Partially Fixed Structure Determinantal Assignment Problems

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***Abstract*—We deal with the study of the Determinantal Assignment Problem (DAP) when the parameters of the compensator are not entirely free, but some of them are fixed. The problem is reduced to a restricted form of an exterior algebra problem (decomposability of multi-vectors) which is referred to as Partial Decomposability problem. We study this problem and in case that this problem has no solution, we examine the problem of approximate partial decomposability. We treat the problem of exact or partial decomposability into a vector and a multi-vector of lower dimension. If this procedure is repeated then this results in an approximation of the initial multi-vector into a decomposable vector. The approximation of a vector by an optimal decomposable multi-vector is a non linear procedure and has been solved completely using the Power Method. The method developed in this paper although it produces a sub- optimal solution, can be used alternatively for the solution of DAP or the Approximate DAP, as a shorter and easier approach, because it is based on known tools as the Singular Value Decomposition (SVD). We apply these results to treat the Restricted - Approximate Decomposability problem, which leads to approximate solutions to the pole placement and zero assignment problems.**

***Index Terms*—Algebraic Control, Pole Assignment, Control Systems Design.**

I. INTRODUCTION

The Determinantal Assignment Problem (DAP) has emerged as the abstract problem to which the study of pole, zero assignment of linear systems may be reduced [4], [7], [6], [8]. The different versions of DAP have been introduced as the abstract unifying descriptions of frequency assignment problems (pole,zero) [12] that arise in linear systems theory. This problem has been treated so far under the conditions that the compensators are not restricted. The aim of this paper is to provide a methodology for handling the case of partially fixed compensators. The multilinear nature of DAP suggests that the natural framework for its study is that of exterior algebra [1], [2]. The study of DAP [4] may be reduced to a linear problem of zero assignment of polynomial combinants [11] and a standard problem of multi-linear algebra, that is the decomposability of multi-vectors [1]. The solution of the linear subproblem, whenever it exists, defines a linear space in a projective space $\mathscr{P}^{\rho}(\mathbb{R})$ whereas decomposability is characterised by the set of Quadratic Plücker Relations (QPR), which define the Grassmann variety of $\mathscr{P}^{\rho}(\mathbb{R})$ [3]. The general Determinantal Assignment Problem (DAP) is expressed as finding a constant matrix H of dimension $l\times m$ such that for an appropriate system description $M(s)\in\mathbb{R}^{m\times l}[s]$, $l\leq m$ we have:

$$\det(H\cdot M(s))=f(s) \tag{1}$$

where $f(s)$ is a desirable polynomial. Thus, the different classes of DAP are clearly of multi-linear nature, as far as the parameters in H and thus the natural setting for its study is that of exterior algebra [1], [2]. Let h_i^T, be the rows of $H\in\mathbb{R}^{l\times m}$ and $m_i(s)$ be the columns of $M(s)$, $i=1,2,...l$ respectively. Then, if we denote by

$$C_l(H)=h_1^T\wedge\ldots\wedge h_l^T=\underline{h}^T\in\mathbb{R}^{1\times q}, q=\binom{m}{l}$$

$$C_l(M(s))=m_1(s)\wedge\ldots\wedge m_l(s)=\underline{m}(s)\in R^q[s]$$

then the Binet-Cauchy Theorem [2] leads to:

$$f(s)=C_l(H)\cdot C_l(M(s))=\langle\underline{h},\underline{m}(s)\rangle=\sum_{\omega\in Q_{l,m}}h_\omega m_\omega(s) \tag{2}$$

where $\langle\bullet,\bullet\rangle$ denotes scalar product, $\omega=(i_1,...,i_l)\in Q_{l,m}$ [2] and h_ω, m_ω are the entries in $\underline{h}$, $\underline{m}(s)$ respectively. Note that h_ω is the $l\times l$ minor of H, which corresponds to the ω set of columns of H and thus is a multi-linear alternating function of the h_{ij} entries of H.

Let $\underline{z}\in\wedge^m\mathbb{R}^n$, then the problem of decomposability of $\underline{z}$ is to find $\underline{z}_i\in\mathbb{R}^n$, $i=1,2,...,m$ such that:

$$\underline{z}=\underline{z}_1\wedge\underline{z}_2\wedge\ldots\wedge\underline{z}_m$$

In this case we call $\underline{z}$ decomposable. For a multi-vector $\underline{z}$ to be decomposable there is a necessary and sufficient condition which states that $\underline{z}$ must satisfy certain quadratic relations called Quadratic Plücker Relations (QPR), which characterise the Grassman variety of the corresponding projective space [1], [3]. The general DAP assumes that the row vectors of H are free. However, for most of the applications part of the rows of H are fixed and this leads to the problem of partial decomposability considered here. We make the assumption that l is different than 1 or $m-1$. In these two cases the DAP problem is reduced to a linear problem and its solutions, as well as the restricted versions of DAP are trivial.

In fact, if l is different than 1, or $m-1$ a generic $\underline{z}\in\wedge^m\mathbb{R}^n$ is not decomposable; however, it might be possible to decompose a vector into a product of lower dimensional multi-vectors in which case we refer to partial decomposability. The simplest case of partial decomposability of a vector $\underline{z}\in\wedge^m\mathbb{R}^n$ is to decompose it as:

$$\underline{z}=\underline{z}_{m-1}\wedge\underline{z}_1$$

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where $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$ and $\underline{z}_1 \in \mathbb{R}^n$. In case that this is not possible we approximate $\underline{z}$ by $\underline{z}_{m-1} \in \wedge \underline{z}_1$, i.e. we solve the minimization problem:

$$\min \left\|\underline{z} - \underline{z}_{m-1} \wedge \underline{z}_1\right\|, \text{ where } \underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n \text{ and } \underline{z}_1 \in \mathbb{R}^n \quad (3)$$

This is the approximate partial decomposability problem. Here we solve this problem completely by matrix methods, i.e. we find the closest multi-vector of the form $\underline{z}_{m-1} \wedge \underline{z}$ to $\underline{z}$. For the solvability of this problem two matrices related to the multi-vector $\underline{z}$, the so-called Grassmann and Hodge-Grassmann matrix [10] play a crucial role.

The singular value decomposition of these matrices plays an important role to the solution of the problem [16]. In fact, the maximum singular value and the corresponding singular vectors can be used to construct the solution. Note that Grassmann matrices [5],[10] as opposed to tensor unfolding matrices presented in [14], [15] are unique and do not depend on the mode used to define the unfolding. Furthermore Grassmann matrices incorporate the skew symmetry of the tensor as well as information of how exterior product operates and in this respect are more suitable for tensor representations in the Grassmann algebra [10].

The results from the partial decomposition may be used to consecutively construct a decomposable multivector approximating $\underline{z}$. This does not constitute an optimal but a suboptimal solution, which may be used as a starting point for the construction of the best decomposable approximation.

We apply these results to treat the Restricted - Approximate Decomposability problem, which leads to approximate solutions of pole placement and zero assignment problems [10],[13].

II. The Problem of Exact Partial Decomposability

Here we consider the problem that given a $\underline{z} \in \wedge^m\mathbb{R}^n$ to derive conditions that $\underline{z}$ can be written as $\underline{z} = \underline{z}_{m-1} \wedge \underline{z}_1$ where $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$ and $\underline{z}_1 \in \mathbb{R}^n$. We will use the following lemma:

Lemma 1: [14] The following statements are equivalent:

a) $\underline{z} = \underline{z}_{m-1} \wedge \underline{z}_1$, $\underline{z}_1 \in \mathbb{R}^n$
b) $\underline{z}_1 \wedge \underline{z} = 0$, $\underline{z}_1 \in \mathbb{R}^n$

Definition 1: Given $\underline{z} \in \wedge^m\mathbb{R}^n$ define the Grassmann matrix [10] $\Phi_n^m(\underline{z})$ as the representation matrix of the map [10]:

$$T : \mathbb{R}^n \to \wedge^{m+1}\mathbb{R}^n$$

such that $T(\underline{z}_1) = \underline{z}_1 \wedge \underline{z}$.

By utilizing this matrix and Lemma 1 we have the following results for partial decomposability.

Theorem 1: [18] [10] A multi-vector $\underline{z} \in \wedge^m\mathbb{R}^n$ is partially decomposable as:

$$\underline{z} = \underline{z}_{m-1} \wedge \underline{z}_1 \quad (4)$$

with $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$ and $\underline{z}_1 \in \mathbb{R}^n$ if and only if the following equivalent statements are true:

a) $rank(\Phi_n^m(\underline{z})) < n$
b) $\det(\Phi_n^m(\underline{z})^T \cdot \Phi_n^m(\underline{z})) = 0$
c) $\Phi_n^m(\underline{z})$ has at least one singular value equal to zero.

Proof 1: Due to lemma 1 $\underline{z}$ is written as in (4) is equivalent to:

$$\Phi_n^m(\underline{z}) \cdot \underline{z}_1 = 0$$

and departing from this, Theorem is evidently proved.

Corollary 1: If multi-vector $\underline{z} \in \wedge^m\mathbb{R}^n$ is partially decomposable as $\underline{z} = \underline{z}_{m-1} \wedge \underline{z}_1$ then $\underline{z}_1$ may be constructed as the right singular vector corresponding to 0 singular value of $\Phi_n^m(\underline{z})$.

III. The Approximate Partial Decomposability Problem

Consider the problem:

$$\min \left\|\underline{z} - \underline{z}_{m-1} \wedge \underline{z}_1\right\|, \text{ where } \underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n \text{ and } \underline{z}_1 \in \mathbb{R}^n \quad (5)$$

One can easily see [13] that this is equivalent to solve:

$$\min_{\substack{\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n \\ \underline{z}_1 \in \mathbb{R}^n}} \quad \|\underline{z}\|^2 - \frac{\left\langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \right\rangle^2}{\left\|\underline{z}_{m-1} \wedge \underline{z}_1\right\|^2}$$

Therefore we may solve the problem (M1)

$$\max_{\substack{\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n \\ \underline{z}_1 \in \mathbb{R}^n}} \quad \left\langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \right\rangle \text{ subject to } \left\|\underline{z}_{m-1} \wedge \underline{z}_1\right\| = 1. \quad (6)$$

We denote the solution of (6) or (M1) a pair $(\sigma, \underline{z}_{m-1} \wedge \underline{z}_1)$, where $\underline{z}_{m-1} \wedge \underline{z}_1$ is the multi-vector that maximizes $\left\langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \right\rangle$ and σ is the attained maximum value. If $\left(\sigma, \underline{z}_{m-1} \wedge \underline{z}_1\right)$ is a solution of the latter (M1) then $\left(\sqrt{\|\underline{z}\|^2 - \sigma^2}, \underline{z}_{m-1} \wedge \underline{z}_1\right)$ is a solution of the original problem (5). To solve (M1) we require the following Lemma:

Lemma 2: If $\left\|\underline{z}_{m-1} \wedge \underline{z}_1\right\| = 1$, $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$, $\underline{z}_1 \in \mathbb{R}^n$ then there exists $\underline{z}_1^{'} \in \mathbb{R}^n$, $\underline{z}_{m-1}^{'} \in \wedge^{m-1} span\left[\underline{z}_1\right]^{\perp}$ such that $\underline{z}_{m-1}^{'} \wedge \underline{z}_1^{'} = \underline{z}_{m-1} \wedge \underline{z}_1$ and $\left\|\underline{z}_{m-1}^{'}\right\| = \left\|\underline{z}_1^{'}\right\| = 1$.

Proof 2: Set $\underline{z}_1^{'} = \frac{\underline{z}_1}{\|\underline{z}_1\|}$. Then consider $z_1^{'}, z_2^{'}, ..., z_n^{'}$ an oriented basis for $\mathbb{R}^n$. In this basis $\underline{z}_{m-1}$ can be written as

$$\begin{aligned}\underline{z}_{m-1} &= \sum_{\omega \in Q_n^{m-1}} \alpha_\omega \underline{z}_\omega = \\ &= \sum_{\substack{\omega^{'} \in Q_n^{m-1} \\ \omega^{'} \subseteq \{2,3,...,n\}}} \alpha_{\omega^{'}} \underline{z}_{\omega^{'}} + \underline{z}_1^{'} \wedge \sum_{\substack{\omega^{''} \in Q_n^{m-2} \\ \omega^{''} \subseteq \{2,3,...,n\}}} \alpha_{\omega^{''}} \underline{z}_{\omega^{''}}\end{aligned}$$

We set $\underline{z}_{m-1}^{'} = \|\underline{z}_1\| \sum_{\substack{\omega \in Q_n^{m-1} \\ \omega^{'} \subseteq \{2,3,...,n\}}} \alpha_{\omega^{'}} \underline{z}_{\omega^{'}}$. In this setting

$$\begin{aligned}\underline{z}_{m-1} \wedge \underline{z}_1 &= \|\underline{z}_1\| \sum_{\substack{\omega^{'} \in Q_n^{m-1} \\ \omega^{'} \subseteq \{2,3,...,n\}}} \alpha_{\omega^{'}} \underline{z}_{\omega^{'}} \wedge \underline{z}_1^{'} = \\ &= \underline{z}_{m-1}^{'} \wedge \underline{z}_1^{'}\end{aligned}$$

and $1 = \left\|\underline{z}_m \wedge \underline{z}_1\right\|^2 = \left\|\underline{z}_1\right\|^2$. Moreover, we have that $\sum \left(\alpha_{\omega^{'}}\right)^2 = \left\|\underline{z}_{m-1}^{'}\right\|^2$, where the sum is taken for all $\omega^{'} \in Q_n^{m-1}$ and $\omega^{'} \subseteq \{2,3,...,n\}$ Hence, $\underline{z}_{m-1}^{'} \wedge \underline{z}_1^{'} = \underline{z}_{m-1} \wedge \underline{z}_1$ and $\left\|\underline{z}_1^{'}\right\| = \left\|\underline{z}_{m-1}^{'}\right\| = 1$.

Definition 2: Let $\underline{z} \in \wedge^m \mathbb{R}^n$, define the $\binom{n}{m-1} \times n$ matrix $\Phi_n^m(\underline{z}^*)^*$ [10], [13] as:

$$\underline{z}_{m-1}^T \Phi_n^{n-m}(\underline{z}^*)^* \underline{z}_1 = \langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \rangle$$

Proposition 1: Let $\underline{z} \in \wedge^m \mathbb{R}^n$, then $\Phi_n^{n-m}(\underline{z}^*)^*$ is the matrix representation of the map:

$$T : \mathbb{R}^n \to \wedge^{m-1}\mathbb{R}^n$$

such that $T(\underline{z}_1) = (-1)^{(n+1)\times(m-1)}(\underline{z}_1 \wedge \underline{z}^*)^*$.

Proof 3:

$$\begin{aligned}
&\langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \rangle = \langle \underline{z}_{m-1} \wedge \underline{z}_1 \wedge \underline{z}^* \rangle^* = \\
&= (-1)^{(n+1)\times(m-1)} (\underline{z}_{m-1} \wedge (\underline{z}_1 \wedge \underline{z}^*)^{**})^* = \\
&= \left\langle \underline{z}_{m-1}, (-1)^{(n+1)\times(m-1)} (\underline{z}_1 \wedge \underline{z}^*)^* \right\rangle = \\
&= \underline{z}_{m-1}^t \Phi_n^{n-m}(\underline{z}^*)^* \underline{z}_1
\end{aligned}$$

And this completes the proof.

Define the (M2) optimization problem: $\max \langle \underline{z}, \underline{z}_{m-1} \wedge \underline{z}_1 \rangle$ where $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$ and $\underline{z}_1 \in \mathbb{R}^n$ such that $\|\underline{z}_{m-1}\| = \|\underline{z}_1\| = 1$.

Theorem 2: [18] Both maximization problems (M1) and (M2) can be solved and they attain the same maximum σ.

Proof 4: Indeed, (M1) is defined on the set:

$$\left\{ \ \underline{z}_m \in \mathbb{R}^{\binom{n}{m}}, \|\underline{z}_m\| = 1 \quad and \quad \det(\Phi_n^m(\underline{z}_m)^T \cdot \Phi_n^m(\underline{z}_m)) = 0 \ \right\}$$

and (M2) is defined on the set:

$$\left\{ \ (\underline{z}_{m-1}, \underline{z}_1) \in \mathbb{R}^{\binom{n}{m-1}} \times \mathbb{R}^n, \quad \|\underline{z}_{m-1}\| = \|\underline{z}_1\| = 1 \ \right\}$$

which are both compact therefore both objective functions attain global maxima. Let $\left(\sigma_1, \underline{z}_{m-1}, \underline{z}_1\right), \left(\sigma_2, \underline{z}'_{m-1}, \underline{z}'_1\right)$ are solutions for (M1), (M2) respectively.

By lemma 2 $\underline{z}_{m-1}, \underline{z}_1$ can be corresponded to a $\underline{z}''_{m-1}, \underline{z}''_1$ such that $\underline{z}''_{m-1} \wedge \underline{z}''_1 = \underline{z}_{m-1} \wedge \underline{z}_1$ and $\left\|\underline{z}''_{m-1}\right\| = \left\|\underline{z}''_1\right\| = 1$, this proves that $\sigma_1 \leq \sigma_2$. On the other hand as $\left\|\underline{z}'_{m-1} \wedge \underline{z}_1\right\| \leq \left\|\underline{z}'_{m-1}\right\| \|\underline{z}_1\|' = 1$, both (M1), (M2) may be relaxed to (M3):

$$\max_{\substack{z_{m-1} \in \wedge^{m-1}\mathbb{R}^n \\ z_1 \in \mathbb{R}^n}} \langle z, z_{m-1} \wedge z_1 \rangle$$

such that $\|\underline{z}_{m-1} \wedge \underline{z}_1\| \leq 1$.

If $(\sigma_3, \underline{z}'''_{m-1}, \underline{z}'''_1)$ is a solution to (M3) by a re-scaling argument must satisfy $\left\|\underline{z}'''_{m-1} \wedge \underline{z}'''_1\right\| = 1$, thus $\sigma_2 \leq \sigma_3 = \sigma_1$. And this completes the proof.

Now, (M2) may be solved via a singular value decomposition of the matrix $\Phi_n^{n-m}(\underline{z}^*)^*$.

Theorem 3: [18] Let $\underline{z} \in \wedge^m \mathbb{R}^n$.

a) The problem (M2) has a solution $(\sigma, \underline{z}_{m-1}, \underline{z}_1)$, where σ is the highest singular value of $\Phi_n^{n-m}(\underline{z}^*)^*$ and $\underline{z}_{m-1}, \underline{z}_1$ are the corresponding left and right singular vectors;
b) Furthermore, $\|z_{m-1} \wedge z_1\| = 1$.

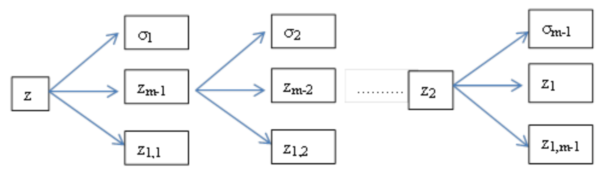
Proof 5:

a) Evident from the properties of the singular value decomposition;
b) Since, $\underline{z}_{m-1} = (-1)^{(n+1)\times(m-1)}\left(\underline{z}_1 \wedge \underline{z}^*\right)^*$ we have that $\underline{z}_{m-1} \in \wedge^{m-1} span(\underline{z}_1)^{\perp}$, hence $\|\underline{z}_{m-1} \wedge \underline{z}_1\| = \|\underline{z}_{m-1}\| \|\underline{z}_1\| = 1$.

IV. Applications to the Problem of Decomposability

The previous results may be applied to the problem of approximate decomposability i.e. approximating an m-vector $\underline{z} \in \wedge^m \mathbb{R}^n$ by a product of $m \times 1$-vectors. This can be done iteratively as follows:

We start by the vector $\underline{z}_m = \underline{z} \in \wedge^m \mathbb{R}^n$ and using the matrix $\Phi_n^{n-m}(\underline{z}^*)^*$ we approximate it by $\sigma_1 \underline{z}_{m-1} \wedge \underline{z}_{1,1}$, $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n, \underline{z}_{1,1} \in \mathbb{R}^n$. Then we repeat the same procedure for $\underline{z}_{m-1} \in \wedge^{m-1}\mathbb{R}^n$ utilizing the matrix $\Phi_n^{n-m+1}(\underline{z}_{m-1}^*)^*$ approximating by $\sigma_2 \underline{z}_{m-2} \wedge \underline{z}_{1,2}$, $\underline{z}_{m-2} \in \wedge^{m-2}\mathbb{R}^n, \underline{z}_{1,2} \in \mathbb{R}^n$. We follow the same steps until we reach a two vector $\underline{z}_2$ which we approximate by a product of two one vectors. This may be described graphically as follows: In this setting the (subopti-



mal) decomposable vector approximating $\underline{z}$ is given by:

$$(\prod_{i=1}^{m-1} \sigma_i) \underline{z}_1 \wedge \underline{z}_{1,p-1} \wedge \underline{z}_{1,p-2} \wedge \cdots \wedge \underline{z}_{1,1} \tag{7}$$

Furthermore if $u_1 \wedge u_2 \wedge \cdots \wedge u_m$ is a solution of the best decomposable approximation of $\underline{z}$, with $\|u_1 \wedge u_2 \wedge \cdots \wedge u_m\| = \sigma$, we have that:

$$(\prod_{i=1}^{m-1} \sigma_i) \leq \sigma \leq \sigma_1$$

where σ corresponds to the optimal solution, i.e. the best decomposable approximation, $(\prod_{i=1}^{m-1} \sigma_i)$ corresponds to the partial decomposability in cascade of the present technical note, i.e. suboptimal solution. Finally, σ_1 represents the first level of partial decomposability, which although approximates closer the original vector does not correspond to a full decomposable solution and σ_1 only serves as an upper bound.

We may apply this approach to $\wedge^3 \mathbb{R}^n$ in which case we consider $\wedge^3 \mathbb{R}^n$ and it is required to approximate this vector by a product of three 1-vectors. In this case the above algorithm consists of the following two steps:

Step 1 Consider the SVD of the matrix $\Phi_n^3(\underline{z}^*)^*$ and approximate $\underline{z}$ via the highest singular value and the corresponding singular vectors as $\sigma_1 \underline{z}_2 \wedge \underline{z}_{1,1}$, $\underline{z}_2 \in \wedge^2 \mathbb{R}^n, \underline{z}_{1,1} \in \mathbb{R}^n$

Step 2 Consider the SVD of the matrix $\Phi_n^4(\underline{z}_2{}^*)^*$ and approximate $\underline{z}_2$ via the highest singular value and the corresponding singular vectors as $\sigma_2 \underline{z}_1 \wedge \underline{z}_{1,2}$, $\underline{z}_1, \underline{z}_{1,1} \in \mathbb{R}^n$.
The sub-optimal decomposable approximation of z is then given by:

$$(\sigma_1 \sigma_2) \underline{z}_1 \wedge \underline{z}_{1,2} \wedge \underline{z}_{1,1}$$

Additionally the norm of the best decomposable approximation σ is related to the previous data by the following inequality:

$$\sigma_1\sigma_2 \leq \sigma \leq \sigma_1$$

V. Application of Partial Decomposability of Multivectors in DAP

In this subsection we demonstrate how partial decomposability can be utilized to solve the DAP, such as a pole / zero assignment problem. These particular problems are decomposed into a linear and a multilinear problem [10] as mentioned in Section I. We start by solving the linear problem, that is, the solution of linear equations and we continue by approximating the solution by a decomposable one, using the methodology described in Section IV. To calculate a controller that stabilizes a system, we compute a vector $\underline{z}$ in $\wedge^m\mathbb{R}^{p+m}$ that assigns the desired stable controller and then we approximate $\underline{z}$ by $\hat{\underline{z}}$ decomposable, according to the methodology of the present technical note. If $\hat{\underline{z}}$ is adequately close to $\underline{z}$, then the polynomial assigned by $\hat{\underline{z}}$ will be stable. The metric relationship that describes the above approximate problem can be found in [13] (Theorem 4.2). A methodology for constructing an approximate stabilizing controller K is described below, along with an example.

A. Methodology for Pole Placement & Zero Assignment Problems

Let a system with m inputs, p outputs and ρ states. The polynomial matrix of the system is defined by:

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

and in terms of transfer function is given by:

$$G(s) = N(s) \cdot D^{-1}(s)$$

where $D(s)$ and $N(s)$ are the denominator and numerator of a coprime MFD of the transfer function of the system. The aim is to stabilize the system, by assigning the poles to the left half plane, using a static output feedback controller K, i.e. $det([I,K] \cdot M(s)) = f(s)$, where $f(s)$ is the desired polynomial.

Step 1: Compute the Plücker matrix P [4], which is defined as the coefficient matrix of all $m\times m$ minors of $M(s)$, or equivalently:

$$C_m(M(s)) = P \cdot \begin{bmatrix} s^n & s^{n-1} & \cdots & s^1 & 1 \end{bmatrix}^T$$

where $C_m[M(s)]$ denotes the m-th compound matrix [2] of $M(s)$. The Plücker matrix would then be of $\binom{m+p}{m} \times (\rho+1)$ dimension.

Step 2: We solve the linear problem:

$$\underline{z} \cdot P = \underline{f} \tag{8}$$

where $\underline{f} = \text{coeff.vector}(f(s))$.

Step 3: We seek for a solution $\underline{z}$ to the linear problem (8) that has the minimum norm, or equivalently a multi-vector $\underline{z}$ that is perpendicular to the linear variety, such that $\underline{z} \cdot P = \underline{f}$. The solution $\underline{z} = \underline{z}_{min}$ is then computed as follows:

$$\underline{z}_{\min} = \underline{f} \cdot \left(P^T \cdot P\right)^{-1} \cdot P^T$$

and results to be a $\binom{m+p}{m} \times 1$ multi-vector, which is *not necessarily decomposable*.

Step 4: We aim to find a *decomposable vector* $\hat{\underline{z}} = z_1 \wedge z_2 \wedge \ldots \wedge z_m$ of dimension $\binom{m+p}{m} \times 1$ which is close to $\underline{z}_{min}$, using the partial decomposability method.
We follow the procedure presented in section IV, according to which we find an approximate decomposition of $\underline{z}$ as a product of $p\times 1$ vectors using the iterative procedure described in that section. The expression in (7) produces an approximate decomposition of $\underline{z}$, that is $z_1 \wedge z_2 \wedge \ldots \wedge z_m$.

Step 5: In the final step we compute the controller K that assigns the poles of the system as follows:
We form the matrix:

$$X^T = \begin{bmatrix} z_1 & z_2 & \cdots & z_{m-1} & z_m \end{bmatrix}^T$$

where X is of dimension $m\times(p+m)$ and we partition it as:

$$X = \begin{bmatrix} \underbrace{A}_{m\times m} & \underbrace{K_1}_{m\times p} \end{bmatrix}$$

where A is the leftmost $m\times m$ submatrix of X. In this setting the required controller K is given by:

$$K = A^{-1} \cdot K_1$$

In the case of strictly proper systems the leading coefficient of $f(s)$ is calculated to be equal to $\det A$. This means that it is compulsory that the solution $\begin{bmatrix} A & K_1 \end{bmatrix}$ has the property $\det A \neq 0$, i.e. A is non-singular, otherwise at least one of the closed loop poles goes to infinity. This excludes any sub-optimal solution that $\det A = 0$.

B. example

We consider a system of $m=3$ inputs, $p=3$ outputs and $\rho=6$ states. The arbitrary pole placement problem cannot be solved by dyadic linearization (Kimura's condition $m+p-1 \geq \rho$ [19], [20], [21]). Therefore, more sophisticated methods have to be employed. This can be either the Global Linearization Method [9], [22], which can be applied when $mp > \rho$ (this is the case here) or the methodology of the present technical note. This system has transfer function the following 3×3 rational matrix:

$$G(s) = \begin{bmatrix} \frac{-1-s^2-s^3+s^4}{s^5} & \frac{1+s^2}{s^3} & \frac{-1+s}{s^2} \\ -\frac{2(1+s)}{s^3} & \frac{1}{s} & \frac{1}{s} \\ -\frac{(-1+s)(1+s)^2}{s^5} & \frac{-1-s}{s^3} & \frac{1+s}{s^2} \end{bmatrix} = \begin{bmatrix} 1+s & 1+s & -1+s \\ 0 & 1+s & s \\ 0 & 0 & 1+s \end{bmatrix} \cdot \begin{bmatrix} s^2 & 0 & 0 \\ s+1 & s^2 & 0 \\ s+1 & s & s^2 \end{bmatrix}^{-1} = N(s) \cdot D(s)^{-1}$$

where $D(s)$ and $N(s)$ are the denominator and numerator of the transfer function $G(s)$ and $M(s) = \begin{bmatrix} D(s)^T, & N(s)^T \end{bmatrix}^T$ denotes the composite MFD polynomial matrix fraction decomposition of $G(s)$. This particular system is unstable as all of its poles are located at 0. The stabilisation of this system by real static output feedback pole placement is equivalent to finding a 3×3 real matrix K such that:

$$\det\left(\begin{matrix} [I_3, K] & M(s) \end{matrix} \right) = f(s)$$

where $f(s)$ is a given 6-degree polynomial with zeroes located at the left-half plane. In this case, we select $f(s) = (s+1)^6$. i.e. all the closed loop poles to be located at -1. To do so, we proceed as follows:

Step 1: We compute the Plücker matrix P, which is defined as the coefficient matrix of all 3×3 minors of $M(s)$, or equivalently: $C_3(M(s)) = P \cdot \begin{bmatrix} s^6 & s^5 & s^4 & s^3 & s^2 & s^1 & 1 \end{bmatrix}^T$ where $C_3(M(s))$ denotes the 3rd compound matrix of $M(s)$. The Plücker matrix P is a 20×7 matrix and is:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 \end{bmatrix}$$

Step 2: We solve the linear problem $\underline{z} \cdot P = \underline{f}$, where

$$\underline{f} = \text{Coeff.vector}((s+1)^6)$$

and this is $\underline{f} = \begin{bmatrix} 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$.

Step3: We seek for a solution $\underline{z}$ that has a minimum norm, or equivalently a multi-vector $\underline{z}$ that is perpendicular to the linear variety, such that $\underline{z} \cdot P = \underline{f}$. This solution $\underline{z} = \underline{z}_{min}$ is computed as:

$$\underline{z}_{\min} = \underline{f} \cdot (P^T \cdot P)^{-1} \cdot P^T$$

and results to be a 20×1 vector, which is not necessarily decomposable:

$$\underline{z}_{\min} = [1, -0.93, 1.30, 3.53, -0.13, 5 - 1.30, 1.06, \\ 0.89, 1.95, 1.95, 0.66, -2.12, 0.58, -2.09, 0.44, 1.51, \\ 0.03, -1.03, 1.51, 1.51]$$

Step 4: We aim to find a decomposable vector $\hat{\underline{z}} = z_1 \wedge z_2 \wedge z_3$ of dimension 20×1 which is close to $\underline{z}_{min}$, using the partial decomposability method.

We start by the vector $\underline{z} = \underline{z}_{min}$, that we calculated in Step 3. We compute the matrix $\Phi_6^3(\underline{z}^*)^*$ and this results to be the following 15×6 matrix:

$\Phi_6^3(\underline{z}^*)^* =$

$$= \begin{bmatrix} 0 & 0 & 1 & -0.93 & 1.30 & 3.53 \\ 0 & -1 & 0 & -0.13 & -1.30 & 1.06 \\ 0 & 0.93 & 0.13 & 0 & 0.89 & 1.95 \\ 0 & -1.30 & 1.30 & 0.89 & 0 & 1.95 \\ 0 & -3.53 & -1.06 & -1.95 & -1.95 & 0 \\ 1 & 0 & 0 & 0.66 & -2.12 & 0.58 \\ -0.93 & 0 & -0.66 & 0 & -2.09 & -0.44 \\ 1.30 & 0 & 2.12 & 2.09 & 0 & 1.51 \\ 3.53 & 0 & -0.58 & 0.44 & -1.51 & 0 \\ -0.13 & 0.66 & 0 & 0 & 0.33 & -1.03 \\ -1.30 & -2.12 & 0 & -0.03 & 0 & 1.51 \\ 1.06 & 0.58 & 0 & 1.03 & -1.51 & 0 \\ 0.89 & -2.09 & 0.03 & 0 & 0 & 1.51 \\ 1.95 & -0.44 & -1.03 & 0 & -1.51 & 0 \\ 1.95 & 1.51 & 1.51 & 1.51 & 0 & 0 \end{bmatrix}$$

We factorize $\Phi_6^3(\underline{z}^*)^*$ using the SVD approach and we choose the highest singular value σ_1. For this singular value we find the left and right corresponding singular vectors $\underline{z}_2$ and $\underline{z}_1$. These are the following:

$\hat{z}_1^T = [1.18, 0.21, -0.03, 0.27, 0.61, -0.01, 0.09, -0.17, -0.13, -0.12, 0.38, -0.14, 0.27, 0.03, -0.4]^T$ and

$z_1^T = [-0.29, -0.75, -0.17 - 0.43 - 0.160.332]^T$

We keep the z_1^T vector and we follow the same procedure for $\hat{z}_1^T$.

Step 5: We compute the 6×6 matrix $\Phi_6^4(\hat{z}_1^*)^* =$

$$= \begin{bmatrix} 0 & 0.18 & 0.21 & -0.03 & 0.27 & 0.61 \\ -0.18 & 0 & -0.01 & 0.09 & -0.18 & -0.13 \\ -0.21 & 0.01 & 0 & -0.12 & 0.38 & -0.14 \\ 0.03 & 0.09 & 0.12 & 0 & 0.27 & 0.03 \\ -0.27 & 0.17 & -0.38 & -0.27 & 0 & -0.4 \\ -0.61 & 0.13 & 0.14 & -0.03 & 0.4 & 0 \end{bmatrix}.$$

We factorize $\Phi_6^4(\hat{z}_1^*)^*$ using the SVD approach and we choose the highest singular value σ_2. For this singular value we find the left and right corresponding singular vectors, $\underline{z}_2$ and $\underline{z}_3$ These are the following: $z_2^T = \left[-3.44 \cdot 10^{-16}, -0.07, -0.46, -0.09, -0.47, -0.74\right]^T$ and $z_3^T = [0.74, -0.19, 0.06, 0.27, -0.53, 0.28]^T$.

The decomposable vector that we are seeking is the

$$\hat{\underline{z}} = z_1 \wedge z_2 \wedge z_3 = C_3 \begin{bmatrix} z_1 \\ z_3 \\ z_4 \end{bmatrix}$$

and the angle between the decomposable vector $\hat{\underline{z}}$ and $\underline{z}_{min}$ is equal to

$$\arccos \frac{\left\langle \hat{\underline{z}}, \underline{z}_{\min} \right\rangle}{\|\hat{\underline{z}}\| \, \|\underline{z}_{\min}\|} = 25,79^o.$$

Step 6: We calculate the resulting $\underline{f}'$ (the polynomial that results from the approximate controller) with the new $\hat{\underline{z}}$ as

$\underline{\hat{z}} \cdot P = \underline{f}^{'}$ and we divide by the highest coefficient to make it monic. The resulting polynomial $\underline{f}^{'}$ is given by:

$$\underline{f}^{'} = s^6 + 3.92s^5 + 7.1s^4 + 11.28s^3 + 9.54s^2 + 5.64s + 1.87$$

Finally, we compute the roots of $\underline{f}^{'}$,

$$\begin{array}{lr} s_1 \rightarrow & -2.38 \\ s_2 \rightarrow & -0.66 \\ s_3 \rightarrow & -0.26 - 0.71i \\ s_4 \rightarrow & -0.26 + 0.71i \\ s_5 \rightarrow & -0.18 - 1.42i \\ s_6 \rightarrow & -0.18 + 1.42i \end{array}$$

which all have negative real parts, hence the polynomial is stable.

Step 7: Finally, the controller K of (9)

$$\det\left(\begin{bmatrix} I & K \end{bmatrix} M(s) \right) \tag{9}$$

that stabilizes the system, can be computed from the matrix

$$X = \begin{bmatrix} A & K_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

as shown below:

$$K = A^{-1}K_1 \tag{10}$$

Hence, the controller K is given by the 3×3 matrix:

$$K = \begin{bmatrix} 0.38 & -0.72 & 0.01 \\ 0.39 & 0.27 & -0.82 \\ 0.14 & 0.98 & 1.74 \end{bmatrix}$$

VI. Restricted - Approximate Decomposability of multi-vectors

We can additionally consider the problem of partial decomposability of a multi-vector $\underline{z} \in \wedge^p \mathbb{R}^n$ [13], where we require that the approximate decomposition is of the form:

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge x_{k+1} \wedge x_{k+2} \wedge \ldots \wedge x_p \tag{11}$$

where $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal set of $k < p$ fixed vectors. To do so we set as

$$V = span\{v_1, v_2, \ldots, v_k\}$$

and let

$$V^T = span\{v_{k+1}, v_{k+2}, \ldots, v_n\}$$

where $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is an orthonormal basis of V^T. We then rewrite $\underline{z}$ in terms of the orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ and we then factorize this expression as:

$$\underline{z} = v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge \underline{z}_1 + \underline{z}_2$$

with $\underline{z}_1 \in \wedge^{p-k} V^T$ and $\underline{z}_2$ satisfies $v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge \underline{z}_2^* = 0$. Then as in section IV we approximate $\underline{z}_1$ as $x_{k+1} \wedge x_{k+2} \wedge \ldots \wedge x_p$ and the final restricted approximate decomposition of $\underline{z}$ is given by:

$$\underline{z} = v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge x_{k+1} \wedge x_{k+2} \wedge \ldots \wedge x_p \tag{12}$$

VII. Conclusions

We have examined the problem of partially fixed structure Determinantal Assignment Problem which is reduced to a problem of decomposability of skew symmetric tensors and this is expressed in terms of the exact or approximate partial decomposability. This problem may be treated with matrix methods in terms of the Hodge and Grassmann matrices [10]. Consecutive applications of this method derive sub-optimal solutions of the approximate decomposability problem. We have applied these results to treat the Restricted - Approximate Decomposability problem, which leads to approximate solutions of pole placement and zero assignment problems.

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