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RESEARCH ARTICLE

Structured singular value of Implicit Systems

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Abstract

Implicit systems provide a general framework in which many important properties of dynamic systems can be studied. Implicit systems are especially relevant to Behavioural Systems Theory, the analysis and synthesis of Complex Interconnected Systems, Systems Identification and Robust Control. By incorporating algebraic constraints, implicit models provide additional versatility relative to the standard input-output framework. Problems of robust stability in implicit systems lead in a natural way to non-standard structured singular value (μ) formulations. In this note it is shown that for a class of uncertainty structures involving repeated scalar parameters these problems reduce to a standard μ problem which is well studied and for the solution of which several numerical algorithms are available. Our results are based on a matrix dilation technique and the redefinition of the uncertainty structure of the transformed problem. The main results of the paper are illustrated with a numerical example.

KEYWORDS:
structured singular value, robust control, implicit systems, distance problems

1 | INTRODUCTION

Throughout its history, the input-output framework has dominated the Systems and Control Theory paradigm. Building on early work of Rosenberg [1], the behavioural framework was introduced in [2] as an alternative. In this framework, the system is described by the collection of its trajectories rather than input-output relations. Implicit systems play an important role in this context. By incorporating constraints in the model description, they provide additional versatility relative to standard input-output models. Areas where implicit systems have proved useful include the analysis and synthesis of complex Interconnected Systems, Systems Identification and Robust Control [3,4].

A formal definition of implicit systems is given next [5]. This is closely related to the behavioural definition given by Willems [6]. Note that no distinction is made between input and output variables and that the system equations are defined as equality constraints restricting the values variables can take. This set of values is the behaviour of the system.

Definition 1. An implicit system (W, E, G) is defined by two vector spaces, the variable space W and the equation space E, and an equation operator $G : W \rightarrow E$. The behaviour of the implicit system is the set $B = \text{Ker}(G) = \{w \in W : Gw = 0\}$. The system is called linear if G is a linear map.

Following [5] a more concrete formulation can be developed by restricting attention to discrete-time dynamic systems (time index set is $\mathbb{Z}$ or $\mathbb{Z}_+$) and restricting G to the class of linear bounded operators $G \in \mathcal{L}(l_2^n, l_2^m)$ (or simply $G \in \mathcal{L}$), i.e. $G : l_2^n \rightarrow l_2^m$. Here l_2^n denotes the Hilbert space of square-summable $\mathbb{R}^n$-valued sequences over $\mathbb{Z}$ or $\mathbb{Z}_+$. Alternatively G may be characterised as a causal, finite-gain operator, $G : l_{2e}^n \to l_{2e}^m$, equivalently denoted as $G \in \mathcal{L}_e(l_{2e}^n, l_{2e}^m)$ (or simply $G \in \mathcal{L}_e$). Here l_{2e} is the extended l_2 space over $\mathcal{Z}_+$ corresponding to positive-indexed discrete sequences (signals) which do not "blow-up" in finite time (see[5] for a more formal definition). The analysis can be further simplified by restricting G to the class of linear time-invariant (LTI) operators, i.e. $G\lambda = \lambda G$ where λ is a delay operator[7] . l_2 stability for the linear implicit system (W, E, G) over l_2 is defined as the left invertibility of G in $\mathcal{L}(l_2)$, i.e. the existence of $L \in \mathcal{L}(l_2)$ such that $LG = I$. The corresponding notion of stability for causal implicit systems is well posedness, i.e. G has a causal left inverse $L : E \to W$ and the requirement that L has a finite gain over the range of G[5].

Uncertain systems description can be incorporated in the theory of implicit systems via a set framework, similarly to standard systems defined as input-output maps. Robust Control analysis of implicit systems is useful in the formulation of uncertainty models of various types, e.g. linear time-invariant (LTI), linear time-varying (LTV), polytopic, delay, sector-nonlinearity, etc. All these descriptions can be unified via Integral Quadratic Constraints (IQC) resulting in constrained system descriptions[7]. The main idea of IQC's is to replace uncertain operators which are difficult to model by integral quadratic constraints on the input-output pair which applies for all uncertain conditions. This results in robust stability and performance criteria which can be checked via semi-definite programming. Considerable work has been done recently in formulating and solving distance problems involving uncertain systems[5]. Some examples of constrained H_∞ and H_2 problems are presented in[4,3,8] providing systematic methods for the control structure selection.

Uncertainty can be incorporated into the implicit modelling framework by replacing the equation map G by a parametrised map $G(\mathbf{\Delta})$ where $\mathbf{\Delta}$ represents a structured set of uncertainty descriptions of the form:

$$\Delta = \text{block-diag}(\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}, \Delta_{s+1}, \dots \Delta_{s+F}) \tag{1}$$

In this equation δ_i represent real uncertain parameters and Δ_i linear dynamic non-causal (causal) perturbations $\Delta \subseteq \mathcal{L}(l_2^n)$ ($\Delta \subseteq \mathcal{L}_e(l_{2e}^n)$). More general classes of block perturbations can also be defined. The uncertainty set can be normalised to a unit ball $\mathbf{B\Delta} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}$ where $\|\cdot\|$ is the operator norm. Uncertain implicit systems can be studied in the setting of linear fractional transformations on Δ[5] depicted in Figure 1 in which

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(l_2^{n+q}, l_2^{m+p}) \;\text{ or }\; M \in \mathcal{L}_e(l_{2e}^{n+q}, l_{2e}^{m+p}) \tag{2}$$

The equations corresponding to Figure 1 can be written in the form:

$$\varphi(\Delta, M) \begin{bmatrix} z \\ w \end{bmatrix} = 0, \quad \varphi(\Delta, M) = \begin{bmatrix} I - \Delta A & -\Delta B \\ C & D \end{bmatrix} \tag{3}$$

The implicit system with $M \in \mathcal{L}_2$ and $\mathbf{\Delta} \subseteq \mathcal{L}_2$ is now defined as robust l_2 stable if it is l_2 stable for every $\Delta \in \mathbf{B\Delta}$. It can be shown that a necessary and sufficient condition for robust l_2 stability is that D is l_2 stable with bounded left inverse L and that the implicit system

$$\begin{bmatrix} I - \Delta\hat{A} \\ \hat{C} \end{bmatrix}, \;\text{ where }\; \hat{A} = A - BLC, \;\text{ and }\; \hat{C} = C - DLC \tag{4}$$

is l_2 robust stable. Note that equation (4) is obtained by eliminating variable $w = -LCz$ in equation (3). A similar definition and conditions for robust stability apply for an implicit system $M \in \mathcal{L}_{2e}$ and $\mathbf{\Delta} \subseteq \mathcal{L}_{2e}$[5].

The problem is considerably simplified if we assume that $\hat{A}$, $\hat{C}$ and Δ are finite-dimensional linear time-invariant maps in $\mathcal{L}(l_2)$. In this case $\hat{A}$ and $\hat{C}$ are rational functions in λ and necessary and sufficient condition for the implicit system (4) to be l_2 robust stable is that:

$$\text{null} \begin{bmatrix} I - \Delta\hat{A}(e^{i\omega}) \\ \hat{C}(e^{i\omega}) \end{bmatrix} = 0 \;\text{ for all }\; \Delta \in \mathbf{B}_\Delta \;\text{ for all }\; \omega \in [0\ 2\pi) \tag{5}$$

Here Δ is a constant matrix with the block-diagonal structure of (1) and $\mathbf{B}_\Delta$ contains all constant matrices with this structure with $|\delta_i| \leq 1$, $i = 1, 2, \dots, s$ and $\sigma_1(\Delta_{s+i}) \leq 1$, $i = 1, 2, \dots, F$, where $\sigma_1(\cdot)$ denotes the largest singular value (spectral norm) of a matrix. In the remaining part of the paper we will consider the (single-frequency) problem of calculating the minimum norm of a matrix Δ with block-diagonal structure (1) such that

$$\text{null} \begin{bmatrix} I + \Delta M \\ N \end{bmatrix} \neq 0$$

This problem follows directly from (5) by identifying $M = -\hat{A}(e^{i\omega_0})$ and $N = \hat{C}(e^{i\omega_0})$ for a fixed $\omega_0 \in [0, 2\pi)$. We will simplify the problem further by considering only real repeated scalar perturbations δ_i in the uncertainty structure (1), i.e. we will restrict

$\mathbf{\Delta}$ to the set:

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1}, \dots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R}, i = 1, 2, \dots, s\} \tag{6}$$

Alternatively we can define the uncertain implicit system via the state-space description in which:

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_M & D_M \\ C_N & D_N \end{array}\right] \tag{7}$$

in which A, B, C_M, C_N, D_M and D_N are constant matrices of appropriate dimensions. Let λ be a delay operator, then the implicit form can be obtained as

$$\left(I + \begin{bmatrix} \lambda I & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} A & B \\ C_M & D_M \end{bmatrix} \right) \begin{bmatrix} x \\ z \end{bmatrix} = 0, \tag{8}$$

$$\begin{bmatrix} C_N & D_N \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = 0. \tag{9}$$

that is equivalent to

$$\mathrm{null} \begin{bmatrix} I + \Delta_S M \\ N \end{bmatrix} = 0, \tag{10}$$

where $\Delta_S = \mathrm{diag}(\lambda, \Delta)$ is the uncertainty set.

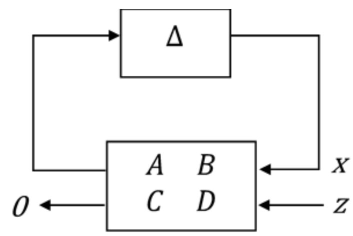

FIGURE 1 An upper LFT of plant

The structured singular value (μ) is a fundamental analysis and synthesis tool of Robust Control theory of input-output models with several applications in model validation and the characterisation of robust stability and performance of dynamic systems; see [9,4,10,11,12,13,7] and references therein. The analysis of structured perturbations introduced in [14] has been developed and extended to various problems including general structured singular value problems [9], mixed μ-value problems [10,11], lower and upper bound estimation [11], minimal magnitude norm perturbations in polynomials [15,16], stability radii analysis [17,12], pseudo spectrum [4], and many others.

Calculation of the exact μ-value is an NP-hard problem. Thus, most optimisation techniques aim to evaluate tight upper and lower bounds. Most upper bound calculations rely on convex relaxations and can be performed efficiently via Linear Matrix Inequalities or other convex programming techniques.

In the Linear Time Invariant (LTI) case robust stability properties of an implicit system defined by the pair (M, N) can be analysed using a Scaled Linear Fractional Transformation (SLFT). This leads to a generalised μ-problem involving additional constraints in the form of equation (10) which is formally defined in the next section. Since exact calculations are numerically hard, it is more appropriate to develop optimisation procedures for lower and upper bounds. Calculation of an upper bound can be achieved via a convex relaxation technique using Linear Matrix Inequalities, similar to the non-negative scaling approach used for the standard problem [18]. Unfortunately, however, in contrast to the standard version of the problem, there is not sufficient numerical experience to assess the effectiveness of this method, i.e. the proximity of the upper bound to the exact generalised μ value.

In this paper we focus on a special class of generalised μ problems arising in robust stability analysis of uncertain implicit systems involving repeated real scalar perturbations. It is shown that in this case the problem is equivalent to a standard μ problem arising in robust control of uncertain LTI input-output models which is well studied and for which several numerical algorithms are applicable. Our results are based on a matrix dilation technique and the redefinition of the uncertainty structure of the transformed problem. The results of the paper are illustrated with a simple numerical example.

The paper is organised as follows. Section 2 contains some background results and highlights the importance of the implicit framework. Section 3 adapts a technique established in a previous paper by the authors [16] and introduces an alternative approach for solving the generalised μ-value problem arising in the analysis and synthesis of uncertain implicit systems. It is shown that in the case of real repeated uncertain parameters, the problem can be expressed in the standard μ setting using a matrix dilation technique and by redefining the uncertainty structure. Both cases of real and complex matrix pair (M, N) are considered.

The notation used in paper is standard and is included here for convenience. $\mathbb{R}, \mathbb{C}$ are the sets of real and complex numbers, respectively. For a given matrix $A \in \mathbb{C}^{n\times m}$ the complex-conjugate transpose is denoted by A'. The characteristic polynomial of $A \in \mathbb{C}^{n\times n}$, $\det(\lambda I_n - A)$, is denoted as $\phi_A(\lambda)$ and the spectrum of A as $\sigma(A)$. We define the right null-space (kernel) and the range (column span) of $A \in \mathbb{C}^{n\times m}$ as $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. We also define $\text{rank}(A) = \dim(\mathcal{R}(A))$ and $\text{null}(A) = \dim(\mathcal{N}(A))$. The rank-nullity theorem then says that $\text{rank}(A) + \text{null}(A) = m$. $N_\perp$ is a maximal rank right annihilator of N, i.e. $NN_\perp = 0$. The singular values of $A \in \mathbb{C}^{n\times m}$ are denoted by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{min(n,m)} \geq 0$. The spectral norm of A is defined as the largest singular value, i.e. $\|A\| = \sigma_1(A)$. $A \otimes B$ denotes the Kronecker product of matrices A and B.

2 | PRELIMINARIES AND PROBLEM DEFINITION

Robust stability of an implicit system can be evaluated via a Linear Fractional Transformation (LFT) of a nominal system and an uncertainty structure[7]. Its canonical form is given by equation (10) in which $M \in \mathbb{C}^{n\times n}$, $N \in \mathbb{C}^{m\times n}$ are given matrices that characterise the system and $\boldsymbol{\Delta}$ is a set of perturbations which in general combines different types of structured uncertainty. This canonical representation can be derived either from transfer function or from a state-space description in discrete time (see previous section)[19,7] and leads to stability analysis under performance constraints.

The generalised structured singular value for implicit systems is defined as follows (see[8,5,7] and[19] for more details). Note that the set $\boldsymbol{\Delta}$ in the definition is restricted to the set of real repeated scalar parameters, which corresponds to the structure considered in this paper. Although more general diagonal structures can be defined (combining real/complex and scalar/matrix perturbations) these will not be considered here.

Definition 2. [[7]] Let M and N be real (or complex) matrices of dimension $n \times n$ and $m \times n$, respectively, with $m \leq n$ Then, the generalised structured singular value $\hat{\mu}_\Delta(M, N)$ of the implicit system is defined as:

$$\hat{\mu}_\Delta(M,N) = \left(\min\left\{\|\Delta\| \,:\, \Delta \in \boldsymbol{\Delta}, \text{null}\begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0\right\}\right)^{-1} \tag{11}$$

where

$$\boldsymbol{\Delta} = \{\text{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) \,:\, \delta_i \in \mathbb{R}, i = 1, 2, \ldots, , s\} \tag{12}$$

in which $\sum_{i=1}^{s} r_i = n$. If for all $\Delta \in \boldsymbol{\Delta}$ the nullity of the pencil is zero, then $\hat{\mu}_\Delta(M, N) = 0$.

The standard definition of the structured singular value is also included below for completeness:

Definition 3. Let $M \in \mathbb{R}^{n\times n}$ (or $M \in \mathbb{C}^{n\times n}$). Then the structured singular value of M is defined as:

$$\mu_{\boldsymbol{\Delta}}(M) = \left(\min\{\|\Delta\| \,:\, \Delta \in \boldsymbol{\Delta}, \det(I_n + \Delta M) = 0\}\right)^{-1} \tag{13}$$

where $\boldsymbol{\Delta}$ is a set of structured perturbations defined in equation (12). If for all $\Delta \in \boldsymbol{\Delta}$ $\det(I_n + \Delta M) \neq 0$, then we define $\mu_{\boldsymbol{\Delta}}(M) = 0$.

Note again that although more general structures can be defined (combining real/complex and scalar/matrix perturbations), we restrict Δ to the set defined in equation (12). It is clear that $\hat{\mu}_\Delta(M, N)$ in Definition 2 is a generalisation of $\mu_{\boldsymbol{\Delta}}(M)$ in Definition 3 and that the two definitions coincide if $N = 0$. The rank of the matrices involved in both problems is full and equal to n when $\Delta = 0$. As the norm of Δ increases, while Δ varies within the structured set, the rank of the two matrices may eventually drop. The minimum norm of Δ at which this occurs (if it does) defines the structured and generalised structured singular value, respectively. If the rank remains equal to n irrespective of the size of the norm, then the structured singular value (resp. generalised structured singular value) is defined as zero.

The computational difficulties of the standard μ-problem are amplified in the general case. Thus, it is more promising to rely on the lower and upper bound calculations in this case as well. The lower bound can be obtained by maximising the modulus of a generalised eigenvalue of a pair of matrices over a bounded structured set. This procedure inherits the problems of algorithms corresponding to the standard μ-problem by exhibiting multiple local maxima.[9]. Fundamental work on upper-bound estimation of the generalised μ-value has been reported in[19,7,5] and[8]. Reference[8] uses the right annihilator of N to transform the nullity constraint:

$$\text{null}\begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0 \Longleftrightarrow \text{null}(N_\perp + \Delta M N_\perp) \neq 0 \tag{14}$$

and then applying convex relaxations to calculate the upper bound of $\hat{\mu}(M,N)$[8] based on a scaling technique. Consider a set $\mathbf{X}$ of positive scaling matrices X that commute with the structure of $\mathbf{\Delta}$, i.e.

$$X = \text{diag}(X_1, X_2, \dots, X_s), X_i = X_i' > 0, X_i \in \mathbb{R}^{r_i \times r_i} \tag{15}$$

Then an upper bound of $\mu_\Delta(M,N)$ is obtained as:

$$\hat{\mu}_{\mathbf{\Delta}}(M,N) = \inf\{\beta > 0 : \exists X \in \mathbf{X} : M'XM - \beta^2 X - N'N < 0\} \tag{16}$$

Unfortunately the class of μ-simple structures for which this convex bound coincides with the actual value of $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ is rather restricted as shown in the following result:

Theorem 1 ([8])**.** In the implicit case the following are simple-μ structures:

(i) $\{\Delta = \delta I : \delta \in \mathbb{C}\}$.

(ii) Full real blocks: $\Delta = \text{diag}(\Delta_1, \dots, \Delta_F) : \Delta_i \in \mathbb{R}^{n_i \times n_i}, F \leq 2$ for M, N real.

(iii) Full complex blocks: $\Delta = \text{diag}(\Delta_1, \dots, \Delta_F) : \Delta_i \in \mathbb{C}^{n_i \times n_i}, F \leq 3$

The following section develops an alternative methodology for calculating $\hat{\mu}_{\mathbf{\Delta}}(M,N)$. This is based on a matrix dilation technique and the transformation of the uncertainty structure $\mathbf{\Delta}$. It is shown that the transformed problem is equivalent to a standard μ-problem. This is potentially significant since, in contrast to $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ computational experience with μ calculations is extensive.

3 | EQUIVALENCE BETWEEN μ AND GENERALISED μ PROBLEM ARISING IN UNCERTAIN IMPLICIT SYSTEMS

3.1 | Implicit structured singular value

Let $M \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^{m \times n}$ with $m \leq n$. We consider the $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ problem defined in equation (11). The uncertainty structure $\mathbf{\Delta}$ throughout the rest of the paper is assumed to be as defined in equation (12). Before stating the main result of the paper we need the following lemma.

Lemma 1. Let $M \in \mathbb{R}^{n \times m}$, $n \geq m$ and define:

$$A = \begin{bmatrix} I_n & M \\ M' & 0_m \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{17}$$

Then $\text{null}(M) = \text{null}(A)$. In particular M has full column rank if and only if matrix A is nonsingular.

Proof. Let $\text{Rank}(M) = r$ and $M = U\Sigma V'$ be the singular value decomposition of M with $UU' = U'U = I_n$, $VV' = V'V = I_m$ and

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_r = \text{diag}(\Sigma_r) \in \mathbb{R}^{r \times r}, \quad \det(\Sigma_r) \neq 0$$

Then:

$$\begin{bmatrix} U' & 0 \\ 0 & V' \end{bmatrix} A \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} I_r & 0 & \Sigma_r & 0 \\ 0 & I_{n-r} & 0 & 0 \\ \Sigma_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence:

$$\text{Rank}(A) = n - r + \text{Rank} \begin{bmatrix} I_r & \Sigma_r \\ \Sigma_r & 0 \end{bmatrix} = n + r$$

Hence from the rank-nullity theorem $\text{null}(A) = n + m - (n + r) = m - r = \text{null}(M)$ as required. The equivalence between the conditions: (i) M is full column rank, and (ii) A is nonsingular follows in the special case $r = m$. □

The following Lemma is standard but is included here for ease of reference.

Lemma 2. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ with $n \geq m$. Then $\text{null}(I_n + BA) = \text{null}(I_m + AB)$.

Proof. First note that

$$\begin{bmatrix} I_n & B \\ 0_{m,n} & I_m \end{bmatrix} \begin{bmatrix} 0_{n,n} & 0_{n,m} \\ A & AB \end{bmatrix} \begin{bmatrix} I_n & -B \\ 0_{m,n} & I_m \end{bmatrix} = \begin{bmatrix} BA & 0_{n,m} \\ A & 0_{m,m} \end{bmatrix}$$

Thus the two matrices:

$$\begin{bmatrix} 0_{n,n} & 0_{n,m} \\ A & AB \end{bmatrix} \text{ and } \begin{bmatrix} BA & 0_{n,m} \\ A & 0_{m,m} \end{bmatrix}$$

are similar and hence:

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(I_n - BA) \Leftrightarrow \lambda^{n-m}\phi_{AB}(\lambda) = \phi_{BA}(\lambda)$$

and hence

$$\phi_{I_m+AB}(\lambda) = \lambda^q \prod_{i\in Z}(\lambda - \lambda_i(AB) - 1),\ Z = \{i : \lambda_i(AB) \neq -1\}$$

and

$$\phi_{I_n+BA}(\lambda) = \lambda^q (\lambda - 1)^{n-m} \prod_{i\in Z}(\lambda - \lambda_i(AB) - 1)$$

Hence $\text{null}(I_m + AB) = \text{null}(I_n + BA) = q$, the algebraic multiplicity of the eigenvalue $\lambda = -1$ of AB (or BA). □

The following Theorem which is our main result shows the equivalence between μ and generalised μ in the real-valued case.

Theorem 2. Consider $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{m\times n}$ with $m \leq n$. Then,

$$\hat{\mu}_{\mathbf{\Delta}}(N, M) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}), \tag{18}$$

where $\hat{\mu}_{\mathbf{\Delta}}(N, M)$ denotes the generalised structured singular value of the pair (N, M) with respect to structure $\mathbf{\Delta}$ as defined in Definition 2 and $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ denotes the structured singular value of $\tilde{M}$ with respect to structure $\tilde{\mathbf{\Delta}}$ as defined in Definition 3 in which

$$\tilde{M} = P' \begin{bmatrix} (I + N'N)^{-1}M' & N'N(I + N'N)^{-1} \\ -M(I + N'N)^{-1}M' & M(I + N'N)^{-1} \end{bmatrix} P \tag{19}$$

Further,

$$\mathbf{\Delta} = \{\text{diag}(\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R},\ i = 1, 2, \ldots, s\} \subseteq \mathbb{R}^{n\times n} \tag{20}$$

$$\tilde{\mathbf{\Delta}} = \{\text{diag}(\delta_1 I_{2r_1}, \ldots, \delta_s I_{2r_s}) : \delta_i \in \mathbb{R},\ i = 1, 2, \ldots, s\} \subseteq \mathbb{R}^{2n\times 2n} \tag{21}$$

and P is a permutation matrix such that $\text{diag}(\mathbf{\Delta}, \mathbf{\Delta}) = P\tilde{\mathbf{\Delta}}P'$.

Proof. From Lemma 1 it follows that

$$v = \text{null} \begin{bmatrix} I + \Delta M \\ N \end{bmatrix} = \text{null} \begin{bmatrix} I_n & 0 & I_n + \Delta M \\ 0 & I_m & N \\ I_n + M'\Delta & N' & 0 \end{bmatrix}$$

which is equivalent to:

$$\nu = \text{null}\left\{ \underbrace{\begin{bmatrix} I_n & 0 & I_n \\ 0 & I_m & N \\ I_n & N' & 0 \end{bmatrix}}_{\Phi} + \begin{bmatrix} 0 & I_n \\ 0 & 0 \\ M' & 0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \right\} \tag{22}$$

Let $\Delta \in \boldsymbol{\Delta}$ and introduce a permutation matrix P such that $\text{diag}(\Delta, \Delta) = P\text{diag}(\tilde{\Delta})P'$, $\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$. (This is just a “reshufling” of $\text{diag}(\Delta, \Delta)$). Let Φ be as defined in equation (22). From Lemma 1, Φ is non-singular. Then, using Lemma 2,

$$\nu = \text{null}\left\{ I_{2n} + P' \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \Phi^{-1} \begin{bmatrix} 0 & I_n \\ 0 & 0 \\ M' & 0 \end{bmatrix} P\tilde{\Delta} \right\}$$

On noting (after some algebra) that

$$\Phi^{-1} = \begin{bmatrix} N'N(I+N'N)^{-1} & -(I+N'N)^{-1}N' & (I+N'N)^{-1} \\ -N(I+N'N)^{-1} & I - N(I+N'N)^{-1}N' & N(I+N'N)^{-1} \\ (I+N'N)^{-1} & (I+N'N)^{-1}N' & -(I+N'N)^{-1} \end{bmatrix}$$

we conclude that:

$$\nu = \text{null} \begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} = \text{null}(I_{2n} + \tilde{M}\tilde{\Delta})$$

where

$$\tilde{M} = P' \begin{bmatrix} (I+N'N)^{-1}M' & N'N(I+N'N)^{-1} \\ -M(I+N'N)^{-1}M' & M(I+N'N)^{-1} \end{bmatrix} P$$

and $\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$. Furthermore, $\Delta \in \boldsymbol{\Delta} \iff \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$ and $\|\Delta\| = \|\tilde{\Delta}\|$. Since

$$\left\{ \text{null} \begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0, \forall \Delta \in \boldsymbol{\Delta} \right\} \iff \left\{ \det(I_{2n} + \tilde{\Delta}\tilde{M}) \neq 0, \forall \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}} \right\}$$

we conclude that

$$\hat{\mu}_{\boldsymbol{\Delta}}(N, M) = 0 \iff \mu_{\tilde{\boldsymbol{\Delta}}}(\tilde{M}) = 0.$$

Similarly, if $\hat{\mu}_{\boldsymbol{\Delta}}(N, M) \neq 0$ we conclude that

$$\begin{aligned} \hat{\mu}_{\boldsymbol{\Delta}}^{-1}(N, M) &= \min \left\{ \|\Delta\| : \ \Delta \in \boldsymbol{\Delta}, \ \text{null} \begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0 \right\} \\ &= \min \left\{ \|\tilde{\Delta}\| : \ \tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}, \ \det(I_{2n} + \tilde{\Delta}\tilde{M}) = 0 \right\} \\ &= \mu_{\tilde{\boldsymbol{\Delta}}}^{-1}(\tilde{M}) \end{aligned}$$

and hence $\hat{\mu}_{\boldsymbol{\Delta}}(N, M) = \mu_{\tilde{\boldsymbol{\Delta}}}(\tilde{M})$ as required. □

Remark 1. Note that the augmentation (18) doubles the number of uncertain parameters computational complexity of the μ calculation may be significant. For the complex data case considered in the next section the number of uncertain parameters increase by a factor of four.

Remark 2. In the standard robust interpretation of the implicit-systems problem matrices M and N are in general complex since they arise by evaluating rational functions of the Z-transform variable z on the unit circle. Real data matrices M and N result only when these two transfer functions are evaluated at $z = 1$ or $z = \pi$ (equivalently at the zero or half-Nyquist angular frequency π rads per sample). There are, however, other constant problems with real data which do not involve this frequency-response interpretation, e.g. model validation problems, see Paganini, Analysis of Implicit Uncertain Systems part II: Constant Matrix Problems and Application to Robust H_2 Analysis. In reference [16] we analyse an approximate GCD problem for multiple polynomials with real coefficients and perturbations via a structured Sylvester resultant approach which has a very similar structure to the implicit systems problem considered here.

Theorem 2 shows that the standard μ-problem and the generalised-μ problem are essentially equivalent. In particular, all theoretical results and numerical algorithms developed for the solution of the former can also be applied directly to the later. This is significant since μ is a well studied problem, with an extensive range of theoretical and numerical work devoted to its calculation.

3.2 | The case of complex matrices M and N

Consider now $M \in \mathbb{C}^{n\times n}$ and $\Delta \in \mathbb{R}^{n\times n}$. Then we have the following result:

Lemma 3. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$ and $\Delta \in \mathbb{R}^{n\times n}$. Then $\det(I + \Delta M) = 0$ if and only if

$$\det\left\{ I_{2n} + \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right\} = 0. \tag{23}$$

Proof. Assume that $\det(I + \Delta M) = 0$. Then there exists $\xi = \alpha + i\beta \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{R}^n$, $\xi \neq 0$ such that

$$\Delta M\xi = -\xi \iff \Delta(X + iY)(\alpha + i\beta) = -\alpha - i\beta, \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq 0$$

This is further equivalent to:

$$\begin{aligned} \Delta M\xi = -\xi \iff & \left. \begin{array}{c} \Delta X\alpha - \Delta Y\beta = -\alpha \\ \Delta Y\alpha + \Delta X\beta = -\beta \end{array} \right\}, \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq 0 \\ \iff & \begin{bmatrix} I + \Delta X & -\Delta Y \\ \Delta Y & I + \Delta X \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0, \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq 0 \\ \iff & \det\left\{ I_{2n} + \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right\} = 0. \end{aligned}$$

This completes the proof. □

The following Lemma is a direct consequence of Lemma 3:

Lemma 4. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $N = A + iB \in \mathbb{C}^{m\times n}$, $A \in \mathbb{R}^{m\times n}$, $Y \in \mathbb{R}^{m\times n}$ with $m \leq n$ and $\Delta \in \mathbb{R}^{n\times n}$. Then:

$$\text{null}\begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0 \iff \text{null}\begin{bmatrix} I_n + \Delta X & -\Delta Y \\ \Delta Y & I_n + \Delta X \\ A & -B \\ B & A \end{bmatrix} \neq 0 \tag{24}$$

Proof. Note that:

$$\text{null}\begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0 \iff \exists x \neq 0, x \in \mathcal{N}(I + \Delta M) \cap \mathcal{N}(N)$$

From Lemma 3 this is equivalent to the existence of a vector $\xi \neq 0$ such that:

$$\left(I_{2n} + (I_2 \otimes \Delta) \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right) \xi = 0 \ \text{ and } \ \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \xi = 0$$

which is also equivalent to the condition stated in the RHS of equation (24). □

The next Theorem generalised Theorem 2 to the case of complex M and N matrices. The solution involves only real data. This is achieved at the expense of dilating the uncertainty structure.

Theorem 3. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $N = A + iB \in \mathbb{C}^{m\times n}$, where $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$ and $m \leq n$. Then,

$$\hat{\mu}_{\mathbf{\Delta}}(M, N) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \tag{25}$$

where

$$\tilde{M} = P' \begin{bmatrix} (I_{2m} + \tilde{N}'\tilde{N})^{-1}\Omega & \tilde{N}'\tilde{N}(I_{2m} + \tilde{N}'\tilde{N})^{-1} \\ -\Omega(I_{2m} + \tilde{N}'\tilde{N})^{-1}\Omega' & \Omega(I_{2m} + \tilde{N}'\tilde{N})^{-1} \end{bmatrix} P, \tag{26}$$

$\mathbf{\Delta}$ is defined in equation (12), $\tilde{\mathbf{\Delta}} = \{\mathrm{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}) : \delta_i \in \mathbb{R}\}$, P is a permutation matrix such that $P(I_4 \otimes \mathbf{\Delta})P' = \tilde{\mathbf{\Delta}}$,

$$\tilde{N} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \quad \text{and} \quad \tilde{\Omega} = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

Proof. Applying Lemma 4 gives:

$$v = \mathrm{null}\begin{bmatrix} I_n + \Delta M \\ N \end{bmatrix} \neq 0 \iff \mathrm{null}\begin{bmatrix} I_n + \Delta X & -\Delta Y \\ \Delta Y & I_n + \Delta X \\ A & -B \\ B & A \end{bmatrix} \neq 0$$

and hence from Lemma 1, $v \neq 0$ if and only if, assuming that $\Delta \in \mathbf{\Delta}$:

$$v = \mathrm{null}\begin{bmatrix} I_n & 0 & 0 & 0 & I_n + \Delta X & -\Delta Y \\ 0 & I_n & 0 & 0 & \Delta Y & I_n + \Delta X \\ 0 & 0 & I_m & 0 & A & -B \\ 0 & 0 & 0 & I_m & B & A \\ I_n + X'\Delta & Y'\Delta & A' & B' & 0 & 0 \\ -Y'\Delta & I_n + X'\Delta & -B' & A' & 0 & 0 \end{bmatrix} \neq 0$$

This is equivalent to:

$$\mathrm{null}\left\{\Phi + \begin{bmatrix} 0 & 0 & 0 & 0 & \Delta X & -\Delta Y \\ 0 & 0 & 0 & 0 & \Delta Y & \Delta X \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X'\Delta & Y'\Delta & 0 & 0 & 0 & 0 \\ -Y'\Delta & X'\Delta & 0 & 0 & 0 & 0 \end{bmatrix}\right\} \neq 0 \tag{27}$$

where we have defined:

$$\Phi = \begin{bmatrix} I_n & 0 & 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & I_n \\ 0 & 0 & I_m & 0 & A & -B \\ 0 & 0 & 0 & I_m & B & A \\ I_n & 0 & A' & B' & 0 & 0 \\ 0 & I_n & -B' & A' & 0 & 0 \end{bmatrix}$$

Note that from Lemma 1 matrix Φ is nonsingular. The second matrix in equation (27) can be factored as:

$$\begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{bmatrix} (I_4 \otimes \Delta) \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & -Y \\ 0 & 0 & 0 & 0 & Y & X \end{bmatrix}$$

Next define permutation matrix P such that

$$P(I_4 \otimes \Delta)P' = \tilde{\Delta} = \mathrm{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s})$$

(this is just a reshuffling of the diagonal entries of $I_4 \otimes \Delta$). Using Lemma (2) then shows that $v \neq 0$ if and only if $\mathrm{null}(I_{4n} + \tilde{M}\tilde{\Delta}) \neq 0$ where

$$\tilde{M} = P' \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & -Y \\ 0 & 0 & 0 & 0 & Y & X \end{bmatrix} \Phi^{-1} \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{bmatrix} P$$

Using the definitions of μ and $\hat{\mu}$ establishes (25). The expression of $\tilde{M}$ given in equation (26) follows after several intricate but straightforward calculations. □

4 | NUMERICAL EXAMPLE

In this section we illustrate our results with a simple numerical example. Consider an implicit dynamic system defined by the pair $(\hat{N}(\lambda), \hat{M}(\lambda))$ where λ is the unit delay operator,

$$\hat{M}(\lambda) = \begin{bmatrix} \lambda & \lambda^2 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad \hat{N}(\lambda) = \begin{bmatrix} \lambda & -1 \end{bmatrix}$$

Uncertainty in the model is introduced as an upper linear fractional transformation of $\begin{bmatrix} \hat{M}(\lambda) \\ \hat{N}(\lambda) \end{bmatrix}$ with $\mathrm{diag}(\delta_1, \delta_2)$ where δ_1 and δ_2 are real uncertain parameters (see Figure 1). Thus the set of all perturbations is

$$\boldsymbol{\Delta} = \{\mathrm{diag}(\delta_1, \delta_2) : \delta_i \in \mathbb{R}, i = 1, 2\}$$

Evaluating $\hat{M}(\lambda)$ and $\hat{N}(\lambda)$ at zero frequency ($\lambda = 1$) gives:

$$M := \hat{M}(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad N := \hat{N}(1) = [1 \;\; -1]$$

Then for $\Delta \in \boldsymbol{\Delta}$:

$$\begin{bmatrix} I_2 + \Delta M \\ N \end{bmatrix} = \begin{bmatrix} 1+\delta_1 & \delta_1 \\ 0 & 1+\delta_2 \\ 1 & -1 \end{bmatrix}$$

loses rank if and only if there exists $(x, y)' \in \mathbb{R}^2, (x, y) \neq 0$ such that

$$\begin{bmatrix} 1+\delta_1 & \delta_1 \\ 0 & 1+\delta_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which implies that $x = y \neq 0, \delta_1 = -\frac{1}{2}$ and $\delta_2 = -1$. Thus $\hat{\mu}_{\boldsymbol{\Delta}}(N, M) = 1$. Next write

$$\left[\begin{array}{cc|c} I_2 & 0_{2,1} & I_2 + \Delta M \\ 0_{1,2} & 1 & N \\ \hline I_2 + M'\Delta & N' & 0 \end{array}\right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1+\delta_1 & \delta_1 \\ 0 & 1 & 0 & 0 & 1+\delta_2 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1+\delta_1 & 0 & 1 & 0 & 0 \\ \delta_1 & 1+\delta_2 & -1 & 0 & 0 \end{array}\right]$$

which can be decomposed as

$$\left[\begin{array}{cc|c} I_2 & 0_{2,1} & I_2 + \Delta M \\ 0_{1,2} & 1 & N \\ \hline I_2 + M'\Delta & N' & 0 \end{array}\right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc|cc} 0 & 0 & 0 & \delta_1 & \delta_1 \\ 0 & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \delta_1 & 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & 0 & 0 & 0 \end{array}\right]$$

Then Theorem 2 says that $\hat{\mu}_{\boldsymbol{\Delta}}(N, M) = \mu_{\tilde{\boldsymbol{\Delta}}}(\tilde{M})$ where

$$\tilde{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\tilde{\boldsymbol{\Delta}} = \{\mathrm{diag}(\delta_1, \delta_1, \delta_2, \delta_2); \delta_1, \delta_2 \in \mathbb{R}\}$$

The singular values of $\tilde{M}$ are:

$$\{3.29623546, \;\; 0.78314066, \;\; 0.69807278, \;\; 0.18497798\}$$

Setting $\Delta = \text{diag}(-\frac{1}{2}, -\frac{1}{2}, -1, -1)$ verifies that $\det(I_4 + \tilde{M}\Delta) = 0$ so that $\mu_{\tilde{\Delta}}(\tilde{M}) \geq 1$. Using Matlab's *mussv* function gives the convex upper bound of $\mu_{\tilde{\Delta}}(\tilde{M}) \leq 1.000599$. Thus,

$$1 \leq \mu_{\tilde{\Delta}}(\tilde{M}) \leq 1.000599 \tag{28}$$

Using the symbolic Matlab toolbox the solutions of equation $\det(I_4 + \tilde{M}\Delta) = 0$ are obtained as

$$\delta_2 = -1 \pm i \frac{2\delta_1 + 1}{\sqrt{\delta_1^2 + 2\delta_1 + 2}}$$

and hence the only real solutions are $\delta_1 = -\frac{1}{2}$ and $\delta_2 = -1$ as expected. Thus, $\mu_{\tilde{\Delta}}(\tilde{M})$ coincides with the lower bound given in equation (28) and is equal to $\hat{\mu}_{\mathbf{\Delta}}(N, M)$.

5 | CONCLUSION

The paper has defined and developed a framework for calculating the generalised structured singular value of implicit systems in the case of real scalar repeated perturbations. By dilating the problem it is shown that it is equivalent to a standard structured singular value calculation for which extensive numerical algorithms are available. Our approach applies to implicit systems described by both real and complex data. Extensions to more general classes of uncertainty are currently under investigation.

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$\square$