FREE VIBRATION OF THICK RECTANGULAR ORTHOTROPIC PLATES WITH CLAMPED EDGES- USING ASYMPTOTIC ANALYSIS OF INFINITE SYSTEMS

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Abstract

Exact solutions for free vibration of thick rectangular orthotropic plates when their all edges are clamped are sought through asymptotic analysis of infinite systems without resorting to the usual truncation of series solution. The use of modified trigonometric functions made it possible to obtain a general solution for the problem which has the same form for all four cases of symmetry of the quarter plate. Thus, an infinite system of linear algebraic equations is derived for the unknown coefficients of the series representing the solution for each case. This is in sharp contrast to previous publications based on series-solution which does not allow the satisfaction of the quasi-regularity condition of the corresponding infinite system, and therefore, the method used earlier, was not amenable to asymptotic solution of the infinite system. In this investigation, the quasi-regularity of the infinite system is proved, but importantly, an algorithm for determining the natural frequencies of the plate based on the theorem of the existence of the solution for the quasi-regular system is presented. The asymptotic behaviour of the non-trivial solution of the homogeneous quasi-regular infinite system is ascertained by generalising the asymptotic law of Koialovich which essentially led to the development of the algorithm. Numerical examples are given with significant conclusions drawn.

***Keywords*:** thick orthotropic plate, free vibration, clamped edges, exact solution, infinite system of linear equations, natural modes

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1. Introduction

A large volume of publications on vibration of thick plates can be found in the literature because of their importance as building blocks in modelling engineering structures. However, it should be recognized that for a substantial majority of cases for rectangular plates, explicit analytical solutions in the form of trigonometric series have been presented in the literature only when the two opposite edges of the plates are simply supported. For other boundary conditions, variational and/or numerical methods have generally been used. In particular, the free vibration of thick isotropic plates with clamped edges within the framework of the seventy years old Mindlin plate theory [1] was analysed around four decades ago by Dawe and Roufaeil [2] who used the Rayleigh-Ritz method. Some years later, Liew et al. [3] developed the pb-2 Rayleigh-Ritz method to analyse the free vibration behaviour of thick plates with different boundary conditions. By contrast, Cheung and Zhou [4] used static beam functions as the basis functions when applying the Rayleigh-Ritz method for free vibration analysis of thick plates. Another related, but different variational approach, namely, the DSC Element Method, was proposed by Xiang et al. [5] to investigate such problems. With the growing interest in the dynamic stiffness method (DSM) for free vibration analysis of structures, in which the Wittrick-Williams algorithm is generally used as solution technique, the investigation of clamped ended natural frequencies of structural elements which is an essential part of the algorithm has become very important to ensure that no natural frequency of the structure is missed. Exact free vibration analysis using DSM allows an infinite number of natural frequencies to be accounted for when all the nodes of the structure or structural elements are fully clamped. This has provided the main motivation for the current research which focuses on the free vibration analysis of thick orthotropic plates with clamped edges. In the context of a bending-torsional coupled beam, a similar attempt was made by Banerjee and Williams [6] who published the theory and the computational procedure for the computation of clamped-clamped natural frequencies of such beams. They emphasised the need for the computation of clamped ended natural frequencies of structural elements when the DSM in conjunction with the Wittrick-Williams algorithm [7-13] is used in free vibration analysis of structures.

The vibration of a rectangular plate is often related to parametric optimization problems when analysing vibration resistant technical systems, and also when solving frequency attenuation problems for which the finite element method (FEM) is generally used. The FEM with sufficiently fine mesh allows us to express the dynamic characteristics of structure through the application of the approximating functions which are essentially shape functions or interpolating functions. With the increasing order of the natural frequency of vibrations, the number of elements needed as building blocks when representing and analysing the structure may become excessive. Because of this reason, in some industries, such as mechanical engineering and aerospace, the application of the FEM is sometimes limited to the low frequency range. Additionally, the classical FEM gives rather an obscure error in the case when the plate is significantly thick (the so-called shear-locking phenomenon [14]). In this respect, considerable efforts have been expended by many investigators to develop methods for thick rectangular plates within the framework of FEM to simulate the mechanical properties of the plate more accurately. Bathe and Dvorkin [15] developed a refined plate element (the so-called MITC element) based on the Ritz method which significantly enhanced the solution of the problem. Durán et al. [16] presented a detailed mathematical justification for the convergence of his method in solving the free vibration of thick plates including the case when all plate edges were clamped. On the other hand, Kolarevic et al [17] developed the dynamic stiffness method (DSM) to analyse the free vibration behaviour of an assembly of thick isotropic plates. Gorman's superposition method [18] to solve such problems is notably a significant contribution to the literature. Following the work of Kolarevic et al [17], the DSM has recently been proposed by Papkov and Banerjee [19] to deal with the free vibration behaviour of an assembly of thick orthotropic plates. In most of these approaches, approximations of some kind or other are evident, despite the analytical forms of the solution given by the investigators. It should be noted that the DSM solution, in contrast with the solution from variational approaches, is generally constructed from an infinite series of the solution of the governing differential equations. Of course, for practical purposes, only a finite number of terms are used for numerical implementation. Thus, the approximation in DSM solution arises from neglecting the remainders of the terms in the infinite series which represents solution.

In this current study, an exact solution for the problem of natural vibration of thick orthotropic plates with clamped edges is presented for the first time, using a novel approach. The boundary value problem is essentially reduced to an infinite system of linear algebraic equations. This is achieved by taking advantage of some aspects of the superposition method. A detailed investigation of the infinite system rooted in the solution is carried out which enabled the determination of the asymptotic behaviour of the unknown coefficients of the general solution which has not been attempted before. This novel procedure allowed the use of the untruncated infinite series solution of the problem and subsequent development of an algorithm to compute the natural frequencies of thick orthotropic plates with clamped edges and recover the corresponding mode shapes. The investigation carried out is particularly relevant when the dynamic stiffness method is used in free vibration analysis of structures by applying the Wittrick-Williams algorithm [7] as solution technique for which the number of clamped ended natural frequencies of structural elements that exists below an arbitrarily chosen trial frequency is an essential prerequisite. The algorithm has featured in literally hundreds of papers, see for example [7-13].

2. General solution of the governing differential equations and generation of the infinite systems

Let us consider a rectangular orthotropic plate $\{(x,y)\in[-a;a]\times[-b;b]\}$ with thickness h in a right-handed rectangular Cartesian coordinate system. The theory for free vibration analysis of thick plates was originally given by Mindlin [1] and subsequently the term Mindlin-plate was coined as it is well-known. The displacement field for a Mindlin-plate relative to its mid-surface in the usual notation is given by

$$\vec{\mathbf{u}} = \begin{pmatrix} u^0(x,y,t) + z\phi_x^0(x,y,t) \\ v^0(x,y,t) + z\phi_y^0(x,y,t) \\ w^0(x,y,t) \end{pmatrix} \quad (1)$$

Equation (1) leads to the derivation of the governing differential equations of motion for thick plates in free vibration which can be found in numerous papers and in some specialised texts. Essentially the derivation constitutes a system of three partial differential equations with respect to three functional variables, namely, the mid-plane plate deflection W^0 and the angles of rotation of the mid-plane ϕ_x^0, ϕ_y^0 about the X and Y axes, respectively. For harmonic oscillation, one can assume $\phi_x^0(x,y,t) = \phi_x(x,y)\mathrm{e}^{i\omega t}$, $\phi_y^0(x,y,t) = \phi_y(x,y)\mathrm{e}^{i\omega t}$ and $w^0(x,y,t) = W(x,y)\mathrm{e}^{i\omega t}$ in which case the time dependent terms can be eliminated and the equations of motion in free vibration become

$$\frac{\partial^2 \phi_x}{\partial x^2} + k_6 \frac{\partial^2 \phi_x}{\partial y^2} + \tilde{k}\frac{\partial^2 \phi_y}{\partial x \partial y} - k_4\left(\frac{\partial W}{\partial x} + \phi_x\right) + \Omega_h^4 \phi_x = 0 \quad (2)$$

$$k_2 \frac{\partial^2 \phi_y}{\partial y^2} + k_6 \frac{\partial^2 \phi_y}{\partial x^2} + \tilde{k}\frac{\partial^2 \phi_x}{\partial x \partial y} - k_5\left(\frac{\partial W}{\partial y} + \phi_y\right) + \Omega_h^4 \phi_y = 0 \quad (3)$$

$$k_4\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + k_5\left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial \phi_y}{\partial y}\right) + \Omega^4 W = 0 \quad (4)$$

where $\Omega^4 = \frac{\rho h \omega^2}{D_1} a^4$, $\Omega_h^4 = \frac{h^2}{12 a^4}\Omega^4$ are the frequency parameters and $k_2 = \frac{D_2}{D_1}$, $k_4 = \frac{A_{44}}{D_1} a^4$, $k_5 = \frac{A_{55}}{D_1} a^4$, $k_6 = \frac{D_{66}}{D_1}$, $\tilde{k} = \frac{D_{66} + D_{12}}{D_1}$, κ is the shear correction factor, $D_1, D_2, D_{12}, D_{66}, A_{44}$ are elastic constants usually obtained from the classical lamination theory that are related to the material properties $E_1, E_2, G_{12}, \nu_1, \nu_2$ as follows:

$$D_1 = \frac{h^3 E_1}{12(1-\nu_1\nu_2)};\ D_2 = \frac{h^3 E_2}{12(1-\nu_1\nu_2)};\ D_{12} = \frac{h^3 \nu_1 E_2}{12(1-\nu_1\nu_2)};$$

$$D_{66} = \frac{h^3 G_{12}}{12};\ A_{44} = \kappa h G_{12};\ A_{55} = \kappa h G_{12}. \quad (5)$$

It should be noted that according to Betty's principle and Maxwell's reciprocal theorem $E_2\nu_1 = E_1\nu_2$, i.e. the plate material properties can be described by only four elastic constants.

For clamped edges, the boundary conditions can be prescribed as

$$x = \pm a:\quad W = \phi_x = \phi_y = 0 \quad (6)$$

$$y = \pm b:\quad W = \phi_x = \phi_y = 0 \quad (7)$$

The approach used here to obtain the general solution is similar, but not the same as the one proposed in [19] because the selected displacement functions are different. The procedure is briefly summarised below. Essentially, the general solution of Eqs. (2) - (4) can be represented as a sum of even and odd components of W, ϕ_x, ϕ_y, denoted W_{kj}, $\phi_{x,kj}$, $\phi_{y,kj}$ with k and j being 0 and 1 as follows

$$\begin{pmatrix} W \\ \phi_x \\ \phi_y \end{pmatrix} = \sum_{k,j=0}^{1} \begin{pmatrix} W_{kj} \\ \phi_{x,kj} \\ \phi_{y,kj} \end{pmatrix} \tag{8}$$

Thus, in Eq. (8), W_{00} is an even function of both X and Y coordinates, W_{01} is an even function of the coordinate X, but an odd function of the coordinate Y, etc. Furthermore, the indices k and j denote the symmetry with respect to the X and Y axes, respectively, so that an index '0' denotes an even function and '1' denotes an odd function.

A formal solution for the differential equations, i.e. Eqs. (2) – (4), can be obtained by the method of separation of variables for each of the four cases of symmetry defined by (k, j), i.e. symmetric-symmetric (0, 0), symmetric-antisymmetric (0, 1), antisymmetric-symmetric (1, 0) and antisymmetric-antisymmetric (1, 1) and then summing up the solutions for all of the four individual cases. To achieve this objective, two trigonometric series H_k , T_j and H_j , T_k which are defined and explained later, with undefined coefficients X_{ln} and Y_{ln} are proposed as follows

$$W_{kj} = \sum_{n=1}^{\infty}\sum_{l=1}^{3} X_{ln} A_{l,nk} H_j(p_{l,nk}y) T_k(\alpha_{nk}x) + \sum_{n=1}^{\infty}\sum_{l=1}^{3} Y_{ln} E_{l,nj} H_k(q_{l,nj}x) T_j(\beta_{nj}y) \tag{9}$$

$$\phi_{x,kj} = \sum_{n=1}^{\infty}\sum_{l=1}^{3} X_{ln} B_{l,nk} H_j(p_{l,nk}y) T'_k(\alpha_{nk}x) + \sum_{n=1}^{\infty}\sum_{l=1}^{3} Y_{ln} H'_k(q_{l,nj}x) T_j(\beta_{nj}y) \tag{10}$$

$$\phi_{y,kj} = \sum_{n=1}^{\infty}\sum_{l=1}^{3} X_{ln} H'_j(p_{l,nk}y) T_k(\alpha_{nk}x) + \sum_{n=1}^{\infty}\sum_{l=1}^{3} Y_{ln} F_{l,nj} H_k(q_{l,nj}x) T'_j(\beta_{nj}y) \tag{11}$$

where $\alpha_{nk} = \frac{\pi}{a}\left(n + \frac{k-1}{2}\right)$, $\beta_{nj} = \frac{\pi}{b}\left(n + \frac{j-1}{2}\right)$ are separation constants; $p^2_{l,nk}$ and $q^2_{l,nj}$ are the branches (l= 1, 2, 3) of the roots the following characteristic equations

$$c_0 p^6 + c_{1n} p^4 + c_{2n} p^2 + c_{3n} = 0 \tag{12}$$

$$d_0 q^6 + d_{1n} q^4 + d_{2n} q^2 + d_{3n} = 0 \tag{13}$$

It should be noted that although there are similarities in the approach used here with that used in Ref. [19] leading to the general solution (see Eqs. (9) – (11)) there are significant differences between the two approaches in the representation of displacement functions. In particular Eqs. (9) – (11) here represent expansions of plate displacements with respect to the trigonometric functions $\cos\frac{\pi z}{h}\left(n-\frac{1}{2}\right)$ and $\sin\frac{\pi z n}{h}$ when $z\in[-h;h]$, whereas in [19] the expansions used for plate displacements were based on the trigonometric functions $\cos\frac{\pi z n}{h}$ and $\sin\frac{\pi z}{h}\left(n-\frac{1}{2}\right)$. The former representation allows the construction of quasi-regular infinite systems for unknown coefficients leading to an exact solution for the boundary value problem. The latter representation used in [19] does not permit such construction.

The roots $p_{l,nk}$ and $q_{l,nj}$ of the characteristic equations (12) – (13) are related to $A_{l,nk}$, $B_{l,nk}$, $E_{l,nj}$, $F_{l,nj}$ of Eqs. (9)-(11), which can be expressed as

$$A_{l,nk}=\frac{\tilde{k}k_5p_{l,nk}^2-k_4\left(k_2p_{l,nk}^2-k_6\alpha_{nk}^2-k_5+\Omega_h^4\right)}{C_{l,nk}}, \tag{14}$$

$$B_{l,nk}=\frac{k_5^2p_{l,nk}^2+\left(k_2p_{l,nk}^2-k_6\alpha_{nk}^2-k_5+\Omega_h^4\right)\left(k_5p_{l,nk}^2-k_4\alpha_{nk}^2+\Omega^4\right)}{\alpha_{nk}C_{l,nk}}, \tag{15}$$

$$E_{l,nj}=\frac{\tilde{k}k_4q_{l,nj}^2-k_5\left(q_{l,nj}^2-k_6\beta_{nj}^2-k_4+\Omega_h^4\right)}{G_{l,nj}}, \tag{16}$$

$$F_{l,nj}=\frac{k_4^2q_{l,nj}^2+\left(q_{l,nj}^2-k_6\beta_{nj}^2-k_4+\Omega_h^4\right)\left(k_4q_{l,nj}^2-k_5\beta_{nj}^2+\Omega^4\right)}{\beta_{nj}G_{l,nj}}. \tag{17}$$

where $C_{l,nk}=p_{l,nk}\left(\tilde{k}k_5p_{l,nk}^2-\tilde{k}k_4\alpha_{nk}^2+k_4k_5+\tilde{k}\Omega^4\right);G_{l,nj}=q_{l,nj}\left(\tilde{k}k_4q_{l,nj}^2-\tilde{k}k_5\beta_{nj}^2+k_4k_5+\tilde{k}\Omega^4\right)$ (18)

Clearly the coefficients of Eqs. (12) and (13) are dependent on the separation constants α_{nk} and β_{nj}, the elastic constants, thickness of plate and the frequency parameters. The sixth order polynomial equations of Eqs. (12) and (13) can be transformed into cubic equations whose roots can be found using standard procedure. The procedure is facilitated by the application of Viet's theorem [20] , see Appendix A.

In Eqs. (9)-(11), the following notations for trigonometric and hyperbolic functions are used depending on the type of symmetry

$$H_0=\cosh(z)\quad H_1=\sinh(z),\ T_0=\cos(z),\ T_1=\sin(z). \tag{19}$$

The choice of the notation given by Eq. (19) and its usefulness can be explained by the fact that the hyperbolic functions and trigonometric series provide the following relationships for different types of symmetry [21]

$$\frac{H_k(qx)}{H_k(qa)}=\frac{2}{a}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\alpha_{mk}T_k(\alpha_{mk}x)}{\alpha_{mk}^2+q^2},\ \frac{H_j(py)}{H_j(pb)}=\frac{2}{b}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\beta_{mj}T_j(\beta_{mj}y)}{\beta_{mj}^2+p^2} \tag{20}$$

In this way, the notation given in Eqs. (20) allows one to have consistent uniformity of relationship for all four types of symmetry of the quarter plate.

For the chosen trigonometric system of functions, the following identity will be always true

$$T_k(\alpha_{nk}a) = T_j(\beta_{nj}b) = 0 \tag{21}$$

Furthermore, the boundary conditions $W(\pm a, y) = W(x, \pm b) = \phi_y(\pm a, y) = \phi_x(x, \pm b) = 0$ corresponding to all clamped edges of the plate can be satisfied exactly by asserting that the unknown coefficients of the series given by Eqs. (9) - (11) are related as follow

$$\left.\begin{aligned} X_{2n} &= \frac{A_{3,nk}B_{1,nk} - A_{1,nk}B_{3,nk}}{A_{2,nk}B_{3,nk} - A_{3,nk}B_{2,nk}}\frac{H_j(p_{1,nk}b)}{H_j(p_{2,nk}b)}X_{1n}, \quad X_{3n} = \frac{A_{1,nk}B_{2,nk} - A_{2,nk}B_{1,nk}}{A_{2,nk}B_{3,nk} - A_{3,nk}B_{2,nk}}\frac{H_j(p_{1,nk}b)}{H_j(p_{3,nk}b)}X_{1n} \\ Y_{2n} &= \frac{E_{3,nj}F_{1,nj} - E_{1,nj}F_{3,nj}}{E_{2,nj}F_{3,nj} - E_{3,nj}F_{2,nj}}\frac{H_k(q_{1,nj}a)}{H_k(q_{2,nj}a)}Y_{1n}, \quad Y_{3n} = \frac{E_{1,nj}F_{2,nj} - E_{2,nj}F_{1,nj}}{E_{2,nj}F_{3,nj} - E_{3,nj}F_{2,nj}}\frac{H_k(q_{1,nj}a)}{H_k(q_{3,nj}a)}Y_{1n} \end{aligned}\right\} \tag{22}$$

Substituting the expressions of Eqs. (10)- (11) into the remaining two boundary conditions $\phi_y(x, \pm b) = \phi_x(\pm a, y) = 0$, other than $W(\pm a, y) = W(x, \pm b) = \phi_y(\pm a, y) = \phi_x(x, \pm b) = 0$,and using the expansion of hyperbolic functions in terms of the system of trigonometric functions given by Eqs. (20), and after rearranging the order of summation on the left-hand side of the equalities, we obtain an infinite system of linear algebraic equations from the equality to zero of the coefficients at $\{T_k(\alpha_{nk}x)\}$ and $\{T_j(\beta_{nj}y)\}$ as shown below

$$\left.\begin{aligned} x_m &= \frac{2\alpha_{mk}}{\Delta_{1m}}\sum_{n=1}^{\infty}\frac{\alpha_{mk}^2\xi_n^x + \eta_n^x}{(\beta_{nj}^2 + p_{1,mk}^2)(\beta_{nj}^2 + p_{2,mk}^2)(\beta_{nj}^2 + p_{3,mk}^2)}y_n \\ y_m &= \frac{2\beta_{mj}}{\Delta_{2m}}\sum_{n=1}^{\infty}\frac{\beta_{mj}^2\xi_n^y + \eta_n^y}{(\alpha_{nk}^2 + q_{1,mj}^2)(\alpha_{nk}^2 + q_{2,mj}^2)(\alpha_{nk}^2 + q_{3,mj}^2)}x_n \end{aligned} \quad (m = 1,2,\ldots)\right\} \tag{23}$$

where

$$\left.\begin{aligned} & x_m = \frac{(-1)^m k_2 k_5}{A_{2,mk}B_{3,mk} - A_{3,mk}B_{2,mk}}H_j(p_{1,mk}b)X_{1m}, \; y_m = \frac{(-1)^m k_4}{E_{2,mj}F_{3,mj} - E_{3,mj}F_{2,mj}}H_k(q_{1,mj}a)Y_{1m}; \\ & \xi_n^x = F_{1,nj}E_{2,nj}F_{3,nj}(q_{3,nj}^2 - q_{1,nj}^2) + F_{1,nj}E_{3,nj}F_{2,nj}(q_{1,nj}^2 - q_{2,nj}^2) + F_{2,nj}E_{1,nj}F_{3,nj}(q_{2,nj}^2 - q_{3,nj}^2); \\ & \eta_n^x = F_{1,nj}E_{2,nj}F_{3,nj}q_{2,nj}^2(q_{3,nj}^2 - q_{1,nj}^2) + F_{1,nj}E_{3,nj}F_{2,nj}q_{3,nj}^2(q_{1,nj}^2 - q_{2,nj}^2) + F_{2,nj}E_{1,nj}F_{3,nj}q_{1,nj}^2(q_{2,nj}^2 - q_{3,nj}^2); \\ & \xi_n^y = B_{1,nj}A_{2,nj}B_{3,nj}(p_{3,nk}^2 - p_{1,nk}^2) + B_{1,nj}A_{3,nj}B_{2,nj}(p_{1,nk}^2 - p_{2,nk}^2) + B_{2,nj}A_{1,nj}B_{3,nj}(p_{2,nk}^2 - p_{3,nk}^2); \\ & \eta_n^y = B_{1,nj}A_{2,nj}B_{3,nj}p_{2,nk}^2(p_{3,nk}^2 - p_{1,nk}^2) + B_{1,nj}A_{3,nj}B_{2,nj}p_{3,nk}^2(p_{1,nk}^2 - p_{2,nk}^2) + B_{2,nj}A_{1,nj}B_{3,nj}p_{1,nk}^2(p_{2,nk}^2 - p_{3,nk}^2); \\ & \Delta_{1m}/a = (A_{2,mk}B_{3,mk} - A_{3,mk}B_{2,mk})\mathrm{Cth}_j(p_{1,mk}b) + (A_{3,mk}B_{1,mk} - A_{1,mk}B_{3,mk})\mathrm{Cth}_j(p_{2,mk}b) + \\ & + (A_{1,mk}B_{2,mk} - A_{2,mk}B_{1,mk})\mathrm{Cth}_j(p_{3,mk}b); \\ & \Delta_{2m}/b = (E_{2,mj}F_{3,mj} - E_{3,mj}F_{2,mj})\mathrm{Cth}_k(q_{1,mj}a) + (E_{3,mj}F_{1,mj} - E_{1,mj}F_{3,mj})\mathrm{Cth}_k(q_{2,mj}a) + \\ & + (E_{1,mj}F_{2,mj} - E_{2,mj}F_{1,mj})\mathrm{Cth}_k(q_{3,mj}a); \end{aligned}\right\} \tag{24}$$

with

$$\mathrm{Cth}_j(z) = \frac{H_j'(z)}{H_j(z)} \cdot \tag{25}$$

It should be noted that when deriving the infinite system of equations, see Eqs. (23), the following identity arising from Vieta's theorem [20] for the roots of characteristic equations of Eqs. (12)-(13) was used

$$k_5 k_2 (\alpha_{mk}^2 + q_{1,nj}^2)(\alpha_{mk}^2 + q_{2,nj}^2)(\alpha_{mk}^2 + q_{3,nj}^2) = k_4 (\beta_{nj}^2 + p_{1,mk}^2)(\beta_{nj}^2 + p_{2,mk}^2)(\beta_{nj}^2 + p_{3,mk}^2) \tag{26}$$

The infinite system of Eqs. (23) can now be used for an approximate, but sufficiently accurate computation of the eigenfrequencies and mode shapes of thick orthotropic Mindlin-plates by using well-established and standard reduction methods as follows. (It should be noted that the results can be computed to any desired accuracy.) The first N equations of the system are solved relatively in terms of the first N unknowns, while setting the other unknowns to zero. Then the determinant of the reduced finite system is assumed to be an approximate dispersion equation. However, for a more detailed and accurate analysis of the infinite system, its regularity, and its asymptotic behavior for the solution, the system needs to be transformed into a more convenient form as explained below.

First, in order to evaluate the regularity of the system, it is necessary to sum the series of the coefficients of the infinite system analytically. Quite obviously, this cannot be accomplished easily if the infinite system is in the form of Eqs. (23). This is because the expressions for ξ_n^x, η_n^x, ξ_n^y and η_n^y include the roots of the bicubic equations Eqs. (12) - (13). To overcome this difficulty, the expressions for ξ_n^x, η_n^x, ξ_n^y and η_n^y are transformed algebraically with help of Vieta's theorem [20] and then making use of the roots of the characteristic equations (see Appendix A). By substituting Eqs. (14) – (15) for $A_{l,nk}$ and $B_{l,nk}$ we can write after collecting terms, the expression for ξ_n^y as follows

$$\begin{aligned}\xi_n^y = {} & \frac{k_5 (p_{1,nk}^2 - p_{2,nk}^2)(p_{1,nk}^2 - p_{3,nk}^2)(p_{2,nk}^2 - p_{3,nk}^2)}{\alpha_{nk}^2 C_{1,nk} C_{2,nk} C_{3,nk}} \left(\frac{c_{3n}}{c_0} k_2^2 k_5 (k_4 k_2 - k_5 \widetilde{k}) + \left(\frac{c_{2n}}{c_0} k_2^2 k_4 k_5 + \right.\right. \\ & + k_2 \left(k_4^2 \alpha_{nk}^2 (\widetilde{k} \alpha_{nk}^2 - k_5) + \widetilde{k} \Omega^8 + k_4 \Omega^4 (k_5 - 2\widetilde{k} \alpha_{nk}^2) + \frac{c_{1n}}{c_0} k_5 \left(k_4 \left(\alpha_{nk}^2 (k_6 + \widetilde{k}) - \Omega_h^4 \right) - \widetilde{k} \Omega^4 \right) \right) + \\ & \left.\left. + k_5 \left(k_4 \left(\alpha_{nk}^2 (k_6 + \widetilde{k}) - \Omega_h^4 \right) - \widetilde{k} \Omega^4 \right) \right) \left(k_6 \alpha_{nk}^2 - \Omega_h^4 \right) \left(k_6 \alpha_{nk}^2 - \Omega_h^4 + k_5 \right) \right)\end{aligned} \tag{27}$$

After regrouping the roots $p_{l,nk}$ of the characteristic equation in the expression given by equation (27) for ξ_n^y, it is possible to express them by means of the coefficients c_{ln} of Eq. (12), so that they depend only on α_{nk}, thickness h and the elastic parameters of the plate. Next, the expression for ξ_n^y is rewritten in the following form.

$$\xi_n^y = \frac{k_5\tilde{k}\,(p_{1,nk}^2 - p_{2,nk}^2)(p_{1,nk}^2 - p_{3,nk}^2)(p_{2,nk}^2 - p_{3,nk}^2)}{k_6\alpha_{nk}^2 C_{1,nk} C_{2,nk} C_{3,nk}}\left(\gamma_0^y\alpha_{nk}^4 + \gamma_1^y\alpha_{nk}^2 + \gamma_2^y\right), \tag{28}$$

where coefficients γ_l^y are given in Appendix B.

Similar transformation for η_n^y allows us to obtain the following expression.

$$\eta_n^y = \frac{(p_{1,nk}^2 - p_{2,nk}^2)(p_{1,nk}^2 - p_{3,nk}^2)(p_{2,nk}^2 - p_{3,nk}^2)}{k_6 C_{1,nk} C_{2,nk} C_{3,nk}}\left(\gamma_0^y\alpha_{nk}^4 + \gamma_1^y\alpha_{nk}^2 + \gamma_2^y\right)\left(k_4\tilde{k}\alpha_{nk}^2 - k_4k_5 - \tilde{k}\Omega^4\right). \tag{29}$$

It should be noted that transformed expressions given by equations (28) - (29) for ξ_n^y and η_n^y have common multiplier, which makes it possible to represent the coefficients of the infinite system as follows.

$$\begin{aligned} \frac{2\beta_{mj}}{\Delta_{2m}}\frac{\beta_{mj}^2\xi_n^y + \eta_n^y}{(\alpha_{nk}^2 + q_{1,mj}^2)(\alpha_{nk}^2 + q_{2,mj}^2)(\alpha_{nk}^2 + q_{3,mj}^2)} &= \frac{2\beta_{mj}}{\Delta_{2m}}\frac{(p_{1,nk}^2 - p_{2,nk}^2)(p_{1,nk}^2 - p_{3,nk}^2)(p_{2,nk}^2 - p_{3,nk}^2)}{k_6 C_{1,nk} C_{2,nk} C_{3,nk}}\times \\ &\times\left(\gamma_0^y\alpha_{nk}^4 + \gamma_1^y\alpha_{nk}^2 + \gamma_2^y\right)\frac{k_5\tilde{k}\beta_{mj}^2 + k_4\tilde{k}\alpha_{nk}^2 - k_4k_5 - \tilde{k}\Omega^4}{(\alpha_{nk}^2 + q_{1,mj}^2)(\alpha_{nk}^2 + q_{2,mj}^2)(\alpha_{nk}^2 + q_{3,mj}^2)} \end{aligned} \tag{30}$$

Similarly, for the second part of the equations of the system can be rewritten as

$$\begin{aligned} \frac{2\alpha_{mk}}{\Delta_{1m}}\frac{\alpha_{mk}^2\xi_n^x + \eta_n^x}{(\beta_{nj}^2 + p_{1,mk}^2)(\beta_{nj}^2 + p_{2,mk}^2)(\beta_{nj}^2 + p_{3,mk}^2)} &= \frac{2\alpha_{mk}}{\Delta_{1m}}\frac{(q_{1,nj}^2 - q_{2,nj}^2)(q_{1,nj}^2 - q_{3,nj}^2)(q_{2,nj}^2 - q_{3,nj}^2)}{k_6 G_{1,nj} G_{2,nj} G_{3,nj}}\times \\ &\times\left(\gamma_0^x\beta_{nj}^4 + \gamma_1^x\beta_{nj}^2 + \gamma_2^x\right)\frac{k_5\tilde{k}\beta_{nj}^2 + k_4\tilde{k}\alpha_{mk}^2 - k_4k_5 - \tilde{k}\Omega^4}{(\beta_{nj}^2 + p_{1,mk}^2)(\beta_{nj}^2 + p_{2,mk}^2)(\beta_{nj}^2 + p_{3,mk}^2)} \end{aligned} \tag{31}$$

The form of Eqs. (30) - (31) of the coefficients of infinite system originally represented by Eqs.(23) suggests the following changes in the unknown variables can be made

$$\frac{a}{b}\sqrt{\frac{k_4}{k_2k_5}}\frac{(p_{1,nk}^2 - p_{2,nk}^2)(p_{1,nk}^2 - p_{3,nk}^2)(p_{2,nk}^2 - p_{3,nk}^2)}{C_{1,nk} C_{2,nk} C_{3,nk}}\left(\gamma_0^y\alpha_{nk}^4 + \gamma_1^y\alpha_{nk}^2 + \gamma_2^y\right)x_n = Z_{2n-1} \tag{32}$$

$$\frac{(q_{1,nj}^2 - q_{2,nj}^2)(q_{1,nj}^2 - q_{3,nj}^2)(q_{2,nj}^2 - q_{3,nj}^2)}{k_6 G_{1,nj} G_{2,nj} G_{3,nj}}\left(\gamma_0^x\beta_{nj}^4 + \gamma_1^x\beta_{nj}^2 + \gamma_2^x\right)y_n = Z_{2n}, \tag{33}$$

Then the infinite system of Eqs. (23) may be rewritten in the following canonical form

$$Z_m = \sum_{n=1}^{\infty} M_{mn} Z_n \quad (m = 1, 2, \ldots) \tag{34}$$

where

$$M_{2m-1,2n-1} = M_{2m,2n} = 0 \tag{35}$$

and

$$\left.\begin{aligned}
M_{2m-1,2n} &= \frac{a}{b}\sqrt{\frac{k_4}{k_2k_5}}\,\frac{2\alpha_{mk}\left(\gamma_0^y\alpha_{mk}^4+\gamma_1^y\alpha_{mk}^2+\gamma_2^y\right)(p_{1,mk}^2-p_{2,mk}^2)(p_{1,mk}^2-p_{3,mk}^2)(p_{2,mk}^2-p_{3,mk}^2)}{k_6\Delta_{1m}C_{1,mk}C_{2,mk}C_{3,mk}}\times\\
&\quad\times\frac{k_5\tilde{k}\beta_{nj}^2+k_4\tilde{k}\alpha_{mk}^2-k_4k_5-\tilde{k}\Omega^4}{(\beta_{nj}^2+p_{1,mk}^2)(\beta_{nj}^2+p_{2,mk}^2)(\beta_{nj}^2+p_{3,mk}^2)}.\\[2ex]
M_{2m,2n-1} &= \frac{b}{a}\sqrt{\frac{k_5k_2}{k_4}}\,\frac{2\beta_{mj}\left(\gamma_0^x\beta_{mj}^4+\gamma_1^x\beta_{mj}^2+\gamma_2^x\right)(q_{1,mj}^2-q_{2,mj}^2)(q_{1,mj}^2-q_{3,mj}^2)(q_{2,mj}^2-q_{3,mj}^2)}{k_6\Delta_{2m}G_{1,mj}G_{2,mj}G_{3,mj}}\times\\
&\quad\times\frac{k_5\tilde{k}\beta_{mj}^2+k_4\tilde{k}\alpha_{nk}^2-k_4k_5-\tilde{k}\Omega^4}{(\alpha_{nk}^2+q_{1,mj}^2)(\alpha_{nk}^2+q_{2,mj}^2)(\alpha_{nk}^2+q_{3,mj}^2)}.
\end{aligned}\right\} \quad (36)$$

It should be also noted that even though the roots of the characteristic equations Eqs. (12) - (13) may be complex, the coefficients M_{mn} of the infinite system of Eqs. (34) will be always real. Moreover, it is interesting to note that M_{mn} will be of the same sign for sufficiently large values of the indices m.

3. Analysis of the infinite system of linear algebraic equations

It is known [22, 23] that a homogeneous infinite system of linear algebraic equations of the canonical form such as the one in Eqs. (34) constitutes a regular system if the series of the moduli of the coefficients in each of the equations is less than one, i.e.

$$\sum_{n=1}^{\infty}|M_{mn}| = 1-\rho_m < 1, \quad (m = 1,2,\ldots) \tag{37}$$

If this series does not exceed a value, which is less than one

$$\sum_{n=1}^{\infty}|M_{mn}| = 1-\rho_m \le 1-\theta < 1, \quad (m= 1,2,\ldots) \tag{38}$$

in this case, the infinite system is called fully regular.

The regular (or fully regular) infinite systems can be considered [23] as functional equations in the space of bounded sequences ℓ^{∞} . It can be proved [22, 23] that the fully regular system with bounded free members always has a unique bounded solution. Obviously, then the homogeneous infinite system will have only trivial solution, i.e., $Z_m = 0$, which leads to trivial zero-solution $W = \phi_x = \phi_y = 0$ for the boundary value problem which corresponds to the case of fully clamped plate. Thus, for infinite systems arising in dynamical problems, the regularity conditions are not satisfied over the full frequency range. For such type of systems, a generalization of the regularity conditions is needed, and the following definition is proposed [23].

An infinite system is called quasi-regular if there exists a number N_R such that

$$\sum_{n=1}^{\infty} | M_{mn} | < 1 \ (m = N_R + 1, N_R + 2, \ldots); \ \sum_{n=1}^{\infty} | M_{mn} | < \infty \ (m = 1, 2, \ldots, N_R) \tag{39}$$

Therefore, for quasi-regular systems the condition of regularity is necessarily fulfilled for some number $m > N_R$ and the analysis of quasi-regular infinite system can be reduced to a finite system of order N_R with help of the following change of variables

$$Z_m = \sum_{l=1}^{N_R} \xi_m^l Z_l \ (m>N_R) \tag{40}$$

Because of the change in variable shown in Eq. (40), the homogeneous quasi-regular infinite system of Eqs. (34) reduces to a set of regular infinite systems with the same infinite matrix and different free members, but with the new unknowns $\left\{\xi_m^l\right\}_{m=N_R+1}^{\infty}$ (l = 1, 2,..., N_R) given by

$$\xi_m^l = \sum_{n=N_R+1}^{\infty} M_{mn} \xi_n^l + M_{ml}, \quad (m = N_R + 1, N_R + 2, \ldots) \tag{41}$$

For the first N_R equations of the infinite system in Eqs. (34), the substitution of Eqs. (40) gives the following finite system of equations in terms of the first N_R unknowns $Z_1, Z_2, \ldots, Z_{N_R}$ as

$$Z_m = \sum_{n=1}^{N_R} \left(M_{mn} + \sum_{l=N_R+1}^{\infty} M_{ml} \xi_l^n \right) Z_n, \ (m = 1, 2, \ldots, N_R) \tag{42}$$

The infinite series within the condition of regularity can be calculated analytically using established techniques, see for example, Prudnikov et al.[21]. In this respect, the notation for hyperbolic functions used in Eqs. (19) is helpful so that the following infinite series can be written as

$$S_k(z) = \sum_{n=1}^{\infty} \frac{1}{\pi^2 (n + k/2 - 1/2)^2 + z^2} = \frac{1}{2z} \frac{H_k'(z)}{H_k(z)} - \frac{k}{2z^2} \ (k = 0, 1) \tag{43}$$

Then for $m > N^*$ we have the following analytical representation for series within the condition of regularity

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| M_{2m-1,2n} \right| = \frac{a}{b} \sqrt{\frac{k_4}{k_2 k_5}} \frac{2\alpha_{mk} \left(\gamma_0^y \alpha_{mk}^4 + \gamma_1^y \alpha_{mk}^2 + \gamma_2^y \right) (p_{1,mk}^2 - p_{2,mk}^2)(p_{1,mk}^2 - p_{3,mk}^2)(p_{2,mk}^2 - p_{3,mk}^2)}{k_6 \Delta_{1m} C_{1,mk} C_{2,mk} C_{3,mk}} \times \\
\times \left(\frac{\tilde{k} k_5 p_{1,mk}^2 - \tilde{k} k_4 \alpha_{mk}^2 + k_4 k_5 + \tilde{k} \Omega^4}{(p_{1,mk}^2 - p_{2,mk}^2)(p_{1,mk}^2 - p_{3,mk}^2)} S_j(p_{1,mk} b) + \frac{\tilde{k} k_5 p_{2,mk}^2 - \tilde{k} k_4 \alpha_{mk}^2 + k_4 k_5 + \tilde{k} \Omega^4}{(p_{1,mk}^2 - p_{2,mk}^2)(p_{2,mk}^2 - p_{3,mk}^2)} S_j(p_{2,mk} b) + \right. \\
\left. + \frac{\tilde{k} k_5 p_{3,mk}^2 - \tilde{k} k_4 \alpha_{mk}^2 + k_4 k_5 + \tilde{k} \Omega^4}{(p_{1,mk}^2 - p_{3,mk}^2)(p_{2,mk}^2 - p_{3,mk}^2)} S_j(p_{3,mk} b) \right) + \sum_{n=1}^{N^*} \left(\left| M_{2m-1,2n} \right| - M_{2m-1,2n} \right);
\end{aligned} \tag{44}$$

$$\sum_{n=1}^{\infty}\left|M_{2m,2n-1}\right| = \frac{b}{a}\sqrt{\frac{k_5k_2}{k_4}}\,\frac{2\beta_{mj}\left(\gamma_0^x\beta_{mj}^4+\gamma_1^x\beta_{mj}^2+\gamma_2^x\right)(q_{1,mj}^2-q_{2,mj}^2)(q_{1,mj}^2-q_{3,mj}^2)(q_{2,mj}^2-q_{3,mj}^2)}{k_6\Delta_{2m}G_{1,mj}G_{2,mj}G_{3,mj}}\times$$

$$\times\left(\frac{\tilde{k}k_4q_{1,mj}^2-\tilde{k}k_5\beta_{mj}^2+k_4k_5+\tilde{k}\Omega^4}{(q_{1,mj}^2-q_{2,mj}^2)(q_{1,mj}^2-q_{3,mj}^2)}S_k\left(q_{1,mj}a\right)+\frac{\tilde{k}k_4q_{2,mj}^2-\tilde{k}k_5\beta_{mj}^2+k_4k_5+\tilde{k}\Omega^4}{(q_{1,mj}^2-q_{2,mj}^2)(q_{2,mj}^2-q_{3,mj}^2)}S_k\left(q_{2,mj}a\right)+\right. \tag{45}$$

$$\left.+\frac{\tilde{k}k_4q_{3,mj}^2-\tilde{k}k_5\beta_{mj}^2+k_4k_5+\tilde{k}\Omega^4}{(q_{1,mj}^2-q_{3,mj}^2)(q_{2,mj}^2-q_{3,mj}^2)}S_k\left(q_{3,mj}a\right)\right)+\sum_{n=1}^{N*}\left(\left|M_{2m,2n-1}\right|-M_{2m,2n-1}\right)$$

where N^* is such a number which starts from the coefficients of system M_{mn} that have constant sign.

To evaluate the upper bound of the series shown in Eqs. (44) - (45), one can use the following asymptotic equalities for the roots of the characteristic equations when $m\to\infty$

$$q_{l,mj}=Q_l\beta_{mj};\quad p_{l,mk}=P_l\alpha_{mk}\ \ (l=1,2,3) \tag{46}$$

where Q_l^2 and P_l^2 correspond to the three different branches of following cubic (limiting) equations

$$k_4k_6Q^6+\left(\tilde{k}^2k_4-k_4k_6^2-k_5k_6-k_2k_4\right)Q^4+\left(k_2k_5+k_2k_4k_6+k_5k_6^2-\tilde{k}^2k_5\right)Q^2-k_2k_5k_6=0, \tag{47}$$

$$k_2k_5k_6P^6+\left(\tilde{k}^2k_5-k_2k_4k_6^2-k_2k_4k_6-k_2k_5\right)P^4+\left(k_2k_4+k_5k_6+k_4k_6^2-\tilde{k}^2k_4\right)P^2-k_4k_6=0 \tag{48}$$

Furthermore, using the asymptotic behavior of the relationships (Note that the coefficients Δ_{1m} and Δ_{2m} appear in Eqs. (23) and defined in Eqs. (24)), one can write

$$\Delta_{1m}C_{1,mk}C_{2,mk}C_{3,mk}=\Delta_1\alpha_{mk}^8 \text{ and } \Delta_{2m}G_{1,mj}G_{2,mj}G_{3,mj}=\Delta_2\beta_{mj}^8,\text{ when } m\to\infty \tag{49}$$

The asymptotic behaviour of the series in Eqs. (44) –(45) can now be evaluated as follows.

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}\left|M_{2m-1,2n}\right|=\lim_{m\to\infty}\sum_{n=1}^{\infty}\left|M_{2m,2n-1}\right|=\mu<1 \tag{50}$$

where

$$\begin{aligned}\mu&=\sqrt{\frac{k_5k_2}{k_4}}\frac{\gamma_0^x}{k_6\Delta_2}\left(-\frac{\left(k_4Q_1^2-k_5\right)\left(Q_2^2-Q_3^2\right)}{Q_1}+\frac{\left(k_4Q_2^2-k_5\right)\left(Q_1^2-Q_3^2\right)}{Q_2}-\frac{\left(k_4Q_3^2-k_5\right)\left(Q_1^2-Q_2^2\right)}{Q_3}\right)=\\&=\sqrt{\frac{k_4}{k_5k_2}}\frac{\gamma_0^y}{k_6\Delta_1}\left(-\frac{\left(k_5P_1^2-k_4\right)\left(P_2^2-P_3^2\right)}{P_1}+\frac{\left(k_5P_2^2-k_4\right)\left(P_1^2-P_3^2\right)}{P_2}-\frac{\left(k_5P_3^2-k_4\right)\left(P_1^2-P_2^2\right)}{P_3}\right).\end{aligned} \tag{51}$$

The validity of Eqs. (50) can be confirmed by further calculations using Eqs. (51) for any combinations of the problem parameters. To serve as an illustration, Fig. 1 shows the dependence of $\mu=\mu(\nu)$ on Poisson's ratio ν for an isotropic material. It is clear from the existence of limit shown in Eqs. (50) suggest that the condition of regularity is necessarily fulfilled for some number $m>N_R$ .Thus, it can be concluded that the homogeneous infinite system given by Eqs. (34) is a quasi-regular infinite system.

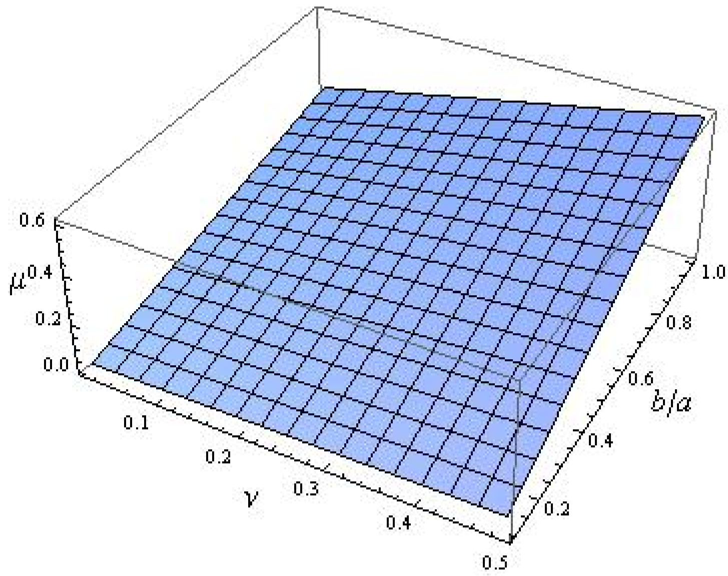

Fig. 1. The variation of the function μ against ν and *b*/*a* for *h* = 0.1*a*

It should be noted that number N_R of the first set of non-regular equations of the infinite system of Eqs. (34) is essentially dependent upon the frequency parameter Ω, wherein for some frequency range, the infinite system is fully regular ($N_R = 0$). For instance, on the basis of the data presented in Fig. 2 it can be argued that, for $\Omega \leq 1.6$, there are no natural frequencies of the plate. Actually, for this frequency range, the infinite system Eqs. (34), being fully regular, has a unique zero solution. As a consequence, there is no non-trivial solution for the boundary value problem of Eqs. (2) - (4), (6) - (7).This idea was first developed by Papkov and Banerjee [24]. They showed that the quasi-regular infinite system can be reduced to a regular infinite system if there exists a number N such that the coefficients M_{mn} of the system satisfy the following criterion

$$T_N = 1 - \max_{\mathrm{j=1..N}} \sum_{i=1}^{N} \left|\sigma_{ji}\right| \left(1 - \rho_i - \sum_{n=1}^{N} \left|M_{in}\right|\right) + \inf_{k>N} \frac{\rho_k - \vartheta}{\sum_{n=1}^{N} \left|M_{kn}\right|} > 0 \qquad (52)$$

where $\{\sigma_{k,n}\}_{k,n=1}^{N}$ is the inverse of the matrix $\{\delta_{kn} - M_{kn}\}_{k,n=1}^{N}$, δ_{kn} is Kronecker delta, ρ_k and θ are sequence and constant from the regularity conditions Eqs. (38) and $0 < \vartheta < \theta$.

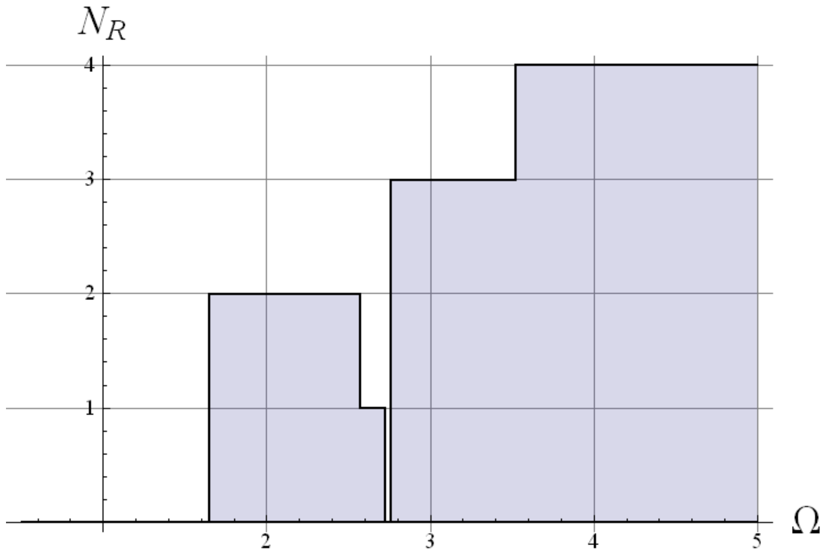

Fig. 2. The effect of the number N_R of non-regular equations in the infinite system of Eqs.(34) for $h/a = 0.1$, $b/a = 1$, $k = j = 0$, $\kappa = 0.8601$, $E_1 = 60.7\,GPa$, $G_{12} = 12\,GPa$, $\nu_{12} = 0.23$, $\nu_{21} = 0.094$

The fulfilment of the criterion given by Eqs. (52) has essentially meant that after eliminating the first N unknowns $Z_1, Z_2, ..., Z_N$ from the system, the infinite system with respect to the remaining unknowns has become fully regular. Thus, for the homogeneous infinite system Eqs. (34), the fulfilment of Eqs. (52) at a certain frequency Ω guarantees that this frequency is not a natural frequency. From a practical standpoint, localization of the natural frequency according to criterion of Eqs. (52) requires only the inversion of the matrix $\{\delta_{mn} - M_{mn}\}_{m,n=1}^{N}$ and the summation of the series with help of Eqs. (44) -(45).

Representative behavior for the localization of the first three natural frequencies according to the proposed approach for a thick square orthotropic plate (epoxy glass) is shown in Table 1. For comparison purposes, the last column of the table shows the values of the natural frequencies found according to the method of simple reduction from the approximate equation given by

$$\det\left(\delta_{mn} - M_{mn}(\Omega)\right)_{m,n=1}^{N_{red}} = 0 \tag{53}$$

using N_{red} = 32. It may be seen that the results obtained with help of both methods are very close, when the value of N = 8. For this illustration, the lower and upper bounds of the natural frequency parameter agree up to three significant figure for N=4, and for N = 8, the two bounds are extremely close and the results using the average values of the two, are in almost complete agreement with the ones obtained by using the method of simple reduction.

Table 1. The first three non-dimensional natural frequency parameters Ω_n for $h/a = 0.1$, $b/a = 1$, $\kappa = 0.8601$, $E_1 = 60.7\ GPa$, $G_{12} = 12\ GPa$, $\nu_{12} = 0.23$, $\nu_{21} = 0.094$.

Ω_n	Lower and upper bounds of Ω_n for $T_N(\Omega) < 0$			Method of simple reduction
	$N = 2$	$N = 4$	$N = 8$	
Ω_1	2.6376 - 2.6631	2.6511 – 2.6548	2.6528	2.6527
Ω_2	3.4698 – 3.4976	3.4851 – 3.4892	3.4873	3.4875
Ω_3	3.9496 – 3.9675	3.9587 - 3.9614	3.9602	3.9602

After the natural frequency parameter is computed, it is necessary to recover the mode shapes from the non-trivial solution of the quasi-regular infinite system of Eq. (34).This is can be accomplished in an approximate, but sufficiently accurate manner, i.e. when the first unknowns $Z_1, Z_2, ..., Z_N$ are found as the non-trivial solution of reduced finite system according to the approach of simple reduction method

$$Z_m = \sum_{n=1}^{N_{red}} M_{mn} Z_n \quad (m = 1, 2, \ldots, N_{red}) \tag{54}$$

Alternatively, all infinite sequence of unknowns can be found with the help of the asymptotic behaviour of the non-trivial solution quasi-regular infinite system of Eq. (34). Appropriate result for such investigation with respect to quasi-regular systems was published earlier by the first author [26] (The theorem described in [26] extended the approach of Koialovich’s Asymptotic Law [25]).

By using the above decomposition of Eqs. (40)–(42) of infinite system in relation to Eqs. (34) into one finite linear system and the set of N_R regular infinite systems, it can be observed that each of the infinite systems from this set will satisfy the conditions of regular systems if the following changes in unknowns are made

$$\xi^l_{2m-1} = \Gamma \beta^{-\lambda}_{mj} y^l_m, \quad \xi^l_{2m} = \alpha^{-\lambda}_{mk} x^l_m \tag{55}$$

where $\lambda \in [0; 1)$.

Then, the unique solutions x^l_m and y^l_m for the transformed infinite systems will have the common non-zero limit as given below

$$\lim_{m\to\infty} y^l_m = \lim_{m\to\infty} x^l_m = K_l > 0 \tag{56}$$

The constants λ and Γ in Eq. (55) are to be chosen in such a manner that the transformed infinite systems in respect to x^l_m and y^l_m remain regular, but they no longer satisfy the condition of fully regularity of Eqs. (38), that is, the series under regularity conditions will have to tend to unity from below, i.e. from the bottom end.

Using the following asymptotical relationships when $m \to \infty$

$$\sum_{n=N_R+1}^{\infty} \frac{\alpha_{nk}^{-\lambda}}{\alpha_{nk}^2 + q_{l,mj}^2} = \frac{aQ_l^{-\lambda-1}}{2\cos\frac{\pi\lambda}{2}} \cdot \frac{1}{\beta_{mj}^{\lambda+1}} + O\left(\frac{1}{\beta_{mj}^2}\right), \quad \sum_{n=N_R+1}^{\infty} \frac{\beta_{nj}^{-\lambda}}{\beta_{nj}^2 + p_{l,mk}^2} = \frac{bP_l^{-\lambda-1}}{2\cos\frac{\pi\lambda}{2}} \cdot \frac{1}{\alpha_{mk}^{\lambda+1}} + O\left(\frac{1}{\alpha_{mk}^2}\right) \quad (57)$$

from this condition we obtain the following transcendental equation for finding of parameter λ

$$\left(\frac{\tilde{k}}{k_6}\right)^2 \left|\frac{\gamma_0^x \gamma_0^y}{\Delta_1 \Delta_2}\right| \left(\frac{(k_5 P_1^2 - k_4)(P_2^2 - P_3^2)}{P_1^{\lambda+1}} - \frac{(k_5 P_2^2 - k_4)(P_1^2 - P_3^2)}{P_2^{\lambda+1}} + \frac{(k_5 P_3^2 - k_4)(P_1^2 - P_2^2)}{P_3^{\lambda+1}}\right) \times$$
$$\times \left(\frac{(k_4 Q_1^2 - k_5)(Q_2^2 - Q_3^2)}{Q_1^{\lambda+1}} - \frac{(k_4 Q_2^2 - k_5)(Q_1^2 - Q_3^2)}{Q_2^{\lambda+1}} + \frac{(k_4 Q_3^2 - k_5)(Q_1^2 - Q_2^2)}{Q_3^{\lambda+1}}\right) = \cos^2 \frac{\pi\lambda}{2} \quad (58)$$

It should also be noted that Eq. (58) has unique real solution in the range $\lambda \in [0; 1)$.

Therefore, it is possible to obtain the asymptotic behaviour of the non-trivial solution of system of Eqs. (34) as follows

$$Z_{2m-1} = \frac{a_1}{\alpha_{mk}^{\lambda}}; \quad Z_{2m} = \frac{a_2}{\beta_{mj}^{\lambda}} \quad (m \to \infty) \quad (59)$$

where a_1 and a_2 are some limiting constants.

Using the Eqs. (59) we can write the solution for CCCC boundary conditions in the form of an un-truncated infinite series as

$$W_{kj} = \sum_{n=1}^{\infty} \left(A_{1,nk} \left(A_{2,nk} B_{3,nk} - A_{3,nk} B_{2,nk}\right) \frac{H_j(p_{1,nk} y)}{H_j(p_{1,nk} b)} + A_{2,nk} \left(A_{3,nk} B_{1,nk} - A_{1,nk} B_{3,nk}\right) \frac{H_j(p_{2,nk} y)}{H_j(p_{2,nk} b)} + \right.$$
$$\left. + A_{3,nk} \left(A_{1,nk} B_{2,nk} - A_{2,nk} B_{1,nk}\right) \frac{H_j(p_{3,nk} y)}{H_j(p_{3,nk} b)} \right) \frac{(-1)^n x_n}{k_2 k_5} T_k(\alpha_{nk} x) + \sum_{n=1}^{\infty} \left(E_{1,nj} \left(E_{2,nj} F_{3,nj} - E_{3,nj} F_{2,nj}\right) \frac{H_k(q_{1,nj} x)}{H_k(q_{1,nj} a)} + \right. \quad (60)$$
$$\left. + E_{2,nj} \left(E_{3,nj} F_{1,nj} - E_{1,nj} F_{3,nj}\right) \frac{H_k(q_{2,nj} x)}{H_k(q_{2,nj} a)} + E_{3,nj} \left(E_{1,nj} F_{2,nj} - E_{2,nj} F_{1,nj}\right) \frac{H_k(q_{1,nj} x)}{H_k(q_{1,nj} a)} \right) \frac{(-1)^n y_n}{k_4} T_j(\beta_{nj} y)$$

$$\phi_{x,kj} = \sum_{n=1}^{\infty} \left(B_{1,nk} \left(A_{2,nk} B_{3,nk} - A_{3,nk} B_{2,nk}\right) \frac{H_j(p_{1,nk} y)}{H_j(p_{1,nk} b)} + B_{2,nk} \left(A_{3,nk} B_{1,nk} - A_{1,nk} B_{3,nk}\right) \frac{H_j(p_{2,nk} y)}{H_j(p_{2,nk} b)} + \right.$$
$$\left. + B_{3,nk} \left(A_{1,nk} B_{2,nk} - A_{2,nk} B_{1,nk}\right) \frac{H_j(p_{3,nk} y)}{H_j(p_{3,nk} b)} \right) \frac{(-1)^n x_n}{k_2 k_5} T'_k(\alpha_{nk} x) + \sum_{n=1}^{\infty} \left(\left(E_{2,nj} F_{3,nj} - E_{3,nj} F_{2,nj}\right) \frac{H'_k(q_{1,nj} x)}{H_k(q_{1,nj} a)} + \right. \quad (61)$$
$$\left. + \left(E_{3,nj} F_{1,nj} - E_{1,nj} F_{3,nj}\right) \frac{H'_k(q_{2,nj} x)}{H_k(q_{2,nj} a)} + \left(E_{1,nj} F_{2,nj} - E_{2,nj} F_{1,nj}\right) \frac{H'_k(q_{1,nj} x)}{H_k(q_{1,nj} a)} \right) \frac{(-1)^n y_n}{k_4} T_j(\beta_{nj} y)$$

$$\phi_{y,kj}=\sum_{n=1}^{\infty}\left(\left(A_{2,nk}B_{3,nk}-A_{3,nk}B_{2,nk}\right)\frac{H_j'(p_{1,nk}y)}{H_j(p_{1,nk}b)}+\left(A_{3,nk}B_{1,nk}-A_{1,nk}B_{3,nk}\right)\frac{H_j'(p_{2,nk}y)}{H_j(p_{2,nk}b)}+\right.$$
$$\left.+\left(A_{1,nk}B_{2,nk}-A_{2,nk}B_{1,nk}\right)\frac{H_j'(p_{3,nk}y)}{H_j(p_{3,nk}b)}\right)\frac{(-1)^n x_n}{k_2k_5}T_k(\alpha_{nk}x)+\sum_{n=1}^{\infty}\left(F_{1,nj}\left(E_{2,nj}F_{3,nj}-E_{3,nj}F_{2,nj}\right)\frac{H_k(q_{1,nj}x)}{H_k(q_{1,nj}a)}+\right. \quad (62)$$
$$\left.+F_{2,nj}\left(E_{3,nj}F_{1,nj}-E_{1,nj}F_{3,nj}\right)\frac{H_k(q_{2,nj}x)}{H_k(q_{2,nj}a)}+F_{3,nj}\left(E_{1,nj}F_{2,nj}-E_{2,nj}F_{1,nj}\right)\frac{H_k(q_{1,nj}x)}{H_k(q_{1,nj}a)}\right)\frac{(-1)^n y_n}{k_4}T_j'(\beta_{nj}y)$$

where coefficients of series x_n and y_n are defined with help of Eqs. (32), (33) and they can be described for large indexes as

$$x_n=\frac{x_\infty}{\alpha_{nk}^{\lambda+1}};\quad y_n=\frac{y_\infty}{\beta_{nj}^{\lambda+1}}\quad (n\to\infty) \quad (63)$$

$$x_\infty=\frac{b}{a}\sqrt{\frac{k_2k_5}{k_4}}\frac{\tilde{k}^3P_1P_2P_3(k_5P_1^2-k_4)(k_5P_2^2-k_4)(k_5P_3^2-k_4)}{\left(P_1^2-P_2^2\right)\left(P_1^2-P_3^2\right)\left(P_2^2-P_3^2\right)\gamma_0^y}a_1;$$
$$y_\infty=\frac{k_6\tilde{k}^3Q_1Q_2Q_3(k_4Q_1^2-k_5)(k_4Q_2^2-k_5)(k_4Q_3^2-k_5)}{\left(Q_1^2-Q_2^2\right)\left(Q_1^2-Q_3^2\right)\left(Q_2^2-Q_3^2\right)\gamma_0^x}a_2 \quad (64)$$

It should be noted that the above series of Eqs. (61) and (62) representing the solution of the boundary value problem converge sufficiently rapidly everywhere inside the plate region. Indeed, for $y \geq 0$, taking into account that $P_l > 0$, we can obtain the following asymptotic estimates when $n \to \infty$

$$\frac{H_j(p_{l,nk}y)}{H_j(p_{l,nk}b)}=\frac{e^{p_{l,nk}y}+(-1)^j e^{-p_{l,nk}y}}{e^{p_{l,nk}b}+(-1)^j e^{-p_{l,nk}b}}=e^{-P_l(b-y)\alpha_{nk}},$$

$$\frac{H_j'(p_{l,nk}y)}{H_j(p_{l,nk}b)}=\frac{e^{p_{l,nk}y}-(-1)^j e^{-p_{l,nk}y}}{e^{p_{l,nk}b}+(-1)^j e^{-p_{l,nk}b}}=e^{-P_l(b-y)\alpha_{nk}},$$

i.e. for the calculation of values given in the series represented by Eqs. (60) – (62) for $0 \leq y < b$, it is sufficient to know only the first few of the terms, and the reminders of the infinite series have no significant effect on the result. Similar estimates are valid also for the series that contain hyperbolic functions of x.

Nevertheless, one of the series represented by Eqs. (61) - (62) has a weak convergence on the plate boundary. In particular, Eq. (61) when $x = a$ can be written as

$$\phi_{x,kj}(a,y)=\sum_{n=1}^{\infty}\left(B_{1,nk}\left(A_{2,nk}B_{3,nk}-A_{3,nk}B_{2,nk}\right)\frac{H_j(p_{1,nk}y)}{H_j(p_{1,nk}b)}+B_{2,nk}\left(A_{3,nk}B_{1,nk}-A_{1,nk}B_{3,nk}\right)\frac{H_j(p_{2,nk}y)}{H_j(p_{2,nk}b)}+\right.$$
$$\left.+B_{3,nk}\left(A_{1,nk}B_{2,nk}-A_{2,nk}B_{1,nk}\right)\frac{H_j(p_{3,nk}y)}{H_j(p_{3,nk}b)}\right)\frac{x_n}{k_2k_5}+\sum_{n=1}^{\infty}\Delta_{2n}\frac{(-1)^n y_n}{bk_4}T_j(\beta_{nj}y) \quad (65)$$

By considering the asymptotic formulas $y_n = y_\infty\beta_{nj}^{-\lambda-1}$ and $\Delta_{2n}/b = \Delta_\infty\beta_{nj}$ $(n\to\infty)$, it maybe been seen that the second trigonometric series in the expression of Eq. (65) has a weak convergence. Obviously, the first few terms of the series do not provide the value of the function $\phi_{x,kj}$ (a, y) with sufficient accuracy. An improvement of the convergence of the series is now sought analytically using well-established technique [22] to give the following series with improved convergence related to the asymptotic behavior shown in Eq. (63).

$$\frac{1}{k_4}\sum_{n=1}^{\infty}(-1)^n\frac{\Delta_{2n}y_n}{b}T_j(\beta_{nj}y)=\frac{1}{k_4}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\Delta_{2n}y_n}{b}-\frac{\Delta_{\infty}y_{\infty}}{\beta_{nj}^{\lambda}}\right)T_j(\beta_{nj}y)+\frac{\Delta_{\infty}y_{\infty}}{k_4}\sum_{n=1}^{\infty}\frac{(-1)^nT_j(\beta_{nj}y)}{\beta_{nj}^{\lambda}} \quad (66)$$

for which the value of the second series for j = 0 and j = 1 can be found analytically using special functions as follows.

$$j=0: \quad \sum_{n=1}^{\infty}\frac{(-1)^nT_j(\beta_{nj}y)}{\beta_{nj}^{\lambda}}=\left(\frac{b}{\pi}\right)^{\lambda}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\pi\left(n-\frac{1}{2}\right)\frac{y}{b})}{\left(n-\frac{1}{2}\right)^{\lambda}}=\left(\frac{b}{\pi}\right)^{\lambda}\operatorname{Re}e^{-\frac{i\pi y}{2b}}\sum_{n=1}^{\infty}\frac{\left(-e^{\frac{i\pi y}{b}}\right)^n}{\left(n-\frac{1}{2}\right)^{\lambda}}= \quad (67)$$

$$=-\left(\frac{b}{\pi}\right)^{\lambda}\operatorname{Re}e^{\frac{i\pi y}{2b}}\Phi\left(-e^{\frac{i\pi y}{b}},\lambda,\frac{1}{2}\right)$$

where $\Phi(z,\lambda,a)=\sum_{n=0}^{\infty}\frac{z^k}{(n+a)^{\lambda}}$ is Lerch Phi function [27].

$$j=1: \sum_{n=1}^{\infty}\frac{(-1)^nT_j(\beta_{nj}y)}{\beta_{nj}^{\lambda}}=\left(\frac{b}{\pi}\right)^{\lambda}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\frac{\pi ny}{b})}{n^{\lambda}}=\left(\frac{b}{\pi}\right)^{\lambda}\operatorname{Im}\sum_{n=1}^{\infty}\frac{\left(-e^{\frac{i\pi y}{b}}\right)^n}{n^{\lambda}}=\left(\frac{b}{\pi}\right)^{\lambda}L_{\lambda}\left(-e^{\frac{i\pi y}{b}}\right) \quad (68)$$

where $L_{\lambda}(z)=\sum_{n=1}^{\infty}\frac{z^k}{n^{\lambda}}$ is polylogarithm function [27].

Thus Eqs. (67) – (68) allow us to improve the convergence of series for $\phi_{x,kj}(a,y)$ and obtain more accurate solution for the boundary value problem, particularly near the plate edges in comparison with previous methods of simple reduction [19]. The convergence of series for $\phi_{y,kj}(x,b)$ is also improved in a similar manner. To illustrate this fact, Table 2 shows fulfilling of the boundary condition $\phi_{x,kj}(a,y)=0$ for the first natural mode of a thick square orthotropic plate (epoxy glass) when using the first eight terms of the series using the previous methods and the current method. Clearly, the convergence is significantly improved in the current method.

Table 2. The fulfillment of the boundary condition $\phi_{x,kj}(a,y)=0$ for Ω_1=2.6527; $h/a=0.1$, $b/a=1$, $\kappa=0.8601$, $E_1=60.7\,GPa$, $G_{12}=12\,GPa$, $\nu_{12}=0.23$, $\nu_{21}=0.094$.

Comparison between current method and Ref [19]	$\phi_{x,kj}(a,y)$				
	y/b				
	0.0	0.2	0.4	0.6	0.8
Without improving the convergence (Simple reduction) [19]	-0.000158	0.000156	-0.000109	0.000037	-0.000007
With improving the convergence (current method)	0.000002	0.000002	0.000002	0.000001	0.000001

4. Numerical results and discussion

The theory and the methodology described above were implemented in a computer program in the Mathematica package. The natural frequencies were computed with the help of the criteria based on Eqs. (52). The corresponding non-trivial solutions of the infinite system were determined using the decomposition of Eqs. (40) – (42), and also by taking into account the asymptotic behaviour of non-trivial solution of Eqs. (59). The first set of results is obtained for validation purposes. Table 3 shows the first seven natural frequencies of a fully clamped isotropic plate alongside the results reported in [16] that are based on the Ritz method. For the convenience of comparison, the frequencies are given in circular or angular frequency ω (rad/sec). Table 3 shows that the agreement between the results computed using the present method and the ones using the Ritz method of [16] is excellent, the discrepancies in the natural frequencies, being less than 0.3%. The natural frequencies obtained by the Ritz method turn out to be a little bit overestimated. This is consistent with similar observations made by previous investigator [28].

Table 3. Validation of the first seven natural frequencies for isotropic thick rectangular clamped plate with $a=2$ m; $b=1$ m; $E_1=143\,GPa$; $\nu=0.35, \kappa=5/6, h/a=0.1$

Case study	ω_n(rad/s)						
	1	2	3	4	5	6	7
$T_N(\Omega)<0$	3011.78	3847.28	5301.29	7236.97	7292.36	7954.69	9169.55
Ref.[16]	3022.32	3860.63	5319.84	7262.14	7317.79	7982.47	9201.45

Following the validation of results given in Table 3, further investigation was carried out. Tables 4-6 show the first five non-dimensional natural frequencies (Ω_n, n = 1, 2,….5) of thick rectangular plates with all-round clamped boundary condition, computed by applying the proposed method using three different materials and with varying plate geometries. Table 4 presents results when the plate is made of isotropic material (quartz glass) whereas the results shown in Table 5 are for orthotropic plate material (epoxy glass). By contrast, Table 6 shows results for an auxetic material. The authors have noted with great interest that there is a growing interest in unconventional materials that have negative Poisson's ratios. Paradoxically such materials are not some imaginary abstractions, but they can be found among natural materials, and they can also be created artificially as auxetic materials. The first five natural frequencies for a thick all round clamped plate made of an auxetic material which in fact is one of the forms of α-cristobalite quartz are shown in Table 6.

Table 4. The natural frequencies Ω for CCCC isotropic plate with $E_1 = 73.6\,GPa$, $\nu = 0.17, \kappa = 0.8601, h/a = 0.1$

Ω_n	$b/a = 1$	$b/a = 2$	$b/a = 3$
Ω_1	2.9638	2.4578	2.3888
Ω_2	4.1949	2.7917	2.5201
Ω_3	4.1949	3.3024	2.7447
Ω_4	5.0530	3.9137	3.0511
Ω_5	5.5479	3.9273	3.4176

Table 5. The natural frequencies Ω for CCCC orthotropic plate with $E_1 = 60.7\,GPa$, $G_{12} = 12\,GPa$, $\nu_{12} = 0.23$, $\nu_{21} = 0.094$, $\kappa = 0.8601, h/a = 0.1$

Ω_n	$b/a = 1$	$b/a = 2$	$b/a = 3$
Ω_1	2.6528	2.3809	2.3481
Ω_2	3.4873	2.5566	2.4123
Ω_3	3.9602	2.8680	2.5311
Ω_4	4.4539	3.2865	2.7102
Ω_5	4.5223	3.7697	2.9444

Table 6. The natural frequencies Ω for CCCC auxetic plate $E_1 = 73.6\,GPa$, $\nu = -0.16, \kappa = 0.8601, h/a = 0.1$

Ω_n	$b/a = 1$	$b/a = 2$	$b/a = 3$
Ω_1	2.9120	2.4350	2.3781
Ω_2	4.1320	2.7318	2.4840
Ω_3	4.1320	3.2262	2.6834
Ω_4	4.9199	3.8380	2.9745
Ω_5	5.4976	3.9087	3.3361

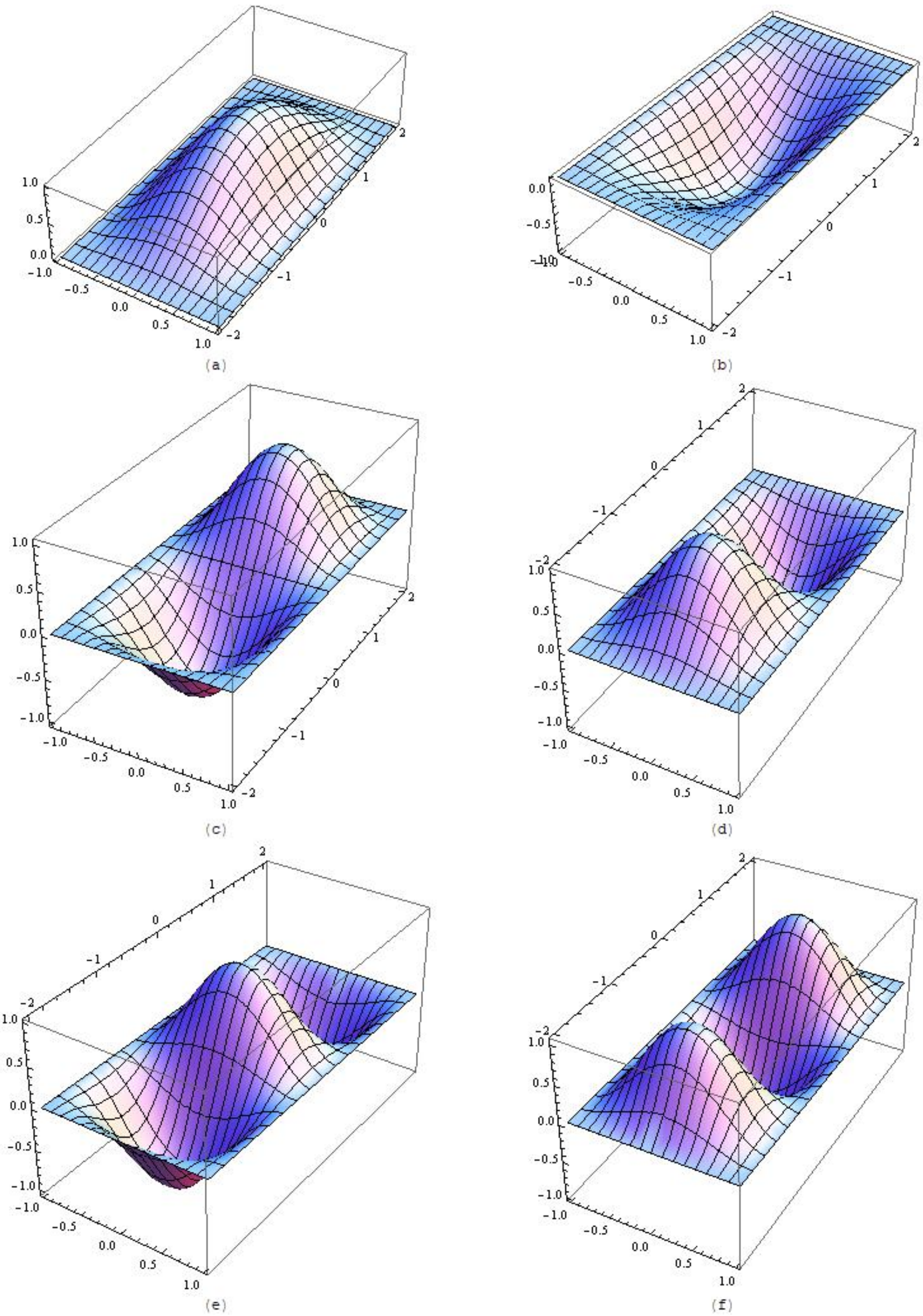

Fig. 3. The first three natural modes of rectangular plate with clamped edges for $h/a = 0.1$, $b/a = 2$, $\kappa = 0.8601$; (a), (c) , (e) – quartz glass plate; (b), (d) , (f) – α-cristobalite plate.

It can be observed that for all of the constitutive three materials of the plate, the natural frequencies decrease with increasing ratio of the sides of the plate, as expected, but at the same time, there are no noticeable differences in the behaviour of the spectrum of the auxetic plate from the frequency spectrum of the plate made of isotropic materials that have a positive Poisson's ratio. It is noted that the natural frequencies of a α-cristobalite plate are a little lower than those of a quartz plate. Meanwhile the data of Table 5 show that for an epoxy glass plate, the natural frequencies are significantly lower. It should be noted that the first free vibration mode is always symmetric, as expected. Thus, from the presented numerical results it follows that the spectrum of the natural vibrations of a thick plate is more influenced by the pronounced orthotropy of the material rather than its auxeticity.

For completeness, the first five mode shapes for the thick rectangular plate made of isotropic material (quartz glass) and auxetic material (α-cristobalite) and with CCCC boundary conditions are shown in Fig.3. The corresponding natural frequencies of vibration have already been given in Tables 4 and 6. It should be noted that the mode shapes differ very little. This has meant that the auxeticity of the material does not produce significant effects on the mode shapes for free vibration problems of clamped thick plates.

5. Conclusions

Through the analysis and solution of an infinite system of linear algebraic equations, an elegant theory together with an associated algorithm is developed in this paper to determine within any desired accuracy, the natural frequencies and mode shapes of rectangular thick orthotropic plates with clamped edges, comprising different geometric and/or elastic properties. The theory is validated by published results showing excellent agreement.

The investigation has shown that the spectrum of natural frequencies of vibrations for thick clamped plates are more influenced by the orthotropic properties of the plate materials rather than the auxeticity of the material. The differences in natural frequencies and shapes for isotropic and orthotropic plates are most clearly manifested when the ratio of the two sides of the plate is increased. It is in the context of the use of the dynamic stiffness method in conjunction with the application Wittrtick-Williams algorithm which requires the knowledge of the clamped ended natural frequencies of structural elements, the proposed investigation is expected to be most effective and important.

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Appendix A

COEFFICIENTS OF BICUBIC CHARACTERISTIC EQUATIONS AND RELATIONSHIPS OF VIET'S THEOREM

For characteristic equation of Eq. (12) can be written as

$$c_0 p^6 + c_{1n} p^4 + c_{2n} p^2 + c_{3n} = 0 \tag{A.1}$$

where

$$c_0 = k_2 k_5 k_6,$$

$$c_{1n} = \alpha_{nk}^2 \left(k_5\left(\tilde{k}^2 - k_6^2\right) - k_2\left(k_5 + k_4 k_6\right)\right) + \Omega_h^4 k_5\left(k_2 + k_6\right) + \Omega^4 k_2 k_6 - k_2 k_4 k_5,$$

$$c_{2n} = \alpha_{nk}^4\left(k_4\left(k_2 + k_6^2 - \tilde{k}^2\right) + k_5 k_6\right) + \alpha_{nk}^2\left(\Omega^4\left(\tilde{k}^2 - k_6^2\right) - k_5\Omega_h^4(1 + k_6) - k_2\left(\Omega^4 + k_4\Omega_h^4\right) + k_4\left(2k_5\left(k_6 + \tilde{k}\right) - k_6\Omega_h^4\right)\right)$$
$$+ \Omega^4\left(\Omega_h^4\left(k_2 + k_6\right) - k_2 k_4\right) + k_5\left(\Omega_h^4\left(\Omega_h^4 - k_4\right) - k_6\Omega^4\right),$$

$$c_{3n} = -\alpha_{nk}^6 k_4 k_6 + \alpha_{nk}^4\left(-k_4 k_5 + k_6\Omega^4 + k_4\Omega_h^4(1 + k_6)\right) + \alpha_{nk}^2\left(\Omega^4\left(k_5 + k_4 k_6 - \Omega_h^4\right) + \right.$$
$$\left. + \Omega_h^4\left(k_4 k_5 - k_6\Omega^4 - k_4\Omega_h^4\right)\right) + \Omega^4\left(\Omega_h^4 - k_4\right)\left(\Omega_h^4 - k_5\right). \tag{A.2}$$

We can write the following relationships using Viet's theorem [20]

$$\begin{aligned} &p_{1,nk}^2 + p_{2,nk}^2 + p_{3,nk}^2 = -c_{1n} / c_0 \\ &p_{1,nk}^2 p_{2,nk}^2 + p_{1,nk}^2 p_{3,nk}^2 + p_{2,nk}^2 p_{3,nk}^2 = c_{2n} / c_0 \\ &p_{1,nk}^2 p_{2,nk}^2 p_{3,nk}^2 = -c_{3n} / c_0 \end{aligned} \tag{A.3}$$

Similarly, the characteristic equation of Eq. (13) can be written as

$$d_0 q^6 + d_{1n} q^4 + d_{2n} q^2 + d_{3n} = 0 \tag{A.4}$$

where

$$d_0 = k_4 k_6,$$

$$d_{1n} = \beta_{nj}^2\left(k_4\left(\tilde{k}^2 - k_6^2 - k_2\right) - k_5 k_6\right) + \Omega_h^4 k_4\left(1 + k_6\right) + \Omega^4 k_6 - k_4 k_5,$$

$$d_{2n} = \beta_{nj}^4\left(k_5\left(k_2 + k_6^2 - \tilde{k}^2\right) + k_2 k_4 k_6\right) + \beta_{nj}^2\left(2k_4 k_5\left(k_6 + \tilde{k}\right) - \Omega^4\left(k_2 + k_6^2 - \tilde{k}^2\right) - \Omega_h^4\left(k_2 k_4 + k_5 + k_4 k_6 + k_5 k_6\right)\right)$$
$$+ \Omega^4\left(\Omega_h^4\left(1 + k_6\right) - k_5 - k_4 k_6\right) + k_4\Omega_h^4\left(\Omega_h^4 - k_5\right),$$

$$d_{3n} = -\beta_{nj}^6 k_2 k_5 k_6 + \beta_{nj}^4\left(-k_2 k_4 k_5 + k_2 k_6 \Omega^4 + k_5\Omega_h^4\left(k_2 + k_6\right)\right) + \beta_{nj}^6\left(\Omega^4\left(k_2 k_4 + k_5 k_6 - k_2\Omega_h^4\right) + \right.$$
$$\left. + \Omega_h^4\left(k_4 k_5 - k_6\Omega^4 - k_5\Omega_h^4\right)\right) + \Omega^4\left(\Omega_h^4 - k_4\right)\left(\Omega_h^4 - k_5\right). \tag{A.5}$$

these relationships can be written as

$$\begin{aligned} &q_{1,nj}^2 + q_{2,nj}^2 + q_{3,nj}^2 = -d_{1n} / d_0 \\ &q_{1,nj}^2 q_{2,nj}^2 + q_{1,nj}^2 q_{3,nj}^2 + q_{2,nj}^2 q_{3,nj}^2 = d_{2n} / d_0 \\ &q_{1,nj}^2 q_{2,nj}^2 q_{3,nj}^2 = -d_{3n} / d_0 \end{aligned} \tag{A.6}$$

Appendix B

COEFFICIENTS OF γ_l^x, γ_l^y, Δ_1 AND Δ_2

$$\gamma_0^x = \tilde{k}k_5k_6(k_4(\tilde{k}+k_6)-k_5) \quad \text{(B.1)}$$

$$\gamma_1^x = k_5\left(\tilde{k}k_6\Omega^4 + k_5\left(k_4k_6 - \tilde{k}k_4 + \tilde{k}\Omega_h^4\right)\right) + \tilde{k}k_4\left(k_5(k_4k_6 + \tilde{k}k_4 - \Omega_h^4(2k_6+\tilde{k})) - \tilde{k}k_6\Omega^4\right) \quad \text{(B.2)}$$

$$\gamma_2^x = \left(k_4 - \Omega_h^4\right)\left(k_5\left(k_4k_5 + \tilde{k}\Omega^4\right) - \tilde{k}k_4\left(\tilde{k}\Omega^4 + k_5\Omega_h^4\right)\right) \quad \text{(B.3)}$$

$$\gamma_0^y = \tilde{k}k_4k_6(k_5(\tilde{k}+k_6)-k_2k_4) \quad \text{(B.4)}$$

$$\gamma_1^y = k_2k_4\left(\tilde{k}k_6\Omega^4 + k_4\left(k_5k_6 - \tilde{k}k_5 + \tilde{k}\Omega_h^4\right)\right) + \tilde{k}k_5\left(k_4(k_5k_6 + \tilde{k}k_5 - \Omega_h^4(2k_6+\tilde{k})) - \tilde{k}k_6\Omega^4\right) \quad \text{(B.5)}$$

$$\gamma_2^y = \left(k_5 - \Omega_h^4\right)\left(k_2k_4\left(k_4k_5 + \tilde{k}\Omega^4\right) - \tilde{k}k_5\left(\tilde{k}\Omega^4 + k_4\Omega_h^4\right)\right) \quad \text{(B.6)}$$

$$\Delta_1 = \tilde{k}\,\det\begin{pmatrix} P_1(k_5P_1^2-k_4) & P_2(k_5P_2^2-k_4) & P_3(k_5P_3^2-k_4) \\ (k_5P_1^2-k_4)(k_2P_1^2-k_6) & (k_5P_2^2-k_4)(k_2P_2^2-k_6) & (k_5P_3^2-k_4)(k_2P_3^2-k_6) \\ (\tilde{k}k_5-k_2k_4)P_1^2+k_4k_6 & (\tilde{k}k_5-k_2k_4)P_2^2+k_4k_6 & (\tilde{k}k_5-k_2k_4)P_3^2+k_4k_6 \end{pmatrix} \quad \text{(B.7)}$$

$$\Delta_2 = \tilde{k}\,\det\begin{pmatrix} Q_1(k_4Q_1^2-k_5) & Q_2(k_4Q_2^2-k_5) & Q_3(k_4Q_3^2-k_5) \\ (k_4Q_1^2-k_5)(Q_1^2-k_6) & (k_4Q_2^2-k_5)(Q_2^2-k_6) & (k_4Q_3^2-k_5)(Q_3^2-k_6) \\ (\tilde{k}k_4-k_5)Q_1^2+k_5k_6 & (\tilde{k}k_4-k_5)Q_2^2+k_5k_6 & (\tilde{k}k_4-k_5)Q_3^2+k_5k_6 \end{pmatrix} \quad \text{(B.8)}$$

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