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Quantum Spectral Curve for AdS_3/CFT_2: a proposal

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ABSTRACT: We conjecture the Quantum Spectral Curve equations for string theory on $AdS_3 \times S^3 \times T^4$ with RR charge and its CFT$_2$ dual. We show that in the large-length regime, under additional mild assumptions, the QSC reproduces the Asymptotic Bethe Ansatz equations for the massive sector of the theory, including the exact dressing phases found in the literature. The structure of the QSC shares many similarities with the previously known AdS$_5$ and AdS$_4$ cases, but contains a critical new feature – the branch cuts are no longer quadratic. Nevertheless, we show that much of the QSC analysis can be suitably generalised producing a self-consistent system of equations. While further tests are necessary, particularly outside the massive sector, the simplicity and self-consistency of our construction suggests the completeness of the QSC.

Contents

1 Introduction

The Quantum Spectral Curve (QSC) has become an indispensable tool of precision spectroscopy in $\mathrm{AdS}_5/\mathrm{CFT}_4$ and $\mathrm{AdS}_4/\mathrm{CFT}_3$ holographic models [1–20]. For a review on the QSC, see [21]. In this paper, we shall take a step towards extending this powerful method to the spectral problem in another important holographic duality, namely planar $\mathrm{AdS}_3/\mathrm{CFT}_2$.

It is believed that $\mathrm{AdS}_3/\mathrm{CFT}_2$ dual pairs with 8+8 supersymmetries are integrable [22–24].[1] This is the maximal amount of supersymmetry that is allowed for string theory backgrounds of the form $\mathrm{AdS}_3 \times \mathcal{M}_7$, with $\mathcal{M}_7 = \mathrm{S}^3 \times \mathrm{T}^4$ or $\mathcal{M}_7 = \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. The symmetries of these two backgrounds are respectively the small and large $(4,4)$ superconformal symmetries, whose Lie sub-algebras are $\mathrm{psu}(1,1|2)^2$ and $\mathrm{d}(2,1;\alpha)^2$. The exact S matrices can be found by imposing compatibility with the (centrally extended) vacuum-preserving symmetry algebras of the two theories [27–32], much like what can be done in higher-dimensional cases [33]. In this paper, we will focus on string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ supported by R-R charge.

An important difference between these theories and higher-dimensional integrable string backgrounds is the presence of massless excitations in the worldsheet theory, in addition to the more familiar massive ones. The resulting integrable 2-to-2 S matrix breaks up into independent pieces for the scattering of massless/massless, massive/massive and mixed mass excitations. Expressed in terms of Zhukovsky variables, the S matrices resemble those of higher-dimensional integrable holographic theories, with the mass entering through the shortening conditions. This resemblance is particularly striking in the case of massive excitations [22, 28], where in the weak-coupling limit the Bethe Equations (BEs) reduce to those of a homogeneous nearest-neighbour $\mathrm{psu}(1,1|2) \times \mathrm{psu}(1,1|2)$ spin-chain, with the two factors only connected by the level-matching condition. Away from the weak-coupling limit, the BEs for each $\mathrm{psu}(1,1|2)$ wing bear a striking similarity to the corresponding part of the $\mathrm{psu}(2,2|4)$ BEs of $\mathrm{AdS}_5/\mathrm{CFT}_4$. These observations suggest that (at least a part of) the $\mathrm{AdS}_3/\mathrm{CFT}_2$ Q-system can be constructed using two sets of $\mathrm{psu}(1,1|2)$ Q-functions, one for each wing, and coupling them together in a way that is consistent with the crossing. The Q-system is an important part of any known QSC [2, 14]. In this paper, instead of deriving the QSC following a long route from TBA equations, we use the Q-system as a starting point supplying it with the analyticity properties following closely the previously known cases. However, fairly quickly we realise that one of the analyticity assumptions must be relaxed in our case – namely we no longer assume the square-root type of singularity near the branch points. This new feature is inherently connected with the properties of the *dressing factors* of [30–32].

Each S matrix block comes with a dressing factor which is not fixed by symmetry requirements. Dressing factors satisfy crossing equations [30–32] that follow from the Hopf algebra structure of the theory [34–36]. In the case of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ supported by R-R charge only, dressing factors which solve these crossing equations have been found [37, 38]. There are two independent dressing phases that enter the massive S

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[1]For earlier work in this direction see [25, 26].

matrix, corresponding to either scattering excitations in the same psu$(1,1|2)$ wing or in different wings. Their sum is equal to (twice) the Beisert-Eden-Staudacher (BES) phase [39], while their difference is a new phase, which appears only at the so-called Hernandez-Lopez order [40]. The relative simplicity of this latter factor is related to the fact that bound-states in the theory can only be made from massive constituent excitations from the same psu$(1,1|2)$ wing. As with all solutions of crossing equations, there is potential for CDD ambiguities due to homogeneous solutions of crossing. The absence of such additional factors was demonstrated in [41], where it was shown that the proposed dressing factors have exactly the required Dorey-Hofman-Maldacena (DHM) double poles and zeros [42].

In the case of massive modes, crossing maps the two psu$(1,1|2)$ wings into one another. This suggest that, as a consequence of crossing, the two copies of the psu$(1,1|2)$ Q-systems should be related by a suitable analytic continuation. Analogous gluing conditions, which can be traced back to crossing, are known to exist in the AdS$_5$/CFT$_4$ and AdS$_4$/CFT$_3$ QSC and are needed in addition to the QQ-relations to constrain the system to a closed system of equations, which can be treated analytically [6, 7] in some limits and by means of numerical analysis [4] in general. Furthermore, the simple gluing of the Q-functions can be shown [2, 14] to produce a rather involved expression for the BES dressing phase when considering the large-volume solution.

While a number of ingredients for the current construction are borrowed from the known cases, the new type of near-branch point singularity is a crucial novel ingredient. As a test of our proposal we demonstrated how the ABA equations for the massive sector are precisely reproduced in the large-length limit including the dressing phases. In these considerations, we had to make an additional simplifying assumption about the monodromy of μ-function in the asymptotic limit, which we have not managed to prove. At the same time, we only reproduced the massive sector equations, which suggests that removing this assumption could revive all the massless degrees of freedom, but we leave this question for future work. Another important direction is to verify the completeness of our system of equations by solving it either numerically as in [4] or in a near BPS limits like in [3, 15].

An intuitive way in which to understand the effect of massless modes is that the massless dispersion relation can be viewed in an approximate sense as the large coupling limit of the massive one, as long as the particle momentum is kept fixed. In the QSC formalism, the coupling usually controls the distance between the cuts in the rescaled spectral parameter u/g. As a result, in the zero mass limit, one might expect this to lead to a number of quadratic cuts collapsing on top of one another. This suggests that, in models with massless modes, the QSC may have a more general singularity structure near the branch points, rather than the conventional square root behaviour seen in higher-dimensional cases. We also expect the analyticity to be simplified in the purely massless sector by employing the pseudo-relativistic variable of [43, 44].

In fact, the assumption of a square-root singularity is over-restrictive in AdS$_3$ because it gives rise to a new algebraic constraint on the Q-functions in addition to the QQ-relations. In turn, such a condition collapses the two wings of Q-functions into one, likely leading to drastically simpler analytic properties such as those seen in the Hubbard model [45], based on a *single su*$(2|2)$ symmetry.

The rest of the paper is organised as follows. In section 2, we collect pre-existing results on integrability for the AdS$_3$/CFT$_2$ duality, which will inspire our conjecture, and describe the algebraic structure of the Q-system for $psu(1,1|2)$. Section 3 presents our main proposal for the Quantum Spectral Curve, and describes the unique features of these equations as compared to the previous cases. In section 2.1, we study the large-volume limit of these equations, reproducing precisely the full Asymptotic Bethe Ansatz for massive modes. Finally, we present our conclusions and discuss some future directions. The paper also contains three appendices collecting some notations and technical details.

Note added: The work described here begun before the epidemic. Shortly after the first wave was coming to an end in Europe, we concluded that the large-length limit was incompatible with square-root cuts as described in section 4.3.1. During the recent "Integrability in Lower Dimensional AdS/CFT" online workshop we learnt that Simon Ekhammar and Dima Volin had also independently come to a similar conclusion. We are grateful to Dima and Simon for informing us of their findings and coordinating on the release date of the manuscripts to the arXiv. Motivated by these discussions, we revisited our construction and found that relaxing the branch-cut condition allows for a consistent definition of the QSC together with a large-length limit that reproduces the all-loop massive ABA equations found in the literature. Our proposal for the QSC seems to be fully consistent with the one published simultaneously in [46].

2 Data on the AdS$_3$/CFT$_2$ integrable system

In this section we assemble together the known facts about the AdS$_3$/CFT$_2$ integrable system. This includes the asymptotic Bethe ansatz for massive modes, classical algebraic curve and the $psu(1,1|2)$ Q-system.

2.1 Asymptotic Bethe Ansatz

The massive Asymptotic Bethe Ansatz (ABA) equations which we will be referring to are those presented in [28]. The symmetry controlling the Bethe equations is psu$(1,1|2)^2$. Each copy of psu$(1,1|2)$ has associated one momentum carrying root and two auxiliary roots. These are called x, y_1 and y_3 for one copy of psu$(1,1|2)$ and $\bar{x}, y_{\bar{1}}$ and $y_{\bar{3}}$, respectively, for

the other copy. The explicit form of the BEs is:

$$1 = \prod_{j=1}^{K_2} \frac{y_{1,k} - x_j^+}{y_{1,k} - x_j^-} \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{y_{1,k}\bar x_j^-}}{1 - \frac{1}{y_{1,k}\bar x_j^+}},$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K_2} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \sigma^2(x_k, x_j) \prod_{j=1}^{K_1} \frac{x_k^- - y_{1,j}}{x_k^+ - y_{1,j}} \prod_{j=1}^{K_3} \frac{x_k^- - y_{3,j}}{x_k^+ - y_{3,j}}$$

$$\times \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{x_k^+ \bar x_j^+}}{1 - \frac{1}{x_k^- \bar x_j^-}} \frac{1 - \frac{1}{x_k^+ \bar x_j^-}}{1 - \frac{1}{x_k^- \bar x_j^+}} \tilde\sigma^2(x_k, \bar x_j) \prod_{j=1}^{K_{\bar 1}} \frac{1 - \frac{1}{x_k^- y_{\bar 1,j}}}{1 - \frac{1}{x_k^+ y_{\bar 1,j}}} \prod_{j=1}^{K_{\bar 3}} \frac{1 - \frac{1}{x_k^- y_{\bar 3,j}}}{1 - \frac{1}{x_k^+ y_{\bar 3,j}}},$$

$$1 = \prod_{j=1}^{K_2} \frac{y_{3,k} - x_j^+}{y_{3,k} - x_j^-} \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{y_{3,k}\bar x_j^-}}{1 - \frac{1}{y_{3,k}\bar x_j^+}}, \tag{2.1}$$

$$1 = \prod_{j=1}^{K_{\bar 2}} \frac{y_{\bar 1,k} - \bar x_j^-}{y_{\bar 1,k} - \bar x_j^+} \prod_{j=1}^{K_2} \frac{1 - \frac{1}{y_{\bar 1,k} x_j^+}}{1 - \frac{1}{y_{\bar 1,k} x_j^-}},$$

$$\left(\frac{\bar x_k^+}{\bar x_k^-}\right)^L = \prod_{j\neq k}^{K_{\bar 2}} \frac{\bar x_k^- - \bar x_j^+}{\bar x_k^+ - \bar x_j^-} \frac{1 - \frac{1}{\bar x_k^+ \bar x_j^-}}{1 - \frac{1}{\bar x_k^- \bar x_j^+}} \sigma^2(\bar x_k, \bar x_j) \prod_{j=1}^{K_{\bar 1}} \frac{\bar x_k^+ - y_{\bar 1,j}}{\bar x_k^- - y_{\bar 1,j}} \prod_{j=1}^{K_{\bar 3}} \frac{\bar x_k^+ - y_{\bar 3,j}}{\bar x_k^- - y_{\bar 3,j}}$$

$$\times \prod_{j=1}^{K_2} \frac{1 - \frac{1}{\bar x_k^- x_j^-}}{1 - \frac{1}{\bar x_k^+ x_j^+}} \frac{1 - \frac{1}{\bar x_k^+ x_j^-}}{1 - \frac{1}{\bar x_k^- x_j^+}} \tilde\sigma^2(\bar x_k, x_j) \prod_{j=1}^{K_1} \frac{1 - \frac{1}{\bar x_k^+ y_{1,j}}}{1 - \frac{1}{\bar x_k^- y_{1,j}}} \prod_{j=1}^{K_3} \frac{1 - \frac{1}{\bar x_k^+ y_{3,j}}}{1 - \frac{1}{\bar x_k^- y_{3,j}}},$$

$$1 = \prod_{j=1}^{K_{\bar 2}} \frac{y_{\bar 3,k} - \bar x_j^-}{y_{\bar 3,k} - \bar x_j^+} \prod_{j=1}^{K_2} \frac{1 - \frac{1}{y_{\bar 3,k} x_j^+}}{1 - \frac{1}{y_{\bar 3,k} x_j^-}}. \tag{2.2}$$

The Bethe equations are written in the grading described in Figure 3 of [28]. The massless modes will not be included in our analysis, and they do not feature anywhere in the Bethe equations we write. There is a further level-matching constraint on the solutions to the Bethe equations, in the form of

$$1 = \prod_{j=1}^{K_2} \frac{x_j^+}{x_j^-} \prod_{j=1}^{K_{\bar 2}} \frac{\bar x_j^+}{\bar x_j^-} \tag{2.3}$$

(once more disregarding massless modes). The Zhukovsky variables satisfy the familiar constraint given by (suppressing the particle index)

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{h}, \qquad \frac{x^+}{x^-} = e^{ip}, \tag{2.4}$$

where h is the coupling constant of the theory and p is the particle momentum. The same holds for the barred variables. The dispersion relation that gives the energy of a particle of momentum p reads

$$E(p) = \sqrt{1 + 16\, h^2 \sin^2 \frac{p}{2}}, \tag{2.5}$$

and the anomalous dimension of the state associated to a solution of the ABA is given by

$$\delta\Delta \equiv \gamma = 2h \sum_{k=1}^{K_2} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right) + 2h \sum_{j=1}^{K_{\bar{2}}} \left(\frac{i}{\bar{x}_j^+} - \frac{i}{\bar{x}_j^-} \right). \tag{2.6}$$

The explicit form of the dressing phases from [37] is given by

$$\sigma(p_1, p_2) = \mathrm{e}^{i\theta(p_1,p_2)}, \qquad \tilde{\sigma}(p_1, p_2) = \mathrm{e}^{i\tilde{\theta}(p_1,p_2)}, \tag{2.7}$$

with the familiar splitting

$$\theta(p_1, p_2) = \chi(x_1^+, x_2^+) + \chi(x_1^-, x_2^-) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+), \tag{2.8}$$

with similar expressions for $\tilde{\sigma}$. The individual blocks read

$$\chi(x,y) = \chi^{\mathrm{BES}}(x,y) - \frac{1}{2}\Big[\chi^{\mathrm{HL}}(x,y) - \chi^-(x,y)\Big], \tag{2.9}$$

$$\tilde{\chi}(x,y) = \chi^{\mathrm{BES}}(x,y) - \frac{1}{2}\Big[\chi^{\mathrm{HL}}(x,y) + \chi^-(x,y)\Big].$$

The part denoted by BES is the Beisert-Eden-Staudacher [39] dressing phase - its expression can be found reproduced in the review [47]. The same holds for the HL part, referring to the Hernandez-Lopez phase [48]

$$\chi^{\mathrm{HL}}(x,y) = \left(\int_{C^+} - \int_{C^-} \right) \frac{dw}{4\pi} \frac{1}{x-w} \Big[\log(y-w) - \log(y - \frac{1}{w}) \Big]. \tag{2.10}$$

The new ingredient which was constructed in [37] is given by

$$\chi^-(x,y) = \left(\int_{C^+} - \int_{C^-} \right) \frac{dw}{8\pi} \frac{1}{x-w} \log\Big[(y-w)\Big(1 - \frac{1}{yw}\Big) \Big] - x \leftrightarrow y, \tag{2.11}$$

where the contours $C^\pm$ denote the upper (resp., lower) half semicircle in the complex w-plane, both running anti-clockwise. These expressions are valid in the physical region $|x| > 1, |y| > 1$. The notation χ^- is commonly used in the AdS$_3$ literature for this portion of the phase. The minus sign should not be confused with a shift in the spectral parameter - as it will otherwise always be meant in this paper.

Since we will be merely concerned with the massive modes, it is expected that the Asymptotic Bethe equations which we have written above should be valid exactly in the coupling h but only asymptotically in the length L. In other words, wrapping corrections are expected to be exponentially suppressed [49]. This situation would be rather different were we to include massless modes, whose impact on the TBA is not exponentially suppressed - they are expected to be polynomially suppressed in the presence of mixed massive-massless interactions [50], or require exact solutions as in the case of the conformal TBA of [44, 51] (see also [52, 53]).

Notice also that $4h$ gives the size of the branch cut which goes to zero at weak coupling. Since all interaction between the two psu$(1,1|2)$ wings go through the branch-cut, the two wings become completely decoupled in the limit of small coupling constant $h \to 0$, except for the level-matching condition.

2.2 Main features of the classical curve

The Quantum Spectral Curve is a quantum version of the classical curve, which thus contains crucial structural hints. We shall from now on denote with un-dotted/dotted indices the variables pertaining to the first/second wing, respectively, of the Dynkin diagram – corresponding to the first/second copy of $\mathrm{psu}(1,1|2)$.

Here we present a short description of some aspects of the algebraic curve describing the integrability of classical solutions of string theory on $AdS_3 \times S^3 \times T^4$, following the discussion in [54]. This description is based on 4+4 quasimomenta, associated to the fundamental representations of the two $\mathrm{psu}(1,1|2)$'s. They will be denoted by $(p_1^A, p_2^A, p_1^S, p_2^S)$ and $(p_{\dot{1}}^A, p_{\dot{2}}^A, p_{\dot{1}}^S, p_{\dot{2}}^S)$. Each quasimomentum naturally parametrises motion in one of the factors of the target space, which is marked by the superscripts A, S for AdS_3 and S^3, respectively. They are very important quantities which are expected to arise in a WKB-type approximation of the Q-functions in the classical limit of the quantum spectral curve.

The p's are naturally seen as functions of the Zhukovsky variables, and contain the symmetry charges of the solution in their asymptotics:

$$
\begin{pmatrix} p_1^A \\ p_2^A \\ p_1^S \\ p_2^S \end{pmatrix} \simeq \frac{1}{2hx} \begin{pmatrix} -\Delta - S - \hat{B} \\ +\Delta + S - \hat{B} \\ -J - K - \hat{B} \\ +J + K - \hat{B} \end{pmatrix} = \frac{1}{2hx} \begin{pmatrix} -\gamma - 2K_1 - L \\ +\gamma + 2K_3 + L \\ -2K_1 + 2K_2 - L \\ -2K_2 + 2K_3 + L \end{pmatrix} , \tag{2.12}
$$

$$
\begin{pmatrix} p_{\dot{1}}^A \\ p_{\dot{2}}^A \\ p_{\dot{1}}^S \\ p_{\dot{2}}^S \end{pmatrix} \simeq \frac{1}{2hx} \begin{pmatrix} +\Delta - S - \check{B} \\ -\Delta + S - \check{B} \\ +J - K - \check{B} \\ -J + K - \check{B} \end{pmatrix} = \frac{1}{2hx} \begin{pmatrix} +\gamma - 2K_{\dot{1}} + 2K_{\dot{2}} + L \\ -\gamma - 2K_{\dot{2}} + 2K_{\dot{3}} - L \\ -2K_{\dot{1}} + L \\ 2K_{\dot{3}} - L \end{pmatrix} , \tag{2.13}
$$

where on the rhs we used the explicit expression of the charges in terms of Bethe roots numbers. Finally, the classical curve tells us how the quasimomenta in the two wings are related. In particular, for the quasimomenta describing motion in AdS_3, the relation is extremely simple and consists in analytic continuation

$$
p_a^A\left(\frac{1}{x}\right) = p_{\dot{a}}^A(x), \quad a = 1, 2, \tag{2.14}
$$

as described in equations (7.13) and (7.38) of [22]. We will lift this property to the quantum case.

2.3 Algebra of the $\mathrm{psu}(1,1|2)$ Q-system

The sets of functional relations between the Q-functions (known as Q-systems) take a universal form depending only on the symmetry algebra of the integrable system. Since our model contains two copies of $\mathrm{psu}(1,1|2)$, important input for our construction comes from the structure of QQ relations for this algebra.

The $\mathrm{psu}(1,1|2)$ Q-system contains 16 independent Q-functions depending on the spectral parameter u. They can be labelled as $Q_{A|I}$, where A, I are completely anti-symmetric strings of indices made from $\{1, 2\}$

$$
A, I \in \{\emptyset, 1, 2, (12)\} , \tag{2.15}
$$

interrelated by the QQ relations

$$Q_{aA|I i}Q_{A|I i} = Q_{aA|I i}^{+}Q_{A|I}^{-} - Q_{aA|I i}^{-}Q_{A|I}^{+} \,, \tag{2.16}$$

$$Q_{12|I}Q_{\emptyset|I} = Q_{1|I}^{+}Q_{2|I}^{-} - Q_{1|I}^{-}Q_{2|I}^{+} \,, \tag{2.17}$$

$$Q_{A|12}Q_{A|\emptyset} = Q_{A|1}^{+}Q_{A|2}^{-} - Q_{A|1}^{-}Q_{A|2}^{+} \,, \tag{2.18}$$

where $a, i \in \{1, 2\}$ are single indices, and A, I are anti-symmetric multi-indices defined above. The first type of relation (2.16) is usually called fermionic, and the remaining two bosonic. In these equations, we are using the notation adopted in the whole paper for shifts in the spectral parameter u: for any function g,

$$g^{[\pm n]}(u) \equiv g(u \pm \tfrac{in}{2}) \,, \qquad g^{\pm}(u) \equiv g^{[\pm 1]}(u). \tag{2.19}$$

In our proposal, the QSC will contain two copies of these relations, which we will denote by distinguishing between dotted and undotted indices (giving us 16+16 Q-functions). In this section, we focus on one wing, and elaborate on some consequences of (2.16)-(2.18).

We will make a simple special choice for the Q-functions with the extremal combinations of indices:

$$Q_{\emptyset|\emptyset} = Q_{12|12} = 1, \tag{2.20}$$

which is analogous to the choice made in the other known QSC cases, and so far seems fully consistent with the description of the AdS_3 integrable system. From now on, we assume the validity of (2.20).

We will adopt a special notation for some of the Q-functions,

$$\mathbf{Q}_k \equiv Q_{\emptyset|k}, \quad \mathbf{P}_a \equiv Q_{a|\emptyset}, \quad \mathbf{Q}^k \equiv \epsilon^{kl}Q_{12|l}, \quad \mathbf{P}^a \equiv \epsilon^{ab}Q_{b|12}, \tag{2.21}$$

as well as $Q^{a|i} \equiv \epsilon^{ab}\epsilon^{ij}Q_{b|j}$. Explicitly,

$$Q^{a|i} = \begin{pmatrix} Q_{2|2} & -Q_{2|1} \\ -Q_{1|2} & Q_{1|1} \end{pmatrix}, \tag{2.22}$$

such that

$$Q_{a|i}Q^{b|i} = \delta_a^b, \quad Q_{a|i}Q^{a|j} = \delta_i^j, \tag{2.23}$$

due to the unimodularity property

$$\det\left(Q_{a|i}\right) = 1, \tag{2.24}$$

which is a consequence of the Q-system with the boundary conditions (2.20). Let us write explicitly some of the fermionic equations, which will be used extensively,

$$Q_{a|i}^{+} - Q_{a|i}^{-} = \mathbf{P}_a \mathbf{Q}_i, \tag{2.25}$$

together with $Q_{a|i}^{-} - Q_{a|i}^{+} = Q_{12|i}Q_{a|12}$, which can be rewritten in Hodge-dual notation as

$$Q^{a|i\,+} - Q^{a|i\,-} = -\mathbf{P}^a \mathbf{Q}^i. \tag{2.26}$$

Further useful consequences of the QQ relations are:

$$\mathbf{P}_a \mathbf{P}^a = \mathbf{Q}_i \mathbf{Q}^i = 0, \tag{2.27}$$

and the following relations

$$Q_{a|i}^{\pm} \mathbf{Q}^i = \mathbf{P}_a \ , \ \ Q_{a|i}^{\pm} \mathbf{P}^a = \mathbf{Q}_i \ , \ \ Q^{a|i\,\pm} \mathbf{Q}_i = \mathbf{P}^a \ , \ \ Q^{a|i\,\pm} \mathbf{P}_a = \mathbf{Q}^i \ , \tag{2.28}$$

where the equations with $\pm$ signs are compatible due to (2.25)–(2.27).

A useful rewriting of (2.25), (2.26) incorporating $Q_{a|i}$ is

$$Q_{a|i}^- = Q_{a|j}^+ \left(\delta_i^j - \mathbf{Q}^j \mathbf{Q}_i \right), \ \ Q^{a|i\,-} = Q^{a|j\,+} \left(\delta_j^i + \mathbf{Q}^i \mathbf{Q}_j \right), \tag{2.29}$$

or alternatively,

$$Q_{a|i}^- = Q_{b|i}^+ \left(\delta_a^b - \mathbf{P}^b \mathbf{P}_a \right), \ \ Q^{a|i\,-} = Q^{b|i\,+} \left(\delta_a^b + \mathbf{P}^b \mathbf{P}_a \right). \tag{2.30}$$

So far, most of these relations are structurally similar to the ones found for psu(2, 2|4) - the AdS_5 case. In this case of lower rank, however, there is an interesting new feature, which follows from the fact that $Q_{a|i}$ and $Q^{a|i}$ are related in a simple manner by (2.22). The compatibility of (2.25) and (2.26) then gives

$$\mathbf{Q}^k \mathbf{P}^a = -\epsilon^{kl} \epsilon^{ab} \mathbf{Q}_l \mathbf{P}_b, \tag{2.31}$$

or explicitly,

$$\mathbf{Q}^1 \mathbf{P}^1 = -\mathbf{Q}_2 \mathbf{P}_2 \ , \ \ \mathbf{Q}^1 \mathbf{P}^2 = +\mathbf{Q}_2 \mathbf{P}_1 \ , \ \ \mathbf{Q}^2 \mathbf{P}^1 = +\mathbf{Q}_1 \mathbf{P}_2 \ , \ \ \mathbf{Q}^2 \mathbf{P}^2 = +\mathbf{Q}_1 \mathbf{P}_1, \tag{2.32}$$

which imply the equalities of certain ratios of $\mathbf{P}$ or $\mathbf{Q}$ functions:

$$\frac{\mathbf{Q}^1}{\mathbf{Q}_2} = -\frac{\mathbf{Q}^2}{\mathbf{Q}_1} = -\frac{\mathbf{P}_2}{\mathbf{P}^1} = +\frac{\mathbf{P}_1}{\mathbf{P}^2} \equiv r. \tag{2.33}$$

The quantity $r(u)$ defined above will have an interesting role in our system. Notice that it allows to raise or lower the indices

$$\mathbf{Q}^k = +r\epsilon^{kl} \mathbf{Q}_l \ , \ \ \mathbf{Q}_k = -\frac{1}{r} \epsilon_{kl} \mathbf{Q}^l \ , \ \ \mathbf{P}^k = -\frac{1}{r} \epsilon^{kl} \mathbf{P}_l \ , \ \ \mathbf{P}_k = +r\epsilon_{kl} \mathbf{P}^l \ . \tag{2.34}$$

Finally, a useful consequence of the Q-system is the existence of a 2nd order finite difference equation, describing the $\mathbf{Q}$ functions in terms of the $\mathbf{P}$ functions (and vice versa). These Baxter-type equations are described in appendix C.

Q-system and Bethe ansatz. An important consequence of a Q-system is that it immediately implies the existence of Bethe-like equations restricting the positions of the zeros of the Q-functions, which play the role of Bethe roots. In this argument, we anticipate a crucial assumption on the Q-functions, namely that they do not have any poles.

One such system of Bethe equations constrains the zeros of the Q-functions

$$Q_{\emptyset|1} = \mathbf{Q}_1, \ \ Q_{1|1}, \ \ Q_{12|1} = -\mathbf{Q}^2. \tag{2.35}$$

For instance, from (2.26) we learn that

$$Q_{1|1}^+ - Q_{1|1}^-\Big|_{u \in \{\text{zeros of } \mathbf{Q}_1\}} = \mathbf{P}_1\mathbf{Q}_1|_{u \in \{\text{zeros of } \mathbf{Q}_1\}} = 0, \tag{2.36}$$

while, since $\mathbf{Q}_1\mathbf{P}_1 = -\mathbf{P}^2\mathbf{Q}^2$, it is also true that

$$Q_{1|1}^+ - Q_{1|1}^-\Big|_{u \in \{\text{zeros of } \mathbf{P}_1\}} = Q_{1|1}^+ - Q_{1|1}^-\Big|_{u \in \{\text{zeros of } \mathbf{P}^2\}} = Q_{1|1}^+ - Q_{1|1}^-\Big|_{u \in \{\text{zeros of } \mathbf{Q}^2\}} = 0. \tag{2.37}$$

Shifting the bosonic equation $Q_{1|1}^+ Q_{2|1}^- - Q_{1|1}^- Q_{2|1}^+ = -\mathbf{Q}^2\mathbf{Q}_1$ by $\pm i/2$, we also obtain

$$Q_{1|1}^{++} Q_{2|1}\Big|_{u \in \{\text{zeros of } Q_{1|1}\}} = -\mathbf{Q}^{2+}\mathbf{Q}_1^+|_{u \in \{\text{zeros of } Q_{1|1}\}}, \tag{2.38}$$

$$Q_{1|1}^{--} Q_{2|1}\Big|_{u \in \{\text{zeros of } Q_{1|1}\}} = +\mathbf{Q}^{2-}\mathbf{Q}_1^-|_{u \in \{\text{zeros of } Q_{1|1}\}}. \tag{2.39}$$

The above constraints can be recast as the exact Bethe equations[2]

$$\frac{Q_{1|1}^+}{Q_{1|1}^-}\Bigg|_{u \in \{\text{zeros of } \mathbf{Q}_1\}} = 1 \tag{2.40}$$

$$\frac{Q_{1|1}^{++}\mathbf{Q}_1^-\mathbf{Q}^{2-}}{Q_{1|1}^{--}\mathbf{Q}_1^+\mathbf{Q}^{2+}}\Bigg|_{u \in \{\text{zeros of } Q_{1|1}\}} = -1, \tag{2.41}$$

$$\frac{Q_{1|1}^+}{Q_{1|1}^-}\Bigg|_{u \in \{\text{zeros of } \mathbf{Q}^2\}} = 1, \tag{2.42}$$

where the middle relation comes from the ratio of (2.38),(2.39). In a similar way one can deduce several other systems of Bethe equations. For instance, relations of the same form are valid for the zeros of the functions $\mathbf{P}_1$, $Q_{1|1}$, $\mathbf{P}^2$. We write these relations with a dot, anticipating that they will be relevant for the second wing:

$$\frac{Q_{\dot1|\dot1}^+}{Q_{\dot1|\dot1}^-}\Bigg|_{u \in \{\text{zeros of } \mathbf{P}_{\dot1}\}} = 1 \tag{2.43}$$

$$\frac{Q_{\dot1|\dot1}^{++}\mathbf{P}_{\dot1}^-\mathbf{P}^{\dot2-}}{Q_{\dot1|\dot1}^{--}\mathbf{P}_{\dot1}^+\mathbf{P}^{\dot2+}}\Bigg|_{u \in \{\text{zeros of } Q_{\dot1|\dot1}\}} = -1, \tag{2.44}$$

$$\frac{Q_{\dot1|\dot1}^+}{Q_{\dot1|\dot1}^-}\Bigg|_{u \in \{\text{zeros of } \mathbf{P}^{\dot2}\}} = 1, \tag{2.45}$$

In a system like the ones arising in AdS/CFT, the Q-functions are in general complicated functions not known explicitly, therefore such exact Bethe equations have limited practical usefulness when analysing generic solutions of the QSC. However, for certain

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[2]In the case where the Q-functions have cuts, such as will be our system, the relation will be valid on the main Riemann sheet where the Q-system is defined.

classes of solutions, such as those with large charges or near special points in the moduli space of the holographic theory the Q-functions do simplify. In the last section of the paper, we find the explicit large-volume limit of some Q-functions, arising from our QSC equations. Exact Bethe equations such as the ones given above will then reduce to the ABA equations. Additionally, $\text{AdS}_3/\text{CFT}_2$ dual pairs have multiple moduli, which preserve integrability [55] and at special points in the moduli space of each holographic pair additional simplifications to the exact Bethe equations may occur. For example, the weakly coupled RR-charged theory is expected to describe a nearest-neighbour integrable spin chain [56].

3 Proposal for the QSC

In this section we describe the structure of the proposed Quantum Spectral Curve for AdS_3. In the absence of the general TBA equations we cannot follow the usual route of [2, 57, 58] to derive the QSC from TBA. Instead we will be guided by the common properties of the known QSCs for AdS_5 and AdS_4.

If we summarise the known QSCs there are two main ingredients: QQ-relations, and analytical properties of Q-functions. We consider these components in turn in the following.

3.1 Introducing the Q-functions

QQ-relations. In the known case, the QQ-relations follow from the structure of the symmetry of the system. In AdS_3 we have two copies of $\text{psu}(1,1|2)$ and a natural assumption would be to have two copies of QQ-relations for $\text{psu}(1,1|2)$, described in the previous section. To distinguish the two copies we will use dotted indices for one of them, so we will use the following sets of indices ($a = 1, 2$, $k = 1, 2$ and same for dotted indices)

$$\mathbf{Q}_k,\ \mathbf{P}_a,\ Q_{a|k} \leftrightarrow Q^{a|k},\ \mathbf{Q}^k,\ \mathbf{P}^a\ , \tag{3.1}$$

$$\mathbf{Q}_{\dot{k}},\ \mathbf{P}_{\dot{a}},\ Q_{\dot{a}|\dot{k}} \leftrightarrow Q^{\dot{a}|\dot{k}},\ \mathbf{Q}^{\dot{k}},\ \mathbf{P}^{\dot{a}}\ . \tag{3.2}$$

The above Q-functions are related by the QQ-relations. A distinguished subset of them, from which one can recover the remaining Q-functions are

$$\mathbf{P}_a\ ,\ \mathbf{P}^a\ \text{ and }\ \mathbf{P}_{\dot{a}}\ ,\ \mathbf{P}^{\dot{a}}\ \text{ constrained by }\ \mathbf{P}_a\mathbf{P}^a = \mathbf{P}_{\dot{a}}\mathbf{P}^{\dot{a}} = 0\ . \tag{3.3}$$

For example, $\mathbf{Q}_k$ can be reconstructed from $\mathbf{P}_a$ and $\mathbf{P}^a$ by solving the second order finite-difference equation

$$\mathbf{Q}_k^{++}D_1^- - \mathbf{Q}_k D_2 + \mathbf{Q}_k^{--}D_1^+ = 0\ , \tag{3.4}$$

with the coefficients depending solely on $\mathbf{P}$'s:

$$D_1 = \epsilon_{ab}\mathbf{P}^{a-}\mathbf{P}^{b+}\ ,\quad D_2 = \epsilon_{ab}\mathbf{P}^{a--}\mathbf{P}^{b++} - \mathbf{P}_c\mathbf{P}^{c--}\ \epsilon_{ab}\mathbf{P}^a\mathbf{P}^{b++}\ . \tag{3.5}$$

The above relation, derived in appendix C, is a consequence of the QQ-relations, so an identical equation holds for the dotted Q-functions. Equally one can interchange $\mathbf{Q} \leftrightarrow \mathbf{P}$ in (3.4) and (3.5).

Classical correspondence. In the classical limit, described by strong coupling $h \to \infty$ and large quantum numbers scaling as $\sim h$, we expect that the quasimomenta appear in a WKB approximation of some of the Q-functions. In particular, they should be directly related to the Q-functions living in the fundamental representation of each $\mathrm{psu}(1,1|2)$ algebra. With the notation borrowed from the other cases, we link $\mathbf{P}$'s with the quasi-momenta associated with S^3 and $\mathbf{Q}$'s with the ones for AdS_3.

For the first wing, we will take this correspondence to be the following:

$$(\mathbf{Q}_1, \mathbf{Q}_2 | \mathbf{P}_1, \mathbf{P}_2) \sim \left(e^{-\int^u p_1^A}, e^{-\int^u p_2^A} | e^{+\int^u p_1^S}, e^{+\int^u p_2^S} \right), \tag{3.6}$$

$$\left(\mathbf{Q}^1, \mathbf{Q}^2 | \mathbf{P}^1, \mathbf{P}^2 \right) \sim \left(e^{\int^u p_1^A}, e^{\int^u p_2^A} | e^{-\int^u p_1^S}, e^{-\int^u p_2^S} \right), \tag{3.7}$$

which is structurally the same as in AdS_5. For the second wing, we take[3]

$$(\mathbf{Q}_{\dot 1}, \mathbf{Q}_{\dot 2} | \mathbf{P}_{\dot 1}, \mathbf{P}_{\dot 2}) \sim \left(e^{-\int^u p_2^A}, e^{-\int^u p_1^A} | e^{\int^u p_2^S}, e^{\int^u p_1^S} \right), \tag{3.8}$$

$$\left(\mathbf{Q}^{\dot 1}, \mathbf{Q}^{\dot 2} | \mathbf{P}^{\dot 1}, \mathbf{P}^{\dot 2} \right) \sim \left(e^{\int^u p_2^A}, e^{\int^u p_1^A} | e^{-\int^u p_2^S}, e^{-\int^u p_1^S} \right). \tag{3.9}$$

Large-u asymptotics. Consistently with the quasi-classical identifications (3.9) and the asymptotics of the quasimomenta described in section 2.2, the Q-functions should exhibit power-law asymptotics at large u, with behaviour characterised by the charges. In particular, we assume

$$\mathbf{P}_a \simeq A_a u^{M_a}, \quad \mathbf{P}^a \simeq A^a u^{-M_a-1}, \quad \mathbf{Q}_i \simeq B_i u^{\hat{M}_i}, \quad \mathbf{Q}^i \simeq B^i u^{-\hat{M}_i-1}, \tag{3.10}$$

for large u, where

$$M_a \equiv \left(-\frac{L}{2} + K_2 - K_1 - 1, \ \frac{L}{2} - K_2 + K_3 \right), \quad \hat{M}_k \equiv \left(\frac{\gamma}{2} + \frac{L}{2} + K_1, \ -\frac{\gamma}{2} - \frac{L}{2} - K_3 - 1 \right), \tag{3.11}$$

$$M_{\dot a} \equiv \left(-\frac{L}{2} + K_{\dot 3}, \ \frac{L}{2} - K_{\dot 1} - 1 \right), \quad \hat{M}_{\dot k} \equiv \left(\frac{\gamma}{2} + \frac{L}{2} + K_{\dot 2} - K_{\dot 3} - 1, \ -\frac{\gamma}{2} - \frac{L}{2} - K_{\dot 2} + K_{\dot 1} \right). \tag{3.12}$$

In the following sections, we will see that some of the Q-functions have horizontal cuts connecting to infinity. In this case, the asymptotic behaviour above will be assumed to be valid for $\mathrm{Im}(u) > 0$.

Notice that the classical identification is valid in a regime of large quantum numbers, so that it only fixes the structure of (3.10) up to finite shifts. However, those can be fine-tuned by the match with the ABA which will be described in the last section of the paper. We will take the exact asymptotics of the Q-functions to be as above.

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[3]Comparing (3.6) and (3.8), the reader will notice that we reordered some of the labels in the second wing. This is just an arbitrary choice with no loss of generality at this stage (notice that $1 \leftrightarrow 2$ in the indices is a trivial symmetry of the Q-system), but it will be convenient for the future, as it will make the discussion more symmetric between the two wings.

Constraints on the constant prefactors and shortening conditions. The prefactors A and B in $\mathbf{P}$ and $\mathbf{Q}$ functions (3.10) usually play an important role. They can be determined by plugging the large u expansion into the QQ-relations or Baxter equation. This leads to the following identities

$$r_0\, h^{\hat{B}}\frac{\prod(-y_{\dot{3},i})}{\prod(-y_{\dot{1},i})} = \frac{B^1}{B_2} = \frac{A_1}{A^2} \qquad r_0\, h^{\check{B}}\frac{\prod(-y_{3,i})}{\prod(-y_{1,i})} = \frac{B^{\dot{1}}}{B_{\dot{2}}} = \frac{A_{\dot{1}}}{A^{\dot{2}}}\,. \tag{3.13}$$

The Baxter equation then implies

$$B_1 B^1 = -B_2 B^2 = \frac{i}{4}\frac{(\Delta - J - K + S)(\Delta + J + K + S + 2)}{\Delta + S + 1}, \tag{3.14}$$

$$A_1 A^1 = -A_2 A^2 = \frac{i}{4}\frac{(\Delta - J - K + S)(\Delta + J + K + S + 2)}{J + K + 1}, \tag{3.15}$$

and with dots

$$B_{\dot{1}} B^{\dot{1}} = -B_{\dot{2}} B^{\dot{2}} = \frac{i}{4}\frac{(\Delta - J + K - S)(\Delta + J - K - S - 2)}{\Delta - S - 1}, \tag{3.16}$$

$$A_{\dot{1}} A^{\dot{1}} = -A_{\dot{2}} A^{\dot{2}} = \frac{i}{4}\frac{(\Delta - J + K - S)(\Delta + J - K - S - 2)}{J - K - 1}. \tag{3.17}$$

Above we used the following relation between the charges and the Bethe root numbers:

$$\begin{aligned}
\Delta &= \gamma + L + K_{\dot{2}} + \frac{K_1}{2} + \frac{K_3}{2} - \frac{K_{\dot{1}}}{2} - \frac{K_{\dot{3}}}{2}\,,\\
S &= \frac{K_1}{2} + \frac{K_3}{2} + \frac{K_{\dot{1}}}{2} + \frac{K_{\dot{3}}}{2} - K_{\dot{2}}\,,\\
K &= \frac{K_1}{2} + \frac{K_3}{2} + \frac{K_{\dot{1}}}{2} + \frac{K_{\dot{3}}}{2} - K_2\,,\\
J &= L - K_2 + \frac{K_1}{2} + \frac{K_3}{2} - \frac{K_{\dot{1}}}{2} - \frac{K_{\dot{3}}}{2}\,,\\
\hat{B} &= K_1 - K_3\,,\\
\check{B} &= K_{\dot{1}} - K_{\dot{3}}\,.
\end{aligned} \tag{3.18}$$

The half-BPS shortening condition $\Delta = J$ and $S = K$ follows from requiring for A and B to vanish. This is an integrability-based derivation of a non-renormalization result for theories with small $(4,4)$ super-conformal symmetry. In such theories, there are left or right sub-algebra shortening conditions: $\Delta_L = J_L$ or $\Delta_R = J_R$. It is well-known that at generic points in the moduli space states which are short with respect to only one such sub-algebra (*i.e.* quarter-BPS states) are not protected, while states which satisfy both shortening conditions (half-BPS states) do not receive quantum corrections [59–61]. An independent derivation of these results was found using ABA methods [62, 63] which are valid in the large L limit. The QSC derivation presented here, showing that only half-BPS states are protected, is valid for all lengths L.

3.2 Analytic properties

As in all other studied cases, we assume that all 4 types of $\mathbf{P}$'s have only one branch cut $(-2h, 2h)$ on the real axis and no other singularities on either sheet of their Riemann
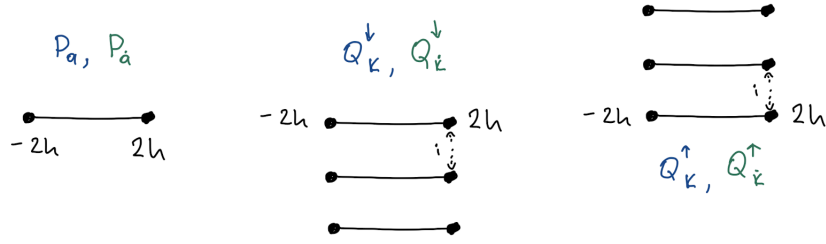
Figure 1. Standard analytic structure of **P**'s with one branch cut. As a consequence of this, **Q** functions will have an infinite ladder of cuts separated by i in the lower or upper half of the analytic plane.

surface, as shown on Figure 1. Since the **Q**-functions are determined in terms of **P**'s by means of equation (3.4), the analytic properties of **Q** can be deduced from those of **P**. Before describing them let us introduce two different bases of solutions of (3.4):

$$\mathbf{Q}_k^\downarrow \quad - \quad \text{Upper Half Plane analytic (UHPA) solutions} \tag{3.19}$$

$$\mathbf{Q}_k^\uparrow \quad - \quad \text{Lower Half Plane analytic (LHPA) solutions .} \tag{3.20}$$

As the coefficients of (3.4) only have a few cuts near the real axis, and are analytic otherwise, we can always find two solutions of (3.4) $\mathbf{Q}_k^\downarrow$ which do not have cuts in the UHP, and another pair of solutions $\mathbf{Q}_k^\uparrow$ which are analytic in the LHP. Rewriting (3.4) as

$$\mathbf{Q}_k^\downarrow = \frac{\mathbf{Q}_k^{\downarrow++} D_2^{++} - \mathbf{Q}_k^{\downarrow[+4]} D_1^+}{D_1^{[+3]}} \ , \tag{3.21}$$

and assuming that $\mathbf{Q}_k^\downarrow$ is analytic for Im $u > 0$ we see that the highlighted terms in the r.h.s. will produce a branch cut on the real axis. Iterating further (3.21) with shifts $u \to u - 2in$ in general we generate a ladder of cuts going down the complex plane like on Figure 1.

At the same time, since there are only two linearly independent (with periodic coefficients) solutions of a second order equation (3.4) there must exist an i-periodic function (with short cuts) $\Omega_k{}^l$ which relates the two sets of solutions

$$\mathbf{Q}_k^\uparrow = \Omega_k{}^m \mathbf{Q}_m^\downarrow \ , \quad \Omega_k{}^m(u + i) = \Omega_k{}^m(u) \ . \tag{3.22}$$

In fact one can write $\Omega_k{}^m$ explicitly in terms of **Q**'s

$$\Omega_k{}^m = \epsilon^{ml} \frac{\mathbf{Q}_k^\uparrow \mathbf{Q}_l^{\downarrow--} - \mathbf{Q}_k^{\uparrow--} \mathbf{Q}_l^\downarrow}{\mathbf{Q}_1^\downarrow \mathbf{Q}_2^{\downarrow--} - \mathbf{Q}_1^{\downarrow--} \mathbf{Q}_2^\downarrow} \tag{3.23}$$

and the periodicity can be verified using (3.4). There are identical equations for the dotted indices. Furthermore, in Appendix C we show that the Hodge-dual Q-functions also satisfy

$$\mathbf{Q}^{\uparrow k} = \Omega^k{}_m \mathbf{Q}^{\downarrow m} \qquad \Omega^k{}_m \Omega_k{}^l = \delta_m^l \ . \tag{3.24}$$
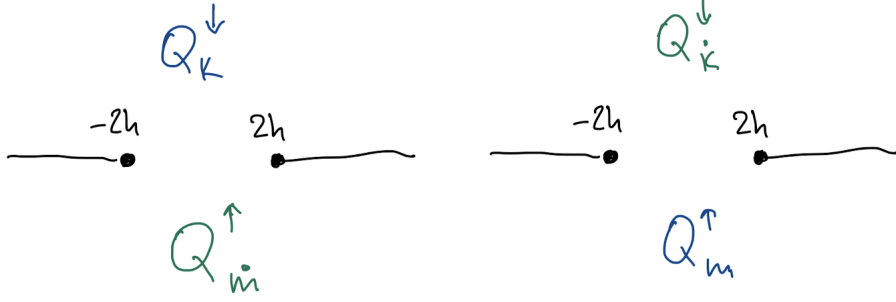
Figure 2. Two **Q**'s from different Q-systems are glued together.

Gluing conditions. So far, the two Q-systems were existing independently. Here we propose a particular way of joining them together. The underlying idea is to fix the apparent asymmetry between the analytic properties of **Q** and **P** (see Figure 1). Whereas **P** has only one branch-cut, as we argued above, **Q** should have a ladder of cuts going either up or down from the real axis. Following the observation in other QSCs, we notice that a section of the Riemann surface of **Q**'s with long cut i.e. $(-\infty, -2h) \cup (2h, \infty)$ on the real axis should not have any other cuts. More specifically we require that

$$\mathbf{Q}_k^{\downarrow}(u+i0) = G_k{}^{\dot{n}} \, \mathbf{Q}_{\dot{n}}^{\uparrow}(u-i0) \ , \quad \mathbf{Q}_{\dot{k}}^{\downarrow}(u+i0) = G_{\dot{k}}{}^{n} \, \mathbf{Q}_n^{\uparrow}(u-i0) \ , \quad u \in (-2h, 2h) \quad (3.25)$$

where $G_k{}^{\dot{n}}$ and $G_{\dot{k}}{}^{n}$ are two different independent constant matrices. In the studied cases of QSC they have several zero components, but in our case their exact form is still to be deduced. However, one can make a first guess by looking at the classical counterpart of the gluing relations (2.14). Using the identification (3.6) we see that it suggests $G_1^{\dot{2}}$ and $G_2^{\dot{1}}$ to be the only non-zero elements of $G_k{}^{\dot{n}}$.

For the Hodge-dual Q-functions, the gluing conditions take a similar form

$$\mathbf{Q}^{k\downarrow}(u+i0) = G^k{}_{\dot{n}} \, \mathbf{Q}^{\dot{n}\uparrow}(u-i0) \ , \quad \mathbf{Q}^{\dot{k}\downarrow}(u+i0) = G^{\dot{k}}{}_n \, \mathbf{Q}^{n\uparrow}(u-i0) \ , \quad u \in (-2h, 2h). \quad (3.26)$$

Like in the known cases, we assert that gluing is a symmetry of the Q-system

$$G^k{}_{\dot{n}} = \epsilon^{kl} \epsilon_{\dot{n}\dot{m}} G_l{}^{\dot{m}} \ , \quad G^{\dot{k}}{}_n = \epsilon^{\dot{k}\dot{l}} \epsilon_{nm} G_{\dot{l}}{}^m \ , \quad \det G = 1 \ . \quad (3.27)$$

In the following, we will choose a basis of Q-functions with specified large-u asymptotics on the first sheet, described in (3.10). after this choice is made, we are not free to diagonalise the gluing matrix with a linear transformation. For this reason, we will keep track of it explicitly throughout. We leave for future work the discussion of the matrix structure of G in this special basis, but as we argued above the classical limit suggests an off-diagonal structure for this matrix.

Properties of the r-function. The r-function, which was defined in section 2.3 and allows to lower and raise indices, has interesting analyticity properties. From (2.33) we note that $r = \mathbf{P}^1/\mathbf{P}_2$, meaning that r (and $\dot{r}$) has at most one cut on the main sheet
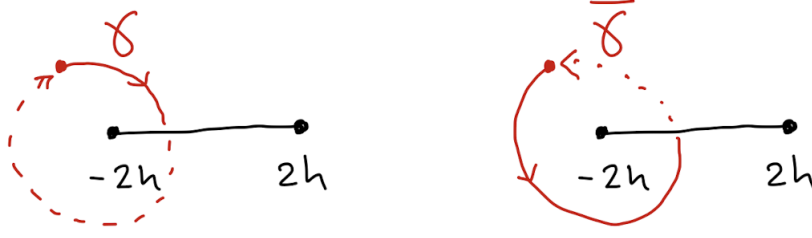
Figure 3. Two contours we use for analytic continuation

$(-2h, 2h)$. At the same time $r = \mathbf{Q}_1^\downarrow / \mathbf{Q}^{2\downarrow}$, meaning that it has only one long cut at the same time. In other words the analytic continuation from above r^γ is analytic in the LHP.

$$(\mathbf{Q}^{k\downarrow})(u + i0) = r(u + i0)\epsilon^{kl}\mathbf{Q}_l^\downarrow(u + i0) = r(u + i0)\epsilon^{kl}G_l{}^{\dot{n}}\mathbf{Q}_{\dot{n}}^\uparrow(u - i0), \qquad (3.28)$$

at the same time the l.h.s. can be expressed as

$$G^k{}_{\dot{n}}\,\mathbf{Q}^{\dot{n}\uparrow}(u - i0) = G^k{}_{\dot{n}}\,\epsilon^{\dot{n}\dot{m}}\dot{r}(u - i0)\mathbf{Q}_{\dot{m}}^\uparrow(u - i0) = \dot{r}(u - i0)\epsilon^{kl}G_l{}^{\dot{n}}\mathbf{Q}_{\dot{n}}^\uparrow(u - i0), \quad (3.29)$$

from where we deduce that $r(u + i0) = \dot{r}(u - i0)$. Similarly, we can start from the dotted version of the derivation above to get $r(u - i0) = \dot{r}(u + i0)$. From this consideration we see that $r(u)$ has a single quadratic cut, which connects it to $\dot{r}(u)$. This branch cut can be rationalised with the help of the Zhukovsky variable $x(u) = \frac{u + \sqrt{u-2h}\sqrt{u+2h}}{2h}$ so we can write r explicitly in terms of its zeros/poles[4]

$$r(u) = r_0 \frac{\prod_i^{K_3 - N_R}(x(u) - y_{3,i})\prod_i^{K_{\dot{3}} - N_R}(1/x(u) - y_{\dot{3},i})}{\prod_i^{K_1 - N_B}(x(u) - y_{1,i})\prod_i^{K_{\dot{1}} - N_B}(1/x(u) - y_{\dot{1},i})} \ , \qquad (3.30)$$

and $\dot{r}(u)$ is $r(u)$ with $x(u)$ replaced by $1/x(u)$. r_0 is a constant. In the above expression we assume $|y_{...}| \geq 1$. Finding such a simple expression for a combination of $\mathbf{P}$'s is an interesting novel feature of the AdS$_3$ QSC.

3.3 On analytic continuation

We now deduce several consequences of the discussion in the previous section. We will see that the simple set of constraints given above implies the existence of a rich mathematical structure. The Q-functions live on a Riemann surface with infinitely many sheets, but the equations we will now deduce allow us to map any one of these sheets to the first one, as is the case also for the other examples of QSCs.

As anticipated in the introduction, it will turn out that the branch cuts in this system of QSC equations *cannot be* quadratic. This means that, for any branch point on the Riemann surface, we can go around it in two ways, and in principle this yields two different results.

We will introduce the analytic continuation paths γ and its inverse γ^{-1}, which we will also denote by $\bar{\gamma}$. The path γ goes around a branch point at $2h$ in anticlockwise sense,

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[4]The number of poles and zeros K_n is introduced to match later the notations in the ABA. N_R and N_B are introduced to allow for different types of Bethe roots to coincide and consequently cancel in the ratio.

or alternatively, it goes around the branch point at $-2h$ in clockwise sense. Since in this section we think in terms of short cuts for all the Q-functions, we can say that γ goes through the short cut $(-2h, +2h)$ from above, while $\bar\gamma$ crosses it from below. The two paths are represented in figure 3. We denote the analytic continuation of any function of u along these paths as f^γ or $f^{\gamma^{-1}} \equiv f^{\bar\gamma}$. In this notation, (3.25) and (3.26) become

$$(\mathbf{Q}_k^\downarrow)^\gamma = G_k{}^{\dot m}\mathbf{Q}_{\dot n}^\uparrow \ , \ \ (\mathbf{Q}_{\dot k}^\downarrow)^\gamma = G_{\dot k}{}^n\mathbf{Q}_n^\uparrow, \ \ \ (\mathbf{Q}^{k\downarrow})^\gamma = G^k{}_{\dot n}\bar{\mathbf{Q}}^{\dot n\uparrow}, \ \ \ (\mathbf{Q}^{\dot n\downarrow})^\gamma = G^{\dot a}{}_b\mathbf{Q}^{b\uparrow}. \ \ (3.31)$$

3.3.1   The Qω-system

By defining i-periodic functions ω:

$$\omega_k{}^{\dot n} = G_k{}^{\dot m}\Omega_{\dot m}{}^{\dot n}\,, \qquad\qquad \omega_{\dot k}{}^n = G_{\dot k}{}^m\Omega_m{}^n\,, \tag{3.32}$$

where Ω's are the matrices relating LHPA and UHPA bases in (3.24), (3.22), the system of equations (3.31) can be conveniently rewritten in the form

$$\mathbf{Q}_k^{\downarrow\gamma} = \omega_k{}^{\dot m}\mathbf{Q}_{\dot m}^\downarrow\,, \qquad\qquad \mathbf{Q}_{\dot k}^{\downarrow\gamma} = \omega_{\dot k}{}^m\mathbf{Q}_m^\downarrow. \tag{3.33}$$

Notice also that by construction, it follows from the properties of the gluing matrix and Ω function that

$$\omega_k{}^{\dot m}\omega^l{}_{\dot m} = \delta_k^l\,, \qquad\qquad \det(\omega) = 1. \tag{3.34}$$

Similarly, one can introduce

$$\bar\omega_{\dot n}{}^m = G^k{}_{\dot n}\Omega^m{}_k, \tag{3.35}$$

such that

$$\mathbf{Q}_{\dot k}^{\uparrow\gamma^{-1}} = \bar\omega_{\dot k}{}^m\mathbf{Q}_m^\uparrow. \tag{3.36}$$

In what follows, we adopt a simplified notation[5], where $\mathbf{Q}^\downarrow$ is denoted by $\mathbf{Q}$ and $\mathbf{Q}^\uparrow$ is denoted by $\bar{\mathbf{Q}}$. So (3.33) and (3.36) become

$$\mathbf{Q}_k^\gamma = \omega_k{}^{\dot m}\mathbf{Q}_{\dot m}\,, \qquad\qquad \bar{\mathbf{Q}}_{\dot k}^{\bar\gamma} = \bar\omega_{\dot k}{}^{\dot m}\bar{\mathbf{Q}}_{\dot m}\ . \tag{3.37}$$

Now let us understand the analytic continuation under the cuts of ω, focusing on $\omega_{\dot k}{}^l$ first. Notice that the matrix $\Omega_k{}^l$ can be expressed as $\Omega_k{}^l = \bar Q^+_{a|k}Q^{a|l\,+}$ (see (C.7)) and since $Q^{a|i\,+}$ has no cut on the real axis, we only need to understand the analytic continuation of $\bar Q^+_{a|k}$. The defining relation of this function is

$$\bar Q_{a|k}^+ - \bar Q_{a|k}^- = \mathbf{P}_a\bar{\mathbf{Q}}_k, \tag{3.38}$$

where $\bar Q^-_{a|k}$ is now analytic and invariant under the analytic continuation along $\bar\gamma$. Computing the discontinuity we obtain

$$\bar Q_{a|k}^{+\bar\gamma} - \bar Q_{a|k}^+ = \mathbf{P}_a^{\bar\gamma}\bar{\mathbf{Q}}_k^{\bar\gamma} - \mathbf{P}_a\bar{\mathbf{Q}}_k\,, \tag{3.39}$$

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[5]Notice that this notation does not necessarily mean complex conjugation of the Q-functions; however, we expect that for real parameters there will be a simple relation.

which, multiplied by $Q^{a|l\,+}$ on the left, leads to

$$(\Omega_k{}^l)^{\bar\gamma} - \Omega_k{}^l = \bar{\mathbf{Q}}_{\dot k}^{\bar\gamma}\mathbf{Q}^{l\,\bar\gamma} - \bar{\mathbf{Q}}_{\dot k}\mathbf{Q}^l \ . \tag{3.40}$$

Next, multiplying these equations by $G_{\dot k}{}^k$, and using (3.31), we find:

$$(\omega_{\dot k}{}^l)^{\bar\gamma} - \omega_{\dot k}{}^l = \mathbf{Q}_{\dot k}\mathbf{Q}^{l\,\bar\gamma} - \mathbf{Q}_{\dot k}^{\gamma}\mathbf{Q}^l \ . \tag{3.41}$$

This expression generalises a similar relation found in AdS_5 and AdS_4 cases, but now we distinguish two different directions for the analytic continuation on the r.h.s.. As usual one can replace dotted to undotted indices to get a similar identity for $\omega_k{}^l$.

We can use (3.41) to determine the double continuation of $\mathbf{Q}_k$ along the contour γ – we will then see explicitly that there may be an obstruction to the cuts being quadratic. We start by continuing (3.37) along the inverse path $\bar\gamma$, which gives

$$\mathbf{Q}_{\dot k} = (\omega_{\dot k}{}^l)^{\bar\gamma}\mathbf{Q}_l^{\bar\gamma} = \omega_{\dot k}{}^l\mathbf{Q}_l^{\bar\gamma} - \mathbf{Q}_{\dot k}^{\gamma}\mathbf{Q}^l\mathbf{Q}_l^{\bar\gamma} = \omega_{\dot k}{}^l\left(\delta_l^p - \mathbf{Q}_l\mathbf{Q}^p\right)\mathbf{Q}_p^{\bar\gamma}, \tag{3.42}$$

where the second equality is obtained by using (3.41), and recalling that $\mathbf{Q}_i\mathbf{Q}^i = 0$. Inverting the factor on the r.h.s., we get

$$\mathbf{Q}_p^{\bar\gamma} = \left(\delta_p^l + \mathbf{Q}_p\mathbf{Q}^l\right)\omega^{\dot k}{}_l\mathbf{Q}_{\dot k} \ . \tag{3.43}$$

From this we can compute directly the difference of the analytic continuation of the Q-function along γ and $\bar\gamma$:

$$\mathbf{Q}_k^{\gamma} - \mathbf{Q}_k^{\bar\gamma} = \mathbf{Q}_{\dot m}\left(\omega_k{}^{\dot m} - \omega^{\dot m}{}_k\right) + \mathbf{Q}_k\mathbf{Q}^l\omega_l{}^{\dot m}\mathbf{Q}_{\dot m}. \tag{3.44}$$

In the case of AdS_5, the two terms on the r.h.s. would vanish separately, due to the symmetry properties of the analogue of ω, ensuring that the branch cuts are quadratic. In our case, that does not need to be the case, since ω connects different kinds of indices and there is no reason $a\ priori$ to expect any symmetry between them.

We make a further interesting observation by rewriting (3.41) in the form

$$(\omega_{\dot k}{}^l)^{\bar\gamma} - \mathbf{Q}_{\dot k}\mathbf{Q}^{l\,\bar\gamma} = \omega_{\dot k}{}^l - \mathbf{Q}_{\dot k}^{\gamma}\mathbf{Q}^l \ . \tag{3.45}$$

This shows immediately that the combination $\omega_{\dot k}{}^l - \mathbf{Q}_{\dot k}^{\gamma}\mathbf{Q}^l$ is equal to its analytic continuation, and therefore the cut on the real axis disappears in this combination. We can also write it as $\omega_{\dot k}{}^m(\delta_m^l - \mathbf{Q}_m\mathbf{Q}^l)$. Then taking (3.42) along γ, we get

$$\mathbf{Q}_{\dot k}^{\gamma} = \omega_{\dot k}{}^l\left(\delta_l^p - \mathbf{Q}_l\mathbf{Q}^p\right)\mathbf{Q}_p = \omega_{\dot k}{}^l\mathbf{Q}_l \tag{3.46}$$

with the final equality being in agreement with (3.32). The first equality allows us to find the expression for $\mathbf{Q}_{\dot k}$ continued a second time along γ:

$$\mathbf{Q}_{\dot k}^{\gamma^2} = \omega_{\dot k}{}^l\left(\delta_l^p - \mathbf{Q}_l\mathbf{Q}^p\right)\mathbf{Q}_p^{\gamma} = \omega_{\dot k}{}^l\left(\delta_l^p - \mathbf{Q}_l\mathbf{Q}^p\right)\omega_p{}^{\dot h}\mathbf{Q}_{\dot h} \ . \tag{3.47}$$

This expression confirms the potential obstruction to the cuts being quadratic. In particular, we can repeatedly iterate this continuation and obtain in general

$$(\mathbf{Q}_{\dot{k}})^{\gamma^n} = U_{\dot{k}}{}^p (\mathbf{Q}_p)^{\gamma^{n-1}}, \qquad (\mathbf{Q}_k)^{\gamma^n} = \dot{U}_k{}^{\dot{p}}(\mathbf{Q}_{\dot{p}})^{\gamma^{n-1}}, \qquad (3.48)$$

$$(\mathbf{Q}_{\dot{k}})^{\bar{\gamma}^n} = \bar{U}_{\dot{k}}{}^p (\mathbf{Q}_p)^{\bar{\gamma}^{n-1}}, \qquad (\mathbf{Q}_k)^{\bar{\gamma}^n} = \dot{\bar{U}}_k{}^{\dot{p}}(\mathbf{Q}_{\dot{p}})^{\bar{\gamma}^{n-1}}, \qquad (3.49)$$

where

$$U_{\dot{k}}{}^p \equiv \omega_{\dot{k}}{}^l \left(\delta_l^p - \mathbf{Q}_l \mathbf{Q}^p \right), \qquad \bar{U}_{\dot{k}}{}^p \equiv \left(\dot{U}^{-1} \right)_{\dot{k}}{}^p = \left(\delta_{\dot{k}}^{\dot{m}} + \mathbf{Q}_{\dot{k}} \mathbf{Q}^{\dot{m}} \right) \omega^p{}_{\dot{m}}, \qquad (3.50)$$

$$\dot{U}_k{}^{\dot{p}} \equiv \omega_k{}^{\dot{m}} \left(\delta_{\dot{m}}^{\dot{p}} - \mathbf{Q}_{\dot{m}} \mathbf{Q}^{\dot{p}} \right), \quad \dot{\bar{U}}_k{}^{\dot{p}} \equiv \left(U^{-1} \right)_k{}^{\dot{p}} = \left(\delta_k^m + \mathbf{Q}_k \mathbf{Q}^m \right) \omega^{\dot{p}}{}_m. \qquad (3.51)$$

In general, following the path γ^n produces a concatenation of monodromies $U \cdot \dot{U} \cdot U \cdot \dot{U} \ldots$, but since there is no reason to expect $(U \cdot \dot{U})$ to be the identity matrix (or a root of the latter), this is nontrivial, meaning that each branch point has infinite order and connects to infinitely many sheets.

Notice that, while in general we expect the branch points to be non-quadratic, there are some special combinations of Q-functions that do exhibit this property. We already showed that this is the case for the ratio r defined in (2.33). We now consider

$$\mathbf{Q}^l \mathbf{Q}_l^{\bar{\gamma}} = \mathbf{Q}^l \omega^{\dot{k}}{}_l \mathbf{Q}_{\dot{k}} = \mathbf{Q}^{\dot{k}\,\gamma} \mathbf{Q}_{\dot{k}}, \qquad (3.52)$$

where we used (3.43) and the analogous equation to (3.37) with (raised, dotted) indices. Lowering the indices with (2.33), and remembering that, as deduced above, $r^\gamma = r^{\bar{\gamma}} = \dot{r}$, the same relations (and their dotted version) can be written as

$$\epsilon^{kl} \mathbf{Q}_k \mathbf{Q}_l^{\bar{\gamma}} = -\epsilon^{\dot{k}\dot{l}} \mathbf{Q}_{\dot{k}} \mathbf{Q}_{\dot{l}}^\gamma, \qquad \epsilon^{kl} \mathbf{Q}_k \mathbf{Q}_l^\gamma = -\epsilon^{\dot{k}\dot{l}} \mathbf{Q}_{\dot{k}} \mathbf{Q}_{\dot{l}}^{\bar{\gamma}}. \qquad (3.53)$$

Continuing the first equation above along γ, we get

$$\epsilon^{kl} \mathbf{Q}_k^\gamma \mathbf{Q}_l = -\epsilon^{\dot{k}\dot{l}} \mathbf{Q}_{\dot{k}}^\gamma \mathbf{Q}_{\dot{l}}^{\gamma^2}, \qquad (3.54)$$

but due to the second equation in (3.53), the l.h.s. is also equal to $-\epsilon^{\dot{k}\dot{l}} \mathbf{Q}_{\dot{k}}^{\bar{\gamma}} \mathbf{Q}_{\dot{l}}$, meaning that the combination $\epsilon^{\dot{k}\dot{l}} \mathbf{Q}_{\dot{k}}^{\bar{\gamma}} \mathbf{Q}_{\dot{l}} = \epsilon^{kl} \mathbf{Q}_k \mathbf{Q}_l^\gamma$ comes back after γ^2!

As a final observation, we notice that, continuing the two sides of (3.45) along γ, one can also obtain an explicit equation for ω^γ in terms of quantities on the first sheet.

The main results of this section can be summarised in the following equations:[6]

$$(\mathbf{Q})_{\dot{k}}^\gamma = \omega_{\dot{k}}{}^l \mathbf{Q}_l, \quad (\mathbf{Q}^{\dot{k}})^\gamma = \omega^{\dot{k}}{}_l \mathbf{Q}^l, \qquad (3.55)$$

and

$$\left((\omega)^{\bar{\gamma}} - \omega \right)_{\dot{k}}{}^l = \mathbf{Q}_{\dot{k}} (\mathbf{Q}^l)^{\bar{\gamma}} - (\mathbf{Q}_{\dot{k}})^\gamma \mathbf{Q}^l, \quad \left((\omega)^{\bar{\gamma}} - \omega \right)^{\dot{k}}{}_l = -\mathbf{Q}^{\dot{k}} (\mathbf{Q}_l)^{\bar{\gamma}} + (\mathbf{Q}^{\dot{k}})^\gamma \mathbf{Q}_l. \qquad (3.56)$$

Here, as usual, we understand that for every equation there is its double obtained by interchanging dotted and undotted indices. Together with $\mathbf{Q}_i \mathbf{Q}^i = 0$, $\omega_{\dot{k}}{}^l \omega^{\dot{k}}{}_m = \delta_m^l$, and the

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[6]Results for $\mathbf{Q}$ and ω functions with raised indices can be found using the same steps.
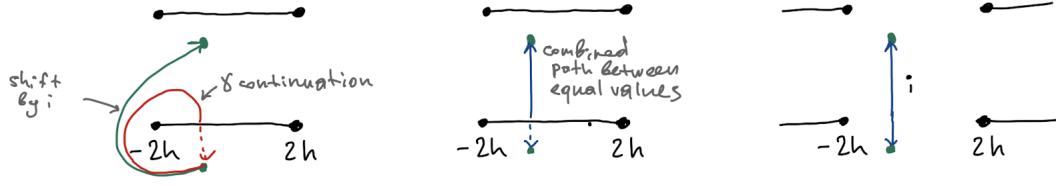
Figure 4. Periodicity of μ as a function with long cuts is identical to the property $\mu^{++} = \mu^\gamma$ for a section with short cuts.

periodicity of ω, the relations (3.55),(3.56) may be taken as a self-consistent description of the QSC, which is usually dubbed $\mathbf{Q}\omega$-system.[7] Bouncing back and forth between these equations, and using the fact that ω is i-periodic, one can obtain the result of any analytic continuation of the $\mathbf{Q}$-functions and ω functions, inside any cut, and express it in terms of their values on the first sheet. This is the same feature that was observed in the other examples of QSC, see the discussion in [1]. It is encouraging that this property is still valid here, even though the analytic structure is more complicated due to the branch points having infinite order.

3.3.2 The Pμ-system

We now describe the constraints on the analytic continuation of $\mathbf{P}$ functions. Analogously to [2], the main object in this case is the matrix μ defined as

$$\mu_a{}^{\dot{b}} \equiv Q_{a|c}^{-}\,\omega^c{}_{\dot{d}}\,Q^{\dot{b}|\dot{d}\ -}\,, \qquad \mu^a{}_{\dot{b}} \equiv Q^{a|c\ -}\,\omega_c{}^{\dot{d}}\,Q_{\dot{b}|\dot{d}}^{-}, \tag{3.57}$$

which will play a role similar to ω. Notice that just like in the case of ω, μ has unit determinant and $\mu_a{}^{\dot{b}}\mu^c{}_{\dot{b}} = \delta_a^c$. We also notice the alternative expression

$$\mu_a{}^{\dot{b}} = \bar{Q}_{a|i}^{-}(G_{\dot{k}}{}^{i}\,\omega^{\dot{k}}{}_l\,G^l{}_{\dot{m}})\bar{Q}^{\dot{b}|\dot{m}\ -}\,, \tag{3.58}$$

which is obtained through the relations (C.6), and will become useful in the discussion of the next section.

While ω is an i-periodic function on the Riemann section with short cuts, μ has a periodicity on the section with long cuts, as depicted in figure 4. Expressed in terms of a section with short cuts, this "mirror periodicity" becomes

$$\mu^{++} = (\mu)^\gamma. \tag{3.59}$$

To prove this relation (and thus also long-cut periodicity of μ), we continue it along $\bar{\gamma}$ and show that the combination $(\mu^{++})^{\bar{\gamma}} - \mu$ vanishes. We can rewrite such a difference as

$$\left((\mu^{++})^{\bar{\gamma}} - \mu\right)_a{}^{\dot{b}} = Q_{a|i}^{+}\,(\omega^i{}_{\dot{k}})^{\bar{\gamma}}\,Q^{\dot{b}|\dot{k}\ +} - Q_{a|i}^{-}\,\omega^i{}_{\dot{k}}\,Q^{\dot{b}|\dot{k}\ -}\,. \tag{3.60}$$

---

[7]As we saw in this section, these relations can be used to deduce algebraically all remaining properties, including the effect of crossing the cuts in the opposite directions.

We can now plug in $(\omega)^{\gamma^{-1}}$ from the (undotted version of) (3.56), and, in the second term, use the identities (2.29) to relate $Q^-_{a|i}$ and $Q^+_{a|i}$. We get

$$Q^+_{a|k} \left(\omega^k_{\dot{m}} + (\mathbf{Q}^k)^\gamma \mathbf{Q}_{\dot{m}} - \mathbf{Q}^k (\mathbf{Q}_{\dot{m}})^{\bar{\gamma}} \right) Q^{\dot{b}|\dot{m}\,+} + \tag{3.61}$$

$$-Q^+_{a|k} \left(\delta^k_m - \mathbf{Q}^k \mathbf{Q}_m \right) \omega^m_{\dot{m}} \left(\delta^{\dot{m}}_{\dot{k}} + \mathbf{Q}^{\dot{m}} \mathbf{Q}_{\dot{k}} \right) Q^{\dot{b}|\dot{k}\,+} = 0, \tag{3.62}$$

where a perfect cancellation occurs due to (3.55),(3.56), establishing (3.59). We can use μ to compute the values of $\mathbf{P}$ on the second sheet. In particular, the definition (3.57), together with $\mathbf{P}_a = Q^+_{a|i} \mathbf{Q}^i$, immediately implies

$$(\mathbf{P}_a)^\gamma = \mu^{\dot{b}\,++}_a \mathbf{P}_{\dot{b}} = (\mu^{\dot{b}}_a)^\gamma \mathbf{P}_{\dot{b}}, \tag{3.63}$$

which is conveniently rewritten as

$$(\mathbf{P}_a)^{\bar{\gamma}} = \mathbf{P}_{\dot{b}} \mu^{\dot{b}}_{\ a}. \tag{3.64}$$

This equation, compared to (3.37), highlights the symmetry of the construction between $\mathbf{P}$ and $\mathbf{Q}$ functions. From (3.41), it is also immediate to derive

$$(\mu^{\dot{b}}_a)^\gamma - \mu^{\dot{b}}_a = Q^+_{a|k} \left(\omega^k_{\dot{m}} - (\omega^k_{\dot{m}})^{\bar{\gamma}} \right) Q^{\dot{b}|\dot{m}+} = \mathbf{P}_a (\mathbf{P}^b)^{\bar{\gamma}} - (\mathbf{P}_a)^\gamma \mathbf{P}^b, \tag{3.65}$$

which shows that the combination $\mu^{\dot{b}}_a + \mathbf{P}_a (\mathbf{P}^{\dot{b}})^{\bar{\gamma}} = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b \right) \mu^{\dot{b}}_b$ has no cut on the real axis. From this observation and (3.64) we also deduce

$$(\mathbf{P}_a)^\gamma = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b \right) \mu^{\dot{b}}_b \mathbf{P}_{\dot{b}}, \quad (\mathbf{P}_a)^{\gamma^2} = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b \right) \mu^{\dot{b}}_b (\mathbf{P}_{\dot{b}})^\gamma, \tag{3.66}$$

and we obtain, similar to the previous discussion, that the branch points are in general connected to an infinite series of sheets, which can be reached by iterating

$$(\mathbf{P}_a)^{\bar{\gamma}^n} = \bar{W}^{\dot{b}}_a (\mathbf{P}_{\dot{b}})^{\bar{\gamma}^{n-1}}, \quad (\mathbf{P}_a)^{\gamma^n} = W^{\dot{b}}_a (\mathbf{P}_{\dot{b}})^{\gamma^{n-1}}, \tag{3.67}$$

with $W, \bar{W}$ defined by

$$W^{\dot{b}}_a = \left(\delta^b_a + \mathbf{P}_a \mathbf{P}^b \right) \mu^{\dot{b}}_b, \quad \bar{W}^{\dot{b}}_a = \mu^{\dot{c}}_a \left(\delta^{\dot{b}}_{\dot{c}} - \mathbf{P}_{\dot{c}} \mathbf{P}^{\dot{b}} \right), \tag{3.68}$$

with $\dot{W} = (\bar{W})^{-1}$, $\bar{\dot{W}} = (W)^{-1}$ defined similarly by dotting/undotting all indices. As in the previous paragraph, we see that going around the branch point many times keeps leading to new sheets, since we expect in general that $(W \cdot \dot{W})^b_a \neq \delta^b_a$, being there no reason to expect otherwise.

We can summarise the finding of this section in a set of $\mathbf{P}\mu$ equations. For the first wing they read,

$$(\mathbf{P}_a)^{\bar{\gamma}} = \mathbf{P}_{\dot{b}} \mu^{\dot{b}}_{\ a}, \quad (\mathbf{P}^a)^{\bar{\gamma}} = \mathbf{P}^{\dot{b}} \mu^{\ a}_{\dot{b}}, \tag{3.69}$$

and

$$(\mu^{\dot{b}}_a)^\gamma - \mu^{\dot{b}}_a = \mathbf{P}_a (\mathbf{P}^b)^{\bar{\gamma}} - (\mathbf{P}_a)^\gamma \mathbf{P}^b, \quad (\mu^a_{\ \dot{b}})^\gamma - \mu^a_{\ \dot{b}} = -\mathbf{P}^a (\mathbf{P}_b)^{\bar{\gamma}} + (\mathbf{P}^a)^\gamma \mathbf{P}_b. \tag{3.70}$$

Together with the mirror-periodicity of μ, this can also be taken as a self-consistent description of the QSC. As remarked for the $\mathbf{Q}\omega$-system, these equations contain enough information to map the values of $\mathbf{P}$ and μ functions on any sheet, back to the first main one.

4 The ABA limit

In this section, we will find an asymptotic solution for some of the Q-functions in the large-L limit. This will lead us to a perfect match with the Asymptotic Bethe Ansatz for massive states, including the dressing phases.

4.1 Large-volume scaling of the QSC

To deduce the large-L solution, we will use arguments developed for the AdS_5 case in [2] and then also successfully used for AdS_4 case to derive the ABA in [14]. The crucial observation is that, for large L, some Q-functions are exponentially suppressed/enhanced, following the pattern of their large-u asymptotics (3.10). Following the notation of [2], we introduce a parameter $\epsilon \propto e^{-L}$ to keep track of this scaling. We then see that for large L (i.e., $\epsilon \sim 0$),

$$Q_{a|i} \sim \begin{pmatrix} 1 & \epsilon^2 \\ \frac{1}{\epsilon^2} & 1 \end{pmatrix}, \quad Q^{a|i} \sim \begin{pmatrix} 1 & \frac{1}{\epsilon^2} \\ \epsilon^2 & 1 \end{pmatrix}, \tag{4.1}$$

$$\mathbf{Q}_i \sim (\epsilon^{-1}, \epsilon), \ \mathbf{Q}^i \sim (\epsilon, \epsilon^{-1}), \quad \mathbf{P}_i \sim (\epsilon, \epsilon^{-1}), \ \mathbf{P}^i \sim (\epsilon^{-1}, \epsilon). \tag{4.2}$$

In the second wing, we would have exactly the same pattern for the dotted Q-functions. In addition, since the ω functions are periodic on a Riemann section with short cuts, they have constant asymptotics. We will then assume that they all scale as

$$\omega_k^{\dot m} \sim O(1), \ \omega_m^k \sim O(1), \ \omega_{\dot k}^m \sim O(1), \ \omega_m^{\dot k} \sim O(1), \quad \epsilon \to 0. \tag{4.3}$$

We then notice that some of the QQ relations, $\mathbf{P}\mu$ and $\mathbf{Q}\omega$ equations simplify significantly. Dropping the subleading terms for $\epsilon \to 0$ we find for instance, from (3.57),

$$\mu_1^{\dot 2} = Q_{1|k}^- \omega_l^k Q^{\dot 2|\dot l\,-} \sim Q_{1|1}^- \omega_{\dot 2}^1 Q^{\dot 2|\dot 2\,-} = Q_{1|1}^- \omega_{\dot 2}^1 Q_{\dot 1|\dot 1}^-, \tag{4.4}$$

and similarly we get to

$$\mu^2_{\dot 1} \sim Q_{1|1}^- \omega_2^{\dot 1} Q_{\dot 1|\dot 1}^-, \tag{4.5}$$

where we recalled that by definition $Q^{2|2} = Q_{1|1}$. Another important equation is obtained starting from $\mathbf{P}_1 = (Q_{1|i}^+)\mathbf{Q}^i$, and considering the analytic continuation along γ (recall that $Q_{a|1}^+$ has no cut on the real axis). Using the $\mathbf{Q}\omega$-system, and then considering the large-L scaling, we get

$$(\mathbf{P}_1)^\gamma = Q_{1|l}^+ \omega_{\dot k}^l \mathbf{Q}^{\dot k} \sim Q_{1|1}^+ \omega_{\dot 2}^1 \mathbf{Q}^{\dot 2}, \tag{4.6}$$

which will play an important role in the following derivation of the ABA.

We now proceed to deduce the form of some of the elements of the QSC in the ABA scaling. To do that, we will take as a working hypothesis the property that, for the functions $\mu_1^{\dot 2}, \mu_{\dot 1}^2, \mu^{\dot 1}_2, \mu^1_{\dot 2}$, the cut on the real axis becomes quadratic in the large-L limit. We will

see that all the solutions for massive states fall into this category.[8]

We will also make an assumption that the gluing matrix follows the pattern one can deduce from the gluing equations in the classical limit, namely that all the diagonal elements vanish. Our derivation assumes that this is true at least in the ABA limit, but we suspect it may be true even at finite L (this is what happens in AdS_5).

Finally, we will use the expressions obtained from (3.58) in the ABA limit, such as

$$\mu_1^{\dot{2}} \sim \bar{Q}_{1|1}^-(G_{\dot{k}}^{\;1}\omega^{\dot{k}}_{\;l}G^l_{\;\dot{2}})\bar{Q}_{\dot{1}|\dot{1}}^- \propto \bar{Q}_{1|1}^-\omega^{\dot{2}}_{\;1}\bar{Q}_{\dot{1}|\dot{1}}^-. \tag{4.7}$$

4.2 Fixing Q-functions on the first sheet

Finding $Q_{1|1}$, $\mu_1^{\dot{2}}$ and $\omega^1_{\;\dot{2}}$. To determine these functions, we use the assumption on the quadratic nature of the branch point of $\mu_1^{\dot{2}}$ in the ABA limit. Even though this assumption could appear to be too restrictive, we will nevertheless show that in the ABA limit this extra restriction does not lead to any inconsistencies. The simplification of the analytic structure of μ is quite typical in the ABA limit – for instance in the AdS_5 case the discontinuity of $\log \mu$ appears to be a simple rational function of x, whereas in general it would have an infinity tower of cuts. With that in mind, we can follow closely [2], and this part may be skimmed through by the reader familiar with that paper. The surprises begin from section 4.3, where the non-quadratic nature of the branch points pops up again in a crucial way.

We start by considering the function $\mu_1^{\dot{2}}(u+i/2)$. We take it to have a finite number of zeros on the first Riemann sheet with short cuts, and we store such zeros in a polynomial $\mathbb{Q}(u) = \prod_i(u-u_i)$. We then consider

$$(F)^2 \equiv \frac{\mu_1^{\dot{2}}}{\mu_1^{\dot{2}++}}\frac{\mathbb{Q}^+}{\mathbb{Q}^-} = \frac{\mu_1^{\dot{2}}}{(\mu_1^{\dot{2}})^\gamma}\frac{\mathbb{Q}^+}{\mathbb{Q}^-} \tag{4.8}$$

which by definition has no zeros or poles on the first Riemann sheet with short cuts. Since by our assumption the branch points of μ become quadratic in the ABA limit, using the property $(\mu)^\gamma = \mu^{++} \sim \mu^{\bar{\gamma}}$, it is simple to obtain the same equations as in [2]:

$$F(F)^\gamma \sim F(F)^{\bar{\gamma}} \sim \frac{\mathbb{Q}^+}{\mathbb{Q}^-}. \tag{4.9}$$

All the other cuts in F must disappear in the ABA limit. In fact, using (4.5), and the periodicity of ω, we see that F^2 can be rewritten as

$$F^2 = \frac{Q_{1|1}^- Q_{\dot{1}|\dot{1}}^-}{Q_{1|1}^+ Q_{\dot{1}|\dot{1}}^+}\frac{\mathbb{Q}^+}{\mathbb{Q}^-}, \tag{4.10}$$

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[8]It is tempting to speculate that asymptotic solutions including massless modes might be found by relaxing this assumption on the behaviour at large L. On the other hand the massless modes suffer from stronger wrapping effects, which limits the range of validity of the corresponding ABA regime, which may mean that the approach of [2] is not sensitive enough to detect those power-like effects, and the ABA should be recovered via a different route. We reserve these questions for future studies.

which does not have cuts in the upper half plane, while (4.7) leads us to the expression

$$F^2 = \frac{\bar{Q}_{1|1}^- \bar{Q}_{\dot{1}|\dot{1}}^-}{\bar{Q}_{1|1}^+ \bar{Q}_{\dot{1}|\dot{1}}^+} \frac{\mathbb{Q}^+}{\mathbb{Q}^-}, \tag{4.11}$$

which shows that there are no cuts in the lower half plane either. Taking into account that F has constant asymptotics at large u on the first sheet, we have a simple Riemann-Hilbert problem (4.9), with the standard solution

$$F = \pm e^{i\frac{\mathcal{P}}{2}} \frac{B_{(+)}}{B_{(-)}}, \tag{4.12}$$

with $e^{i\mathcal{P}} \equiv \prod_i \frac{x_i^+}{x_i^-}$, and $B_{(\pm)}(u) \equiv \sqrt{\frac{h}{x_i^{\mp}}}(\frac{1}{x(u)} - x_i^{\mp})$. The constant factor will not be very important in the current considerations.[9]

Setting $Q_{1|1}Q_{\dot{1}|\dot{1}} \equiv \mathbb{Q}(f^+)^2$, equation (4.10) then gives us a difference equation

$$\frac{f^{++}}{f} = \frac{B_{(-)}}{B_{(+)}}, \tag{4.13}$$

where by construction f should have neither poles nor zeros in the upper half plane, and power-like asymptotics. Up to a multiplicative constant, the solution is

$$f(u) \propto \exp\left(-\int_{-2h}^{2h} \frac{dz}{2\pi i} \log \frac{B_{(-)}(z)R_{(+)}(z)}{B_{(+)}(z)R_{(-)}(z)} \partial_z \log \Gamma(iz - iu)\right), \tag{4.14}$$

where we use $\propto$ to indicate that there could be an irrelevant constant factor in the equation. With the explicit form of f in (4.14), we have fixed $Q_{1|1}Q_{\dot{1}|\dot{1}}$ completely. Noticing that $\mu_1{}^{\dot{2}} = Q_{1|1}^- Q_{\dot{1}|\dot{1}}^- \omega^1{}_{\dot{2}} \propto \bar{Q}_{1|1}^- \bar{Q}_{\dot{1}|\dot{1}}^- \omega^1{}_{\dot{2}}$, where ω should be i-periodic, we can also find

$$\mu_1{}^{\dot{2}} \propto \mathbb{Q}^- f \bar{f}^{--}, \quad \omega^1{}_{\dot{2}} \propto \frac{\bar{f}^{--}}{f}, \quad Q_{1|1}Q_{\dot{1}|\dot{1}} \propto \mathbb{Q}(f^+)^2, \tag{4.15}$$

where $\bar{f}$ is solution of $\bar{f}/\bar{f}^{--} = \frac{B_{(-)}}{B_{(+)}}$ with no cuts in the lower half plane and constant asymptotics.[10] From the expression (4.10), we also see that the set of zeros of $\mathbb{Q}$ must coincide with the union of the zeros of $Q_{1|1}$ and $Q_{\dot{1}|\dot{1}}$. Therefore we split this polynomial as $\mathbb{Q}(u) \equiv \mathbb{Q}_2(u)\mathbb{Q}_{\dot{2}}(u)$, with $\mathbb{Q}_2(u) \equiv \prod_{i=1}^{K_2}(u - u_{2,i})$, $\mathbb{Q}_{\dot{2}}(u) \equiv \prod_{i=1}^{K_{\dot{2}}}(u - u_{\dot{2},i})$, with the understanding that $\mathbb{Q}_2$ contains zeros of $Q_{1|1}$, and $\mathbb{Q}_{\dot{2}}$ zeros of $Q_{\dot{1}|\dot{1}}$. This notation is chosen in anticipation of the role of the zeros in the ABA. With these conventions, we have

$$\mu_1{}^{\dot{2}} \propto \mathbb{Q}_2^- \mathbb{Q}_{\dot{2}}^- f_2 f_{\dot{2}} \bar{f}_2^{--} \bar{f}_{\dot{2}}^{--}, \quad \omega^1{}_{\dot{2}} \propto \frac{\bar{f}_2^{--} \bar{f}_{\dot{2}}^{--}}{f_2 f_{\dot{2}}}, \quad Q_{1|1} \propto \mathbb{Q}_2 f_2^+ f_{\dot{2}}^+ P, \quad Q_{\dot{1}|\dot{1}} \propto \mathbb{Q}_{\dot{2}} f_2^+ f_{\dot{2}}^+ \frac{1}{P}, \tag{4.16}$$

---

[9]In any case, one can establish by an argument parallel to the one in [2], that $e^{i\mathcal{P}} = 1$, which can be recognised as the level matching condition in the ABA interpretation.

[10]We have that $\bar{f}$ is simply the complex conjugate of f for real roots, and otherwise it is given by a simple integral representation similar to (4.14).

with the obvious notation that f_α are solutions of $f_\alpha^{++}/f_\alpha = \frac{B_{\alpha,(-)}}{B_{\alpha,(+)}}$, with $\alpha \in \{2, \dot 2\}$ (see appendix A), and where P is a yet unfixed function of u coming from splitting the product $Q_{1|1}Q_{\dot 1|\dot 1}$. This function should have neither zeros nor poles, and moreover P^- cannot have any cuts in the upper half plane. On the other hand, the quantity

$$\bar Q_{1|1}^+ = Q_{1|j}^+ \Omega_1{}^j \sim Q_{1|1}^+ \Omega_1{}^1 \quad \text{for } \epsilon \to 0, \tag{4.17}$$

should be analytic in the lower half plane, where the matrix Ω is defined by $\omega_{\dot a}{}^b = G_{\dot a}{}^c \Omega_c{}^b$. Using the assumed classics-inspired off-diagonal property of the gluing matrix, we see that $\Omega_1{}^1 \propto \omega_{\dot 2}{}^1$. Then, from (4.17) and the above found solution for $\omega_{\dot 2}{}^1$, we deduce that $\bar Q_{1|1}^+$ - which should be analytic in the lower half plane - can also be written as $\mathbb{Q}_2 \frac{B_{2,(-)}B_{\dot 2,(-)}}{B_{2,(+)}B_{\dot 2,(+)}} \bar f_2^{--} \bar f_{\dot 2}^{--} P^+$. Since all the other factors already have this property, we conclude that P^+ should have no cuts in the lower half plane. All together, we found that the function P cannot have any singularities or zeroes and thus is a constant (due to regularity at infinity). In conclusion, we found

$$\begin{aligned}
Q_{1|1} &\propto \mathbb{Q}_2\, f_{\dot 2}^+ f_2^+, \quad Q_{\dot 1|\dot 1} \propto \mathbb{Q}_{\dot 2}\, f_{\dot 2}^+ f_2^+, \\
\mu_1{}^{\dot 2} &\propto \mu_{\dot 1}{}^2 \propto \mathbb{Q}_2^- \mathbb{Q}_{\dot 2}^-\, f_2\, \bar f_2^{--} f_{\dot 2}\, \bar f_{\dot 2}^{--}, \\
\omega^1{}_{\dot 2} &\propto \omega^{\dot 1}{}_2 \propto \frac{\bar f_2^{--}}{f_2} \frac{\bar f_{\dot 2}^{--}}{f_{\dot 2}},
\end{aligned} \tag{4.18}$$

where we included the values of more ω, μ functions, obtained by obvious generalisations of the argument above.

Parametrising P and Q functions. From the AdS_5 and AdS_4 cases, we expect that a special subset of **P** and **Q** functions will converge to simple explicit expressions in the ABA limit. This is the subset of the **P** functions which are *small*, together with the **Q** functions that are *large*, for $\epsilon \to 0$. From (4.2), we see that those are $\mathbf{P}_1$, $\mathbf{P}^2$, $\mathbf{Q}_1$, $\mathbf{Q}^2$, and their dotted counterparts. We expect that their zeros on the first sheet will acquire the meaning of Bethe roots.

With this in mind, we make the following ansatz:

$$\mathbf{P}_1 \propto x^{-L/2}\, \mathcal{A} \times R_{\tilde 1} B_{\tilde 1}\, B_{2,(-)}, \qquad \mathbf{P}^2 \propto x^{-L/2}\, \mathcal{A} \times R_{\tilde 3} B_{\tilde 3} B_{2,(-)}, \tag{4.19}$$

$$\mathbf{Q}_1 \propto \frac{x^{L/2}}{\mathcal{A}'} \times R_1 B_{\dot 1} f_2 \frac{f_{\dot 2}}{B_{\dot 2,(+)}}, \qquad \mathbf{Q}^2 \propto \frac{x^{L/2}}{\mathcal{A}'} \times R_3 B_{\dot 3}\, f_2 \frac{f_{\dot 2}}{B_{\dot 2,(+)}}\,. \tag{4.20}$$

Above, we have stored the zeros of the **P** and **Q** functions on the first sheet inside the Zhukovsky polynomials R_α, defined[11] in appendix A (again, the notation anticipates the role of these zeros in the ABA, but for now they are generic parameters). The other B_α and f_α factors (also defined in the appendix) are chosen for future convenience, but they do not have zeros on the first sheet. Notice that the ansatz above is fully general, because

---

[11]In the definitions (A.6),(A.5), we take the zeros to satisfy $|x_{\alpha,j}| > 1$, which means the zeros of R_α (B_α) are on the first (second) sheet in terms of the spectral parameter u.

it contains the arbitrary functions of u $\mathcal{A}(u)$, $\mathcal{A}'(u)$. By construction they should have no poles or zeros on the first sheet, and moreover $\mathcal{A}$, which appears in the $\mathbf{P}$ functions, can have only a single cut.

Comparing with (2.33), we see that we can write the important function r in two alternative ways as

$$r \propto \frac{R_{\tilde{1}} B_{\tilde{1}}}{R_{\tilde{3}} B_{\tilde{3}}} \propto \frac{R_3 B_3}{R_1 B_1} \, , \tag{4.21}$$

which means that $R_{\tilde{3}}$ and $R_{\tilde{1}}$ could have common zeroes.

Likewise we make a similar ansatz for the second wing:

$$\mathbf{P}_{\dot{1}} \propto x^{-L/2} \, \dot{\mathcal{A}} \times R'_{\dot{3}} B'_3 \, B_{\dot{2},(-)}, \qquad\qquad \mathbf{P}^{\dot{2}} \propto x^{-L/2} \, \dot{\mathcal{A}} \times R'_{\dot{1}} B'_1 \, B_{\dot{2},(-)}, \tag{4.22}$$

$$\mathbf{Q}_{\dot{1}} \propto \frac{x^{L/2}}{\dot{\mathcal{A}}'} \times R'_{\tilde{3}} B'_3 f_{\dot{2}} \, \frac{f_2}{B_{\dot{2},(+)}}, \qquad\qquad \mathbf{Q}^{\dot{2}} \propto \frac{x^{L/2}}{\dot{\mathcal{A}}'} \times R'_{\tilde{1}} B'_{\tilde{1}} \, f_{\dot{2}} \, \frac{f_2}{B_{\dot{2},(+)}} \, , \tag{4.23}$$

with functions $\dot{\mathcal{A}}(u)$, $\dot{\mathcal{A}}'(u)$ having no zeroes on the main sheet. In (4.22),(4.23), we have introduced polynomials in x vs $\frac{1}{x}$, R'_α and B'_α, respectively. They are defined just like in (A.5),(A.6), but where the zeros of these polynomials (and their number) are in principle unrelated to the ones appearing in the first wing. We will however soon see that there is a simple identification. From (2.33) we again get

$$\dot{r} \propto \frac{R'_{\dot{3}} B'_3}{R'_{\dot{1}} B'_1} \propto \frac{R'_{\tilde{1}} B'_{\tilde{1}}}{R'_{\tilde{3}} B'_{\tilde{3}}} \, . \tag{4.24}$$

Furthermore, recalling that $\dot{r}^\gamma = r$ we get

$$\frac{R_{\tilde{1}} B_{\tilde{1}}}{R_{\tilde{3}} B_{\tilde{3}}} \propto \frac{R_3 B_3}{R_1 B_1} \propto \frac{B'_3 R'_3}{B'_1 R'_1} \propto \frac{B'_{\tilde{1}} R'_{\tilde{1}}}{B'_{\tilde{3}} R'_{\tilde{3}}} \, . \tag{4.25}$$

One can for example deduce that $R_{\tilde{1}} R_1 = R_3 R_{\tilde{3}}$ etc. from the above equation.

Fermionic duality equation. An important constraint comes from one of the QQ relations

$$Q_{1|1}^+ - Q_{1|1}^- = \mathbf{Q}_1 \mathbf{P}_1 \, , \tag{4.26}$$

where we see the appearance of $Q_{1|1}$ determined in (4.18). Plugging in that value, and the ansatz (4.19),(4.20), we find, from the first equality in (4.26),

$$R_{2,(+)} B_{\dot{2},(-)} - R_{2,(-)} B_{\dot{2},(+)} \propto R_1 \, R_{\tilde{1}} B_{\dot{1}} B_{\tilde{1}} \frac{\mathcal{A}}{\mathcal{A}'} \, , \tag{4.27}$$

where we used the property that $\mathbb{Q}_\alpha^\pm = \pm B_{\alpha,(\pm)} R_{\alpha,(\pm)}$. Since the left hand side is a rational function in $1/x(u)$, and $\mathcal{A}$, $\mathcal{A}'$ should have no zeros or poles on the first sheet, the ratio $\mathcal{A}(u)/\mathcal{A}'(u)$ can only be a polynomial in the variable $\frac{1}{x(u)}$. But we can absorb any such function in a redefinition of the $B_{\tilde{1}}$, $B_{\dot{1}}$ polynomials (which are so far completely unconstrained), so without loss of generality we can take $\mathcal{A}/\mathcal{A}' = 1$. Similar considerations arise from considering (4.26) in the second wing. From now on, therefore we take

$$\mathcal{A}(u) = \mathcal{A}'(u) \, , \, \, \dot{\mathcal{A}}(u) = \dot{\mathcal{A}}'(u). \tag{4.28}$$

Notice that we still have two undetermined functions, which will be fixed in the next section. Using that $\mathcal{A} = \mathcal{A}'$, from (4.27) we obtain

$$R_{2,(+)}B_{\dot{2},(-)} - R_{2,(-)}B_{\dot{2},(+)} \propto R_1\,R_{\tilde{1}}B_{\dot{1}}B_{\dot{\tilde{1}}}\,. \tag{4.29}$$

The analogous constraint obtained by considering the second wing reads

$$B_{2,(-)}R_{\dot{2},(+)} - B_{2,(+)}R_{\dot{2},(+)} \propto B_1'\,B_{\tilde{1}}'R_1'R_{\tilde{1}}'\,, \tag{4.30}$$

and analytically continuing this equation to another sheet we find the identity

$$R_1\,R_{\tilde{1}}B_{\dot{1}}B_{\dot{\tilde{1}}} \propto R_1'\,R_{\tilde{1}}'B_{\dot{1}}'B_{\dot{\tilde{1}}}' \tag{4.31}$$

which implies

$$R_1\,R_{\tilde{1}} = R_1'\,R_{\tilde{1}}'\,. \tag{4.32}$$

Equations of the form (4.29) are examples of fermionic duality relations. They imply that the sets of roots with labels 1, 2, 3 (or alternatively the "dual" set obtained with $1 \leftrightarrow \tilde{1}$, $3 \leftrightarrow \tilde{3}$) satisfy the auxiliary ABA equations of the form

$$1 = \left.\frac{\mathbb{Q}_2^+\,B_{2,(-)}\,B_{\dot{2},(-)}}{\mathbb{Q}_2^-\,B_{2,(+)}\,B_{\dot{2},(+)}}\right|_{u \in \{\text{roots of type } 1, \tilde{1}, 3, \tilde{3}\}}\,. \tag{4.33}$$

4.3 Going inside the cut: fixing the dressing phases

So far we reduced the ansatz for $\mathbf{P}$'s and $\mathbf{Q}$'s to just two unknown functions with one cut and no zeroes $\mathcal{A}$ and $\dot{\mathcal{A}}$ on the main sheet. In order to constrain them further, we need to go to the next sheet of their Riemann surfaces.

This will bring us to the most interesting part of the analysis, where things will be radically different than in AdS_5 and AdS_4. By studying equations of the form (4.6), which we repeat here,

$$(\mathbf{P}_1)^\gamma \sim Q_{1|1}^+\omega_{\dot{2}}^1\mathbf{Q}^{\dot{2}}\ ,\ \ (\mathbf{P}_{\dot{1}})^\gamma \sim Q_{\dot{1}|\dot{1}}^+\omega_2^{\dot{1}}\mathbf{Q}^2 \tag{4.34}$$

we will find that the $\mathbf{P}$ and $\mathbf{Q}$ functions *cannot have* a quadratic cut even in the ABA limit. We will also be able to fix the form of the yet undetermined functions $\mathcal{A}(u)$, $\dot{\mathcal{A}}(u)$ and relate them to the dressing phases of [37].

4.3.1 The cuts cannot be quadratic

The strategy will be to compare the r.h.s. of each of the equations (4.34), with the analytic continuation of $\mathbf{P}$ functions, starting from their form in (4.19),(4.23).[12] From the first equation in (4.6), in particular, we obtain:

$$\mathbf{P}_1^\gamma = x^{L/2}(\mathcal{A})^\gamma B_{\tilde{1}}R_{\tilde{1}}\,R_{2,(-)} = \left(\mathbb{Q}_2^+ f_{\dot{2}}^{++} f_2^{++}\right)\left(\frac{\bar{f}_2\bar{f}_{\dot{2}}}{f_2^{++}f_{\dot{2}}^{++}}\right)\left(\frac{x^{L/2}}{\dot{\mathcal{A}}}R_{\tilde{1}}'B_{\tilde{1}}'\frac{f_2 f_{\dot{2}}}{B_{2,(+)}}\right)\,. \tag{4.35}$$

---

[12]Here a comment is in order: in principle, the analytic continuation through the cut might not commute with the large-L limit, due to the presence of Stokes-type phenomena - where a subleading correction on the first sheet might become large on the second sheet invalidating the result. However, as discussed in [2], one can expect that it is safe to analytically continue the ABA limit of a Q-function that is already *small* on the first sheet. This is the case of the $\mathbf{P}$ functions we consider which are of order ϵ.

We noticed in the previous section that the roots of $R_{\bar{1}}$ and $R'_{\bar{1}}$ satisfy the same BAE equation (4.33). The same is true for the roots of $R_{\tilde{1}}$ and $R'_{\tilde{1}}$. Whereas this does not necessarily mean that all roots coincide, we will assume $R_{\bar{1}} = R'_{\bar{1}}$ and $R_{\tilde{1}} = R'_{\tilde{1}}$. In this case we get a nice cancellation in the above equation, which further supports this requirement. Then we get a simple relation

$$(\mathcal{A})^\gamma \dot{\mathcal{A}} = \left(\frac{R_{2,(+)}}{R_{2,(-)}} \right) \left(\bar{f}_2^{--} f_2^{++} \bar{f}_{\dot{2}}^{--} f_{\dot{2}}^{++} \right) . \tag{4.36}$$

It is striking to compare this with the consequence of the second relation in (4.34), which yields

$$\mathcal{A} (\dot{\mathcal{A}})^\gamma = \left(\frac{R_{\dot{2},(+)}}{R_{\dot{2},(-)}} \right) \left(\bar{f}_2^{--} f_2^{++} \bar{f}_{\dot{2}}^{--} f_{\dot{2}}^{++} \right) . \tag{4.37}$$

Now we continue this relation along the reverse path $\bar{\gamma}$: the result on the l.h.s. is $(\mathcal{A})^{\bar{\gamma}}\dot{\mathcal{A}}$, and the analytic continuation of the r.h.s. is simple to compute, since the f^{++}, $\bar{f}^{--}$ functions have no cut on the real axis, so are left unchanged. By comparing the result with (4.36), we get the following "double-discontinuity" relations

$$\frac{\mathcal{A}^\gamma}{\mathcal{A}^{\bar{\gamma}}} = \frac{R_{2,(+)}}{R_{2,(-)}} \frac{B_{\dot{2},(-)}}{B_{\dot{2},(+)}} \;\; , \;\; \frac{\dot{\mathcal{A}}^\gamma}{\dot{\mathcal{A}}^{\bar{\gamma}}} = \frac{R_{\dot{2},(+)}}{R_{\dot{2},(-)}} \frac{B_{2,(-)}}{B_{2,(+)}} \;\; , \tag{4.38}$$

where the r.h.s. clearly cannot vanish (except for the vacuum) since the R_α and B_α functions have zeros on different sheets. We will now solve (4.36) and (4.37).

4.3.2 Relation to the dressing phases

In order to find the solution, without lack of generality we introduce the following ansatz in terms of ρ and $\dot{\rho}$

$$\mathcal{A} = \sqrt{\frac{B_{2,(+)}}{B_{2,(-)}}} \, \sigma_2^1 \sigma_{\dot{2}}^1 \rho \;\; , \;\; \dot{\mathcal{A}} = \sqrt{\frac{B_{\dot{2},(+)}}{B_{\dot{2},(-)}}} \, \sigma_2^1 \sigma_{\dot{2}}^1 \dot{\rho} \;\; , \tag{4.39}$$

where, using notation from [2], $\sigma_\alpha^{1,\text{BES}}$ denote natural building blocks of the Beisert-Eden-Staudacher dressing factor. They satisfy

$$(\sigma_\alpha^{1,\text{BES}})^\gamma \, \sigma_\alpha^{1,\text{BES}} = (\sigma_\alpha^{1,\text{BES}})^{\bar{\gamma}} \, \sigma_\alpha^{1,\text{BES}} \propto f_\alpha^{++} \bar{f}_\alpha^{--} \;\; , \;\; \alpha = 2, \dot{2} \;\; , \tag{4.40}$$

and are related to the product of the BES dressing factors via

$$\sigma_{\text{BES}}(u) = \frac{\sigma^{1,\text{BES}}(u + \frac{i}{2})}{\sigma^{1,\text{BES}}(u - \frac{i}{2})} \;\; , \tag{4.41}$$

with the notation explained in appendix A. With this redefinition, (4.36), (4.37) become

$$(\rho)^\gamma \dot{\rho} \propto \sqrt{\frac{R_{2,(+)}}{R_{2,(-)}} \frac{B_{\dot{2},(-)}}{B_{\dot{2},(+)}}} \;\; , \;\; (\dot{\rho})^\gamma \rho \propto \sqrt{\frac{R_{\dot{2},(+)}}{R_{\dot{2},(-)}} \frac{B_{2,(-)}}{B_{2,(+)}}} \;\; . \tag{4.42}$$

In appendix B, we define the functions $\sigma^{1,\text{extra}}$ and $\tilde{\sigma}^{1,\text{extra}}$ which are related to the two independent dressing phases appearing in ABA equations of section 2.1 in the following way:

$$\sigma(u) = \sigma_{\text{BES}}(u)\frac{\sigma^{1,\text{extra}}(u + \frac{i}{2})}{\sigma^{1,\text{extra}}(u - \frac{i}{2})}, \quad \tilde{\sigma}(u) = \sigma_{\text{BES}}(u)\frac{\tilde{\sigma}^{1,\text{extra}}(u + \frac{i}{2})}{\tilde{\sigma}^{1,\text{extra}}(u - \frac{i}{2})}. \tag{4.43}$$

In the same appendix, we also show that these extra pieces satisfy the following identities

$$(\sigma_\alpha^{1,\text{extra}})^\gamma \, \tilde{\sigma}_\alpha^{1,\text{extra}} = \sqrt{\frac{R_{\alpha,(+)}}{R_{\alpha,(-)}}}, \quad (\tilde{\sigma}_\alpha^{1,\text{extra}})^\gamma \, \sigma_\alpha^{1,\text{extra}} = \sqrt{\frac{B_{\alpha,(-)}}{B_{\alpha,(+)}}} \ , \ \alpha \in \{2, \dot{2}\} \ , \tag{4.44}$$

which we both verify directly and also independently deduce from crossing via functional arguments.

Using those building blocks, we can write

$$\rho = \sigma_2^{1,\text{extra}}\tilde{\sigma}_{\dot{2}}^{1,\text{extra}}\rho_0 \ , \ \dot{\rho} = \sigma_{\dot{2}}^{1,\text{extra}}\tilde{\sigma}_2^{1,\text{extra}}\dot{\rho}_0 \ , \tag{4.45}$$

where ρ_0 and $\dot{\rho}_0$ should be functions with square-root branch cut on the real axis satisfying

$$\rho_0^\gamma = 1/\dot{\rho}_0 \ , \ \dot{\rho}_0^\gamma = 1/\rho_0 \ . \tag{4.46}$$

This equation tells us that ρ_0^γ is a function with a single cut and neither zeroes nor poles, and likewise $\dot{\rho}_0$ and ρ_0. In other words it can only be a power of x, which can be included into a re-definition of L. So without reducing the generality we can set $\dot{\rho}_0 = \rho_0 = 1$. This completes the derivation of the asymptotic limit of our QSC.

4.4 Summary of results for the asymptotic limit

Let us summarise what we found for the expressions of $\mathbf{P}$ and $\mathbf{Q}$ functions. In the first wing we have:

$$\mathbf{P}_1 \propto x^{-L/2} \, R_{\tilde{1}}B_{\tilde{1}} \, \sqrt{B_{2,(+)}B_{2,(-)}} \, \sigma_2^1 \, \tilde{\sigma}_{\dot{2}}^1, \quad \mathbf{P}^2 \propto x^{-L/2} \, R_{\tilde{3}}B_{\tilde{3}}\sqrt{B_{2,(+)}B_{2,(-)}} \, \sigma_2^1 \, \tilde{\sigma}_{\dot{2}}^1, \tag{4.47}$$

$$\mathbf{Q}_1 \propto x^{L/2}R_1 B_{\dot{1}} \, \sqrt{\frac{B_{2,(-)}}{B_{2,(+)}}} \, \frac{f_2 \, f_{\dot{2}}}{B_{\dot{2},(+)} \, \sigma_2^1 \, \tilde{\sigma}_{\dot{2}}^1}, \quad \mathbf{Q}^2 \propto x^{L/2}R_3 B_{\dot{3}} \, \sqrt{\frac{B_{2,(-)}}{B_{2,(+)}}} \, \frac{f_2 \, f_{\dot{2}}}{B_{\dot{2},(+)} \, \sigma_2^1 \, \tilde{\sigma}_{\dot{2}}^1}, \tag{4.48}$$

and in the second wing:

$$\mathbf{P}_{\dot{1}} \propto x^{-L/2} \, R_{\dot{3}}B_3 \, \sqrt{B_{\dot{2},(+)}B_{\dot{2},(-)}} \, \sigma_{\dot{2}}^1 \, \tilde{\sigma}_2^1, \quad \mathbf{P}^{\dot{2}} \propto x^{-L/2} \, R_{\dot{1}}B_1 \, \sqrt{B_{\dot{2},(+)}B_{\dot{2},(-)}} \, \sigma_{\dot{2}}^1 \, \tilde{\sigma}_2^1, \tag{4.49}$$

$$\mathbf{Q}_{\dot{1}} \propto x^{L/2} \, R_{\tilde{3}}B_{\tilde{3}} \, \sqrt{\frac{B_{\dot{2},(-)}}{B_{\dot{2},(+)}}} \, \frac{f_2 f_{\dot{2}}}{B_{2,(+)} \, \sigma_{\dot{2}}^1 \, \tilde{\sigma}_2^1}, \quad \mathbf{Q}^{\dot{2}} \propto x^{L/2} \, R_{\tilde{1}}B_{\tilde{1}} \, \sqrt{\frac{B_{\dot{2},(-)}}{B_{\dot{2},(+)}}} \, \frac{f_2 f_{\dot{2}}}{B_{2,(+)} \, \sigma_{\dot{2}}^1 \, \tilde{\sigma}_2^1}. \tag{4.50}$$

Having an asymptotic solution for all relevant $\mathbf{P}$ and $\mathbf{Q}$ functions we can plug them into the exact Bethe ansatz equations (2.40) and compare the result with the ABA (2.1).

4.5 Match with the Asymptotic Bethe Ansatz

We have finally arrived at a full specification of the Q-functions $\mathbf{P}_1$, $\mathbf{P}^2$, $\mathbf{Q}_1$, $\mathbf{Q}^2$, $Q_{1|1}$, and their dotted cousins, in the ABA limit. To obtain the Asymptotic Bethe Ansatz, we can just plug their values in the exact Bethe equations following from the Q-system (2.40).

We will have the following correspondence between the zeros appearing on the first sheet of the Q-functions, and the Bethe roots appearing in the Asymptotic Bethe Ansatz: for the first wing

$$
\begin{array}{ll}
\text{Roots:} & u_{1,k}\ u_{2,k}\ u_{3,k} \\
\text{Q-function:} & \mathbf{Q}_1\ \ Q_{1|1}\ \ \mathbf{Q}^2
\end{array}
,\qquad
\begin{array}{ll}
\text{Dual roots:} & u_{\bar{1},k}\ u_{\bar{3},k} \\
\text{Q-function:} & \mathbf{P}_1\ \ \ \mathbf{P}^2
\end{array}
,
\tag{4.51}
$$

and for the second wing:

$$
\begin{array}{ll}
\text{Root:} & u_{\dot{1},k}\ u_{\dot{2},k}\ u_{\dot{3},k} \\
\text{Q-function:} & \mathbf{P}_{\dot{1}}\ \ Q_{\dot{1}|\dot{1}}\ \ \mathbf{P}^{\dot{2}}
\end{array}
,\qquad
\begin{array}{ll}
\text{Dual roots:} & u_{\tilde{\dot{1}},k}\ u_{\tilde{\dot{3}},k} \\
\text{Q-function:} & \mathbf{Q}_{\dot{1}}\ \ \mathbf{Q}^{\dot{2}}
\end{array}
.
\tag{4.52}
$$

In particular, the exact Bethe equations (2.40) for the first wing reduce exactly the ABA equations (A.14)-(A.17) using the Q-functions (4.47)-(4.48). Similarly, using the exact Bethe equations of the form (2.43), but for the dotted Q-functions, we reproduce the ABA equations (A.18)-(A.21) using the asymptotic values (4.49)-(4.50).

As an example to demonstrate the procedure, we display the case of the middle-node equation for the first wing. At the roots of $Q_{1|1}$ i.e. at $u = u_{2,i}$ we have

$$
\begin{aligned}
-1 &= \frac{Q_{1|1}^{++}Q_{\emptyset|1}^{-}Q_{12|1}^{-}}{Q_{1|1}^{--}Q_{\emptyset|1}^{+}Q_{12|1}^{+}} = \frac{Q_{1|1}^{++}\mathbf{Q}_1^{-}\mathbf{Q}^{2-}}{Q_{1|1}^{--}\mathbf{Q}_1^{+}\mathbf{Q}^{2+}} \\
&= \frac{(x^-)^L(\sigma_2^{1+}\tilde{\sigma}_{\dot{2}}^{1+})^2\mathbb{Q}_2^{++}R_1^{-}B_{\dot{1}}^{-}R_3^{-}B_{\dot{3}}^{-}f_2^{[+3]}f_{\dot{2}}^{[+3]}f_2^{-}f_{\dot{2}}^{-}B_{2(-)}^{-}B_{2(+)}^{+}[B_{\dot{2}(+)}^{+}]^2}{(x^+)^L(\sigma_2^{1-}\tilde{\sigma}_{\dot{2}}^{1-})^2\mathbb{Q}_2^{--}R_1^{+}B_{\dot{1}}^{+}R_3^{+}B_{\dot{3}}^{+}f_2^{+}\ \ f_{\dot{2}}^{+}\ \ f_2^{+}f_{\dot{2}}^{+}B_{2(-)}^{+}B_{2(+)}^{-}[B_{\dot{2}(+)}^{-}]^2},
\end{aligned}
$$

where we have cancelled some terms repeated in the numerator and denominator. Next we have to use the defining property of the function f_α: $\frac{f_\alpha^{++}}{f_\alpha} = \frac{B_{\alpha,(-)}}{B_{\alpha,(+)}}$ in appropriate shifted version, to re-create various B functions, some of which then cancel out and some remain. At the end of this massive simplification what is left is exactly the middle-node ABA equation for the first wing, where one needs to recall how the dressing phases are reconstructed from σ^1 and $\tilde{\sigma}^1$ via $\sigma_\alpha = \frac{\sigma^{1+}}{\sigma^{1-}}$ and $\tilde{\sigma}_\alpha = \frac{\tilde{\sigma}_\alpha^{1+}}{\tilde{\sigma}_\alpha^{1-}}$:

$$
\begin{aligned}
-1 = &\left(\frac{x^-}{x^+}\right)^L \times \frac{\mathbb{Q}_2^{++}}{\mathbb{Q}_2^{--}} \times (\sigma_2)^2 \times \frac{R_1^{-}R_3^{-}}{R_1^{+}R_3^{+}} \\
&\times \frac{B_{\dot{2},(-)}^{+}B_{\dot{2},(+)}^{+}}{B_{\dot{2},(-)}^{-}B_{\dot{2},(+)}^{-}} \times (\tilde{\sigma}_{\dot{2}})^2 \times \left.\frac{B_{\dot{1}}^{-}B_{\dot{3}}^{-}}{B_{\dot{1}}^{+}B_{\dot{3}}^{+}}\right|_{u=u_{2,i}}, \quad i = 1,\dots,K_2.
\end{aligned}
\tag{4.53}
$$

Finally, since the ABA equations (2.1) in the classical regime, $h \to \infty$, $L \sim K_\alpha \to \infty$ with fixed $\mathcal{L} \equiv L/h$ reproduce the classical limit (3.6)-(3.9) via condensation of roots into cuts in the standard way [54], it follows that we also reproduce the classical limit from the QSC, similarly to [2]. Thus we see that our QSC successfully reproduces all the data from section 2.

5 Discussion and outlook

The QSCs for AdS_5 and AdS_4 have a lot in common with one another – both are based on QQ-relations dictated by the global symmetries and have similar additional analyticity constraints. We use these general features to propose a QSC for string theory on $AdS_3 \times S^3 \times T^4$ with RR charge and its CFT$_2$ dual. However, in contrast to the higher-dimensional QSCs, the assumption of square-root singularity near the branch points needs to be dropped. While we reproduced successfully the ABA equations for massive modes, we should still emphasise that, unlike in the previous cases [2, 14], we do not have the luxury of TBA equations which can be used as a starting point to derive the QSC equations. Instead, we use a bottom-up approach where we guess the QSC based on the symmetries and analogy with previous cases, and then verify it in some limits.

On the important point of the order of the cuts, we notice that if we assume them to be of the usual square-root type, unlike in previous cases, we get a further nontrivial algebraic constraint (3.47) on the Q-functions in addition to the QQ-relations, resulting in a too restrictive set of equations. So to some extent the absence of square-root behaviour is dictated by symmetries.

To be fully confident in the self-consistency and completeness of equations proposed here, we need to perform further tests beyond the matching with the ABA presented here. For example, constructing the perturbative weak coupling solution at several loop orders would be useful, which can be done with the methods of [6, 7]. The QSC should also reproduce the protected spectrum of the theory [59], accounting for all-order wrapping corrections not considered in the ABA analysis [62, 63]. Further, it would be interesting to consider near-BPS limits where one can expect a non-trivial analytic solution at finite coupling [3, 15]. Finally, one should try to solve the system numerically with high precision like in [4]. Another potential way to test our equations would be to re-derive the TBA equations for the massless modes [44, 51]. Note that at the moment we do not have any evidence that the massless modes are contained in our QSC. However, the QSC structure is usually very rigid and does not allow for much more freedom. One potential option would be to relax the $Q_{12|12} = Q_{\dot1\dot2|\dot1\dot2} = 1$ or modify the gluing condition, but we cannot at this stage exclude the possibility that neither of these options will be necessary and the QSC presented here is already complete.

If these additional tests can be satisfactorily performed, one can hope that AdS_3 would become an ideal background for application of SoV program for correlators [18]. Further, combining the AdS_3 QSC spectral methods with Conformal Bootstrap [20] techniques could provide a simpler testing ground for these ideas compared to the $\mathcal{N} = 4$ SYM case.

Following these tests of our conjecture, it would be interesting to extend the AdS_3 QSC construction to $AdS_3 \times S^3 \times T^4$ backgrounds supported by combinations of RR and NSNS charges. The ABA for these theories is also known [31] and solutions to the crossing equations have recently been found [64], which should provide a further testing ground for the QSC analysis. String theory on $AdS_3 \times S^3 \times S^3 \times S^1$ is also expected to be integrable [32]. Finding the QSC for this model would be particularly interesting since the global symmetry algebra is $d(2, 1; \alpha)^2$, for which the Q-system should exhibit novel features.

It would be interesting to see whether similar techniques to the ones we have employed here can be extended to the AdS_2/CFT_1 integrable system [65], which also features the presence of massless modes and has an algebraic structure of a similar complexity. The issue of long vs short representations, which is relevant in that case, is likely to represent an additional novelty and a reason for adapting the method even further.

6 Acknowledgments

A Rewriting the ABA equations

Notations. We introduce some useful notations for the ABA equations. First, using the Zhukovsky map,

$$x(u) = \frac{1}{2h}\left(u + \sqrt{u - 2h}\sqrt{u + 2h}\right), \tag{A.1}$$

we reparametrise the roots in terms of $u_{2,k}$, $u_{1,k}$, $u_{3,k}$, such that $x_k^\pm \equiv x(u_{2,k} \pm \frac{i}{2})$, $1 \le k \le K_2$, $y_{1,k} \equiv x(u_{1,k})$, $1 \le k \le K_1$, $y_{3,k} \equiv x(u_{3,k})$, $1 \le k \le K_3$, and similarly for the other wing introducing $u_{\dot{2},k}$, $u_{\dot{1},k}$, $u_{\dot{3},k}$. We also accordingly rename $K_{\bar\alpha} \equiv K_{\dot\alpha}$, $\alpha = 1,2,3$, as compared to the notations of section 2.1.

It is convenient to introduce the generalised Baxter polynomials

$$B_{\alpha,(\pm)}(u) \equiv \prod_{j=1}^{K_\alpha} \sqrt{\frac{h}{x_{\alpha,j}^{\mp}}} \left(\frac{1}{x(u)} - x_{\alpha,j}^{\mp}\right), \quad \alpha \in \{2, \dot{2}\} \tag{A.2}$$

$$R_{\alpha,(\pm)}(u) \equiv \prod_{j=1}^{K_\alpha} \sqrt{\frac{h}{x_{\alpha,j}^{\mp}}} \left(x(u) - x_{\alpha,j}^{\mp}\right), \quad \alpha \in \{2, \dot{2}\} \tag{A.3}$$

$$\mathbb{Q}_\alpha(u) = \prod_{j=1}^{K_\alpha} (u - u_{\alpha,j}), \quad \alpha \in \{2, \dot{2}\}, \tag{A.4}$$

$$B_\alpha(u) = \prod_{j=1}^{K_\alpha} \left(\frac{1}{x(u)} - y_{\alpha,j}\right), \quad \alpha \in \{1, 3, \dot{1}, \dot{3}\} \tag{A.5}$$

$$R_\alpha(u) = \prod_{j=1}^{K_\alpha} (x(u) - y_{\alpha,j}), \quad \alpha \in \{1, 3, \dot{1}, \dot{3}\}. \tag{A.6}$$

Notice that $B_{\alpha,(\pm)} R_{\alpha,(\pm)} \propto \mathbb{Q}_\alpha^\pm$, where the shift of a function of u is defined as $g^{[\pm n]}(u) \equiv g(u + i\frac{n}{2})$, $g^\pm \equiv g^{[\pm 1]}$. Through the Zhukovsky map, we also consider the dressing phase a function of rapidities:

$$\sigma(u,v) \equiv \exp\left(i\chi(x^+(u), x^+(v)) - i\chi(x^+(u), x^-(v)) + i\chi(x^-(u), x^-(v)) - i\chi(x^-(u), x^+(v))\right), \tag{A.7}$$

and we introduce the notation:

$$\sigma_\alpha(u) \equiv \prod_{i=1}^{K_\alpha} \sigma(u, u_{\alpha,i}), \quad \alpha = 2, \dot{2}, \tag{A.8}$$

and the same conventions are taken for $\tilde{\sigma}$. We also use the same notation for the BES dressing phase. We also introduce useful building blocks

$$\sigma^1(u,v) \equiv \exp\left(i\chi(x(u), x^+(v)) - i\chi(x(u), x^-(v))\right), \tag{A.9}$$

and similarly for $\tilde{\sigma}^1$, and $\sigma^{1\mathrm{BES}}$, and denote again the products over roots as

$$\sigma_\alpha^1(u) \equiv \prod_{i=1}^{K_\alpha} \sigma^1(u, u_{\alpha,i}), \tag{A.10}$$

with the analogous definitions made for $\tilde{\sigma}_\alpha^1(u)$ and $\sigma^{1,\mathrm{BES}}(u)$. We then have the relation

$$\sigma_\alpha(u) = \frac{\sigma_\alpha^1(u + \frac{i}{2})}{\sigma_\alpha^1(u - \frac{i}{2})}, \tag{A.11}$$

and its generalisations. It will also be useful for some of our discussions to define $\sigma_\alpha^{1,\mathrm{extra}}(u)$, $\tilde{\sigma}_\alpha^{1,\mathrm{extra}}(u)$ through

$$\sigma_\alpha^1(u) \equiv \sigma_\alpha^{1,\mathrm{BES}}(u)\, \sigma_\alpha^{1,\mathrm{extra}}(u), \quad \tilde{\sigma}_\alpha^1(u) \equiv \sigma_\alpha^{1,\mathrm{BES}}(u)\, \tilde{\sigma}_\alpha^{1,\mathrm{extra}}(u). \tag{A.12}$$

Finally, for the reader's convenience we collect the defining relations for the functions f_α, $\bar{f}_\alpha$ appearing in the large-volume solution of the QSC:

$$\frac{f_\alpha^{++}}{f_\alpha} = \frac{B_{\alpha,(-)}}{B_{\alpha,(+)}}, \quad \frac{\bar{f}_\alpha}{\bar{f}_\alpha^{--}} = \frac{B_{\alpha,(-)}}{B_{\alpha,(+)}}, \tag{A.13}$$

where f_α is assumed analytic in the upper half plane and $\bar{f}_\alpha$ in the lower half plane, and both are free of poles everywhere. These functions are given explicitly (up to an arbitrary multiplicative constant) by DHM-type integral representations similar to (4.14).

Compact rewriting of the ABA equations. With the notations above, the ABA equations can be rewritten as

$$1 = \left.\frac{\mathbb{Q}_2^- \, B_{2,(+)} \, B_{\dot{2},(+)}}{\mathbb{Q}_2^+ \, B_{2,(-)} \, B_{\dot{2},(-)}}\right|_{u=u_{1,i}} , \quad i = 1, \ldots, K_1, \tag{A.14}$$

$$-1 = \left(\frac{x^{[-]}}{x^{[+]}}\right)^L \times \frac{\mathbb{Q}_2^{++}}{\mathbb{Q}_2^{--}} \times (\sigma_2)^2 \times \frac{R_1^- R_3^-}{R_1^+ R_3^+} \tag{A.15}$$

$$\times \left.\frac{B_{\dot{2},(-)}^+ \, B_{\dot{2},(+)}^+}{B_{\dot{2},(-)}^- \, B_{\dot{2},(+)}^-} \times (\tilde{\sigma}_{\dot{2}})^2 \times \frac{B_{\dot{1}}^- B_{\dot{3}}^-}{B_{\dot{1}}^+ B_{\dot{3}}^+}\right|_{u=u_{2,i}} , \quad i = 1, \ldots, K_2 \tag{A.16}$$

$$1 = \left.\frac{\mathbb{Q}_2^- \, B_{2,(+)} \, B_{\dot{2},(+)}}{\mathbb{Q}_2^+ \, B_{2,(-)} \, B_{\dot{2},(-)}}\right|_{u=u_{3,i}} , \quad i = 1, \ldots, K_3 \tag{A.17}$$

for the first wing, and

$$1 = \left.\frac{\mathbb{Q}_{\dot{2}}^- \, B_{\dot{2},(+)} \, B_{2,(+)}}{\mathbb{Q}_{\dot{2}}^+ \, B_{\dot{2},(-)} \, B_{2,(-)}}\right|_{u=u_{\dot{1},i}} , \quad i = 1, \ldots, K_{\dot{1}}, \tag{A.18}$$

$$-1 = \left(\frac{x^{[-]}}{x^{[+]}}\right)^{-L} \times \frac{\mathbb{Q}_{\dot{2}}^{++}}{\mathbb{Q}_{\dot{2}}^{--}} \times \left(\frac{B_{\dot{2},(-)}^-}{B_{\dot{2},(+)}^+}\right)^2 (\sigma_{\dot{2}})^{-2} \times \frac{R_{\dot{1}}^- R_{\dot{3}}^-}{R_{\dot{1}}^+ R_{\dot{3}}^+} \tag{A.19}$$

$$\times \left.\frac{B_{2,(-)}^- \, B_{2,(-)}^+}{B_{2,(+)}^- \, B_{2,(+)}^+} \times (\tilde{\sigma}_2)^{-2} \times \frac{B_1^- B_3^-}{B_1^+ B_3^+}\right|_{u=u_{\dot{2},i}} , \quad i = 1, \ldots, K_{\dot{2}} \tag{A.20}$$

$$1 = \left.\frac{\mathbb{Q}_{\dot{2}}^- \, B_{\dot{2},(+)} \, B_{2,(+)}}{\mathbb{Q}_{\dot{2}}^+ \, B_{\dot{2},(-)} \, B_{2,(-)}}\right|_{u=u_{\dot{3},i}} , \quad i = 1, \ldots, K_{\dot{3}} \tag{A.21}$$

for the second wing.

B Functional equations for the building blocks of dressing factors

In this section, we decompose the two types of dressing factors appearing in the ABA as

$$\sigma(u,v) = \sigma_{\text{BES}}(u,v)\, \sigma_{\text{extra}}(u,v)\,, \qquad \tilde{\sigma}(u,v) = \sigma_{\text{BES}}\, \tilde{\sigma}_{\text{extra}}(u,v)\,, \tag{B.1}$$

and similarly

$$\sigma^1(u,v) = \sigma^{1,\mathrm{BES}}(u,v)\,\sigma^{1,\mathrm{extra}}(u,v)\,, \qquad \tilde{\sigma}^1(u,v) = \sigma^{1,\mathrm{BES}}\,\tilde{\sigma}^{1,\mathrm{extra}}(u,v), \qquad \text{(B.2)}$$

see section A for notation. The goal of this appendix is to establish the functional relations

$$\sigma^{1,\mathrm{extra}}(u^\gamma,v)\,\tilde{\sigma}^{1,\mathrm{extra}}(u,v) = \sqrt{\frac{R_{(+)}(u,v)}{R_{(-)}(u,v)}}, \quad \tilde{\sigma}^{1,\mathrm{extra}}(u^\gamma,v)\,\sigma^{1,\mathrm{extra}}(u,v) = \sqrt{\frac{B_{(-)}(u,v)}{B_{(+)}(u,v)}},$$
$$\text{(B.3)}$$

where in this appendix we denote

$$R_{(\pm)}(u,v) = \frac{x(u) - x^{\mp}(v)}{\sqrt{x^{\mp}(v)}}, \quad B_{(\pm)}(u,v) = R_{(\pm)}(u^\gamma,v) = \frac{\frac{1}{x(u)} - x^{\mp}(v)}{\sqrt{x^{\mp}(v)}}. \qquad \text{(B.4)}$$

These relations are important for deriving the ABA from the QSC, as they imply the crucial equation (4.44). In presenting their proof here, we will also deduce

$$\sigma^{1,\mathrm{extra}}(u^{\gamma^{-1}},v)\,\tilde{\sigma}^{1,\mathrm{extra}}(u,v) = \sqrt{\frac{R_{(-)}(u,v)}{R_{(+)}(u,v)}}, \quad \tilde{\sigma}^{1,\mathrm{extra}}(u^{\gamma^{-1}},v)\,\sigma^{1,\mathrm{extra}}(u,v) = \sqrt{\frac{B_{(+)}(u,v)}{B_{(-)}(u,v)}}.$$
$$\text{(B.5)}$$

B.1 Direct derivation

We start by verifying these relations directly, based on the expressions for the dressing phases of [37]. From the results of this paper we deduce

$$\sigma^{1,\mathrm{extra}}(x, x_2^\pm) = \exp i\Lambda^{LL}(x, x_2^\pm), \qquad \tilde{\sigma}^{1,\mathrm{extra},RL}(x, x_2^\pm) = \exp i\Lambda^{RL}(x, x_2^\pm), \qquad \text{(B.6)}$$

where we have defined[13]

$$\Lambda^{LL}(x, x_2^\pm) = -\frac{1}{2}\chi^{\mathrm{HL}}(x, x_2^+) + \frac{1}{2}\chi^{\mathrm{HL}}(x, x_2^-) + \frac{1}{2}\chi^-(x, x_2^+) - \frac{1}{2}\chi^-(x, x_2^-),$$
$$\Lambda^{RL}(x, x_2^\pm) = -\frac{1}{2}\chi^{\mathrm{HL}}(x, x_2^+) + \frac{1}{2}\chi^{\mathrm{HL}}(x, x_2^-) - \frac{1}{2}\chi^-(x, x_2^+) + \frac{1}{2}\chi^-(x, x_2^-), \quad \text{(B.7)}$$

and we have the integral representations

$$\chi^{\mathrm{HL}}(x,y) = \frac{\pi}{2}\oint \frac{dw}{2\pi i}\oint \frac{dw'}{2\pi i}\frac{\mathrm{sign}(w' + 1/w' - w - 1/w)}{(x-w)(y-w')},$$
$$\chi^-(x,y) = \left(\int_{C^+} - \int_{C^-}\right)\frac{dw}{8\pi}\frac{1}{x-w}\log\left[(y-w)\left(1 - \frac{1}{yw}\right)\right] - x \leftrightarrow y, \qquad \text{(B.8)}$$

where the full circles run counterclockwise and the contours $C^\pm$ denote the upper (resp., lower) half semicircle in the complex w-plane, running counterclockwise.

We can write

$$t(x, x_2^\pm) \equiv \left(\sigma^{1,\mathrm{extra}}\right)^2(x, x_2^\pm) = \exp\left[2i\chi_{\mathrm{extra}}(x, x_2^+) - 2i\chi_{\mathrm{extra}}(x, x_2^-)\right], \qquad \text{(B.9)}$$

---

[13]We use the notation χ^- of [37], where the minus does not denote a shift in the spectral parameter but is just a label.

and the same for the other block denoted with tilde, where

$$\chi_{\text{extra}} = -\frac{1}{2}\Big(\chi^{\text{HL}} - \chi^-\Big), \qquad \tilde{\chi}_{\text{extra}} = -\frac{1}{2}\Big(\chi^{\text{HL}} + \chi^-\Big), \tag{B.10}$$

and χ^{HL} χ^- are given by an explicit integral representation.

Equations (3.8) and (3.9) of [37] are consistent with

$$\chi^{\text{HL}} + (\chi^{\text{HL}})^{\gamma^{-1}} = \frac{i}{2}\log \ell^{\text{HL}}, \qquad \ell^{\text{HL}}(x,y) = \frac{x-y}{1-\frac{1}{xy}}, \tag{B.11}$$

while equations (3.14) and (3.15) in the same paper are consistent with

$$\chi^- - (\chi^-)^{\gamma^{-1}} = \frac{i}{2}\log \ell^-, \qquad \ell^-(x,y) = (x-y)(1-\frac{1}{xy}). \tag{B.12}$$

Therefore, we can assemble

$$(\sigma^{1,\text{extra}})^{\gamma^{-1}}\, \tilde{\sigma}^{1,\text{extra}} = \exp\frac{1}{4}\log\frac{\ell^{\text{HL}}(x,x_2^+)\ell^-(x,x_2^+)}{\ell^{\text{HL}}(x,x_2^-)\ell^-(x,x_2^-)} = \Big(\frac{x_2^-}{x_2^+}\Big)^{\frac{1}{4}}\sqrt{\frac{(x-x_2^+)}{(x-x_2^-)}}. \tag{B.13}$$

Recalling the definition of the function $R_{(\pm)}$, we can reproduce the first equation in (B.5). Likewise, we can compute

$$(\tilde{\sigma}^{1,\text{extra}})^{\gamma^{-1}}\, \sigma^{1,\text{extra}} = \exp\frac{1}{4}\log\frac{\ell^{\text{HL}}(x,x_2^+)\ell^-(x,x_2^-)}{\ell^{\text{HL}}(x,x_2^-)\ell^-(x,x_2^+)} = \Big(\frac{x_2^+}{x_2^-}\Big)^{\frac{1}{4}}\sqrt{\frac{\frac{1}{x}-x_2^-}{\frac{1}{x}-x_2^+}}, \tag{B.14}$$

which reproduces the second equation in (B.5) if we recall the definition of the function $B_{(\pm)}$. The other relations in (B.26) also follow: since the cut is of logarithmic type, we get the reciprocal results on the r.h.s. if we cross it in the other direction.

B.2 Functional argument

Here we establish the same relations starting from the crossing equation, and assuming certain minimality requirements on its solution. The crossing equation can be decomposed into the crossing satisfied by the BES part,

$$\sigma_{\text{BES}}(u_1^{\gamma_{\text{cross}}}, u_2)\sigma_{\text{BES}}(u_1, u_2) = \frac{x_2^-}{x_2^+}\frac{x_1^- - x_2^+}{x_1^- - x_2^-}\frac{1-\frac{1}{x_1^+ x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}, \tag{B.15}$$

and the crossing relations for the extra pieces:

$$\sigma_{\text{extra}}(u_1^{\gamma_{\text{cross}}}, u_2)^2\, \tilde{\sigma}_{\text{extra}}(u_1, u_2)^2 = \frac{(x_1^+ - x_2^+)(x_1^- - x_2^-)}{(x_1^- - x_2^+)(x_1^+ - x_2^-)}, \tag{B.16}$$

$$\sigma_{\text{extra}}(u_1, u_2)^2\, \tilde{\sigma}_{\text{extra}}(u_1^{\gamma_{cross}}, u_2)^2 = \frac{\Big(1-\frac{1}{x_1^+ x_2^-}\Big)\Big(1-\frac{1}{x_1^- x_2^+}\Big)}{\Big(1-\frac{1}{x_1^+ x_2^+}\Big)\Big(1-\frac{1}{x_1^- x_2^-}\Big)}. \tag{B.17}$$

The path γ_{cross} is depicted in figure 5, and it can be decomposed as the concatenation of the path γ_+, entering the lower cut, followed by γ_- which enters the upper cut. Notice
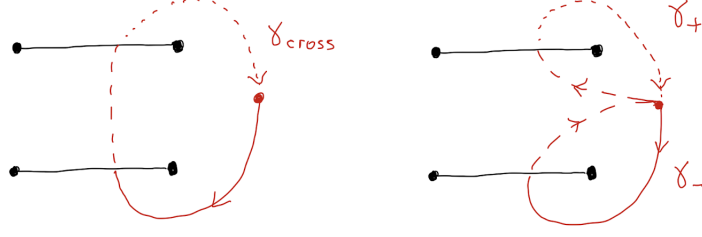
Figure 5. The analytic continuation path used in the crossing relation. It crosses the cuts of the dressing factors at $(-2h, 2h) \pm \frac{i}{2}$, and can be decomposed into γ_+ and γ_-, which cross only one cut each.

that in our notations, different from some of the literature, both $\gamma^{\pm}$ cross one cut from below.

We now follow a similar route to the one described in [47], and disentangle the path γ_{cross} to derive a simpler equation for a natural building block of the solution to the crossing constraints. We will assume that, for the minimal solution, the crossing path is equivalent to the one obtained by concatenating γ_- and γ_+ in opposite order, $\gamma_{\text{cross}} \simeq \gamma_+ \cdot \gamma_- \simeq \gamma_- \cdot \gamma_+$. Under this assumption, analytically continuing along the inverse path γ_+^{-1} the crossing relations (B.16),(B.17), we get:

$$s(u_1^{\gamma_-}, u_2)\hat{s}(u_1^{\gamma_+^{-1}}, u_2) = \frac{R_{(-)}^+ B_{(+)}^-}{R_{(+)}^+ B_{(-)}^-} \equiv A, \tag{B.18}$$

$$\hat{s}(u_1^{\gamma_-}, u_2)s(u_1^{\gamma_+^{-1}}, u_2) = \frac{R_{(-)}^- B_{(+)}^+}{R_{(+)}^- B_{(-)}^+} \equiv C, \tag{B.19}$$

while continuing the same variable along γ_-^{-1}, we get:

$$s(u_1^{\gamma_+}, u_2)\hat{s}(u_1^{\gamma_-^{-1}}, u_2) = \frac{B_{(-)}^+ R_{(+)}^-}{B_{(+)}^+ R_{(-)}^-} \equiv B, \tag{B.20}$$

$$\hat{s}(u_1^{\gamma_+}, u_2)s(u_1^{\gamma_-^{-1}}, u_2) = \frac{B_{(-)}^- R_{(+)}^+}{B_{(+)}^- R_{(-)}^+} \equiv D, \tag{B.21}$$

where for simplicity of the next expressions, we denoted $s(u_1, u_2) \equiv \sigma_{\text{extra}}(u_1, u_2)^2$, $\hat{s}(u_1, u_2) \equiv \tilde{\sigma}_{\text{extra}}(u_1, u_2)^2$.

From now on, we omit the second variable, since it is simply a spectator in all these functional relations, and use the notation $g^{[n]}$, described in the main text, to shift the first variable of various functions. We proceed by making the ansatz

$$s \equiv \frac{t^+}{t^-}, \quad \tilde{s} \equiv \frac{\tilde{t}^+}{\tilde{t}^-}, \tag{B.22}$$

where $t, \tilde{t}$ are assumed to be functions with a single cut $(-2h, 2h)$. The relations between these blocks and the ones introduced above is simply $t \propto (\sigma^{1,\text{extra}})^2$, $\tilde{t} \propto (\tilde{\sigma}^{1,\text{extra}})^2$.

Now we notice that,

$$\left(\frac{t^+}{t^-}\right)^{\gamma_-} = \frac{(t^{\gamma^{-1}})^+}{t^-}, \quad \left(\frac{t^+}{t^-}\right)^{\gamma_+} = \frac{t^+}{(t^{\gamma^{-1}})^-}, \tag{B.23}$$

and there are similar relations if we do analytic continuations along the inverse paths $\gamma_\pm^{-1}$, which are simply obtained by replacing $\gamma^{-1} \to \gamma$ on the r.h.s. Taking the product of (B.18),(B.20), we arrive at

$$\left((t)^{\gamma^{-1}} \tilde{t} \, t \, (\tilde{t})^{\gamma}\right)^{\hat{D}-\hat{D}^{-1}} = AB = \left(\frac{R_{(-)}B_{(-)}}{R_{(+)}B_{(+)}}\right)^{\hat{D}-\hat{D}^{-1}}, \tag{B.24}$$

where $\hat{D} \equiv \frac{i}{2}\partial_u$, so that in this notation $g^{n\hat{D}} \equiv g^{[n]}$. Since we look for the minimal solution to crossing, we take the simplest solution to the previous functional relation:

$$(t)^{\gamma^{-1}} \tilde{t} \, t \, (\tilde{t})^{\gamma} = \frac{R_{(-)}B_{(-)}}{R_{(+)}B_{(+)}}. \tag{B.25}$$

Similarly, considering the ratio of the same two equations, and assuming the minimal solution, we obtain

$$\left((t)^{\gamma^{-1}} \tilde{t}\right) \Big/ \left(t \, (\tilde{t})^{\gamma}\right) = \frac{R_{(-)}B_{(+)}}{R_{(+)}B_{(-)}}, \tag{B.26}$$

and finally from (B.25),(B.26) we read:

$$(t)^{\gamma^{-1}} \tilde{t} = \frac{R_{(-)}}{R_{(+)}}, \quad t \, (\tilde{t})^{\gamma} = \frac{B_{(-)}}{B_{(+)}}. \tag{B.27}$$

By the same arguments from the remaining two equations we extract:

$$(t)^{\gamma} \tilde{t} = \frac{R_{(+)}}{R_{(-)}}, \quad (t) \, (\tilde{t})^{\gamma^{-1}} = \frac{B_{(+)}}{B_{(-)}}. \tag{B.28}$$

Taking into account that, in the notations of the main text, $t \equiv (\sigma^{1,\text{extra}})^2$, $\tilde{t} \equiv (\tilde{\sigma}^{1,\text{extra}})^2$, we have therefore deduced the relations (B.3), (B.5).

C Baxter equations

Baxter equations for Q and P functions. The obvious identities

$$\mathbf{Q}_k^{++}\epsilon^{ij}\mathbf{Q}_i^{--}\mathbf{Q}_j - \mathbf{Q}_k\epsilon^{ij}\mathbf{Q}_i^{--}\mathbf{Q}_j^{++} + \mathbf{Q}_k^{--}\epsilon^{ij}\mathbf{Q}_i\mathbf{Q}_j^{++} = 0 \;, \quad k = 1, 2, \tag{C.1}$$

can be recast as the Baxter equations

$$\mathbf{Q}_k^{++}D_1^- - \mathbf{Q}_k D_2 + \mathbf{Q}_k^{--}D_1^+ = 0 \;, \quad k = 1, 2, \tag{C.2}$$

where the coefficients can also be rewritten in terms of $\mathbf{P}$ functions using the QQ relations:

$$D_1 \equiv \epsilon^{ij}\mathbf{Q}_i^-\mathbf{Q}_j^+ = \epsilon_{ab}\mathbf{P}^{a-}\mathbf{P}^{b+} \tag{C.3}$$

$$D_2 \equiv \epsilon^{ij}\mathbf{Q}_i^{--}\mathbf{Q}_j^{++} = \epsilon_{ab}\mathbf{P}^{a--}\mathbf{P}^{b++} - \mathbf{P}_c\mathbf{P}^{c--}\epsilon_{ab}\mathbf{P}^a\mathbf{P}^{b++} \tag{C.4}$$

$$= \epsilon_{ab}\mathbf{P}^{a--}\mathbf{P}^{b++} - \mathbf{P}_c\mathbf{P}^{c++}\epsilon_{ab}\mathbf{P}^a\mathbf{P}^{b--}$$

(the last equality follows from $\mathbf{P}_a\mathbf{P}^a = 0$). These equations, supplemented by the large-u asymptotics, give a way to compute the $\mathbf{Q}$ functions starting from the knowledge of the $\mathbf{P}$ functions.

There are also equations of the same form, obtained by replacing $\mathbf{P} \leftrightarrow \mathbf{Q}$, which may be used to compute the $\mathbf{P}$ functions starting from the $\mathbf{Q}$'s.

Finite difference relations for $Q_{a|i}$. We close this appendix by noticing that also the middle node Q-functions can be defined as the solutions of a system of finite-difference equations, which are simply obtained from the Q-system.

One such system of relations is

$$Q_{a|i}^+ - Q_{a|i}^- = \mathbf{P}_a\mathbf{P}^b Q_{b|i}^+. \tag{C.5}$$

These relations can be used to construct $Q_{a|i}$ from the knowledge of the $\mathbf{P}$ functions. The solution is specified by requiring the appropriate asymptotic behaviour, and the region of analyticity. Solutions analytic in the upper half plane are denoted as $Q_{a|i}^{\downarrow}$. The solutions analytic in the lower half plane form an alternative basis of solutions, denoted by $Q_{a|i}^{\uparrow}$. The numerical method to compute $Q_{a|i}^{\downarrow}$ and $Q_{a|i}^{\uparrow}$ in terms of the $\mathbf{P}$ functions is described in [4].

The two bases of solutions of the same finite-difference equations are related by an i-periodic matrix

$$Q_{a|i}^{\uparrow+} = \Omega_i{}^j Q_{a|j}^{\downarrow+}, \quad Q^{a|i\uparrow+} = \Omega^i{}_j Q^{a|j\downarrow+}, \tag{C.6}$$

which imply

$$\Omega_k{}^l = Q_{a|k}^{\uparrow+} Q^{a|l\downarrow+}, \quad \Omega^k{}_l = Q^{a|k\uparrow+} Q_{a|l}^{\downarrow+}. \tag{C.7}$$

Multiplying the first equation in (C.6) by $\mathbf{P}^a$ on the left, we see immediately that Ω is the same matrix relating $\mathbf{Q}^{\uparrow}$ and $\mathbf{Q}^{\downarrow}$ in (3.22). Similarly, the second equation show that

$$\mathbf{Q}^{i\uparrow} = \Omega^i{}_j \mathbf{Q}^{j\downarrow}. \tag{C.8}$$

Since $Q_{a|i}$ has unit determinant and $Q_{a|i} Q^{a|j} = \delta_i^j$, from (C.7) we see immediately that Ω has unit determinant as well, and

$$\Omega^i{}_j \Omega_k{}^j = \delta_k^i. \tag{C.9}$$

Finally, another useful form of (C.5) is

$$Q_{a|i}^+ - Q_{a|i}^- = \mathbf{Q}_i \mathbf{Q}^j Q_{a|j}^+, \tag{C.10}$$

which can be used to determine $Q_{a|i}$ from the knowledge of the $\mathbf{Q}$ functions.

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