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Ordering dynamics of the multi-state voter model

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Abstract. The voter model is a paradigm of ordering dynamics. At each time step, a random node is selected and copies the state of one of its neighbors. Traditionally, this state has been considered as a binary variable. Here, we address the case in which the number of states is a parameter that can assume any value, from 2 to ∞, in the thermodynamic limit. We derive mean-field analytical expressions for the exit probability, the consensus time, and the number of different states as a function of time for the case of an arbitrary number of states. We finally perform a numerical study of the model in low dimensional lattices, comparing the case of multiple states with the usual binary voter model. Our work sheds light on the role of the parameter accounting for the number of states.

1. Introduction

Models of ordering dynamics have since long been considered as paradigms of opinion dynamics and consensus formation in social systems (1). Most of them share the fundamental feature that order results from the self-organization of local and usually short-range pairwise interactions between agents, as it is well illustrated by the simplest and most analyzed of them, the so-called voter model (2; 3). In its basic formulation, the voter model is defined as follows: Each individual in a population (agent) is endowed with a binary spin variable, representing two alternative opinions, and taking values $\sigma = \pm 1$. At each time step, an agent i is selected at random together with one nearest neighbor j and the state of the system is updated as $\sigma_i := \sigma_j$, the first agent copying the opinion of its neighbor. Starting from a disordered initial state, this dynamics leads in finite systems to a uniform state with all individuals sharing the same opinion (the so-called consensus).

From the point of view of social dynamics, the interest in this kind of models is mainly focused on the way in which consensus is reached. The approach to this state is characterized in terms of the exit probability $E(x)$ and the consensus time $T_N(x)$, defined as the probability that the final state corresponds to all agents in the state $+1$ and the average time needed to reach consensus in a system of size N, respectively, when starting from a homogeneous initial condition with a fraction x of agents in state $+1$ (1). Due to its simplicity, the voter model dynamics can be exactly solved in regular lattices for any number of dimensions (4; 5). Thus, considering the average conservation of magnetization $m = \sum_{i=1}^{N} \sigma_i / N$, it can be shown that the exit probability is always a linear function, $E(x) = x$. On the other hand, the consensus time starting from the homogeneous symmetric initial condition $x = 1/2$ scales with system size N as $T_N(1/2) \sim N_{\text{eff}}$, with $N_{\text{eff}} \sim N^2$ in $d = 1$, $N_{\text{eff}} \sim N \log N$ in $d = 2$, and $N_{\text{eff}} \sim N$ in $d > 2$ (at the mean-field level) (5). Finally, the dependence of consensus time with the initial density of $+1$ spins, starting from homogeneous initial conditions, takes the form

$$T_N(x) = -N_{\text{eff}} \left[x \ln(x) + (1 - x) \ln(1 - x) \right], \tag{1}$$

for $d \geq 2$ (6).

Different variants of the voter model have been considered in the past, including the presence of quenched disorder in the form of "zealots" which do not change opinion (7; 8), memory and noise reduction (9), inertia (10), non-conservative voters (11), non-linear interactions (12; 13), etc.; see Ref. (1) for an extended bibliography on this subject. A variant that has been considered in several contexts is the *multi-state* voter model, in which each agent can be in one of S different exclusive states or opinions, in analogy of the Potts model (14). The multi-state voter model has been considered in the past theoretically in terms of mappings of coarsening of the Potts model on the Ising model with constant magnetization (15) or in terms of duality properties (16) and has found applications in understanding the fragmentation transition in adaptive networks (17), in neutral models of biodiversity (18; 19) or in ecological models (20). Variants of the pure

multi-state voter model, introducing non-equivalent states, have also been discussed in the literature (21; 22; 23).

In this paper we focus in the study the ordering dynamics of the symmetric multi-state voter model, focusing in particular on the limit of a large number of initial states. Each agent can be in one of S different but dynamically equivalent states. Agents follow the same dynamical update rules than in the binary version, with time being update at every dynamical step as $t \to t + 1/N$. Expressions for the consensus time in this model at the mean-field level have already been provided in the literature (24; 25; 26). The derivations presented so far rely however on heavy mathematics. Here, building on the Fokker-Planck formalism presented in Ref. (27), we rederive in a simple way the expressions for the exit probability and the consensus times in the general case of S states. We find that the consensus time increases very slowly with the number of states, and its difference with the binary case saturates as $S \to \infty$; we rationalize this finding by comparing it with the case of the mutant invasion in the ordinary two-state voter model. We also investigate the decay of the number of states as a function of time, providing a mean-field expression in excellent agreement with simulations. We finally consider the dynamics of the multi-state voter model on low dimensional lattices. Lacking of specific analytical insights, we compare the numerically observed phenomenology to the mean-field case, and point out similarities and differences, focusing on the effect of the number of initial states and their configuration.

The paper is structured as follows. Sec. 2 is devoted to the analysis of the mean-field multi-state voter model. Sec. 3 reports on numerical experiments concerning the low-dimensional case. Finally, Sec. 4 presents our conclusions.

2. Mean-field analysis

The form of the consensus time in the multi-state voter model at the mean-field level has been discussed in the past, mainly in the context of population genetics dynamics (24; 25; 26). Here we present a simple derivation of this expression, based in the Fokker-Plank formalism developed in Ref. (27). The Fokker-Plank equation for the multi-state voter model can be simply obtained as follows: Let us denote $\vec{n}$ as a generic configuration of the system (not unique) with n_i voters in state i, $\vec{n} = \{n_1, n_2, \ldots, n_S\}$, with a normalization $\sum_i n_i = N$. The probability of finding the system in the configuration $\vec{n}$ at time t, $P(\vec{n}, t)$, evolves in terms of a master equation that is defined by the transition rates $w(\vec{n}' \to \vec{n})$ from the state $\vec{n}'$ to the state $\vec{n}$. At each time step only one voter changes its state, consequently we can write a new configuration $\vec{n}'$ of a transition $\vec{n} \to \vec{n}'$ as $\vec{n}' = \vec{n}_{i+j-} = \{n_1, \ldots, n_j - 1, \ldots, n_i + 1, \ldots, n_S\}$, being j and i the state of the voter before and after the transition, respectively. The transition rates $\vec{n} \to \vec{n}_{i+j-}$ and $\vec{n} \to \vec{n}_{i-j+}$ are given by

$$w(\vec{n} \to \vec{n}_{i+j-}) = w(\vec{n} \to \vec{n}_{i-j+}) = \frac{1}{\Delta} \frac{n_i}{N} \frac{n_j}{N}, \tag{2}$$

where $\Delta = 1/N$ is the natural microscopic time step of the model, while the transitions rates $\vec{n}_{i+j-} \to \vec{n}$ and $\vec{n}_{i-j+} \to \vec{n}$ have the form

$$w(\vec{n}_{i+j-} \to \vec{n}) = \frac{1}{\Delta}\frac{n_i + 1}{N}\frac{n_j - 1}{N}, \quad w(\vec{n}_{i-j+} \to \vec{n}) = \frac{1}{\Delta}\frac{n_i - 1}{N}\frac{n_j + 1}{N}. \quad (3)$$

It is now possible to derive the associated master equation. Under the diffusion approximation (28), valid for large N, we consider the frequencies of the states $x_i = n_i/N$ and we rescale the time by a factor $1/N$, so that one time step t measures N updates of the voters. A generic configuration is therefore denoted by $\vec{x} = \{x_1, x_2, \ldots, x_S\}$, and lies in the standard simplex $\mathcal{S}_S = \{\vec{x} \in \mathbb{R}^S | \sum_i^S x_i = 1\}$. The set of the S vertices of the simplex, $\mathcal{B}_S = \{\vec{e}^i \in \mathbb{R}^S | e_j^i = \delta_{ij}, \ i = 1, \ldots, S\}$ is the absorbing boundary of the dynamics. We note that the constraint $\sum_i x_i = 1$ reduces the number of independent variables from S to $S-1$, so we can choose x_S to be dependent on the others. Expanding the master equation in terms of $1/N$ we finally obtain, up to order N^{-2}, the final Fokker-Plank equation in continuous time (27)

$$\partial_t P(\vec{x}, t) = \frac{1}{N}\sum_{i=1}^{S-1}\partial_i^2\left[x_i(1-x_i)P(\vec{x}, t)\right] - \frac{2}{N}\sum_{j<i}^{S-1}\partial_i\partial_j\left[x_i x_j P(\vec{x}, t)\right], \quad (4)$$

where $\partial_i \equiv \partial/\partial x_i$.

The Fokker-Plank equation for the multi-state voter model at mean-field level does not have a drift term. This implies that the ensemble average density of each state $\langle x_i \rangle$ is constant in time. This observation allows to extend the calculation of the exit probability in the standard voter model to general boundary conditions in the multi-state case. Let us define the generalized exit probability $E_{\mathcal{A}}(\vec{x})$ as the probability that the system, starting in some random initial configuration $\vec{x}$, orders in some configuration $\vec{e}^i \in \mathcal{A}$, being $\mathcal{A}$ an arbitrary subset of the absorbing boundary $\mathcal{B}_S$. During the evolution of the system, the average densities are conserved. Let us define the quantity $\phi_{\mathcal{A}} = \sum_{i|\vec{e}^i \in \mathcal{A}} x_i$, which is also conserved. In the final consensus state, $\phi_{\mathcal{A}}$ will have a value 1 with probability $E_{\mathcal{A}}(\vec{x})$, and a value 0 with probability $1 - E_{\mathcal{A}}(\vec{x})$. We hence obtain the generalized exit probability

$$E_{\mathcal{A}}(\vec{x}) = \phi_{\mathcal{A}}. \quad (5)$$

The consensus time for a given initial condition $\vec{x}$ is given, on the other hand, by the general equation (28)

$$-N = \sum_{i=1}^{S-1}x_i(1-x_i)\partial_i^2 T_N(\vec{x}) - 2\sum_{j<i}^{S-1}x_i x_j\partial_i\partial_j T_N(\vec{x}), \quad (6)$$

subject to the boundary conditions

$$T_N(\vec{x} \in \mathcal{B}_S) = 0. \quad (7)$$

We can solve in a simple way this equation by noting that the consensus time $T_N(\vec{x})$ has to be symmetric under any exchange $x_i \leftrightarrow x_j$ with $i, j = 1, \ldots S$. Thus, we can impose the ansatz form

$$T_N(\vec{x}) = \sum_{i=1}^{S}\mathcal{F}(x_i), \quad (8)$$

where the function $\mathcal{F}(x)$ is independent of S. Introducing this ansatz into Eq. (6) we obtain

$$1 = -\frac{1}{N} \sum_{i=1}^{S} x_i(1-x_i)\partial_i^2 \mathcal{F}(x_i) = \sum_{i=1}^{S} \mathcal{G}(x_i), \qquad (9)$$

where we have defined

$$\mathcal{G}(x) = -\frac{1}{N}x(1-x)\mathcal{F}''(x). \qquad (10)$$

From Eq. (9) and the normalization condition for x_i, se can see that the only possible values of $\mathcal{G}(x)$ are $\mathcal{G}(x) = \text{const} \equiv 1/S$ or $\mathcal{G}(x) = x$. Considering now the solution for the $S = 2$ case in Eq. (1), we can see that the correct solution is given by the second case, which, after integration of Eq. (10), applying the boundary conditions $\mathcal{F}(1) = \mathcal{F}(0) = 0$, leads to

$$T_N(\vec{x}) = -N \sum_{i=1}^{S} (1-x_i)\ln(1-x_i). \qquad (11)$$

This solution generalizes the "entropic" form corresponding to the standard voter model, Eq. (1), recovering in a considerably simpler way the formal result previously obtained in (24; 25; 26).

From Eq. (11), we can analyze the behavior of the system in the limit of a large number of initial states. In particular, considering the homogeneous initial conditions $x_i = 1/S$, we have

$$T_N^H(S) = N(S-1)\ln\left(\frac{S}{S-1}\right). \qquad (12)$$

That is, as we could naively expect, the consensus time increases with the number of states allowed (the system is initially more disordered and therefore requires more time to order), but its growth is very slow and saturates in the limit $S \gg 1$. In fact, in the worst case scenario in a finite system, in which $S = N$, we have $T_N^H(S = N) = N(N-1)\ln[N/(N-1)] \to N$ in the limit of large N, being only a factor $1/\ln(2) \simeq 1.44$ larger that the binary case $S = 2$. This result can be rationalized considering that, when $S = N$, we are effectively describing an initial condition in which every agent has a different state, and the ordering occurs when one of this individuals manages to impose its state at the population level. It is therefore not surprising that we recover the N behavior observed in the binary model when the initial condition consists of a given state of one species in a population of individuals of the opposite state, and, crucially, only the runs in which the state of the mutant gets fixated are considered. From Eq. 12 we can obtain the form in which the saturation at large S is reached, namely,

$$1 - \frac{T_N^H(S)}{T_N^H(S=N)} = 1 - (S-1)\ln\left(\frac{S}{S-1}\right) \simeq \frac{1}{2S} + \mathcal{O}(S^{-2}), \qquad (13)$$

where the last expression is asymptotically valid in the limit of large S.

Another interesting property of the ordering dynamics of the multi-state voter model is the number of different states at time t, starting from an initial condition

with S states. We define the number of surviving states as $s(t) = \sum_i \delta_i(t)$, where $\delta_i(t) = 0$ if $x_i(t) = 0$, and $\delta_i(t) = 1$ otherwise. Expressions for this quantity have been given in the past in an implicit form (24; 26). Here we present a transparent derivation of its explicit form, based on the form of the consensus time, Eq. (11). We start by considering the average consensus time $\langle T_N(s) \rangle$ for a random initial configuration $\vec{x}$ with s different states that can be computed by averaging the consensus time $T_N(\vec{x})$ over all the initial conditions in the simplex $\mathcal{S}_s$,

$$\langle T_N(s) \rangle = \frac{1}{|\mathcal{S}_s|} \int_{\mathcal{S}_s} d\vec{x}\, T(\vec{x}), \tag{14}$$

where $|\mathcal{S}_s| = \frac{1}{(s-1)!}$ is the volume of the standard simplex $\mathcal{S}_s$. The integral in Eq. (14) can be computed using the variables $\sigma_n = \sum_i^n x_i$, which respect the constraint $0 \le \sigma_1 \le \sigma_2 \le \ldots \le \sigma_{s-1} \le \sigma_s = 1$, and noting that

$$\int_0^1 \sigma_{s-1} \log(\sigma_{s-1}) d\sigma_{s-1} \ldots \int_0^{\sigma_2} d\sigma_1 = \int_0^1 \frac{(\sigma_{s-1})^{s-1}}{(s-2)!} \log(\sigma_{s-1}) d\sigma_{s-1} = \frac{-1}{s^2(s-2)!}. \tag{15}$$

From here it follows that

$$\langle T_N(s) \rangle = \frac{N(s-1)}{s}, \tag{16}$$

Now, assuming that the average time to go from $s+1$ to s states is $\Delta T = \langle T_N(s+1) \rangle - \langle T_N(s) \rangle \simeq N s^{-2}$, for $s \gg 1$, we have

$$\frac{d}{dt} s(t) \simeq \frac{s(t + \Delta T) - s(t)}{\Delta T} = -\frac{1}{\Delta T} \simeq -\frac{s^2}{N}. \tag{17}$$

By integrating and inverting this relation we obtain that the number of surviving states $s(t)$ starting with random initial conditions with $s(0) = S$ states decays as

$$s(t) = \left(\frac{t}{N} + \frac{1}{S} \right)^{-1} \qquad \text{for} \qquad t \ll N, \tag{18}$$

expression which is valid far from the ordering time of the system $\sim N$. In the case $t \gg N$, we expect $s(t) \sim$ const in surviving runs; that is, averaging over dynamical realizations that have not reached consensus at time t. In this case, we will assume that $s(t)$, averaged over all runs, will decay as the survival probability (29). Assuming the exponential form derived in Ref. (29) for the standard voter model, we will expect to observe

$$s(t) \sim \exp(-2t/N) \qquad \text{for} \qquad t \gg N. \tag{19}$$

In Fig. 1 we check this prediction by means of numerical simulations of the multi-state voter model on a complete graph. The plot shows the behavior of $s(t)$ for homogeneous initial conditions, which is fully compatible with the analytical predictions in Eqs. (18) and (19).

3. Numerical results in finite dimensional lattices

In this section we present and discuss the results of numerical simulations of the multi-state voter model on lattices of dimension $d = 1$ and $d = 2$, comparing them with the analytical results obtained at the mean-field level.
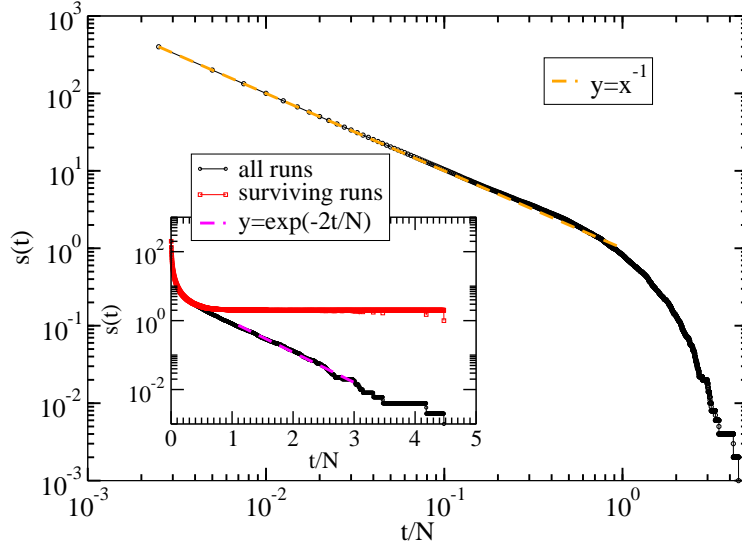
Figure 1. Number of surviving states $s(t)$ as a function of time in the multi-state voter model on a complete graph of size $N = 400$, starting with homogeneous condition and $s(0) = S = N$ states. We compare the result with eq. (18). In the inset we show $s(t)$ for $t \gg N$, averaged over all runs, compared with eq. (19). We observe $s(t) \sim$ const if averaged only over surviving runs.

3.1. Consensus time

We focus in the first place on the behavior of the consensus time $T_N(\vec{x})$ as a function of the initial densities of the different states. We consider the simplest case $S = 3$, parametrizing the initial configuration as $\vec{x} = \{x_1, x_2, x_3\} \equiv \{x, \alpha(1-x), (1-\alpha)(1-x)\}$, with $x \in [0, 1]$ and $\alpha \in [0, 1]$. This parametrization preserves the normalization, $\sum_i x_i = 1$, and has the advantage that, for a given value of α, the whole range of values of x can be explored. In Fig. 2 we plot the consensus time $T_N(\alpha, x)$ computed in lattices of dimension $d = 1$ and $d = 2$ as a function of x, and for different fixed values of α. In order to get rid of size-dependent prefactors due to the dimensionality in the consensus time, we normalized it by its value at $x = 0$, which takes the form $T_N(\alpha) = -N[\alpha \ln(\alpha) + (1-\alpha) \ln(1-\alpha)]$ at mean-field level. The numerical simulations for $d = 2$ fit quite precisely the theoretical mean-field prediction of the consensus time dependence on the initial configuration $\vec{x}$, developed in Sec. 2, with only slight deviations for small α and close to $x \sim 0.5$. On the other hand, strong deviations are noticeable in dimension $d = 1$, specially for small values of α. This result is in agreement with the expectation for the standard voter model, in which the mean-field consensus time Eq. (1) is expected to be exact only for $d \geq 2$ (6).
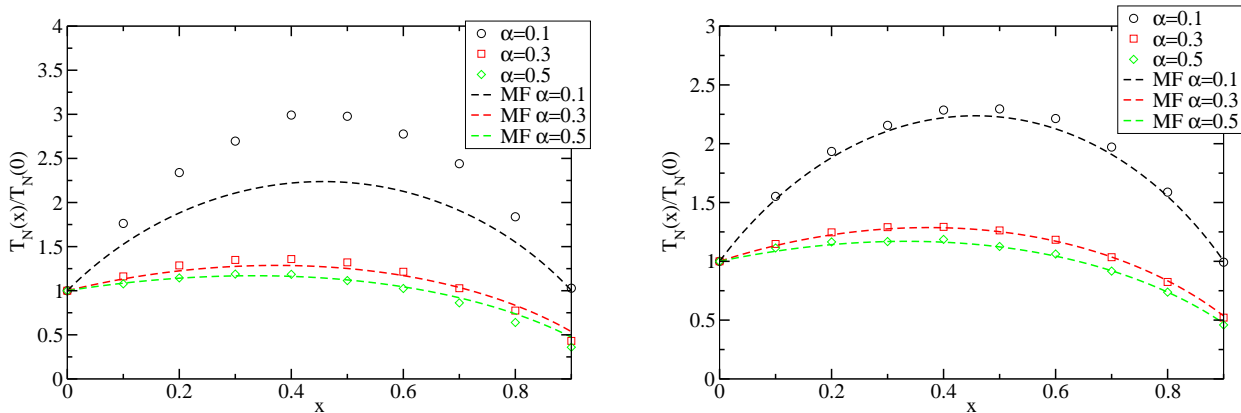
Figure 2. Normalized consensus time $T_N(x)/T_N(0)$ as a function of x for the initial configuration $\vec{x} = \{x, \alpha(1-x), (1-\alpha)(1-x)\}$ in regular lattices of dimension $d = 1$ (left) and $d = 2$ (right) of size $N = 400$ sites, compared with the analytical mean-field prediction Eq. (11).
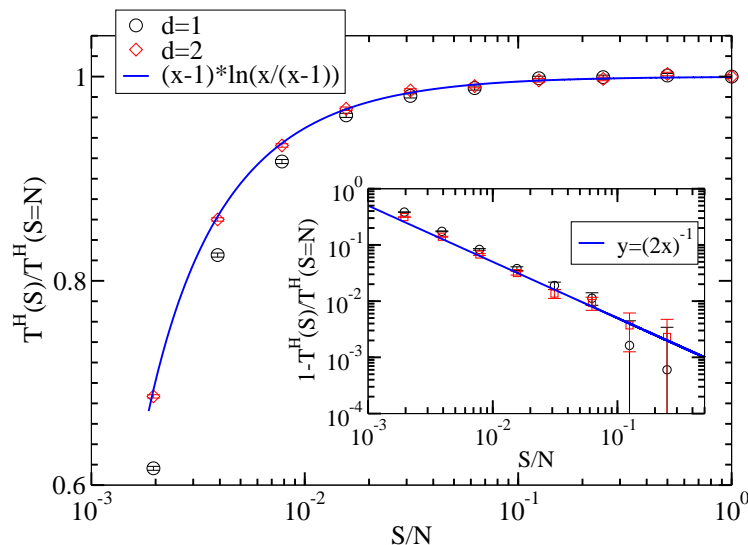
3.2. Effect of the number of states

We have seen that, at the mean field level, and for homogeneous initial conditions $x_i = 1/S$ for $i = 1, \ldots, S$, the consensus time increases with S towards it limit value $T_N^H(S = N)$ with a power-law form, as given by Eq. (13). In Fig. 3 we plot the rescaled consensus time $T_N^H(S)/T_N^H(S = N)$ as a function of S for lattices of dimension $d = 1$ and $d = 2$ and fixed size $N = 10^3$. From this figure we observe once again that the $d = 2$ behavior is well fitted by the mean-field prediction, while the $d = 1$ case shows deviations for small S. Interestingly, increasing the number of initial states S reduces the deviation from the mean-field theory, in a way that the behavior for $S \rightarrow N$ is very well fitted by Eq. (13).

3.3. Number of surviving states $s(t)$

At the mean field level, the number of surviving states $s(t)$, starting from maximally heterogeneous conditions $S = N$, decays as $s(t) \sim S t^{-\beta}$ in the initial time regime, with an exponent $\beta = 1$, crossing over to a exponential decay at large times. In Fig. 4 we show the number of surviving states $s(t)$ as a function of time corresponding to numerical simulations on $d = 1$ and $d = 2$ lattices.

From the results of Fig. 4, it is clear that the initial decay of the density of surviving states follows, as expected, a power-law form. The decay is, however, slower than the mean-field prediction. In particular, we see that in $d = 1$, $s(t) \sim N t^{-1/2}$, while in $d = 2$ numerical data can be fitted to the form $s(t) \sim N t^{-1} \log t$, corresponding to mean-field behavior with a logarithmic correction (shown in the inset of Fig. 4). On the other hand, we observe that the tail of the density of surviving states is again exponential; in

Figure 3. Rescaled consensus time $T_N^H(S)/T_N^H(S = N)$ starting from homogenous initial conditions, as a function of the number of states S/N in dimension $d = 1$ and $d = 2$, for $N = 10^3$, compared with the theoretical mean-field prediction Eq. (12). In the inset we plot the quantity $1 - T_N^H(S)/T_N^H(S = N)$, showing the power-law decay with S. Error bars represent the standard deviation error on the average of the distribution. Each point is averaged over 10^5 runs.

particular, we find that in the large time regime we can fit $s(t) \sim \exp(-1.5t/N_{\text{eff}})$ for $d = 1$, while $s(t) \sim \exp(-2t/N_{\text{eff}})$ for $d = 2$.

The origin of the slowing down in the decay of the number of surviving states in low dimensions can be attributed to the formation of spatial domains of sites in the same state, which have to annihilate diffusively in order to reach the consensus state. In this line, the behavior of the number of surviving states in $d = 1$ can be understood by means of a simple argument: At a given time $t > 1$, there will be a number of surviving states $s(t)$. Assuming that sites with the same state form clusters, the activity will be driven by the diffusive fluctuation of the boundaries of those clusters, which will have a length $\ell \propto t^{1/2}$. The number of different clusters will thus be $s(t) \propto N/\ell \sim Nt^{-1/2}$, recovering the observed time dependence.

3.4. Effects of correlated initial configurations

Considering the multi-state voter model on a finite lattice allows to investigate the effects of correlated initial configurations in the dynamical approach to the consensus state, which should be particularly important in one-dimensional lattices. We have thus simulated the multi-state voter model in a $d = 1$ lattice, stating from an initial configuration of S states arranged in S contiguous blocks on length N/S in a lattice of size N. In Fig. 5 we plot the consensus time for $T_N^C(S)$ in this correlated initial
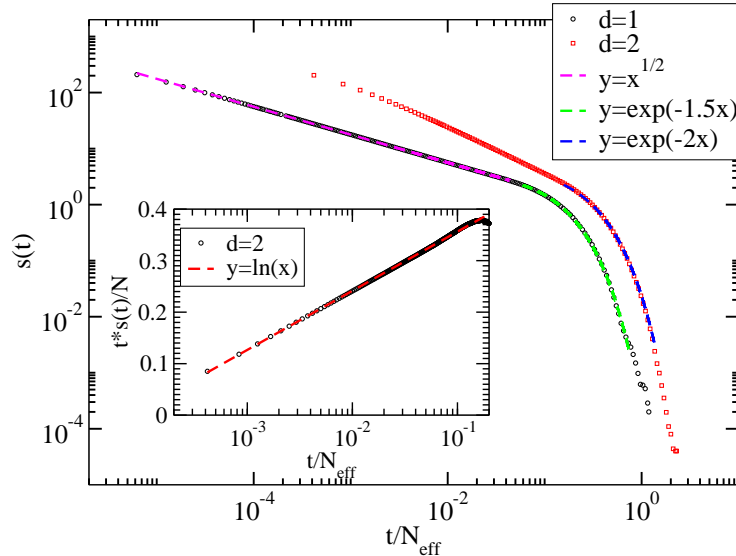
Figure 4. Surviving states $s(t)$ as a function of rescaled time t/N_{eff} for a $d = 1$ and $d = 2$ lattice of $N = 400$ nodes, starting with homogeneous condition and $s(0) = S = N$ states. For $t \ll N_{\text{eff}}$, the number of surviving states decays as $s(t) \sim t^{-1/2}$ for $d = 1$ and $s(t) \sim t^{-1} \log t$ for $d = 2$ (inset). For large t, $s(t)$ decays exponentially in both cases.

conditions as a function of S, comparing it with the consensus time starting with uncorrelated homogeneous initial conditions, $T_N^H(S)$. We find that the effect of starting with correlated initial condition strongly slows down the achievement of consensus. As expected, the difference between the consensus time with correlated and homogeneous initials conditions approaches zero for $S \to N$ with a behavior compatible with a power-law form of exponent -1, i.e.

$$\frac{T_N^C(S)}{T_N(S)} - 1 \sim \left(\frac{S}{N} - 1\right)^{-1} \qquad \text{for } S \to N. \tag{20}$$

4. Conclusions

In this paper we have addressed the general scenario of the voter model in which the number of different states allowed in the model can be larger than two, and, in the thermodynamic limit, even unlimited. At the mean-field level, we have presented derivations for the expression of the exit probability, the consensus time (which generalizes naturally the 'entropic' form observed for the two-states case), and the density of surviving states as a function of time. We have highlighted that in the limit of $S \to \infty$ the ordering time is only $1/\ln 2$ times bigger than in the binary voter model, and with a simple analytic argument we have found the decay of the number of surviving states in times. Finally, we have studied numerically the behavior of the
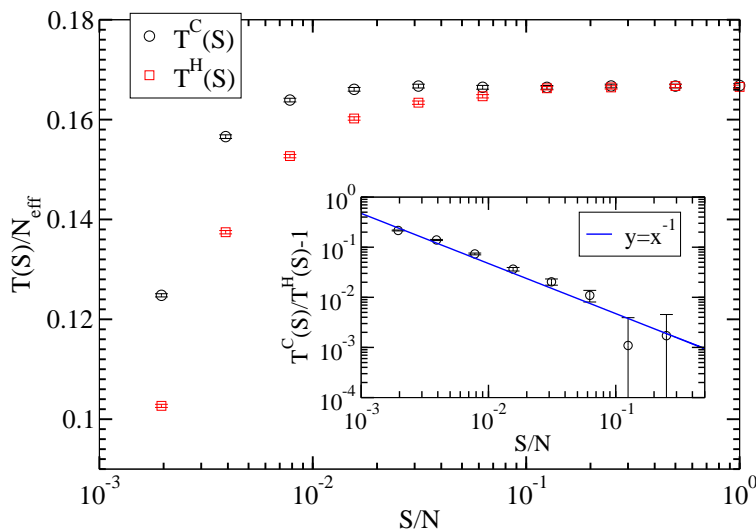
Figure 5. Consensus time $T(S)/N_{\mathrm{eff}}$ as a function of the number of initial states S on a $d = 1$ lattice with $N = 10^3$ nodes with a correlated initial configuration made of blocks of voters in the same state, of initial length N/S, $T^C(S)$, compared with the consensus time obtained with random homogeneous initial conditions $T^H(S)$w. In the inset we show that the quantity $T^c(S)/T^H(S) - 1$ goes to zero with a power law behavior with exponent -1. Error bars are obtained as in Figure 3. Each point is averaged over 10^5 runs.

multi state voter model on $1-$ and $2-$dimensional lattices, and compared the results with the binary case. The consensus time in the $d = 2$ case is well predicted by the mean-field theory, while the uni-dimensional case behaves differently. Remarkably, it increases with the number of initial states with a power-law form as predicted by the mean-field theory, for both $d = 1$ and $d = 2$. We have also addressed the effect of correlated initial conditions on the consensus time, finding out that the relevance of this effect decreases with the number of initial states with a power law behavior. In summary, our results show that the number of states is not a trivial parameter in the voter model, and it affects the overall dynamics in subtle ways.

Acknowledgments

References

[1] Castellano C, Fortunato S and Loreto V 2009 *Rev. Mod. Phys.* **81** 591–646

[2] Clifford P and Sudbury A 1973 *Biometrika* **60** 581–588

[3] Holley R A and Liggett T M 1975 *Annals of Probability* **3** 643–663

[4] Liggett T M 1999 *Stochastic interacting particle systems: Contact, Voter, and Exclusion processes* (New York: Springer-Verlag)

[5] Krapivsky P, Redner S and Ben-Naim E 2010 *A Kinetic View of Statistical Physics* (Cambridge: Cambridge University Press)

[6] Blythe R A 2010 *Journal of Physics A: Mathematical and Theoretical* **43** 385003

[7] Mobilia M 2003 *Phys. Rev. Lett.* **91** 028701

[8] Mobilia M, Petersen A and Redner S 2007 *Journal of Statistical Mechanics: Theory and Experiment* **2007** P08029

[9] Dall'Asta L and Castellano C 2007 *Europhys. Lett.* **77** 60005

[10] Stark H, Tessone C J and Schweitzer F 2008 *Phys. Rev. Lett.* **101** 018701

[11] Lambiotte R and Redner S 2008 *Europhys. Lett.* **82** 18007

[12] Cox J T and Durret R 1991 Nonlinear voter models *Random Walks, Brownian Motion, and Interacting Particle Systems* ed Durrett R and Kesten H (Boston: Birkhauser) pp 189–202

[13] de Oliveira M, Mendes J and Santos M 1993 *J. Phys. A* **26** 2317–2324

[14] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235–268

[15] Sire C and Majumdar S N 1995 *Phys. Rev. E* **52** 244–254

[16] López F J, Sanz G and Sobottka M 2008 *Journal of Statistical Mechanics: Theory and Experiment* **2008** P05006

[17] Böhme G A and Gross T 2012 *Phys. Rev. E* **85** 066117

[18] Hubbell S 2001 *The Unified Neutral Theory of Biodiversity and Biogeography* Monographs in Population Biology (Princeton, NJ: Princeton University Press)

[19] McKane A J, Alonso D and Solé R V 2004 *Theoretical Population Biology* **65** 67 – 73

[20] Pigolotti S, Flammini A, Marsili M and Maritan A 2005 *Proc. Natl. Acad. Sci. USA* **102** 15747

[21] Volovik D, Mobilia M and Redner S 2009 *Europhy. Lett.* **85** 48003

[22] Vázquez F, Krapivsky P L and Redner S 2003 *Journal of Physics A: Mathematical and General* **36** L61

[23] Castelló X, Eguíluz V M and Miguel M S 2006 *New Journal of Physics* **8** 308

[24] Tavaré S 1984 *Theoretical Population Biology* **26** 119–164

[25] Cox J T 1989 *The Annals of Probability* **17** 1333–1366

[26] Baxter G J, Blythe R A and McKane A J 2007 *Mathematical Biosciences* **209** 124–170

[27] Blythe R A and McKane A J 2007 *J. Stat. Mech.* P07018

[28] Gardiner C W 1985 *Handbook of stochastic methods* 2nd ed (Berlin: Springer)

[29] F Slanina and H Lavicka 2003 *Eur. Phys. J. B* **35** 279–288