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Enhancement of δ-SPH for ocean engineering applications through incorporation of a background mesh scheme

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ABSTRACT

The improved Smoothed Particle Hydrodynamics (SPH) method with numerical diffusive terms, which is referred to as δ-SPH, has been widely applied into the studies of engineering problems, especially in the fields of ocean and coastal engineering. Nonetheless, there are still some drawbacks associated with δ-SPH model as in case of other particle methods. These drawbacks correspond to presence of unphysical pressure noise as well as imprecise satisfaction of dynamic free-surface boundary condition in numerical simulations. Through derivation and implementation of a so-called Background Mesh (BM) scheme and enforcing the spatial continuity of calculated source term of pressure equation (Equation of State; EOS), we present enhancements for δ-SPH in providing a more accurate pressure field as well as a more precise reproduction of free-surface. The enhancing effects of the newly proposed δ-SPH+BM are portrayed through several ocean engineering-related test cases including dam break and violent sloshing flows in both two and three dimensions. The incorporation of BM scheme is shown to result in enhanced calculation of pressure field as well as improved satisfaction of dynamic free-surface boundary condition as a consequence of enhanced density and pressure fields.

Keywords: Smoothed Particle Hydrodynamics; δ-SPH method; BM scheme; Pressure calculation; Free-surface flows

1. Introduction

The phenomenon of violent free-surface flows can be observed frequently in the field of ocean engineering, such as liquid sloshing, water on deck flows, and wave impact on coastal/offshore structures. During these processes, there are global and local impact loads which are caused by the interactions between violent flows and structure, and evaluation of such impact loads would be of prime importance for design of ocean and coastal structures to ensure the security in operation.

To date, lots of numerical studies on the violent free-surface flows have been carried out based on Eulerian grids, such as FEM (Finite Element Method) [1] and FVM (Finite Difference Method) [2]. However, there exist significant challenges related to the grid-based methods due to the extremely non-linear and complex characteristics of violent flows. Such challenges include complexities for precise and efficient treatment of moving interfaces including free-surfaces by both Eulerian and Lagrangian grid-based methods [3]. As new generation computational methods, the Lagrangian meshfree or particle methods provide robust and versatile computational framework to precisely and efficiently simulate violent free-surface flows as well as their interactions with surrounding structures. As a well-known particle method, the Smoothed Particle Hydrodynamics (SPH) method [4] has been applied into a wide range of simulations including violent free-surface flows, such as multiphase flow [5-7], sloshing flow [8,9], wave impact [10-14], and fluid-structure interactions [15,16].

Despite its wide range of applicability and potential robustness, the conventional SPH method may bring about substantial unphysical pressure oscillations [3, 17]. The issue related to unphysical pressure noise exists in both Weakly Compressible SPH (WCSPH) [74,18,19] as well as projection SPH or Incompressible SPH (ISPH) [8,20]. In case of WCSPH, the solution process is fully explicit as the pressure is calculated explicitly from an Equation of State (EOS). Hence, the challenge of presence of unphysical pressure fluctuations may even become more difficult to be precisely and consistently treated without deterioration of, e.g. conservation properties of the method. There have been great numbers of studies for enhancement of pressure field in both ISPH (e.g. [21-24]) as well as WCSPH (e.g. [25,26]) contexts.

In the WCSPH context, the so-called Riemann SPH [27,28] and δ-SPH [29-31] variants have been presented to enhance the accuracy and stability of SPH and in particular, to tackle the challenge associated with unphysical pressure noise. The δ-SPH method, which is a popular and well-studied variant of SPH, is developed by adding a proper numerical diffusion term of density δ into the continuity equation [29,31] to minimize the spurious high-frequency numerical oscillations in the pressure field. Many studies in recent years have highlighted the reliability of δ-SPH scheme for the simulation of violent flows [31-33]. As a result, δ-SPH method has been widely applied in different engineering fields [34-36]. There have been some improvements for δ-SPH method in terms of stability in reproduction of flow fields with negative pressure and tensile stress [37-39], and extensions for dynamic adjustment of the parameter of the diffusion term [40].

Despite significant advances in SPH as well as its well-known variant δ-SPH, there are still some remaining challenges associated with particle methods [3,41]. These challenges do not only correspond to the presence of unphysical pressure noise, but also imprecise satisfaction of dynamic boundary condition at free-surfaces in numerical implementations. Considering the fact that one of the main sources of numerical noise in particle methods can be attributed to numerical inaccuracies in estimations of density as in case of WCSPH [75] or time variation of density as in case of ISPH [3,20], even small-scale perturbations in particle motions would lead to unphysical pressure noise. Accordingly, some researchers proposed to use the characteristics of mesh-based methods (fixed and regularly distributed calculation points) to improve the calculated pressure field and minimize the pressure oscillations [45-48]. However, in most cases, the pressure equation and thus pressure field were calculated at the fixed grid points, therefore, the Lagrangian feature of particle methods could not be well preserved. Wang et al. [17] proposed a Background Mesh (BM) scheme to enhance the pressure calculation in a projection particle method, namely, MPS (Moving Particle Semi-implicit) method. In this scheme, the background mesh is only applied for the calculation of source term of pressure equation; thus, the pressure equation is still calculated at the moving calculation points under the Lagrangian framework without any Eulerian treatments. Therefore, the advantageous features of Lagrangian particle methods could be preserved.

In this paper, a BM scheme, comparable to that by Wang et al. [17], is proposed and applied for the δ-SPH method to enhance the calculation of pressure field and, as a consequence of enhanced density and pressure fields, to improve the imposition of dynamic free-surface boundary condition in numerical simulations. In the paper of Wang et al. [17], the BM scheme was applied for projection MPS method, in which the pressure field is obtained by solving a Poisson Pressure Equation (PPE). Comparing MPS with δ-SPH, the difference is that the pressure field in δ-SPH method is calculated explicitly by utilizing an EOS and the source term of pressure equation (EOS) is a function of density field. Therefore, a BM scheme is implemented to provide a more accurate and spatially continuous time variation of density for calculation of density field at the moving calculation points (particles). Through achieving a more accurate and spatially continuous density field for the source term of pressure equation, a more accurate pressure field would be obtained. The BM scheme would also lead to a better resolution of the material interfaces (free-surface) as more calculation points will be set at and in the vicinity of interfaces (e.g.

free-surface) to provide time variation of density and then density field, leading to an enhanced pressure field and improved satisfaction of dynamic free-surface boundary condition.

The paper is organized as follows. In section 2 the considered δ-SPH method is concisely described. The δ-SPH method considered in this study corresponds to that presented by Antuono et al. [49], and to date, this version of δ-SPH is the most well-known, well-studied and commonly applied one [29,31]. The proposed Background Mesh (BM) scheme for δ-SPH (which is also applicable to all WCSPH methods) is presented in section 3. Section 4 is dedicated to presentation of validations and numerical investigations. Validations are first carried out in two-dimensions through consideration of i) a simple uniform flow with exact analytical solution, ii) dam break flow and iii) violent sloshing flow. Although Wang et al. [17] portrayed the effectiveness of BM scheme only in two-dimensions, our study considers three-dimensional validations as well. Three-dimensional validations consist of i) a hydrostatic test with exact analytical solution, ii) dam break flow and iii) sloshing flows. Through the presented validations, the enhancement of δ-SPH in providing more accurate reproductions of violent free surface flows for ocean engineering applications will be demonstrated.

2. δ-SPH method

2.1 Governing equations

In the δ-SPH method the fluid is assumed to be weakly compressible and the continuity and momentum equations are considered as follows in the continuous form:

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{u}=0 \tag{1}$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho}\nabla P+\boldsymbol{f} \tag{2}$$

where ρ is the fluid density; t is the time; $\boldsymbol{u}$ is the velocity vector; P is pressure; $\boldsymbol{g}$ is the gravitational acceleration; $\boldsymbol{f}$ corresponds to the body force.

The basic principle of SPH method corresponds to an integral representation of a typical function $f(x)$ over a domain Ω:

$$\langle f(x)\rangle=\int_{\Omega} f(x')w(x-x',h)dx' \tag{3}$$

with h being the smoothing length and $w(x-x',h)$ a kernel function. Accordingly, the discretized particle-based approximation of a field function $f(x)$ at a typical target particle i may be written as:

$$\langle f(x_i)\rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) w(x_i - x_j, h) \tag{4}$$

where particle j is a neighboring particle within the support domain of the target particle i; N is the total number of neighboring particles j for target particle i; m_j is the mass of particle j; ρ_j is the density of particle j. In the present study, a (fifth-order) Wendland C2 kernel function [50] is applied:

$$w(r) = \begin{cases} \left(1 - \frac{r}{r_e}\right)^4 (1 + 4\frac{r}{r_e}), & r < r_e \\ 0, & r \geq r_e \end{cases} \tag{5}$$

where r_e is the influence radius of target particle, and it is set as 2.4dx (i.e. r_e = 2h = 2.4dx), r denotes the distance between the target particle and neighboring particle. For the simulations conducted in this study we consider h/dx, i.e. ratio of smoothing length to particle spacing, to be 1.2. In Appendix B, we will investigate the effect of h/dx on simulation results. It should be noted here that in some of the important articles on δ-SPH, h/dx has been considered as 2.0 [38,76].

One of the issues related to particle methods including SPH corresponds to presence of non-physical numerical noise in pressure field. In the context of Weakly Compressible SPH (WCSPH), the pressure noise is due to weak compressibility assumption (acoustic noise) as well as numerical inaccuracies [74]. In order to minimize the numerical noise in density and thus pressure field, an effective approach corresponds to careful introduction of conservative diffusive terms in the continuity equation which leads to a robust variant of SPH, referred to as δ-SPH [49,29]. The conservation of mass and momentum equations for particle i in the δ-SPH framework are expressed as follows [29,31]:

$$\frac{D\rho_i}{Dt} = \rho_i \sum_{j=1}^{N} \frac{m_j}{\rho_j} \boldsymbol{u}_{ij} \cdot \nabla_i W_{ij} + \delta h C_0 \sum_{j=1}^{N} \frac{m_j}{\rho_j} \boldsymbol{\Psi}_{ij} \cdot \nabla_i W_{ij} \tag{6}$$

$$\frac{D\boldsymbol{u}_i}{Dt} = \boldsymbol{g} - \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij}\right) \nabla_i W_{ij} \tag{7}$$

$$\boldsymbol{\Psi}_{ij} = 2(\rho_j - \rho_i) \frac{\boldsymbol{r}_{ji}}{|\boldsymbol{r}_{ij}|^2} - (\langle\nabla\rho\rangle_i^L + \langle\nabla\rho\rangle_j^L) \tag{8}$$

$$\langle\nabla\rho\rangle_i^L = \sum_{j=1}^{N} \frac{m_j}{\rho_j} (\rho_j - \rho_i) \boldsymbol{L}_i \cdot \nabla_i W_{ij} \tag{9}$$

$$\boldsymbol{L}_i = \left[\sum_{j=1}^{N} \frac{m_j}{\rho_j} \boldsymbol{r}_{ji} \otimes \nabla_i W_{ij}\right]^{-1} \tag{10}$$

$$\Pi_{ij} = \begin{cases} \dfrac{-\alpha_\Pi c_{ij} \phi_{ij} + \beta_\Pi \phi_{ij}^2}{\rho_{ij}}, & \boldsymbol{u}_{ij} \cdot \boldsymbol{r}_{ij} < 0 \\ 0, & \boldsymbol{u}_{ij} \cdot \boldsymbol{r}_{ij} \geq 0 \end{cases} \tag{11}$$

$$\phi_{ij} = \frac{h_{ij} \boldsymbol{u}_{ij} \cdot \boldsymbol{r}_{ij}}{|\boldsymbol{r}_{ij}|^2 + 0.01 h_{ij}^2} \tag{12}$$

where the coefficient δ controls the intensity of the diffusion for density and in all simulations of this study δ is set as 0.1 [72,39]. As shown in Antuono et al. [29] and as stated in several papers (e.g. [39]), the range of variability of δ is quite narrow and it cannot be considered as a tuning variable. In Eq. (6), C_0 represents the artificial sound speed in water and the value of C_0 is set equal to 10 times of the maximum fluid velocity [29]. In the above equations, $\boldsymbol{r}$ is the position vector, Π_{ij} is the artificial viscosity term, α_Π and β_Π are artificial viscosity coefficients that can be modified according to the targeted problem. In this study we deal with Newtonian fluid flows and thus β_Π would be considered as zero [4]. With regard to the inviscid fluid flow simulations, the artificial viscosity term α_Π would be set as zero and in the viscous fluid flow simulations of this study α_Π is considered as 0.1. In Eq. (10), $\langle\nabla\rho\rangle^L$ is the renormalized density gradient with consideration of corrective matrix $\boldsymbol{L}$ [31].

The pressure corresponding to each target particle i can be calculated by the equation of state (EOS) described as follows [31]:

$$P_i = {C_0}^2(\rho_i - \rho_0) \tag{13}$$

where ρ_0 is the reference density of fluid particles. Through consideration of Eq. (13), the pressure at target particle i at time step n, $P_i^{\,n}$, would be obtained from the calculated particle density at the same time step, $\rho_i^{\,n}$. In other words, the source term of the EOS would be a function of density field. The value of $\rho_i^{\,n+1}$ is calculated from the time variation of density $D\rho_i/Dt$ at time step n and n+1/2 on the basis of the time integration scheme presented in the next section (section 2.2), where $D\rho_i/Dt$ at every time step is obtained by using Eq. (6). Hence, $D\rho/Dt$ is clearly linked to the source term of the EOS, and obtaining a more accurate time variation of density and thus a more precise satisfaction of continuity equation would be crucial for a precise calculation of pressure field.

2.2 Time integration scheme

In the present work, a predictor-corrector scheme [44, 67-70] is used for the time integration with *CFL* number of 0.3, as previously stated. It should be noted that generally in δ-SPH a fourth-order Runge-Kutta time integration scheme is implemented [29,35,49,72] which would allow a relatively larger *CFL* number (e.g. *CFL* = 2.0 [72]) and thus, faster calculations. In the predictor-corrector scheme which is applied in the present work, the values of density, velocity and position at time step n+1/2 are firstly predicted as follows:

$$\rho^{n+\frac{1}{2}} = \rho^n + \frac{\Delta t}{2}\left(\frac{d\rho}{dt}\right)^n \tag{14}$$

$$\boldsymbol{u}^{n+\frac{1}{2}} = \boldsymbol{u}^n + \frac{\Delta t}{2}\left(\frac{d\boldsymbol{u}}{dt}\right)^n \tag{15}$$

$$\boldsymbol{x}^{n+\frac{1}{2}} = \boldsymbol{x}^n + \frac{\Delta t}{2}\,\boldsymbol{u}^{n+\frac{1}{2}} \tag{16}$$

Then, the predicted values $\boldsymbol{u}^{n+\frac{1}{2}}$, $\rho^{n+\frac{1}{2}}$ and $\boldsymbol{x}^{n+\frac{1}{2}}$ are used to calculate the $\left(\frac{d\boldsymbol{u}}{dt}\right)^{n+\frac{1}{2}}$ and $\left(\frac{d\rho}{dt}\right)^{n+\frac{1}{2}}$. Next, the predicted values are corrected as follows:

$$\rho^{n+\frac{1}{2}*} = \rho^n + \frac{\Delta t}{2}\left(\frac{d\rho}{dt}\right)^{n+\frac{1}{2}} \tag{17}$$

$$\boldsymbol{u}^{n+\frac{1}{2}*} = \boldsymbol{u}^n + \frac{\Delta t}{2}\left(\frac{d\boldsymbol{u}}{dt}\right)^{n+\frac{1}{2}} \tag{18}$$

$$\boldsymbol{x}^{n+\frac{1}{2}*} = \boldsymbol{x}^n + \frac{\Delta t}{2}\,\boldsymbol{u}^{n+\frac{1}{2}*} \tag{19}$$

Finally, the values at time step n+1 are obtained:

$$\rho^{n+1} = 2\rho^{n+\frac{1}{2}*} - \rho^n \tag{20}$$

$$\boldsymbol{u}^{n+1} = 2\boldsymbol{u}^{n+\frac{1}{2}*} - \boldsymbol{u}^n \tag{21}$$

$$\boldsymbol{x}^{n+1} = 2\boldsymbol{x}^{n+\frac{1}{2}*} - \boldsymbol{x}^n \tag{22}$$

The time step Δt is determined by the *CFL* (Courant–Friedrichs–Levy) condition:

$$\Delta t \le CFL\; h/C_0 \tag{23}$$

where *CFL* is a coefficient and in the present work $CFL = 0.3$. In Eq. (23), h corresponds to the smoothing length and C_0 symbolizes the artificial sound speed.

2.3 Solid boundary

In this work, the treatment of solid boundaries is carried out through applying the “fixed ghost particles” [31,71] scheme. In this scheme the distribution of ghost particles always remains uniform and does not depend on the fluid particle positions. A brief description of this scheme is presented as follows.

Firstly, the fixed solid boundary particles are arranged at the beginning of the simulation. Then, as shown in the **Fig. 1**, the interpolation points are generated at the fluid domain by mirroring the positions of the fixed solid particles. The physical quantities at the interpolation points (pressure, velocity) are obtained through the interpolation of the information at corresponding neighboring fluid particles. Finally, the quantities at the fixed solid particles are calculated by considering the corresponding values of physical quantities at interpolation points to satisfy the prescribed wall boundary conditions, i.e.

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{24}$$

$$\frac{\partial P}{\partial \boldsymbol{n}} = \rho \boldsymbol{f} \cdot \boldsymbol{n} \tag{25}$$

where $\boldsymbol{u}$ is the velocity, $\boldsymbol{n}$ is the unit normal vector to the solid boundary, P is the pressure, ρ is the density and $\boldsymbol{f}$ is the body force. For example, the pressure of a fixed solid particle P_w can be obtained as follows:

$$P_w = \frac{\sum_j P_j W_{ij} + \boldsymbol{f} \cdot \boldsymbol{n} d \sum_j \rho_j W_{ij}}{\sum_j W_{ij}} \tag{26}$$

where particle j belongs to the fluid particles, d represents the normal distance between the interpolation point and the fixed solid particle.

3. Background mesh scheme

3.1. Challenges of perturbations in particles' motions and material discontinuity

In the δ-SPH method, the pressure at a particle, P, is obtained from the corresponding particle density ρ by utilizing the EOS, and the density ρ is obtained through consideration of the time variation of density $D\rho/Dt$, i.e. the source term of EOS would be a function of density ρ (Eq. 13) and density is calculated with consideration of $D\rho/Dt$ (Eqs. 14, 17 and 20). In the δ-SPH method, similar to other particle methods, irregularities in instantaneous particle positions would lead to errors in calculation of velocity divergence and thus time variation of density, density and pressure fields. In addition, in δ-SPH, similar to other well-known SPH variants, particle motions are described in a Lagrangian manner with the utilization of Eulerian kernels (kernels that are functions of spatial coordinates rather than material coordinates [73]). Thus, perturbations in motion of particles close to kernel domain boundaries would add to spatial discontinuities in calculation of time variation of density $D\rho/Dt$, which would again affect the accuracy of the calculated density and thus source term of EOS, adding to numerical noise in the calculated pressure field.

To present the situation regarding noise associated with particle perturbations two simple scenarios are considered here. As shown in **Fig. 2(a)**, r_e represents the influence radius of a target particle i. When calculation for target particle i is at step n, a typical particle k is involved in particle-based approximations for particle i as a neighboring particle, and the corresponding $D\rho/Dt$ for particle i at time step n can be expressed by Eq. (27). However, as shown in **Fig. 2(b)**, at step n+1, because of a small-scale perturbation, particle k would no longer be at the neighboring region or influence domain of target particle i. In Fig. 2(b), the position of virtual particle k' represents the original position of particle k at step n. Then the time variation of density, $D\rho/Dt$, for particle i at time step n+1 would be written by Eq. (28). The second term on the right-hand side of Eq. (28) corresponds to the perturbation noise in calculated time variation of density field for target particle i at time step n+1.

$$\frac{D\rho_i}{Dt}^n = \left(\rho_i \sum_j \frac{m_j}{\rho_j} \boldsymbol{u}_{ij} \cdot \nabla_i W_{ij} + \delta h C_0 \sum_j \frac{m_j}{\rho_j} \boldsymbol{\Psi}_{ij} \cdot \nabla_i W_{ij}\right)^n \tag{27}$$

$$\frac{D\rho_i}{Dt}^{n+1} = \left(\rho_i \sum_j \frac{m_j}{\rho_j} \boldsymbol{u}_{ij} \cdot \nabla_i W_{ij} + \delta h C_0 \sum_j \frac{m_j}{\rho_j} \boldsymbol{\Psi}_{ij} \cdot \nabla_i W_{ij} + \rho_i \frac{m_{k'}}{\rho_{k'}} \boldsymbol{u}_{ik'} \cdot \nabla_i W_{ik'} + \delta h C_0 \frac{m_{k'}}{\rho_{k'}} \boldsymbol{\Psi}_{ik'} \cdot \nabla_i W_{ik'}\right)^{n+1}$$
$$- \left(\rho_i \frac{m_{k'}}{\rho_{k'}} \boldsymbol{u}_{ik'} \cdot \nabla_i W_{ik'} + \delta h C_0 \frac{m_{k'}}{\rho_{k'}} \boldsymbol{\Psi}_{ik'} \cdot \nabla_i W_{ik'}\right)^{n+1} \tag{28}$$

Fig. 3 illustrates another scenario, where a neighboring particle q at time step n (Fig. 3a) is perturbed towards target particle i at time step n+1 (Fig. 3b). The position of virtual particle q' in Fig. 3(b) represents the original position of particle q at step n. The $D\rho/Dt$ for target particle i at time step n+1 would be expressed by Eq. (29) and the second term on the right-hand side of this equation represents the noise in calculated time variation of density for target particle i at time step n+1.

$$\frac{D\rho_i}{Dt}^{n+1} = \left(\rho_i \sum_j^{j\neq q} \frac{m_j}{\rho_j} \boldsymbol{u}_{ij} \cdot \nabla_i W_{ij} + \delta h C_0 \sum_j^{j\neq q} \frac{m_j}{\rho_j} \boldsymbol{\Psi}_{ij} \cdot \nabla_i W_{ij} + \rho_i \frac{m_{q'}}{\rho_{q'}} \boldsymbol{u}_{iq'} \cdot \nabla_i W_{iq'} + \delta h C_0 \frac{m_{q'}}{\rho_{q'}} \boldsymbol{\Psi}_{iq'} \cdot \nabla_i W_{iq'}\right)^{n+1}$$
$$+ \left[\rho_i \left(\frac{m_q}{\rho_q} \boldsymbol{u}_{iq} \cdot \nabla_i W_{iq} - \frac{m_{q'}}{\rho_{q'}} \boldsymbol{u}_{iq'} \cdot \nabla_i W_{iq'}\right) + \delta h C_0 \left(\frac{m_q}{\rho_q} \boldsymbol{\Psi}_{iq} \cdot \nabla_i W_{iq} - \frac{m_{q'}}{\rho_{q'}} \boldsymbol{\Psi}_{iq'} \cdot \nabla_i W_{iq'}\right)\right]^{n+1} \tag{29}$$

Afterwards, the noises resulted from perturbations in particle motion may lead to growth of errors in calculated pressure field affecting the overall accuracy and stability of the numerical method.

Another issue corresponding to SPH-based methods, especially those formulated in WCSPH context, corresponds to the challenge of precise imposition of dynamic free-surface boundary condition in numerical simulations. This challenge is mainly due to presence of a material discontinuity and corresponding truncated kernel support domains. In the context of projection particle methods, some efforts have been made to enhance volume conservation at free-surfaces and improve imposition of dynamic free-surface boundary condition (constant or zero gauge pressure at free-surface particles).

Among these efforts we can name the SPP (Space Potential Particle) concept by Tsuruta et al. [51] and incorporation of so-called conceptual particles by Chen et al. [52]. In the WCSPH context, Colagrossi et al. [53], clearly proved that in the SPH context free-surface boundary conditions are satisfied in integral form. However, approximations due to particle-based discretizations would not necessarily guarantee this important aspect.

In the SPH context, in calculations corresponding to particles located at and in the vicinity of free-surfaces, there are no particles in the vacuum region above the free-surface and thus, there would be no related information. In other words, we need to deal with an abrupt material discontinuity. In addition, the surface integrals for calculation of pressure gradients [15] are ignored through assumption of zero gauge pressure at free-surfaces. Therefore, the introduced material discontinuity, i.e. the free-surface, would bring about numerical errors as well as imprecise satisfaction of dynamic free-surface boundary condition.

3.2 Concept and implementation of BM scheme in δ-SPH

The key concept of BM scheme corresponds to enhancement of pressure field through calculation of a more accurate and spatially continuous density field. The key point here is that through application of a set of fixed and spatially continuous background nodes a more accurate estimation of velocity divergence can be obtained leading to a more accurate time variation of density at the background nodes. Then through extrapolation of calculated $D\rho/Dt$ from fixed background nodes to moving calculation points (particles), spatially smoother and more accurate time variation of density can be achieved at particles, leading to more accurate density and thus pressure and velocity fields. It should be highlighted that only time variation of density would be extrapolated from nodes to particles and all other physical fields, i.e. density, pressure and velocity would be calculated at the particles. **Fig. 4** presents a schematic sketch of the key concept of the BM scheme, i.e. enhancing the pressure field through improving calculation of density field by obtaining more accurate and spatially continuous time variation of density.

The algorithm of the δ-SPH method with BM scheme (δ-SPH+BM) can be divided into four steps, and the flowchart is presented by **Fig. 5**. In the ***first step***, the initial background mesh is set according to the whole computational domain, and the mesh spacing is set equal to the initial particle spacing dx. In the ***second step***, velocities and densities at all mesh nodes are calculated by kernel-based interpolations by using the information of neighboring particles as expressed by Eq. (30). For kernel-based interpolations in Eq. (30), the same kernel used for other calculations, i.e. a fifth-order Wendland kernel function $w(r)$ expressed by Eq. (30) [50] is applied.

$$f_M = \frac{\sum_i f_i\, w(r_{iM})}{\sum_i w(r_{iM})} \tag{30}$$

$$w(r_{iM}) = \begin{cases} \left(1-\frac{r_{iM}}{r_{eM}}\right)^4 \left(1+4\frac{r_{iM}}{r_{eM}}\right), & r_{iM} < r_{eM} \\ 0\ , & r_{iM} \geq r_{eM} \end{cases} \tag{31}$$

In Eq. (31), r_{eM} is the influence radius of every mesh node during the second step of BM implementation, and r_{eM} is set as $1.2dx$ (i.e. $r_{eM} = h = 1.2dx$), corresponding to that in Wang et al. [17]. In Eq. (31), r_{iM} denotes the distance between the target mesh node and its neighboring particle.

In the ***third step***, $D\rho/Dt$ at each target mesh node M_i is calculated from the information at its neighboring nodes M_j located at the influence circle of target node i with $r_e = 2h = 2.4dx$ by using Eq. (32):

$$\frac{D\rho_{M_i}}{Dt} = \rho_{M_i} \sum_{M_j} \frac{m_{M_j}}{\rho_{M_j}} \boldsymbol{u}_{M_iM_j} \cdot \nabla_{M_i} W_{M_iM_j} + \delta h C_0 \sum_{M_j} \frac{m_{M_j}}{\rho_{M_j}} \boldsymbol{\Psi}_{M_iM_j} \cdot \nabla_{M_i} W_{M_iM_j} \tag{32}$$

With regard to the mesh nodes near the free-surface, as shown in **Fig. 6**, there will be an identification that whether there are particles in the influence domain r_{eM} of the target mesh node, if not, then the value of weight function at that mesh node is set as zero (mesh nodes with cyan color in Fig. 6). Mesh nodes with zero value of weight function will not be considered in the calculations during the corresponding time step. For mesh nodes that have particles in their influence domain r_M (mesh nodes denoted by red color in Fig 6), the weight function at those mesh nodes will be evaluated and they will be involved in calculations being conducted at the corresponding time step.

The ***fourth and final step*** is to calculate the $D\rho/Dt$ at each target particle i from the corresponding values at its four nearest mesh nodes which are the corner nodes of the grid occupied by the target particle (four nodes in 2D and eight nodes in 3D). For the final stage related to extrapolation of $D\rho/Dt$ from mesh nodes to target particles, kernel-based approximations through utilization of the fifth-order Wendland C2 kernel $w(r)$ are conducted.

It should be noted through consideration and implementation of the BM scheme and achievement of more accurate density and pressure fields at particles, dynamic free-surface boundary condition would also be better satisfied in the δ-SPH+BM method. In fact, implementation of the BM scheme would lead to a better resolution of the material interfaces including the free-surface because for every target interface particle four (in 2D) or eight (in 3D) nearest nodes would contribute to evaluation of time variation of density at that target particle. Through incorporation of calculated information of $D\rho/Dt$ at the background nodes in the free-surface region, a more precise and spatially continuous estimation of $D\rho/Dt$ would be obtained for free-surface and their neighboring particles. Accordingly, the calculated density and thus, the calculated pressure field would be more accurate and spatially continuous. Hence, enhancements of calculated density and pressure fields at the free-surface lead to more precise pressure gradient evaluation and velocity field, resulting in a more precise imposition of dynamic free-surface boundary condition.

In brief, the BM scheme contributes to (i) enhancement of pressure field through achievement of a more accurate and spatially continuous density field and (ii) a more accurate reproduction of free-surface flows through an enhanced resolution of material interfaces and improved satisfaction of dynamic free-surface boundary condition. These enhancements are schematically depicted in **Fig. 7**.

4. Validations and numerical investigations

In this section, the enhancing effect of the proposed BM scheme for δ-SPH will be verified through both two-dimensional and three-dimensional test cases presented in sections 4.1 and 4.2, respectively. The enhancing effects correspond to improvement of pressure field including impact pressure in violent dam break as well as sloshing flows. In addition, the dynamic free-surface boundary condition is shown to be better resolved through the implementation of the BM scheme. We also show that the BM scheme does not result in possible adverse effects such as artificial decay of volume (Appendix A) or energy (Appendix B). In Appendix C, a concise discussion is presented on increase of computational time (CPU time) due to incorporation of the BM scheme.

4.1 Two-dimensional validations

4.1.1 Uniform flow

In this section, a simple test case of uniform flow is considered for which analytical solutions exist. The performances of δ-SPH with and without the BM scheme in reproduction of this simple benchmark test will be compared. **Fig. 8** presents a schematic sketch of the considered uniform flow benchmark test. At the beginning of the simulation, the distribution of flow particles is regular as shown in the figure. The particles flow along the positive direction of *x* axis, and the initial velocity of flow is 0.5 m/s, the pressure at the *xy* plane is 500.0 Pa. In the SPH computation, a particle diameter dx = 2.0E-3 m and time step dt = 1.0E-4 s is used, and the periodic boundary is employed to simulate the infinite flow.

Figs. 9 and **10** present the distributions of particle density and pressure at the instant t = 0.1 s, calculated by the δ-SPH method excluding and including the BM scheme, respectively. It can be seen from Fig. 9(a) that there is some noise in the density distribution after a 0.1 second of flow motion, which leads to substantial spurious fluctuations in the pressure field calculated by EOS, and the result is shown in Fig. 10(a). Figs. 9(b) and 10(b) depict the results calculated by incorporation of the BM scheme, demonstrating more accurate and stable results of both density and pressure fields.

Fig. 11 presents the pressure time histories at the center of flow field (0.1, 0.1) calculated by δ-SPH method with and without the BM scheme, and the results are compared with the analytical solution. The calculation of pressure by δ-SPH method excluding the BM scheme is obviously unstable, which clearly shows unphysical pressure oscillations. On the other hand, incorporation of the BM scheme has resulted in more precise and stable pressure field. In order to provide a quantitative presentation of the accuracy enhancement by BM scheme, the Root Mean Square Errors (RMSE) of numerical results with respect to analytical solution are presented in **Table 1**. From the presented table, it is evident that the incorporation of BM scheme has been effective in enhancing the accuracy of the pressure field.

Fig. 12 shows the pressure time histories calculated by the δ-SPH method with the BM scheme using three different initial particle spacings and the RMSE of numerical results of Fig. 12 with respect to the analytical solution are shown in **Table 2**. From Fig. 12 and Table 2, refinement of spatial resolution (particle diameter or initial particle spacing) has clearly enhanced the accuracy of the δ-SPH+BM.

4.1.2 Dam break flow

In this section, a benchmark test of dam break flow is considered corresponding to the experiment conducted by Lobovský et al. [54]. A schematic view of this benchmark test is shown in **Fig. 13**, where the geometry dimensions and location of pressure sensor are provided. In the experiment, the length of the water column is L and its height is H; L_w is the length of the tank. There is a pressure sensor P installed on the left vertical wall; the height of P from the bottom is 3.0E-3 m.

The considered dam break flow is simulated by δ-SPH method with and without the BM scheme, and the results are compared with the experimental data by Lobovský et al. [54]. In the SPH computations, the particle diameter is set equal to dx = 5.0E-3 m and the computational time step is considered as dt = 1.0E-4 s. **Fig. 14** provides qualitative comparisons of numerical snapshots of the dam break flow with their corresponding experimental photo at t = 0.320 s, 0.415 s and 0.455 s. Fig. 14(b) portrays the free-surface profiles of dam break flow at three instants in the experiment; (a) and (c) display the corresponding free-surface profiles together with pressure field reproduced by δ-SPH method without and with the BM scheme, respectively. From the comparisons with experimental results, the overall features in the dam break process have been well captured by δ-SPH. Generally speaking, comparing Fig. 14(a) with Fig. 14(c), it is clear δ-SPH+BM scheme has provided an enhanced pressure field. In specific, from Fig. 14(a) and at t = 0.415 s, the pressure field by δ-SPH is shown to be discontinuous in the shallow region of the flow close to the wall. From Fig. 14(c), application of BM scheme has enhanced the pressure field leading to a more continuous and qualitatively accurate pressure field at this instant.

To quantify the accuracy of numerical results, the pressure time histories at the measuring point P are compared with the experimental data in **Fig. 15**. The comparison shows that the impact pressure calculated with BM scheme compares better with the experimental data in comparison with the result obtained without the BM scheme. The peak instant is also relatively better estimated through incorporation of the BM scheme.

4.1.3 Two-dimensional sloshing flow

In this section, simulation of a sloshing flow is considered as the third benchmark test. **Fig. 16** illustrates the rectangular sloshing tank corresponding to the experiment by Kishev et al. [55]. The presented figure provides a schematic sketch of the geometrical dimensions of the sloshing tank; the length of the tank is 0.6 m, the height is 0.3 m, the initial depth of water is 0.12 m. The tank is subjected to a sway excitation; the amplitude and period of excitation are set as 0.05 m and 1.5 s, respectively. Point P denotes the location of a pressure sensor to record the pressure time histories.

Based on the sloshing experiment of Kishev et al. [55], the sloshing flow is simulated by the δ-SPH method with and without the BM scheme. In the conducted SPH calculations, the particle diameter is set as dx = 4.0E-3 m and the calculation time step is considered as dt = 1.0E-4 s. **Fig. 17** illustrates a qualitative comparison in between the results by δ-SPH with and without BM at three typical instants of t = 0.91 s, 1.65 s and 2.01 s. The figure shows the generation and propagation of sloshing wave in the tank. From the presented figure, enhancement of pressure field by incorporation of BM scheme is well portrayed. In specific, from the enlarged portions of the snapshots corresponding to t = 0.91 s, the enhancing effect of BM in providing a qualitatively more accurate pressure field is evident. Another interesting and important matter which is clearly recognizable from Fig. 17 corresponds to enhanced reproduction of the free-surface in the considered sloshing flow. In snapshots by δ-SPH excluding the BM scheme (Fig. 17a), at some instants, a non-physical gap exists in between free-surface particles and the inner fluid particles. In addition, free-surface particles tend to be non-physically dispersed, some containing non-zero gauge pressures. Through implementation of BM scheme, such non-physical dispersiveness has been clearly minimized and a more integrated free-surface profile without any non-physical gap is reproduced.

Fig. 18 presents the comparisons of calculated pressure histories at measuring point P with the experimental data [55]. **Fig. 18(a)** indicates that the peak pressure instants and magnitudes have been relatively well captured by both numerical methods, i.e. δ-SPH and δ-SPH+BM. **Fig. 18(b)** presents a more detailed comparison through consideration of an enlarged portion of Fig. 18(a). From Fig. 18(b), improvement of pressure field by incorporation of BM scheme is well portrayed. In addition, from Fig. 18(a) and (b), in overall, application of the BM scheme has also resulted in peak pressures that are closer to their corresponding experimental values. In order to check the sensitivity of results by δ-SPH+BM to initial particle spacing, pressure time histories corresponding to three different particle diameters are presented in **Fig. 19**. The presented figure shows that by refinement of spatial computational resolution, better estimations of peak pressure magnitudes and instants are obtained.

4.2 Three-dimensional validations

Test cases considered in previous section, namely, section 4.1, were all corresponding to two-dimensions. However, in practical engineering cases, the hydrodynamic flows exist in three-dimensions, as highlighted in many previous conducted studies [31,56,57]. Hence, the enhancing effect of BM scheme must be carefully investigated for three-dimensional simulations as well, including test cases that are directly related to the focus of this paper, i.e. ocean engineering applications. In this section, three categories of test cases are considered, namely, a hydrostatic test with designed gravity, dam break flow in three-dimensions, and three-dimensional sloshing flows including a 3D violent sloshing test case under resonant excitation.

4.2.1 Hydrostatic test with designed gravity

In this section, a 3D water column with exponentially excited sinusoidal pressure variations [56] is simulated to show the enhancing performance of BM scheme in 3D δ-SPH simulations with nonlinear variations of external forces. In this test, the gravitational acceleration is modified as Eq. (33) to impose a designed external excitation on the fluid particles [58,59].

$$\boldsymbol{g}_d = \boldsymbol{g} + (\boldsymbol{g}/2)\sin(2\pi t/T)\exp(0.03\pi t/T) \tag{33}$$

where $\boldsymbol{g}_d$ denotes the designed gravitational acceleration, $\boldsymbol{g}$ stands for the original gravitational acceleration, t is the time at the current step, and T represents excitation period, which is 0.02 s in this test. This test is considered to highlight the need for rigorous consistency of schemes in providing accurate and spatially continuous source term of pressure equation in presence of considerable external excitations. In addition, for this test exact analytical solutions exist for pressure, density and volume fields.

Fig. 20 depicts a schematic sketch of the considered numerical tank; the length, width and height of the tank are 0.18 m and the initial depth of water is 0.16 m. A single measuring point A (at one particle) at the center bottom of the tank is placed to record the pressure time variations. In the conducted simulations for this test, the particle diameter is set as dx = 4.0E-3 m and the calculation time step is considered as dt = 5.0E-5 s. There are about 81000 fluid particles in the computational domain.

Fig. 21 presents a qualitative comparison of δ-SPH without and with the BM scheme. The figure presents two snapshots of particles together with pressure field at t = 0.205 s. At this instant, there are some clear noise and discontinuities in the spatial pressure distribution calculated by δ-SPH. On the other hand, the pressure field reproduced by the δ-SPH method incorporating the BM scheme, i.e. δ-SPH+BM, is smoother and spatially more continuous.

Fig. 22 presents a quantitative comparison through providing time histories of calculated pressure at measuring point A by 3D δ-SPH with and without BM, being compared with the corresponding analytical solution. From the presented figure, it can be seen that both numerical methods show good agreements with the analytical solution at the first several periods. Nonetheless, there are obvious unphysical oscillations in the result by δ-SPH excluding BM after $t \approx 0.2$ s. In contrast, δ-SPH including the BM scheme has provided a quantitatively accurate time history of pressure until the end of the considered simulation. For further quantification of the errors and comparison of performances of the considered numerical methods, the RMSE and the Normalized RMSE (NRMSE) are calculated and shown in **Table 3**. The NRMSE is obtained by normalization of the RMSE with respect to the standard deviation of the analytical solution. By comparing RMSE and NRMSE of two considered numerical methods, it is obvious that the incorporation of the BM scheme would be effective for enhancing the accuracy of the pressure calculations.

Fig. 23 presents the pressure time histories at point A calculated by 3D δ-SPH+BM with three different spatial resolutions corresponding to particle spacings of 1.0E-2 m, 6.0E-3 m and 4.0E-3 m. From the presented figure, the pressure time history corresponding to particle size of 1.0E-2 m is characterized by slight overestimation of the peak values. Through refinement of the spatial resolution, the accuracy of results has been clearly improved with the peak values becoming closer to the analytical solution. It also can be seen from **Table 4** that the numerical errors have decreased with refinement of the spatial resolution, illustrating a good convergence property for 3D δ-SPH+BM.

An important matter with respect to incorporation of BM scheme corresponds to possible adverse effects that the scheme may bring about with respect to conservation of energy or volume. Hence, these important matters need to be carefully investigated. **Fig. 24** shows quantitative comparisons of time histories of the kinetic energy, gravitational potential energy and total mechanical energy by 3D δ-SPH with and without the BM scheme. The total mechanical energy corresponds to sum of kinetic and gravitation energy in simulation of an incompressible fluid. From **Fig. 24(a)**, due to imposition of a considerable external excitation, very slight increases of kinetic energy are seen in results by both 3D δ-SPH with and without the BM scheme. Such slight variations remain to be within the order or 10^{-8}. From **Fig. 24(b)**, the time variation of potential energy is shown to be closer to the analytical solution in case of 3D δ-SPH+BM. Accordingly, as shown in **Fig. 24(c)**, the time variation of total mechanical energy by 3D δ-SPH+BM is found to be closer to the reference solution with respect to 3D δ-SPH without the BM scheme. Hence, in the considered simulation, incorporation of BM scheme even led to a more accurate time variation of total mechanical energy. As for investigations on volume conservation, a distinct section

(Appendix A) is dedicated to this matter.

4.2.2 Dam break flow in three-dimensions

In this section, the enhancing performance of BM scheme for 3D δ-SPH is further investigated through the simulation of a dam break flow in three-dimensions. In section 4.1.2, the experiment conducted by Lobovský et al. [54] is used as a benchmark test for 2D dam break flow. However, this experiment is carried out in three-dimensions, thus a 3D simulation is conducted in this section according to this experiment.

A schematic view of the experimental tank is shown in **Fig. 25**. The length and height of the water column are both 0.6 m, and its width is 0.15 m. The distance between two vertical walls is 1.61 m. The pressure sensor L_1 is installed on the centerline of the vertical wall at the height of 3.0E-3 m from the bottom wall. The 3D dam break flow is simulated by 3D δ-SPH including and excluding the BM scheme. For each simulation, the particle size is considered as dx = 6.0E-3 m, resulting in a total number of about 250,000 fluid particles.

Fig. 26 illustrates a qualitative comparison of numerical results with their corresponding experimental photographs [54] at t = 0.320 s, 0.415 s and 0.455 s. The figure portrays the enhancing effect of BM scheme in providing smoother and qualitatively more accurate pressure field. In specific, similar pressure discontinuities seen in 2D are observed in 3D in results by 3D δ-SPH excluding the BM scheme. Incorporation of the BM scheme has eliminated such non-physical discontinuous pressure field and has been effective in improving the hydrodynamic pressure field.

Fig. 27 presents the time histories of calculated pressure at measuring point L_1 by 3D δ-SPH with and without the BM scheme with respect to the corresponding experimental data [54]. The comparison manifests that the result by 3D δ-SPH+BM better corresponds to the experiment. Comparing Fig. 27 with its 2D version (Fig. 15), the enhancing effect of BM scheme for three-dimensional simulations appears to be even superior with respect to that for two-dimensional simulations. This is evident from the overall smoothness of pressure field, as well as correspondence of numerical and experimental pressure time histories. From Figs. 27 and 15, as a result of reduced numerical noise, the δ-SPH+BM has also resulted in impact instants closer to those observed in the experiment.

Fig. 28 depicts a simple check of convergence by consideration of three different computational spatial resolutions for pressure time history at measuring point L_1. The figure shows that through refinement of spatial resolution, the correspondence of numerical results with respect to reference experimental data is enhanced. This enhancement is in particular better observed in terms of instant and magnitude of peak pressure.

In order to present the enhancing effect of BM scheme along with effect of spatial resolution on reproduction of free-surface in three-dimensional simulations, a set of particle snapshots with pressure contours at instant t = 0.455 s are shown in **Fig. 29**. The presented figure clearly shows that the incorporation of the BM scheme has resulted in a more continuous and integrated free-surface. Such continuity and integrity are further enhanced through refinement of spatial resolution. The presented figure also shows that application of BM scheme has clearly enhanced the satisfaction of the dynamic free-surface boundary condition in a violent three-dimensional fluid flow.

4.2.3 Three-dimensional sloshing flows

In this section, two 3D sloshing flow cases are simulated by 3D δ-SPH with and without the BM scheme. The corresponding experiments are carried out by Luo et al. [60] using a prismatic tank, which is scaled down, from a real LNG (Liquefied Natural Gas) tank [61]. A schematic view of the tank used in the experiments is shown in **Fig. 30**, with exact geometry and locations of pressure sensors being provided.

In the first sloshing flow case, the sloshing tank experienced a translational motion with an oblique angle θ as shown in **Fig. 30**(**b**) and $\theta = 45°$ is considered in this experiment. The amplitude and frequency of the excitation are 0.005 m and 6.618 rad/s, respectively. In the SPH computations, the initial particle spacing is set as dx = 4.0E-3 m, resulting in a total number of 628,324 fluid particles.

Fig. 31 presents qualitative comparisons of three typical snapshots of the 3D sloshing flow with pressure field at t = 8.02 s, 9.92 s and 13.72 s with their corresponding experimental photographs [60]. **Fig. 31**(**a**) and (**c**) portray the numerical results by 3D δ-SPH excluding and including the BM scheme, respectively; and **Fig. 31**(**b**) presents the corresponding experimental photos. The presented figure and its containing enlarged portions clearly portray the enhancing effect of BM scheme in providing a more accurate pressure field as well as resulting in better satisfaction of dynamic boundary condition at the free-surface.

Fig. 32 illustrates the enhanced reproductions by BM scheme from a quantitative perspective. The figure shows the time histories of calculated pressure at point P_1 by 3D δ-SPH including and excluding the BM scheme, and the pressure variations are compared with the corresponding experimental data [60]. The experimental data [60] indicates that the impact pressure shows a two-peak pattern in the reproduced sloshing process. The comparison shows that the two-peak pattern has been well captured by 3D δ-SPH including the BM scheme. In contrast, the results calculated by 3D δ-SPH excluding BM show a single-peak pattern for this 3D sloshing flow. **Fig. 33** portrays the pressure time histories calculated by 3D δ-SPH+BM by considering three different spatial resolutions corresponding to particle sizes of 8.0E-2 m, 6.0E-3 m and 4.0E-3 m. The figure shows that for all considered spatial resolutions acceptable agreements are achieved and through refinement of spatial resolution, the simulation-experiment agreement has been relatively improved.

The second considered three-dimensional sloshing flow corresponds to a violent sloshing under resonant excitation [60]. In this test case the excitation frequencies of roll and pitch rotations are set to be the same as the corresponding natural frequencies as explained by Luo et al. [60]. For this sloshing experiment, the initial depth of water is 0.118 m, the tank is subjected to roll and pitch rotational excitations as shown in **Fig. 34**. In this figure, κ represents the rotations corresponding to the roll and pitch motions. For the conducted SPH computations of resonant sloshing, the initial particle spacing is set as dx = 4.0E-3 m resulting in a total number of 380,000 fluid particles.

Fig. 35 presents a set of typical snapshots of particles together with pressure field by 3D δ-SPH excluding and including the BM scheme at three different instants along with the corresponding experimental photographs [60]. The figure clearly portrays the advantageous features of BM scheme in enhancing the pressure field as well as improving the continuity and dynamic boundary condition at free-surface for this violent sloshing under resonant excitation. In particular, incorporation of BM scheme has well improved the pressure field at the local impact zones, as highlighted by the enlarged portions of the snapshots.

Fig. 36 provides a quantitative comparison of pressure time histories at point P_1 by 3D δ-SPH including and excluding the BM scheme. The figure well portrays that through incorporation of BM scheme the agreement of reproduced pressure field with respect to the experiment [60] has been clearly improved. The BM scheme has not only minimized the existing pressure noise but also has resulted in more precise pressure time histories including the magnitude and instants of peak pressures. In order to present this improvement more clearly, an enlarged portion of **Fig. 36**(**a**) is presented in **Fig. 36**(**b**), where accuracy enhancement by incorporation of BM scheme is clearly portrayed.

Fig. 37 presents a comparison of the results by 3D δ-SPH+BM and the numerical data by Luo et al. [60] that corresponds to their projection-based Consistent Particle Method (CPM). It must be noted that the CPM results correspond to a coarser particle size of dx = 8.0E-3 m (compared with dx = 4.0E-3 m in 3D δ-SPH+BM simulation). The figure shows the potential practical applicability of 3D δ-SPH+BM for precise simulation of highly violent 3D sloshing flows. In contrast to projection-based CPM that includes a semi-implicit solution process, the δ-SPH possesses a fully explicit solution process, permitting flexibilities for fast and highly parallelizable computations, resulting in relatively precise predictions of pressure time histories in 3D violent sloshing flows.

5. Concluding remarks

The paper presents an improvement for a well-known and commonly applied variant of SPH, namely, the δ-SPH method [29,31]. The proposed improvement corresponds to incorporation of a so-called BM (Background Mesh) scheme to enhance the density and pressure calculation and to improve the satisfaction of dynamic free-surface boundary condition at the free-surface region. The improvements brought about by the BM scheme are due to enhancements of the source term of the pressure equation (the EOS or equation of state). Through incorporation of the BM scheme, the fixed and regularly distributed background nodes located at the BM, provide a more accurate and spatially continuous calculations of instantaneous time variation of density through a more accurate velocity divergence field calculation by utilizing the velocity and density field at the moving particles. The calculated time variations of density are then extrapolated to the particles being surrounded by four (in 2D) or eight (in 3D) nearest nodes. Through consideration of extrapolated time variations of density, a more accurate and spatially continuous density field would be obtained for the computational elements (particles) resulting in a more precise pressure field. Simultaneously, incorporation of the BM scheme results in a better resolution of the material interfaces, e.g. free-surfaces by providing more accurate time variation of density, density, pressure as well as velocity field for the free-surface and their vicinity particles. In other words, the BM scheme provides more precise information of the space (void and matter) and its time evolution, hence, resulting in better reproductions of free-surfaces and improved satisfaction of dynamic free-surface boundary condition at the free-surface region.

The enhancing effects of the BM scheme for δ-SPH are shown through both two-dimensional and three-dimensional test cases including simple test cases with exact analytical solutions as well as test cases related to ocean engineering with experimental reference solutions (dam break and violent sloshing flows). In all conducted test cases, δ-SPH incorporating the BM scheme, i.e. δ-SPH+BM, has shown to provide more accurate pressure field along with better resolution of dynamic free-surface boundary condition at free-surface. At the same time, the conducted numerical investigations clearly portrayed that incorporation of BM scheme does not result in possible adverse effects, such as excessive dissipations in energy or volume. On the contrary, implementation of the BM scheme has been shown to minimize dissipations in mechanical energy (through providing more accurate density, pressure and velocity fields). In addition, as discussed in Appendix C, the extra computational cost corresponding to incorporation of BM scheme is found to be moderate with only 20% to 30% of increase for the conducted simulations of this study.

The results of this study show that through incorporation of a simple and effective BM scheme, the fully explicit δ-SPH method, that guarantees relatively fast and highly parallelizable computations (with respect to implicit or semi-implicit particle methods), can be more reliably applied for practical engineering applications corresponding to ocean engineering including problems involving violent flows and large hydrodynamic impact pressures as in case of wave impact on coastal/offshore structures or violent sloshing flows. Future works include incorporation of BM scheme for enhanced multi-phase [62,63] and multi-physics [64,65] simulations involving discontinuous material interfaces as well as more systematic investigations on conservation properties including energy and momentum. In this paper, the BM scheme was shown to improve imposition of dynamic free-surface boundary condition in SPH-based numerical simulations. Possible contributions of BM scheme for enhancements related to all so-called SPH Grand Challenges [77] need to be systematically studied. In addition, the present study considered incorporation of BM for δ-SPH as a representative fully explicit particle method. However, the concept of BM scheme is expected to bring improvements for other variants of SPH as well (e.g. δ-VSPH [66] or other advanced SPH-based methods [78]).

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Appendix A Effect of BM scheme on volume variation

The effect of BM scheme on variation of volume is investigated by two test cases, namely, the 2D dam break flow test case in section 4.1.2 and 3D hydrostatic test with designed gravity in section 4.2.1. In the 2D dam break flow test, as shown in **Fig. A1**, there are two measuring points A and B at the tank wall. The time histories of calculated density at points A and B are considered by utilizing two types of equations for the density calculation in SPH context. These two equations are presented by Eqs. (A.1) and (A.2) as follows:

$$\rho_i^{n+1} = \rho_i^{n} + \left(\frac{D\rho_i}{Dt}\right)^n \Delta t$$

$$\frac{D\rho_i}{Dt}^n = \left(\rho_i \sum_j \frac{m_j}{\rho_j} \boldsymbol{u}_{ij} \cdot \nabla_i W_{ij} + \delta h C_0 \sum_j \frac{m_j}{\rho_j} \boldsymbol{\Psi}_{ij} \cdot \nabla_i W_{ij}\right)^n \tag{A.1}$$

$$\rho_i = \sum_j m_j W_{ij} \tag{A.2}$$

Figs. A2 and **A3** present the calculated time histories of density at points A and B by δ-SPH including and excluding the BM scheme. Through comparison of results with the theoretical value, it is clear that both methods, i.e. δ-SPH with and without BM, result in very slight level of compressibility of maximum 0.2% obtained by either Eq. (A.1) or (A.2). The presented figures clearly show that the application of BM scheme has not adversely affected the volume conservation and has resulted in smoother time variations of density field with the overall trend and magnitudes similar to those in δ-SPH excluding the BM scheme.

In the 3D hydrostatic test, as shown in **Fig. A4**, measuring point A is placed at the center bottom of the tank, and point B is placed at the center point of the water. **Figs. A5** and **A6** show the comparisons of the time histories of density at points A and B calculated by δ-SPH with and without BM by utilizing either Eq. (A.1) or Eq. (A.2), respectively. The figures also demonstrate that the same density variations have been obtained by using two types of calculations for both δ-SPH with and without the BM scheme. Therefore, the results manifest that the incorporation of BM scheme does not adversely affect the volume conservation feature.

Appendix B Effect of BM scheme on energy conservation

The energy conservation property of δ-SPH+BM is investigated by an oscillating drop test [6]. In this test, the drop is subjected to a central conservative force. The force field is given as follows:

$$\begin{cases} f_x = -\Omega^2 x \\ f_y = -\Omega^2 y \end{cases} \tag{B.1}$$

The drop initially forms a circle with a radius of R = 0.5 m and its initial velocity field is considered as follows:

$$\begin{cases} u_0 = \Gamma_0\, x \\ v_0 = -\Gamma_0\, y \end{cases} \tag{B.2}$$

In this test, Ω and Γ_0 are set to 1.2 and 0.4, respectively. The fluid is considered to be incompressible and inviscid (no artificial viscosity). The initial particle spacing is considered as 0.005 m (R/dx = 100), and artificial sound speed is set as 10 m/s. **Figs. B1** and **B2** depict snapshots of particles together with pressure field by standard SPH, standard SPH+BM, δ-SPH and δ-SPH+BM at t = 0.9 s and t = 5.4 s, respectively. Here, standard SPH simply refers to δ-SPH without the density diffusion term in the continuity equation, i.e. δ term or second term on the right-hand side of Eq. (6). Comparisons of the snapshots in **Figs. B1** and **B2** indicate that the incorporation of BM scheme clearly enhances the reproduced pressure field, resulting in smoother and spatially more continuous pressure field.

Fig. B3 presents a quantitative comparison through providing time histories of calculated semi-major and minor axes by considered SPH-based methods, along with the corresponding analytical solution. From **Fig. B3**, the results by standard SPH show clear deviation from the analytical solutions. Application of δ-SPH has enhanced the accuracy of results through minimization of non-physical pressure noise. Further enhancement is achieved by δ-SPH+BM through application of the BM scheme. The presented figure clearly portrays the enhancing effects of both δ-SPH and BM schemes in improving the accuracy of simulations and suggests enhancements in preservation of mechanical energy.

In order to quantitatively investigate the enhancements corresponding to mechanical energy preservation, time histories of kinetic energy and potential energy of results by standard SPH, δ-SPH and δ-SPH+BM methods are presented in **Fig. B4**. From the presented figure, clear dissipations of kinetic and potential energies by standard SPH are minimized through application of δ-SPH and then BM scheme. **Fig. B5** shows that incorporation of BM also minimizes the energy dissipation observed in standard SPH results. In **Fig. B6**, time histories of total mechanical energy, composed of kinetic and potential energy for the incompressible, inviscid fluid flow simulations considered here, are presented. The figure portrays clear enhancing effect of the BM scheme in minimizing the non-physical energy dissipation mainly through providing a more accurate pressure field. It should be noted that the theoretical solution in **Fig. B6** is based on the assumptions of incompressible and inviscid fluid. Although we have not used any form of viscosity here and the fluid is thus reproduced as an inviscid one, the incompressibility condition is not perfectly guaranteed due to weakly-compressible assumption. A more rigorous energy analysis for quantification of energy dissipation requires consideration of total energy including compressible energy similar to that performed by Antuono et al. [72].

As previously stated, through implementation of the BM scheme the material interfaces including free-surfaces would be better resolved as for every target particle located at and in the vicinity of the free-surface four (in 2D) or eight (in 3D) nearest nodes would contribute to calculation of $D\rho/Dt$ for that target particle. For every nearest neighboring node of a target particle, divergence of velocity has been calculated based on relevant information at corresponding neighboring nodes that are regularly distributed in space and have received information of instantaneous spatial distribution of matter and corresponding velocity field from their neighboring particles. As a consequence, the BM scheme would result in more accurate evaluation of velocity divergence field, in particular at free-surfaces. This would lead to a better estimation of $D\rho/Dt$, density and thus pressure/velocity fields. In order to better demonstrate this key feature of BM scheme, **Figs. B7** and **B8** present the spatial distribution of velocity divergence at t = 4.55 s and t = 10.45 s by standard SPH, standard SPH+BM, δ-SPH and δ-SPH+BM. In order to ensure that enhancements by BM scheme would not be dependent on h/dx ratio, the results in Figs. B7 and B8 correspond to h/dx = 2.0 (which is considered in several δ-SPH articles, e.g. [38,76]). The figure clearly illustrates an enhanced calculation of velocity divergence field by incorporation of the BM scheme. In particular, divergence of velocity is obtained as almost zero for free-surface particles through the application of BM scheme. This would correspond to almost zero time variation of density and thus almost invariant density field at the free-surface. As a result, the pressure at free-surface particles would be very close to zero (enhancement of dynamic free-surface boundary condition) as based on Eq. (13) pressure is a direction function of variation of density with respect to reference density in WCSPH context. The improvement of imposition of dynamic free-surface boundary condition in SPH-based numerical simulations would not only contribute to more accurate free-surface reproductions, but also possibly bringing about enhanced conservation of mechanical energy through minimization of volume variation at free-surfaces in simulations of incompressible fluid flows in WCSPH context.

The considered test case of oscillating drop is a classical benchmark that has been frequently considered for validation of the SPH-based methods. Antuono et al. [72] also conducted this test case by using the δ-SPH method [72]. In their simulations, excellent results were obtained by the δ-SPH method, in particular for R/dx = 200 with h/dx = 2.0. The simulations of this section are conducted by consideration of R/dx = 100 with h/dx = 1.2. Besides, Antuono et al. [72] applied a slightly different pressure gradient model together with a fourth-order Runge-Kutta time integration scheme. Indeed, in the conducted simulations of this study, the computational conditions for all four considered SPH-based methods are exactly the same and thus, these simulations clearly present the enhancing effect of the BM scheme.

In order to investigate the effect of BM scheme on simulation results corresponding to different h/dx ratios, a new set of simulations are conducted through consideration of h/dx being equal to 1.5 and 2. **Figs. B9** and **B10** portray clear enhancements in simulation results for both δ-SPH as well as δ-SPH+BM as the ratio of h/dx is increased from 1.2 to 1.5 and then to 2.0. In addition, **Tables B1** and **B2** present the RMSE of data in **Figs**. **B9** and **B10**, showing enhancements due to increase of h/dx as well as implementation of BM scheme. The normalized time variations of mechanical energy for oscillating drop simulations by δ-SPH and δ-SPH+BM for different h/dx are presented in **Fig. B11**. The figure clearly portrays the enhancing effect of the BM scheme in minimizing the dissipations in mechanical energy for all considered h/dx ratios.

Appendix C Effect of BM scheme on computational time

In order to study the effect of BM scheme on computational cost, the test case of dam break flow in two and three-dimensions are considered with different particle numbers. **Tables C1** and **C2** present the corresponding required computational time for simulations corresponding to different computational spatial resolutions by δ-SPH including and excluding the BM scheme in two-dimensions and three-dimensions, respectively. The last columns in the tables show the percentage of the increased computational time because of the incorporation the BM scheme. As it is demonstrated in the tables, the percentage of the increased computational cost due to incorporation of the BM scheme is slightly increased through refinement of particle size. This is mainly due to increase of computational cost corresponding to the search of neighboring particles for the mesh nodes. However, the increase in CPU time is found to be inconsiderable for both 2D and 3D simulations with a maximum of about 20% for the conducted 2D and 30% for the conducted 3D simulations.

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