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Decision Support

Risk sharing with multiple indemnity environments

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ABSTRACT

Optimal risk sharing arrangements have been substantially studied in the literature, from the aspects of generalizing objective functions, incorporating more business constraints, and investigating different optimality criteria. This paper proposes an insurance model with multiple risk environments. We study the case where the two agents are endowed with the Value-at-Risk or the Tail Value-at-Risk, or when both agents are risk-neutral but have heterogeneous beliefs regarding the underlying probability distribution. We show that layer-type indemnities, within each risk environment, are Pareto optimal, which may be environment-specific. From Pareto optimality, we get that the premium can be chosen in a given interval, and we propose to allocate the gains from risk sharing equally between the buyer and seller.

1. Introduction

This paper studies an optimal (re)insurance contract design problem, where there are different indemnity environments. In the insurance market, indemnity contracts are often allowed to depend on an exogenous realization of mutually exclusive possible events, or the so-called triggers; such triggers are not necessarily a function of the underlying loss. Examples include multiple-peril and index-linked insurances; they are also common in the market for catastrophe (CAT) bonds and other risk-linked securities (see, e.g., Cummins, 2008). In this paper, we focus on insurance contracts, while our model applies also to an optimal reinsurance setting.

Multiple-peril insurance contract bundles together different coverages arising from various mutually exclusive perils. For instance, a homeowner insurance policy package may include coverages due to fire and smoke, theft, lightning strikes, as well as windstorms and hail. Another instance would be the federally subsidized multiple-peril crop insurance (MPCI) program operated by the Federal Crop Insurance Corporation in the United States (see, e.g., Smith & Baquet, 1996). Index-linked insurance contract usually writes on a single-peril, with various coverage levels among different realized values of certain index, such as the Catastrophe Loss Index (CLI); such a contract is expected to be increas-

ingly more prevalent in the currently pressing climate change. Both multiple-peril and index-linked insurance contracts have a characteristic of mutually exclusive and verifiable triggers; after an independent third party confirms that the one and only one trigger is met, insurance company provides an indemnity coverage to the insured for that particular trigger. Multiple indemnity environments are also observed in cyber insurance policies with exclusions, such as, criminal activity, disregard for computer security, act of terrorism or war, and so on.

There have been only a few studies where the optimal indemnity is not a function only of the underlying loss, but can also depend on other exogenous risk factors. For instance, Mahul & Wright (2003) show that indemnity functions depend not only on the underlying loss, but also on other factors such as the individual yield and/or price for crop revenues. Moreover, Dana & Scarsini (2007) and Chi & Wei (2020) show that optimal indemnities can depend on exogenous background risk. In all three papers, it is thus shown that exogenous events may influence the optimality of indemnity contracts. It requires us to study conditional probability distributions of losses, which is studied empirically by Ker & Coble (2003) for crop insurance. Albrecher & Cani (2019) show that if the Value-at-Risk (VaR) is used for holding capital of the insurer, then randomized reinsurance contracts can be optimal by "creating" a random event that triggers a non-coverage of the indemnity. In such contracts, there is a trigger that leads to no coverage for the insurer. For instance, if the reinsurer faces default risk, then limited liability enables the reinsurer not to fully pay

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the reinsurance indemnity. In this manner, there is for the insurer an exogenous event that yields an adjustment in the recovery of the indemnity. This exogenous event is allowed to be correlated with the underlying loss that the insurer seeks reinsurance for. In this paper, we propose a very general setting where there is an exogenous event that can trigger different indemnity contracts.

Originally, optimal (re)insurance contract theory focuses on the unilateral maximization of the utility of the insurer, where there is a given premium principle for (re)insurance (Arrow, 1963; Borch, 1960a). With preferences based on risk measures, this is more recently studied by Balbás, Balbás, & Heras (2011) and Tan, Wei, Wei, & Zhuang (2020). A bilateral bargaining approach is proposed by Raviv (1979) and Aase (2009) for the case where both the insurer and the reinsurer are endowed with expected utilities. This is extended by Boonen, Tan, & Zhuang (2016) to a class of comonotonic additive risk measures. In such a setting, Pareto optimal indemnities have been characterized by Asimit & Boonen (2018) as the contract profile that minimizes the sum of risk measures. We extend this approach in this paper to the case with multiple indemnity environments. Thus, our focus in this paper is on Pareto optimality, which implies that there is no profile of contracts that is better for both agents, and strictly better for at least one agent.

The primary risk holder (buyer) approaches an insurance seller, and bargains for such an optimal contract in this bilateral setting. The two agents seek to find an acceptable profile of indemnity contracts and the corresponding premium paid by the buyer to the seller. Moreover, we allow the insurer to include a compensation (a bonus) in the case that the trigger for insurance coverage is not met, and thus no indemnities need to be covered. In particular, we show optimal indemnity profiles that yield Pareto optimality. The corresponding premiums are usually non-uniquely determined by Pareto optimality alone.

Pareto optimality leads to a rather specific structure on the profile of indemnity contracts, while the corresponding premium can be chosen in a flexible manner. In particular, like Asimit & Boonen (2018), optimal indemnities follow from a sum-minimization. Here, risk is perceived by the agents with the indemnity environment being *also* unknown. We first show Pareto optimal insurance indemnity contracts in the case that the agents use the VaR or the Tail Value-at-Risk (TVaR), or the agents are risk-neutral but have heterogeneous beliefs regarding the underlying probability distribution. In the case that the VaR is used by both agents, then Pareto optimality leads stop-loss or dual stop-loss type indemnities, within each risk environment. On the other hand, when both agents use TVaR or when both agents are risk-neutral and have heterogeneous beliefs, then layer-type indemnities, within each environment, are optimal.[1] The corresponding parameters may depend on the specific indemnity environment. In particular, the effect on these optimal indemnity profiles by environment probabilities are rigorously investigated. As a second step, the premium can be chosen in a flexible manner, and thus we allow for reciprocal reinsurance contracts as in Borch (1960b); in particular, we propose to allocate the gains from risk sharing equally between the buyer and seller, which coincides with the Nash-bargaining solution.[2]

This paper is set out as follows. Section 2 defines the Pareto optimality problem with multiple indemnity environments. Pareto optimal solutions of this problem in the case that the two agents are endowed with VaR or TVaR are shown in Sections 3 and 4, respectively. A constructive example is provided in Section 5. In the

case that the two agents are risk-neutral but are endowed with heterogeneous beliefs regarding the underlying probability distribution, Pareto optimal insurance contracts are studied in Section 6. Section 7 discusses the selection of an insurance premium. Finally, Section 8 concludes. The proofs are delegated to the appendices.

2. Problem formulation

Let $(\Omega, \mathscr{F})$ be a measurable space, and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathscr{F})$. We consider a one-period economy, where a primary risk holder is endowed with a loss X, which is payable at a fixed future reference time $T > 0$. The risk X is defined on $(\Omega, \mathscr{F})$, and we assume that the loss X is a non-negative random variable with $0 < \mathbb{E}^{\mathbb{P}}[X] < \infty$.

The primary risk holder, or buyer, intends to share the loss at time T with another party, or seller, and accepts to pay a premium at time 0. Both parties agree to achieve an optimality in terms of their risk positions by choosing appropriate amounts of indemnity and premium at the present (time 0). However, unlike classical risk sharing problems, this paper considers a setting such that the indemnity level depends upon an external factor, which cannot be influenced by both parties, yet can be precisely observed and measured at time T. We will provide a practical example in Section 5.

To this end, let Y, defined on the same measurable space $(\Omega, \mathscr{F})$, be the trigger to characterize the exogenous environment such that the sample space Ω is partitioned into finitely many, more precisely $m + 1$, disjoint subsets, which are given by $\{\omega \in \Omega : Y(\omega) = k\}$, for $k = 0, 1, \ldots, m$. Moreover, for any $\omega \in \Omega$, if $Y(\omega) = 0$, then $X(\omega) = 0$. For each remaining environment $k = 1, \ldots, m$, the loss $(X | Y = k)$ is risky, in the sense that $\mathbb{P}(X > 0 | Y = k) > 0$. Thus, we explicitly assume that the random variables X and Y are *not* independent.

If the realized environment is non-risky, i.e. given that $Y = 0$, since the loss becomes void, no indemnity transfer is required. We assume that, instead, a bonus $b \in [0, \bar{b}]$ (also called an experience refund in the insurance industry) will then be paid by the seller to the buyer at time T, where $\bar{b} \geq 0$.[3] Moreover, if the realized environment is risky, i.e. given that $Y = k$, for some $k = 1, \ldots, m$, the buyer will transfer $I_k(X)$ to the seller at time T, where $I_k(\cdot)$ is called an indemnity function. Note that both parties have to agree at time 0 on a bonus b and a profile of indemnity functions $(I_1, \ldots, I_m)$ since the exogenous environment is not realized until time T. Moreover, the buyer also agrees to pay the seller a premium $\pi \geq 0$ at time 0. We refer to a tuple $(b, (I_1, \ldots, I_m), \pi)$ as a *contract*.

Any admissible profile of indemnity functions is composed of risk transfers that are comonotonic *within* the risky environment Y:

$$\mathcal{I} := \{(I_1, \ldots, I_m) : 0 \leq I_k \leq \mathrm{Id},$$

$$I_k \text{ and } R_k \text{ are non-decreasing for all } k = 1, \ldots, m\},$$

where Id denotes the identity function and R_k, $k = 1, \ldots, m$, is called a retention function, which is defined by $R_k := \mathrm{Id} - I_k$. For each $k = 1, \ldots, m$, the first condition is motivated by the fact that the indemnity loss $I_k(X)$ paid by the seller is at least non-negative and is at most the loss X; the second condition precludes ex post moral hazard from both parties, as suggested by Huberman, Mayers, & Smith (1983). Note that the realization of Y at time T is not affected by decisions of any of the two parties at time 0. For each admissible bonus $b \in [0, \bar{b}]$ and indemnity profile $\mathbf{I} := (I_1, \ldots, I_m) \in \mathcal{I}$, the realized risk positions of the buyer and seller are respec-

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[1] In absence of multiple indemnity environments, optimal risk sharing with heterogeneous beliefs and risk-neutral insurance agents have been studied by Boonen & Ghossoub (2019).

[2] The use of game-theoretic arguments to understand insurance transactions is also proposed by Dutang, Albrecher, & Loisel (2013).

[3] In practice, b is usually a fraction of the premium, which is a partial return of the premium to the insurance buyer in the case that there are no losses at all in the industry. In this paper we model the bonus as a decision variable.

tively given by:

$$\mathbf{B}(b, \mathbf{R}; X, Y) := -b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}}, \tag{2.1}$$

and

$$\mathbf{S}(b, \mathbf{I}; X, Y) := b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}}, \tag{2.2}$$

where $\mathbf{R} := (R_1, \ldots, R_m)$, and $\mathbb{I}_A$ is an indicator function of an event A. Notice that, due to the exogenous environment, by definition, the risk positions of the buyer and seller are not necessarily monotonic functions with respect to the underlying loss X.

Let ρ_1 and ρ_2 be two risk measures for the buyer and seller respectively to order their risk preferences at time 0. Together with the agreed premium payment, the post-transfer risk positions of the buyer and seller are respectively given by $\rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi)$ and $\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi)$. Unless otherwise specified, the following assumption holds throughout this paper.

Assumption 2.1. The risk measures ρ_1 and ρ_2 are:

- translational invariant: for any $m \in \mathbb{R}$ and $Z \in \mathcal{X}$, $\rho_i(Z + m) = \rho_i(Z) + m$;
- monotonic: for any $Z_1, Z_2 \in \mathcal{X}$ with $Z_1 \leq Z_2$, $\mathbb{P}$-a.s., $\rho_i(Z_1) \leq \rho_i(Z_2)$;
- such that $\rho_i(0) = 0$ and $\rho_i(X) < \infty$,

where $\mathcal{X}$ is the linear space of finite random variables.

It is well-known that the VaR and the TVaR under the probability measure $\mathbb{P}$ satisfy the conditions in Assumption 2.1. These two risk measures will be recalled and discussed in Sections 3 and 4.

To ensure the risk sharing being viable, both buyer and seller expect that it does not create any extra risk at time 0. In other words, the following individual rationality constraints have to be held:

$$\rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi) \leq \rho_1(X) \text{ and } \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi) \leq \rho_2(0) = 0.$$

Together with translation invariance, these can be rewritten as additional premium constraints:

$$\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)) \leq \pi \leq \rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y)). \tag{2.3}$$

Therefore, the joint admissible set $\mathcal{A}$ of contracts contains any bonus $b \in [0, \bar{b}]$, indemnity profile $(I_1, \ldots, I_m) \in \mathcal{I}$, and premium $\pi \geq 0$ such that (2.3) holds. Notice that the joint admissible set $\mathcal{A}$ is non-empty; in particular, no risk sharing is feasible: $b = 0$, $I_1(X) = \cdots = I_m(X) = 0$, and $\pi = 0$.

At time 0, both parties negotiate to choose the design of bonus, indemnity profiles, and premium payments in the admissible set $\mathcal{A}$. We require that such a contract is Pareto optimal, which implies it is impossible to find another contract that reduces the post-transfer risk position of either of them, without increasing the risk position of counterparty. Pareto optimality is formally defined as follows.

Definition 2.1. A bonus, indemnity profile, and premium payment tuple $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in [0, \bar{b}] \times \mathcal{I} \times [0, \infty)$ is called Pareto optimal in $\mathcal{A}$, if $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{A}$, and there is no admissible tuple $(b, (I_1, \ldots, I_m), \pi) \in \mathcal{A}$ such that

$$\rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi) \leq \rho_1(\mathbf{B}(b^*, \mathbf{R}^*; X, Y) + \pi^*);$$

$$\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi) \leq \rho_2(\mathbf{S}(b^*, \mathbf{I}^*; X, Y) - \pi^*),$$

with at least one of the two inequalities being strict.

It holds that $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{A}$ is Pareto optimal if and only if $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{S}_G$, where

$$\mathcal{S}_G := argmin_{(b, (I_1, \ldots, I_m), \pi) \in \mathcal{A}} \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y)) + \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)), \tag{2.4}$$

which follows by similar arguments as in Theorem 3.1 of Asimit & Boonen (2018).[4] Since the objective function of the minimization problem in (2.4) does not depend on premium π, this problem can be solved sequentially by:

Step 1: solving $\min_{b \in [0, \bar{b}], (I_1, \ldots, I_m) \in \mathcal{I}} \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y)) + \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y));$

Step 2: for each optimal $b^* \in [0, \bar{b}]$ and $(I_1^*, \ldots, I_m^*) \in \mathcal{I}$ from Step 1, choose $\pi^* \geq 0$ such that (2.3) holds, i.e.,

$$\rho_2(\mathbf{S}(b^*, \mathbf{I}^*; X, Y)) \leq \pi^* \leq \rho_1(X) - \rho_1(\mathbf{B}(b^*, \mathbf{R}^*; X, Y)).$$

Notice that in Step 1, we omit the constraint:

$$\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)) \leq \rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y)).$$

This is because an optimizer of the minimization problem in Step 1 must satisfy this constraint. Indeed, with the optimal $b^* \in [0, \bar{b}]$ and $(I_1^*, \ldots, I_m^*) \in \mathcal{I}$, it holds

$$\rho_1(\mathbf{B}(b^*, \mathbf{R}^*; X, Y)) + \rho_2(\mathbf{S}(b^*, \mathbf{I}^*; X, Y)) \leq \rho_1(\mathbf{B}(0, \mathbf{X}; X, Y)) + \rho_2(\mathbf{S}(0, \mathbf{0}; X, Y)) = \rho_1(X),$$

where $\mathbf{R} = \mathbf{X}$ and $\mathbf{I} = \mathbf{0}$ is defined as $R_k = \text{Id}$ and $I_k = 0$ for all $k = 1, \ldots, m$, respectively. The *main objective* of this paper is Step 1: the structure of the indemnities in Pareto optimal contracts. Section 7 will discuss Step 2: selecting the premium.

One might conjecture that any optimizer of the minimization problem in Step 1 satisfies $I_1^* = \cdots = I_m^*$. If this is the case, then the Pareto optimal risk sharing exercise with multiple indemnity environments proposed in this paper reduces to the classical problem with a single indemnity environment, with a simple extension of bonus inclusion. We will however demonstrate that for the cases of VaR and TVaR this conjecture is not true. We provide counterexamples in Sections 3.1 and 5, in which we show that there exists an optimizer of the minimization problem in Step 1 such that $I_i^* \neq I_j^*$ for some $i \neq j$. Another interesting question is under what condition should all Pareto optimal contracts be different among risky environments; this will be addressed in Section 6.

Remark 2.1. In the recent literature on optimal risk sharing in insurance, there have been roughly two approaches towards Pareto optimality. First, Cai, Liu, & Wang (2017); Jiang, Hong, & Ren (2018), and Lo & Tang (2019) assume that the premium follows from a premium principle, i.e. a given function of the insurance indemnity. Then, the only variable that is determined by Pareto optimal risk sharing is the indemnity function. Second, Asimit & Boonen (2018), as well as Asimit, Cheung, Chong, & Hu (2020), study Pareto optimal risk sharing in which the contract is given by the pair (I, π): an indemnity function and a corresponding premium. The premium is then part of the contract that is determined by Pareto optimal risk sharing. This paper follows the second approach.

3. Pareto optimality with VaR preferences

In this section, assume that the risk preferences ρ_1 and ρ_2 of the buyer and seller are both characterized by the VaR under the probability measure $\mathbb{P}$. The VaR under the probability measure $\mathbb{P}$ is

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[4] Asimit & Boonen (2018) assume law invariance of the preferences; that is, the preferences are only functions of distributions of random variables. But this assumption is not needed in the proof of the result herein. Also note that Asimit & Boonen (2018) do not assume convexity of the preferences. Under convexity of the preferences, it is well-known that the Pareto optimal frontier can be obtained as solutions of the minimization of all weighted sums of risk measures, where the weights are positive (see Miettinen, 1999). If the preferences are translation invariant, then Asimit & Boonen (2018) show that it is sufficient to set the weights equal to 1 for both agents.

given by

$$\mathrm{VaR}_\gamma(Z) := \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq \gamma\}, \text{ where } \gamma \in (0, 1).$$

In particular, practical values of γ are close to 0 in banking and insurance regulation.

Let $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ be the respective risk tolerance levels of the buyer and seller. The minimization problem solving *all* Pareto optimal bonus and profiles of indemnity functions is given by:

$$\min_{b \in [0, \bar{b}]; (I_1, \ldots, I_m) \in \mathcal{I}} \mathrm{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$
$$+ \mathrm{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}}\right). \quad (3.1)$$

The risk positions of buyer and seller both involve mutually exclusive loss components herein. Hence, with the mutual exclusivity, we apply a modification argument (see, e.g., Chi, 2012 and Cheung, Chong, & Yam, 2015a; Cheung, Chong, & Yam, 2015b) to identify a sub-class of optimal solutions for the minimization problem in (3.1), which has the *least* finite number of parameters to be determined. However, the modification arguments herein are largely extended from the canonical one, which combines with the definition of VaR, as well as balancing between retained and indemnity losses when a modification is carried out.

To this end, denote the objective function in the minimization problem (3.1) as F. Define a subset of the admissible indemnity profiles as

$$\mathcal{I}_1 := \{(I_1, \ldots, I_m) \in \mathcal{I} : \text{ for each } k = 1, \ldots, m,$$
$$\text{there exists a } d_k \in [0, \operatorname{ess\,sup}(X)]$$
$$\text{such that } I_k(x) = (x - d_k)_+ \text{ or } I_k(x) = x - (x - d_k)_+\},$$

where $\operatorname{ess\,sup}(X)$ is the essential supremum of random variable X under probability measure $\mathbb{P}$, and $(x)_+ = \max\{x, 0\}$. For the sake of a clear exposition of our results, we assume that $\operatorname{ess\,sup}(X) < \infty$, but our results also hold true in the case that $\operatorname{ess\,sup}(X) = \infty$ and we replace $[0, \operatorname{ess\,sup}(X)]$ by $[0, \infty)$ as the range of X. The following theorem provides a functional form of some Pareto optimal indemnities with the VaR, and its proof is delegated to Appendix A.

Theorem 3.1. *Let* $\rho_1 = \mathrm{VaR}_\alpha$ *and* $\rho_2 = \mathrm{VaR}_\beta$. *For any* $b \in [0, \bar{b}]$ *and* $(I_1, \ldots, I_m) \in \mathcal{I}$, *there exists an* $(\tilde{I}_1, \ldots, \tilde{I}_m) \in \mathcal{I}_1$ *such that* $F(b, (\tilde{I}_1, \ldots, \tilde{I}_m)) \leq F(b, (I_1, \ldots, I_m))$.

Theorem 3.1 states that any admissible indemnity profile $(I_1, \ldots, I_m)$ is suboptimal to an indemnity profile $(\tilde{I}_1, \ldots, \tilde{I}_m) \in \mathcal{I}_1$, with the same bonus b, where the indemnity profile $(\tilde{I}_1, \ldots, \tilde{I}_m) \in \mathcal{I}_1$ is composed of the stop-loss or dual stop-loss risk transfers. Due to such a sub-optimality result, the minimization problem (3.1), which is infinite-dimensional, can be reduced to a finite dimensional one:

$$\min_{\substack{b \in [0, \bar{b}]; \\ \theta_1, \ldots, \theta_m \in \{-1, 1\}; \\ d_1, \ldots, d_m \in [0, \operatorname{ess\,sup}(X)]}} F_B(b, \theta_1, d_1, \ldots, \theta_m, d_m) + F_S(b, -\theta_1, d_1, \ldots, -\theta_m, d_m), \quad (3.2)$$

where, for any $b \in [0, \bar{b}]$, $\phi_1, \ldots, \phi_m \in \{-1, 1\}$, and $d_1, \ldots, d_m \in [0, \operatorname{ess\,sup}(X)]$,

$$F_B(b, \phi_1, d_1, \ldots, \phi_m, d_m)$$
$$:= \mathrm{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \left((X - d_k)_+ \mathbb{I}_{\{\phi_k=-1\}} + (X - (X - d_k)_+)\mathbb{I}_{\{\phi_k=1\}}\right)\mathbb{I}_{\{Y=k\}}\right),$$

and

$$F_S(b, \phi_1, d_1, \ldots, \phi_m, d_m)$$

$$:= \mathrm{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \left((X - d_k)_+ \mathbb{I}_{\{\phi_k=-1\}} + (X - (X - d_k)_+)\mathbb{I}_{\{\phi_k=1\}}\right)\mathbb{I}_{\{Y=k\}}\right).$$

By (2.4) it follows that all these solutions of this finite dimensional minimization problem constitute Pareto optimal bonuses and profiles of indemnity functions. As we alluded in Section 2, we shall provide a counterexample in Section 5 that there exists an optimizer such that $I_i^* \neq I_j^*$ for some $i \neq j$ under the case of VaR; see also the section below.

3.1. Explicit optimal indemnities

In this section, the effect on the Pareto optimal indemnity profile by the probabilities of exogenous risky environment is studied under the VaR preferences. In order to do so, the finite dimensional problem (3.2) is first solved explicitly, under some conditions for technical tractability, in the following proposition, and its proof is delegated to Appendix B.

Proposition 3.1. *Let* $\mathbb{P}(Y = 0) = 0$, $\bar{b} = 0$, $m = 2$. *Denote* $p := \mathbb{P}(Y = 1) \in (0, 1)$, *and thus* $\mathbb{P}(Y = 2) = 1 - p \in (0, 1)$. *Denote* $F_{X|Y}(\cdot|1)$ *and* $F_{X|Y}(\cdot|2)$ *as conditional distribution functions of the loss, given that* $Y = 1$ *and* $Y = 2$, *respectively, which are assumed to be strictly increasing and continuous in* $x \in [0, \operatorname{ess\,sup}(X)]$, *with* $F_{X|Y}(0|1) = F_{X|Y}(0|2) = 0$; *denote* $F_{X|Y}^{-1}(\cdot|1)$ *and* $F_{X|Y}^{-1}(\cdot|2)$ *as the inverse functions of* $F_{X|Y}(\cdot|1)$ *and* $F_{X|Y}(\cdot|2)$, *respectively. Assume furthermore that* $\rho_1 = \rho_2 = \mathrm{VaR}_\alpha$ *with* $\alpha \in (0, 1)$. *The optimal indemnity profiles of the finite dimensional problem (3.2) and the minimized value of the objective function are given as follows:*

(1) *If* $p < \alpha$ *and* $p \leq 1 - \alpha$,
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = x \wedge d_2^*$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = (x - d_2^*)_+$,
for any $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_2^*]$ *and* $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$, *or*
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = (x - d_2^*)_+$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = x \wedge d_2^*$,
for any $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ *and* $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$;
- $F(0, (I_1^*, I_2^*)) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(2) *If* $\alpha \leq p \leq 1 - \alpha$ *and* $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = x \wedge d_2^*$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = (x - d_2^*)_+$,
for any $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ *and* $d_2^* \in [\max\{F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*, 0\}, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*]$, *or*
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = (x - d_2^*)_+$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = x \wedge d_2^*$,
for any $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ *and* $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$;
- $F(0, (I_1^*, I_2^*)) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(3) *If* $\alpha \leq p \leq 1 - \alpha$ *and* $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = x \wedge d_2^*$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = (x - d_2^*)_+$,
for any $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)] = [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ *and* $d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*$;
- $F(0, (I_1^*, I_2^*)) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(4) *If* $\alpha \leq p \leq 1 - \alpha$ *and* $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) < F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$,
- $I_1^*(x) = (x - d_1^*)_+$ *and* $I_2^*(x) = x \wedge d_2^*$, *or* $I_1^*(x) = x \wedge d_1^*$ *and* $I_2^*(x) = (x - d_2^*)_+$,
for any $d_1^* \in [\max\{F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_2^*, 0\}, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_2^*]$ *and* $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$, *or*

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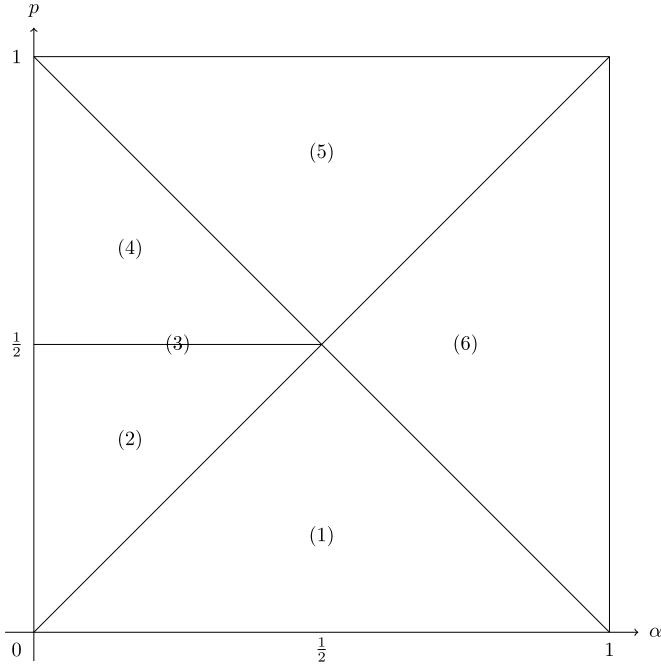
Fig. 1. Optimal indemnity profiles of two risky environments when risk tolerance levels of buyer's and seller's Value-at-Risk are equal; linear and horizontal solution of $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$ for Case (3) is chosen for graphical convenience.

- $I_1^*(x) = (x - d_1^*)_+$ and $I_2^*(x) = (x - d_2^*)_+$, or $I_1^*(x) = x \wedge d_1^*$ and $I_2^*(x) = x \wedge d_2^*$,

  for any $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-(1-p)F_{X|Y}(d_2^*|2)}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$;

- $F(0, (I_1^*, I_2^*)) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(5) If $\alpha \leq p$ and $1 - \alpha < p$,
- $I_1^*(x) = (x - d_1^*)_+$ and $I_2^*(x) = x \wedge d_2^*$, or $I_1^*(x) = x \wedge d_1^*$ and $I_2^*(x) = (x - d_2^*)_+$,

  for any $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*]$, or

- $I_1^*(x) = (x - d_1^*)_+$ and $I_2^*(x) = (x - d_2^*)_+$, or $I_1^*(x) = x \wedge d_1^*$ and $I_2^*(x) = x \wedge d_2^*$,

  for any $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-(1-p)F_{X|Y}(d_2^*|2)}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$;

- $F(0, (I_1^*, I_2^*)) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(6) If $1 - \alpha < p < \alpha$,
- $I_1^*(x) = (x - d_1^*)_+$ and $I_2^*(x) = x \wedge d_2^*$, or $I_1^*(x) = x \wedge d_1^*$ and $I_2^*(x) = (x - d_2^*)_+$,

  for any $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \text{ess sup}(X)]$;

- $F(0, (I_1^*, I_2^*)) = 0$.

Regardless of the conditions on the parameters p and α, and the conditional distributions of the loss, $I_1^* \neq I_2^*$. The six cases in Proposition 3.1 are illustrated in Fig. 1.

Suppose the risk tolerance level of the buyer and seller is such that $\alpha \in (0, \frac{1}{2})$. When the probability of the first risky environment is very small, i.e. $p < \alpha$ and $p \leq 1 - \alpha$, if the buyer and seller each bear a stop-loss in one risky environment and a dual stop-loss in another risky environment, the deductibles d_1^* and d_2^* are flexible, as long as the sum $d_1^* + d_2^*$ is smaller than the conditional VaR of the second risky environment at an adjusted risk tolerance level $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$; if the buyer and seller each bear stop-loss or dual stop-loss in both risky environments, the deductibles d_1^* and d_2^* are

also flexible, but d_1^* is smaller than $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$ while d_2^* is necessarily larger than it. When the probability of the first risky environment increases, but before the conditional VaR of the first risky environment at an adjusted risk tolerance level $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$ is at least $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, same set of optimal indemnity profiles hold except that the flexibility of deductibles shrinks. When the probability of the first risky environment increases to a level such that $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, the buyer and seller each must bear a stop-loss in one risky environment and a dual stop-loss in another risky environment, where the deductibles are least flexible that $d_1^* + d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$. The remaining two Cases (4) and (5) are similar with the probability of the second risky environment being small.

Similar conclusions hold for the case that $\alpha \in [\frac{1}{2}, 1)$. However, in this case, as long as the probability of the first risky environment is such that $p \in (1 - \alpha, \alpha)$, the buyer and seller each must bear a stop-loss in one risky environment and a dual stop-loss in another risky environment, regardless of the conditional distributions of the loss. However, the deductibles d_1^* and d_2^* do depend on the conditional distributions of the loss; specifically, they are flexible, as long as $d_1^* \geq F_{X|Y}^{-1}(\frac{1-\alpha}{p}|1)$ and $d_2^* \geq F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2)$, which are necessarily larger than the unconditional VaR of the loss $F_X^{-1}(1 - \alpha)$.

We close this section with some comments on the assumptions of Proposition 3.1. Our aim is to show explicitly solved optimal indemnity profiles in Proposition 3.1, and to illustrate the effect on the optimal indemnity profile (I_1^*, I_2^*) by the probabilities of exogenous risky environments. A solution of the finite dimensional problem (3.2) under weaker assumptions could be obtained numerically, but there are a large number of cases to be considered. We next summarize two computational issues:

- The assumption that there are only two possible risky environments ($m = 2$) is for mathematical tractability. Indeed, if there are in general m risky environments, there exist 2^m combinations of $\theta_1, \theta_2, \ldots, \theta_m$, which grows exponentially in m. However, we emphasize that even when the number of risky environments is moderately large, the finite dimensional problem should still be computationally tractable.
- The assumption that the risk tolerance levels of buyer and seller are the same ($\rho_1 = \rho_2 = \text{VaR}_\alpha$) is for simplicity. If $\rho_1 = \text{VaR}_\alpha$ and $\rho_2 = \text{VaR}_\beta$ with $\alpha \neq \beta$, we need to distinguish more cases, which yields more challenges to aggregate and compare local objective values. Note also that, under the assumption that $\alpha = \beta$, the Pareto optimal contracts in (3.2) do not "distinguish" the roles of buyer and seller via the optimal indemnity profiles, although it does via the premium (see Section 7). However, if $\alpha \neq \beta$, this no longer holds true.

4. Pareto optimality with TVaR preferences

In this section, we assume that the risk preferences ρ_1 and ρ_2 of the buyer and seller are both characterized by the TVaR under the probability measure $\mathbb{P}$. The TVaR under the probability measure $\mathbb{P}$ is given by

$$\text{TVaR}_\gamma(Z) := \frac{1}{\gamma} \int_0^\gamma \text{VaR}_\eta(Z) d\eta, \text{ where } \gamma \in (0, 1].$$

The TVaR is alternatively called Conditional Value-at-Risk or Expected Shortfall, and has gained practitioner's interest since the introduction of Basel III regulations.

Let $\alpha \in (0, 1]$ and $\beta \in (0, 1]$ be the respective risk tolerance levels of the buyer and seller. The minimization problem that yields all Pareto optimal bonuses and profiles of indemnity functions is

given by:

$$\min_{\substack{b\in[0,\bar{b}]; \\ (I_1,\ldots,I_m)\in\mathcal{I}}} \mathrm{TVaR}_\alpha\left(-b\times\mathbb{I}_{\{Y=0\}}+\sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$+\mathrm{TVaR}_\beta\left(b\times\mathbb{I}_{\{Y=0\}}+\sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right). \tag{4.1}$$

In parallel to Section 3, using mutual exclusivity and extended modification arguments, a sub-class of optimal solutions for the minimization problem (4.1) is identified, which has the *least* finite number of parameters. Hence, the infinite dimensional minimization problem (4.1) is reduced to a finite dimensional one (cf. Eq. (4.2) below); yet the finite dimensional minimization problem (4.2) can only characterize *some* Pareto optimal contracts.

Denote the objective function in the minimization problem (4.1) as G. Define a subset of the admissible indemnity profiles

$\mathcal{I}_2 := \{(I_1,\ldots,I_m)\in\mathcal{I}:$
for each $k=1,\ldots,m$, there exist $d_{k,1}\in[0,\mathrm{ess\,sup}(X)]$
and $d_{k,2}\in\left[d_{k,1},\mathrm{ess\,sup}(X)\right]$ s.t. $I_k(x)=\left(x-d_{k,1}\right)_+-\left(x-d_{k,2}\right)_+$
or $I_k(x)=x-\left(x-d_{k,1}\right)_++\left(x-d_{k,2}\right)_+\}.$

The following theorem provides a functional form of some Pareto optimal indemnities with the TVaR, and its proof is delegated to Appendix C.

Theorem 4.1. *Let* $\rho_1=\mathrm{TVaR}_\alpha$ *and* $\rho_2=\mathrm{TVaR}_\beta$. *For any* $b\in[0,\bar{b}]$ *and* $(I_1,\ldots,I_m)\in\mathcal{I}$, *there exists an* $(\tilde{I}_1,\ldots,\tilde{I}_m)\in\mathcal{I}_2$ *such that* $G(b,(\tilde{I}_1,\ldots,\tilde{I}_m))\leq G(b,(I_1,\ldots,I_m))$.

Theorem 4.1 states that any admissible indemnity profile $(I_1,\ldots,I_m)$ is sub-optimal to an indemnity profile $(\tilde{I}_1,\ldots,\tilde{I}_m)\in\mathcal{I}_2$, with the same bonus b. The indemnity profile $(\tilde{I}_1,\ldots,\tilde{I}_m)\in\mathcal{I}_2$ is composed of single layer or dual single layer risk transfers. Due to such a sub-optimality result, the infinite dimensional minimization problem (4.1) can be reduced to a finite dimensional problem to obtain some Pareto optimal contracts:

$$\min_{\substack{b\in[0,\bar{b}];\\ \theta_1,\ldots,\theta_m\in\{-1,1\};\\ (d_{1,1},d_{1,2}),\ldots,(d_{m,1},d_{m,2})\in[0,\mathrm{ess\,sup}(X)]^2:\\ d_{k,1}\leq d_{k,2},\,\forall k=1,\ldots,m}} G_B(b,\theta_1,d_{1,1},d_{1,2},\ldots,\theta_m,d_{m,1},d_{m,2}) \\ +G_S(b,-\theta_1,d_{1,1},d_{1,2},\ldots,-\theta_m,d_{m,1},d_{m,2}), \tag{4.2}$$

where, for any $b\in[0,\bar{b}]$, $\phi_1,\ldots,\phi_m\in\{-1,1\}$ and $0\leq d_{k,1}\leq d_{k,2}\leq\mathrm{ess\,sup}(X)$,

$G_B(b,\phi_1,d_{1,1},d_{1,2},\ldots,\phi_m,d_{m,1},d_{m,2})$

$:=\mathrm{TVaR}_\alpha\Bigg(-b\times\mathbb{I}_{\{Y=0\}}+\sum_{k=1}^m\Bigg(\Big(\left(X-d_{k,1}\right)_+-\left(X-d_{k,2}\right)_+\Big)\mathbb{I}_{\{\phi_k=-1\}}$

$+\left(X-\left(X-d_{k,1}\right)_++\left(X-d_{k,2}\right)_+\right)\mathbb{I}_{\{\phi_k=1\}}\Bigg)\mathbb{I}_{\{Y=k\}}\Bigg),$

and

$G_S(b,\phi_1,d_{1,1},d_{1,2},\ldots,\phi_m,d_{m,1},d_{m,2})$

$:=\mathrm{TVaR}_\beta\Bigg(b\times\mathbb{I}_{\{Y=0\}}+\sum_{k=1}^m\Bigg(\Big(\left(X-d_{k,1}\right)_+-\left(X-d_{k,2}\right)_+\Big)\mathbb{I}_{\{\phi_k=-1\}}$

$+\left(X-\left(X-d_{k,1}\right)_++\left(X-d_{k,2}\right)_+\right)\mathbb{I}_{\{\phi_k=1\}}\Bigg)\mathbb{I}_{\{Y=k\}}\Bigg).$

We comment on two technical difficulties in explicitly solving the finite dimensional problem (4.2) under the case of TVaR, even under the same set of assumptions as in Proposition 3.1.

- Since an optimal indemnity function is taking the (dual) layer-form (Theorem 4.1), there appears an additional jump in the conditional distribution functions of the retained and indemnity losses. Hence, more cases need to be considered.
- TVaR is defined as an area under the quantile function. Therefore, we need to distinguish considerably more sub-case conditions compared with the proof of Proposition 3.1 in Appendix B. This brings more challenges to aggregate and compare local objective values.

Later, in Section 5, we will provide an example where there exists an optimizer to the finite dimensional problem (4.2) such that $I_i^*\neq I_j^*$ for some $i\neq j$. Here, before closing this section, we first study a situation where there exists an optimizer such that $I_1^*=I_2^*=\cdots=I_m^*$. This is indeed the case if the risk measure of the seller ρ_2 is given by the expectation under $\mathbb{P}$; in other words, the seller is risk-neutral with $\rho_2=\mathbb{E}^{\mathbb{P}}=\mathrm{TVaR}_1$. The proof of the following proposition is delegated to Appendix D.

Proposition 4.1. *Let* $\rho_1=\mathrm{TVaR}_\alpha$ *and* $\rho_2=\mathbb{E}^{\mathbb{P}}$. *Then,* $(b^*,(I_1^*,\ldots,I_m^*),\pi^*)\in\mathcal{A}$ *with* $b^*=0$, $I_1^*=I_2^*=\cdots=I_m^*=\mathrm{Id}$, *i.e.* $R_1^*=R_2^*=\cdots=R_m^*=0$, *is a Pareto optimal contract.*

In particular, note that the Pareto optimal indemnity functions in Proposition 4.1 are not environment-specific provided that $Y\neq 0$. Then, there exists an optimal contract that coincides with an optimal contract in the case that $m=1$.

5. Heterogeneous indemnities among risky environments

In this section, we numerically illustrate a flexible implementation of the indemnity profile with multiple risky environments by Pareto optimality. Consider an insurance company that has $N=1000$ policyholders, who all have hurricane homeowner insurance contracts with the insurance company. The policyholders are assumed to have their houses in the same geological area, and individual claims are assumed to be identical. Therefore, the total loss of the insurance company is $X=N\times X_I$, where X_I is the stochastic individual claim. Note that we assume for simplicity that the insurance claim is the same for every policyholder, which may not be true when the policyholders hold different values of their houses; one should consider to differentiate the costs of various house values in order to avoid excessive under-insurance or over-insurance. Such details are ignored for parsimonious reasoning.

The insurance company seeks to purchase an index-linked hurricane reinsurance contract, which covers two grouped scales, i.e. $m=2$, for example, of Saffir-Simpson Hurricane Wind Scale in the United States. When the scale is either 1,2,3,4, the wind could cause certain degree of structural damage to a well-constructed frame house; when the scale is at the highest level 5, the wind must cause total structural damage to any well-constructed frame house. The advantage of an index-linked reinsurance policy is that the loss adjustment expenses, i.e. claim settling costs, are dramatically reduced, which should reduce the premium. This is possible, since the risky environments are fully identifiable by robust and publicly available weather measurements. For simplicity, assume that only one hurricane per year is covered by this reinsurance contract.

We denote $Y=0$, when there is no hurricane, with probability 0.5; $Y=1$, when the covered hurricane is of scale 1,2,3, or 4, with probability 0.2; $Y=2$, when the covered hurricane is of scale 5, with probability 0.3. If $Y=0$, then, with probability 1, $X_I=0$; if $Y=1$, then, with probability 0.25, $X_I=0$, with probability 0.5, $X_I=1$ million, with probability 0.25, $X_I=2$ millions; if $Y=2$, then, with probability 1, $X_I=2$ millions. To summarize,

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$\mathbb{P}(X = 0|Y = 0) = 1$;

$\mathbb{P}(X = 0|Y = 1) = 0.25$,

$\mathbb{P}(X = 1|Y = 1) = 0.5$, $\mathbb{P}(X = 2|Y = 1) = 0.25$;

$\mathbb{P}(X = 2|Y = 2) = 1$,

in which the units for X are in billions. Assume that the risk tolerance levels of the insurer (buyer) and reinsurer (seller) are respectively $\alpha = 0.01$ and $\beta = 0.02$. Assume further that $\bar{b} = 0$, and hence the only feasible bonus is given by $b^* = 0$.

First, we assume that both the buyer and the seller adopt the VaR as their time-0 risk preferences. The finite dimensional minimization problem (3.2) can be solved explicitly, and the optimal post-transfer objective value is given by 2. There exist multiple solutions, and it is possible that $I_1^* \neq I_2^*$. For instance, a solution is given by $I_1^*(x) = (x - 0.5)_+$ and $I_2^*(x) = x - (x - 1.5)_+$, for any $x \in [0, 2]$.

Second, we assume that both the buyer and the seller adopt the TVaR as their time-0 risk preferences. Then, the finite dimensional minimization problem is given by (4.2), which can also be solved explicitly. Again, the optimal post-transfer objective value is given by 2, and it is possible that $I_1^* \neq I_2^*$. For instance, a solution is given by $I_1^*(x) = (x - 0.5)_+ - (x - 1.3)_+$ and $I_2^*(x) = x - (x - 0.6)_+ + (x - 1.8)_+$, for any $x \in [0, 2]$.

There are two important implications of this numerical illustration. Firstly, it shows that there may exist profiles $(b^*, (I_1^*, \ldots, I_m^*)) \in [0, \bar{b}] \times \mathcal{I}$ solving

$$\min_{b \in [0, \bar{b}]; (I_1, \ldots, I_m) \in \mathcal{I}} \rho_1\big(\mathbf{B}(b, \mathbf{R}; X, Y)\big) + \rho_2\big(\mathbf{S}(b, \mathbf{I}; X, Y)\big),$$

such that $I_i^* \neq I_j^*$ for some $1 \leq i, j \leq m$. Hence, Pareto optimal risk sharing with multiple indemnity environments may not reduce to the classical Pareto optimal risk sharing with a single indemnity environment. Secondly, since the optimal indemnities among risky environments are not necessarily the same, there is much more flexibility in the Pareto optimal contracts than in the classical case with a single risky environment.

6. Heterogeneous indemnities via heterogeneous beliefs

In Section 5, we introduced an example in which there exists a Pareto optimal indemnity profile with heterogeneous indemnities among risky environments, under the cases of VaR and TVaR. However, due to modification arguments in solving the cases of VaR and TVaR, such an example does not necessarily imply that for all Pareto optimal indemnity profiles, the indemnities are heterogeneous among risky environments; indeed, Proposition 4.1 is a counter-statement under the case of TVaR, with seller's preference being expectation, that there exists a Pareto optimal indemnity profile with homogeneous indemnities among risky environments. In this section, we consider a model setting, which is simple enough to provide a sufficient condition of the statement that, for all Pareto optimal indemnity profiles, the indemnities are heterogeneous among risky environments.

We assume that the risk preferences ρ_1 and ρ_2 of the buyer and seller are both simply characterized by the expectation, but the buyer and seller have heterogeneous beliefs on the future states of the world. Specifically, the buyer is endowed with the probabilistic measure $\mathbb{P}$, while the seller is endowed with another probability measure $\mathbb{Q}$. Both $\mathbb{P}$ and $\mathbb{Q}$ are beliefs on the measurable space $(\Omega, \mathscr{F})$. We assume that $\mathbb{E}^{\mathbb{Q}}[X] < \infty$ and that the expectation under $\mathbb{Q}$ is monotonic, i.e., $\mathbb{Q} \ll \mathbb{P}$, or for any $Z_1, Z_2 \in \mathcal{X}$ with $Z_1 \leq Z_2$, $\mathbb{P}$-a.s., it holds that $\mathbb{E}^{\mathbb{Q}}[Z_1] \leq \mathbb{E}^{\mathbb{Q}}[Z_2]$. It thus holds that both the preferences of the buyer and the seller satisfy Assumption 2.1. Moreover, all assumptions on the risky environments and the conditional loss hold under $\mathbb{Q}$, i.e. $\mathbb{Q}(X > 0|Y = k) > 0$ for any risky environment $k = 1, 2, \ldots, m$. Then, according to the discussions in

Section 2, the minimization problem solving *all* Pareto optimal profiles of indemnity functions is given by:

$$\min_{b \in [0, \bar{b}]; (I_1, \ldots, I_m) \in \mathcal{I}} \mathbb{E}^{\mathbb{P}}\left[-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}}\right]$$

$$+ \mathbb{E}^{\mathbb{Q}}\left[b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}}\right]. \quad (6.1)$$

Unlike Sections 3 and 4, the minimization problem (6.1) is solved explicitly in the following theorem, and its proof is delegated to Appendix E.

Theorem 6.1. *Let* $\rho_1 = \mathbb{E}^{\mathbb{P}}$ *and* $\rho_2 = \mathbb{E}^{\mathbb{Q}}$. *An admissible tuple* $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{A}$ *is Pareto optimal if and only if*

$b^* = \bar{b}$ *if* $\mathbb{Q}(Y = 0) < \mathbb{P}(Y = 0)$,

$b^* = \tilde{b}$ *if* $\mathbb{Q}(Y = 0) = \mathbb{P}(Y = 0)$, *and*

$b^* = 0$ *if* $\mathbb{Q}(Y = 0) > \mathbb{P}(Y = 0)$,

where $\tilde{b}$ *is an arbitrary constant in* $[0, \bar{b}]$, *and for any* $k = 1, 2, \ldots, m$, *and for any* $t \in [0, \mathrm{ess\,sup}(X)]$,

$\big(I_k^*\big)'(t) = 1$ *if* $\mathbb{Q}(\{X > t\} \cap \{Y = k\}) < \mathbb{P}(\{X > t\} \cap \{Y = k\})$,

$\big(I_k^*\big)'(t) = h_k(t)$ *if* $\mathbb{Q}(\{X > t\} \cap \{Y = k\}) = \mathbb{P}(\{X > t\} \cap \{Y = k\})$, *and*

$\big(I_k^*\big)'(t) = 0$ *if* $\mathbb{Q}(\{X > t\} \cap \{Y = k\}) > \mathbb{P}(\{X > t\} \cap \{Y = k\})$,

where h_k *is a measurable function taking values in* $[0, 1]$.

This theorem entails that, under each risky environment $k = 1, 2, \ldots, m$, if the seller believes that the loss is more likely to be small while the buyer believes that the loss is more likely to be large, then the environment-specific tail of the loss is transferred from the buyer to the seller. If there is only one indemnity environment, then the optimal contracts in Theorem 6.1 are also shown in Proposition 4.2 of Boonen & Ghossoub (2019). In absence of heterogeneous beliefs and under a single indemnity environment when the insurance agents are endowed with distortion risk measures, the optimal indemnities have a similar layer-type structure as in Theorem 6.1 (see Assa, 2015; Cui, Yang, & Wu, 2013). Note, however, that with multiple indemnity environments, the layer-type indemnity structure only holds *within* a risky environment $k = 1, 2, \ldots, m$. In general, the indemnity is not of a layer-type.

Suppose that $\mathbb{P}(Y = k) = \mathbb{Q}(Y = k)$, for some $k = 1, 2, \ldots, m$. Then, given a realization of $Y = k \neq 0$, the optimal indemnity contracts only depend on the distribution of $X|Y = k$ under $\mathbb{P}$ and $\mathbb{Q}$, and not on the distribution of Y. This is in sharp contrast to what we observed in Section 3.1, in which the shapes of optimal indemnities I_1 and I_2 under homogeneous beliefs, $m = 2$ and $\rho_1 = \rho_2 = \mathrm{VaR}_\alpha$, do explicitly depend on $\mathbb{P}(Y = 1) = p$.

The following corollary is a direct consequence of Theorem 6.1, and provides a sufficient condition of the statement that, for all Pareto optimal indemnity profiles, the indemnities are heterogeneous among risky environments.

Corollary 6.2. *Let* $\rho_1 = \mathbb{E}^{\mathbb{P}}$, $\rho_2 = \mathbb{E}^{\mathbb{Q}}$, *and let a pair of indemnity environments* (i, j) *with* $i \neq j$. *If there exists a* $t \geq 0$ *such that*

$$(\mathbb{Q}(\{X > t\} \cap \{Y = i\}) - \mathbb{P}(\{X > t\} \cap \{Y = i\}))$$
$$\times (\mathbb{Q}(\{X > t\} \cap \{Y = j\}) - \mathbb{P}(\{X > t\} \cap \{Y = j\})) < 0,$$

then for all Pareto optimal contracts it holds that $I_i^* \neq I_j^*$.

Example 6.1. Suppose that $m = 2$, and that the seller and buyer both believe that $Y = k$ happens with probability $p_k > 0$ for $k = 1, 2$, with $p_1 + p_2 < 1$. The seller believes $X|Y = k$ is exponentially distributed with parameter λ_{1k} and the buyer

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believes $X|Y = k$ is exponentially distributed with parameter λ_{2k}, for $k = 1, 2$. Then, $\mathbb{P}(\{X > t\} \cap \{Y = 1\}) = e^{-\lambda_{11}t}p_1$, $\mathbb{P}(\{X > t\} \cap \{Y = 2\}) = e^{-\lambda_{12}t}p_2$, $\mathbb{Q}(\{X > t\} \cap \{Y = 1\}) = e^{-\lambda_{21}t}p_1$, and $\mathbb{Q}(\{X > t\} \cap \{Y = 2\}) = e^{-\lambda_{22}t}p_2$. We get from Theorem 6.1 that $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{A}$ is Pareto optimal if and only if for any $k = 1, 2$,

$$I_k^* = Id \text{ if } \lambda_{2k} > \lambda_{1k}, \text{ and } I_k^* = 0 \text{ if } \lambda_{2k} < \lambda_{1k},$$

which is a direct consequence of $e^{-\lambda_{2k}t}p_k < e^{-\lambda_{1k}t}p_k$ if and only if $\lambda_{2k} > \lambda_{1k}$, for all $t \geq 0$ and $k = 1, 2$. For instance, if $\lambda_{11} > \lambda_{21}$ and $\lambda_{22} > \lambda_{12}$, then $I_1^* = 0$ and $I_2^* = Id$. Thus, there is full coverage if $Y = 2$ and no coverage if $Y = 1$.

Now, assume that there may also be heterogeneous beliefs regarding the distribution of Y. Let $p_k := \mathbb{P}(Y = k)$ and $q_k := \mathbb{Q}(Y = k)$ for $k = 0, 1, 2$. Then, we get from Theorem 6.1 that $(b^*, (I_1^*, \ldots, I_m^*), \pi^*) \in \mathcal{A}$ is Pareto optimal if and only if

$$b^* = \bar{b} \text{ if } q_0 < p_0,$$
$$b^* = \tilde{b} \text{ if } q_0 = p_0, \text{ and}$$
$$b^* = 0 \text{ if } q_0 > p_0,$$

where $\tilde{b}$ is an arbitrary constant in $[0, \bar{b}]$, and for any $k = 1, 2$, and for any $t \in [0, \text{ess} \sup(X)]$,

$$\left(I_k^*\right)'(t) = 1 \text{ if } e^{-\lambda_{2k}t}q_k < e^{-\lambda_{1k}t}p_k,$$
$$\left(I_k^*\right)'(t) = h_k(t) \text{ if } e^{-\lambda_{2k}t}q_k = e^{-\lambda_{1k}t}p_k, \text{ and}$$
$$\left(I_k^*\right)'(t) = 0 \text{ if } e^{-\lambda_{2k}t}q_k > e^{-\lambda_{1k}t}p_k.$$

Thus, b^* is a non-decreasing function of p_0 and a non-increasing function of q_0. Likewise, for any $t \in [0, \text{ess} \sup(X)]$, $I_k(t)$ is a non-decreasing function of p_k and a non-increasing function of q_k. Hence, if an agent's subjective probability of a state $Y = k$ decreases, *ceteris paribus*, then the agent absorbs weakly more risk in any Pareto optimal contract.

7. Range of premiums

Recall from Section 2 that the structure of Pareto optimal contracts is based on two steps. In Step 1, we got a characterization of the indemnity contracts corresponding to Pareto optimal contracts. In this section, we discuss Step 2: selecting the premium. For a fixed bonus b and indemnity profile $(I_1, \ldots, I_m)$ minimizing the sum of risk measures, (and thus constituting a Pareto optimal contract), the aim is to select π in the interval $[\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)), \rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y))]$ so that $(b, (I_1, \ldots, I_m), \pi) \in \mathcal{S}_G$. Recall that this interval is always non-empty. If the interval $[\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)), \rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y))]$ is single-valued, we have no relevant problem as the status quo is then Pareto optimal. Suppose now that this interval is not single-valued, so that $\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)) < \rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y))$.

The set $\mathcal{A}$ captures already the individual rationality conditions. In other words, any contract in $\mathcal{A}$ is weakly preferred by the both agents compared to the status quo. The status quo is reached when both agents do nothing, and thus $b = 0$, $I_k(X) = 0$ for all $k = 1, \ldots, m$, and $\pi = 0$. Clearly, if π is equal to $\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y))$ or $\rho_1(X) - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y))$, then the seller or the buyer is indifferent compared to the status quo, respectively.

When there are more than two agents, it may be of interest to study remaining core-type stability conditions (Asimit & Boonen, 2018). If there is a deep liquid market with many agents, Arrow-Debreu equilibrium concepts are popular in pricing (Arrow & Debreu, 1954). In the absence of knowing other information about the market, a solution is to allocate the gains from sharing risk equally:

$$\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi) = \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi) - \rho_1(X).$$

Translation invariance of ρ_1 and ρ_2 yields directly that

$$\pi = \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y)) + \frac{1}{2}(\rho_1(X) \\ - \rho_1(\mathbf{B}(b, \mathbf{R}; X, Y)) - \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y))). \quad (7.1)$$

The contract $(b, (I_1, \ldots, I_m), \pi) \in \mathcal{S}_G$, where π is given by (7.1), coincides with Nash-bargaining solution (Nash, 1950), as $(b, (I_1, \ldots, I_m), \pi)$ solves:

$$\max_{(b, (I_1, \ldots, I_m), \pi) \in \mathcal{A}} (\rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi) - \rho_1(X)) \cdot \rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi).$$

This equivalence can be shown via the same arguments as in Boonen et al. (2016). Note here that $(b, (I_1, \ldots, I_m), \pi) \in \mathcal{A}$ implies that the two components $\rho_1(\mathbf{B}(b, \mathbf{R}; X, Y) + \pi) - \rho_1(X)$ and $\rho_2(\mathbf{S}(b, \mathbf{I}; X, Y) - \pi)$ are non-positive. Moreover, it is well-known that the Nash-bargaining is necessarily Pareto optimal (Nash, 1950). The Nash-bargaining solution has been characterized by Nash (1950) as the only solution concept that satisfies four properties[5], and alternative characterizations are proposed by Binmore, Rubinstein, & Wolinsky (1986) and Van Damme (1986).

8. Conclusion

Multiple indemnity environments affect the shape of optimal indemnity contracts. Traditionally, with a single indemnity environment, it is well-known that stop-loss and layer-type indemnities are Pareto optimal for the VaR and TVaR, respectively. This paper generalizes this finding to the case of multiple indemnity environments. We show that stop-loss and layer-type indemnities are also optimal for the VaR and TVaR, respectively, but these indemnities may have parameters that are environment-specific; moreover, full insurance is Pareto optimal when the buyer minimizes a TVaR and the seller is risk-neutral. When both the buyer and the seller are risk-neutral but have heterogeneous beliefs regarding the underlying probability distribution, we find all Pareto optimal contracts in closed form.

While there are many applications of VaR, TVaR and the expectation in insurance and its regulatory frameworks as Solvency II and Swiss Solvency Test, we wish to generalize our findings to the more general monetary risk measures. For the monetary risk measure, the technical difficulty of the problem to find Pareto optimal contracts with multiple indemnity environments is to include mutually exclusive background risk into the risk sharing approach. Also, a more realistic situation would be that the seller is endowed with background risk that is due to potential other business lines (see, e.g., Chi & Wei, 2020; Dana & Scarsini, 2007). We leave these two problems open for further research.

---

[5] These four properties are given by (i) invariance with respect to positive affine transformations, (ii) strict individual rationality, (iii) independence of irrelevant alternatives, and (iv) symmetry. For an extensive discussion of these four properties, we refer to Osborne & Rubinstein (1994).

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Appendix A. Proof of Theorem 3.1

For any $b \in [0, \bar{b}]$ and $(I_1, \ldots, I_m) \in \mathcal{I}$, define

$$a := \mathrm{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}}\right),$$

$$c := \mathrm{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}}\right).$$

For any $k = 1, \ldots, m$, define the right-continuous inverse of I_k in c as

$$I_k^{-1+}(c) := \inf\{x \in [0, \mathrm{ess\,sup}(X)] : I_k(x) > c\},$$

where, by convention, $\inf \emptyset = \mathrm{ess\,sup}(X)$. Moreover, define similarly the right-continuous inverse of R_k in a as

$$R_k^{-1+}(a) := \inf\{x \in [0, \mathrm{ess\,sup}(X)] : R_k(x) > a\}.$$

First, we assume $a \geq 0$. For each $k = 1, \ldots, m$, consider the following two cases and the corresponding sub-cases to construct the modification $\tilde{I}_k$.

Case 1: Assume that $I_k^{-1+}(c) \leq R_k^{-1+}(a)$.

Sub-case 1.1: Consider that $I_k^{-1+}(c) < \mathrm{ess\,sup}(X)$. Define $\tilde{I}_k(x) := (x - I_k^{-1+}(c) + c)_+$, and hence $\tilde{R}_k(x) = x - (x - I_k^{-1+}(c) + c)_+$, for any $x \in [0, \mathrm{ess\,sup}(X)]$. Note that, by definition, it holds for any $x \in [0, \mathrm{ess\,sup}(X)]$ that $\tilde{I}_k(x) > c$ if and only if $I_k(x) > c$. Moreover, using the identity that $I_k(I_k^{-1+}(c)) + R_k(I_k^{-1+}(c)) = I_k^{-1+}(c)$, it holds for any $x \in [0, \mathrm{ess\,sup}(X)]$ that $\tilde{R}_k(x) \leq I_k^{-1+}(c) - c = R_k(I_k^{-1+}(c)) \leq a$, where the last inequality is true regardless of the following two sub-sub-cases:

Sub-sub-case 1.1.1: Consider further that $R_k^{-1+}(a) < \mathrm{ess\,sup}(X)$. By the monotonicity of R_k, it follows $R_k(I_k^{-1+}(c)) \leq R_k(R_k^{-1+}(a)) = a$.

Sub-sub-case 1.1.2: Consider further that $R_k^{-1+}(a) = \mathrm{ess\,sup}(X)$. Necessarily, for any $x \in [0, \mathrm{ess\,sup}(X)]$, it holds that $R_k(x) \leq a$; in particular, $R_k(I_k^{-1+}(c)) \leq a$.

Sub-case 1.2: Consider that $I_k^{-1+}(c) = \mathrm{ess\,sup}(X)$. Necessarily, for any $x \in [0, \mathrm{ess\,sup}(X)]$, $I_k(x) \leq c$. By the case condition that $I_k^{-1+}(c) \leq R_k^{-1+}(a)$, necessarily, $R_k^{-1+}(a) = \mathrm{ess\,sup}(X)$, and hence, for any $x \in [0, \mathrm{ess\,sup}(X)]$, $R_k(x) \leq a$. Moreover, $\mathrm{ess\,sup}(X)$ must be finite. Indeed, if $\mathrm{ess\,sup}(X) = \infty$, the facts that, $I_k(2(a+c)) \leq c$ and $R_k(2(a+c)) \leq a$, implies that $2(a+c) = I_k(2(a+c)) + R_k(2(a+c)) \leq a + c$, which leads to a contradiction. Define $\tilde{I}_k(x) := (x - \mathrm{ess\,sup}(X) + I_k(\mathrm{ess\,sup}(X)))_+$, and hence $\tilde{R}_k(x) = x - (x - \mathrm{ess\,sup}(X) + I_k(\mathrm{ess\,sup}(X)))_+$, for any $x \in [0, \mathrm{ess\,sup}(X)]$. Note that, by definition, it holds for any $x \in [0, \mathrm{ess\,sup}(X)]$ that $\tilde{I}_k(x) \leq c$. Moreover, using the identity that $I_k(\mathrm{ess\,sup}(X)) + R_k(\mathrm{ess\,sup}(X)) = \mathrm{ess\,sup}(X)$, it holds for any $x \in [0, \mathrm{ess\,sup}(X)]$ that $\tilde{R}_k(x) \leq \mathrm{ess\,sup}(X) - I_k(\mathrm{ess\,sup}(X)) = R_k(\mathrm{ess\,sup}(X))$. Recall that, for any $x \in [0, \mathrm{ess\,sup}(X)]$, $R_k(x) \leq a$; in particular, $R_k(\mathrm{ess\,sup}(X)) \leq a$. Hence, for any $x \in [0, \mathrm{ess\,sup}(X)]$, $\tilde{R}_k(x) \leq a$.

In both sub-cases 1.1 and 1.2, the constructed $\tilde{I}_k$ satisfies that, for any $x \in [0, \mathrm{ess\,sup}(X)]$, $\tilde{I}_k(x) > c$ if and only if $I_k(x) > c$. Therefore,
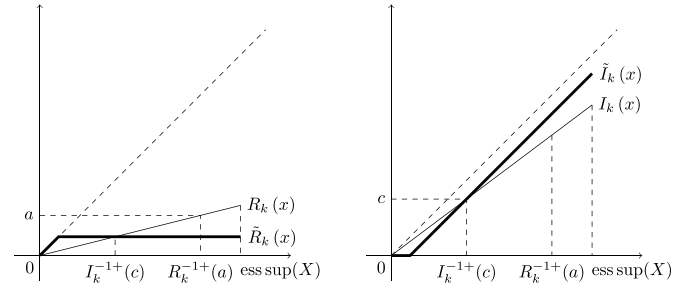

Fig. A.2. Constructions of $\tilde{R}_k$ (left) and $\tilde{I}_k$ (right), under the conditions that $a \geq 0$ and $I_k^{-1+}(c) \leq R_k^{-1+}(a) < \mathrm{ess\,sup}(X)$, for VaR risk preferences (Sub-sub-case 1.1.1); linear R_k and I_k are chosen for graphical convenience, and $\tilde{R}_k$ and $\tilde{I}_k$ are bold-faced.

$$
\begin{aligned}
\mathbb{P}\left(I_k(X)\mathbb{I}_{\{Y=k\}} > c\right) &= \mathbb{P}(\{I_k(X) > c\} \cap \{Y = k\}) \\
&\quad + \mathbb{P}(\{0 > c\} \cap \{Y \neq k\}) \\
&= \mathbb{P}\left(\{\tilde{I}_k(X) > c\} \cap \{Y = k\}\right) \\
&\quad + \mathbb{P}(\{0 > c\} \cap \{Y \neq k\}) \\
&= \mathbb{P}\left(\tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right).
\end{aligned}
$$

Moreover, in both sub-cases 1.1 and 1.2, the constructed $\tilde{R}_k$ satisfies $\tilde{R}_k(x) \leq a$ for any $x \in [0, \mathrm{ess\,sup}(X)]$. Therefore,

$$
\begin{aligned}
\mathbb{P}\left(R_k(X)\mathbb{I}_{\{Y=k\}} > a\right) &= \mathbb{P}(\{R_k(X) > a\} \cap \{Y = k\}) \geq 0 \\
&= \mathbb{P}\left(\tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > a\right).
\end{aligned}
$$

Case 2: Assume that $R_k^{-1+}(a) \leq I_k^{-1+}(c)$. If $R_k^{-1+}(a) < \mathrm{ess\,sup}(X)$, define $\tilde{R}_k(x) := \left(x - R_k^{-1+}(a) + a\right)_+$, and hence $\tilde{I}_k(x) = x - \left(x - R_k^{-1+}(a) + a\right)_+$, for any $x \in [0, \mathrm{ess\,sup}(X)]$. If $R_k^{-1+}(a) = \mathrm{ess\,sup}(X)$, define $\tilde{R}_k(x) := (x - \mathrm{ess\,sup}(X) + R_k(\mathrm{ess\,sup}(X)))_+$, and hence $\tilde{I}_k(x) = x - (x - \mathrm{ess\,sup}(X) + R_k(\mathrm{ess\,sup}(X)))_+$, for any $x \in [0, \mathrm{ess\,sup}(X)]$. By following similar arguments as in Case 1 with interchanging the roles of I_k (or the constructed $\tilde{I}_k$) and R_k (or the constructed $\tilde{R}_k$), as well as of a and c, one can show that the constructed $\tilde{I}_k$ and $\tilde{R}_k$ satisfy

$$\mathbb{P}\left(I_k(X)\mathbb{I}_{\{Y=k\}} > c\right) \geq \mathbb{P}\left(\tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right);$$

$$\mathbb{P}\left(R_k(X)\mathbb{I}_{\{Y=k\}} > a\right) = \mathbb{P}\left(\tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > a\right).$$

For the ease of understanding these cases above, Fig. A.2 illustrates the modification arguments of Sub-sub-case 1.1.1.

Therefore, for any $k = 1, \ldots, m$, the constructed $\tilde{I}_k$ and $\tilde{R}_k$ satisfy

$$\mathbb{P}\left(I_k(X)\mathbb{I}_{\{Y=k\}} > c\right) \geq \mathbb{P}\left(\tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right)$$

and $\mathbb{P}\left(R_k(X)\mathbb{I}_{\{Y=k\}} > a\right) \geq \mathbb{P}\left(\tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > a\right)$.

By definition,

$$a = \inf\left\{z \in \mathbb{R} : \mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}} > z\right) \leq \alpha\right\}$$

and

$$c = \inf\left\{z \in \mathbb{R} : \mathbb{P}\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}} > z\right) \leq \beta\right\}.$$

Therefore, by mutual exclusivity,

$$\mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > a\right)$$

$$= \mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} > a\right) + \sum_{k=1}^{m} \mathbb{P}\left(\tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > a\right)$$

$$\leq \mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} > a\right) + \sum_{k=1}^{m} \mathbb{P}\left(R_k(X)\mathbb{I}_{\{Y=k\}} > a\right)$$

$$= \mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} R_k(X)\mathbb{I}_{\{Y=k\}} > a\right)$$

$$\leq \alpha$$

and

$$\mathbb{P}\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right)$$

$$= \mathbb{P}\left(b \times \mathbb{I}_{\{Y=0\}} > c\right) + \sum_{k=1}^{m} \mathbb{P}\left(\tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right)$$

$$\leq \mathbb{P}\left(b \times \mathbb{I}_{\{Y=0\}} > c\right) + \sum_{k=1}^{m} \mathbb{P}\left(I_k(X)\mathbb{I}_{\{Y=k\}} > c\right)$$

$$= \mathbb{P}\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} I_k(X)\mathbb{I}_{\{Y=k\}} > c\right)$$

$$\leq \beta.$$

By definition, it holds that

$$\text{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$= \inf\left\{z \in \mathbb{R} : \mathbb{P}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} > z\right) \leq \alpha\right\}$$

$$\leq a.$$

Similarly, it holds that

$$\text{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right) \leq c.$$

Hence, $(\tilde{I}_1, \ldots, \tilde{I}_m) \in \mathcal{I}_1$ and

$$F(b, (\tilde{I}_1, \ldots, \tilde{I}_m))$$

$$= \text{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$+ \text{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$\leq a + c$$

$$= F(b, (I_1, \ldots, I_m)).$$

Following, we consider the case $a < 0$. For this case, the definition of VaR together with the nonnegative property of $R_k(X)$ implies

$$a = -b \quad \text{and} \quad \mathbb{P}(Y = 0) \geq 1 - \alpha.$$

We can construct $\tilde{I}_k(x)$ as

$$\tilde{I}_k(x) = \begin{cases} (x - I_k^{-1+}(c) + c)_+, & I_k^{-1+}(c) < \text{ess sup}(X); \\ x - (x - c)_+, & I_k^{-1+}(c) = \text{ess sup}(X), \end{cases}$$

then

$$\mathbb{P}\left(I_k(X)\mathbb{I}_{\{Y=k\}} > c\right) = \mathbb{P}\left(\tilde{I}_k(X)\mathbb{I}_{\{Y=k\}} > c\right), \forall k = 1, \cdots, m,$$

which in turn implies

$$\text{VaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right) \leq c.$$

On the other hand, it is trivial that $\text{VaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^{m} \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right) = -b$ because of $\mathbb{P}(Y = 0) \geq 1 - \alpha$. As a consequence, we have $F(b, (\tilde{I}_1, \ldots, \tilde{I}_m)) \leq F(b, (I_1, \ldots, I_m))$. The proof is finally complete.

Appendix B. Proof of Proposition 3.1

First, note that

$$\max\left\{F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right), F_{X|Y}^{-1}\left(1 - \frac{\alpha}{1-p}\Big|2\right)\right\}$$

$$< F_X^{-1}(1 - \alpha)$$

$$< \min\left\{F_{X|Y}^{-1}\left(\frac{1-\alpha}{p}\Big|1\right), F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)\right\}.$$

Indeed, if $0 < p \leq \alpha < 1$, then $1 - \frac{\alpha}{p} \leq 0$ and $\frac{1-\alpha}{1-p} \leq 1$. Therefore, $F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right) = 0$, while $F_X^{-1}(1 - \alpha) < F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)$ is equivalent to

$$1 - \alpha < F_X\left(F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)\right)$$

$$= F_{X|Y}\left(F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)\Big|1\right)p + F_{X|Y}\left(F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)\Big|2\right)(1-p)$$

$$= F_{X|Y}\left(F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)\Big|1\right)p + 1 - \alpha,$$

which is true. If $0 < \alpha < p < 1$, then $1 - \frac{\alpha}{p} > 0$ and $\frac{1-\alpha}{1-p} > 1$. Therefore, $F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right) = \text{ess sup}(X)$, while $F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right) < F_X^{-1}(1 - \alpha)$ is equivalent to

$$1 - \alpha > F_X\left(F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right)\right)$$

$$= F_{X|Y}\left(F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right)\Big|1\right)p + F_{X|Y}\left(F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right)\Big|2\right)(1-p)$$

$$= p - \alpha + F_{X|Y}\left(F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right)\Big|2\right)(1-p),$$

which is true. These together show that $F_{X|Y}^{-1}\left(1 - \frac{\alpha}{p}\Big|1\right) < F_X^{-1}(1 - \alpha) < F_{X|Y}^{-1}\left(\frac{1-\alpha}{1-p}\Big|2\right)$. Similar arguments yield that $F_{X|Y}^{-1}\left(1 - \frac{\alpha}{1-p}\Big|2\right) < F_X^{-1}(1 - \alpha) < F_{X|Y}^{-1}\left(\frac{1-\alpha}{p}\Big|1\right)$.

Consider the four cases separately that, (i) Case 1: $\theta_1 = 1$ and $\theta_2 = 1$, (ii) Case 2: $\theta_1 = 1$ and $\theta_2 = -1$, (iii) Case 3: $\theta_1 = -1$ and $\theta_2 = 1$, and (iv) Case 4: $\theta_1 = -1$ and $\theta_2 = -1$. Since $\alpha = \beta$, $\mathbb{P}(Y = 0) = 0$, and $\bar{b} = 0$, it follows from a symmetry argument that if (I_1^*, I_2^*) is optimal among Cases 1 and 2, then $(\text{Id} - I_1^*, \text{Id} - I_2^*)$ is also optimal among Cases 4 and 3. Then, (I_1^*, I_2^*) and $(\text{Id} - I_1^*, \text{Id} - I_2^*)$ are both optimal for the finite dimensional problem (3.2), and it thus suffices to consider only the first two cases.

For Case 1, the finite dimensional problem (3.2) is optimized with $I_1(x) = (x - d_1)_+$, with $R_1(x) = x \wedge d_1$, and $I_2(x) = (x - d_2)_+$, with $R_2(x) = x \wedge d_2$, for some $d_1, d_2 \in [0, \text{ess sup}(X)]$. To explicitly optimize the sum of VaR of buyer and seller, there are two sub-cases.

Consider the first sub-case that $d_1 \leq d_2$. The unconditional cumulative distribution functions of the retained loss of buyer $F_{\mathbf{B}(0,R_1,R_2;X,Y)}$ and the indemnified loss of seller $F_{\mathbf{S}(0,I_1,I_2;X,Y)}$ are respectively given by

$$F_{\mathbf{B}(0,R_1,R_2;X,Y)}(x) = \begin{cases} F_X(x) & \text{for } x \in [0, d_1); \\ p + (1-p)F_{X|Y}(x|2) & \text{for } x \in [d_1, d_2); \\ 1 & \text{for } x \in [d_2, \text{ess sup}(X)], \end{cases}$$

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and $F_{\mathbf{S}(0,I_1,I_2;X,Y)}(x) = pF_{X|Y}(x+d_1|1) + (1-p)F_{X|Y}(x+d_2|2)$, for $x \in [0, \operatorname{ess\,sup}(X)]$; recall that $\mathbf{B}(0, R_1, R_2; X, Y)$ and $\mathbf{S}(0, I_1, I_2; X, Y)$ are defined in (2.1) and (2.2). In this sub-case that $d_1 \leq d_2$, there are five further sub-cases to consider in order to explicitly optimize the sum of VaR of buyer and seller. They are listed as follows:

(i) $p + (1-p)F_{X|Y}(d_2|2) \leq 1 - \alpha < 1$;

(ii) $\max\{p + (1-p)F_{X|Y}(d_1|2), pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2)\} \leq 1 - \alpha < p + (1-p)F_{X|Y}(d_2|2)$;

(iii) $\min\{p + (1-p)F_{X|Y}(d_1|2), pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2)\} \leq 1 - \alpha < \max\{p + (1-p)F_{X|Y}(d_1|2), pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2)\}$;

(iv) $F_X(d_1) \leq 1 - \alpha < \min\{p + (1-p)F_{X|Y}(d_1|2), pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2)\}$;

(v) $0 < 1 - \alpha < F_X(d_1)$.

For each of these further sub-cases, the sum of VaR of buyer and seller (that is $F_B(0, 1, d_1, 1, d_2) + F_S(0, -1, d_1, -1, d_2)$) is locally optimized, with the locally optimized d_1^* and d_2^*, as well as the locally optimal objective value, under various conditions on the parameters p and α, and the conditional distribution functions of the loss. More specifically, they are summarized as follows:

(i)
- If $1 - \alpha < p$, (i) cannot hold.
- If $p \leq 1 - \alpha$, $d_1^* = d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_X^{-1}(1 - \alpha)$.

(ii)
- If $1 - \alpha < p$, (ii) cannot hold.
- If $p \leq 1 - \alpha$ and $p < \alpha$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* = F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* = F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) \leq F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$, $d_1^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$ and $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(iii) There are two further sub-sub-cases.

(I) Suppose that $p + (1-p)F_{X|Y}(d_1|2) \leq 1 - \alpha < pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2)$.
- If $1 - \alpha < p$, (iii, I) cannot hold.
- If $p \leq 1 - \alpha$ and $p < \alpha$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in (F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, $d_1^* \in (F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in (F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) \leq F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$, there is no solution.

(II) Suppose that $pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_2|2) \leq 1 - \alpha < p + (1-p)F_{X|Y}(d_1|2)$.
- If $1 - \alpha < p < \alpha$, $d_1^* = 0$ and $d_2^* = F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = 0$.
- If $1 - \alpha < p$ and $\alpha \leq p$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

- If $p \leq 1 - \alpha$ and $p < \alpha$, there is no solution.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) \leq F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, there is no solution.
- If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) < F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$, $d_1^* \in (F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$, $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(iv)
- If $1 - \alpha < p < \alpha$, $d_1^* = 0$ and $d_2^* \in (F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = 0$.
- If $1 - \alpha < p$ and $\alpha \leq p$, there is no solution.
- If $p \leq 1 - \alpha$ and $p < \alpha$, there is no solution.
- If $\alpha \leq p \leq 1 - \alpha$, there is no solution.

(v) $d_1^* \in (F_X^{-1}(1 - \alpha), \operatorname{ess\,sup}(X)]$ and $d_2^* \in [d_1^*, \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_X^{-1}(1 - \alpha)$.

Since the objective function is locally optimized for each of these further sub-cases, the locally optimal objective values under various conditions need to be aggregated and compared. Therefore, the following summarizes the first sub-case that $d_1^* \leq d_2^*$.

(1) If $p < \alpha$ and $p \leq 1 - \alpha$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(2) If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(3) If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, $d_1^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$ and $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(4) If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) < F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$, $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(5) If $\alpha \leq p$ and $1 - \alpha < p$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = \operatorname{ess\,sup}(X)$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(6) If $1 - \alpha < p < \alpha$, $d_1^* = 0$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \operatorname{ess\,sup}(X)]$, with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = 0$.

For the second sub-case that $d_2 \leq d_1$, the above summary also holds by relabeling d_1^* and d_2^*, as well as $p = \mathbb{P}(Y = 1)$ and $1 - p = \mathbb{P}(Y = 2)$.

Again, since the objective function is locally optimized for the two sub-cases that $d_1 \leq d_2$ and $d_2 \leq d_1$, the locally optimal objective values under various conditions need to be aggregated and compared. Therefore, for Case 1 that $\theta_1 = 1$ and $\theta_2 = 1$, it holds that:

(1) If $p < \alpha$ and $p \leq 1 - \alpha$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-pF_{X|Y}(d_1^*|1)}{1-p}|2), \operatorname{ess\,sup}(X)]$, or
- $d_1^* = \operatorname{ess\,sup}(X)$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$,
with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(2) If $\alpha \leq p \leq 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,

- $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha - pF_{X|Y}(d_1^*|1)}{1-p}|2), \text{ess sup}(X)]$, or

- $d_1^* = \text{ess sup}(X)$ and $d_2^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$,

  with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(3) If $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,

- $d_1^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$ and $d_2^* = \text{ess sup}(X)$, or

- $d_1^* = \text{ess sup}(X)$ and $d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,

  with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(4) If $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) < F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$,

- $d_1^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = \text{ess sup}(X)$, or

- $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-(1-p)F_{X|Y}(d_2^*|2)}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$,

  with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(5) If $\alpha \le p$ and $1 - \alpha < p$,

- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = \text{ess sup}(X)$, or

- $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha-(1-p)F_{X|Y}(d_2^*|2)}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$,

  with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(6) If $1 - \alpha < p < \alpha$,

- $d_1^* = 0$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \text{ess sup}(X)]$, or

- $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{p}|1), \text{ess sup}(X)]$ and $d_2^* = 0$,

  with $F_B(0, 1, d_1^*, 1, d_2^*) + F_S(0, -1, d_1^*, -1, d_2^*) = 0$.

For Case 2 that $\theta_1 = 1$ and $\theta_2 = -1$, the finite dimensional problem (3.2) is optimized with $I_1(x) = (x - d_1)_+$, with $R_1(x) = x \wedge d_1$, and $I_2(x) = x \wedge d_2$, with $R_2(x) = (x - d_2)_+$, for some $d_1, d_2 \in [0, \text{ess sup}(X)]$. The unconditional cumulative distribution functions of the retained loss of the buyer $F_{\mathbf{B}(0,R_1,R_2;X,Y)}$ and the indemnified loss of the seller $F_{\mathbf{S}(0,R_1,R_2;X,Y)}$ are respectively given by

$F_{\mathbf{B}(0,R_1,R_2;X,Y)}(x)$
$= \begin{cases} pF_{X|Y}(x|1) + (1-p)F_{X|Y}(x+d_2|2) & \text{for } x \in [0, d_1); \\ p + (1-p)F_{X|Y}(x+d_2|2) & \text{for } x \in [d_1, \text{ess sup}(X)], \end{cases}$

$F_{\mathbf{S}(0,I_1,I_2;X,Y)}(x)$
$= \begin{cases} pF_{X|Y}(x+d_1|1) + (1-p)F_{X|Y}(x|2) & \text{for } x \in [0, d_2); \\ pF_{X|Y}(x+d_1|1) + (1-p) & \text{for } x \in [d_2, \text{ess sup}(X)]. \end{cases}$

There are seven sub-cases to consider in order to explicitly optimize the sum of VaR of buyer and seller, which are listed as follows:

(i) $\max\{p + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1 - p)\} \le 1 - \alpha < 1$;

(ii) $\min\{p + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)\} \le 1 - \alpha < \max\{p + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1 - p)\}$, which induces two further sub-cases for simplifying the minimum and maximum;

(iii) $\max\{pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)F_{X|Y}(d_2|2)\} \le 1 - \alpha < \min\{p + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)\}$;

(iv) $\min\{pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)F_{X|Y}(d_2|2)\} \le 1 - \alpha < \max\{pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)F_{X|Y}(d_2|2)\}$, which induces two further sub-cases for simplifying the minimum and maximum;

(v) $\max\{(1-p)F_{X|Y}(d_2|2), pF_{X|Y}(d_1|1)\} \le 1 - \alpha < \min\{pF_{X|Y}(d_1|1) + (1-p)F_{X|Y}(d_1 + d_2|2), pF_{X|Y}(d_1 + d_2|1) + (1-p)F_{X|Y}(d_2|2)\}$;

(vi) $\min\{(1-p)F_{X|Y}(d_2|2), pF_{X|Y}(d_1|1)\} \le 1 - \alpha < \max\{(1-p)F_{X|Y}(d_2|2), pF_{X|Y}(d_1|1)\}$, which induces two further sub-cases for simplifying the minimum and maximum;

(vii) $0 < 1 - \alpha < \min\{(1-p)F_{X|Y}(d_2|2), pF_{X|Y}(d_1|1)\}$.

For each of these sub-cases, the sum of VaR of buyer and seller is locally optimized (that is $F_B(0, 1, d_1, -1, d_2) + F_S(0, -1, d_1, 1, d_2)$), with the locally optimized d_1^* and d_2^*, as well as the locally optimal objective value, under various conditions on the parameters p and α, and the conditional distributions of the loss. For example, for (i),

- if $p < \alpha$ or $1 - \alpha < p$, (i) cannot hold;
- if $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*$, with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$;
- if $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) \le F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$, $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*$, with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

Since the objective function is locally optimized for each of these sub-cases, the locally optimal objective values under various conditions need to be aggregated and compared. Therefore, for Case 2 that $\theta_1 = 1$ and $\theta_2 = -1$, it holds that:

(1) If $p < \alpha$ and $p \le 1 - \alpha$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*]$, or
- $d_1^* = \text{ess sup}(X)$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \text{ess sup}(X)]$,
  with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(2) If $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) < F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*]$, or
- $d_1^* \in (F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1), F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*]$,
  with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(3) If $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)] = [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^* = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*$,
  with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) = F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)$.

(4) If $\alpha \le p \le 1 - \alpha$ and $F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) < F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2)]$ and $d_2^* \in [F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2) - d_1^*, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*]$, or
- $d_1^* \in (F_{X|Y}^{-1}(1 - \frac{\alpha}{1-p}|2), F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*]$,
  with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(5) If $\alpha \le p$ and $1 - \alpha < p$,
- $d_1^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)]$ and $d_2^* \in [0, F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1) - d_1^*]$, or
- $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{p}|1), \text{ess sup}(X)]$ and $d_2^* = \text{ess sup}(X)$,
  with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = F_{X|Y}^{-1}(1 - \frac{\alpha}{p}|1)$.

(6) If $1 - \alpha < p < \alpha$,
- $d_1^* = 0$ and $d_2^* = 0$, or
- $d_1^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{p}|1), \text{ess sup}(X)]$ and $d_2^* \in [F_{X|Y}^{-1}(\frac{1-\alpha}{1-p}|2), \text{ess sup}(X)]$,

with $F_B(0, 1, d_1^*, -1, d_2^*) + F_S(0, -1, d_1^*, 1, d_2^*) = 0$.

Finally, the result follows by aggregating and comparing the summaries of Case 1 and Case 2.

Appendix C. Proof of Theorem 4.1

For any $b \in [0, \bar{b}]$ and $(I_1, \ldots, I_m) \in \mathcal{I}$, define $a := \text{VaR}_\alpha(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}})$ and $c := \text{VaR}_\beta(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}})$. Recall the right-continuous inverse functions of I_k and R_k from Appendix A.

We first assume $a \geq 0$. For each $k = 1, \ldots, m$, consider the following two cases and the corresponding sub-cases to construct the modification $\tilde{I}_k$.

Case 1: Assume that $I_k^{-1+}(c) \leq R_k^{-1+}(a)$.

Sub-case 1.1: Consider that $R_k^{-1+}(a) < \text{ess sup}(X)$. Define $\tilde{R}_k(x) := x - (x-a)_+ + (x-\tilde{d}_{k,2})_+$, and hence $\tilde{I}_k(x) = (x-a)_+ - (x-\tilde{d}_{k,2})_+$, for any $x \in [0, \text{ess sup}(X)]$, where $\tilde{d}_{k,2} \in [R_k^{-1+}(a), \text{ess sup}(X)]$ such that

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)_+|Y=k] = \mathbb{E}^\mathbb{P}\left[(R_k(X) - a)_+|Y=k\right].$$

This implies

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k]$$
$$= \mathbb{E}^\mathbb{P}[(R_k(X) - a)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k],$$

which can be written as

$$\mathbb{E}^\mathbb{P}[\tilde{R}_k(X)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k]$$
$$= \mathbb{E}^\mathbb{P}[R_k(X)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k],$$

and

$$\mathbb{E}^\mathbb{P}[\tilde{I}_k(X)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k]$$
$$= \mathbb{E}^\mathbb{P}[I_k(X)\mathbb{I}_{\{X \geq R^{-1+}(a)\}}|Y=k].$$

Thus,

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - \tilde{I}_k(R_k^{-1+}(a)))_+|Y=k]$$
$$= \mathbb{E}^\mathbb{P}[(I_k(X) - I_k(R_k^{-1+}(a)))_+|Y=k].$$

By definition, it holds for any $x \in [0, R_k^{-1+}(a)]$ that $\tilde{I}_k(x) \leq I_k(x)$. Furthermore, we have $\tilde{I}_k(R_k^{-1+}(a)) = R_k^{-1+}(a) - a = I_k(R_k^{-1+}(a)) \geq c$. Therefore,

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k]$$
$$= \mathbb{E}^\mathbb{P}\left[(\tilde{I}_k(X) - c)_+\mathbb{I}_{\{X \leq R_k^{-1+}(a)\}}|Y=k\right]$$
$$\quad + \mathbb{E}^\mathbb{P}\left[(\tilde{I}_k(X) - c)_+\mathbb{I}_{\{X > R_k^{-1+}(a)\}}|Y=k\right]$$
$$= \mathbb{E}^\mathbb{P}\left[(\tilde{I}_k(X) - c)_+\mathbb{I}_{\{X \leq R_k^{-1+}(a)\}}|Y=k\right]$$
$$\quad + \mathbb{E}^\mathbb{P}\left[(\tilde{I}_k(X) - \tilde{I}_k(R_k^{-1+}(a)))_+|Y=k\right]$$
$$\quad + (\tilde{I}_k(R_k^{-1+}(a)) - c)\mathbb{P}(X > R_k^{-1+}(a)|Y=k)$$
$$\leq \mathbb{E}^\mathbb{P}\left[(I_k(X) - c)_+\mathbb{I}_{\{X \leq R_k^{-1+}(a)\}}|Y=k\right]$$
$$\quad + \mathbb{E}^\mathbb{P}\left[(I_k(X) - I_k(R_k^{-1+}(a)))_+|Y=k\right]$$
$$\quad + (I_k(R_k^{-1+}(a)) - c)\mathbb{P}(X > R_k^{-1+}(a)|Y=k)$$
$$= \mathbb{E}^\mathbb{P}\left[(I_k(X) - c)_+\mathbb{I}_{\{X \leq R_k^{-1+}(a)\}}|Y=k\right]$$
$$\quad + \mathbb{E}^\mathbb{P}\left[(I_k(X) - c)_+\mathbb{I}_{\{X > R_k^{-1+}(a)\}}|Y=k\right]$$
$$= \mathbb{E}^\mathbb{P}[(I_k(X) - c)_+|Y=k].$$
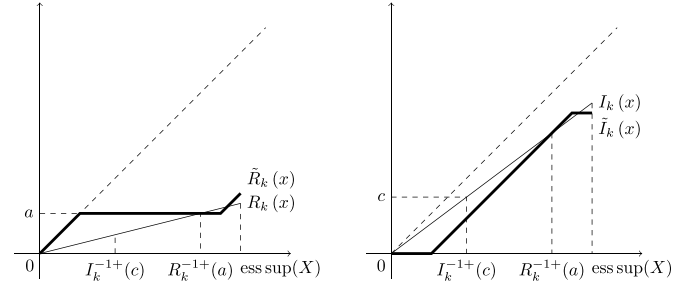

Fig. C.3. Constructions of $\tilde{R}_k$ (left) and $\tilde{I}_k$ (right), under the conditions that $a \geq 0$ and $I_k^{-1+}(c) \leq R_k^{-1+}(a) < \text{ess sup}(X)$, for TVaR risk preferences (Sub-case 1.1); linear R_k and I_k are chosen for graphical convenience, and $\tilde{R}_k$ and $\tilde{I}_k$ are bold-faced.

Sub-case 1.2: Consider that $R_k^{-1+}(a) = \text{ess sup}(X)$. Necessarily, for any $x \in [0, \text{ess sup}(X)]$, $R_k(x) \leq a$. Define $\tilde{R}_k(x) := x - (x-a)_+$, and hence $\tilde{I}_k(x) = (x-a)_+$, for any $x \in [0, \text{ess sup}(X)]$. By definition, it holds for any $x \in [0, \text{ess sup}(X)]$ that $R_k(x) \leq \tilde{R}_k(x) \leq a$, and necessarily $\tilde{I}_k(x) \leq I_k(x)$. Therefore,

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)_+|Y=k] = 0$$
$$= \mathbb{E}^\mathbb{P}\left[(R_k(X) - a)_+|Y=k\right];$$

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k] \leq \mathbb{E}^\mathbb{P}[(I_k(X) - c)_+|Y=k].$$

In both sub-cases 1.1 and 1.2, the constructed $\tilde{I}_k$ and $\tilde{R}_k$ satisfy

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k] \leq \mathbb{E}^\mathbb{P}[(I_k(X) - c)_+|Y=k],$$

and

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)_+|Y=k] = \mathbb{E}^\mathbb{P}\left[(R_k(X) - a)_+|Y=k\right].$$

Case 2: Assume that $R_k^{-1+}(a) \leq I_k^{-1+}(c)$. If $I_k^{-1+}(c) < \text{ess sup}(X)$, define $\tilde{I}_k(x) := x - (x-c)_+ + (x-\tilde{d}_{k,2})_+$, and hence $\tilde{R}_k(x) = (x-c)_+ - (x-\tilde{d}_{k,2})_+$, for any $x \in [0, \text{ess sup}(X)]$, where $\tilde{d}_{k,2} \in [I_k^{-1+}(c), \text{ess sup}(X)]$ such that

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k] = \mathbb{E}^\mathbb{P}\left[(I_k(X) - c)_+|Y=k\right].$$

If $I_k^{-1+}(c) = \text{ess sup}(X)$, define $\tilde{I}_k(x) := x - (x-c)_+$, and hence $\tilde{R}_k(x) = (x-c)_+$, for any $x \in [0, \text{ess sup}(X)]$. By following similar arguments as in Case 1 with interchanging the roles of I_k (or the constructed $\tilde{I}_k$) and R_k (or the constructed $\tilde{R}_k$), as well as of a and c, one can show that the constructed $\tilde{I}_k$ and $\tilde{R}_k$ satisfy that

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k] = \mathbb{E}^\mathbb{P}[(I_k(X) - c)_+|Y=k];$$

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)_+|Y=k] \leq \mathbb{E}^\mathbb{P}\left[(R_k(X) - a)_+|Y=k\right].$$

For the ease of understanding these cases above, Fig. C.3 illustrates the modification arguments of Sub-case 1.1.

Therefore, for any $k = 1, \ldots, m$, the constructed $\tilde{I}_k$ and $\tilde{R}_k$ satisfy that

$$\mathbb{E}^\mathbb{P}[(\tilde{I}_k(X) - c)_+|Y=k] \leq \mathbb{E}^\mathbb{P}[(I_k(X) - c)_+|Y=k];$$

$$\mathbb{E}^\mathbb{P}[(\tilde{R}_k(X) - a)_+|Y=k] \leq \mathbb{E}^\mathbb{P}\left[(R_k(X) - a)_+|Y=k\right].$$

By the dual representation of TVaR (Rockafellar & Uryasev, 2000),

$$\text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$= \inf_{d \geq -b}\left(d + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}} - d\right)_+\right]\right),$$

where the infimum can be attained at $d^* = a$. Moreover,

$$\text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$= \inf_{e \geq 0}\left(e + \frac{1}{\beta}\mathbb{E}^{\mathbb{P}}\left[\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}} - e\right)_+\right]\right),$$

where the infimum is attained at $e^* = c$. The last two relations imply that

$$\text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$= \inf_{d \geq -b}\left(d + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} - d\right)_+\right]\right)$$

$$\leq a + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}} - a\right)_+\right]$$

$$= a + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(-b - a)_+|Y=0\right]\mathbb{P}(Y=0)$$

$$+ \sum_{k=1}^m \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(\tilde{R}_k(X) - a)_+|Y=k\right]\mathbb{P}(Y=k)$$

$$\leq a + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b - a\right)_+|Y=0\right]\mathbb{P}(Y=0)$$

$$+ \sum_{k=1}^m \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(R_k(X) - a)_+|Y=k\right]\mathbb{P}(Y=k)$$

$$= a + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}} - a\right)_+\right]$$

$$= \inf_{d \geq -b}\left(d + \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}} - d\right)_+\right]\right)$$

$$= \text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right).$$

Similarly, we have that

$$\text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$\leq \text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right).$$

Therefore, $(\tilde{I}_1, \ldots, \tilde{I}_m) \in \mathcal{I}_2$ and

$$G(b, (\tilde{I}_1, \ldots, \tilde{I}_m))$$

$$= \text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$+ \text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$\leq \text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$+ \text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$= G(b, (I_1, \ldots, I_m)).$$

Following, we consider the case $a < 0$. For this case, we use the same way as Case 2 to construct $\tilde{I}_k(x)$, then it is easy to find that $\mathbb{E}^{\mathbb{P}}[(\tilde{I}_k(X) - c)_+|Y=k] = \mathbb{E}^{\mathbb{P}}[(I_k(X) - c)_+|Y=k]$ and $\mathbb{E}^{\mathbb{P}}[\tilde{R}_k(X)|Y=k] \leq \mathbb{E}^{\mathbb{P}}[R_k(X)|Y=k]$, which in turn imply

$$\text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{R}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$\leq \text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

and

$$\text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m \tilde{I}_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$\leq \text{TVaR}_\beta\left(b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right).$$

Therefore, $G(b, (\tilde{I}_1, \ldots, \tilde{I}_m)) \leq G(b, (I_1, \ldots, I_m))$, which completes the proof.

Appendix D. Proof of Proposition 4.1

Note that $\text{TVaR}_1 = \mathbb{E}^{\mathbb{P}}$. Thus, the expectation is a special case of TVaR. By Theorem 4.1, it suffices to consider the finite dimensional problem (4.2):

$$\min_{\substack{b \in [0,\bar{b}]; \\ (I_1, \ldots, I_m) \in \mathcal{I}_2}} \text{TVaR}_\alpha\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right)$$

$$+ \mathbb{E}^{\mathbb{P}}\left[b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m I_k(X)\mathbb{I}_{\{Y=k\}}\right].$$

By the dual representation of TVaR (Rockafellar & Uryasev, 2000), the above problem is equivalent to

$$\min_{b \in [0,\bar{b}]} \min_{(I_1, \ldots, I_m) \in \mathcal{I}_2} \inf_{d \geq -b}\left(d + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}} - d\right)_+\right]\right.$$

$$+ \left.\mathbb{E}^{\mathbb{P}}\left[b \times \mathbb{I}_{\{Y=0\}} - \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right]\right)$$

$$= \min_{b \in [0,\bar{b}]} \inf_{d \geq -b} \min_{(I_1, \ldots, I_m) \in \mathcal{I}_2}\left(d + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}\left(-b \times \mathbb{I}_{\{Y=0\}} + \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}} - d\right)_+\right]\right.$$

$$+ \left.\mathbb{E}^{\mathbb{P}}\left[b \times \mathbb{I}_{\{Y=0\}} - \sum_{k=1}^m R_k(X)\mathbb{I}_{\{Y=k\}}\right]\right)$$

$$= \min_{b \in [0,\bar{b}]} \inf_{d \geq -b}\left(d + b \times \mathbb{P}(Y=0)\right.$$

$$+ \left.\min_{(I_1, \ldots, I_m) \in \mathcal{I}_2} \sum_{k=1}^m \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\Big|Y=k\right]\mathbb{P}(Y=k)\right).$$

Let $b \in [0, \bar{b}]$. First, fix any $d \geq 0$. For any $k = 1, 2, \ldots, m$, I_k takes a form either

$$I_k(x) = (x - d_{k,1})_+ - (x - d_{k,2})_+ \quad \text{or} \quad I_k(x) = x - (x - d_{k,1})_+ + (x - d_{k,2})_+,$$

for some $d_{k,1} \in [0, \text{ess sup}(X)]$ and $d_{k,2} \in [d_{k,1}, \text{ess sup}(X)]$, which allows us to directly compute $\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)|Y = k\right]$. In the sequel, we let $d_{k,1}^*$ and $d_{k,2}^*$ be parameters that yields the minimum of this conditional expectation.

Case 1: Suppose that $I_k(x) = (x - d_{k,1})_+ - (x - d_{k,2})_+$, i.e. $R_k(x) = x - (x - d_{k,1})_+ + (x - d_{k,2})_+$.

Sub-case 1.1: Consider that $0 \leq d \leq d_{k,1} \leq d_{k,2} \leq \text{ess sup}(X)$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(X - d)_+ - (X - d_{k,1})_+ + (X - d_{k,2})_+|Y = k\right]$$
$$\quad - \mathbb{E}^{\mathbb{P}}\left[X - (X - d_{k,1})_+ + (X - d_{k,2})_+|Y = k\right]$$
$$= \frac{1}{\alpha}\int_d^{\text{ess sup}(X)} S_{X|Y}(t|k)dt + \left(1 - \frac{1}{\alpha}\right)\int_{d_{k,1}}^{d_{k,2}} S_{X|Y}(t|k)dt$$
$$\quad - \mathbb{E}^{\mathbb{P}}[X|Y = k],$$

which is non-decreasing in $d_{k,1}$ and non-increasing in $d_{k,2}$, where $S_{X|Y}(\cdot|k)$ is the survival function of the loss $X|Y = k$ under $\mathbb{P}$, for $k = 1, 2, \ldots, m$. Therefore, $d_{k,1}^* = d$ and $d_{k,2}^* = \text{ess sup}(X)$.

Sub-case 1.2: Consider that $0 \leq d_{k,1} \leq d \leq d_{k,1} + \text{ess sup}(X) - d_{k,2} \leq \text{ess sup}(X)$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(X - (d_{k,2} + d - d_{k,1}))_+|Y = k\right]$$
$$\quad - \mathbb{E}^{\mathbb{P}}\left[X - (X - d_{k,1})_+ + (X - d_{k,2})_+|Y = k\right].$$

Define $z = d_{k,2} + d - d_{k,1}$, and thus $0 \leq d_{k,1} \leq d \leq z \leq \text{ess sup}(X)$, and

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(X - z)_+|Y = k\right]$$
$$\quad - \mathbb{E}^{\mathbb{P}}\left[X - (X - d_{k,1})_+ + (X - (z - d + d_{k,1}))_+|Y = k\right]$$
$$= \left(\frac{1}{\alpha} - 1\right)\int_z^{\text{ess sup}(X)} S_{X|Y}(t|k)dt - \int_0^{d_{k,1}} S_{X|Y}(t|k)dt$$
$$\quad - \int_{z-d+d_{k,1}}^z S_{X|Y}(t|k)dt,$$

which can be easily shown to be non-increasing in $d_{k,1}$, given a z. Therefore, $d_{k,1}^* = d$, and hence,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \left(\frac{1}{\alpha} - 1\right)\int_z^{\text{ess sup}(X)} S_{X|Y}(t|k)dt - \int_0^d S_{X|Y}(t|k)dt,$$

which is non-increasing in z. Therefore, $z^* = d_{k,2}^* = \text{ess sup}(X)$.

Sub-case 1.3: Consider that $0 \leq d_{k,1} \leq d_{k,1} + \text{ess sup}(X) - d_{k,2} \leq d \leq \text{ess sup}(X)$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= -\mathbb{E}^{\mathbb{P}}\left[X - (X - d_{k,1})_+ + (X - d_{k,2})_+|Y = k\right]$$
$$= \int_{d_{k,1}}^{d_{k,2}} S_{X|Y}(t|k)dt - \mathbb{E}^{\mathbb{P}}[X|Y = k],$$

which is non-increasing in $d_{k,1}$. Therefore, $d_{k,1}^* = d_{k,2} - \text{ess sup}(X) + d$, and hence,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \int_{d_{k,2}-\text{ess sup}(X)+d}^{d_{k,2}} S_{X|Y}(t|k)dt - \mathbb{E}^{\mathbb{P}}[X|Y = k],$$

which can be easily shown to be non-increasing in $d_{k,2}$. Therefore, $d_{k,2}^* = \text{ess sup}(X)$, and thus $d_{k,1}^* = d$.

Sub-case 1.4: Consider that $0 \leq d_{k,1} \leq d_{k,2} \leq \text{ess sup}(X) \leq d$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= -\mathbb{E}^{\mathbb{P}}\left[X - (X - d_{k,1})_+ + (X - d_{k,2})_+|Y = k\right]$$
$$= \int_{d_{k,1}}^{d_{k,2}} S_{X|Y}(t|k)dt - \mathbb{E}^{\mathbb{P}}[X|Y = k],$$

which is non-increasing in $d_{k,1}$ and non-decreasing in $d_{k,2}$. Therefore, $d_{k,1}^* = d_{k,2}^* \leq \text{ess sup}(X) \leq d$.

Case 2: Suppose that $I_k(x) = x - (x - d_{k,1})_+ + (x - d_{k,2})_+$, i.e. $R_k(x) = (x - d_{k,1})_+ - (x - d_{k,2})_+$.

Sub-case 2.1: Consider that $0 \leq d \leq d_{k,2} - d_{k,1}$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[(X - (d_{k,1} + d))_+ - (X - d_{k,2})_+|Y = k\right]$$
$$\quad - \mathbb{E}^{\mathbb{P}}\left[(X - d_{k,1})_+ - (X - d_{k,2})_+|Y = k\right]$$
$$= \frac{1}{\alpha}\int_{d_{k,1}+d}^{\text{ess sup}(X)} S_{X|Y}(t|k)dt - \int_{d_{k,1}}^{\text{ess sup}(X)} S_{X|Y}(t|k)dt$$
$$\quad + \left(1 - \frac{1}{\alpha}\right)\int_{d_{k,2}}^{\text{ess sup}(X)} S_{X|Y}(t|k)dt,$$

which is non-decreasing in $d_{k,2}$. Therefore, $d_{k,2}^* = d_{k,1} + d$, and hence

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= -\int_{d_{k,1}}^{d_{k,1}+d} S_{X|Y}(t|k)dt,$$

which is non-decreasing in $d_{k,1}$. Therefore, $d_{k,1}^* = 0$ and $d_{k,2}^* = d$.

Sub-case 2.2: Consider that $0 \leq d_{k,2} - d_{k,1} \leq d$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= -\mathbb{E}^{\mathbb{P}}\left[(X - d_{k,1})_+ - (X - d_{k,2})_+|Y = k\right]$$
$$= -\int_{d_{k,1}}^{\text{ess sup}(X)} S_{X|Y}(t|k)dt + \int_{d_{k,2}}^{\text{ess sup}(X)} S_{X|Y}(t|k)dt,$$

which is non-increasing in $d_{k,2}$. Therefore, $d_{k,2}^* = d_{k,1} + d$, and hence

$$\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\bigg|Y = k\right]$$
$$= -\int_{d_{k,1}}^{d_{k,1}+d} S_{X|Y}(t|k)dt,$$

which is non-decreasing in $d_{k,1}$. Therefore, $d_{k,1}^* = 0$ and $d_{k,2}^* = d$.

Hence, if $d \geq 0$, in any case, it is optimal to choose $I_k(x) = (x - d)_+$, i.e. $R_k(x) = x - (x - d)_+$, which are independent of the environment $k = 1, 2, \ldots, m$, and the objective function becomes:

$$d - \sum_{k=1}^{m} \mathbb{E}^{\mathbb{P}}\big[X - (X - d)_+ | Y = k\big]\mathbb{P}(Y = k) + b \times \mathbb{P}(Y = 0),$$

which is non-decreasing in d, and thus the minimum is attained at $d^* = 0$, and thus the objective function is $b \times \mathbb{P}(Y = 0)$.

Second, fix any $d \in [-b, 0]$. For any $k = 1, 2, \ldots, m$, $\mathbb{E}^{\mathbb{P}}\big[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X) | Y = k\big] = (\frac{1}{\alpha} - 1)\mathbb{E}^{\mathbb{P}}[R_k(X) | Y = k] - \frac{d}{\alpha}$. In the next two cases, we determine the parameters $d_{k,1}^*$ and $d_{k,2}^*$ that yield the minimum of this conditional expectation.

Case 1: Suppose that $I_k(x) = (x - d_{k,1})_+ - (x - d_{k,2})_+$, i.e. $R_k(x) = x - (x - d_{k,1})_+ + (x - d_{k,2})_+$, for some $d_{k,1} \in [0, \mathrm{ess\,sup}(X)]$ and $d_{k,2} \in [d_{k,1}, \mathrm{ess\,sup}(X)]$. Therefore,

$$\mathbb{E}^{\mathbb{P}}\Big[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\Big| Y = k\Big]$$
$$= \Big(\frac{1}{\alpha} - 1\Big)\Big(\int_0^{d_{k,1}} S_{X|Y}(t|k)dt + \int_{d_{k,2}}^{\mathrm{ess\,sup}(X)} S_{X|Y}(t|k)dt\Big) - \frac{d}{\alpha},$$

which is non-decreasing in $d_{k,1}$ and non-increasing in $d_{k,2}$, and thus $d_{k,1}^* = 0$ and $d_{k,2}^* = \mathrm{ess\,sup}(X)$.

Case 2: Suppose that $I_k(x) = x - (x - d_{k,1})_+ + (x - d_{k,2})_+$, i.e. $R_k(x) = (x - d_{k,1})_+ - (x - d_{k,2})_+$, for some $d_{k,1} \in [0, \mathrm{ess\,sup}(X)]$ and $d_{k,2} \in [d_{k,1}, \mathrm{ess\,sup}(X)]$. Therefore,

$$\mathbb{E}^{\mathbb{P}}\Big[\frac{1}{\alpha}(R_k(X) - d)_+ - R_k(X)\Big| Y = k\Big]$$
$$= \Big(\frac{1}{\alpha} - 1\Big)\int_{d_{k,1}}^{d_{k,2}} S_{X|Y}(t|k)dt - \frac{d}{\alpha},$$

and thus $0 \leq d_{k,1}^* = d_{k,2}^* \leq \mathrm{ess\,sup}(X)$.

Hence, if $d \in [-b, 0]$, in any case, $I_k(x) = x$, i.e. $R_k(x) = 0$, which are independent of the environment $k = 1, 2, \ldots, m$, and the objective function becomes:

$$\Big(1 - \frac{1}{\alpha}\sum_{k=1}^{m}\mathbb{P}(Y = k)\Big)d + b \times \mathbb{P}(Y = 0).$$

If $\alpha \geq \sum_{k=1}^{m}\mathbb{P}(Y = k)$, the objective function is non-decreasing in d, and thus it is minimized at $d^* = -b$. If $\alpha \leq \sum_{k=1}^{m}\mathbb{P}(Y = k)$, the objective function is non-increasing in d, and thus it is minimized at $d^* = 0$.

Therefore, there are two cases to consider on determining the optimal bonus $b^* \in [0, \bar{b}]$.

Case 1: Suppose that $\alpha \geq \sum_{k=1}^{m}\mathbb{P}(Y = k)$. Therefore, $d^* = -b$, and the objective function becomes, for any $b \in [0, \bar{b}]$, $\big(\mathbb{P}(Y = 0) + \frac{1}{\alpha}\sum_{k=1}^{m}\mathbb{P}(Y = k) - 1\big)b$, which is non-decreasing in b, and thus it is minimized at $b^* = 0$.

Case 2: Suppose that $\alpha \leq \sum_{k=1}^{m}\mathbb{P}(Y = k)$. Therefore, $d^* = 0$, and the objective function is, for any $b \in [0, \bar{b}]$, $b \times \mathbb{P}(Y = 0)$, which is also non-decreasing in b, and thus it is minimized at $b^* = 0$.

Hence, in any case, the bonus $b^* = 0$ and the optimal indemnity functions $I_k^* = \mathrm{Id}$, for $k = 1, 2, \ldots, m$, solve Problem (4.2), and are thus Pareto optimal.

Appendix E. Proof of Theorem 6.1

The minimization problem (6.1) can be rewritten as

$$\mathbb{E}^{\mathbb{P}}[X] + \min_{b \in [0, \bar{b}]} b(\mathbb{Q}(Y = 0) - \mathbb{P}(Y = 0))$$

$$+ \min_{(I_1, \ldots, I_m) \in \mathcal{I}} \sum_{k=1}^{m}(\mathbb{E}^{\mathbb{Q}}[I_k(X) | Y = k]\mathbb{Q}(Y = k)$$
$$- \mathbb{E}^{\mathbb{P}}[I_k(X) | Y = k]\mathbb{P}(Y = k)).$$

Here, $\min_{b \in [0, \bar{b}]} b(\mathbb{Q}(Y = 0) - \mathbb{P}(Y = 0))$ is solved by

$$b\begin{cases} = \bar{b} & \text{if } \mathbb{Q}(Y = 0) < \mathbb{P}(Y = 0); \\ \in [0, \bar{b}] & \text{if } \mathbb{Q}(Y = 0) = \mathbb{P}(Y = 0); \\ = 0 & \text{if } \mathbb{Q}(Y = 0) > \mathbb{P}(Y = 0). \end{cases}$$

By Assa (2015), $(I_1, \ldots, I_m) \in \mathcal{I}$ implies $I_k'(t) \in [0, 1]$ for all $t \geq 0$ almost everywhere and all $k = 1, \ldots, m$. Moreover, for any $k = 1, 2, \ldots, m$,

$$\mathbb{E}^{\mathbb{Q}}[I_k(X) | Y = k]\mathbb{Q}(Y = k) - \mathbb{E}^{\mathbb{P}}[I_k(X) | Y = k]\mathbb{P}(Y = k)$$

$$= \int_0^{\infty} \mathbb{Q}(X > t | Y = k)dI_k(t)\mathbb{Q}(Y = k)$$
$$- \int_0^{\infty} \mathbb{P}(X > t | Y = k)dI_k(t)\mathbb{P}(Y = k)$$
$$= \int_0^{\infty} (\mathbb{Q}(\{X > t\} \cap \{Y = k\}) - \mathbb{P}(\{X > t\} \cap \{Y = k\}))dI_k(t)$$
$$= \int_0^{\infty} (\mathbb{Q}(\{X > t\} \cap \{Y = k\}) - \mathbb{P}(\{X > t\} \cap \{Y = k\}))I_k'(t)dt,$$

where the last equality is due to the fact that I_k is absolutely continuous, since $(I_1, \ldots, I_m) \in \mathcal{I}$ (see Cheung & Lo, 2017). Hence, the result follows directly.

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