Semiclassical Analysis of AdS_3 String Solitons

Ádám Varga

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Department of Mathematics

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Declaration

The work contained in this thesis was carried out by the author while studying for the degree of Doctor of Philosophy at City, University of London. Parts of this work were previously published in the single author papers [1] and [2].

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Abstract

Solitons provide a window into regimes of integrable quantum field theories not directly accessible by the perturbative degrees of freedom. In this thesis we develop techniques for the semiclassical analysis of string solitons on two of the AdS_3 backgrounds with maximal amount of supersymmetry, $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. As the main application of these techniques, we explicitly construct the four and two fermion zero modes for the mixed-flux AdS_3 generalization of the Hofman-Maldacena giant magnon, and show how to match the semiclassically quantized zero modes to the odd generators of the centrally extended $\mathfrak{psu}(1|1)^4$ and $\mathfrak{su}(1|1)^2$ off-shell residual symmetry algebras. We further obtain explicit formulas for the eight bosonic and eight fermionic fluctuations around the mixed-flux magnon, confirming that the semiclassical quantization of these fluctuations leads to a vanishing one-loop correction to the magnon energy, as expected from symmetry based arguments. Lastly, we consider the fermion zero modes for an $AdS_3 \times \mathbb{R}$ string soliton and a simple scattering state of two magnons, confirming the relation between fermion zero modes and representations of the residual algebras.

Chapter 1

Introduction

One of the fundamental challenges of theoretical physics in the last 50 years has been to understand strongly coupled quantum systems, arising for example in the description of nuclear forces (QCD) or condensed-matter systems. Despite all of its successes, the framework of perturbative QFT has been unable to provide analytical results away from the weakly coupled (high-energy) regime, and we have had to rely on numerical methods to investigate important non-perturbative effects, such as the confinement of quarks. A new and promising development in this direction is in terms of gauge/string dualities, stating the equivalence of certain strongly coupled gauge theories to higher dimensional quantum gravity. In broad terms, string theories may provide insight and a new set of tools for understanding strongly coupled quantum systems.

In 1997 Maldacena conjectured a special class of gauge/gravity dualitiy, the AdS/CFT correspondence [3], which relates string theories on backgrounds that contain the anti-de Sitter space-time AdS_{d+1} as a factor, and strongly coupled conformal field theories (CFTs) formulated on the d-dimensional conformally flat boundary of the AdS_{d+1}. The duality provides a dictionary between the two theories, for example by matching the scaling dimensions of gauge-invariant operators to energies of the corresponding closed string states [4]. A more general class of dualities is often referred to as holography [5], where processes in the bulk space are encoded on the boundary. AdS/CFT is a concrete example of holography, where both sides of the duality are well-defined, specific theories, and Maldacena's derivation in terms a decoupling limit of D-branes gives an explanation as to why and how the duality holds.

1.1 $\mathrm{AdS}_5/\mathrm{CFT}_4$

The most renowned example, with maximal amount of supersymmetry, is $\mathrm{AdS}_5/\mathrm{CFT}_4$, the equivalence of four-dimensional $\mathcal{N} = 4$ super Yang-Mills

(SYM) theory [6] with gauge group $\mathrm{SU}(N)$ to type IIB supersting theory on[1] $\mathrm{AdS}_5 \times \mathrm{S}^5$. The parameters of the gauge theory are the 't Hooft coupling[2] $\lambda = g_{YM}^2 N$ and the number of colors N (which is effectively same as the rank $N-1$ of the gauge group for large N), while on the string side we have the effective string tension $\mathrm{h} = R^2/2\pi\alpha'$ (where α' is the string tension and R is the radius of the AdS space) and the string coupling g_s. The AdS/CFT correspondence relates the two sets of parameters by

$$\lambda = 4\pi^2 \mathrm{h}^2 , \qquad \frac{1}{N} = \frac{g_s}{4\pi^2 \mathrm{h}^2} . \tag{1.1}$$

At the core of the duality is the equivalence of the string partition function with vertex operator sources ϕ, taking value J on the boundary of AdS_{d+1}, to the CFT_d partition function with sources J for local operators

$$Z_{\text{string}}[\phi|_{\partial \text{AdS}} = J] = Z_{\text{CFT}}[J]. \tag{1.2}$$

The CFT, however, is fully determined by the scaling dimensions of gauge invariant superconformal primary operators and three-point correlators, all other observables being computable using operator product expansions. The spectrum of the CFT consists of the scaling dimensions Δ, which are the eigenvalues of the dilatation operator $\mathfrak{D}$, one of the Casimirs of the $3+1$ dimensional conformal algebra $\mathrm{SO}(2,4)$

$$\mathfrak{D}\hat{\mathcal{O}}(x) = \Delta(\lambda, N)\hat{\mathcal{O}}(x). \tag{1.3}$$

The duality then relates these to the spectrum energies on the string side

$$\mathcal{H}_{\text{string}} \left|\mathcal{O}\right\rangle = E(\mathrm{h}, g_s) \left|\mathcal{O}\right\rangle , \tag{1.4}$$

according to

$$\Delta(\lambda, N) = E(\mathrm{h}, g_s), \tag{1.5}$$

[1] The geometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$ and its Green-Schwartz string action is reviewed in chapter 2.

[2] Naively g_{YM} and N are the expansion parameters of $\mathcal{N}=4$ SYM, but this rearrangement is customary due to the fact that the large-N limit allows an expansion in $1/N$ if one keeps λ fixed [7]. This *planar* limit is explained below.

where the parameters are related by (1.1).

From the gauge theory perspective the region of small λ is generally called the *weak coupling regime*, and is exactly where perturbative QFT with its Feynman diagrams provides reliable results. Higher loop calculations might give more accurate results for small finite values of the coupling, but conventional methods only allow for the calculation of the first handful of terms in practice, and loop-expansions cannot capture large-λ (non-perturbative) effects. Perturbative string theory, on the other hand, applies in the region around the point $\mathrm{h} = \infty$ and $g_s = 0$. Even though the strings here are weakly coupled, this region is called the *strong coupling regime*, referring to $\lambda = \infty$. The double expansion around this point is characterised by the two directions: finite λ accuracy is increased by adding quantum corrections to the worldsheet sigma model (curvature expansion or "worldsheet loops"), while finite g_s corrections are obtained with a genus expansion of the worldsheet itself ("string loops"). Just like in the gauge theory, both expansions are highly non-trivial, and give unreliable results far away from the point $\lambda = \infty$, $g_s = 0$.

We see that the perturbative regimes of the two models do not overlap, AdS/CFT is a weak/strong duality. This is a very exciting premise, enabling us to understand non-perturbative phenomena on each side in terms of perturbative calculations on the other. At the same time, it makes verifying the duality a daunting task. In the early years tests of the conjecture were only possible for a limited class of operators. Superconformal chiral primaries (protected by supersymmetry from renormalisation) and their descendants were investigated with regard to both their anomalous dimensions [4, 8] and three-point functions [9, 10]. It was also argued that operators with large global charges are dual to semiclassical string states [11, 12].

Planar limit. Keeping λ fixed, $\mathcal{N} = 4$ SYM admits an expansion in $1/N$ [7]. Feynman diagrams in the perturbative expansion can be grouped according to their genus: graphs that can be drawn on a plane without crossing are called planar, and the ones with crossing lines are suppressed. On the string side this is a weak coupling expansion $g_s \sim \lambda/N$, and in the *planar limit*

$$N \to \infty \ , \qquad \lambda = 4\pi^2 \mathrm{h}^2 \ \text{ fixed,} \tag{1.6}$$

we get a free string theory. And as for the duality in this limit, we can be encouraged by the natural appearance of two-dimensional surfaces on which the diagrams are drawn on the gauge side, reminescent of the string worldsheet. Despite the technical difficulties arising from the weak/strong nature of the duality, it turns out that in the planar limit we can find the spectrum exactly (to all loops) on both sides, with the help of *integrability*, a sort of hidden symmetry.

A key indicator of integrability is the presence of a sufficient, in the case of field theories infinite, number of conserved quantities in involution[3]. In the context of the AdS/CFT correspondence, quantum integrabilty (i.e. integrabilty at all loops) is a computational tool-kit for planar SYM at arbitrary coupling, predicting the spectrum of scaling dimension for local gauge-invariant operators as a function of λ. This opens the way for robust tests of the conjecture: in the weak coupling regime one can check agreement with perturbative gauge theory results, while in the strong coupling regime comparison to perturbative string spectrum is possible. Ultimately, integrability can give us valuable insights into a truly quantum gauge and/or string theory at intermediate coupling strengths.

1.1.1 Integrability on the gauge side: spin-chain description

The full symmetry group of $\mathcal{N} = 4$ SYM is $\mathrm{PSU}(2, 2|4)$, also known as the $\mathcal{N} = 4$ superconformal group. The superalgebra $\mathfrak{psu}(2, 2|4)$ has 16 fermionic supercharges, and the bosonic subalgebra is $\mathfrak{su}(2, 2) \oplus \mathfrak{su}(4)$. The $\mathfrak{su}(2, 2) \simeq \mathfrak{so}(2, 4)$ factor is the four dimensional conformal algebra, and $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ is the R-symmetry. Note that $\mathrm{SO}(2, 4)$ and $\mathrm{SO}(6)$ correspond to the global isometry groups of AdS_5 and S^5 on the string side. This $\mathrm{PSU}(2, 2|4)$ symmetry is unbroken by quantum corrections, putting significant constraints on the theory.

Operators fall into representations of the global symmetry group, labelled by the 6 Casimir eigenvalues

$$(\Delta, S_1, S_2; J_1, J_2, J_3), \tag{1.7}$$

[3] Phase space functions f and g are said to be in involution if their Poisson bracket vanishes: $\{f, g\} = 0$, and a quantity is conserved if it is in involution with the Hamitonian.

where J_i are the 3 angular momenta of $SO(6)$, (S_1, S_2) are spins of $\mathrm{SO}(2,4)$ which has a third Casimir, the dilatation operator $\mathfrak{D}$, whose eigenvalues are the scaling dimensions. The highest weight state in each multiplet has the lowest dimension[4], and is called *primary*. The task of finding the spectrum is equivalent to diagonalizing the dilatation operator, which can be written as an expansion in λ

$$\mathfrak{D} = \sum_{n=0}^{\infty} \lambda^n \, \mathfrak{D}^{(2n)}. \tag{1.8}$$

The eigenvalue of $\mathfrak{D}^{(0)}$ is the bare (classical) dimension Δ_0, while the relation $\Delta = \Delta_0 + \gamma(\lambda)$ defines the anomalous scaling dimension γ. In general $\mathfrak{D}$ introduces operator mixing, but this only occurs between operators with the same R-charges, spins and bare dimensions, since the $\mathfrak{D}^{(2n)}$ commute with $\mathfrak{D}^{(0)}$ as well as all other Casimirs. This fact can be used to show the existence of closed sectors. One such sector consists of operators made up from two complex scalars X, Z with classical charges $(1,0,0;1,0,0)$ and $(1,0,0;0,1,0)$, often called the SU(2) sector, since X and Z form a doublet of an SU(2) subgroup of the global SO(6).

In $\mathcal{N} = 4$ SYM the gauge invariant local operators can be constructed as products of traces of the fields that transform covariantly under the gauge group. Furthermore, in the large N limit the dimension of a product of single trace operators is equal to the sum of the individual dimensions, and it is sufficient to understand the spectrum of single trace operators

$$\mathcal{O}(x) = \mathrm{Tr}[\chi_1(x)\chi_2(x)...\chi_L(x)]\,, \tag{1.9}$$

where $\chi_i(x)$ refers to any covariant field. Conformal symmetry fixes the form of two-point correlators

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta}}\,, \tag{1.10}$$

and in the perturbative approach ($\lambda \ll 1$) the dilatation operator can be diagonalized by computing the Feynman diagrams for these correlators.

[4] In a unitary quantum field theory all operators (apart from the identity) must have positive dimension, and elements of the algebra change the scaling dimension in quantized units, hence there is an operator of lowest dimension in each multiplet.

The first hint that planar $\mathcal{N} = 4$ SYM might be integrable was discovered by Minahan and Zarembo [13]. They showed that the 1-loop spectral problem of single-trace scalar operators (forming the so(6) sector) is equivalent to a spin chain with nearest neighbour interactions. The length L of the chain is given by the bare dimension of the operators[5], and the 1-loop dilatation operator is identified with the spin-chain Hamiltonian

$$\mathfrak{D}^{\text{planar}} = L + \frac{\lambda}{4\pi^2}\,\mathcal{H} + O(\lambda^2). \tag{1.11}$$

The spectrum of (1-loop) anomalous dimensions is then equivalent to the energy spectrum of this spin-chain Hamiltonian. Importantly, the spin chain is integrable, hence the planar spectrum is efficiently solvable by the corresponding Bethe Ansatz (BA) [14] (for a modern formulation see [15]).

Let us review this method for the simpler $\mathfrak{su}(2)$ sector, where we only have the scalars X and Z inside the trace. Since these two fields transform in a doublet of $\mathfrak{su}(2)$, we can label them as spin up ($X = \uparrow$) and spin down ($Z = \downarrow$)

$$\mathcal{O} = \text{tr}(XXZX \cdots XZX) \quad \Leftrightarrow \quad |\Psi\rangle = |\uparrow\uparrow\downarrow\uparrow \cdots \uparrow\downarrow\uparrow\rangle\ . \tag{1.12}$$

As we have already seen, mixing must preserve the global charges, so a single trace operator made up of of $L - M$ X fields and M Z fields, with classical charges (1.7) $(L, 0, 0; L - M, M, 0)$ will only mix with operators having the exact same number of X and Z fields, possibly rearranged. This is also reflected in the (1-loop) spin-chain Hamiltonian

$$\mathcal{H}_{\mathfrak{su}(2)} = \frac{1}{2}\sum_{\ell=1}^{L}(1 - P_{\ell,\ell+1})\,, \tag{1.13}$$

where $P_{\ell,\ell+1}$ exchanges the spins at sites ℓ and $\ell + 1$. The ground state for this Hamiltonian is

$$|0\rangle = |\uparrow\uparrow\uparrow \cdots \uparrow\rangle \quad \Leftrightarrow \quad \text{tr}(X^L)\,, \tag{1.14}$$

with zero energy. This is in fact a superconformal chiral primary operator, i.e it commutes with half of the supercharges of $\mathfrak{psu}(2, 2|4)$, and transforms

[5] The bare dimension of a single trace operator made up of scalar fields is equal to the number of fields in the trace.

in a short representation. From general theorems about non-renormalization of short multiplets [10], it follows that this state is protected from quantum corrections, leading to vanishing anomalous dimension to all loop orders (while the spin chain argument only guarantees this at 1-loop).

Let us now diagonalize the Hamiltonian restricting to the case of a single down spin. For such a state the Hamiltonian (1.13) acts like a constant plus a hopping term, moving the down spin one site either to the left or the right

$$\begin{aligned}&\mathcal{H}_{\mathfrak{su}(2)}\,|{\uparrow}\cdots{\uparrow}\overset{\ell}{\downarrow}{\uparrow}\cdots{\uparrow}\rangle\\&\quad=\frac{1}{2}\left(2\,|{\uparrow}\cdots{\uparrow}\overset{\ell}{\downarrow}{\uparrow}\cdots{\uparrow}\rangle-|{\uparrow}\cdots\overset{\ell-1}{\downarrow}{\uparrow}{\uparrow}\cdots{\uparrow}\rangle-|{\uparrow}\cdots{\uparrow}{\uparrow}\overset{\ell+1}{\downarrow}\cdots{\uparrow}\rangle\right).\end{aligned}\tag{1.15}$$

It is then easy to see that the eigenstates are

$$|\mathrm{p}\rangle=\frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}e^{i\mathrm{p}\ell}\,|{\uparrow}\cdots{\uparrow}\overset{\ell}{\downarrow}{\uparrow}\cdots{\uparrow}\rangle\tag{1.16}$$

with

$$\mathcal{H}_{\mathfrak{su}(2)}\,|\mathrm{p}\rangle=\varepsilon(\mathrm{p})\,|\mathrm{p}\rangle\,,\qquad\varepsilon(p)=\frac{\lambda}{2\pi^2}\,\sin^2\frac{\mathrm{p}}{2}\,.\tag{1.17}$$

The state $|\mathrm{p}\rangle$ is called a single magnon with momentum p. Invariance under the shifts $\ell\to\ell+L$ (i.e. periodicity of the spin chain) implies the quantization condition

$$e^{i\mathrm{p}L}=1.\tag{1.18}$$

Due to the cyclicity of single trace operators, we also need to impose invariance under single shifts $\ell\to\ell+1$, and we find that the only physical single-magnon state is the trivial one, with $\mathrm{p}=0$.

The simplest nontrivial physical state has two down spins. The trick is then to suppose that we have a chain of infinite length $L\to\infty$, where we can consider the scattering of two (mostly) well-separated asymptotic magnon states

$$\begin{aligned}|\mathrm{p}_1,\mathrm{p}_2\rangle=&\sum_{\ell_1<\ell_2}e^{i\mathrm{p}_1\ell_1+i\mathrm{p}_2\ell_2}\,|\cdots\overset{\ell_1}{\downarrow}\cdots\overset{\ell_2}{\downarrow}\cdots\rangle\\&+e^{i\phi}\sum_{\ell_1>\ell_2}e^{i\mathrm{p}_1\ell_1+i\mathrm{p}_2\ell_2}\,|\cdots\overset{\ell_2}{\downarrow}\cdots\overset{\ell_1}{\downarrow}\cdots\rangle\,,\end{aligned}\tag{1.19}$$

where we assume that $\mathrm{p}_1 > \mathrm{p}_2$. The first term represents the incoming magnons, and the second, outgoing term appears with the phase factor $e^{i\phi}$, which is the S-matrix S_{12} for their scattering. Requiring that this is an eigenstate we get

$$e^{i\phi} = S_{12} = -\frac{e^{i\mathrm{p}_1+i\mathrm{p}_2} - 2e^{i\mathrm{p}_2} + 1}{e^{i\mathrm{p}_1+i\mathrm{p}_2} - 2e^{i\mathrm{p}_1} + 1}. \tag{1.20}$$

Back on the cyclic spin chain of length L, the trace condition imposes zero total momentum

$$\mathrm{p}_1 + \mathrm{p}_2 = 0. \tag{1.21}$$

The two-particle state should also be invariant under transporting the first magnon around the chain once. In this process we pick up a phase $e^{i\phi}$ when passing the second magnon, and the periodicity condition becomes

$$e^{i\mathrm{p}_1 L} S_{12} = 1. \tag{1.22}$$

Zero total momentum in (1.20) gives $e^{i\phi} = e^{-i\mathrm{p}_1}$, and the allowed quantized momenta are $\mathrm{p}_1 = 2\pi n/(L-1)$, with total anomalous dimension

$$\gamma = \frac{\lambda}{\pi^2} \sin^2 \frac{\pi n}{L-1}\,. \tag{1.23}$$

For multi-particle states it is convenient to define rapidity variables u_j

$$e^{i\mathrm{p}_j} = \frac{u_j + i/2}{u_j - i/2}, \tag{1.24}$$

leading to the simple form of two-particle S-matrices and dispersion relations

$$S_{jk} = \frac{u_j - u_k - i}{u_j - u_k + i}\,, \qquad \varepsilon(u) = \frac{\lambda}{8\,\pi^2}\,\frac{1}{u^2 + 1/4}\,. \tag{1.25}$$

Writing out the M-particle analogue of (1.19) one can show that the multi-particle S-matrices factorize into two-particle S-matrices, so the Bethe equations and cyclicity condition become

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \qquad \prod_{j=1}^{M} \frac{u_j + i/2}{u_j - i/2} = 1\,, \tag{1.26}$$

The 1-loop anomalous dimension of such an M-particle states is

$$\gamma = \sum_{j=1}^{M} \varepsilon(u_j). \tag{1.27}$$

The initial observation [13] was restricted to the $\mathfrak{so}(6)$ scalar sector of operators at one loop. However, shortly after, strong evidence was found that integrability extends to higher loops, with the interaction range of the corresponding spin chain Hamiltonian increasing with loop order [16], and also to the full $\mathcal{N} = 4$ theory (not just its subsectors), first at one loop [17], and later to all loops in the asymptotic limit [18].

1.1.2 Integrability on the string side: coset sigma-model

In a parallel, intertwined development integrable structures were observed in the worldsheet theory of strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$. With 32 supercharges, this is one of the three maximally supersymmetric type IIB 10-d superstring backgrounds, along with flat Minkowski space and the plane-wave background [19]. The group of superisometries for $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory is $\mathrm{PSU}(2, 2|4)$, the same as the full symmetry group of $\mathcal{N} = 4$ SYM, which is most explicit in the Metsaev-Tseytlin formulation [20, 21], where the action is written as a sigma-model on the coset superspace

$$\frac{\mathrm{PSU}(2, 2|4)}{\mathrm{SO}(1, 4) \times \mathrm{SO}(5)} \,. \tag{1.28}$$

Note that the bosonic subspace is isomorphic to $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$\frac{\mathrm{SO}(2, 4)}{\mathrm{SO}(1, 4)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)} \;\simeq\; \mathrm{AdS}_5 \times \mathrm{S}^5 \,. \tag{1.29}$$

It is important to realize that rather than any details of the geometry, it is the $\mathbb{Z}_4$ grading [22] of the supercoset that guarantees integrability [23].

Let us review the construction of the Metsaev-Tseytlin action and its integrability. A semi-symmetric superspace is a coset G/H_0 of a supergroup G over a bosonic subgroup H_0, such that the Lie algebra $\mathfrak{g}$ admits a $\mathbb{Z}_4$ decomposition. This is equivalent to the existence of an order-four automorphism

$\Omega : \mathfrak{g} \to \mathfrak{g}$ that satisfies

$$[\Omega(X), \Omega(Y)\} = \Omega([X, Y\}), \qquad \Omega^4 = \mathrm{id}. \tag{1.30}$$

The worldsheet embedding in G/H_0 is parameterized by a coset representative $g(x) \in G$, up to the gauge transformations $g(x) \to g(x)h(x)$ with $h(x) \in H_0$. The left-invariant current

$$J_{\mathbf{a}} = g^{-1}\partial_{\mathbf{a}} g = J^{(0)}_{\mathbf{a}} + J^{(1)}_{\mathbf{a}} + J^{(2)}_{\mathbf{a}} + J^{(3)}_{\mathbf{a}} \tag{1.31}$$

maps to the superalgebra, where the grading is given by

$$\Omega(J^{(n)}_{\mathbf{a}}) = i^n J^{(n)}_{\mathbf{a}} \; . \tag{1.32}$$

The fermion number is the $\mathbb{Z}_4$ charge mod 2, i.e. $J^{(0)}_{\mathbf{a}}$, $J^{(2)}_{\mathbf{a}}$ describe the bosonic degrees of freedom, while $J^{(1)}_{\mathbf{a}}$, $J^{(3)}_{\mathbf{a}}$ describe fermions. Under gauge transformations $J^{(0)}_{\mathbf{a}}$ transforms as a connection $J^{(0)}_{\mathbf{a}} \to h^{-1}J^{(0)}_{\mathbf{a}}h + h^{-1}\partial_{\mathbf{a}}h$, while all other fields transform in the adjoint representation $J^{(i)}_{\mathbf{a}} \to h^{-1}J^{(i)}_{\mathbf{a}}h$. In terms of these matter fields, the action is given by[6]

$$S = \frac{1}{2}\int \mathrm{d}^2x \; \mathrm{Str}\left(\sqrt{-\gamma}\gamma^{\mathbf{ab}} J^{(2)}_{\mathbf{a}} J^{(2)}_{\mathbf{b}} + \kappa\, \varepsilon^{\mathbf{ab}} J^{(1)}_{\mathbf{a}} J^{(3)}_{\mathbf{b}}\right) \; , \tag{1.33}$$

while the gauge field $J^{(0)}_{\mathbf{a}}$ is absent. It turns out that one needs to take $\kappa = 1$ in order to have κ-symmetry. Varying the action with respect to g, one finds the equations of motion

$$\partial_{\mathbf{a}}\Lambda^{\mathbf{a}} - [J_{\mathbf{a}}, \Lambda^{\mathbf{a}}] = 0, \quad \text{where} \;\; \Lambda^{\mathbf{a}} = \gamma^{\mathbf{ab}} J^{(2)}_{\mathbf{b}} - \frac{1}{2}\varepsilon^{\mathbf{ab}}\left(J^{(1)}_{\mathbf{b}} - J^{(3)}_{\mathbf{b}}\right). \tag{1.34}$$

Expanding these, and the Mauer-Cartan equations

$$\partial_{\mathbf{a}} J_{\mathbf{b}} - \partial_{\mathbf{b}} J_{\mathbf{a}} - [J_{\mathbf{a}}, J_{\mathbf{b}}] = 0, \tag{1.35}$$

[6] Here $\mathrm{Str}(\cdot\,\cdot)$ denotes the G and $\mathbb{Z}_4$ invariant bilinear form on $\mathfrak{g}$.

in terms of the $\mathbb{Z}_4$ components we get

$$\begin{aligned} 2D_{\mathbf{a}}\left(\sqrt{-\gamma}\gamma^{\mathbf{ab}}J^{(2)}_{\mathbf{b}}\right) - \varepsilon^{\mathbf{ab}}[J^{(1)}_{\mathbf{a}}, J^{(1)}_{\mathbf{b}}] + \varepsilon^{\mathbf{ab}}[J^{(3)}_{\mathbf{a}}, J^{(3)}_{\mathbf{b}}] &= 0, \\ \left(\sqrt{-\gamma}\gamma^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right)[J^{(2)}_{\mathbf{a}}, J^{(1)}_{\mathbf{b}}] &= 0, \\ \left(\sqrt{-\gamma}\gamma^{\mathbf{ab}} - \varepsilon^{\mathbf{ab}}\right)[J^{(2)}_{\mathbf{a}}, J^{(3)}_{\mathbf{b}}] &= 0, \\ \varepsilon^{\mathbf{ab}}\left(2D_{\mathbf{a}}J^{(2)}_{\mathbf{b}} + [J^{(1)}_{\mathbf{a}}, J^{(1)}_{\mathbf{b}}] + [J^{(3)}_{\mathbf{a}}, J^{(3)}_{\mathbf{b}}]\right) &= 0, \\ \varepsilon^{\mathbf{ab}}\left(D_{\mathbf{a}}J^{(1)}_{\mathbf{b}} + [J^{(2)}_{\mathbf{a}}, J^{(3)}_{\mathbf{b}}]\right) &= 0, \\ \varepsilon^{\mathbf{ab}}\left(D_{\mathbf{a}}J^{(3)}_{\mathbf{b}} + [J^{(2)}_{\mathbf{a}}, J^{(1)}_{\mathbf{b}}]\right) &= 0, \\ F_{\mathbf{ab}} + [J^{(2)}_{\mathbf{a}}, J^{(2)}_{\mathbf{b}}] + [J^{(1)}_{\mathbf{a}}, J^{(3)}_{\mathbf{b}}] + [J^{(3)}_{\mathbf{a}}, J^{(1)}_{\mathbf{b}}] &= 0, \end{aligned} \tag{1.36}$$

where $D_{\mathbf{a}} = \partial_{\mathbf{a}} + [J^{(0)}_{\mathbf{a}}, \cdot]$ and $F_{\mathbf{ab}} = \partial_{\mathbf{a}}J^{(0)}_{\mathbf{b}} - \partial_{\mathbf{b}}J^{(0)}_{\mathbf{a}} + [J^{(0)}_{\mathbf{a}}, J^{(0)}_{\mathbf{b}}]$. These equations are equivalent to the flatness of the Lax connection [23]

$$\begin{aligned} L_{\mathbf{a}} = J^{(0)}_{\mathbf{a}} + \frac{\mathtt{x}^2+1}{\mathtt{x}^2-1}J^{(2)}_{\mathbf{a}} - \frac{2\mathtt{x}}{\mathtt{x}^2-1}\frac{\gamma_{\mathbf{ab}}\varepsilon^{\mathbf{bc}}}{\sqrt{-\gamma}}J^{(2)}_{\mathbf{c}} + \\ \sqrt{\frac{\mathtt{x}+1}{\mathtt{x}-1}}J^{(1)}_{\mathbf{a}} + \sqrt{\frac{\mathtt{x}-1}{\mathtt{x}+1}}J^{(3)}_{\mathbf{a}} \ , \end{aligned} \tag{1.37}$$

where the spectral parameter $\mathtt{x}$ is an arbitrary complex number $\mathtt{x} \neq \pm 1$. If the currents $J_{\mathbf{a}}$ satisfy the equations of motion, the Lax connection is flat

$$\partial_{\mathbf{a}}L_{\mathbf{b}} - \partial_{\mathbf{b}}L_{\mathbf{a}} - [L_{\mathbf{a}}, L_{\mathbf{b}}] = 0, \tag{1.38}$$

and conversely, if $L_{\mathbf{a}}$ is flat for all values of $\mathtt{x}$, the currents satisfy the equations of motion.

It is a requirement for classical integrability that there are an infinite number of conserved charges. As a consequence of the zero curvature condition (1.38), the *monodromy matrix*, i.e. path ordered exponential (or Wilson loop) of the Lax connection $L_{\mathbf{a}}(\sigma, \tau, \mathtt{x})$

$$T(\tau, \mathtt{x}) = \mathrm{Pexp}\int_0^{2\pi} \mathrm{d}\sigma\, L_{\mathbf{1}}(\sigma, \tau, \mathtt{x}) \tag{1.39}$$

satisfies the equation

$$\partial_{\mathbf{0}}T(\tau, \mathtt{x}) = [L_{\mathbf{0}}, T(\tau, \mathtt{x})]. \tag{1.40}$$

Therefore its eigenvalues, which depend on the complex spectral parameter, are independent of τ, and form an infinite set of conserved quantities.

Given this Lax connection, Kazakov, Marshakov, Minahan and Zarembo used the finite-gap method to write the spectral problem for (classical) string states in the compact sector in terms of a simpler set of integral equations [24], and this construction was later generalized to non-compact sectors [25]. In fact the same integral equations can be obtained from the spin chain, by taking the thermodynamic limit of its Bethe Equations. In an attempt to find the quantum spectrum of the sigma-model, an all-loop BA was reversed engineered from the finite-gap equations, and an approximate S-matrix was proposed in [26]. The finite-gap method proved to be an indispensable tool in the AdS/CFT integrability machinery, and it did not take long until the complete classical algebraic curve for $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory was understood [27].

While the coset action emphasizes the underlying geometry of superisometries in the most elegant form, it is also important to understand the symplectic structure of string theory. To this end, the Hamiltonian of the classical bosonic string propagating on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was shown to be integrable by constructing, in a special gauge, the corresponding Lax representation [28].

1.1.3 Shifting focus to the S-matrix

Quantum integrability is deeply tied to the concept of diffractionless, factorized scattering [14, 15]. It means that the elementary excitations of a quantum many-body system interact only through a sequence of two-body scattering processes which may lead to the exchange of quantum numbers and momenta, but do not alter the magnitudes of the latter. This is the next-best thing to a free system: the only effect of interactions is the permutation of a fixed set of momenta (and other quantum numbers). As we have seen above the early results strongly suggest that both sides of the duality are integrable, and motivated by this, Staudacher proposed that the key to solving the problem is not necessarily to find and diagonalise the full dilatation operator (which was increasingly harder at higher loops), but instead to concentrate on the S-matrix for elementary excitations [29]. On very long operators one can define asymptotic states consisting of magnons at distances exceeding the interaction range, and write down an *asymptotic* Bethe Ansatz (ABA).

Subsequently, Beisert showed, assuming integrability, that the spin-chain S-matrix is completely determined by the $\mathrm{SU}(2|2) \times \mathrm{SU}(2|2)$ symmetries[7], up to an overall phase [30, 31]. In relativistic theories this dressing phase is determined by unitarity and crossing symmetry, which relates scattering of particles to scattering of their antiparticle partners [32]. Adapting this argument to the non-Lorentzian case of $\mathrm{AdS}_5/\mathrm{CFT}_4$, Janik wrote down the crossing equations for Beisert's undetermined phase [33]. The physically relevant solution to Janik's equations is given by the Beisert-Eden-Staudacher (BES) phase [34, 35], whose pole structure can also be explained on physical grounds [36]. On the string side semiclassical (AFS) [26] and 1-loop Hernandez-Lopez (HL) [37] dressing phases were established using the algebraic curve, and were found to be in agreement with the expansion of the all-loop BES phase [35]. The off-shell symmetry algebra and S-matrix were also reproduced for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ (asymptotic) worldsheet excitations, using the Green-Schwarz formalism [38, 39].

1.1.4 Giant magnons

Based on supersymmetry, the BMN limit and periodicity in the momentum p of the excitations, the all-loop dispersion relation for the spin-chain magnon was determined to be [40]

$$\epsilon = \sqrt{1 + 4\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}\ , \tag{1.41}$$

where $4\pi^2\mathrm{h}^2 = \lambda$. This formula is a BPS bound saturation condition, and must be valid for all values of the gauge coupling λ. In particular at strong coupling, where semiclassical string theory is a good description of the integrable theory, we expect to find a classical string configuration with this energy. This solution is the $\mathbb{R} \times \mathrm{S}^2$ giant magnon, found by Hofman and Maldacena [41], with dispersion relation

$$\epsilon = 2\mathrm{h} \sin \frac{\mathrm{p}}{2} \tag{1.42}$$

[7] Subgroup of the $\mathrm{PSU}(2,2|4)$ symmetry group preserved by the spin-chain vacuum.

in agreement with the large h limit of (1.41). On the spin chain side a single magnon is only a physical state in the asymptotic limit, while on the string theory side the giant magnon is an open string state on the decompactified worldsheet, and physical states satisfying the level-matching condition (1.21) can be built by adding other magnons "at infinity". In other words, a closed string can be constructed by gluing together giant magnons with zero total momentum. The periodic dependence on the momentum is natural on a discrete spin chain, and quite interestingly, for the giant magnon it is related to the opening angle between the endpoints on S^2. A detailed presentation of the HM giant magnon can be found in chapter 2.

The asymptotic spectrum of the spin-chain contains an infinite tower of BPS states [42], labelled by a positive integer Q and their momentum p. They have the dispersion relation

$$\epsilon = \sqrt{Q^2 + 4\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}\,, \tag{1.43}$$

where $Q = 1$ corresponds to the elementary magnon (1.41), while states with $Q > 1$ are bound states of these elementary magnons. The classical strings dual to these states are generalizations of the HM giant magnon with an extra angular momentum on the S^5 [43]. These *dyonic* giant magnons live on $\mathbb{R} \times \mathrm{S}^3$ and satisfy the dispersion relation

$$E - J_1 = \sqrt{J_2^2 + 4\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}\,. \tag{1.44}$$

Moreover, after semiclassical quantization J_2 takes integer values, and we reproduce the bound state spectrum (1.43). Let us also mention that there is an interesting S-duality based argument for the non-renormalization of the $J_2 = 1$ single-magnon disperison relation [44]. The giant magnon was further generalized to other configurations with various non-zero R-charges [45, 46, 47, 48].

There are a number of calculations one can perform to check that the giant magnon is indeed the large coupling limit of the elementary excitation of the quantum theory. In an integrable theory the multi-particle S-matrix factorizes, and this was explicitly shown for magnon bound-states on the spin chain [49], dyonic giant magnons on the worldsheet [50], and general light-cone gauge string excitations [51]. A semiclassical analysis of the worldsheet scattering of dyonic giant magnons [52] shows that their 1-loop S-matrix agrees with the Hernandez-Lopez phase [34], and also that the 1-loop correction to the giant magnon energy vanishes. From an algebraic perspective the magnon is a BPS state of the $\mathfrak{su}(2|2)^2_{\text{c.e.}}$ superalgebra, and accordingly, must be part of a 16 dimensional short multiplet [31]. As a consequence the giant magnon should have eight fermionic zero modes, as Hofman and Maldacena argued in [41]. These zero modes were explicitly constructed by Minahan [53], starting from the quadratic fermionic part of the Green-Schwarz action expanded around the giant magnon. Quantizing these modes he was also able to reproduce the odd generators of the residual algebra. Subsequently, building on Minahan's work, an explicit basis of the magnon's fluctuation spectrum was found by Papathanasiou and Spradlin [54], once again confirming that the dispersion relation receives no 1-loop corrections. In chapter 2 we present both the fermion zero modes and the complete fluctuation spectrum of the $\text{AdS}_5 \times \text{S}^5$ giant magnon.

1.2 $\text{AdS}_3/\text{CFT}_2$

Remarkably, integrability persists to other, less symmetric classes of AdS/CFT duals. With less supersymmetry, these models have more realistic properties and understanding them is likely to teach us more general lessons about the equivalence of gauge and gravity theories. One example is $\text{AdS}_4/\text{CFT}_3$, the duality between ABJM Super Chern-Simons gauge theory [55, 56] and type IIA string theory on $\text{AdS}_4 \times CP^3$ [57, 58, 59] with 24 supersymemtries, for a review and more references see [60]. The main focus of this thesis is $\text{AdS}_3/\text{CFT}_2$, and in particular two[8] backgrounds with maximal supersymmetry allowed for such geometries (16 supercharges), $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. While the moduli of T^4 and S^1 are free parameters, supergravity equations for

[8] There is a third maximally supersymmetric AdS_3 background, $\text{AdS}_3 \times \text{S}^3 \times \text{K3}$. It should be possible to apply integrable methods to this background, at least in the orbifold limit of K3, and then it would be interesting to see what the effect of turning on the blow-up modes is.

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ constrain the radii of AdS_3 and S^3 to be equal

$$R_{\mathrm{AdS}_3} = R_{\mathrm{S}^3} \ , \tag{1.45}$$

while for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ the AdS radius R and the radii of the two 3-spheres $R_\pm$ satisfy [61]

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{R^2}. \tag{1.46}$$

This allows for the parametrization of the radii

$$\frac{R^2}{R_+^2} \equiv \cos^2\varphi = \alpha \ , \quad \frac{R^2}{R_-^2} \equiv \sin^2\varphi = 1 - \alpha \ . \tag{1.47}$$

by an angle $\varphi \in (0, \frac{\pi}{2})$, or $\alpha \in (0,1)$. In fact the $\varphi \to 0$ limit of this parametrization covers the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ geometry too, once the blown up sphere is compactified on a torus.

Historically, these backgrounds were considered in two different settings: either supported by Ramond-Ramond (R-R) or Neveu–Schwarz-Neveu–Schwarz (NS-NS) three-form fluxes. The pure NS-NS theory is relatively well understood, the free string spectrum can be solved using a chiral decomposition [62, 63, 64]. No such method exists for the pure R-R case, where the spectrum is believed[9] to be best described by an integrable machinery similar to the AdS_5 case. The type IIB supergravity equations also allow these backgrounds to be supported by a mixture of R-R and NS-NS fluxes

$$\begin{aligned} F &= 2\tilde{q}\big(\,\mathrm{Vol}(\mathrm{AdS}_3) + \cos\varphi\,\mathrm{Vol}(\mathrm{S}^3_+) + \sin\varphi\,\mathrm{Vol}(\mathrm{S}^3_-)\big) \ , \\ H &= 2q\big(\,\mathrm{Vol}(\mathrm{AdS}_3) + \cos\varphi\,\mathrm{Vol}(\mathrm{S}^3_+) + \sin\varphi\,\mathrm{Vol}(\mathrm{S}^3_-)\big) \ , \end{aligned} \tag{1.48}$$

where $q \in [0,1]$ and $\tilde{q} = \sqrt{1-q^2}$. This mixed-flux background interpolates between the qualitatively different pure R-R theory at $q = 0$ and the pure NS-NS theory at $q = 1$. String theory on the above AdS_3 backgrounds was shown to be classically integrable both in the pure R-R [69, 70, 71] and mixed flux [72] cases.

[9] Although it is worth noting that there have been attempts to understand the pure R-R theory using the hybrid formalism of Berkovits, Vafa and Witten [65, 66, 67, 68].

1.2.1 Brane picture and CFT duals

The pure R-R $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ background arises as a near-horizon limit of a stack of N_1 D1- and N_5 D5-branes, with the D5 containing the D1, and the remaining four transverse directions of the D5 branes compactified on a T^4 [3]. The AdS_3 and S^3 radii, as well as the volume of the torus are fixed to be

$$R^2_{\text{AdS}_3} = R^2_{\text{S}^3} = \sqrt{N_1 N_5}\,, \qquad \text{Vol}\,(\text{T}^4) = \frac{N_1}{N_5}\,. \tag{1.49}$$

On the brane intersection this results in a 1+1 dimensional $\text{U}(N_1) \times \text{U}(N_5)$ supersymmetric gauge theory. With the 16 supercharges chirally decomposed under the $\mathfrak{so}(1,1)$ symmetry algebra of boosts along the intersection, the theory has $\mathcal{N} = (4,4)$ supersymmetry. In contrast to $\mathcal{N} = 4$ SYM this gauge theory has, in addition to the adjoint-valued vector multiplet, a number of fundamental- and adjoint-valued hypermultiplets. While this UV gauge theory is not conformal, it flows to a two-dimensional CFT in the low energy limit.

The moduli space of the UV theory has two branches: the Coulomb branch [73] and the Higgs branch [74], with non-zero vacuum expectation of the scalars in the vector- and hyper-multiplets, respectively. In the UV description the Higgs branch represents the dynamics of D1 branes inside the D5 branes, while on the Coulomb branch the D1 branes separate from the D5 branes. In the low energy limit the Higgs branch CFT can be understood in terms of the D1-branes becoming instantons (of instanton number N_1) in the $\text{SU}(N_5)$ gauge theory living on the D5 branes [75]. The CFT is then given by a sigma-model on the instanton moduli space, which is a deformation of the symmetric product orbifold [76, 74, 77, 78]

$$(\text{T}^4)^{N_1 N_5}/S_{N_1 N_5}\,, \tag{1.50}$$

where S_N is the symmetic group. This CFT is conjectured to be dual to string theory on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$.

Similarly to $\mathcal{N} = 4$ SYM, the Higgs branch CFT admits a 't Hooft expansion, now in powers of $\lambda = \frac{N_1}{N_5}$, with non-planar diagrams suppressed by factors of $1/N_1^2$ [79]. In the planar limit

$$N_1 \to \infty\,, \qquad \lambda = \frac{N_1}{N_5} = \text{fixed} \tag{1.51}$$

integrability manifests in terms of a spin chain description, as one would expect from $\mathrm{AdS}_5/\mathrm{CFT}_4$. It was found that in the scalar sector, the 1-loop dilatation operator of single-trace operators corresponds to the Hamiltonian of an integrable homogeneous $\mathfrak{so}(4)$ spin-chain.

Much less is known about the gauge theory duals beyond the pure R-R case. $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with pure NS-NS flux is the near-horizon limit of a F1/NS5 brane system, and for the special case of $k = 1$, i.e. the smallest amount of quantized NS-NS charge, it was recently argued that the CFT dual is a symmetric product orbifold [80, 81, 82, 83]. The conventional interpretation of the mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background is in terms of the near-horizon limit of bound states of D1/D5- and F1/NS5-branes carrying R-R and NS-NS charges, respectively [84]. However, [85] offers an alternative picture: the mixed-flux action is equivalent to the pure NS-NS theory with an R-R modulus turned on, upon identifying q and $\tilde{q}$ as

$$q = k\frac{\alpha'}{R^2}\ , \qquad \tilde{q} = -g_s c_0 k\frac{\alpha'}{R^2}\ . \tag{1.52}$$

where k is quantized in integer units and c_0 is continuous.

Finding the CFT_2 dual of IIB strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ also proved to be a difficult problem [86, 87]. The R-symmetry of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ is $\mathfrak{su}(2)^4$, and the superconformal algebra is enhanced to a *large* $\mathcal{N} = (4,4)$, as opposed to the *small* $\mathcal{N} = (4,4)$ algebra of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with R-symmetry $\mathfrak{su}(2)^2$ [88]. The most promising candidate is based on a brane system with $\mathcal{N} = (0,4)$ supersymmetry. The IR fixed-point of the corresponding gauge theory is conjectured to be a CFT with large $\mathcal{N} = (4,4)$ superconformal symmetry, and a central charge matching that of the holographic dual of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ [89]. More recently it was proposed that, for a special case of the brane charges, the dual CFT is of symmetric orbifold type [90, 91].

1.2.2 Coset models and Integrability

Encouraged by the success of the integrability approach for $\mathrm{AdS}_5/\mathrm{CFT}_4$, it is natural to ask whether Bethe ansatz techniques could be used to calculate the quantum spectrum of AdS_3 theories. As we have seen above, integrability, at least at the classical level, follows from the formulation of string theory as a supercoset sigma-model [92] with a $\mathbb{Z}_4$ grading (1.31) [23]. It turns out that for a special class of cosets, where the superisometry group is the direct product of two identical supergroups $G \cong H \times H$, there is a natural $\mathbb{Z}_4$ automorphism of the Lie superalgebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, given by a combination of the fermion parity operator $(-1)^F$ and the permutation of the two factors [69]:

$$\Omega = \begin{pmatrix} 0 & \mathrm{id} \\ (-1)^F & 0 \end{pmatrix} . \tag{1.53}$$

Furthermore, we see that the invariant subspace is the diagonal bosonic subalgebra, which is isomorphic to the bosonic part of a single factor

$$\mathfrak{h}_0 = \{(X, X) | X \in \mathfrak{h}_{\mathrm{bos}}\} \cong \mathfrak{h}_{\mathrm{bos}} . \tag{1.54}$$

Consequently, the bosonic sector of such a supercoset is isomorphic to the bosonic subgroup of a single factor

$$(H \times H/H_0)_{\mathrm{bos}} \cong H_{\mathrm{bos}} \times H_{\mathrm{bos}}/H_{\mathrm{bos}} \cong H_{\mathrm{bos}} . \tag{1.55}$$

In fact, the maximally supersymmetric AdS_3 string theories can be written as supercoset sigma-models of this direct product type. String theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ gives rise to the small $\mathcal{N} = (4, 4)$ superconformal algebra with rigid part $\mathfrak{psu}(1, 1|2)^2$ [78], while string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ leads to the large $\mathcal{N} = (4, 4)$ superconformal algebra, whose rigid part is $\mathfrak{d}(2, 1; \alpha)^2$ [86, 61], with the parameter determined by the geometry $\alpha = \cos^2\varphi$. We have already pointed out that in the $\varphi \to 0$ limit one of the spheres is blown up, and after recompactification on a torus we get the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background. It is worth mentioning that this limit has a clear interpretation in terms of the superisometry algebras: the $\alpha \to 1$ degeneration of $\mathfrak{d}(2, 1; \alpha)$ contracts to $\mathfrak{psu}(1, 1|2)$ plus some abelian factors.

This motivates the formulation of superstring theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ as a supersymmetric coset model on [93, 92]

$$\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SU}(1,1) \times \mathrm{SU}(2)} \ , \tag{1.56}$$

while the equivalent description of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3$ strings is in terms of the coset representation [69]

$$\frac{D(2,1;\alpha) \times D(2,1;\alpha)}{\mathrm{SU}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)} \ . \tag{1.57}$$

As a quick check, we see that the bosonic subspaces, or equivalently the manifolds of the bosonic subgroups of a single factor (1.55), are isomorphic to

$$\mathrm{SU}(1,1) \times \mathrm{SU}(2) \Big(\times \mathrm{SU}(2) \Big) \ \simeq \ \mathrm{AdS}_3 \times \mathrm{S}^3 \Big(\times \mathrm{S}^3 \Big) \ . \tag{1.58}$$

Note that in view of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, this is not surprising: the two-dimensional conformal algebra is a two-fold tensor product of independent algebras acting on left-, and right-movers, hence we expect all appropriate cosets to be of this two-fold product form.

Missing modes. It is not immediately clear if the supercoset models (1.56), (1.57) are capable of describing the type IIB Green-Schwarz superstring on the ten-dimensional AdS_3 backgrounds, because of the missing flat directions. These missing bosonic modes have to be added by hand, by way of an independent worldsheet CFT on top of the coset sigma-model. In the hybrid formalism, assuming conformal gauge, this is permitted as long as the total central charge vanishes [65]. In the general GS action, however, two issues arise. Firstly, a priori all (10d) bosons couple to all fermions through the kinetic term, hence the bosons corresponding to the flat directions must decouple in a non-trivial way. Secondly, there are 16 fermions (twice the number of supercharges in the $\mathfrak{psu}(1,1|2)$ or $\mathfrak{d}(2,1;\alpha)$ superalgebras) in either of the supercosets compared to the 32 fermions in the ten-dimensional type IIB GS action.

The solution to both of these problems lies in realising that, because of the fermionic κ-symmetry, half of the 32 GS fermions are unphysical anyway. In a paper essentially kicking off the analysis of AdS_3/CFT_2 integrability, Babichenko, Stefanski and Zarembo showed that in a special κ-gauge the S^1 factor decouples from the rest of $AdS_3 \times S^3 \times S^3 \times S^1$ and the resulting κ-fixed GS action is equivalent to the coset (1.57) plus one free boson [69]. The corresponding result for $AdS_3 \times S^3 \times T^4$ can be obtained by taking the $\alpha \to 1$ limit.

Mixed flux. A special property of AdS_3 string theories, when compared to their higher dimensional counterparts, is that the three-form preventing AdS_3 from collapse can be an arbitrary combination of R-R and NS-NS fluxes. In the sigma-model action the NS-NS flux should correspond to a topological Wess-Zumino (WZ) term [94, 95]. We have seen that the $\mathbb{Z}_4$-grading of the coset ensures integrability, however, this alone will not guarantee that the WZ action can be defined [96].

A superspace is called a *permutation coset* if its bosonic section is also a group manifold, and in this case the WZ term can always be constructed [72]. As we have seen above, the supercosets for both AdS_3 backgrounds are of two-fold product form, and the bosonic subspace is isomorphic to the bosonic subgroup of a single factor (1.55), therefore it is indeed a group manifold. The WZ term that one needs to add to the coset action (1.33) takes the form

$$S_{WZ} = q \int_{\mathcal{B}} \mathrm{d}^3x \; \varepsilon^{\mathbf{abc}} \, \mathrm{Str} \left(\frac{2}{3} J^{(2)}_{\mathbf{a}} J^{(2)}_{\mathbf{b}} J^{(2)}_{\mathbf{c}} + J^{(1)}_{\mathbf{a}} J^{(3)}_{\mathbf{b}} J^{(2)}_{\mathbf{c}} + J^{(3)}_{\mathbf{a}} J^{(1)}_{\mathbf{b}} J^{(2)}_{\mathbf{c}} \right) \,, \tag{1.59}$$

where $\mathcal{B}$ is a three-dimensional manifold whose boundary is the string world-sheet. It was shown by Cagnazzo and Zarembo [72] that the sigma-model remains integrable[10] after the addition of the Wess-Zumino term to the action if the parameters satisfy

$$\kappa^2 + q^2 = 1 \,. \tag{1.60}$$

Moreover, the conditions for integrability, κ-symmetry and conformal invari-

[10] For integrability to hold in the more general $\mathbb{Z}_4$-invariant sense, one needs to slightly modify the definition of the supertrace [97]. This is explained in more detail a few paragraphs below.

ance are equivalent to each other. From here on we will use the more customary mixed-flux notation for this specific κ value

$$\kappa = \tilde{q} \equiv \sqrt{1 - q^2}. \tag{1.61}$$

1.2.3 Massive and massless modes

A novel feature of AdS_3 string theories is the presence of massless as well as massive excitations. Near-BMN expansion [69] reveals that string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ has elementary excitations with masses $m = 0,\, 1$, while for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ background one finds states of masses $m = 0,\, \sin^2\varphi$, $\cos^2\varphi,\, 1$. The massless bosons of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ come from the T^4 directions, while for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ theory one massless boson comes from the S^1 and the other results from the freedom of choosing a relative angle between the three-spheres at which the light-like BMN geodesic is taken.

Massive modes. As techniques allowing for incorporation of the massless modes into the the AdS_3 integrability scheme were not immediately available, initial efforts were limited to the more straightforward adaptation of well-established AdS_5 methods for the massive modes of the spectrum. For the pure R-R $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ background, the (massive) finite-gap equations were derived entirely in terms of the group-theory data [69]. Building on this, an all-loop BA was proposed for strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, valid for all values α, and also for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ [70]. In the same paper, investigating the weakly-coupled limit of these BAs, short-range integrable $\mathfrak{d}(2,1;\alpha)^2$ and $\mathfrak{psu}(1,1)^2$ spin-chains were constructed. These spin-chains are *alternating* and *homogeneous*, respectively, and provide valuable hints about the CFT_2 duals[11].

The all-loop S-matrix (for massive modes) of the alternating $\mathfrak{d}(2,1;\alpha)^2$ spin-chain can be bootstrapped from its symmetries [98]. This procedure is conceptually very similar to the construction for $\mathcal{N} = 4$ SYM in $\mathrm{AdS}_5/\mathrm{CFT}_4$ [30], the main difference being that the spin-chain vacuum in this case has a (centrally extended) residual $\mathfrak{su}(1|1)^2$ symmetry algebra. In fact the S-matrix and representations are just projections from $\mathfrak{su}(2|2)$ to $\mathfrak{su}(1|1)^2$. Then,

[11] As opposed to the AdS_5 case, these spin-chains are not derived from the actual CFTs (in the planar limit), but rather conjectured from the symmetries.

using this all-loop S-matrix, a set of modified quantum Bethe equations were proposed for the system [99]. Both the S-matrix and the BA involve four undetermined scalar factors that play a role similar to the dressing phase of $\mathrm{AdS}_5 \times \mathrm{S}^5$. Imposing crossing symmetry it was found that these scalar factors must differ from the BES dressing phase [35], but in the semiclassical limit they reduce to a suitable generalisation of the AFS phase [26]. Furthermore, these phases introduce non-trivial processes between magnons of different masses, a feature unaccounted for in [69, 70]. It is important to mention that these all-loop results are in agreement with the semiclassical calculations using the $\mathfrak{d}(2,1;\alpha)$ algebraic curve of [69]. In particular, it was found that the one-loop S-matrix agrees with the non-perturbative result [98], and so does the one-loop dressing phase [100, 101] with the outcome of [99].

In a similar manner, the all-loop S-matrix and Bethe equations for the massive modes of IIB string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ were derived from the symmetries of the homogeneous $\mathfrak{psu}(1,1|2)^2$ spin-chain, up to two antisymmetric scalar factors in [102]. These dressing phases were then determined by solving the crossing relations and studying their singularity structure [103]. Just like for the $\mathfrak{d}(2,1;\alpha)$ spin-chain, the solutions differ from the BES phase, but at strong coupling to leading order they both reduce to the AFS phase. At next-to-leading order, however, they differ from one-another, only their sum reproducing the HL phase [37].

While integrability of AdS_3 string theories with mixed flux was demonstrated in [72], the WZ term (1.59) brakes the explicit $\mathbb{Z}_4$ symmetry, obscuring the generalisation of standard integrability techniques, such as the algebraic curve and the finite-gap equations, to these backgrounds. A way around this issue is to introduce a non-dynamic, yet $\mathbb{Z}_4$-graded factor into the supercoset action, such that the Lax connection still obeys standard $\mathbb{Z}_4$ relations [97], allowing for the construction of finite-gap equations for the massive sector of mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. In the same paper an all-loop BA was proposed using the massive worldsheet S-matrix of [104], and taking the thermodynamic limit, this BA was shown to reproduce the finite-gap equations. Using semi-classical quantisation of the algebraic curve, the one-loop dressing phases were also predicted.

The fact that the coset sigma-model only describes the AdS_3 GS actions in a specific fermionic gauge [69] somewhat obscures the universality of integrability. In fact, it was found that the non-gauge fixed $AdS_3 \times S^3 \times S^3 \times S^1$ GS string is also integrable to quadratic order in fermions, by constructing the flat Lax connection from the worldsheet Noether currents [71]. Subsequently, the direct worldsheet calculation of the S-matrix [105] was found to be in agreement with the exact results [98], while for the dressing phase it managed to reproduce the results of [101]. However, comparing the perturbative (worldsheet) results of [106, 101, 105] to the exact dressing phases of [103], one finds a discrepancy. This was later argued to be due to the effect of wrapping interactions[12] of massless modes [107]. Although the exact difference was not reproduced, this is a remarkable thought: it is necessary to include massless particles in order to understand the massive sector in the quantum theory.

Perturbative worldsheet calculations were also carried out for the mixed-flux backgrounds. Assuming that integrability holds at the quantum level, Hoare and Tseytlin first calculated the tree-level S-matrix for the massive spectrum of mixed-flux $AdS_3 \times S^3 \times T^4$ in uniform light-cone gauge [108], then, by analysing the constraints of symmetry, they proposed an exact massive worldsheet S-matrix [104], generalising the results of [102] to $q \neq 0$.

Massless modes. Massless excitations move at the speed of light on the worldsheet, therefore scattering between particles of the same chirality cannot take place, and in general a relativistic treatment requires a more abstract notion of an S-matrix [109]. This difficulty in incorporating massless modes into the integrability scheme presented an early challenge to understanding the complete AdS_3/CFT_2 duality using integrable methods. Initial speculations focused on the $\alpha \to 0$ limit of the alternating $\mathfrak{d}(2,1;\alpha)^2$ spin-chains [70, 110], since in this limit two of the light modes of the $AdS_3 \times S^3 \times S^3 \times S^1$ theory (from the worldsheet sigma model perspective at least) smoothly transform into massless modes of $AdS_3 \times S^3 \times T^4$.

It turns out that the traditional way in which the Virasoro constraint had

[12] These interactions "wrap around" the spin-chain. In the decompactification limit they are exponentially supressed for massive modes, but (surprisingly) sub-leading order for the massless sector.

been imposed in the finite gap construction [69] is indeed too strict. After identifying a precise, yet less restrictive condition, the massless modes were successfully incorporated into the finite-gap equations [111]. For strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ this new condition reduces to the old one previously used in the literature. With this revised $\mathfrak{d}(2,1;\alpha)$ algebraic curve, it became possible to study not only spinning string configurations probing the flat directions of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, but also the massless fermions [112].

Complete spectrum. The method that managed to fit both massive and massless excitations into the integrability machinery was the (worldsheet) off-shell symmetry algebra construction for the string S-matrix. Based on the work of Arutyunov, Frolov and Zamaklar for $\mathrm{AdS}_5 \times \mathrm{S}^5$ [38, 39], it was first applied to the pure R-R $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ in [113, 114]. Instead of using the coset sigma model, the starting point of this construction is the full GS action, which automatically includes the massless modes. The off-shell (i.e. lifting the level-matching condition that would rule out single-magnon states) symmetry algebra is constructed from the worldsheet Noether currents, and from this symmetry it is then possible to determine the non-perturbative S-matrix, up to four independent dressing factors. This analysis was further extended to $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed three-form flux in [115].

The corresponding calculations for strings on mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ were carried out in [116]. Fixing light-cone gauge results in the centrally extended $\mathfrak{psu}(1|1)^2_{\text{c.e.}}$ off-shell symmetry algebra, from which the non-perturbative S-matrix of worldsheet excitations can be derived, as usual, up to a number of phase factors satisfying the crossing equations. In contrast to the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ theory, the presence of lighter excitations ($m = \alpha, 1-\alpha$) prompts the question whether the heavy modes ($m = 1$) could be bound states or composites. In fact, this question has not been satisfactorily answered to date.

1.2.4 Giant magnons

Quantum integrability, as we mentioned before, is characterised by diffractionless factorised scattering of asymptotic excitations on the string worldsheet.

From this perspective, giant magnons in AdS_3/CFT_2 were first investigated by David and Sahoo [117]. They noted that the $AdS_5 \times S^5$ classical string solution of Hofman and Maldacena [41] is also a solution on $AdS_3 \times S^3$ (as long as the latter is supported purely by R-R flux through the 3-sphere), since they only require an $\mathbb{R} \times S^2$ subspace that is available in both geometries. Arguing that it is a BPS states in a centrally extended $\mathfrak{su}(1|1)^2$ superalgebra, they found the giant magnon dispersion relation, which was periodic in the worldsheet momentum. In a follow-up paper [118] they derived, from the symmetries of the system, the S-matrix for magnon scattering up to a phase. Using semiclassical methods this phase was calculated to sub-leading order in the strong coupling expansion, also demonstrating in the process that the dispersion relation is one-loop exact, in accordance with the BPS nature of the giant magnons.

The giant magnon on mixed-flux AdS_3 backgrounds will be presented in great technical detail in chapter 2, here we just give a brief summary of the relevant literature. Similarly to the AdS_5/CFT_4 duality, the symmetry algebra can be used to determine both the S-matrix and the all-loop magnon disperison relation [102, 104, 115]

$$\epsilon_\pm = \sqrt{\left(m \pm q\sqrt{\lambda}\frac{\mathrm{p}}{2\pi}\right)^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2\frac{\mathrm{p}}{2}}\ . \tag{1.62}$$

One of the main differences compared to AdS_5/CFT_4 is that the coupling h will receive quantum corrections, and

$$\mathrm{h} = \frac{\sqrt{\lambda}}{2\pi} \tag{1.63}$$

only in the classical string limit. The excitations are of mass $m = 1, 0$ for $AdS_3 \times S^3 \times T^4$ and $m = 0, \sin^2\varphi, \cos^2\varphi, 1$ for $AdS_3 \times S^3 \times S^3 \times S^1$. The mixed-flux $AdS_3 \times S^3 \times T^4$ dyonic giant magnon was found by Hoare, Stepanchuk and Tseytlin [119], with the dispersion relation

$$E - J_1 = \sqrt{(J_2 \pm q\mathrm{hp})^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2\frac{\mathrm{p}}{2}}\ , \tag{1.64}$$

where E is the spacetime energy and J_1, J_2 are two angular momenta on the S^3. They also noted that upon semiclassical quantization J_2 takes integer values, and the lowest $J_2 = 1$ matches the quantum dispersion relation (1.62) if we take the classical value for h. Just like in the AdS_5 case, there are a number of semiclassical checks on these string solutions. The 1-loop worldsheet S-matrix has been determined from multi-soliton scattering states in [120], in agreement with the finite-gap calculations of [97], and the unitarity-cut based conjectures for the 1-loop phases in [121, 122]. The 1-loop correction to the magnon energy can also be calculated from the algebraic curve [123], or directly from the GS action [71, 124].

The off-shell residual symmetry algebras of $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ are the centrally extended $\mathfrak{psu}(1|1)^4$ [102, 114, 113, 115] and the centrally extended $\mathfrak{su}(1|1)^2$ [98, 116], and as a BPS state, the magnon must transform in 4 and 2 dimensional short multiplets of these superalgebras, respectively. Therefore, the mixed-flux magnon on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ should have 4 and 2 fermion zero modes. In chapter 3 we are going to adapt the arguments of Miinahan [53] to the mixed-flux AdS_3 backgrounds to find these fermion zero modes, and use them to construct the odd generators of the residual algebras. In chapter 4 we consider the complete spectrum of fluctuations around the mixed-flux magnon, similar to the AdS_5 calculations of [54].

1.2.5 Recent developments

The AdS_3/CFT_2 duality is an area of active research with many open questions. Our understanding of the CFT duals, especially beyond the pure R-R case, is still somewhat lacking, for the pure NS-NS theory with smallest quantized charge ($k = 1$) they have only recently been argued to be symmetric product orbifolds [80, 81, 82, 83]. Semiclassical methods continue to be useful in probing the string theory side, present thesis being one example, or the calculation of one-loop corrections to rigid spinning string dispersion relations in [125], it seems, however, that massless modes cannot be captured in the semiclassical limit. Instead, to understand these elusive modes we need non-perturbative methods, like the low-energy integrable massless S-matrix and TBA for $AdS_3 \times S^3 \times T^4$ [126, 127, 128], based on the earlier observation of non-trivial massless scattering in the BMN limit [129].

There have been recent advances in our understanding of the protected spectrum of AdS_3/CFT_2 using integrable methods [130, 131]. The protected spectrum for $AdS_3 \times S^3 \times T^4$ agrees with the older results of [132], while the $AdS_3 \times S^3 \times S^3 \times S^1$ case was independently derived using supergravity and WZW methods in [133].

Chapter 2

Solitons of integrable AdS/CFT theories

Solitons are particle-like solutions of integrable field theories, whose dynamics can be captured by a small number of collective degrees of freedom. Quantization of these collective coordinates [134, 135, 136, 137] provides a window into regimes of the quantum theory not directly accessible to perturbation methods. Our main focus in this chapter will be the giant magnon, a classical string solution corresponding to the massive elementary excitations in various instances of AdS/CFT. The chapter is structured as follows.

In section 2.1 we present a detailed semiclassical analysis of the Hofman-Maldacena giant magnon [41], a soliton of the integrable $\mathrm{AdS}_5 \times \mathrm{S}^5$ worldsheet sigma-model [20], together with its fermion zero modes and complete fluctuation spectrum. This serves as a basis of comparison for section 2.2, where we give a classical description of the giant magnon on mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ backgrounds. The semiclassical analysis of this mixed-flux AdS_3 magnon is the main topic of this dissertation, and will be presented in subsequent chapters.

2.1 $\mathrm{AdS}_5/\mathrm{CFT}_4$

In this section we first review the symmetries of $\mathrm{AdS}_5/\mathrm{CFT}_4$, paying particular attention to the representations of the off-shell residual symmetry algebra. We then write down the type IIB superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$, and after a short general discussion of its classical solutions, move on to the detailed presentation of the giant magnon of Hofman and Maldacena [41]. Finally, we give a summary of two important papers dealing with the semiclassical quantization of the giant magnon: the fermion zero mode analysis of Minahan [53], and the calculation of the complete perturbation spectrum, carried out by Papathanasiou and Spradlin [54].

2.1.1 Symmetries

The full symmetry algebra of $\mathrm{AdS}_5/\mathrm{CFT}_4$ is $\mathfrak{psu}(2,2|4)$, but in the excitation picture, considering asymptotic states in the infinite spin limit, the spin-chain/BMN vacuum is manifestly invariant only under the residual subalgebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$. The excitations transform in a $(2|2)$ representation of each $\mathfrak{psu}(2|2)$ factor, which we will write down below. It turns out, however, that $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$ is too limited to describe these off-shell single-particle states. We can get around this problem by enlarging the residual algebra by two unphysical central charges (that will vanish on-shell), essentially introducing a free continuous degree of freedom into the $(2|2)$ representation, capturing the arbitrary momentum of an off-shell particle [30]. An excellent review of this construction can be found in [138], below we just present the centrally extended $\mathfrak{su}(2|2)^2$ algebra and its short representations.

The centrally extended $\mathfrak{su}(2|2)$ algebra

The superalgebra $\mathfrak{su}(2|2)^2$ consists of the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ rotation generators $\mathbf{R}_{ab}$, $\mathbf{L}_{\alpha\beta}$, the supersymmetry generators $\mathbf{Q}_{\alpha b}$, $\mathbf{S}_{a\beta}$, and the central charge $\mathbf{C}$, and its non-trivial commutators are

$$
\begin{aligned}
[\mathbf{R}_{ab}, \mathbf{J}_c] &= \epsilon_{cb}\mathbf{J}_a - \tfrac{1}{2}\epsilon_{ab}\mathbf{J}_c \ , \\
[\mathbf{L}_{\alpha\beta}, \mathbf{J}_\gamma] &= \epsilon_{\gamma\beta}\mathbf{J}_\alpha - \tfrac{1}{2}\epsilon_{\alpha\beta}\mathbf{J}_\gamma \ , \\
\{\mathbf{Q}_{\alpha a}, \mathbf{S}_{b\beta}\} &= \epsilon_{\alpha\beta}\mathbf{R}_{ab} - \epsilon_{ab}\mathbf{L}_{\alpha\beta} - \epsilon_{\alpha\beta}\epsilon_{ab}\mathbf{C} \ ,
\end{aligned}
\tag{2.1}
$$

where $\mathbf{J}$ is any generator with the appropriate index. We can extend this algebra by two additional central charges $\mathbf{P}$, $\mathbf{P}^\dagger$ to get $\mathfrak{su}(2|2) \ltimes \mathbb{R}$

$$
\begin{aligned}
\{\mathbf{Q}_{\alpha a}, \mathbf{Q}_{\beta b}\} &= \epsilon_{\alpha\beta}\epsilon_{ab}\mathbf{P} \ , \\
\{\mathbf{S}_{a\alpha}, \mathbf{S}_{b\beta}\} &= \epsilon_{\alpha\beta}\epsilon_{ab}\mathbf{P}^\dagger \ .
\end{aligned}
\tag{2.2}
$$

Short representation. The $\mathfrak{su}(2|2)^2_{\text{c.e.}}$ superalgebra has a $2|2$-dimensional representation with two bosons $|\phi_a\rangle$ and two fermions $|\psi_\alpha\rangle$ transforming as

$$
\begin{aligned}
\mathbf{R}_{ab}\,|\phi_c\rangle &= \epsilon_{cb}\,|\phi_a\rangle - \tfrac{1}{2}\epsilon_{ab}\,|\phi_c\rangle \, , & \mathbf{L}_{\alpha\beta}\,|\psi_\gamma\rangle &= \epsilon_{\gamma\beta}\,|\psi_\alpha\rangle - \tfrac{1}{2}\epsilon_{\alpha\beta}\,|\psi_\gamma\rangle \ , \\
\mathbf{Q}_{\alpha a}\,|\phi_b\rangle &= a\,\epsilon_{ba}\,|\psi_\alpha\rangle \, , & \mathbf{Q}_{\alpha a}\,|\psi_\beta\rangle &= b\,\epsilon_{\alpha\beta}\,|\phi_a\rangle \, , \\
\mathbf{S}_{a\alpha}\,|\phi_b\rangle &= c\,\epsilon_{ab}\,|\psi_\alpha\rangle \, , & \mathbf{S}_{a\alpha}\,|\psi_\beta\rangle &= d\,\epsilon_{\beta\alpha}\,|\phi_a\rangle \, ,
\end{aligned}
\tag{2.3}
$$

where the closure of the algebra further requires $ad - bc = 1$. For all four states $\chi = \phi_a, \psi_\alpha$ the eigenvalues of the central charges are

$$\begin{aligned} \mathbf{C}\,|\chi\rangle &= \tfrac{1}{2}(ad+bc)\,|\chi\rangle\,, \\ \mathbf{P}\,|\chi\rangle &= ab\,|\chi\rangle\,, \\ \mathbf{P}^\dagger\,|\chi\rangle &= cd\,|\chi\rangle\,, \end{aligned} \tag{2.4}$$

and they satisfy the *shortening condition*

$$\mathbf{C}^2 - \mathbf{P}\mathbf{P}^\dagger = \frac{1}{4}. \tag{2.5}$$

On physical states, transforming under the $\mathfrak{su}(2|2)$ algebra, the central charges $\mathbf{P}, \mathbf{P}^\dagger$ have zero eigenvalues, i.e. $ab = cd = 0$. The two solutions satisfying these conditions have $\mathbf{C} = \pm\frac{1}{2}$, which is too restrictive to capture asymptotic states. However, multiparticle states built from these off-shell excitations can easily be made physical, since only the overall action of $\mathbf{P}, \mathbf{P}^\dagger$ must be trivial.

Representation coefficients. Using the fact that total momentum of physical states vanishes, it can be shown that the values of the central charges for an off-shell one-particle representation are given by [30]

$$\begin{aligned} \mathbf{P} &= \frac{\varsigma \mathrm{h}}{2}\left(e^{-i\mathrm{p}} - 1\right), \\ \mathbf{P}^\dagger &= \frac{\mathrm{h}}{2\varsigma}\left(e^{+i\mathrm{p}} - 1\right), \end{aligned} \tag{2.6}$$

where p is the momentum, and h is the effective string tension, and ς is an arbitrary complex factor, which can be scaled away for single-particle representations, but plays an important role for multi-particle tensor-product representations. Then, from the shortening condition (2.5) we get

$$\mathbf{C} = \pm\frac{1}{2}\sqrt{1 + 4\mathrm{h}^2 \sin^2\tfrac{\mathrm{p}}{2}}. \tag{2.7}$$

Noting that $\mathbf{C}$ is half the Hamiltonian in the full theory, this is equivalent to the celebrated magnon disperison relation

$$\epsilon = \sqrt{1 + 4\mathrm{h}^2 \sin^2 \tfrac{\mathrm{p}}{2}}. \tag{2.8}$$

These formulas can also be derived from the light-cone gauge $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring with the level matching condition relaxed [38].

The centrally extended $\mathfrak{su}(2|2)^2$ algebra

The centrally extended $\mathfrak{su}(2|2)^2$ algebra consists of two copies of $\mathfrak{su}(2|2)^2_{\text{c.e.}}$ (2.1)–(2.2) sharing the same three central charges

$$\{\mathbf{R}_{ab}, \mathbf{L}_{\alpha\beta}, \mathbf{Q}_{\alpha b}, \mathbf{S}_{a\beta}, \mathbf{R}_{\dot{a}\dot{b}}, \mathbf{L}_{\dot{\alpha}\dot{\beta}}, \mathbf{Q}_{\dot{\alpha}\dot{b}}, \mathbf{S}_{\dot{a}\dot{\beta}}, \mathbf{C}, \mathbf{P}, \mathbf{P}^\dagger\}. \tag{2.9}$$

In other words

$$\mathfrak{su}(2|2)^2_{\text{c.e.}} \simeq \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3 \ . \tag{2.10}$$

It has 16 dimensional short representations that are tensor products of two copies of (2.3), sharing $\mathbf{C}, \mathbf{P}$ and $\mathbf{P}^\dagger$. The construction of a similar tensor product representation will be presented for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ off-shell residual algebra $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ in section 2.2.1.

2.1.2 Bosonic string action

As we discussed in chapter 1, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Green-Schwarz action [139] can be written in the explicitly $\mathrm{PSU}(2,2|4)$-symmetric Metsaev-Tseytlin formulation [20], as a sigma model on the supercoset

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)} \ . \tag{2.11}$$

While the explicit integrability of this formalism allows one to describe a large class of (finite-gap) classical string solutions in terms of the associated spectral curve [24, 27], it also somewhat obscures the physical interpretation of the solutions. It is therefore useful to look directly at the Green-Schwartz action, its symmetries and solutions. This is the aim of this subsection.

Geometry of AdS_5 and S^5

Before we write down the action let us look at the definition of AdS_5 and S^5 geometries, and their coordinate parametrizations.

AdS_5. The five dimensional anti-de Sitter space can be represented as a hyperboloid

$$\eta_{\mu\nu} Y^\mu Y^\nu = -Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 = -1, \tag{2.12}$$

in $\mathbb{R}^{2,4}$ with the metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}Y^\mu \mathrm{d}Y^\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1, +1, -1), \tag{2.13}$$

or equivalently in $\mathbb{C}^3$

$$(Z_0, Z_1, Z_2) = (Y_5 + iY_0, Y_1 + iY_2, Y_3 + iY_4) \quad : \quad |Z_0|^2 - |Z_1|^2 - |Z_2|^2 = 1. \tag{2.14}$$

It is useful to solve this in terms of 5 independent global coordinates

$$Z_0 = \cosh\rho\; e^{it}, \quad Z_1 = \sinh\rho \cos\gamma\; e^{i\psi_1}, \quad Z_2 = \sinh\rho \sin\gamma\; e^{i\psi_2}, \tag{2.15}$$

where (γ, ψ_1, ψ_2) have standard S^3 periodicities, and AdS radius takes values $\rho \in [0, \infty)$. Note that $t \in [0, 2\pi)$ already covers the hyperboloid once, and in the context of AdS/CFT it is standard to decompactify the t direction to avoid closed time-like curves. In other words we cut the AdS space open and take $t \in (-\infty, \infty)$. For the simpler case of AdS_2 this is depicted in Figure 2.1. With these coordinates the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - \cosh^2\!\rho\; \mathrm{d}t^2 + \sinh^2\!\rho \left(\mathrm{d}\gamma^2 + \cos^2\!\gamma\; \mathrm{d}\psi_1^2 + \sin^2\!\gamma\; \mathrm{d}\psi_2^2\right). \tag{2.16}$$

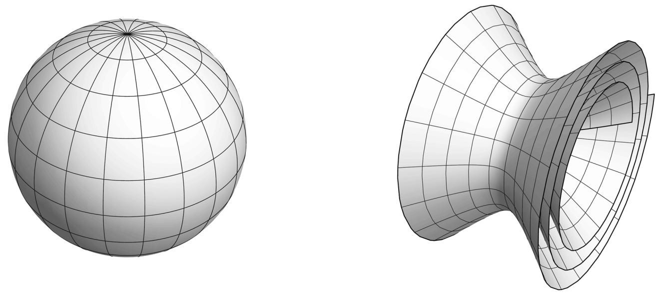

Figure 2.1: Image of a sphere and universal cover of AdS space

S^5. The 5-sphere is the hypersurface

$$\eta_{ij}X^iX^j = X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 1, \tag{2.17}$$

in $\mathbb{R}^6$ with the metric

$$\mathrm{d}s^2 = \eta_{ij}\ \mathrm{d}X^i\mathrm{d}X^j, \qquad \eta_{ij} = \mathrm{diag}(+1,+1,+1,+1,+1,+1). \tag{2.18}$$

Equivalently, in $\mathbb{C}^4$ it is given by

$$(Z_1, Z_2, Z_3) = (X_1 + iX_2, X_3 + iX_4, X_5 + iX_6) \quad : \quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1, \tag{2.19}$$

which can be solved in terms of the Hopf coordinates

$$Z_1 = \sin\theta\cos\varphi\ \ e^{i\phi_1}, \quad Z_2 = \sin\theta\sin\varphi\ e^{i\phi_2}, \quad Z_3 = \cos\theta\ e^{i\phi_3}, \tag{2.20}$$

where the ranges of θ and φ are both $[0, \pi/2]$, and (ϕ_1, ϕ_2, ϕ_3) all take values in $[0, 2\pi)$. With these coordinates the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \cos^2\theta\ \mathrm{d}\phi_3^2 + \sin^2\theta\left(\mathrm{d}\varphi^2 + \cos^2\varphi\ \mathrm{d}\phi_1^2 + \sin^2\varphi\ \mathrm{d}\phi_1^2\right). \tag{2.21}$$

Bosonic action

The general form of the bosonic string sigma-model action is

$$S_B = -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\gamma}\gamma^{ab}G_{MN}(X)\,\partial_aX^M\partial_bX^N, \tag{2.22}$$

where $X^M(\tau,\sigma)$, $M = 0,\dots,9$ are the embedding coordinates, G_{MN} is the target space metric with signature $(-,+,\dots,+)$, and γ_{ab} is the independent 2d metric on the worldsheet $\mathcal{M}$ with signature $(-,+)$. The ranges of σ and τ are taken to be $(-\pi,\pi)$ and $(-\infty,\infty)$ respectively, with periodic boundary conditions on σ.

Equations of motion for γ_{ab} are equivalent to a vanishing worldsheet energy-momentum tensor and give the Virasoro constraints

$$\begin{aligned} T^{ab} &= -\frac{1}{\sqrt{-\gamma}}\frac{\delta S_B}{\delta\gamma_{ab}} = \frac{\mathrm{h}}{2}\Big(g^{ab} - \frac{1}{2}\gamma^{ab}\gamma_{cd}g^{cd}\Big) = 0, \\ g_{ab} &\equiv G_{MN}\partial_a Y^M \partial_b Y^N. \end{aligned} \tag{2.23}$$

Conformal gauge. The action (2.22) is invariant under diffeomorphisms $(\sigma,\tau)\to(\tilde\sigma,\tilde\tau)$ and this is a gauge symmetry of the worldsheet sigma-model. It is a unique feature of 2-dimensional (Lorentzian or Riemannian) manifolds that there exist diffeomorphisms transforming the metric to a globally conformally flat form[1]

$$\begin{aligned} \gamma_{ab}(\sigma,\tau) &= \Omega^2(\sigma,\tau)\eta_{ab}, \\ \sqrt{-\gamma}\gamma^{ab} &= \eta^{ab}. \end{aligned} \tag{2.24}$$

This choice is referred to as *conformal gauge*.

Conformal gauge action. Fixing conformal gauge, the action (2.22) for $AdS_5\times S^5$ bosonic strings can be written as

$$S = -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2x\Big[\eta^{ab}\partial_a Y^\mu \partial_b Y_\mu + \tilde\Lambda\,(Y^2+1)\Big] + \Big[\eta^{ab}\partial_a X^i\partial_b X_i + \Lambda\,(X^2-1)\Big], \tag{2.25}$$

where $\eta^{ab} = \mathrm{diag}(-1,+1)$, and the Lagrange multipliers $\tilde\Lambda,\Lambda$ enforce the embedding coordinates $Y\in\mathbb{R}^{4,2}$, $X\in\mathbb{R}^6$

$$Y^2 = -1, \quad X^2 = 1, \tag{2.26}$$

[1] This can be understood heuristically by noting that γ_{ab}, being symmetric, has three functions' worth of information, while a diffeomorphisms has two functions' worth. One can be used to eliminate the off-diagonal part of γ_{ab} while the other might fix the ratio of the diagonal elements, leaving us with a single functional degree of freedom, $\Omega^2(\sigma,\tau)$.

to lie on AdS_5 and S^5 respectively. The solutions must satisfy the sigma-model equations of motion

$$\begin{aligned}(\partial^2 - \tilde{\Lambda})\, Y = (\partial^2 - \Lambda)\, X = 0,\\ \tilde{\Lambda} = -Y \cdot \partial^2 Y,\\ \Lambda = +X \cdot \partial^2 X,\end{aligned} \tag{2.27}$$

and the conformal gauge Virasoro constraints (2.23)

$$\begin{aligned}(\partial_0 Y)^2 + (\partial_1 Y)^2 + (\partial_0 X)^2 + (\partial_1 X)^2 = 0,\\ \partial_0 Y \cdot \partial_1 Y + \partial_0 X \cdot \partial_1 X = 0.\end{aligned} \tag{2.28}$$

Closed strings are defined on a cylinder, with periodic boundary conditions

$$Y^\mu(\tau, \sigma + 2\pi) = Y^\mu(\tau, \sigma), \qquad X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma). \tag{2.29}$$

The action (2.25) is invariant under the $SO(2,4)$ rotations of Y^μ and $SO(6)$ rotations of X^i with conserved charges

$$S_{\mu\nu} = \mathrm{h} \int_{-\pi}^{\pi} d\sigma \left(Y_\mu \dot{Y}_\nu - Y_\nu \dot{Y}_\mu \right), \qquad J_{ij} = \mathrm{h} \int_{-\pi}^{\pi} d\sigma \left(X_i \dot{X}_j - X_j \dot{X}_i \right), \tag{2.30}$$

There is a natural choice for the Cartan basis of $SO(2,4) \times SO(6)$, corresponding to the 3+3 linear isometries of the $AdS_5 \times S^5$ metric (2.16), (2.21)

$$\begin{aligned}E = S_{50}, \quad S_1 = S_{12}, \quad S_2 = S_{34},\\ J_1 = J_{12}, \quad J_2 = J_{34}, \quad J_3 = J_{56},\end{aligned} \tag{2.31}$$

namely E is the spacetime energy for translations in t, S_i are AdS_5 spins for rotations in ψ_i, and J_i are S^5 angular momenta corresponding to ϕ_i.

Strings on $\mathbb{R} \times S^3$. We will mostly be interested in strings moving on the $\mathbb{R} \times S^3$ subspace of $AdS_5 \times S^5$, where $\mathbb{R}$ refers to AdS_3 time t, and S^3 is a great 3-sphere within S^5. It turns out that there is a residual gauge freedom even after fixing conformal gauge (2.24), corresponding to conformal rescalings of the metric, which in this case we can use to fix (for some constant κ)

$$t = \kappa\tau. \tag{2.32}$$

This is often referred to as *static* gauge, and with Hopf coordinates for S^3 the action can be written as

$$S = -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2\sigma \Big[\ \partial_a\theta\partial^a\theta + \sin^2\theta\ \partial_a\phi_1\partial^a\phi_1 + \cos^2\theta\ \partial_a\phi_2\partial^a\phi_2\Big]\ . \qquad (2.33)$$

The equations of motion read

$$\begin{aligned} \ddot\theta - \theta'' - \sin\theta\cos\theta\left(\dot\phi_1^2 - \phi_1'^2 - \dot\phi_2^2 + \phi_2'^2\right) &= 0\ , \\ \sin^2\theta\left(\ddot\phi_1 - \phi_1''\right) + 2\sin\theta\cos\theta\left(\dot\theta\dot\phi_1 - \theta'\phi_1'\right) &= 0\ , \\ \cos^2\theta\left(\ddot\phi_2 - \phi_2''\right) - 2\sin\theta\cos\theta\left(\dot\theta\dot\phi_2 - \theta'\phi_2'\right) &= 0\ , \end{aligned} \qquad (2.34)$$

which need to be supplemented with the Virasoro constraints

$$\begin{aligned} &\dot\theta^2 + \theta'^2 + \sin^2\theta(\dot\phi_1^2 + \phi_1'^2) + \cos^2\theta(\dot\phi_2^2 + \phi_2'^2) = \kappa^2\ , \\ &\dot\theta\theta' + \sin^2\theta\ \dot\phi_1\phi_1' + \cos^2\theta\ \dot\phi_2\phi_2' = 0\ . \end{aligned} \qquad (2.35)$$

The energy E of $\mathbb{R}\times\mathrm{S}^3$ solutions is fixed by the static gauge condition (2.32)

$$E = 2\pi\mathrm{h}\kappa, \qquad (2.36)$$

and the conserved angular momenta J_1, J_2 are given by

$$J_1 = \mathrm{h}\int_{-\pi}^{\pi}\mathrm{d}\sigma\sin^2\theta\ \dot\phi_1, \qquad J_2 = \mathrm{h}\int_{-\pi}^{\pi}\mathrm{d}\sigma\cos^2\theta\ \dot\phi_2. \qquad (2.37)$$

BMN string. The BMN geodesic is a point-like string moving along a great circle of S^3

$$\theta = \frac{\pi}{2}, \quad \phi_1 = \kappa\tau, \quad \phi_2 = 0, \qquad (2.38)$$

with conserved charges

$$E = J_1 = 2\pi\mathrm{h}\kappa, \quad J_2 = 0. \qquad (2.39)$$

This solution can be regarded as the classical string vacuum, above which the giant magnons represents the elementary excitations. It satisfies

$$E - J_1 = 0, \tag{2.40}$$

corresponding to zero energy[2] in the excitation picture.

Approaches to constructing solutions. The simplest way of finding solutions to the string equations (2.27) is to start with certain natural ansatze. The semiclassical analysis of rigid multi-spin strings constructed using this method served as a useful early test of the AdS/CFT correspondence [140, 141], and in fact this is how Hofman and Maldacena first found the giant magnon [41].

There are also a number of integrability based methods for finding classical strings. A large class of finite gap solutions can be constructed in terms of theta-functions [142] using the spectral curve of the theory [24, 27]. From a given (simple) solution one might generate new non-trivial ones using either the dressing method[3] [143], or Bäcklund transformations [144]. Examples include scattering and bound states of giant magnons with several spins [145, 146], or single-spike strings [147, 148] and their scattering states with multiple spikes [149].

Yet another approach to construct $\mathrm{AdS}_5 \times \mathrm{S}^5$ bosonic strings is to use their equivalence to generalized sine-Gordon (non-abelian Toda) theories based on the Pohlmeyer-reduction [150, 151, 152, 153, 154]. The basic idea is to introduce, instead of the embedding coordinates (Y^μ, X^i), a set of current-type variables that by their definition solve the Virasoro constraints (2.27). Given well-studied solitonic solutions of these integrable generalized sine-Gordon models, one can then invert the currents to find the corresponding string solution. For example, the Pohlmeyer-reduced model for $\mathbb{R} \times \mathrm{S}^3$ is the complex SG model, while AdS_5 strings are equivalent to generalized sinh-Gordon models. The two-spin generalization of the giant magnon was first found using this method [43], but various other examples (scattering and bound states, spiky strings)

[2] In uniform light-cone gauge the Hamiltonian is given by $E - J_1$. This argument is presented in section 2.2.2 for the more complicated case of mixed-flux AdS_3, where the action also includes a Wess-Zumino term.

[3] A detailed explanation of the dressing method can be found in section 2.2.3.

can also be obtained via Pohlmeyer-reduction [151, 155, 156, 157, 158, 159].

2.1.3 Giant magnon

As we have explained in section 1.1.4, the giant magnon is the string dual of the elementary magnon excitation of the spin-chain. A single magnon is only a physical state in the asymptotic limit, which on the string theory side corresponds to the the Hofman-Maldacena limit [41]

$$E,\ J_1 \to \infty \ , \qquad\qquad E - J_1,\ J_2 \ = \ \text{fixed} \ , \tag{2.41}$$

with decompactified worldsheet coordinates[4]

$$x = \kappa\sigma \ , \ t = \kappa\tau \ , \qquad\qquad \kappa \to \infty \ , \ x \in (-\infty, +\infty) \ . \tag{2.42}$$

Changing to these coordinates, we can write the finite combinations of $\mathbb{R} \times \mathrm{S}^3$ charges as

$$\begin{aligned} E - J_1 &= \mathrm{h} \int_{-\infty}^{\infty} \mathrm{d}x \left(1 - \sin^2\theta \ \partial_t \phi_1\right) , \\ J_2 &= \mathrm{h} \int_{-\infty}^{\infty} \mathrm{d}x \ \cos^2\theta \ \partial_t \phi_2 . \end{aligned} \tag{2.43}$$

Hofman-Maldacena giant magnon

The Hofman-Maldacena giant magnon is the $\mathbb{R} \times \mathrm{S}^2$ solution given by

$$Z_1 = \frac{e^{it} \left[b + i \ \tanh \mathcal{X}\right]}{\sqrt{1 + b^2}} \ , \qquad Z_2 = \frac{\mathrm{sech}\mathcal{X}}{\sqrt{1 + b^2}} \ , \tag{2.44}$$

or in terms of the Hopf coordinates

$$\theta = \mathrm{arccot}\left[\frac{\mathrm{sech}\mathcal{X}}{\sqrt{1 + b^2}}\right], \quad \phi_1 = t + \arctan\left[\frac{\tanh \mathcal{X}}{b}\right], \quad \phi_2 = 0, \tag{2.45}$$

where

[4] Slightly abusing notation, the target-space time coordinate is functionally the same as the rescaled worldsheet time.

$$\mathcal{X} = \gamma(x - ut), \quad \mathcal{T} = \gamma(t - ux), \qquad \gamma = \frac{1}{\sqrt{1-u^2}}, \tag{2.46}$$

and the parameters are related by

$$b = u\gamma. \tag{2.47}$$

The magnon is a kink moving on the worldsheet with speed $u \in (0, 1)$. Decompactification opens up the string, and the two endpoints of the magnon move on the equator at the speed of light

$$x \to \pm\infty : \qquad Z_1 \to \exp\left(it \pm i\frac{\Delta\phi_1}{2}\right) , \qquad Z_2 \to 0 , \tag{2.48}$$

where

$$\Delta\phi_1 = 2 \arctan b^{-1} \quad \in (0, \pi) \tag{2.49}$$

is the angle between the string endpoints. A light-cone gauge argument, which we will present for the mixed-flux AdS_3 strings in section 2.2.2, shows that this opening angle is in fact the worldsheet momentum of the magnon $\mathrm{p} = \Delta\phi_1$, and the magnon can also be written as

$$\begin{aligned} Z_1 &= e^{it}\left[\cos\tfrac{\mathrm{p}}{2} + i\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{X}\right], \\ Z_2 &= \sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{X}. \end{aligned} \tag{2.50}$$

Conserved charges. Substituting the solution (2.45) into (2.43) we get

$$E - J_1 = \frac{2\mathrm{h}}{\sqrt{1+b^2}}, \qquad J_2 = 0. \tag{2.51}$$

Which we can express in terms of the opening angle/worldsheet momentum (2.49)

$$E - J_1 = 2\mathrm{h}\,|\sin\tfrac{\mathrm{p}}{2}\,| . \tag{2.52}$$

This agrees with the dispersion relation (2.8) derived from supersymmetry, exactly in the strong coupling, i.e. classical string limit.

Dyonic giant magnon

Generalizing the HM giant magnon to have a second non-zero angular momentum on $\mathbb{R} \times \mathrm{S}^3$, we get the dyonic giant magnon [43]

$$Z_1 = \frac{e^{it}\left[b + i\,\tanh U\right]}{\sqrt{1+b^2}}\;, \qquad Z_2 = \frac{e^{iV}\,\mathrm{sech} U}{\sqrt{1+b^2}}\;, \tag{2.53}$$

where, using the boosted coordinates (2.46)

$$U = \cos\rho\,\mathcal{X}, \quad V = \sin\rho\,\mathcal{T}, \qquad b = u\gamma\,\sec\rho. \tag{2.54}$$

The conserved charges (2.43) are

$$E - J_1 = 2\mathrm{h}\gamma\sec\rho\,\sin^2\tfrac{\mathrm{p}}{2}, \qquad J_2 = 2\mathrm{h}\tan\rho\,\sin^2\tfrac{\mathrm{p}}{2}. \tag{2.55}$$

The extra parameter $\rho \in (0, \frac{\pi}{2})$ controls the amount of J_2 angular momentum, and eliminating it we get the dispersion relation

$$E - J_1 = \sqrt{J_2^2 + 4\mathrm{h}^2\sin^2\tfrac{\mathrm{p}}{2}}. \tag{2.56}$$

Upon semiclassical quantization[5] J_2 takes integer values, and we recover the exact dispersion relation (2.8).

2.1.4 Fermion zero modes of the giant magnon

In this subsection we review the construction and semiclassical quantization of the fermion zero modes of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ giant magnon, following [53]. The conformal gauge quadratic action describing fermion fluctuations about classical string configurations is given by[6] [160]

$$S_{\mathrm{F}} = \mathrm{h}\int \mathrm{d}^2\sigma\;\mathcal{L}_{\mathrm{F}}\;, \qquad \mathcal{L}_{\mathrm{F}} = i\left(\eta^{ab}\delta^{IJ} - \epsilon^{ab}\sigma_3^{IJ}\right)\,\bar{\vartheta}^I\rho_a\mathcal{D}_b\vartheta^J\;, \tag{2.57}$$

[5] The argument is the same, up to trivial modifications, as the semiclassical quantization of the second angular momentum J_2 for the mixed-flux AdS_3 magnon, which we present in section 2.2.4.

[6] For clarity, here we adopt the action as written in [53], which is related to the action in [160] by a linear redefinition of the two spinors ϑ^1, ϑ^2. In later chapters, when considering fermions of the AdS_3 superstring, we take the action as given in [160] .

where $I, J = 1, 2$, the ϑ^I are ten-dimensional Majorana-Weyl spinors, and ρ_a are projections of the ten-dimensional Dirac matrices

$$\rho_a \equiv e_a^A \Gamma_A \ , \qquad e_a^A \equiv \partial_a X^\mu E_\mu^A(X) \ . \tag{2.58}$$

X^μ are the coordinates of AdS_5 for $\mu = 0, 1, 2, 3, 4$ and the coordinates of S^5 for $\mu = \phi, \theta, 7, 8, 9$, as evaluated on the magnon solution (2.45). The covariant derivative is given by

$$\mathcal{D}_a \vartheta^I = \left(\mathrm{D}_a \delta^{IJ} - \frac{i}{2} \epsilon^{IJ} \Gamma_* \rho_a \right) \vartheta^J, \qquad \Gamma_* \equiv i\Gamma_{01234}, \quad \Gamma_*^2 = 1, \tag{2.59}$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, and $\omega_a^{AB} \equiv \partial_a X^\mu \omega_\mu^{AB}$ is the spin connection pulled back to the worldsheet. Changing coordinates to

$$\mathcal{X} = \gamma(x - ut), \qquad \mathcal{T} = \gamma(t - ux), \tag{2.60}$$

the equations of motion for (2.57) become

$$\begin{aligned} (\rho_0 - \rho_1) \left[(1-u)\gamma(D + \partial_{\mathcal{T}})\vartheta^1 - \frac{i}{2}\Gamma_*(\rho_0 + \rho_1)\vartheta^2 \right] &= 0 \ , \\ (\rho_0 + \rho_1) \left[(1+u)\gamma(\tilde{D} - \partial_{\mathcal{T}})\vartheta^2 - \frac{i}{2}\Gamma_*(\rho_0 - \rho_1)\vartheta^1 \right] &= 0 \ , \end{aligned} \tag{2.61}$$

where

$$\begin{aligned} D &= \partial_{\mathcal{X}} + \frac{1}{2} G \, \Gamma_{\phi\theta}, \quad \tilde{D} = \partial_{\mathcal{X}} + \frac{1}{2} \tilde{G} \, \Gamma_{\phi\theta}, \\ G &= \frac{(1+u)\cosh^2\mathcal{X} - 1 + u^2}{\sinh^2\mathcal{X} + u^2} \mathrm{sech}\mathcal{X}, \\ \tilde{G} &= -\frac{(1-u)\cosh^2\mathcal{X} - 1 + u^2}{\sinh^2\mathcal{X} + u^2} \mathrm{sech}\mathcal{X}. \end{aligned} \tag{2.62}$$

Fixing kappa symmetry. Kappa symmetry [161] is a local fermionic symmetry of the Green-Schwarz superstring action [139], ensuring supersymmetry of the physical spectrum. As a result, half of the fermions in the action (2.57) are unphysical. It can be shown that the nilpotent operators

$$(\rho_0 + \rho_1)^2 = (\rho_0 - \rho_1)^2 = 0 \ , \tag{2.63}$$

are exactly half rank, and commute with the covariant derivatives

$$[(\rho_0 - \rho_1), D] = [(\rho_0 + \rho_1), \tilde{D}] = 0. \tag{2.64}$$

Consequently, they can be used to fix kappa-symmetry[7]. The spinors

$$\Psi^1 = i(\rho_0 - \rho_1)\vartheta^1, \quad \Psi^2 = i(\rho_0 + \rho_1)\vartheta^2, \tag{2.65}$$

have only half the degrees of freedom of ϑ^1, ϑ^2, but capture all the dynamics in (2.61) as

$$\begin{aligned} (1-u)\gamma(D + \partial_{\mathcal{T}})\Psi^1 - \frac{i}{2}\Gamma_*(\bar{\rho}_0 - \rho_0)\Psi^2 = 0\ , \\ (1+u)\gamma(\tilde{D} - \partial_{\mathcal{T}})\Psi^2 - \frac{i}{2}\Gamma_*(\bar{\rho}_0 - \rho_0)\Psi^1 = 0\ , \end{aligned} \tag{2.66}$$

where $\bar{\rho}_0 = \Gamma_*\rho_0\Gamma_* = -\rho_0^\dagger$.

Normalizable zero modes. The zero mode solutions satisfy $\partial_{\mathcal{T}}\Psi^J = 0$. Inverting the first equation for Ψ^2, and substituting back into the second, we get the compact expression

$$\left(\frac{1}{\tanh\mathcal{X}}\,D\right)^2\Psi^1 - \Psi^1 = 0. \tag{2.67}$$

Since kappa-fixing commutes with the covariant derivative (2.1.4), this is equivalent to

$$(\rho_0 - \rho_1)\left(\left(\frac{1}{\tanh\mathcal{X}}\,D\right)^2\vartheta^1 - \vartheta^1\right) = 0. \tag{2.68}$$

In other words, we can first solve the equation without worrying about κ-symmetry, and impose the projection $\Psi^1 = i(\rho_0 - \rho_1)\vartheta^1$ at the end of the calculation, to get the kappa-fixed normalizable zero modes

$$\begin{aligned} \Psi^1 &= \frac{i\,\mathrm{sech}\mathcal{X}}{4\sqrt{1-u}}\left(\left(e^{i\chi}\Gamma_0 + e^{-i\chi}\Gamma_\phi\right)U_+ + \left(e^{-i\chi}\Gamma_0 + e^{i\chi}\Gamma_\phi\right)U_-\right), \\ \Psi^2 &= -\frac{i\,\mathrm{sech}\mathcal{X}}{4\sqrt{1+u}}\Gamma_*\Gamma_\phi\left(\left(e^{i\tilde{\chi}}\Gamma_0 + e^{-i\tilde{\chi}}\Gamma_\phi\right)U_+ + \left(e^{-i\tilde{\chi}}\Gamma_0 + e^{i\tilde{\chi}}\Gamma_\phi\right)U_-\right), \end{aligned} \tag{2.69}$$

[7] Kappa symmetry is discussed in much more detail in chapter 3, in relation to the fermion zero modes of the AdS_3 magnon.

where the constant spinors satisfy $\Gamma_{\phi\theta}U_{\pm} = \pm i U_{\pm}$, and the phases are given by

$$\begin{aligned} e^{i\chi} &= \left(\frac{\sinh\mathcal{X}+iu}{\sinh\mathcal{X}-iu}\right)^{1/4}(\tanh\mathcal{X}+i\,\mathrm{sech}\mathcal{X})^{1/2}\,, \\ e^{i\tilde{\chi}} &= \left(\frac{\sinh\mathcal{X}-iu}{\sinh\mathcal{X}+iu}\right)^{1/4}(\tanh\mathcal{X}+i\,\mathrm{sech}\mathcal{X})^{1/2}\,. \end{aligned} \tag{2.70}$$

Zero mode quantization. The Majorana condition implies $U_- = U_+^*$, and we can parametrize the solution in terms of the single Majorana-Weyl spinor $U = \frac{1}{\sqrt{2}}(U_+ + U_-)$. Letting U depend on $\mathcal{T}$, substituting the zero modes (2.69) back into the action (2.57), and integrating over $\mathcal{X}$ we get the zero mode action

$$S_{\mathrm{F},0} = \mathrm{h}\gamma\int \mathrm{d}\mathcal{T}\,\left(-\frac{i}{4}U^T(\Gamma_0+\Gamma_\phi)^T(\Gamma_0+\Gamma_\phi)\partial_{\mathcal{T}}U\right). \tag{2.71}$$

The zero modes are parametrized by U. A general unconstrained 10-d MW spinor has 16 real degrees of freedom, but U further satisfies the light-cone condition $(\Gamma_0 - \Gamma_\phi)U = 0$, which means that there are 8 real fermion zero modes, as expected from representation theory. Writing U in terms of the $\mathrm{SU}(2|2)^2$ representations preserved by the the light cone condition, we can group its components as $U_{\alpha a}$ and $\tilde{U}_{\dot{\alpha}\dot{a}}$ where the $\alpha, \dot{\alpha}$ correspond to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ isometry of the transverse piece of AdS_5, while the indices $a, \dot{a}$ correspond to the $\mathrm{SO}(4)\simeq\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry of the transverse part of S^5. Quantization of these modes leads to the anti-commutators

$$\begin{aligned} \{U_{\alpha a}, U_{\beta b}\} &= \frac{1}{\mathrm{h}\gamma}\varepsilon_{\alpha\beta}\varepsilon_{ab}, \\ \{\tilde{U}_{\dot{\alpha}\dot{a}}, \tilde{U}_{\dot{\beta}\dot{b}}\} &= \frac{1}{\mathrm{h}\gamma}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{a}\dot{b}}, \\ \{U_{\alpha a}, \tilde{U}_{\dot{\beta}\dot{b}}\} &= 0. \end{aligned} \tag{2.72}$$

It is then possible to construct the odd generators of the off-shell residual algebra $\mathfrak{su}(2|2)^2_{\mathrm{c.e.}}$ from these quantized zero modes. For the first $\mathfrak{su}(2|2)_{\mathrm{c.e.}}$ we can take

$$\begin{aligned} \mathbf{Q}_{\alpha a} &= \quad \varsigma^{1/2}\left(\mathcal{A}-\mathcal{B}(-1)^F\right)U_{\alpha a}, \\ \mathbf{S}_{a\alpha} &= -\varsigma^{-1/2}\left(\mathcal{A}+\mathcal{B}(-1)^F\right)U_{\alpha a}, \end{aligned} \tag{2.73}$$

where F is the fermion number operator,

$$\mathcal{A} = \frac{\mathrm{h}}{\sqrt{2}}\left(\sqrt{\frac{\gamma^2}{4\mathrm{h}^2}+1}+1\right)^{1/2}, \qquad \mathcal{B} = \frac{\mathrm{h}}{\sqrt{2}}\left(\sqrt{\frac{\gamma^2}{4\mathrm{h}^2}+1}-1\right)^{1/2}. \tag{2.74}$$

In agreement with (2.6) the central charges take the values[8]

$$\mathbf{P} = \varsigma \mathrm{h} \sin\frac{\mathrm{p}}{2}, \qquad \mathbf{P}^\dagger = \varsigma^{-1}\mathrm{h}\sin\frac{\mathrm{p}}{2}, \tag{2.75}$$

and of course

$$\mathbf{C} = \frac{1}{2}\sqrt{1+4\mathrm{h}^2\sin^2\frac{\mathrm{p}}{2}}. \tag{2.76}$$

The generators for the other $\mathfrak{su}(2|2)_{\text{c.e.}}$ can be constructed from $\tilde{U}_{\dot{\alpha}\dot{a}}$ in the same way.

2.1.5 Semiclassical quantization of the giant magnon

We finish the semiclassical analysis of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ giant magnon by presenting explicit formulas for its complete spectrum of bosonic and fermionic perturbations, as found in [54]. We will also discuss how these solutions can be used to show that the one-loop correction to the magnon energy vanishes.

Bosonic fluctuations

We start from the $\mathrm{AdS}_5 \times \mathrm{S}^5$ bosonic action in the form (2.25)

$$S = -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2x \Big[\eta^{ab}\partial_a Y^\mu \partial_b Y_\mu + \tilde{\Lambda}\,(Y^2+1)\Big] + \Big[\eta^{ab}\partial_a X^i \partial_b X_i + \Lambda\,(X^2-1)\Big], \tag{2.77}$$

where the embedding coordinates $Y \in \mathbb{R}^{4,2}$, $X \in \mathbb{R}^6$ satisfy the equations of motion

$$\begin{aligned} (\partial^2 - \tilde{\Lambda})\, Y = 0, \quad Y^2 = -1, \\ (\partial^2 - \Lambda)\, X = 0, \quad X^2 = 1, \end{aligned} \tag{2.78}$$

and the Lagrange multipliers take the classical values

$$\tilde{\Lambda} = -Y\cdot\partial^2 Y, \qquad \Lambda = +X\cdot\partial^2 X. \tag{2.79}$$

[8] Note that the arbitrary scalings ς are different in (2.6) and (2.75).

The giant magnon. As our bosonic background solution we take the Hofman-Maldacena giant magnon (2.44), but with a general polarization vector $\vec{n}$ in the transverse part of S^5. We choose Y^0 and Y^5 to be the the timelike directions on $\mathbb{R}^{4,2}$, take the equator of S^5 along which the endpoints of the magnon move to lie in the $X^5 - X^6$ plane, and denote the transverse coordinates $X^1, ..., X^4$ by the vector $\vec{X}$. The magnon solution then takes the form

$$\begin{aligned} Y^0 + iY^5 &= e^{it}, \\ \vec{X} &= \vec{n}\,\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{X}, \\ Z \equiv X^5 + iX^6 &= e^{it}\left[\cos\tfrac{\mathrm{p}}{2} + i\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{X}\right], \end{aligned} \tag{2.80}$$

where the magnon's speed on the worldsheet is $u = \cos\frac{\mathrm{p}}{2}$ and the boosted coordinates are

$$\mathcal{X} = \left(x - t\cos\tfrac{\mathrm{p}}{2}\right)\csc\tfrac{\mathrm{p}}{2}, \qquad \mathcal{T} = \left(t - x\cos\tfrac{\mathrm{p}}{2}\right)\csc\tfrac{\mathrm{p}}{2}. \tag{2.81}$$

The Lagrange multipliers (2.79) evaluate to the classical values

$$\tilde{\Lambda} = 1, \qquad \Lambda = 1 - 2\,\mathrm{sech}^2\mathcal{X}. \tag{2.82}$$

AdS_5 fluctuation spectrum. For a classical solution Y, let us write the perturbed solution as

$$Y + \delta\,\tilde{y}, \tag{2.83}$$

where $\delta \ll 1$ and the perturbation $\tilde{y} \in \mathbb{R}^{4,2}$ is a bounded function of the worldsheet coordinates. Substituting into (2.78), (2.79), the terms first order in δ give us the perturbation equation

$$(\partial^2 - 1)\,\tilde{y}^\mu + (Y\cdot\partial^2\tilde{y})\,Y^\mu = 0, \tag{2.84}$$

subject to the orthogonality constraint (to preserve the norm)

$$Y_\mu\,\tilde{y}^\mu = 0. \tag{2.85}$$

For the giant magnon (2.80), orthogonality is automatic if we restrict to the transverse coordinates in AdS_5, and the equations become

$$(\partial^2 - 1)\,\vec{\tilde{y}} = 0, \tag{2.86}$$

giving four free bosons of mass $m = 1$. Besides these, we find a fifth solution along the time-like directions, satisfying

$$\tilde{y}^0 = -f \sin t, \quad \tilde{y}^5 = f \cos t, \qquad \partial^2 f = 0. \tag{2.87}$$

This massless mode, together with a similar S^5 mode, is analogous to longitudinal fluctuations in light-cone gauge, and in a proper quantization of the $AdS_5 \times S^5$ action (2.77) these would be cancelled by ghosts. For our purposes it is sufficient to simply omit them.

S^5 fluctuation spectrum. Substituting the perturbed solution $X + \delta\,\tilde{x}$ into (2.78) we get the perturbation equation

$$\left(\partial^2 - 1 + 2\,\mathrm{sech}^2\mathcal{Y}\right)\tilde{x}^i - (X \cdot \partial^2 \tilde{x})\,X^i = 0, \tag{2.88}$$

subject to

$$X_i\,\tilde{x}^i = 0. \tag{2.89}$$

We will write the solutions in terms of the transverse perturbation vector $\vec{\tilde{x}}$ and two complexified fluctuations

$$z = \tilde{x}^5 + i\tilde{x}^6, \qquad \bar{z} = \tilde{x}^5 - i\tilde{x}^6, \tag{2.90}$$

where $\bar{z}$ is not necessarily the complex conjugate of z, since $\tilde{x}^i$ can themselves be complex. The fluctuations are of the plane-wave form

$$e^{ik\mathcal{X} - i\omega\mathcal{T}} f(\mathcal{X}), \tag{2.91}$$

where $f(\mathcal{Y})$ is a bounded profile, moving together with the magnon. Instead of solving the equations (2.88) directly, the authors of [54] suggest that one can construct the fluctuations using the dressing method[9] [143, 144, 145]. A magnon-breather scattering state may be constructed by dressing the breather

[9] The dressing method will be presented in detail in section 2.2.3.

solution, and expanding this scattering state in the breather momentum, we get the perturbation as the subleading term.

One of the plane-wave solutions is the massless

$$\begin{aligned}
\tilde{\vec{x}} &= e^{ik\mathcal{X}-i\omega\mathcal{T}}\vec{n}\,(k+\omega\cos\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{X}\tanh\mathcal{X}, \\
z &= -ie^{ik\mathcal{X}-i\omega\mathcal{T}}e^{+it}\left[k-\omega\,\sinh\mathcal{X}\sinh(\mathcal{X}+i\tfrac{\mathrm{p}}{2})\right]\mathrm{sech}^2\mathcal{X}, \\
\bar{z} &= +ie^{ik\mathcal{X}-i\omega\mathcal{T}}e^{-it}\left[k-\omega\,\sinh\mathcal{X}\sinh(\mathcal{X}-i\tfrac{\mathrm{p}}{2})\right]\mathrm{sech}^2\mathcal{X},
\end{aligned} \tag{2.92}$$

with dispersion relation

$$\omega^2 = k^2. \tag{2.93}$$

This mode, as explained beneath (2.87), will be omitted in string theory. The remaining four physical fluctuations are parametrized by a polarization vector $\vec{m}$ in the transverse $\mathbb{R}^4$, and are given by

$$\begin{aligned}
\tilde{\vec{x}} &= e^{ik\mathcal{X}-i\omega\mathcal{T}}\left[\vec{m}(k+i\tanh\mathcal{X})-\vec{n}(n\cdot m)\,(k+\omega\cos\tfrac{\mathrm{p}}{2})\,\mathrm{sech}^2\mathcal{X}\right], \\
z &= -ie^{ik\mathcal{X}-i\omega\mathcal{T}}e^{+it}(n\cdot m)\left[k\sinh\mathcal{X}+\omega\sinh(\mathcal{X}+i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right]\mathrm{sech}^2\mathcal{X}, \\
\bar{z} &= +ie^{ik\mathcal{X}-i\omega\mathcal{T}}e^{-it}(n\cdot m)\left[k\sinh\mathcal{X}+\omega\sinh(\mathcal{X}-i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right]\mathrm{sech}^2\mathcal{X},
\end{aligned} \tag{2.94}$$

with dispersion relation

$$\omega^2 = 1+k^2. \tag{2.95}$$

Fermionic fluctuations

The fermion fluctuations are described by the same action (2.57) and equations of motion (2.61) that we wrote down in section 2.1.4. The derivation of fermion fluctuations deviates from the zero mode calculation after the equation (2.66) for kappa-fixed spinors

$$\Psi^1 = i(\rho_0-\rho_1)\vartheta^1, \quad \Psi^2 = i(\rho_0+\rho_1)\vartheta^2, \tag{2.96}$$

which we present here as our starting point

$$\begin{aligned}
(1-u)\gamma(D+\partial_{\mathcal{T}})\Psi^1 - \frac{i}{2}\Gamma_*(\bar{\rho}_0-\rho_0)\Psi^2 = 0\ , \\
(1+u)\gamma(\tilde{D}-\partial_{\mathcal{T}})\Psi^2 - \frac{i}{2}\Gamma_*(\bar{\rho}_0-\rho_0)\Psi^1 = 0\ .
\end{aligned} \tag{2.97}$$

Since all operators and covariant derivatives in these equations are explicitly independent of $\mathcal{T}$, the simplification of (2.67) persists even for the case of $\partial_{\mathcal{T}}\Psi^J \neq 0$, and we get the second order PDE

$$(\rho_0 - \rho_1)\left[\frac{1}{\tanh\mathcal{X}}\left(D - \partial_{\mathcal{T}}\right)\frac{1}{\tanh\mathcal{X}}\left(D + \partial_{\mathcal{T}}\right)\vartheta^1 - \vartheta^1\right] = 0. \tag{2.98}$$

Just like for the zero modes, kappa-fixing commutes with this equation, so we can find solutions ϑ^1 first, and apply the projection (2.96) at the very end.

Solving the equation for ϑ^1. To solve the unprojected equation

$$\frac{1}{\tanh\mathcal{X}}\left(D - \partial_{\mathcal{T}}\right)\frac{1}{\tanh\mathcal{X}}\left(D + \partial_{\mathcal{T}}\right)\vartheta^1 - \vartheta^1 = 0, \tag{2.99}$$

we transform it into a first order system

$$\begin{aligned}\frac{1}{\tanh\mathcal{X}}\left(D + \partial_{\mathcal{T}}\right)\vartheta^1 &= \tilde{\vartheta}^1,\\ \frac{1}{\tanh\mathcal{X}}\left(D - \partial_{\mathcal{T}}\right)\tilde{\vartheta}^1 &= \vartheta^1,\end{aligned} \tag{2.100}$$

where we intorduced a new field $\tilde{\vartheta}^1$. Making a Fourier ansatz for the $\mathcal{T}$-dependence

$$\begin{pmatrix}\vartheta^1\\ \tilde{\vartheta}^1\end{pmatrix} = e^{-i\omega\mathcal{T}}\Theta(\mathcal{X}), \tag{2.101}$$

and decomposing Θ into $\Gamma_{\phi\theta}$ eigenspinors

$$\Theta = \Theta_+ + \Theta_-, \qquad \Gamma_{\phi\theta}\Theta_\pm = \pm i\Theta_\pm, \tag{2.102}$$

the equations (2.100) can be written in the matrix form

$$(\partial_{\mathcal{X}} - A_\pm)\Theta_\pm = 0, \qquad A_\pm = \begin{pmatrix} i(\omega \mp \frac{G}{2}) & \tanh\mathcal{X}\\ \tanh\mathcal{X} & i(\omega \pm \frac{G}{2})\end{pmatrix}, \tag{2.103}$$

where G was defined in (2.62). The trick is to find an invertible transformation

$$\Theta_\pm \to \Theta'_\pm = S\Theta_\pm \tag{2.104}$$

such that the transformed equation

$$(\partial_{\mathcal{X}} - A'_{\pm})\Theta'_{\pm} = 0, \qquad A'_{\pm} = (\partial_{\mathcal{X}} S + S A_{\pm})S^{-1}, \tag{2.105}$$

has a diagonal $A'_{\pm}$. For the details of this calculation the reader is referred to the discussion in [54], here we just mention that, remarkably, there is a choice of S that diagonalizes both A'_{+} and A'_{-}

$$A'_{\pm} = \begin{pmatrix} i(\omega \mp \frac{G}{2}) & 0 \\ 0 & i(\omega \mp \frac{G}{2}) \end{pmatrix}, \tag{2.106}$$

and the most general solution to (2.100) is

$$\vartheta^1 = \mathrm{sech}\mathcal{X}\sqrt{\omega \cosh 2\mathcal{X} + k}\, e^{i\alpha} \left[e^{+i\chi} U_+ + e^{-i\chi} U_- \right], \tag{2.107}$$

where $U_{\pm}$ are constant Weyl spinors satisfying $\Gamma_{\phi\theta} U_{\pm} = \pm i U_{\pm}$, χ is the same as (2.70), α is given in (2.110), and the dispersion relation is

$$\omega^2 = 1 + k^2. \tag{2.108}$$

The κ-fixed solution. After applying the projection (2.96) and substituting back into (2.97) we get

$$\begin{aligned}
\Psi^1 &= i \csc \tfrac{\mathrm{p}}{4} \sqrt{\omega + k}\, \mathrm{sech}\mathcal{X}\sqrt{\omega \cosh 2\mathcal{X} + k}\, e^{i\alpha} \left[e^{+i\chi}\Gamma_0 + e^{-i\chi}\Gamma_\phi \right] \mathcal{P}U, \\
\Psi^2 &= \sec \tfrac{\mathrm{p}}{4} \sqrt{\omega - k}\, \mathrm{sech}\mathcal{X}\sqrt{\omega \cosh 2\mathcal{X} - k}\, e^{i\beta}\Gamma_*\Gamma_\phi \left[e^{+i\tilde{\chi}}\Gamma_0 + e^{-i\tilde{\chi}}\Gamma_\phi \right] \mathcal{P}U,
\end{aligned} \tag{2.109}$$

where the phases $\chi, \tilde{\chi}$ can be found in (2.70), α and β are given by

$$\begin{aligned}
e^{i\alpha} &= e^{ik\mathcal{X} - i\omega\mathcal{T}} \left(\frac{1 + i\omega \sinh 2\mathcal{X}}{1 - i\omega \sinh 2\mathcal{X}} \frac{1 - ik \tanh 2\mathcal{X}}{1 + ik \tanh 2\mathcal{X}} \right)^{1/4}, \\
e^{i\beta} &= e^{ik\mathcal{X} - i\omega\mathcal{T}} \left(\frac{1 - i\omega \sinh 2\mathcal{X}}{1 + i\omega \sinh 2\mathcal{X}} \frac{1 - ik \tanh 2\mathcal{X}}{1 + ik \tanh 2\mathcal{X}} \right)^{1/4},
\end{aligned} \tag{2.110}$$

the dispersion relation is still

$$\omega^2 = 1 + k^2, \tag{2.111}$$

and half of the 16 (complex) degrees of freedom of the constant Weyl spinor U are projected out by

$$\mathcal{P} = \frac{1}{2}\left[(1 - i\Gamma_{\phi\theta}) - \Gamma_0\Gamma_\phi(1 + i\Gamma_{\phi\theta})\right]. \tag{2.112}$$

It is shown in [54] that the Majorana condition can be consistently applied, leaving in total 8 different kinds of fermionic fluctuations (each an infinite family parametrized by the wavenumber k).

The 1-loop functional determinant

Using the explicit form of the fluctuations we can calculate the leading order quantum correction to the classical magnon energy, based on well-established techniques for the semiclassical quantization of solitons [162, 137, 136, 163]. The classical energy of the giant magnon is

$$\epsilon_{cl} = \frac{\sqrt{\lambda}}{\pi} \,|\sin \tfrac{\mathrm{p}}{2}\,|, \tag{2.113}$$

and expanding the exact relation

$$\epsilon = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2 \tfrac{\mathrm{p}}{2}} = \epsilon_{cl}\left(1 + \mathcal{O}(1/\lambda)\right) \tag{2.114}$$

we see that the first $\mathcal{O}(1/\sqrt{\lambda})$ correction is zero, and we should expect a vanishing one-loop correction from our fluctuation calculation.

The semiclassical one-loop correction comes from evaluating the functional determinant $\ln\det|\delta^2 S|$ around the classical background, and can be calculated as

$$\frac{1}{2}\sum_{i,k}(-1)^F\nu_i, \tag{2.115}$$

where F is the fermion number operator, and ν_i are frequencies of small oscillations around the classical solution, also called *stability angles*. We can calculate these by following the method of Dashen, Hasslacher and Neveu [162]. Putting the system in a box of length $L \gg 1$, with periodic boundary conditions $x \cong x + L$, we see that our magnon solution will also be periodic in worldsheet time, with period $T = L/u$, and the stability angle of a generic fluctuation $\delta\phi$ can be read off from

$$\delta\phi(t + T, x) = e^{-i\nu}\delta\phi(t, x). \tag{2.116}$$

It is then a straightforward to determine the stability angles for the four massive AdS_5 bosons $\tilde{y}$ (these are simple plane waves $e^{ik\mathcal{X}-i\omega\mathcal{T}}$)

$$\nu_k(\tilde{y}) = \frac{L}{u}\frac{\omega + uk}{\sqrt{1-u^2}}, \tag{2.117}$$

four physical S^5 bosons $\tilde{x}$ (2.94)

$$\nu_k(\tilde{x}) = \frac{L}{u}\frac{\omega + uk}{\sqrt{1-u^2}} + 2\cot^{-1} k, \tag{2.118}$$

and 8 fermion fluctuations Ψ (2.109)

$$\nu_k(\Psi) = \frac{L}{u}\frac{\omega + uk}{\sqrt{1-u^2}} + \cot^{-1} k. \tag{2.119}$$

Summing these, with a minus sign for fermions, gives the expected result that the one-loop correction to the magnon energy is zero

$$\sum_i (-1)^F \nu_i = 4\,\nu_k(\tilde{y}) + 4\,\nu_k(\tilde{x}) - 8\,\nu_k(\Psi) = 0. \tag{2.120}$$

2.2 AdS_3/CFT_2

The first half of this section is structured similarly to section 2.1. We start by reviewing the symmetries of AdS_3/CFT_2, and write down the IIB superstring action on mixed-flux $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. We then present the dressing method, a systematic way of generating soliton solutions of these integrable string theories. Applying the dressing method we construct the $AdS_3 \times S^3 \times T^4$ mixed-flux giant magnon, first found by Hoare, Stepanchuk and Tseytlin [119], and also a string soliton on $AdS_3 \times S^1$. The only original contribution in this chapter is the identification of a one-parameter restriction of the mixed-flux magnon, that we call *stationary*. This stationary magnon can be considered the mixed-flux generalization of the HM giant magnon, and will be the starting point of the semiclassical analysis performed in subsequent chapters. Finally, we outline how these solutions can be put on $AdS_3 \times S^3 \times S^3 \times S^1$.

2.2.1 Symmetries

The symmetry algebra of $AdS_3 \times S^3 \times T^4$ string theory is $\mathfrak{psu}(1,1|2)^2$ [78], while superstrings on $AdS_3 \times S^3 \times S^3 \times S^1$ possess a $\mathfrak{d}(2,1;\alpha)^2$ symmetry [86, 61]. Just like in the AdS_5 case, the elementary excitations transform under the off-shell residual symmetry algebras: the centrally extended $\mathfrak{su}(1|1)^2$ superalgebra for $AdS_3 \times S^3 \times S^3 \times S^1$ [116], and centrally extended $\mathfrak{psu}(1|1)^4$ superalgebra for $AdS_3 \times S^3 \times T^4$ [114]. Here we review these algebras, together with their short representations.

The $\mathfrak{su}(1|1)$ algebra

To build up the (centrally extended) $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, we start with the simple $\mathfrak{psu}(1|1)$ algebra, consisting of two anticommuting supercharges $\mathbf{Q}$ and $\mathbf{S}$. Adding the central charge $\mathbf{H}$, i.e. introducing the non-trivial anticommutation relation

$$\{\mathbf{Q}, \mathbf{S}\} = \mathbf{H} \;, \tag{2.121}$$

we get the $\mathfrak{su}(1|1)$ algebra. In its simplest non-trivial representation a bosonic state $|\phi\rangle$ and a fermionic state $|\psi\rangle$ transform under the charges according to

$$\begin{aligned} &\mathbf{Q}\,|\phi\rangle = a\,|\psi\rangle \;, \qquad &&\mathbf{S}\,|\phi\rangle = 0 \;, \qquad &&\mathbf{H}\,|\phi\rangle = H\,|\phi\rangle \;, \\ &\mathbf{Q}\,|\psi\rangle = 0 \;, \qquad &&\mathbf{S}\,|\psi\rangle = b\,|\phi\rangle \;, \qquad &&\mathbf{H}\,|\psi\rangle = H\,|\psi\rangle \;. \end{aligned} \tag{2.122}$$

For closure of the algebra the eigenvalue of the central charge must be $H = ab$. In fact the representation is labelled by H alone, the ratio of a and b is physically irrelevant, it only parametrizes the difference in normalization of the states $|\phi\rangle$ and $|\psi\rangle$. Let us denote this representation by $(\mathbf{1}|\mathbf{1})_H$.

The $\mathfrak{su}(1|1)^2$ algebra

In the direct product of two $\mathfrak{su}(1|1)$ algebras we have two copies (left and right) of each charge, satisfying

$$\begin{aligned} &\{\mathbf{Q}_{\text{L}}, \mathbf{S}_{\text{L}}\} = \mathbf{H}_{\text{L}}, \qquad &&\{\mathbf{Q}_{\text{L}}, \mathbf{Q}_{\text{R}}\} = 0, \qquad &&\{\mathbf{Q}_{\text{L}}, \mathbf{S}_{\text{R}}\} = 0, \\ &\{\mathbf{Q}_{\text{R}}, \mathbf{S}_{\text{R}}\} = \mathbf{H}_{\text{R}}, \qquad &&\{\mathbf{S}_{\text{L}}, \mathbf{S}_{\text{R}}\} = 0, \qquad &&\{\mathbf{Q}_{\text{R}}, \mathbf{S}_{\text{L}}\} = 0. \end{aligned} \tag{2.123}$$

When coupling these two systems, we can introduce the total Hamiltonian $\mathbf{H}$ and the angular momentum $\mathbf{M}$

$$\mathbf{H} = \mathbf{H}_{\mathrm{L}} + \mathbf{H}_{\mathrm{R}}, \qquad \mathbf{M} = \mathbf{H}_{\mathrm{L}} - \mathbf{H}_{\mathrm{R}}. \tag{2.124}$$

In terms of these generators we have

$$\{\mathbf{Q}_{\mathrm{L}}, \mathbf{S}_{\mathrm{L}}\} = \tfrac{1}{2}(\mathbf{H} + \mathbf{M}), \qquad \{\mathbf{Q}_{\mathrm{R}}, \mathbf{S}_{\mathrm{R}}\} = \tfrac{1}{2}(\mathbf{H} - \mathbf{M}). \tag{2.125}$$

Representations. Irreducible representations will be tensor products of a left-moving and a right-moving part, since the algebra is a direct product. For later convenience we take $\mathbf{S}_{\mathrm{L}}$ and $\mathbf{Q}_{\mathrm{R}}$ to be raising operators, while $\mathbf{Q}_{\mathrm{L}}$ and $\mathbf{S}_{\mathrm{R}}$ will be lowering operators. A highest weight state then satisfies

$$\mathbf{S}_{\mathrm{L}} |\text{h.w.}\rangle = 0, \qquad \mathbf{Q}_{\mathrm{R}} |\text{h.w.}\rangle = 0. \tag{2.126}$$

In a *short* representation a highest weight state will be annihilated by additional supercharges. For the $\mathfrak{su}(1|1)^2$ algebra the two shortening conditions are $H_{\mathrm{L}} = 0$ and $H_{\mathrm{R}} = 0$. A representation where the h.w. state has vanishing H_{R}, and is therefore annihilated by $\mathbf{S}_{\mathrm{R}}$, is called a *left-moving* representation. The simplest non-trivial example is given by $(\mathbf{1}|\mathbf{1})_H \otimes \mathbf{1}$, with a bosonic state $|\phi\rangle$ and a fermionic state $|\psi\rangle$ transforming as

$$\begin{aligned}
&\mathbf{Q}_{\mathrm{L}} |\phi\rangle = a |\psi\rangle \ , && \mathbf{S}_{\mathrm{L}} |\phi\rangle = 0 \ , && \mathbf{H}_{\mathrm{L}} |\phi\rangle = H |\phi\rangle \ , \\
&\mathbf{Q}_{\mathrm{L}} |\psi\rangle = 0 \ , && \mathbf{S}_{\mathrm{L}} |\psi\rangle = b |\phi\rangle \ , && \mathbf{H}_{\mathrm{L}} |\psi\rangle = H |\psi\rangle \ , \\
&\mathbf{Q}_{\mathrm{R}} |\phi\rangle = 0 \ , && \mathbf{S}_{\mathrm{R}} |\phi\rangle = 0 \ , && \mathbf{H}_{\mathrm{R}} |\phi\rangle = 0 \ , \\
&\mathbf{Q}_{\mathrm{R}} |\psi\rangle = 0 \ , && \mathbf{S}_{\mathrm{R}} |\psi\rangle = 0 \ , && \mathbf{H}_{\mathrm{R}} |\psi\rangle = 0 \ .
\end{aligned} \tag{2.127}$$

with $H = ab$. We also have *right-moving* representations with $H_{\mathrm{L}} = 0$, whose highest weight states are annihilated by $\mathbf{Q}_{\mathrm{L}}$. An example is $\mathbf{1} \otimes (\mathbf{1}|\mathbf{1})_H$, in which the right generators act on the two states $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$ as in (2.122), and all the left generators annihilate them.

The centrally extended $\mathfrak{su}(1|1)^2$ algebra

We can extend the $\mathfrak{su}(1|1)^2$ algebra by introducing two additional central charges $\mathbf{C}$ and $\overline{\mathbf{C}}$. These appear in anticommutators between the two sectors, and we take the choice[10]

$$\begin{aligned} \{\mathbf{Q}_{\text{L}}, \mathbf{S}_{\text{L}}\} &= \mathbf{H}_{\text{L}} \ , & \{\mathbf{Q}_{\text{L}}, \mathbf{Q}_{\text{R}}\} &= \mathbf{C} \ , & \{\mathbf{Q}_{\text{L}}, \mathbf{S}_{\text{R}}\} &= 0 \ , \\ \{\mathbf{Q}_{\text{R}}, \mathbf{S}_{\text{R}}\} &= \mathbf{H}_{\text{R}} \ , & \{\mathbf{S}_{\text{L}} \ , \mathbf{S}_{\text{R}} \ \} &= \overline{\mathbf{C}} \ , & \{\mathbf{Q}_{\text{R}}, \mathbf{S}_{\text{L}}\} &= 0 \ . \end{aligned} \tag{2.128}$$

Note that $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ is not of direct product form, i.e. we cannot construct its irreducible representations from irreps of the two sectors. To make connection to the physics, from now on we use the subscript p on the states and representation parameters, indicating that these depend on the momentum of the excitation. Let us now consider the short representations of this algebra.

The left-moving representation. The generalization of (2.127) compatible with the above deformation is given by

$$\boxed{\varrho_{\text{L}}:} \qquad \begin{aligned} \mathbf{Q}_{\text{L}} |\phi^{\text{L}}_p\rangle &= a_p |\psi^{\text{L}}_p\rangle \ , & \mathbf{Q}_{\text{L}} |\psi^{\text{L}}_p\rangle &= 0, \\ \mathbf{S}_{\text{L}} \ |\phi^{\text{L}}_p\rangle &= 0, & \mathbf{S}_{\text{L}} \ |\psi^{\text{L}}_p\rangle &= b_p |\phi^{\text{L}}_p\rangle \ , \\ \mathbf{Q}_{\text{R}} |\phi^{\text{L}}_p\rangle &= 0, & \mathbf{Q}_{\text{R}} |\psi^{\text{L}}_p\rangle &= c_p |\phi^{\text{L}}_p\rangle \ , \\ \mathbf{S}_{\text{R}} \ |\phi^{\text{L}}_p\rangle &= d_p |\psi^{\text{L}}_p\rangle \ , & \mathbf{S}_{\text{R}} \ |\psi^{\text{L}}_p\rangle &= 0, \end{aligned} \tag{2.129}$$

with central charges

$$\begin{aligned} \mathbf{H}_{\text{L}} |\phi^{\text{L}}_p\rangle &= a_p b_p |\phi^{\text{L}}_p\rangle \ , & \mathbf{C} |\phi^{\text{L}}_p\rangle &= a_p c_p |\phi^{\text{L}}_p\rangle \ , \\ \mathbf{H}_{\text{R}} |\phi^{\text{L}}_p\rangle &= c_p d_p |\phi^{\text{L}}_p\rangle \ , & \overline{\mathbf{C}} |\phi^{\text{L}}_p\rangle &= b_p d_p |\phi^{\text{L}}_p\rangle \ . \end{aligned} \tag{2.130}$$

The highest weight state $|\phi^{\text{L}}_p\rangle$ is annihilated by the raising operators $\mathbf{S}_{\text{L}}$ and $\mathbf{Q}_{\text{R}}$, but also satisfies the condition

$$(\mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}} - \mathbf{C}\mathbf{S}_{\text{R}}) |\phi^{\text{L}}_p\rangle = (a_p c_p d_p - a_p c_p d_p) |\psi^{\text{L}}_p\rangle = 0. \tag{2.131}$$

[10] Alternatively we could have taken $\{\mathbf{Q}_{\text{L}}, \mathbf{S}_{\text{R}}\} = \mathbf{C}$, but this deformation was ruled out for the case of $\text{AdS}_3/\text{CFT}_2$, by considering the length-changing effects on the spin-chain [98].

Since this particular combination of the lowering operators $\mathbf{Q}_{\text{L}}$ and $\mathbf{S}_{\text{R}}$ annihilates the h.w. state, the representation is short. The state $|\phi_p^{\text{L}}\rangle$ must also be annihilated by the anticommutator $\{\mathbf{S}_{\text{L}}, \mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}} - \mathbf{C}\mathbf{S}_{\text{R}}\} = \mathbf{H}_{\text{L}}\mathbf{H}_{\text{R}} - \mathbf{C}\overline{\mathbf{C}}$, but this is a central charge, implying that

$$(\mathbf{H}_{\text{L}}\mathbf{H}_{\text{R}} - \mathbf{C}\overline{\mathbf{C}})\,|\chi_p^{\text{L}}\rangle = 0 \tag{2.132}$$

for all states $\chi_p^{\text{L}} = \phi_p^{\text{L}},\ \psi_p^{\text{L}}$ in the representation. This *shortening condition*, when applied to physical states, will play the role of the dispersion relation.

The right-moving representation For this representation the role of $\mathbf{Q}_{\text{L}}$, $\mathbf{Q}_{\text{R}}$ and $\mathbf{S}_{\text{L}}$, $\mathbf{S}_{\text{R}}$ is exchanged, and the right-movers $|\phi_p^{\text{R}}\rangle$ and $|\psi_p^{\text{R}}\rangle$ transform according to

$$\boxed{\varrho_{\text{R}}:}\qquad \begin{aligned} &\mathbf{Q}_{\text{R}}\,|\phi_p^{\text{R}}\rangle = a_p\,|\psi_p^{\text{R}}\rangle\,, && \mathbf{Q}_{\text{R}}\,|\psi_p^{\text{R}}\rangle = 0,\\ &\mathbf{S}_{\text{R}}\,\,|\phi_p^{\text{R}}\rangle = 0, && \mathbf{S}_{\text{R}}\,\,|\psi_p^{\text{R}}\rangle = b_p\,|\phi_p^{\text{R}}\rangle\,,\\ &\mathbf{Q}_{\text{L}}\,|\phi_p^{\text{R}}\rangle = 0, && \mathbf{Q}_{\text{L}}\,|\psi_p^{\text{R}}\rangle = c_p\,|\phi_p^{\text{R}}\rangle\,,\\ &\mathbf{S}_{\text{L}}\,\,|\phi_p^{\text{R}}\rangle = d_p\,|\psi_p^{\text{R}}\rangle\,, && \mathbf{S}_{\text{L}}\,\,|\phi_p^{\text{R}}\rangle = 0, \end{aligned} \tag{2.133}$$

with the central charges acting as

$$\begin{aligned} &\mathbf{H}_{\text{L}}\,|\phi_p^{\text{R}}\rangle = c_p d_p\,|\phi_p^{\text{R}}\rangle\,, && \mathbf{C}\,|\phi_p^{\text{R}}\rangle = a_p c_p\,|\phi_p^{\text{R}}\rangle\,,\\ &\mathbf{H}_{\text{R}}\,|\phi_p^{\text{R}}\rangle = a_p b_p\,|\phi_p^{\text{R}}\rangle\,, && \overline{\mathbf{C}}\,|\phi_p^{\text{R}}\rangle = b_p d_p\,|\phi_p^{\text{R}}\rangle\,. \end{aligned} \tag{2.134}$$

The highest weight state, which is $|\psi_p^{\text{R}}\rangle$ in this case, again satisfies the condition

$$(\mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}} - \mathbf{C}\mathbf{S}_{\text{R}})\,|\psi_p^{\text{R}}\rangle = 0, \tag{2.135}$$

and the representation is short. The state $|\psi_p^{\text{R}}\rangle$ must also be annihilated by the anticommutator $\{\mathbf{S}_{\text{L}}, \mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}} - \mathbf{C}\mathbf{S}_{\text{R}}\} = \mathbf{H}_{\text{L}}\mathbf{H}_{\text{R}} - \mathbf{C}\overline{\mathbf{C}}$, and we have the same shortening condition in terms of the central charges as for the left-movers

$$(\mathbf{H}_{\text{L}}\mathbf{H}_{\text{R}} - \mathbf{C}\overline{\mathbf{C}})\,|\chi_p^{\text{R}}\rangle = 0 \tag{2.136}$$

for all states $\chi_p^{\text{R}} = \phi_p^{\text{R}}, \psi_p^{\text{R}}$.

The centrally extended $\mathfrak{psu}(1|1)^4$ algebra

If we take two copies of the $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ algebra (2.128) that share the four central charges, we get the centrally extended $\mathfrak{psu}(1|1)^4$ [102] with generators

$$\{\mathbf{Q}_{\text{L}}{}^{\dot{a}}, \mathbf{S}_{\text{L}\dot{a}}, \mathbf{Q}_{\text{R}\dot{a}}, \mathbf{S}_{\text{R}}{}^{\dot{a}}, \mathbf{H}_{\text{L}}, \mathbf{H}_{\text{R}}, \mathbf{C}, \overline{\mathbf{C}}\}_{\dot{a}=1,2} \tag{2.137}$$

satisfying the anticommutation relations

$$\begin{aligned} \{\mathbf{Q}_{\text{L}}{}^{\dot{a}}, \mathbf{S}_{\text{L}\dot{b}}\} &= \delta^{\dot{a}}{}_{\dot{b}}\, \mathbf{H}_{\text{L}}\ , & \{\mathbf{Q}_{\text{L}}{}^{\dot{a}}, \mathbf{Q}_{\text{R}\dot{b}}\} &= \delta^{\dot{a}}{}_{\dot{b}}\, \mathbf{C}\ , \\ \{\mathbf{Q}_{\text{R}\dot{a}}, \mathbf{S}_{\text{R}}{}^{\dot{b}}\} &= \delta_{\dot{a}}{}^{\dot{b}}\, \mathbf{H}_{\text{R}}\ , & \{\mathbf{S}_{\text{L}\dot{a}}, \mathbf{S}_{\text{R}}{}^{\dot{b}}\} &= \delta_{\dot{a}}{}^{\dot{b}}\, \overline{\mathbf{C}}\ . \end{aligned} \tag{2.138}$$

In other words,

$$\mathfrak{psu}(1|1)^4 \simeq \mathfrak{psu}(1|1)^4 \ltimes \mathfrak{u}(1)^4\ . \tag{2.139}$$

Equivalently, we can consider a tensor product of two copies of (2.128)

$$\begin{aligned} \mathbf{Q}_{\text{L}}{}^{1} &= \mathbf{Q}_{\text{L}} \otimes \mathbf{1}\ , & \mathbf{S}_{\text{L}1} &= \mathbf{S}_{\text{L}} \otimes \mathbf{1}\ , \\ \mathbf{Q}_{\text{R}1} &= \mathbf{Q}_{\text{R}} \otimes \mathbf{1}\ , & \mathbf{S}_{\text{R}}{}^{1} &= \mathbf{S}_{\text{R}} \otimes \mathbf{1}\ , \\ \mathbf{Q}_{\text{L}}{}^{2} &= \mathbf{1} \otimes \mathbf{Q}_{\text{L}}\ , & \mathbf{S}_{\text{L}2} &= \mathbf{1} \otimes \mathbf{S}_{\text{L}}\ , \\ \mathbf{Q}_{\text{R}2} &= \mathbf{1} \otimes \mathbf{Q}_{\text{R}}\ , & \mathbf{S}_{\text{R}}{}^{2} &= \mathbf{1} \otimes \mathbf{S}_{\text{R}}\ , \end{aligned} \tag{2.140}$$

also for the central elements

$$\begin{aligned} \mathbf{H}_{\text{L}}{}^{1} &= \mathbf{H}_{\text{L}} \otimes \mathbf{1}\ , & \mathbf{H}_{\text{L}}{}^{2} &= \mathbf{1} \otimes \mathbf{H}_{\text{L}}\ , \\ \mathbf{H}_{\text{R}}{}^{1} &= \mathbf{H}_{\text{R}} \otimes \mathbf{1}\ , & \mathbf{H}_{\text{R}}{}^{2} &= \mathbf{1} \otimes \mathbf{H}_{\text{R}}\ , \\ \mathbf{C}\ ^{1} &= \mathbf{C} \otimes \mathbf{1}\ , & \mathbf{C}\ ^{2} &= \mathbf{1} \otimes \mathbf{C}\ , \\ \overline{\mathbf{C}}\ ^{1} &= \overline{\mathbf{C}} \otimes \mathbf{1}\ , & \overline{\mathbf{C}}\ ^{2} &= \mathbf{1} \otimes \overline{\mathbf{C}}\ . \end{aligned} \tag{2.141}$$

After identifying the central charges as

$$\mathbf{H}_{\text{L}}{}^{1} = \mathbf{H}_{\text{L}}{}^{2}, \quad \mathbf{H}_{\text{R}}{}^{1} = \mathbf{H}_{\text{R}}{}^{2}, \quad \mathbf{C}^{1} = \mathbf{C}^{2}, \quad \overline{\mathbf{C}}^{1} = \overline{\mathbf{C}}^{2}, \tag{2.142}$$

and consequently dropping the indices $1, 2$, we are left with $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$. Looking at the algebra this way will be helpful in constructing its short representations.

Bi-fundamental representations. It was shown, first for the spin-chain and later for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ superstring, that the massive off-shell excitations in both of the left- and right-moving sectors transform in *short* (four-dimensional) bi-fundamental representations of the centrally extended $\mathfrak{psu}(1|1)^4$. That is, we can obtain the relevant representations by tensoring the fundamental representations ϱ_{L} (2.129) and ϱ_{R} (2.133) of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$.

Left module. Borrowing notation from [114], the four left-movers can be written as

$$\begin{aligned} Y^{\mathrm{L}} &= \phi^{\mathrm{L}} \otimes \phi^{\mathrm{L}} \;, \quad & Z^{\mathrm{L}} &= \psi^{\mathrm{L}} \otimes \psi^{\mathrm{L}} \;, \\ \eta^{\mathrm{L}1} &= \psi^{\mathrm{L}} \otimes \phi^{\mathrm{L}} \;, \quad & \eta^{\mathrm{L}2} &= \phi^{\mathrm{L}} \otimes \psi^{\mathrm{L}} \;, \end{aligned} \tag{2.143}$$

and they transform under the tensor product of two left-moving representations ϱ_{L}:

$$\boxed{\varrho_{\mathrm{L}} \otimes \varrho_{\mathrm{L}} :} \qquad \begin{aligned} \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \, |Y^{\mathrm{L}}_p\rangle &= a_p \, |\eta^{\mathrm{L}\dot{a}}_p\rangle \;, & \mathbf{Q}_{\mathrm{L}}{}^{\dot{a}} \, |\eta^{\mathrm{L}\dot{b}}_p\rangle &= \epsilon^{\dot{a}\dot{b}} \, a_p \, |Z^{\mathrm{L}}_p\rangle \;, \\ \mathbf{S}_{\mathrm{L}\dot{a}} \, |Z^{\mathrm{L}}_p\rangle &= -\epsilon_{\dot{a}\dot{b}} \, b_p \, |\eta^{\mathrm{L}\dot{b}}_p\rangle \;, & \mathbf{S}_{\mathrm{L}\dot{a}} \, |\eta^{\mathrm{L}\dot{b}}_p\rangle &= \delta_{\dot{a}}{}^{\dot{b}} \, b_p \, |Y^{\mathrm{L}}_p\rangle \;, \\ \mathbf{Q}_{\mathrm{R}\dot{a}} \, |Z^{\mathrm{L}}_p\rangle &= -\epsilon_{\dot{a}\dot{b}} \, c_p \, |\eta^{\mathrm{L}\dot{b}}_p\rangle \;, & \mathbf{Q}_{\mathrm{R}\dot{a}} \, |\eta^{\mathrm{L}\dot{b}}_p\rangle &= \delta_{\dot{a}}{}^{\dot{b}} \, c_p \, |Y^{\mathrm{L}}_p\rangle \;, \\ \mathbf{S}_{\mathrm{R}}{}^{\dot{a}} \, |Y^{\mathrm{L}}_p\rangle &= d_p \, |\eta^{\mathrm{L}\dot{a}}_p\rangle \;, & \mathbf{S}_{\mathrm{R}}{}^{\dot{a}} \, |\eta^{\mathrm{L}\dot{b}}_p\rangle &= \epsilon^{\dot{a}\dot{b}} \, d_p \, |Z^{\mathrm{L}}_p\rangle \;. \end{aligned} \tag{2.144}$$

The representation coefficients of the two ϱ_{L} must match, since the central charges are shared, and we get a minus sign when charges of the second type act on a state with a fermion in the first part of the tensor product. Each central charge acts uniformly across all states

$$\begin{aligned} \mathbf{H}_{\mathrm{L}} \, |\chi^{\mathrm{L}}\rangle &= a_p b_p \, |\chi^{\mathrm{L}}\rangle \,, & \mathbf{C} \, |\chi^{\mathrm{L}}\rangle &= a_p c_p \, |\chi^{\mathrm{L}}\rangle \,, \\ \mathbf{H}_{\mathrm{R}} \, |\chi^{\mathrm{L}}\rangle &= c_p d_p \, |\chi^{\mathrm{L}}\rangle \,, & \overline{\mathbf{C}} \, |\chi^{\mathrm{L}}\rangle &= b_p d_p \, |\chi^{\mathrm{L}}\rangle \,. \end{aligned} \tag{2.145}$$

Right module. Similarly we can introduce the right-moving excitations

$$\begin{aligned} Y^{\mathrm{R}} &= \phi^{\mathrm{R}} \otimes \phi^{\mathrm{R}} \;, \quad & Z^{\mathrm{R}} &= \psi^{\mathrm{R}} \otimes \psi^{\mathrm{R}} \;, \\ \eta^{\mathrm{R}}{}_1 &= \psi^{\mathrm{R}} \otimes \phi^{\mathrm{R}} \;, \quad & \eta^{\mathrm{R}}{}_2 &= \phi^{\mathrm{R}} \otimes \psi^{\mathrm{R}} \;, \end{aligned} \tag{2.146}$$

and these will transform in the representation

$$\boxed{\varrho_{\text{R}} \otimes \varrho_{\text{R}}:} \qquad \begin{aligned} \mathbf{Q}_{\text{R}\dot{a}} \,|Y_p^{\text{R}}\rangle &= \epsilon_{\dot{a}\dot{b}}\, a_p \,|\eta_p^{\text{R}\dot{b}}\rangle \;, & \mathbf{Q}_{\text{R}\dot{a}} \,|\eta_p^{\text{R}\dot{b}}\rangle &= \delta_{\dot{a}}{}^{\dot{b}}\, a_p \,|Z_p^{\text{R}}\rangle \;, \\ \mathbf{S}_{\text{R}}{}^{\dot{a}} \,|Z_p^{\text{R}}\rangle &= b_p \,|\eta_p^{\text{R}\dot{a}}\rangle \;, & \mathbf{S}_{\text{R}}{}^{\dot{a}} \,|\eta_p^{\text{R}\dot{b}}\rangle &= -\epsilon^{\dot{a}\dot{b}}\, b_p \,|Y_p^{\text{R}}\rangle \;, \\ \mathbf{Q}_{\text{L}}{}^{\dot{a}} \,|Z_p^{\text{R}}\rangle &= c_p \,|\eta_p^{\text{R}\dot{a}}\rangle \;, & \mathbf{Q}_{\text{L}}{}^{\dot{a}} \,|\eta_p^{\text{R}\dot{b}}\rangle &= -\epsilon^{\dot{a}\dot{b}}\, c_p \,|Y_p^{\text{R}}\rangle \;, \\ \mathbf{S}_{\text{L}\dot{a}} \,|Y_p^{\text{R}}\rangle &= \epsilon_{\dot{a}\dot{b}}\, d_p \,|\eta_p^{\text{R}\dot{b}}\rangle \;, & \mathbf{S}_{\text{L}\dot{a}} \,|\eta_p^{\text{R}\dot{b}}\rangle &= \delta_{\dot{a}}{}^{\dot{b}}\, d_p \,|Z_p^{\text{R}}\rangle \;, \end{aligned} \tag{2.147}$$

and for all right-movers

$$\begin{aligned} \mathbf{H}_{\text{L}} \,|\chi^{\text{R}}\rangle &= c_p d_p \,|\chi^{\text{R}}\rangle \,, & \mathbf{C} \,|\chi^{\text{R}}\rangle &= a_p c_p \,|\chi^{\text{R}}\rangle \,, \\ \mathbf{H}_{\text{R}} \,|\chi^{\text{R}}\rangle &= a_p b_p \,|\chi^{\text{R}}\rangle \,, & \overline{\mathbf{C}} \,|\chi^{\text{R}}\rangle &= b_p d_p \,|\chi^{\text{R}}\rangle \,. \end{aligned} \tag{2.148}$$

Shortening condition. Naturally extending the choice made for $\mathfrak{su}(1|1)^2$, we take $\mathbf{S}_{\text{L}\dot{a}}$ and $\mathbf{Q}_{\text{R}\dot{a}}$ as our raising operators, while $\mathbf{Q}_{\text{L}}{}^{\dot{a}}$ and $\mathbf{S}_{\text{R}}{}^{\dot{a}}$ will be lowering operators. The highest weight states for $\varrho_{\text{L}} \otimes \varrho_{\text{L}}$ and $\varrho_{\text{R}} \otimes \varrho_{\text{R}}$ are $|Y_p^{\text{L}}\rangle$ and $|Z_p^{\text{R}}\rangle$ respectively, but they are also annihilated by two combinations of lowering operators, as should be the case for short representations

$$\begin{aligned} (\mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}}{}^{\dot{a}} - \mathbf{C}\mathbf{S}_{\text{R}}{}^{\dot{a}}) \,|Y_p^{\text{L}}\rangle &= (a_p c_p d_p - a_p c_p d_p) \,|\eta_p^{\text{R}\dot{a}}\rangle = 0 \;, \\ (\mathbf{H}_{\text{R}}\mathbf{Q}_{\text{L}}{}^{\dot{a}} - \mathbf{C}\mathbf{S}_{\text{R}}{}^{\dot{a}}) \,|Z_p^{\text{R}}\rangle &= (a_p b_p c_p - a_p b_p c_p) \;|\eta_p^{\text{R}\dot{a}}\rangle = 0 \;. \end{aligned} \tag{2.149}$$

Similarly to the case of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, the anticommutator of this with $\mathbf{S}_{\text{L}\dot{b}}$ still annihilates the highest weight states, and in fact any state across both sectors, since it is a central element of the algebra

$$(\mathbf{H}_{\text{L}}\mathbf{H}_{\text{R}} - \mathbf{C}\overline{\mathbf{C}}) \,|\chi_p^{\text{L,R}}\rangle = 0. \tag{2.150}$$

Note that this is the same as (2.132) and (2.136).

Representation coefficients

Similarly to the case of AdS_5/CFT_4, an argument based on supersymmetry, physical state conditions and quantization determines the values of the central charges in terms of the momentum p of the off-shell particle [115]

$$\begin{aligned} \mathbf{C} &= \frac{\varsigma \mathrm{h}}{2}\left(e^{+i\mathrm{p}}-1\right), \\ \overline{\mathbf{C}} &= \frac{\mathrm{h}}{2\varsigma}\left(e^{-i\mathrm{p}}-1\right), \\ \mathbf{M} &= m + q\sqrt{\lambda}\frac{\mathrm{p}}{2\pi}, \end{aligned} \tag{2.151}$$

where h is the effective string tension, λ is the 't Hooft coupling, and ς is an arbitrary complex factor, which can be scaled away for single-particle representations. From the shortening conditions (2.132), (2.150) we get

$$\mathbf{H} = \sqrt{\left(m + q\sqrt{\lambda}\frac{\mathrm{p}}{2\pi}\right)^2 + 4\,\tilde{q}^2\,\mathrm{h}^2\sin^2\tfrac{\mathrm{p}}{2}}\ . \tag{2.152}$$

In the classical limit

$$\mathrm{h} = \frac{\sqrt{\lambda}}{2\pi}\ , \tag{2.153}$$

and we have the dispersion relation

$$\epsilon = \sqrt{(m + q\mathrm{h}\mathrm{p})^2 + 4\,\tilde{q}^2\,\mathrm{h}^2\sin^2\tfrac{\mathrm{p}}{2}}\ . \tag{2.154}$$

The masses of elementary excitations are $m = 1, 0$ for $AdS_3 \times S^3 \times T^4$, and $m = 0, \sin^2\varphi, \cos^2\varphi, 1$ for $AdS_3 \times S^3 \times S^3 \times S^1$.

2.2.2 Bosonic string action

Supergravity equations fix the AdS_3 and S^3 radii to be equal for the $AdS_3 \times S^3 \times T^4$ background, and for $AdS_3 \times S^3 \times S^3 \times S^1$ they relate the the AdS radius R and the radii of the two 3-spheres $R_\pm$ by [61]

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{R^2}. \tag{2.155}$$

Setting unit radius for AdS_3, this geometry can be parametrized by an angle $\varphi \in (0, \frac{\pi}{2})$

$$\frac{1}{R_+^2} = \cos^2\varphi = \alpha, \qquad \frac{1}{R_-^2} = \sin^2\varphi = 1 - \alpha. \tag{2.156}$$

The $\varphi \to 0$ limit blows up the second sphere, which can be compactified on a torus to recover the $AdS_3 \times S^3 \times T^4$ geometry. For this reason in the following discussion we mostly focus on the $AdS_3 \times S^3 \times S^3 \times S^1$ bosonic action.

The main difference compared to $AdS_5 \times S^5$, apart from the appearance of flat directions, is that the AdS_3 backgrounds can be supported by a mixture of Ramond-Ramond (R-R) and Neveu–Schwarz-Neveu–Schwarz (NS-NS) fluxes

$$\begin{aligned} F &= 2\tilde{q}\big(\,\mathrm{Vol}(AdS_3) + \cos\varphi\,\mathrm{Vol}(S_+^3) + \sin\varphi\,\mathrm{Vol}(S_-^3)\big), \\ H &= 2q\big(\,\mathrm{Vol}(AdS_3) + \cos\varphi\,\mathrm{Vol}(S_+^3) + \sin\varphi\,\mathrm{Vol}(S_-^3)\big), \end{aligned} \tag{2.157}$$

where $q \in [0, 1]$ and $\tilde{q} = \sqrt{1 - q^2}$. The maximally supersymmetric AdS_3 is in fact a two-parameter family of theories, providing both richness compared to AdS_5, and, as we will see, technical challenges when trying to derive analogous semiclassical results.

Geometry of AdS_3 and S^3

Although AdS_3 and S^3 are simple subspaces of AdS_5 and S^5, both described in section 2.1.2, let us, for completeness, write down their coordinate parametrizations again, this time together with their vielbein and spin-connection, both of which will appear in the fermionic perturbation equations in the following chapters. For a manifold with metric $g_{\mu\nu}$ the *vielbein* are (non-uniquely) defined by

$$E_\mu^A E_\nu^B \eta_{AB} = g_{\mu\nu}\ , \tag{2.158}$$

where A, B are tangent-space indices and η_{AB} is the flat (Minkowski or Euclidean) metric. The vielbein provides the most tractable construction of curved-space Dirac matrices Γ_μ from those of flat space Γ_A:

$$\Gamma_\mu \equiv E_\mu^A \Gamma_A \quad \Rightarrow \quad \{\Gamma_\mu, \Gamma_\nu\} = E_\mu^A E_\nu^B \underbrace{\{\Gamma_A, \Gamma_B\}}_{2\eta_{AB}} = 2g_{\mu\nu}\ , \tag{2.159}$$

hence its appearance in the fermionic Lagrangian. Another object of similar importance is the *spin connection* ω_μ^{AB}, as it appears in the construction of the covariant derivative for spinors. It is given by the formula

$$\omega_\mu^{AB} = E_\nu^A \partial_\mu E^{\nu B} + E_\nu^A \Gamma^\nu_{\sigma\mu} E^{\sigma B} \ , \tag{2.160}$$

where the Greek indices are raised by the inverse metric $g^{\mu\nu}$, and the Christoffel symbols are given by the usual $\Gamma^\nu_{\sigma\mu} = \frac{1}{2} g^{\nu\rho} \left(\partial_\sigma g_{\mu\rho} + \partial_\mu g_{\sigma\rho} - \partial_\rho g_{\sigma\mu}\right)$.

AdS$_3$. Embedded in flat $Y \in \mathbb{R}^{2,2}$, the AdS$_3$ space can be parametrized as

$$Y_3 + iY_0 = \cosh\rho \, e^{it}, \quad Y_1 + iY_2 = \sinh\rho \, e^{i\psi}, \tag{2.161}$$

where $\rho \in [0, \infty)$, $\psi \in [0, 2\pi)$ and the temporal direction t is cut open $t \in (-\infty, \infty)$, exactly as explained below (2.15). Ordering the coordinates as (t, ρ, ψ), the metric is

$$g_{\mu\nu} = \text{diag}(-\cosh^2\!\rho, \ 1, \ \sinh^2\!\rho) \ , \tag{2.162}$$

from which we can immediately read off the natural choice of the vielbein

$$E_\mu^A = \text{diag}(\cosh\rho, \ 1, \ \sinh\rho). \tag{2.163}$$

A straightforward calculation gives the non-trivial Christoffel symbols

$$\Gamma^\tau_{\tau\rho} = \Gamma^\tau_{\rho\tau} = \tanh\rho \ , \quad \Gamma^\psi_{\psi\rho} = \Gamma^\psi_{\rho\psi} = \coth\rho \ , \quad \Gamma^\rho_{\tau\tau} = -\Gamma^\rho_{\psi\psi} = \cosh\rho \, \sinh\rho \ , \tag{2.164}$$

and non-zero spin connection components

$$\omega_\tau^{01} = -\omega_\tau^{10} = \sinh\rho \ , \tag{2.165}$$

$$\omega_\psi^{21} = -\omega_\psi^{12} = \cosh\rho \ , \tag{2.166}$$

where the tangent space indices $0, 1, 2$ correspond to the directions t, ρ, ψ respectively.

$\mathbf{S}^3$. We parametrize the $X \in \mathbb{R}^4$ embedding of the 3-sphere by the Hopf coordinates

$$X_1 + iX_2 = \sin\theta\, e^{i\phi_1}, \quad X_3 + iX_4 = \cos\theta\, e^{i\phi_2}, \tag{2.167}$$

where $\theta \in [0, \pi/2]$ and ϕ_1, ϕ_2 take values in $[0, 2\pi)$. Ordering the coordinates as (θ, ϕ_1, ϕ_2), the metric and vielbein are

$$\begin{aligned} g_{\mu\nu} &= \mathrm{diag}(1, \sin^2\theta, \cos^2\theta), \\ E^A_\mu &= \mathrm{diag}(1,\ \sin\theta,\ \cos\theta). \end{aligned} \tag{2.168}$$

It is then a simple exercise to obtain the non-trivial Christoffel symbols

$$\Gamma^{\phi_1}_{\phi_1\theta} = \Gamma^{\phi_1}_{\theta\phi_1} = \cot\theta, \qquad \Gamma^{\phi_2}_{\phi_2\theta} = \Gamma^{\phi_2}_{\theta\phi_2} = -\tan\theta, \qquad \Gamma^{\theta}_{\phi_1\phi_1} = -\Gamma^{\theta}_{\phi_2\phi_2} = -\cos\theta\,\sin\theta, \tag{2.169}$$

and non-zero spin connection

$$\omega^{12}_{\phi_1} = -\omega^{21}_{\phi_1} = -\cos\theta, \tag{2.170}$$

$$\omega^{31}_{\phi_2} = -\omega^{13}_{\phi_2} = -\sin\theta, \tag{2.171}$$

where the tangent space components $1, 2, 3$ correspond to the directions θ, ϕ_1, ϕ_2 respectively.

Bosonic action

The mixed-flux AdS_3 bosonic sigma-model action

$$S_B = -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}\sigma\mathrm{d}\tau\Big(\sqrt{-\gamma}\gamma^{ab}G_{MN} + \epsilon^{ab}B_{MN}\Big)\partial_a X^M \partial_b X^N \tag{2.172}$$

differs from the AdS_5 action (2.22) only in the Wess-Zumino term, encoded by the antisymmetric tensor B_{MN}, related to the three-form NS-NS flux (2.157) by

$$H = \mathrm{d}B. \tag{2.173}$$

The worldsheet metric γ_{ab} doesn't couple to the NS-NS flux, and we can fix conformal gauge $\gamma_{ab} = \eta_{ab}$ in much the same way, with unchanged Virasoro constraints (2.23).

Using Hopf coordinates[11] (2.161) for AdS_3 and (2.167) for the two spheres $S^3_\pm$

$$\begin{aligned}\tilde{\Omega}(\mathbf{x}) &= \Big(T(\tau,\sigma),\, \rho(\tau,\sigma),\, \psi(\tau,\sigma)\Big),\\ \Omega_\pm(\mathbf{x}) &= \Big(\theta^\pm(\tau,\sigma),\, \phi_1^\pm(\tau,\sigma),\, \phi_2^\pm(\tau,\sigma)\Big),\end{aligned} \tag{2.174}$$

and omitting the flat S^1, the mixed-flux $AdS_3 \times S^3 \times S^3$ conformal gauge action is

$$S_B = S_{\text{AdS}}[\tilde{\Omega}] + \frac{1}{\cos^2\varphi}\, S_{\text{S}}[\Omega_+] + \frac{1}{\sin^2\varphi}\, S_{\text{S}}[\Omega_-]\ , \tag{2.175}$$

with unit-radius AdS_3 and S^3 components

$$\begin{aligned} S_{\text{AdS}}[\tilde{\Omega}] = -\frac{\text{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2\sigma \Big[& -\cosh^2\rho\ \partial_a T\,\partial^a T + \partial_a\rho\,\partial^a\rho + \sinh^2\rho\ \partial_a\psi\,\partial^a\psi \\ & + q(\cosh 2\rho + \tilde{c})(\dot{T}\psi' - \dot{\psi}T')\Big]\ , \\ S_{\text{S}}[\Omega] = -\frac{\text{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2\sigma \Big[& \partial_a\theta\partial^a\theta + \sin^2\theta\ \partial_a\phi_1\partial^a\phi_1 + \cos^2\theta\ \partial_a\phi_2\partial^a\phi_2 \\ & + q(\cos 2\theta + c)(\dot{\phi}_1\phi_2' - \dot{\phi}_2\phi_1')\Big]\ . \end{aligned} \tag{2.176}$$

The indices $a = 0, 1$ correspond to τ, σ, with derivatives $\dot{} = \partial_\tau$, $' = \partial_\sigma$, and the c-terms, first introduced in [119], are total derivatives that drop out of the equations of motion, but change the value of conserved charges for string solutions with non-periodic boundary conditions, e.g. the giant magnon. The equations of motion for AdS_3 are

$$\begin{aligned} \cosh^2\rho\left(\ddot{T}_1 - T''\right) + 2\sinh\rho\cosh\rho\left(\dot{T}\dot{\rho} - T'\rho' + q\left(\dot{\rho}\psi' - \rho'\dot{\psi}\right)\right) &= 0\ , \\ \ddot{\rho} - \rho'' + \sinh\rho\cosh\rho\left(\dot{T}^2 - T'^2 - \dot{\psi}^2 + \psi'^2 + 2q\left(\dot{T}\psi' - \dot{\psi}T'\right)\right) &= 0\ , \\ \sinh^2\rho\left(\ddot{\psi} - \psi''\right) + 2\sinh\rho\cosh\rho\left(\dot{\rho}\dot{\psi} - \rho'\psi' + q\left(\dot{\rho}T' - \rho'\dot{T}\right)\right) &= 0\ , \end{aligned} \tag{2.177}$$

[11] We use T for the temporal coordinate on AdS_3, to avoid confusion with the decompactified worldsheet coordinate $t = \kappa\tau$, since we are relaxing the static gauge condition $T = \kappa\tau$.

while $\Omega_\pm$ both satisfy

$$
\begin{aligned}
\ddot\theta - \theta'' - \sin\theta\cos\theta\left(\dot\phi_1^2 - \phi_1'^2 - \dot\phi_2^2 + \phi_2'^2 + 2q\left(\dot\phi_1\phi_2' - \dot\phi_2\phi_1'\right)\right) &= 0 \ , \\
\sin^2\theta\left(\ddot\phi_1 - \phi_1''\right) + 2\sin\theta\cos\theta\left(\dot\theta\dot\phi_1 - \theta'\phi_1' + q\left(\dot\theta\phi_2' - \theta'\dot\phi_2\right)\right) &= 0 \ , \\
\cos^2\theta\left(\ddot\phi_2 - \phi_2''\right) - 2\sin\theta\cos\theta\left(\dot\theta\dot\phi_2 - \theta'\phi_2' + q\left(\dot\theta\phi_1' - \theta'\dot\phi_1\right)\right) &= 0 \ .
\end{aligned}
\tag{2.178}
$$

The equations for the three components decouple due to the block-diagonal spacetime metric, but they are connected via the Virasoro constraints

$$
\begin{aligned}
\tilde V_1[\tilde\Omega] + \frac{1}{\cos^2\varphi}\, V_1[\Omega_+] + \frac{1}{\sin^2\varphi}\, V_1[\Omega_-] = 0 \ , \\
\tilde V_2[\tilde\Omega] + \frac{1}{\cos^2\varphi}\, V_2[\Omega_+] + \frac{1}{\sin^2\varphi}\, V_2[\Omega_-] = 0 \ ,
\end{aligned}
\tag{2.179}
$$

where

$$
\begin{aligned}
&\tilde V_1[\tilde\Omega] \equiv -\cosh^2\rho\,(\dot T_1^2 + T'^2) + \dot\rho^2 + \rho'^2 + \sinh^2\rho\,(\dot\psi^2 + \psi'^2) \ , \\
&\tilde V_2[\tilde\Omega] \equiv -\cosh^2\rho\ \dot T T' + \dot\rho\rho' + + \sinh^2\rho\ \dot\psi\psi' \ . \\
&V_1[\Omega] \equiv \dot\theta^2 + \theta'^2 + \sin^2\theta\,(\dot\phi_1^2 + \phi_1'^2) + \cos^2\theta\,(\dot\phi_2^2 + \phi_2'^2) \ , \\
&V_2[\Omega] \equiv \dot\theta\theta' + \sin^2\theta\ \dot\phi_1\phi_1' + \cos^2\theta\ \dot\phi_2\phi_2' \ .
\end{aligned}
\tag{2.180}
$$

We get six conserved charges from the symmetries of the action: the spacetime energy E due to invariance in AdS time T, an AdS spin J_0 for translations in ψ and two angular momenta J_1 and J_2 on each sphere, due to invariance under shifts in ϕ_1 and ϕ_2

$$
\begin{aligned}
E &= \mathrm{h}\int_{-\pi}^{\pi} \mathrm{d}\sigma\big[\cosh^2\rho\ \dot T + \frac{q}{2}(\cosh 2\rho + \tilde c)\psi'\big] \ , \\
J_0 &= \mathrm{h}\int_{-\pi}^{\pi} \mathrm{d}\sigma\big[\sinh^2\rho\ \dot\psi + \frac{q}{2}(\cosh 2\rho + \tilde c)T'\big] \ , \\
J_1^\pm &= R_\pm^2\ \mathrm{h}\int_{-\pi}^{\pi} \mathrm{d}\sigma\big[\sin^2\theta^\pm\ \dot\phi_1^\pm - \frac{q}{2}(\cos 2\theta^\pm + c)\phi_2^{\pm\prime}\big] \ , \\
J_2^\pm &= R_\pm^2\ \mathrm{h}\int_{-\pi}^{\pi} \mathrm{d}\sigma\big[\cos^2\theta^\pm\ \dot\phi_2^\pm + \frac{q}{2}(\cos 2\theta^\pm + c)\phi_1^{\pm\prime}\big] \ .
\end{aligned}
\tag{2.181}
$$

BMN string

The $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ BMN string is a point-like solution moving at some angle along the equators of $\mathrm{S}^3 \times \mathrm{S}^3$. Suppose that this motion is generated by $A\mathcal{J}_1^+ + B\mathcal{J}_1^-$, where the charges $\mathcal{J}_1^\pm$ generate (unit) shifts along $\phi_1^\pm$. On the classical level the only requirement is that the trajectory is a light-cone geodesic, i.e. that the $\mathbb{R}^{1,2}$ vector $(\delta t, R_+\delta\phi_1^+, R_-\delta\phi_1^-) = \delta t(1, AR_+, BR_-)$ is null

$$\frac{A^2}{\cos^2\varphi} + \frac{B^2}{\sin^2\varphi} = 1 \ . \tag{2.182}$$

The above construction also gives the definition of the physical BMN angular momentum

$$J_1 = AJ_1^+ + BJ_1^- . \tag{2.183}$$

Explicitly, the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ BMN geodesic is given by

$$\begin{aligned} &T = \kappa\tau, \quad \rho = 0, \quad \psi = 0, \\ &\theta^\pm = \frac{\pi}{2}, \quad \phi_1^+ = A\kappa\tau, \quad \phi_1^+ = B\kappa\tau, \quad \phi_2^\pm = 0, \end{aligned} \tag{2.184}$$

and (2.182) is the Virasoro constraint for the BMN string. The conserved charges (2.181)

$$E = 2\pi \mathrm{h}\kappa, \quad J_1^+ = \frac{2\pi\mathrm{h}A\kappa}{\cos^2\varphi}, \quad J_1^- = \frac{2\pi\mathrm{h}B\kappa}{\cos^2\varphi}, \quad J_0 = J_2^\pm = 0 \tag{2.185}$$

satisfy the dispersion relation

$$E - J_1 = 0, \tag{2.186}$$

just like the $\mathbb{R}\times\mathrm{S}^3$ BMN string (2.40), further justifying the definition (2.183) for the physical angular momentum.

Maximally SUSY BMN vacuum. While we found a family of light-cone geodesics, the true BMN vacuum of the theory preserves maximal amount of supersymmetry, and the excitations above this true vacuum transform under the residual symmetry algebra. This condition leads to the (up to signs) unique choice [69]

$$A = \cos^2\varphi, \qquad B = \sin^2\varphi, \tag{2.187}$$

which is often referred to as the *maximally SUSY* BMN solution.

BMN angle. Below we will use a light-cone gauge argument to show that the worldsheet momentum of an off-shell (open) string is related to the opening of BMN angle between the two endpoints. In the $\mathbb{R}\times\mathrm{S}^3$ case (2.38) the BMN angle is simply $\phi = \phi_1$, and here we find its generalisation $\phi(\phi_1^+, \phi_1^-)$ to the $\mathbb{R}\times\mathrm{S}^3\times\mathrm{S}^3$ geometry. The momenta $p_{\phi_1^\pm}$, conjugate to the angles $\phi_1^\pm$, are related to the conserved charges (2.181) by

$$J_1^\pm = \int_{-\pi}^{\pi} \mathrm{d}\sigma \; p_{\phi_1^\pm} \ . \tag{2.188}$$

We want a canonical transformation such that the BMN momentum p_ϕ corresponds to the physical angular momentum (2.183)

$$J_1 = AJ_1^+ + BJ_1^- = \int_{-\pi}^{\pi} \mathrm{d}\sigma \; p_\phi \ . \tag{2.189}$$

This fixes

$$p_\phi = A\, p_{\phi_1^+} + B\, p_{\phi_1^-} \ , \tag{2.190}$$

and we need to find the BMN angle conjugate to this momentum

$$\phi = C_+\phi_1^+ + C_-\phi_1^- \ . \tag{2.191}$$

For the transformation to be canonical, the Poisson brackets need to be unchanged, imposing

$$A\, C_+ + B\, C_- = 1 \ . \tag{2.192}$$

The complete transformation of course includes another, orthogonal pair of phase-space coordinates $(\tilde\phi, p_{\tilde\phi})$, and there is a unique (up to rescalings) canonical choice consistent with the definition of ϕ and p_ϕ:

$$\tilde\phi = B\,\phi_1^+ - A\,\phi_1^- \ , \qquad p_{\tilde\phi} = C_- p_{\phi_1^+} - C_+ p_{\phi_1^-} \ . \tag{2.193}$$

To fix $C_\pm$ we require that the total angular momentum corresponding to $\tilde{\phi}$ should vanish for the BMN string (2.185)

$$\int_{-\pi}^{\pi} \mathrm{d}\sigma \; p_{\tilde{\phi}} = C_- J_1^+ - C_+ J_1^- = \frac{A}{\cos^2\varphi} C_- - \frac{B}{\sin^2\varphi} C_+ = 0 \; . \tag{2.194}$$

Recalling the Virasoro constraint (2.182), the solution to (2.192), (2.194) is

$$C_+ = \frac{A}{\cos^2\varphi}, \qquad C_- = \frac{B}{\sin^2\varphi}, \tag{2.195}$$

the BMN angle becomes

$$\phi = \frac{A}{\cos^2\varphi}\phi_1^+ + \frac{B}{\sin^2\varphi}\phi_1^- , \tag{2.196}$$

and in particular for the maximally SUSY case (2.187)

$$\phi = \phi_1^+ + \phi_1^- . \tag{2.197}$$

Worldsheet momentum

To understand the relation between the worldsheet momentum and opening angle of off-shell string solutions, we need to consider the bosonic string in light-cone gauge, as described in [45, 164] and for $q \neq 0$ in [116]. We start with the Green-Schwarz action[12] with WZ-term (2.172)

$$S_B = -\frac{\mathrm{h}}{2} \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \Big(\sqrt{-\gamma}\gamma^{ab} G_{MN} + \epsilon^{ab} B_{MN} \Big) \partial_a X^M \partial_b X^N . \tag{2.198}$$

To fix light-cone gauge we must first rewrite this action in Hamiltonian form. Introducing the conjugate momenta

$$p_M = \frac{\delta S}{\delta \dot{Y}^M} = -\mathrm{h}\gamma^{0b} G_{MN} \partial_b X^N - \mathrm{h} B_{MN} X'^N \; , \tag{2.199}$$

the bosonic action takes the first-order form

$$S_B = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_M \dot{X}^M + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2\mathrm{h}\gamma^{00}} C_2 \right) , \tag{2.200}$$

[12] Here we take $\sigma \in (-r, r)$ instead of $(-\pi, \pi)$, and fix r later from the consistency of conserved charges.

with

$$
\begin{aligned}
C_1 &= p_M X'^M \ , \\
C_2 &= G^{MN} p_M p_N + \mathrm{h}^2 G_{MN} X'^M X'^N + 2\mathrm{h} G^{MN} B_{NK} p_M X'^K \\
&\quad + \mathrm{h}^2 G^{MN} B_{MK} B_{NL} X'^K X'^L \ .
\end{aligned}
\tag{2.201}
$$

Reparametrisation invariance (i.e. vanishing variation under changes in γ^{ab}) leads to the Virasoro constraints, now written as

$$
C_1 = 0 \ , \qquad C_2 = 0 \ . \tag{2.202}
$$

Assuming the invariance of the string action under shifts of the time coordinate T and BMN angle ϕ we have the conserved Noether charges

$$
E = -\int_{-r}^{r} \mathrm{d}\sigma \ p_T, \qquad J = \int_{-r}^{r} \mathrm{d}\sigma \ p_\phi. \tag{2.203}
$$

Let us introduce light-cone coordinates and momenta[13]

$$
x^- = \phi - T \ , \quad x^+ = T \ , \qquad p_- = p_\phi \ , \quad p_+ = p_\phi + p_T \ , \tag{2.204}
$$

with all other (transverse) directions x^i unchanged. Note that this is a canonical transformation, hence the from of the action is unchanged

$$
S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_+ \dot{x}^+ + p_- \dot{x}^- + p_i \dot{x}^i + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2\mathrm{h}\gamma^{00}} C_2 \right) \ , \tag{2.205}
$$

where

$$
C_1 = p_+ x'^+ + p_- x'^- + p_i x'^i \ , \tag{2.206}
$$

and C_2 is a quadratic polynomial in p_+. Using the definitions (2.203), the light-cone gauge momenta are

$$
P_- = \int_{-r}^{r} \mathrm{d}\sigma \ p_- = J \ , \qquad P_+ = \int_{-r}^{r} \mathrm{d}\sigma \ p_+ = J - E \ . \tag{2.207}
$$

[13] In fact, one might consider a one-parameter family of light-cone gauges with $x^+ = (1-a)t + a\phi$ and $p_- = (1-a)p_\phi - a p_t$. For $a = \frac{1}{2}$ this reduces to the usual light-cone gauge $x_+ = \frac{1}{2}(t+\phi) = \tau$, however, in this section we restrict our attention to the $a = 0$ case.

We then fix the uniform light-cone gauge by imposing

$$x^+ = \tau \ , \qquad p_- = 1 \ , \tag{2.208}$$

where "uniform" refers to the uniform distribution of light-cone momentum along the string. Consistency of this gauge fixes the value of r

$$r = \frac{1}{2} P_- \ . \tag{2.209}$$

Gauge fixing completely determines the dynamics in the light-cone directions. The first Virasoro constraint can be solved for

$$x'^- = -p_i x'^i , \tag{2.210}$$

then substituting this into $C_2 = 0$ we obtain $p_+ = p_+(p_i, x^i, x'^i)$. The gauge-fixed action is

$$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_i \dot{x}^i - \mathcal{H} \right) \ , \tag{2.211}$$

where

$$\mathcal{H} = -p_+(p_i, x^i, \acute{x}^i) \tag{2.212}$$

is the worldsheet Hamiltonian density depending only on the physical (transverse) fields. Note that the light-cone gauge worldsheet Hamiltonian is, as we mentioned before, given by the difference of space-time energy and angular momentum

$$H = \int_{-r}^{r} \mathrm{d}\sigma \ \mathcal{H} = E - J \ . \tag{2.213}$$

Closed string solutions should satisfy periodicity in the transverse directions: $x^i(r) = x^i(-r)$ and an additional level-matching condition:

$$\Delta x^- = \int_{-r}^{r} \mathrm{d}\sigma x'^- = -\int_{-r}^{r} \mathrm{d}\sigma p_i x'^i = 2\pi m \ , \tag{2.214}$$

where the winding number m is integer-valued since ϕ is an angular coordinate. Invariance of the gauge-fixed action under the shifts of the worldsheet coordinate σ results in the conservation of worldsheet momentum[14]

[14] Deriving the expression for the worldsheet momentum is greatly simplified if one makes use of the Hamiltonian field equations $\dot{x}^i = \frac{\delta \mathcal{H}}{\delta p_i}$, $\dot{p}_i = -\frac{\delta \mathcal{H}}{\delta x^i}$.

$$\mathrm{p_{ws}} = -\int_{-r}^{r} \mathrm{d}\sigma p_i x'^i = \Delta x^- \ . \tag{2.215}$$

In the zero-winding sector the level-matching condition implies a vanishing worldsheet momentum for physical, closed string states $\mathrm{p_{ws}} = 0$. However, the giant magnon is a solution in the decompactification limit, where $P_+ \to \infty$ with h kept fixed. This naturally opens up the worldsheet ($r \to \infty$) and the closed string level-matching can be relaxed to give non-zero worldsheet momentum

$$\mathrm{p} = \Delta x^- = \Delta\phi - \Delta T \ . \tag{2.216}$$

For $\mathrm{AdS_3 \times S^3 \times S^3 \times S^1}$ with maximally SUSY BMN angle (2.197) this becomes

$$\mathrm{p} = \Delta x^- = \Delta\phi_1^+ + \Delta\phi_1^- - \Delta T \ . \tag{2.217}$$

Note that for solutions in static gauge, e.g. the giant magnon, $\Delta T = 0$ and p is simply the opening angle, but we will also consider $\mathrm{AdS_3 \times S^1}$ solitons with $\Delta T \neq 0$.

$\mathbf{AdS_3 \times S^3 \times S^3}$ strings from $\mathbf{AdS_3 \times S^3}$ solutions

One can construct $\mathrm{AdS_3 \times S^3 \times S^3 \times S^1}$ strings from simple $\mathrm{AdS_3 \times S^3 \times T^4}$ solutions, by combining them with appropriate scaling of the wordlsheet coordinates. This method will be applied below, where we generate $\mathrm{AdS_3 \times S^3} \subset \mathrm{AdS_3 \times S^3 \times T^4}$ solitons using the dressing method, and write down their $\mathrm{AdS_3 \times S^3 \times S^3 \times S^1}$ generalizations.

If $\tilde{\Omega}_0(\mathbf{x})$ is a solution on $\mathrm{AdS_3 \times S^1}$ with point-like BMN motion along the $\mathrm{S^1}$, i.e. it satisfies (2.177) and

$$\tilde{V}_1[\tilde{\Omega}_0] = -\kappa^2, \qquad \tilde{V}_2[\tilde{\Omega}_0] = 0, \tag{2.218}$$

and $\Omega_1(\mathbf{x})$, $\Omega_2(\mathbf{x})$ are solutions on $\mathbb{R} \times \mathrm{S^3}$ with point-like BMN motion along $\mathbb{R}$, i.e. they satisfy (2.178) and

$$V_1[\Omega_i] = \kappa^2, \qquad V_2[\Omega_i] = 0 \ , \tag{2.219}$$

the configuration

$$\left(\tilde{\Omega}, \Omega_+, \Omega_-\right) = \left(\tilde{\Omega}_0(\mathbf{x}), \Omega_1(A\mathbf{x}), \Omega_2(B\mathbf{x})\right) \tag{2.220}$$

will be a valid solution on $AdS_3 \times S^3 \times S^3$, as long as the constants A, B satisfy the combined Virasoro constraint

$$\frac{A^2}{\cos^2\varphi} + \frac{B^2}{\sin^2\varphi} = 1. \tag{2.221}$$

In the simplest application of this prescription one can take $\tilde{\Omega}_0, \Omega_1, \Omega_2$ all to be the $AdS_3 \times S^3$ BMN solution, and get the $AdS_3 \times S^3 \times S^3$ BMN string (2.184) as a result, noting that the conditions (2.221) and (2.182) are the same.

2.2.3 Dressing method

In this subsection we review the dressing method [143, 144] for the construction of solitons of classically integrable systems, following the discussion of Spradlin and Volovich [145]. A mixed-flux generalization of the dressing method is presented in [120], however, in a form that breaks down at the special point $q = 1$. Below we present a slight variation that is capable of handling the pure NS-NS case. We consider strings moving on the $AdS_3 \times S^3$ subspace of the $AdS_3 \times S^3 \times T^4$ background, which then can be lifted to $AdS_3 \times S^3 \times S^3 \times S^1$ using (2.220).

The $SU(1,1) \times SU(2)$ principal chiral model

Conformal gauge bosonic string theory on mixed-flux $AdS_3 \times S^3$ is equivalent to the principal chiral model with Wess-Zumino term on $SU(1,1) \times SU(2)$ with action

$$S = S_{\mathrm{PCM}}[g_{\mathrm{AdS}}] + S_{\mathrm{PCM}}[g_{\mathrm{S}}], \tag{2.222}$$

where, in terms of the left currents $\mathfrak{J}_a = g^{-1}\partial_a g$

$$S_{\mathrm{PCM}}[g] = -\frac{\mathrm{h}}{2}\Big[\int_{\mathcal{M}} \mathrm{d}^2\sigma\, \tfrac{1}{2}\mathrm{tr}(\bar{\mathfrak{J}}\,\mathfrak{J}) - q\int_{\mathcal{B}} \mathrm{d}^3\sigma\, \tfrac{1}{3}\varepsilon^{abc}\mathrm{tr}(\mathfrak{J}_a\mathfrak{J}_b\mathfrak{J}_c)\Big] . \tag{2.223}$$

Here $\mathcal{M}$ is the string worldsheet, $\mathcal{B}$ is a 3d manifold with boundary $\mathcal{M}$, and $\mathfrak{J} = g^{-1}\partial g$, $\bar{\mathfrak{J}} = g^{-1}\bar{\partial} g$, where the partial derivatives are with respect to $z = \frac{1}{2}(\tau - \sigma)$ and $\bar{z} = \frac{1}{2}(\tau + \sigma)$. Also introducing the right current $\mathfrak{K}_a = \partial_a g g^{-1}$, the PCM equations of motion can be written in the two equivalent forms

$$\begin{aligned} &(1+q)\partial\bar{\mathfrak{J}} + (1-q)\bar{\partial}\mathfrak{J} = 0, \\ &(1-q)\partial\bar{\mathfrak{K}} + (1+q)\bar{\partial}\mathfrak{K} = 0. \end{aligned} \tag{2.224}$$

These are the equations for both matrix fields $g = g_{\text{AdS}}(z, \bar{z}) \in \text{SU}(1,1)$ and $g = g_{\text{S}}(z, \bar{z}) \in \text{SU}(2)$, and need to be supplemented by the conformal gauge Virasoro constraints

$$\begin{aligned} &\text{tr}\left[\mathfrak{J}_{\text{AdS}}^2\right] - \text{tr}\left[\mathfrak{J}_{\text{S}}^2\right] = 0 \ , \\ &\text{tr}\left[\bar{\mathfrak{J}}_{\text{AdS}}^2\right] - \text{tr}\left[\bar{\mathfrak{J}}_{\text{S}}^2\right] = 0 \ . \end{aligned} \tag{2.225}$$

Equivalence to the coordinate space GS action (2.176) can be established via the embedding

$$g_{\text{AdS}} = \begin{pmatrix} Z_1 & -iZ_2 \\ i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \text{SU}(1,1), \qquad g_{\text{S}} = \begin{pmatrix} Z_3 & -iZ_4 \\ -i\bar{Z}_4 & \bar{Z}_3 \end{pmatrix} \in \text{SU}(2), \tag{2.226}$$

and Hopf coordinates

$$\begin{aligned} &Z_1 = \cosh\rho \ e^{iT} \ , \quad Z_2 = \sinh\rho \ e^{i\psi} \ , \\ &Z_3 = \sin\theta \ e^{i\phi_1} \ , \quad Z_4 = \cos\theta \ e^{i\phi_2} \ . \end{aligned} \tag{2.227}$$

From the action we get the left-invariant and right-invariant conserved currents

$$L_a = \mathfrak{J}_a - q\epsilon_{ab}\mathfrak{J}^b \ , \quad R_a = \mathfrak{K}_a + q\epsilon_{ab}\mathfrak{K}^b \ , \quad \partial_a L^a = \partial_a R^a = 0 \ , \tag{2.228}$$

which give rise to the conserved charges

$$Q_L = \text{h}\int \text{d}\sigma\,(\mathfrak{J}_0 + q\mathfrak{J}_1) \ , \quad Q_R = \text{h}\int \text{d}\sigma\,(\mathfrak{K}_0 - q\mathfrak{K}_1) \ , \tag{2.229}$$

again, both for $\text{SU}(1,1)$ and $\text{SU}(2)$.

From a given solution g to the equations (2.224) the dressing method allows us to generate a new solution g' by

$$g \to g' = \chi g \tag{2.230}$$

for some appropriately chosen $\chi(z, \bar{z})$. We might apply this method to either (or both) of the SU$(1,1)$ and SU(2) components independently, as long as the resulting solution still satisfies the Virasoro constraints. Most of the discussion below holds for both SU$(1,1)$ and SU(2), any differences will be explicitly pointed out.

Auxiliary problem

The construction starts by considering the system of equations for the matrix field $\Psi(\lambda)$

$$\bar{\partial}\Psi = \frac{A\Psi}{1+(1+q)\lambda}, \qquad \partial\Psi = \frac{B\Psi}{1-(1-q)\lambda}, \tag{2.231}$$

where the matrices A and B are independent of the complex auxiliary variable λ, also known as the *spectral parameter*. This is an overdetermined system, whose compatibility ($\partial\bar{\partial}\Psi = \bar{\partial}\partial\Psi$) is guaranteed for all values of λ by the conditions

$$\begin{aligned} \partial A - \bar{\partial}B + [A, B] &= 0, \\ (1-q)\partial A + (1+q)\bar{\partial}B &= 0. \end{aligned} \tag{2.232}$$

The dressing method exploits the following relation between the principal chiral model (2.224) and the auxiliary problem (2.231). Given any solution g to (2.224)

$$A = \bar{\partial}g\, g^{-1}, \qquad B = \partial g\, g^{-1} \tag{2.233}$$

will satisfy the compatibility conditions (2.232), and we can solve (2.231) to find $\Psi(\lambda)$ subject to

$$\Psi(0) = g. \tag{2.234}$$

Conversely, for any any collection $(\Psi(\lambda), A, B)$ satisfying (2.231) for all values of λ, $g = \Psi(0)$ satisfies (2.224), as a direct consequence of (2.232). We further impose the unitarity constraint[15]

$$\Psi^\dagger(\bar{\lambda})M\Psi(\lambda) = M \tag{2.235}$$

with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for SU}(1,1) \text{ and } M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for SU}(2). \tag{2.236}$$

Dressed solution

Consider the analogue of the "gauge" transformation (2.230) for the auxiliary system (2.231), now with a λ-dependent matrix field $\chi(\lambda)$,

$$\begin{aligned} &\Psi \to \Psi' = \chi\Psi, \\ &A \to A' = \chi A \chi^{-1} + i(1 + (1+q)\lambda)\bar{\partial}\chi\chi^{-1}, \\ &B \to B' = \chi B \chi^{-1} + i(1 - (1-q)\lambda)\partial\chi\chi^{-1}. \end{aligned} \tag{2.237}$$

If we can find a $\chi(\lambda)$ such that the transformed A' and B' continue to be independent of λ, then the triplet $(\Psi'(\lambda), A', B')$ is another legitimate solution of (2.231), and $g' = \Psi'(0)$ is a new solution of the principal chiral model with WZ term.

The λ-independence of A' and B' can be easily achieved by imposing constraints on the analytic properties of $\chi(\lambda)$ in the complex λ-plane. We start by requiring that $\chi(\lambda)$ is meromorphic, and that[16] $\chi(\lambda) \to 1$ as $\lambda \to \infty$. In order to preserve the unitarity condition (2.235), $\chi(\lambda)$ should satisfy

$$\chi^\dagger(\bar{\lambda})M\chi(\lambda) = M. \tag{2.238}$$

The simplest such $\chi(\lambda)$ has a single pole at some location λ_1, and is fixed, up to a constant phase, by the above conditions to be

$$\chi(\lambda) = \mathbf{1} + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P, \tag{2.239}$$

[15] By this notation we mean $\Psi^\dagger(\bar{\lambda}) = (\Psi(\bar{\lambda}))^\dagger$, which is a function of λ only.

[16] Any constant matrix could have been chosen as the limit at infinity, all being an unphysical field-redefinition away from our convenient choice of the unit matrix.

where the projector P satisfies $P^2 = P$ and $P^\dagger = MPM$. Since A' and B' asymptotically become A and B, a standard theorem in complex analysis guarantees their λ-independence if they have no poles. The only possible locations for poles are λ_1 or $\bar{\lambda}_1$, but one can check that the residues at these putative poles vanish if we choose

$$P = \frac{v_1 v_1^\dagger M}{v_1^\dagger M v_1} \ , \quad v_1 = \Psi(\bar{\lambda}_1) e \tag{2.240}$$

for a constant vector e. The overall scale of e drops out of (2.240), and we can take

$$e = (w, 1/w) \tag{2.241}$$

for some complex parameter w. The dressing factor (2.239) has the determinant

$$\det \chi(\lambda) = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \ . \tag{2.242}$$

Requiring that the dressed solution $\chi(0)\Psi(0)$ has unit determinant fixes the constant phase in front of $\chi(\lambda)$ to be $(\lambda_1/\bar{\lambda}_1)^{1/2}$. With this, the dressed solution is

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left(\mathbb{1} - \left(1 - \frac{\bar{\lambda}_1}{\lambda_1} \right) P \right) g \ . \tag{2.243}$$

Below we apply both the SU(2) and SU(1,1) dressing method to the BMN string, to get the mixed flux giant magnon on $\mathbb{R} \times \mathrm{S}^3$ and another soliton on $\mathrm{AdS}_3 \times \mathrm{S}^1$.

2.2.4 Mixed-flux AdS_3 giant magnon

The Hofman-Maldacena giant magnon (2.45) is, quite naturally, also a solution on the $\mathbb{R} \times \mathrm{S}^3$ subspace of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with pure R-R background flux. Its mixed-flux generalization was first found by Hoare, Stepanchuk and Tseytlin [119], using a clever q-deformation of the SU(2) currents. The same solution can be obtained from a rigidly rotating string ansatz [165], and using the dressing method [120]. In this subsection we apply the SU(2) dressing method, as presented above, to construct the mixed-flux magnon.

Dressing the giant magnon

We dress the BMN vacuum

$$Z_1 = e^{it}, \qquad Z_2 = 0, \tag{2.244}$$

for which the auxiliary system (2.231) reads

$$g_0 = \begin{pmatrix} e^{-i(z-\bar{z})} & 0 \\ 0 & e^{i(z-\bar{z})} \end{pmatrix}, \qquad A_0 = -B_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.245}$$

The solution to this system, subject to (2.234), is

$$\begin{aligned} \Psi_0(\lambda) &= \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & e^{iZ(\lambda)} \end{pmatrix}, \\ Z(\lambda) &= \frac{z}{1-(1-q)\lambda} - \frac{\bar{z}}{1+(1+q)\lambda}. \end{aligned} \tag{2.246}$$

To fix the projector (2.240) we need to specify $e \in \mathbb{P}^1$, which we can parametrize as

$$e = (w, 1/w) \tag{2.247}$$

for $w \in \mathbb{C}^*$. Since e only appears in the projector as part of

$$\Psi_0(\bar{\lambda}_1)e = \begin{pmatrix} w\, e^{-iZ(\bar{\lambda}_1)} \\ \frac{1}{w}\, e^{+iZ(\bar{\lambda}_1)} \end{pmatrix}, \tag{2.248}$$

any non-trivial w just amounts to a shift $Z(\bar{\lambda}_1) \to Z(\bar{\lambda}_1) - i\log w$, which is equivalent to a translation of the worldsheet coordinates. This will not result in a physically different solution, and without loss of generality we can set $w = 1$. With this, the projector becomes

$$P = \frac{1}{1+e^{2i(Z(\bar{\lambda}_1)-Z(\lambda_1))}} \begin{pmatrix} 1 & e^{-2iZ(\lambda_1)} \\ e^{+2iZ(\bar{\lambda}_1)} & e^{2i(Z(\bar{\lambda}_1)-Z(\lambda_1))} \end{pmatrix}, \tag{2.249}$$

and, including the phase factor explained under (2.242), the dressed one-soliton solution is

$$\Psi_1(\lambda) = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \right] \Psi_0(\lambda). \tag{2.250}$$

Plugging in the vacuum (2.246), we can read off the solution from (2.226)

$$\begin{aligned} Z_1 &= \frac{e^{+it}}{|\lambda_1|} \frac{\lambda_1 e^{2iZ(\bar{\lambda}_1)} + \bar{\lambda}_1 e^{2iZ(\lambda_1)}}{e^{2iZ(\lambda_1)} + e^{2iZ(\bar{\lambda}_1)}}, \\ Z_2 &= \frac{e^{-it}}{|\lambda_1|} \frac{i(\bar{\lambda}_1 - \lambda_1)}{e^{2iZ(\lambda_1)} + e^{2iZ(\bar{\lambda}_1)}}. \end{aligned} \tag{2.251}$$

If we rewrite the location of the pole

$$\lambda_1 = r e^{ip/2}, \tag{2.252}$$

and introduce

$$\begin{aligned} U &= i \left(Z(\bar{\lambda}_1) - Z(\lambda_1) \right), \\ V &= -Z(\bar{\lambda}_1) - Z(\lambda_1) - t, \end{aligned} \tag{2.253}$$

the solution (2.251) can be expressed as

$$\begin{aligned} Z_1 &= e^{it} \left[\cos \tfrac{\mathrm{p}}{2} + i \sin \tfrac{\mathrm{p}}{2} \, \tanh U \right], \\ Z_2 &= e^{iV} \sin \tfrac{\mathrm{p}}{2} \, \mathrm{sech} U. \end{aligned} \tag{2.254}$$

From

$$x \to \pm\infty : \qquad Z_1 \to \exp\left(it \pm i \frac{\Delta\phi_1}{2} \right), \qquad Z_2 \to 0, \tag{2.255}$$

we can read off the opening angle between the string endpoints $\Delta\phi_1 = \mathrm{p}$, and see that p is indeed the worldsheet momentum (2.217) of the mixed-flux magnon. Further substituting (2.252) and (2.246) into (2.253) we get

$$\begin{aligned} U &= \cos \varrho \, \tilde{q} \, \gamma (x - ut), \\ V &= \sin \varrho \, \tilde{q} \, \gamma (t - ux) - qx, \end{aligned} \tag{2.256}$$

where $\tilde{q} = \sqrt{1-q^2}$, $\gamma = (1-u^2)^{-1/2}$, and

$$\begin{aligned} u &= \frac{-q(1-\tilde{q}^2 r^2) + 2\tilde{q}^2 r \cos\frac{\mathrm{p}}{2}}{1+\tilde{q}^2 r^2}, \\ \cot\varrho &= \frac{2r\sin\frac{\mathrm{p}}{2}}{1-\tilde{q}^2 r^2 + 2qr\cos\frac{\mathrm{p}}{2}}. \end{aligned} \tag{2.257}$$

This solution agrees with the mixed-flux dyonic giant magnon found by Hoare, Stepanchuk and Tseytlin [119], if one rewrites the magnon speed u as

$$u = \frac{v-q}{1-qv}. \tag{2.258}$$

Conserved charges and dispersion relation

Defining

$$b = \cot\tfrac{\mathrm{p}}{2} = \tilde{q}\,\gamma u\,\sec\varrho + q\tan\varrho, \tag{2.259}$$

we can write the magnon (2.254) in Hopf coordinates

$$\begin{aligned} &\theta = \arccos\left(\frac{\operatorname{sech}[\tilde{q}\cos\varrho\,\mathcal{X}]}{\sqrt{1+b^2}}\right), \\ &\phi_1 = t + \arctan\left(b^{-1}\tanh[\tilde{q}\cos\varrho\,\mathcal{X}]\right), \qquad \phi_2 = \tilde{q}\sin\varrho\,\mathcal{T} - qx\,, \\ &\mathcal{X} = \gamma(x-ut), \quad \mathcal{T} = \gamma(t-ux). \end{aligned} \tag{2.260}$$

In the decompactification limit (2.41), (2.42), the $\mathbb{R}\times\mathrm{S}^3$ conserved charges (2.181) are

$$\begin{aligned} E - J_1 &= \mathrm{h}\int_{-\infty}^{\infty}\mathrm{d}x\,\left(1 - \left[\sin^2\theta\;\partial_t\phi_1 - \frac{q}{2}(\cos 2\theta + c)\partial_x\phi_2\right]\right), \\ J_2 &= \mathrm{h}\int_{-\infty}^{\infty}\mathrm{d}x\,\left[\cos^2\theta\;\partial_t\phi_2 + \frac{q}{2}(\cos 2\theta + c)\partial_x\phi_1\right]. \end{aligned} \tag{2.261}$$

For the magnon solution (2.260) these integrals evaluate to

$$E - J_1 = 2\mathrm{h}\frac{\sqrt{1-q^2+(b\cos\varrho - q\sin\varrho)^2}}{(1+b^2)\cos\varrho} + \frac{1}{2}\mathrm{h}q(c-1)\Delta\phi_2\,, \tag{2.262}$$

$$J_2 = M + \frac{1}{2}(c+1)\mathrm{h}q\Delta\phi_1 \; , \qquad M = 2\mathrm{h}\frac{\tan\varrho - qb}{1+b^2} \; , \tag{2.263}$$

where the opening angles $\Delta\phi_i = \phi_i(x=\infty) - \phi_i(x=-\infty)$ are

$$\begin{aligned} \Delta\phi_1 &= 2\,\mathrm{arccot}\, b, \\ \Delta\phi_2 &= -\cos\varrho\;(q\cos\varrho + b\sin\varrho)\; x\big|_{-\infty}^{\infty}. \end{aligned} \tag{2.264}$$

We see that $\Delta\phi_2$ is divergent, and the only way to keep $E - J_1$ finite is to fix the total derivative ambiguity in the Wess-Zumino term (2.176)

$$c = 1. \tag{2.265}$$

With this choice, and recalling that $\Delta\phi_1$ is the worldsheet momentum, the charges become

$$\begin{aligned} E - J_1 &= 2\mathrm{h}\tilde{q}\gamma\sec\varrho\,\sin^2\tfrac{\mathrm{p}}{2}, \\ J_2 &= M + \mathrm{h}q\mathrm{p}, \qquad M = 2\mathrm{h}\sin^2\tfrac{\mathrm{p}}{2}\left(\tan\varrho - q\cot\tfrac{\mathrm{p}}{2}\right), \end{aligned} \tag{2.266}$$

and the mixed-flux magnon satisfies the disperison relation

$$E - J_1 = \sqrt{(J_2 - \mathrm{h}q\mathrm{p})^2 + 4\mathrm{h}^2\tilde{q}^2\,\sin^2\tfrac{\mathrm{p}}{2}}. \tag{2.267}$$

Semiclassical quantization of J_2. The similarity between (2.267) and the quantum dispersion relation (2.154) derived from the symmetries of the action can be taken one step further by considering the semiclassical quantization of the second angular momentum J_2, as presented in [119]. The mixed-flux magnon solution (2.260) is time-periodic in ϕ_2, assuming the shift in t is compensated by a shift in x so that θ is kept constant. In fact, we might treat θ as a spatial coordinate ($\cos\theta$ changes from zero to its maximal value then back) and write the other two angles as a function of t and θ

$$\begin{aligned} \phi_1(t,\theta) &= t + \arctan\left[\, b^{-1}\sqrt{1-(1+b^2)\cos^2\theta}\,\right] , \\ \phi_2(t,\theta) &= wt + r\,\mathrm{arccosh}\left[\,\left(\sqrt{1+\mathrm{b}^2}\cos\theta\right)^{-1}\right] , \end{aligned} \tag{2.268}$$

$$w = \frac{\tilde{q}\sin\varrho}{\gamma} - qu \ , \qquad \mathrm{T}_{\phi_2} \equiv \frac{2\pi}{|w|} \ , \tag{2.269}$$

$$r = w\frac{q + b\tan\varrho}{qb - \tan\varrho} \ , \tag{2.270}$$

where T_{ϕ_2} is the period of the motion. This periodicity implies the existence of an associated action variable which takes integer values upon semiclassical quantisation. Applying Liouville's theory of integrable Hamiltonian systems, the action variable I is given by

$$2\pi I = S - \mathrm{T}_{\phi_2}\frac{\partial S}{\partial \mathrm{T}_{\phi_2}}\bigg|_{\mathrm{p}} \ , \tag{2.271}$$

where $S = S(\mathrm{T}_{\phi_2}, \mathrm{p})$ is the light-cone gauge string action for the giant magnon[17] computed over one period T_{ϕ_2}. Since the string action is reparametrization-invariant we can evaluate $S(\mathrm{T}_{\phi_2}, \mathrm{p})$ in conformal gauge coordinates, over $t \in [0, \mathrm{T}_{\phi_2}]$ and $x \in (-\infty, \infty)$. Substituting (2.260) back into (2.175) we get

$$S = 2\pi\mathrm{h}\Big[-\frac{2(1-q^2)}{\tan\varrho - q\tilde{b}} + \frac{1}{2}q(c+1)\Delta\phi_1 - \frac{1}{4\pi}\mathrm{T}q(c-1)\Delta\phi_2\Big], \tag{2.272}$$

Since $\Delta\phi_2$ is divergent we need $c = 1$ for the action finite, a choice consistent with the finiteness of $E - J_1$ (2.265). Eliminating ϱ this becomes

$$\frac{S}{2\pi\mathrm{h}} = \frac{2\tilde{q}^2\sqrt{1-w^2}}{qb\sqrt{1-w^2} - \left(\tilde{q}^2 + b^2 - \left[\sqrt{\tilde{q}^2(1+b^2)(1-w^2)} - qwb\right]^2\right)^{1/2}} + q\mathrm{p} \ , \tag{2.273}$$

and from (2.271) we find that the action variable conjugate to ϕ_2 is indeed the second angular momentum (2.262)

$$I = J_2 \ . \tag{2.274}$$

We conclude that upon semiclassical quantization J_2 takes integer values, and for $J_2 = 1$ the dispersion relation (2.267) exactly matches (2.154).

[17] Note that for this calculation one needs to express the parameters ϱ and b in terms of T_{ϕ_2} and p.

Stationary magnon on $\mathbb{R}\times\mathbf{S}^3$

What made the the fermion zero mode calculation [53] for the AdS_5 magnon relatively simple is the fact that the shape of the HM magnon is time-independent, unlike the dyonic magnon (2.53), which is $\mathcal{T}$-dependent for general values of ϱ. We want to make the same simplification before attempting the semiclassical analysis of the mixed-flux magnon. Requiring that ϕ_2 in (2.260) only depends on $\mathcal{X}$ fixes the value of ϱ

$$\sin\varrho = \frac{qu}{\sqrt{1-q^2}\sqrt{1-u^2}} = \frac{\gamma u q}{\tilde{q}}, \tag{2.275}$$

and we get the *stationary* mixed-flux magnon

$$Z_1 = \frac{e^{it}\left[b + i\,\tanh\left(\gamma\sqrt{\tilde{q}^2-u^2}\mathcal{X}\right)\right]}{\sqrt{1+b^2}},$$
$$Z_2 = \frac{e^{-iq\gamma\mathcal{X}}\,\mathrm{sech}\left(\gamma\sqrt{\tilde{q}^2-u^2}\mathcal{X}\right)}{\sqrt{1+b^2}}, \qquad b = \frac{u}{\sqrt{\tilde{q}^2-u^2}}. \tag{2.276}$$

For this solution the dispersion relation (2.267) takes the simpler form

$$E - J_1 = 2\mathrm{h}\tilde{q}\,\sin\tfrac{\mathrm{p}}{2}, \tag{2.277}$$

much like the dispersion relation of the HM giant magnon (2.52), further pointing to the special role our stationary magnon plays among mixed-flux magnons.

Parameter ranges Since u is the worldsheet speed of the magnon, one might expect it to take values in the range $(-1, 1)$. This is certainly true for the general solution (2.260), but the stationary condition (2.275) further restricts

$$\sin^2\varrho \leq 1 \quad\Rightarrow\quad |u| \leq \tilde{q}. \tag{2.278}$$

It might look like some solutions are missing, but we do have a stationary magnon for all values of the worldsheet momentum p, a fact that becomes clear once the condition (2.275) is rewritten using (2.259) as

$$u = \tilde{q}\cos\frac{\mathrm{p}}{2}, \qquad \tan\rho = q\cot\frac{\mathrm{p}}{2}. \tag{2.279}$$

Mixed-flux giant magnon on $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$

We can construct the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ mixed-flux magnon using the prescription (2.220), by putting the magnon (2.260) on S^3_+ and the BMN string on S^3_-. In Hopf coordinates for the two spheres $\mathrm{S}^3_\pm$

$$Z_1^\pm = \sin\theta^\pm \; e^{i\phi_1^\pm} \;, \quad Z_2^\pm = \cos\theta^\pm \; e^{i\phi_2^\pm} \;, \tag{2.280}$$

the solution is given by

$$\begin{aligned}
&\theta^+ = \arccos\left(\frac{\operatorname{sech}\left[A\cos\varrho\;\tilde{q}\mathcal{X}\right]}{\sqrt{1+b^2}}\right) \;,\\
&\phi_1^+ = At + \arctan\left(b^{-1}\tanh\left[A\cos\varrho\;\tilde{q}\mathcal{X}\right]\right) \;, \qquad \phi_2^+ = A\sin\varrho\;\tilde{q}\,\mathcal{T} - Aqx \;,\\
&\theta^- = \frac{\pi}{2} \;, \qquad \phi_1^- = Bt \;, \qquad \phi_2^- = 0 \;.\\
&\gamma^2 = \frac{1}{1-u^2} \;, \qquad b = \tilde{q}\,\gamma u\,\sec\rho + q\tan\varrho \;, \qquad u \in (0,1) \;,
\end{aligned} \tag{2.281}$$

where the parameters A, B satisfy the Virasoro constraint (2.221)

$$\frac{A^2}{\cos^2\varphi} + \frac{B^2}{\sin^2\varphi} = 1 \;. \tag{2.282}$$

Noether charges. Recalling that the physical combination of first angular momenta is $J_1 = AJ_1^+ + BJ_1^-$ (2.183), the conserved charges (2.181) for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ mixed-flux magnon become

$$\begin{aligned}
E - J_1 &= \frac{A}{\cos^2\varphi} 2\mathrm{h}\tilde{q}\gamma\sec\varrho\,\sin^2\tfrac{\mathrm{p}}{2},\\
J_2 &= \frac{1}{\cos^2\varphi}\left(M + \mathrm{h}q\mathrm{p}\right), \qquad M = 2\mathrm{h}\sin^2\tfrac{\mathrm{p}}{2}\left(\tan\varrho - q\cot\tfrac{\mathrm{p}}{2}\right),
\end{aligned} \tag{2.283}$$

with dispersion relation

$$E - J_1 = \frac{A}{\cos^2\varphi}\sqrt{\left(\cos^2\varphi J_2 - \mathrm{h}q\mathrm{p}\right)^2 + 4\mathrm{h}^2\tilde{q}^2\sin^2\tfrac{\mathrm{p}}{2}}. \tag{2.284}$$

There are two conclusions to be made. Firstly, to match the correct dispersion relation derived from symmetry (2.154) we need to take $A = \cos^2\varphi$, in agreement with the maximal SUSY condition (2.187). Secondly, an argument similar to (2.274) shows that J_2 is quantized in integer units, and (2.281) represent one of the light magnons with mass $m = \cos^2\varphi$. We can get the other light magnon of mass $\sin^2\varphi$ by switching the two spheres, but this construction doesn't give us the massless ($m = 0$) or heavy ($m = 1$) classical string excitations.

Stationary magnon. As discussed above (2.275), for the purposes of the semiclassical analysis we will focus on the maximally SUSY $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ generalization of the stationary magnon (2.276)

$$\begin{aligned}
&\theta^+ = \arccos\left(\frac{\operatorname{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right) ,\\
&\phi_1^+ = \cos^2\varphi\, t + \arctan\left(b^{-1}\tanh\mathcal{Y}\right) , \qquad \phi_2^+ = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2 - u^2}} ,\\
&\theta^- = \frac{\pi}{2} , \qquad \phi_1^- = \sin^2\varphi\, t , \qquad \phi_2^- = 0 .\\
&\gamma^2 = \frac{1}{1-u^2} , \qquad b = \frac{u}{\sqrt{\tilde{q}^2 - u^2}} , \qquad u \in (-\tilde{q}, \tilde{q}) ,
\end{aligned} \tag{2.285}$$

where we further defined the scaled and boosted worldsheet coordinate

$$\mathcal{Y} = \cos^2\varphi\ \gamma\sqrt{\tilde{q}^2 - u^2}\,\mathcal{X}. \tag{2.286}$$

Double magnon. Another simple application of (2.220) is to put an $\mathbb{R}\times\mathrm{S}^3$ magnon on both of the spheres, corresponding to the diffractionless scattering of of two light magnons with different masses. Since this state is not in a short representation of the residual algebra, comparing its fermion zero modes to those of the single magnon reveals some key differences between short and long representations. In chapter 6 we perform this analysis for a highly symmetric special case of the double magnon solution, where the two spheres have equal radius

$$\cos^2\varphi = \sin^2\varphi = \frac{1}{2}, \tag{2.287}$$

and the two magnons are stationary with the same speed, i.e.

$$\begin{aligned}
&\theta^+ = \theta^- = \arccos\left(\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right) , \\
&\phi_1^+ = \phi_1^- = \frac{1}{2}\,t + \arctan\left(b^{-1}\tanh\mathcal{Y}\right) , \qquad \phi_2^+ = \phi_2^- = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2-u^2}} , \\
&\gamma^2 = \frac{1}{1-u^2} , \quad b = \frac{u}{\sqrt{\tilde{q}^2-u^2}} , \quad u \in (-\tilde{q},\tilde{q}) ,
\end{aligned} \tag{2.288}$$

with

$$\mathcal{Y} = \frac{1}{2}\gamma\sqrt{\tilde{q}^2-u^2}\mathcal{X}. \tag{2.289}$$

2.2.5 Mixed-flux $AdS_3 \times S^1$ soliton

The mixed magnon from section 2.2.4 is an $\mathbb{R}\times S^3$ string solution, but one can equally consider solitons living on the $AdS_3 \times S^1$ subspace of either $AdS_3 \times S^3 \times T^4$ or $AdS_3 \times S^3 \times S^3 \times S^1$. For AdS_5 such a solution was first described as part of a three-spin giant magnon [166], that is also a valid string state on the pure R-R AdS_3 backgrounds. Using a rigidly rotating string ansatz, the corresponding mixed-flux solutions were later found in [165, 167]. The dispersion relation of these 3-spin magnons is consistent with the fact that they are made up of two particles: a dyonic giant magnon on S^3, and an AdS_3 soliton. In this subsection we study this mixed-flux $AdS_3 \times S^1$ soliton in isolation.

Dressing

We can construct the $AdS_3 \times S^1$ soliton by applying the $SU(1,1)$ dressing method, as presented in section 2.2.3, to the BMN-vacuum. The auxiliary system is the same as for the $SU(2)$ case (2.245), with solution

$$\Psi_0(\lambda) = \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & e^{iZ(\lambda)} \end{pmatrix},$$
$$Z(\lambda) = \frac{z}{1-(1-q)\lambda} - \frac{\bar{z}}{1+(1+q)\lambda}. \tag{2.290}$$

As explained above (2.249), without loss of generality we can take

$$e = (1,1), \tag{2.291}$$

and this fixes the projector (2.240)

$$P = \frac{1}{1-e^{2i(Z(\bar{\lambda}_1)-Z(\lambda_1))}} \begin{pmatrix} 1 & -e^{-2iZ(\lambda_1)} \\ e^{+2iZ(\bar{\lambda}_1)} & -e^{2i(Z(\bar{\lambda}_1)-Z(\lambda_1))} \end{pmatrix}. \tag{2.292}$$

The dressed soliton can be read off from

$$\Psi_1(\lambda) = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P\right] \Psi_0(\lambda), \tag{2.293}$$

at zero spectral parameter

$$\Psi_1(0) = \begin{pmatrix} Z_1 & -iZ_2 \\ i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}, \tag{2.294}$$

as

$$Z_1 = \frac{e^{+it}}{|\lambda_1|} \frac{\bar{\lambda}_1 e^{2iZ(\lambda_1)} - \lambda_1 e^{2iZ(\bar{\lambda}_1)}}{e^{2iZ(\lambda_1)} - e^{2iZ(\bar{\lambda}_1)}},$$
$$Z_2 = \frac{e^{-it}}{|\lambda_1|} \frac{i(\lambda_1 - \bar{\lambda}_1)}{e^{2iZ(\lambda_1)} - e^{2iZ(\bar{\lambda}_1)}}. \tag{2.295}$$

Using the parametrization (2.252), (2.253), the solution can be written as

$$Z_1 = e^{it}\left[\cos\tfrac{\mathrm{p}}{2} + i\sin\tfrac{\mathrm{p}}{2}\,\coth U\right],$$
$$Z_2 = e^{iV}\sin\tfrac{\mathrm{p}}{2}\,\operatorname{csch} U, \tag{2.296}$$

where

$$
\begin{aligned}
U &= \cos\varrho\ \tilde{q}\,\gamma(x - ut),\\
V &= \sin\varrho\ \tilde{q}\,\gamma(t - ux) - qx,
\end{aligned}
\tag{2.297}
$$

$\tilde{q} = \sqrt{1-q^2}$, $\gamma = (1-u^2)^{-1/2}$, and u is the soliton speed on the worldsheet.

Conserved charges and dispersion relation

Let us write the $\mathrm{AdS}_3 \times \mathrm{S}^1$ soliton (2.297) in Hopf coordinates

$$
\begin{aligned}
&\rho = \operatorname{arcsinh}\left(\frac{\operatorname{csch} U}{\sqrt{1+b^2}}\right),\\
&T = t + \arctan\left(b^{-1}\coth U\right), \qquad \psi = V - qx\\
&\theta = \frac{\pi}{2}, \quad \phi_1 = t, \quad \phi_2 = 0,
\end{aligned}
\tag{2.298}
$$

where the parameters are related by

$$
b = \tilde{q}\,\gamma u\ \sec\varrho + q\tan\varrho. \tag{2.299}
$$

In the decompactification limit the charges (2.181) of a general $\mathrm{AdS}_3 \times \mathrm{S}^1$ solution are

$$
\begin{aligned}
E - J_1 &= \mathrm{h}\int_{-\infty}^{\infty} \mathrm{d}x\left[\cosh^2\!\rho\ \dot{T} + q\sinh^2\!\rho\ \psi' - 1\right]\ + \frac{1+\tilde{c}}{2}\mathrm{h}q\Delta\psi,\\
J_0 &= \mathrm{h}\int_{-\infty}^{\infty} \mathrm{d}x\left[\sinh^2\!\rho\ \dot{\psi} + q\cosh^2\!\rho\ T'\right]\ - \frac{1-\tilde{c}}{2}\mathrm{h}q\Delta T.
\end{aligned}
\tag{2.300}
$$

For the solution (2.298) the difference in AdS time between the string end-points is

$$
\Delta T = 2\arctan b^{-1}, \tag{2.301}
$$

while $\Delta\psi$ is infinite. Calculating the conserved charges (2.300) we find both IR and UV divergences. The IR divergence, which is essentially the same as for the $\mathbb{R} \times \mathrm{S}^3$ magnon, appears due to the infinite worldsheet volume, and can be easily removed by adjusting the boundary term to

$$
\tilde{c} = -1, \tag{2.302}
$$

since the only IR divergent term in (2.300) is $\Delta\psi$. The UV divergence is the result of the string (2.298) stretching to the boundary of AdS_3 at $U = 0$, and can be removed using a simple cutoff regularization as prescribed in [166]. After regularization we have

$$\begin{aligned} E - J_1 &= -2\mathrm{h}\frac{\tilde{q}\gamma \sec\varrho}{1+b^2}, \\ J_0 &= -2\mathrm{h}\frac{\tan\rho - qb}{1+b^2} - \mathrm{h}q\Delta T, \end{aligned} \tag{2.303}$$

and recalling that the wordsheet momentum is $\mathrm{p} = -\Delta T$ (2.217) we get the dispersion relation

$$E - J_1 = -\sqrt{(J_0 - \mathrm{h}q\mathrm{p})^2 + 4\mathrm{h}^2\tilde{q}^2 \cos^2\tfrac{\mathrm{p}}{2}}. \tag{2.304}$$

This expression is similar to the dispersion relation of the $\mathbb{R} \times S^3$ magnon (2.267), with the main differences being the negative sign and the appearance of $\cos\frac{\mathrm{p}}{2}$ instead of $\sin\frac{\mathrm{p}}{2}$.

Semiclassical quantization

Similarly to the second angular momentum J_2 of the giant magnon, the AdS spin J_0 of the $AdS_3 \times S^1$ soliton will be quantized. Treating ρ as a spatial coordinate (ρ goes from zero to infinity then back), we can rewrite (2.298) in the form

$$\begin{aligned} T(t,\rho) &= t \pm \arctan\left[\, b^{-1}\sqrt{1+(1+b^2)\sinh^2\!\rho}\,\right] , \\ \psi(t,\rho) &= wt + r\,\mathrm{arcsinh}\left[\, \left(\sqrt{1+\mathrm{b}^2}\sinh\rho\right)^{-1} \right] , \end{aligned} \tag{2.305}$$

$$w = \frac{\tilde{q}\sin\varrho}{\gamma} - qu\ , \qquad \mathrm{T}_\psi \equiv \frac{2\pi}{|w|}\ , \tag{2.306}$$

$$r = w\frac{q + b\tan\varrho}{qb - \tan\varrho}\ , \tag{2.307}$$

where the $\pm$ correspond to to the sign of ρ. In particular, we see that the solution is time-periodic in the ψ direction, assuming $x - ut$ thus ρ is kept fixed, with period T_ψ. Note that the periodicity of the solution is even more explicit in light-cone coordinates $x_+ = (1-a)T + a\phi_1$, $x_- = \phi_1 - T$, where, with $a = 1$, we fix $x_+ = t$ and x^- only depends on ρ. This periodicity has an associated action variable I

$$2\pi I = S - \mathrm{T}_\psi \frac{\partial S}{\partial \mathrm{T}_\psi}\bigg|_{\mathrm{p}}, \tag{2.308}$$

which should take integer values upon semiclassical quantization. Here $S = S(\mathrm{T}_\psi, \mathrm{p})$ is the action evaluated on the ranges $t \in [0, \mathrm{T}_\psi]$ and $x \in (-\infty, \infty)$. Substituting (2.298) back into (2.175) we get

$$\begin{aligned} S = &-\mathrm{h}\tilde{q}^2 \cos^2\varrho \left(1 + b^2\right) \int_0^{\mathrm{T}_\psi} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \ \sinh^2\rho \\ &+ 2\pi \mathrm{h} \left(\frac{1-\tilde{c}}{2} q\Delta T - \frac{\mathrm{T}_\psi}{2\pi}\frac{1+\tilde{c}}{2} q\Delta\psi\right). \end{aligned} \tag{2.309}$$

Just like the conserved charges (2.300), this expression exhibits both IR and UV divergences. The IR divergence is easily removed by setting $\tilde{c} = -1$ (agreeing with our choice above), while the integral needs to be UV regularized. Changing coordinates to $z = \cosh\rho$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}x \ \sinh^2\rho &= 2\int_1^{\infty} \mathrm{d}z \left(\frac{\partial z}{\partial x}\right)^{-1} \left(z^2 - 1\right) \\ &= \frac{2}{\tilde{q}\gamma \cos\varrho \sqrt{1+b^2}} \int_1^{\infty} \mathrm{d}z \frac{z}{\sqrt{z^2 - z_0^2}}, \end{aligned} \tag{2.310}$$

with $z_0 = b/\sqrt{1+b^2}$. Introducing a simple cutoff (i.e. not letting the string reach the boundary of AdS space) this integral evaluates to

$$\frac{2}{\tilde{q}\gamma \cos\varrho \sqrt{1+b^2}} \left(\Lambda - \sqrt{1 - z_0^2}\right), \tag{2.311}$$

which we regularize by subtracting the infinite term, leading to

$$S = 2\pi \mathrm{h} \left(\frac{2\tilde{q}^2}{\tan\varrho - qb} + q\Delta T\right). \tag{2.312}$$

After eliminating ϱ and recalling that $b = \cot\frac{\Delta T}{2} = -\cot\frac{\mathrm{p}}{2}$, we obtain

$$\frac{S}{2\pi\mathrm{h}} = \frac{2\tilde{q}^2\sqrt{1-w^2}}{\left(\tilde{q}^2+b^2-\left[\sqrt{\tilde{q}^2(1+b^2)(1-w^2)}-qwb\right]^2\right)^{1/2}-qb\sqrt{1-w^2}} - q\mathrm{p}\,. \tag{2.313}$$

Substituting back into (2.308) we get

$$I = J_0, \tag{2.314}$$

i.e. upon semiclassical quantization J_0 takes integer values, and (2.304) can be interpreted as an excitation of unit mass $m = 1$.

Shape of the solution

The target-space shape of the $\mathrm{AdS}_3 \times \mathrm{S}^1$ soliton is not immediately obvious from (2.298), since it is not in static gauge. Eliminating t in favour of T, and using (2.305), we have

$$\begin{aligned}\phi_1(T,\rho) &= T \mp \arctan\left[\,b^{-1}\sqrt{1+(1+b^2)\sinh^2\rho}\,\right]\,, \\ \psi(T,\rho) &= wT \mp w\arctan\left[\,b^{-1}\sqrt{1+(1+b^2)\sinh^2\rho}\,\right] \\ &\quad + r\,\mathrm{arcsinh}\left[\,\left(\sqrt{1+\mathrm{b}^2}\sinh\rho\right)^{-1}\right],\end{aligned} \tag{2.315}$$

where the $\mp$ signs correspond to positive/negative values of ρ. As we traverse the string, $U \in (-\infty, \infty)$, we have ρ going from 0 to $-\infty$, then from ∞ back to 0. For fixed T, ψ winds around $\rho = 0$ infinitely many times at both ends of the string, while at $\rho \to \pm\infty$ there are two fixed asymptotic angles. The BMN angle ϕ_1 takes a topological kink form, but with a jump at the middle ($\rho = \infty$). The opening angle is related to (2.301) by

$$\Delta\phi_1 = -\Delta T, \tag{2.316}$$

giving more physical meaning to the worldsheet momentum $\mathrm{p} = -\Delta T = \Delta\phi_1$. A typical configuration is shown in Figure 2.2. Letting T evolve, this string rotates[18] in both ϕ_1 and ψ, such that the endpoints move along the BMN

[18] Unless $w = 0$, in which case ψ is stationary, see the solution (2.318).

geodesic at the speed of light.

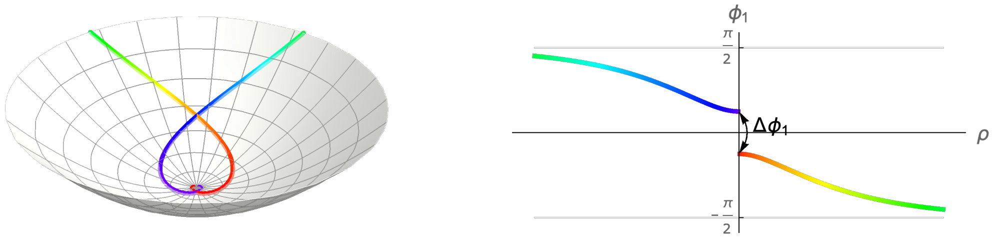

Figure 2.2: Spacetime image of a typical $AdS_3 \times S^1$ soliton configuration for fixed T. The two ends of the string are in purple and red, while green represents the middle. On the left we see part of the AdS_3 space with embedding $X+iY = \sinh\rho\, e^{i\psi}$, $Z = \cosh\rho$. The picture on the right shows ϕ_1 as a function of the target-space coordinate ρ.

Stationary soliton

An important special case of (2.298) is the stationary $AdS_3 \times S^1$ soliton, which has a time-independent shape. Setting $w = 0$ in (2.305) fixes the value of ϱ

$$\sin\varrho = \frac{\gamma u q}{\tilde{q}}. \tag{2.317}$$

This is the exact same condition we had for the $\mathbb{R} \times S^3$ stationary magnon (2.275). We can write the stationary $AdS_3 \times S^1$ soliton in Hopf coordinates as

$$\rho = \operatorname{arcsinh}\left(\frac{\operatorname{csch}\mathcal{Y}}{\sqrt{1+b^2}}\right), \quad T = t + \arctan\left(\frac{\coth\mathcal{Y}}{b}\right), \quad \psi = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2 - u^2}}, \tag{2.318}$$

where

$$\mathcal{Y} = \gamma\sqrt{\tilde{q}^2 - u^2}\,\mathcal{X}, \quad b = \frac{u}{\sqrt{\tilde{q}^2 - u^2}}. \tag{2.319}$$

The AdS_3 spin satisfies $J_0 + \mathrm{h}q\Delta T = 0$, and the dispersion relation takes the simpler form

$$E - J_1 = -2\mathrm{h}\tilde{q}\cos\tfrac{\mathrm{p}}{2}. \tag{2.320}$$

The parameter range is restricted to

$$\sin^2 \varrho \leq 1 \quad \Rightarrow \quad |u| \leq \tilde{q}, \tag{2.321}$$

but in fact we have a stationary soliton for all values of p, since

$$u = -\tilde{q} \cos \tfrac{\mathrm{p}}{2}. \tag{2.322}$$

$\mathbf{AdS_3 \times S^1 \times S^1}$ stationary soliton. We can put the $\mathrm{AdS}_3 \times \mathrm{S}^1$ soliton (2.298) on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, by applying (2.220), with the BMN string on both S^3_+ and S^3_-. The physically relevant case is of course the maximally SUSY solution with $A = \cos^2\varphi$, $B = \sin^2\varphi$, given by the Hopf coordinate parametrization

$$\begin{aligned}
&\rho = \operatorname{arcsinh}\left(\frac{\operatorname{csch}\mathcal{Y}}{\sqrt{1+b^2}}\right), \\
&T = t + \arctan\left(b^{-1}\coth\mathcal{Y}\right), \qquad \psi = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2 - u^2}}, \\
&\phi_1^+ = \cos^2\varphi\; t, \quad \phi_1^- = \sin^2\varphi\; t, \\
&\theta^\pm = \frac{\pi}{2}, \quad \phi_2^\pm = 0, \qquad b = \frac{u}{\sqrt{\tilde{q}^2 - u^2}},
\end{aligned} \tag{2.323}$$

with scaled and boosted worldsheet coordinate

$$\mathcal{Y} = \gamma\sqrt{\tilde{q}^2 - u^2}\,\mathcal{X}. \tag{2.324}$$

In chapter 6 we find the fermion zero modes of this mixed-flux stationary $\mathrm{AdS}_3 \times \mathrm{S}^1 \times \mathrm{S}^1$ soliton, and see that they are consistent with the string state being part of a short representation. The semiclassical quantization of J_0 is unchanged compared to (2.314), and the $\mathrm{AdS}_3 \times \mathrm{S}^1 \times \mathrm{S}^1$ soliton has mass $m = 1$.

Chapter 3

Fermion zero modes for the mixed-flux AdS_3 giant magnon

The residual (off-shell) symmetry algebra of the BMN ground state of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ superstring theory is the centrally extended $\mathfrak{psu}(1|1)^4$ superalgebra [102, 114, 113, 115], while on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ the elementary excitations transform under the centrally extended $\mathfrak{su}(1|1)^2$ algebra [98, 116]. The giant magnon is a BPS state, i.e. part of the 4 and 2 dimensional short multiplets[1] of $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ and $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, respectively. To reproduce these representations, the mixed-flux giant magnon on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ should have 4 and 2 fermion zero modes. These numbers come from the broken supersymmetries of the BMN vacuum, and can also be argued by matching the quantized zero modes to the odd generators of the residual algebra, see appendix A. This chapter presents our original work [1], where we explicitly construct these fermion zero modes, based on the AdS_5 calculation of Minahan [53].

As the starting point of the fermion zero mode analysis we take the mixed-flux AdS_3 *stationary* magnon, which can be considered the mixed-flux generalization of the HM giant magnon. A detailed classical analysis of this string solution can be found in section 2.2.4, here we just repeat the Hopf-coordinate form (2.285)

$$
\begin{aligned}
&\theta^+ = \arccos\left(\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right) , \\
&\phi_1^+ = \cos^2\!\varphi\; t + \arctan\left(b^{-1}\tanh\mathcal{Y}\right) , \qquad \phi_2^+ = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2-u^2}} , \\
&\theta^- = \frac{\pi}{2} , \qquad \phi_1^- = \sin^2\!\varphi\; t , \qquad \phi_2^- = 0 . \\
&\gamma^2 = \frac{1}{1-u^2} , \qquad b = \frac{u}{\sqrt{\tilde{q}^2-u^2}} , \qquad u \in (-\tilde{q},\tilde{q}) ,
\end{aligned}
\tag{3.1}
$$

[1] A detailed description of the $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ and $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ Lie superalgebras and their short representations can be found in section 2.2.1.

where, in terms of the boosted worldsheet coordinates

$$\mathcal{X} = \gamma(x - ut), \qquad \mathcal{T} = \gamma(t - ux), \tag{3.2}$$

we have

$$\mathcal{Y} = \cos^2\varphi \; \gamma\sqrt{\tilde{q}^2 - u^2}\,\mathcal{X}, \tag{3.3}$$

and the speed u of the magnon is related to the worldsheet momentum p by

$$u = \tilde{q}\cos\frac{\mathrm{p}}{2}. \tag{3.4}$$

The rest of this chapter is structured as follows.

In section 3.1 we discuss the quadratic fermionic action, which is obtained from the GS action by considering perturbations around the giant magnon as background. We will look at the zero mode condition and kappa-gauge fixing, before arriving at the zero mode equations of motion. These equations are then solved in section 3.2, to get the expected number of normalizable zero modes. After semiclassical quantization, we construct the fermionic generators of the corresponding superalgebras.

In section 3.3 we consider the special case of $q = 1$. In agreement with the chiral nature of the background, we find that all of the zero modes are non-normalizable. Since the notion of stationary magnon breaks down, we cannot simply take the $q \to 1$ limit of the zero modes found for $q < 1$, and the issue of semiclassical quantization also needs further attention. We conclude in section 3.4 and present some of the more technical details in appendices.

3.1 Fermion zero mode equations

In this section we look at the equations of motion describing fermion perturbations around the stationary giant magnon (3.1). Note that this treatment includes both the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ (for $\varphi = 0$) cases. We explain what is meant by zero modes, and describe in some detail the fixing of fermionic kappa-gauge. Finally, we write down the zero mode equations for kappa-fixed spinors, that will be solved in the next section.

3.1.1 Fermionic equations of motion

The quadratic fermionic action in conformal gauge is given by [160]

$$S_{\mathrm{F}} = \mathrm{h} \int \mathrm{d}^2\sigma \; \mathcal{L}_{\mathrm{F}} \;, \qquad \mathcal{L}_{\mathrm{F}} = -i \left(\eta^{ab} \delta^{IJ} + \epsilon^{ab} \sigma_3^{IJ} \right) \bar{\vartheta}^I \rho_a \mathcal{D}_b \vartheta^J \;, \tag{3.5}$$

where $I, J = 1, 2$, the ϑ^I are ten-dimensional Majorana-Weyl spinors, and ρ_a are projections of the ten-dimensional Dirac matrices

$$\rho_a \equiv e_a^A \, \Gamma_A \;, \qquad e_a^A \equiv \partial_a X^\mu E_\mu^A(X) \;. \tag{3.6}$$

X^μ are the coordinates of the target spacetime $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. The giant magnon solution has non-constant components for $\mu = t, \theta^+, \phi_1^+, \phi_2^+, \phi_1^-$ corresponding to the tangent space components $A = 0, 3, 4, 5, 7$ respectively. The covariant derivative is given by

$$\mathcal{D}_a \vartheta^I = \left(\mathrm{D}_a \delta^{IJ} + \frac{1}{48} \not{F} \rho_a \sigma_1^{IJ} + \frac{1}{8} \not{H}_a \sigma_3^{IJ} \right) \vartheta^J \;, \tag{3.7}$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4} \omega_a^{AB} \Gamma_{AB}$ with the pullback of the spin connection $\omega_a^{AB} \equiv \partial_a X^\mu \omega_\mu^{AB}$. For a detailed review of the vielbein and spin connection the reader is referred to section 2.2.2, while explicit expressions for the pullbacks e_a^A, ω_a^{AB} can be found in appendix B. The tangent space components of the fluxes (2.157) are given by

$$F_{012} = 2\tilde{q} \;, \qquad F_{345} = 2\tilde{q} \cos\varphi \;, \qquad F_{678} = 2\tilde{q} \sin\varphi \;, \tag{3.8}$$

$$H_{012} = 2q \;, \qquad H_{345} = 2q \cos\varphi \;, \qquad H_{678} = 2q \sin\varphi \;. \tag{3.9}$$

Introducing

$$\Gamma_* \equiv \Gamma^{012} \;, \qquad (\Gamma_*)^2 = \mathbb{1} \;, \tag{3.10}$$

$$\Gamma_+ \equiv \Gamma^{345} \;, \qquad (\Gamma_+)^2 = -\mathbb{1} \;, \tag{3.11}$$

$$\Gamma_- \equiv \Gamma^{678} \;, \qquad (\Gamma_-)^2 = -\mathbb{1} \;, \tag{3.12}$$

the contractions of the fluxes with the Dirac matrices are

$$\begin{aligned} \not{F} &= 12\tilde{q} \, \left(\Gamma_* + \cos\varphi \, \Gamma_+ + \sin\varphi \, \Gamma_- \right) \;, \\ \not{H} &= 12q \, \left(\Gamma_* + \cos\varphi \, \Gamma_+ + \sin\varphi \, \Gamma_- \right) \;. \end{aligned} \tag{3.13}$$

The equations of motion derived from (3.5) are

$$\begin{aligned}(\rho_0+\rho_1)(\mathcal{D}_0-\mathcal{D}_1)\ \vartheta^1 &= 0\ ,\\ (\rho_0-\rho_1)(\mathcal{D}_0+\mathcal{D}_1)\ \vartheta^2 &= 0\ .\end{aligned} \tag{3.14}$$

After expanding the covariant derivatives $\mathcal{D}_a$ we get

$$\begin{aligned}(\rho_0+\rho_1)\left[(\mathrm{D}_1-\mathrm{D}_0)\ \vartheta^1-\frac{1}{48}\not{F}(\rho_0-\rho_1)\ \vartheta^2-\frac{1}{8}(\not{H}_0-\not{H}_1)\ \vartheta^1\right] &= 0\ ,\\ (\rho_0-\rho_1)\left[(\mathrm{D}_1+\mathrm{D}_0)\ \vartheta^2+\frac{1}{48}\not{F}(\rho_0+\rho_1)\ \vartheta^1-\frac{1}{8}(\not{H}_0+\not{H}_1)\ \vartheta^2\right] &= 0\ .\end{aligned} \tag{3.15}$$

At this point it is natural to change variables to the scaled and boosted worldsheet coordinates (3.3)

$$\mathcal{Y}=\cos^2\varphi\,\zeta\mathcal{X},\quad \mathcal{S}=\cos^2\varphi\,\zeta\mathcal{T},\qquad \zeta=\gamma\sqrt{\tilde{q}^2-u^2}, \tag{3.16}$$

satisfying

$$\partial_1\mp\partial_0=\cos^2\varphi\,\zeta(1\pm u)\gamma(\partial_{\mathcal{Y}}\mp\partial_{\mathcal{S}}). \tag{3.17}$$

With this, the equations become

$$\begin{aligned}(\rho_0+\rho_1)\left[\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\ \vartheta^1+\mathcal{O}\vartheta^2\right] &= 0\ ,\\ (\rho_0-\rho_1)\left[\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\ \vartheta^2+\tilde{\mathcal{O}}\vartheta^1\right] &= 0\ ,\end{aligned} \tag{3.18}$$

where

$$\mathcal{O}=-\frac{1}{48\cos^2\varphi}\not{F}(\rho_0-\rho_1)\ ,\quad \tilde{\mathcal{O}}=\ \frac{1}{48\cos^2\varphi}\not{F}(\rho_0+\rho_1)\ , \tag{3.19}$$

and the fermion derivatives are

$$\begin{aligned}D &= \partial_{\mathcal{Y}}+\frac{1}{2}G\ \Gamma_{34}+\frac{1}{2}Q\ \Gamma_{35}-\frac{(1-u)\gamma}{48\cos^2\varphi\,\zeta}\left(\not{H}(\rho_0-\rho_1)+(\rho_0-\rho_1)\not{H}\right)\ ,\\ \tilde{D} &= \partial_{\mathcal{Y}}+\frac{1}{2}\tilde{G}\ \Gamma_{34}+\frac{1}{2}Q\ \Gamma_{35}-\frac{(1+u)\gamma}{48\cos^2\varphi\,\zeta}\left(\not{H}(\rho_0+\rho_1)+(\rho_0+\rho_1)\not{H}\right)\ .\end{aligned} \tag{3.20}$$

A detailed derivation can be found in appendix D, together with explicit expressions for the scalar functions $G, \tilde{G}, Q$ in (D.5).

The operators in front of the equations (3.18) are nilpotent

$$(\rho_0 + \rho_1)^2 = (\rho_0 - \rho_1)^2 = 0 \ . \tag{3.21}$$

If we further define

$$\bar{\rho}_0 \equiv e_0^0 \, \Gamma_0 - e_0^3 \, \Gamma_3 - e_0^4 \, \Gamma_4 - e_0^5 \, \Gamma_5 + e_0^7 \, \Gamma_7 \ , \tag{3.22}$$

which turns out to be $\bar{\rho}_0 = -\rho_0^\dagger$ for the gamma matrices described in appendix C, we get another set of nilpotent operators $(\bar{\rho}_0 + \rho_1)^2 = (\bar{\rho}_0 - \rho_1)^2 = 0$. However, the two sets differ by the nonsingular operator $\bar{\rho}_0 - \rho_0$, which squares to

$$(\bar{\rho}_0 - \rho_0)^2 = 4 \cos^2\!\varphi \, \tilde{q}^{-2} \left(\zeta^2 \tanh^2\!\mathcal{Y} + q^2 u^2 \gamma^2 \right) \ \mathbb{1} \ . \tag{3.23}$$

The kernel of a $2m$-dimensional nilpotent operator is of at least m dimensions since all its eigenvalues are zero. If the sum of two nilpotent operators is full-rank, as above, the kernels must be disjoint, therefore the sum of their nullities is at most the full $2m$. From this we see that the $(\rho_0 \pm \rho_1)$ are half-rank, an important observation for subsection 3.1.3.

3.1.2 Zero mode condition

Note that the fermion Lagrangian (3.5) has a dependence on the worldsheet coordinates only through the vielbein and spin connection. These quantities, on the other hand, depend only on $\mathcal{Y}$, i.e. the Lagrangian is independent of the temporal coordinate $\mathcal{S}$

$$\mathcal{L}_{\mathrm{F}} = \mathcal{L}_{\mathrm{F}} \left(\mathcal{Y}, \vartheta^J, \partial_{\tilde{a}} \, \vartheta^J \right) \ , \tag{3.24}$$

where $\tilde{a} = 0, 1$ correspond to the variables $\mathcal{S}$ and $\mathcal{Y}$, respectively.

Translations in $\mathcal{S}$ can be equivalently described as a transformations of the fields

$$\delta\vartheta^J = \varepsilon \ \partial_{\mathcal{S}} \vartheta^J \ , \qquad \delta(\partial_{\tilde{a}} \, \vartheta^J) = \varepsilon \ \partial_{\mathcal{S}} (\partial_{\tilde{a}} \, \vartheta^J) \ , \tag{3.25}$$

and accordingly

$$\delta\mathcal{L}_{\mathrm{F}} = \varepsilon \left(\frac{\partial \mathcal{L}_{\mathrm{F}}}{\partial \vartheta^J} \partial_{\mathcal{S}} \vartheta^J + \frac{\partial \mathcal{L}_{\mathrm{F}}}{\partial \left(\partial_{\tilde{a}}\, \vartheta^J \right)} \partial_{\mathcal{S}} (\partial_{\tilde{a}}\, \vartheta^J) \right) \tag{3.26}$$

$$= \varepsilon\ \partial_{\mathcal{S}} \mathcal{L}_{\mathrm{F}} = \varepsilon\ \partial_{\tilde{a}} \left(\delta^{\tilde{a}}_{\tilde{0}} \mathcal{L}_{\mathrm{F}} \right)\ . \tag{3.27}$$

The change in the Lagrangian is a total derivative, and applying Noether's theorem we get a conserved current

$$j^{\tilde{a}} = \frac{\partial \mathcal{L}_{\mathrm{F}}}{\partial \left(\partial_{\tilde{a}}\, \vartheta^J \right)}\ \partial_{\mathcal{S}}\, \vartheta^J - \delta^{\tilde{a}}_{\tilde{0}} \mathcal{L}_{\mathrm{F}}\ , \tag{3.28}$$

where summation over $J = 1, 2$ is understood. However, for the fermionic action we have $\mathcal{L}_{\mathrm{F}} = 0$ on-shell, and the current simply reduces to

$$j^{\tilde{a}} = \frac{\partial \mathcal{L}_{\mathrm{F}}}{\partial \left(\partial_{\tilde{a}}\, \vartheta^J \right)}\ \partial_{\mathcal{S}}\, \vartheta^J\ , \tag{3.29}$$

The explicit form of this current is unimportant for the present argument.

Since $\mathcal{S}$ is a time-like worldsheet coordinate, we might interpret the corresponding conserved quantity as the energy of the fermionic perturbation above the giant magnon background

$$E_{\mathrm{F}} = \int \mathrm{d}\mathcal{X}\ j^{\tilde{0}} = \int \mathrm{d}\mathcal{X}\ \frac{\partial \mathcal{L}_{\mathrm{F}}}{\partial \left(\partial_{\mathcal{S}}\, \vartheta^J \right)}\ \partial_{\mathcal{S}}\, \vartheta^J\ . \tag{3.30}$$

Zero modes, by definition, are zero energy fluctuations above the giant magnon, i.e. $E_{\mathrm{F}} = 0$. Henceforth, we will take the zero mode condition to be

$$\partial_{\mathcal{S}}\, \vartheta^J = 0\ , \tag{3.31}$$

and with this, the equations for the fermion zero modes are

$$\begin{aligned} (\rho_0 + \rho_1) \left[\zeta(1+u)\gamma D\ \vartheta^1 + \mathcal{O}\vartheta^2 \right] &= 0\ , \\ (\rho_0 - \rho_1) \left[\zeta(1-u)\gamma \tilde{D}\ \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1 \right] &= 0\ . \end{aligned} \tag{3.32}$$

3.1.3 Fixing kappa symmetry

The Green-Schwarz superstring has a local fermionic symmetry, the so-called kappa-symmetry, that ensures spacetime supersymmetry of the physical spectrum. Let us take another look at the quadratic fermionic Lagrangian (3.5)

$$\mathcal{L}_{\rm F} = -i\left(\eta^{ab}\delta^{IJ} + \epsilon^{ab}\sigma_3^{IJ}\right)\bar{\vartheta}^I\rho_a\mathcal{D}_b\vartheta^J\ , \tag{3.33}$$

$$= i\,\bar{\vartheta}^1(\rho_0+\rho_1)(\mathcal{D}_0-\mathcal{D}_1)\vartheta^1 + i\,\bar{\vartheta}^2(\rho_0-\rho_1)(\mathcal{D}_0+\mathcal{D}_1)\vartheta^2\ , \tag{3.34}$$

$$\begin{aligned} &= -i\cos^2\!\varphi\,\bar{\vartheta}^1(\rho_0+\rho_1)\Big(\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\vartheta^1 + \mathcal{O}\vartheta^2\Big) \\ &\quad + i\cos^2\!\varphi\,\bar{\vartheta}^2(\rho_0-\rho_1)\Big(\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\Big)\ , \end{aligned} \tag{3.35}$$

where $D, \tilde{D}, \mathcal{O}$ and $\tilde{\mathcal{O}}$ are defined in (3.19) –(3.20). We see the nilpotent operators $(\rho_0 \pm \rho_1)$ acting on the conjugate spinors: components of ϑ^1 and ϑ^2 that are projected out by $(\rho_0+\rho_1)$ and $(\rho_0-\rho_1)$, respectively, do not contribute to the action, we can consider them non-dynamical.

To fully fix kappa-gauge, however, not only do we need to project out non-dynamical degrees of freedom, but also specify what happens to the rest, i.e. we need actual projectors:

$$K_1 = \frac{1}{2}\Pi(\rho_0+\rho_1)\ , \qquad K_2 = \frac{1}{2}\Pi(\rho_0-\rho_1)\ , \tag{3.36}$$

for some invertible Π, that has to satisfy a number of conditions. A straightforward, albeit somewhat cumbersome,[2] calculation gives $[\rho_0+\rho_1, D] = [\rho_0-\rho_1, \tilde{D}] = 0$, so we have

$$[K_1, D] = 0\ , \qquad [K_2, \tilde{D}] = 0\ , \tag{3.37}$$

provided $[\Pi, D] = [\Pi, \tilde{D}] = 0$. Another condition of course, is that the K_J have to be genuine projectors — i.e. $K_J^2 = K_J$ — , which, with (3.19), would

[2] One can easily convince themselves that it is sufficient to check the Γ_3, Γ_4 and Γ_5 components of the operator equations, simplifying matters a great deal.

imply that

$$\mathcal{O} = \mathcal{O} K_2 \ , \qquad \tilde{\mathcal{O}} = \tilde{\mathcal{O}} K_1 \ . \tag{3.38}$$

The most obvious choice would be $\Pi = \Gamma^0$, but taking this route one encounters technical difficulties when considering the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ geometry, arising from the appearance of Γ_7 in $(\rho_0 \pm \rho_1)$. Noting that in both of these operators Γ_7 only appears in the combination $\Gamma_0 + \sin\varphi\,\Gamma_7$, it is tempting to "rotate" our gamma matrices in the 0-7 directions

$$\hat{\Gamma}^0 = \sec\varphi\left(\Gamma^0 - \sin\varphi\ \Gamma^7\right) \ , \qquad \hat{\Gamma}^7 = \sec\varphi\left(\Gamma^7 - \sin\varphi\ \Gamma^0\right) \ , \tag{3.39}$$

leaving unchanged all the others $\hat{\Gamma}^A = \Gamma^A$, $A \neq 0, 7$. One can easily check that these satisfy the Clifford algebra. We lower the index on $\hat{\Gamma}^A$ with the Minkowski metric, in particular $\hat{\Gamma}_0 = -\hat{\Gamma}^0 = \sec\varphi(\Gamma_0 + \sin\varphi\ \Gamma_7)$ soaks up all the Γ_7 dependence in $(\rho_0 \pm \rho_1)$

$$\rho_0 \pm \rho_1 = \cos\varphi\left(\hat{\Gamma}_0 + \hat{e}^3_\pm\ \hat{\Gamma}_3 + \hat{e}^4_\pm\ \hat{\Gamma}_4 + \hat{e}^5_\pm\ \hat{\Gamma}_5\right) \tag{3.40}$$

where $\hat{e}^A_\pm = \sec\varphi\,(e^A_0 \pm e^A_1)$. All of this is good motivation for the choice of $\Pi = \sec\varphi\,\hat{\Gamma}^0$, which can be easily shown to satisfy our conditions. Henceforth, we will take

$$K_1 = \frac{1}{2}\sec\varphi\ \hat{\Gamma}^0(\rho_0 + \rho_1) \ , \qquad K_2 = \frac{1}{2}\sec\varphi\ \hat{\Gamma}^0(\rho_0 - \rho_1) \ . \tag{3.41}$$

The advantages of this choice will become obvious in the next subsection.

If we take a basis of gamma matrices such that $\hat{\Gamma}^A$ have definite hermiticity, e.g. the one described in appendix C, the projectors are Hermitian $K_J^\dagger = K_J$. Furthermore, in such a basis the Hermitian conjugate intertwiner (see app. C) is given by $\hat{\Gamma}^0$, hence the Dirac conjugate is $\bar{\vartheta} = \vartheta^\dagger\ \hat{\Gamma}^0$. With this, and the properties listed above, we can write the Lagrangian as

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} = \ & -2i\cos^3\!\varphi\,(\Psi^1)^\dagger\Big(\zeta(1+u)\gamma(D - \partial_{\mathcal{S}})\ \Psi^1 + \mathcal{O}\Psi^2\Big) \\ & + 2i\cos^3\!\varphi\,(\Psi^2)^\dagger\Big(\zeta(1-u)\gamma(\tilde{D} + \partial_{\mathcal{S}})\ \Psi^2 + \tilde{\mathcal{O}}\Psi^1\Big) \ , \end{aligned} \tag{3.42}$$

where we introduced the notation $\Psi^J = K_J \vartheta^J$ for the projected spinors, and we indeed see that only these components are dynamical.

Using the kappa-projectors, the zero mode equations (3.32) can be written as

$$\begin{aligned} &K_1 \left[\zeta(1+u)\gamma D \; \vartheta^1 + \mathcal{O}\vartheta^2 \right] = 0 \; , \\ &K_2 \left[\zeta(1-u)\gamma \tilde{D} \; \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1 \right] = 0 \; . \end{aligned} \tag{3.43}$$

For the kappa-fixed spinors $\Psi^J = K_J \vartheta^J$, using (3.37) –(3.38), these equations become

$$\begin{aligned} &\zeta(1+u)\gamma D \; \Psi^1 + K_1 \mathcal{O}\Psi^2 = 0 \; , \\ &\zeta(1-u)\gamma \tilde{D} \; \Psi^2 + K_2 \tilde{\mathcal{O}}\Psi^1 = 0 \; . \end{aligned} \tag{3.44}$$

3.1.4 Zero mode equations

With the choice of kappa projectors (3.41) we get a commuting 6d chirality projector for free[3]

$$P_{\pm} = \frac{1}{2}\left(\mathbb{1} \pm \hat{\Gamma}_* \hat{\Gamma}_+\right) \; , \qquad [P_{\pm}, K_J] = 0 \; . \tag{3.45}$$

Using this we can rewrite the contraction of the background fluxes $\not{F}$, $\not{H}$ (3.13)

$$\begin{aligned} \Gamma_* + \cos\varphi\; \Gamma_+ + \sin\varphi\; \Gamma_- &= \cos\varphi\Big(\left(\sec\varphi\; \Gamma^0 + \tan\varphi\; \Gamma^{1268}\; \Gamma^7 \right) \Gamma^{12} + \Gamma_+ \Big) \\ &= \cos\varphi \left(\hat{\Gamma}^0 \Gamma^{12} + \Gamma_+ \right) - 2\cos\varphi\; \Delta\; \Gamma^{12} \\ &= 2\cos\varphi \left(\hat{\Gamma}_*\; P_+ - \Delta\; \hat{\Gamma}^{12} \right) , \end{aligned} \tag{3.46}$$

where

$$\Delta = -\frac{1}{2}\tan\varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right) \Gamma^7 \equiv \; \Delta_0\, \hat{\Gamma}^0 + \Delta_7\, \hat{\Gamma}^7 \; , \tag{3.47}$$

with

$$\Delta_0 = -\frac{1}{2}\tan^2\!\varphi \left(\hat{\Gamma}^{1268} + \mathbb{1} \right) \; , \qquad \Delta_7 = \csc\varphi\, \Delta_0 \; . \tag{3.48}$$

[3] In any spinor operator M, replace Γ^A by $\hat{\Gamma}^A$ to get $\hat{M}$.

Even though Δ_0 and Δ_7 are matrices, we can essentially treat them as scalars, since they commute with the equations of motion.

Recalling $\bar{\rho}_0$ from (3.22), which also satisfies $\rho_0 \hat{\Gamma}^0 = \hat{\Gamma}^0 \bar{\rho}_0$, we can define an invertible operator from (3.23)

$$R = \frac{1}{2\tilde{A}} \hat{\Gamma}_*(\bar{\rho}_0 - \rho_0) \qquad : \qquad R^2 = -\tilde{q}^{-2} \left(\zeta^2 \tanh^2\mathcal{Y} + q^2u^2\gamma^2\right) \mathbb{1} \ . \quad (3.49)$$

With all of this, the fermion derivatives (3.20) can be rewritten as (see appendix D)

$$D = \partial_{\mathcal{Y}} + \frac{1}{2}G\,\hat{\Gamma}_{34} + \frac{1}{2}Q\,\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\left(RP_- - (R + \hat{\Gamma}_{12})\,P_+ + \Delta_0\,\hat{\Gamma}_{12}\right) \ ,$$

$$\tilde{D} = \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\,\hat{\Gamma}_{34} + \frac{1}{2}Q\,\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\left(RP_- - (R + \hat{\Gamma}_{12})\,P_+ + \Delta_0\,\hat{\Gamma}_{12}\right) \ , \quad (3.50)$$

however, these expressions are only valid when acting on kappa-fixed spinors, i.e. in the form DK_1 and $\tilde{D}K_2$. As for the terms (3.19) mixing the two spinors in the equations of motion, we have

$$\begin{aligned} \mathcal{O} &= -\tilde{q}\left(\hat{\Gamma}^{12}P_- + \Delta\,\hat{\Gamma}_*\right)K_2 \ , \\ \tilde{\mathcal{O}} &= \tilde{q}\left(\hat{\Gamma}^{12}P_- + \Delta\,\hat{\Gamma}_*\right)K_1 \ . \end{aligned} \quad (3.51)$$

Using the nilpotency relations $(\rho_0 \pm \rho_1)^2 = 0$, it is easy to see that

$$\Gamma^{12}K_1K_2 = -RK_2 \ , \qquad \Gamma^{12}K_2K_1 = -RK_1 \ , \quad (3.52)$$

and the equations of motion (3.44) become

$$\begin{aligned} &\zeta(1+u)\gamma D\ \Psi^1 + \tilde{q}\left(R\ P_- - K_1\Delta\,\hat{\Gamma}_*\right)\Psi^2 = 0 \ , \\ &\zeta(1-u)\gamma\tilde{D}\ \Psi^2 - \tilde{q}\left(R\ P_- - K_2\Delta\,\hat{\Gamma}_*\right)\Psi^1 = 0 \ . \end{aligned} \quad (3.53)$$

Equations for $\Delta = 0$

Equation (3.46) might seem arbitrary at first, so let us elaborate on the advantages of this rearrangement. Our goal was to have $(\hat{\Gamma}_* + \hat{\Gamma}_+)$ — instead of

$\not{F}$ — in the equations, since $P_\pm$ commutes with K_J. After this rewriting we are left with an extra term $K\Delta K$, which does not in general commute with $P_\pm$. However, in the following two cases we have $\Delta = 0$

- $\varphi = 0$: corresponding to the $\mathbf{AdS_3} \times \mathbf{S}^3 \times \mathbf{T}^4$ **geometry**.
- "$\hat{\Gamma}^{1268} = -1$" : i.e. the $\mathbf{AdS_3} \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ **geometry, restricted to the -1 eigenspace of** Γ^{1268}. Note that this is compatible with the equations, since $\hat{\Gamma}^{1268}$ commutes with all the terms.

Assuming $\Delta = 0$, the fermion derivatives take the simpler form

$$
\begin{aligned}
D &= \partial_{\mathcal{Y}} + \frac{1}{2} G\,\hat{\Gamma}_{34} + \frac{1}{2} Q\,\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\Big(RP_- - (R + \hat{\Gamma}_{12})\,P_+\Big)\ , \\
\tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G}\,\hat{\Gamma}_{34} + \frac{1}{2} Q\,\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\Big(RP_- - (R + \hat{\Gamma}_{12})\,P_+\Big)\ .
\end{aligned}
\tag{3.54}
$$

Also note that the equations of motion have no explicit dependence on φ, only an implicit one via the rescaled variable $\mathcal{Y}$ (3.16). In other words, the following equations apply in both geometries

$$
\begin{aligned}
&\zeta(1+u)\gamma D\ \Psi^1 + \tilde{q}\ R\ P_- \Psi^2 = 0\ , \\
&\zeta(1-u)\gamma \tilde{D}\ \Psi^2 - \tilde{q}\ R\ P_- \Psi^1 = 0\ .
\end{aligned}
\tag{3.55}
$$

as long as we impose the extra condition $\hat{\Gamma}^{1268}\vartheta^J = -\vartheta^J$ in the S^1 geometry.

The case of $\Delta \neq 0$

As we have seen above, we can treat the $AdS_3 \times S^3 \times S^3 \times S^1$ fermion zero modes in much the same way as those of the $AdS_3 \times S^3 \times T^4$ giant magnon, provided $\Delta = 0$. In section 3.2 this will allow us to find solutions for both geometries and general values of q in a single calculation. However, we need to make sure there are no zero modes that we are missing by restricting to $\Delta = 0$.

We can get an intuition for why this must be the case by looking at the fermion fluctuations around the BMN string on $AdS_3 \times S^3 \times S^3 \times S^1$. In appendix E we show that the mass of near-BMN fermions is determined by their Γ_{1235} and Γ_{1268} eigenvalues, with $m = \cos^2\varphi$ corresponding to $\Gamma_{1235} =$

$+1$, $\hat{\Gamma}_{1268} = \Gamma_{1268} = -1$. Taking the BMN limit of the fermion zero modes themselves, they must become superpartners of the magnon, with all the same mass $m = \cos^2\varphi$, hence definite chirality $\hat{\Gamma}_{1268} = -1$. This is equivalent to $\Delta = 0$, and we expect no zero modes for $\Delta \neq 0$ ($\hat{\Gamma}_{1268} = +1$). In appendix F we show that there are in fact no normalizable solutions to (3.53) for $\Delta \neq 0$.

3.2 Mixed-flux fermion zero modes

In this section we find exact solutions for the ($\Delta = 0$) zero mode equations (3.55). Our main aim is to write down the normalizable solutions, representing the perturbative zero modes over the giant magnon background. Using these normalizable zero modes, we then perform semiclassical quantization, and reproduce the the algebra that the fermion excitations must satisfy.

3.2.1 Fixing kappa-gauge

We start by noting that the kappa-projectors (3.41) can be written as

$$\begin{aligned} K_1 &= \frac{1}{2}\Big(\mathbb{1} - \sin(2\chi)\cos\upsilon_+\,\hat{\Gamma}_{03} - \cos(2\chi)\cos\upsilon_+\,\hat{\Gamma}_{04} + \sin\upsilon_+\,\hat{\Gamma}_{05}\Big)\ , \\ K_2 &= \frac{1}{2}\Big(\mathbb{1} + \sin(2\tilde{\chi})\cos\upsilon_-\,\hat{\Gamma}_{03} + \cos(2\tilde{\chi})\cos\upsilon_-\,\hat{\Gamma}_{04} - \sin\upsilon_-\,\hat{\Gamma}_{05}\Big)\ , \end{aligned} \tag{3.56}$$

where

$$\begin{aligned} \chi(\mathcal{Y}) &= \frac{1}{2}\left(\operatorname{arccot}\left(\frac{u\,\operatorname{csch}\mathcal{Y}}{\tilde{q}}\right) - \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1 - Q_+^2\,\operatorname{sech}^2\mathcal{Y}}}\right)\right)\ , \\ \tilde{\chi}(\mathcal{Y}) &= \frac{1}{2}\left(\operatorname{arccot}\left(\frac{u\,\operatorname{csch}\mathcal{Y}}{\tilde{q}}\right) + \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1 - Q_-^2\,\operatorname{sech}^2\mathcal{Y}}}\right)\right)\ , \end{aligned} \tag{3.57}$$

and we also introduced

$$Q_\pm = \frac{q\sqrt{\tilde{q}^2 - u^2}}{\tilde{q}(1 \pm u)}, \qquad \upsilon_\pm = \arcsin\left(Q_\pm\,\operatorname{sech}\mathcal{Y}\right)\ . \tag{3.58}$$

Ansatz. Since K_J, $\hat{\Gamma}_{12}$ and $\hat{\Gamma}_*\hat{\Gamma}_+$ all mutually commute, as our starting point we can take shared eigenvectors U

$$\hat{\Gamma}_{12}U^J = \lambda_{12}U^J \ , \qquad \hat{\Gamma}_*\hat{\Gamma}_+U^J = \lambda_P U^J \ , \tag{3.59}$$

where $\lambda_{12} = \pm i$, and $\lambda_P = \pm 1$ correspond to the $P_\pm$ projections. Accordingly, there are no restrictions on these eigenvalues for the kappa-fixed spinor. The operator $\hat{\Gamma}_{34}$ does not commute with K_J, hence a suitable combination of its opposite eigenvectors makes a good candidate for the general gauge-fixed spinor. This motivates the further restriction of $\hat{\Gamma}_{34}U^J = iU^J$ and the ansatz

$$\Psi^J = \left(\alpha_+^J(\mathcal{Y}) + \alpha_-^J(\mathcal{Y})\,\hat{\Gamma}_{45}\right) U^J \tag{3.60}$$

Solution. Substituting this into the equations $K_1\Psi^1 = \Psi^1$, and using the various eigenvector relations of U^1, we get

$$\begin{aligned} \lambda e^{2i\chi}\cos\upsilon_+\,\alpha_-^1 - \lambda\sin\upsilon_+\,\alpha_+^1 &= \alpha_+^1 \ , \\ \lambda e^{-2i\chi}\cos\upsilon_+\,\alpha_+^1 + \lambda\sin\upsilon_+\,\alpha_-^1 &= \alpha_-^1 \ , \end{aligned} \tag{3.61}$$

where $\lambda = i\lambda_{12}\lambda_P = \pm 1$. What we have here are two equations for the single variable α_-/α_+, corresponding to the fact that the norm of the eigenvector is not fixed. The equations are consistent, and a symmetric solution is given by

$$\alpha_+^1 = e^{i\chi}\sqrt{1-\lambda Q_+\,\mathrm{sech}\mathcal{Y}} \ , \qquad \alpha_-^1 = e^{-i\chi}\lambda\sqrt{1+\lambda Q_+\,\mathrm{sech}\mathcal{Y}} \ . \tag{3.62}$$

A similar calculation gives

$$\alpha_+^2 = e^{i\tilde{\chi}}\sqrt{1+\lambda Q_-\,\mathrm{sech}\mathcal{Y}} \ , \qquad \alpha_-^2 = -e^{-i\tilde{\chi}}\lambda\sqrt{1-\lambda Q_-\,\mathrm{sech}\mathcal{Y}} \ . \tag{3.63}$$

Written in a single expression, the most general gauge-fixed spinors are

$$\begin{aligned} \Psi^1 &= \sum_{\lambda=\pm}\left(e^{i\chi}\sqrt{1-\lambda Q_+\,\mathrm{sech}\mathcal{Y}} + e^{-i\chi}\lambda\sqrt{1+\lambda Q_+\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45}\right)U_\lambda^1 \ , \\ \Psi^2 &= \sum_{\lambda=\pm}\left(e^{i\tilde{\chi}}\sqrt{1+\lambda Q_-\,\mathrm{sech}\mathcal{Y}} - e^{-i\tilde{\chi}}\lambda\sqrt{1-\lambda Q_-\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45}\right)U_\lambda^2 \ , \end{aligned} \tag{3.64}$$

where $\hat{\Gamma}_{34}U^J_\pm = +iU^J_\pm$ and $i\hat{\Gamma}_{12}\hat{\Gamma}_*\hat{\Gamma}_+U^J_\pm = i\hat{\Gamma}_{0345}U^J_\pm = \pm U^J_\pm$. The above analysis shows that these are kappa eigenvectors, and by counting the degrees of freedom (components of U^J) we see that there are no others.

3.2.2 Zero mode solutions

The projectors $P_\pm$ commute with the equations of motion (3.55), therefore we can consider solutions of definite $P_\pm$ "chirality". In the following we obtain solutions on the two subspaces in turn, by letting $U^J_\pm$ depend on $\mathcal{Y}$, and substituting (3.64) into the equations. The identities listed in appendix G were useful in simplifying some of the more complicated expressions.

Solutions on the P_+ subspace

For this projection the spinors decouple

$$D\ \Psi^1 = 0\ , \qquad \tilde{D}\ \Psi^2 = 0\ , \tag{3.65}$$

and we get

$$\begin{aligned} \sum_{\lambda=\pm}\left(\alpha^1_+ + \alpha^1_-\,\hat{\Gamma}_{45}\right)(\partial_{\mathcal{Y}} + C_+)\,U^1_\lambda = 0\ , \\ \sum_{\lambda=\pm}\left(\alpha^2_+ + \alpha^2_-\,\hat{\Gamma}_{45}\right)(\partial_{\mathcal{Y}} + C_-)\,U^2_\lambda = 0\ , \end{aligned} \tag{3.66}$$

with c-numbers

$$C_\pm \ = \ \frac{i\lambda q}{2\sqrt{\tilde{q}^2 - u^2}} + \frac{i\lambda Q_\pm\sqrt{1-Q_\pm^2}}{2\left(\cosh^2\mathcal{Y} - Q_\pm^2\right)}\ , \tag{3.67}$$

and this simple form of the equations is a consequence (or proof in itself) of the fact that kappa-fixing commutes with the fermion derivative operators. The solution is

$$\begin{aligned} U^1_\lambda &= e^{-\frac{i\lambda q}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y} - \frac{i}{2}\lambda\arctan\left(\frac{Q_+\tanh\mathcal{Y}}{\sqrt{1-Q_+^2}}\right)}\ V^1_\lambda\ , \\ U^2_\lambda &= e^{-\frac{i\lambda q}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y} - \frac{i}{2}\lambda\arctan\left(\frac{Q_-\tanh\mathcal{Y}}{\sqrt{1-Q_-^2}}\right)}\ V^2_\lambda\ , \end{aligned} \tag{3.68}$$

where V^J are (independent) constant MW spinors with $\hat{\Gamma}_{34}V^J_\pm = +iV^J_\pm$, $\hat{\Gamma}_{12}V^J_\pm = \mp iV^J_\pm$ and $P_+V^J_\pm = V^J_\pm$. However, with these, the spinors (3.64) are not normalizable and we discard them as perturbative zero modes.

Solutions on the P_- subspace

The equations on this subspace become

$$\begin{aligned} &\zeta(1+u)\gamma D\ \Psi^1 + \tilde{q}R\ \Psi^2 = 0\ , \\ &\zeta(1-u)\gamma \tilde{D}\ \Psi^2 - \tilde{q}R\ \Psi^1 = 0\ , \end{aligned} \tag{3.69}$$

with fermion derivatives

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\ \hat{\Gamma}_{34} + \frac{1}{2}Q\ \hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\ R\ , \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\ \hat{\Gamma}_{34} + \frac{1}{2}Q\ \hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\ R\ . \end{aligned} \tag{3.70}$$

After substitution, and a considerable amount of simplification, we get

$$\begin{aligned} &\sum_{\lambda=\pm}\left(\alpha^1_+ + \alpha^1_-\ \hat{\Gamma}_{45}\right)\left[\left(\partial_{\mathcal{Y}} + C_{11}\right)U^1_\lambda + C_{12}U^2_\lambda\right] = 0\ , \\ &\sum_{\lambda=\pm}\left(\alpha^2_+ + \alpha^2_-\ \hat{\Gamma}_{45}\right)\left[\left(\partial_{\mathcal{Y}} + C_{21}\right)U^2_\lambda + C_{22}U^1_\lambda\right] = 0\ , \end{aligned} \tag{3.71}$$

with

$$\begin{aligned} C_{11} &= -\frac{i\lambda q(1-2u)}{2\sqrt{\tilde{q}^2-u^2}} + \frac{i\lambda Q_+\sqrt{1-Q_+^2}}{2\left(\cosh^2\mathcal{Y} - Q_+^2\right)}\ , \\ C_{21} &= -\frac{i\lambda q(1+2u)}{2\sqrt{\tilde{q}^2-u^2}} + \frac{i\lambda Q_-\sqrt{1-Q_-^2}}{2\left(\cosh^2\mathcal{Y} - Q_-^2\right)}\ , \\ C_{12} &= (1-u)\gamma\ e^{\int(C_{21}-C_{11})\mathrm{d}\mathcal{Y}}\ e^{+i2\lambda\xi\mathcal{Y}}\ \left(\lambda\tanh\mathcal{Y} - i\xi\right)\ , \\ C_{22} &= (1+u)\gamma\ e^{\int(C_{11}-C_{12})\mathrm{d}\mathcal{Y}}\ e^{-i2\lambda\xi\mathcal{Y}}\ \left(\lambda\tanh\mathcal{Y} + i\xi\right)\ , \end{aligned} \tag{3.72}$$

where we also defined

$$\xi = \frac{qu}{\sqrt{\tilde{q}^2-u^2}}\ . \tag{3.73}$$

The motivation for writing C_{12} and C_{22} in the above form becomes clear once we make the ansatz

$$U^1_\lambda = \frac{e^{-\int C_{11}\mathrm{d}\mathcal{Y}}}{\sqrt{1+u}}\ \tilde{U}^1_\lambda = \frac{1}{\sqrt{1+u}}e^{\frac{i\lambda q(1-2u)}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y} - \frac{i}{2}\lambda\arctan\left(\frac{Q_+\tanh\mathcal{Y}}{\sqrt{1-Q_+^2}}\right)}\ \tilde{U}^1_\lambda\ , \tag{3.74}$$

$$U^2_\lambda \;=\; \frac{e^{-\int C_{12}\mathrm{d}\mathcal{Y}}}{\sqrt{1-u}}\,\tilde{U}^2_\lambda \;=\; \frac{1}{\sqrt{1-u}}e^{\frac{i\lambda q(1+2u)}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}-\frac{i}{2}\lambda\arctan\left(\frac{Q_-\tanh\mathcal{Y}}{\sqrt{1-Q_-^2}}\right)}\,\tilde{U}^2_\lambda\;, \quad (3.75)$$

and the equations in brackets (3.71) reduce to

$$\begin{aligned} \partial_{\mathcal{Y}}\tilde{U}^1_\lambda \;+\; e^{+i2\lambda\xi\mathcal{Y}}\,(\lambda\tanh\mathcal{Y}-i\xi)\,\tilde{U}^2_\lambda \;&=\; 0\;, \\ \partial_{\mathcal{Y}}\tilde{U}^2_\lambda \;+\; e^{-i2\lambda\xi\mathcal{Y}}\,(\lambda\tanh\mathcal{Y}+i\xi)\,\tilde{U}^1_\lambda \;&=\; 0\;. \end{aligned} \quad (3.76)$$

Inverting the first equation and substituting into the second we get a second-order ODE for $\tilde{U}^1_\lambda$

$$\partial^2_{\mathcal{Y}}\tilde{U}^1_\lambda - \left(2i\lambda\xi + \frac{\mathrm{sech}^2\mathcal{Y}}{\tanh\mathcal{Y}-i\lambda\xi}\right)\partial_{\mathcal{Y}}\tilde{U}^1_\lambda - \left(\tanh^2\mathcal{Y}+\xi^2\right)\tilde{U}^1_\lambda = 0 \quad (3.77)$$

with solutions

$$\tilde{U}^1_\lambda = \Big(\mathrm{sech}\mathcal{Y}\;V_\lambda + (\cosh\mathcal{Y} - i\lambda\xi\sinh\mathcal{Y} - i\lambda\xi\,\mathcal{Y}\,\mathrm{sech}\mathcal{Y})\;\tilde{V}_\lambda\Big)\;e^{i\lambda\xi\mathcal{Y}}. \quad (3.78)$$

Taking $\tilde{V}_\lambda = 0$, we obtain the normalizable solutions

$$\tilde{U}^1_\lambda = \mathrm{sech}\mathcal{Y}\;e^{i\lambda\xi\mathcal{Y}}\;V_\lambda\;, \qquad \tilde{U}^2_\lambda = \lambda\;\mathrm{sech}\mathcal{Y}\;e^{-i\lambda\xi\mathcal{Y}}\;V_\lambda\;, \quad (3.79)$$

and the (kappa-fixed) fermion zero modes are given by

$$\begin{aligned} \Psi^1 &= \sum_{\lambda=\pm}\frac{\mathrm{sech}\mathcal{Y}}{4\sqrt{1+u}}\;e^{i\lambda\omega_+}\Big(e^{i\chi}\sqrt{1-\lambda Q_+\,\mathrm{sech}\mathcal{Y}}\,+ \\ &\qquad\qquad e^{-i\chi}\lambda\sqrt{1+\lambda Q_+\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45}\Big)V_\lambda\;, \\ \Psi^2 &= \sum_{\lambda=\pm}\frac{\lambda\;\mathrm{sech}\mathcal{Y}}{4\sqrt{1-u}}\;e^{i\lambda\omega_-}\,(e^{i\tilde{\chi}}\sqrt{1+\lambda Q_-\,\mathrm{sech}\mathcal{Y}}\,- \\ &\qquad\qquad e^{-i\tilde{\chi}}\lambda\sqrt{1-\lambda Q_-\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45})V_\lambda\;, \end{aligned} \quad (3.80)$$

where

$$\omega_\pm(\mathcal{Y}) = \frac{q\,\mathcal{Y}}{2\sqrt{\tilde{q}^2-u^2}} - \frac{1}{2}\arctan\left(\frac{Q_\pm\tanh\mathcal{Y}}{\sqrt{1-Q_\pm^2}}\right)\;, \quad (3.81)$$

and the constant MW spinors $V_\pm$ satisfy $P_-V_\pm = V_\pm$, $\hat{\Gamma}_{34}V_\pm = +iV_\pm$, and $\hat{\Gamma}_{12}V_\pm = \pm iV_\pm$.

Counting the zero modes. The normalizable zero mode solutions above are parametrized by the constant spinor $V = V_+ + V_-$. An unconstrained 10-d MW spinor has 16 real degrees of freedom, but kappa-fixing (which in our parametrisation translates to $\hat{\Gamma}_{34} V = +iV$) and 6d-chirality ($P_- V = V$) both reduce the number of components by half. Recalling the further restriction $\hat{\Gamma}^{1268} V = -V$ for the S^1 case, we conclude that there are 4 and 2 normalizable solutions for the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ backgrounds, respectively, i.e. we get the expected number of fermion zero modes.

3.2.3 Zero mode action

Now letting $V = V_+ + V_-$ depend on $\mathcal{T}$ and substituting these zero modes into (3.42) we get

$$\mathcal{L}_{F,0} = 2i \cos\varphi \, (1+u)\gamma \, {\Psi^1}^\dagger \partial_{\mathcal{T}} \Psi^1 + 2i\tilde{A}(1-u)\gamma \, {\Psi^2}^\dagger \partial_{\mathcal{T}} \Psi^2 \, , \tag{3.82}$$

$$= \frac{i \cos\varphi \, \gamma}{2} \operatorname{sech}^2 \mathcal{Y} \; V^\dagger \partial_{\mathcal{T}} V \, , \tag{3.83}$$

where, going to the second line, we implicitly used the fact that $V = \frac{1}{2}(\mathbb{1} - i\Gamma_{34}) V$, and $(\mathbb{1} - i\Gamma_{34})\Gamma_{45}(\mathbb{1} - i\Gamma_{34}) = 0$. Integrating over $\mathcal{X}$ we get the zero mode action

$$S_{F,0} = \mathrm{h}\tilde{\gamma} \sec\varphi \int d\mathcal{T} \left(i \, V^\dagger \partial_{\mathcal{T}} V \right) \, , \tag{3.84}$$

with

$$\tilde{\gamma} = \frac{\gamma}{\zeta} = \frac{1}{\sqrt{\tilde{q}^2 - u^2}} \, . \tag{3.85}$$

We can further simplify this by considering a Majorana basis, where all (rotated) gamma-matrices are purely imaginary $\hat{\Gamma}^*_A = -\hat{\Gamma}^*_A$, and the Majorana condition reduces to reality of the spinors ${\Psi^I}^* = \Psi^I$. Applying this to the solutions (3.80), we get

$$V_- = {V_+}^* \quad \Rightarrow \quad V^* = V \, , \tag{3.86}$$

and the zero mode action becomes

$$S_{\mathrm{F},0} = \mathrm{h}\tilde{\gamma}\sec\varphi \int \mathrm{d}\mathcal{T}\left(i\,V^T\partial_{\mathcal{T}}V\right) . \tag{3.87}$$

As we have noted above, there are 2 and 4 real fermion zero modes for the giant magnons on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ respectively. Quantization of these real fermions leads to the anticommutators

$$\{V_{\alpha a}, V_{\beta b}\} = \delta_{\alpha\beta}\,\delta_{ab}\,\frac{\cos\varphi}{\mathrm{h}\tilde{\gamma}} , \tag{3.88}$$

where $a, \alpha = 1, 2$, and for $\varphi \neq 0$ only the $a = 1$ modes are present. After complexifying

$$V_{\mathrm{L}a} = \frac{1}{\sqrt{2}}\left(V_{1a} + i\,V_{2a}\right) , \quad V_{\mathrm{R}a} = \frac{1}{\sqrt{2}}\left(V_{1a} - i\,V_{2a}\right) , \tag{3.89}$$

the only non-trivial zero-mode anticommutator is

$$\{V_{\mathrm{L}a}, V_{\mathrm{R}b}\} = \delta_{ab}\,\frac{\cos\varphi}{\mathrm{h}\tilde{\gamma}} . \tag{3.90}$$

In the remaining part of this section we will see, for both geometries, how the symmetry superalgebra of the ground state (BMN vacuum) arises from these zero modes.

3.2.4 Zero-mode algebra for $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$

By considering the corresponding spin-chain, it was argued that the fundamental excitations transform in the 2 dimensional short representations of the centrally extended $\mathfrak{su}(1|1)^2$ algebra [98]. This superalgebra has 4 fermionic generators and 4 central charges satisfying[4]

$$\begin{aligned} \{\mathbf{Q}_{\mathrm{L}}, \mathbf{S}_{\mathrm{L}}\} &= \mathbf{H}_{\mathrm{L}} , \qquad & \{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} &= \mathbf{C} , \\ \{\mathbf{Q}_{\mathrm{R}}, \mathbf{S}_{\mathrm{R}}\} &= \mathbf{H}_{\mathrm{R}} , \qquad & \{\mathbf{S}_{\mathrm{L}}, \mathbf{S}_{\mathrm{R}}\} &= \overline{\mathbf{C}} . \end{aligned} \tag{3.91}$$

Consequently, the symmetry algebra of light-cone gauge superstring theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ was shown to take the same form, after lifting the level-

[4] For a detailed description of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ and its representations see section 2.2.1.

matching condition [116]. Thus, it is important to see how the supercharges of the algebra can be constructed from the zero modes (3.90).

For an off-shell one-particle representation the values of the central charges are given by

$$\begin{aligned} \mathbf{H}_{\mathrm{L}} &= \frac{1}{2}(\epsilon + M)\ , \qquad \mathbf{C} = \frac{\mathrm{h}\varsigma}{\tilde{\gamma}}\ , \\ \mathbf{H}_{\mathrm{R}} &= \frac{1}{2}(\epsilon - M)\ , \qquad \overline{\mathbf{C}} = \frac{\mathrm{h}}{\tilde{\gamma}\varsigma}\ , \end{aligned} \tag{3.92}$$

where $M = m \pm qhp$, with mass m, ϵ is the energy of the magnon

$$\epsilon = \sqrt{M^2 + \frac{4\mathrm{h}^2}{\tilde{\gamma}^2}}\ , \tag{3.93}$$

and ς can be removed by rescaling for a one-particle state, but plays an important role in constructing multi-particle representations [39]. Note that the momentum of the excitation enters into these expressions through (3.4)

$$(\tilde{q}\tilde{\gamma})^{-1} = \sin\frac{p}{2}\ . \tag{3.94}$$

These values satisfy the shortening condition $\mathbf{H}_{\mathrm{L}}\mathbf{H}_{\mathrm{R}} - \mathbf{C}\overline{\mathbf{C}} = 0$, therefore on this representation the supercharges must be related to each other. Assuming only $\{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} = \frac{\mathrm{h}\varsigma}{\gamma}$, it is not too hard to justify[5] that the rest of (3.91) will follow from

$$\mathbf{S}_{\mathrm{L,R}} \ = \ \varsigma^{-1}\left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4\mathrm{h}^2} + 1} + \frac{\tilde{\gamma} M}{2\mathrm{h}}(-1)^F\right)\mathbf{Q}_{\mathrm{R,L}}\ , \tag{3.95}$$

where F is the fermion number operator, i.e. $(-1)^F$ anticommutes with the supercharges. This leaves us with the task of expressing $\mathbf{Q}_{\mathrm{L,R}}$ in terms of the zero modes. We can make the general ansatz

$$\mathbf{Q}_{\mathrm{L,R}} \ = \ \varsigma^{1/2}\left(\mathcal{A} - \mathcal{B}(-1)^F\right) V_{\mathrm{L,R}}\ , \tag{3.96}$$

where $\mathcal{A}$ and $\mathcal{B}$ are some c-numbers, and (3.90) guarantees that the condition $\{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} = \frac{h\varsigma}{\tilde{\gamma}}$ will be satisfied as long as $\mathcal{A}^2 - \mathcal{B}^2 = \sec\varphi\,\mathrm{h}^2$. Our freedom in

[5] In doing so, one might find useful the fact that acting on the short representation, the supercharges satisfy: $[\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}] = -(-1)^F\mathbf{C}$, $[\mathbf{S}_{\mathrm{L}}, \mathbf{S}_{\mathrm{R}}] = (-1)^F\overline{\mathbf{C}}$.

choosing $\mathcal{A}$ is just basis dependence, and a symmetric identification is given by

$$\begin{aligned}\mathcal{A} &= \sqrt{\frac{\sec\varphi\,\mathrm{h}^2}{2}}\left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4\mathrm{h}^2}+1}+1\right)^{1/2}, \\ \mathcal{B} &= \sqrt{\frac{\sec\varphi\,\mathrm{h}^2}{2}}\left(\sqrt{\frac{\tilde{\gamma}^2 M^2}{4\mathrm{h}^2}+1}-1\right)^{1/2},\end{aligned} \tag{3.97}$$

with the supercharges taking the form

$$\begin{aligned}\mathbf{Q}_{\mathrm{L,R}} &= \varsigma^{1/2}\left(\mathcal{A}-\mathcal{B}(-1)^F\right)V_{\mathrm{L,R}}\,, \\ \mathbf{S}_{\mathrm{L,R}} &= \varsigma^{-1/2}\left(\mathcal{A}+\mathcal{B}(-1)^F\right)V_{\mathrm{R,L}}\,.\end{aligned} \tag{3.98}$$

3.2.5 Zero-mode algebra for $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$

The off-shell symmetry algebra of superstring theory on this background is the centrally extended $\mathfrak{psu}(1|1)^4$ [115], which is essentially a tensor product of two $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ algebras with matching central charges.[6] The giant magnon is part of a 4 dimensional short representation, and we should be able to match the supercharges to the zero modes.

Having noted the tensor product structure of the algebra, the construction is trivial, since (3.90) gives us two non-interacting copies of $U_{\mathrm{L,R}}$. The central charges take the same values as in (3.92), hence everything from the previous subsection holds for each copy of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, and the supercharges of $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ are simply

$$\begin{aligned}\mathbf{Q}_{\mathrm{L,R}\ a} &= \varsigma^{1/2}\left(\mathcal{A}-\mathcal{B}(-1)^F\right)V_{\mathrm{L,R}\ a}\,, \\ \mathbf{S}_{\mathrm{L,R}\ a} &= \varsigma^{-1/2}\left(\mathcal{A}+\mathcal{B}(-1)^F\right)V_{\mathrm{R,L}\ a}\,.\end{aligned} \tag{3.99}$$

where $\mathcal{A}$ and $\mathcal{B}$ are still given by (3.97).

3.2.6 Zero modes in the $\alpha \to 0, 1$ limits

The parameter $\alpha \in [0,1]$ determines the radii of the 3-spheres in the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ geometry (1.47), and in the limits $\alpha \to 0, 1$, blowing up either of the spheres, we are left with—up to compactification of the flat directions—$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. It is interesting to see what happens to the fermion zero

[6] See section 2.2.1 for the construction and short representations of $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$.

modes (3.80) in the process.

Taking $\alpha \to 1$ (or $\varphi \to 0$) blows up S^3_-, the sphere on which we have the BMN-like leg of the magnon (3.1). In this limit $\mathcal{Y} \to \gamma\sqrt{\tilde{q}^2 - u^2}\mathcal{X}$, i.e. the magnon becomes the T^4 magnon, and the zero modes reduce to two of the four real T^4 zero modes, the ones on the $\Gamma_{1268} = -1$ subspace. The remaining two we will find on the $\Gamma_{1268} = +1$ eigenspace, where Δ also becomes zero (3.47).

On the other hand, $\alpha \to 0$ (or $\varphi \to \frac{\pi}{2}$) blows up S^3_+ with the stationary magnon on it, and the bosonic solution becomes a BMN string on S^3_-. Since the rescaled coordinate

$$\mathcal{Y} = \cos^2\varphi\, \gamma\sqrt{\tilde{q}^2 - u^2}\mathcal{X} \to 0 \tag{3.100}$$

for all points on the string, the zero mode solution (3.80) reduces to constant spinors. The highest weight state of the massless magnon is fermionic [115] and should correspond to the limit of our fermion fluctuations, but it appears we are unable to learn more about these modes from a semiclassical analysis. This shows that some aspects of the massless modes can only be captured by exact in α' results, in agreement with similar findings in the spin chain limit [110].

3.3 Fermion zero modes for $q = 1$

In this section we take a look at the special case of $q = 1$, as there are some subtleties not captured by our general discussion. The $q = 1$ fermion zero modes on the two AdS_3 backgrounds are more closely related than for $q < 1$, hence we will first focus on the $AdS_3 \times S^3 \times T^4$ case, then briefly describe the differences for $AdS_3 \times S^3 \times S^3 \times S^1$.

3.3.1 Bosonic solution

For $q = 1$, the giant magnon

$$\begin{aligned} Z_1 &= e^{it}\left[\cos\tfrac{\mathrm{p}}{2} + i\sin\tfrac{\mathrm{p}}{2}\,\tanh U\right], \\ Z_2 &= e^{iV}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}U, \end{aligned} \tag{3.101}$$

found in [119] is still a valid solution, but the magnon speed on the world-sheet is actually fixed to be the speed of light, and we will use a different parametrization[7]

$$
\begin{aligned}
U &= \cos\rho\ \beta\,(x+t), \\
V &= \sin\rho\ \beta\,(x+t) - x,
\end{aligned}
\tag{3.102}
$$

where $\beta > 0$, $\rho \in [0, 2\pi)$, and the parameters are related via

$$
b \equiv \cot\tfrac{\mathrm{p}}{2} = \frac{\sin\rho - \beta}{\cos\rho}. \tag{3.103}
$$

Already from this representation of the solution it seems like the main dependence is on the light-cone coordinate $x^+ = \frac{1}{2}(t+x)$. This is hinting at the magnon having a definite chirality, not completely unexpectedly considering that bosonic theory reduces to the conformal WZW model at $q = 1$. This statement will be made more precise shortly.

Conserved charges. For the above solution the conserved charges are

$$
\begin{aligned}
E - J_1 = M &= 2\mathrm{h}\sin^2\tfrac{\mathrm{p}}{2}\left(\tan\rho - \cot\tfrac{\mathrm{p}}{2}\right), \\
J_2 &= M + \mathrm{hp},
\end{aligned}
\tag{3.104}
$$

with dispersion relation

$$
E - J_1 = J_2 - \mathrm{hp}. \tag{3.105}
$$

The SU(2) principal chiral model for $q = 1$. The SU(2) PCM with WZ term (see section 2.2.3) simplifies significantly for the case of $q = 1$, with the equations of motion (2.224) now reading

$$
\partial_- \mathfrak{J}_+ = 0\ , \quad \partial_+ \mathfrak{K}_- = 0\ . \tag{3.106}
$$

The degrees of freedom separate based on chirality: the left-movers are described by $\mathfrak{J}_+(x^+)$, while $\mathfrak{K}_-(x^-)$ describes right-movers. Looking at the magnon's SU(2) currents, listed in appendix H, we note that $\mathfrak{K}_-$ is in fact

[7] In the $q \to 1$ limit the parameter v of [119] is meaningless, instead we will use $\beta = \sqrt{\frac{1-v}{1+v}}$.

constant with no dynamical information (i.e. it can be gauged away). It is in this sense that the classical bosonic solution has a definite chirality.

3.3.2 Zero mode equations for $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$

The derivation of the fermion equations of motion is analogous to the $q \in [0,1)$ case presented in section 3.1, and we omit the details here. In terms of the light-cone coordinates $x^{\pm} = \frac{1}{2}(t \pm x)$, we have

$$\begin{aligned} \left(\partial_- - 2\beta \cos\rho \, M(x^+)\right) \Psi^1 &= 0 \ , \\ \left(\partial_+ + 2\beta \cos\rho \, \tilde{M}(x^+)\right) \Psi^2 &= 0 \ , \end{aligned} \tag{3.107}$$

with

$$\begin{aligned} M &= \frac{1}{2\beta\cos\rho}\left(\frac{1}{2}G\,\Gamma_{34} + \frac{1}{2}Q\,\Gamma_{35} + R\,P_- - (R+\Gamma_{12})\,P_+\right) \\ \tilde{M} &= \frac{1}{2\beta\cos\rho}\left(\frac{1}{2}\tilde{G}\,\Gamma_{34} + \frac{1}{2}\tilde{Q}\,\Gamma_{35} + R\,P_- - (R+\Gamma_{12})\,P_+\right) \end{aligned} \tag{3.108}$$

where all the dependence is on x^+ via

$$\mathcal{Y} = 2\beta\cos\rho\, x^+ \ . \tag{3.109}$$

The expressions for $G, \tilde{G}, Q$ and $\tilde{Q}$, along with the pullbacks of the vielbein and spin connection can be found in Appendix I. These equations are the $q = 1$ versions of (3.18), but also after commuting the kappa projectors through. Note however, that they cannot be obtained as limits of the $q < 1$ analogues. In this general setting for $q = 1$ surely not (there are two parameters β, ρ here versus the one parameter u in section 3.1), but not even for any special case, since there is no $q = 1$ stationary magnon (see towards the end of this section).

Zero mode condition. As we have seen above, the bosonic background is itself chiral ($\partial_- \mathfrak{J}_+ = 0$), and it is reasonable to expect this to carry through to the fermionic zero modes, i.e. $\partial_- \vartheta^J = 0$. This can be viewed as the extension of the zero mode condition for $q \in [0,1)$, and forces the first spinor to be trivial

$$\Psi^1 = 0 \ . \tag{3.110}$$

Changing to the variable $\mathcal{Y}$, the remaining equation for Ψ^2 reads

$$\left(\partial_{\mathcal{Y}} + \tilde{M}\right)\Psi^2 = 0 \ . \tag{3.111}$$

3.3.3 Zero mode solutions for $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$

We can find the solutions for Ψ^2 in much the same way we did in section 3.2. First we solve for the general kappa-fixed spinor, then substituting it into (3.111) we get a set of simpler equations on the $P_\pm$ subspaces, that we can easily solve.

Fixing kappa-gauge. The projector can be written as

$$K_2 = \frac{1}{2}\left(\mathbb{1} - \cos\upsilon\,\Gamma_{04} - \sin\upsilon\,\Gamma_{05}\right) \ , \tag{3.112}$$

with

$$\upsilon = \arcsin\left(\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right) \ . \tag{3.113}$$

Making the ansatz

$$\Psi^2 = \left(\alpha_+(\mathcal{Y}) + \alpha_-(\mathcal{Y})\,\Gamma_{45}\right) U \ , \tag{3.114}$$

with $\Gamma_{34}U = iU$ and $i\Gamma_{12}\Gamma_*\Gamma_+U = \lambda U$, the equation $K_2\Psi^2 = \Psi^2$ reduces to

$$\begin{aligned} \lambda\sin\upsilon\,\alpha_+ + \lambda\cos\upsilon\,\alpha_- &= \alpha_+ \ , \\ \lambda\cos\upsilon\,\alpha_+ - \lambda\sin\upsilon\,\alpha_- &= \alpha_- \ . \end{aligned} \tag{3.115}$$

A symmetric solution is given by

$$\alpha_+ = \sqrt{1+\lambda\sin\upsilon} \ , \qquad \alpha_- = \lambda\sqrt{1-\lambda\sin\upsilon} \ , \tag{3.116}$$

and the most general gauge-fixed spinor is

$$\Psi^2 = \sum_{\lambda=\pm}\left(\sqrt{1+\lambda\sin\upsilon} + \lambda\sqrt{1-\lambda\sin\upsilon}\,\Gamma_{45}\right) U_\lambda \ , \tag{3.117}$$

where still $\Gamma_{34}U_\pm = +iU_\pm$ and $i\Gamma_{12}\Gamma_*\Gamma_+U_\pm = i\Gamma_{0345}U_\pm = \pm U_\pm$.

Solutions on the $P_\pm$ subspaces. Now letting U_λ depend on $\mathcal{Y}$ and substituting (3.117) into the $P_\pm$ projections of (3.111), after a considerable amount of simplification, we get

$$\begin{aligned} &\sum_{\lambda=\pm} (\alpha_+ + \alpha_- \Gamma_{45}) (\partial_\mathcal{Y} + C_+) U_\lambda = 0 \qquad \text{on} \quad P_+ \ , \\ &\sum_{\lambda=\pm} (\alpha_+ + \alpha_- \Gamma_{45}) (\partial_\mathcal{Y} + C_-) U_\lambda = 0 \qquad \text{on} \quad P_- \ , \end{aligned} \tag{3.118}$$

with the scalars[8]

$$C_\pm = -\frac{i\lambda}{4} \left(\frac{2b\,\mathrm{sech}^2\mathcal{Y}}{b^2 + \tanh^2\mathcal{Y}} \pm \frac{\sec^2\rho}{b - \tan\rho} + 2\tan\rho \right) \ . \tag{3.119}$$

It is now a simple exercise to arrive at the solutions Ψ^{2+}, Ψ^{2-} on the P_+ and P_- subspaces, respectively,

$$\Psi^{2\pm} = \sum_{\lambda=\pm} e^{i\lambda\omega_\pm(\mathcal{Y})} \left(\sqrt{1 + \lambda \sin\upsilon} + \lambda\sqrt{1 - \lambda\sin\upsilon}\,\Gamma_{45} \right) V_\lambda^\pm \ , \tag{3.120}$$

where

$$\omega_\pm(\mathcal{Y}) = \frac{1}{2}\arctan\left(\frac{\tanh\mathcal{Y}}{b} \right) + \frac{1}{4}\left(2\tan\rho \pm \frac{\sec^2\rho}{b - \tan\rho} \right)\mathcal{Y} \ , \tag{3.121}$$

and the constant spinors V_λ^a satisfy $\Gamma_{34} V_\lambda^a = +iV_\lambda^a$, $P_\pm V_\lambda^\pm = V_\lambda^\pm$ and $i\Gamma_{0345} V_\pm^a = \pm V_\pm^a$. Starting with 16 (unconstrained) real MW spinors, these conditions leave us with 4+4 real zero modes on the P_+ and P_- subspaces. We see that none of the solutions are normalizable, which is to be expected given the chiral nature of the background. However, only looking at the solutions, and not extrapolating from the $q < 1$ case, it is unclear which 4 of these should be included in semiclassical quantization and the construction of the algebra.

3.3.4 Zero modes for $\mathbf{AdS_3 \times S^3 \times S^3 \times S^1}$

We can put the magnon (3.101) on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ using the prescription from section 2.2.2. Just like above, the zero modes satisfy $\Psi^1 = 0$ and

$$\left(\partial_\mathcal{Y} + \tilde{M} \right) \Psi^2 = 0 \ , \tag{3.122}$$

[8] Note that these are different from (3.67).

where, similarly to (3.50)

$$\tilde{M} = \frac{1}{2\beta\cos\rho}\left(\frac{1}{2}\tilde{G}\,\hat{\Gamma}_{34} + \frac{1}{2}\tilde{Q}\,\hat{\Gamma}_{35} + R\,P_- - (R + \hat{\Gamma}_{12})\,P_+ + \Delta_0\,\hat{\Gamma}_{12}\right) . \tag{3.123}$$

The boosted worldsheet coordinate is $\mathcal{Y} = 2\cos^2\!\varphi\,\beta\cos\rho\,x^+$, the scalar functions $G, \tilde{G}, Q, \tilde{Q}$ are still as given in Appendix I, and from (3.47)

$$\Delta_0 = -\frac{\kappa^2}{2}\left(\hat{\Gamma}^{1268} + \mathbb{1}\right) , \qquad \kappa \equiv \tan\varphi . \tag{3.124}$$

$\boldsymbol{\hat{\Gamma}^{1268} = -1.}$ On the -1 eigenspace of $\hat{\Gamma}^{1268}$ the solutions are the same as for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ (3.120), with all Γ_A replaced by $\hat{\Gamma}_A$ (including $P_\pm = \frac{1}{2}(\mathbb{1} \pm \hat{\Gamma}_*\hat{\Gamma}_+)$) and imposing the extra condition $\hat{\Gamma}^{1268}V^a_\lambda = -V^a_\lambda$.

$\boldsymbol{\hat{\Gamma}^{1268} = +1.}$ On this subspace $\Delta_0 = -\kappa^2$, and after making the ansatz (3.117) we get

$$\begin{aligned} \left(\partial_{\mathcal{Y}} + C_+ + \frac{i\lambda\kappa^2}{2\beta\cos\rho}\right) U_\lambda &= 0 \qquad \text{on } \; P_+ \; , \\ \left(\partial_{\mathcal{Y}} + C_- - \frac{i\lambda\kappa^2}{2\beta\cos\rho}\right) U_\lambda &= 0 \qquad \text{on } \; P_- \; . \end{aligned} \tag{3.125}$$

The zero mode solutions are

$$\Psi^{2\pm} = \sum_{\lambda=\pm} e^{i\lambda\tilde{\omega}_\pm(\mathcal{Y})}\left(\sqrt{1+\lambda\sin\upsilon} + \lambda\sqrt{1-\lambda\sin\upsilon}\,\Gamma_{45}\right) V^\pm_\lambda \; , \tag{3.126}$$

where

$$\tilde{\omega}_\pm(\mathcal{Y}) = \frac{1}{2}\arctan\left(\frac{\tanh\mathcal{Y}}{b}\right) + \frac{1}{4}\left(2\tan\rho \pm \frac{(1+\kappa^2)\sec^2\rho}{b - \tan\rho}\right)\mathcal{Y} \; , \tag{3.127}$$

and $\Gamma_{34}V^a_\lambda = +iV^a_\lambda$, $P_\pm V^\pm_\lambda = V^\pm_\lambda$, $i\Gamma_{0345}V^a_\pm = \pm V^a_\pm$, $\hat{\Gamma}^{1268}V^a_\lambda = +V^a_\lambda$.

We have 8 real solutions in total, 2+2 for $P_\pm$ on each eigenspace of $\hat{\Gamma}^{1268}$. Once again, all of these zero modes are non-normalizable, and without extrapolating from the $q < 1$ analysis, we have not been able to find any distinguishing features of the 2 that would enter into canonical quantization.

3.3.5 The $q \to 1$ limit

We can go from the $q < 1$ dyonic magnon (2.260) to the $q = 1$ solution (3.101) by taking

$$\tilde{q} \to 0, \quad u \to -1, \qquad \text{with} \quad \tilde{q}\gamma = \beta \text{ fixed.} \tag{3.128}$$

However, to compare the zero modes above to those found in section 3.2, we need the $q = 1$ version of the stationary magnon (3.1) we used as a background for the $q < 1$ fermions. There are two natural ways of taking the $q \to 1$ limit, let us look at them in turn.

Our first instinct would be to take the same limit (3.128) for the stationary magnon (3.1), but this is not compatible with the condition (3.4), restricting $|u| \leq \tilde{q}$. Equivalently, we cannot make V in (3.102) only depend on x^+ (technically one could take $\beta \to \infty$, but this results in a discontinuous bosonic solution).

Alternatively, we can impose the second form of the stationary condition, and require the SU(2) charge M to be zero. This would mean $\beta = 0$, and then $U \equiv 0$, with the endpoints not on the equator any more. Furthermore, the parameter p in (3.101) would not be the worldsheet momentum, as $\Delta\phi_1 = 0$.

Lacking a suitable generalization of the Hofman-Maldacena magnon for $q = 1$, it is not immediately clear how we can apply the analysis of previous sections. It would be interesting to further investigate the relation between the $q \to 1$ limit of zero modes found in section 3.2, to the $q = 1$ fermion fluctuations found here.

3.4 Chapter conclusions and outlook

In this chapter we have seen how the fermion zero mode equations of the mixed-flux stationary AdS_3 giant magnon can be solved explicitly by exploiting the symmetries of the system. We found that there are 4 and 2 zero modes for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ magnons, respectively, in agreement with the algebraic structure. We also showed how to get the generators of the centrally extended $\mathfrak{psu}(1|1)^4$ and $\mathfrak{su}(1|1)^2$ algebras from the semiclassically quantized fermion zero modes.

We treated the $q = 1$ limit separately, and found that there is no stationary magnon in this case. As expected from the chiral nature of the magnons at the $q = 1$ point, all of the zero modes we found are non-normalizable. We have the same number of fermionic generators in the off-shell algebra as for $q < 1$, and with the excess number of solutions, the issue of canonical quantization needs to be further addressed.

Another interesting question was whether we can learn something about the massless modes from this semiclassical analysis. Considering that in the T^4 theory the massless modes' highest weight state is a fermion [114], taking the $\alpha \to 0$ limit of the fermion zero modes might have been a good way to arrive at the solutions. The fact that this did not work indicates that the fermionic massless mode is inherently non-perturbative in nature. This is also in agreement with [110], where it was found that the $\alpha \to 0$ limit fails to capture the massless mode at the spin chain point (i.e. at the opposite limit of the duality). To understand this elusive mode we need non-perturbative methods, like the low-energy integrable massless S-matrix and TBA for $AdS_3 \times S^3 \times T^4$ [126, 127, 128].

There are a number of natural directions for future research. Given the lack of stationary magnon for $q = 1$, we need to better understand the pure NS-NS classical string solitons, and their fermion zero modes. As we have seen, the fermion zero modes tie in nicely with the residual symmetry algebra in the decompactification limit, and it would be interesting to perform a similar analysis for the finite size giant magnons, either on $AdS_5 \times S^5$ [45, 168] or the mixed-flux AdS_3 backgrounds [169, 170]. Lastly, in this chapter we restricted our attention to the zero energy fluctuations, and an obvious next step would be to consider the full fluctuation spectrum, along the lines of the AdS_5 calculation [54]. In fact, we have carried out this analysis [2], and the results are presented in the next chapter.

Chapter 4

Semiclassical quantization of the mixed-flux AdS$_3$ giant magnon

Having found the zero mode solutions to the perturbation equations of the mixed-flux AdS$_3$ giant magnon in chapter 3, we now present the derivation of the complete fluctuation spectrum [2], based on the AdS$_5$ calculation [54]. The fluctuations can be used to determine the 1-loop correction to the soliton energy [162, 137, 136, 163], and showing that this correction is zero provided a simple check on the explicit AdS$_5$ fluctuation solutions. The same is going to be true for AdS$_3$, although with an important difference. The dispersion relation determined from symmetry [102, 104, 115]

$$\epsilon_{\pm} = \sqrt{\left(m \pm q\sqrt{\lambda}\frac{\mathrm{p}}{2\pi}\right)^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2\frac{\mathrm{p}}{2}} \tag{4.1}$$

does receive quantum corrections, it is only in the classical string limit that

$$\mathrm{h} = \frac{\sqrt{\lambda}}{2\pi}, \tag{4.2}$$

and other physical inputs are necessary to determine the expansion of $\mathrm{h}(\lambda)$. A more detailed discussion can be found in section 4.4. This chapter is structured as follows.

In section 4.1 we first review the mixed-flux stationary giant magnon on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, then write down the spectrum of small bosonic fluctuations around the classical solution. Although the perturbation equations are rather complicated, one can construct explicit solutions algebraically using the dressing method, which we adapt to be more suited to the fluctuation analysis. In section 4.2 we find the fermionic fluctuations, closely following the methods developed in [53, 1] extended to non-zero angular frequencies. Using the symmetries of the system and an explicit kappa-fixed ansatz, the full 2×32

component spinor equations are reduced to a 4 dimensional system, which we can solve explicitly.

Finally in section 4.3 we read off the stability angles of the fluctuations, and use them to evaluate the 1-loop functional determinant around the soliton background, following the method of Dashen, Hasslacher and Neveu [162]. We find that, in agreement with our expectations based on the superalgebra, the leading order quantum correction vanishes. We conclude in section 4.4 and present some of the lengthier or more technical details in appendices.

4.1 Bosonic sector

In this section we review the mixed-flux AdS_3 stationary magnon, and solve for its bosonic fluctuations using a similar approach employed to study the AdS_5 magnon in [54]. We consider the case of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ background in our calculations, and comment on how the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ modes can be obtained at the end of the section.

We start by rewriting the conformal gauge bosonic action (2.175) in the form

$$S = \tilde{S}[Y] + \frac{1}{\cos^2\varphi}\, S_+[X^+] + \frac{1}{\sin^2\varphi}\, S_-[X^-], \tag{4.3}$$

with AdS_3 and $\mathrm{S}^3_\pm$ components

$$\begin{aligned} \tilde{S}[Y] &= -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2x \Big[\eta^{ab}\partial_a Y^i \partial_b Y_i + \tilde{\Lambda}\,(Y^2+1)\Big] \\ &\qquad - \frac{\mathrm{h}q}{3}\int_{\mathcal{B}} \mathrm{d}^3x\, \epsilon^{abc}\epsilon_{\mu\nu\rho\sigma} Y^\mu \partial_a Y^\nu \partial_b Y^\rho \partial_c Y^\sigma \\ S_\pm[X] &= -\frac{\mathrm{h}}{2}\int_{\mathcal{M}} \mathrm{d}^2x \Big[\eta^{ab}\partial_a X^i \partial_b X_i + \Lambda_\pm\,(X^2-1)\Big] \\ &\qquad - \frac{\mathrm{h}q}{3}\int_{\mathcal{B}} \mathrm{d}^3x\, \epsilon^{abc}\epsilon_{ijkl} X^i \partial_a X^j \partial_b X^k \partial_c X^l \end{aligned} \tag{4.4}$$

where $\eta^{ab} = \mathrm{diag}(-1,+1)$, the embedding coordinates $Y \in \mathbb{R}^{2,2}$, $X^\pm \in \mathbb{R}^4$ are enforced to lie on the unit-radius surfaces

$$Y^2 = -1, \quad (X^\pm)^2 = 1 \tag{4.5}$$

by the Lagrange multipliers $\tilde{\Lambda}, \Lambda_{\pm}$, and the Wess-Zumino term is defined on a 3d manifold $\mathcal{B}$ such that its boundary is the worldsheet $\partial\mathcal{B} = \mathcal{M}$. The equations of motion

$$
\begin{aligned}
&(\partial^2 - \tilde{\Lambda}\ \ ) Y_\mu\ \ - q\, \tilde{K}_\mu\ = 0, \qquad \tilde{K}_\mu = \epsilon^{ab} \epsilon_{\mu\nu\rho\sigma} Y^\nu \partial_a Y^\rho \partial_b Y^\sigma, \\
&(\partial^2 - \Lambda_\pm)\, X_i^\pm - q\, K_i^\pm = 0, \qquad K_i^\pm = \epsilon^{ab} \epsilon_{ijkl} X_j^\pm \partial_b X_k^\pm \partial_c X_l^\pm,
\end{aligned}
\tag{4.6}
$$

need to be supplemented by the conformal gauge Virasoro constraints

$$
\begin{aligned}
&(\partial_0 Y)^2 + (\partial_1 Y)^2 + \frac{1}{\cos^2\varphi} \Big((\partial_0 X^+)^2 + (\partial_1 X^+)^2 \Big) \\
&\qquad\qquad + \frac{1}{\sin^2\varphi} \Big((\partial_0 X^-)^2 + (\partial_1 X^-)^2 \Big) = 0,
\end{aligned}
\tag{4.7}
$$

$$
\partial_0 Y \cdot \partial_1 Y + \frac{1}{\cos^2\varphi} \partial_0 X^+ \cdot \partial_1 X^+ + \frac{1}{\sin^2\varphi} \partial_0 X^- \cdot \partial_1 X^- = 0.
$$

Taking scalar products of (4.6) with $Y, X^\pm$, it follows from (4.5) and

$$
Y^\mu \tilde{K}_\mu = 0, \qquad X^{\pm\, i} K_i^\pm = 0, \tag{4.8}
$$

that the Lagrange multipliers take the classical values

$$
\tilde{\Lambda} = -Y \cdot \partial^2 Y, \qquad \Lambda_\pm = X^\pm \cdot \partial^2 X^\pm. \tag{4.9}
$$

4.1.1 The stationary giant magnon

Just like in chapter 3, we take the classical background to be the stationary mixed-flux magnon (2.285), written in terms of the embedding coordinates as

$$
\begin{aligned}
Y^0 + i Y^1 &= e^{it} \\
X_1^- + i X_2^- &= e^{i \sin^2\varphi\, t} \\
Z_1 \equiv X_1^+ + i X_2^+ &= e^{i \cos^2\varphi\, t} \left[\cos \tfrac{\mathrm{p}}{2} + i \sin \tfrac{\mathrm{p}}{2}\, \tanh \mathcal{Y}\right] \\
Z_2 \equiv X_3^+ + i X_4^+ &= e^{-\frac{i q}{\sqrt{\hat{q}^2 - u^2}} \mathcal{Y}} \sin \tfrac{\mathrm{p}}{2}\, \mathrm{sech} \mathcal{Y}
\end{aligned}
\tag{4.10}
$$

where the scaled and boosted worldsheet coordinate is

$$\mathcal{Y} = \cos^2\varphi\,\gamma\sqrt{\tilde{q}^2 - u^2}\mathcal{X}, \qquad \mathcal{X} = \gamma(x - ut) \tag{4.11}$$

and

$$\tilde{q} = \sqrt{1 - q^2}, \quad \gamma^2 = \frac{1}{1 - u^2}. \tag{4.12}$$

The parameter u, restricted to $u \in (-\tilde{q}, \tilde{q})$, can be regarded as the velocity of the magnon. The worldhseet momentum $\mathrm{p} \in [0, 2\pi)$ is not a Noether charge of the action, rather a topological charge of the soliton, corresponding to the longitudinal distance between the two endpoints of the magnon on the equator of S^3_+ ($Z_2 = 0$). The parameters further satisfy

$$u = \tilde{q}\cos\tfrac{\mathrm{p}}{2}. \tag{4.13}$$

This is a special case of the dyonic mixed-flux magnon, which was first constructed in [119] for the $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ background. The stationary magnon was identifed in [1] as the mixed-flux equivalent of the Hofman-Maldacena magnon [41], as compared to the more general AdS_5 dyonic magnon of [43]. The dispersion relation[1]

$$E - J_1 = 2\mathrm{h}\tilde{q}\,\sin\tfrac{\mathrm{p}}{2}, \tag{4.14}$$

bears witness to this analogy, to be compared to the similarly simple $E - J_1 = 2\mathrm{h}\sin\frac{\mathrm{p}}{2}$ for the $q = 0$ HM magnon. The Lagrange multipliers (4.9) evaluate to the classical values

$$\tilde{\Lambda} = 1, \quad \Lambda_- = -\sin^4\varphi, \quad \Lambda_+ = \cos^4\varphi\left(1 - 2\,\tilde{q}^{-2}\gamma^2(\tilde{q}^2 - u^2)\,\mathrm{sech}^2\mathcal{Y}\right). \tag{4.15}$$

4.1.2 AdS_3 fluctuation spectrum

Let us now determine the spectrum of fluctuations around the mixed-flux magnon (4.10), starting with the AdS_3 bosons. We denote the perturbed solution by

$$Y + \delta\,\tilde{y} \tag{4.16}$$

[1] E is the spacetime energy, J_1 is the angular momentum corresponding to the maximally supersymmetric geodesic along the equators of $\mathrm{S}^3_\pm$.

where Y is the classical solution, $\delta \ll 1$ and the perturbation $\tilde{y} \in \mathbb{R}^{2,2}$ is bounded. Substituting into the equation (4.6) and expanding to first order in δ (note that $\tilde{\Lambda}$ also receives corrections) we get the perturbation equation

$$(\partial^2 - 1)\,\tilde{y}_\mu + (Y \cdot \partial^2 \tilde{y} + q\tilde{K} \cdot \tilde{y})\,Y_\mu - q\tilde{k}_\mu = 0 \tag{4.17}$$

where $\tilde{K}_\mu$ is as in (4.6) and

$$\tilde{k}_\mu = \epsilon^{ab}\epsilon_{\mu\nu\rho\sigma}\,(\tilde{y}^\nu \partial_a Y^\rho \partial_b Y^\sigma + 2Y^\nu \partial_a \tilde{y}^\rho \partial_b Y^\sigma)\,. \tag{4.18}$$

Furthermore, to preserve the norm (4.5), the perturbation must be orthogonal to the classical solution

$$Y_\mu \tilde{y}^\mu = 0. \tag{4.19}$$

These equations have one massless and two massive solutions. To get the massless perturbation we make the ansatz

$$\tilde{y}^0 = -f \sin t, \quad \tilde{y}^1 = f \cos t, \tag{4.20}$$

for which (4.17) reduces to the free wave equation

$$\partial^2 f = 0 \quad \Rightarrow \quad f = e^{ikx - i\omega t} \tag{4.21}$$

satisfying the massless dispersion relation $\omega^2 = k^2$. The remaining two massive solutions lie in the transverse directions ($\tilde{y}^0 = \tilde{y}^1 = 0$) of AdS_3, automatically satisfying (4.19). A simple plane-wave ansatz gives

$$\tilde{y}^2 = e^{ikx - i\omega t}, \quad \tilde{y}^3 = \mp i e^{ikx - i\omega t}, \qquad \omega^2 = (1 \pm qk)^2 + \tilde{q}^2 k^2. \tag{4.22}$$

Note that this is the small p, fixed $k = \mathrm{hp}$ limit of the mixed-flux $AdS_3 \times S^3 \times S^3 \times S^1$ dispersion relation [115]

$$\epsilon_\pm = \sqrt{(m \pm q\mathrm{hp})^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}\,. \tag{4.23}$$

with mass $m = 1$.

4.1.3 S^3_- fluctuation spectrum

The S^3_- fluctuations are very similar to the ones on AdS_3. Substituting the perturbed solution

$$X^- + \delta\,\tilde{x}^- \tag{4.24}$$

into (4.6), we get the first order equations

$$(\partial^2 + \sin^4\varphi)\,\tilde{x}_i^- + (X^- \cdot \partial^2 \tilde{x}^- - qK^- \cdot \tilde{x}^-)\,X_i - qk_i^- = 0 \tag{4.25}$$

where K_i^- is as in (4.6) and

$$k_i^- = \epsilon^{ab}\epsilon_{ijkl}\left(\tilde{x}_j^-\partial_b X_k^-\partial_c X_l^- + 2X_j^-\partial_b\tilde{x}_k^-\partial_c X_l^-\right), \tag{4.26}$$

which needs to be supplemented by $X_i^- \tilde{x}^{-i} = 0$ to preserve the norm. Just like on AdS_3, these equations admit a massless solution

$$\begin{aligned} \tilde{x}_1^- &= -e^{ikx - i\omega t}\sin\left(\sin^2\varphi t\right), \\ \tilde{x}_2^- &= e^{ikx - i\omega t}\cos\left(\sin^2\varphi t\right), \qquad \omega^2 = k^2, \end{aligned} \tag{4.27}$$

and two perturbations of mass $m = \sin^2\varphi$

$$\begin{aligned} \tilde{x}_3^- &= e^{ikx - i\omega t}, \\ \tilde{x}_4^- &= \mp e^{ikx - i\omega t}, \qquad \omega^2 = (\sin^2\varphi \pm qk)^2 + \tilde{q}^2 k^2. \end{aligned} \tag{4.28}$$

4.1.4 S^3_+ fluctuation spectrum

For the S^3_+ perturbed solution we write

$$X^+ + \delta\,\tilde{x}^+, \tag{4.29}$$

and also introduce the complex coordinates

$$z_1 = \tilde{x}_1^+ + i\tilde{x}_2^+, \quad z_2 = \tilde{x}_3^+ + i\tilde{x}_4^+, \tag{4.30}$$

so that the perturbed S^3_+ component of (4.10) can be written as

$$Z_1 + \delta\,z_1, \quad Z_2 + \delta\,z_2. \tag{4.31}$$

The equations of motion for the S^3_+ fluctuations read

$$\begin{aligned}\left(\partial^2 - \cos^4\varphi \left(1 - 2\,\tilde{q}^{-2}\gamma^2(\tilde{q}^2 - u^2)\,\mathrm{sech}^2\mathcal{Y}\right)\right)\,\tilde{x}_i^+ \\ + (X^+\cdot\partial^2\tilde{x}^+ - qK^+\cdot\tilde{x}^+)\,X_i - qk_i^+ = 0\end{aligned} \tag{4.32}$$

where K_i^+ is as in (4.6),

$$k_i^+ = \epsilon^{ab}\epsilon_{ijkl}\left(\tilde{x}_j^+\partial_b X_k^+\partial_c X_l^+ + 2X_j^+\partial_b\tilde{x}_k^+\partial_c X_l^+\right), \tag{4.33}$$

and to preserve the embedding norm

$$X_i^+\tilde{x}^{+i} = 0. \tag{4.34}$$

These equations have two different classes of solutions.

Firstly, there are the zero modes, representing collective coordinates of the magnon. The BMN limit fixes the orientation of the magnon in the (X_1^+, X_2^+) plane, but there is a rotational freedom in traverse coordinates (X_3^+, X_4^+) leading to the zero mode

$$\begin{aligned} z_1 &= 0, \\ z_2 &= ie^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\mathrm{sech}\mathcal{Y}. \end{aligned} \tag{4.35}$$

Furthermore, the magnon breaks the x-translation symmetry of the BMN vacuum, leading to the zero mode

$$\begin{aligned} z_1 &= ie^{i\cos^2\varphi\, t}\mathrm{sech}^2\mathcal{Y}, \\ z_2 &= -e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\mathrm{sech}\mathcal{Y}\tanh\mathcal{Y}. \end{aligned} \tag{4.36}$$

These two normalizable zero modes are presented for completeness, but will not play any further role in our analysis.

The solutions we are interested in are plane-wave fluctuations of the form

$$e^{ikx - i\omega t}f(\mathcal{Y}), \tag{4.37}$$

where $f(\mathcal{Y})$ is a bounded profile that is stationary in the magnon's frame. The equations are too complicated for us to find solutions by substituting the plane-wave ansatz into (4.32), we need to look for another strategy. The authors of [54] suggest using the dressing method [143, 144, 145] to construct the scattering state of a magnon and a breather, only then to expand this solution in the breather momentum to find the fluctuation as the subleading term. We find, instead, that it is simpler to apply the dressing method to the perturbed BMN vacuum, i.e. the point-like string moving along the equator together with fluctuations like (4.27)–(4.28), which results in the perturbed magnon. The details of this calculation can be found in appendix J, here we just present the solutions. As further confirmation of the validity of our approach, we show in appendix K that applying our method in the $\varphi = q = 0$ limit we recover the expected subset of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ fluctuations found in [54].

The massles plane-wave solution is given by

$$
\begin{aligned}
z_1 &= -ie^{ikx-i\omega t}e^{+i\cos^2\varphi\, t}\Big(\tilde{q}k - \omega\cos\tfrac{\mathrm{p}}{2} \\
&\qquad - i\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\left(\omega - \tilde{q}k\,\cosh(\mathcal{Y}+i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\right)\Big), \\
\bar{z}_1 &= ie^{ikx-i\omega t}e^{-i\cos^2\varphi\, t}\Big(\tilde{q}k - \omega\cos\tfrac{\mathrm{p}}{2} \\
&\qquad + i\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\left(\omega - \tilde{q}k\,\cosh(\mathcal{Y}-i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\right)\Big), \\
z_2 &= ie^{ikx-i\omega t}\sin\tfrac{\mathrm{p}}{2}\,e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\mathrm{sech}\mathcal{Y}\left(qk - i\tilde{q}k\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\right), \\
\bar{z}_2 &= -ie^{ikx-i\omega t}\sin\tfrac{\mathrm{p}}{2}\,e^{+\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\mathrm{sech}\mathcal{Y}\left(qk + i\tilde{q}k\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\right),
\end{aligned}
\tag{4.38}
$$

with

$$\omega^2 = k^2. \tag{4.39}$$

Here $\bar{z}_i$ are not the complex conjugates of z_i, rather[2]

$$\bar{z}_1 = \tilde{x}_1^+ - i\tilde{x}_2^+, \quad \bar{z}_2 = \tilde{x}_3^+ - i\tilde{x}_4^+, \tag{4.40}$$

[2] To preserve the (relative) simplicity of the formulas we consider $\tilde{x}_i^+$ to be complex themselves. Real solutions to (4.32) can be readily obtained by taking the real parts of these fluctuations.

The two massive modes both have $m = \cos^2\varphi$. One of them is

$$
\begin{aligned}
z_1 &= -ie^{ikx-i\omega t} e^{\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\cos^2\varphi\, t} \sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{Y}\times \\
&\qquad \Big(\omega + \cos^2\varphi + qk - \tilde{q}k\,\cosh(\mathcal{Y}+i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\Big), \\
\bar{z}_1 &= -ie^{ikx-i\omega t} e^{\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\cos^2\varphi\, t} \sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{Y}\times \\
&\qquad \Big(\omega - \cos^2\varphi - qk - \tilde{q}k\,\cosh(\mathcal{Y}-i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\Big), \\
z_2 &= \quad ie^{ikx-i\omega t}\Big(\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{Y} - 2(\tilde{q}k - \omega\cos\tfrac{\mathrm{p}}{2}) - \\
&\qquad 2i(\cos^2\varphi + qk)\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\Big) \\
\bar{z}_2 &= \quad ie^{ikx-i\omega t} e^{\frac{2iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{Y},
\end{aligned}
\tag{4.41}
$$

with

$$
\omega^2 = (\cos^2\varphi + qk)^2 + \tilde{q}^2k^2, \tag{4.42}
$$

while the other one is

$$
\begin{aligned}
z_1 &= -ie^{ikx-i\omega t} e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\cos^2\varphi\, t} \sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{Y}\times \\
&\qquad \Big(\omega + \cos^2\varphi - qk - \tilde{q}k\,\cosh(\mathcal{Y}+i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\Big), \\
\bar{z}_1 &= -ie^{ikx-i\omega t} e^{-\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\cos^2\varphi\, t} \sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}\mathcal{Y}\times \\
&\qquad \Big(\omega - \cos^2\varphi + qk - \tilde{q}k\,\cosh(\mathcal{Y}-i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}\mathcal{Y}\Big), \\
z_2 &= \quad ie^{ikx-i\omega t} e^{-\frac{2i\ q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\,\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{Y}, \\
\bar{z}_2 &= \quad ie^{ikx-i\omega t}\Big(\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{Y} - 2(\tilde{q}k - \omega\cos\tfrac{\mathrm{p}}{2}) - \\
&\qquad 2i(\cos^2\varphi - qk)\sin\tfrac{\mathrm{p}}{2}\,\tanh\mathcal{Y}\Big)
\end{aligned}
\tag{4.43}
$$

with

$$
\omega^2 = (\cos^2\varphi - qk)^2 + \tilde{q}^2k^2. \tag{4.44}
$$

4.1.5 Bosonic modes in $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ string theory

In addition to the fluctuations we found above, there is of course the massless S^1 mode

$$
e^{ikx-i\omega t} \qquad \omega^2 = k^2. \tag{4.45}
$$

However, in a proper quantization of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ string theory the sigma-model action (4.3) would need to be supplemented by ghosts, cancelling the massless AdS_3 mode (4.21), and also a combination of the massless $\mathrm{S}^3_\pm$ modes (4.27), (4.38), corresponding to the $\mathrm{S}^3_+ \times \mathrm{S}^3_-$ leg of the BMN geodesic. These are analogous to the longitudinal modes in light-cone gauge, and in our semiclassical analysis we will simply omit them [140, 171].

In summary, the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ magnon has two massless modes (one on the flat S^1 and another one perpendicular to the BMN angle on $\mathrm{S}^3_+ \times \mathrm{S}^3_-$), two $m = 1$ fluctuations on AdS_3, two $m = \cos^2\varphi$ modes on S^3_+, and two $m = \sin^2\varphi$ modes on S^3_-, all with the dispersion relations

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2. \tag{4.46}$$

4.1.6 Bosonic modes in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ string theory

Taking the $\varphi \to 0$ limit of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ blows up the S^3_- factor, which we can recompactify on a T^3 to get the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ geometry. In this limit the AdS_3 and S^1 fluctuations are unchanged, the S^3_+ modes take the same form but become $m = 1$, while on S^3_- the massless mode becomes the one unaffected by the ghosts, and the two $m = \sin^2\varphi$ modes become massless T^4 modes. In summary, the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ magnon has four massless, and four mass 1 bosonic fluctuations.

4.2 Fermionic sector

In this section we solve for the complete fermion fluctuation spectrum around the mixed-flux stationary magnon (4.10). Our approach will be very similar to chapter 3, but rather than normalizable zero modes, we will be looking for solutions with plane-wave asymptotes. The leading order (quadratic) action for fermion fluctuations around a general bosonic string solution $X^\mu(t,x)$ is given by [160]

$$S_{\mathrm{F}} = \mathrm{h} \int \mathrm{d}^2 x \; \mathcal{L}_{\mathrm{F}} \ , \qquad \mathcal{L}_{\mathrm{F}} = -i \left(\eta^{ab} \delta^{IJ} + \epsilon^{ab} \sigma_3^{IJ} \right) \, \bar{\vartheta}^I \rho_a \mathcal{D}_b \vartheta^J \ . \tag{4.47}$$

The ϑ^I are two ten-dimensional Majorana-Weyl spinors, σ_3^{IJ} is the Pauli matrix diag$(+1,-1)$, and ρ_a are projections of the ten-dimensional Dirac matrices

$$\rho_a \equiv e_a^A \Gamma_A \ , \qquad e_a^A \equiv \partial_a X^\mu E_\mu^A(X) \ . \tag{4.48}$$

Note the difference in notation compared to the previous section, X^μ are now the curved space coordinates of $AdS_3\times S^3\times S^3\times S^1$, and not coordinates of a flat embedding space. For the remainder of this section we use Hopf coordinates, where the only non-constant components of the stationary magnon are along $\mu = t, \theta^+, \phi_1^+, \phi_2^+, \phi_1^-$ corresponding to the tangent space components $A = 0, 3, 4, 5, 7$, respectively. The covariant derivative is

$$\mathcal{D}_a\vartheta^I = \left(\delta^{IJ}(\partial_a + \tfrac{1}{4}\omega_\mu^{AB}\partial_a X^\mu \Gamma_{AB}) + \tfrac{1}{48}\sigma_1^{IJ}\not{F}\rho_a + \tfrac{1}{8}\sigma_3^{IJ}\not{H}_a\right)\vartheta^J \ , \tag{4.49}$$

where ω_μ^{AB} is the usual spin-connection,

$$\not{H}_a \equiv \ e_a^A H_{ABC}\Gamma^{BC} = \tfrac{1}{6}(\rho_a\not{H} + \not{H}\rho_a) \ , \tag{4.50}$$

and the contracted 3-form fluxes are

$$\begin{aligned} \not{F} &= 12\tilde{q}\ \left(\Gamma^{012} + \cos\varphi\ \Gamma^{345} + \sin\varphi\ \Gamma^{678}\right) \ , \\ \not{H} &= 12q\ \left(\Gamma^{012} + \cos\varphi\ \Gamma^{345} + \sin\varphi\ \Gamma^{678}\right) \ . \end{aligned} \tag{4.51}$$

4.2.1 The fluctuation equations

The equations of motion for (4.47) are

$$\begin{aligned} (\rho_0+\rho_1)(\mathcal{D}_0-\mathcal{D}_1)\ \vartheta^1 &= 0, \\ (\rho_0-\rho_1)(\mathcal{D}_0+\mathcal{D}_1)\ \vartheta^2 &= 0, \end{aligned} \tag{4.52}$$

We proceed by changing variables to the more natural scaled and boosted worldsheet coordinates (4.11) of the magnon

$$\mathcal{Y} = \cos^2\!\varphi\,\zeta\mathcal{X}, \quad \mathcal{S} = \cos^2\!\varphi\,\zeta\mathcal{T}, \qquad \zeta = \gamma\sqrt{\tilde{q}^2 - u^2}, \tag{4.53}$$

yielding

$$
\begin{aligned}
&(\rho_0+\rho_1)\left[\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\ \vartheta^1+\mathcal{O}\vartheta^2\right]=0,\\
&(\rho_0-\rho_1)\left[\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\ \vartheta^2+\tilde{\mathcal{O}}\vartheta^1\right]=0.
\end{aligned}
\tag{4.54}
$$

Here we defined the mixing operators

$$
\mathcal{O}=-\frac{1}{48\cos^2\varphi}\not{F}(\rho_0-\rho_1)\ ,\qquad \tilde{\mathcal{O}}=\frac{1}{48\cos^2\varphi}\not{F}(\rho_0+\rho_1)\ , \tag{4.55}
$$

and fermion derivatives

$$
\begin{aligned}
D&=\partial_{\mathcal{Y}}+\frac{1}{2}G\ \Gamma_{34}+\frac{1}{2}Q\ \Gamma_{35}-\frac{(1-u)\gamma}{48\cos^2\varphi\,\zeta}\left(\not{H}(\rho_0-\rho_1)+(\rho_0-\rho_1)\not{H}\right)\ ,\\
\tilde{D}&=\partial_{\mathcal{Y}}+\frac{1}{2}\tilde{G}\ \Gamma_{34}+\frac{1}{2}Q\ \Gamma_{35}-\frac{(1+u)\gamma}{48\cos^2\varphi\,\zeta}\left(\not{H}(\rho_0+\rho_1)+(\rho_0+\rho_1)\not{H}\right)\ ,
\end{aligned}
\tag{4.56}
$$

with

$$
\begin{aligned}
G&=\ \frac{\tilde{q}^2(1-u)\cosh^2\mathcal{Y}-\tilde{q}^2+u^2}{\tilde{q}\left(\tilde{q}^2\sinh^2\mathcal{Y}+u^2\right)}\mathrm{sech}\mathcal{Y}\ ,\\
\tilde{G}&=-\frac{\tilde{q}^2(1+u)\cosh^2\mathcal{Y}-\tilde{q}^2+u^2}{\tilde{q}\left(\tilde{q}^2\sinh^2\mathcal{Y}+u^2\right)}\mathrm{sech}\mathcal{Y}\ ,\\
Q&=-\frac{q}{\tilde{q}\sqrt{\tilde{q}^2-u^2}}\ \sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}\ \mathrm{sech}\mathcal{Y}\ .
\end{aligned}
\tag{4.57}
$$

The full Green-Schwarz superstring has a local fermionic symmetry (κ-symmetry), that we need to fix for physical solutions. Noting that the operators $(\rho_0\pm\rho_1)$ are half-rank, nilpotent and commute with the fermion derivatives D and $\tilde{D}$, it is clear that the projectors

$$
K_1=\frac{1}{2}\sec\varphi\ \hat{\Gamma}^0(\rho_0+\rho_1)\ ,\qquad K_2=\frac{1}{2}\sec\varphi\ \hat{\Gamma}^0(\rho_0-\rho_1)\ , \tag{4.58}
$$

can be used to fix κ-gauge. Here we introduced a set of "boosted" gamma matrices

$$
\hat{\Gamma}^0=\sec\varphi\left(\Gamma^0-\sin\varphi\ \Gamma^7\right),\quad \hat{\Gamma}^7=\sec\varphi\left(\Gamma^7-\sin\varphi\ \Gamma^0\right),\quad \hat{\Gamma}^A=\Gamma^A\ (A\neq 0,7), \tag{4.59}
$$

that simplify the notation in what follows. The kappa-fixed spinors $\Psi^J = K_J\vartheta^J$ then satisfy

$$\begin{aligned}&\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\Psi^1 + K_1\mathcal{O}\Psi^2 = 0,\\ &\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\Psi^2 + K_2\tilde{\mathcal{O}}\Psi^1 = 0.\end{aligned} \tag{4.60}$$

Introducing the 6d chirality projector

$$P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \hat{\Gamma}^{012345}\right), \qquad [P_\pm, K_J] = 0, \tag{4.61}$$

and, with $\bar{\rho}_0 = -\hat{\Gamma}_0\,\rho_0\,\hat{\Gamma}_0$, the invertible matrix

$$R = \frac{1}{2}\sec\varphi\;\hat{\Gamma}^{012}\left(\bar{\rho}_0 - \rho_0\right), \tag{4.62}$$

we can rewrite the equations, using the boosted gamma matrix basis

$$\begin{aligned}&\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\Psi^1 + \tilde{q}\left(RP_- - K_1\Delta\;\hat{\Gamma}^{012}\right)\Psi^2 = 0,\\ &\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\Psi^2 - \tilde{q}\left(RP_- - K_2\Delta\;\hat{\Gamma}^{012}\right)\Psi^1 = 0.\end{aligned} \tag{4.63}$$

The fermion differential operators are

$$\begin{aligned}D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\;\hat{\Gamma}_{34} + \frac{1}{2}Q\;\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\left(RP_- - \left(R+\hat{\Gamma}_{12}\right)P_+ + \Delta_0\,\hat{\Gamma}_{12}\right),\\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\;\hat{\Gamma}_{34} + \frac{1}{2}Q\;\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\left(RP_- - \left(R+\hat{\Gamma}_{12}\right)P_+ + \Delta_0\,\hat{\Gamma}_{12}\right),\end{aligned} \tag{4.64}$$

and we define

$$\Delta = -\frac{1}{2}\tan\varphi\left(\hat{\Gamma}^{1268} + \mathbb{1}\right)\Gamma^7 \equiv\; \Delta_0\,\hat{\Gamma}^0 + \Delta_7\,\hat{\Gamma}^7\;, \tag{4.65}$$

with

$$\Delta_0 = -\frac{1}{2}\tan^2\!\varphi\left(\hat{\Gamma}^{1268} + \mathbb{1}\right)\;, \qquad \Delta_7 = \csc\varphi\,\Delta_0\;. \tag{4.66}$$

Note that the only source of structural difference between the equations for $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ and $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ is a non-zero Δ, and in fact this was our main reason to introduce the boosted gamma matrix basis. A much more detailed derivation of these equations, together with a thorough explanation of κ-gauge fixing, can be found in chapter 3.

4.2.2 Ansatz and reduced equations

To reduce the seemingly complicated (4.63) to a more manageable set of equations we will make an ansatz that reflects the symmetries of the system. Firstly, all of $\hat{\Gamma}^{012345}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ commute with the kappa projectors (4.58), so the kappa-fixed spinors can be written as

$$\Psi^J = \sum_{\lambda_P,\lambda_{12},\lambda_{68}\in\{\pm\}} \mathcal{K}_J(\lambda_P\lambda_{12}) V^J_{\lambda_P,\lambda_{12},\lambda_{68}}(\mathcal{S},\mathcal{Y}), \tag{4.67}$$

where the eigenvalues of $V^J_{\lambda_P,\lambda_{12},\lambda_{68}}$ under $\hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $i\lambda_{12}, i\lambda_{68}$ and λ_P, respectively. Note that $\lambda_{12}, \lambda_{68}, \lambda_P$ all take values in ± 1. There are multiple ways to make the above ansatz satisfy $K_J\Psi^J = \Psi^J$, in chapter 3 we chose to impose the additional constraint[3] $\hat{\Gamma}^{34}V^J = +iV^J$ and found

$$\begin{aligned} \mathcal{K}_1(\lambda) &= e^{+i\chi}\sqrt{1+\lambda Q_+ \,\mathrm{sech}\mathcal{Y}} - \lambda e^{-i\chi}\sqrt{1-\lambda Q_+\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45}, \\ \mathcal{K}_2(\lambda) &= e^{+i\tilde{\chi}}\sqrt{1-\lambda Q_- \,\mathrm{sech}\mathcal{Y}} + \lambda e^{-i\tilde{\chi}}\sqrt{1+\lambda Q_-\,\mathrm{sech}\mathcal{Y}}\,\hat{\Gamma}_{45}, \end{aligned} \tag{4.68}$$

where

$$Q_\pm = \frac{q\sqrt{\tilde{q}^2-u^2}}{\tilde{q}(1\pm u)}, \tag{4.69}$$

and

$$\begin{aligned} \chi(\mathcal{Y}) &= \frac{1}{2}\left(\mathrm{arccot}\left(\frac{u\,\mathrm{csch}\,\mathcal{Y}}{\tilde{q}}\right) - \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_+^2\,\mathrm{sech}^2\mathcal{Y}}}\right)\right), \\ \tilde{\chi}(\mathcal{Y}) &= \frac{1}{2}\left(\mathrm{arccot}\left(\frac{u\,\mathrm{csch}\,\mathcal{Y}}{\tilde{q}}\right) + \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_-^2\,\mathrm{sech}^2\mathcal{Y}}}\right)\right). \end{aligned} \tag{4.70}$$

While the zero modes are time-independent in the magnon's frame $\partial_\mathcal{S}\Psi^J = 0$, for the $\mathcal{S}$-dependence of the non-zero modes we make a Fourier ansatz

$$V^J(\mathcal{S},\mathcal{Y}) = e^{-i\tilde{\omega}\mathcal{S}}V^J(\mathcal{Y}). \tag{4.71}$$

[3] Note that kappa-fixing reduces the degrees of freedom by half, and in our ansatz this is done at the level of the projections $\hat{\Gamma}^{34}V^J = +iV^J$, since $\mathcal{K}_J(\lambda)$ are invertible.

As opposed to the kappa-projectors, the equations of motion (4.63) only commute with $\hat{\Gamma}^{12}$ and $\hat{\Gamma}^{68}$, and the solutions will not have definite chirality under $\hat{\Gamma}^{012345}$, unless $\Delta = 0$, i.e. for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background, or on the $\hat{\Gamma}^{1268} = -1$ spinor subspace for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ background. With this in mind, we take the general ansatz

$$\Psi^J = e^{-i\tilde{\omega}\mathcal{S}} \Big(f_J(\mathcal{Y})\ \mathcal{K}_J(-\lambda_{12}) + g_J(\mathcal{Y})\ \mathcal{K}_J(\lambda_{12})\hat{\Gamma}_{07} \Big)\, U, \tag{4.72}$$

where the constant Weyl[4] spinor U, that is shared between Ψ^1 and Ψ^2, has eigenvalues $i\lambda_{12}, i\lambda_{68}, +i, -1$ under $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$, $\hat{\Gamma}^{34}$, $\hat{\Gamma}^{012345}$, respectively. The P_- part of the solution is represented by the scalar functions f_1, f_2, while g_1, g_2 correspond to the P_+ components. The validity of such an ansatz is further justified by a quick counting of the degrees of freedom. A general Weyls spinor has 16 complex components, and after 4 mutually commuting projections, there is a single free component left, hence we can capture the $\mathcal{Y}$-dependence with a single function f_J multiplying U. Substituting (4.72) into (4.63), after a considerable amount of simplification we get

$$\begin{aligned} &e^{-i\tilde{\omega}\mathcal{S}} \bigg[\Big(\big(\partial_{\mathcal{Y}} + C_{f_1 f_1}\big)\, f_1 + C_{f_1 f_2} f_2 + C_{f_1 g_2} g_2 \Big) \mathcal{K}_1(-\lambda_{12}) \\ &\qquad \Big(\big(\partial_{\mathcal{Y}} + C_{g_1 g_1}\big)\, g_1 + C_{g_1 g_2} g_2 + C_{g_1 f_2} f_2 \Big) \mathcal{K}_1(\lambda_{12})\hat{\Gamma}_{07} \bigg] U = 0\ , \\ &e^{-i\tilde{\omega}\mathcal{S}} \bigg[\Big(\big(\partial_{\mathcal{Y}} + C_{f_2 f_2}\big)\, f_2 + C_{f_2 f_1} f_1 + C_{f_2 g_1} g_1 \Big) \mathcal{K}_2(-\lambda_{12}) \\ &\qquad \Big(\big(\partial_{\mathcal{Y}} + C_{g_2 g_2}\big)\, g_2 + C_{g_2 g_1} g_1 + C_{g_2 f_1} f_1 \Big) \mathcal{K}_2(\lambda_{12})\hat{\Gamma}_{07} \bigg] U = 0\ , \end{aligned} \tag{4.73}$$

with coefficients $C_{..}$ listed in appendix L. The matrix structure matches that of the general kappa-fixed spinors, confirming that the kappa-projectors com-

[4] We postpone the analysis of the Majorana condition until later, see the discussion around (4.102).

mute with the fermion derivatives D, $\tilde{D}$. Further substituting

$$\begin{aligned}
f_1 &= \frac{1}{\sqrt{1+u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\left(\frac{1}{2}+p_{1268}\tan^2\varphi\right)\mathcal{Y}} e^{-\frac{i}{2}\lambda_{12}\arctan\left(\frac{Q_+\tanh\mathcal{Y}}{\sqrt{1-Q_+^2}}\right)} \tilde{f}_1, \\
g_1 &= \frac{i\lambda_{12}}{\sqrt{1+u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\left(\frac{1}{2}+p_{1268}\tan^2\varphi\right)\mathcal{Y}} e^{+\frac{i}{2}\lambda_{12}\arctan\left(\frac{Q_+\tanh\mathcal{Y}}{\sqrt{1-Q_+^2}}\right)} \tilde{g}_1, \\
f_2 &= \frac{\lambda_{12}}{\sqrt{1-u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\left(\frac{1}{2}+p_{1268}\tan^2\varphi\right)\mathcal{Y}} e^{-\frac{i}{2}\lambda_{12}\arctan\left(\frac{Q_-\tanh\mathcal{Y}}{\sqrt{1-Q_-^2}}\right)} \tilde{f}_2, \\
g_2 &= \frac{i}{\sqrt{1-u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\left(\frac{1}{2}+p_{1268}\tan^2\varphi\right)\mathcal{Y}} e^{+\frac{i}{2}\lambda_{12}\arctan\left(\frac{Q_-\tanh\mathcal{Y}}{\sqrt{1-Q_-^2}}\right)} \tilde{g}_2,
\end{aligned} \tag{4.74}$$

where p_{1268} is the eigenvalue of the projector $\frac{1}{2}(\mathbb{1}+\hat{\Gamma}^{1268})$

$$p_{1268} = \tfrac{1}{2}(1-\lambda_{12}\lambda_{68}), \tag{4.75}$$

and defining

$$\xi = \frac{qu}{\sqrt{\tilde{q}^2-u^2}}\ , \tag{4.76}$$

we arrive at the reduced equations

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{f}_1 + i(\tilde{\omega}+(1+p_{1268}\tan^2\varphi)\lambda_{12}\xi)\tilde{f}_1 \\
&\qquad + (1+p_{1268}\tan^2\varphi)(\tanh\mathcal{Y} - i\lambda_{12}\xi)\tilde{f}_2 \\
&\qquad - \lambda_{12}\,p_{1268}\tan\varphi\sec\varphi\,\mathrm{sech}\mathcal{Y}\tilde{g}_2 = 0,
\end{aligned} \tag{4.77}$$

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{f}_2 - i(\tilde{\omega}+(1+p_{1268}\tan^2\varphi)\lambda_{12}\xi)\tilde{f}_2 \\
&\qquad + (1+p_{1268}\tan^2\varphi)(\tanh\mathcal{Y} + i\lambda_{12}\xi)\tilde{f}_1 \\
&\qquad + \lambda_{12}\,p_{1268}\tan\varphi\sec\varphi\,\mathrm{sech}\mathcal{Y}\tilde{g}_1 = 0,
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{g}_1 + i(\tilde{\omega}+p_{1268}\tan^2\varphi\lambda_{12}\xi)\tilde{g}_1 \\
&\qquad + \lambda_{12}\,p_{1268}\tan\varphi\sec\varphi\,\mathrm{sech}\mathcal{Y}\tilde{f}_2 \\
&\qquad + p_{1268}\tan^2\varphi(\tanh\mathcal{Y} + i\lambda_{12}\xi)\tilde{g}_2 = 0,
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{g}_2 - i(\tilde{\omega} + p_{1268}\tan^2\varphi\lambda_{12}\xi)\tilde{g}_2 \\
&\quad - \lambda_{12}\, p_{1268}\tan\varphi\sec\varphi\,\mathrm{sech}\mathcal{Y}\tilde{f}_1 \\
&\quad + p_{1268}\tan^2\varphi(\tanh\mathcal{Y} - i\lambda_{12}\xi)\tilde{g}_1 = 0.
\end{aligned} \tag{4.80}$$

4.2.3 Solutions

Let us first find the solutions for $\varphi > 0$, i.e. for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ geometry. For $p_{1268} = 0$ the P_+ components $\tilde{g}_1$ and $\tilde{g}_2$ decouple, and we have the two solutions

$$\tilde{g}_1 = e^{i\tilde{k}\mathcal{Y}}, \quad \tilde{g}_2 = 0, \qquad \tilde{k} = -\tilde{\omega}, \tag{4.81}$$

$$\tilde{g}_2 = e^{i\tilde{k}\mathcal{Y}}, \quad \tilde{g}_1 = 0, \qquad \tilde{k} = +\tilde{\omega}, \tag{4.82}$$

while on the P_- subspace we have the equations

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{f}_1 + i(\tilde{\omega} + \lambda_{12}\xi)\tilde{f}_1 + (\tanh\mathcal{Y} - i\lambda_{12}\xi)\tilde{f}_2 = 0, \\
&\partial_{\mathcal{Y}}\tilde{f}_2 - i(\tilde{\omega} + \lambda_{12}\xi)\tilde{f}_2 + (\tanh\mathcal{Y} + i\lambda_{12}\xi)\tilde{f}_1 = 0,
\end{aligned} \tag{4.83}$$

with the two solutions

$$\begin{aligned}
\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}}\left(\tanh\mathcal{Y} - i(\tilde{k} - \tilde{\omega})\right), \\
\tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}}\left(\tanh\mathcal{Y} - i(\tilde{k} + \tilde{\omega})\right), \\
\tilde{k} &= \pm\sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi\tilde{\omega} - 1}.
\end{aligned} \tag{4.84}$$

The case of $p_{1268} = 1$ is a bit more complicated, but solving the first two equations of (4.77) for $\tilde{g}_J$ and substituting into the second two, we get two second order differential equations for $\tilde{f}_1, \tilde{f}_2$. The difference of those two equations is

$$\partial_{\mathcal{Y}}^2\left(\tilde{f}_1 - \tilde{f}_2\right) + \left(\tilde{\omega}^2 + 2\lambda_{12}\xi\sec^2\varphi\,\tilde{\omega} - \sec^4\varphi\right)\left(\tilde{f}_1 - \tilde{f}_2\right) = 0, \tag{4.85}$$

which is easily solved, and inserting the solution into the $\left(\tilde{f}_1 + \tilde{f}_2\right)$ equation we find

$$
\begin{aligned}
\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}}\lambda_{12}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y},\\
\tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}}\lambda_{12}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y},\\
\tilde{g}_1 &= -e^{i\tilde{k}\mathcal{Y}}\left(\tan^2\varphi\tanh\mathcal{Y} + i(\tilde{k}-\tilde{\omega})\right),\\
\tilde{g}_2 &= \quad e^{i\tilde{k}\mathcal{Y}}\left(\tan^2\varphi\tanh\mathcal{Y} + i(\tilde{k}+\tilde{\omega})\right),\\
\tilde{k} &= \pm\sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi\tan^2\varphi\tilde{\omega} - \tan^4\varphi},
\end{aligned}
\tag{4.86}
$$

and

$$
\begin{aligned}
\tilde{f}_1 &= e^{i\tilde{k}\mathcal{Y}}\left(\sec^2\varphi\ \tanh\mathcal{Y} - i(\tilde{k}-\tilde{\omega})\right),\\
\tilde{f}_2 &= e^{i\tilde{k}\mathcal{Y}}\left(\sec^2\varphi\tanh\mathcal{Y} - i(\tilde{k}+\tilde{\omega})\right),\\
\tilde{g}_1 &= \quad e^{i\tilde{k}\mathcal{Y}}\lambda_{12}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y},\\
\tilde{g}_2 &= -e^{i\tilde{k}\mathcal{Y}}\lambda_{12}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y},\\
\tilde{k} &= \pm\sqrt{\tilde{\omega}^2 + 2\lambda_{12}\xi\sec^2\varphi\,\tilde{\omega} - \sec^4\varphi}.
\end{aligned}
\tag{4.87}
$$

Dispersion relation. The observant reader might have already noted that all of the these solutions come with a plane-wave factor $e^{i\tilde{k}\mathcal{Y}-i\tilde{\omega}\mathcal{S}}$, satisfying

$$
\tilde{k}^2 = \tilde{\omega}^2 \pm 2\xi(\sec^2\varphi\, m)\,\tilde{\omega} - (\sec^2\varphi\, m)^2, \tag{4.88}
$$

with masses $m = 0, \cos^2\varphi, \sin^2\varphi$, and 1. This is not quite the expected dispersion relation, and there are two reasons why. Firstly, $(\mathcal{S},\mathcal{Y})$ are scaled versions of the boosted worldsheet coordinates $(\mathcal{T},\mathcal{X})$, but more importantly, the dispersion relation (4.46) is not relativistically invariant. We therefore need to rewrite the fermion fluctuations in the form

$$
e^{i\tilde{k}\mathcal{Y}-i\tilde{\omega}\mathcal{S}}\vartheta(\mathcal{Y}) = e^{i(\tilde{k}+\alpha)\mathcal{Y}-i\tilde{\omega}\mathcal{S}}e^{-i\alpha\mathcal{Y}}\vartheta(\mathcal{Y}) = e^{ikx-i\omega t}e^{-i\alpha\mathcal{Y}}\vartheta(\mathcal{Y}), \tag{4.89}
$$

where α will be necessary to match (4.46). From (4.53) it follows that

$$
\tilde{k} = \frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}(k-u\omega) - \alpha, \qquad \tilde{\omega} = \frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}(\omega-uk), \tag{4.90}
$$

and substituting these into (4.88) we get the expected relation

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2, \tag{4.91}$$

provided that

$$\alpha = \frac{\sec^2\varphi}{\sqrt{\tilde{q}^2 - u^2}} \lambda_{12} q\, m. \tag{4.92}$$

Using this transformation we can parametrize the fluctuations by their wavenumber k, and we find that for a given wavenumber there are two positive frequency, and two negative frequency solutions of each mass, $m = 0, \cos^2\varphi, \sin^2\varphi$, and 1. Further defining

$$\hat{w}_\pm = \frac{1}{2} \arctan\left(\frac{Q_\pm \tanh \mathcal{Y}}{\sqrt{1 - Q_\pm^2}} \right), \tag{4.93}$$

we collect these solutions below.

Fermion fluctuations with $m = 0$. The massless perturbations are somewhat special, with the positive and negative frequency solutions exciting only one of the two spinors Ψ^J. Writing the solutions as

$$\Psi^J = e^{ikx - i\omega t} \hat{g}_J(\mathcal{Y})\ \mathcal{K}_J(\lambda) V_\lambda, \tag{4.94}$$

the positive and negative frequency fluctuations are

$$\begin{aligned} \hat{g}_2 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2 - u^2}}\mathcal{Y}} e^{i\lambda \hat{w}_-}, \quad \hat{g}_1 = 0, \qquad \omega = +k, \\ \hat{g}_1 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2 - u^2}}\mathcal{Y}} e^{i\lambda \hat{w}_+}, \quad \hat{g}_2 = 0, \qquad \omega = -k, \end{aligned} \tag{4.95}$$

and the eigenvalues of the (k-dependent) constant Weyl spinor V_λ under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $+i, i\lambda, i\lambda$ and $+1$, respectively.

Fermion fluctuations with $m = \cos^2\varphi$. These solutions live on the same subspace as the normalizable zero modes ($\lambda_P = -1$, $\lambda_{12}\lambda_{68} = 1$) and are given by

$$\Psi^J = e^{ikx - i\omega t} \hat{f}_J(\mathcal{Y})\ \mathcal{K}_J(-\lambda) U_\lambda, \tag{4.96}$$

where

$$
\begin{aligned}
\hat{f}_1 &= \frac{1}{\sqrt{1+u}}\left(\tanh\mathcal{Y} - i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1+u)(k-\omega) - \lambda q\cos^2\varphi\right)\right)\times \\
&\qquad e^{-\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{-i\lambda\hat{w}_+}, \\
\hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}}\left(\tanh\mathcal{Y} - i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1-u)(k+\omega) - \lambda q\cos^2\varphi\right)\right)\times \\
&\qquad e^{-\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{-i\lambda\hat{w}_-}, \\
w &= \pm\sqrt{(\cos^2\varphi - \lambda qk)^2 + \tilde{q}^2k^2},
\end{aligned}
\tag{4.97}
$$

and the (k-dependent) constant Weyl spinor U_λ has eigenvalues $+i, i\lambda, i\lambda$ and -1 under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$, respectively.

Fermion fluctuations with $m = \sin^2\varphi$. These fluctuations live on the $\hat{\Gamma}^{1268} = 1$ subspace, and do not have a definite chirality under $P_\pm$

$$
\Psi^J = e^{ikx-i\omega t}\left(\hat{f}_J(\mathcal{Y})\ \mathcal{K}_J(-\lambda) + \hat{g}_J(\mathcal{Y})\ \mathcal{K}_J(\lambda)\hat{\Gamma}_{07}\right)W_\lambda, \tag{4.98}
$$

$$
\begin{aligned}
\hat{f}_1 &= \frac{1}{\sqrt{1+u}}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y}\ e^{\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{-i\lambda\hat{w}_+}, \\
\hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y}\ e^{\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{-i\lambda\hat{w}_-} \\
\hat{g}_1 &= \frac{i}{\sqrt{1+u}}\left(\tan^2\varphi\ \tanh\mathcal{Y} + i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1+u)(k-\omega) - \lambda q\sin^2\varphi\right)\right)\times \\
&\qquad e^{\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{i\lambda\hat{w}_+}, \\
\hat{g}_2 &= \frac{-i\lambda}{\sqrt{1-u}}\left(\tan^2\varphi\ \tanh\mathcal{Y} + i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1-u)(k+\omega) - \lambda q\sin^2\varphi\right)\right)\times \\
&\qquad e^{\frac{1}{2}\frac{i\,\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}}\, e^{i\lambda\hat{w}_-}, \\
\omega &= \pm\sqrt{(\sin^2\varphi - \lambda qk)^2 + \tilde{q}^2k^2},
\end{aligned}
\tag{4.99}
$$

and the eigenvalues of the (k-dependent) constant Weyl spinor W_λ under $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ are $+i, i\lambda, -i\lambda$ and -1, respectively.

Fermion fluctuations with $m = 1$. Finally, the heaviest fermions are

$$\Psi^J = e^{ikx-i\omega t}\left(\hat{f}_J(\mathcal{Y})\ \mathcal{K}_J(-\lambda) + \hat{g}_J(\mathcal{Y})\ \mathcal{K}_J(\lambda)\hat{\Gamma}_{07}\right) W_\lambda, \tag{4.100}$$

with

$$\begin{aligned}
\hat{f}_1 &= \frac{1}{\sqrt{1+u}}\left(\sec^2\varphi\ \tanh\mathcal{Y} - i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1+u)(k-\omega)-\lambda q\right)\right)\times \\
&\qquad e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}_+}, \\
\hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}}\left(\sec^2\varphi\ \tanh\mathcal{Y} - i\frac{\sec^2\varphi}{\sqrt{\tilde{q}^2-u^2}}\left((1-u)(k+\omega)-\lambda q\right)\right)\times \\
&\qquad e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}_-} \\
\hat{g}_1 &= \frac{-i}{\sqrt{1+u}}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y}\ e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}_+}, \\
\hat{g}_2 &= \frac{i\lambda}{\sqrt{1-u}}\tan\varphi\sec\varphi\ \mathrm{sech}\mathcal{Y}\ e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}_-}, \\
\omega &= \pm\sqrt{(1-\lambda q k)^2 + \tilde{q}^2 k^2},
\end{aligned} \tag{4.101}$$

and the constant spinor W_λ satisfies the same conditions as for $m = \sin^2\varphi$.

Majorana condition. In a Majorana basis $(\hat{\Gamma}^A)^* = -\hat{\Gamma}^A$ and the Majorana condition is $(\Psi^J)^* = \Psi^J$. To impose this condition we need to consider linear combinations of two solutions (from the same mass group) such that the wavenumbers are k and $-k$, the frequencies are of opposite sign (apart from the massless case), and so are the λ eigenvalues. Noting that the dispersion relation is invariant under $(k \to -k, \lambda \to -\lambda)$, and

$$\mathcal{K}_1(\lambda)^* = -\lambda\mathcal{K}_1(-\lambda)\hat{\Gamma}_{45}, \quad \mathcal{K}_2(\lambda)^* = \lambda\mathcal{K}_2(-\lambda)\hat{\Gamma}_{45}, \tag{4.102}$$

it follows that $(\Psi^J)^* = \Psi^J$ will simply relate the constant spinor multipliers of the two components. We show explicitly how to construct solutions satisfying the Majorana condition in the massless case. Analogous expressions for the massive modes can also be found, but these are quite lengthy. Since they do not play any role in the subsequent analysis we do not write them explicitly here. We start with the linear combination

$$\begin{aligned}\Psi^1 &= e^{+ik(x+t)} e^{+\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\hat{w}_+}\ \mathcal{K}_1(+1)V_+^1 \\ &\quad + e^{-ik(x+t)} e^{-\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\hat{w}_+}\ \mathcal{K}_1(-1)V_-^2,\end{aligned} \tag{4.103}$$

where the two components have opposite k, ω, λ. Its complex conjugate is

$$\begin{aligned}(\Psi^1)^* &= -e^{-ik(x+t)} e^{-\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\hat{w}_+}\ \mathcal{K}_1(-1)\hat{\Gamma}_{45}(V_+^1)^* \\ &\quad + e^{+ik(x+t)} e^{\frac{1}{2}\frac{iq}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{+i\hat{w}_+}\ \mathcal{K}_1(+1)\hat{\Gamma}_{45}(V_-^2)^*,\end{aligned} \tag{4.104}$$

and $(\Psi^J)^* = \Psi^J$ as long as

$$\hat{\Gamma}_{45}(V_+^1)^* = -V_-^2 \quad \text{and} \quad \hat{\Gamma}_{45}(V_-^2)^* = V_+^1. \tag{4.105}$$

These two conditions are equivalent, and consistent with the $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}, \hat{\Gamma}^{68}$ and $\hat{\Gamma}^{012345}$ eigenvalues of V_+^1 and V_-^2. We have found an explicit Majorana solution.

Solutions for $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$. Again, this geometry corresponds to the $\varphi \to 0$ limit, the reduced equations (4.77) decouple for the $P_\pm$ subspaces, and all of the solutions are the same form as the $p_{1268} = 0$ fluctuations above. In particular, we have four massless fermions

$$\begin{aligned}\Psi^J &= e^{ikx-i\omega t}\hat{g}_J(\mathcal{Y})\ \mathcal{K}_J(\lambda)V_\lambda, \\ \hat{g}_2 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}_-}, \quad \hat{g}_1 = 0, \qquad \omega = +k, \\ \hat{g}_1 &= e^{\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{i\lambda\hat{w}_+}, \quad \hat{g}_2 = 0, \qquad \omega = -k,\end{aligned} \tag{4.106}$$

and four massive fermions

$$\begin{aligned}\Psi^J &= e^{ikx-i\omega t}\hat{f}_J(\mathcal{Y})\ \mathcal{K}_J(-\lambda)U_\lambda, \qquad \omega = \pm\sqrt{(1-\lambda qk)^2 + \tilde{q}^2k^2}, \\ \hat{f}_1 &= \frac{1}{\sqrt{1+u}}\Big(\tanh\mathcal{Y} - i\frac{1}{\sqrt{\tilde{q}^2-u^2}}\left((1+u)(k-\omega)-\lambda q\right)\Big) e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}_+}, \\ \hat{f}_2 &= \frac{\lambda}{\sqrt{1-u}}\Big(\tanh\mathcal{Y} - i\frac{1}{\sqrt{\tilde{q}^2-u^2}}\left((1-u)(k+\omega)-\lambda q\right)\Big) e^{-\frac{1}{2}\frac{i\lambda q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{-i\lambda\hat{w}_-},\end{aligned} \tag{4.107}$$

where under the operators $\hat{\Gamma}^{34}, \hat{\Gamma}^{12}$ and $\hat{\Gamma}^{012345}$ the constant spinor V_λ has eigenvalues $+i, i\lambda$ and $+1$, while U_λ has eigenvalues $+i, i\lambda$ and -1, respectively. The difference compared to (4.95), (4.97) is that the $\hat{\Gamma}^{68}$ eigenvalues of U_λ, V_λ are no longer constrained.

4.3 The 1-loop functional determinant

Using the fluctuations found in the previous two sections we now calculate the leading order quantum corrections to the energy of the stationary giant magnon. We follow a similar argument in [54], which is based on well-established quantization techniques for solitons [162, 137, 136, 163]. By energy we mean the Noether charge combination $E - J_1$, where E is the conserved charge associated with translations in global AdS_3 time, while J_1 is the U(1) charge associated with rotations along the BMN geodesic. In light-cone gauge, the quantity $E - J_1$ can be identified with the (transverse) Hamiltonian of physical string excitations [140]. In conformal gauge the sigma-model action has to be supplemented by ghosts to cancel two unphysical bosons, however, for the purposes of our semiclassical analysis it is sufficient to simply omit two of the massless bosonic modes, as disucssed in Section 4.1.

A detailed presentation of the mixed-flux AdS_3 magnon can be found in chapter 2, here we just recall that the classical conserved charges satisfy

$$E - J_1 = \sqrt{\left(\cos^2\varphi\, J_2 - \mathrm{h}q\mathrm{p}\right)^2 + 4\mathrm{h}^2\tilde{q}^2 \sin^2 \tfrac{\mathrm{p}}{2}}, \tag{4.108}$$

where $\cos^2\varphi$ is the mass of the magnon and J_2 is its second angular momentum. Remarkably, this classical expression is in agreement with the exact dispersion relation of elementary excitations

$$\epsilon = \sqrt{(m \pm q\mathrm{hp})^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}\ , \tag{4.109}$$

determined from supersymmetry [102, 104, 115], hence we expect no quantum corrections. The one-loop correction to the energy can be calculated as the functional determinant $\ln\det|\delta^2 S|$ around the classical background, and is given by

$$\frac{1}{2}\sum_{i,k}(-1)^F \nu_i, \tag{4.110}$$

where F is the fermion number operator, ν_i are the so-called stability angles, frequencies of small oscillations around the classical solution, and the sum is over excitations i and wavenumbers k. For a non-static soliton, like the giant magnon, we can apply the method of Dashen, Hasslacher and Neveu [162] to calculate these stability angles. We put the system in a box of length $L \gg 1$, with periodic boundary conditions $x \cong x + L$. It is clear from the form of the solution (4.10) that the system is also periodic in worldsheet time, with period $T = L/u$. Then, the stability angle ν of a generic fluctuation $\delta\phi$ can be read off from

$$\delta\phi(t + T, x) = e^{-i\nu}\delta\phi(t, x). \tag{4.111}$$

Although we had to write the oscillations in the original worldsheet coordinates (x, t) to get the correct dispersion relations, the magnon's stationary frame $(\mathcal{X}, \mathcal{T})$ is better suited to the analysis of stability angles. In Sections 4.1 and 4.2 we found fluctuations with oscillatory terms

$$e^{ikx-i\omega t} \tag{4.112}$$

parametrized by mass m and an additional eigenvalue $\lambda = \pm 1$, and satisfying dispersion relations

$$\omega = \sqrt{(m - \lambda q k)^2 + \tilde{q}^2 k^2}. \tag{4.113}$$

Rewriting the plane-wave terms as[5]

$$e^{ikx-i\omega t} = e^{i\hat{k}\mathcal{X} - i\hat{\omega}\mathcal{T}} e^{i\lambda q m\gamma\mathcal{X}}, \tag{4.114}$$

the new frequency and wavenumber satisfy

$$\hat{\omega} = -\lambda q u\gamma m + \sqrt{\tilde{q}^2 m^2 + \hat{k}^2}, \tag{4.115}$$

while $e^{i\lambda q m\gamma\mathcal{X}}$ can be absorbed into the rest of the $\mathcal{Y}$-dependent solution.

[5] Note that this is the inverse of the transformation (4.89) that we applied to the fermion fluctuations.

4.3.1 1-loop correction in $\mathbf{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ string theory

For each excitation, the stability angle can be further decomposed as

$$\nu_i(m,\lambda) = \nu_i^{(0)}(m,\lambda) + \nu_i^{(1)}(m,\lambda) + \lambda\nu_i^{(2)}(m), \tag{4.116}$$

where $\nu_i^{(0)}$ comes from the pure plane-wave $e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}$, $\nu_i^{(2)}$ from terms like $e^{i\lambda f(\mathcal{Y})}$ and $\nu_i^{(1)}$ corresponds to the rest. Since we have exactly one boson and one fermion for each of the 8 combinations of (m,λ), and the first terms are the same

$$\nu_{bos}^{(0)}(m,\lambda) = \nu_{ferm}^{(0)}(m,\lambda) = \frac{L}{u}\gamma\left(\hat{\omega} + u\hat{k}\right), \tag{4.117}$$

the contribution form these terms vanishes even before integrating over $\hat{k}$

$$\sum_{m,\lambda} \nu_{bos}^{(0)}(m,\lambda) - \nu_{ferm}^{(0)}(m,\lambda) = 0. \tag{4.118}$$

Furthermore, summing over $\lambda = \pm 1$ pairs of the same excitation the $\nu_i^{(2)}$ terms cancel, leaving us with the total correction

$$\sum_{i,k}(-1)^F \nu_i = \int \mathrm{d}\hat{k} \sum_{m,\lambda}\left(\nu_{bos}^{(1)}(m,\lambda) - \nu_{ferm}^{(1)}(m,\lambda)\right). \tag{4.119}$$

Under the transformation (4.114) we have

$$k = \gamma(\hat{k} + u\hat{\omega}) + \lambda q\gamma^2 m, \qquad \omega = \gamma(\hat{\omega} + u\hat{k}) + \lambda q u\gamma^2 m, \tag{4.120}$$

and it is then straightforward to read off the $\nu_i^{(1)}$ stability angles for the fluctuations in sections 4.1 and 4.2. The excitations with non-zero $\nu_i^{(1)}$ are the two $m = \cos^2\varphi$ bosons (4.41), (4.43) with

$$e^{\nu_{bos}^{(1)}(\cos^2\varphi,\lambda)} = E_{bos}(\cos^2\varphi,\lambda), \tag{4.121}$$

and six massive fermions (4.97), (4.99), (4.101) with

$$\begin{aligned} e^{\nu_{ferm}^{(1)}(\cos^2\varphi,\lambda)} &= E_{ferm}(\cos^2\varphi,\lambda), \\ e^{\nu_{ferm}^{(1)}(\sin^2\varphi,\lambda)} &= 1/E_{ferm}(\sin^2\varphi,\lambda), \\ e^{\nu_{ferm}^{(1)}(1,\lambda)} &= E_{ferm}(1,\lambda), \end{aligned} \tag{4.122}$$

where we have defined

$$E_{bos}(m,\lambda) = \frac{\hat{k} - \frac{q^2 u}{\tilde{q}^2 - u^2}\,\hat{\omega} + \lambda q\gamma m + i\left(\gamma\sqrt{\tilde{q}^2 - u^2}m - \frac{\lambda q}{\sqrt{\tilde{q}^2 - u^2}}\,(\hat{k} + u\hat{\omega})\right)}{\hat{k} - \frac{q^2 u}{\tilde{q}^2 - u^2}\,\hat{\omega} + \lambda q\gamma m - i\left(\gamma\sqrt{\tilde{q}^2 - u^2}m - \frac{\lambda q}{\sqrt{\tilde{q}^2 - u^2}}\,(\hat{k} + u\hat{\omega})\right)},$$

$$E_{ferm}(m,\lambda) = \frac{\hat{k} - \hat{\omega} + i\gamma\sqrt{\tilde{q}^2 - u^2}m}{\hat{k} - \hat{\omega} - i\gamma\sqrt{\tilde{q}^2 - u^2}m}. \tag{4.123}$$

With these, the integrand of (4.119) becomes

$$\sum_{m,\lambda}\left(\nu^{(1)}_{bos}(m,\lambda) - \nu^{(1)}_{ferm}(m,\lambda)\right) = - i\log\left(\prod_{\lambda=\pm 1}\frac{E_{bos}(\cos^2\varphi,\lambda)E_{ferm}(\sin^2\varphi,\lambda)}{E_{ferm}(\cos^2\varphi,\lambda)E_{ferm}(1,\lambda)}\right). \tag{4.124}$$

Since

$$\frac{E_{bos}(m,+1)E_{bos}(m,-1)}{\left(E_{ferm}(m,+1)E_{ferm}(m,-1)\right)^2} = 1 \tag{4.125}$$

holds for general m, (4.119) simplifies to

$$\sum_{i,k}(-1)^F\nu_i = -i\int \mathrm{d}\hat{k}\log\left(\prod_{\lambda=\pm 1}\frac{E_{ferm}(\cos^2\varphi,\lambda)E_{ferm}(\sin^2\varphi,\lambda)}{E_{ferm}(1,\lambda)}\right). \tag{4.126}$$

Further noting that

$$E_{ferm}(m,+1)E_{ferm}(m,-1) = \frac{\hat{k} + i\gamma\sqrt{\tilde{q}^2 - u^2}m}{\hat{k} - i\gamma\sqrt{\tilde{q}^2 - u^2}m} \tag{4.127}$$

it is clear that the integrand is antisymmetric in $\hat{k}$. Moreover, we have the asymptotic expansion around $\hat{k} = \pm\infty$

$$\prod_{\lambda=\pm 1}\frac{E_{ferm}(\cos^2\varphi,\lambda)E_{ferm}(\sin^2\varphi,\lambda)}{E_{ferm}(1,\lambda)} = 1 + \frac{i}{2}\gamma^3(\tilde{q}^2 - u^2)^{3/2}\sin^2 2\varphi\,\frac{1}{\hat{k}^3} + O\!\left(\frac{1}{\hat{k}^5}\right), \tag{4.128}$$

and taking logarithm, the integrand of (4.126) is $O\!\left(\hat{k}^{-3}\right)$, hence the integral itself is bounded and well-defined. We conclude that the integral is zero, and, in agreement with our expectations, the giant magnon energy receives no corrections at one loop, providing another check on our results.

4.3.2 1-loop correction in $\mathbf{AdS_3 \times S^3 \times T^4}$ string theory

On $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ the situation is even simpler. We have two bosons and two fermions for each of the 4 combinations of $m = 0, 1,\ \lambda = \pm 1$. Paring these up, the $\nu_i^{(0)}$ contribution vanishes, while the $\nu_i^{(2)}$ terms cancel between $\lambda = \pm 1$ pairs. With two of the massive bosons and four of the massive fermions contributing, the integrand of (4.119) becomes

$$\begin{aligned}\sum_\lambda \left(\nu^{(1)}_{bos}(1,\lambda) - 2\nu^{(1)}_{ferm}(1,\lambda)\right) = \\ - i \log\left(\frac{E_{bos}(1,+1)E_{bos}(1,-1)}{\left(E_{ferm}(1,+1)E_{ferm}(1,-1)\right)^2}\right),\end{aligned} \tag{4.129}$$

which is the same as the $\varphi \to 0$ limit of (4.124). Using (4.125) we arrive at the expected zero one-loop correction result even before integrating over $\hat{k}$.

4.4 Chapter conclusions and outlook

In this chapter we found the full spectrum of fluctuations around the mixed-flux AdS_3 stationary giant magnon, the $q > 0$ generalisation of the Hofman-Maldacena giant magnon. To obtain the non-trivial bosonic fluctuations, we adapted the method used in [54]. Rather than dressing the vacuum twice to get a complicated breather-soliton superposition (only then to expand in small breather momentum), we dress the perturbed BMN vacuum once, keeping terms up to subleading order throughout the calculation. The leading order term in the dressed solution is the giant magnon, so the subleading term must be its perturbation. The fermionic fluctuations are obtained as solutions of the equations derived from the quadratic fermionic action, using the formalism presented in chapter 3, which builds on the original developments of [53] for AdS_5.

We find that all of the fluctuations can be written in the form

$$e^{ikx - i\omega t} f(x - ut) \tag{4.130}$$

where u is the magnon's speed on the worldsheet (4.13), and k, ω satisfy

$$\omega^2 = (m \pm qk)^2 + \tilde{q}^2 k^2, \tag{4.131}$$

which is the small-momentum limit of the exact dispersion relation (4.1). Furthermore, the fluctuations can be arranged into short multiplets of the residual symmetry algebras, according to mass and chirality ($\pm$ sign in the dispersion relation). On $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ there are four 4 dimensional multiplets of $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ with two bosons and two fermions, while $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ has eight 2 dimensional multiplets of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, with a boson and a fermion each.

Finally, from the explicit form of each fluctuation we read off the so called stability angles, which sum to the one-loop functional determinant. In both of the geometries we were able to show that this one-loop determinant is zero, or in other words, the one-loop correction to the magnon energy vanishes. It is interesting to compare this result with other calculations of the one-loop correction to energies of AdS_3 string states. The expansion of the coupling h around the classical string limit

$$\mathrm{h}(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{4.132}$$

is equivalent to the expansion of the energy (4.1)

$$\epsilon = \epsilon_0 + \frac{4\,\tilde{q}^2\,\mathrm{h}_0^2 \sin^2\frac{\mathrm{p}}{2}}{\mathrm{h}_0 \epsilon_0} c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{4.133}$$

where the subscript 0 refers to the classical (string) values, and we see that our results translate to $c = 0$ for both geometries. The one-loop correction to the giant magnon energy on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ with pure R-R flux was derived in [71] directly from the GS action, and in [123] from the algebraic curve. They both found that the correction is dependent on the chosen regularisation scheme, with two naturally emerging prescriptions: in the *physical* regularisation the cutoff is at the same mode number for all excitations, while in the *new* prescription the cutoff is proportional to the mass of the polarisation. The two prescriptions both give zero correction $c_{phys} = c_{new} = 0$ on the

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background, but differ for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ theory

$$c_{phys} = \frac{\alpha \log \alpha + (1-\alpha)\log(1-\alpha)}{2\pi}, \qquad c_{new} = 0. \tag{4.134}$$

For the mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background the direct GS action calculation [124] shows that there is no one-loop correction, $c = 0$, and the same conclusion can be drawn by considering the worldsheet scattering of giant magnons [120]. Our results are in agreement with the *new* prescription, although it is not clear that we work in either of the regularisation schemes, as in (4.126) we have an implicit cutoff[6] on the mode numbers $\hat{k}$ in the magnon's frame.

There are a number of interesting directions for future research. The above calculations only apply to $q < 1$, and one could extend these results to the pure NS-NS backgrounds, although our understanding of the solitons of the $q = 1$ theory is somewhat limited. One could also, instead of the infinite-spin giant magnon, consider the finite-size magnon [169, 170] as the bosonic background and attempt a similar fluctuation analysis.

[6] The integrals should be computed separately for each mass before summing, instead we first sum, then compute the integral, which is equivalent to having the same cutoff on $\hat{k}$ for each mass.

Chapter 5

Fermion zero modes for other AdS_3 classical strings

The semiclassical analysis of chapters 3 and 4 focused exclusively on the mixed-flux $\mathbb{R} \times S^3 \times S^3$ magnon, and although the giant magnon is, arguably, the most important string soliton in any AdS string theory, it is instructive to perform the same calculations for other solutions. In this chapter we explicitly construct the fermion zero modes for two more classical strings of the mixed flux AdS_3 backgrounds, both of which we described in chapter 2. The methods developed in chapter 3 are directly applicable, although with a few key differences.

In section 5.1 we take the mixed-flux $AdS_3 \times \mathbb{R}$ soliton (2.323) as the bosonic background. Stretching to the boundary of AdS_3, this string has infinite target space length, and we have to be careful when removing the resulting UV divergences. Once these are taken care of, we find the same number of zero modes as for the magnon (4 and 2 on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ respectively), implying that the $AdS_3 \times \mathbb{R}$ soliton also transforms in a short multiplet of the residual algebra, although the particle interpretation in this case is not so straightforward.

In section 5.2 we obtain the fermion zero modes of the $AdS_3 \times S^3 \times S^3 \times S^1$ double magnon (2.288), which is a simple (and tractable) special case of the more general scattering state of two magnons, one on each S^3. We find 4 zero modes, twice as many as for the magnon, in line with the fact that scattering states do not transform in short multiplets, or alternatively, that the double magnon breaks all residual supersymmetries of the BMN vacuum. Concluding remarks are in section 5.3, and some of the more technical details are presented in the appendices. The contents of this chapter have not been published.

5.1 Fermion zero modes for the mixed-flux $\mathrm{AdS}_3 \times \mathbb{R}$ soliton

In this section we find the fermion zero modes for the stationary mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^1$ soliton (2.318)

$$\begin{aligned} &\rho = \operatorname{arcsinh}\left(\frac{\operatorname{csch}\mathcal{Y}}{\sqrt{1+b^2}}\right), \\ &T = t + \arctan\left(b^{-1}\coth\mathcal{Y}\right), \qquad \psi = -\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2-u^2}}, \end{aligned} \tag{5.1}$$

$$\phi_1^+ = \cos^2\!\varphi\; t, \quad \phi_1^- = \sin^2\!\varphi\; t,$$

$$\theta^\pm = \frac{\pi}{2}, \quad \phi_2^\pm = 0, \qquad b = \frac{u}{\sqrt{\tilde{q}^2-u^2}}.$$

where

$$\mathcal{Y} = \gamma\sqrt{\tilde{q}^2-u^2}\,\mathcal{X}, \tag{5.2}$$

with boosted worldsheet coordinates

$$\mathcal{X} = \gamma(x-ut), \qquad \mathcal{T} = \gamma(t-ux), \tag{5.3}$$

and the parameters are related to the worldsheet momentum p via

$$u = -\tilde{q}\cos\tfrac{\mathrm{p}}{2}. \tag{5.4}$$

A detailed classical analysis of this solution is presented in section 2.2.5.

5.1.1 Fermion zero mode equations

The quadratic action of fermion perturbations, as presented at the beginning of section 3.1.1, is valid for any mixed-flux $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ bosonic string background, and to avoid repeating ourselves, here we just write down the equations of motion (3.15)

$$\begin{aligned} (\rho_0+\rho_1)\left[(\mathrm{D}_1-\mathrm{D}_0)\,\vartheta^1 - \frac{1}{48}\not{F}(\rho_0-\rho_1)\,\vartheta^2 - \frac{1}{8}(\not{H}_0-\not{H}_1)\,\vartheta^1\right] = 0\ , \\ (\rho_0-\rho_1)\left[(\mathrm{D}_1+\mathrm{D}_0)\,\vartheta^2 + \frac{1}{48}\not{F}(\rho_0+\rho_1)\,\vartheta^1 - \frac{1}{8}(\not{H}_0+\not{H}_1)\,\vartheta^2\right] = 0\ . \end{aligned} \tag{5.5}$$

The pull-back of the Dirac matrices $\rho_a = e_a^A \Gamma_A$ and the covariant derivatives $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$ are given in terms of the pull-back of the vielbein and spin-connection to the worldsheet

$$e_a^A \equiv \partial_a X^\mu E_\mu^A(X), \qquad \omega_a^{AB} \equiv \partial_a X^\mu \omega_\mu^{AB}. \tag{5.6}$$

These expressions need be evaluated on the classical solution (5.1), which has non-constant components for $\mu = T, \rho, \psi, \phi_1^+, \phi_1^-$ corresponding to the tangent space indices $A = 0, 1, 2, 4, 7$, respectively. The spacetime vielbein E_μ^A and spin-connection ω_μ^{AB} can be found in section 2.2.2, and the pull-backs e_a^A, ω_a^{AB} for this specific classical background are presented in appendix M. Changing variables to the scaled and boosted worldsheet coordinates (5.2)

$$\mathcal{Y} = \zeta\mathcal{X}, \quad \mathcal{S} = \zeta\mathcal{T}, \qquad \zeta = \gamma\sqrt{\tilde{q}^2 - u^2}, \tag{5.7}$$

the equations can be written as

$$\begin{aligned} &(\rho_0 + \rho_1)\left[\zeta(1+u)\gamma(D - \partial_{\mathcal{S}})\,\vartheta^1 + \mathcal{O}\vartheta^2\right] = 0\ , \\ &(\rho_0 - \rho_1)\left[\zeta(1-u)\gamma(\tilde{D} + \partial_{\mathcal{S}})\,\vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\right] = 0\ , \end{aligned} \tag{5.8}$$

where the mixing terms are

$$\mathcal{O} = -\frac{1}{48}\not{F}(\rho_0 - \rho_1)\ , \qquad \tilde{\mathcal{O}} = \frac{1}{48}\not{F}(\rho_0 + \rho_1)\ , \tag{5.9}$$

and we introduced the fermion derivatives

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\,\Gamma_{01} + \frac{1}{2}Q\,\Gamma_{12} - \frac{(1-u)\gamma}{48\zeta}\left(\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1)\not{H}\right), \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\,\Gamma_{01} + \frac{1}{2}Q\,\Gamma_{12} - \frac{(1+u)\gamma}{48\zeta}\left(\not{H}(\rho_0 + \rho_1) + (\rho_0 + \rho_1)\not{H}\right). \end{aligned} \tag{5.10}$$

For a detailed derivation see appendix N, where we also list explicit expressions for $G, \tilde{G}, Q$ in (N.4).

Zero mode condition. One can consider solutions to (5.8) oscillating at various frequencies, but we will focus on the zero-frequency modes, zero modes for short, which are time-independent in the magnon's frame

$$\partial_{\mathcal{S}} \vartheta^J = 0 \ . \tag{5.11}$$

Hence, the zero mode equations are

$$\begin{aligned} &(\rho_0 + \rho_1) \left[\zeta(1+u)\gamma D \, \vartheta^1 + \mathcal{O}\vartheta^2\right] = 0 \ , \\ &(\rho_0 - \rho_1) \left[\zeta(1-u)\gamma \tilde{D} \, \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\right] = 0 \ . \end{aligned} \tag{5.12}$$

A more in-depth analysis of the zero mode condition can be found in chapter 3.

Fixing kappa symmetry

Let us rotate the gamma matrices, aligning them with the BMN geodesic,

$$\hat{\Gamma}^4 = \cos\varphi \Gamma^4 + \sin\varphi \Gamma^7, \quad \hat{\Gamma}^7 = \cos\varphi \Gamma^7 - \sin\varphi \Gamma^4, \tag{5.13}$$

and $\hat{\Gamma}^A = \Gamma^A$ for the rest. With this we have

$$\rho_0 = \hat{\Gamma}_4 + e^0_0 \, \hat{\Gamma}_0 + e^1_0 \, \hat{\Gamma}_1 + e^2_0 \, \hat{\Gamma}_2, \qquad \rho_1 = \hat{e}^0_1 \, \hat{\Gamma}_0 + e^1_1 \, \hat{\Gamma}_1 + e^2_1 \, \hat{\Gamma}_2, \tag{5.14}$$

and just like in chapter 3, the operators in front of the equations (5.12) are nilpotent

$$(\rho_0 + \rho_1)^2 = (\rho_0 - \rho_1)^2 = 0 \ . \tag{5.15}$$

In fact, $(\rho_0 \pm \rho_1)$ both have half maximal rank. To see this first define

$$\bar{\rho}_0 \equiv \hat{\Gamma}_4 - e^0_0 \, \hat{\Gamma}_0 - e^1_0 \, \hat{\Gamma}_1 - e^2_0 \, \hat{\Gamma}_2, \tag{5.16}$$

giving another pair of nilpotent operators $(\bar{\rho}_0 \pm \rho_1)^2 = 0$. However, the difference $(\bar{\rho}_0 - \rho_0)$ between the two sets is non-singular

$$(\bar{\rho}_0 - \rho_0)^2 = -4\tilde{q}^{-2} \left(\zeta^2 \coth^2\mathcal{Y} + q^2 u^2 \gamma^2\right) \ \mathbb{1} \ , \tag{5.17}$$

which can only be the case if the nilpotent operators are exactly half rank.

Kappa-symmetry is a fermionic gauge symmetry of the Green-Schwarz superstring action ensuring spacetime SUSY of the physical spectrum. Half of the fermionic degrees of freedom are projected out in fixing kappa-gauge, making the half-rank $(\rho_0 \pm \rho_1)$ good candidates for this role. In fact, they commute with the corresponding covariant derivatives $[\rho_0+\rho_1, D] = [\rho_0-\rho_1, \tilde{D}] = 0$, and appear in the right mixing operators (5.9), so that the components of ϑ^1 and ϑ^2 projected out by $(\rho_0 + \rho_1)$ and $(\rho_0 - \rho_1)$, respectively, are non-dynamical. Also considering the matrix structure of (5.14), we take the kappa-projectors to be

$$K_1 = \frac{1}{2}\hat{\Gamma}^4(\rho_0 + \rho_1) \ , \qquad K_2 = \frac{1}{2}\hat{\Gamma}^4(\rho_0 - \rho_1) \ , \tag{5.18}$$

satisfying $K_J^2 = K_J$,

$$[K_1, D] = 0 \ , \qquad [K_2, \tilde{D}] = 0 \ , \tag{5.19}$$

$$\mathcal{O} = \mathcal{O}K_2 \ , \qquad \tilde{\mathcal{O}} = \tilde{\mathcal{O}}K_1 \ . \tag{5.20}$$

We can write the zero mode equations (5.12) for the kappa-fixed spinors $\Psi^J = K_J\vartheta^J$ as

$$\begin{aligned} &\zeta(1+u)\gamma D\,\Psi^1 + K_1\mathcal{O}\Psi^2 = 0 \ , \\ &\zeta(1-u)\gamma\tilde{D}\,\Psi^2 + K_2\tilde{\mathcal{O}}\Psi^1 = 0 \ . \end{aligned} \tag{5.21}$$

Zero mode equations

The kappa projectors (5.18) commute with the 6d chirality projector[1]

$$P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \hat{\Gamma}_*\hat{\Gamma}_+\right) \ , \qquad [P_\pm, K_J] = 0 \ . \tag{5.22}$$

We can rewrite the contracted fluxes $\not{F}$, $\not{H}$ (3.13), relating them to this new projector

$$\begin{aligned} \Gamma_* + \cos\varphi\ \Gamma_+ + \sin\varphi\ \Gamma_- &= \hat{\Gamma}_* + \hat{\Gamma}_+ + \sin\varphi\Gamma^7\Gamma^{35}\left(\mathbb{1} + \Gamma^{3568}\right) \\ &= 2\left(\hat{\Gamma}_+\ P_+ - \Delta\ \hat{\Gamma}^{35}\right) , \end{aligned} \tag{5.23}$$

[1] Replacing all gamma matrices with their hatted versions $\hat{\Gamma}_* = \hat{\Gamma}^{012}$, $\hat{\Gamma}_+ = \hat{\Gamma}^{345}$.

where

$$\Delta = -\frac{1}{2}\sin\varphi\left(\mathbb{1} + \hat{\Gamma}^{3568}\right)\Gamma^7 \equiv \Delta_4\,\hat{\Gamma}^4 + \Delta_7\,\hat{\Gamma}^7\ , \tag{5.24}$$

with

$$\Delta_4 = -\frac{1}{2}\sin^2\!\varphi\left(\mathbb{1} + \hat{\Gamma}^{3568}\right)\ , \qquad \Delta_7 = \cot\varphi\,\Delta_4\ . \tag{5.25}$$

Note that we can essentially treat Δ_4 and Δ_7 as scalars, since they commute with the equations of motion.

Recalling the relation (5.17), we define the invertible operator

$$R = \frac{1}{2}\hat{\Gamma}_+(\bar{\rho}_0 - \rho_0) \quad : \quad R^2 = -\tilde{q}^{-2}\left(\zeta^2\coth^2\mathcal{Y} + q^2u^2\gamma^2\right)\,\mathbb{1}\ . \tag{5.26}$$

Then, the fermion derivatives can be written as (see appendix N)

$$\begin{aligned} D &= \partial_\mathcal{Y} + \frac{1}{2}G\,\hat{\Gamma}_{01} + \frac{1}{2}Q\,\hat{\Gamma}_{12} + \frac{q(1-u)\gamma}{\zeta}\left(RP_- - (R - \hat{\Gamma}_{35})\,P_+ + \Delta_4\,\hat{\Gamma}_{35}\right)\ , \\ \tilde{D} &= \partial_\mathcal{Y} + \frac{1}{2}\tilde{G}\,\hat{\Gamma}_{01} + \frac{1}{2}Q\,\hat{\Gamma}_{12} + \frac{q(1+u)\gamma}{\zeta}\left(RP_- - (R - \hat{\Gamma}_{35})\,P_+ + \Delta_4\,\hat{\Gamma}_{35}\right)\ , \end{aligned} \tag{5.27}$$

while, using the relations

$$\Gamma^{35}K_1K_2 = RK_2\ , \qquad \Gamma^{35}K_2K_1 = RK_1\ , \tag{5.28}$$

the mixing terms (5.9) become

$$\begin{aligned} K_1\mathcal{O} &= \tilde{q}\left(RP_- - K_1\Delta\,\hat{\Gamma}_+\right)K_2\ , \\ K_2\tilde{\mathcal{O}} &= -\tilde{q}\left(RP_- - K_2\Delta\,\hat{\Gamma}_+\right)K_1\ . \end{aligned} \tag{5.29}$$

The final form of the kappa-fixed zero mode equations (5.21) is therefore

$$\begin{aligned} &\zeta(1+u)\gamma D\,\Psi^1 + \tilde{q}\left(R\ P_- - K_1\Delta\,\hat{\Gamma}_+\right)\Psi^2 = 0\ , \\ &\zeta(1-u)\gamma\tilde{D}\,\Psi^2 - \tilde{q}\left(R\ P_- - K_2\Delta\,\hat{\Gamma}_+\right)\Psi^1 = 0\ . \end{aligned} \tag{5.30}$$

Equations for $\Delta = 0$. From a technical perspective, the difference between $\Delta = 0$ and $\Delta \neq 0$ is that $P_\pm$ only commutes with the equations for $\Delta = 0$. From a physical point of view $\Delta = 0$ corresponds to either

- $\varphi = 0$, i.e. the $\mathbf{AdS_3 \times S^3 \times T^4}$ **geometry**, or
- "$\hat{\Gamma}^{3568} = -1$" : i.e. the $\mathbf{AdS_3 \times S^3 \times S^3 \times S^1}$ **geometry, with fermions restricted to the -1 eigenspace of $\hat{\Gamma}^{3568}$**.

In these cases the covariant derivatives take the simpler form

$$\begin{aligned} D &= \partial_\mathcal{Y} + \frac{1}{2} G\, \hat{\Gamma}_{01} + \frac{1}{2} Q\, \hat{\Gamma}_{12} + \frac{q(1-u)\gamma}{\zeta} \Big(R P_- - (R - \hat{\Gamma}_{35})\, P_+ \Big) \ , \\ \tilde{D} &= \partial_\mathcal{Y} + \frac{1}{2} \tilde{G}\, \hat{\Gamma}_{01} + \frac{1}{2} Q\, \hat{\Gamma}_{12} + \frac{q(1+u)\gamma}{\zeta} \Big(R P_- - (R - \hat{\Gamma}_{35})\, P_+ \Big) \ , \end{aligned} \tag{5.31}$$

and there equations become

$$\begin{aligned} &\zeta(1+u)\gamma D\ \Psi^1 + \tilde{q}\ R\ P_- \Psi^2 = 0\ , \\ &\zeta(1-u)\gamma \tilde{D}\ \Psi^2 - \tilde{q}\ R\ P_- \Psi^1 = 0\ . \end{aligned} \tag{5.32}$$

Note that these apply in both geometries as long as we impose $\hat{\Gamma}^{3568}\Psi^J = -\Psi^J$ on the solutions in the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ case.

The case of $\Delta \neq 0$. In section 5.1.2 we will show that the $\Delta = 0$ equations give the right number of normalizable zero modes for a short representation, and indeed there are no normalizable solutions at all for $\Delta \neq 0$, as we argue in appendix P. To get some intuition about the importance of Δ, we can look at the fermion fluctuations around the BMN string on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. In appendix E we find that the $m = 1$ fermions live on the eigenspace $\Gamma_{1235} = +1$, $\Gamma_{1268} = +1$. If one accepts that in the BMN limit the zero modes of the $\mathrm{AdS}_3 \times \mathbb{R}$ soliton should converge to these $m = 1$ fermions (although this is less clear than the similar argument for the giant magnon), then it follows that $\Gamma_{3568} = \hat{\Gamma}_{3568} = +1$ for the zero modes, corresponding to $\Delta = 0$.

5.1.2 Mixed-flux fermion zero modes

In this section we find the normalizable solutions for the ($\Delta = 0$) equations (5.32), representing the perturbative zero modes above the $\mathrm{AdS}_3 \times \mathbb{R}$ soliton. Just like in chapter 3, the trick is to reduce the degrees of freedom of the problem by first fixing kappa-gauge with a suitable ansatz. After removing the UV divergences, we also see that the quantized zero modes can be mapped onto the odd generators of the residual algebra.

Fixing kappa-gauge

Due to the nilpotency of $(\rho_0 \pm \rho_1)$ we can parametrize the kappa-projectors (5.18) as

$$\begin{aligned} K_1 &= \frac{1}{2}\Big(\mathbb{1} - \cosh(2\chi)\cosh\upsilon_+\,\hat{\Gamma}_{04} - \sinh(2\chi)\cosh\upsilon_+\,\hat{\Gamma}_{14} + \sinh\upsilon_+\,\hat{\Gamma}_{24}\Big), \\ K_2 &= \frac{1}{2}\Big(\mathbb{1} - \cosh(2\tilde{\chi})\cosh\upsilon_-\,\hat{\Gamma}_{04} - \sinh(2\tilde{\chi})\cosh\upsilon_-\,\hat{\Gamma}_{14} - \sinh\upsilon_-\,\hat{\Gamma}_{24}\Big). \end{aligned} \tag{5.33}$$

Using the exponential identities presented in appendix O, and introducing

$$Q_\pm = \frac{q\sqrt{\tilde{q}^2 - u^2}}{\tilde{q}(1 \pm u)}, \tag{5.34}$$

the parameters are found to be

$$\upsilon_\pm = \operatorname{arcsinh}\left(Q_\pm \operatorname{csch}\mathcal{Y}\right)\ , \tag{5.35}$$

$$\begin{aligned} \chi(\mathcal{Y}) &= \frac{1}{2}\left[\operatorname{arctanh}\left(\frac{u\operatorname{sech}\mathcal{Y}}{\tilde{q}}\right) - \operatorname{arccosh}\left(\frac{\coth\mathcal{Y}}{\sqrt{1+Q_+^2\operatorname{csch}^2\mathcal{Y}}}\right)\right]\ , \\ \tilde{\chi}(\mathcal{Y}) &= \frac{1}{2}\left[\operatorname{arctanh}\left(\frac{u\operatorname{sech}\mathcal{Y}}{\tilde{q}}\right) + \operatorname{arccosh}\left(\frac{\coth\mathcal{Y}}{\sqrt{1+Q_-^2\operatorname{csch}^2\mathcal{Y}}}\right)\right]\ . \end{aligned} \tag{5.36}$$

Ansatz. We take the kappa-fixed ansatz to be

$$\Psi^J = \Big(\alpha_+^J(\mathcal{Y}) + \alpha_-^J(\mathcal{Y})\,\hat{\Gamma}_{02}\Big)\, U^J, \tag{5.37}$$

where U^J satisfies

$$\hat{\Gamma}_{01} U^J = +U^J, \qquad \hat{\Gamma}_{35} U^J = i\lambda_{35} U^J, \qquad \hat{\Gamma}_* \hat{\Gamma}_+ U^J = \lambda_P U^J, \tag{5.38}$$

with $\lambda_{35}, \lambda_P \in \{\pm 1\}$. First of all, the definite $\hat{\Gamma}_{35}$ and $\hat{\Gamma}_*\hat{\Gamma}_+$ eigenvalues are motivated by the fact that these operators mutually commute with K_J, so we can take the shared eigenvector U^J as a starting point to the ansatz. Then, to capture the $\mathcal{Y}$-dependent projection of the K_J, we project U down to the $+1$ eigenspace of $\hat{\Gamma}_{01}$, relating it to the complement eigenspace at the level of Ψ via the operator $\hat{\Gamma}_{02}$. Note that there is some freedom in this construction, instead of $\hat{\Gamma}_{01}$ and $\hat{\Gamma}_{02}$ we could have chosen any two anti-commuting operators that do not commute with K_J.

Solution. Substituting the ansatz (5.37) into $K_1 \Psi^1 = \Psi^1$, we get an equation on each eigenspace of $\hat{\Gamma}_{01}$

$$\begin{aligned} i\lambda \sinh \upsilon_+ \, \alpha_+^1 + i\lambda e^{+2\chi} \cosh \upsilon_+ \, \alpha_-^1 &= \alpha_+^1 \ , \\ -i\lambda \sinh \upsilon_+ \, \alpha_-^1 - i\lambda e^{-2\chi} \cosh \upsilon_+ \, \alpha_+^1 &= \alpha_-^1 \ , \end{aligned} \tag{5.39}$$

where $\lambda = \lambda_{35}\lambda_P = \pm 1$. Since the overall scale of eigenvectors are unfixed, both of these equations are for the single variable α_-/α_+. They are consistent, with a symmetric solution

$$\alpha_+^1 = e^{\chi} \sqrt{1 - i\lambda Q_+ \operatorname{csch} \mathcal{Y}} \ , \qquad \alpha_-^1 = i\lambda e^{-\chi} \sqrt{1 + i\lambda Q_+ \operatorname{csch} \mathcal{Y}} \ . \tag{5.40}$$

The same analysis for Ψ^2 gives

$$\alpha_+^2 = e^{\tilde{\chi}} \sqrt{1 + i\lambda Q_+ \operatorname{csch} \mathcal{Y}} \ , \qquad \alpha_-^2 = i\lambda e^{-\tilde{\chi}} \sqrt{1 - i\lambda Q_+ \operatorname{csch} \mathcal{Y}} \ . \tag{5.41}$$

The most general form of the gauge-fixed spinors is therefore

$$
\begin{aligned}
\Psi^1 &= \sum_{\lambda_{35},\lambda_P} \Big(e^{\chi}\sqrt{1 - i\lambda Q_+ \operatorname{csch}\mathcal{Y}} + \\
&\qquad\qquad i\lambda e^{-\chi}\sqrt{1 + i\lambda Q_+ \operatorname{csch}\mathcal{Y}}\,\hat{\Gamma}_{02}\Big)U^1 \ , \\
\Psi^2 &= \sum_{\lambda_{35},\lambda_P} \Big(e^{\tilde{\chi}}\sqrt{1 + i\lambda Q_+ \operatorname{csch}\mathcal{Y}} + \\
&\qquad\qquad i\lambda e^{-\tilde{\chi}}\sqrt{1 - i\lambda Q_+ \operatorname{csch}\mathcal{Y}}\,\hat{\Gamma}_{02}\Big)U^2 \ ,
\end{aligned}
\tag{5.42}
$$

where $\hat{\Gamma}_{01}U^J = +U^J$, $\hat{\Gamma}_{35}U^J = i\lambda_{35}U^J$, and $P_+U^J = \frac{1+\lambda_P}{2}\,U^J$. The derivation above tells us that these are kappa-fixed, and with a simple counting of the degrees of freedom we can convince ourselves that there are no others.

Reduced zero mode equations

Due to the symmetries of (5.32), we can find solutions on smaller invariant subspaces. Starting with 32 complex components in total for the two Weyl-spinors ϑ^1 and ϑ^2 (not worrying about the Majorana condition for now), first we make the kappa-fixed ansatz (5.37) that leaves us with the 16 physical degrees of freedom. Then, the operators $\hat{\Gamma}^{35}$, $\hat{\Gamma}^{68}$ and $\hat{\Gamma}_*\hat{\Gamma}_+$ all mutually commute with the equations, and we can consider solutions on their mutual eigenspaces. Each of these symmetries reduce dimensionality by half, leaving us with a reduced ODE system that only involves 2 complex scalar functions. Therefore the ansatz we make is

$$
\begin{aligned}
\Psi^1 &= \Big(\alpha^1_+ + \alpha^1_-\,\hat{\Gamma}_{02}\Big)\,f_1(\mathcal{Y})\,V, \\
\Psi^2 &= \Big(\alpha^2_+ + \alpha^2_-\,\hat{\Gamma}_{02}\Big)\,f_2(\mathcal{Y})\,V,
\end{aligned}
\tag{5.43}
$$

where the constant spinor V has eigenvalues $i\lambda_{35}, \lambda_P$ and $+1$ under $\hat{\Gamma}_{35}, \hat{\Gamma}_*\hat{\Gamma}_+$ and $\hat{\Gamma}_{01}$, respectively. It turns out, the $\hat{\Gamma}_{68}$ eigenvalue does not enter into the reduced equations for T^4, while on the S^1 geometry it is fixed in terms of λ_{35}, since we have $\hat{\Gamma}_{3568} = -1$ for $\Delta = 0$. Substituting (5.43) into (5.32), and after a considerable amount of simplification, using identities like the ones in appendix O, we get

$$\begin{aligned}
&\left(\alpha_+^1 + \alpha_-^1\,\hat{\Gamma}_{02}\right)\left[\left(\partial_{\mathcal{Y}} + C_{11}\right) f_1 + C_{12} f_2\right] V = 0\ , \\
&\left(\alpha_+^2 + \alpha_-^2\,\hat{\Gamma}_{02}\right)\left[\left(\partial_{\mathcal{Y}} + C_{22}\right) f_2 + C_{21} f_1\right] V = 0\ ,
\end{aligned} \tag{5.44}$$

with

$$\begin{aligned}
C_{11} &= \frac{i}{2}\frac{\lambda_{35} q\left(1-(1-\lambda_P)u\right)}{\sqrt{\tilde{q}^2-u^2}} - \frac{i}{2}\frac{\lambda_{35}\lambda_P\, Q_+\sqrt{1-Q_+^2}}{\sinh^2\mathcal{Y}+Q_+^2}\ , \\
C_{22} &= \frac{i}{2}\frac{\lambda_{35} q\left(1+(1-\lambda_P)u\right)}{\sqrt{\tilde{q}^2-u^2}} - \frac{i}{2}\frac{\lambda_{35}\lambda_P\, Q_-\sqrt{1-Q_-^2}}{\sinh^2\mathcal{Y}+Q_-^2}\ ,
\end{aligned} \tag{5.45}$$

$$\begin{aligned}
C_{12} &= i\lambda_{35}\frac{1-\lambda_P}{2}\,(1-u)\gamma\; e^{\int(C_{22}-C_{11})\mathrm{d}\mathcal{Y}}\; e^{-2i\xi\mathcal{Y}}\left(\coth\mathcal{Y}+i\xi\right), \\
C_{21} &= -i\lambda_{35}\frac{1-\lambda_P}{2}\,(1+u)\gamma\; e^{\int(C_{11}-C_{22})\mathrm{d}\mathcal{Y}}\; e^{+2i\xi\mathcal{Y}}\left(\coth\mathcal{Y}-i\xi\right),
\end{aligned} \tag{5.46}$$

where we also defined

$$\xi = \frac{\lambda_{35}\, qu}{\sqrt{\tilde{q}^2-u^2}}\ . \tag{5.47}$$

Note the kappa-fixing factors $(\alpha_+^J + \alpha_-^J\,\hat{\Gamma}_{02})$ multiplying (5.44) from the left, once again confirming that kappa-projection commutes with the equations of motion. We have already written the mixing terms C_{12}, C_{21} in a form that lends itself to the substitution

$$\begin{aligned}
f_1 &= \frac{1}{\sqrt{1-u}}e^{-\int C_{11}\mathrm{d}\mathcal{Y}}\,\tilde{f}_1 = \frac{1}{\sqrt{1+u}}e^{-\frac{i\lambda_{35}q(1-(1-\lambda_P)u)}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}+\frac{i}{2}\lambda_{35}\lambda_P\,\mathrm{arccot}\left(\frac{Q_+\coth\mathcal{Y}}{\sqrt{1-Q_+^2}}\right)}\tilde{f}_1\ , \\
f_2 &= \frac{-i\lambda_{35}}{\sqrt{1-u}}e^{-\int C_{22}\mathrm{d}\mathcal{Y}}\,\tilde{f}_2 = \frac{-i\lambda_{35}}{\sqrt{1-u}}e^{-\frac{i\lambda_{35}q(1+(1-\lambda_P)u)}{2\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}+\frac{i}{2}\lambda_{35}\lambda_P\,\mathrm{arccot}\left(\frac{Q_-\coth\mathcal{Y}}{\sqrt{1-Q_-^2}}\right)}\tilde{f}_2\ ,
\end{aligned} \tag{5.48}$$

upon which the equations in brackets (5.44) reduce to

$$\begin{aligned}
\partial_{\mathcal{Y}}\tilde{f}_1 &+ \frac{1-\lambda_P}{2}\,e^{-2i\xi\mathcal{Y}}\left(\coth\mathcal{Y}+i\xi\right)\tilde{f}_2 = 0\ , \\
\partial_{\mathcal{Y}}\tilde{f}_2 &+ \frac{1-\lambda_P}{2}\,e^{+2i\xi\mathcal{Y}}\left(\coth\mathcal{Y}-i\xi\right)\tilde{f}_1 = 0\ .
\end{aligned} \tag{5.49}$$

Solutions on the $\mathbf{P}_+$ subspace. On the P_+ subspace ($\lambda_P = +1$) the two spinors decouple, and at the level of the reduced equations (5.49) the solution is simply

$$\tilde{f}_1 = c_1, \qquad \tilde{f}_2 = c_2, \tag{5.50}$$

for some constants c_i. With this, the spinors (5.43) are not normalizable, and we can not interpret them as perturbative zero modes.

Solutions on the $\mathbf{P}_-$ subspace. Setting $\lambda_P = -1$ in (5.49) we can invert the first equation for $\tilde{f}_2$, and substitute into the second equation to get the second-order ODE

$$\partial_{\mathcal{Y}}^2 \tilde{f}_1 + \left(2i\xi + \frac{\operatorname{csch}^2 \mathcal{Y}}{\coth \mathcal{Y} + i\xi}\right) \partial_{\mathcal{Y}} \tilde{f}_1 + \left(\coth^2 \mathcal{Y} + \xi^2\right) \tilde{f}_1 = 0, \tag{5.51}$$

with solutions

$$\tilde{f}_1 = \Big(c_1 \operatorname{csch} \mathcal{Y} + c_2 (\sinh \mathcal{Y} + i\xi \cosh \mathcal{Y} - i\xi\, \mathcal{Y} \operatorname{csch} \mathcal{Y})\Big)\, e^{-i\xi \mathcal{Y}}. \tag{5.52}$$

Picking the solution that is normalizable as $\mathcal{Y} \to \pm\infty$, we have

$$\tilde{f}_1 = \operatorname{csch} \mathcal{Y}\; e^{-i\xi\mathcal{Y}}, \qquad \tilde{f}_2 = \operatorname{csch} \mathcal{Y}\; e^{+i\xi\mathcal{Y}}. \tag{5.53}$$

and the kappa-fixed normalizable fermion zero modes are given by

$$\begin{aligned} \Psi^1 &= \sum_{\lambda_{35}=\pm} \qquad \frac{\operatorname{csch} \mathcal{Y}}{4\sqrt{1+u}}\; e^{i\omega_+} \left(\alpha_+^1 + \alpha_-^1\, \hat{\Gamma}_{02}\right) V_{\lambda_{35}}\ , \\ \Psi^2 &= \sum_{\lambda_{35}=\pm} -i\lambda_{35} \frac{\operatorname{csch} \mathcal{Y}}{4\sqrt{1-u}}\; e^{i\omega_-} \left(\alpha_+^2 + \alpha_-^2\, \hat{\Gamma}_{02}\right) V_{\lambda_{35}}\ , \end{aligned} \tag{5.54}$$

where

$$\omega_\pm = -\frac{q\, \lambda_{35} \mathcal{Y}}{2\sqrt{\tilde{q}^2 - u^2}} - \frac{\lambda_{35}}{2} \operatorname{arccot}\left(\frac{Q_\pm \coth \mathcal{Y}}{\sqrt{1 - Q_\pm^2}}\right)\ , \tag{5.55}$$

the constant MW spinors $V_\pm$ satisfy $P_- V_\pm = V_\pm$, $\hat{\Gamma}_{01} V_\pm = V_\pm$, $\hat{\Gamma}_{35} V_\pm = \pm i V_\pm$, and the factors of 4 have been introduced for later convenience.

Counting the zero modes. The degrees of freedom of the normalizable zero modes are encoded in the constant spinor $V = V_+ + V_-$, which has eigenvalues -1 and $+1$ under $\hat{\Gamma}_*\hat{\Gamma}_+$ and $\hat{\Gamma}_{01}$, respectively. These projections leave 4 of the 16 real components of an unconstrained 10-d MW spinor. Recalling that we further need to impose $\hat{\Gamma}_{3568}V = -V$ for the S^1 case, we conclude that there are 4 and 2 normalizable fermion zero modes for the $\mathrm{AdS}_3 \times \mathrm{S}^1(\times \mathrm{S}^1)$ soliton on the $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ and $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ backgrounds, respectively.

Zero mode action

We can rewrite the fermion Lagrangian (3.5), similarly to (5.8), as

$$\mathcal{L}_{\mathrm{F}} = i\,\bar{\vartheta}^1(\rho_0+\rho_1)(\mathcal{D}_0-\mathcal{D}_1)\vartheta^1 + i\,\bar{\vartheta}^2(\rho_0-\rho_1)(\mathcal{D}_0+\mathcal{D}_1)\vartheta^2\ , \tag{5.56}$$

$$\begin{aligned} &= -i\,\bar{\vartheta}^1(\rho_0+\rho_1)\Big(\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\vartheta^1 + \mathcal{O}\vartheta^2\Big) \\ &\quad + i\,\bar{\vartheta}^2(\rho_0-\rho_1)\Big(\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\Big)\ . \end{aligned} \tag{5.57}$$

In a basis where the (rotated) gamma matrices have definite hermiticity, the kappa-projectors satisfy $\hat{\Gamma}_{04}K_J = K_J^\dagger\hat{\Gamma}_{04}$. Furthermore, the Hermitian conjugate intertwiner is $\hat{\Gamma}^0$, hence $\bar{\vartheta} = \vartheta^\dagger\,\hat{\Gamma}^0$. We then have

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} = &-2i\,(\Psi^1)^\dagger\,\hat{\Gamma}_{04}\Big(\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\ \Psi^1 + \mathcal{O}\Psi^2\Big) \\ &+ 2i\,(\Psi^2)^\dagger\,\hat{\Gamma}_{04}\Big(\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\ \Psi^2 + \tilde{\mathcal{O}}\Psi^1\Big)\ . \end{aligned} \tag{5.58}$$

Letting $V = V_+ + V_-$ depend on $\mathcal{T}$, and substituting the zero modes (5.54) into the above Lagrangian, we get

$$\mathcal{L}_{\mathrm{F},0} = 2i(1+u)\gamma\,\Psi^{1^\dagger}\,\hat{\Gamma}_{04}\,\partial_{\mathcal{T}}\Psi^1 + 2i(1-u)\gamma\,\Psi^{2^\dagger}\,\hat{\Gamma}_{04}\,\partial_{\mathcal{T}}\Psi^2\ , \tag{5.59}$$

$$= -\frac{i}{2}\gamma\ \mathrm{csch}^2\mathcal{Y}\ V^\dagger\partial_{\mathcal{T}}V\ , \tag{5.60}$$

Integrating over $\mathcal{X}$, we run into the same UV divergences that we mentioned during the classical analysis in section 2.2.5. We can regularize by changing variables to $z = \cosh\rho$

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{d}\mathcal{X}\ \ \mathrm{csch}^2\mathcal{Y} &= 2\int_1^{\infty}\mathrm{d}z\left(\frac{\mathrm{d}z}{\mathrm{d}\mathcal{X}}\right)^{-1}\mathrm{csch}^2\mathcal{Y} \\ &= 2\,\zeta^{-1}\sqrt{1+b^2}\int_1^{\infty}\mathrm{d}z\frac{z}{\sqrt{z^2-z_0^2}}, \end{aligned} \tag{5.61}$$

where $z_0 = b/\sqrt{1+b^2}$, and applying a simple cutoff so that the integral evaluates to

$$2\,\zeta^{-1}\sqrt{1+b^2}\left(\Lambda - \sqrt{1-z_0^2}\right). \tag{5.62}$$

Removing the divergent term we get

$$\left(\int_{-\infty}^{\infty} \mathrm{d}\mathcal{X}\ \operatorname{csch}^2\mathcal{Y}\right)_{\mathrm{reg}} = -2\,\zeta^{-1}, \tag{5.63}$$

and the regularized zero mode action is

$$\left(S_{\mathrm{F},0}\right)_{\mathrm{reg}} = \ \mathrm{h}\tilde{\gamma}\int \mathrm{d}\mathcal{T}\left(i\,V^{\dagger}\partial_{\mathcal{T}}V\right)\,, \tag{5.64}$$

with

$$\tilde{\gamma} = \frac{\gamma}{\zeta} = \frac{1}{\sqrt{\tilde{q}^2 - u^2}}\ . \tag{5.65}$$

To apply the Majorana condition, let us consider a Majorana basis, where all gamma matrices are purely imaginary $\hat{\Gamma}_A^* = -\hat{\Gamma}_A^*$. In such a basis the Majorana-spinors satisfy the reality condition ${\Psi^I}^* = \Psi^I$, which, after examining the solutions (5.54), reduces to

$$V_- = {V_+}^* \quad \Rightarrow \quad V^* = V\ , \tag{5.66}$$

and the zero mode action becomes

$$S_{\mathrm{F},0} = \ \mathrm{h}\tilde{\gamma}\int \mathrm{d}\mathcal{T}\left(i\,V^{T}\partial_{\mathcal{T}}V\right)\,. \tag{5.67}$$

Comparing this action to (3.87), it is clear that the zero mode quantization argument from chapter 3 applies here in the exact same way. In particular, the semiclassically quantized zero modes can be matched to the odd generators of the residual symmetry algebra.

5.2 Fermion zero modes for the double magnon

Lastly, we consider the fermion zero modes of the double magnon, an $\mathbb{R}\times S^3\times S^3$ string on the mixed-flux $AdS_3\times S^3\times S^3\times S^1$ background with equal 3-sphere radii, i.e. $\cos^2\varphi=\sin^2\varphi=1/2$, given in Hopf coordinates by (2.288)

$$
\begin{aligned}
&\theta^+=\theta^-=\arccos\left(\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}}\right) ,\\
&\phi_1^+=\phi_1^-=\frac{1}{2}t+\arctan\left(b^{-1}\tanh\mathcal{Y}\right) , \qquad \phi_2^+=\phi_2^-=-\frac{q\,\mathcal{Y}}{\sqrt{\tilde{q}^2-u^2}} , \qquad (5.68)\\
&\gamma^2=\frac{1}{1-u^2} , \quad b=\frac{u}{\sqrt{\tilde{q}^2-u^2}} , \quad u\in(-\tilde{q},\tilde{q}) ,
\end{aligned}
$$

where

$$
\mathcal{Y}=\frac{1}{2}\gamma\sqrt{\tilde{q}^2-u^2}\mathcal{X}, \qquad \mathcal{X}=\gamma(x-ut), \quad \mathcal{T}=\gamma(t-ux). \tag{5.69}
$$

5.2.1 Fermion zero mode equations

Just like in the previous section, we start with the fermion perturbation equations (3.15) for the ten-dimensional Majorana-Weyl spinors ϑ^1,ϑ^2

$$
\begin{aligned}
&(\rho_0+\rho_1)\left[(D_1-D_0)\,\vartheta^1-\frac{1}{48}\not{F}(\rho_0-\rho_1)\,\vartheta^2-\frac{1}{8}(\not{H}_0-\not{H}_1)\,\vartheta^1\right]=0 ,\\
&(\rho_0-\rho_1)\left[(D_1+D_0)\,\vartheta^2+\frac{1}{48}\not{F}(\rho_0+\rho_1)\,\vartheta^1-\frac{1}{8}(\not{H}_0+\not{H}_1)\,\vartheta^2\right]=0 .
\end{aligned} \tag{5.70}
$$

The worldsheet Dirac matrices and covariant derivatives

$$
\begin{aligned}
\rho_a&=e_a^A\,\Gamma_A, & e_a^A&\equiv\partial_aX^\mu E_\mu^A(X),\\
D_a&=\partial_a+\frac{1}{4}\omega_a^{AB}\Gamma_{AB}, & \omega_a^{AB}&\equiv\partial_aX^\mu\omega_\mu^{AB},
\end{aligned} \tag{5.71}
$$

should be evaluated on the classical solution (5.68), which has non-constant components for $\mu=t,\theta^+,\phi_1^+,\phi_2^+,\theta^-,\phi_1^-,\phi_2^-$, corresponding to the tangent space indices $A=0,3,4,5,6,7,8$. The vielbein and spin-connection were described in section 2.2.2, while their pull-backs e_a^A, ω_a^{AB} for the double magnon can be found in appendix Q. Changing variables to (5.69)

$$\mathcal{Y} = \frac{1}{2}\zeta\mathcal{X}, \quad \mathcal{S} = \frac{1}{2}\zeta\mathcal{T}, \qquad \zeta = \gamma\sqrt{\tilde{q}^2 - u^2}, \tag{5.72}$$

the equations become

$$\begin{aligned} &(\rho_0+\rho_1)\left[\zeta(1+u)\gamma(D-\partial_{\mathcal{S}})\ \vartheta^1 + \mathcal{O}\vartheta^2\right] = 0\ , \\ &(\rho_0-\rho_1)\left[\zeta(1-u)\gamma(\tilde{D}+\partial_{\mathcal{S}})\ \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\right] = 0\ , \end{aligned} \tag{5.73}$$

where

$$\mathcal{O} = -\frac{1}{24}\not{F}(\rho_0-\rho_1)\ , \qquad \tilde{\mathcal{O}} = \frac{1}{24}\not{F}(\rho_0+\rho_1)\ , \tag{5.74}$$

and the fermion derivatives are

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\ (\Gamma_{34}+\Gamma_{67}) + \frac{1}{2}Q\ (\Gamma_{35}+\Gamma_{68}) \\ &\qquad - \frac{(1-u)\gamma}{24\zeta}\left(\not{H}(\rho_0-\rho_1) + (\rho_0-\rho_1)\not{H}\right)\ , \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\ (\Gamma_{34}+\Gamma_{67}) + \frac{1}{2}Q\ (\Gamma_{35}+\Gamma_{68}) \\ &\qquad - \frac{(1+u)\gamma}{24\zeta}\left(\not{H}(\rho_0+\rho_1) + (\rho_0+\rho_1)\not{H}\right)\ . \end{aligned} \tag{5.75}$$

Details of this calculation can be found in appendix R, together with explicit expression for the scalar functions $G, \tilde{G}, Q$ in (R.3).

Zero mode condition. Zero modes are stationary perturbations, in other words, they are time-independent in the magnon's frame

$$\partial_{\mathcal{S}}\,\vartheta^J = 0\ . \tag{5.76}$$

With this, the fermion zero mode equations are

$$\begin{aligned} &(\rho_0+\rho_1)\left[\zeta(1+u)\gamma D\ \vartheta^1 + \mathcal{O}\vartheta^2\right] = 0\ , \\ &(\rho_0-\rho_1)\left[\zeta(1-u)\gamma\tilde{D}\ \vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\right] = 0\ . \end{aligned} \tag{5.77}$$

A more detailed explanation of the zero mode condition can be found in section 3.1.2.

Rotated gamma matrices

Due to the symmetry of the bosonic background between the two spheres, the fermion perturbation equations (at least structurally) simplify if we introduce a set of rotated gamma matrices

$$\begin{aligned} \hat{\Gamma}^3 &= \frac{1}{\sqrt{2}}\left(\Gamma^3+\Gamma^6\right), \quad \hat{\Gamma}^6 = \frac{1}{\sqrt{2}}\left(\Gamma^6-\Gamma^3\right), \\ \hat{\Gamma}^4 &= \frac{1}{\sqrt{2}}\left(\Gamma^4+\Gamma^7\right), \quad \hat{\Gamma}^7 = \frac{1}{\sqrt{2}}\left(\Gamma^7-\Gamma^4\right), \\ \hat{\Gamma}^5 &= \frac{1}{\sqrt{2}}\left(\Gamma^5+\Gamma^8\right), \quad \hat{\Gamma}^8 = \frac{1}{\sqrt{2}}\left(\Gamma^8-\Gamma^5\right), \end{aligned} \tag{5.78}$$

with $\hat{\Gamma}^A = \Gamma^A$ left unchanged for $A = 0, 1, 2, 9$. Noting that

$$e_a^3 = e_a^6, \quad e_a^4 = e_a^7, \quad e_a^5 = e_a^8, \tag{5.79}$$

and defining $\hat{e}_a^A = \sqrt{2} e_a^A$, we have

$$\rho_a = e_a^0\,\hat{\Gamma}_0 + \hat{e}_a^3\,\hat{\Gamma}_3 + \hat{e}_a^4\,\hat{\Gamma}_4 + \hat{e}_a^5\,\hat{\Gamma}_5, \tag{5.80}$$

i.e. the symmetric combinations appear naturally.

Fixing kappa symmetry

Once again, the operators in front of the equations (5.73) are nilpotent

$$(\rho_0+\rho_1)^2 = (\rho_0-\rho_1)^2 = 0 \ , \tag{5.81}$$

and in fact half-rank. Kappa-gauge can be fixed with the projectors

$$K_1 = \frac{1}{2}\hat{\Gamma}^0(\rho_0+\rho_1) \ , \qquad K_2 = \frac{1}{2}\hat{\Gamma}^0(\rho_0-\rho_1) \ , \tag{5.82}$$

satisfying $K_J^2 = K_J$,

$$[K_1, D] = 0 \ , \qquad [K_2, \tilde{D}] = 0 \ , \tag{5.83}$$

and, using (5.74)

$$\mathcal{O} = \mathcal{O}K_2 \ , \qquad \tilde{\mathcal{O}} = \tilde{\mathcal{O}}K_1 \ . \tag{5.84}$$

Introducing the kappa-fixed spinors $\Psi^J = K_J \vartheta^J$, the zero mode equations (5.77) are equivalent to

$$\begin{aligned} \zeta(1+u)\gamma D\ \Psi^1 + K_1 \mathcal{O}\Psi^2 = 0\ , \\ \zeta(1-u)\gamma \tilde{D}\ \Psi^2 + K_2 \tilde{\mathcal{O}}\Psi^1 = 0\ . \end{aligned} \tag{5.85}$$

Zero mode equations

The kappa projectors (5.82) come with the commuting 6d chirality projectors[2]

$$P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \hat{\Gamma}_* \hat{\Gamma}_+\right)\ , \qquad [P_\pm, K_J] = 0\ , \tag{5.86}$$

and the contracted fluxes $\not{F}$, $\not{H}$ can be written as

$$\Gamma_* + \tfrac{1}{\sqrt{2}}\ \Gamma_+ + \tfrac{1}{\sqrt{2}}\ \Gamma_- = \hat{\Gamma}_* \left(P_+ + \Pi_+\right) = \left(P_- + \Pi_-\right) \hat{\Gamma}_*, \tag{5.87}$$

where

$$\Pi_\pm = \frac{1}{2}\Big(\mathbb{1} \pm \hat{\Gamma}_* \left(\hat{\Gamma}^{378} + \hat{\Gamma}^{648} + \hat{\Gamma}^{675}\right)\Big). \tag{5.88}$$

With this, we can write

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\ (\hat{\Gamma}_{34} + \hat{\Gamma}_{67}) + \frac{1}{2}Q\ (\hat{\Gamma}_{35} + \hat{\Gamma}_{68}) \\ &\qquad - \frac{q(1-u)\gamma}{\zeta}\,\hat{\Gamma}^{12}\Big(\left(P_- + \Pi_-\right) K_2 + \left(\mathbb{1} - K_2\right)\left(P_+ + \Pi_+\right)\Big)\ , \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\ (\hat{\Gamma}_{34} + \hat{\Gamma}_{67}) + \frac{1}{2}Q\ (\hat{\Gamma}_{35} + \hat{\Gamma}_{68}) \\ &\qquad - \frac{q(1+u)\gamma}{\zeta}\,\hat{\Gamma}^{12}\Big(\left(P_- + \Pi_-\right) K_1 + \left(\mathbb{1} - K_1\right)\left(P_+ + \Pi_+\right)\Big) \end{aligned} \tag{5.89}$$

the fermion derivatives (5.75), and

$$\begin{aligned} \mathcal{O} &= -\tilde{q}\ \hat{\Gamma}^{12}\left(P_- + \Pi_-\right) K_2\ , \\ \tilde{\mathcal{O}} &= \quad \tilde{q}\ \hat{\Gamma}^{12}\left(P_- + \Pi_-\right) K_1 \end{aligned} \tag{5.90}$$

[2] With hatted indices $\hat{\Gamma}_* = \hat{\Gamma}^{012}$, $\hat{\Gamma}_+ = \hat{\Gamma}^{345}$.

for the mixing terms (5.74), and the equations of motion for the kappa-fixed zero modes (5.85) become

$$\begin{aligned}&\zeta(1+u)\gamma D\ \Psi^1-\tilde{q}\ \hat{\Gamma}^{12}K_1\,(P_-+\Pi_-)\,\Psi^2=0\ ,\\&\zeta(1-u)\gamma\tilde{D}\ \Psi^2+\tilde{q}\ \hat{\Gamma}^{12}K_2\,(P_-+\Pi_-)\,\Psi^1=0\ .\end{aligned}\tag{5.91}$$

5.2.2 Mixed-flux fermion zero modes

In this section we find the normalizable solutions for equations (5.91), representing the perturbative zero modes over the double magnon background. We first construct a kappa-fixed ansatz, which is then substituted into the zero-mode equations. Identifying further ($\mathcal{Y}$-independent) symmetries of these equations, we obtain a minimal set of coupled ODEs, which are simple enough to solve.

Fixing kappa-gauge

The kappa-projectors (5.82) are exactly the same form (although with $\hat{\Gamma}^A$ defined differently) as for the single-magnon (3.56)

$$\begin{aligned}K_1&=\frac{1}{2}\Big(\mathbb{1}-\sin(2\chi)\cos\upsilon_+\,\hat{\Gamma}_{03}\ -\cos(2\chi)\cos\upsilon_+\,\hat{\Gamma}_{04}+\sin\upsilon_+\,\hat{\Gamma}_{05}\Big)\ ,\\K_2&=\frac{1}{2}\Big(\mathbb{1}+\sin(2\tilde{\chi})\cos\upsilon_-\,\hat{\Gamma}_{03}\ +\cos(2\tilde{\chi})\cos\upsilon_-\,\hat{\Gamma}_{04}-\sin\upsilon_-\,\hat{\Gamma}_{05}\Big)\ ,\end{aligned}\tag{5.92}$$

with

$$\begin{aligned}\chi(\mathcal{Y})&=\frac{1}{2}\left(\operatorname{arccot}\left(\frac{u\operatorname{csch}\mathcal{Y}}{\tilde{q}}\right)-\arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_+^2\operatorname{sech}^2\mathcal{Y}}}\right)\right)\ ,\\\tilde{\chi}(\mathcal{Y})&=\frac{1}{2}\left(\operatorname{arccot}\left(\frac{u\operatorname{csch}\mathcal{Y}}{\tilde{q}}\right)+\arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_-^2\operatorname{sech}^2\mathcal{Y}}}\right)\right)\ ,\\\upsilon_\pm&=\arcsin\left(Q_\pm\ \operatorname{sech}\mathcal{Y}\right)\ ,\qquad Q_\pm=\frac{q\sqrt{\tilde{q}^2-u^2}}{\tilde{q}(1\pm u)}.\end{aligned}\tag{5.93}$$

For a detailed construction of the kappa-fixed spinors Ψ^J, satisfying $K_J\Psi^J=\Psi^J$ for $J=1,2$, the reader is referred to section 3.2.1, here we just present the most general kappa-fixed ansatz[3]

$$\Psi^J = \sum_{\lambda_{12},\lambda_P} \left(\alpha_+^J(\mathcal{Y}) + \alpha_-^J(\mathcal{Y})\,\hat{\Gamma}_{45}\right) U^J_{\lambda_{12},\lambda_P}, \tag{5.94}$$

where

$$\begin{aligned} \alpha_+^1 &= e^{i\chi}\sqrt{1+\lambda Q_+ \operatorname{sech}\mathcal{Y}}\,, \qquad & \alpha_-^1 &= -\lambda e^{-i\chi}\sqrt{1-\lambda Q_+ \operatorname{sech}\mathcal{Y}}\,, \\ \alpha_+^2 &= e^{i\tilde{\chi}}\sqrt{1-\lambda Q_- \operatorname{sech}\mathcal{Y}}\,, \qquad & \alpha_-^2 &= \lambda e^{-i\tilde{\chi}}\sqrt{1+\lambda Q_- \operatorname{sech}\mathcal{Y}}\,, \end{aligned} \tag{5.95}$$

the spinors have eigenvalues

$$\begin{aligned} \hat{\Gamma}_{12} U^J_{\lambda_{12},\lambda_P} &= i\lambda_{12} U^J_{\lambda_{12},\lambda_P}, \\ \hat{\Gamma}_* \hat{\Gamma}_+ U^J_{\lambda_{12},\lambda_P} &= \lambda_P U^J_{\lambda_{12},\lambda_P}, \\ \hat{\Gamma}_{34} U^J &= +iU^J, \end{aligned} \tag{5.96}$$

and we defined $\lambda = \lambda_P \lambda_{12}$. The sums in (5.94) are over $\lambda_{12}, \lambda_P = \pm 1$.

Reduced zero mode equations

Our aim is to reduce the equations (5.91) to a system of ODEs involving the smallest number of (complex) scalar functions possible. We start with 16 complex components for each of the Weyl-projected spinors ϑ^1 and ϑ^2 (not worrying about the Majorana condition for now). The operators $\hat{\Gamma}^{12}$ and $\hat{\Gamma}_*\hat{\Gamma}_+$ both commute with the equations, and we can consider solutions with definite λ_{12}, λ_P, reducing dimensionality by a factor of 2 each, leaving us with 4+4 components. Making the kappa-fixed ansatz (5.94), this is further reduced to a system of 2+2=4 ODEs. For the single-magnon in chapter 3 we had an additional symmetry, namely $\hat{\Gamma}^{68}$ commuting with the equations, and the final minimal coupled system had 2 degrees of freedom, but for the double-magnon it can be shown that this is not possible with a $\mathcal{Y}$-independent transformation.

Putting all of this into practice, we make the ansatz

$$\begin{aligned} \Psi^1 &= \left(\alpha_+^1 + \alpha_-^1\,\hat{\Gamma}_{45}\right)\left(f_1(\mathcal{Y}) + g_1(\mathcal{Y})\,\hat{\Gamma}_{78}\right) U, \\ \Psi^2 &= \left(\alpha_+^2 + \alpha_-^2\,\hat{\Gamma}_{45}\right)\left(f_2(\mathcal{Y}) + g_2(\mathcal{Y})\,\hat{\Gamma}_{78}\right) U, \end{aligned} \tag{5.97}$$

[3] Note that the definition of λ_{12} is different to the one used in chapter 3, resulting in a sign change $\lambda \to -\lambda$ in the kappa-fixed ansatz.

where the shared constant spinor U has eigenvalues $i\lambda_{12}, \lambda_P, +i$ and $+i$ under $\hat{\Gamma}_{12}, \hat{\Gamma}_*\hat{\Gamma}_+, \hat{\Gamma}_{34}$ and $\hat{\Gamma}_{67}$, respectively. In effect, these projections (to eigenspaces of mutually commuting operators) leave a single free (complex) component in U, which we can normalize, since the physical degrees of freedom are carried by the four scalar functions f_1, g_1, f_2, g_2.

Substituting (5.97) into (5.91), after a considerable amount of simplification we get

$$\begin{aligned}
&\left(\alpha^1_+ + \alpha^1_- \hat{\Gamma}_{45}\right) \Bigg[\left(\left(\partial_{\mathcal{Y}} + C_{f_1 f_1}\right) f_1 + C_{f_1 g_1} g_1 + C_{f_1 f_2} f_2 + C_{f_1 g_2} g_2 \right) \\
&\qquad \left(\left(\partial_{\mathcal{Y}} + C_{g_1 g_1}\right) g_1 + C_{g_1 f_1} f_1 + C_{g_1 g_2} g_2 + C_{g_1 f_2} f_2 \right) \hat{\Gamma}_{78} \Bigg] U = 0 \ , \\
&\left(\alpha^2_+ + \alpha^2_- \hat{\Gamma}_{45}\right) \Bigg[\left(\left(\partial_{\mathcal{Y}} + C_{f_2 f_2}\right) f_2 + C_{f_2 g_2} g_2 + C_{f_2 f_1} f_1 + C_{f_2 g_1} g_1 \right) \\
&\qquad \left(\left(\partial_{\mathcal{Y}} + C_{g_2 g_2}\right) g_2 + C_{g_2 f_2} f_2 + C_{g_2 g_1} g_1 + C_{g_2 f_1} f_1 \right) \hat{\Gamma}_{78} \Bigg] U = 0 \ ,
\end{aligned} \tag{5.98}$$

where the coefficients $C_{..}$ are listed in appendix S. It is worth noting that we have overall kappa-fixing by K_1 and K_2, again confirming the fact that the kappa-projectors commute with the fermion derivatives D, $\tilde{D}$. Further substituting

$$\begin{aligned}
f_1 &= \frac{1}{\sqrt{1+u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{\frac{i}{2}\lambda_{12}\lambda_P \arctan\left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1-Q_+^2}}\right)} e^{+\frac{i}{2}\operatorname{arccot}\left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}}\right)} \tilde{f}_1, \\
g_1 &= \frac{i}{\sqrt{1+u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{\frac{i}{2}\lambda_{12}\lambda_P \arctan\left(\frac{Q_+ \tanh \mathcal{Y}}{\sqrt{1-Q_+^2}}\right)} e^{-\frac{i}{2}\operatorname{arccot}\left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}}\right)} \tilde{g}_1, \\
f_2 &= \frac{1}{\sqrt{1-u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{\frac{i}{2}\lambda_{12}\lambda_P \arctan\left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1-Q_-^2}}\right)} e^{+\frac{i}{2}\operatorname{arccot}\left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}}\right)} \tilde{f}_2, \\
g_2 &= \frac{i}{\sqrt{1-u}} e^{i\frac{\lambda_{12} q}{\sqrt{\tilde{q}^2-u^2}}\mathcal{Y}} e^{\frac{i}{2}\lambda_{12}\lambda_P \arctan\left(\frac{Q_- \tanh \mathcal{Y}}{\sqrt{1-Q_-^2}}\right)} e^{-\frac{i}{2}\operatorname{arccot}\left(\frac{u \operatorname{csch} \mathcal{Y}}{\tilde{q}}\right)} \tilde{g}_2,
\end{aligned} \tag{5.99}$$

the equations become

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{f}_1 + iM_1^+\tilde{f}_1 + M_2\tilde{g}_1 + M_3^+\tilde{f}_2 + \left(\frac{1}{2} + \lambda_P\lambda_{12}M_2\right)\tilde{g}_2 = 0,\\
&\partial_{\mathcal{Y}}\tilde{g}_1 + iM_1^-\tilde{g}_1 - \bar{M}_2\tilde{f}_1 + M_3^-\tilde{g}_2 + \left(\frac{1}{2} - \lambda_P\lambda_{12}\bar{M}_2\right)\tilde{f}_2 = 0,\\
&\partial_{\mathcal{Y}}\tilde{f}_2 - iM_1^+\tilde{f}_2 - M_2\tilde{g}_2 + \bar{M}_3^+\tilde{f}_1 + \left(\frac{1}{2} - \lambda_P\lambda_{12}M_2\right)\tilde{g}_1 = 0,\\
&\partial_{\mathcal{Y}}\tilde{g}_2 - iM_1^-\tilde{g}_2 + \bar{M}_2\tilde{f}_2 + \bar{M}_3^-\tilde{g}_1 + \left(\frac{1}{2} + \lambda_P\lambda_{12}\bar{M}_2\right)\tilde{f}_1 = 0,
\end{aligned} \tag{5.100}$$

with

$$\begin{aligned}
M_1^\pm &= \frac{1}{2}\left(\lambda_{12}(2-\lambda_P)\xi \pm \tilde{q}\,\mathrm{sech}\mathcal{Y}\right),\\
M_2 &= \frac{-q}{2\sqrt{\tilde{q}^2 - u^2}}\left(\tilde{q}\,\mathrm{sech}\mathcal{Y} - iu\tanh\mathcal{Y}\right),\\
M_3^\pm &= \frac{\lambda_{12}}{2}(2-\lambda_P)\tanh\mathcal{Y} + i\lambda_P\lambda_{12}M_1^\pm.
\end{aligned} \tag{5.101}$$

An exhaustive list of solutions to these equations can be found in appendix T, here we just present the normalizable ones, all of which are on the P_- subspace ($\lambda_P = -1$)

$$\begin{aligned}
\tilde{f}_1 &= \mathrm{sech}\mathcal{Y}\left(c_1 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})} + c_2 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{g}_1 &= i\lambda_{12}\,\mathrm{sech}\mathcal{Y}\left(c_1 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} - c_2 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{f}_2 &= \lambda_{12}\,\mathrm{sech}\mathcal{Y}\left(c_1 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})} + c_2 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{g}_2 &= i\,\mathrm{sech}\mathcal{Y}\left(c_1 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} - c_2 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),
\end{aligned} \tag{5.102}$$

for some integration constants c_1, c_2. Consequently, we can write the normalizable zero mode solutions to (5.91) as

$$\begin{aligned}
\Psi^1 &= \sum_{\lambda_{12},\lambda_{67}} \frac{\mathrm{sech}\mathcal{Y}}{8\sqrt{1+u}}\, e^{i\omega_+}\left(\alpha_+^1 + \alpha_-^1\,\hat{\Gamma}_{45}\right)\left(e^{i\hat{\omega}} - \lambda_{12}e^{-i\hat{\omega}}\,\hat{\Gamma}_{78}\right)V_{\lambda_{12},\lambda_{67}}\ ,\\
\Psi^2 &= \sum_{\lambda_{12},\lambda_{67}} \lambda_{12}\frac{\mathrm{sech}\mathcal{Y}}{8\sqrt{1-u}}\, e^{i\omega_-}\left(\alpha_+^2 + \alpha_-^2\,\hat{\Gamma}_{45}\right)\left(e^{i\hat{\omega}} - \lambda_{12}e^{-i\hat{\omega}}\,\hat{\Gamma}_{78}\right)V_{\lambda_{12},\lambda_{67}}\ ,
\end{aligned} \tag{5.103}$$

where

$$\omega_\pm = \frac{q\,\lambda_{12}\mathcal{Y}}{2\sqrt{\tilde{q}^2-u^2}} - \frac{\lambda_{12}}{2}\arctan\left(\frac{Q_\pm \tanh\mathcal{Y}}{\sqrt{1-Q_\pm^2}}\right) ,$$

$$\hat{\omega} = -\frac{1}{2}\arctan(\sinh\mathcal{Y}) + \frac{\lambda_{67}}{2}\operatorname{arccot}\left(\frac{u\operatorname{csch}\mathcal{Y}}{\tilde{q}}\right) , \tag{5.104}$$

the constant MW spinors V_{ab} satisfy $P_- V_{ab} = V_{ab}$, $\hat{\Gamma}_{34}V_{ab} = +iV_{ab}$, $\hat{\Gamma}_{12}V_{\pm\lambda} = \pm iV_{\pm\lambda}$, $\hat{\Gamma}_{67}V_{\lambda\pm} = \pm iV_{\lambda\pm}$, and the factors of 8 have been introduced for later convenience.

Counting the zero modes. A general MW spinor has 16 real degrees of freedom, but V_{ab} in (5.103) is further constrained to have eigenvalues $-1, +i$ under $\hat{\Gamma}_*\hat{\Gamma}_+$ and $\hat{\Gamma}_{34}$. These leave 4 degrees of freedom, one for each pair of the indices a, b, i.e. each combination of λ_{12} and λ_{67}. The double magnon has a total of 4 fermion zero modes.

Zero mode action

Let us write the quadratic fermionic Lagrangian (3.5) as

$$\mathcal{L}_\mathrm{F} = -i\left(\eta^{ab}\delta^{IJ} + \epsilon^{ab}\sigma_3^{IJ}\right)\bar{\vartheta}^I\rho_a\mathcal{D}_b\vartheta^J , \tag{5.105}$$

$$= i\,\bar{\vartheta}^1(\rho_0+\rho_1)(\mathcal{D}_0-\mathcal{D}_1)\vartheta^1 + i\,\bar{\vartheta}^2(\rho_0-\rho_1)(\mathcal{D}_0+\mathcal{D}_1)\vartheta^2 , \tag{5.106}$$

$$= -\frac{i}{2}\,\bar{\vartheta}^1(\rho_0+\rho_1)\Big(\zeta(1+u)\gamma(D-\partial_\mathcal{S})\vartheta^1 + \mathcal{O}\vartheta^2\Big) + \frac{i}{2}\,\bar{\vartheta}^2(\rho_0-\rho_1)\Big(\zeta(1-u)\gamma(\tilde{D}+\partial_\mathcal{S})\vartheta^2 + \tilde{\mathcal{O}}\vartheta^1\Big) . \tag{5.107}$$

Taking a basis of gamma matrices such that $\hat{\Gamma}^A$ have definite hermiticity, the kappa-projectors will be Hermitian $K_J^\dagger = K_J$, and the Hermitian conjugate intertwiner is given by $\hat{\Gamma}^0$, i.e. the Dirac conjugate is $\bar{\vartheta} = \vartheta^\dagger\,\hat{\Gamma}^0$. With this,

$$\mathcal{L}_\mathrm{F} = -i\,(\Psi^1)^\dagger\Big(\zeta(1+u)\gamma(D-\partial_\mathcal{S})\ \Psi^1 + \mathcal{O}\Psi^2\Big) + i\,(\Psi^2)^\dagger\Big(\zeta(1-u)\gamma(\tilde{D}+\partial_\mathcal{S})\ \Psi^2 + \tilde{\mathcal{O}}\Psi^1\Big) . \tag{5.108}$$

Now letting $V = V_{++} + V_{+-} + V_{-+} + V_{--}$ depend on $\mathcal{T}$ and substituting the zero modes (5.103) into the above Lagrangian we get

$$\begin{aligned}\mathcal{L}_{\mathrm{F},0} =&\ 2i(1+u)\gamma\, {\Psi^1}^\dagger \partial_{\mathcal{T}}\Psi^1 + 2i(1-u)\gamma\, {\Psi^2}^\dagger \partial_{\mathcal{T}}\Psi^2, \\ =&\ \frac{i}{4}\gamma\,\mathrm{sech}^2\mathcal{Y}\Big(V^\dagger\partial_{\mathcal{T}}V \\ &- i\tanh\mathcal{Y}\sum_{\lambda_{12},\lambda_{67}}\lambda_{12}\,V^\dagger_{\lambda_{12},\lambda_{67}}\hat{\Gamma}_{78}\,\partial_{\mathcal{T}}V_{\lambda_{12},-\lambda_{67}}\Big).\end{aligned} \tag{5.109}$$

Integrating over $\mathcal{X}$ we arrive at the zero mode action

$$S_{\mathrm{F},0} = \ \mathrm{h}\tilde{\gamma}\int \mathrm{d}\mathcal{T}\,\Big(\,i\,V^\dagger\partial_{\mathcal{T}}V\,\Big)\ , \tag{5.110}$$

where

$$\tilde{\gamma} = \frac{\gamma}{\zeta} = \frac{1}{\sqrt{\tilde{q}^2 - u^2}}\ . \tag{5.111}$$

To apply the Majorana condition, in Majorana basis where $\hat{\Gamma}^*_A = -\hat{\Gamma}^*_A$, we need to impose reality of the spinors ${\Psi^I}^* = \Psi^I$. For the solutions (5.103) this is equivalent to

$$V_{-b} = {V_{+b}}^* \quad \Rightarrow \quad V^* = V\ , \tag{5.112}$$

and the zero modes action takes the real form

$$S_{\mathrm{F},0} = \ \mathrm{h}\tilde{\gamma}\int \mathrm{d}\mathcal{T}\,\Big(\,i\,V^T\partial_{\mathcal{T}}V\,\Big)\ . \tag{5.113}$$

Quantizing the corresponding Poisson brackets, we get the zero-mode anti-commutators

$$\{V_{\alpha a}, V_{\beta b}\} = \delta_{\alpha\beta}\,\delta_{ab}\,\frac{1}{\mathrm{h}\tilde{\gamma}}\ , \tag{5.114}$$

where $a, \alpha = 1, 2$. After complexifying

$$V_{\mathrm{L}a} = \frac{1}{\sqrt{2}}\left(V_{1a} + i\,V_{2a}\right)\ , \quad V_{\mathrm{R}a} = \frac{1}{\sqrt{2}}\left(V_{1a} - i\,V_{2a}\right)\ , \tag{5.115}$$

the only non-zero anticommutator is

$$\{V_{\mathrm{L}a}, V_{\mathrm{R}b}\} = \delta_{ab}\,\frac{1}{\mathrm{h}\tilde{\gamma}}\ . \tag{5.116}$$

We can in fact match these 4 quantized zero modes to the 4 odd generators of the residual symmetry algebra $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, using the general prescription described in appendix A.

5.3 Chapter conclusions

In this chapter we wrote down the fermion zero modes for the mixed-flux $\text{AdS}_3 \times \mathbb{R}$ soliton and the $\mathbb{R} \times \text{S}^3 \times \text{S}^3$ double magnon. For the $\text{AdS}_3 \times \mathbb{R}$ soliton we found 4 and 2 zero modes on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, respectively, which would imply that the corresponding quantum state transforms in a short representations of the off-shell residual symmetry algebras $\mathfrak{psu}(1|1)^4_{\text{c.e.}}$ and $\mathfrak{su}(1|1)^2_{\text{c.e.}}$. After removing the UV divergence, stemming from the infinite length of the string, we managed to match the quantized zero modes to the odd generators of the residual algebras, just like for the giant magnon in chapter 3. Since the AdS spin J_0 is quantized in integer units, the dispersion relation (2.304)

$$E - J_1 = -\sqrt{(J_0 - \text{h}q\text{p})^2 + 4\text{h}^2\tilde{q}^2\cos^2\tfrac{\text{p}}{2}} \tag{5.117}$$

is reminiscent of the energy of an $m = 1$ magnon. An important difference, though, is the negative sign (which is the result of the UV regularization), and in fact the $\text{AdS}_3 \times \mathbb{R}$ soliton does not represent a physical state of the theory, neither does it reduce to the BMN string in the zero momentum limit. Such a solution would, however, be physical in the mirror theory [172], and these results might be applicable to the study of supersymmetries there.

For the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ double magnon we found 4 fermion zero modes, as expected, since the scattering state transforms in a long multiplet of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$. This two-magnon solution and its perturbations encode information about mixed-mass scattering on $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, and could be compared to the expansion of the exact S-matrix.

Chapter 6

Conclusions

The AdS/CFT duality offers a completely new perspective on strongly coupled gauge theories and quantum gravity, stating that these seemingly very different theories actually describe the same physics. In the maximally supersymmetric case of $AdS_5 \times S^5$ integrability proved to be an invaluable tool, allowing one to explicitly calculate the energies of closed string states, or equivalently the anomalous dimensions of gauge theory operators, at all values of the coupling.

Although integrability is unlikely to explain the inner workings of generic gauge/gravity dualities, there are valuable lessons to be learnt from instances of AdS/CFT that are less supersymmetric than AdS_5/CFT_4, but still integrable. In this thesis we focussed on the string theory side of AdS_3/CFT_2, and in particular on two backgrounds with 16 supercharges, $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. On the classical level, these string backgrounds were observed to be integrable when supported by pure R-R [69, 70, 71] or mixed R-R and NS-NS fluxes [72]. One can approach the richness of these theories compared to AdS_5 from many angles. Firstly, less (super)symmetry usually leaves more room for non-trivial behaviour in the quantum theory, and a prominent example of this is the (quantum) dispersion relations of elementary excitations in the two theories. In AdS_5/CFT_4, supersymmetry and the BMN limit determine the all-loop magnon dispersion relation to be [40]

$$\epsilon = \sqrt{1 + \frac{\lambda^2}{\pi^2} \sin^2 \frac{\mathrm{p}}{2}} \,, \tag{6.1}$$

while for the mixed-flux AdS_3 backgrounds [102, 104, 115]

$$\epsilon = \sqrt{\left(m \pm q\sqrt{\lambda}\frac{\mathrm{p}}{2\pi}\right)^2 + 4\,\tilde{q}^2\,\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}} \,, \tag{6.2}$$

where $\mathrm{h} = \frac{\sqrt{\lambda}}{2\pi}$ only in the classical string limit, and in general h will receive quantum corrections. Secondly, AdS_3 has massless modes, for which the very core of the integrable machinery had the be revised [111]. And lastly, the AdS_3 theories we consider have two continuous parameters, the angle φ describing the relative radii of the three-spheres of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ (also capturing $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ in the $\varphi \to 0$ limit), and $q \in [0, 1]$ corresponding to the strength of the NS-NS flux. Even if we assume some kind of similarity between $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the pure R-R $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background, turning on these two extra parameters provides both richness, and as we have seen in this thesis, technical challenges when carrying out analogous calculations.

Understanding $\mathrm{AdS}_3/\mathrm{CFT}_2$ is an ambitious project, and we devoted this thesis only to a small segment of it, the semiclassical analysis of its string solitons. The quantization of these particle-like solutions of integrable field theories [134, 135, 136, 137] provides a window into regimes of the quantum theory not directly accessible to perturbation methods. Without a doubt, the most important soliton of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ worldsheet sigma-model is the Hofman-Maldacena giant magnon [41], and as a BPS state of the off-shell residual algebra $\mathfrak{su}(2|2)^2_{\text{c.e.}}$, it should have 8 fermion zero modes. These were explicitly constructed by Minahan [53], who also managed to match the quantized zero modes to the odd generators of the algebra. Subsequently, a basis of the complete fluctuation spectrum of the magnon was found by Papathanasiou and Spradlin [54], and from these fluctuations the 1-loop correction to the magnon energy was confirmed to be zero, in agreement with (6.1). The main goal of this thesis had been to perform these two calculations for the mixed-flux AdS_3 giant magnon. Importantly, we first had to identify the right classical background for the analysis. In chapter 2 we found that among all the 2-spin magnons, the *stationary* magnon (2.285) can be regarded as the mixed-flux generalization of the HM magnon, and as such, represents a suitable classical background.

The off-shell residual symmetry algebras of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ are the centrally extended $\mathfrak{psu}(1|1)^4$, and the centrally extended $\mathfrak{su}(1|1)^2$, and the BPS magnon should transform in the 4 and 2 dimensional short representations of these superalgebras, respectively. We confirmed this in chapter 3 by explicitly constructing the 4 and 2 fermion zero modes of the stationary mixed-flux magnon on these backgrounds. Furthermore, we managed to match the odd generators of the residual algebras to the semi-classically quantized fermion zero modes. In chapter 5 we performed the same zero mode analysis on two other classical solutions. It turns out that, just like the stationary magnon, the $\mathrm{AdS}_3 \times \mathrm{S}^1$ soliton has 4 and 2 fermion zero modes, confirming that it transforms in short representations, while for the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ double magnon we found 8 zero modes, in agreement with the fact that general scattering states are part of long representations.

In chapter 5 we wrote down the full spectrum of fluctuations around the mixed-flux AdS_3 stationary magnon. For the fermions we used the techniques developed in chapter 3 extended to the case of non-zero frequencies, and we applied the dressing method to construct the bosons. The fluctuations naturally arrange into short multiplets of the residual symmetry algebras, according to mass and chirality. Reading off the stability angle of each fluctuation, we determined that the one-loop functional determinant vanishes for both geometries, or in other words, that

$$\mathrm{h}(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \tag{6.3}$$

and the subleading $\mathcal{O}(1)$ correction to the dispersion relation (6.2) is zero. This result is in agreement with the 1-loop corrections calculated directly from the GS action [71, 124], using the algebraic curve [123], or considering the worldsheet scattering of giant magnons [120].

In chapter 3 we found that the stationary magnon cannot be defined in the $q = 1$ limit, and the only zero modes we managed to construct were not normalizable. The pure NS-NS string theory has been long known to be solvable using a chiral decomposition [62, 63, 64], but it would be interesting to see a soliton/integrability based analysis of these backgrounds. In more recent developments, the CFT dual of the $k = 1$ WZW model, i.e. $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with minimal quantized NS-NS flux, has been identified as a symmetric product orbifold [80, 81, 82, 83].

Semiclassical methods continue to provide valuable insight into the string theory side of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, this thesis being one example, or the calculation of 1-loop corrections to the rigid spinning string dispersion relations

[125]. Where they seem to fail is the description of massless modes. In the $\alpha \to 0$ limit our zero modes and fluctuations simply reduce to the plane-wave perturbations of the BMN vacuum, shedding no further light on the nature of the massless soliton of the theory, in agreement with the fact that the $\alpha \to 0$ limit of the spin-chain fails to capture these inherently non-perturbative modes on the other side of the duality [110]. Furthermore, massless modes render a perturbative computation of wrapping corrections impossible, once the theory is put on a compactified worldsheet[1] [107]. Instead, wrapping corrections may be computed from a non-perturbative TBA using an alternative low-momentum expansion [126, 127, 128], based on the earlier observation of non-trivial massless scattering in the BMN limit [129].

Our semiclassical calculations probe the giant magnon in the decompactification limit, expanded in powers of $1/\sqrt{\lambda}$ for large h. If instead we want stay in the classical limit, but consider the theory on a closed worldsheet we need to introduce another type of corrections, often referred to as *finite size*. As we have seen, the fermion zero modes of the magnon on the decompactified worldsheet can be matched to the residual symmetries, and it would be interesting to perform a similar analysis for the finite size giant magnons, either on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [45, 168] or the mixed-flux AdS_3 backgrounds [169, 170].

Yet another direction for future research would be to consider semiclassical soliton analysis on other, even less supersymmetric AdS theories. An interesting example is the string theory on $\mathrm{AdS}_3 \times \left(\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1\right)/\mathbb{Z}_2$ [174]. This background can be obtained from $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ by orbifolding the $\mathbb{Z}_2$ action that exchanges the two three-spheres and simultaneously reflects the circle. At the level of the supercharges of the large $\mathcal{N} = 4$ superconformal algebra, this orbifold projection can be taken to either reduce the spacetime supersymmetry to $\mathcal{N} = 3$ or $\mathcal{N} = 1$. This is independently true for the left- and right-movers, and the orbifolded theory admits $\mathcal{N} = (3, 3)$, $\mathcal{N} = (3, 1)$, $\mathcal{N} = (1, 3)$, or $\mathcal{N} = (1, 1)$ supersymmetry. Due to its richness and tractability (non-maximal but still sufficient SUSY) the $\mathcal{N} = (3, 3)$ case was further examined in [175], where its CFT_2 dual was proposed to be the symmetric product orbifold $\mathrm{Sym}^N(\mathcal{S}_0/\mathbb{Z}_2)$.

[1] This is to be compared with the finite-size AdS_5 giant magnon, where wrapping interactions give the right correction to the dispersion relation [45, 173].

Appendix A

Fermion zero modes and the off-shell residual algebra

An important piece in the fermion zero mode puzzle is the relation between the (quantized) zero modes and the off-shell residual symmetry algebra of the (BMN) ground state. In this section we focus on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$, with residual algebra $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, the case of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ is a trivial extension from the algebraic perspective, its residual algebra being the direct product of two copies of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ with matching central charges.

The normalizable fermion zero modes represent *fermionic collective coordinates* of the bosonic solution, and the corresponding quantized fermion zero mode operators transform the bosonic state into fermions in the same multiplet. This multiplet forms a representation of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$, and we should be able to recover the generators of this algebra from the zero mode operators.

The $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ has 4 odd generators and 4 even central charges satisfying

$$\begin{aligned} \{\mathbf{Q}_{\mathrm{L}}, \mathbf{S}_{\mathrm{L}}\} &= \mathbf{H}_{\mathrm{L}} \ , \qquad & \{\mathbf{Q}_{\mathrm{L}}, \mathbf{Q}_{\mathrm{R}}\} &= \mathbf{C} \ , \\ \{\mathbf{Q}_{\mathrm{R}}, \mathbf{S}_{\mathrm{R}}\} &= \mathbf{H}_{\mathrm{R}} \ , \qquad & \{\mathbf{S}_{\mathrm{L}}, \mathbf{S}_{\mathrm{R}}\} &= \overline{\mathbf{C}} \ , \end{aligned} \tag{A.1}$$

and, accordingly, we expect at most 4 fermion zero modes. In a general long representation of $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ there are four states, two bosons and two fermions, and we do in fact need 4 fermion operators[1] to generate the multiplet. In a short representation, however, there are only two states, and the generators satisfy the shortening condition

$$\mathbf{H}_{\mathrm{L}}\mathbf{H}_{\mathrm{R}} - \mathbf{C}\overline{\mathbf{C}} = 0. \tag{A.2}$$

It is reasonable to expect that capturing the action of the algebra on such a representation does not require the full set of 4 independent generators. In

[1] Two to annihilate the highest weight state boson, and one each to transform it into the two independent fermions.

the rest of this section we show how to construct (A.1) for general central charges using 4 fermion operators, and at the same time demonstrate that for a short representation, i.e. when (A.2) holds, one only needs 2 odd operators to reproduce the algebra.

We start with 4 odd (zero mode) operators $\mathbf{V}_{\mathrm{L}a}, \mathbf{V}_{\mathrm{R}a}$ ($a = 1, 2$), normalized such that

$$\{\mathbf{V}_{\mathrm{L}a}, \mathbf{V}_{\mathrm{R}b}\} = \delta_{ab} \ , \tag{A.3}$$

and make the ansatz

$$\begin{aligned} \mathbf{Q}_{\mathrm{L}} &= x_1\,\mathbf{V}_{\mathrm{L}1} + x_2\,\mathbf{V}_{\mathrm{L}2} \ , \qquad & \mathbf{S}_{\mathrm{L}} &= x_3\,\mathbf{V}_{\mathrm{R}2} + x_4\,\mathbf{V}_{\mathrm{R}1} \ , \\ \mathbf{Q}_{\mathrm{R}} &= y_1\,\mathbf{V}_{\mathrm{R}1} + y_2\,\mathbf{V}_{\mathrm{R}2} \ , \qquad & \mathbf{S}_{\mathrm{R}} &= y_3\,\mathbf{V}_{\mathrm{L}2} + y_4\,\mathbf{V}_{\mathrm{L}1} \ . \end{aligned} \tag{A.4}$$

The zero anti-commutators $\{\mathbf{Q}_{\mathrm{L}}, \mathbf{S}_{\mathrm{R}}\} = \{\mathbf{Q}_{\mathrm{R}}, \mathbf{S}_{\mathrm{L}}\} = 0$ are automatically satisfied, and (A.1) are equivalent to

$$\begin{aligned} x_1x_4 + x_2x_3 &= \mathrm{H}_{\mathrm{L}} \ , \qquad & x_1y_1 + x_2y_2 &= \mathrm{C} \ , \\ y_1y_4 + y_2y_3 &= \mathrm{H}_{\mathrm{R}} \ , \qquad & x_3y_3 + x_4y_4 &= \bar{\mathrm{C}} \ , \end{aligned} \tag{A.5}$$

where H_{L}, H_{R}, C and $\bar{\mathrm{C}}$ are the eigenvalues of the central charges parametrizing the representation. Assuming[2] $\mathrm{C}\bar{\mathrm{C}} \neq 0$ and making the ansatz symmetric

$$\begin{aligned} \hat{x}_1 &\equiv x_1\mathrm{C}^{-1/2} = x_3\mathrm{C}^{+1/2} \ , \qquad & \hat{x}_2 &\equiv x_2\mathrm{C}^{-1/2} = x_4\mathrm{C}^{+1/2} \ , \\ \hat{y}_1 &\equiv y_1\bar{\mathrm{C}}^{+1/2} = y_3\bar{\mathrm{C}}^{-1/2} \ , \qquad & \hat{y}_2 &\equiv y_2\bar{\mathrm{C}}^{+1/2} = y_4\bar{\mathrm{C}}^{-1/2} \ , \end{aligned} \tag{A.6}$$

we are left with the equations

$$\begin{aligned} \hat{x}_1\hat{y}_1 + \hat{x}_2\hat{y}_2 &= \sqrt{\mathrm{C}\bar{\mathrm{C}}} \ , \\ 2\hat{x}_1\hat{x}_2 &= \mathrm{H}_{\mathrm{L}} \ , \\ 2\hat{y}_1\hat{y}_2 &= \mathrm{H}_{\mathrm{R}} \ . \end{aligned} \tag{A.7}$$

This system is still under-determined, but imposing the additional constraint

$$\frac{\hat{x}_2}{\hat{x}_1} = \frac{\hat{y}_2}{\hat{y}_1} \tag{A.8}$$

[2] This is going to be the case for the the magnon solutions we consider. It is a simple exercise to write down solutions for the case $\mathrm{C}\bar{\mathrm{C}} = 0$, but we omit further discussion on this point.

leads to the solution

$$\begin{aligned} \hat{x}_1^2 &= \sqrt{\frac{\mathrm{H}_{\mathrm{L}}}{\mathrm{H}_{\mathrm{R}}}}\,\frac{\sqrt{\mathrm{C}\bar{\mathrm{C}}}+\sqrt{\mathrm{D}}}{2}\,, & \hat{x}_2^2 &= \sqrt{\frac{\mathrm{H}_{\mathrm{L}}}{\mathrm{H}_{\mathrm{R}}}}\,\frac{\sqrt{\mathrm{C}\bar{\mathrm{C}}}-\sqrt{\mathrm{D}}}{2}\,, \\ \hat{y}_1^2 &= \sqrt{\frac{\mathrm{H}_{\mathrm{R}}}{\mathrm{H}_{\mathrm{L}}}}\,\frac{\sqrt{\mathrm{C}\bar{\mathrm{C}}}+\sqrt{\mathrm{D}}}{2}\,, & \hat{y}_2^2 &= \sqrt{\frac{\mathrm{H}_{\mathrm{R}}}{\mathrm{H}_{\mathrm{L}}}}\,\frac{\sqrt{\mathrm{C}\bar{\mathrm{C}}}-\sqrt{\mathrm{D}}}{2}\,, \end{aligned} \tag{A.9}$$

where

$$\mathrm{D} = \mathrm{C}\bar{\mathrm{C}} - \mathrm{H}_{\mathrm{L}}\mathrm{H}_{\mathrm{R}}. \tag{A.10}$$

In terms of this solution the generators (A.4) can be written as

$$\begin{aligned} \mathbf{Q}_{\mathrm{L}} &= \hat{x}_1 \mathrm{C}^{+1/2}\left(\mathbf{V}_{\mathrm{L}1} + \frac{\hat{x}_2}{\hat{x}_1}\mathbf{V}_{\mathrm{L}2}\right), & \mathbf{S}_{\mathrm{L}} &= \hat{x}_2 \mathrm{C}^{-1/2}\left(\mathbf{V}_{\mathrm{R}1} + \frac{\hat{x}_1}{\hat{x}_2}\mathbf{V}_{\mathrm{R}2}\right), \\ \mathbf{Q}_{\mathrm{R}} &= \hat{y}_1 \bar{\mathrm{C}}^{-1/2}\left(\mathbf{V}_{\mathrm{R}1} + \frac{\hat{y}_2}{\hat{y}_1}\mathbf{V}_{\mathrm{R}2}\right), & \mathbf{S}_{\mathrm{R}} &= \hat{y}_2 \bar{\mathrm{C}}^{+1/2}\left(\mathbf{V}_{\mathrm{L}1} + \frac{\hat{y}_1}{\hat{y}_2}\mathbf{V}_{\mathrm{L}2}\right). \end{aligned} \tag{A.11}$$

For a short representation $\mathrm{D} = 0$,

$$\hat{x}_1^2 = \hat{x}_2^2 = \frac{\mathrm{H}_{\mathrm{L}}}{2}\,, \qquad \hat{y}_1^2 = \hat{y}_2^2 = \frac{\mathrm{H}_{\mathrm{R}}}{2}\,, \tag{A.12}$$

and we can write

$$\begin{aligned} \mathbf{Q}_{\mathrm{L}} &= \sqrt{\frac{\mathrm{H}_{\mathrm{L}}\mathrm{C}}{2}}\,(\mathbf{V}_{\mathrm{L}1} + \mathbf{V}_{\mathrm{L}2})\,, & \mathbf{S}_{\mathrm{L}} &= \sqrt{\frac{\mathrm{H}_{\mathrm{L}}}{2\mathrm{C}}}\,(\mathbf{V}_{\mathrm{R}1} + \mathbf{V}_{\mathrm{R}2})\,, \\ \mathbf{Q}_{\mathrm{R}} &= \sqrt{\frac{\mathrm{H}_{\mathrm{R}}}{2\bar{\mathrm{C}}}}\,(\mathbf{V}_{\mathrm{R}1} + \mathbf{V}_{\mathrm{R}2})\,, & \mathbf{S}_{\mathrm{R}} &= \sqrt{\frac{\mathrm{H}_{\mathrm{R}}\bar{\mathrm{C}}}{2}}\,(\mathbf{V}_{\mathrm{L}1} + \mathbf{V}_{\mathrm{L}2})\,. \end{aligned} \tag{A.13}$$

Note that only the combinations $(\mathbf{V}_{\mathrm{L}1} + \mathbf{V}_{\mathrm{L}2})$ and $(\mathbf{V}_{\mathrm{R}1} + \mathbf{V}_{\mathrm{R}2})$ appear, in other words, acting on a short representation, the algebra can indeed be constructed from 2 zero modes.

In conclusion, a long representation requires four, while a short representation requires two fermion zero modes to reproduce the residual symmetry algebra. The giant magnon is the string dual of the elementary magnon excitations transforming in a short representation, and we expect it to have exactly two fermion zero modes on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. On $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ the residual algebra is the direct product of two $\mathfrak{su}(1|1)^2_{\text{c.e.}}$ algebras, and accordingly, the giant magnon on this background should have exactly four fermion zero modes.

Appendix B

Pullback of the vielbein and spin connection for the mixed-flux $\mathbb{R} \times \mathbf{S}^3$ magnon

Putting the giant magnon (3.1) as background, one finds the following components for the pulled-back vielbein $e_a^A = E_\mu^A(X)\partial_a X^\mu$

$$
\begin{aligned}
e_0^0 &= 1 \ , \qquad e_0^7 = \sin\varphi \ , \\
e_0^3 &= -\cos\varphi \ \frac{u\gamma^2\left(\tilde{q}^2 - u^2\right)\tanh\mathcal{Y}}{\sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}} \ , \\
e_0^4 &= \cos\varphi \ \frac{\left(q^2u^2\gamma^2 + \tilde{q}^2\sinh^2\mathcal{Y}\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}} \ , \\
e_0^5 &= \cos\varphi \ \frac{qu\gamma^2\sqrt{\tilde{q}^2 - u^2}\operatorname{sech}\mathcal{Y}}{\tilde{q}} \ ,
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
e_1^0 &= 0 \ , \qquad e_1^7 = 0 \ , \\
e_1^3 &= \cos\varphi \ \frac{\gamma^2\left(\tilde{q}^2 - u^2\right)\tanh\mathcal{Y}}{\sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}} \ , \\
e_1^4 &= \cos\varphi \ \frac{u\gamma^2\left(\tilde{q}^2 - u^2\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}} \ , \\
e_1^5 &= -\cos\varphi \ \frac{q\gamma^2\sqrt{\tilde{q}^2 - u^2}\operatorname{sech}\mathcal{Y}}{\tilde{q}} \ ,
\end{aligned}
\tag{B.2}
$$

while the only non-zero components of the spin connection (pulled back to the worldsheet) are

$$
\begin{aligned}
\omega_0^{34} = -\omega_0^{43} &= -\frac{\cos^2\varphi\sqrt{\tilde{q}^2 - u^2}}{\tilde{q}} \frac{\left(q^2u^2\gamma^2 + \tilde{q}^2\sinh^2\mathcal{Y}\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}, \\
\omega_1^{34} = -\omega_1^{43} &= -\frac{\cos^2\varphi\, u\gamma^2\left(\tilde{q}^2 - u^2\right)^{3/2}}{\tilde{q}} \frac{\operatorname{sech}\mathcal{Y}}{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}, \\
\omega_0^{35} = -\omega_0^{53} &= \frac{\cos^2\varphi\, qu\gamma^2}{\tilde{q}} \sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}\,\operatorname{sech}\mathcal{Y}, \\
\omega_1^{35} = -\omega_1^{53} &= -\frac{\cos^2\varphi\, q\gamma^2}{\tilde{q}} \sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}\,\operatorname{sech}\mathcal{Y}.
\end{aligned}
\tag{B.3}
$$

Appendix C

Gamma matrices for the mixed-flux $\mathbb{R}\times\mathbf{S}^3$ magnon

In chapter 3 we use a set of boosted gamma matrices, related to the original 10d Dirac matrices Γ^A, $A=0,1,...,9$, by

$$
\begin{aligned}
&\hat{\Gamma}^0 = \sec\varphi\left(\Gamma^0 - \sin\varphi\;\Gamma^7\right)\;,\quad \hat{\Gamma}^7 = \sec\varphi\left(\Gamma^7 - \sin\varphi\;\Gamma^0\right)\;,\\
&\hat{\Gamma}^A = \Gamma^A \quad \text{for } A\neq 0,7\;.
\end{aligned} \tag{C.1}
$$

We pick the representation of Γ^A that yields the following forms for $\hat{\Gamma}^A$:

$$
\begin{aligned}
\hat{\Gamma}^\mu &= \sigma^1\otimes\gamma^\mu\otimes\mathbb{1}\otimes\sigma^2\otimes\mathbb{1}\,, && \mu=0,1,2\\
\hat{\Gamma}^n &= \sigma^1\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma^1\otimes\gamma^n\,, && n=3,4,5\\
\hat{\Gamma}^{\dot{n}} &= \sigma^1\otimes\mathbb{1}\otimes\gamma^{\dot{n}}\otimes\sigma^3\otimes\mathbb{1}\,, && \dot{n}=6,7,8\\
\hat{\Gamma}^9 &= -\sigma^2\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\;,
\end{aligned} \tag{C.2}
$$

where, in terms of the Pauli matrices σ^i

$$
\begin{aligned}
\gamma^\mu &= (-i\sigma^3,\sigma^1,\sigma^2)\;,\\
\gamma^n &= (\sigma^1,\sigma^2,\sigma^3)\;,\\
\gamma^{\dot{n}} &= (\sigma^2,-\sigma^3,-\sigma^1)\;.
\end{aligned} \tag{C.3}
$$

In this basis,

$$
\begin{aligned}
\hat{\Gamma} &= \sigma^3\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\;,\\
\hat{\Gamma}^{12} &= \mathbb{1}\otimes(i\sigma^3)\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\;,\\
\hat{\Gamma}^{68} &= \mathbb{1}\otimes\mathbb{1}\otimes(i\sigma^3)\otimes\mathbb{1}\otimes\mathbb{1}\;,\\
\hat{\Gamma}^{012345} &= \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma^3\otimes\mathbb{1}\;,\\
\hat{\Gamma}^{34} &= \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes(i\sigma^3)\;,\\
\hat{\Gamma}^{35} &= \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes(-i\sigma^2)\;,
\end{aligned} \tag{C.4}
$$

and in particular we see that $\hat{\Gamma}$ (Weyl matrix), $\hat{\Gamma}^{12}$, $\hat{\Gamma}^{68}$ and the projectors $\hat{P}_{\pm} = \frac{1}{2}\left(\mathbb{1} \pm \hat{\Gamma}^{012345}\right)$ are simultaneously diagonalized.

Note that in this representation, instead of Γ^A, it is $\hat{\Gamma}^A$ that have definite hermiticity: $\hat{\Gamma}^0$ is anti-hermitian, while $\hat{\Gamma}^i$ is hermitian for $i = 1, 2, ..., 9$. Accordingly, for the intertwiners B, T and C, defined by the relations[1]

$$\begin{aligned}(\Gamma^A)^* &= \quad B\,\Gamma^A\,B^{-1},\\ (\Gamma^A)^\dagger &= -T\,\Gamma^A\,T^{-1},\\ (\Gamma^A)^T &= -C\,\Gamma^A\,C^{-1},\end{aligned} \tag{C.5}$$

we have

$$B = \Gamma^{1469}, \quad T = \hat{\Gamma}^0, \quad C = T\;B. \tag{C.6}$$

[1] These relations must hold for Γ^A, not the rotated $\hat{\Gamma}^A$.

Appendix D

Fermion derivatives for the mixed-flux $\mathbb{R}\times\mathbf{S}^3$ magnon

Looking at equation (3.15) we can define the following fermion derivatives

$$
\begin{aligned}
D &= \frac{(1-u)\gamma}{\cos^2\varphi\,\zeta}\left(\mathrm{D}_1-\mathrm{D}_0-\frac{1}{8}(\not{H}_0-\not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0}\ ,\\
\tilde{D} &= \frac{(1+u)\gamma}{\cos^2\varphi\,\zeta}\left(\mathrm{D}_1+\mathrm{D}_0-\frac{1}{8}(\not{H}_0+\not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0}\ ,
\end{aligned}
\tag{D.1}
$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, and $A\zeta(1\pm u)\gamma$ were introduced to normalize the $\partial_{\mathcal{Y}}$ term. The NS-NS flux appears as $\not{H}_a \equiv\ e_a^A H_{ABC}\Gamma^{BC}$, which we can rewrite

$$
\begin{aligned}
\not{H}_a &= \frac{1}{3}H_{ABC}\left(e_a^A\Gamma^{BC}+e_a^B\Gamma^{CA}+e_a^C\Gamma^{AB}\right)\\
&= \sum_{ABC}\frac{1}{6}H_{ABC}\Big(e_a^A(\Gamma_A\Gamma^{ABC}+\Gamma^{BCA}\Gamma_A)+e_a^B(\Gamma_B\Gamma^{BCA}+\Gamma^{CAB}\Gamma_B)\\
&\qquad\qquad + e_a^C(\Gamma_C\Gamma^{CAB}+\Gamma^{ABC}\Gamma_C)\Big)\\
&= \sum_{ABC}\frac{1}{6}H_{ABC}\sum_{D\in A,B,C}e_a^D(\Gamma_D\Gamma^{ABC}+\Gamma^{ABC}\Gamma_D)\\
&= \frac{1}{6}H_{ABC}\sum_D e_a^D(\Gamma_D\Gamma^{ABC}+\Gamma^{ABC}\Gamma_D)\\
&= \frac{1}{6}(\rho_a\not{H}+\not{H}\rho_a)\ .
\end{aligned}
\tag{D.2}
$$

On the first line we used the antisymmetry of H, going to the second that $\Gamma_A\Gamma^A = \mathbb{1}$ (no summation), on the third the antisymmetry of Γ^{ABC}, and lastly on the fourth line the fact that for $D\notin\{A,B,C\}$

$$
\Gamma_D\Gamma^{ABC}+\Gamma^{ABC}\Gamma_D = 0\ . \tag{D.3}
$$

Hence we have

$$D = \partial_{\mathcal{Y}} + \frac{1}{2} G\ \Gamma_{34} + \frac{1}{2} Q\ \Gamma_{35} - \frac{(1-u)\gamma}{48 \cos^2\varphi\, \zeta} \left(\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1)\not{H} \right)\ ,$$

$$\tilde{D} = \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G}\ \Gamma_{34} + \frac{1}{2} Q\ \Gamma_{35} - \frac{(1+u)\gamma}{48 \cos^2\varphi\, \zeta} \left(\not{H}(\rho_0 + \rho_1) + (\rho_0 + \rho_1)\not{H} \right)\ , \tag{D.4}$$

$$G = \frac{\omega_1^{34} - \omega_0^{34}}{\cos^2\varphi\, \zeta(1+u)\gamma} = \frac{\tilde{q}^2(1-u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}\left(\tilde{q}^2 \sinh^2\mathcal{Y} + u^2\right)} \operatorname{sech}\mathcal{Y}\ ,$$

$$\tilde{G} = \frac{\omega_1^{34} + \omega_0^{34}}{\cos^2\varphi\, \zeta(1-u)\gamma} = -\frac{\tilde{q}^2(1+u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}\left(\tilde{q}^2 \sinh^2\mathcal{Y} + u^2\right)} \operatorname{sech}\mathcal{Y}\ , \tag{D.5}$$

$$Q = \frac{\omega_1^{35} \mp \omega_0^{35}}{\cos^2\varphi\, \zeta(1 \pm u)\gamma} = -\frac{q}{\tilde{q}\sqrt{\tilde{q}^2 - u^2}} \sqrt{\tilde{q}^2 \sinh^2\mathcal{Y} + u^2}\ \operatorname{sech}\mathcal{Y}\ .$$

The next step is to take (3.46)

$$\not{H} = 24 q \cos\varphi \left(\hat{\Gamma}_*\ P_+ - \Delta\ \hat{\Gamma}^{12} \right) \tag{D.6}$$

and substitute into (D.4), with the further restriction that the derivatives act on kappa fixed spinors, as in (3.44). For DK_1 the relevant term is

$$\begin{aligned}
&\frac{1}{48\cos^2\varphi}\left(\not{H}(\rho_0 - \rho_1) + (\rho_0 - \rho_1)\not{H}\right) K_1 \\
&\quad = -\frac{1}{24\cos\varphi}\left(\not{H}\hat{\Gamma}^0 K_2 + \hat{\Gamma}^0 K_2 \not{H}\right) K_1 \\
&\quad = -q\ \Big(\hat{\Gamma}_* P_+ \hat{\Gamma}^0 K_2 K_1 + \hat{\Gamma}^0 K_2 \hat{\Gamma}_* P_+ K_1 \\
&\qquad\qquad - \Delta\hat{\Gamma}^{12}\hat{\Gamma}^0 K_2 K_1 - \hat{\Gamma}^0 K_2 \Delta \hat{\Gamma}^{12} K_1\Big) \\
&\quad = q\ \Big(P_- \hat{\Gamma}^{12} K_2 K_1 + (-K_2 + \mathbb{1})\, P_+ \hat{\Gamma}^{12} K_1 \\
&\qquad\qquad - \left(\Delta_0 + \Delta_7\, \hat{\Gamma}^{07}\right) \hat{\Gamma}^{12} K_2 K_1 \\
&\qquad\qquad + \left(\Delta_0 + \Delta_7\, \hat{\Gamma}^{07}\right) \hat{\Gamma}^{12} K_2 K_1 - \Delta_0\, \hat{\Gamma}^{12} K_1\Big) \\
&\quad = -q\left(RP_- - \left(R + \hat{\Gamma}_{12}\right) P_+ + \Delta_0\, \hat{\Gamma}_{12}\right) K_1,
\end{aligned} \tag{D.7}$$

where we have also used the definition of the kappa projectors (3.41), the form of Δ in (3.47), the relation $K_J\hat{\Gamma}^0 = -\hat{\Gamma}^0 K_J + \hat{\Gamma}^0$, and (3.52). Similarly, for $\tilde{D}K_2$ we have

$$\frac{1}{48\cos^2\varphi}\Big(\not{H}(\rho_0+\rho_1)+(\rho_0+\rho_1)\not{H}\Big)\,K_2 \tag{D.8}$$

$$= -q\left(RP_- - \left(R+\hat{\Gamma}_{12}\right)P_+ + \Delta_0\,\hat{\Gamma}_{12}\right)K_2, \tag{D.9}$$

and with this, the fermion derivatives take the final form

$$\begin{aligned} D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\,\hat{\Gamma}_{34} + \frac{1}{2}Q\,\hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\left(RP_- - (R+\hat{\Gamma}_{12})\,P_+ + \Delta_0\,\hat{\Gamma}_{12}\right)\ , \\ \tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\,\hat{\Gamma}_{34} + \frac{1}{2}Q\,\hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\left(RP_- - (R+\hat{\Gamma}_{12})\,P_+ + \Delta_0\,\hat{\Gamma}_{12}\right)\ . \end{aligned} \tag{D.10}$$

Let us stress one last time, that these forms are only valid when acting on kappa-fixed spinors.

Appendix E

Fermions fluctuations around the AdS_3 BMN vacuum

In this appendix we are going to derive the fermion perturbation spectrum around the maximally SUSY $AdS_3 \times S^3 \times S^3 \times S^1$ BMN string

$$\theta^{\pm} = \frac{\pi}{2}, \quad \phi_1^+ = \cos^2\varphi \; t, \quad \phi_1^- = \sin^2\varphi \; t, \quad \phi_2^{\pm} = 0. \tag{E.1}$$

Fermion equations and kappa-fixing. For this simple bosonic background all spin-connections are zero, $\omega_a^{AB} = 0$, and

$$\rho_0 = \Gamma_0 + \cos\varphi\,\Gamma_4 + \sin\varphi\,\Gamma_7, \qquad \rho_1 = 0. \tag{E.2}$$

With this, the fluctuation equations (3.15) simplify to

$$\begin{aligned} \rho_0 \left[(\partial_0 - \partial_1)\; \vartheta^1 + \frac{1}{48} (\rho_0 \not{H} + \not{H} \rho_0)\; \vartheta^1 + \frac{1}{48} \not{F} \rho_0 \; \vartheta^2 \right] = 0 \;, \\ \rho_0 \left[(\partial_0 + \partial_1)\; \vartheta^2 - \frac{1}{48} (\rho_0 \not{H} + \not{H} \rho_0)\; \vartheta^2 + \frac{1}{48} \not{F} \rho_0 \; \vartheta^1 \right] = 0 \;. \end{aligned} \tag{E.3}$$

We further note that $\rho_0^2 = 0$, and

$$K = \frac{1}{2}\,\Gamma^0 \rho_0 = \frac{1}{2} \left(\mathbb{1} + \cos\varphi \; \Gamma^{04} + \sin\varphi \; \Gamma^{07} \right) \tag{E.4}$$

can be used as a kappa-projector for both ϑ^1 and ϑ^2, leading to the equations

$$\begin{aligned} (\partial_0 - \partial_1)\,\Psi^1 + \frac{1}{24}\; K \not{H}\,\Gamma_0 K \;\Psi^1 + \frac{1}{24}\; K \not{F}\,\Gamma_0 K \;\Psi^2 = 0 \;, \\ (\partial_0 + \partial_1)\,\Psi^2 - \frac{1}{24}\; K \not{H}\,\Gamma_0 K \;\Psi^2 + \frac{1}{24}\; K \not{F}\,\Gamma_0 K \;\Psi^1 = 0 \;, \end{aligned} \tag{E.5}$$

for the kappa-fixed spinors $\Psi^J = K \vartheta^J$.

Invariant subspaces. Next, we introduce the 4 projectors

$$\mathcal{P}_{ab} = \frac{1}{4}\left(\mathbb{1} + a\,\Gamma^{1235}\right)\left(\mathbb{1} + b\,\Gamma^{1268}\right) \tag{E.6}$$

for $a, b = \pm 1$. These commute with the equation of motion, therefore we can look for solutions restricted to each of the 4 subspaces, i.e. $\Psi^J = \mathcal{P}_{ab}\Psi^J$. It is a simple exercise to show that on these subspaces

$$\frac{1}{24}K\not{H}\,\Gamma_0 K\,\mathcal{P}_{ab} = q\,m_{ab}\,\Gamma_{12}K\,\mathcal{P}_{ab}, \qquad \frac{1}{24}K\not{F}\,\Gamma_0 K\,\mathcal{P}_{ab} = \tilde{q}\,m_{ab}\,\Gamma_{12}K\,\mathcal{P}_{ab}, \tag{E.7}$$

with constants (as we will later see, masses)

$$m_{++} = 1, \quad m_{+-} = \cos^2\varphi, \quad m_{-+} = \sin^2\varphi, \quad m_{--} = 0. \tag{E.8}$$

We can further restrict Ψ^J with definite Γ_{12} eigenvalue. In other words, we have the equations

$$\begin{aligned} &(\partial_0 - \partial_1)\,\Psi^1 + i\,q\,m_{ab}\,\lambda_{12}\;\Psi^1 + i\,\tilde{q}\,m_{ab}\,\lambda_{12}\;\Psi^2 = 0\ , \\ &(\partial_0 + \partial_1)\,\Psi^2 - i\,q\,m_{ab}\,\lambda_{12}\;\Psi^2 + i\,\tilde{q}\,m_{ab}\,\lambda_{12}\;\Psi^1 = 0\ , \end{aligned} \tag{E.9}$$

where $\mathcal{P}_{ab}\Psi^J = \Psi^J$ and $\Gamma_{12}\Psi^J = i\lambda_{12}\Psi^J$ ($\lambda_{12} = \pm 1$).

Solutions. We look for plane-wave solutions of the form $e^{i(\omega t - kx)}$. There are two qualitatively distinct scenarios, determined by whether the equations for Ψ^1 and Ψ^2 decouple or not. Let us first consider the decoupled case, which happens for $q = 1$ or $m_{ab} = 0$, implying $q\,m_{ab} = m_{ab}$ and $\tilde{q}\,m_{ab} = 0$. Substituting the plane-wave ansatz we get the solutions

$$\omega = \mp(k + \lambda_{12} m_{ab}), \tag{E.10}$$

with the $\mp$ signs corresponding to the Ψ^1 and Ψ^2 solutions, respectively.

In all other cases we can invert the first equation for Ψ^2

$$\Psi^2 = \frac{i\lambda_{12}}{\tilde{q}\,m_{ab}}\,(\partial_0 - \partial_1 + i\,q\,m_{ab}\,\lambda_{12})\,\Psi^1\ , \tag{E.11}$$

and substitute into the second equation to get a second order PDE for Ψ^1

$$(\partial_1 + \partial_0 - i\, q\, m_{ab}\, \lambda_{12})\, (\partial_1 - \partial_0 - i\, q\, m_{ab}\, \lambda_{12})\, \Psi^1 - \tilde{q}^2\, m_{ab}^2\, \Psi^1 = 0. \qquad \text{(E.12)}$$

Equivalently,

$$\partial^2 \Psi^1 - m_{ab}^2\, \Psi^1 - 2i\, q\, m_{ab}\, \lambda_{12}\, \partial_1 \Psi^1 = 0, \qquad \text{(E.13)}$$

which is a q-deformed version of the massive wave equation, with solutions satisfying

$$\omega^2 = m_{ab}^2 + k^2 + 2\lambda_{12}\, q\, m_{ab}\, k. \qquad \text{(E.14)}$$

Finally, let us note that (E.10) and (E.14) together can be written as

$$\omega^2 = (m_{ab} \pm q\, k)^2 + \tilde{q}^2 k^2 \qquad \text{(E.15)}$$

in agreement with the dispersion relation of elementary $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ excitations of masses $m = 1$, $\cos^2\varphi$, $\sin^2\varphi$, 0.

Conclusion. We have found that the fermion perturbations around the BMN string correspond to the elementary fermion excitations of masses m_{ab} living on the 4 subspaces with projectors

$$\mathcal{P}_{ab} = \frac{1}{4} \left(\mathbb{1} + a\, \Gamma^{1235} \right) \left(\mathbb{1} + b\, \Gamma^{1268} \right). \qquad \text{(E.16)}$$

The corresponding masses are $m_{++} = 1$, $m_{+-} = \cos^2\varphi$, $m_{-+} = \sin^2\varphi$, and $m_{--} = 0$.

Appendix F

No normalizable solutions for $\Delta \neq 0$

In section 3.2 we found the expected number of normalizable solutions in an analytic form for $\Delta = 0$. However, to complete the counting argument for fermion zero modes, it is necessary to demonstrate that there are no normalizable solutions at all for $\Delta \neq 0$. This happens for the maximally SUSY $AdS_3 \times S^3 \times S^3 \times S^1$ giant magnon, on the $\Gamma_{1268} = +1$ spinor subspace:

$$\begin{aligned} \Delta = -\tan^2\varphi\ \hat{\Gamma}^0 - \tan\varphi \sec\varphi\ \hat{\Gamma}^7 = \left(\kappa^2 - \kappa\tilde{\kappa}\ \hat{\Gamma}_{07}\right)\hat{\Gamma}_0\ , \\ \kappa = \tan\varphi\ , \quad \tilde{\kappa} = \sqrt{1+\kappa^2} = \sec\varphi\ . \end{aligned} \tag{F.1}$$

The equations of motion are

$$\begin{aligned} \zeta(1+u)\gamma D\ \Psi^1 + \tilde{q}\left(R\ P_- - K_1\Delta\,\hat{\Gamma}_*\right)\Psi^2 = 0\ , \\ \zeta(1-u)\gamma\tilde{D}\ \Psi^2 - \tilde{q}\left(R\ P_- - K_2\Delta\,\hat{\Gamma}_*\right)\Psi^1 = 0\ . \end{aligned} \tag{F.2}$$

with fermion derivatives

$$\begin{aligned} D = \partial_{\mathcal{Y}} + \frac{1}{2}G\ \hat{\Gamma}_{34} + \frac{1}{2}Q\ \hat{\Gamma}_{35} + \frac{q(1-u)\gamma}{\zeta}\left(RP_- - (R+\hat{\Gamma}_{12})\,P_+ - \kappa^2\,\hat{\Gamma}_{12}\right)\ , \\ \tilde{D} = \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\ \hat{\Gamma}_{34} + \frac{1}{2}Q\ \hat{\Gamma}_{35} + \frac{q(1+u)\gamma}{\zeta}\left(RP_- - (R+\hat{\Gamma}_{12})\,P_+ - \kappa^2\,\hat{\Gamma}_{12}\right)\ . \end{aligned} \tag{F.3}$$

Our approach will be similar to section 3.2. First we write down general kappa-fixed spinors, which we then substitute into the equations of motion to get a system of simpler ODEs.

Kappa fixing. The main difference from $\Delta = 0$ is that the solutions will not have definite $P_\pm$ chirality, since Δ mixes the P_+ and P_- subspaces. Accordingly, the kappa-fixed ansatz generalizing (3.60) will have to relate the two projections. This is achieved by

$$\Psi^J = \sum_{\lambda=\pm} \Big[\Big(\alpha_+^J + \alpha_-^J \, \hat\Gamma_{45} \Big) \, f_J(\mathcal{Y}) + \Big(\bar\alpha_+^J + \bar\alpha_-^J \, \hat\Gamma_{45} \Big) \, g_J(\mathcal{Y}) \, \hat\Gamma_{07} \Big] \; U_\lambda \ , \tag{F.4}$$

where the constant spinor U_λ is shared between $I = 1$ and 2, and has eigenvalues $\Gamma_{34} U_\lambda = +i U_\lambda$, $P_- U_\lambda = U_\lambda$ and $\hat\Gamma_{12} U_\lambda = i\lambda U_\lambda$. The functions f_J, g_J represent the parts of the solution on the P_- and P_+ subspaces respectively, and $\hat\Gamma_{07}$ transforms U_λ between the two. We take $\alpha_\pm^J$ to be defined by (3.62), (3.63), and

$$\bar\alpha_\pm^J \;\equiv\; \alpha_\pm^J|_{\lambda \to -\lambda} \ . \tag{F.5}$$

This is because the definition of λ here differs from that in section 3.2, the two agree on the P_- subspace, while on P_+ they are related by a minus sign.

The $\mathbf{K \Delta \hat\Gamma_*}$ terms. For the most part, substitution yields equations that are familiar from section 3.2, the only new terms being $K_1 \Delta \, \hat\Gamma_* \Psi^2$ and $K_2 \Delta \, \hat\Gamma_* \Psi^1$. It is easy to see that

$$\Delta \, \hat\Gamma_* \Psi^J \;=\; \sum_{\lambda=\pm} \Big[\Big(\alpha_+^J + \alpha_-^J \, \hat\Gamma_{45} \Big) \, i\lambda \, \Big(\kappa^2 - \kappa\tilde\kappa \; \hat\Gamma_{07} \Big) \, f_J \tag{F.6}$$

$$+ \Big(\bar\alpha_+^J + \bar\alpha_-^J \, \hat\Gamma_{45} \Big) \, i\lambda \, \Big(\kappa^2 \; \hat\Gamma_{07} - \kappa\tilde\kappa \Big) \, g_J \Big] \, U_\lambda \ . \tag{F.7}$$

On the other hand, from (3.56) and the definitions (3.62)–(3.63) one can derive the action of the kappa-projectors on a general spinor $V = V_+ + V_-$ on the P_- subspace, with components $\hat\Gamma_{34} V_\pm = \pm i V_\pm$

$$\begin{aligned} K_1 V = \Big(\alpha_+^1 + \alpha_-^1 \, \hat\Gamma_{45} \Big) \Big[& \frac{1}{2} e^{-i\chi} \sqrt{1 - \lambda Q_+ \, \mathrm{sech}\mathcal{Y}} \; V_+ \\ & - \frac{1}{2} \lambda e^{i\chi} \sqrt{1 + \lambda Q_+ \, \mathrm{sech}\mathcal{Y}} \; \hat\Gamma_{45} V_- \Big] \ , \\ K_2 V = \Big(\alpha_+^2 + \alpha_-^2 \, \hat\Gamma_{45} \Big) \Big[& \frac{1}{2} e^{-i\tilde\chi} \sqrt{1 + \lambda Q_- \, \mathrm{sech}\mathcal{Y}} \; V_+ \\ & + \frac{1}{2} \lambda e^{i\tilde\chi} \sqrt{1 - \lambda Q_- \, \mathrm{sech}\mathcal{Y}} \; \hat\Gamma_{45} V_- \Big] \ . \end{aligned} \tag{F.8}$$

The corresponding expressions for the P_+ subspace are obtained by sending $\lambda \to -\lambda$. Putting these together we get

$$\begin{aligned} K_1 \Delta\, \hat{\Gamma}_* \Psi^2 \;&=\; \sum_{\lambda=\pm} \Big[\Big(\alpha^1_+ + \alpha^1_-\, \hat{\Gamma}_{45}\Big) \Big[i\lambda\tilde{q}\kappa^2 \delta_1\, f_2 - i\lambda\tilde{q}\kappa\tilde{\kappa}\delta_2\, g_2 \Big] \\ &\qquad + \Big(\bar{\alpha}^1_+ + \bar{\alpha}^1_-\, \hat{\Gamma}_{45}\Big) \Big[i\lambda\tilde{q}\kappa^2 \bar{\delta}_1\, g_2 - i\lambda\tilde{q}\kappa\tilde{\kappa}\bar{\delta}_2\, f_2 \Big] \hat{\Gamma}_{07} \Big] U_\lambda \;, \\ K_2 \Delta\, \hat{\Gamma}_* \Psi^1 \;&=\; \sum_{\lambda=\pm} \Big[\Big(\alpha^2_+ + \alpha^2_-\, \hat{\Gamma}_{45}\Big) \Big[-i\lambda\tilde{q}\kappa^2 \bar{\delta}_1\, f_1 - i\lambda\tilde{q}\kappa\tilde{\kappa}\delta_2\, g_1 \Big] \\ &\qquad + \Big(\bar{\alpha}^2_+ + \bar{\alpha}^2_-\, \hat{\Gamma}_{45}\Big) \Big[-i\lambda\tilde{q}\kappa^2 \delta_1\, g_1 - i\lambda\tilde{q}\kappa\tilde{\kappa}\bar{\delta}_2\, f_1 \Big] \hat{\Gamma}_{07} \Big] U_\lambda \;, \end{aligned} \tag{F.9}$$

with

$$\begin{aligned} \delta_1 &= \frac{1}{2}\bigg(e^{i(\tilde{\chi}-\chi)} \sqrt{(1-\lambda Q_+\,\mathrm{sech}\mathcal{Y})\,(1+\lambda Q_-\,\mathrm{sech}\mathcal{Y})} \\ &\qquad - e^{-i(\tilde{\chi}-\chi)} \sqrt{(1+\lambda Q_+\,\mathrm{sech}\mathcal{Y})\,(1-\lambda Q_-\,\mathrm{sech}\mathcal{Y})} \bigg) \;, \\ \delta_2 &= \frac{1}{2}\bigg(e^{i(\tilde{\chi}-\chi)} \sqrt{(1-\lambda Q_+\,\mathrm{sech}\mathcal{Y})\,(1-\lambda Q_-\,\mathrm{sech}\mathcal{Y})} \\ &\qquad + e^{-i(\tilde{\chi}-\chi)} \sqrt{(1+\lambda Q_+\,\mathrm{sech}\mathcal{Y})\,(1+\lambda Q_-\,\mathrm{sech}\mathcal{Y})} \bigg) \;, \end{aligned} \tag{F.10}$$

and $\bar{\delta}_J = \delta_J|_{\lambda\to-\lambda}$.

Reduced equations. Substituting (F.4) into (F.2) we get

$$\begin{aligned} \sum_{\lambda=\pm} \Big[&\Big(\alpha^1_+ + \alpha^1_-\, \hat{\Gamma}_{45}\Big) \Big[\partial_{\mathcal{Y}} f_1 + \Big(C_{11} - i\lambda q\kappa^2 \frac{(1-u)\gamma}{\zeta} \Big) f_1 \\ &\quad + \Big(C_{12} - i\lambda\tilde{q}\kappa^2\delta_1 \frac{(1-u)\gamma}{\zeta} \Big) f_2 + i\lambda\tilde{q}\kappa\tilde{\kappa}\delta_2 \frac{(1-u)\gamma}{\zeta}\, g_2 \Big] \\ &+ \Big(\bar{\alpha}^1_+ + \bar{\alpha}^1_-\, \hat{\Gamma}_{45}\Big) \Big[\partial_{\mathcal{Y}} g_1 - \Big(C_+ + i\lambda q\kappa^2 \frac{(1-u)\gamma}{\zeta} \Big) g_1 \\ &\quad + i\lambda\tilde{q}\kappa\tilde{\kappa}\bar{\delta}_2 \frac{(1-u)\gamma}{\zeta}\, f_2 - i\lambda\tilde{q}\kappa^2\bar{\delta}_1 \frac{(1-u)\gamma}{\zeta}\, g_2 \Big] \hat{\Gamma}_{07} \Big] U_\lambda \;=\; 0 \;, \end{aligned} \tag{F.11}$$

$$\sum_{\lambda=\pm}\Bigg[\left(\alpha_+^2+\alpha_-^2\,\hat\Gamma_{45}\right)\Bigg[\partial_{\mathcal{Y}} f_2+\left(C_{21}-i\lambda q\kappa^2\frac{(1+u)\gamma}{\zeta}\right)f_2$$
$$+\left(C_{22}-i\lambda\tilde q\kappa^2\bar\delta_1\frac{(1+u)\gamma}{\zeta}\right)f_1-i\lambda\tilde q\kappa\tilde\kappa\delta_2\frac{(1+u)\gamma}{\zeta}\,g_1\Bigg]$$
$$+\left(\bar\alpha_+^2+\bar\alpha_-^2\,\hat\Gamma_{45}\right)\Bigg[\partial_{\mathcal{Y}} g_2-\left(C_-+i\lambda q\kappa^2\frac{(1+u)\gamma}{\zeta}\right)g_2$$
$$-i\lambda\tilde q\kappa\tilde\kappa\bar\delta_2\frac{(1+u)\gamma}{\zeta}\,f_1-i\lambda\tilde q\kappa^2\delta_1\frac{(1+u)\gamma}{\zeta}\,g_1\Bigg]\hat\Gamma_{07}\Bigg]U_\lambda \;=\; 0\;, \tag{F.12}$$

where $C_\pm$ and C_{ij} are as defined in (3.67), (3.72). If we make the ansatz

$$f_1=\frac{e^{-\int C_{11}\mathrm{d}\mathcal{Y}}}{\sqrt{1+u}}\,\tilde f_1\;,\qquad g_1=\frac{e^{\int C_+\mathrm{d}\mathcal{Y}}}{\sqrt{1+u}}\,\tilde g_1\;,$$
$$f_2=\frac{e^{-\int C_{21}\mathrm{d}\mathcal{Y}}}{\sqrt{1-u}}\,\tilde f_2\;,\qquad g_2=\frac{e^{\int C_-\mathrm{d}\mathcal{Y}}}{\sqrt{1-u}}\,\tilde g_2\;, \tag{F.13}$$

we get the following four equations

$$\left(\partial_{\mathcal{Y}}-i\lambda q\kappa^2\frac{(1-u)\gamma}{\zeta}\right)\tilde f_1+i\lambda\tilde q\kappa\tilde\kappa\frac{\delta_2}{\zeta}e^{\int(C_{11}+C_-)\mathrm{d}\mathcal{Y}}\,\tilde g_2$$
$$+\left(e^{+i2\lambda\xi\mathcal{Y}}\left(\lambda\tanh\mathcal{Y}-i\xi\right)-i\lambda\tilde q\kappa^2\frac{\delta_1}{\zeta}e^{\int(C_{11}-C_{21})\mathrm{d}\mathcal{Y}}\right)\tilde f_2 \;=\; 0\;, \tag{F.14}$$

$$\left(\partial_{\mathcal{Y}}-i\lambda q\kappa^2\frac{(1+u)\gamma}{\zeta}\right)\tilde f_2-i\lambda\tilde q\kappa\tilde\kappa\frac{\delta_2}{\zeta}e^{\int(C_{21}+C_+)\mathrm{d}\mathcal{Y}}\,\tilde g_1$$
$$+\left(e^{-i2\lambda\xi\mathcal{Y}}\left(\lambda\tanh\mathcal{Y}+i\xi\right)-i\lambda\tilde q\kappa^2\frac{\bar\delta_1}{\zeta}e^{\int(C_{21}-C_{11})\mathrm{d}\mathcal{Y}}\right)\tilde f_1 \;=\; 0\;, \tag{F.15}$$

$$\left(\partial_{\mathcal{Y}}-i\lambda q\kappa^2\frac{(1-u)\gamma}{\zeta}\right)\tilde g_1+i\lambda\tilde q\kappa\tilde\kappa\frac{\bar\delta_2}{\zeta}e^{-\int(C_{21}+C_+)\mathrm{d}\mathcal{Y}}\,\tilde f_2$$
$$-i\lambda\tilde q\kappa^2\frac{\bar\delta_1}{\zeta}e^{\int(C_--C_+)\mathrm{d}\mathcal{Y}}\,\tilde g_2 \;=\; 0\;, \tag{F.16}$$

$$\left(\partial_{\mathcal{Y}}-i\lambda q\kappa^2\frac{(1+u)\gamma}{\zeta}\right)\tilde g_2-i\lambda\tilde q\kappa\tilde\kappa\frac{\bar\delta_2}{\zeta}e^{-\int(C_{11}+C_-)\mathrm{d}\mathcal{Y}}\,\tilde f_1$$
$$-i\lambda\tilde q\kappa^2\frac{\delta_1}{\zeta}e^{\int(C_+-C_-)\mathrm{d}\mathcal{Y}}\,\tilde g_1 \;=\; 0\;. \tag{F.17}$$

The first thing to observe is that setting $\kappa=0$ the functions $\tilde f_1,\tilde f_2$ decouple from $\tilde g_1,\tilde g_2$, and indeed we recover the $\Delta=0$ solutions found in section 3.2.

Pure R-R background. We have not been able to find exact solutions at general values of q and κ, nonetheless, we can give an argument for their non-normalizability if we consider an expansion in powers of q and κ. It turns out we can already see non-normalizability at leading order in q, i.e. at $q = 0$, with the equations simplifying to

$$\begin{aligned}
\partial_{\mathcal{Y}} \tilde{f}_1 \; + \; \lambda \tilde{\kappa}^2 \tanh \mathcal{Y} \; \tilde{f}_2 \; + \; i \, \lambda \kappa \tilde{\kappa} \, \mathrm{sech} \mathcal{Y} \; \tilde{g}_2 \; &= \; 0 \; , \\
\partial_{\mathcal{Y}} \tilde{f}_2 \; + \; \lambda \tilde{\kappa}^2 \tanh \mathcal{Y} \; \tilde{f}_1 \; - \; i \, \lambda \kappa \tilde{\kappa} \, \mathrm{sech} \mathcal{Y} \; \tilde{g}_1 \; &= \; 0 \; , \\
\partial_{\mathcal{Y}} \tilde{g}_1 \; + \; \lambda \kappa^2 \tanh \mathcal{Y} \; \tilde{g}_2 \; + \; i \, \lambda \kappa \tilde{\kappa} \, \mathrm{sech} \mathcal{Y} \; \tilde{f}_2 \; &= \; 0 \; , \\
\partial_{\mathcal{Y}} \tilde{g}_2 \; + \; \lambda \kappa^2 \tanh \mathcal{Y} \; \tilde{g}_1 \; - \; i \, \lambda \kappa \tilde{\kappa} \, \mathrm{sech} \mathcal{Y} \; \tilde{f}_1 \; &= \; 0 \; .
\end{aligned} \tag{F.18}$$

Zeroth order in κ. The first thing to observe is that setting $\kappa = 0$ leads to a significant simplification of the equations. $\tilde{f}_1, \tilde{f}_2$ decouple from $\tilde{g}_1, \tilde{g}_2$, and the solutions take the form

$$\begin{aligned}
\tilde{f}_1 &= \quad c_1 \mathrm{sech} \mathcal{Y} + \quad c_2 \cosh \mathcal{Y} \; , \quad \tilde{g}_1 = c_3 \; , \\
\tilde{f}_2 &= \lambda c_1 \mathrm{sech} \mathcal{Y} - \lambda c_2 \cosh \mathcal{Y} \; , \quad \tilde{g}_2 = c_4 \; .
\end{aligned} \tag{F.19}$$

This limit corresponds to the case of $\Delta = 0$, and the solutions match those found in section 3.2, after we set $q = 0$. Let us denote the only normalizable solution in the $\kappa \to 0$ limit by

$$\begin{aligned}
\tilde{f}_1^{(0)} &= \quad C_0 \, \mathrm{sech} \mathcal{Y} \; , \qquad \tilde{g}_1^{(0)} = 0 \; , \\
\tilde{f}_2^{(0)} &= \lambda C_0 \, \mathrm{sech} \mathcal{Y} \; , \qquad \tilde{g}_2^{(0)} = 0 \; .
\end{aligned} \tag{F.20}$$

Expansion in κ. Introducing the vector notation $\mathbf{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2)^\top$, the equations above, for general values of κ, can be written as

$$\partial_{\mathcal{Y}} \mathbf{f} + M_\kappa(\mathcal{Y}) \mathbf{f} = \mathbf{0} \; . \tag{F.21}$$

Since $M_\kappa(\mathcal{Y})$ is regular at $\kappa = 0$, we can make the ansatz

$$\mathbf{f} = \sum_{n=0}^{\infty} \kappa^n \mathbf{f}^{(n)} \; , \tag{F.22}$$

where $\mathbf{f}^{(n)} = (\tilde{f}_1^{(n)}, \tilde{f}_2^{(n)}, \tilde{g}_1^{(n)}, \tilde{g}_2^{(n)})^\top$ are independent of κ. Substituting this into the equations, then expanding in κ, we get a system of ODEs for each power of κ: for all n the $\mathbf{f}^{(n)}$ equations will have the same homogeneous part as the $\kappa = 0$ system, and the forcing terms will be given by some linear combination of lower order solutions

$$\partial_{\mathcal{Y}} \mathbf{f}^{(n)} + M_0(\mathcal{Y}) \mathbf{f}^{(n)} = \sum_{k=0}^{n-1} {F^n}_k \, \mathbf{f}^{(k)} \ . \tag{F.23}$$

We need to solve these order-by-order, and for normalizability at generic values of κ, we would need all $\mathbf{f}^{(n)}$ to be normalizable.

First order in κ. At zeroth order we simply have the homogeneous $\kappa = 0$ equations, and the normalizable $\mathbf{f}^{(0)}$ solution is (F.20). The first subleading solution $\mathbf{f}^{(1)}$ is obtained from (F.23) with

$${F^1}_0 = i\lambda \, \mathrm{sech}\mathcal{Y} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{F.24}$$

and is given by

$$\begin{aligned} \tilde{f}_1^{(1)} &= c_3 \mathrm{sech}\mathcal{Y} + c_4 \cosh\mathcal{Y} \ , \qquad & \tilde{g}_1^{(1)} &= c_1 - \ \ i \, C_0 \tanh\mathcal{Y} \ , \\ \tilde{f}_2^{(1)} &= c_3 \mathrm{sech}\mathcal{Y} - c_4 \cosh\mathcal{Y} \ , \qquad & \tilde{g}_2^{(1)} &= c_2 - i\lambda \ C_0 \tanh\mathcal{Y} \ . \end{aligned} \tag{F.25}$$

The terms with C_0 are fixed, they are the response to the zeroth order ($\kappa = 0$) solution (F.20), while the integration constants c_j for $j = 1, ..., 4$ parametrize the homogeneous solution. We see that there is no combination of c_j that would make all components normalizable, in particular, $\tilde{g}_J^{(1)}$ can be chosen to decay at either $\mathcal{Y} \to \infty$ or $\mathcal{Y} \to -\infty$, but not both.

It is already impossible to find a decaying solution at first order in κ, and we conclude that there are no normalizable solutions for $\Delta \neq 0$.

Appendix G

Phase identities for the mixed-flux $\mathbb{R} \times S^3$ magnon

The following formulae are useful when deriving the reduced equations of motion (3.66) and (3.71). Using simple trigonometric and hyperbolic identities and Euler's formula it is easy to see that

$$e^{i\,\mathrm{arccot}(\alpha\,\mathrm{csch}\,\mathcal{Y})} = i\left(\frac{\sinh\mathcal{Y} - i\alpha}{\sinh\mathcal{Y} + i\alpha}\right)^{1/2} , \tag{G.1}$$

$$e^{i\arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-\alpha^2\,\mathrm{sech}^2\mathcal{Y}}}\right)} = i\,\frac{\tanh\mathcal{Y} - i\sqrt{1-\alpha^2}\mathrm{sech}\mathcal{Y}}{\sqrt{1-\alpha^2\,\mathrm{sech}^2\mathcal{Y}}} , \tag{G.2}$$

and with these we have

$$\begin{aligned} e^{i\chi} &= \left(\frac{\tilde{q}\sinh\mathcal{Y} - iu}{\tilde{q}\sinh\mathcal{Y} + iu}\right)^{1/4}\left(\frac{\tanh\mathcal{Y} + i\sqrt{1-Q_+^2}\mathrm{sech}\mathcal{Y}}{\sqrt{1-Q_+^2\,\mathrm{sech}^2\mathcal{Y}}}\right)^{1/2} , \\ e^{i\tilde{\chi}} &= i\left(\frac{\tilde{q}\sinh\mathcal{Y} - iu}{\tilde{q}\sinh\mathcal{Y} + iu}\right)^{1/4}\left(\frac{\tanh\mathcal{Y} - i\sqrt{1-Q_-^2}\mathrm{sech}\mathcal{Y}}{\sqrt{1-Q_-^2\,\mathrm{sech}^2\mathcal{Y}}}\right)^{1/2} , \end{aligned} \tag{G.3}$$

where, as defined in (3.57),

$$\begin{aligned} \chi &= \frac{1}{2}\left(\mathrm{arccot}\left(\frac{u\,\mathrm{csch}\,\mathcal{Y}}{\tilde{q}}\right) - \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_+^2\,\mathrm{sech}^2\mathcal{Y}}}\right)\right) , \\ \tilde{\chi} &= \frac{1}{2}\left(\mathrm{arccot}\left(\frac{u\,\mathrm{csch}\,\mathcal{Y}}{\tilde{q}}\right) + \arcsin\left(\frac{\tanh\mathcal{Y}}{\sqrt{1-Q_-^2\,\mathrm{sech}^2\mathcal{Y}}}\right)\right) . \end{aligned} \tag{G.4}$$

Appendix H

SU(2) currents for the $q = 1$ giant magnon

Using the usual SU(2) embedding (2.226), it is a relatively simple exercise to derive the left- and right-currents for the $q = 1$ giant magnon (3.101):

$$\begin{aligned} \mathfrak{J}_+ &= \begin{pmatrix} ia & b \\ -b^* & -ia \end{pmatrix} , \quad \mathfrak{J}_- = \begin{pmatrix} ic & d \\ -d^* & -ic \end{pmatrix} , \\ \mathfrak{K}_+ &= \begin{pmatrix} ie & f \\ -f^* & -ie \end{pmatrix} , \quad \mathfrak{K}_- = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \end{aligned} \tag{H.1}$$

where

$$\begin{aligned} a &= 1 - 2\beta^2 \sin^2\tfrac{p}{2} \,\mathrm{sech}^2\mathcal{Y} , \\ b &= 2i\beta \sin^2\tfrac{p}{2} \,\mathrm{sech}\mathcal{Y} (\sec\rho - \beta\tan\rho - i\beta\tanh\mathcal{Y}) e^{-2i(1-\beta\sin\rho)x^+} , \\ c &= 1 - 2\sin^2\tfrac{p}{2} \,\mathrm{sech}^2\mathcal{Y} , \\ d &= 2i\sin\tfrac{p}{2} \,\mathrm{sech}\mathcal{Y} \sqrt{1 - \sin^2\tfrac{p}{2}\,\mathrm{sech}^2\mathcal{Y}}\, e^{-2i(1-\beta\sin\rho)x^+ - i\arctan\left(\tan\frac{p}{2}\tanh\mathcal{Y}\right)} , \\ e &= 1 - 2\cos^2\rho \,\mathrm{sech}^2\mathcal{Y} , \\ f &= 2\cos^2\rho \,\mathrm{sech}\mathcal{Y} (\tanh\mathcal{Y} - i\tan\rho) e^{2i\left(\beta\sin\rho x^+ + x^-\right)} . \end{aligned} \tag{H.2}$$

Appendix I

Terms appearing in the $q = 1$ fermion equations for the $\mathbb{R} \times \mathbf{S}^3$ magnon

With the bosonic solution from section 3.3 as background, the following are the components of the pulled-back vielbein $e_a^A = E_\mu^A(X)\partial_a X^\mu$

$$\begin{aligned}
e_0^0 &= 1 \,, \\
e_0^3 &= \frac{\beta\cos\rho\,\tanh\mathcal{Y}}{\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}} \,, \qquad e_0^5 = \beta\sin\rho\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,, \\
e_0^4 &= \frac{b\beta\cos\rho+b^2+(1+b^2)\sinh^2\mathcal{Y}}{\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,,
\end{aligned} \tag{I.1}$$

$$\begin{aligned}
e_1^0 &= 0 \,, \\
e_1^3 &= \frac{\beta\cos\rho\,\tanh\mathcal{Y}}{\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}} \,, \qquad e_1^5 = (\beta\sin\rho-1)\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,, \\
e_1^4 &= \frac{b\beta\cos\rho}{\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,,
\end{aligned} \tag{I.2}$$

while the only non-zero components of the spin connection (pulled back to the worldsheet) are

$$\begin{aligned}
\omega_0^{34} = -\omega_0^{43} &= -\frac{b\beta\cos\rho+b^2+(1+b^2)\sinh^2\mathcal{Y}}{b^2+(1+b^2)\sinh^2\mathcal{Y}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,, \\
\omega_1^{34} = -\omega_1^{43} &= -\frac{b\beta\cos\rho}{b^2+(1+b^2)\sinh^2\mathcal{Y}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,, \\
\omega_0^{35} = -\omega_0^{53} &= \beta\sin\rho\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,, \\
\omega_1^{35} = -\omega_1^{53} &= (\beta\sin\rho-1)\sqrt{b^2+(1+b^2)\sinh^2\mathcal{Y}}\,\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \,.
\end{aligned} \tag{I.3}$$

Note that $\mathcal{Y} = 2\beta \cos\rho\, x^+$ and the three parameters are related by $\beta = -(b\cos\rho - \sin\rho)$. The combinations appearing in the fermion derivatives are

$$
\begin{aligned}
G &= \omega_1^{34} - \omega_0^{34} = \frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \ , \\
Q &= \omega_1^{35} - \omega_0^{35} = -\sqrt{b^2 + (1+b^2)\sinh^2\mathcal{Y}}\frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \ , \\
\tilde{G} &= \omega_1^{34} + \omega_0^{34} = -\frac{2b\beta\cos\rho + b^2 + (1+b^2)\sinh^2\mathcal{Y}}{b^2 + (1+b^2)\sinh^2\mathcal{Y}}\ \frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \ , \\
\tilde{Q} &= \omega_1^{35} + \omega_0^{35} = (2\beta\sin\rho - 1)\sqrt{b^2 + (1+b^2)\sinh^2\mathcal{Y}}\ \frac{\mathrm{sech}\mathcal{Y}}{\sqrt{1+b^2}} \ .
\end{aligned}
\tag{I.4}
$$

Appendix J

Dressing the perturbed BMN string

In this appendix we apply the $SU(2)$ dressing method [143, 144, 145] to perturbations of the BMN strings to generate the three S^3 fluctuations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ giant magnon. The S^3_+ perturbations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ magnon can be obtained from these, simply by scaling the worldsheet coordinates by $\cos^2\varphi$. For a detailed description of the dressing method the reader is referred to section 2.2.3, here we just repeat the key points, in order to lay down the notation for the rest of the section.

J.1 Review of the SU(2) dressing method

The sigma-model action for $\mathbb{R} \times \mathrm{S}^3$ strings in static conformal gauge is equivalent to the SU(2) principal chiral model with Wess-Zumino term (2.223). This formulation uses the embedding

$$g = \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \mathrm{SU}(2), \tag{J.1}$$

where, in terms of the $\mathbb{R}^4$ coordinates of (4.3)

$$\begin{aligned} Z_1 &= X_1 + iX_2, \qquad \bar{Z}_1 = X_1 - iX_2, \\ Z_2 &= X_3 + iX_4, \qquad \bar{Z}_2 = X_3 - iX_4. \end{aligned} \tag{J.2}$$

Note that $\bar{Z}_i$ are the complex conjugates of Z_i for the real classical solution, but not necessarily for the perturbation that we will write as complex functions. Starting with a solution g, the dressing method aims to find the appropriate dressing factor $\chi(z, \bar{z})$ such that

$$g \to g' = \chi g \tag{J.3}$$

is a new solution. The equations of motion for the principal chiral model are equivalent to the compatibility condition of the overdetermined *auxiliary system*

$$\bar{\partial}\Psi = \frac{A\Psi}{1+(1+q)\lambda}, \qquad \partial\Psi = \frac{B\Psi}{1-(1-q)\lambda}, \tag{J.4}$$

via

$$A = \bar{\partial}g\, g^{-1}, \qquad B = \partial g\, g^{-1}. \tag{J.5}$$

Given the solution $\Psi(\lambda)$ for general spectral parameter λ, satisfying

$$\Psi(0) = g, \tag{J.6}$$

the simplest non-trivial dressing factor is

$$\chi(\lambda) = \mathbf{1} + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P, \tag{J.7}$$

with the projector

$$P = \frac{v_1 v_1^\dagger}{v_1^\dagger v_1}, \quad v_1 = \Psi(\bar{\lambda}_1)e, \quad e = (1,1). \tag{J.8}$$

Below we will also refer to the matrix X and scalar y

$$X = v_1 v_1^\dagger, \quad y = v_1^\dagger v_1 \quad : \quad P = \frac{X}{y}. \tag{J.9}$$

In order for $\chi(0)\Psi(0)$ to have unit determinant, we need to introduce an additional constant phase $(\lambda_1/\bar{\lambda}_1)^{1/2}$, and with this, the dressed solution becomes

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left(\mathbb{1} - \left(1 - \frac{\bar{\lambda}_1}{\lambda_1}\right) P \right) g\ . \tag{J.10}$$

J.2 Dressing the unperturbed BMN string

To set the scene and some notation, let us quickly run through the application of the dressing method to the BMN string $Z_1 = e^{it}, Z_2 = 0$. We solve the auxiliary problem

$$g_{\text{BMN}} = \begin{pmatrix} e^{-i(z-\bar{z})} & 0 \\ 0 & e^{i(z-\bar{z})} \end{pmatrix}, \qquad A_{\text{BMN}} = -B_{\text{BMN}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{J.11}$$

to find

$$\Psi_{\mathrm{BMN}}(\lambda) = \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & e^{iZ(\lambda)} \end{pmatrix}, \qquad Z(\lambda) = \frac{z}{1-(1-q)\lambda} - \frac{\bar{z}}{1+(1+q)\lambda}. \tag{J.12}$$

Introducing the real variables

$$U = i\left(Z(\bar{\lambda}_1) - Z(\lambda_1)\right), \qquad V = -Z(\bar{\lambda}_1) - Z(\lambda_1) - t, \tag{J.13}$$

the projector (J.8) becomes

$$P_{\mathrm{BMN}} = \frac{X_{\mathrm{BMN}}}{y_{\mathrm{BMN}}} \tag{J.14}$$

with

$$y_{\mathrm{BMN}} = 2\cosh U, \quad X_{\mathrm{BMN}} = \begin{pmatrix} e^{-U} & e^{i(t+V)} \\ e^{-i(t+V)} & e^{U} \end{pmatrix}. \tag{J.15}$$

Pametrizing the pole as $\lambda_1 = re^{i\frac{\mathrm{p}}{2}}$, the dressing (J.10) yields the giant magnon

$$g_{\mathrm{GM}} = \begin{pmatrix} e^{it}\left[\cos\frac{p}{2} + i\sin\frac{p}{2}\,\tanh U\right] & -ie^{iV}\sin\frac{p}{2}\,\mathrm{sech}U \\ -ie^{-iV}\sin\frac{p}{2}\,\mathrm{sech}U & e^{-it}\left[\cos\frac{p}{2} - i\sin\frac{p}{2}\,\tanh U\right] \end{pmatrix}. \tag{J.16}$$

Furthermore, setting $r = \tilde{q}^{-1}$, we get the stationary magnon

$$U = \gamma\sqrt{\tilde{q}^2 - u^2}\,\mathcal{X}, \quad V = -q\gamma\,\mathcal{X}, \qquad \mathcal{X} = \gamma(x - ut), \tag{J.17}$$

where

$$\gamma^2 = \frac{1}{1-u^2}, \quad \cot\tfrac{\mathrm{p}}{2} = \frac{u}{\sqrt{\tilde{q}^2 - u^2}}. \tag{J.18}$$

J.3 Dressing the perturbed BMN string

To apply the dressing method to the perturbed BMN string

$$g_0 = g_{\mathrm{BMN}} + \delta\, g_{\mathrm{pert}}, \tag{J.19}$$

in each step we keep terms up to first order in δ. For example

$$g_0^{-1} = g_{\rm BMN}^{-1} - \delta\, g_{\rm BMN}^{-1}\, g_{\rm pert}\, g_{\rm BMN}^{-1}. \tag{J.20}$$

The auxiliary problem can be written as

$$A_0 = A_{\rm BMN} + \delta\, A_{\rm pert}, \quad B_0 = B_{\rm BMN} + \delta\, B_{\rm pert}, \tag{J.21}$$

and its solution

$$\Psi_0(\lambda) = \Psi_{\rm BMN}(\lambda) + \delta\, \Psi_{\rm pert}(\lambda). \tag{J.22}$$

Then we expand the projector

$$P_0 = \frac{X_0}{y_0} = \frac{X_{\rm BMN} + \delta\, X_{\rm pert}}{y_{\rm BMN} + \delta\, y_{\rm pert}} \equiv P_{\rm BMN} + \delta\, P_{\rm pert}, \tag{J.23}$$

i.e.

$$P_{\rm pert} = \frac{X_{\rm pert}}{y_{\rm BMN}} - \frac{y_{\rm pert}}{y_{\rm BMN}}\, P_{\rm BMN}, \tag{J.24}$$

and the dressing factor (J.10)

$$\chi_0 = \chi_{\rm BMN} + \delta\, \chi_{\rm pert} \;\; : \qquad \chi_{\rm pert} = \frac{\bar\lambda_1 - \lambda_1}{|\lambda_1|} P_{\rm pert}. \tag{J.25}$$

Finally, the dressed solution is

$$g_1 = \chi_0\, g_0 \approx \chi_{\rm BMN}\, g_{\rm BMN} + \delta\left(\chi_{\rm pert}\, g_{\rm BMN} + \chi_{\rm BMN}\, g_{\rm pert}\right) \tag{J.26}$$

from which we can read off the perturbation as the first order term. Let us now apply these steps to the three perturbations we found in[1] (4.27)–(4.28).

Massless fluctuation.

The massless BMN perturbation is

$$g_{\rm pert} = e^{ikx - i\omega t} \begin{pmatrix} ie^{it} & 0 \\ 0 & -ie^{-it} \end{pmatrix}, \qquad \omega^2 = k^2, \tag{J.27}$$

[1] Setting $\sin\varphi = 1$ for the S^3_- perturbations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ magnon gives the S^3 fluctuations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ BMN string.

for which the auxiliary problem has perturbations

$$\begin{aligned} A_{\text{pert}} &= i(\omega - k)e^{ikx - i\omega t} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \\ B_{\text{pert}} &= i(\omega + k)e^{ikx - i\omega t} \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \end{aligned} \tag{J.28}$$

and

$$\Psi_{\text{pert}}(\lambda) = \frac{i\ (k(1+q\lambda) + \omega\lambda)e^{ikx - i\omega t}}{k(1-(1-q)\lambda)(1+(1+q)\lambda)} \begin{pmatrix} e^{-iZ(\lambda)} & 0 \\ 0 & -e^{iZ(\lambda)} \end{pmatrix}. \tag{J.29}$$

Substituting into (J.24), (J.25) we can read off the fluctuation components

$$\begin{aligned} z_1 = &-ie^{ikx - i\omega t}e^{+it}\Big(\tilde{q}k - \omega\cos\tfrac{p}{2} \\ &\qquad - i\sin\tfrac{p}{2}\,\tanh U\left(\omega - \tilde{q}k\,\cosh(U + i\tfrac{p}{2})\,\text{sech}U\right)\Big), \\ \bar{z}_1 = &\quad ie^{ikx - i\omega t}e^{-it}\Big(\tilde{q}k - \omega\cos\tfrac{p}{2} \\ &\qquad + i\sin\tfrac{p}{2}\,\tanh U\left(\omega - \tilde{q}k\,\cosh(U - i\tfrac{p}{2})\,\text{sech}U\right)\Big), \\ z_2 = &\quad ie^{ikx - i\omega t}\sin\tfrac{p}{2}\,e^{+iV}\,\text{sech}U\left(qk - i\tilde{q}k\sin\tfrac{p}{2}\,\tanh U\right), \\ \bar{z}_2 = &-ie^{ikx - i\omega t}\sin\tfrac{p}{2}\,e^{-iV}\,\text{sech}U\left(qk + i\tilde{q}k\sin\tfrac{p}{2}\,\tanh U\right), \end{aligned} \tag{J.30}$$

Massive fluctuation (1).

The first massive BMN fluctuation is

$$g_{\text{pert}} = e^{ikx - i\omega t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \omega^2 = (1 + qk)^2 + \tilde{q}^2k^2, \tag{J.31}$$

for which the auxiliary problem has perturbations

$$\begin{aligned} A_{\text{pert}} &= +(\omega + 1 - k)e^{it}e^{ikx - i\omega t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ B_{\text{pert}} &= -(\omega + 1 + k)e^{it}e^{ikx - i\omega t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{J.32}$$

and
$$\Psi_{\text{pert}}(\lambda) = \frac{e^{it} e^{ikx-i\omega t}}{1+(1+q)\frac{\omega-1-k}{\omega+1-k}\lambda} \begin{pmatrix} 0 & e^{iZ(\lambda)} \\ 0 & 0 \end{pmatrix}. \tag{J.33}$$

Further substituting and using the identity

$$\frac{1}{1+\frac{\omega-1-k}{\omega+1-k}\sqrt{\frac{1+q}{1-q}}e^{i\frac{\mathrm{p}}{2}}} = \frac{1}{2}\frac{\omega+1+qk-\tilde{q}ke^{-i\frac{\mathrm{p}}{2}}}{\omega-\tilde{q}k\cos\frac{\mathrm{p}}{2}} \tag{J.34}$$

one gets the magnon fluctuation (rescaled by a constant)

$$\begin{aligned}
z_1 &= -ie^{ikx-i\omega t}e^{-iV}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}U\times \\
&\qquad \Big(\omega+1+qk-\tilde{q}k\,\cosh(U+i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}U\Big), \\
\bar{z}_1 &= -ie^{ikx-i\omega t}e^{-iV}e^{-it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}U\times \\
&\qquad \Big(\omega-1-qk-\tilde{q}k\,\cosh(U-i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}U\Big), \\
z_2 &= \quad ie^{ikx-i\omega t}\Big(\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2U-2(\tilde{q}k-\omega\cos\tfrac{\mathrm{p}}{2})- \\
&\qquad 2i(1+qk)\sin\tfrac{\mathrm{p}}{2}\,\tanh U\Big) \\
\bar{z}_2 &= \quad ie^{ikx-i\omega t}e^{-2iV}\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2U.
\end{aligned} \tag{J.35}$$

Massive fluctuation (2).
The other massive BMN fluctuation is

$$g_{\text{pert}} = e^{ikx-i\omega t}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \omega^2 = (1-qk)^2 + \tilde{q}^2k^2, \tag{J.36}$$

for which the auxiliary problem has perturbations

$$A_{\text{pert}} = +(\omega-1-k)e^{-it}e^{ikx-i\omega t}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{J.37}$$

$$B_{\text{pert}} = -(\omega-1+k)e^{-it}e^{ikx-i\omega t}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{J.38}$$

and
$$\Psi_{\text{pert}}(\lambda) = \frac{e^{-it}e^{ikx-i\omega t}}{1+(1+q)\frac{\omega+1-k}{\omega-1-k}\lambda}\begin{pmatrix} 0 & 0 \\ e^{-iZ(\lambda)} & 0 \end{pmatrix}. \tag{J.39}$$

Further substituting, using the identity

$$\frac{1}{1+\frac{\omega+1-k}{\omega-1-k}\sqrt{\frac{1+q}{1-q}}e^{i\frac{\mathrm{p}}{2}}} = \frac{1}{2}\frac{\omega-1+qk-\tilde{q}ke^{-i\frac{\mathrm{p}}{2}}}{\omega-\tilde{q}k\cos\frac{\mathrm{p}}{2}} \tag{J.40}$$

and after constant rescaling, one can read off the magnon fluctuation

$$\begin{aligned}
z_1 &= -ie^{ikx-i\omega t}e^{iV}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}U\times \\
&\qquad \Big(\omega+1-qk-\tilde{q}k\,\cosh(U+i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}U\Big), \\
\bar{z}_1 &= -ie^{ikx-i\omega t}e^{iV}e^{-it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}U\times \\
&\qquad \Big(\omega-1+qk-\tilde{q}k\,\cosh(U-i\tfrac{\mathrm{p}}{2})\,\mathrm{sech}U\Big), \\
z_2 &= \quad ie^{ikx-i\omega t}e^{2iV}\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2U, \\
\bar{z}_2 &= \quad ie^{ikx-i\omega t}\Big(\tilde{q}k\,\sin^2\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2U-2(\tilde{q}k-\omega\cos\tfrac{\mathrm{p}}{2})- \\
&\qquad 2i(1-qk)\sin\tfrac{\mathrm{p}}{2}\,\tanh U\Big).
\end{aligned} \tag{J.41}$$

Appendix K

Comparison to $\mathrm{AdS}_5 \times \mathrm{S}^5$ fluctuations

In this appendix we compare our solutions, in the $\varphi = q = 0$ limit, to the fluctuations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ giant magnon found in [54]. To harmonize notation, we need to write the frequency and wavenumber in the boosted worldsheet basis

$$
\begin{aligned}
e^{ikx-i\omega t} &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}, \\
k &= \gamma(\hat{k}+u\hat{\omega}) = \csc\tfrac{\mathrm{p}}{2}(\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega}), \\
\omega &= \gamma(\hat{\omega}+u\hat{k}) = \csc\tfrac{\mathrm{p}}{2}(\hat{\omega}+\cos\tfrac{\mathrm{p}}{2}\hat{k}),
\end{aligned}
\tag{K.1}
$$

where we also used the $q = 0$ version of (4.13).

K.1 Bosonic fluctuations

Although in the $q = 0$ limit the stationary magnon reduces to the HM giant magnon, due to obvious differences in the geometry we will only match a subset of our fluctuations to a subset of the ones found in [54]. The magnon on $\mathrm{AdS}_5 \times \mathrm{S}^5$ has four massive and one (unphysical) massless fluctuations on AdS_5, and, four massive and one (unphysical) massless fluctuations on S^5. Out of these, we will match both unphysical and four of the massive modes (two each on AdS_3 and S^3), while our massless modes on the T^4 have no counterparts on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The pure plane-wave AdS_3 bosons (4.21), (4.22) are trivially the same as the AdS_5 bosons (2.11) of [54] (restricted to the $\mathrm{AdS}_3 \subset \mathrm{AdS}_5$ subspace), so let us focus on the S^3 fluctuations. Substituting (K.1), the massless solution (4.38) becomes

$$
\begin{aligned}
z_1 &= -ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{+it}\sin\tfrac{\mathrm{p}}{2}\left(\hat{k}-\hat{\omega}\sinh\mathcal{X}\sinh(\mathcal{X}+i\tfrac{\mathrm{p}}{2})\right), \\
\bar{z}_1 &= \quad ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{-it}\sin\tfrac{\mathrm{p}}{2}\left(\hat{k}-\hat{\omega}\sinh\mathcal{X}\sinh(\mathcal{X}-i\tfrac{\mathrm{p}}{2})\right), \\
z_2 &= \bar{z}_2 = e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}(\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega})\,\mathrm{sech}\mathcal{Y}\tanh\mathcal{Y},
\end{aligned}
\tag{K.2}
$$

which, up to a factor of $\sin\frac{\mathrm{p}}{2}$, matches[1] equation (2.19) of [54]. In this limit the massive boson (4.41) reduces to

$$\begin{aligned}
z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X}+\hat{\omega}\sinh(\mathcal{X}+i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right),\\
\bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X}+\hat{\omega}\sinh(\mathcal{X}-i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right),\\
z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}\left((\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega})\,\mathrm{sech}^2\mathcal{X}-2(\hat{k}+i\tanh\mathcal{X})\right),\\
\bar{z}_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}(\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega})\,\mathrm{sech}^2\mathcal{X},
\end{aligned} \tag{K.3}$$

while (4.43) becomes

$$\begin{aligned}
z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X}+\hat{\omega}\sinh(\mathcal{X}+i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right),\\
\bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}e^{+it}\sin\tfrac{\mathrm{p}}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X}+\hat{\omega}\sinh(\mathcal{X}-i\tfrac{\mathrm{p}}{2})+i\cosh\mathcal{X}\right),\\
z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}(\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega})\,\mathrm{sech}^2\mathcal{X},\\
\bar{z}_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}\left((\hat{k}+\cos\tfrac{\mathrm{p}}{2}\hat{\omega})\,\mathrm{sech}^2\mathcal{X}-2(\hat{k}+i\tanh\mathcal{X})\right).
\end{aligned} \tag{K.4}$$

Although the two $m=1$ bosons do not mix for $q>0$, as can be seen from their dispersion relations $\omega^2=(1\pm qk)^2+\tilde{q}^2k^2$, in the pure R-R limit they become degenerate, and one can take linear combinations to match the specific solutions of [54]. The difference $\frac{1}{2}((\mathrm{K.4})-(\mathrm{K.3}))$

$$\begin{aligned}
z_1 &= \bar{z}_1 = 0\\
z_2 &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}(\hat{k}+i\tanh\mathcal{X}),\\
\bar{z}_2 &= -ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\sin\tfrac{\mathrm{p}}{2}(\hat{k}+i\tanh\mathcal{X}),
\end{aligned} \tag{K.5}$$

[1] Note that δZ, $\delta\vec{X}$ of [54] are related to our notation by $z_1=\delta Z$, $z_2=\delta X_3+i\delta X_4$, and we have chosen the magnon-polarization vector $\vec{n}$ to point in the X_3 direction.

reproduces the solution (2.22) of [54], with $\vec{m}$ pointing along the X_4 direction, while the sum $\frac{1}{2}\big((\text{K.3}) + (\text{K.4})\big)$

$$\begin{aligned}
z_1 &= -e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin\tfrac{p}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X} + \hat{\omega}\sinh(\mathcal{X}+i\tfrac{p}{2}) + i\cosh\mathcal{X}\right),\\
\bar{z}_1 &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} e^{+it} \sin\tfrac{p}{2}\,\mathrm{sech}^2\mathcal{X}\left(\hat{k}\sinh\mathcal{X} + \hat{\omega}\sinh(\mathcal{X}-i\tfrac{p}{2}) + i\cosh\mathcal{X}\right),\\
z_2 &= \bar{z}_2 = ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}} \sin\tfrac{p}{2}\left((\hat{k}+\cos\tfrac{p}{2}\hat{\omega})\,\mathrm{sech}^2\mathcal{X} - (\hat{k}+i\tanh\mathcal{X})\right),
\end{aligned} \tag{K.6}$$

matches the solution (2.20) of [54], with $\vec{m} = \vec{n}$ pointing along the X_3 direction.

K.2 Fermionic fluctuations

Since $\mathrm{AdS}_5 \times \mathrm{S}^5$ is supported by 5-form fluxes, while $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ is supported by 3-form fluxes, the spinor structure of fermion fluctuations on the two backgrounds will be quite different, however, it is reasonable to expect similar functional forms. The kappa-fixed solutions (3.35), (3.37) in [54] are of the form

$$\begin{aligned}
\Psi^1 &\sim \csc\tfrac{p}{4}\sqrt{\hat{\omega}+\hat{k}}\,\mathrm{sech}\mathcal{X}\sqrt{\hat{\omega}\cosh 2\mathcal{X}+\hat{k}}\,e^{i\alpha}e^{\pm i\chi}U,\\
\Psi^2 &\sim \sec\tfrac{p}{4}\sqrt{\hat{\omega}-\hat{k}}\,\mathrm{sech}\mathcal{X}\sqrt{\hat{\omega}\cosh 2\mathcal{X}-\hat{k}}\,e^{i\beta}e^{\pm i\tilde{\chi}}U,
\end{aligned} \tag{K.7}$$

where $\chi, \tilde{\chi}$ are the same as our (4.70) and

$$\begin{aligned}
e^{i\alpha} &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\frac{1+i\hat{\omega}\sinh 2\mathcal{X}}{1-i\hat{\omega}\sinh 2\mathcal{X}}\frac{1-i\hat{k}\tanh 2\mathcal{X}}{1+i\hat{k}\tanh 2\mathcal{X}}\right)^{1/4},\\
e^{i\beta} &= e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\frac{1-i\hat{\omega}\sinh 2\mathcal{X}}{1+i\hat{\omega}\sinh 2\mathcal{X}}\frac{1-i\hat{k}\tanh 2\mathcal{X}}{1+i\hat{k}\tanh 2\mathcal{X}}\right)^{1/4}.
\end{aligned} \tag{K.8}$$

At first glance these solutions seem rather different from (4.107), but for $\hat{\omega} = \sqrt{\hat{k}^2+1}$

$$\begin{aligned}
\sqrt{\hat{\omega}+\hat{k}}\,\mathrm{sech}\mathcal{X}\sqrt{\hat{\omega}\cosh 2\mathcal{X}+\hat{k}}\,e^{i\alpha} &= ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\tanh\mathcal{X} - i(\hat{k}+\hat{\omega})\right),\\
\sqrt{\hat{\omega}-\hat{k}}\,\mathrm{sech}\mathcal{X}\sqrt{\hat{\omega}\cosh 2\mathcal{X}-\hat{k}}\,e^{i\beta} &= -ie^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\tanh\mathcal{X} - i(\hat{k}-\hat{\omega})\right).
\end{aligned} \tag{K.9}$$

$$\csc\tfrac{p}{4} = \sqrt{\frac{2}{1-u}}, \qquad \sec\tfrac{p}{4} = \sqrt{\frac{2}{1+u}}, \tag{K.10}$$

and we can rewrite (K.7) as

$$\begin{aligned}\Psi^1 &\sim \frac{1}{\sqrt{1-u}}\, e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\tanh\mathcal{X} - i(\hat{k}+\hat{\omega})\right) e^{\pm i\chi} U,\\ \Psi^2 &\sim \frac{1}{\sqrt{1+u}}\, e^{i\hat{k}\mathcal{X}-i\hat{\omega}\mathcal{T}}\left(\tanh\mathcal{X} - i(\hat{k}-\hat{\omega})\right) e^{\pm i\tilde{\chi}} U,\end{aligned} \tag{K.11}$$

in agreement with the $q=0$ limit of (4.107), with the caveat that in [54] the spinors are swapped $\Psi^1 \leftrightarrow \Psi^2$ compared to our notation.

Appendix L

Coefficients in the reduced fluctuation equations for the mixed-flux $\mathbb{R}\times\mathbf{S}^3$ magnon

Here we present the coefficients of the reduced equations (4.73), and to do so in a relatively compact form we need to introduce the shorthands

$$p_{1268} = \tfrac{1}{2}(1-\lambda_{12}\lambda_{68}), \tag{L.1}$$

$$\xi = \frac{qu}{\sqrt{\tilde{q}^2-u^2}}\ , \tag{L.2}$$

and define

$$N_{ab} = \frac{i}{2}\lambda_{12}\left(\frac{aq}{\sqrt{\tilde{q}^2-u^2}} + \frac{Q_b\sqrt{1-Q_b^2}\ \mathrm{sech}^2\mathcal{Y}}{1-Q_b^2\ \mathrm{sech}^2\mathcal{Y}}\right), \qquad a,b\in\{\pm\}. \tag{L.3}$$

With these, we have

$$\begin{aligned}
C_{f_1f_1} &= +N_{-+} + i(\tilde{\omega}+\lambda_{12}(1+p_{1268}\tan^2\varphi)\xi) - i\lambda_{12}q\,\gamma\zeta^{-1}\,p_{1268}\tan^2\varphi,\\
C_{g_1g_1} &= -N_{++} + i(\tilde{\omega}+\lambda_{12}p_{1268}\tan^2\varphi\,\xi) - i\lambda_{12}q\,\gamma\zeta^{-1}\,p_{1268}\tan^2\varphi,\\
C_{f_2f_2} &= +N_{--} - i(\tilde{\omega}+\lambda_{12}(1+p_{1268}\tan^2\varphi)\xi) - i\lambda_{12}q\,\gamma\zeta^{-1}\,p_{1268}\tan^2\varphi,\\
C_{g_2g_2} &= -N_{+-} - i(\tilde{\omega}+\lambda_{12}p_{1268}\tan^2\varphi\,\xi) - i\lambda_{12}q\,\gamma\zeta^{-1}\,p_{1268}\tan^2\varphi,
\end{aligned} \tag{L.4}$$

$$\begin{aligned}
C_{f_1f_2} &= (1-u)\gamma\, e^{\int(-N_{-+}+N_{--})\mathrm{d}\mathcal{Y}}\ (1+p_{1268}\tan^2\varphi)(\lambda_{12}\tanh\mathcal{Y}-i\xi),\\
C_{g_1g_2} &= (1-u)\gamma\, e^{\int(+N_{++}-N_{+-})\mathrm{d}\mathcal{Y}}\ p_{1268}\tan^2\varphi(\lambda_{12}\tanh\mathcal{Y}+i\xi),\\
C_{f_2f_1} &= (1+u)\gamma\, e^{\int(-N_{--}+N_{-+})\mathrm{d}\mathcal{Y}}\ (1+p_{1268}\tan^2\varphi)(\lambda_{12}\tanh\mathcal{Y}+i\xi),\\
C_{g_2g_1} &= (1+u)\gamma\, e^{\int(+N_{+-}-N_{++})\mathrm{d}\mathcal{Y}}\ p_{1268}\tan^2\varphi(\lambda_{12}\tanh\mathcal{Y}-i\xi),
\end{aligned} \tag{L.5}$$

$$
\begin{aligned}
C_{f_1 g_2} &= (1-u)\gamma\, e^{\int(-N_{-+}-N_{+-})\mathrm{d}\mathcal{Y}}\ (i\lambda_{12}\, p_{1268} \tan\varphi \sec\varphi \,\mathrm{sech}\mathcal{Y}),\\
C_{g_1 f_2} &= (1-u)\gamma\, e^{\int(+N_{++}+N_{--})\mathrm{d}\mathcal{Y}}\ (i\lambda_{12}\, p_{1268} \tan\varphi \sec\varphi \,\mathrm{sech}\mathcal{Y}),\\
C_{f_2 g_1} &= (1+u)\gamma\, e^{\int(-N_{--}-N_{++})\mathrm{d}\mathcal{Y}}\ (-i\lambda_{12}\, p_{1268} \tan\varphi \sec\varphi \,\mathrm{sech}\mathcal{Y}),\\
C_{g_2 f_1} &= (1+u)\gamma\, e^{\int(+N_{+-}+N_{-+})\mathrm{d}\mathcal{Y}}\ (-i\lambda_{12}\, p_{1268} \tan\varphi \sec\varphi \,\mathrm{sech}\mathcal{Y}).
\end{aligned}
\tag{L.6}
$$

Note that p_{1268} is the eigenvalue of the ansatz with respect to the projector $\frac{1}{2}(\mathbb{1}+\hat{\Gamma}^{1268})$, and $\Delta = 0$ exactly when $p_{1268}\tan\varphi = 0$. In this case we see that the last block of coefficients are zero, the $P_\pm$ parts of the equations decouple and we have solutions with definite $\hat{\Gamma}^{012345}$ chirality.

Appendix M

Pullback of the vielbein and spin connection for the mixed-flux $\mathrm{AdS}_3 \times \mathbb{R}$ soliton

For the AdS_3 soliton background (5.1) the c omponents of the pulled-back vielbein $e^A_a = E^A_\mu(X)\partial_a X^\mu$ are

$$e^0_0 = \frac{\left(\tilde{q}^2\cosh^2\mathcal{Y} - q^2u^2\gamma^2\right)\operatorname{csch}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}}\ , \quad e^0_1 = -\frac{u\gamma^2\left(\tilde{q}^2 - u^2\right)\operatorname{csch}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}}\ , \tag{M.1}$$

$$e^1_0 = \frac{u\gamma^2\left(\tilde{q}^2 - u^2\right)\coth\mathcal{Y}}{\sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}}\ , \quad e^1_1 = -\frac{\gamma^2\left(\tilde{q}^2 - u^2\right)\coth\mathcal{Y}}{\sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}}\ , \tag{M.2}$$

$$e^2_0 = \frac{qu\gamma^2\sqrt{\tilde{q}^2 - u^2}\operatorname{csch}\mathcal{Y}}{\tilde{q}}\ , \quad e^2_1 = -\frac{q\gamma^2\sqrt{\tilde{q}^2 - u^2}\operatorname{csch}\mathcal{Y}}{\tilde{q}}\ , \tag{M.3}$$

$$e^4_0 = \cos\varphi\ , \quad e^4_1 = 0\ , \tag{M.4}$$

$$e^7_0 = \sin\varphi\ , \quad e^7_1 = 0\ , \tag{M.5}$$

and the non-zero components of the pulled-back spin connection can be expressed as

$$\begin{aligned}
\omega^{01}_0 = -\omega^{10}_0 &= \frac{\sqrt{\tilde{q}^2 - u^2}}{\tilde{q}}\frac{\left(\tilde{q}^2\cosh^2\mathcal{Y} - q^2u^2\gamma^2\right)\operatorname{csch}\mathcal{Y}}{\tilde{q}^2\cosh^2\mathcal{Y} - u^2},\\
\omega^{01}_1 = -\omega^{10}_1 &= -\frac{u\gamma^2\left(\tilde{q}^2 - u^2\right)^{3/2}}{\tilde{q}}\frac{\operatorname{csch}\mathcal{Y}}{\tilde{q}^2\cosh^2\mathcal{Y} - u^2},\\
\omega^{12}_0 = -\omega^{21}_0 &= -\frac{qu\gamma^2}{\tilde{q}}\ \sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}\operatorname{csch}\mathcal{Y},\\
\omega^{12}_1 = -\omega^{21}_1 &= \frac{q\gamma^2}{\tilde{q}}\ \sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}\operatorname{csch}\mathcal{Y}.
\end{aligned} \tag{M.6}$$

Appendix N

Fermion derivatives for the mixed-flux $\mathrm{AdS}_3 \times \mathbb{R}$ soliton

Grouping the operators acting on each spinor ϑ^J in (5.5), we get the fermion derivatives

$$
\begin{aligned}
D &= \frac{(1-u)\gamma}{\zeta}\left(\mathrm{D}_1 - \mathrm{D}_0 - \frac{1}{8}(\not{H}_0 - \not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0} , \\
\tilde{D} &= \frac{(1+u)\gamma}{\zeta}\left(\mathrm{D}_1 + \mathrm{D}_0 - \frac{1}{8}(\not{H}_0 + \not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0} ,
\end{aligned}
\tag{N.1}
$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, and the constants were introduced to normalize the $\partial_{\mathcal{Y}}$ term. The NS-NS flux appears in the partial contraction (D.2)

$$
\not{H}_a = \frac{1}{6}(\rho_a \not{H} + \not{H}\rho_a) , \tag{N.2}
$$

and, writing out (N.1), we have

$$
\begin{aligned}
D &= \partial_{\mathcal{Y}} + \frac{1}{2}G\,\Gamma_{01} + \frac{1}{2}Q\,\Gamma_{12} - \frac{(1-u)\gamma}{48\zeta}\left(\not{H}(\rho_0-\rho_1) + (\rho_0-\rho_1)\not{H}\right) , \\
\tilde{D} &= \partial_{\mathcal{Y}} + \frac{1}{2}\tilde{G}\,\Gamma_{01} + \frac{1}{2}Q\,\Gamma_{12} - \frac{(1+u)\gamma}{48\zeta}\left(\not{H}(\rho_0+\rho_1) + (\rho_0+\rho_1)\not{H}\right) ,
\end{aligned}
\tag{N.3}
$$

with

$$
\begin{aligned}
G &= \frac{\omega_1^{01} - \omega_0^{01}}{\zeta(1+u)\gamma} = -\frac{\tilde{q}^2(1-u)\sinh^2\mathcal{Y} + \tilde{q}^2 - u^2}{\tilde{q}\left(\tilde{q}^2\cosh^2\mathcal{Y} - u^2\right)}\operatorname{csch}\mathcal{Y} , \\
\tilde{G} &= \frac{\omega_1^{01} + \omega_0^{01}}{\zeta(1-u)\gamma} = \frac{\tilde{q}^2(1+u)\sinh^2\mathcal{Y} + \tilde{q}^2 - u^2}{\tilde{q}\left(\tilde{q}^2\cosh^2\mathcal{Y} - u^2\right)}\operatorname{csch}\mathcal{Y} , \\
Q &= \frac{\omega_1^{12} \mp \omega_0^{12}}{\zeta(1\pm u)\gamma} = \frac{q}{\tilde{q}\sqrt{\tilde{q}^2 - u^2}}\sqrt{\tilde{q}^2\cosh^2\mathcal{Y} - u^2}\operatorname{csch}\mathcal{Y} .
\end{aligned}
\tag{N.4}
$$

To further expand these fermion derivatives, we take (5.23)

$$\not{H} = 24q\left(\hat{\Gamma}_+ \, P_+ - \Delta\,\hat{\Gamma}^{35}\right), \tag{N.5}$$

and substitute into (N.3), but also letting $D, \tilde{D}$ act on the kappa-fixed spinors, like in (5.21). The interesting term in DK_1 is

$$\begin{aligned}
&\frac{1}{48}\Big(\not{H}(\rho_0-\rho_1)+(\rho_0-\rho_1)\not{H}\Big)\,K_1\\
&\quad= \frac{1}{24}\Big(\not{H}\hat{\Gamma}^4K_2+\hat{\Gamma}^4K_2\not{H}\Big)K_1\\
&\quad= q\,\Big(\hat{\Gamma}_+P_+\hat{\Gamma}^4K_2K_1+\hat{\Gamma}^4K_2\hat{\Gamma}_+P_+K_1\\
&\qquad\qquad -\Delta\hat{\Gamma}^{35}\hat{\Gamma}^4K_2K_1-\hat{\Gamma}^4K_2\Delta\hat{\Gamma}^{35}K_1\Big)\\
&\quad= -q\,\Big(P_-\hat{\Gamma}^{35}K_2K_1+(-K_2+\mathbb{1})\,P_+\hat{\Gamma}^{35}K_1\\
&\qquad\qquad +\Big(\Delta_4-\Delta_7\,\hat{\Gamma}^{47}\Big)\,\hat{\Gamma}^{35}K_2K_1\\
&\qquad\qquad -\Big(\Delta_4-\Delta_7\,\hat{\Gamma}^{47}\Big)\,\hat{\Gamma}^{35}K_2K_1+\Delta_4\,\hat{\Gamma}^{35}K_1\Big)\\
&\quad= -q\,\Big(RP_- -\Big(R-\hat{\Gamma}_{35}\Big)\,P_+ +\Delta_4\,\hat{\Gamma}_{35}\Big)\,K_1,
\end{aligned} \tag{N.6}$$

where we used the definition of kappa-projectors (5.18), Δ from (5.24), the identity $K_J\hat{\Gamma}^4 = -\hat{\Gamma}^4K_J + \hat{\Gamma}^4$, and (5.28). Similarly,

$$\begin{aligned}
&\frac{1}{48}\Big(\not{H}(\rho_0+\rho_1)+(\rho_0+\rho_1)\not{H}\Big)\,K_2\\
&\quad= -q\,\Big(RP_- -\Big(R-\hat{\Gamma}_{35}\Big)\,P_+ +\Delta_4\,\hat{\Gamma}_{35}\Big)\,K_2,
\end{aligned} \tag{N.7}$$

and with this, the fermion derivatives (acting on kappa-ficed spinors) take the final form

$$\begin{aligned}
D &= \partial_{\mathcal{y}} + \frac{1}{2}G\,\hat{\Gamma}_{01} + \frac{1}{2}Q\,\hat{\Gamma}_{12} + \frac{q(1-u)\gamma}{\zeta}\,\Big(RP_- - (R-\hat{\Gamma}_{35})\,P_+ + \Delta_4\,\hat{\Gamma}_{35}\Big)\;,\\
\tilde{D} &= \partial_{\mathcal{y}} + \frac{1}{2}\tilde{G}\,\hat{\Gamma}_{01} + \frac{1}{2}Q\,\hat{\Gamma}_{12} + \frac{q(1+u)\gamma}{\zeta}\,\Big(RP_- - (R-\hat{\Gamma}_{35})\,P_+ + \Delta_4\,\hat{\Gamma}_{35}\Big)\;.
\end{aligned} \tag{N.8}$$

Appendix O

Phase identities for the mixed-flux $AdS_3 \times \mathbb{R}$ soliton

Let us collect here some identities that we used when deriving the hyperbolic parametrization of the kappa-projectors (5.33) and the simplified form of the reduced equations (5.44). From the definitions of hyperbolic functions in terms of exponentials it follows that

$$e^{\mathrm{arctanh}(\alpha\,\mathrm{sech}\mathcal{Y})} = \left(\frac{\cosh\mathcal{Y}+\alpha}{\cosh\mathcal{Y}-\alpha}\right)^{1/2} , \tag{O.1}$$

$$e^{\mathrm{arccosh}\left(\frac{\coth\mathcal{Y}}{\sqrt{1+Q^2\,\mathrm{csch}^2\mathcal{Y}}}\right)} = \left(\frac{\coth\mathcal{Y}+\sqrt{1-Q^2}\,\mathrm{csch}\,\mathcal{Y}}{\coth\mathcal{Y}-\sqrt{1-Q^2}\,\mathrm{csch}\,\mathcal{Y}}\right)^{1/2} , \tag{O.2}$$

as long as $|\alpha| < 1$. In particular, with $\alpha = u/\tilde{q}$ and $Q = Q_\pm$, we have

$$\begin{aligned} e^{\pm\chi} &= \left(\frac{\tilde{q}\cosh\mathcal{Y}\pm u}{\sqrt{\tilde{q}^2\cosh^2\mathcal{Y}-u^2}}\right)^{1/2}\left(\frac{\coth\mathcal{Y}\mp\sqrt{1-Q_+^2}\,\mathrm{csch}\,\mathcal{Y}}{\sqrt{1+Q_+^2\,\mathrm{csch}^2\mathcal{Y}}}\right)^{1/2}, \\ e^{\pm\tilde{\chi}} &= \left(\frac{\tilde{q}\cosh\mathcal{Y}\pm u}{\sqrt{\tilde{q}^2\cosh^2\mathcal{Y}-u^2}}\right)^{1/2}\left(\frac{\coth\mathcal{Y}\pm\sqrt{1-Q_-^2}\,\mathrm{csch}\,\mathcal{Y}}{\sqrt{1+Q_-^2\,\mathrm{csch}^2\mathcal{Y}}}\right)^{1/2}, \end{aligned} \tag{O.3}$$

where, as defined in (5.36),

$$\begin{aligned} \chi &= \frac{1}{2}\left[\mathrm{arctanh}\left(\frac{u\,\mathrm{sech}\mathcal{Y}}{\tilde{q}}\right) - \mathrm{arccosh}\left(\frac{\coth\mathcal{Y}}{\sqrt{1+Q_+^2\,\mathrm{csch}^2\mathcal{Y}}}\right)\right] , \\ \tilde{\chi} &= \frac{1}{2}\left[\mathrm{arctanh}\left(\frac{u\,\mathrm{sech}\mathcal{Y}}{\tilde{q}}\right) + \mathrm{arccosh}\left(\frac{\coth\mathcal{Y}}{\sqrt{1+Q_-^2\,\mathrm{csch}^2\mathcal{Y}}}\right)\right] . \end{aligned} \tag{O.4}$$

Appendix P

No normalizable solutions for $\Delta \neq 0$

In section 5.1.2 we obtain solutions to the fermions zero mode equations (5.30) for the case of $\Delta = 0$. For completeness, and to get the right number of zero modes, here we are going to argue that there are no normalizable solutions for $\Delta \neq 0$. We restrict to the $q = 0$ case, which can be regarded as the leading term in a q-expansion. For $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ on the $\hat{\Gamma}_{3568} = +1$ spinor subspace, (5.24) becomes

$$\Delta = -\kappa^2 \, \hat{\Gamma}_4 - \kappa\tilde{\kappa} \, \hat{\Gamma}_7, \tag{P.1}$$

with

$$\kappa = \sin\varphi \ , \quad \tilde{\kappa} = \sqrt{1-\kappa^2} = \cos\varphi. \tag{P.2}$$

The equations of motion are

$$\begin{aligned} &(1+u)\gamma D \, \Psi^1 + \Big(R \; P_- - K_1 \Delta \; \hat{\Gamma}_+ \Big) \, \Psi^2 = 0 \ , \\ &(1-u)\gamma \tilde{D} \, \Psi^2 - \Big(R \; P_- - K_2 \Delta \; \hat{\Gamma}_+ \Big) \, \Psi^1 = 0 \ . \end{aligned} \tag{P.3}$$

with ($q = 0$) fermion derivatives

$$D = \partial_{\mathcal{Y}} + \frac{1}{2} G \; \hat{\Gamma}_{01} \ , \quad \tilde{D} = \partial_{\mathcal{Y}} + \frac{1}{2} \tilde{G} \; \hat{\Gamma}_{01} \ . \tag{P.4}$$

We take the same approach as in section 5.1.2. First we write down a suitable kappa-fixed ansatz, which we substitute back into (P.3) to get the reduced system of ODEs. Finally we demonstrate that the reduced system has no normalizable solutions.

Kappa-gauge fixing Turning on non-zero Δ, the projectors $P_\pm$ do not commute with the equations any more, and the kappa-fixed ansatz generalizing (5.43) needs to relate the two chiralities. One such ansatz is

$$\Psi^J = \left[\left(\alpha_+^J + \alpha_-^J\,\hat\Gamma_{02}\right) f_J(\mathcal{Y}) + \left(\bar\alpha_+^J + \bar\alpha_-^J\,\hat\Gamma_{02}\right) g_J(\mathcal{Y})\,\hat\Gamma_{47}\right] U_\lambda\ , \tag{P.5}$$

where, with $\alpha_\pm^J$ defined in (5.40)–(5.41),

$$\bar\alpha_\pm^J \;\equiv\; \alpha_\pm^J|_{\lambda\to-\lambda}\ , \tag{P.6}$$

and the constant spinor satisfies $\hat\Gamma_{01}U_\lambda = +U_\lambda$, $P_-U_\lambda = U_\lambda$ and $\hat\Gamma_{35}U_\lambda = -i\lambda U_\lambda$ (note that we previously had $\lambda = \lambda_{35}\lambda_P$, and $\lambda_P = -1$ on U, hence $\lambda_{35} = -\lambda$). In the above expression f_J, g_J correspond to the P_- and P_+ chiralities, respectively, and $\hat\Gamma_{47}$ transforms between the two subspaces.

Reduced equations for $q = 0$ Substituting (P.5) into (P.3), most of the terms we get are the same as in section 5.1.2, with the exceptions being $K_1\Delta\,\hat\Gamma_+\Psi^2$ and $K_2\Delta\,\hat\Gamma_+\Psi^1$. It is easy to see that

$$\Delta\,\hat\Gamma_+\Psi^J \;=\; -i\lambda\Big[\left(\alpha_+^J + \alpha_-^J\,\hat\Gamma_{02}\right)\left(\kappa^2 - \kappa\tilde\kappa\,\hat\Gamma_{47}\right) f_J \tag{P.7}$$

$$+\left(\bar\alpha_+^J + \bar\alpha_-^J\,\hat\Gamma_{02}\right)\left(\kappa\tilde\kappa + \kappa^2\,\hat\Gamma_{47}\right) g_J\Big] U_\lambda\ , \tag{P.8}$$

and after some simplification we get

$$\begin{aligned}
K_1\Delta\,\hat\Gamma_+\Psi^2 \;=\; & \Big[\left(\alpha_+^1 + \alpha_-^1\,\hat\Gamma_{02}\right)\left[-i\lambda\kappa^2\coth\mathcal{Y}\,f_2 - i\lambda\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\,g_2\right] \\
& + \left(\bar\alpha_+^1 + \bar\alpha_-^1\,\hat\Gamma_{45}\right)\left[-i\lambda\kappa^2\coth\mathcal{Y}\,g_2 + i\lambda\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\,f_2\right]\hat\Gamma_{47}\Big] U_\lambda\ , \\
K_2\Delta\,\hat\Gamma_+\Psi^1 \;=\; & \Big[\left(\alpha_+^2 + \alpha_-^2\,\hat\Gamma_{02}\right)\left[-i\lambda\kappa^2\coth\mathcal{Y}\,f_1 + i\lambda\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\,g_1\right] \\
& + \left(\bar\alpha_+^2 + \bar\alpha_-^2\,\hat\Gamma_{02}\right)\left[-i\lambda\kappa^2\coth\mathcal{Y}\,g_1 - i\lambda\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\,f_1\right]\hat\Gamma_{47}\Big] U_\lambda\ .
\end{aligned} \tag{P.9}$$

With this, the combined result of substitution is

$$\begin{aligned}\Big[\Big(\alpha_+^1 + \alpha_-^1\,\hat\Gamma_{02}\Big)\Big[\partial_{\mathcal{Y}} f_1 + \Big(C_{12} + i\lambda(1-u)\gamma\,\kappa^2 \coth\mathcal{Y}\Big)\, f_2 \\ + i\lambda(1-u)\gamma\,\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\, g_2\Big] \\ + \Big(\bar\alpha_+^1 + \bar\alpha_-^1\,\hat\Gamma_{02}\Big)\Big[\partial_{\mathcal{Y}} g_1 + i\lambda(1-u)\gamma\,\kappa^2\coth\mathcal{Y} g_2 \\ - i\lambda(1-u)\gamma\,\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\, f_2\Big]\hat\Gamma_{47}\Big]\, U_\lambda \;=\; 0\;,\end{aligned} \tag{P.10}$$

$$\begin{aligned}\Big[\Big(\alpha_+^2 + \alpha_-^2\,\hat\Gamma_{02}\Big)\Big[\partial_{\mathcal{Y}} f_2 + \Big(C_{21} - i\lambda(1+u)\gamma\,\kappa^2 \coth\mathcal{Y}\Big)\, f_1 \\ + i\lambda(1+u)\gamma\,\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\, g_1\Big] \\ + \Big(\bar\alpha_+^2 + \bar\alpha_-^2\,\hat\Gamma_{02}\Big)\Big[\partial_{\mathcal{Y}} g_2 - i\lambda(1+u)\gamma\,\kappa^2\coth\mathcal{Y} g_1 \\ - i\lambda(1+u)\gamma\,\kappa\tilde\kappa\,\mathrm{csch}\,\mathcal{Y}\, f_1\Big]\hat\Gamma_{47}\Big]\, U_\lambda \;=\; 0\;,\end{aligned} \tag{P.11}$$

where, from (5.46)

$$\begin{aligned} C_{12} &= -i\lambda\,(1-u)\gamma\,\coth\mathcal{Y}, \\ C_{21} &= i\lambda\,(1+u)\gamma\,\coth\mathcal{Y}. \end{aligned} \tag{P.12}$$

Making the ansatz

$$\begin{aligned} f_1 &= \frac{1}{\sqrt{1+u}}\,\tilde f_1, \quad g_1 = \frac{1}{\sqrt{1+u}}\,\tilde g_1, \\ f_2 &= \frac{i\,\lambda}{\sqrt{1-u}}\,\tilde f_2, \quad g_2 = \frac{i\,\lambda}{\sqrt{1-u}}\,\tilde g_2, \end{aligned} \tag{P.13}$$

we arrive at the following four equations

$$\begin{aligned} \partial_{\mathcal{Y}}\tilde f_1 \;+\; \tilde\kappa^2\coth\mathcal{Y}\;\tilde f_2 \;-\; \kappa\tilde\kappa\;\mathrm{csch}\,\mathcal{Y}\;\tilde g_2 &= 0\;, \\ \partial_{\mathcal{Y}}\tilde f_2 \;+\; \tilde\kappa^2\coth\mathcal{Y}\;\tilde f_1 \;+\; \kappa\tilde\kappa\;\mathrm{csch}\,\mathcal{Y}\;\tilde g_1 &= 0\;, \\ \partial_{\mathcal{Y}}\tilde g_1 \;-\; \kappa^2\coth\mathcal{Y}\;\tilde g_2 \;+\; \kappa\tilde\kappa\;\mathrm{csch}\,\mathcal{Y}\;\tilde f_2 &= 0\;, \\ \partial_{\mathcal{Y}}\tilde g_2 \;-\; \kappa^2\coth\mathcal{Y}\;\tilde g_1 \;-\; \kappa\tilde\kappa\;\mathrm{csch}\,\mathcal{Y}\;\tilde f_1 &= 0\;. \end{aligned} \tag{P.14}$$

Note that for $\kappa = 0$ the functions $\tilde f_1, \tilde f_2$ decouple from $\tilde g_1, \tilde g_2$, and we recover the $q = 0$ limit of the $\Delta = 0$ solutions found in section 5.1.2.

Expansion in κ. We have not been able to find closed form solutions to the system (P.14) for general values of κ, instead we take a series expansion to subleading order

$$\tilde{f}_i = \tilde{f}_i^{(0)} + \kappa \tilde{f}_i^{(1)} + O(\kappa^2), \qquad \tilde{g}_i = \tilde{g}_i^{(0)} + \kappa \tilde{g}_i^{(1)} + O(\kappa^2), \tag{P.15}$$

where $\tilde{f}_i^{(n)}, \tilde{g}_i^{(n)}$ are independent of κ. To zeroth order $\tilde{f}_1, \tilde{f}_2$ decouple from $\tilde{g}_1, \tilde{g}_2$, and the solutions are

$$\begin{aligned} \tilde{f}_1^{(0)} &= c_1 \operatorname{csch} \mathcal{Y} + c_2 \sinh \mathcal{Y} \ , \quad \tilde{g}_1^{(0)} = c_3 \ , \\ \tilde{f}_2^{(0)} &= c_1 \operatorname{csch} \mathcal{Y} - c_2 \sinh \mathcal{Y} \ , \quad \tilde{g}_2^{(0)} = c_4 \ , \end{aligned} \tag{P.16}$$

matching the solutions we found in section 5.1.2. The only normalizable solution is

$$\begin{aligned} \tilde{f}_1^{(0)} &= C_0 \operatorname{csch} \mathcal{Y} \ , \qquad \tilde{g}_1^{(0)} = 0 \ , \\ \tilde{f}_2^{(0)} &= C_0 \operatorname{csch} \mathcal{Y} \ , \qquad \tilde{g}_2^{(0)} = 0 \ , \end{aligned} \tag{P.17}$$

and these will be the forcing term in the first order equations

$$\begin{aligned} \partial_{\mathcal{Y}} \tilde{f}_1^{(1)} \ + \ \coth \mathcal{Y} \ \tilde{f}_2^{(1)} \ &= \ + \ \operatorname{csch} \mathcal{Y} \ \tilde{g}_2^{(0)}, \\ \partial_{\mathcal{Y}} \tilde{f}_2^{(1)} \ + \ \coth \mathcal{Y} \ \tilde{f}_1^{(1)} \ &= \ - \ \operatorname{csch} \mathcal{Y} \ \tilde{g}_1^{(0)}, \\ \partial_{\mathcal{Y}} \tilde{g}_1^{(1)} \ &= \ - \ \operatorname{csch} \mathcal{Y} \ \tilde{f}_2^{(0)} \ . \\ \partial_{\mathcal{Y}} \tilde{g}_2^{(1)} \ &= \ + \ \operatorname{csch} \mathcal{Y} \ \tilde{f}_1^{(0)} \ . \end{aligned} \tag{P.18}$$

The solution at this subleading order is

$$\begin{aligned} \tilde{f}_1^{(1)} &= c_1 \operatorname{csch} \mathcal{Y} + c_2 \sinh \mathcal{Y} \ , \quad \tilde{g}_1^{(1)} = c_3 + C_0 \coth \mathcal{Y} \ , \\ \tilde{f}_2^{(1)} &= c_1 \operatorname{csch} \mathcal{Y} - c_2 \sinh \mathcal{Y} \ , \quad \tilde{g}_2^{(1)} = c_4 - C_0 \coth \mathcal{Y} \ , \end{aligned} \tag{P.19}$$

where the c_i are independent of those in (P.16) (which have been fixed), and C_0 is from the leading order solution (P.17). We see that there is no combination of the constants c_i that would make this solution normalizable at both $\mathcal{Y} \to \pm\infty$. As it is already impossible to find decaying solutions at subleading order in κ, we conclude that there are no normalizable solutions for $\Delta \neq 0$.

Appendix Q

Pullback of the vielbein and spin connection for the mixed-flux double magnon

For the double-magnon background (5.68) the pulled-back vielbein has components $e_a^A = E_\mu^A(X)\partial_a X^\mu$

$$\begin{aligned}
e_0^0 &= 1\ ,\\
e_0^3 = e_0^6 &= -\frac{1}{\sqrt{2}}\frac{u\gamma^2\left(\tilde{q}^2-u^2\right)\tanh\mathcal{Y}}{\sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}}\ ,\\
e_0^4 = e_0^7 &= \frac{1}{\sqrt{2}}\frac{\left(q^2u^2\gamma^2+\tilde{q}^2\sinh^2\mathcal{Y}\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}}\ ,\\
e_0^5 = e_0^8 &= \frac{1}{\sqrt{2}}\frac{qu\gamma^2\sqrt{\tilde{q}^2-u^2}\operatorname{sech}\mathcal{Y}}{\tilde{q}}\ ,
\end{aligned} \tag{Q.1}$$

$$\begin{aligned}
e_1^0 &= 0\ ,\\
e_1^3 = e_1^6 &= \frac{1}{\sqrt{2}}\frac{\gamma^2\left(\tilde{q}^2-u^2\right)\tanh\mathcal{Y}}{\sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}}\ ,\\
e_1^4 = e_1^7 &= \frac{1}{\sqrt{2}}\frac{u\gamma^2\left(\tilde{q}^2-u^2\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}\sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}}\ ,\\
e_1^5 = e_1^8 &= -\frac{1}{\sqrt{2}}\frac{q\gamma^2\sqrt{\tilde{q}^2-u^2}\operatorname{sech}\mathcal{Y}}{\tilde{q}}\ ,
\end{aligned} \tag{Q.2}$$

while the non-zero components of the pulled-back spin connection are

$$\omega_0^{34} = \omega_0^{67} = -\frac{\sqrt{\tilde{q}^2-u^2}}{2\tilde{q}}\frac{\left(q^2u^2\gamma^2+\tilde{q}^2\sinh^2\mathcal{Y}\right)\operatorname{sech}\mathcal{Y}}{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}, \tag{Q.3}$$

$$\omega_1^{34} = \omega_1^{67} = -\frac{u\gamma^2\left(\tilde{q}^2-u^2\right)^{3/2}}{2\tilde{q}}\frac{\operatorname{sech}\mathcal{Y}}{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}, \tag{Q.4}$$

$$\omega_0^{35} = \omega_0^{68} = \frac{qu\gamma^2}{2\tilde{q}} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}\ \mathrm{sech}\mathcal{Y}, \tag{Q.5}$$

$$\omega_1^{35} = \omega_1^{68} = -\frac{q\gamma^2}{2\tilde{q}} \sqrt{\tilde{q}^2 \sinh^2 \mathcal{Y} + u^2}\ \mathrm{sech}\mathcal{Y}. \tag{Q.6}$$

Appendix R

Fermion derivatives for the mixed-flux double magnon

Grouping the operators acting on each spinor ϑ^J in (5.73), we get the fermion derivatives

$$
\begin{aligned}
D &= \frac{(1-u)\gamma}{\zeta/2}\left(\mathrm{D}_1 - \mathrm{D}_0 - \frac{1}{8}(\not{H}_0 - \not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0} , \\
\tilde{D} &= \frac{(1+u)\gamma}{\zeta/2}\left(\mathrm{D}_1 + \mathrm{D}_0 - \frac{1}{8}(\not{H}_0 + \not{H}_1)\right)_{\partial_{\mathcal{S}}\to 0} ,
\end{aligned} \tag{R.1}
$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, and the constants are chosen to normalize the $\partial_{\mathcal{Y}}$ term. The NS-NS flux contraction can be rewritten as $\not{H}_a = \frac{1}{6}(\rho_a \not{H} + \not{H}\rho_a)$, see (D.2), hence we have

$$
\begin{aligned}
D = \partial_{\mathcal{Y}} &+ \frac{1}{2}G\left(\Gamma_{34}+\Gamma_{67}\right) + \frac{1}{2}Q\left(\Gamma_{35}+\Gamma_{68}\right) \\
&- \frac{(1-u)\gamma}{24\zeta}\left(\not{H}(\rho_0-\rho_1) + (\rho_0-\rho_1)\not{H}\right) , \\
\tilde{D} = \partial_{\mathcal{Y}} &+ \frac{1}{2}\tilde{G}\left(\Gamma_{34}+\Gamma_{67}\right) + \frac{1}{2}Q\left(\Gamma_{35}+\Gamma_{68}\right) \\
&- \frac{(1+u)\gamma}{24\zeta}\left(\not{H}(\rho_0+\rho_1) + (\rho_0+\rho_1)\not{H}\right) .
\end{aligned} \tag{R.2}
$$

with

$$
\begin{aligned}
G &= \frac{\omega_1^{34} - \omega_0^{34}}{\frac{1}{2}\zeta(1+u)\gamma} = \frac{\tilde{q}^2(1-u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}\left(\tilde{q}^2\sinh^2\mathcal{Y} + u^2\right)}\operatorname{sech}\mathcal{Y} , \\
\tilde{G} &= \frac{\omega_1^{34} + \omega_0^{34}}{\frac{1}{2}\zeta(1-u)\gamma} = -\frac{\tilde{q}^2(1+u)\cosh^2\mathcal{Y} - \tilde{q}^2 + u^2}{\tilde{q}\left(\tilde{q}^2\sinh^2\mathcal{Y} + u^2\right)}\operatorname{sech}\mathcal{Y} , \\
Q &= \frac{\omega_1^{35} \mp \omega_0^{35}}{\frac{1}{2}\zeta(1\pm u)\gamma} = -\frac{q}{\tilde{q}\sqrt{\tilde{q}^2 - u^2}}\sqrt{\tilde{q}^2\sinh^2\mathcal{Y} + u^2}\,\operatorname{sech}\mathcal{Y} .
\end{aligned} \tag{R.3}
$$

Appendix S

Coefficients in the reduced zero mode equations for the double magnon

Here we present the coefficients appearing in the reduced zero mode equations (5.98). Defining[1]

$$
\begin{aligned}
M_1^{\pm} &= \frac{1}{2}\left(\lambda_{12}(2-\lambda_P)\xi \pm \tilde{q}\,\mathrm{sech}\mathcal{Y}\right), \\
M_2 &= \frac{-q}{2\sqrt{\tilde{q}^2-u^2}}\left(\tilde{q}\,\mathrm{sech}\mathcal{Y} - iu\tanh\mathcal{Y}\right), \\
M_3^{\pm} &= \frac{\lambda_{12}}{2}(2-\lambda_P)\tanh\mathcal{Y} + i\lambda_P\lambda_{12}M_1^{\pm},
\end{aligned}
\tag{S.1}
$$

$$
\begin{aligned}
N_1^{\pm}(u) = \frac{i}{2}\Bigg(&\pm \frac{u\,\mathrm{sech}\mathcal{Y}}{\tilde{q}} \pm \frac{u\left(\tilde{q}^2-u^2\right)\mathrm{sech}\mathcal{Y}}{\tilde{q}\left(\tilde{q}^2\sinh^2\mathcal{Y}+u^2\right)} \\
&- \frac{2\lambda_{12}q}{\sqrt{\tilde{q}^2-u^2}} - \frac{\lambda_{12}\lambda_P Q_+\sqrt{1-Q_+^2}}{\cosh^2\mathcal{Y}-Q_+^2}\Bigg),
\end{aligned}
\tag{S.2}
$$

$$
N_2^{\pm} = \frac{q}{2\sqrt{\tilde{q}^2-u^2}}\frac{i\tilde{q}u\cosh\mathcal{Y} \pm \left(\tilde{q}^2-u^2\right)\tanh\mathcal{Y}}{\sqrt{\tilde{q}^2\sinh^2\mathcal{Y}+u^2}},
$$

the coefficients can be written as

$$
\begin{aligned}
C_{f_1f_1} &= N_1^-(u) + iM_1^+, \quad & C_{g_1g_1} &= N_1^+(u) + iM_1^-, \\
C_{f_2f_2} &= N_1^+(-u) - iM_1^+, \quad & C_{g_2g_2} &= N_1^-(-u) - iM_1^-,
\end{aligned}
\tag{S.3}
$$

$$
\begin{aligned}
C_{f_1g_1} &= N_2^-, \quad & C_{g_1f_1} &= N_2^+, \\
C_{f_2g_2} &= -N_2^-, \quad & C_{g_2f_2} &= -N_2^+,
\end{aligned}
\tag{S.4}
$$

[1] Note that changing $u \to -u$ in N_1, we also have $Q_+ \to Q_-$.

$$
\begin{aligned}
C_{f_1f_2} &= (1-u)\gamma\, e^{\int (N_1^+(-u)-N_1^-(u))\mathrm{d}\mathcal{Y}}\ M_3^+ ,\\
C_{g_1g_2} &= (1-u)\gamma\, e^{\int (N_1^-(-u)-N_1^+(u))\mathrm{d}\mathcal{Y}}\ M_3^- ,\\
C_{f_2f_1} &= (1+u)\gamma\, e^{\int (N_1^-(u)-N_1^+(-u))\mathrm{d}\mathcal{Y}}\ \bar{M}_3^+ ,\\
C_{g_2g_1} &= (1+u)\gamma\, e^{\int (N_1^+(u)-N_1^-(-u))\mathrm{d}\mathcal{Y}}\ \bar{M}_3^- ,
\end{aligned}
\tag{S.5}
$$

$$
\begin{aligned}
C_{f_1g_2} &= -i\,(1-u)\gamma\, e^{\int (N_1^-(-u)-N_1^-(u))\mathrm{d}\mathcal{Y}}\ \left(\frac{1}{2}+\lambda_P\lambda_{12}M_2\right),\\
C_{g_1f_2} &= +i\,(1-u)\gamma\, e^{\int (N_1^+(-u)-N_1^+(u))\mathrm{d}\mathcal{Y}}\ \left(\frac{1}{2}-\lambda_P\lambda_{12}\bar{M}_2\right),\\
C_{f_2g_1} &= -i\,(1+u)\gamma\, e^{\int (N_1^+(u)-N_1^+(-u))\mathrm{d}\mathcal{Y}}\ \left(\frac{1}{2}-\lambda_P\lambda_{12}M_2\right),\\
C_{g_2f_1} &= +i\,(1+u)\gamma\, e^{\int (N_1^-(u)-N_1^-(-u))\mathrm{d}\mathcal{Y}}\ \left(\frac{1}{2}+\lambda_P\lambda_{12}\bar{M}_2\right),
\end{aligned}
\tag{S.6}
$$

Appendix T

Solving the reduced zero mode equations for the double magnon

Here we solve the reduced equations

$$\begin{aligned}
&\partial_{\mathcal{Y}}\tilde{f}_1 + \frac{i}{2}M_+^{(1)}\tilde{f}_1 + \frac{1}{2}M^{(2)}\tilde{g}_1 + \frac{1}{2}M_+^{(3)}\tilde{f}_2 + \frac{1}{2}\left(1+\lambda_P\lambda_{12}M^{(2)}\right)\tilde{g}_2 = 0,\\
&\partial_{\mathcal{Y}}\tilde{g}_1 + \frac{i}{2}M_-^{(1)}\tilde{g}_1 - \frac{1}{2}\bar{M}^{(2)}\tilde{f}_1 + \frac{1}{2}M_-^{(3)}\tilde{g}_2 + \frac{1}{2}\left(1-\lambda_P\lambda_{12}\bar{M}^{(2)}\right)\tilde{f}_2 = 0,\\
&\partial_{\mathcal{Y}}\tilde{f}_2 - \frac{i}{2}M_+^{(1)}\tilde{f}_2 - \frac{1}{2}M^{(2)}\tilde{g}_2 + \frac{1}{2}\bar{M}_+^{(3)}\tilde{f}_1 + \frac{1}{2}\left(1-\lambda_P\lambda_{12}M^{(2)}\right)\tilde{g}_1 = 0,\\
&\partial_{\mathcal{Y}}\tilde{g}_2 - \frac{i}{2}M_-^{(1)}\tilde{g}_2 + \frac{1}{2}\bar{M}^{(2)}\tilde{f}_2 + \frac{1}{2}\bar{M}_-^{(3)}\tilde{g}_1 + \frac{1}{2}\left(1+\lambda_P\lambda_{12}\bar{M}^{(2)}\right)\tilde{f}_1 = 0,
\end{aligned} \tag{T.1}$$

where

$$\begin{aligned}
M_\pm^{(1)} &= \lambda_{12}(2-\lambda_P)\xi \pm \tilde{q}\,\mathrm{sech}\mathcal{Y},\\
M^{(2)} &= \frac{-q}{\sqrt{\tilde{q}^2-u^2}}\left(\tilde{q}\,\mathrm{sech}\mathcal{Y} - iu\tanh\mathcal{Y}\right),\\
M_\pm^{(3)} &= \lambda_{12}(2-\lambda_P)\tanh\mathcal{Y} + i\lambda_P\lambda_{12}M_\pm^{(1)},
\end{aligned} \tag{T.2}$$

and $\bar{M}^{(i)}$ is the complex conjugate of $M^{(i)}$. Introducing the rotated basis F_1, F_2, G_1, G_2 via the transformation

$$\begin{aligned}
\tilde{f}_1 &= \frac{1}{2}(F_1+F_2+G_1+G_2), & \tilde{f}_2 &= \frac{\lambda_P\lambda_{12}}{2}(F_1+F_2-G_1-G_2),\\
\tilde{g}_1 &= \frac{1}{2}(F_1-F_2+G_1-G_2), & \tilde{g}_2 &= \frac{\lambda_P\lambda_{12}}{2}(F_1-F_2-G_1+G_2),
\end{aligned} \tag{T.3}$$

the reduced equations become

$$\begin{aligned}
&\partial_{\mathcal{Y}}F_1 + \frac{\lambda_P}{2}\left((2-\lambda_P)\tanh\mathcal{Y} + \lambda_{12}\right)F_1 = 0,\\
&\partial_{\mathcal{Y}}F_2 + \frac{\lambda_P}{2}\left((2-\lambda_P)\tanh\mathcal{Y} - \lambda_{12}\right)F_2 = 0,
\end{aligned} \tag{T.4}$$

$$\partial_{\mathcal{Y}} G_1 - \frac{\lambda_P}{2} \left((2-\lambda_P) \tanh \mathcal{Y} + \lambda_{12} \right) G_1 + i\xi \left(\lambda_{12}(2-\lambda_P) + \tanh \mathcal{Y} \right) F_1 + \tilde{q} \left(i + \frac{q}{\sqrt{\tilde{q}^2 - u^2}} \right) \operatorname{sech}\mathcal{Y}\, F_2 = 0,$$
$$\partial_{\mathcal{Y}} G_2 - \frac{\lambda_P}{2} \left((2-\lambda_P) \tanh \mathcal{Y} - \lambda_{12} \right) G_2 + i\xi \left(\lambda_{12}(2-\lambda_P) - \tanh \mathcal{Y} \right) F_2 + \tilde{q} \left(i - \frac{q}{\sqrt{\tilde{q}^2 - u^2}} \right) \operatorname{sech}\mathcal{Y}\, F_1 = 0. \tag{T.5}$$

It is now a simple exercise to solve these ODEs, and we look at the general solutions on the $\hat{P}_\pm$ subspaces separately.

T.1 Solutions on the $\hat{P}_+$ subspace.

Setting $\lambda_P = 1$ in (T.4), the solution is

$$F_1 = C_1 e^{-\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}}, \qquad F_2 = C_2 e^{+\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}},$$
$$G_1 = C_3 e^{+\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\cosh \mathcal{Y}} + i C_1 \xi e^{-\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}} - C_2 \tilde{q} \left(i + \frac{q}{\sqrt{\tilde{q}^2 - u^2}} \right) e^{+\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}} \sinh \mathcal{Y}, \tag{T.6}$$
$$G_2 = C_4 e^{-\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\cosh \mathcal{Y}} - i C_2 \xi e^{+\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}} - C_1 \tilde{q} \left(i - \frac{q}{\sqrt{\tilde{q}^2 - u^2}} \right) e^{-\frac{\lambda_{12}}{2}\mathcal{Y}} \sqrt{\operatorname{sech}\mathcal{Y}} \sinh \mathcal{Y},$$

for some integration constants C_i. Using the identities

$$e^{\pm\frac{\lambda_{12}}{2}\mathcal{Y}} = \frac{1}{2} e^{-\frac{i}{2}\arctan(\sinh \mathcal{Y})} \Big((1 \pm i\lambda_{12}) \cosh \mathcal{Y} + (1 \mp i\lambda_{12})(1 + i \sinh \mathcal{Y}) \Big) \sqrt{\operatorname{sech}\mathcal{Y}}$$
$$= \frac{1}{2} e^{+\frac{i}{2}\arctan(\sinh \mathcal{Y})} \Big((1 \mp i\lambda_{12}) \cosh \mathcal{Y} + (1 \pm i\lambda_{12})(1 - i \sinh \mathcal{Y}) \Big) \sqrt{\operatorname{sech}\mathcal{Y}} \tag{T.7}$$

we can transform this solution back to the original basis (T.3) to get

$$\begin{aligned}
\tilde{f}_1 &= e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}(c_1 + c_3\cosh\mathcal{Y} + c_4\sinh\mathcal{Y}) \\
&\quad + e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\frac{\lambda_{12}q(\tilde{q}-u)}{(1+\tilde{q})\sqrt{\tilde{q}^2-u^2}}(c_1 + ic_4), \\
\tilde{g}_1 &= i\lambda_{12}e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}(c_1 - c_3\cosh\mathcal{Y} - c_4\sinh\mathcal{Y}) \\
&\quad - i\lambda_{12}e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\frac{\lambda_{12}q(\tilde{q}-u)}{(1+\tilde{q})\sqrt{\tilde{q}^2-u^2}}(c_1 + ic_4), \\
\tilde{f}_2 &= i\lambda_{12}e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}(c_2 + c_4\cosh\mathcal{Y} + c_3\sinh\mathcal{Y}) \\
&\quad - i\lambda_{12}e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\frac{\lambda_{12}q(\tilde{q}+u)}{(1+\tilde{q})\sqrt{\tilde{q}^2-u^2}}(c_2 - ic_3), \\
\tilde{g}_2 &= e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}(c_2 - c_4\cosh\mathcal{Y} - c_3\sinh\mathcal{Y}) \\
&\quad + e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\frac{\lambda_{12}q(\tilde{q}+u)}{(1+\tilde{q})\sqrt{\tilde{q}^2-u^2}}(c_2 - ic_3).
\end{aligned} \tag{T.8}$$

The two sets of integration constants are related by

$$\begin{aligned}
c_1 = \frac{1}{4}\Big(&(1-i\lambda_{12})(1+i\xi)C_1 + (1+i\lambda_{12})(1-i\xi)C_2 \\
&+ (1-i\lambda_{12})C_3 + (1+i\lambda_{12})C_4\Big),
\end{aligned} \tag{T.9}$$

$$\begin{aligned}
c_2 = \frac{\lambda_{12}}{4}\Big(&(1-i\lambda_{12})(1-i\xi)C_1 - (1+i\lambda_{12})(1+i\xi)C_2 \\
&- (1-i\lambda_{12})C_3 + (1+i\lambda_{12})C_4\Big),
\end{aligned} \tag{T.10}$$

$$\begin{aligned}
c_3 = \frac{i\lambda_{12}}{4}\Bigg(&\tilde{q}(1-i\lambda_{12})\Big(1 + \frac{iq}{\sqrt{\tilde{q}^2-u^2}}\Big)C_1 \\
&- \tilde{q}(1+i\lambda_{12})\Big(1 - \frac{iq}{\sqrt{\tilde{q}^2-u^2}}\Big)C_2 \\
&+ (1-i\lambda_{12})C_3 - (1+i\lambda_{12})C_4\Bigg),
\end{aligned} \tag{T.11}$$

$$\begin{aligned}
c_4 = -\frac{i}{4}\Bigg(&\tilde{q}(1-i\lambda_{12})\Big(1 + \frac{iq}{\sqrt{\tilde{q}^2-u^2}}\Big)C_1 \\
&+ \tilde{q}(1+i\lambda_{12})\Big(1 - \frac{iq}{\sqrt{\tilde{q}^2-u^2}}\Big)C_2 \\
&- (1-i\lambda_{12})C_3 - (1+i\lambda_{12})C_4\Bigg).
\end{aligned} \tag{T.12}$$

We see that there are no normalizable solutions on this subspace.

T.2 Solutions on the $\hat{P}_-$ subspace.

On the other hand, if we set $\lambda_P = -1$ in (T.4), the solutions are

$$
\begin{aligned}
F_1 &= C_1 e^{+\frac{\lambda_{12}}{2}\mathcal{Y}}(\cosh\mathcal{Y})^{3/2}, \qquad F_2 = C_2 e^{-\frac{\lambda_{12}}{2}\mathcal{Y}}(\cosh\mathcal{Y})^{3/2},\\
G_1 &= C_3 e^{-\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}\\
&\quad - \frac{1}{2}C_2\tilde{q}\left(i + \frac{q}{\sqrt{\tilde{q}^2 - u^2}}\right) e^{-\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}(\mathcal{Y} + \cosh\mathcal{Y}\sinh\mathcal{Y})\\
&\quad - \frac{i}{8}C_1\lambda_{12}\xi e^{+\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}\Big(\sinh(3\mathcal{Y}) + 5\lambda_{12}e^{\lambda_{12}\mathcal{Y}} + 8\mathcal{Y}e^{-\lambda_{12}\mathcal{Y}}\Big),\\
G_2 &= C_4 e^{+\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}\\
&\quad - \frac{1}{2}C_1\tilde{q}\left(i - \frac{q}{\sqrt{\tilde{q}^2 - u^2}}\right) e^{+\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}(\mathcal{Y} + \cosh\mathcal{Y}\sinh\mathcal{Y})\\
&\quad - \frac{i}{8}C_2\lambda_{12}\xi e^{-\frac{\lambda_{12}}{2}\mathcal{Y}}(\mathrm{sech}\mathcal{Y})^{3/2}\Big(\sinh(3\mathcal{Y}) - 5\lambda_{12}e^{-\lambda_{12}\mathcal{Y}} + 8\mathcal{Y}e^{\lambda_{12}\mathcal{Y}}\Big),
\end{aligned}
\tag{T.13}
$$

where C_i are arbitrary integration constants. Rotating back to the original basis (T.3), again using (T.7), we get

$$
\begin{aligned}
\tilde{f}_1 = \Bigg(& c_1\sec\mathcal{Y} + c_3\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Big(c_3 - \frac{\lambda_{12}qc_4}{\sqrt{\tilde{q}^2-u^2}}\Big)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y})\\
&+ \frac{5}{4}\lambda_{12}\xi c_4 - i\lambda_{12}\xi c_4\mathcal{Y}\mathrm{sech}\mathcal{Y} - \frac{i}{4}\lambda_{12}\xi c_3\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\\
+ \Bigg(& c_2\sec\mathcal{Y} + c_4\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Big(c_4 + \frac{\lambda_{12}qc_3}{\sqrt{\tilde{q}^2-u^2}}\Big)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y})\\
&- \frac{5}{4}\lambda_{12}\xi c_3 - i\lambda_{12}\xi c_3\mathcal{Y}\mathrm{sech}\mathcal{Y} - \frac{i}{4}\lambda_{12}\xi c_4\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})},
\end{aligned}
\tag{T.14}
$$

$$\begin{aligned}\tilde{g}_1 = i\lambda_{12}\Bigg(& c_1 \sec\mathcal{Y} - c_3\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Bigg(c_3 - \frac{\lambda_{12} q c_4}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & - \frac{5}{4}\lambda_{12}\xi c_4 - i\lambda_{12}\xi c_4\mathcal{Y}\mathrm{sech}\mathcal{Y} + \frac{i}{4}\lambda_{12}\xi c_3\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} \\ - i\lambda_{12}\Bigg(& c_2\sec\mathcal{Y} - c_4\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Bigg(c_4 + \frac{\lambda_{12} q c_3}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & + \frac{5}{4}\lambda_{12}\xi c_3 - i\lambda_{12}\xi c_3\mathcal{Y}\mathrm{sech}\mathcal{Y} + \frac{i}{4}\lambda_{12}\xi c_4\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})},\end{aligned} \tag{T.15}$$

$$\begin{aligned}\tilde{f}_2 = \lambda_{12}\Bigg(& c_2 \sec\mathcal{Y} - c_4\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Bigg(c_4 + \frac{\lambda_{12} q c_3}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & - \frac{5}{4}\lambda_{12}\xi c_3 - i\lambda_{12}\xi c_3\mathcal{Y}\mathrm{sech}\mathcal{Y} - \frac{i}{4}\lambda_{12}\xi c_4\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} \\ + \lambda_{12}\Bigg(& c_1\sec\mathcal{Y} - c_3\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Bigg(c_3 - \frac{\lambda_{12} q c_4}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & + \frac{5}{4}\lambda_{12}\xi c_4 - i\lambda_{12}\xi c_4\mathcal{Y}\mathrm{sech}\mathcal{Y} - \frac{i}{4}\lambda_{12}\xi c_3\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})},\end{aligned} \tag{T.16}$$

$$\begin{aligned}\tilde{g}_2 = -i\Bigg(& c_2 \sec\mathcal{Y} + c_4\cosh^2\mathcal{Y} - \frac{i}{2}\tilde{q}\Bigg(c_4 + \frac{\lambda_{12} q c_3}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & + \frac{5}{4}\lambda_{12}\xi c_3 - i\lambda_{12}\xi c_3\mathcal{Y}\mathrm{sech}\mathcal{Y} - \frac{i}{4}\lambda_{12}\xi c_4\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})} \\ + i\Bigg(& c_1\sec\mathcal{Y} + c_3\cosh^2\mathcal{Y} + \frac{i}{2}\tilde{q}\Bigg(c_3 - \frac{\lambda_{12} q c_4}{\sqrt{\tilde{q}^2 - u^2}}\Bigg)(\mathcal{Y}\mathrm{sech}\mathcal{Y} + \sinh\mathcal{Y}) \\ & - \frac{5}{4}\lambda_{12}\xi c_4 - i\lambda_{12}\xi c_4\mathcal{Y}\mathrm{sech}\mathcal{Y} + \frac{i}{4}\lambda_{12}\xi c_3\mathrm{sech}\mathcal{Y}\sinh(3\mathcal{Y})\Bigg)e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})},\end{aligned} \tag{T.17}$$

where the new integration constants are given by

$$
\begin{aligned}
c_3 &= \frac{1}{4}\Big((1+i\lambda_{12})C_1 + (1-i\lambda_{12})C_2\Big),\\
c_4 &= \frac{-i\lambda_{12}}{4}\Big((1+i\lambda_{12})C_1 - (1-i\lambda_{12})C_2\Big),\\
c_1 &= \frac{1}{4}\Big((1-i\lambda_{12})C_3 + (1+i\lambda_{12})C_4\Big) + \frac{5}{8}\lambda_{12}\xi c_3,\\
c_2 &= \frac{i\lambda_{12}}{4}\Big((1-i\lambda_{12})C_3 - (1+i\lambda_{12})C_4\Big) - \frac{5}{8}\lambda_{12}\xi c_4.
\end{aligned}
\tag{T.18}
$$

For $c_3 = c_4 = 0$ we get the normalisable solutions

$$
\begin{aligned}
\tilde{f}_1 &= \operatorname{sech}\mathcal{Y}\left(c_1 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})} + c_2 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{g}_1 &= i\lambda_{12}\operatorname{sech}\mathcal{Y}\left(c_1 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} - c_2 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{f}_2 &= \lambda_{12}\operatorname{sech}\mathcal{Y}\left(c_1 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})} + c_2 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right),\\
\tilde{g}_2 &= i\operatorname{sech}\mathcal{Y}\left(c_1 e^{+\frac{i}{2}\arctan(\sinh\mathcal{Y})} - c_2 e^{-\frac{i}{2}\arctan(\sinh\mathcal{Y})}\right).
\end{aligned}
\tag{T.19}
$$

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