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Forward Mortality Rates in Discrete Time II: Longevity Risk and Hedging Strategies

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Abstract

Longevity risk has emerged as an important risk in the early 21^{st} century for the providers of pension benefits and annuities. Any changes in the assumptions for future mortality rates can have a major financial impact on the valuation of these liabilities and motivates many of the longevity-linked securities that have been proposed to hedge this risk. Using the framework developed in Hunt and Blake (2015b), we investigate how these assumptions can change over a one-year period and the potential for hedging longevity risk in an illustrative annuity

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portfolio, and find that relatively simple hedging strategies can significantly mitigate longevity risk over a one-year period.
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1 Introduction

Longevity risk has emerged as an important risk in the early 21^{st} century for the providers of pension benefits and annuities. It is often defined as the risk that life expectancy increases at a faster rate than assumed, or conversely, that mortality rates decrease faster than assumed. However, in many contexts, the major financial impact of longevity risk is not the difference between assumed and actual mortality rates. Instead, it is the impact of changes in the assumptions for future mortality rates that has the greatest impact on the valuation of longevity-linked liabilities and securities.

The efficient valuation of these liabilities and securities requires us to make an assessment of what mortality rates are expected to be in future, i.e., a forward framework for mortality rates, such as that described in Hunt and Blake (2015b). This framework builds on the structure of age/period/cohort models of the force of mortality in discrete time, in order to give a forward surface of mortality from which values of longevity-linked liabilities and securities could be calculated. Market information is incorporated into this surface via a change of measure using the Esscher transformation, enabling all the values calculated from this surface to be both internally consistent as well as consistent with any market prices for extant longevity-linked securities.

The measurement of longevity risk then requires us to investigate how our assessment of future mortality rates changes in response to new information and, thus, how values linked to longevity change with time. To do this, this study develops the dynamics of the framework described in Hunt and Blake (2015b) to investigate how forward mortality rates can change over a one-year period. Because the values of all longevity-linked liabilities and securities are then calculated from this updated forward mortality surface,

their changes in value are consistent with each other. Consequently, we use our forward mortality framework to investigate the longevity risk present in annuities and other longevity-linked securities, and also the potential for portfolios of these securities to hedge longevity risk.

The structure of this paper is as follows. In Section 2, we first review the forward mortality rate framework used to value longevity-linked liabilities and securities in Hunt and Blake (2015b). We then consider how the forward surface of mortality will evolve over a one year period in Section 3 by examining the processes assumed to be generating the period and cohort parameters. This is then applied in Section 4 to examine the longevity risk in an illustrative annuity book, various longevity-linked securities and, finally, the impact of using these securities to hedge longevity risk. Finally, Section 5 concludes.

2 Forward mortality rates in discrete time I: A recap

In Hunt and Blake (2015b), we introduced the concept of forward mortality rates in discrete time. Assuming the existence of a market in "longevity zeros",[1] we defined the forward mortality rate, $\nu^{\mathbb{Q}}_{x,t}(\tau)$, for age x and future time t, as

$$\nu^{\mathbb{Q}}_{x,t}(\tau) = -\ln\left(\frac{{}_{t-\tau+1}P^{\mathbb{Q}}_{x-t+\tau,\tau}(\tau)}{{}_{t-\tau}P^{\mathbb{Q}}_{x-t+\tau,\tau}(\tau)}\right) \tag{1}$$

where τ is the current time, $\mathbb{Q}$ is the "market-consistent" measure and ${}_sP^{\mathbb{Q}}_{x,\tau}(\tau)$ is the market expectation of the probability that an individual aged x at time τ survives a further s years, conditional on information at τ. From this we found[2]

$$\nu^{\mathbb{Q}}_{x,t}(\tau) = \mathbb{E}^{\mathbb{Q}}_{\tau}\,\mu_{x,t} \tag{2}$$

[1] Longevity zeros are zero-coupon bonds whose principal is proportional to the survivorship of a specified cohort of individuals.

[2] We adopt the convention that the subscript on operators $\mathbb{E}_\tau(.)$, $\mathbb{V}ar_\tau(.)$ or $\mathbb{C}ov_\tau(.)$ denotes conditioning on the information available at time τ, i.e., $\mathcal{F}_\tau$.

where $\mu_{x,t}$ is the stochastic force of mortality.[3] We assumed that the dynamics of the short rate is described by an age/period/cohort (APC) model of the form considered in Hunt and Blake (2015e)

$$\ln(\mu_{x,t}) = \eta_{x,t} = \alpha_x + \boldsymbol{\beta}_x^\top \boldsymbol{\kappa}_t + \gamma_{t-x} \tag{3}$$

where

- we have historical data for ages, x, in the range $[1, X]$ and periods, t, in the range $[1, \tau]$ and therefore observations of cohorts born in years, $y = t - x$, in the range $[1 - X, \tau - 1]$;
- α_x is a static function of age;
- $\boldsymbol{\kappa}_t = \left(\kappa_t^{(i)}, \quad \ldots \quad \kappa_t^{(N)}\right)^\top$ are N period functions governing the evolution of mortality with time;
- $\boldsymbol{\beta}_x = \left(\beta_x^{(i)}, \quad \ldots \quad \beta_x^{(N)}\right)^\top$ are corresponding age functions modulating the impact of the period function dynamics over the age range;[4] and
- γ_y is a cohort function describing mortality effects which depend upon a cohort's year of birth and follow that cohort through life as it ages.

We find

$$\nu_{x,t}^{\mathbb{P}}(\tau) = \exp\left(\alpha_x + \boldsymbol{\beta}_x^\top \mathbb{E}^{\mathbb{P}}{}_\tau \boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x^\top \mathbb{V}ar_\tau^{\mathbb{P}}(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \mathbb{E}^{\mathbb{P}}{}_\tau \gamma_{t-x} + \frac{1}{2}\mathbb{V}ar_\tau^{\mathbb{P}}(\gamma_{t-x})\right) \tag{4}$$

if the period and cohort functions are projected independently using time series processes with normally distributed innovations in the real-world measure, $\mathbb{P}$. We assume that a "well-identified"[5] multivariate random walk is

[3] This also assumed that $\mathbb{E}_\tau \exp(\mu_{x,t}) = \exp(\mathbb{E}_\tau\ \mu_{x,t})$, which is the upper bound given by Jensen's inequality. When this assumption was tested in Hunt and Blake (2015b), it was found to be reasonable across almost all ages and years of interest.

[4] These can be either non-parametric in the sense of Hunt and Blake (2015e) as being one fitted without any imposing any a priori structure across ages, or parametric, in the sense of having a specific functional form, $\beta_x^{(i)} = f^{(i)}(x; \theta^{(i)})$, selected a priori.

[5] In the sense of Hunt and Blake (2015c,d) that the trends, X_t, are selected so that the projected mortality rates do not depend on any identifiability constraints imposed to fit the model in Equation 3 to historical data.

used to project the period functions

$$\boldsymbol{\kappa}_t = \mu X_t + \boldsymbol{\kappa}_{t-1} + \boldsymbol{\epsilon}_t \tag{5}$$

whilst the cohort parameters are modelled using the Bayesian approach developed in Hunt and Blake (2015a). Together, these mean that

$$\mathbb{E}^{\mathbb{P}}{}_{\tau}\boldsymbol{\kappa}_t = \boldsymbol{\kappa}_\tau + \mu \sum_{s=\tau+1}^{t} X_s \tag{6}$$

$$\mathbb{V}ar^{\mathbb{P}}_{\tau}(\boldsymbol{\kappa}_t) = (t-\tau)\Sigma \tag{7}$$

$$\text{Þ}_{\tau-y,s} \equiv \prod_{r=0}^{s-1}(1 - D_{\tau-y+r}) \tag{8}$$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}{}_{\tau}\gamma_y &\equiv M(y,\tau) \\ &= \sum_{s=0}^{\infty} \text{Þ}_{\tau-y,s}\rho^s \left[D_{\tau-y}\overline{\gamma}_y(\tau) + (1 - D_{\tau-y+s})\beta(\tilde{X}_{y-s} - \rho\tilde{X}_{y-s-1})\right]\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbb{V}ar^{\mathbb{P}}_{\tau}(\gamma_y) &\equiv V(y,\tau) \\ &= \sum_{s=0}^{\infty} \text{Þ}^2_{\tau-y,s}(1 - D_{\tau-y+s})\rho^{2s}\sigma^2\end{aligned} \tag{10}$$

where

- X_t is a set of deterministic functions ("trends") chosen to ensure identifiability in the random walk process and μ is a matrix of "drift" coefficients corresponding to these trends;[6]
- Σ is the covariance matrix of the random walk innovations in Equation 5, i.e., $\Sigma = \mathbb{V}ar(\boldsymbol{\epsilon_t})$;
- D_x is the proportion of a cohort assumed to still be alive by age x (assumed to be constant in time);

[6]For example, the classic random walk with drift process has a constant trend, $X_t = 1$, with the "drift", μ, found be regressing $\Delta\boldsymbol{\kappa}_t$ on this trend. Similarly, the random walk with linear drift introduced in Hunt and Blake (2015d) and Hunt and Blake (2016b) has constant and linear trends, $X_t = \begin{pmatrix}1, & t\end{pmatrix}^\top$, with the drifts found by regressing $\Delta\boldsymbol{\kappa}_t$ against X_t in a similar fashion.

- ρ and σ^2 are the autocorrelation and variance of the AR(1) process assumed to be driving the evolution of the cohort parameters;
- $\tilde{X}_y$ are a set of deterministic functions introduced in order to ensure that our projections of the cohort parameters are well-identified;
- β is a set of drifts for the cohort parameters with respect to $\tilde{X}_y$;[7] and
- $\overline{\gamma}_y(\tau)$ are the cohort parameters fitted by the mortality model at time τ.

Together, these allow us to define forward mortality rates consistently across all ages $[1, X]$ and for all future time period, $t > \tau$. We refer to this complete set of consistent forward mortality rates as the "forward mortality surface", since it can be treated as a single object governed by the dynamics of the underlying period and cohort functions.

Finally, in order to price longevity-linked liabilities and securities, we transform the forward mortality rates to a market-consistent measure, $\mathbb{Q}$, from those in the real-world measure, $\mathbb{P}$, given by Equation 4, by using the Esscher transformation

$$\mathbb{E}^{\mathbb{Q}}{}_{\tau} \exp(\eta_{x,t}) = \frac{\mathbb{E}^{\mathbb{P}}{}_{\tau} \exp(Z_{x,t}\eta_{x,t})}{\mathbb{E}^{\mathbb{P}}{}_{\tau} \exp(Z_{x,t})} \tag{11}$$

with $Z_{x,t} = \boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t + \lambda^{\gamma}\gamma_{t-x}$ to obtain

$$\nu^{\mathbb{Q}}_{x,t}(\tau) = \exp\left(-\boldsymbol{\beta}_x^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} - \lambda^{\gamma}\mathbb{V}ar^{\mathbb{P}}_{\tau}(\gamma_{t-x})\right)\nu^{\mathbb{P}}_{x,t}(\tau) \tag{12}$$

where the N free parameters of the vector $\boldsymbol{\lambda}$ and $\lambda^{(\gamma)}$ are market prices of longevity risk,[8] associated with the different terms in the underlying APC model. These are found using whatever market prices are available. In Hunt and Blake (2015b), we calibrated the forward mortality surface in an illustrative market-consistent measure. To do this, we first assumed that we had an "external" market price for an index-based longevity swap, but also needed "internal" market prices for the probabilities of death, derived from the implicit price for longevity risk embedded in the mortality assumptions used by

[7] These depend upon our identifiability constraints. In practice, we impose a set of identifiability constraints such that $\beta = 0$ to simplify matters considerably.

[8] Collectively, we denote these market prices of longevity risk as $\{\lambda^{(j)}\ j = 1, \ldots, N, \gamma\}$.

the insurer for accounting and reserving purposes. In this study, we use the same market-consistent measure to illustrate the risk arising from changes in the market prices of longevity-linked liabilities and securities.

In Hunt and Blake (2015b), we used the above framework with a number of different APC models, including the Lee-Carter model (Lee and Carter (1992)), the classic APC model of Hobcraft et al. (1982) and the model developed in Hunt and Blake (2016a) using the "general procedure" (GP) of Hunt and Blake (2014). In this paper, we only use the GP model as it provides a good fit to the historical data and possesses most of the features of more complicated mortality models such as multiple age/period terms and a cohort term. However, it is important to note that the techniques we propose could be used in combination with any mortality model within the class of APC models discussed in Hunt and Blake (2015e).

3 One-year updates of the forward mortality surface

The mortality forward rate framework discussed in Hunt and Blake (2015b) and Section 2 enables us to value longevity-linked liabilities and securities values in a market-consistent fashion. However, for many risk measurement purposes we are also interested in how these values change with time. There will be three components to such changes:

1. Changes in value due to changing conditions in financial markets not linked to longevity, for instance, due to changes in interest or inflation rate expectations. Changes in these quantities have been widely studied and a range of models have been developed for interest rates and inflation that could be used to deal with the impact of these changes on longevity-linked liabilities and securities values. Accordingly, we do not study the impact of these changes in this paper.[9]

[9]We also implicitly assume that processes governing the evolution of mortality rates are independent of other financial risks. This is in common with the majority of studies, such as Cairns et al. (2006) and Bauer et al. (2008) and with the available evidence to date, as discussed in Loeys et al. (2007). Although there may be some situations where longevity risk is not independent of other financial risks in the real-world measure, as in the examples of Miltersen and Persson (2005), we believe that these situations are relatively extreme

2. Changes due to new mortality data. Mortality data is released relatively infrequently, typically annually, and would be used to refit the underlying APC mortality model. Such changes will be considered further in this paper.

3. Changes due to changing market preferences for longevity risk. These would result in changes in the values of traded securities not explainable in terms of new mortality data or changes in other non-demographic market indicators, and would be incorporated into the forward mortality rate model as time-dependent market prices of longevity risk, $\lambda^{(j)}(\tau)$. With the traded market in longevity-linked securities in the very early stage of development, there is no reliable information available to determine how these changes should be modelled. As Blake et al. (2006) said *"sophisticated assumptions about the dynamics of the market price of longevity risk are pointless"*, given the absence of market data to calibrate them. We therefore assume that the market prices for longevity risk are constant and do not consider them further.

To investigate the second component of these changes, we are, therefore, interested in the random variables

$$\nu^{\mathbb{Q}}_{x,t}(\tau+1)|\mathcal{F}_\tau$$

i.e., the distribution of the forward mortality rates at $\tau+1$ conditional on information at time τ. This is equivalent to studying the "updating factors"

$$\frac{\nu^{\mathbb{Q}}_{x,t}(\tau+1)}{\nu^{\mathbb{Q}}_{x,t}(\tau)}$$

which underpins the models of Cairns (2007) and Zhu and Bauer (2011b).

and are better considered by scenario analysis rather than through a stochastic model. Furthermore, Dhaene et al. (2013) show that independence between longevity risk and financial risks in the real-world measure does not automatically ensure independence in the market-consistent measure. However, more complicated models are required in order to allow for any dependence between longevity and investment risks, which require more market information for calibration. Therefore, we believe that the assumption of independence between longevity risk and other financial risks is both necessary and justifiable at this early stage of development of the longevity risk market.

In reality, the process of determining the forward surface of mortality would involve acquiring death counts and exposures to risk across all ages for year $\tau+1$, re-estimating the chosen mortality model with a revised dataset which included this new information to obtain new estimates of the various age, period and cohort parameters and then using these revised estimates within the framework of Hunt and Blake (2015b). However, this process is not practical for risk management purposes, since the process of generating new death counts and exposures to risk and refitting the model can be sufficiently time consuming that it is not viable to perform it thousands of times. Instead, we note the key new information which the additional data gives us:[10]

1. We can use the new data to estimate for the first time the value of $\boldsymbol{\kappa}_{\tau+1}$.

2. We can use the new data to re-estimate the cohort parameters, and so revise the old fitted cohort parameters, $\overline{\gamma}_y(\tau)$, to a new set of fitted cohort parameters, $\overline{\gamma}_y(\tau+1)$.

Accordingly, to avoid the need to simulate death counts and exposures for $\tau+1$ and refit the model, we instead generate new "observations" of $\boldsymbol{\kappa}_{\tau+1}$ and $\overline{\gamma}_y(\tau+1)$ based on the assumed time series dynamics which underlie the forward mortality framework. The procedures for doing this are discussed in Sections 3.1 and 3.2 for the period and the cohort functions, respectively.

In following this procedure, it is important to ensure that our updated forward mortality surface is "self-consistent", as defined in Zhu and Bauer (2011b), namely that *"that expected values of future forecasts should align with the current forecasts"*. This means that forward mortality rates should be martingales. Such a condition is similar to "no arbitrage" conditions in forward interest rate models. However, because the market for longevity risk is not complete and is likely to cover a more diverse range of securities,[11] we cannot rule out the possibility of arbitrage opportunities even in a self-consistent framework. Given the definition of the forward mortality rates in

[10] A similar line of reasoning can be found in Tan et al. (2014), which used the "time invariant" property of the period functions in some mortality models to investigate the hedging of longevity risk.

[11] Such as longevity zeros (based on survivorship), q-forwards (based on probabilities of death), e-forwards (based on period life expectancy) and other securities based on bespoke indices.

Equation 2, we note that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}{}_{\tau}\ \nu^{\mathbb{P}}_{x,t}(\tau+1) &= \mathbb{E}^{\mathbb{P}}{}_{\tau}\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\ \mu_{x,t} \\ &= \mathbb{E}^{\mathbb{P}}{}_{\tau}\ \mu_{x,t} \\ &= \nu^{\mathbb{P}}_{x,t}(\tau) \end{aligned} \tag{13}$$

by the tower property of conditional expectations. This means that real-world measure forward mortality rates are self-consistent in the real-world measure. We can verify this by considering the period and cohort functions separately, which is done in Section 3.1 for the period parameters and Appendix A.1 for the cohort parameters.

A similar line of reasoning leads to

$$\mathbb{E}^{\mathbb{Q}}{}_{\tau}\ \nu^{\mathbb{Q}}_{x,t}(\tau+1) = \nu^{\mathbb{Q}}_{x,t}(\tau)$$

i.e., market-consistent forward mortality rates are self-consistent in the market-consistent measure. This result is verified algebraically in Appendix A.2 and provides a useful and important check on the validity of the modelling approach and ensures that there are no internal contradictions.

For the hedging purposes discussed in Section 4, what is of interest is how values of liabilities and securities change in the real-world measure (i.e., to investigate whether changes in the value of longevity-linked securities can offset the changes in value of an annuity book). Since these values are calculated using market-consistent forward mortality rates, the value of liabilities and securities are not self-consistent in the real-world measure. However, this is not surprising and is similar to other results in finance.[12] Nevertheless, it will have a number of consequences for the behaviour of longevity-linked liabilities and securities, as discussed in the following sections.

[12]For example, the Black-Scholes stock option price is a martingale in the risk-neutral measure by construction. When performing risk management on stock options in the real-world measure, the options prices will not be martingales (in general, we would expect to see the value of a call option increase with time, since the share price is expected to grow at a faster rate than the risk-free rate).

3.1 Period parameters

Consider first the period functions. From Equation 6 and 7, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\boldsymbol{\kappa}_t &= \boldsymbol{\kappa}_{\tau+1} + \mu \sum_{s=\tau+2}^{t} X_s \\ \mathbb{V}ar^{\mathbb{P}}_{\tau+1}\left(\boldsymbol{\kappa}_t\right) &= (t-\tau-1)\Sigma\end{aligned}$$

Therefore, by generating a value of $\boldsymbol{\kappa}_{\tau+1}$ using the random walk with drift process underlying the projections, we can update the means and variances of the future period functions (and hence the forward surface of mortality) from those found at τ to a (stochastic) update at $\tau+1$:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\boldsymbol{\kappa}_t &= \boldsymbol{\kappa}_{\tau+1} + \mu \sum_{s=\tau+2}^{t} X_s \\ &= \left[\boldsymbol{\kappa}_\tau + \mu X_{\tau+1} + \boldsymbol{\epsilon}_{\tau+1}\right] + \mu \sum_{s=\tau+2}^{t} X_s \\ &= \boldsymbol{\kappa}_\tau + \mu \sum_{s=\tau+1}^{t} X_s + \boldsymbol{\epsilon}_{\tau+1} \\ &= \mathbb{E}^{\mathbb{P}}{}_{\tau}\boldsymbol{\kappa}_t + \boldsymbol{\epsilon}_{\tau+1} \\ \mathbb{V}ar^{\mathbb{P}}_{\tau+1}\left(\boldsymbol{\kappa}_t\right) &= (t-\tau)\Sigma - \Sigma \\ &= \mathbb{V}ar^{\mathbb{P}}_{\tau}\left(\boldsymbol{\kappa}_t\right) - \Sigma\end{aligned}$$

Hence we see that the expectation of future period parameters changes by the innovation $\boldsymbol{\epsilon}_{\tau+1}$ for all future times, whilst the variance of the future period parameters reduces to reflect that, at $\tau+1$, they will be projected for one fewer year than at τ.

Figure 1 shows the 95% prediction interval for $\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\,\kappa_t^{(1)}|\mathcal{F}_\tau$ from the GP model. As can be seen, it is the value of $\kappa_{\tau+1}^{(1)}$ which generates the uncertainty in the later period functions, which shift in parallel as a result of this new information.[13]

[13]Note that, as the drift of the random walk process, μ, is assumed to be known, the

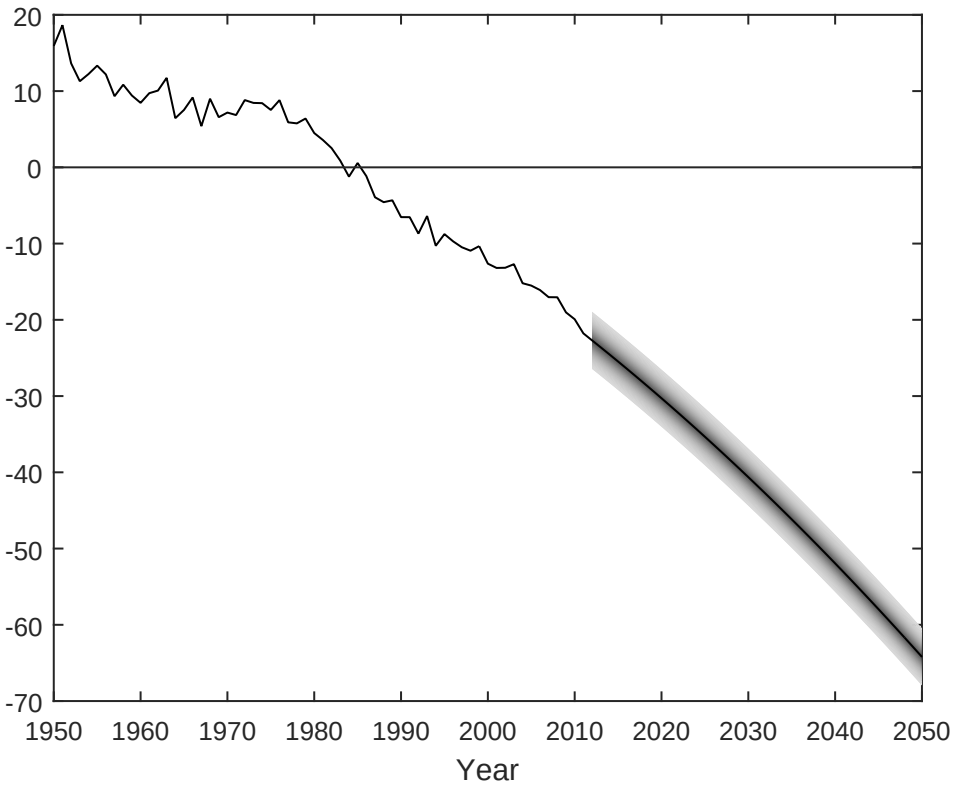


Figure 1: Distribution of $\mathbb{E}_{\tau+1}\kappa_t^{(1)}|\mathcal{F}_\tau$

To demonstrate the impact of this update of the period functions on the

forward mortality framework does not allow for what was termed "recalibration" risk in Cairns et al. (2013), i.e., the risk that one year's new information will cause a reappraisal of the drift term. This may have the effect of understating the risk in long-term projections of mortality rates. We leave the inclusion of recalibration risk in the framework as future work.

forward mortality rates, we see that

$$\begin{aligned}\nu^{\mathbb{P}}_{x,t}(\tau+1)|\mathcal{F}_\tau &= \exp\left(\alpha_x + \boldsymbol{\beta}^\top_x \mathbb{E}_{\tau+1}\boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x \mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\beta}^\top_x\right)|\mathcal{F}_\tau \\ &= \exp\left(\alpha_x + \boldsymbol{\beta}^\top_x\left(\mathbb{E}_\tau\boldsymbol{\kappa}^\top_t + \boldsymbol{\epsilon}_{\tau+1}\right) + \frac{1}{2}\boldsymbol{\beta}^\top_x\left(\mathbb{V}ar_\tau\left(\boldsymbol{\kappa}_t\right) - \Sigma\right)\boldsymbol{\beta}_x\right)|\mathcal{F}_\tau \\ &= \exp\left(\boldsymbol{\beta}^\top_x\boldsymbol{\epsilon}_{\tau+1} - \frac{1}{2}\boldsymbol{\beta}^\top_x\Sigma\boldsymbol{\beta}_x\right)\nu^{\mathbb{P}}_{x,t}(\tau)\end{aligned}$$

if the underlying mortality model of the mortality short rate does not possess a cohort term. Hence, generating random values of $\boldsymbol{\epsilon}_{\tau+1}$ (the time-series innovations for the period parameters) can therefore be used to update stochastically the forward mortality surface at $\tau+1$, conditional on information to time τ in a relatively straightforward fashion. In addition, we see that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}{}_\tau\,\nu^{\mathbb{P}}_{x,t}(\tau+1) &= \exp\left(\boldsymbol{\beta}^\top_x\mathbb{E}^{\mathbb{P}}{}_\tau\boldsymbol{\epsilon}_{\tau+1} + \frac{1}{2}\boldsymbol{\beta}^\top_x\mathbb{V}ar_\tau(\boldsymbol{\epsilon}_{\tau+1})\boldsymbol{\beta}_x - \frac{1}{2}\boldsymbol{\beta}^\top_x\Sigma\boldsymbol{\beta}_x\right)\nu^{\mathbb{P}}_{x,t}(\tau) \\ &= \nu^{\mathbb{P}}_{x,t}(\tau)\end{aligned}$$

and, hence, the real-world forward mortality rates are martingales in the $\mathbb{P}$-measure as expected.

3.2 Cohort parameters

As discussed above, the impact of new data for year $\tau+1$ has a fundamentally different impact on the cohort parameters compared with the period parameters in a mortality model. For the period parameters, new data would allow us to estimate a value for $\boldsymbol{\kappa}_{\tau+1}$. To approximate this, we use the time series dynamics of the period functions to project $\boldsymbol{\kappa}_{\tau+1}$ stochastically, and use this to update the forward surface of mortality.

In contrast, new death count and exposure to risk data allows us to:

1. update the cohort parameters estimated by the model to allow for one additional observation on each cohort which is alive at $\tau+1$;

$$\overline{\gamma}_y(\tau) \to \overline{\gamma}_y(\tau+1) \quad \text{for } \tau+1-X \leq y \leq Y$$

2. estimate for the first time the cohort parameter for year of birth $Y+1$, i.e., $\overline{\gamma}_{Y+1}(\tau+1)$, which we did not have sufficient information to do the year before; and

3. revise our forecasts of cohort parameters for future years of birth due to an adjusted boundary condition for the cohort parameter at $Y+1$.

Unlike with the period functions, the new data does not give us a complete observation of any new, single year of birth. It is this fundamental difference in the information that new data provides that means that we need to adopt a fundamentally different approach when updating the cohort parameters in the forward mortality framework.

To explain why this is important, we need to first consider the problems with using more classical approaches to projecting the cohort parameters. In Hunt and Blake (2015b), we found that classical approaches, such as those using ARIMA models, are not suitable in a forward mortality framework. This was because there is a discontinuity in the variance of the parameters when we move from the estimated parameters based on historical data to the projected parameters. This discontinuity would give rise to pricing anomalies. In the context of updating the forward mortality surface, we also find that using these classical approaches will lead to certain irregularities, as we now show.

Classical time series processes assume that the cohort parameters for which we have observations at time τ (up to and including γ_Y, say) are known with certainty and will not be revised and updated to reflect the new information received at $\tau+1$. Therefore, they do not allow for the first impact of new data described above. Instead, new information at $\tau+1$ is assumed to be sufficient to estimate γ_{Y+1}, i.e., the second impact above. This value of γ_{Y+1} is then used as the last observation (i.e., the boundary condition between observations and projections) and so adjusts the forecasts of the future cohort parameters, i.e., the third impact described above. The pattern of updated cohort parameters which would be observed using such as model is shown in Figure 2.

However, this is inconsistent with the impact new data would be expected to have on the previously estimated cohort parameters. In addition, using these classical approaches generates unfeasible patterns of uncertainty in the

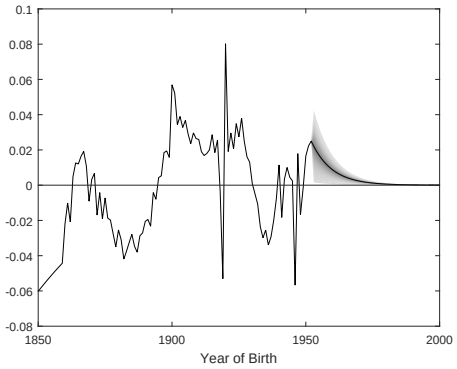


Figure 2: One-year update of projected γ_y using AR(1) process

forward mortality surface, with a sharp discontinuity between cohort parameters which are estimated from historical data and those which are projected, as discussed previously in Hunt and Blake (2015b).

In order to update the cohort parameters in a manner which is consistent with how they would actually update in response to new data, we instead need to use an approach which combines the time series dynamics of the cohort parameters with the partial observations we have of them to date. With such an approach, we can model the updating of this partial information to reflect the impact of new data, and then combine this updated set of observations with the time series dynamics to revise our forecast cohort parameters. In Hunt and Blake (2015a), we developed a Bayesian modelling approach which can be used for this purpose. In particular, we assumed that we had two sources of information for estimating the "ultimate" cohort parameter, γ_y, which would only be known fully once all members of the cohort had died. These were, first, the underlying time series dynamics for the cohort parameters, which acted as a prior assumption for their distribution, and, second, the "interim" cohort parameters estimated by the mortality model, $\overline{\gamma}_y(\tau)$, which were based on partial information to time τ. Hence, the impact of new data on the cohort parameters can be modelled by generating updates of the estimated cohort parameters, $\overline{\gamma}_y(\tau+1)$, which reflect new observations of the relevant cohorts.

In Hunt and Blake (2015a), we assumed that the ultimate cohort param-

eters were generated by independent discrete packets, γ_y^x, for each age of observation for the cohort, i.e.,

$$\gamma_y = \sum_{x=1}^{X} d_x \gamma_y^x \tag{14}$$

where d_x is the proportion of the total cohort which dies at age x (assumed to be the same for all cohorts). However, at any specific time, we would only have received an incomplete set of observations of any cohort where members of that cohort were still alive, i.e., we would have received packets of information γ_y^x for $x \in [1, \tau - y]$ by time τ. These partial observations are combined to give us the estimated cohort parameters fitted by a mortality model based on data to time τ:

$$\underline{\gamma}_y(\tau) = \sum_{x=1}^{\tau-y} d_x \gamma_y^x \tag{15}$$

$$\overline{\gamma}_y(\tau) = \frac{1}{D_{\tau-y}} \underline{\gamma}_y(\tau) \tag{16}$$

where $D_x = \sum_{\xi=1}^{x} d_\xi$, i.e., the proportion of a cohort expected to die before age, x, as defined in Section 2.

Hence, we can replicate the process of updating the fitted cohort parameters to reflect new information for year $\tau + 1$ to allow for the first impact of new data discussed at the beginning of this section. This process is equivalent to generating new packets of information to represent the new observations of each of the still living cohorts at time $\tau + 1$, and incorporating these into the existing estimates of the cohort parameters at time τ

$$\underline{\gamma}_y(\tau + 1) = \underline{\gamma}_y(\tau) + d_{\tau+1-y} \gamma_y^{\tau+1-y} \tag{17}$$

$$\begin{aligned} \overline{\gamma}_y(\tau + 1) &= \frac{1}{D_{\tau+1-y}} \underline{\gamma}_y(\tau + 1) \\ &= \frac{1}{D_{\tau+1-y}} \left[\underline{\gamma}_y(\tau) + d_{\tau+1-y} \gamma_y^{\tau+1-y} \right] \\ &= \frac{1}{D_{\tau+1-y}} \left[D_{\tau-y} \overline{\gamma}_y(\tau) + d_{\tau+1-y} \gamma_y^{\tau+1-y} \right] \end{aligned} \tag{18}$$

This can be compared to the results of a credibility analysis, as described in in Chapter 7 of Kaas et al. (2001), since the updated estimate of the

cohort parameter is a weighted average of the previous estimate and the new observation of the cohort. Because of this, our ability to update the forward mortality surface for new cohort information rests on our ability to simulate new packets of information, $\gamma_y^{\tau+1-y}$. To do this, we know from Hunt and Blake (2015a) and the well-identified AR(1) process underlying the cohort parameters that

$$\gamma_y^x|\gamma_{y-1},\beta,\rho,\sigma^2 \sim N\left(\beta\tilde{X}_y + \rho(\gamma_{y-1} - \beta\tilde{X}_{y-1}), \frac{\sigma^2}{d_x}\right)$$

where β, $\tilde{X}_y$, ρ and σ^2 are defined in Section 2. However, the ultimate cohort parameter for year of birth $y-1$, γ_{y-1}, will not, in general, be known at time τ (as individuals born in year $y-1$ will still be alive), but we do know the distribution of γ_{y-1} at τ from Equations 9 and 10. Therefore, in order to find the distribution of $\gamma_y^{\tau+1-y}|\mathcal{F}_\tau$, we use Bayes Theorem and the distribution of γ_{y-1} to give

$$\gamma_y^{\tau+1-y}|\mathcal{F}_\tau,\beta,\rho,\sigma^2 \sim N\left(\beta X_y + \rho(M(y-1,\tau) - \beta X_{y-1}), \rho^2 V(y-1,\tau) + \frac{\sigma^2}{d_{\tau+1-y}}\right) \tag{19}$$

In addition, we assume

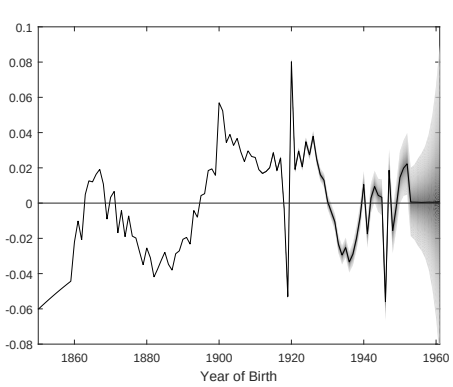
$$\mathbb{C}ov_\tau(\gamma_y^{\tau+1-y}, \gamma_{y-s}^{\tau+1-y+s}) = \rho^s \text{Þ}_{\tau+1-y,s}\frac{\sigma^2}{d_{\tau+1-y+s}} \tag{20}$$

in order for the forward mortality rates to be self-consistent in the $\mathbb{P}$-measure, which is demonstrated in Appendix A.1.

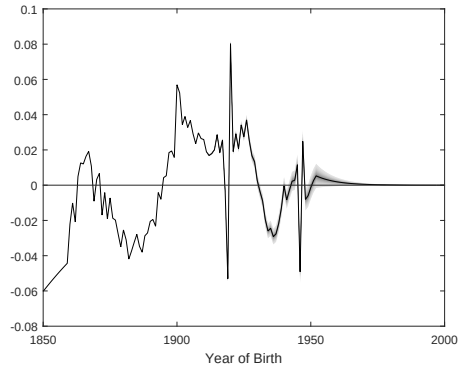
Hence, by generating new packets of information, γ_y^x, in respect of the cohorts that we would have observed in the new data for year $\tau+1$, we can update the values of $\overline{\gamma}_y(\tau)$ consistent with how they would update in response to actual new data.

To summarise, the process for updating the cohort parameters is:

1. generate new cohort information packets, $\gamma_y^{\tau+1-y}$ for $y \in [\tau+1-X, Y+1]$, randomly using the distribution in Equations 19 and 20;
2. update partial sums using Equation 18 without refitting the APC mortality model, to give $\overline{\gamma}_y(\tau) \to \overline{\gamma}_y(\tau+1)$;



(a) One-year update of $\overline{\gamma}_y(\tau+1)$



(b) One-year update of $\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\gamma_y = M(y,\tau+1)$

Figure 3: Updating the cohort parameters

3. use Equation 9 to find $M(y,\tau+1)$ (the updated estimate of the mean of the ultimate cohort parameters);

4. use Equation 10 to find $V(y,\tau+1)$ (the updated estimate of the variance of the ultimate cohort parameters);

5. use these to calculate $\nu^{\mathbb{P}}_{x,t}(\tau+1)$ in conjunction with the updated period parameters;

6. use Equation 12 to transform the real-world-measure forward mortality rates to the market-consistent measure, for use in valuing liabilities and securities.

The 95% prediction interval of the "interim" cohort parameters, $\overline{\gamma}_y(\tau+1)|\mathcal{F}_\tau$ is shown in Figure 3a, and the 95% prediction interval of the updated expectation of the ultimate cohort parameters, $M(y,\tau+1)|\mathcal{F}_\tau$ is shown in Figure 3b.[14] We observe the following:

[14]Note that we use indicator variables to remove the large outliers due to the cohort anomalies in 1919/20 and 1946/47 when estimating the time series dynamics for the cohort process. This is because we believe them to be artefacts of the data collection process (see Richards (2008) and Cairns et al. (2015)), rather than genuine features of mortality for these cohorts.

- New data for $\tau + 1$ does not update the interim or ultimate cohort parameters for cohorts where we have assumed all members have died by time $\tau + 1$, i.e., for $y \leq \tau - X$.

- For years of birth $\tau + 1 - X \leq y \leq Y$, the new information allows us to update the interim cohort parameter, $\overline{\gamma}_y(\tau)$, and hence the expectation of the ultimate cohort parameter, $M(y, \tau)$. The importance of this new information for the interim cohort parameters is relatively greater for more recent years of birth than more distant years of birth. However, the Bayesian approach implies that the ultimate cohort parameters can be thought of as weighted averages of the prior distribution (given by the time series dynamics) and the partial information received by observing the cohorts to date, which is represented by the interim parameters, $\overline{\gamma}_y$. For more recent years of birth, this approach gives greater weight to the prior distribution and less to the observations already collected. Accordingly, for recent years of birth, the impact of the new data updating the partial observations of the cohort (i.e., updating $\overline{\gamma}_y(\tau)$ to $\overline{\gamma}_y(\tau + 1)$) has only a limited impact on the distribution of the ultimate cohort parameters. This satisfies the first impact of new data discussed above.

- We make our first estimate of the cohort parameters for year of birth $Y + 1$. This gives a very high variability for the estimated cohort parameter, $\overline{\gamma}_{Y+1}(\tau + 1)$, as this is based on very little information. However, since the Bayesian approach gives most weight to the time series dynamics for this cohort, this variability does not result in large changes in the expectation of the ultimate cohort parameter. This satisfies the second impact of new data discussed above.

- For $y \geq Y + 2$, we are still making projections of cohort parameters for which we have no observations at $\tau + 1$. However, these projections will have changed slightly because of the updated boundary condition at $y = Y + 1$, i.e., our revised estimate for $\gamma_{Y+1}(\tau + 1)$. This satisfies the third impact of new data discussed above. Since we have assumed that the cohort parameters follow a well-identified AR(1) process, updating the distribution of these parameters also updates the prior distribution for the ultimate cohort parameters for $y \geq Y + 2$. However, these changes do not persist indefinitely and, over time, the impact of the new information decreases exponentially. This is understandable, since

> we would not expect to update our estimates of the lifelong mortality characteristics of the cohort born in 2050 (say), based on observations of their parents and grandparents.

In these respects, the Bayesian framework has replicated what we would expect to see if we actually had new death counts and exposures for $\tau + 1$ and used them to refit the model as summarised at the beginning of this section. In addition, in Appendix A.1, we check to ensure that the Bayesian framework for the cohort parameters gives self-consistent forward mortality rates in the real-world measure.

Cohort effects are a feature of many of the more recent mortality models in use, and their robust estimation is of vital importance in the calculation of liabilities, such as annuities, and many of the longevity-linked securities which have been proposed. However, as discussed in Hunt and Blake (2015a), the projection of cohort parameters is difficult, and made more complicated by the nature of the partial information we have regarding them at any specific date. In part because of this, the forward mortality models proposed to date, such as those in the Heath-Jarrow-Morton framework in Barbarin (2008), Bauer et al. (2008) and Tappe and Weber (2013), the semi-parametric factor model of Zhu and Bauer (2011a,b, 2014), or the Olivier-Smith model developed in Olivier and Jeffrey (2004), Smith (2005), Cairns (2007) and Alai et al. (2013), have not been able to incorporate cohort effects.

We believe that a key advantage of the forward mortality framework developed in Hunt and Blake (2015b) and in this paper is that it can give biologically reasonable[15] dynamics for the forward surface of mortality, as it is based on the dynamics of APC models of the mortality hazard rate, which are well understood and easy to estimate from historical data. Since cohort parameters are an important feature of such models, we believe that the successful application of the forward mortality framework proposed in Hunt and Blake (2015b) and which will be used in the present study for risk management purposes is, ultimately, dependent upon using the Bayesian approach of Hunt and Blake (2015a).

[15]Introduced in Cairns et al. (2006) and defined as "*a method of reasoning used to establish a causal association (or relationship) between two factors that is consistent with existing medical knowledge*".

4 Assessing and hedging longevity risk

Based on the results of Section 3, we are able to generate random realisations of the forward mortality surface, which can then be used to value longevity-linked liabilities and securities in a consistent fashion. We first investigate the impact of longevity risk on the value of an illustrative annuity book in Section 4.1, before considering the longevity risk in some of the longevity-linked securities that have been proposed in Section 4.2. Finally, we bring the two together by considering the effectiveness of using these securities to hedge longevity risk over a one-year period in Section 4.3.

4.1 Liability values

We begin by considering an illustrative annuity book. At initial time, τ, the value of an annuity at age x is calculated as

$$a_x(\tau) = \sum_{t=0}^{\infty} {}_tP^{\mathbb{Q}}_{x,\tau}(\tau) B(\tau, \tau + t) \tag{21}$$

where ${}_tP^{\mathbb{Q}}_{x,\tau}(\tau)$ is the market-consistent forward survival probability from time τ to time $\tau + t$ (as evaluated at time τ), as defined in Hunt and Blake (2015b) and used in Equation 1, and $B(\tau, \tau + t)$ is the price at time τ of a risk-free zero coupon bond maturing at time $\tau + t$.[16] For these and all future calculations, we assume a constant risk free real rate of interest of 1% p.a. and extrapolate forward mortality rates beyond the maximum age in the data, $X = 100$, using the topping out procedure of Denuit and Goderniaux (2005).

This assumes that the lives on which the annuities are written are not systematically different from the national population, data for which was used to calibrate the forward mortality surface. Accordingly, we do not allow for potential basis risk in our annuity portfolio. We leave to future work the extension of the forward mortality framework to include basis risk, for example, using the relative modelling approaches of Villegas and Haberman (2014) or Hunt and Blake (2016a). However, the results of Hunt and Blake (2016a) indicate that the impact of basis risk on systematic longevity risk

[16] We therefore see that an annuity is equal to a portfolio of longevity zeros, as defined in Blake et al. (2006) and used in Hunt and Blake (2015b).

may be limited in many situations.

The individual annuities at different ages are then combined into our illustrative annuity book, consisting of annuities written on male lives equally distributed across ages 60 to 80. At time τ, this has present value $\mathcal{L}(\tau)$, given by

$$\mathcal{L}(\tau) = \sum_{x=60}^{80} a_x(\tau) \tag{22}$$

In order to assess the longevity risk in the annuity book over a one-year period, we first need to update the forward surface of mortality to time $\tau + 1$ using the techniques of Section 3 and then use this updated surface to calculate updated annuity values and, hence, and updated value for the annuity book. The updated annuity values are given by

$$a_x(\tau + 1) = \sum_{t=0}^{\infty} {}_tP^{\mathbb{Q}}_{x,\tau+1}(\tau + 1)B(\tau + 1, \tau + 1 + t) \tag{23}$$

However, a direct comparison between these updated annuity values and those in Equation 21 is not valid. This is because $a_x(\tau + 1)$ is not directly comparable to $a_x(\tau)$, since it relates to the cohort born in $\tau+1-x$ as opposed to the cohort born in $\tau - x$. Alternatively, if one tries to compare $a_{x+1}(\tau+1)$ with $a_x(\tau)$ (which do refer to the same cohort), we note that this comparison is also not valid, since the former includes one fewer year of benefits and is discounted to a different point in time compared with the latter. The only valid comparison is between $a_x(\tau)$ and[17]

$$B(\tau, \tau + 1)_1p_{x,\tau} \left(1 + a_{x+1}(\tau + 1)\right) \tag{24}$$

Doing so values the same set of cashflows for the same cohort, discounted to the same point in time and therefore ensures that the two quantities are comparable. The difference between them arises from:

1. replacing the time τ market-consistent forward mortality rates in year $\tau + 1$ with simulated "observed" rates for that year; and

[17] In Equation 24 and subsequently, ${}_tp_{x,\tau}$ is the probability that an individual aged x at τ has survived to age $x+t$ at $\tau+t$, which is realised at time $\tau+t$ but is a random variable before then.

2. replacing the time τ market-consistent forward mortality rates in years $t \geq \tau + 2$ with the time $\tau + 1$ market-consistent forward mortality rates for the same years.

Hence the only differences arise from changes in the evolution of mortality over the year and the consequent updating of the forward surface of mortality and, therefore, solely reflect longevity risk.

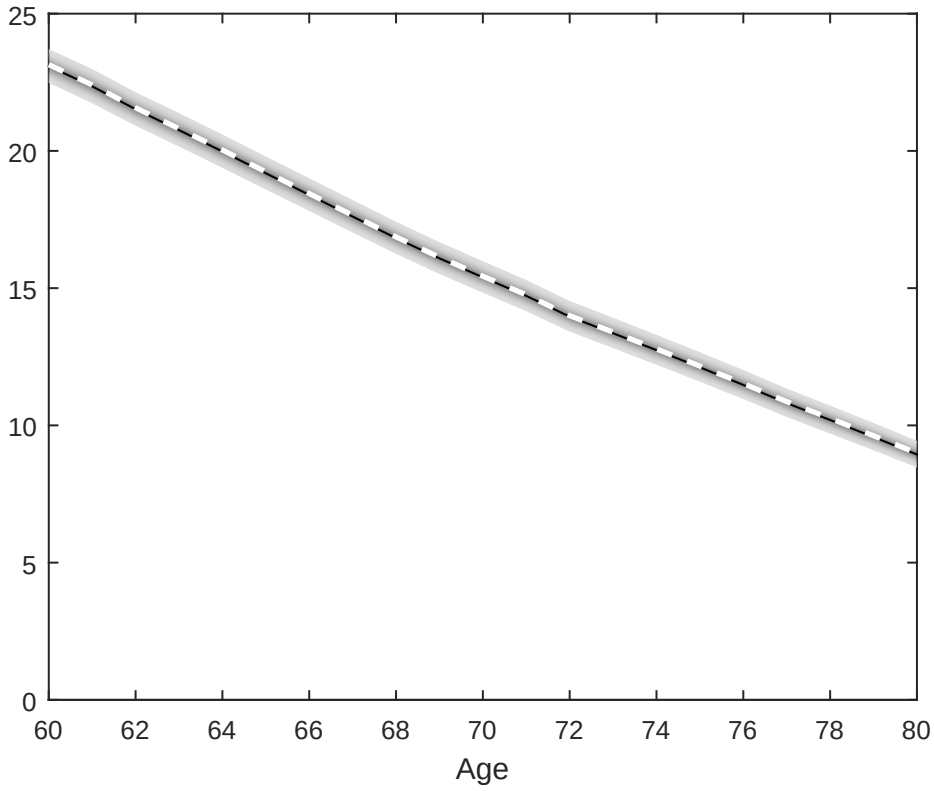


Figure 4: Projected annuity values at $\tau + 1$

Figure 4 shows the 95% fan chart of simulated annuity values at different ages in one year's time. The coefficients of variation[18] of the projected annuity values increase with age, from around 1.4% of the current annuity

[18]The standard deviation of the annuity value divided by its expectation.

value at age 60 to approximately 2.6% at age 80.

Figure 4 also shows the time τ annuity values, $a_x(\tau)$, as a dashed white line. It, therefore, illustrates that $\mathbb{E}^{\mathbb{P}}{}_{\tau}\ a_x(\tau+1) \approx a_x(\tau)$. However, it is important to note, however, that $\mathbb{E}^{\mathbb{P}}{}_{\tau}\ a_x(\tau+1) \neq\ a_x(\tau)$, i.e., the annuity values are not martingales in the real-world measure. The reason for this is that $a_x(\tau+1)$ is calculated using market-consistent forward mortality rates at time $\tau+1$, which are themselves not martingales in the real-world measure, as discussed in Section 3.

In Hunt and Blake (2015b), we said that the marginal participant in the market for longevity-linked securities would probably be a life insurer seeking to hedge longevity risk. Such a life insurer would be averse to longevity risk, and so, we expected that the market-consistent forward mortality rates would be lower than those in the real-world measure

$$\nu^{\mathbb{Q}}_{x,t}(\tau) \leq \nu^{\mathbb{P}}_{x,t}(\tau)$$

Thus, we expect to replace the expected survival probabilities for the period $[\tau, \tau+1)$ under the market-consistent measure with their projected values in the real-world measure, which are lower on average, i.e.,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}{}_{\tau}\ {}_1p_{x,\tau} &= \mathbb{E}^{\mathbb{P}}{}_{\tau} \exp\left(-\mu_{x,\tau+1}\right) \\
&= \exp\left(-\nu^{\mathbb{P}}_{x,\tau+1}(\tau)\right) \\
&< \exp\left(-\nu^{\mathbb{Q}}_{x,\tau+1}(\tau)\right) = {}_1P^{\mathbb{Q}}_{x,\tau}(\tau)
\end{aligned}$$

Therefore, we find $\mathbb{E}^{\mathbb{P}}{}_{\tau}\ a_x(\tau+1) < a_x(\tau)$ across ages, indicating that annuity values would be expected to fall. In simulations, we find this has an impact of around 1% of the value of an annuity. In an insurance context, this would give an "expected return" due to the "release of reserves" in respect of the annuity, caused by holding reserves for the policy higher than the expected value of the benefits in the real-world measure. This expected return on longevity-linked liabilities and securities has important consequences, which will impact the measurement of risk in liabilities and longevity-linked securities, as discussed in the following sections.

In addition to looking at the annuity values at different ages in isolation, we also need to assess their dependence upon each other in order to achieve

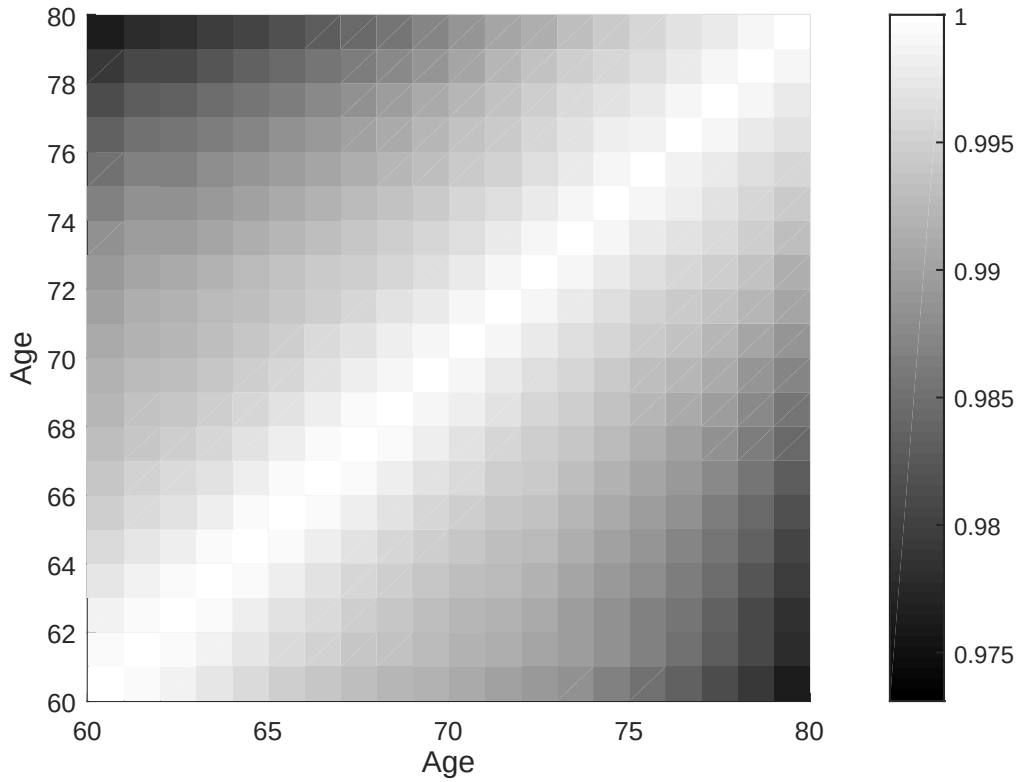


Figure 5: Correlations between annuity values at $\tau + 1$

a full assessment of the longevity risk in our illustrative annuity book. To do this, Figure 5 shows the correlations between annuity values at different ages. From this, we see that there is substantial correlation between annuity values at different ages, typically between 95% and 100%. This is due to the structure of the underlying APC mortality model, since the evolution of the forward surface of mortality over the year is driven by the same few factors, namely the four age/period terms with a limited contribution from the cohort term. This leads to relatively low diversification of longevity risk across different ages. In contrast, there could be apparently large benefits in risk reduction due to "natural hedging", i.e., writing life assurance policies as the value of these would be expected to be negatively correlated with annuity values under longevity risk, as discussed in Cox and Lin (2007). However, as argued in Zhu and Bauer (2014), these benefits are largely model dependent,

although these criticisms can be partly assuaged by using APC mortality models with a sufficient number of terms (in this case, four) to fully capture the dynamics of mortality.

We then combine the time $\tau+1$ annuity values into our illustrative annuity book to calculate the liability value at time $\tau + 1$

$$\mathcal{L}(\tau + 1) = \sum_{x=60}^{80} B(\tau, \tau + 1)_1 p_{x,\tau} \left(1 + a_{x+1}(\tau + 1)\right) \tag{25}$$

Doing so, we find that:

- $\mathbb{E}_\tau \mathcal{L}(\tau + 1) = 99.7\% \times \mathcal{L}(\tau)$ and
- $\mathbb{S}t\mathbb{D}ev_\tau(\mathcal{L}(\tau + 1)) = 1.4\% \times \mathcal{L}(\tau)$.

The slight decrease in the expectation of the liabilities is due to the "release of reserves" effect discussed above. In practice, this effect would be dwarfed by the impact of the benefits being paid, which we have not included for the reasons also discussed above. We also note that the longevity risk in the portfolio, as measured by the standard deviation over a one-year period, is approximately 1.4% of the nominal value, which means that longevity risk is likely to be less important than interest rate, investment and inflation risks over the same period. However, longevity risk is a long-term risk, which will compound annually and, hence, is significant for annuity policies with terms of several decades. In addition, unlike other financial risks, the instruments for managing systematic longevity risk are far less developed, due to the absence of liquid markets for longevity-linked securities.

4.2 Longevity-linked securities

In Hunt and Blake (2015b) the forward mortality framework was used to value a number of potential longevity-linked securities. For capital efficiency, most of these have taken the form of forward contracts, written on various indices of mortality. A number of different mortality indices for use in forward contracts have been proposed to date:

- q-forwards: as discussed in Coughlan et al. (2007), these are forward contracts on future probabilities of death, $q_{x,t}$ (see also Li and Luo (2012)).

- s-forwards: as proposed in Dowd (2003), Blake et al. (2006) and by the Life and Longevity Markets Association,[19] these are forward contracts on the probability of survival of a cohort from inception at time t_0 to maturity.

- e-forwards: as discussed in Denuit (2009), period life expectancy is a natural index to use for summarising the evolution of mortality rates in a population, and therefore we consider the potential of a forward market in period life expectancy (which we refer to as "e-forwards" from the demographic symbol for period life expectancy) at age x in future year t for hedging purposes.

In each of these cases, we assume that the reference population for the index is the national population used to estimate the APC model underpinning the forward mortality model. Hence, the value of the mortality index at time τ is calculated as:[20]

$$\text{q-forward:} \quad \mathcal{Q}_{x,t}(\tau) = 1 - \exp\left(-\nu^{\mathbb{Q}}_{x,t}(\tau)\right) \tag{26}$$

$$\text{s-forward:} \quad \mathcal{S}_{x,t_0,t}(\tau) = {}_{\tau-t_0}p_{x,t_0} \times {}_{t-\tau}P^{\mathbb{Q}}_{x+\tau-t_0,\tau} \tag{27}$$

$$\text{e-forward:} \quad \mathcal{E}_{x,t}(\tau) = 0.5 + \sum_{u=0}^{\infty} \exp\left(-\sum_{v=0}^{u} \nu^{\mathbb{Q}}_{x+v,t}(\tau)\right) \tag{28}$$

Thus, we can see that these mortality measures are qualitatively different from each other, and range from q-forwards which are very simple securities based on only one forward mortality rate, to more complex securities which look at forward mortality rates across a number of different ages and years.

For a general forward contract, linked to mortality index $\mathcal{I}_{x,t}$, the forward price specified by the contract must be equal to the time τ value of the mortality measure, i.e., $\mathcal{I}_{x,t}(\tau)$, in order for the contract to have zero value at inception. We assume that the buyer of the contract will receive a floating payment and pay a fixed amount at time t. Hence, the value of the forward

[19] `http://www.llma.org/`

[20] Note that the s-forward is defined on a reference cohort aged x at the inception data, $t_0 \leq \tau$, and therefore the survivorship of this cohort is a product of the observed survivorship from t_0 to τ, given by ${}_{\tau-t_0}p_{x,t_0}$, and the anticipated survivorship from τ to maturity, t, given by ${}_{t-\tau}P^{\mathbb{Q}}_{x+\tau-t_0,\tau}$. For the purposes of this paper, we assume $t_0 = \tau$.

contract at time $\tau + 1$, will be

$$B(\tau + 1, t) \left[\mathcal{I}_{x,t}(\tau + 1) - \mathcal{I}_{x,t}(\tau)\right]$$

and, therefore, we are interested in the distribution of the change in the index of mortality over time

$$[\mathcal{I}_{x,t}(\tau + 1)|\mathcal{F}_\tau] - \mathcal{I}_{x,t}(\tau)$$

Although longevity risk is a long-term risk which will materialise over a number of decades, it is likely that longevity-linked securities will need to be considerably shorter-term contracts in order to appeal to speculators. Hence, we only consider forward contracts with maturities of 5, 10 and 15 years, i.e. $t = 5, 10, 15$. Specifically, we investigate the time $\tau+1$ values of the following forward contracts entered into at time τ:

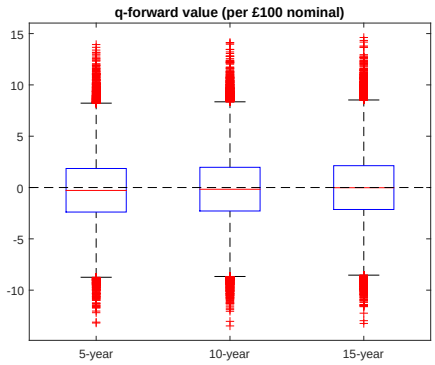
- a q-forward at age 65 and maturity $\tau + t$, i.e., $\mathcal{Q}_{65,\tau+t}$;
- an s-forward with maturity date $\tau + t$, specified on a reference cohort aged 65 at time τ i.e., $\mathcal{S}_{65,\tau,\tau+t}$; and
- an e-forward at age 65 with maturity $\tau + t$, i.e., $\mathcal{E}_{65,\tau+t}$.

Boxplots showing the time $\tau + 1$ distribution of these forward contracts per £100 of nominal value are shown in Figure 6.

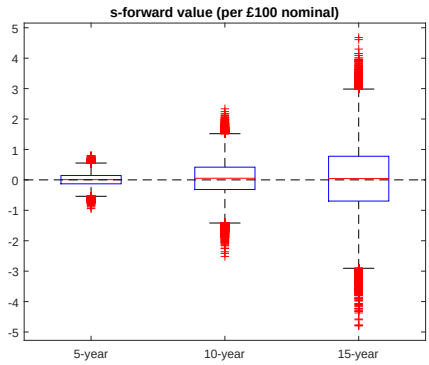
As discussed earlier in the context of annuity values, we note from Figure 6 that

$$\mathbb{E}^{\mathbb{P}}{}_\tau \left[\mathcal{I}(\tau + 1)|\mathcal{F}_\tau\right] - \mathcal{I}(\tau) \neq 0$$

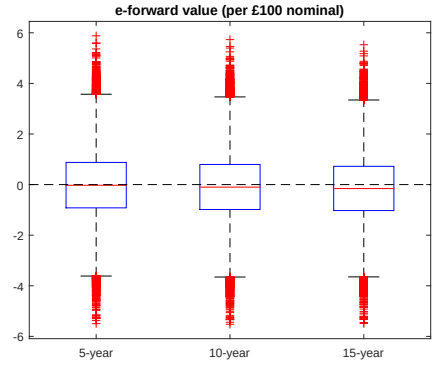
i.e., the expected value of the forward contract at time $\tau + 1$ is not equal to zero, the value at inception. This is, again, due to the prices of securities in the market-consistent measure not being martingales under one-year updates of the forward mortality surface in the real-world measure. Hence, there will be an expected return from trading in longevity-linked forwards, i.e., the premium that the current holders of longevity risk are willing to pay to transfer the risk to the capital markets. This is equivalent to the expected release of reserves for annuities, as discussed in Section 4.1.



(a) q-Forward



(b) s-Forward



(c) e-Forward

Figure 6: Distribution of $\tau + 1$ values of different longevity-linked securities

We also see that for q-forwards and e-forwards, the one-year riskiness of the contract does not change significantly with its term. In contrast, the riskiness of an s-forward increases rapidly with the term of the contract. The reason for this is that the nominal value of the mortality index for q-forwards and e-forwards (probability of dying and period life expectancy, respectively) does not change much with term, whilst that of the s-forward (survivorship of a cohort) decreases rapidly. This means that long-term q-forward and e-forward contracts could, potentially, be written, with the risk in them managed by annually rebalancing the portfolio. However, this may be more difficult for long-term s-forward contracts which might reduce the willingness of speculators to trade (and hence create liquidity) in long-term versions of these particular contracts.

Figure 6 also shows that the q-forward contracts are, however, significantly riskier per £100 nominal than the alternatives. This is because the nominal value of the mortality measure is relatively small,[21] and hence the value of the contract is proportionally more affected by new information. In addition, the q-forward is specified on mortality rates at one specific age and time – rather than across a range of ages and years, as in the case of the s-forward and e-forward – which is likely to be more volatile than s-forwards or e-forwards.

4.3 Hedging

Having measured the longevity risk in an annuity book in Section 4.1, it is natural to consider how this risk could be managed and reduced. In practice, this can be achieved through reinsurance, securitisation (e.g., Cowley and Cummins (2005)) or natural hedging (e.g., Cox and Lin (2007)). Another method which has been proposed (but not yet widely implemented) is to hedge the longevity risk in a liability portfolio using standardised, tradable longevity-linked securities.[22]

[21] Typically, $q_{x,t}$ will be in the range $[0.005, 0.05]$ for most ages of interest, whilst ${}_{t-t_0}p_{x,t_0}$ will be in the range $[0.1, 0.9]$ and $e_{x,t}$ will be in the range $[10, 30]$.

[22] We draw a slight distinction between such a strategy and purchasing a single, customised asset without the intention of rebalancing the hedge in future. Examples of these customised assets include bespoke longevity swaps, as considered in Hunt and Blake (2016e), and highly customised bespoke options on mortality, such as those discussed in

To illustrate the potential effectiveness of hedging these illustrative liabilities, we consider using each of the different securities discussed in Section 4.2 in turn. We adopt a simple mean-variance hedging strategy and select the portfolio whose value at time $\tau + 1$ has smallest variance, i.e., we find the hedged portfolio

$$\mathcal{L}^* = \mathcal{L} - \tilde{\theta}\mathcal{I}_{x,t}$$

where $\tilde{\theta}$ is chosen by minimising the variance

$$\begin{aligned}
\tilde{\theta} &= argmin_\theta \mathbb{V}ar_\tau^{\mathbb{P}} \left(\mathcal{L}(\tau+1) - \theta\mathcal{I}_{x,t}(\tau+1)\right) \\
\Rightarrow \tilde{\theta} &= \frac{\mathbb{C}ov_\tau^{\mathbb{P}} \left(\mathcal{L}(\tau+1), \mathcal{I}_{x,t}(\tau+1)\right)}{\mathbb{V}ar_\tau^{\mathbb{P}} \left(\mathcal{I}_{x,t}(\tau+1)\right)} \\
\mathbb{V}ar_\tau \left(\mathcal{L}^*(\tau+1)\right) &= \left(1 - \rho_{\mathcal{L},\mathcal{I}}^2\right) \mathbb{V}ar_\tau \left(\mathcal{L}(\tau+1)\right)
\end{aligned}$$

Hence we see that such a strategy depends critically upon the correlation between the liabilities and the hedging instrument, $\rho_{\mathcal{L},\mathcal{I}}$, at time $\tau + 1$, with correlations closer to ± 1 giving more effective hedges. The measured correlations for the four securities considered are shown in Table 1. Because we wish to minimise the variability of the value of the portfolio at time $\tau + 1$, this approach investigates "value" hedging strategies as opposed to "cashflow" hedging strategies, which seek to minimise the uncertainty in the realised cashflows.

Security		q-forward	s-forward	e-forward
Term	5	-94.1%	83.9%	99.2%
	10	-94.1%	88.9%	99.4%
	15	-94.0%	93.3%	99.6%

Table 1: Correlation between $\mathcal{L}(\tau + 1)$ and security values with different terms

As can be seen from Table 1, most of the securities being considered give very high correlations with the liabilities. In the case of q-forwards, this

Michaelson and Mulholland (2014). However, we feel that this alternative strategy has more in common with a reinsurance policy than truly hedging risk using capital market securities.

correlation is negative, since higher than anticipated reductions in mortality rates have the effect of increasing liability values, but triggering net payments from the buyer to the seller of the q-forward, giving a negative value under the convention adopted in Section 4.2. This means that a holder of longevity risk will want to receive the floating leg of a q-forward, as opposed to wanting to receive the fixed legs of the other forward contracts.

The high correlations shown in Table 1 arise from the same reasons that we observed high correlations between annuity values at different ages in Section 4.1. This was because relatively few factors (i.e., the age/period terms in the model, and mainly $\kappa_t^{(1)}$) drive the changes in mortality rates.

We also note that, for q-forwards and e-forwards, the correlation between the forward contract and the liabilities is roughly independent of the term of the contract. In contrast, the s-forward value becomes more highly correlated with the liability value as the term of the contract increases. This in unsurprising, since longer term s-forward contracts are more exposed to the cumulative effects of longevity risk and will behave more like annuity contracts by their nature. However, as discussed in Section 4.2 and shown in Figure 6, longer term s-forwards are also more risky. This might unfortunately limit the development of the market in long-term s-forwards which are more useful for hedging longevity risk than short-term contracts.

Figure 7 shows the empirical distribution of the value of the unhedged and hedged liabilities (using the three different hedging securities with maturities of ten years) based on 50,000 Monte Carlo simulations. As expected, all the hedging strategies considered appear to substantially reduce the variability of the portfolio value at time $\tau + 1$. This is shown by considering the standard deviations (as percentages of $\mathcal{L}(\tau)$) of the hedged and unhedged portfolios in Table 2.

It is noticeable from Figure 7 and Table 2 that the strategy based on an e-forward is significantly more effective at reducing risk than the other two. This is because the values of the period life expectancy at the maturity date is calculated in a similar manner to the calculation of an annuity but over a range of different cohorts, and therefore this security is sensitive to the same risk factors as the annuities we are trying to hedge. In contrast, the

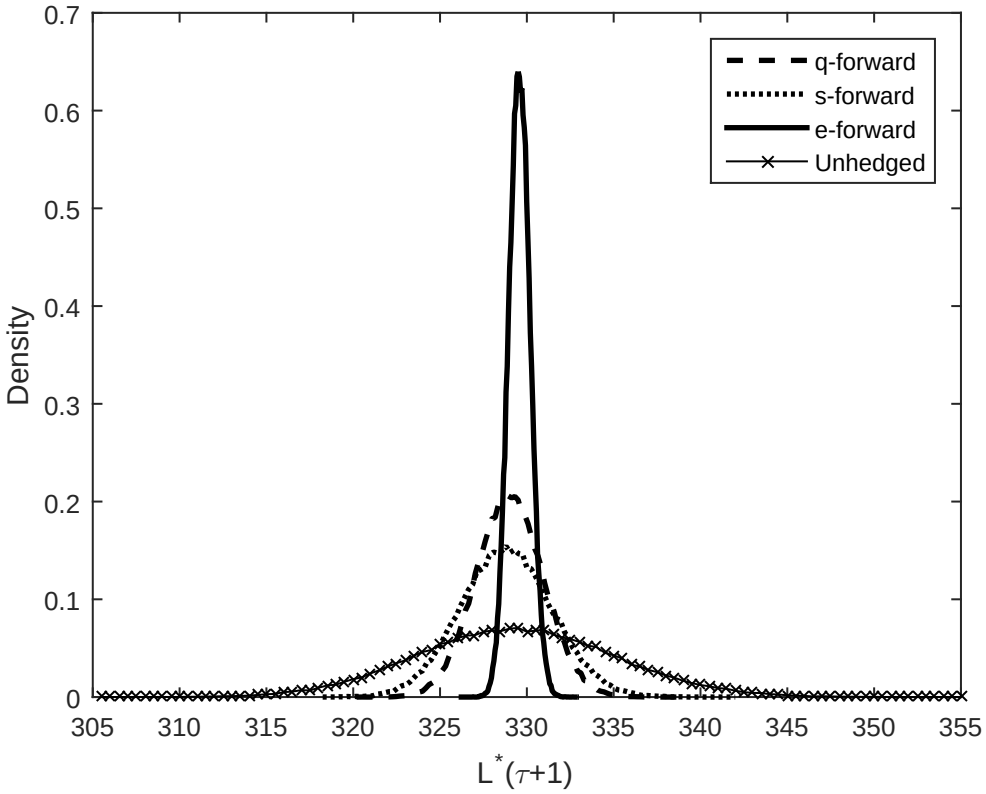


Figure 7: Empirical distribution of liability values under different hedging strategies

q-forward is sensitive to mortality rates at a single selected age, whilst the s-forward considers only a single cohort, and consequently both are poorer at hedging risk.

As can be seen, the reduction in longevity risk with even relatively simple hedging strategies over a one-year period is very high. These are "value hedges", in the sense that the strategy has been chosen to minimise the variance of the total portfolio value, as opposed to "cashflow" hedges that minimise the variability of the net cashflows from the portfolio.[23] Longer term hedges could potentially be achieved by rebalancing the portfolio at

[23]For examples of using q-forwards as cashflow hedges, see Li and Luo (2012).

	Unhedged	Term of instrument (yrs)		
		5	10	15
q-forward	1.74%	0.59%	0.59%	0.59%
s-forward		0.94%	0.80%	0.63%
e-forward		0.22%	0.19%	0.16%

Table 2: Standard deviation of hedged portfolios (as percentage of time τ liability value) using instruments of different terms

least annually to reflect the actual experience of the annuity book. However, such a strategy is dependent upon the existence of a relatively liquid market in the underlying longevity-linked securities.

One potential criticism is that these results are all model dependent. It does not seem likely that the high correlations shown in Table 1 could be achieved in practice and, therefore, such large reductions in risk may not be feasible. In particular, the use of relatively simple APC mortality models to underpin the forward mortality framework might be felt to give correlation structures for future mortality rates which are overly simplistic, and so overstate the effectiveness of any hedging strategy. However, we note that our underlying model for the force of mortality has four age/period terms and a cohort term, making it relatively complex compared with many more commonly used mortality models, and so it is unlikely that using a more complicated model for the short rate would materially affect our results.[24] In addition, the impact of hedging would be lower if the market prices of risk change during the year. However, since the market for longevity risk is just emerging, assuming constant market prices of risk is unavoidable at present, for the reasons discussed in Section 3 and, accordingly, all liability and securities values will be model-dependent for the foreseeable future. Furthermore, high correlations between the liabilities and hedging instruments are required in order to recognise the hedge under some accounting standards. Accordingly, we argue that reductions in risk, even if they are only

[24]We have tested the hedging strategies using the simpler models of the short rate of mortality discussed in Hunt and Blake (2015b) and obtain even higher reductions in risk. In particular, we observed perfect correction between the liabilities and securities, and therefore perfect hedges, when using the Lee-Carter model as the underlying mortality model, since this model only possesses one age/period term and hence only one source of risk.

mark-to-model, are still beneficial for many purposes.

In addition, the results presented above do not allow for potential basis risk between populations or for idiosyncratic risk in the number of deaths observed in an actual annuity book, and so will overstate the potential effectiveness of hedging strategies which could be obtained in practice. We leave the addition of both of these sources of risk to future work.

5 Conclusions

In Hunt and Blake (2015b), we defined a static forward surface of mortality for the purpose of valuing liabilities and longevity-linked securities. In this study, we extend this framework by investigating the dynamics of the forward mortality surface to show how these values might change with time. This involves understanding the processes we use to project the underlying parameters in the mortality model and how these update to reflect new information. In particular, an understanding of how the cohort parameters in the model update in response to new information is critical in assessing the dynamics of the forward mortality surface. We use this understanding to show that the forward mortality rates are martingales in the real-world and market-consistent measures, and therefore are "self-consistent" in the terminology of Zhu and Bauer (2011b).

We then apply this dynamic framework to investigate longevity risk in an illustrative annuity book and various longevity-linked securities. By combining these, we then consider the potential for longevity-linked securities to hedge longevity risk in an annuity portfolio, and find that relatively simple hedging strategies can significantly mitigate the longevity risk in a set of illustrative liabilities over a one-year period. Such strategies can be extended over multiple years by rebalancing the hedging portfolio if there is a liquid market in the hedging instruments.

We believe that the forward mortality framework discussed here has numerous other applications, relating to the assessment and management of longevity risk. One particular application we investigate in further, in Hunt and Blake (2016d), is the use of the framework to assess the capital requirements for longevity risk under modern solvency regimes such as Solvency II

in the EU.

However, the forward mortality framework described here and in Hunt and Blake (2015b) contains some notable omissions. For example, it cannot be used to value options on mortality rates (although we address this in Hunt and Blake (2016c)), it does not allow for potential basis risk between populations or recalibration risk. We leave the last two of these problems for future work, but are confident that they are solvable.

In Hunt and Blake (2015b), we stated our belief that the forward mortality rates are the way forward in answer to the question posed in Norberg (2010). This study reaffirms this conclusion and demonstrates the many practical uses a forward mortality framework can have in completing the framework for measuring and managing longevity risk.

A Self consistency

In Section 3, we discussed the self-consistency property of Zhu and Bauer (2011b) and argued that $\mathbb{P}$-measure forward mortality rates should be self-consistent in the real-world measure and $\mathbb{Q}$-measure forward mortality rates should be self-consistent in the market-consistent measure since they are defined as conditional expectations. However, it is helpful to confirm this explicitly in order to ensure that there are no inconsistencies in the modelling framework. This was done for age/period models of the short rate in Section 3.1, where the time series process updating the period parameters was relatively simple. In this Appendix, we first verify the martingale property for models that include a cohort term and then verify that forward mortality rates are self-consistent in the market consistent $\mathbb{Q}$-measure.

A.1 Self consistency of the cohort parameters

For simplicity, consider a model of the force of mortality with no age/period terms, i.e.,

$$\ln \mu_{x,t} = \alpha_x + \gamma_{t-x}$$

In this case

$$\nu^{\mathbb{P}}_{x,t}(\tau) = \exp\left(\alpha_x + M(t-x,\tau) + \frac{1}{2}V(t-x,\tau)\right)$$

and trivially therefore

$$\nu^{\mathbb{P}}_{x,t}(\tau+1) = \exp\left(\alpha_x + M(t-x,\tau+1) + \frac{1}{2}V(t-x,\tau+1)\right)$$

First, we observe that

$$V(y,\tau+1) = V(y-1,\tau) \tag{29}$$

from the definition of the variance function in Equation 10. Then, using Equation 29 and dropping the superscript $\mathbb{P}$ (since all expectations and variances are in the real-world measure), we see that self-consistency implies

$$\begin{aligned}
&\exp\left(\alpha_x + M(t-x,\tau) + \frac{1}{2}V(t-x,\tau)\right) \\
&\quad = \mathbb{E}_\tau\ \exp\left(\alpha_x + M(t-x,\tau+1) + \frac{1}{2}V(t-x,\tau+1)\right) \\
&\quad = \exp\left(\alpha_x + \mathbb{E}_\tau M(t-x,\tau+1) + \frac{1}{2}\left(\mathbb{V}ar_\tau(M(t-x,\tau+1)) + V(t-x-1,\tau)\right)\right)
\end{aligned}$$

Therefore, we require

$$\mathbb{E}_\tau M(y,\tau+1) = M(y,\tau) \tag{30}$$

$$\mathbb{V}ar_\tau\left(M(y,\tau+1)\right) = V(y,\tau) - V(y-1,\tau) \tag{31}$$

It is important to note that these are direct consequences on the laws of conditional expectation and variance, and therefore that the following is merely a check of whether the Bayesian process underpinning the cohort parameter is internally consistent.

For simplicity, we assume that we have chosen a set of identifiability constraints such that $\beta = 0$. From Hunt and Blake (2015a), we have the following recursive relationships which define the mean and variance functions

(and which were solved to give the closed forms of $M(y,\tau)$ and $V(y,\tau)$ in Equations 9 and 10)

$$M(y,t) = \underline{\gamma}_y(t) + (1 - D_{t-y})\rho M(y-1,t) \tag{32}$$
$$V(y,t) = (1 - D_{t-y})\sigma^2 + (1 - D_{t-y})^2\rho^2 V(y-1,t)) \tag{33}$$

Starting with Equation 30

$$\begin{aligned}\mathbb{E}_\tau M(y,\tau+1) &= \mathbb{E}_\tau \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s}\rho^s \underline{\gamma}_{y-s}(\tau+1) \\ &= \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s}\rho^s \mathbb{E}_\tau \underline{\gamma}_{y-s}(\tau+1)\end{aligned}$$

where $\text{Þ}_{\tau-y,s}$ is defined in Equation 8 and $\text{Þ}_{\tau-y,0} = 1$ by definition, as per Hunt and Blake (2015a). From this definition, we note the following

$$\begin{aligned}\text{Þ}_{\tau-y,s+1} &= (1 - D_{\tau-y+s})\text{Þ}_{\tau-y,s} \\ \text{Þ}_{\tau-y+1,s} &= \frac{(1 - D_{\tau-y+s})}{(1 - D_{\tau-y})}\text{Þ}_{\tau-y,s}\end{aligned}$$

From Equation 19 we have

$$\begin{aligned}\mathbb{E}_\tau \gamma_y^{\tau+1-y} &= \rho M(y-1,\tau) \\ \mathbb{V}ar_\tau(\gamma_y^{\tau+1-y}) &= \rho^2 V(y-1,\tau) + \frac{\sigma^2}{d_{\tau+1-y}}\end{aligned}$$

Using this with Equation 17 gives us

$$\begin{aligned}\mathbb{E}_\tau \underline{\gamma}_y(\tau+1) &= \underline{\gamma}_y(\tau) + d_{\tau-y+1}\mathbb{E}_\tau[\gamma_y^{\tau-y+1}] \\ &= \underline{\gamma}_y(\tau) + d_{\tau-y+1}\rho M(y-1,\tau) \\ &= M(y,\tau) - (1 - D_{\tau-y})\rho M(y-1,\tau) + d_{\tau-y+1}\mathbb{E}_\tau \rho M(y-1,\tau) \\ &= M(y,\tau) - (1 - D_{\tau-y+1})\rho M(y-1,\tau)\end{aligned}$$

where we have used Equation 32 to remove the dependence on $\underline{\gamma}_y(\tau)$.

It therefore follows that

$$\begin{aligned}
\mathbb{E}_\tau M(y,\tau+1) &= \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s}\rho^s \left(M(y-s,\tau) - (1-D_{\tau-y+1})\rho M(y-s-1,\tau)\right) \\
&= \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s}\rho^s M(y-s,\tau) - \sum_{s=0}^{\infty}(1-D_{\tau-y+1})\text{Þ}_{\tau+1-y,s}\rho^{s+1} M(y-s-1,\tau) \\
&= \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s}\rho^s M(y-s,\tau) - \sum_{s=0}^{\infty} \text{Þ}_{\tau+1-y,s+1}\rho^{s+1} M(y-s-1,\tau) \\
&= \text{Þ}_{\tau+1-y,0}\rho^0 M(y,\tau) \\
&= M(y,\tau)
\end{aligned}$$

as required.

Perhaps unsurprisingly, demonstrating Equation 31 is trickier. We start by showing that it is true when $y = \tau + 1 - X$, i.e., the cohort is one year away from being fully run off. Trivially $V(\tau+1-X,\tau+1) = 0$, since at time $\tau+1$, everyone in the cohort born at $\tau+1-X$ has died and so the cohort parameter $\gamma_{\tau+1-X} = \underline{\gamma}_{\tau+1-X}(\tau+1)$ is known with certainty. Therefore

$$\begin{aligned}
\mathbb{V}ar_\tau(M(\tau+1-X,\tau+1)) &= \mathbb{V}ar_\tau(\underline{\gamma}_{\tau+1-X}(\tau+1)) \\
&= \mathbb{V}ar_\tau(\underline{\gamma}_{\tau+1-x}(\tau) + d_X\gamma^X_{\tau+1-X}) \\
&= d_X^2\frac{\sigma^2}{d_X} \\
&= d_X\sigma^2 = (1-D_{X-1})\sigma^2 = V(\tau+1-X,\tau)
\end{aligned}$$

using Equations 19 and 10. This is the first step in an induction argument, enabling us to work forwards in y to prove that Equation 31 holds true.

$$\begin{aligned}
\mathbb{V}ar_\tau(M(y,\tau+1)) &= \mathbb{V}ar_\tau\left(\underline{\gamma}_y(\tau+1) + (1-D_{\tau-y+1})\rho M(y-1,\tau+1)\right) \\
&= \mathbb{V}ar_\tau(\underline{\gamma}_y(\tau+1)) + (1-D_{\tau-y+1})^2\rho^2\mathbb{V}ar_\tau(M(y-1,\tau+1)) \\
&\quad + 2(1-D_{\tau-y+1})\rho\mathbb{C}ov_\tau(\underline{\gamma}_y(\tau+1), M(y-1,\tau+1))
\end{aligned}$$

using Equation 32 and expanding the variance. Looking at the first of these

parts, we see

$$\begin{aligned}\mathbb{V}ar_\tau(\underline{\gamma}_y(\tau+1)) &= \mathbb{V}ar_\tau(\underline{\gamma}_y(\tau) + d_{\tau-y+1}\gamma_y^{\tau-y+1}) \\ &= d_{\tau-y+1}^2\mathbb{V}ar_\tau(\gamma_y^{\tau-y+1}) \\ &= d_{\tau-y+1}\sigma^2 + \rho^2 d_{\tau-y+1}^2 V(y-1,\tau)\end{aligned}$$

from Equation 19. For the second part, we assume that Equation 31 holds for $y-1$, using the inductive argument, and therefore

$$\mathbb{V}ar_\tau(M(y-1,\tau+1)) = V(y-1,\tau) - V(y-2,\tau)$$

Consequently

$$\begin{aligned}&\mathbb{V}ar_\tau(\underline{\gamma}_y(\tau+1)) + (1-D_{\tau-y+1})^2\rho^2\mathbb{V}ar_\tau(M(y-1,\tau+1)) \\ &\quad= d_{\tau-y+1}\sigma^2 + \rho^2\left(d_{\tau-y+1}^2 + (1-D_{\tau-y+1})^2\right)V(y-1,\tau) \\ &\quad\quad - (1-D_{\tau-y+1})^2\rho^2 V(y-2,\tau) \\ &\quad= d_{\tau-y+1}\sigma^2 + \rho^2\left((1-D_{\tau-y+1}+d_{\tau-y+1})^2 - 2(1-D_{\tau-y+1})d_{\tau-y+1}\right)V(y-1,\tau) \\ &\quad\quad - (1-D_{\tau-y+1})\sigma^2 - V(y-1,\tau) \quad \text{using Equation 33 on } V(y-2,\tau) \\ &\quad= (1-D_{\tau-y})\sigma^2 + \rho^2(1-D_{\tau-y})^2 V(y-1,\tau) - V(y-1,\tau) \\ &\quad\quad - 2\rho^2(1-D_{\tau-y+1})d_{\tau-y+1}V(y-1,\tau) \\ &\quad= V(y,\tau) - V(y-1,\tau) - 2\rho^2(1-D_{\tau-y+1})d_{\tau-y+1}V(y-1,\tau)\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{V}ar_\tau(M(y,\tau+1)) = V(y,\tau) - V(y-1,\tau) \\ &+2(1-D_{\tau-y+1})\rho\left(\mathbb{C}ov_\tau(\underline{\gamma}_y(\tau+1), M(y-1,\tau+1)) - \rho d_{\tau-y+1}V(y-1,\tau)\right)\end{aligned}$$

and so Equation 31 will hold if and only if

$$\mathbb{C}ov_\tau(\underline{\gamma}_y(\tau+1), M(y-1,\tau+1)) = \rho d_{\tau-y+1}V(y-1,\tau)$$

To show that this calculation holds, we decompose the covariance as

$$\begin{aligned}
\mathbb{C}ov_\tau(\underline{\gamma}_y(\tau+1), M(y-1,&\tau+1)) = d_{\tau+1-y}\mathbb{C}ov_\tau(\gamma_y^{\tau+1-y}, M(y-1,\tau+1)) \\
&= d_{\tau+1-y}\sum_{s=0}^{\infty} \text{Þ}_{\tau-y+2,s}\rho^s \mathbb{C}ov_\tau(\gamma_y^{\tau+1-y}, \underline{\gamma}_{y-1-s}(\tau+1)) \\
&= d_{\tau+1-y}\sum_{s=0}^{\infty} \text{Þ}_{\tau-y+2,s}\rho^s d_{\tau+2-y+2}\mathbb{C}ov_\tau(\gamma_y^{\tau+1-y}, \gamma_{y-s-1}^{\tau+2-y+s}) \\
&= d_{\tau+1-y}\sum_{s=0}^{\infty} \text{Þ}_{\tau-y+2,s}\rho^s d_{\tau+2-y+s}\rho^{s+1}\text{Þ}_{\tau+1-y,s+1}\frac{\sigma^2}{d_{\tau+2-y+s}} \\
&\quad \text{from Equation 20} \\
&= \rho d_{\tau+1-y}\sum_{s=0}^{\infty}(1-D_{\tau+1-y+s})\text{Þ}^2_{\tau+1-y,s+1}\rho^{2s}\sigma^2 \\
&= \rho d_{\tau+1-y}V(y-1,\tau)
\end{aligned}$$

from the definition of $V(y,\tau)$ in Equation 10. Therefore, Equation 31 does indeed hold and models involving a set of cohort parameters are self-consistent in the real-world $\mathbb{P}$-measure.

A.2 Self-consistency in the market-consistent measure

Together, the results of Section 3.1 and Appendix A.1 show that the forward mortality rates are self-consistent in the historical $\mathbb{P}$-measure, as expected. We now demonstrate that they are self-consistent in the market-consistent $\mathbb{Q}$-measure, i.e.,

$$\mathbb{E}^{\mathbb{Q}}{}_\tau\ \nu^{\mathbb{Q}}_{x,t}(\tau+1) = \nu^{\mathbb{Q}}_{x,t}(\tau)$$

From Equation 12, we have

$$\begin{aligned}
\nu^{\mathbb{Q}}_{x,t}(\tau+1) &= \exp\left(\boldsymbol{\beta}_x^\top \mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^\gamma \mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x})\right) \times \nu^{\mathbb{P}}_{x,t}(\tau+1) \\
&= \exp\Bigg(\alpha_x + \boldsymbol{\beta}_x^\top \mathbb{E}^{\mathbb{P}}{}_{\tau+1}\boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x^\top \mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \mathbb{E}^{\mathbb{P}}{}_{\tau+1}\gamma_{t-x} \\
&\quad + \frac{1}{2}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x}) + \boldsymbol{\beta}_x^\top \mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^\gamma \mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x})\Bigg)
\end{aligned}$$

and also from Equation 11

$$\mathbb{E}^{\mathbb{Q}}{}_{\tau}\ \nu^{\mathbb{Q}}_{x,t}(\tau+1) = \frac{\mathbb{E}^{\mathbb{P}}{}_{\tau}\left[\exp\left(-\boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t - \lambda^{\gamma}\gamma_{t-x}\right)\nu^{\mathbb{Q}}_{x,t}(\tau+1)\right]}{\mathbb{E}^{\mathbb{P}}\exp\left(-\boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t - \lambda^{\gamma}\gamma_{t-x}\right)}$$

Looking first at the denominator

$$\left[\mathbb{E}^{\mathbb{P}}\exp\left(-\boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t - \lambda^{\gamma}\gamma_{t-x}\right)\right]^{-1} = \exp\left(\boldsymbol{\lambda}^{\top}\mathbb{E}^{\mathbb{P}}{}_{\tau}\boldsymbol{\kappa}_t - \frac{1}{2}\boldsymbol{\lambda}^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^{\gamma}\mathbb{E}^{\mathbb{P}}{}_{\tau}\gamma_{t-x} - \frac{1}{2}\lambda^{\gamma\,2}\mathbb{V}ar^{\mathbb{P}}_{\tau}(\gamma_{t-x})\right.$$

Next, let us consider the numerator

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}{}_{\tau}\left[\exp\left(-\boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t - \lambda^{\gamma}\gamma_{t-x}\right)\nu^{\mathbb{Q}}_{x,t}(\tau+1)\right] = \\
&\quad \exp\left(\alpha_x + \boldsymbol{\beta}_x^{\top}\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \mathbb{E}^{\mathbb{P}}{}_{\tau+1}\gamma_{t-x}\right. \\
&\quad \left. + \frac{1}{2}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x}) + \boldsymbol{\beta}_x^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^{\gamma}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x}) - \boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t - \lambda^{\gamma}\gamma_{t-x}\right) \\
&= \exp\left(\alpha_x + \frac{1}{2}\boldsymbol{\beta}_x^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \frac{1}{2}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x}) + \boldsymbol{\beta}_x^{\top}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^{\gamma}\mathbb{V}ar^{\mathbb{P}}_{\tau+1}(\gamma_{t-x})\right) \\
&\quad \times \mathbb{E}^{\mathbb{P}}{}_{\tau}\exp\left(\boldsymbol{\beta}_x\mathbb{E}^{\mathbb{P}}{}_{\tau+1}\boldsymbol{\kappa}_t - \boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t + \mathbb{E}^{\mathbb{P}}{}_{\tau+1}\gamma_{t-x} - \lambda^{\gamma}\gamma_{t-x}\right)
\end{aligned}$$

Since all expectations and variances are under the measure $\mathbb{P}$ (unless stated otherwise), we drop the superscripts for simplicity. Considering only the expectation

$$\begin{aligned}
&\mathbb{E}_{\tau}\exp\left(\boldsymbol{\beta}_x\mathbb{E}_{\tau+1}\boldsymbol{\kappa}_t - \boldsymbol{\lambda}^{\top}\boldsymbol{\kappa}_t + \mathbb{E}_{\tau+1}\gamma_{t-x} - \lambda^{\gamma}\gamma_{t-x}\right) = \\
&\quad \exp\left(\boldsymbol{\beta}_x\mathbb{E}_{\tau}\boldsymbol{\kappa}_t - \boldsymbol{\lambda}^{\top}\mathbb{E}_{\tau}\boldsymbol{\kappa}_t + \mathbb{E}_{\tau}\gamma_{t-x} - \lambda^{\gamma}\mathbb{E}_{\tau}\gamma_{t-x} + \frac{1}{2}\boldsymbol{\beta}_x^{\top}\mathbb{V}ar_{\tau}(\mathbb{E}_{\tau+1}\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \frac{1}{2}\boldsymbol{\lambda}^{\top}\mathbb{V}ar_{\tau}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda}\right. \\
&\quad \left. + \boldsymbol{\beta}_x^{\top}\mathbb{C}ov_{\tau}(\mathbb{E}_{\tau+1}\boldsymbol{\kappa}_t, \boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \frac{1}{2}\mathbb{V}ar_{\tau}(\mathbb{E}_{\tau+1}\gamma_{t-x}) + \frac{1}{2}\lambda^{\gamma\,2}\mathbb{V}ar_{\tau}(\gamma_{t-x}) - \lambda^{\gamma}\mathbb{C}ov_{\tau}(\mathbb{E}_{\tau+1}\gamma_{t-x}, \gamma_{t-x})\right)
\end{aligned}$$

Looking at each of the variance terms, we use the results

$$\begin{aligned}
\mathbb{V}ar_{\tau}(\mathbb{E}_{\tau+1}X) &= \mathbb{V}ar_{\tau}(X) - \mathbb{V}ar_{\tau+1}(X) \\
\mathbb{C}ov_{\tau}(X, \mathbb{E}_{\tau+1}X) &= \mathbb{E}_{\tau}\mathbb{C}ov_{\tau+1}(X, \mathbb{E}_{\tau+1}X) + \mathbb{C}ov_{\tau}(\mathbb{E}_{\tau+1}X, \mathbb{E}_{\tau+1}X) \\
&= 0 + \mathbb{V}ar_{\tau}(\mathbb{E}_{\tau+1}X) \\
&= \mathbb{V}ar_{\tau}(X) - \mathbb{V}ar_{\tau+1}(X)
\end{aligned}$$

to give

$$\begin{aligned}
&\mathbb{E}_\tau \exp\left(\boldsymbol{\beta}_x \mathbb{E}_{\tau+1}\boldsymbol{\kappa}_t - \boldsymbol{\lambda}^\top \boldsymbol{\kappa}_t + \mathbb{E}_{\tau+1}\gamma_{t-x} - \lambda^\gamma \gamma_{t-x}\right) = \\
&\quad \exp\Bigg(\boldsymbol{\beta}_x \mathbb{E}_\tau \boldsymbol{\kappa}_t - \boldsymbol{\lambda}^\top \mathbb{E}_\tau \boldsymbol{\kappa}_t + \mathbb{E}_\tau \gamma_{t-x} - \lambda^\gamma \mathbb{E}_\tau \gamma_{t-x} + \frac{1}{2}\boldsymbol{\beta}_x^\top \left[\mathbb{V}ar_\tau(\boldsymbol{\kappa}_t) - \mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t)\right]\boldsymbol{\beta}_x \\
&\quad + \frac{1}{2}\boldsymbol{\lambda}^\top \mathbb{V}ar_\tau(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \boldsymbol{\beta}_x^\top \left[\mathbb{V}ar_\tau(\boldsymbol{\kappa}_t) - \mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t)\right]\boldsymbol{\lambda} + \frac{1}{2}\mathbb{V}ar_\tau(\gamma_{t-x}) - \frac{1}{2}\mathbb{V}ar_{\tau+1}(\gamma_{t-x}) \\
&\quad + \frac{1}{2}\lambda^{\gamma\,2}\mathbb{V}ar_\tau(\gamma_{t-x}) - \lambda^\gamma \mathbb{V}ar_\tau(\gamma_{t-x}) + \lambda^\gamma \mathbb{V}ar_{\tau+1}(\gamma_{t-x})\Bigg)
\end{aligned}$$

Putting all three parts together and cancelling terms, we find

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}{}_\tau\, \nu^{\mathbb{Q}}_{x,t}(\tau+1) &= \exp\Bigg(\alpha_x + \boldsymbol{\beta}_x^\top \mathbb{E}_\tau \boldsymbol{\kappa}_t + \frac{1}{2}\boldsymbol{\beta}_x^\top \mathbb{V}ar_\tau(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \mathbb{E}_\tau \gamma_{t-x} + \frac{1}{2}\mathbb{V}ar_\tau(\gamma_{t-x}) \\
&\qquad + \boldsymbol{\beta}_x^\top \mathbb{V}ar_\tau(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^\gamma \mathbb{V}ar_\tau(\gamma_{t-x})\Big) \\
&= \exp\left(\boldsymbol{\beta}_x^\top \Lambda \mathbb{V}ar_\tau(\boldsymbol{\kappa}_t)\boldsymbol{\beta}_x + \lambda^\gamma \mathbb{V}ar_\tau(\gamma_{t-x})\right)\nu^{\mathbb{P}}_{x,t}(\tau) \\
&= \nu^{\mathbb{Q}}_{x,t}(\tau)
\end{aligned}$$

i.e., that forward mortality rates are self-consistent martingales under the market-consistent $\mathbb{Q}$-measure. From this, we also see that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}{}_\tau \nu^{\mathbb{Q}}_{x,t}(\tau+1) &= \mathbb{E}^{\mathbb{P}}{}_\tau \exp\left(\boldsymbol{\beta}^\top \mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^\gamma \mathbb{V}ar_{\tau+1}(\gamma_{t-x})\right)\nu^{\mathbb{P}}_{x,t}(\tau+1) \\
&= \exp\left(\boldsymbol{\beta}^\top \mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t)\boldsymbol{\lambda} + \lambda^\gamma \mathbb{V}ar_{\tau+1}(\gamma_{t-x})\right)\nu^{\mathbb{P}}_{x,t}(\tau) \\
&= \exp\left(\boldsymbol{\beta}^\top \left[\mathbb{V}ar_{\tau+1}(\boldsymbol{\kappa}_t) - \mathbb{V}ar_\tau(\boldsymbol{\kappa}_t)\right]\boldsymbol{\lambda} + \lambda^\gamma \left[\mathbb{V}ar_{\tau+1}(\gamma_{t-x}) - \mathbb{V}ar_\tau(\gamma_{t-x})\right]\right)\nu^{\mathbb{Q}}_{x,t}(\tau)
\end{aligned}$$

i.e., the change of measure introduces a distortion which prevents market consistent forward rates being self-consistent in the real-world $\mathbb{P}$-measure.

References

Alai, D. H., Ignatieva, K., Sherris, M., 2013. Modelling longevity risk: Generalizations of the Olivier-Smith Model. Tech. rep., University of New South Wales.

Barbarin, J., 2008. Heath-Jarrow-Morton modelling of longevity bonds and the risk minimization of life insurance portfolios. Insurance: Mathematics and Economics 43 (1), 41–55.

Bauer, D., Börger, M., Ruß, J., Zwiesler, H., 2008. The volatility of mortality. Asia-Pacific Journal of Risk and Insurance 3 (10), 2153–3792.

Blake, D., Cairns, A. J., Dowd, K., 2006. Living with mortality: Longevity bonds and other mortality-linked securities. British Actuarial Journal 12 (1), 153–197.

Cairns, A. J., 2007. A multifactor generalisation of the Olivier-Smith model for stochastic mortality. Tech. rep., Heriot-Watt University, Edinburgh.

Cairns, A. J., Blake, D., Dowd, K., 2006. Pricing death: Frameworks for the valuation and securitization of mortality risk. ASTIN Bulletin 36 (1), 79–120.

Cairns, A. J., Blake, D., Dowd, K., Kessler, A., 2015. Phantoms never die: Living with unreliable population data. Journal of the Royal Statistical Society: Series A (Statistics in Society), Forthcoming.

Cairns, A. J., Dowd, K., Blake, D., Coughlan, G. D., 2013. Longevity hedge effectiveness: A decomposition. Quantitative Finance 14 (2), 217–235.

Coughlan, G. D., Epstein, D., Sinha, A., Honig, P., 2007. q-forwards: Derivatives for transferring longevity and mortality risks. JPMorgan Pension Advisory Group.

Cowley, A., Cummins, J., 2005. Securitization of life insurance assets and liabilities. Journal of Risk and Insurance 72 (2), 193–226.

Cox, S. H., Lin, Y., 2007. Natural hedging of life and annuity mortality risks. North American Actuarial Journal 11 (3), 1–15.

Denuit, M. M., 2009. An index for longevity risk transfer. Journal of Computational and Applied Mathematics 230 (2), 411–417.

Denuit, M. M., Goderniaux, A.-M., 2005. Closing and projecting life tables using log-linear models. Bulletin of the Swiss Association of Actuaries 1, 29–49.

Dhaene, J., Kukush, A., Luciano, E., Schoutens, W., Stassen, B., 2013. On the (in-)dependence between financial and actuarial risks. Insurance: Mathematics and Economics 52 (3), 522–531.

Dowd, K., 2003. Survivor bonds: A comment on Blake and Burrows. Journal of Risk and Insurance 70 (2), 339–348.

Hobcraft, J., Menken, J., Preston, S. H., 1982. Age, period and cohort effects in demography: A review. Population Index 48 (1), 4–43.

Hunt, A., Blake, D., 2014. A general procedure for constructing mortality models. North American Actuarial Journal 18 (1), 116–138.

Hunt, A., Blake, D., 2015a. A Bayesian approach to modelling and projecting cohort effects. Tech. rep., Pensions Institute PI 15-10, Cass Business School, City University London.

Hunt, A., Blake, D., 2015b. Forward mortality rates in discrete time I: Calibration and securities pricing. Tech. rep., Pensions Institute PI 15-11, Cass Business School, City University London.

Hunt, A., Blake, D., 2015c. Identifiability in age/period mortality models. Tech. rep., Pensions Institute PI 15-08, Cass Business School, City University London.

Hunt, A., Blake, D., 2015d. Identifiability in age/period/cohort mortality models. Tech. rep., Pensions Institute PI 15-09, Cass Business School, City University London.

Hunt, A., Blake, D., 2015e. On the structure and classification of mortality models. Tech. rep., Pensions Institute PI 15-06, Cass Business School, City University London.

Hunt, A., Blake, D., 2016a. Basis risk and pension schemes: A relative modelling approach. Tech. rep., Pensions Institute PI 16-01, Cass Business School, City University London.

Hunt, A., Blake, D., 2016b. Consistent mortality projections allowing for trend changes. Work in Progress.

Hunt, A., Blake, D., 2016c. Forward mortality rates in discrete time III: Longevity-linked options. Work in Progress.

Hunt, A., Blake, D., 2016d. Solvency II and longevity risk: An application of forward mortality rates. Work in Progress.

Hunt, A., Blake, D., 2016e. Transferring mortality and longevity risks in pension schemes via bespoke longevity swaps. Work in Progress.

Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M. M., 2001. Modern actuarial risk theory. Kluwer Academic Publishers.

Lee, R. D., Carter, L. R., 1992. Modeling and forecasting U.S. mortality. Journal of the American Statistical Association 87 (419), 659– 671.

Li, J. S.-H., Luo, A., 2012. Key q-duration: A framework for hedging longvevity risk. ASTIN Bulletin 42 (2), 413–452.

Loeys, J., Panigirtzoglou, N., Ribeiro, R., 2007. Longevity: A market in the making. JPMorgan Pension Advisory Group.

Michaelson, A., Mulholland, J., 2014. Strategy for increasing the global capacity for longevity risk transfer: Developing transactions that attract capital markets investors. Journal of Alternative Investments 17 (1), 18–27.

Miltersen, K. R., Persson, S.-A., 2005. Is mortality dead? Stochastic forward force of mortality rate determined by no arbitrage. Tech. rep., University of Ulm.

Norberg, R., 2010. Forward mortality and other vital rates - Are they the way forward? Insurance: Mathematics and Economics 47 (2), 105–112.

Olivier, P., Jeffrey, T., 2004. Stochastic mortality models.
URL http://www.actuaries.ie/EventsandPapers/Events2004/2004-06-01{_}Pensione

Richards, S. J., 2008. Detecting year-of-birth mortality patterns with limited data. Journal of the Royal Statistical Society: Series A (Statistics in Society) 171 (1), 279–298.

Smith, A., 2005. Stochastic mortality modelling.
URL http://www.icms.org.uk/archive/meetings/2005/quantfinance/sci{_}prog.html

Tan, C. I., Li, J., Li, J. S.-H., Balasooriya, U., 2014. Parametric mortality indexes: From index construction to hedging strategies. Insurance: Mathematics and Economics 59, 285–299.

Tappe, S., Weber, S., 2013. Stochastic mortality models: An infinite-dimensional approach. Finance and Stochastics 18 (1), 209–248.

Villegas, A. M., Haberman, S., 2014. On the modeling and forecasting of socioeconomic mortality differentials: An application to deprivation and mortality in England. North American Actuarial Journal 18 (1), 168–193.

Zhu, N., Bauer, D., 2011a. Applications of forward mortality factor models in life insurance practice. Geneva Papers on Risk and Insurance Issues and Practice 36, 567–594.

Zhu, N., Bauer, D., 2011b. Coherent modeling of the risk in mortality projections: A semi parametric approach. Tech. rep., Georgia State University.

Zhu, N., Bauer, D., 2014. A cautionary note on natural hedging of longevity risk. North American Actuarial Journal 18 (1), 104–115.