\mu\cup[i,j,m]} \boxtimes y_1 e_{r,r} \boxtimes e_{\mathsf{T}_{\lambda\cup\alpha}{\downarrow}_{>a}} \qquad \underbrace{e_{\mathsf{T}_\lambda{\downarrow}_{<a-2}} \boxtimes y_1 e_r}_{\mu\cup[i,j-1,m]} \boxtimes e_{r,r+1} \boxtimes e_{\mathsf{T}_{\lambda\cup\alpha}{\downarrow}_{>a}} \tag{1.13}$$

respectively. An example of the idempotents on the righthand-side of 1.13 is given in the second step of Figure 4; the corresponding elements on righthand-side of equation are depicted in Figure 7.

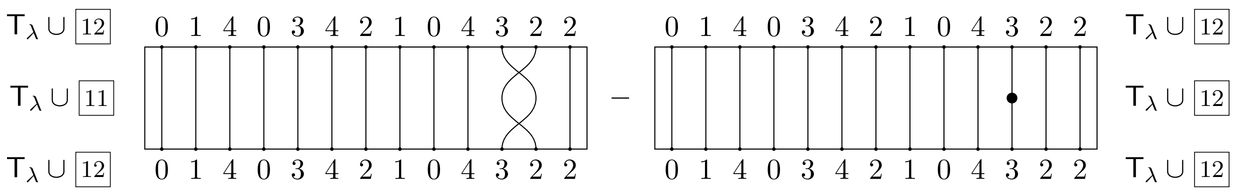


FIGURE 7. The righthand-side of equation (1.12) for $\lambda = (2^6, 1^3)$ and $\alpha = [8,5,0]$, $a = 12$. The residue sequence of the centre of the former diagram is given by $\mathsf{T}_{\lambda\cup\beta}$ for $\beta = [7,4,0]$, $b = 11$.

- We first consider the latter term on the righthand-side of equation (1.12) (which we will see, is the required non-zero term). We note that $[i, j-1, m]$ and α have the same residue and so $Y_{[i,j-1,m]}(\lambda\cup\alpha) = Y_\alpha(\lambda\cup\alpha) = \lambda\cup\beta$. Therefore, by induction and 1.4, we have that

$$e_{\mathsf{T}_\lambda{\downarrow}_{<a-2}} \boxtimes y_1 e_r = y_{a-2} e_{\mathsf{T}_\mu} = \psi^{a-2}_b e_{\mathsf{T}_{\mu\cup\beta}} \psi^b_{a-2} + \mathscr{H}^{(\not\succ\mu)\cup\beta} \tag{1.14}$$

  Substituting this back into the second term of 1.13 we obtain

$$\psi^{a-2}_b e_{\mathsf{T}_{\mu\cup\beta}} \psi^b_{a-2} \boxtimes e_{r,r+1} \boxtimes e_{\mathsf{T}_{\lambda\cup\alpha}{\downarrow}_{>a}} = \psi^{a-2}_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_{a-2} + \mathscr{H}^{(\not\succ\lambda)\cup\beta}$$

  and then substituting into the second term of 1.12 we obtain

$$\begin{aligned} e_{\mathsf{T}_{\lambda\cup\alpha}} \psi_{a-1}\psi_{a-2} y_{a-2} \psi_{a-2}\psi_{a-1} e_{\mathsf{T}_{\lambda\cup\alpha}} &= \psi_{a-1}\psi_{a-2}\psi^{a-2}_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_{a-2}\psi_{a-2}\psi_{a-1} + \mathscr{H}^{(\not\succ\lambda)\cup\beta} \\ &= \psi^a_b e_{\mathsf{T}_{\lambda\cup\beta}} \psi^b_a + \mathscr{H}^{(\not\succ\lambda)\cup\beta} \end{aligned}$$

  as required.

- We now consider the former term of 1.12 (which, we will see, is zero modulo the ideal). We have that $Y_{[i,j,m]}(\mu\cup[i,j,m]) = \mu\cup\gamma \succcurlyeq \mu\cup[i,j,m]$ for $\gamma\succ[i,j,m]$ a box of residue $r+1\in\mathbb{Z}/e\mathbb{Z}$. We set $c=\mathsf{T}^{-1}_{\mu\cup\gamma}(\gamma)$. We have that
$$e_{\mathsf{T}_\mu}\boxtimes e_{r+1} = e_{\mathsf{T}_{\mu\cup[i,j,m]}} = \psi_c^{a-2} e_{\mathsf{T}_{\mu\cup\gamma}}\psi_c^{a-2}$$
by the commuting KLR relations and Lemma 1.16. We now consider the concatenation with y_1e_r. We have that $Y_{[i,j-1,m]}(Y_{[i,j,m]}(\mu\cup[i,j,m])) = Y_{[i,j-1,m]}(\mu\cup\gamma)=\mu\cup\gamma\cup\delta$ for $\gamma\succ\delta\succ[i,j-1,m]$ a box of residue $r\in\mathbb{Z}/e\mathbb{Z}$. We have that
$$e_{\mathsf{T}_{\mu\cup\gamma}}\boxtimes y_1e_r = y_{a-1}e_{\mathsf{T}_{\mu\cup\gamma\cup[i,j-1,m]}} = \psi_{c+1}^{a-1}e_{\mathsf{T}_{\mu\cup\gamma\cup\delta}}\psi^{c+1}_{a-1} + \mathcal{H}^{(\succcurlyeq\mu)\cup\gamma\cup\delta}$$
by induction and 1.3. Finally, we concatenate again to obtain
$$e_{\mathsf{T}_{\lambda\downarrow_{<a-2}}}\boxtimes e_{r+1}\boxtimes y_1e_{r,r}\boxtimes e_{\mathsf{T}_{\lambda\cup\alpha\downarrow_{>a}}} = \psi_c^{a-2}\psi_{c+1}^{a-1}e_{\mathsf{T}_{\lambda\cup\gamma\cup\delta-[i,j,m]}}\psi^{c+1}_{a-1}\psi^c_{a-2} + \mathcal{H}^{(\succcurlyeq\lambda)\cup\delta}$$
and we note that the idempotent on the righthand-side is labelled by $\lambda\cup\gamma\cup\delta-[i,j,m]\succcurlyeq\lambda+\mathsf{J}^{-1}(n)$. In particular, this element belongs to $\mathscr{H}^{(\succcurlyeq\lambda)\cup\beta}$ as required.

It remains to consider the case that $[i,j-1,m]\notin\lambda$ (in other words $j=1$). Since λ is a multipartition, the $(i+1)$th row of the mth component is the singleton with a box of residue r or is empty. If the row is empty, then the process would have already terminated with $\lambda+[i+1,1,m]$. Therefore, the $(i+1)$th row of the mth component is non-empty and so consists of the single box, α, of residue $r\in\mathbb{Z}/e\mathbb{Z}$ as in case (i). However, case (i) moves the box α through $(\ell+1)$ successive rows (rather than the usual single row). Thus case (ii) with $[i,j-1,m]\notin\lambda$ does not need to be considered explicitly as it does not appear (except as an intermediate step in case (i)).

(iii) Now suppose $\mathsf{T}^{-1}_{\lambda\cup\alpha}(a-1)$ has residue $d\in\mathbb{Z}/e\mathbb{Z}$ such that $|d-r|>1$. We set $\mu=\mathrm{Shape}(\mathsf{T}_{\lambda\downarrow_{<a-1}})$. We have that
$$e_{\mathsf{T}_{\lambda\cup\alpha}} = \psi_{a-1}\big(\underbrace{e_{\mathsf{T}_{\lambda\downarrow_{<a-1}}}\boxtimes e_r}_{\mu\cup\alpha}\boxtimes e_d\boxtimes e_{\mathsf{T}_{\lambda\cup\alpha\downarrow_{>a}}}\big)\psi_{a-1} \tag{1.15}$$
by case 2 of relation R4. By induction, we have that
$$e_{\mathsf{T}_{\lambda\downarrow_{<a-1}}}\boxtimes e_r = \psi_b^{a-1}e_{\mathsf{T}_{\mu\cup\beta}}\psi^b_{a-1} + \mathscr{H}^{(\succcurlyeq\mu)\cup\beta}$$
and so, as in the case (ii) above, we concatenate to deduce the result. Two examples of the visualisation of the idempotents on the righthand-side of equation (1.15) are given in the final two steps of Figure 4; the corresponding elements are depicted in Figure 8.

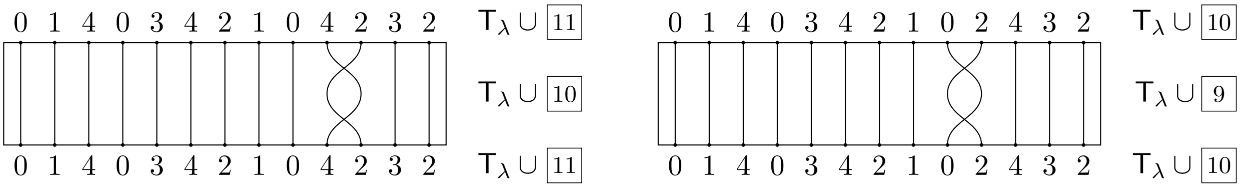


FIGURE 8. The first diagram is the righthand-side of equation (1.15) for $\lambda=(2^6,1^3)$ and $a=11$ and $a=10$ respectively.

(iv) Suppose $\mathsf{T}_\lambda(a-1)=[i,j,m]$ has residue $r-1\in\mathbb{Z}/e\mathbb{Z}$ and that $[i-1,j+1,m]\notin\lambda$ (in other words $\lambda\cup\alpha$ is not a partition — note that if $\lambda\cup\alpha$ is a partition, then we terminate the process and we have the required result). We set $c=\mathsf{T}_\lambda(i-1,j,m)$ and let $\mu=\mathrm{Shape}(\mathsf{T}_{\lambda\downarrow_{<a-1}})\setminus\{[i-1,j,m]\}$ (see Figure 10 for an example). We have that
$$e_{\mathsf{T}_{\lambda\cup\alpha}} = e_{\mathsf{T}_{\lambda\cup\alpha}}\psi^c_{a-2}\psi_c^{a-2}e_{\mathsf{T}_{\lambda\cup\alpha}} \tag{1.16}$$
$$= e_{\mathsf{T}_{\lambda\cup\alpha}}\psi^c_{a-1}\psi^a_c e_{\mathsf{T}_{\lambda\cup\alpha}} - e_{\mathsf{T}_{\lambda\cup\alpha}}\psi_{a-1}\psi^c_{a-1}\psi_c^{a-2}\psi_{a-1}e_{\mathsf{T}_{\lambda\cup\alpha}} \tag{1.17}$$
$$= \psi^c_{a-1}(e_{\mathsf{T}_\mu}\boxtimes e_{r-1,r,r}\boxtimes e_{\mathsf{T}_{\lambda\cup\alpha\downarrow_{>a}}})\psi^a_c - \psi^a_{c+1}e_{\mathsf{T}_{\lambda\cup\beta}}\psi^c_a \tag{1.18}$$
where the first and third equalities follow from the commuting case 2 of relation R4 and Lemma 1.16, and the second equality follows from case 1 of relation R5. For our continuing example, the righthand-side of equation (1.18) is depicted in Figure 9; the box-configurations labelling the idempotents on the left and righthand-sides of equation (1.18) are depicted in Figure 10.

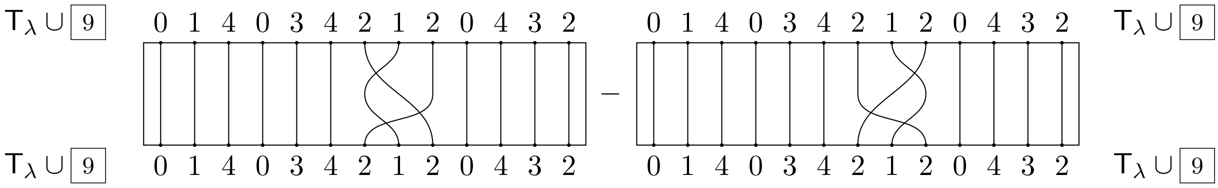


FIGURE 9. The righthand-side of equation (1.18) for $\lambda = (2^6, 1^3)$ and $\square = [5, 2, 0]$.

We now consider the first term on the righthand-side of equation (1.18). We have that

$$e_{\mathsf{T}_\mu} \boxtimes e_{r-1,r,r} = e_{\mathsf{T}_{\mu\cup[i,j,m]\cup[i,j+1,m]\cup\alpha}}$$

By induction, 1.3, and Lemma 1.16, we have that: $e_{\mathsf{T}_{\mu\cup[i,j,m]}}\mathscr{H}^{\succeq\nu}$ for $\nu = Y^{\ell+1}_{[i,j,m]}(\mu+[i,j,m])$; thus $e_{\mathsf{T}_{\mu\cup[i,j,m]\cup[i,j+1,m]}} \in \mathscr{H}^{\succeq\nu\cup[i-1,j,m]}$; and thus $e_{\mathsf{T}_{\mu\cup[i,j,m]\cup[i,j+1,m]\cup[i,j+e+1,m]}} \in \mathscr{H}^{\succeq\nu\cup[i-1,j,m]\cup\beta}$; thus

$$e_{\mathsf{T}_\mu} \boxtimes e_{r-1,r,r} \boxtimes e_{\mathsf{T}_{\lambda\cup\alpha\downarrow_{>a}}} \in \mathscr{H}^{(\not\succ\lambda)\cup\beta}$$

as required. See Figure 11 for an example.

Thus we have shown that 1.8 holds and so we can add the nth box/strand of residue $r \in \mathbb{Z}/e\mathbb{Z}$ through each row of the Young diagram of λ/the element e_{T_λ} as in Figure 5 one at a time (modulo error terms which are higher in the reverse lexicographic ordering) until it reaches the points $\mathsf{J}^{-1}(n)$. Thus equation (1.7) holds and so too does 1.3.

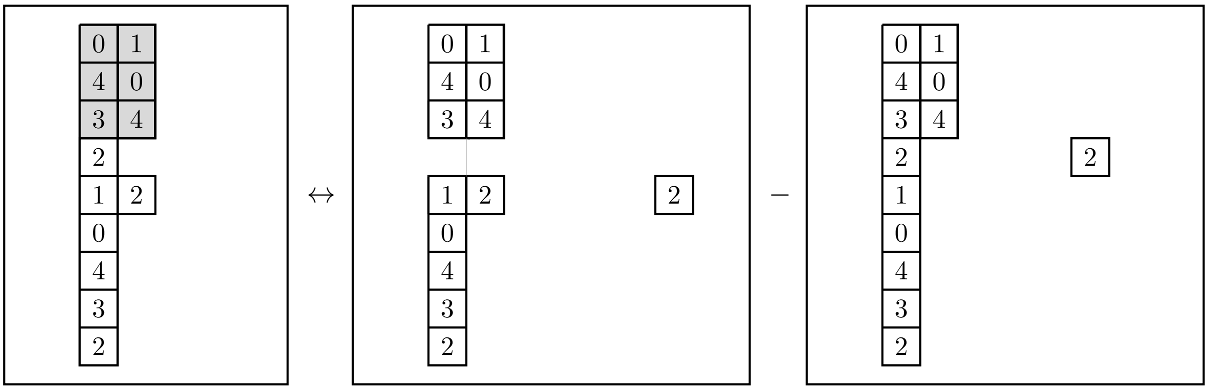


FIGURE 10. Let $e = 5$ and $\ell = 1$. The left hand-side is the final diagram of Figure 5 (with μ shaded grey and $c = 7$). The righthand-side labels the idempotents obtained from applying equation (1.18).

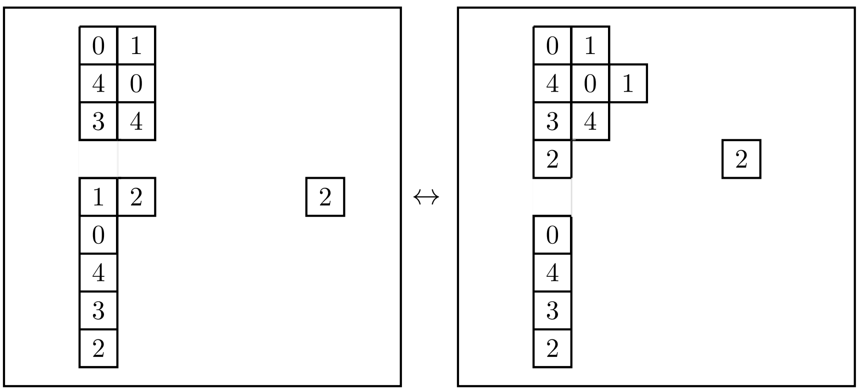


FIGURE 11. Rewriting the first term after the equality in equation (1.18). We have moved the 1-box using case (iii) and this leaves us free to move the 2-boxes up their corresponding diagonals. The diagram on the righthand-side of this figure is clearly higher in the $\succ$-ordering than the rightmost diagram in Figure 10.

Claim (2). We assume that 1.4 holds for $\lambda \in \mathscr{P}_{h,\ell}(n-1)$. We set $\lambda' = \lambda \cup \alpha$ and we assume that α is the lexicographically least removable box of λ' and that this box is of residue $r \in \mathbb{Z}/e\mathbb{Z}$. Let $1 \leqslant a < n$, by induction, we know that

$$y_a e_{\mathsf{T}_{\lambda'}} \in \pm \psi^a_b e_{\mathsf{T}_\beta} \psi^b_a + \sum_{\substack{\mu \succ \lambda \\ \beta \notin \mu}} \mathscr{H}^{\succeq \mu + \beta} \quad \text{for} \quad \beta = \begin{cases} Y_\alpha(\lambda') = \lambda + \beta & \text{for } j > 1 \\ Y_\alpha^{\ell+1}(\lambda') = \lambda + \beta & \text{for } j = 1 \end{cases} \tag{1.19}$$

It remains to prove 1.19 for $a = n$. We have that

$$y_n e_{\mathsf{T}_{\lambda'}} = \begin{cases} y_{n-1} e_{\mathsf{T}_{\lambda'}} - e_{\mathsf{T}_{\lambda'}} \psi^n_b \psi^b_n e_{\mathsf{T}_{\lambda'}} & \text{if } \alpha = [i, j, m] \text{ for some } j > 1 \\ y_{b+1} e_{\mathsf{T}_{\lambda'}} + e_{\mathsf{T}_{\lambda'}} e_{\mathsf{T}_{\lambda \cup \beta}} \psi^n_b \psi^b_n e_{\mathsf{T}_{\lambda'}} & \text{if } \alpha = [i, 1, m] \end{cases}. \tag{1.20}$$

In the former case, this follows from case 3 of relation R4 and the commutativity relations, to see this note that the $(n-1)$th strand has residue $r - 1 \in \mathbb{Z}/e\mathbb{Z}$ (and the kth strands for $b < k < n-1$ are of non-adjacent residue by Lemma 1.16). In the latter case, this follows from case 2 of relation R4 and the commutativity relations, to see this note that $[i-1, 1, m] = \mathsf{T}_\lambda^{-1}(b+1)$ is the first strand of adjacent residue (namely, $r + 1 \in \mathbb{Z}/e\mathbb{Z}$) that we encounter when we pull the nth strand leftwards (by Lemma 1.16). By induction and 1.19, the dotted terms on the righthand-side of equation (1.20) are belong to the ideal in 1.19. Thus the result follows. □

We let $\lambda^{[0]} \succ \lambda^{[1]} \succ \cdots \succ \lambda^{[m]}$ denote the complete set of elements of $\mathscr{P}_{h,\ell}(n)$ enumerated according to the total ordering $\succ$. (We use square brackets in order to not confuse the enumeration of multipartitions with the notation for components of a given multipartition.)

Corollary 1.18. *The $\mathbb{Z}$-algebra $\mathscr{H}_n^\sigma$ has a filtration*

$$0 = \mathscr{H}^0 \subset \mathscr{H}^{\succ \lambda^{[0]}} \subset \mathscr{H}^{\succ \lambda^{[1]}} \subset \cdots \subset \mathscr{H}^{\succ \lambda^{[m]}} = \mathscr{H}_n^\sigma.$$

Proof. This follows immediately from Proposition 1.17 □

The following is an amalgamation of Lemma 2.5 and Proposition 2.5 of [BKW11]. We set $\mathcal{Y} = \langle e_{\underline{i}}, y_k \mid \underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n, 1 \leqslant k \leqslant n \rangle$.

Proposition 1.19. *We let $\underline{w}$, $\underline{w}'$ be any two choices of reduced expression for w and let $\underline{v}$ be any non-reduced expression for w. We have that*

$$e_{\underline{i}} \psi_{\underline{w}} e_{\underline{j}} = e_{\underline{i}} \psi_{\underline{w}'} e_{\underline{j}} + \textstyle\sum_{\underline{x} < \underline{w}, \underline{w}'} e_{\underline{i}} \psi_{\underline{x}} f_{\underline{x}}(y) e_{\underline{j}} \tag{1.21}$$

$$e_{\underline{i}} \psi_{\underline{v}} e_{\underline{j}} = \textstyle\sum_{\underline{x} < \underline{v}} e_{\underline{i}} \psi_{\underline{x}} e_{\underline{j}} g_{\underline{x}}(y) \tag{1.22}$$

$$y_k e_{\underline{i}} \psi_{\underline{w}} e_{\underline{j}} = e_{\underline{j}} \psi_{\underline{w}} e_{\underline{i}} y_{w(k)} + \textstyle\sum_{\underline{x} < \underline{w}} e_{\underline{i}} \psi_{\underline{x}} e_{\underline{j}} \tag{1.23}$$

for some $f_{\underline{x}}(y), g_{\underline{x}}(y) \in \mathcal{Y}$.

Theorem 1.20. *Let $\Bbbk$ be an integral domain. Let $\underline{x}$ and $\underline{y}$ be arbitrary fixed choices of reduced expression for $w^{\mathsf{S}}_{\mathsf{T}_\lambda}$ and $w^{\mathsf{T}_\lambda}_{\mathsf{T}}$ respectively. The $\Bbbk$-algebra $\mathscr{H}_n^\sigma$ is a graded cellular algebra with basis*

$$\{\psi_{\underline{x}} e_{\mathsf{T}_\lambda} \psi_{\underline{y}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{Std}(\lambda), \lambda \in \mathscr{P}_{h,\ell}(n)\} \tag{1.24}$$

anti-involution $*$ and the degree function $\mathsf{deg} : \mathrm{Std} \to \mathbb{Z}$. For $\Bbbk$ a field, $\mathscr{H}_n^\sigma$ is quasi-hereditary.

Proof. Given $x \in \mathfrak{S}_n$ we let a_x denote any element of $\langle \psi_i, y_j \mid 1 \leqslant i < n, 1 \leqslant j \leqslant n \rangle$ whose underlying permutation is $x \in \mathfrak{S}_n$. We have already seen that any $d \in \mathscr{H}_n^\sigma$ can be written as a linear combination of elements of the form $d = a_x e_{\mathsf{T}_\lambda} a_y$ for some $a_x, a_y \in \mathscr{H}_n^\sigma$ which trace out the bijections $x, y \in \mathfrak{S}_n$ (but are possibly decorated with dots and need not be reduced). We proceed by induction along the Bruhat order by working modulo the span of elements

$$\mathrm{Span}_\Bbbk \{\psi_{\underline{u}} e_{\mathsf{T}_\lambda} \psi_{\underline{v}} \mid u \leqslant x \text{ or } v \leqslant y\} + \mathscr{H}^{\succ \lambda}. \tag{1.25}$$

We first note that if a_x or a_y has a dot or a double crossing, then $a_x e_{\mathsf{T}_\lambda} a_y$ is zero modulo equation (1.25) by Proposition 1.19. Given two choices $\underline{x}, \underline{x}'$ of reduced expression for $x \in \mathfrak{S}_n$ we have that $a_{\underline{x}} \psi_{\underline{x}} - a_{\underline{x}'} \psi_{\underline{x}'}$ belongs to equation (1.25) by 1.19. Thus $\mathcal{H}^{\succeq \lambda} / \mathcal{H}^{\succ \lambda}$ is spanned by elements of the form

$$\{\psi_{\underline{x}} e_{\mathsf{T}_\lambda} \psi_{\underline{y}} \mid \text{for } \underline{x}, \underline{y} \text{ arbitrary choices of fixed reduced expressions of } x, y \in \mathfrak{S}_n\} + \mathscr{H}^{\succ \lambda}.$$

It remains to show that a spanning set is given by the elements $x = w^{\mathsf{S}}_{\mathsf{T}_\lambda}$, $y = w^{\mathsf{T}_\lambda}_{\mathsf{T}}$ for $\mathsf{S}, \mathsf{T} \in \mathrm{Std}(\lambda)$.

Given $\mathsf{T} \in \mathrm{CStd}(\lambda) \setminus \mathrm{Std}(\lambda)$, we have that $w^{\mathsf{T}_\lambda}_{\mathsf{T}}$ has a pair of crossing strands from $1 \leqslant i < j \leqslant n$ to $1 \leqslant w^{\mathsf{T}_\lambda}_{\mathsf{T}}(j) < w^{\mathsf{T}_\lambda}_{\mathsf{T}}(j) \leqslant n$ such that $\mathsf{T}^{-1}_\lambda(i) = [r, c, m]$ and $\mathsf{T}^{-1}_\lambda(j) = [r, c+1, m]$ are in the same row and in particular so that $i = j - 1$. By the above, it suffices to show that $\psi_{\underline{x}} e_{\mathsf{T}_\lambda} \psi_{\underline{y}}$ belongs to 1.25 for a preferred choice of $\underline{y}$; we choose $\underline{y} = s_i \underline{w}$ (for some $w \in \mathfrak{S}_n$ such that $s_i w = y$). Thus it remains to show that $\psi_{s_i \underline{w}}$ belongs to 1.25. However, this immediately follows from Proposition 1.17 because $e_{\mathsf{T}_\lambda} \psi_{s_i} = \psi_{s_i} e_{s_i(\mathsf{T}_\lambda)}$ and we have that $e_{s_i(\mathsf{T}_\lambda)} \in \mathscr{H}^{\succeq \alpha}$ for $\alpha = Y_{\mathsf{T}^{-1}_\lambda(i+1)}(\lambda) \succ \lambda$.

Finally, given any $\mathsf{T} \in \mathrm{RStd}(\lambda) \setminus \mathrm{Std}(\lambda)$, we have that $w^{\mathsf{T}}_{\mathsf{T}_\lambda}$ can be written in the form $w^{\mathsf{T}}_{\mathsf{T}_\lambda} = w^{\mathsf{T}}_{\mathsf{G}} w^{\mathsf{G}}_{\mathsf{T}_\lambda}$ for some Garnir tableau $\mathsf{G} \in \mathrm{CStd}(\lambda) \setminus \mathrm{Std}(\lambda)$. (This is a routine argument, see for example [Mat99, 3.14 Lemma].) We first fix the Garnir box to be $[i, j, m] \in \lambda$ and let $1 \leqslant k \leqslant n$ be such that $\mathsf{T}_\lambda[i, j, m] = k$. We will show that $\psi^{\mathsf{G}}_{\mathsf{T}_\lambda} \in \mathscr{H}^{\succ \lambda}$ and thus deduce the result. However, this is not difficult to see as $\psi^{\mathsf{G}}_{\mathsf{T}_\lambda} = e_{\mathsf{G}} \psi^{\mathsf{G}}_{\mathsf{T}_\lambda} e_{\mathsf{T}_\lambda}$ and $e_{\mathsf{G}} = e_{\mathsf{G}_{\leqslant k}} \boxtimes e_{\mathsf{G}_{>k}}$ where $\mathrm{Shape}(e_{\mathsf{G}_{\leqslant k}})$ is a composition and so $e_{\mathsf{G}} \in \mathscr{H}^{\succ \lambda}$ by Proposition 1.17.

Therefore 1.24 is a spanning set of our algebra. By [Bow, Theorem 12.2] we have that $\mathscr{H}^\sigma_n$ is of rank $\sum_{\lambda \in \mathscr{P}_{h,\ell}(n)} |\mathrm{Std}(\lambda)|^2$. Thus the algebra is cellular (by its construction via idempotent ideals) with the stated basis (by the fact that we have a spanning set of the required rank). Finally, we note that each layer of the cell chain contains an idempotent e_{T_λ} and so the algebra is quasi-hereditary, as required. □

Let $\Bbbk$ be an integral domain. We define the standard or Specht modules of $\mathscr{H}^\sigma_n$ as follows,

$$\mathbf{S}_\Bbbk(\lambda) = \{\psi^{\mathsf{S}}_{\mathsf{T}_\lambda} + \mathscr{H}^{\succ \lambda} \mid \mathsf{S} \in \mathrm{Std}(\lambda)\} \tag{1.26}$$

for $\lambda \in \mathscr{P}_{h,\ell}(n)$. We immediately deduce the following corollary of Theorem 1.20.

Corollary 1.21. *The Specht module $\mathbf{S}_\Bbbk(\lambda)$ is the cyclic module generated by e_{T_λ} subject to the following relations:*

- $e_{\underline{i}} e_{\mathsf{T}_\lambda} = \delta_{\underline{i}, \mathsf{T}_\lambda} e_{\mathsf{T}_\lambda}$ *for* $\underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n$;
- $y_k e_{\mathsf{T}_\lambda} = 0$ *for* $1 \leqslant k \leqslant n$;
- $\psi_k e_{\mathsf{T}_\lambda} = 0$ *for any* $1 \leqslant k < n$ *such that* $s_k(\mathsf{T}_\lambda)$ *is not row standard;*
- $\psi^{\mathsf{G}}_{\mathsf{T}_\lambda} e_{\mathsf{T}_\lambda} = 0$ *for* G *any Garnir* λ*-tableau.*

Proof. We have already checked that all of these relations hold (and so one can define a homomorphism from the abstractly defined module with this presentation to $\mathbf{S}_\Bbbk(\lambda)$) it only remains to check that these relations will suffice (i.e. the homomorphism is surjective). We know that $\mathbf{S}_\Bbbk(\lambda)$ has a basis indexed by standard tableaux and so it will suffice to show that $\psi^{\mathsf{S}}_{\mathsf{T}_\lambda}$ for any non-standard tableau S can be written in the form $\psi_{\underline{w}} \psi^{\mathsf{T}}_{\mathsf{T}_\lambda}$ for T equal to either $s_k(\mathsf{T}_\lambda)$ for some $1 \leqslant k \leqslant n$ or a Garnir tableaux. For a $\mathsf{S} \notin \mathrm{RStd}(\lambda)$ this is trivial. For a $\mathsf{S} \notin \mathrm{CStd}(\lambda)$ this is a standard calculation which utilises the fact that the $\psi^{\mathsf{S}}_{\mathsf{T}_\lambda}$ for $\mathsf{S} = \mathsf{G}_{[i,j,m]}$ are of minimal length. □

We now recall that the cellular structure allows us to define bilinear forms, for each $\lambda \in \mathscr{P}_{h,\ell}(n)$, there is a bilinear form $\langle \ , \ \rangle^\lambda$ on $\mathbf{S}_\Bbbk(\lambda)$, which is determined by

$$\psi^{\mathsf{T}_\lambda}_{\mathsf{S}} \psi^{\mathsf{T}}_{\mathsf{T}_\lambda} \equiv \langle \psi^{\mathsf{S}}_{\mathsf{T}_\lambda}, \psi^{\mathsf{T}}_{\mathsf{T}_\lambda} \rangle^\lambda \, e_{\mathsf{T}_\lambda} \pmod{\mathcal{H}^{\succ \lambda}} \tag{1.27}$$

for any $\mathsf{S}, \mathsf{T} \in \mathrm{Std}(\lambda)$. Let $\Bbbk$ be a field of arbitrary characteristic. Factoring out by the radicals of these forms, we obtain a complete set of non-isomorphic simple $\mathscr{H}^\sigma_n$-modules

$$\mathbf{D}_\Bbbk(\lambda) = \mathbf{S}_\Bbbk(\lambda)/\mathrm{rad}(\mathbf{S}_\Bbbk(\lambda)), \ \lambda \in \mathscr{P}_{h,\ell}(n).$$

Proposition 1.22. *Let $\lambda \in \mathscr{P}_{h,\ell}(n)$ and let $A_1 \succ A_2 \succ \cdots \succ A_z$ denote the removable boxes of λ, totally ordered according to the $\succeq$-ordering. The restriction of $\mathbf{S}_\Bbbk(\lambda)$ has an $\mathscr{H}^\sigma_{n-1}$-module filtration*

$$0 = \mathbf{S}^{z+1,\lambda} \subset \mathbf{S}^{z,\lambda} \subset \cdots \subset \mathbf{S}^{1,\lambda} = \mathrm{Res}_{\mathscr{H}^\sigma_{n-1}}(\mathbf{S}_\Bbbk(\lambda)) \tag{1.28}$$

given by

$$\mathbf{S}^{x,\lambda} = \Bbbk\{\psi^{\mathsf{S}}_{\mathsf{T}_\lambda} \mid \mathrm{Shape}(\mathsf{S}_{\leqslant n-1}) = \lambda - A_y \textit{ for some } z \geqslant y \geqslant x\}.$$

For each $1 \leqslant r \leqslant z$, we have that

$$\varphi_r : \mathbf{S}(\lambda - A_r)\langle \deg(A_r)\rangle \cong \mathbf{S}^{r,\lambda}/\mathbf{S}^{r+1,\lambda} \qquad : \psi_{\mathsf{S}_{\leqslant n-1}} \mapsto \psi_{\mathsf{S}_{\leqslant n-1}} \circ \psi^n_{\mathsf{T}_\lambda(A_r)} \tag{1.29}$$

Proof. On the level of $\mathbb{Z}$-modules, this is clear. Lifting this to $\mathscr{H}^{\sigma}_{n-1}$-modules is a standard argument which proceeds by checking the Garnir relations in a routine manner. □

2. General light leaves bases for quiver Hecke algebras

The principal idea of categorical Lie theory is to replace existing structures (combinatorics, bases, and presentations of Hecke algebras) with richer structures which keep track of more information. In this section, we replace the classical tableaux combinatorics of symmetric groups (and quiver Hecke algebras) with that of paths in an alcove geometry. This will allow us to construct "light leaves" bases of these algebras, for which p-Kazhdan–Lusztig is baked-in to the very definition. The light leaves bases of $\mathbf{S}_{\Bbbk}(\lambda)$ are constructed in such a way as to keep track of not just the point $\lambda \in \mathbb{E}_{h,\ell}$ (or rather the single path, T_λ, to the point λ) but of the many different ways we can get to the point λ by a reduced path/word in the alcove geometry. This extra generality is essential when we wish to write bases in terms of "2-generators" of the algebras of interest.

2.1. The alcove geometry. For each $1 \leqslant i \leqslant h$ and $0 \leqslant m < \ell$ we let ε_{hm+i} denote a formal symbol, and define an ℓh-dimensional real vector space

$$\mathbb{E}_{h,\ell} = \bigoplus_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant m < \ell}} \mathbb{R}\varepsilon_{hm+i}$$

and $\overline{\mathbb{E}}_{h,\ell}$ to be the quotient of this space by the one-dimensional subspace spanned by

$$\sum_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant m < \ell}} \varepsilon_{hm+i}.$$

We have an inner product $\langle\ ,\ \rangle$ on $\mathbb{E}_{h,\ell}$ given by extending linearly the relations

$$\langle \varepsilon_{hp+i}, \varepsilon_{hq+j} \rangle = \delta_{i,j}\delta_{p,q}$$

for all $1 \leqslant i, j \leqslant h$ and $0 \leqslant p, q < \ell$, where $\delta_{i,j}$ is the Kronecker delta. We identify $\lambda \in \mathcal{C}_{h,\ell}(n)$ with an element of the integer lattice inside $\mathbb{E}_{h,\ell}$ via the map

$$\lambda \longmapsto \sum_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant m < \ell}} (\lambda^{(m)})^T_i \varepsilon_{hm+i}$$

where $(-)^T$ is the transpose map. We let Φ denote the root system of type $A_{h\ell-1}$ consisting of the roots

$$\{\varepsilon_{hm+i} - \varepsilon_{ht+j} : 1 \leqslant i, j \leqslant h \text{ and } 0 \leqslant m, t < \ell \text{ with } (i, m) \neq (j, t)\}$$

and Φ_0 denote the root system of type $A_{h-1} \times \cdots \times A_{h-1}$ consisting of the roots

$$\{\varepsilon_{hm+i} - \varepsilon_{hm+j} : 1 \leqslant i, j \leqslant h \text{ and } 0 \leqslant m < \ell \text{ with } i \neq j\}.$$

We choose Δ (respectively Δ_0) to be the set of simple roots inside Φ (respectively Φ_0) of the form $\varepsilon_t - \varepsilon_{t+1}$ for some t. Given $r \in \mathbb{Z}$ and $\alpha \in \Phi$ we define $s_{\alpha,re}$ to be the reflection which acts on $\mathbb{E}_{h,\ell}$ by

$$s_{\alpha,re}x = x - (\langle x, \alpha \rangle - re)\alpha$$

The group generated by the $s_{\alpha,0}$ with $\alpha \in \Phi$ (respectively $\alpha \in \Phi_0$) is isomorphic to the symmetric group $\mathfrak{S}_{h\ell}$ (respectively to $\mathfrak{S}_f := \mathfrak{S}_h \times \cdots \times \mathfrak{S}_h$), while the group generated by the $s_{\alpha,re}$ with $\alpha \in \Phi$ and $r \in \mathbb{Z}$ is isomorphic to $\widehat{\mathfrak{S}}_{h\ell}$, the affine Weyl group of type $A_{h\ell-1}$. We set $\alpha_0 = \varepsilon_{h\ell} - \varepsilon_1$ and $\Pi = \Delta \cup \{\alpha_0\}$. The elements $S = \{s_{\alpha,0} : \alpha \in \Delta\} \cup \{s_{\alpha_0,e}\}$ generate $\widehat{\mathfrak{S}}_{h\ell}$.

Notation 2.1. *We shall frequently find it convenient to refer to the generators in S in terms of the elements of Π, and will abuse notation in two different ways. First, we will write s_α for $s_{\alpha,0}$ when $\alpha \in \Delta$ and s_{α_0} for $s_{\alpha_0,e}$. This is unambiguous except in the case of the affine reflection $s_{\alpha_0,e}$, where this notation has previously been used for the element $s_{\alpha_0,0}$. As the element $s_{\alpha_0,0}$ will not be referred to hereafter this should not cause confusion. Second, we will write $\alpha = \varepsilon_i - \varepsilon_{i+1}$ in all cases; if $i = h\ell$ then all occurrences of $i + 1$ should be interpreted modulo $h\ell$ to refer to the index 1.*

We shall consider a shifted action of the affine Weyl group $\widehat{\mathfrak{S}}_{h\ell}$ on $\mathbb{E}_{h,l}$ by the element

$$\rho := (\rho_1, \rho_2, \dots, \rho_\ell) \in \mathbb{Z}^{h\ell} \quad \text{where} \quad \rho_i := (\sigma_i + h - 1, \sigma_i + h - 2, \dots, \sigma_i) \in \mathbb{Z}^h,$$

that is, given an element $w \in \widehat{\mathfrak{S}}_{h\ell}$, we set $w \cdot x = w(x+\rho) - \rho$. This shifted action induces a well-defined action on $\overline{\mathbb{E}}_{h,\ell}$; we will define various geometric objects in $\mathbb{E}_{h,\ell}$ in terms of this action, and denote the corresponding objects in the quotient with a bar without further comment. We let $\mathbb{E}(\alpha, re)$ denote the affine hyperplane consisting of the points

$$\mathbb{E}(\alpha, re) = \{x \in \mathbb{E}_{h,\ell} \mid s_{\alpha,re} \cdot x = x\}.$$

Note that our assumption that $\sigma \in I^\ell$ is (h,e)-admissible implies that the origin does not lie on any hyperplane. Given a hyperplane $\mathbb{E}(\alpha, re)$ we remove the hyperplane from $\mathbb{E}_{h,\ell}$ to obtain two distinct subsets $\mathbb{E}^>(\alpha, re)$ and $\mathbb{E}^<(\alpha, re)$ where the origin lies in $\mathbb{E}^<(\alpha, re)$. The connected components of

$$\overline{\mathbb{E}}_{h,\ell} \setminus (\cup_{\alpha\in\Phi_0} \overline{\mathbb{E}}(\alpha, 0))$$

are called chambers. The dominant chamber, denoted $\overline{\mathbb{E}}^+_{h,\ell}$, is defined to be

$$\overline{\mathbb{E}}^+_{h,\ell} = \bigcap_{\alpha\in\Phi_0} \overline{\mathbb{E}}^<(\alpha, 0).$$

The connected components of

$$\overline{\mathbb{E}}_{h,\ell} \setminus (\cup_{\alpha\in\Phi, r\in\mathbb{Z}} \overline{\mathbb{E}}(\alpha, re))$$

are called alcoves, and any such alcove is a fundamental domain for the action of the group $\widehat{\mathfrak{S}}_{h\ell}$ on the set **Alc** of all such alcoves. We define the fundamental alcove A_0 to be the alcove containing the origin (which is inside the dominant chamber). We have a bijection from $\widehat{\mathfrak{S}}_{h\ell}$ to **Alc** given by $w \longmapsto wA_0$. Under this identification **Alc** inherits a right action from the right action of $\widehat{\mathfrak{S}}_{h\ell}$ on itself. Consider the subgroup

$$\mathfrak{S}_f := \mathfrak{S}_h \times \mathfrak{S}_h \times \dots \times \mathfrak{S}_h \leqslant \widehat{\mathfrak{S}}_{h\ell}.$$

The dominant chamber is a fundamental domain for the action of $\mathfrak{S}_f$ on the set of chambers in $\overline{\mathbb{E}}_{h,\ell}$. We let $\mathfrak{S}^f$ denote the set of minimal length representatives for right cosets $\mathfrak{S}_f \backslash \widehat{\mathfrak{S}}_{h\ell}$. So multiplication gives a bijection $\mathfrak{S}_f \times \mathfrak{S}^f \to \widehat{\mathfrak{S}}_{h\ell}$. This induces a bijection between right cosets and the alcoves in our dominant chamber. Under this identification, alcoves are partially ordered by the Bruhat-ordering on $\mathfrak{S}^f$ which is a coarsening of the order $\succeq_{\mathrm{opp}}$.

If the intersection of a hyperplane $\overline{\mathbb{E}}(\alpha, re)$ with the closure of an alcove A is generically of codimension one in $\overline{\mathbb{E}}_{h,\ell}$ then we call this intersection a wall of A. The fundamental alcove A_0 has walls corresponding to $\overline{\mathbb{E}}(\alpha, 0)$ with $\alpha \in \Delta$ together with an affine wall $\overline{\mathbb{E}}(\alpha_0, e)$. We will usually just write $\overline{\mathbb{E}}(\alpha)$ for the walls $\overline{\mathbb{E}}(\alpha, 0)$ (when $\alpha \in \Delta$) and $\overline{\mathbb{E}}(\alpha, e)$ (when $\alpha = \alpha_0$). We regard each of these walls as being labelled by a distinct colour (and assign the same colour to the corresponding element of S). Under the action of $\widehat{\mathfrak{S}}_{h\ell}$ each wall of a given alcove A is in the orbit of a unique wall of A_0, and thus inherits a colour from that wall. We will sometimes use the right action of $\widehat{\mathfrak{S}}_{h\ell}$ on **Alc**. Given an alcove A and an element $s \in S$, the alcove As is obtained by reflecting A in the wall of A with colour corresponding to the colour of s. With this observation it is now easy to see that if $w = s_1 \dots s_t$ where the s_i are in S then wA_0 is the alcove obtained from A_0 by successively reflecting through the walls corresponding to s_1 up to s_t. We will call a multipartition regular if its image in $\overline{\mathbb{E}}_{h,l}$ lies in some alcove; those multipartitions whose images lies on one or more walls will be called singular.

2.2. **Paths in the geometry.** We now develop a path combinatorics inside our geometry. Given a map $p : \{1, \dots, n\} \to \{1, \dots, h\ell\}$ we define points $\mathsf{P}(k) \in \mathbb{E}_{h,\ell}$ by

$$\mathsf{P}(k) = \sum_{1\leqslant i\leqslant k} \varepsilon_{p(i)}$$

for $1 \leqslant i \leqslant n$. We define the associated path of length n by

$$\mathsf{P} = (\varnothing = \mathsf{P}(0), \mathsf{P}(1), \mathsf{P}(2), \dots, \mathsf{P}(n))$$

and we say that the path has shape $\pi = \mathsf{P}(n) \in \mathbb{E}_{h,\ell}$. We also denote this path by

$$\mathsf{P} = (\varepsilon_{p(1)}, \dots, \varepsilon_{p(n)}).$$

Given $\lambda \in \mathcal{C}_{h,\ell}(n)$ we let $\mathrm{Path}(\lambda)$ denote the set of paths of length n with shape λ. We define $\mathrm{Path}_{h,\ell}(\lambda)$ to be the subset of $\mathrm{Path}(\lambda)$ consisting of those paths lying entirely inside the dominant chamber; i.e. those P such that $\mathsf{P}(i)$ is dominant for all $0 \leqslant i \leqslant n$. We let $\mathrm{Path}_{h,\ell}(n) = \cup_{\lambda \in \mathscr{P}_{h,\ell}(n)} \mathrm{Path}_{h,\ell}(\lambda)$.

Given a path T defined by such a map p of length n and shape λ we can write each $p(j)$ uniquely in the form $p(j) = hm_j + c_j$ where $0 \leqslant m_j \leqslant l-1$ and $1 \leqslant c_j \leqslant h$. We record these elements in a tableau of shape λ^T by induction on j, where we place the positive integer j in the first empty box in the c_jth column of component m_j. By definition, such a tableau will have entries increasing down columns; if λ is a multipartition then the entries also increase along rows if and only if the given path is in $\mathrm{Path}_{h,\ell}(\lambda)$, and hence there is a bijection between $\mathrm{Path}_{h,\ell}(\lambda)$ and $\mathrm{Std}(\lambda)$. For this reason we will sometimes refer to paths as tableaux, to emphasise that what we are doing is generalising the classical tableaux combinatorics for the symmetric group.

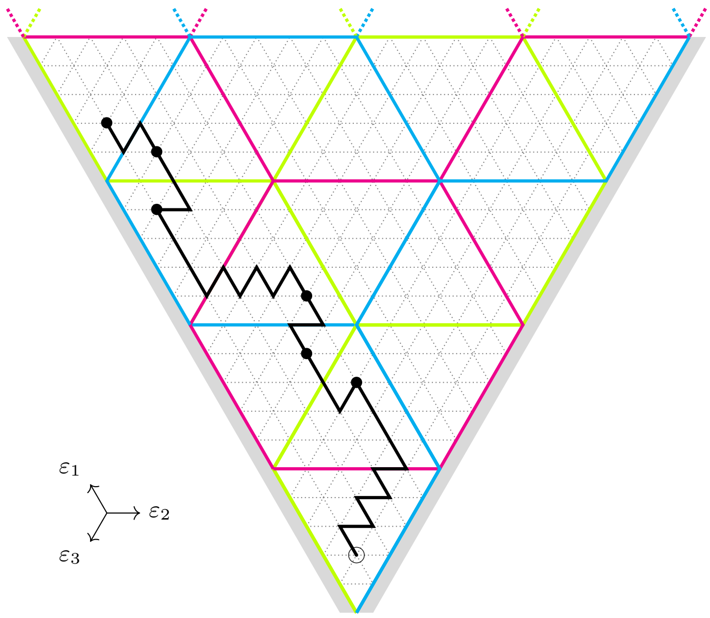
Example 2.2. *Let $h = 2$ and $\ell = 3$ and set $\kappa = (0, 3, 6) \in (\mathbb{Z}/9\mathbb{Z})^3$. For $\lambda = ((2, 1), (2, 1), (1^3))$, the standard λ-tableaux of Example 1.7 correspond to the paths*

$$\mathsf{T}_\lambda = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_1, \varepsilon_3, \varepsilon_5)$$
$$\mathsf{S} = (\varepsilon_1, \varepsilon_1, \varepsilon_3, \varepsilon_5, \varepsilon_4, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_1, \varepsilon_3, \varepsilon_5)$$

Given a path P we define

$$\mathrm{res}(\mathsf{P}) = (\mathrm{res}_{\mathsf{P}}(1), \dots, \mathrm{res}_{\mathsf{P}}(n))$$

where $\mathrm{res}_{\mathsf{P}}(i)$ denotes the residue of the box labelled by i in the tableau corresponding to P.



1	2	10
3	4	17
5	6	19
7	14	20
8	25	29
9		
11		
12		
13		
15		
16		
18		
21		
22		
23		
24		
26		
27		
28		
30		

FIGURE 12. An alcove path in $\overline{\mathbb{E}}^{+}_{3,1}$ in $\mathrm{Path}_{h,\ell}(3^5, 1^{15})$ and the corresponding tableau in $\mathrm{Std}(3^5, 1^{15})$. The black vertices denote vertices on the path in the orbit of the origin.

Given paths $\mathsf{P} = (\varepsilon_{p(1)}, \dots, \varepsilon_{p(n)})$ and $\mathsf{Q} = (\varepsilon_{q(1)}, \dots, \varepsilon_{q(n)})$ we say that $\mathsf{P} \sim \mathsf{Q}$ if there exists an $\alpha = \varepsilon_{ah+i} - \varepsilon_{bh+j} \in \Phi$ and $r \in \mathbb{Z}$ and $s \leqslant n$ such that

$$\mathsf{P}(s) \in \mathbb{E}(\alpha, re) \qquad \text{and} \qquad \varepsilon_{q(t)} = \begin{cases} \varepsilon_{p(t)} & \text{for } 1 \leqslant t \leqslant s \\ s_\alpha \varepsilon_{p(t)} & \text{for } s+1 \leqslant t \leqslant n. \end{cases}$$

In other words the paths P and Q agree up to some point $\mathsf{P}(s) = \mathsf{Q}(s)$ which lies on $\mathbb{E}(\alpha, re)$, after which each $\mathsf{Q}(t)$ is obtained from $\mathsf{P}(t)$ by reflection in $\mathbb{E}(\alpha, re)$. We extend $\sim$ by transitivity to give an equivalence relation on paths, and say that two paths in the same equivalence class are related by a series of wall reflections of paths and given $\mathsf{S} \in \mathrm{Path}_{h,\ell}(n)$ we set $[\mathsf{S}] = \{\mathsf{T} \in \mathrm{Path}_{h,\ell}(n) \mid \mathsf{S} \sim \mathsf{T}\}$.

We recast the degree of a tableau in the path-theoretic setting as follows.

Definition 2.3. *Given a path* $\mathsf{S} = (\mathsf{S}(0), \mathsf{S}(1), \mathsf{S}(2), \ldots, \mathsf{S}(n))$ *we set* $\deg(\mathsf{S}(0)) = 0$ *and define*

$$\deg(\mathsf{S}) = \sum_{1 \leqslant k \leqslant n} d(\mathsf{S}(k), \mathsf{S}(k-1)),$$

where $d(\mathsf{S}(k), \mathsf{S}(k-1))$ *is defined as follows. For* $\alpha \in \Phi$ *we set* $d_\alpha(\mathsf{S}(k), \mathsf{S}(k-1))$ *to be*

- $+1$ *if* $\mathsf{S}(k-1) \in \mathbb{E}(\alpha, re)$ *and* $\mathsf{S}(k) \in \mathbb{E}^{<}(\alpha, re)$;
- -1 *if* $\mathsf{S}(k-1) \in \mathbb{E}^{>}(\alpha, re)$ *and* $\mathsf{S}(k) \in \mathbb{E}(\alpha, re)$;
- 0 *otherwise.*

We let

$$\deg(\mathsf{S}) = \sum_{1 \leqslant k \leqslant n} \sum_{\alpha \in \Phi} d_\alpha(\mathsf{S}(k-1), \mathsf{S}(k)).$$

We say that $\mathsf{P} = (\varepsilon_{p(1)}, \ldots, \varepsilon_{p(n)})$ *is a* reduced path *if* $d_\alpha(\mathsf{P}(k-1), \mathsf{P}(k)) = 0$ *for* $1 \leqslant k \leqslant n$ *and* $\alpha \in \Pi$.

This definition of a reduced path is easily seen to be equivalent to that of [BCH, Section 2.3].

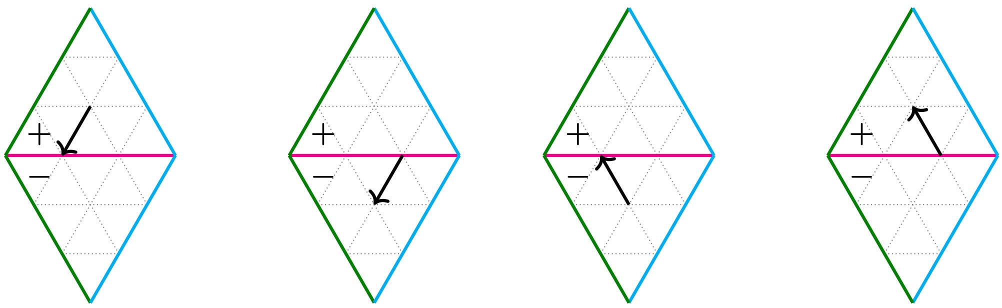


FIGURE 13. The first and second paths have degrees -1 and $+1$ respectively. The third and fourth paths have degree 0.

There exist a unique reduced path in each $\sim$-equivalence class (and, of course, each reduced path belongs to some $\sim$-equivalence class and so $\sim$-classes and reduced paths are in bijection). We remark that T^μ, the maximal path in the reverse lexicographic ordering $\succ$, is an example of a reduced path. Given $\mathsf{S} \in \mathrm{Path}_{h,\ell}(n)$, we let $\min[\mathsf{S}]$ denote the minimal path in the $\sim$-equivalence class containing S. Given a reduced path $\mathsf{P}_\lambda \in \mathrm{Path}_{h,\ell}(\lambda)$, we have that

$$\mathscr{H}_n^\sigma e_{\mathsf{P}_\lambda} = \mathbf{P}(\lambda) \oplus \bigoplus_{\mu \succ \lambda} k^\mu_{\mathsf{P}_\lambda} \mathbf{P}(\mu)$$

decomposes (in a unitriangular fashion) as a sum of projective indecomposable modules for some generalised p-Kostka coefficients $k^\mu_{\mathsf{P}_\lambda} \in \Bbbk$. In general, we do not have an isomorphism

$$\mathscr{H}_n^\sigma e_{\mathsf{P}_\lambda} \ncong \mathscr{H}_n^\sigma e_{\mathsf{Q}_\lambda}$$

for reduced paths $\mathsf{P}_\lambda, \mathsf{Q}_\lambda \in \mathrm{Path}_{h,\ell}(\lambda)$ and so the choice of reduced path does matter. (This is not surprising, the auxiliary steps in Soergel's algorithm for calculating Kazhdan–Lusztig polynomials produces a different pattern depending on the choice of reduced expression.) However, they do agree modulo lexicographically larger terms, as we shall soon see (and indeed, after the cancellations in Soergel's algorithm one obtains that the Kazhdan–Lusztig polynomials are independent of choices of reduced expressions).

Lemma 2.4. *Given* $\lambda \in \mathscr{P}_{h,\ell}(n)$, *let* P_λ, Q_λ, S_λ *be any triple of reduced paths in* $\mathrm{Path}_{h,\ell}(\lambda)$. *The element* e_{P_λ} *generates* $\mathcal{H}^{\succeq\lambda}/\mathcal{H}^{\succ\lambda}$ *and moreover*

$$\psi^{\mathsf{P}_\lambda}_{\mathsf{Q}_\lambda} \psi^{\mathsf{Q}_\lambda}_{\mathsf{S}_\lambda} = k \psi^{\mathsf{P}_\lambda}_{\mathsf{S}_\lambda} + \mathcal{H}^{\succ\lambda}$$

for $k \in \Bbbk \setminus \{0\}$.

Proof. We have already verified the claim in the case that $\mathsf{P}_\lambda = \mathsf{T}_\lambda$ for each $\lambda \in \mathscr{P}_{h,\ell}(n)$. Note that

$$e_{\mathsf{P}_\lambda} \mathbf{S}_{\Bbbk}(\nu) \neq 0 \text{ implies } \nu \rhd \lambda \text{ or } \nu = \lambda \text{ and } e_{\mathsf{P}_\lambda} \mathbf{S}(\lambda) = e_{\mathsf{P}_\lambda} \mathbf{D}_{\Bbbk}(\lambda) = \psi^{\mathsf{P}_\lambda}_{\mathsf{T}_\lambda} \tag{2.1}$$

for P_λ a reduced path in $\mathrm{Path}_{h,\ell}(\lambda)$. This implies that $e_{\mathsf{P}_\lambda} \in \mathcal{H}^{\succeq\lambda}/\mathcal{H}^{\succ\lambda}$ and therefore generates $\mathcal{H}^{\succeq\lambda}/\mathcal{H}^{\succ\lambda}$ and belongs to the simple head of the Specht module; the result follows. □

Definition 2.5. *Given two paths*

$$\mathsf{P} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_p}) \in \mathrm{Path}(\mu) \quad \textit{and} \quad \mathsf{Q} = (\varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_q}) \in \mathrm{Path}(\nu)$$

we define the naive concatenated path

$$\mathsf{P} \boxtimes \mathsf{Q} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_p}, \varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_q}) \in \mathrm{Path}(\mu + \nu).$$

2.3. **Branching coefficients.** We now discuss how one can think of a permutation as a morphism between pairs of paths in the alcove geometries of Subsection 2.1.

Definition 2.6. *Let $\lambda \in \mathcal{C}_{h,\ell}(n)$. Given a pair of paths $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda)$ we write the steps in S and T in sequence along the top and bottom edges of a frame, respectively. We define $w^{\mathsf{S}}_{\mathsf{T}} \in \mathfrak{S}_n$ to be the unique step-preserving permutation with the minimal number of crossings.*

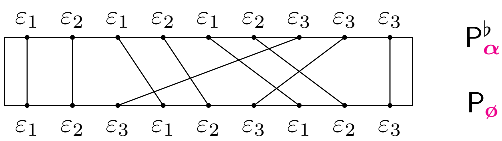
In the following (running) example we label our paths by $\mathsf{P}_{\varnothing} (= \mathsf{T}_{(3^3)})$ and $\mathsf{P}^\flat_{\alpha}$. For this section, we do not need to know what inspires this notation; however, all will become clear in Section 3.

Example 2.7. *We consider $\Bbbk\mathfrak{S}_9$ in the case of $p = 5$. We set $\alpha = \varepsilon_3 - \varepsilon_1 \in \Pi$. Here we have*

$$\mathsf{P}_{\varnothing} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \qquad \textit{and} \qquad \mathsf{P}^\flat_{\alpha} = (\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3, \varepsilon_3)$$

are two examples of paths of shape (3^3). The unique step-preserving permutation of minimal length is given by

$w^{\mathsf{P}^\flat_{\alpha}}_{\mathsf{P}_{\varnothing}} =$



(2.2)

Notice that if two strands have the same step-label, then they do not cross. This is, of course, exactly what it means for a step-preserving permutation to be of minimal length.

Definition 2.8. *Fix (S, T) an ordered pair of paths which both terminate at some point $\lambda \in \mathscr{P}_{h,\ell}(n)$. We now inductively construct a reduced expression for $w^{\mathsf{S}}_{\mathsf{T}}$. We define the branching coefficients*

$$d_p(\mathsf{S}, \mathsf{T}) = w^p_q \qquad \textit{where } q = |\{1 \leqslant i \leqslant p \mid w(i) \leqslant w(p)\}|.$$

These allow us to fix a distinguished reduced expression, $\underline{w}^{\mathsf{S}}_{\mathsf{T}}$, for $w^{\mathsf{S}}_{\mathsf{T}}$ as follows,

$$\underline{w}^{\mathsf{S}}_{\mathsf{T}} = d_1(\mathsf{S}, \mathsf{T}) \ldots d_n(\mathsf{S}, \mathsf{T}).$$

Example 2.9. *We continue to consider $\Bbbk\mathfrak{S}_9$ in the case of $p = 5$. We have that*

$$1_{\mathfrak{S}_9} = d_p(\mathsf{P}_{\varnothing}, \mathsf{P}^\flat_{\alpha})$$

for each $p = 1, 2, 3, 4, 5, 6, 9$ because $w^{\mathsf{P}_{\varnothing}}_{\mathsf{P}^\flat_{\alpha}}(p) \geqslant i$ for all $1 \leqslant i < p$. We have that

$$d_7(\mathsf{P}_{\varnothing}, \mathsf{P}^\flat_{\alpha}) = w^7_3 \quad d_8(\mathsf{P}_{\varnothing}, \mathsf{P}^\flat_{\alpha}) = w^8_6$$

and so our reduced word is depicted in Figure 14.

We can think of the branching coefficients as "one step morphisms" which allow us to mutate the path S into T via a series of n steps (as each branching coefficient moves the position of one step in the path) and so this mutation proceeds via $n + 1$ paths

$$\mathsf{S} = \mathsf{S}_0, \mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_n = \mathsf{T}$$

see Figure 14 for an example. We now lift these branching coefficients to the KLR algebra.

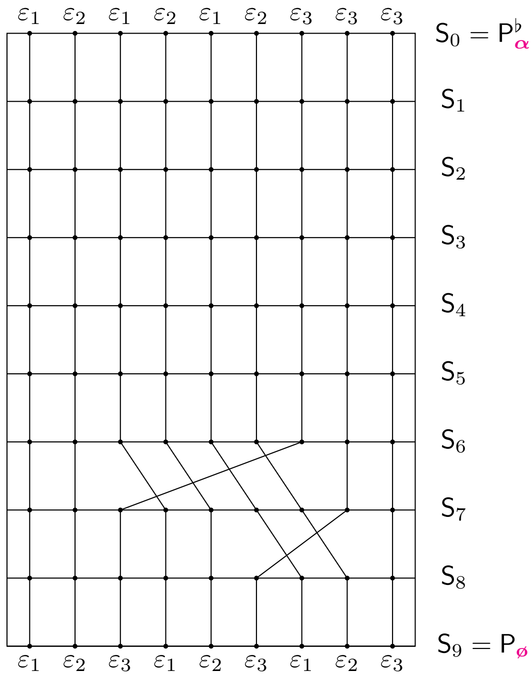


FIGURE 14. The reduced word, $\underline{w}^{\mathsf{P}^\flat_{\alpha}}_{\mathsf{P}_{\varnothing}}$ (see also Examples 2.7 and 2.9).

Definition 2.10. *Given a path* $\mathsf{S} \in \mathrm{Path}(\lambda)$ *we let* e_{S} *denote the KLR idempotent whose residue sequence is given by* $\mathsf{res}(\mathsf{S})$. *Given a pair of paths* $\mathsf{S}, \mathsf{T} \in \mathrm{Path}(\lambda)$, *we define*

$$\psi_p(\mathsf{S}, \mathsf{T}) = \psi_{d_p(\mathsf{S},\mathsf{T})}$$

for $1 \leqslant p \leqslant n$ *and we set*

$$\psi^{\mathsf{S}}_{\mathsf{T}} = e_{\mathsf{S}} \psi_{\underline{w}^{\mathsf{S}}_{\mathsf{T}}} e_{\mathsf{T}} = e_{\mathsf{S}} \psi_1(\mathsf{S},\mathsf{T})\psi_2(\mathsf{S},\mathsf{T}) \dots \psi_n(\mathsf{S},\mathsf{T}) e_{\mathsf{T}}.$$

Now, let's momentarily restrict our attention to pairs of paths of the form $(\mathsf{S}, \mathsf{T}_\lambda)$. In this case, the branching coefficients actually come from the "branching rule" for restriction along the tower $\dots \subset \mathscr{H}^\sigma_{n-1} \subset \mathscr{H}^\sigma_n \subset \dots$. To see this, we note that

$$\underline{w}^{\mathsf{S}}_{\mathsf{T}_\lambda} = w_1(\mathsf{S},\mathsf{T}_\lambda) w_2(\mathsf{S},\mathsf{T}_\lambda) \dots w_n(\mathsf{S},\mathsf{T}_\lambda)$$

where $w_n(\mathsf{S},\mathsf{T}_\lambda) = w^n_{\mathsf{T}_\lambda(\square)}$ for some removable box $\square \in \mathrm{Rem}(\lambda)$ and where

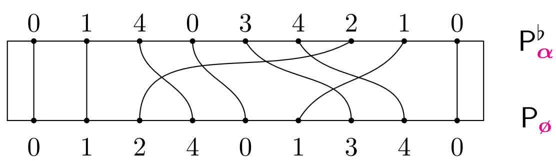
$$w_1(\mathsf{S},\mathsf{T}_\lambda) w_2(\mathsf{S},\mathsf{T}_\lambda) \dots w_{n-1}(\mathsf{S},\mathsf{T}_\lambda) = \underline{w}(\mathsf{S}_{\leqslant n-1}, \mathsf{T}_{\lambda-\square}) \in \mathfrak{S}_{n-1} \leqslant \mathfrak{S}_n. \tag{2.3}$$

By Proposition 1.22, we have that

$$\mathbf{S}_{\Bbbk}(\lambda - \square)\langle \deg(A_r) \rangle \cong \Bbbk\{\psi^{\mathsf{S}}_{\mathsf{T}_\lambda} \mid \mathrm{Shape}(\mathsf{S}_{\leqslant n-1}) = \lambda - \square\}. \tag{2.4}$$

Thus the branching coefficients above provide a factorisation of the cellular basis of Theorem 1.20 which is compatible with the restriction rule.

Example 2.11. *Continuing with Examples 2.7 and 2.9, the lift of the path-morphism to the KLR algebra is as follows,*

$$\psi^{\mathsf{P}^\flat_{\alpha}}_{\mathsf{P}_{\varnothing}} =$$


At each step in the restriction along the tower, there is precisely one removable box of any given residue and so the restriction is, in fact, a direct sum of Specht modules.

We wish to modify the branching coefficients above so that we can consider more general (families of) reduced paths P_λ in place of the path T_λ. Given $\mathsf{S} \in \mathrm{Path}_{h,\ell}(\lambda)$, we can choose a reduced path vector as follows

$$\underline{\mathsf{P}}_{\mathsf{S}} = (\mathsf{P}_{\mathsf{S},0}, \mathsf{P}_{\mathsf{S},1}, \dots, \mathsf{P}_{\mathsf{S},n})$$

such that $\mathrm{Shape}(\mathsf{P}_{\mathsf{S},k}) = \mathrm{Shape}(\mathsf{S}_{\leqslant k})$ for each $0 \leqslant k \leqslant n$. In other words, we choose a reduced path $\mathsf{P}_{\mathsf{S},p}$ for each and every point in the path S. For $0 \leqslant p \leqslant n$ and $\mathrm{Shape}(\mathsf{S}_{\leqslant p}) + \varepsilon_{i_p} = \mathrm{Shape}(\mathsf{S}_{\leqslant p+1})$, we define the modified branching coefficient,

$$d_p(\mathsf{S}, \underline{\mathsf{P}}_{\mathsf{S}}) = \psi_{\mathsf{P}_{\mathsf{S},p}}^{\mathsf{P}_{\mathsf{S},p-1} \boxtimes \mathsf{P}_{i_p}}$$

and we hence define

$$\psi^{\mathsf{S}}_{\underline{\mathsf{P}}_\lambda} = \prod_{1 \leqslant p \leqslant n} d_p(\mathsf{S}, \underline{\mathsf{P}}_{\mathsf{S}}).$$

This element is *not* uniquely defined as there are many choices of reduced word for each $w_{\mathsf{P}_{\mathsf{S},p}}^{\mathsf{P}_{\mathsf{S},p-1} \boxtimes \mathsf{P}_{i_p}}$ labelling a multiplicand in the product. We choose to fix these expression as in Definition 2.8, but any other choice will also work.

Remark 2.12. *For symmetric groups there is a canonical choice of reduced path vector coming from the coset-like combinatorics which has historically been used for studying these groups. For the light leaves construction of Bott–Samelson endomorphism algebras, Libedinsky and Elias–Williamson require very different families of reduced path vectors whose origin can be seen as coming from a basis which can be written in terms of their 2-generators* [Lib08, EW16].

Example 2.13. *Continuing with Example 2.11 we have already noted that $\mathsf{P}_{\varnothing} = \mathsf{T}_{(3^3)}$. The reduced path vector in this example is given by the sequence T_μ for $\mu = \mathrm{Shape}((\mathsf{P}^\flat_{\alpha})_{\leqslant k})$ for $k \geqslant 0$. We record this in tableaux format to help the reader transition between the old and new ways of thinking.*

$$\left(\varnothing,\ \begin{array}{|c|}\hline 1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6&7\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7&8\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7&8&9\\\hline\end{array} \right)$$

The light leaves basis will be given in terms of products $\psi^{\mathsf{S}}_{\underline{\mathsf{P}}_{\mathsf{S}}} \psi^{\underline{\mathsf{P}}_{\mathsf{T}}}_{\mathsf{T}}$ for $\mathsf{S}, \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda)$ and "compatible choices" of $\underline{\mathsf{P}}_{\mathsf{S}}$ and $\underline{\mathsf{P}}_{\mathsf{T}}$. Here the only condition for compatibility is that $\mathsf{P}_{\mathsf{S},n} = \mathsf{Q}_\lambda = \mathsf{P}_{\mathsf{T},n}$ for some fixed choice of reduced path $\mathsf{Q}_\lambda \in \mathrm{Path}_{h,\ell}(\lambda)$, in other words the final choices of reduced path for each of S and T coincide. We remark that if $\mathsf{P}_{\mathsf{S},n} \neq \mathsf{P}_{\mathsf{T},n}$ then the product is clearly equal to zero (by idempotent considerations) and so this is the only sensible choice to make for such a product. In light of the above, we let Q_λ be a reduced path and we say that a reduced path vector $\underline{\mathsf{P}}_{\mathsf{S}}$ terminates at Q_λ if $\underline{\mathsf{P}}_{\mathsf{S},n} = \mathsf{Q}_\lambda$.

Theorem 2.14 (The light leaves basis)**.** *For each $\lambda \in \mathscr{P}_{h,\ell}(n)$ we fix a reduced path $\mathsf{Q}_\lambda \in \mathrm{Path}_{h,\ell}(\lambda)$ and for each $\mathsf{S} \in \mathrm{Path}_{h,\ell}(\lambda)$, we fix an associated reduced path vector $\underline{\mathsf{P}}_{\mathsf{S}}$ terminating with Q_λ. The $\Bbbk$-algebra $\mathscr{H}^\sigma_n$ is a graded cellular algebra with basis*

$$\{\psi^{\mathsf{S}}_{\underline{\mathsf{P}}_{\mathsf{S}}} \psi^{\underline{\mathsf{P}}_{\mathsf{T}}}_{\mathsf{T}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda), \lambda \in \mathscr{P}_{h,\ell}(n)\}$$

anti-involution $*$ and the degree function $\mathsf{deg} : \mathrm{Path}_{h,\ell} \to \mathbb{Z}$.

Proof. By Theorem 1.20, we have that

$$\{\psi^{\mathsf{S}}_{\mathsf{T}_\lambda} \psi^{\mathsf{T}_\lambda}_{\mathsf{T}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda)\}$$

provides a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$. By Lemma 2.4, we have that $\psi^{\mathsf{T}_\lambda}_{\mathsf{Q}_\lambda} e_{\mathsf{Q}_\lambda} \psi^{\mathsf{Q}_\lambda}_{\mathsf{T}_\lambda} = k e_{\mathsf{T}_\lambda}$ for $k \in \Bbbk \setminus \{0\}$ modulo lexicographically higher terms and so

$$\{\psi^{\mathsf{S}}_{\mathsf{T}_\lambda} (\psi^{\mathsf{T}_\lambda}_{\mathsf{Q}_\lambda} \psi^{\mathsf{Q}_\lambda}_{\mathsf{T}_\lambda}) \psi^{\mathsf{T}_\lambda}_{\mathsf{T}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda)\}$$

provides a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$. By equation (2.3) and Subsection 2.3, we have that

$$\{\psi^{\mathsf{s}}_{\mathsf{T}_{\lambda-\varepsilon_i}} \psi^{\mathsf{T}_{\lambda-\varepsilon_i} \boxtimes \mathsf{P}_i}_{\mathsf{T}_\lambda} (\psi^{\mathsf{T}_\lambda}_{\mathsf{Q}_\lambda} \psi^{\mathsf{Q}_\lambda}_{\mathsf{T}_\lambda}) \psi^{\mathsf{T}_\lambda}_{\mathsf{T}} \mid \mathsf{s} \in \mathrm{Std}(\lambda - \varepsilon_i), \varepsilon_i \in \mathrm{Rem}(\lambda), \mathsf{T} \in \mathrm{Path}_{h,\ell}(\lambda)\} \tag{2.5}$$

provides a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$. By Proposition 1.22, we have that

$$\psi^{\mathsf{T}_{\lambda-\varepsilon_i} \boxtimes \mathsf{P}_i}_{\mathsf{T}_\lambda} e_{\mathsf{T}_\lambda}$$

generates a left subquotient of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$ which is isomorphic to $\mathbf{S}_{\Bbbk}(\lambda-\varepsilon_i)$. Now, for each pair $1 \leqslant i \leqslant |\mathrm{Rem}(\lambda)|$ and $\mathsf{s} \in \mathrm{Std}(\lambda-\varepsilon_i)$, we fix a corresponding choice of reduced path $\mathsf{P}_{\mathsf{S},n-1} \in \mathrm{Path}_{h,\ell}(\lambda-\varepsilon_i)$. By Lemma 2.4 and Proposition 1.22, we have that

$$\{\psi^{\mathsf{s}}_{\mathsf{T}_{\lambda-\varepsilon_i}}(\psi^{\mathsf{T}_{\lambda-\varepsilon_i}\boxtimes\mathsf{P}_i}_{\mathsf{P}_{\mathsf{S},n-1}\boxtimes\mathsf{P}_i}\psi^{\mathsf{P}_{\mathsf{S},n-1}\boxtimes\mathsf{P}_i}_{\mathsf{T}_{\lambda-\varepsilon_i}\boxtimes\mathsf{P}_i})\psi^{\mathsf{T}_{\lambda-\varepsilon_i}\boxtimes\mathsf{P}_i}_{\mathsf{T}_\lambda}(\psi^{\mathsf{T}_\lambda}_{\mathsf{Q}_\lambda}\psi^{\mathsf{Q}_\lambda}_{\mathsf{T}_\lambda})\psi^{\mathsf{T}_\lambda}_{\mathsf{T}} \mid \mathsf{s}\in\mathrm{Path}_{h,\ell}(\lambda-\varepsilon_i), \varepsilon_i\in\mathrm{Rem}(\lambda), \mathsf{T}\in\mathrm{Path}_{h,\ell}(\lambda)\}$$

provides a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$. Re-bracketing the above, we have that

$$\{(\psi^{\mathsf{s}}_{\mathsf{T}_{\lambda-\varepsilon_i}}\psi^{\mathsf{T}_{\lambda-\varepsilon_i}}_{\mathsf{P}_{\mathsf{S},n-1}})(\psi^{\mathsf{P}_{\mathsf{S},n-1}\boxtimes\mathsf{P}_i}_{\mathsf{T}_{\lambda-\varepsilon_i}\boxtimes\mathsf{P}_i}\psi^{\mathsf{T}_{\lambda-\varepsilon_i}\boxtimes\mathsf{P}_i}_{\mathsf{T}_\lambda}\psi^{\mathsf{T}_\lambda}_{\mathsf{Q}_\lambda})(\psi^{\mathsf{Q}_\lambda}_{\mathsf{T}_\lambda}\psi^{\mathsf{T}_\lambda}_{\mathsf{T}}) \mid \mathsf{s}\boxtimes\mathsf{P}_i\in\mathrm{Path}_{h,\ell}(\lambda-\varepsilon_i), \varepsilon_i\in\mathrm{Rem}(\lambda), \mathsf{T}\in\mathrm{Path}_{h,\ell}(\lambda)\}$$

provides a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$. Finally, simplifying using Proposition 1.19 we obtain that

$$\{\psi^{\mathsf{s}}_{\mathsf{P}_{\mathsf{S},n-1}}\psi^{\mathsf{P}_{\mathsf{S},n-1}\boxtimes\mathsf{P}_i}_{\mathsf{Q}_\lambda}\psi^{\mathsf{Q}_\lambda}_{\mathsf{T}} \mid \mathsf{s}\boxtimes\mathsf{P}_i\in\mathrm{Path}_{h,\ell}(\lambda-\varepsilon_i), \varepsilon_i\in\mathrm{Rem}(\lambda), \mathsf{T}\in\mathrm{Path}_{h,\ell}(\lambda)\}$$

is a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$ where we note that the middle term in the KLR-product is our modified branching coefficient. Repeating n times, we have that

$$\{\psi^{\mathsf{P}_{\mathsf{S},0}\boxtimes\mathsf{P}_{i_1}}_{\mathsf{P}_{\mathsf{S},1}}\dots\psi^{\mathsf{P}_{\mathsf{S},n-2}\boxtimes\mathsf{P}_{i_{n-1}}}_{\mathsf{P}_{\mathsf{S},n-1}}\psi^{\mathsf{P}_{\mathsf{S},n-1}\boxtimes\mathsf{P}_{i_n}}_{\mathsf{Q}_\lambda}\psi^{\mathsf{Q}_\lambda}_{\mathsf{T}} \mid \mathsf{S},\mathsf{T}\in\mathrm{Path}_{h,\ell}(\lambda)\}$$

is a $\mathbb{Z}$-basis of $\mathscr{H}^{\succeq\lambda}/\mathscr{H}^{\succ\lambda}$; repeating the above for the righthand-side, the result follows. □

In particular, we can set $\underline{\mathsf{P}}_{\mathsf{S}} = (\mathsf{Q}_\lambda{\downarrow_{\leqslant k}})_{k\geqslant 0}$ and obtain the following corollary, which specialises to Theorem 1.20 for $\mathsf{Q}_\lambda = \mathsf{T}_\lambda$.

Corollary 2.15. *For each $\lambda\in\mathscr{P}_{h,\ell}(n)$ we fix a reduced path $\mathsf{Q}_\lambda\in\mathrm{Path}_{h,\ell}(\lambda)$. The $\Bbbk$-algebra $\mathscr{H}^\sigma_n$ is a graded cellular algebra with basis*

$$\{\psi^{\mathsf{S}}_{\mathsf{Q}_\lambda}\psi^{\mathsf{Q}_\lambda}_{\mathsf{T}} \mid \mathsf{S},\mathsf{T}\in\mathrm{Path}_{h,\ell}(\lambda), \lambda\in\mathscr{P}_{h,\ell}(n)\}$$

anti-involution $*$ and the degree function $\mathsf{deg}:\mathrm{Path}_{h,\ell}\to\mathbb{Z}$.

3. Light leaf generators for the principal block

We now restrict our attention to the principal block and illustrate how the constructions of previous sections specialise to be familiar ideas from Soergel diagrammatics. In particular, we provide an exact analogue of Libedinsky's and Elias–Williamson's algorithmic construction of a light leaves basis for such blocks. In order to do this, we provide a short list of path-morphisms which we will show generate the algebra $\mathsf{f}_{n,\sigma}(\mathcal{H}^\sigma_n/\mathcal{H}^\sigma_n\mathsf{e}_h\mathcal{H}^\sigma_n)\mathsf{f}_{n,\sigma}$ (thus proving Theorem B).

3.1. **Alcove paths.** When passing from multicompositions to our geometry $\overline{\mathbb{E}}_{h,l}$, many non-trivial elements map to the origin. One such element is $\delta=((h),\dots,(h))\in\mathscr{P}_{h,\ell}(h\ell)$. (Recall our transpose convention for embedding multipartitions into our geometry.) We will sometimes refer to this as the $\mathsf{determinant}$ as (for the symmetric group) it corresponds to the determinant representation of the associated general linear group. We will also need to consider elements corresponding to powers of the determinant, namely $\delta_n=((h^n),\dots,(h^n))\in\mathscr{P}_{h,\ell}(nh\ell)$. We now restrict our attention to paths between points in the principal linkage class, in other words to paths between points in $\widehat{\mathfrak{S}}_{h\ell}\cdot 0$. Such points can be represented by multicompositions μ in $\widehat{\mathfrak{S}}_{h\ell}\cdot\delta_n$ for some choice of n.

Definition 3.1. *We will associate alcove paths to certain words in the* $\mathsf{alphabet}$

$$S\cup\{1\}=\{s_\alpha \mid \alpha\in\Pi\cup\{\emptyset\}\}$$

where $s_\emptyset=1$. That is, we will consider words in the generators of the affine Weyl group, but enriched with explicit occurrences of the identity in these expressions. We refer to the number of elements in such an expression (including the occurrences of the identity) as the degree *of this expression.*

Given a path P between points in the principal linkage class, the end point lies in the interior of an alcove of the form wA_0 for some $w\in\widehat{\mathfrak{S}}_{h\ell}$. If we write w as a word in our alphabet, and then replace each element s_α by the corresponding non-affine reflection s_α in $\mathfrak{S}_{h\ell}$ to form the element $\overline{w}\in\mathfrak{S}_{h\ell}$ then the basis vectors ε_i are permuted by the corresponding action of $\overline{w}$ to give $\varepsilon_{\overline{w}(i)}$, and there is an isomorphism from $\overline{\mathbb{E}}_{h,l}$ to itself which maps A_0 to wA_0 such that 0 maps to $w\cdot 0$, coloured walls map to walls of the same colour, and each basis element ε_i map to $\varepsilon_{\overline{w}(i)}$. Under this map we can transform a path Q starting at the origin to a path starting at $w\cdot 0$ which passes through the same sequence of coloured walls as Q does.

Definition 3.2. *Given two paths* $\mathsf{P} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_p}) \in \mathrm{Path}(\mu)$ *and* $\mathsf{Q} = (\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_q}) \in \mathrm{Path}(\nu)$ *with the endpoint of* P *lying in the closure of some alcove* wA_0 *we define the* contextualised concatenated path

$$\mathsf{P} \otimes_w \mathsf{Q} = (\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_p}) \boxtimes (\varepsilon_{\overline{w}(j_1)}, \varepsilon_{\overline{w}(j_2)}, \dots, \varepsilon_{\overline{w}(j_q)}) \in \mathrm{Path}(\mu + (w \cdot \nu)).$$

If there is a unique such w *then we may simply write* $\mathsf{P} \otimes \mathsf{Q}$*. If* $w = s_{\alpha}$ *we will simply write* $\mathsf{P} \otimes_{\alpha} \mathsf{Q}$*.*

We now define the building blocks from which all of our distinguished paths will be constructed. We begin by defining certain integers that describe the position of the origin in our fundamental alcove.

Definition 3.3. *Given* $\alpha \in \Pi$ *we define* b_{α} *to be the distance from the origin to the wall corresponding to* α*, and let* $b_{\emptyset} = 1$*. Given our earlier conventions this corresponds to setting*

$$b_{\varepsilon_{hi+j} - \varepsilon_{hi+j+1}} = 1$$

for $1 \leqslant j < h$ *and* $0 \leqslant i < \ell$ *and that*

$$b_{\varepsilon_{hi} - \varepsilon_{hi+1}} = \sigma_{i+1} - \sigma_i - h + 1 \qquad b_{\varepsilon_{h\ell} - \varepsilon_1} = e - \sigma_1 + \sigma_i + h - 1$$

for $0 \leqslant i < \ell - 1$*. We sometimes write* δ_{α} *for the element* $\delta_{b_{\alpha}}$*. Given* $\alpha, \beta \in \Pi$ *we set* $b_{\alpha\beta} = b_{\alpha} + b_{\beta}$*.*

Example 3.4. *Let* $e = 5$*,* $h = 3$ *and* $\ell = 1$ *as in Figure 12. Then* $b_{\varepsilon_2 - \varepsilon_3}$ *and* $b_{\varepsilon_1 - \varepsilon_2}$ *both equal* 1*, while* $b_{\varepsilon_3 - \varepsilon_1} = 3$ *and* $b_{\emptyset} = 1$*.*

Example 3.5. *Let* $e = 7$*,* $h = 2$ *and* $\ell = 2$ *and* $\sigma = (0, 3) \in \mathbb{Z}^2$*. Then* $b_{\varepsilon_1 - \varepsilon_2}$ *and* $b_{\varepsilon_3 - \varepsilon_4}$ *both equal* 1*, while* $b_{\varepsilon_4 - \varepsilon_1} = 3$*,* $b_{\varepsilon_2 - \varepsilon_3} = 2$*, and* $b_{\emptyset} = 1$*.*

We can now define our basic building blocks for paths.

Definition 3.6. *Given* $\alpha = \varepsilon_i - \varepsilon_{i+1} \in \Pi$*, we consider the multicomposition* $s_{\alpha} \cdot \delta_{\alpha}$ *with all columns of length* b_{α}*, with the exception of the* i*th and* $(i+1)$*st columns, which are of length* 0 *and* $2b_{\alpha}$*, respectively. We set*

$$\mathsf{M}_i = (\varepsilon_1, \dots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \dots, \varepsilon_{h\ell}) \quad \textit{and} \quad \mathsf{P}_i = (+\varepsilon_i)$$

where $\widehat{\cdot}$ *denotes omission of a coordinate. Then our distinguished path corresponding to* s_{α} *is given by*

$$\mathsf{P}_{\alpha} = \mathsf{M}_i^{b_{\alpha}} \boxtimes \mathsf{P}_{i+1}^{b_{\alpha}} \in \mathrm{Path}(s_{\alpha} \cdot \delta_{\alpha}).$$

The distinguished path corresponding to $\emptyset$ *is given by*

$$\mathsf{P}_{\emptyset} = (\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_{h\ell}) \in \mathrm{Path}(\delta) = \mathrm{Path}(s_{\emptyset} \cdot \delta)$$

and set $\mathsf{P}_{\emptyset} = (\mathsf{P}_{\emptyset})^{b_{\alpha}}$*.*

Given all of the above, we can finally define our distinguished paths for general words in our alphabet. There will be one such path for each word in our alphabet, and they will be defined by induction on the length of the word, as follows.

Definition 3.7. *We now define a* distinguished path $\mathsf{P}_{\underline{w}}$ *for each word* $\underline{w}$ *in our alphabet* $S \cup \{1\}$ *by induction on the length of* $\underline{w}$*. If* $\underline{w}$ *is* $s_{\emptyset}$ *or a simple reflection* s_{α} *we have already defined the distinguished path in Definition 3.6. Otherwise if* $\underline{w} = s_{\alpha}\underline{w}'$ *then we define*

$$\mathsf{P}_{\underline{w}} := \mathsf{P}_{\alpha} \otimes_{\alpha} \mathsf{P}_{\underline{w}'}.$$

If $\underline{w}$ is a reduced word in $\widehat{\mathfrak{S}}_{hl}$, then the corresponding path $\mathsf{P}_{\underline{w}}$ is a reduced path.

Remark 3.8. *Contextualised concatenation is not associative (if we wish to decorate the tensor products with the corresponding elements* w*). As we will typically be constructing paths as in Definition 3.7 we will adopt the convention that an unbracketed concatenation of* n *terms corresponds to bracketing from the right:*

$$\mathsf{Q}_1 \otimes \mathsf{Q}_2 \otimes \mathsf{Q}_3 \otimes \cdots \mathsf{Q}_n = \mathsf{Q}_1 \otimes (\mathsf{Q}_2 \otimes (\mathsf{Q}_3 \otimes (\cdots \otimes \mathsf{Q}_n) \cdots)).$$

We will also need certain reflections of our distinguished paths corresponding to elements of Π.

Definition 3.9. *Given* $\alpha \in \Pi$ *we set*

$$\mathsf{P}^{\flat}_{\alpha} = \mathsf{M}_i^{b_{\alpha}} \boxtimes \mathsf{P}_i^{b_{\alpha}} = \mathsf{M}_i^{b_{\alpha}} \otimes_{\alpha} \mathsf{P}_{i+1}^{b_{\alpha}} = (+\varepsilon_1, \dots, +\varepsilon_{i-1}, \widehat{+\varepsilon_i}, +\varepsilon_{i+1}, \dots, +\varepsilon_{h\ell})^{b_{\alpha}} \boxtimes (\varepsilon_i)^{b_{\alpha}}$$

the path obtained by reflecting the second part of P_{α} *in the wall through which it passes.*

Example 3.10. *We illustrate these various constructions in a series of examples. In the first two diagrams of Figure 15, we illustrate the basic path $\mathsf{P}_{\boldsymbol{\alpha}}$ and the path $\mathsf{P}^{\flat}_{\boldsymbol{\alpha}}$ and in the rightmost diagram of Figure 15, we illustrate the path $\mathsf{P}_{\emptyset}$. A more complicated example is illustrated in Figure 12, where we show the distinguished path $\mathsf{P}_{\underline{w}}$ for $\underline{w} = s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_3-\varepsilon_2}s_{\varepsilon_3-\varepsilon_1}s_{\varepsilon_2-\varepsilon_1}s_{\varepsilon_3-\varepsilon_2}$ as in Example ??. The components of the path between consecutive black nodes correspond to individual $\mathsf{P}_{\boldsymbol{\alpha}}$s.*

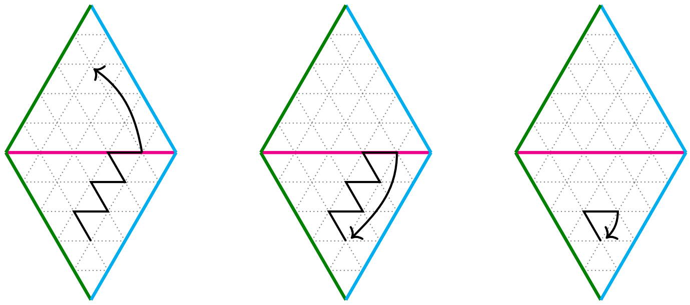


FIGURE 15. The leftmost two diagrams picture the path $\mathsf{P}_{\boldsymbol{\alpha}}$ walking through an $\boldsymbol{\alpha}$-hyperplane in $\overline{\mathbb{E}}^{+}_{1,3}$, and the path $\mathsf{P}^{\flat}_{\boldsymbol{\alpha}}$ which reflects this path through the same $\boldsymbol{\alpha}$-hyperplane. The rightmost diagram pictures the path $\mathsf{P}_{\emptyset}$ in $\overline{\mathbb{E}}^{+}_{1,3}$. We have bent the latter path slightly to make it clearer.

3.2. **The principal block of $\mathscr{H}^{\sigma}_{n}$.** We now restrict our attention to regular blocks of $\mathscr{H}^{\sigma}_{n}$. In order to do this, we first recall that we consider an element of the quiver Hecke algebra to be a morphism between paths. The easiest elements to construct are the idempotents corresponding to the trivial morphism from a path to itself. Given $\boldsymbol{\alpha}$ a simple reflection or $\boldsymbol{\alpha} = \emptyset$, we have an associated path $\mathsf{P}_{\boldsymbol{\alpha}}$, a trivial bijection $w^{\mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}}} = 1 \in \mathfrak{S}_{b_{\boldsymbol{\alpha}}}$, and an idempotent element of the quiver Hecke algebra

$$e_{\mathsf{P}_{\boldsymbol{\alpha}}} := e_{\mathrm{res}(\mathsf{P}_{\boldsymbol{\alpha}})} \in \mathcal{H}^{\sigma}_{b_{\boldsymbol{\alpha}}}.$$

More generally, given any $\underline{w} = s_{\alpha^{(1)}}s_{\alpha^{(2)}}\dots s_{\alpha^{(k)}}$ any expression of breadth $b(\underline{w}) = n$, we have an associated path $\mathsf{P}_{\underline{w}}$, and an element of the quiver Hecke algebra

$$e_{\mathsf{P}_{\underline{w}}} := e_{\mathrm{res}(\mathsf{P}_{\underline{w}})} = e_{\mathsf{P}_{\alpha^{(1)}}} \otimes e_{\mathsf{P}_{\alpha^{(2)}}} \otimes \cdots \otimes e_{\mathsf{P}_{\alpha^{(k)}}} \in \mathcal{H}^{\sigma}_{nh\ell}$$

We let $\mathrm{Std}_{n,\sigma}(\lambda)$, to consist of all standard tableaux which can be obtained by contextualised concatenation of paths from the set

$$\{\mathsf{P}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\} \cup \{\mathsf{P}^{\flat}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Pi\} \cup \{\mathsf{P}_{\emptyset}\},$$

and we let $\mathrm{Std}_{n,\sigma} = \cup_{\lambda \in \mathscr{P}_{h,\ell}(n)} \mathrm{Std}_{n,\sigma}(\lambda)$. For example, the path in Figure 12 is a strict alcove tableau and is equal to $\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\gamma}} \otimes \mathsf{P}_{\boldsymbol{\beta}} \otimes \mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\gamma}} \otimes \mathsf{P}_{\boldsymbol{\beta}}$. We define

$$\mathsf{f}_{n,\sigma} = \sum_{\substack{\mathsf{S} \in \mathrm{Std}_{n,\sigma}(\lambda) \\ \lambda \in \mathscr{P}_{h,\ell}(n)}} e_{\mathsf{S}} \tag{3.1}$$

and the remainder of this paper will be dedicated to understanding the algebra

$$\mathsf{f}_{n,\sigma}(\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{e}_{h}\mathcal{H}^{\sigma}_{n})\mathsf{f}_{n,\sigma}.$$

In fact, we will provide a concise list of generators for this truncated algebra (in the spirit of [EW16]) and rewrite the basis of Theorem 2.14 in terms of these generators. In order to preserve notation between this paper and [BCH], we first set

$$\Upsilon^{p}_{q} = (-1)^{\sharp\{p<k\leqslant q \mid i_k = i_p\}} e_{\underline{i}}\psi^{p}_{q}.$$

This "sign twist" is of no consequence in this paper as we are mostly concerned with constructing generators and bases of quiver Hecke algebras and their truncations. However, in order to match-up our *relations* with those of Elias–Williamson, this sign twist will be necessary and so we introduce it here for the purposes of consistency with [BCH].

In this section, we use our concrete branching coefficients to define the "Soergel 2-generators" of $\mathsf{f}_{n,\sigma}(\mathcal{H}^{\sigma}_{n}/\mathcal{H}^{\sigma}_{n}\mathsf{e}_{h}\mathcal{H}^{\sigma}_{n})\mathsf{f}_{n,\sigma}$ explicitly. In the companion paper [BCH], we will show that these generators are actually independent of these choices of reduced expressions (however, this won't be needed here — we simply make a note, again, for purposes of consistency with [BCH]).

3.3. **Generator morphisms in degree zero.** We first discuss how to pass between paths $\mathsf{P}_{\underline{w}}$ and $\mathsf{P}_{\underline{w}'}$ which are in *different linkage classes* but for which $\underline{w}$ and $\underline{w}'$ have the same underlying permutation. Fix two such paths

$$\mathsf{P}_{\underline{w}} = \mathsf{P}_{\alpha^{(1)}} \otimes \mathsf{P}_{\alpha^{(2)}} \otimes \cdots \otimes \mathsf{P}_{\alpha^{(k)}} \in \mathrm{Path}_{h,\ell}(\lambda) \qquad \mathsf{P}_{\underline{w}'} = \mathsf{P}_{\beta^{(1)}} \otimes \mathsf{P}_{\beta^{(2)}} \otimes \cdots \otimes \mathsf{P}_{\beta^{(k)}} \in \mathrm{Path}_{h,\ell}(\lambda)$$

with $\alpha^{(1)}, \dots, \alpha^{(k)}, \beta^{(1)}, \dots, \beta^{(k)} \in \Pi \cup \{\emptyset\}$. We suppose, only for the purposes of this motivational discussion, that both paths are reduced. In which case, we have that $w \in \widehat{\mathfrak{S}}_{h\ell}$ and so the expressions $\underline{w}$ and $\underline{w}'$ differ only by applying Coxeter relations in of $\widehat{\mathfrak{S}}_{h\ell}$ and the trivial "adjustment" relation $s_i 1 = 1 s_i$ (made necessary by our augmentation of the Coxeter presentation). Moreover, $\underline{w}$ and $\underline{w}'$ are both *reduced* expressions and so we need only apply the "hexagon" relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and the "commutation" relation $s_i s_j = s_j s_i$ for $|i-j| > 1$. The remainder of this subsection will be dedicated to lifting these path-morphisms to the level of generators of the KLR algebra. We stress that one can apply these adjustment/hexagon/commutator path-morphisms to any paths (not just reduced paths) but the reduced paths provide the motivation.

3.3.1. *Adjustment generator.* We will refer to the passage between alcove paths which differ only by occurrences of $s_\emptyset = 1$ (and their associated idempotents) as "adjustment". We define the KLR-adjustment generator to be the element

$$\mathsf{adj}_{\emptyset\boldsymbol{\alpha}}^{\boldsymbol{\alpha}\emptyset} := \Upsilon_{\mathsf{P}_{\emptyset\boldsymbol{\alpha}}}^{\mathsf{P}_{\boldsymbol{\alpha}\emptyset}}.$$

Examples of the paths $\mathsf{P}_{\boldsymbol{\alpha}\emptyset}$, $\mathsf{P}_{\emptyset\boldsymbol{\alpha}}$, and adjustment generators are given in Figures 16 and 17.

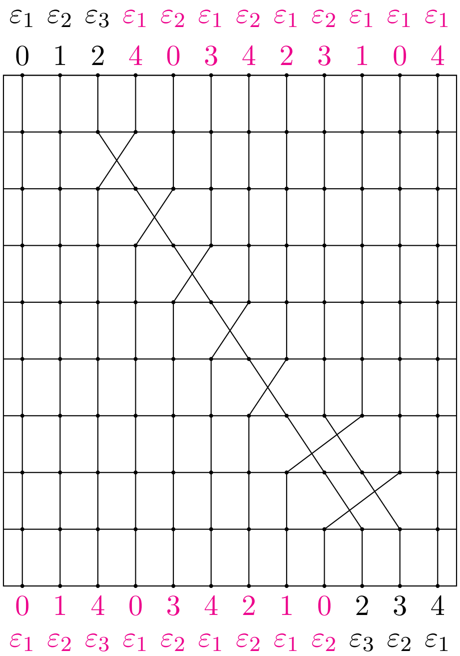


FIGURE 16. We let $h = 3$, $\ell = 1$, $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$. The adjustment term $\mathsf{adj}_{\boldsymbol{\alpha}\emptyset}^{\emptyset\boldsymbol{\alpha}}$ is illustrated.

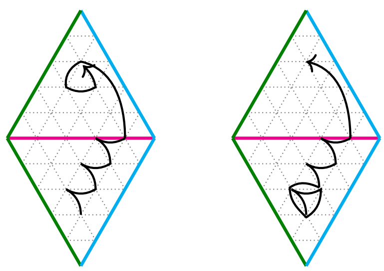


FIGURE 17. We let $h = 3$, $\ell = 1$, $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$. We picture the paths $\mathsf{P}_{\boldsymbol{\alpha}\emptyset}$ and $\mathsf{P}_{\emptyset\boldsymbol{\alpha}}$.

3.3.2. *The KLR hexagon diagram.* We wish to pass between the two distinct paths around a vertex in our alcove geometry which lies at the intersection of two hyperplanes labelled by non-commuting reflections. To this end, we let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Pi$ label a pair of non-commuting reflections. Of course, one path around the vertex may be longer than the other. Thus, we have two cases to consider: if $b_{\boldsymbol{\alpha}} \geqslant b_{\boldsymbol{\beta}}$ then we must pass between the paths $\mathsf{P}_{\boldsymbol{\alpha\beta\alpha}}$ and $\mathsf{P}_{\boldsymbol{\emptyset}-\boldsymbol{\emptyset}} \otimes \mathsf{P}_{\boldsymbol{\beta\alpha\beta}}$ and if $b_{\boldsymbol{\alpha}} \leqslant b_{\boldsymbol{\beta}}$ then we pass between the paths $\mathsf{P}_{\boldsymbol{\emptyset}-\boldsymbol{\emptyset}} \otimes \mathsf{P}_{\boldsymbol{\alpha\beta\alpha}}$ and $\mathsf{P}_{\boldsymbol{\beta\alpha\beta}}$, where here $\boldsymbol{\emptyset} - \boldsymbol{\emptyset} := \emptyset^{b_{\boldsymbol{\alpha}} - b_{\boldsymbol{\beta}}}$.

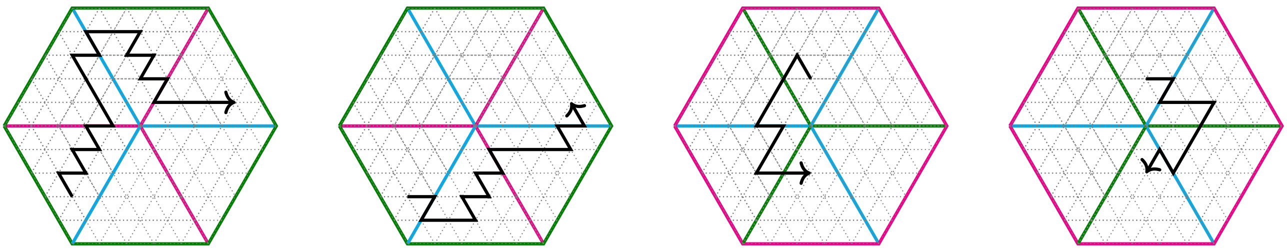


FIGURE 18. We let $h = 3$, $\ell = 1$, $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$ and $\boldsymbol{\beta} = \varepsilon_1 - \varepsilon_2$ and $\boldsymbol{\gamma} = \varepsilon_2 - \varepsilon_3$. The paths $\mathsf{P}_{\boldsymbol{\alpha\beta\alpha}}$, $\mathsf{P}_{\boldsymbol{\beta\alpha\beta}}$, $\mathsf{P}_{\boldsymbol{\gamma\beta\gamma}}$ and $\mathsf{P}_{\boldsymbol{\beta\gamma\beta}}$ are pictured.

We define the $\mathsf{KLR\text{-}hexagon}$ to be the element

$$\mathsf{hex}^{\boldsymbol{\alpha\beta\alpha}}_{\boldsymbol{\beta\alpha\beta}} := \Upsilon^{\mathsf{P}_{\boldsymbol{\alpha\beta\alpha}}}_{\mathsf{P}_{\boldsymbol{\emptyset}-\boldsymbol{\emptyset}} \otimes \mathsf{P}_{\boldsymbol{\beta\alpha\beta}}} \qquad \text{or} \qquad \mathsf{hex}^{\boldsymbol{\alpha\beta\alpha}}_{\boldsymbol{\beta\alpha\beta}} := \Upsilon^{\mathsf{P}_{\boldsymbol{\emptyset}-\boldsymbol{\emptyset}} \otimes \mathsf{P}_{\boldsymbol{\alpha\beta\alpha}}}_{\mathsf{P}_{\boldsymbol{\beta\alpha\beta}}}$$

for $b_{\boldsymbol{\alpha}} \geqslant b_{\boldsymbol{\beta}}$ or $b_{\boldsymbol{\alpha}} \leqslant b_{\boldsymbol{\beta}}$ respectively. Two such pairs of paths are despited in Figure 18. For the latter pair, the corresponding KLR-hexagon element is depicted in Figure 19.

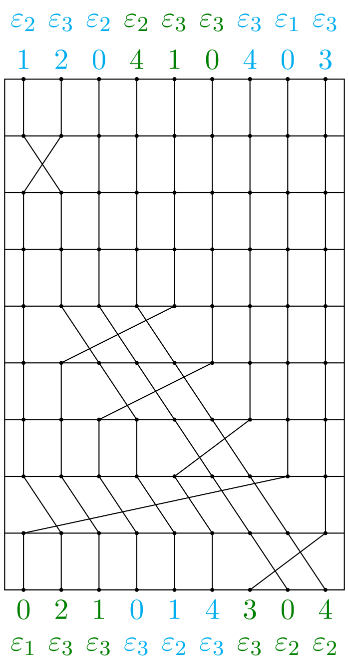


FIGURE 19. We let $h = 3$, $\ell = 1$, $e = 5$ and $\boldsymbol{\beta} = \varepsilon_1 - \varepsilon_2$ and $\boldsymbol{\gamma} = \varepsilon_2 - \varepsilon_3$. We picture $\mathsf{hex}^{\mathsf{P}_{\boldsymbol{\beta\gamma\beta}}}_{\mathsf{P}_{\boldsymbol{\gamma\beta\gamma}}}$.

3.3.3. *The KLR commutator.* Let $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \Pi$ be roots labelling commuting reflections (so that $|k-j| > 1$). We wish to understand the morphism relating the paths $\mathsf{P}_{\boldsymbol{\gamma}} \otimes \mathsf{P}_{\boldsymbol{\beta}}$ to $\mathsf{P}_{\boldsymbol{\beta}} \otimes \mathsf{P}_{\boldsymbol{\gamma}}$. We define the $\mathsf{KLR\text{-}commutator}$ to be the element

$$\mathsf{com}^{\boldsymbol{\gamma\beta}}_{\boldsymbol{\beta\gamma}} := \Upsilon^{\mathsf{P}_{\boldsymbol{\gamma}} \otimes \mathsf{P}_{\boldsymbol{\beta}}}_{\mathsf{P}_{\boldsymbol{\beta}} \otimes \mathsf{P}_{\boldsymbol{\gamma}}}.$$

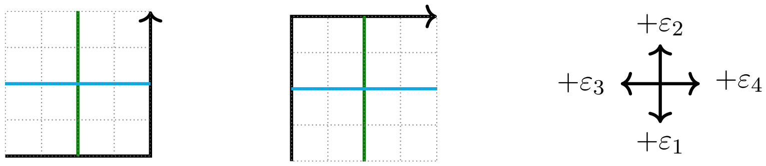


FIGURE 20. We let $h = 1$, $\ell = 4$, $\kappa = (0, 2, 4, 6) \in (\mathbb{Z}/8\mathbb{Z})^4$ and $\beta = \varepsilon_1 - \varepsilon_2$ and $\gamma = \varepsilon_3 - \varepsilon_4$. We picture the paths $\mathsf{P}_{\gamma\beta}$ and $\mathsf{P}_{\beta\gamma}$, the corresponding element $\mathsf{com}^{\beta\gamma}_{\gamma\beta}$ is depicted in Figure 21.

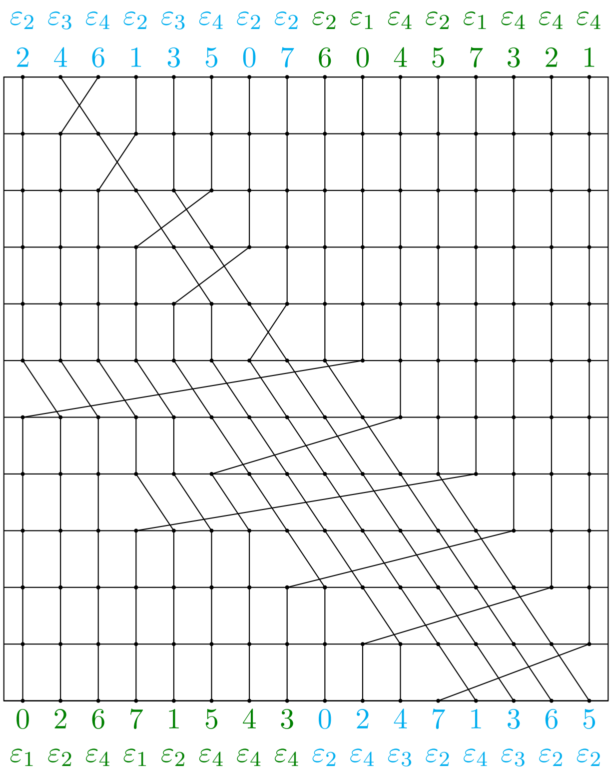


FIGURE 21. We let $h = 1$, $\ell = 4$, $\kappa = (0, 2, 4, 6) \in (\mathbb{Z}/8\mathbb{Z})^4$ and $\beta = \varepsilon_1 - \varepsilon_2$ and $\gamma = \varepsilon_3 - \varepsilon_4$. We picture the element $\mathsf{com}^{\beta\gamma}_{\gamma\beta}$, the corresponding paths are depicted in Figure 20.

3.4. **Generator morphisms in non-zero degree.** We have already seen how to pass between $\mathsf{S}, \mathsf{T} \in \mathrm{Std}_{n,\sigma}(\lambda)$ any two *reduced* paths. We will now see how to inflate a reduced path to obtain a non-reduced path. Given $\mathsf{S}, \mathsf{T} \in \mathrm{Std}_{n,\sigma}(\lambda)$, we suppose that the former is obtained from the latter by inflating by a path through a single hyperplane $\alpha \in \Pi$. Of course, since S and T have the same shape, this inflation must add an $\mathsf{P}^\flat_\alpha$ at some point (and will involve removing an occurrence of $\mathsf{T}_\varnothing$ in order to preserve n). There are two ways which one can approach a hyperplane: from above or from below. The adding a upward/downward occurrence of $\mathsf{P}^\flat_\alpha$ corresponds to the spot/fork Soergel generator.

3.4.1. *The spot morphism.* We now define the morphism which corresponds to reflection towards the origin through the hyperplane labelled by $\alpha \in \Pi$. We consider the paths

$$\mathsf{P}_\varnothing = (\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_{h\ell})^{b_\alpha} \qquad \mathsf{P}^\flat_\alpha = (\varepsilon_1, \dots, \varepsilon_{i-1}, \widehat{\varepsilon_i}, \varepsilon_{i+1}, \dots, \varepsilon_{h\ell})^{b_\alpha} \boxtimes (\varepsilon_i)^{b_\alpha}$$

examples of these paths are depicted in Figure 22. We define the KLR-spot to be the element

$$\mathsf{spot}^\varnothing_\alpha := \Upsilon^{\mathsf{P}_\varnothing}_{\mathsf{P}^\flat_\alpha}$$

which is of degree $+1$ (corresponding to the unique step of off the α-hyperplane). We have already constructed an example of an element $\mathsf{spot}^\varnothing_\alpha$ in great detail over the course of Examples 2.7, 2.9 and 2.11 and Figure 14.

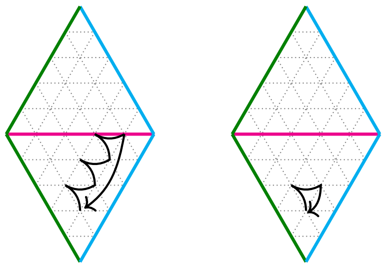


FIGURE 22. We let $h = 3$, $\ell = 1$, $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$ and we depict the paths $\mathsf{P}^{\flat}_{\boldsymbol{\alpha}}$ and $\mathsf{P}_{\varnothing}$ (we actually only depict $\mathsf{P}_{\emptyset}$ which is a third of the path $\mathsf{P}_{\varnothing}$). We have already constructed the corresponding element $\mathsf{spot}^{\varnothing}_{\boldsymbol{\alpha}}$ in great detail over the course of Examples 2.7, 2.9 and 2.11 and Figure 14.

3.4.2. *The fork morphism.* We wish to understand the morphism from $\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^{\flat}_{\boldsymbol{\alpha}}$ to $\mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$. We define the $\mathsf{KLR\text{-}fork}$ to be the elements

$$\mathsf{fork}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} := \Upsilon^{\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\boldsymbol{\alpha}}\otimes\mathsf{P}^{\flat}_{\boldsymbol{\alpha}}}$$

The element $\mathsf{fork}^{\varnothing\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}$ is of degree -1.

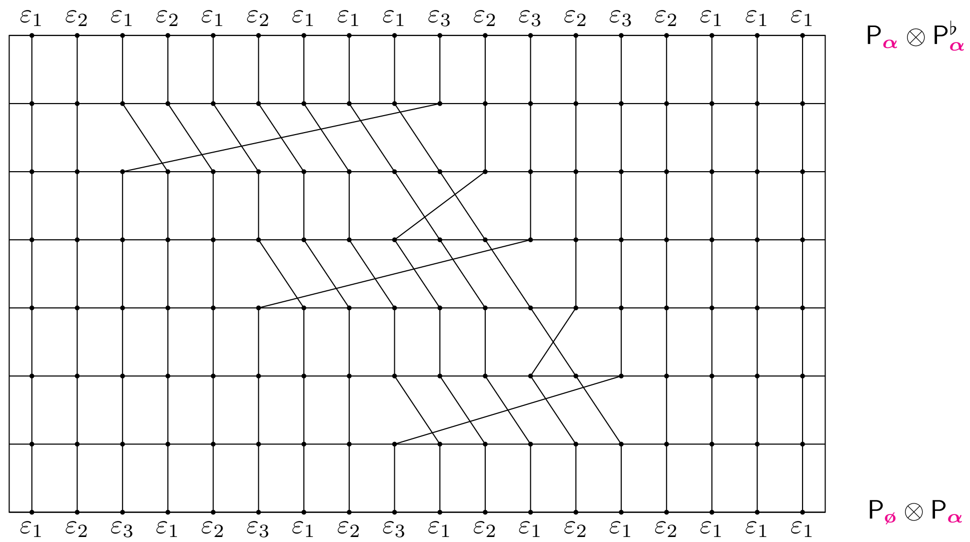


FIGURE 23. Fix $\ell = 1$ and $h = 3$ and $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$ (so that $b_{\boldsymbol{\alpha}} = 3$). We picture the element $\psi^{\mathsf{P}_{\boldsymbol{\alpha}}\otimes\mathsf{P}^{\flat}_{\boldsymbol{\alpha}}}_{\mathsf{P}_{\varnothing}\otimes\mathsf{P}_{\boldsymbol{\alpha}}}$. The first 9 and final 4 of the branching coefficients are trivial and so we do not waste trees by picturing all of them. The corresponding paths are pictured in Figure 24.

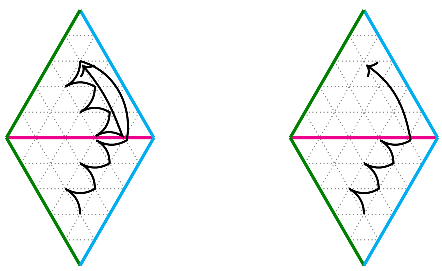


FIGURE 24. Fix $\ell = 1$ and $h = 3$ and $e = 5$ and $\boldsymbol{\alpha} = \varepsilon_3 - \varepsilon_1$ (so that $b_{\boldsymbol{\alpha}} = 3$). We depict the paths $\mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}^{\flat}_{\boldsymbol{\alpha}}$ and $\mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\boldsymbol{\alpha}}$ (although we do not depict the determinant path). The corresponding fork generator is pictured in Figure 23.

3.5. **Light leaves for the Bott–Samelson truncation.** We now rewrite the truncated basis of Theorem 1.20 in terms of the Bott–Samelson generators (thus showing that these are, indeed, generators of the truncated algebra). Of course, the idempotent of equation (3.1) is specifically chosen so that the truncated algebra

$$\mathsf{f}_{n,\sigma}(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma)\mathsf{f}_{n,\sigma}$$

has basis indexed by the (sub)set of alcove-tableaux (and this basis is simply obtained from that of Theorem 1.20 by truncation). It only remains to illustrate how the reduced-path-vectors can be chosen to mirror the construction of paths in $\mathrm{Std}_{n,\sigma}(\lambda)$ via concatenation.

We can extend a path $\mathsf{Q}' \in \mathrm{Std}_{n,\sigma}(\lambda)$ to obtain a new path Q in one of three possible ways

$$\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \qquad \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}} \qquad \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_\emptyset$$

for some $\boldsymbol{\alpha} \in \Pi$. The first two cases each subdivide into a further two cases based on whether $\boldsymbol{\alpha}$ is an upper or lower wall of the alcove containing λ. These four cases are pictured in Figure 25 (for $\mathsf{P}_\emptyset$ we refer the reader to Figure 15). Any two reduced paths $\mathsf{P}_{\underline{w}}, \mathsf{P}_{\underline{v}} \in \mathrm{Std}_{n,\sigma}(\lambda)$ can be obtained from one another by some iterated application of hexagon, adjustment, and commutativity permutations. We let

$$\mathsf{rex}_{\mathsf{P}_{\underline{w}}}^{\mathsf{P}_{\underline{v}}}$$

denote the corresponding path-morphism in the algebras $\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma$ (so-named as they permute reduced expressions). In the following construction, we will assume that the elements $c_{\mathsf{Q}'}^{\mathsf{P}'}$ exist for any choice of reduced path P'. We then extend P' using one of the U_0, U_1, D_0, and D_1 paths (which puts a restriction on the form of the reduced expression) but then use a "rex move" to obtain $c_{\mathsf{Q}}^{\mathsf{P}}$ for P an arbitrary reduced path.

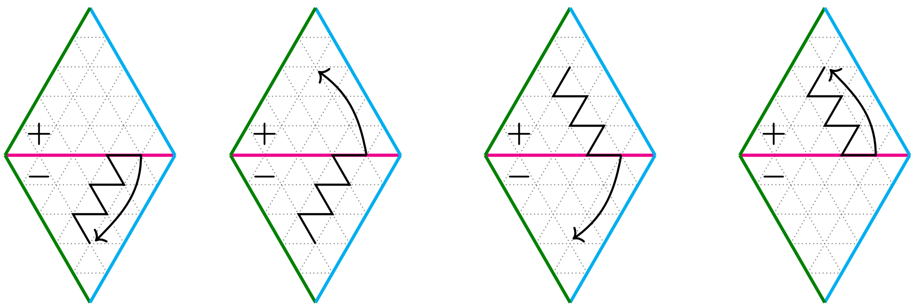


FIGURE 25. The first (respectively last) two paths are $\mathsf{P}_{\boldsymbol{\alpha}}$ and $\mathsf{P}^\flat_{\boldsymbol{\alpha}}$ originating in an alcove with $\boldsymbol{\alpha}$ labelling an upper (respectively lower) wall. The plus and minus signs distinguish the alcoves or greater/lesser length. The degrees of these paths are $1, 0, 0, -1$ respectively. We call these paths U_0, U_1, D_0, and D_1 respectively.

Definition 3.11. *Suppose that λ belongs to an alcove which has a hyperplane labelled by $\boldsymbol{\alpha}$ as an upper alcove wall. Let $\mathsf{Q}' \in \mathrm{Std}_{n,\sigma}(\lambda)$. If $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}}$ then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}')$ and we define*

$$c_{\mathsf{P}}^{\mathsf{Q}} = (c_{\mathsf{P}'}^{\mathsf{Q}'} \;\otimes\; e_{\mathsf{P}_{\boldsymbol{\alpha}}})\mathsf{rex}_{\mathsf{P}}^{\mathsf{P}'\otimes\mathsf{P}_{\boldsymbol{\alpha}}}.$$

If $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}$ then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}') + 1$ and we define

$$c_{\mathsf{P}}^{\mathsf{Q}} = (c_{\mathsf{P}'}^{\mathsf{Q}'} \;\otimes\; \mathsf{spot}_{\emptyset}^{\boldsymbol{\alpha}})\mathsf{rex}_{\mathsf{P}}^{\mathsf{P}'\otimes\mathsf{P}_{\emptyset}}.$$

Now suppose that λ belongs to an alcove which has a hyperplane labelled by $\boldsymbol{\alpha}$ as a lower alcove wall. Thus we can choose $\mathsf{P}_{\underline{v}} \otimes \mathsf{P}_{\boldsymbol{\alpha}} = \mathsf{P}' \in \mathrm{Std}(\lambda)$. For $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}}$, we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}')$ and define

$$c_{\mathsf{P}}^{\mathsf{Q}} = \big(c_{\mathsf{P}'}^{\mathsf{Q}'} \otimes e_{\mathsf{P}_{\boldsymbol{\alpha}}}\big)\big(\mathsf{e}_{\mathsf{P}_{\underline{v}}} \otimes (\mathsf{fork}_{\boldsymbol{\alpha}\emptyset}^{\boldsymbol{\alpha}\boldsymbol{\alpha}} \circ \mathsf{spot}_{\emptyset}^{\boldsymbol{\alpha}})\big)\mathsf{rex}_{\mathsf{P}}^{\mathsf{P}_{\underline{v}\emptyset\emptyset}}$$

and if $\mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}}$ then then we set $\deg(\mathsf{Q}) = \deg(\mathsf{Q}') - 1$ and we define

$$c_{\mathsf{P}}^{\mathsf{Q}} = \big(c_{\mathsf{P}'}^{\mathsf{Q}'} \otimes e_{\mathsf{P}_{\boldsymbol{\alpha}}}\big)\big(e_{\mathsf{P}_{\underline{v}}} \otimes \mathsf{fork}_{\boldsymbol{\alpha}\emptyset}^{\boldsymbol{\alpha}\boldsymbol{\alpha}}\big)\mathsf{rex}_{\mathsf{P}}^{\mathsf{P}_{\underline{v}\boldsymbol{\alpha}\emptyset}}.$$

Theorem 3.12 (The light leaves basis)**.** *For each $\lambda \in \mathscr{P}_{h,\ell}(n,\sigma)$, we choose an arbitrary reduced path $\mathsf{P}_{\underline{w}} \in \mathrm{Std}_{n,\sigma}(\lambda)$. The algebra $f_+(\mathcal{H}_n^\sigma/\mathcal{H}_n^\sigma \mathsf{e}_h \mathcal{H}_n^\sigma)f_+$ is quasi-hereditary with graded integral cellular basis*

$$\{c^{\mathsf{P}}_{\mathsf{P}_{\underline{w}}} c^{\mathsf{P}_{\underline{w}}}_{\mathsf{Q}} \mid \mathsf{P},\mathsf{Q} \in \mathrm{Std}_{n,\sigma}(\lambda), \lambda \in \mathscr{P}_{h,\ell}(n,\sigma)\}$$

with respect to the dominance ordering $\succ$ on $\mathscr{P}_{h,\ell}(n,\sigma)$, the anti-involution $*$ given by flipping a diagram through the horizontal axis and the map $\deg : \mathrm{Std}_{n,\sigma}(\lambda) \to \mathbb{Z}$.

Proof. By induction, we may assume that the element $c^{\mathsf{P}'}_{\mathsf{Q}'}$ has been constructed for all $\mathsf{P}',\mathsf{Q}' \in \mathrm{Std}_{n,\sigma}(\lambda)$ with P' reduced. Suppose that λ belongs to an alcove which has a hyperplane labelled by $\boldsymbol{\alpha}$. We now use Theorem 1.20 to construct the element $c^{\mathsf{P}}_{\mathsf{Q}}$. This simply amounts to fixing our expressions for the modified branching coefficients as follows. If λ belongs to an alcove which has a hyperplane labelled by $\boldsymbol{\alpha}$ as an upper alcove wall, then we set

$$d_{n+p}(\mathsf{P},\mathsf{Q}) = \begin{cases} 1 & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \text{ and } p < b_{\boldsymbol{\alpha}} \\ 1_n \boxtimes d_p(\mathsf{P}^\flat_{\boldsymbol{\alpha}}, \mathsf{P}_{\varnothing}) & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}} \text{ and } p < b_{\boldsymbol{\alpha}} \\ w_{\mathsf{P}}^{\mathsf{P}'\otimes \mathsf{P}_{\boldsymbol{\alpha}}} & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \text{ and } p = b_{\boldsymbol{\alpha}} \\ (1_n \boxtimes d_n(\mathsf{P}^\flat_{\boldsymbol{\alpha}}, \mathsf{P}_{\varnothing}))w_{\mathsf{P}}^{\mathsf{P}'\otimes \mathsf{P}_{\boldsymbol{\alpha}}} & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}} \text{ and } p = b_{\boldsymbol{\alpha}} \end{cases}$$

Now suppose that λ belongs to an alcove which has a hyperplane labelled by $\boldsymbol{\alpha}$ as a lower alcove wall. By induction, we can choose $\mathsf{P}_{\underline{v}} \otimes \mathsf{P}_{\boldsymbol{\alpha}} = \mathsf{P}' \in \mathrm{Std}(\lambda)$ and we can set

$$d_{n+p}(\mathsf{P},\mathsf{Q}) = \begin{cases} 1_{n-b_{\boldsymbol{\alpha}}} \otimes d_p(\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}, \mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\varnothing}) & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \text{ and } p < b_{\boldsymbol{\alpha}} \\ 1_{n-b_{\boldsymbol{\alpha}}} \otimes d_p(\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}, \mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\varnothing}) & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}} \text{ and } p < b_{\boldsymbol{\alpha}} \\ (1_{n-b_{\boldsymbol{\alpha}}} \otimes d_n(\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}, \mathsf{P}_{\varnothing} \otimes \mathsf{P}_{\varnothing}))w_{\mathsf{P}}^{\mathsf{P}_{\underline{v}}\otimes \mathsf{P}_{\varnothing\varnothing}} & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}_{\boldsymbol{\alpha}} \text{ and } p = b_{\boldsymbol{\alpha}} \\ (1_{n-b_{\boldsymbol{\alpha}}} \otimes d_n(\mathsf{P}^\flat_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\boldsymbol{\alpha}}, \mathsf{P}_{\boldsymbol{\alpha}} \otimes \mathsf{P}_{\varnothing}))w_{\mathsf{P}}^{\mathsf{P}_{\underline{v}}\otimes \mathsf{P}_{\boldsymbol{\alpha}\varnothing}} & \text{if } \mathsf{Q} = \mathsf{Q}' \otimes \mathsf{P}^\flat_{\boldsymbol{\alpha}} \text{ and } p = b_{\boldsymbol{\alpha}} \end{cases}.$$

To summarise: we incorporate the "rex" move into the final branching coefficient. Choosing the reduced path vectors in this fashion, we obtain the required basis as a special case of Theorem 1.20 . ∎

We have shown that we can write a basis for our algebra entirely in terms of the elements

$$e_{\mathsf{P}_{\boldsymbol{\alpha}}}, \quad \mathsf{fork}^{\boldsymbol{\alpha}\varnothing}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}, \quad \mathsf{spot}^{\varnothing}_{\boldsymbol{\alpha}}, \quad \mathsf{hex}^{\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}}_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\alpha}}, \quad \mathsf{com}^{\boldsymbol{\gamma}\boldsymbol{\beta}}_{\boldsymbol{\beta}\boldsymbol{\gamma}}, \quad e_{\mathsf{P}_\emptyset}, \text{ and } \mathsf{adj}^{\emptyset\boldsymbol{\alpha}}_{\boldsymbol{\alpha}\emptyset}$$

for $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in \Pi$ such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ label an arbitrary pair of non-commuting reflections and $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ label an arbitrary pair of commuting reflections. Thus we deduce the following:

Corollary 3.13. *Theorem B of the introduction holds.*

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