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Heterogenous mean-field analysis of a generalized voter-like model on networks

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Abstract. We propose a generalized framework for the study of voter models in complex networks at the heterogeneous mean-field (HMF) level that (i) yields a unified picture for existing copy/invasion processes and (ii) allows for the introduction of further heterogeneity through degree-selectivity rules. In the context of the HMF approximation, our model is capable of providing straightforward estimates for central quantities such as the exit probability and the consensus/fixation time, based on the statistical properties of the complex network alone. The HMF approach has the advantage of being readily applicable also in those cases in which exact solutions are difficult to work out. Finally, the unified formalism allows one to understand previously proposed voter-like processes as simple limits of the generalized model.

Key words. Complex networks – Ordering dynamics – Voter models – Mean-field theory

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1 Introduction

A topical problem in the statistical physics approach to social and evolutionary dynamics [1,2] is the study of the mechanisms ruling the formation of consensus in an initially disordered population, in situations implying the opinion about a certain issue, the intention of voting in an election, or the evolutionary competition of different species striving for the same ecological resources. Several stochastic copying/invasion processes have been proposed to represent this kind of problems, the simplest being the voter model [3] and the Moran process [4]. In these models, each individual in a population (agent) is endowed with a binary variable (opinion or state) with value $\sigma = \pm 1$. At each time step, an agent i, together with one nearest neighbor j, are selected at random. In the voter model the system is updated as $\sigma_i := \sigma_j$, the first agent copying the opinion of its neighbor. The Moran process, on the other hand, can be considered as a *reversed* voter model, in which it is the neighboring agent the one who copies the opinion of the first agent, $\sigma_j := \sigma_i$. Starting from a disordered initial state, this kind of dynamics leads in finite systems to a uniform state with all individuals sharing the same state (the so-called consensus). The way in which this final state is reached is usually characterized in terms of the exit probability $E(x)$ and the consensus time $T_N(x)$, defined as the probability that the final state corresponds to all agents in the state $+1$ and the average time needed to reach consensus in a system of size N, respectively, when starting from a homogeneous initial condition with a fraction x of agents in state $+1$ [1].

While simple voter-like models are well understood on regular lattices, even in terms of theorems and exact solutions [5,6], they become more relevant in social and evolutionary contexts when considered on top of complex networks, which act as more realistic representations of social or ecological contact patterns [7,8,9]. The analysis of the voter model in these substrates reveals nontrivial differences with respect to ordered lattices. For example, now the order in which interacting individuals are selected becomes relevant [10], in such a way that voter model and Moran process behave in different ways. Moreover, relevant quantities such as the consensus time turn out to depend on the heterogeneity of the contact pattern, as measured by the degree distribution [11,12,13]. Not only the properties of the interaction substrate are relevant in this case, but also the intrinsic heterogeneity of the actors. Their individual propensity to interact with peers, and change state accordingly, plays a significant role [14,15,16,17]. Other sources of heterogeneity that

have been considered include explicitly the directed nature of the connections [18].

The theoretical understanding of voter-like models (and dynamical processes in general) on complex networks has been traditionally accomplished by application of heterogeneous mean-field (HMF) approaches [19,20], which are based on a twofold assumption: (*i*) The network description is coarse-grained into degree classes, all vertices in the same class having the same degree and sharing the same dynamical properties; (*ii*) The real (*quenched*) network structure is replaced by an *annealed* one [20], which disregards the actual connection pattern and simply assumes that the degree class k is connected to the degree class k' with conditional probability $P(k'|k)$ [21]. Very significant progress has been achieved in the analysis of voter-like models within the HMF approach, which allows one to work out simple analytic expressions for the quantities of interest, showing reasonable agreement with numerical simulations in real quenched networks [11,12,13,14,17,22].

Recently, a generalized formalism for the class of heterogeneous stochastic-copying voter-like models on networks has been proposed [23], in which the process is identified by the copying rate C_{ij}, encoding the full structure of the contact network and the stochastic update rules, and defined as the rate at which vertex i in the network copies the state of vertex j. Thus, for example, the standard voter model corresponds to the choice $C_{ij} = a_{ij}/[Nk_i]$ with $k_i = \sum_r a_{ir}$, where a_{ij} is the adjacency matrix of the network and N the network size. The Moran process, analogously, is given by $C_{ij} = a_{ij}/[Nk_j]$. Within this formalism, it has been shown that both the exit probability and the consensus time can be calculated exactly from the knowledge of the spectral properties of the matrix C_{ij}, provided that certain general conditions are met [23,24], a result that lays the foundation for a mathematical understanding of general copying processes and their mapping to particle-reaction systems. Despite the fact that the formalism in Refs. [23,25] is exact, and provides in some cases more accurate results than HMF theory, it is still useful to consider general stochastic-copying models from the perspective of HMF theory. In this framework, indeed, approximate analytical results can be obtained when the exact solution would be hard to work out in practice. For example, in realistic heterogeneous environments involving large numbers of agents, explicit expressions for C_{ij} might not be readily accessible. Moreover, the spectral properties of the copying rates are in general non-trivial to obtain, unless the matrix C_{ij} has a relatively simple form.

In this paper we pursue this path, proposing a generalized coarse-grained voter-like model on networks and showing how the HMF approach allows us to obtain very simple estimates for central properties such as the exit probability and the consensus time. We check the validity of our approach by considering a simple example of opinion dynamics in a homophilic society, in which vertices with similar degree are more prone to interact than vertices with differing degree.

2 Generalized voter model on networks

Inspired by Ref. [23], we consider a stochastic model on networks defined in terms of a heterogeneous voter model as follows:

- Each vertex i is endowed with a given fitness f_i [12].
- A source vertex i is selected at random, with a probability $f_i/\sum_j f_j$, i.e., proportional to its fitness f_i.
- A nearest neighbor j of i is then selected at random.
- With probability Q_{ij}, i copies the state of vertex j. Otherwise, nothing happens.

With these settings, the microscopic copying rate C_{ij}, as considered in Ref. [23] will be given by

$$C_{ij} = \frac{f_i}{\sum_j f_j}\frac{a_{ij}}{k_i}Q_{ij}. \tag{1}$$

In the spirit of the HMF approximation, we can replace the microscopic copying rate by its degree class average. The quantities f_i and Q_{ij} are simply coarse-grained by averaging them over the set of vertices with a given fixed degree, i.e.

$$f_i \to \frac{1}{NP(k)}\sum_{i\in k} f_i \equiv f_k, \tag{2}$$

$$Q_{ij} \to \frac{1}{NP(k)}\frac{1}{NP(k')}\sum_{i\in k}\sum_{i\in k'} Q_{ij} \equiv Q(k,k'), \tag{3}$$

where $i \in k$ denotes a sum over the degree class k and $P(k)$ is the network's degree distribution. For the term concerning the random choice of a nearest neighbor, we follow Ref. [26] to substitute

$$\frac{a_{ij}}{k_i} \to \frac{[NP(k)]^{-1}\sum_{i\in k}\sum_{j\in k'} a_{ij}}{[NP(k)]^{-1}\sum_{i\in k}\sum_r a_{ir}} \equiv P(k'|k). \tag{4}$$

At the coarse-grained degree level, our generalized voter model is thus defined in terms of the mesoscopic copying rate

$$C(k,k') \equiv \frac{f(k)}{\langle f(k)\rangle}P(k'|k)Q(k,k'), \tag{5}$$

where the function $f(k)$ comes with its proper normalization factor.

3 Heterogeneous mean-field solution

In the HMF approach, ordering processes are quantified by studying the evolution of the density of vertices of degree k in the state $+1$, x_k. In order to determine the rate equation satisfied by these quantities [11,13,17], we consider the probability $\Pi(k;\sigma)$ that a spin in state σ at a vertex of degree k flips its value to $-\sigma$ in a microscopic time step. From the definition of the generalized voter model, this probabilities can be simply written as

$$\Pi(k;\sigma) = \left[\frac{1-\sigma}{2} + \sigma x_k\right]\sum_{k'}\left[\frac{1+\sigma}{2} - \sigma x_{k'}\right]P(k)C(k,k').$$

From the previous expression, the rate equation for x_k can be written as [17]

$$\dot{x}_k = \frac{\Pi(k;-1) - \Pi(k;+1)}{P(k)} \equiv \sum_{k'} C(k,k')(x_{k'} - x_k). \quad (6)$$

The model just posed is still too hard to solve even in the HMF approximation. In the following, we will therefore make two major simplifying assumptions leading to analytically solvable HMF equations: (*i*) Dynamics proceed on uncorrelated networks, i.e. [8]

$$P(k'|k) = \frac{k'P(k')}{\langle k \rangle}; \quad (7)$$

and (*ii*) the interaction probability can be factorized as

$$Q(k,k') = a(k)b(k')s(k,k'), \quad (8)$$

where $s(k,k')$ is any symmetric function of k and k'. As we will see, this simplified form, which permits an analytic HMF solution, still allows for a vastly rich phenomenology, encompassing all voter-like models on complex networks previously proposed. Under this conditions, defining

$$u(k) = \frac{a(k)f(k)}{\langle f(k) \rangle}, \qquad v(k') = \frac{b(k')k'}{\langle k \rangle}, \quad (9)$$

the HMF rate equation can be written as

$$\dot{x}_k = \sum_{k'} P(k')\Gamma(k,k')(x_{k'} - x_k) \quad (10)$$

where $\Gamma(k,k') = u(k)v(k')s(k,k')$,

The HMF analysis proceeds by first determining the corresponding conservation laws [11,12,13]. Conserved quantities for the generalized process in Eq. (10) can be calculated as follows: We define a generic integral of motion $\omega[x_k(t)]$ such that $d\omega/dt = 0$. By definition of time derivative, we have

$$\frac{d\omega}{dt} = \nabla_{\mathbf{x}}\omega \cdot \dot{\mathbf{x}} = \sum_k \frac{\partial \omega}{\partial x_k}\dot{x}_k = 0. \quad (11)$$

In analogy with previous results [11,12,13], we look for conserved quantities that are linear in x_k imposing $\partial\omega/\partial x_k = z_k$ independent of x_k, so that conserved quantities will be given by

$$\omega = \mathbf{z} \cdot \mathbf{x} = \sum_k z_k x_k, \quad (12)$$

where z_k is any solution of $\sum_k z_k \dot{x}_k = 0$ and $\dot{x}_k$ is given by Eq. (10). Considering the explicit form of Eq. (10), the choice $z_k \propto P(k)v(k)/u(k)$ always satisfies the above condition, so that a conserved quantity is found up to multiplicative factors and additive constants. We choose the normalization $\sum_k z_k = 1$, such that the conserved quantity is defined as

$$\omega = \mathbf{z} \cdot \mathbf{x} = \frac{\langle v(k)/u(k)\, x_k \rangle}{\langle v(k)/u(k) \rangle}. \quad (13)$$

As for the usual voter model [11] the conservation law allows the immediate determination of the exit probability E, i.e. the probability that the final state corresponds to all spins in the state $+1$. In the final state with all $+1$ spins we have $\omega = 1$, while $\omega = 0$ is the other possible final state (all -1 spins). Conservation of ω implies then $\omega = E \cdot 1 + [1 - E] \cdot 0$, hence

$$E = \omega = \frac{\langle v(k)/u(k)\, x_k \rangle}{\langle v(k)/u(k) \rangle}. \quad (14)$$

Starting from a homogeneous initial condition, with a given density x of randomly chosen vertices in the state $+1$, we obtain, since $\omega = x$,

$$E_h(x) = x, \quad (15)$$

completely independent of the defining functions a, b, and s, and taking the same form as the standard voter model [1]. On the other hand, with initial conditions consisting of a single $+1$ spin in a vertex of degree k, we have

$$E_1(k) = \frac{v(k)/u(k)}{N\langle v(k)/u(k) \rangle}, \quad (16)$$

which does not depend on the functional form of the symmetric interaction term $s(k,k')$.

By looking at Eq. (10), every choice of x_k constant in k is a solution to the steady state condition $\dot{x}_k = 0$. We can prove that this solution is unique and does not depend on initial conditions if the square matrix $P(k')\Gamma(k,k')$ is irreducible and primitive (it certainly is when working with positive rates, which we will do in the following) [27]. We shall call the solution for the steady state $x_k(t \to \infty) = x^\infty$. Then it is easy to prove that

$$\omega = \sum_{k'} z_{k'} x_{k'} = x^\infty. \quad (17)$$

that is, even in this general case, the steady state value for x_k equals the conserved quantity.

After a rapid exponential convergence to the steady state distribution, the systems starts fluctuating diffusively around this value, until consensus is reached. Such fluctuations characterize finite systems and occur at long time scales, making such two-step relaxation process possible in most cases. The average consensus time $T_N(\mathbf{x})$ for a system in a generic state $\mathbf{x}$ can be derived extending the well known recursive method to our general case [13]. At a given time t, $T_N(\mathbf{x})$ must equal the average consensus time at time $t + \Delta t$ plus the elapsed time $\Delta t = 1/N$ that is, in our notation,

$$T_N(\mathbf{x}) = \bar{\Pi}\, T_N(\mathbf{x}) + \sum_{k,s} \Pi(k;s) T_N(\mathbf{x} + \mathbf{\Delta x}^{(k)}) + \Delta t, \quad (18)$$

where $\bar{\Pi} = 1 - \sum_{k,s} \Pi(k;s)$ is the probability that no state change occurs, while the sum is the weighted average over possible state-updates $\mathbf{x} \to \mathbf{x} + \mathbf{\Delta x}^{(k)}$. The variation $\mathbf{\Delta x}^{(k)}$ is a vector whose all components are zero except for

Model	$f(k)$	$a(k)$	$b(k)$	$s(k,k')$
Voter model [3]	1	1	1	1
Moran process [4]	1	k	k^{-1}	1
Link update [10]	k	1	1	1
Voter weighted [17]	1	1	$\frac{g_s(k)\langle k\rangle}{\langle kg_s(k)\rangle}$	1
Moran weighted [17]	1	$\frac{kg_s(k)}{\langle kg_s(k)\rangle}$	$\frac{\langle k\rangle}{k}$	1
Generalized Voter [14]	1	1	$k^{\alpha-1}$	$\frac{k+k'}{k^\alpha+k'^\alpha}$

Table 1. Summary of the mapping of voter-like models to the present formalism.

the k-th, which equals the update-unit $\Delta_k = [NP(k)]^{-1}$. Expanding to second order in Δ_k, taking $x_k = \omega$ as the initial state and changing variables such that $\partial/\partial x_k = z_k \partial/\partial\omega$ we obtain the backward Kolmogorov equation

$$-1 = \frac{\mathbf{z}^T \Gamma \mathbf{z}}{N}\omega(1-\omega)\frac{\partial^2 T_N}{\partial\omega^2} \tag{19}$$

leading to

$$T_N(\omega) = -N_{\text{eff}}[\omega\ln\omega + (1-\omega)\ln(1-\omega)] \tag{20}$$

where we have defined the effective system size $N_{\text{eff}} = N/\sum_{k,k'} z_k \Gamma(k,k') z_{k'}$, which, in the case of generalized voter dynamics, Eq. (9), becomes

$$N_{\text{eff}} = N\frac{\langle f(k)\rangle\langle k\rangle\left\langle\frac{kb(k)}{f(k)a(k)}\right\rangle^2}{\left\langle\left\langle s(k,k')kb(k)\frac{[k'b(k')]^2}{f(k')a(k')}\right\rangle\right\rangle}, \tag{21}$$

with $\langle\langle\cdot\rangle\rangle = \sum_{kk'} P(k)P(k')(\cdot)$.

Equations (20) and (21), together with the expression for the exit probability, Eq. (14), represent the final HMF solution of the generalized voter model. From these formulas it is easy to recover most of the variations of the voter model considered in the past. For example, the standard voter model is obviously recovered for $a(k) = b(k) = f(k) = s(k,k') = 1$. The invasion process [28,12], also known as the Moran process in the evolutionary literature [4,29], corresponds to $a(k) = k$, $b(k) = k^{-1}$ and $f(k) = s(k,k') = 1$. Link update dynamics [10] is recovered for $a(k) = b(k) = s(k,k') = 1$ and $f(k) = k$. The voter and Moran processes on weighted networks characterized by a symmetric weight between vertices of degree k and k' proportional to $g_s(k)g_s(k')$ [17] are reproduced by imposing $s(k,k') = 1$ and setting $a(k) = f(k) = 1$, $b(k) = g_s(k)\langle k\rangle/\langle kg_s(k)\rangle$ and $f(k) = 1$, $a(k) = kg_s(k)/\langle kg_s(k)\rangle$, $b(k) = \langle k\rangle/k$, respectively. Finally, an HMF implementation of the generalized voter dynamics proposed in [14] is recovered imposing $f(k) = a(k) = 1$, $b(k) = k^{\alpha-1}$ and $s(k,k') = (k+k')/(k^\alpha + k'^\alpha)$. A summary of the above mappings is presented in Table 1.

4 Numerical analysis

In order to show an application of our formalism, we examine a toy model to study the effects that homophily in social networks might have in opinion formation dynamics. Homophily is broadly defined by the tendency of people to interact with similar people [30]. In the absence of information beyond the topological structure of the contact network, the simplest assumption we can make is that homophily is driven by an increased tendency of individuals to copy other individuals with a degree that is not too different from their own. As a simple representation of homophilic behavior we consider the case $s(k,k') = \exp[-R(k-k')/\xi^2]$ and $f(k) = a(k) = b(k) = 1$. Here $R(x)$ is any continuous even function with $R(0) = 0$ and a minimum in $x = 0$, i.e. $R'(0) = 0$ and $R''(0) > 0$. The parameter ξ measures the amplitude of stochastic fluctuations around ideal homophily: For $\xi \approx 1$ the probability of copying neighbors with different degrees is strongly suppressed; for $\xi \gg 1$ fluctuations take over and simple voter behavior is rapidly recovered.

In spite of its apparent simplicity, this problem would be impossible to solve in a realistic network by standard techniques [23]. By applying the HMF result in Eq. (21) instead, we readily find

$$N_{\text{eff}}^{\xi} = N\frac{[\sum_k kP(k)]^3}{\sum_k\sum_{k'} kP(k)k'^2P(k')\exp\left[-\frac{R(k-k')}{\xi^2}\right]}, \tag{22}$$

where we consider complex networks with a scale-free degree distribution $P(k) \sim k^{-\gamma}$, with $\gamma \in]2,3]$, minimum degree m and maximum degree k_c. In the limit of small ξ the denominator of Eq. (22) can be evaluated in the continuous-degree approximation. By applying the Laplace method for the first term in the asymptotic expansion, after an adequate change of variables, it is thus straightforward to find that for small ξ

$$N_{\text{eff}}^{\xi} \simeq \frac{N}{\xi}\left[\frac{R''(0)}{2\pi}\right]^{\frac{1}{2}} \times \\ \times \frac{2(1-\gamma)}{(2-\gamma)^2}\frac{(k_c^{2-\gamma} - m^{2-\gamma})^2}{(k_c^{1-\gamma} - m^{1-\gamma})(k_c^{2-\gamma} + m^{2-\gamma})}, \tag{23}$$

whereas in the opposite limit $\xi \to \infty$ one recovers the simple voter model result, which in our notations reads $N_{\text{eff}}^{\infty} \sim Nk_c^{\gamma-3}$. As a consequence, the consensus time $T_N \propto N_{\text{eff}}^{\xi}$ diverges for small ξ as ξ^{-1} as the selectivity amplitude ξ approaches zero. For increasing ξ, instead, T_N decreases and asymptotically crosses over to a plateau, where simple voter behavior is recovered.

In order to check explicitly the predictions of our formalism, we consider a Gaussian homophily model, given by the simplest choice $R(x) = x^2$. In Fig. 2 we plot the consensus times for homogeneous initial conditions as a function of ξ, computed from the numerical evaluation of Eq. (22). We observe that, starting from large values of ξ, T_N is constant, as expected for simple voter behavior. Upon decreasing the selectivity amplitude ξ, the consensus time starts increasing, asymptotically behaving as ξ^{-1}. As soon as ξ decreases below one, the discrete nature of the degree distribution takes over and T_N reaches a plateau. Interestingly, we find that T_N is an increasing function of

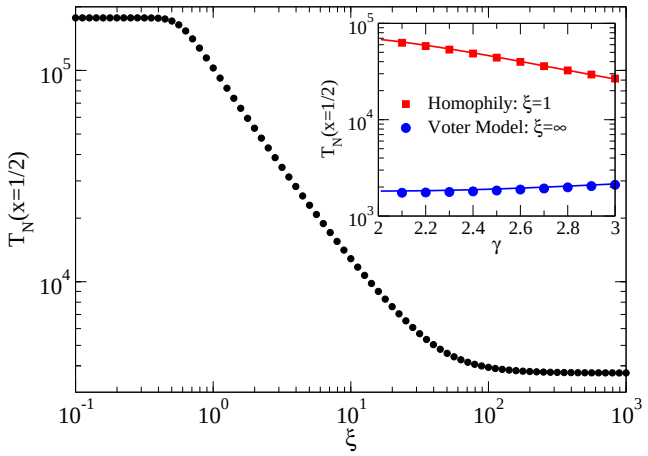

Fig. 1. Consensus time as a function of the selectivity range from numerical evaluation of Eq. (22). The case of homogeneous initial conditions $x_k(0) = 1/2$ (maximum entropy) is considered. Inset: Consensus time as a function of the degree distribution exponent. Numerical estimates (points) and analytical predictions (lines) based on $N_{\rm eff}^{\infty}$ and $N_{\rm eff}^{\xi}$ from the main text are compared.

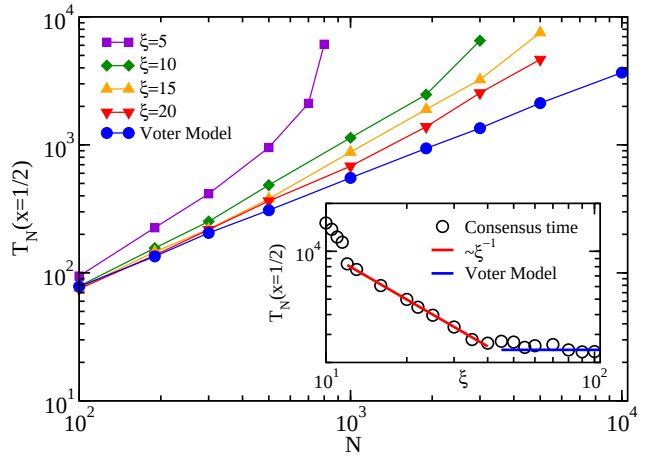

Fig. 2. Consensus time with a Gaussian homophily factor in scale-free networks with $\gamma = 2.5$, $m = 4$ and $k_c \sim N^{1/2}$. Main plot: Consensus time as a function of N, for different values of ξ. Inset: Consensus time at constant $N = 5000$ as a function of ξ. The full line is a guide to the eye with slope -1.

γ for large ξ, while it decreases with γ in the small ξ limit (Fig. 1, inset).

These observations have also been checked against direct numerical simulations of the Gaussian homophily model on uncorrelated scale-free networks, generated with the uncorrelated configuration model [31]. In Fig. 2 we show (main plot) the consensus time as a function of the network size N for different values of ξ. We can see that the overall plot increases at fixed N for decreasing values of ξ, while for large ξ it tends to the limit given by the simple voter model. When considering T_N as a function of ξ in a network of fixed size, inset of Fig. 2, we observe, as expected, an increase for decreasing ξ, followed by a plateau for large ξ. At intermediate values of ξ, a reasonable fit to the form ξ^{-1} can be obtained. For smaller values of ξ, the selectivity range is so narrow that quenched effects of the network topology take over, tending to slow down the dynamics beyond the HMF prediction [17]. Values of T_N fluctuate wildly and deviate from the predictions for annealed-network topologies.

5 Conclusion

In this paper we have studied a generalized voter-like model in the framework of the HMF theory. Remarkably, the HMF formalism we have adopted allows for a straightforward mapping of most of the voter-like models proposed in the past. Not only the the properly said voter model, the Moran process, and their weighted generalization can be reproduced in our formalism, but also the link-update dynamics and recently introduced generalizations of the voter model can be straightforwardly recovered. The HMF approach allows for predictions that are in fair agreement with the exact results whenever they are available, and most importantly provides approximate solutions in the event that exact methods are not viable.

As an instance of the versatility of the HMF approach, we have considered a simple example of a heterogeneous voter model in which vertices with similar degree tend to interact more often among themselves rather than with the rest of the network. This degree selectivity is a natural HMF implementation of the concept of homophily, i.e. the tendency of individuals to associate and bond with similar others. While an exact solution to such problem would be unworkable, we have shown that in this framework homophily ends to slow down consensus dynamics, in reasonable agreement with the predictions of HMF theory.

In the future, it would be interesting to extend the approach to voter-like methods that exhibit a surface tension, as for example the Naming Game [32], or the noise-reduced voter model [33], as well as to improve the quality of our present results by applying more sophisticated HMF approaches, such as the dynamical pair approximation [34], or the master equation approach [35].

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