# The two-spinor formalism in the study of the Bach equations and the massless free-field equations

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# Contents

# List of Figures

# Declaration

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# Abstract

This thesis applies the two-component spinor (two-spinor) calculus to study the Bach equations and the massless free-field equations. We first provide some context for the following chapters by outlining the history of the spinor concept and the motivation for using two-spinor methods as opposed to the better-known tensor calculus. We also give a summary of the original results and introduce the notation to be used throughout the thesis. We continue with the rules of spinor algebra, which are indispensable for carrying out calculations involving spinors, followed by a description of the geometrical picture of a spinor, which provides a useful mental aid to the more abstract algebraic approach. Next, in close analogy to the tensor formalism, we discuss spinor analysis by introducing the spinor covariant derivative and spinor curvature. Then in order to apply the two-spinor calculus in practical calculations, where explicit solutions are sought, we introduce the Newman-Penrose spin-coefficient formalism and its compacted version.

We proceed by applying the compacted spin-coefficient formalism to the Bach equations, which is an example of a conformal theory of gravity. We reconstruct two known exact solutions, namely the PP-wave spacetime and the static spherically-symmetric spacetime. In order to better understand the relationship between solutions of the Bach equations and the Einstein equations, we present the necessary and sufficient conditions, for a certain class of spacetimes to be a conformal Einstein space. As a further application of two-spinor methods, we introduce Lanczos potential theory, including the Lanczos spinor and the Weyl-Lanczos equations. We proceed to solve these equations for the Bach spacetime solutions found earlier.

Next, we discuss duality rotations and helicity in the context of the massless free-field equations. Using concepts from symplectic geometry, we derive an expression for the helicity of an integer spin $s$ field. The helicity expression is given in terms of a three-surface integral over a conserved current density. Moreover, from the conformal invariance of the massless free-field equations, we show that it is conformally invariant. We also utilise concepts from twistor theory in order to better understand the relationship between duality rotations and helicity. We finish with a summary of results and outline future directions related to this work.

# List of Abbreviations

**EFE** Einstein field equations

**NP** Newman-Penrose

**GHP** Geroch-Held-Penrose

**PP-wave** Plane-fronted gravitational wave with parallel rays

**ED** Eastwood-Dighton

**WL** Weyl-Lanczos

**PL** Pauli-Lubanski

# Chapter 1

# Introduction

According to current theory, all physics takes place within spacetime, which can be represented mathematically as a smooth four-dimensional manifold endowed with a smooth Lorentzian $(+, -, -, -)$ metric. The tensor calculus is the most common mathematical formalism used to describe a manifold and its metric. However, in four dimensions and with Lorentzian signature, an alternative exists called the two-component spinor (two-spinor) formalism. Two-spinor techniques have been used to good effect when studying spacetime geometry, making clear some results that would have otherwise been obscure if only tensor descriptions had been employed. The aim of this thesis is to provide further applications of the two-spinor formalism to the study of relativistic field equations in four-dimensional spacetime. In particular, we will apply it to the Bach equations [8] and the massless free-field equations [38, 42], which are both conformally invariant field equations.

The concept of a spinor was first introduced by Cartan [27], however some of the formulas that arise in spinor theory, in the guise of Euler's rotational parameters, date back even further [77]. Much of the important literature on spinors up to 1953 is given in [9]. Spinor algebra and the two-component spinor notation (dotted and undotted indices) was introduced by Van der Waerden, see [108] for an English translation. Soon afterwards, a spinor analysis was provided [54]. Applications to curved spacetime were initiated in [15, 111, 80]. Penrose continued to develop spinor methods in a relativistic context. In [71], Newman and Penrose combined the two-spinor calculus with a Ricci-rotation-coefficient type of formalism, which is an example of a tetrad formalism based on a null-tetrad, now referred to as the Newman-Penrose (NP) spin-coefficient formalism. It was later streamlined into the compacted Geroch-Held-Penrose (GHP) spin-coefficient formalism [45]. Good introductions to two-spinor methods in relativity can be found in [107, 74]. A comprehensive account is provided in [89], and further applications of spinor and twistor methods are given in [90].

A physical motivation for employing the two-spinor calculus is the fundamental importance of the light-cone (null-cone) structure of spacetime. This structure is equivalent to the conformal structure of spacetime and is sufficient for all causal relations. As we will show, two-spinors are null objects. That is, they represent spin-vectors which point along the null-cone. Therefore, they are ideally suited to the study of Lorentzian spacetimes, as the following example shows. We can represent the four-momentum in the following matrix form

$$\boldsymbol{p} = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}, \tag{1.1}$$

where the matrix is Hermitian iff the momentum components are real numbers. If the four-momentum is null, we have the following quadratic condition

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0. \tag{1.2}$$

This condition is equivalent to the vanishing of the determinant of (1.1). Hence its rank must be equal to one, so it can be written as the outer product of two nonzero vectors

$$\boldsymbol{p} = \bar{\boldsymbol{\pi}}\boldsymbol{\pi}. \tag{1.3}$$

The $\bar{\boldsymbol{\pi}}$ and $\boldsymbol{\pi}$ are complex conjugates of each other iff $\boldsymbol{p}$ is Hermitian. They are two-component spinors, often referred to as spin-vectors, because in index notation they have a single spinor index. These spin-vectors exist only if (1.2) holds. Moreover, no further constraints need to be imposed. As spin-vectors, they are automatically null.

Another physical motivation for using two-spinors comes from the standard model of particle physics, where massless fields are treated as fundamental. In order to acquire mass, these fields must interact with the Higgs field. Massless fields only have support on null-momenta, so again a description in terms of two-spinors is more economical. This can be seen explicitly in twistor theory [90], where a twistor can be represented as a pair of two-spinors.

A motivation from a mathematical perspective comes from the fact that two-dimensional objects are simpler to deal with than four-dimensional objects. Furthermore, having two sets of indices (primed and unprimed) brings definite advantages, as we shall see. As one might expect from the above, two-spinors are specifically well-suited to four-dimensional spacetimes with one time-like and three space-like directions. In this case, the spin-space (the space of spin-vectors) splits into two irreducible spin-spaces of complex dimension two. The spin vectors of these spaces are the univalent (one spinor index) two-spinors, also known as Weyl spinors. From (2.6), it follows that two-spinors can be thought of as a square root of a (future-pointing) null-vector. That is, a vector with positive time-like component and zero norm. Because null-vectors are left invariant by a conformal rescaling, two-spinors are particularly useful for studying conformal symmetry and for proving conformal invariance. Furthermore, tensors with antisymmetric (skew) indices are concisely represented in the two-spinor formalism. Since the Maxwell-Faraday tensor is skew and the Riemann tensor has two pairs of skew indices, the equations of electromagnetism and gravitation take a concise form in two-spinor notation [60, 80]. At first sight, the application of the two-spinor calculus in gravitation theory seems to achieve all the same things as the tensor calculus, but with twice as many indices, and therefore feels like an unnecessary complication. However, the ability to manipulate each spinor index separately in a simple fashion is what makes this calculus so efficient. The tensor operations that correspond to these simple spinor index manipulations are often more complicated [108, 89].

In this thesis, we have chosen to investigate the Bach equations and the massless free-field equations for spin $s$ using the two-spinor formalism. For the Bach equations, we aim to show the efficiency of the formalism in finding exact solutions. For the massless free-field equations, we focus of their duality invariance and the associated conserved charge, interpreted physically as the helicity. In both instances, expressing the equations in two-spinor form achieves a certain degree of economy. The main reason is that the relevant field variables can be expressed as totally symmetric complex spinors, which leads to an economic and elegant representation. Before introducing the equations to be studied, we introduce the basics of two-spinor theory, including the notation and the conventions of the abstract-index formalism. Chapter two describes the basics of the two-spinor algebra. Chapter three describes the geometrical picture of a spin-vector, the univalent spinor, which provides a useful mental aid to the more abstract algebraic approach. Chapter four discusses spinor analysis by introducing the spinor covariant derivative and spinor curvature. Chapter five describes the NP spin-coefficient formalism (and its compacted GHP version), which is a powerful calculus based on a null-tetrad or a spinor-dyad.

The main novel results of this thesis are contained in chapters six to nine. In chapter six, we translate the Bach equations into compacted GHP spin coefficient form and apply this formalism to find exact solutions for the Plane-fronted gravitational wave with parallel rays (PP-wave) spacetime and a static spherically-symmetric spacetime. Chapter seven describes the necessary and sufficient conditions for a spacetime to be a conformal Einstein space. For a spherically symmetric spacetime we apply the conditions to calculate the conformal factor that transforms the spacetime to an Einstein space. Chapter eight introduces Lanczos potential theory in its two-spinor formulation. We solve the Weyl-Lanczos (WL) equations to find the Lanczos coefficients for the PP-wave and spherically-symmetric spacetimes. Chapter nine discusses duality rotations and helicity in the context of the massless free-field equations. We use a symplectic two-form approach to derive an expression for the helicity of an integer spin $s$ field. The helicity expression is given in terms of a three-surface integral over a conserved current density. Moreover, from the conformal invariance of the massless free-field equations, we show that it is conformally invariant. We also utilise concepts from twistor theory in order to better understand the relationship between duality rotations and helicity. Chapter ten gives a summary of results and an outline of future directions related to this work.

## 1.1 Summary of results

For the convenience of the reader, we list the original results presented in this thesis:

- The translation of the Bach equations into the compacted GHP spin-coefficient formalism
- Application of the Bach equations in spin-coefficient form to find two exact solutions: PP-wave spacetime and static spherically-symmetric spacetime
- Application of the conformal Einstein space conditions to calculate the conformal factor which transforms the spherically-symmetric Bach solution to the de Sitter-Schwarzschild solution
- Application of the Weyl-Lanczos equations to find the Lanczos coefficients for the PP-wave spacetime and static spherically-symmetric spacetime solutions of the Bach equations
- Applying symplectic methods and the two-spinor formalism to derive an expression for the helicity of an integer spin massless field

## 1.2 Notation

We follow the conventions of the abstract-index formalism [83, 89]. The motivation of this formalism stems from recognising the importance of having a basis or coordinate independent formalism whilst also taking advantage of the familiar index notation, which is very useful for representing operations such as index permutation. The abstract-index formalism retains all the same operations as the familiar index notation, but is less ambiguous and often simpler.
In the abstract-index formalism, the symbol $p^a$ represents an abstract vector, and not its components according to some basis. Similarly $\pi^A$ and $\bar{\pi}_A$ would represent abstract two-spinors. We can think of the index as an abstract marker, it does not take on numerical values. If there is no risk of confusion, when referring to a spinor or tensor within the text, indices may be suppressed.
When calculating solutions to the field equations, we invariably make a choice of basis. In such cases, we use the symbol $p^{\mathbf{a}}$ with a bold index in order to represent the vector components with respect to our vector-basis, where $\mathbf{a} = 0, 1, 2, 3$. Similarly, we would represent spinor components as $\pi^{\mathbf{A}}$ and $\bar{\pi}_{\mathbf{A}'}$, where $\mathbf{A} = 0, 1$ and $\mathbf{A}' = 0', 1'$. The primed and unprimed indices refer to the irreducible spin-spaces, which are mapped to each other under complex conjugation. Lower case Latin indices will always represent vector indices and upper case Latin indices (either primed of unprimed) will represent spinor indices. One advantage of the abstract-index formalism is a greater economy in representing the correspondence between tensors and spinors. We can simply represent an abstract vector index as a pair of abstract spinor indices. For example, (1.3) can be written abstractly as

$$p_a = \bar{\pi}_A \pi_{A'}. \tag{1.4}$$

The equivalent equation in component form is

$$p_{\mathbf{a}} = g_{\mathbf{a}}{}^{\mathbf{A}\mathbf{A}'}\bar{\pi}_{\mathbf{A}}\pi_{\mathbf{A}'}, \tag{1.5}$$

which introduces the mixed quantities $g_{\mathbf{a}}{}^{\mathbf{A}\mathbf{A}'}$ referred to as the Infeld-van der Waerden symbols [54]. These symbols can be viewed as the Kronecker delta tensor $g_a{}^b$ where each index refers to a different type of basis. For each value of **a** they are represented as $2 \times 2$ matrices. For a real vector-basis, the matrices are Hermitian. Our signature convention is such that the Minkowski metric $\eta_{ab}$ is equal to $\mathrm{diag}(+1, -1, -1, -1)$.

# Chapter 2

# Two-Spinor algebra

There are three operations that are basic to the two-spinor algebra, namely scalar multiplication, addition and the inner product. In component notation these are

$$\alpha\left(\nu^0,\nu^1\right)=\left(\alpha\nu^0,\alpha\nu^1\right), \tag{2.1}$$

$$\left(\nu^0,\nu^1\right)+\left(\mu^0,\mu^1\right)=\left(\nu^0+\mu^0,\nu^1+\mu^1\right), \tag{2.2}$$

$$\left\{\left(\nu^0,\nu^1\right),\left(\mu^0,\mu^1\right)\right\}=\nu^0\mu^1-\nu^1\mu^0, \tag{2.3}$$

where $\alpha\in\mathbb{C}$, $\nu^{\mathbf{A}}=\left(\nu^0,\nu^1\right)$ and $\mu^{\mathbf{A}}=\left(\mu^0,\mu^1\right)$ are spin vectors which are elements of spin-space. The first two have analogous operations in the tensor algebra. The third is analogous to the inner product in the tensor algebra. However, the important difference is that it is antisymmetric as opposed to symmetric, as follows from switching the positions of $\nu^A$ and $\mu^A$. In the tensor algebra, the inner product gives rise to a unique symmetric tensor $g_{ab}$. Accordingly, in the spinor algebra there is a unique spinor $\varepsilon_{AB}$, which is antisymmetric

$$\varepsilon_{AB}=-\varepsilon_{BA}, \tag{2.4}$$

which is called the *epsilon* spinor or the *Levi-Civita* spinor. We can write the inner product (2.3) equivalently as

$$\varepsilon_{AB}\nu^A\mu^B. \tag{2.5}$$

Due to the antisymmetry of $\varepsilon_{AB}$, the spinor norm, which is the special case of the inner product of a spin-vector with itself, vanishes. This is in accordance with the fact that spin-vectors are closely connected to null vectors. Finally, there is also the outer product written as follows

$$\nu^{A\dots}_{B\dots}\mu^{C\dots}_{D\dots}, \tag{2.6}$$

which in the abstract index approach is commutative as well as associative and distributive over addition. A particular case of outer multiplication, when one of the spinors is a scalar, is scalar multiplication (2.1).

In order to incorporate tensors into the spinor algebra, we need to include the complex conjugate of a spin-vector. Therefore, we require the operation of complex conjugation

$$\overline{\nu^A} = \bar{\nu}^{A'}, \tag{2.7}$$

which is denoted by a bar over the entire symbol, including the index. The label $A'$ may be regarded as the complex conjugate of the label $A$. The complex conjugate is obtained by replacing all unprimed indices with primed indices and vice versa. We also include a bar over the kernel symbol, whereas some authors omit this. In the case of a scalar, (2.7) is equivalent to the standard complex conjugacy relation. An exception to this convention occurs in the case of the conjugate epsilon spinor $\varepsilon_{A'B'} = -\varepsilon_{B'A'}$, where the over-bar is usually omitted. The condition for a spinor $\xi_{AB'}{}^{CD'}$ to be *Hermitian* is

$$\bar{\xi}_{AB'}{}^{CD'} = \xi_{AB'}{}^{CD'}, \tag{2.8}$$

whereas the term *real* is reserved for when all the spinor indices can be paired off, with one spinor index primed and the other unprimed

$$\bar{\xi}_{AA'}{}^{CC'} = \xi_{AA'}{}^{CC'}. \tag{2.9}$$

The Riemannian metric tensor is a real symmetric two-index tensor, and its relation to the epsilon tensors is

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}. \tag{2.10}$$

From (2.10), we see that $\varepsilon_{AB}$ will play a similar role to the metric tensor of the tensor calculus, however, there are important differences arising from its antisymmetry. It is two-dimensional

$$\varepsilon_A{}^A = 2, \tag{2.11}$$

and raises and lowers spinor indices in the following way

$$\begin{aligned} \varepsilon_{AB}\nu^A &= \nu_B, \\ \varepsilon^{AB}\nu_B &= \nu^A, \\ \varepsilon^{AB}\varepsilon_{CB} &= \varepsilon_C{}^A, \end{aligned} \tag{2.12}$$

where $\nu^A$ is an arbitrary spinor and $\varepsilon^{AB} = -\varepsilon^{BA}$. For raising and lowering primed indices, analogous relations hold for the conjugate spinor. We note the minus sign in the following *see-saw* property

$$\nu_A\mu^A = -\nu^A\mu_A, \tag{2.13}$$

which stands in contrast to the contraction of tensor indices, where a plus sign appears instead. The two-spinor formalism can be used as a practical alternative to the tensor calculus because all tensor indices can be eliminated in favour of spinor indices. In fact, tensors are a particular type of spinor; specifically a spinor whose indices always occur in pairs, one of which is primed, and the other unprimed. That is, we can correlate an abstract tensor index, $a$, to a pair of abstract spinor indices, $AA'$, cf. (1.4). Therefore, the tensor algebra is embedded in the spinor algebra, moreover, the relationship between the algebras can be understood from the following correspondence. Let $V^a$ be a real four-vector in Minkowski spacetime with components

$$V^{\mathbf{a}} = (V^0, V^1, V^2, V^3) \tag{2.14}$$

in some orthonormal frame. Furthermore, let $V^{AA'}$ be the Hermitian matrix with components

$$V^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}. \tag{2.15}$$

Then we have

$$\det\left(X^{AA'}\right) = \tfrac{1}{2}\eta_{ab}V^aV^b, \tag{2.16}$$

where $\eta_{ab}$ is the Minkowski metric. Equation (2.15) gives a real linear correspondence between four-vectors and $2\times 2$ Hermitian matrices. If we multiply (2.15) on the left by a $2\times 2$ complex valued matrix with unit determinant and on the right by its Hermitian conjugate, we have

$$V^{AA'} \to L^A{}_B V^{BB'} \bar{L}^{A'}{}_{B'}, \tag{2.17}$$

where $\left(L^A{}_B\right) \in SL(2,\mathbb{C})$ and $\bar{L}^{A'}{}_{B'} = \overline{L^A{}_B}$. The right hand side of (2.17) is again a Hermitian matrix, and the determinant is invariant. According to (2.15), the transformation (2.17) defines a corresponding linear transformation on the vector

$$V^a \to \Lambda^a{}_b V^b. \tag{2.18}$$

According to (2.16), the transformation (2.18) preserves its length, hence it defines a Lorentz transformation. Every proper orthochronous Lorentz transformation can be represented by (2.17). The isomorphism from $SL(2,\mathbb{C})$ to the identity component of the Lorentz group is two-to-one; the identity Lorentz transformation is represented by both the identity and its minus in $SL(2,\mathbb{C})$[83, 112, 52].

One of the reasons why operations in the two-spinor formalism tend to be simpler than in the tensor formalism is because one only needs to deal with symmetric spinors, which are separately symmetric in all primed and unprimed indices. In order to show this in the simplest case of a two index spinor $\kappa_{CD}$, consider the following identity satisfied by the epsilon tensor

$$\varepsilon_{A[B}\varepsilon_{CD]} = 0, \tag{2.19}$$

where square brackets denote antisymmetrisation over the enclosed indices. This identity is easily seen to be true by the following argument. Spinor indices can only take two values, one value must therefore appear twice in the antisymmetrisation, hence the result must vanish. Expanding out the bracket in (2.19), we find the equivalent relation

$$\varepsilon_{AB}\varepsilon_{CD} + \varepsilon_{AC}\varepsilon_{DB} + \varepsilon_{AD}\varepsilon_{BC} = 0. \tag{2.20}$$

If the indices $C$ and $D$ are raised according to (2.12), we can rearrange (2.20) to find

$$\varepsilon_A{}^C\varepsilon_B{}^D - \varepsilon_A{}^D\varepsilon_B{}^C = \varepsilon_{AB}\varepsilon^{CD}. \tag{2.21}$$

Contracting (2.21) with $\kappa_{CD}$ yields

$$\kappa_{AB} - \kappa_{BA} = \varepsilon_{AB}\kappa_C{}^C. \tag{2.22}$$

Hence, the antisymmetric part of $\kappa_{AB}$ is determined by the trace

$$\kappa_{[AB]} = \frac{1}{2}\varepsilon_{AB}\kappa_C{}^C, \tag{2.23}$$

where the factor of a half arises due to the two-dimensionality of spin-space (2.11). Therefore, we can write $\kappa_{AB}$ as the sum of its irreducible parts,

$$\kappa_{AB} = \kappa_{(AB)} + \frac{1}{2}\varepsilon_{AB}\kappa_C{}^C, \tag{2.24}$$

where round brackets denote symmetrisation over the enclosed indices. The resulting sum consists of a symmetric spinor and the epsilon spinor multiplied by (half) the trace. This result generalises to any multivalent spinor with both primed and unprimed indices. In the general case, it can be decomposed into a sum of a symmetric spinor and outer products of the epsilon tensor with symmetric spinors of lower valence [89]. Due to this result, tensor symmetries are simply represented in the two-spinor formalism. A striking example of this simplification happens in the case of the Weyl or Conformal tensor $C^a{}_{bcd}$, which has the somewhat complicated symmetries

$$C_{abcd} = C_{[ab]cd} = C_{ab[cd]} = C_{cdab}, \qquad C_{a[bcd]} = 0, \qquad C^a{}_{bad} = 0. \tag{2.25}$$

In the two-spinor formalism, the Weyl tensor is represented as

$$C_{abcd} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}, \tag{2.26}$$

where

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \qquad \bar{\Psi}_{A'B'C'D'} = \bar{\Psi}_{(A'B'C'D')}, \tag{2.27}$$

which are referred to as the Weyl conformal spinors. More precisely, $\Psi$ represents the anti-self-dual part of the Weyl tensor, and its complex conjugate $\bar{\Psi}$ represents the self-dual part. Because the Weyl spinors are totally symmetric, the algebraic classification of the Weyl tensor is very systematic in the two-spinor formalism [80, 90].

# Chapter 3

# Geometrical picture of a spin-vector

Many texts introduce spinors in the context of group representation theory [28, 26]. Often the emphasis is on the algebraic rather than the geometric properties of spinors. Unfortunately, this can give the impression that a geometrical picture of a spinor is either non-existent or not very useful. But in many instances, a concrete geometrical picture is a helpful mental aid and provides a conceptually complementary approach to algebraic methods. In this chapter, we provide the geometrical interpretation of a spin-vector, up to a sign, as a null-flag [89, 107]. This is analogous to the way in which a vector can be pictured as a directed line segment or arrow.

Let $\nu^A$ represent a spin-vector. By forming the outer product with its complex conjugate, we define the vector $N^a = \nu^A \bar{\nu}^A$. This vector $N^a$ is real (equal to its complex conjugate). It is also null, since its norm is zero

$$N_a N^a = \nu_A \bar{\nu}_{A'} \nu^A \bar{\nu}^{A'} = \left| \nu_A \nu^A \right|^2 = 0. \tag{3.1}$$

It is also future pointing, which means it lies on the future-pointing null half-cone. If defined instead with a minus sign, it would correspond to a past-pointing null vector. The two cases are distinguished by the sign of $N^0$. According to (3.1), a spinor uniquely determines a future-pointing null vector, which represents the direction of the flagpole. However, $N^a$ is left invariant by a change of phase

$$\nu^A \to e^{i\theta} \nu^A, \tag{3.2}$$

where $\theta$ is real. In order to interpret the phase geometrically, we consider the real bivector

$$M^{ab} = \nu^A \nu^B \varepsilon^{A'B'} + \bar{\nu}^{A'} \bar{\nu}^{B'} \varepsilon^{AB}, \tag{3.3}$$

where $M^{ab} = \bar{M}^{ab} = -M^{ba}$. Furthermore, it is a null-bivector because it satisfies

$$M_{ab} M^{ab} = 0. \tag{3.4}$$

We can simplify (3.3) by introducing the spin-vector $\lambda^A$ satisfying

$$\nu_A \lambda^A = 1. \tag{3.5}$$

From (3.5) and (2.22), the following relation holds

$$\varepsilon^{AB} = \nu^A \lambda^B - \nu^B \lambda^A, \tag{3.6}$$

which is the definition that $\nu^A$ and $\lambda^A$ constitute a spin-frame. Contracting the indices on (3.6) and using (2.11) gives (3.5). Substituting (3.6) into (3.3), we find

$$M^{ab} = N^a L^b - N^b L^a, \tag{3.7}$$

where $L^a$ is defined as follows

$$L^a = \nu^A \bar{\lambda}^{A'} + \lambda^A \bar{\nu}^{A'}. \tag{3.8}$$

The form of (3.7) defines the null-bivector as *simple* because it can be written as the outer product of two vectors. From (3.8) and (3.5), it follows that the vector $L^a$ satisfies the following

$$L^a = \bar{L}^a, \quad L_a L^a = -2,$$

which says that it is real and spacelike of length $\sqrt{2}$. It is also orthogonal to $N^a$

$$L_a N^a = 0.$$

The vector $L^a$ is not uniquely determined by $M^{ab}$ because $\lambda^A$ is not uniquely determined. This is because the transformations

$$\lambda^A \to \lambda^A + \alpha \nu^A, \tag{3.9}$$

leave (3.5) invariant. From (3.8) and (3.9), it follows that $L^a$ transforms as

$$L^a \to L^a + (\alpha + \bar{\alpha}) N^a. \tag{3.10}$$

Therefore, $M^{ab}$ determines a two-space given by the set of vectors

$$a N^a + b L^a, \tag{3.11}$$

where $a$ and $b$ are real numbers. To give significance to the sense of $L$, we restrict $b > 0$. Therefore, the two-space is actually a half-plane. Because it contains only one null direction, namely that of the flagpole $N^a$, it is by definition a null half-plane and is tangent to the null cone along the direction $N^a$. This null half-plane is called the *flag plane* of $\nu^A$, where the direction of the flag is $L^a$, see Fig. 3.1.

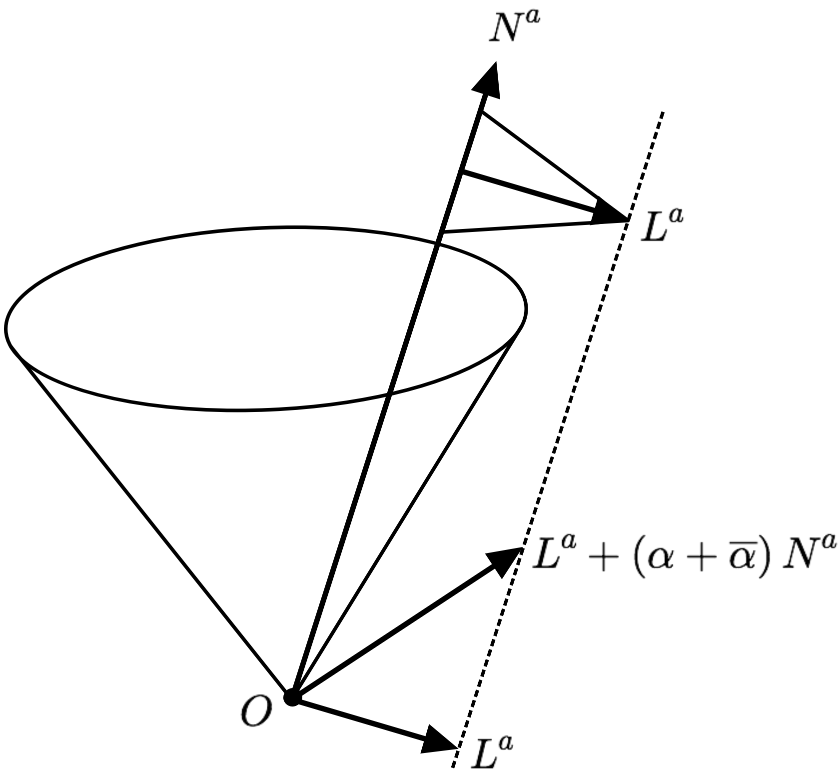


Figure 3.1: The representation of $\nu^A$ as a null flag, exhibiting its relation to the vectors $N^a$ and $L^a$.

We now show how the phase transformation (3.2) affects the direction of the flag $L^a$. In order to preserve (3.5), it is sufficient if $\lambda^A$ transforms as

$$\lambda^A \to e^{-i\theta}\lambda^A. \tag{3.12}$$

From (3.8) it follows that $L^a$ transforms as

$$L^a \to e^{2i\theta}\nu^A\bar{\lambda}^{A'} + e^{-2i\theta}\lambda^A\bar{\nu}^{A'}. \tag{3.13}$$

If we introduce the vector $W^a$ defined as

$$W^a = i\nu^A\bar{\lambda}^{A'} - i\lambda^A\bar{\nu}^{A'}, \tag{3.14}$$

then we can rewrite (3.13) as

$$L^a \to \cos 2\theta L^a + \sin 2\theta W^a. \tag{3.15}$$

The vector $L^a$ is rotated in the $(L^a, W^a)$-plane by the angle $2\theta$. Therefore, the phase transformation (3.2) corresponds to a rotation by $2\theta$ of the flag plane about the flagpole. The corresponding transformation of $M^{ab}$ is

$$M^{ab} \to \cos 2\theta M^{ab} + {}^*M^{ab} \sin 2\theta, \tag{3.16}$$

where

$${}^*M_{ab} = \tfrac{1}{2} e_{abcd} M^{cd}, \tag{3.17}$$

represents the dual of $M_{ab}$ and where $e_{abcd} = e_{[abcd]}$ is the alternating tensor.[1] Accordingly, the transformation (3.16) is referred to as a duality rotation [67]. Substituting the specific value $\theta = \pi$ into (3.2) gives

$$\nu^A \to -\nu^A, \tag{3.18}$$

whereas it follows from (3.15) that $L^a$ is invariant. Hence, whilst the flag plane undergoes a complete revolution through $2\pi$ returning to its original position, the spin-vector $\nu^A$ changes to its negative. Therefore, the null-flag only determines the spin-vector up to a sign, which is related to the well-known fact that spin transformations provide a double-valued representation of Lorentz transformations, cf. (2.17) and (2.18) and the paragraph that follows these equations.

[1] Also known as the four-dimensional Levi-Civita tensor.

# Chapter 4

# Spinor analysis

The basic concepts of spinor analysis are the spinor covariant derivative and the curvature spinors. A quick way to derive these is simply to take the tensor versions and translate them into spinor form according to a standard procedure [89]. Let us then first introduce the basic definitions and relations of Riemannian geometry using the tensor formalism [96]. Denote by $\nabla_a$ the covariant derivative (or connection) which defines covariant differentiation. It satisfies

$$\nabla_a \left(V^b + W^b\right) = \nabla_a V^b + \nabla_a W^b, \tag{4.1}$$

$$\nabla_a \left(f V^b\right) = f \nabla_a V^b + V^b \nabla_a f, \tag{4.2}$$

where $\nabla_a f$ is the ordinary gradient of a scalar $f$. Riemannian geommetry introduces a symmetric non-singular tensor, $g_{ab} = g_{ba}$, called the metric. The covariant derivative is required to be *metric-compatible*,

$$\nabla_a g_{bc} = 0, \tag{4.3}$$

and *torsion-free*

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 0. \tag{4.4}$$

The terminology stems from the fact that (4.4) is equivalent to the vanishing of the *torsion tensor*. If (4.3) and (4.4) are satisfied, the connection is uniquely defined by the metric. We use the following definition of the Riemann tensor in terms of a metric-compatible and torsion-free covariant derivative

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^d = R_{abc}{}^d V^c, \tag{4.5}$$

where $V^a$ is an arbitrary vector. With all indices lowered, it has the following symmetries

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}, \quad R_{[abc]d} = 0. \tag{4.6}$$

The Ricci tensor is formed from the contraction of the Riemann tensor

$$R_{ab} = R_{acb}{}^c = R_{ba}, \tag{4.7}$$

and the Ricci scalar is formed from the contraction of the Ricci tensor

$$R \equiv R_a{}^a = g^{ab} R_{ab}. \tag{4.8}$$

The Weyl tensor is that part of the Riemann tensor which has all the trace parts removed

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - 2R_{[a}{}^{[c} g_{b]}{}^{d]} + \frac{1}{3} R g_{[a}{}^{c} g_{b]}{}^{d}. \tag{4.9}$$

As well as being traceless, it shares all the symmetries of the Riemann tensor, cf. (4.6).
In the spinor calculus, the covariant derivative is written as $\nabla_{AA'}$ and satisfies the following properties when acting on spinors

$$\nabla_{AA'} \left(\xi^B + \omega^B\right) = \nabla_{AA'} \xi^B + \nabla_{AA'} \omega^B, \tag{4.10}$$

$$\nabla_{AA'} \left(f \xi^B\right) = f \nabla_{AA'} \xi^B + \xi^B \nabla_{AA'} f, \tag{4.11}$$

which will agree with (4.1) when acting on tensors. If we demand that the spinor covariant derivative is torsion-free

$$(\nabla_{AA'} \nabla_{BB'} - \nabla_{BB'} \nabla_{AA'}) f = 0, \tag{4.12}$$

and that $\varepsilon_{AB}$ be covariantly constant

$$\nabla_{AA'} \varepsilon_{BC} = 0, \tag{4.13}$$

then it will be uniquely defined. Also, we demand that covariant differentiation commutes with complex conjugation

$$\overline{\nabla_a \xi^{\cdots}_{\cdots}} = \nabla_a \bar{\xi}^{\cdots}_{\cdots}, \tag{4.14}$$

which follows from the following relations

$$\nabla_{AA'} \xi^{B'} = \overline{\nabla_{AA'} \bar{\xi}^B}, \qquad \nabla_{AA'} \omega_{B'} = \overline{\nabla_{AA'} \bar{\omega}_B}. \tag{4.15}$$

Hence, the covariant derivative $\nabla_a$ is a real operator: $\bar{\nabla}_a = \nabla_a$. The Riemann curvature tensor followed from the covariant derivative via (4.5), similarly the curvature spinors follow from the spinor covariant derivative. However, it turns out to be more straightforward to translate the curvature tensor directly into spinor form as follows

$$\begin{aligned} R_{abcd} =& \Psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD} \\ &+ \Phi_{ABC'D'} \varepsilon_{A'B'} \varepsilon_{CD} + \bar{\Phi}_{A'B'CD} \varepsilon_{AB} \varepsilon_{C'D'} \\ &+ 2\Lambda \left(\varepsilon_{AC} \varepsilon_{B'D'} \varepsilon_{A'B'} \varepsilon_{C'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'D'} \varepsilon_{B'D'}\right), \end{aligned} \tag{4.16}$$

where $\Psi_{ABCD}$ is the Weyl conformal spinor, $\Phi_{ABC'D'}$ is the Ricci spinor, and $\Lambda$ is proportional to the Ricci scalar. In fact we have

$$R = 24\Lambda, \tag{4.17}$$

hence $\bar{\Lambda} = \Lambda$. Moreover, the Ricci spinor is proportional to the traceless Ricci tensor

$$R_{ab} - \frac{1}{4}Rg_{ab} = -2\Phi_{ABA'B'} = -2\Phi_{ab}, \tag{4.18}$$

hence $\bar{\Phi}_{ABA'B'} = \Phi_{A'B'AB}$ is Hermitian. The Weyl spinor is related to the Weyl tensor as in (2.26)

$$C_{abcd} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}, \tag{4.19}$$

where $\overline{\Psi_{ABCD}} = \bar{\Psi}_{A'B'C'D'}$ is therefore a complex spinor. All the curvature spinors are symmetric in their respective primed and unprimed spinor indices. This is equivalent to the statement that (4.16) represents a decomposition of the Riemann tensor into its irreducible spinor parts under the group $SL(2, \mathbb{C})$.

The existence of spinors requires only a conformal structure. Moreover, the conformal invariance of a set of field equations follows more straightforwardly if expressed in spinor form. Consider a *conformal rescaling*

$$g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab}, \tag{4.20}$$

where $\Omega$ is a scalar field which is everywhere positive. In order to preserve (2.10) we have

$$\varepsilon_{AB} \mapsto \hat{\varepsilon}_{AB} = \Omega\varepsilon_{AB}. \tag{4.21}$$

More generally, we define $\theta$ (spinor indices suppressed) to be a *conformal density* of weight $k$ if it transforms under (4.20) as

$$\hat{\theta} = \Omega^k\theta. \tag{4.22}$$

Hence, $g_{ab}$, $\varepsilon_{AB}$ would have respective conformal weights 2 and 1. A set of field equations is defined to be *conformally invariant* if it is possible to assign a conformal weight to all field quantities, such that the field equations still hold after the conformal rescaling (4.20). In order to know how a set of field equations transforms under (4.20), we need to know how the covariant derivative transforms. This can be derived from the requirement that both the covariant derivative and the conformally rescaled covariant derivative are torsion-free and satisfy

$$\begin{aligned} \nabla_{AA'}\varepsilon_{AB} &= 0, \\ \hat{\nabla}_{AA'}\hat{\varepsilon}_{AB} &= 0. \end{aligned} \tag{4.23}$$

From (4.23), assuming $\Omega$ is real, the two covariant derivatives are related as follows,

$$\begin{aligned}\hat{\nabla}_{AA'}\xi_{B\ldots D'}{}^{P\ldots R'} =& \nabla_{AA'}\xi_{B\ldots D'}{}^{P\ldots R'} - \Upsilon_{BA'}\xi_{A\ldots D'}{}^{P\ldots R'} - \ldots - \Upsilon_{AD'}\xi_{B\ldots A'}{}^{P\ldots R'} \\ &+ \varepsilon_{A}{}^{P}\Upsilon_{XA'}\xi_{B\ldots D'}{}^{X\ldots R'} + \ldots + \varepsilon_{A'}{}^{R'}\Upsilon_{AX'}\xi_{B\ldots D'}{}^{P\ldots X'},\end{aligned} \tag{4.24}$$

where $\xi_{B\ldots D'}{}^{P\ldots R'}$ is an arbitrary spinor and

$$\Upsilon_{AA'} = \Omega^{-1}\nabla_{AA'}\Omega. \tag{4.25}$$

In the special case when the spinor indices of $\xi$ can be grouped into tensor indices

$$\xi_{BB'\ldots D'D'}{}^{PP'\ldots RR'} = \xi_{b\ldots d}{}^{p\ldots r}, \tag{4.26}$$

we have

$$\begin{aligned}\hat{\nabla}_{a}\xi_{b\ldots d}{}^{p\ldots r} =& \nabla_{a}\xi_{b\ldots d}{}^{p\ldots r} + U_{ax}{}^{p}\xi_{b\ldots d}{}^{x\ldots r} + \ldots + U_{ax}{}^{r}\xi_{b\ldots d}{}^{p\ldots x} \\ &- U_{ab}{}^{x}\xi_{x\ldots d}{}^{p\ldots r} - \ldots - U_{ad}{}^{x}\xi_{b\ldots x}{}^{p\ldots r},\end{aligned} \tag{4.27}$$

where

$$U_{ab}{}^{c} = 2\Upsilon_{(a}g_{b)}{}^{c} - g_{ab}\Upsilon^{c}. \tag{4.28}$$

Due to the symmetrised term, (4.28) is in fact composed of three terms. Consequently, if each term in (4.27) is written out explicitly, it contains many more terms that its equivalent spinor transformation law (4.24). This fact makes proofs of conformal invariance more straightforward in the two-spinor formalism [89].

# Chapter 5

# Spin-coefficient formalism

When explicit calculations are required, all tensor quantities can be expressed in terms of their components with respect to a coordinate system. This requires choosing a coordinate basis (or natural basis) for the tangent space, which is provided by the partial derivatives with respect to the coordinates at a point. If a non-coordinate basis is chosen instead, one introduces a tetrad, which is a set of four linearly independent real vector-fields. This widely used method is called the *tetrad formalism*. Often an orthonormal-tetrad is employed, however, a null-tetrad is more closely related to the spinor calculus and leads to a *Ricci-rotation* type formalism called the NP *spin-coefficient formalism* [71]. Let us therefore introduce a null-tetrad of vectors $l^a$, $n^a$, $m^a$, $\bar{m}^a$,

$$e_{\mathbf{m}}{}^{a} = \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}, \tag{5.1}$$

satisfying

$$\eta_{\mathbf{mn}} = g_{ab} e_{\mathbf{m}}{}^{a} e_{\mathbf{n}}{}^{b}, \tag{5.2}$$

where $\eta_{\mathbf{mn}}$ are the components of the following matrix

$$\eta_{\mathbf{mn}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.3}$$

The null-tetrad can be used to take components of an arbitrary vector in the following way

$$V^{\mathbf{a}} = e_{a}{}^{\mathbf{a}} V^{a}. \tag{5.4}$$

The analogous object for the spinor calculus is a dyad, which is a pair of linearly independent complex vector-fields,

$$\varepsilon_{\mathbf{A}}{}^{A} = (o^A, \iota^A), \tag{5.5}$$

which satisfies the following relations

$$\begin{aligned} \varepsilon_{AB} o^A o^B &= o_A o^A = 0, \\ \varepsilon_{AB} \iota^A \iota^B &= \iota_A \iota^A = 0, \\ \varepsilon_{AB} o^A \iota^B &= o_A \iota^A = 1. \end{aligned} \tag{5.6}$$

The third relation provides a normalisation condition, in which case the dyad is referred to as a spin-frame, cf. (3.6). The dyad may be used to take components of an arbitrary spinor with unprimed indices in the following way

$$\kappa^{\mathbf{A}} = \varepsilon_{A}{}^{\mathbf{A}} \kappa^A. \tag{5.7}$$

For taking components of an arbitrary spinor with primed indices, analogous relations hold for the conjugate dyad. We can express the null tetrad (5.1) in terms of the dyad (5.5) and its conjugate as follows

$$\begin{aligned} l^a &= o^A o^{A'}, \\ n^a &= \iota^A \iota^{A'}, \\ m^a &= o^A \iota^{A'}, \\ \bar{m}^a &= \iota^A o^{A'}. \end{aligned} \tag{5.8}$$

In terms of a spin-frame, twelve complex *spin-coefficients* are defined as follows

$$\begin{aligned} &\kappa = o^A D o_A, && \gamma' = -\iota^A D o_A, && \tau' = -\iota^A D \iota_A, \\ &\rho = o^A \delta' o_A, && \beta' = -\iota^A \delta' o_A, && \sigma' = -\iota^A \delta' \iota_A, \\ &\sigma = o^A \delta o_A, && \beta = \iota^A \delta o_A, && \rho' = -\iota^A \delta \iota_A, \\ &\tau = o^A D' o_A, && \gamma = \iota^A D' o_A, && \kappa' = -\iota^A D' \iota_A, \end{aligned} \tag{5.9}$$

where the *intrinsic derivatives* are the components of the vector covariant derivative in the spinor basis

$$\begin{aligned} D &= o^A o^{A'} \nabla_{AA'}, \\ \delta &= o^A \iota^{A'} \nabla_{AA'}, \\ \delta' &= \iota^A o^{A'} \nabla_{AA'}, \\ D' &= \iota^A \iota^{A'} \nabla_{AA'}. \end{aligned} \tag{5.10}$$

The complex conjugate relations of (5.9) and (5.10) follow by replacing the dyad by its complex conjugate and the fact that the covariant derivative is real (4.14). The significance of the primed symbols is that under a discrete transformation called the *prime operation*, defined below in (5.24), primed symbols become unprimed and vice versa, thus an economy of notation is achieved. Let us also define the following standard shorthand, $\xi_{r,t}$, for the components of a spinor $\xi_{A...D...G'...K'}$ [98, 89]

$$\xi_{r,t} = \xi_{A...D...G'...K'} \underbrace{o^A ..}_{r'} \underbrace{l^D ..}_{r} \underbrace{o^{G'} ..}_{t'} \underbrace{l^{K'} ..}_{t} . \tag{5.11}$$

A particular example of (5.11) for the Weyl and Ricci spinors would be

$$\Psi_2 = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D, \quad \Phi_{11} = \Phi_{ABA'B'}\, o^A \iota^B o^{A'} \iota^{B'}. \tag{5.12}$$

Moreover, the Ricci spinor is real, therefore, we have the following conjugate transpose relations between the components

$$\Phi_{rt} = \bar{\Phi}_{tr}. \tag{5.13}$$

## 5.1 Geroch, Held, Penrose formalism

The GHP formalism [45] is a calculus based on a pair of null directions. It was introduced for situations where two null vectors are physically distinguished by the problem under consideration, but a complete basis is not [45]. More relevant to our purposes, is that the compacted expressions are, in the majority of cases, considerably simpler than their counterparts in the original scheme. This alone justifies our use of the GHP formalism, therefore, the equations may be legitimately regarded as shorthand expressions for full spin-coefficient formulae.
The most general change of spin-frame which leaves two null directions invariant is

$$o^A \mapsto \lambda o^A, \quad \iota^A \mapsto \lambda^{-1} \iota^A, \tag{5.14}$$

where $\lambda$ is an arbitrary (nowhere vanishing) complex scalar field. From (5.14) it follows that

$$o_A \mapsto \lambda o_A, \quad \iota_A \mapsto \lambda^{-1} \iota_A, \tag{5.15}$$

and that

$$o^{A'} \mapsto \bar{\lambda} o^{A'}, \quad \iota^{A'} \mapsto \bar{\lambda}^{-1} \iota^{A'}, o_{A'} \mapsto \bar{\lambda} o_{A'}, \quad \iota_{A'} \mapsto \bar{\lambda}^{-1} \iota_{A'}. \tag{5.16}$$

The formalism deals with scalars $\eta$ associated with a spin-frame or null tetrad, where the scalars transform in the following way

$$\eta \mapsto \lambda^p \bar{\lambda}^q \eta, \tag{5.17}$$

whenever the dyad transforms as in (5.14), where $p$ and $q$ are defined in terms of $r$, $r'$, $t$ and $t'$ from (5.11) as

$$p = r' - r, \qquad q = t' - t. \tag{5.18}$$

A scalar transforming according to (5.17) is called a weighted scalar of type $(p, q)$. We have the following weighted scalars

$$\begin{array}{ll} \Psi_r & (4-2r, 0), \\ \Phi_{rt} & (2-2r, 2-2t), \\ \Lambda & (0,0), \\ \rho & (1,1), \\ \tau & (1,-1), \\ \kappa & (3,1), \\ \sigma & (3,-1), \end{array} \tag{5.19}$$

where $24\Lambda = R$. The remaining weighted scalars are obtained by priming, conjugating or performing both on (5.19), where the prime of a $(p, q)$ scalar is of type $(-p, -q)$, c.f. (5.2), and the complex conjugate of a $(p, q)$ scalar is of type $(q, p)$. As an example let us see how $\sigma$ transforms,

$$\begin{aligned} \sigma &= o^A \delta o_A = o^A o^B \iota^{B'} \nabla_{BB'} o_A \\ &\to \lambda o^A \lambda o^B \bar{\lambda}^{-1} \iota^{B'} \nabla_{BB'} (\lambda o_A) \\ &= \lambda^2 \bar{\lambda}^{-1} o^A o_A o^B \iota^{B'} \nabla_{BB'} \lambda + \lambda^3 \bar{\lambda}^{-1} o^A o^B \iota^{B'} \nabla_{BB'} o_A \\ &= \lambda^3 \bar{\lambda}^{-1} \sigma, \end{aligned} \tag{5.20}$$

where the first term of the second to last line is zero due to (5.6).
The spin coefficients of the middle column of (5.9) are not weighted, nor are the intrinsic derivatives (5.10). They are therefore combined to give the following weighted derivative operators when acting on a $(p, q)$ scalar [45],[1]

$$\begin{array}{ll} þ = D + p\gamma' + q\bar{\gamma}' & (1,1), \\ þ' = D' - p\gamma - q\bar{\gamma} & (-1,-1), \\ ð = \delta - p\beta + q\bar{\beta}' & (1,-1), \\ ð' = \delta' + p\beta' - q\bar{\beta} & (-1,1). \end{array} \tag{5.21}$$

The differential operators (5.21) are of weight $(p, q)$ in the sense that acting on a scalar of type $(u, v)$ produces a scalar of type $(u + p, v + q)$. In terms of (5.21), the following relations follow from (5.9)

$$\begin{array}{ll} þ o^A = -\kappa l^A, & þ l^A = -\tau' o^A, \\ ð' o^A = -\rho l^A, & ð' l^A = -\sigma' o^A, \\ ð o^A = -\sigma l^A, & ð l^A = -\rho' o^A, \\ þ' o^A = -\tau l^A, & þ' l^A = -\kappa' o^A. \end{array} \tag{5.22}$$

Let $\xi_{A \ldots D \ldots G' \ldots K'}$ be a symmetric spinor with components defined by (5.11). Making use of (5.22) and their complex conjugate relations, the corresponding components of the intrinsic

[1] The symbol þ is pronounced 'thorn' and ð is pronounced 'eth'.

derivatives of $\xi_{A\ldots D\ldots G'\ldots K'}$ may be derived

$$\begin{aligned}
(o^A \ldots l^D \ldots o^{G'} \ldots l^{K'} \ldots) D\xi_{A\ldots D\ldots G'\ldots K'} =& \text{þ}\xi_{r,t} + r'\kappa\xi_{r+1,t} + r\tau'\xi_{r-1,t} \\
&+ t'\bar{\kappa}\xi_{r,t+1} + t\bar{\tau}'\xi_{r,t-1}, \\
(o^A \ldots l^D \ldots o^{G'} \ldots l^{K'} \ldots) \delta\xi_{A\ldots D\ldots G'\ldots K'} =& \eth\xi_{r,t} + r'\sigma\xi_{r+1,t} + r\rho'\xi_{r-1,t} \\
&+ t'\bar{\rho}\xi_{r,t+1} + t\bar{\sigma}'\xi_{r,t-1}, \\
(o^A \ldots l^D \ldots o^{G'} \ldots l^{K'} \ldots) \delta'\xi_{A\ldots D\ldots G'\ldots K'} =& \eth'\xi_{r,t} + r'\rho\xi_{r+1,t} + r\sigma'\xi_{r-1,t} \\
&+ t'\bar{\sigma}\xi_{r,t+1} + t\bar{\rho}'\xi_{r,t-1}, \\
(o^A \ldots l^D \ldots o^{G'} \ldots l^{K'} \ldots) D'\xi_{A\ldots D\ldots G'\ldots K'} =& \text{þ}'\xi_{r,t} + r'\tau\xi_{r+1,t} + r\kappa'\xi_{r-1,t} \\
&+ t'\bar{\tau}\xi_{r,t+1} + t\bar{\kappa}'\xi_{r,t-1}.
\end{aligned} \tag{5.23}$$

Equations (5.23) are the key equations which provide an efficient way of translating differential spinor equations into the GHP formalism.

## 5.2 Prime and asterisk operations

The significance of the *primed* symbols, e.g. the primed spin-coefficients occurring in (5.9), is that under the prime operation defined as

$$' : o^A \mapsto i\iota^A, \quad \iota^A \mapsto io^A, \quad o^{A'} \mapsto -i\iota^{A'}, \quad \iota^{A'} \mapsto -io^{A'}, \tag{5.24}$$

which preserves the normalisation condition (5.6), the primed and unprimed spin-coefficients are interchanged. From (5.8), we see that under the prime operation, the null tetrad transforms as follows

$$l^a \mapsto n^a \quad n^a \mapsto l^a \quad m^a \mapsto \bar{m}^a \quad \bar{m}^a \mapsto m^a. \tag{5.25}$$

Furthermore, the curvature spinor components transform as

$$\begin{aligned}
&\Psi_r \mapsto \Psi_s \quad 0 \leftrightarrow 4 \quad 1 \leftrightarrow 3 \quad 2 \leftrightarrow 2, \\
&\Phi_{rs} \mapsto \Phi_{tu} \quad 0 \leftrightarrow 2 \quad 1 \leftrightarrow 1.
\end{aligned} \tag{5.26}$$

Note that the operation of complex conjugation, which interchanges primed indices for unprimed indices, commutes with the prime operation, such that $(\bar{\alpha})' = \overline{(\alpha')}$.
There exists another discrete symmetry possessed by the spin-coefficient formalism, called the asterisk (*) operation, which effects the following transformation

$$* : o^A \mapsto o^A, \quad \iota^A \mapsto \iota^A, \quad o^{A'} \mapsto \iota^{A'}, \quad \iota^{A'} \mapsto -o^{A'}. \tag{5.27}$$

From (5.27) and the definitions (5.8), (5.9), (5.10), (5.11), the corresponding transformation under the asterisk operation of the null tetrad, spin-coefficients, intrinsic derivatives and spinor components follow respectively. The prime and asterisk operations may be used to generate new equations from equations already known, or alternatively to check the correctness and consistency of equations found by other means.

## 5.3 Newman-Penrose equations

Equation (4.5) defines the Riemann tensor in terms of the commutator of the covariant derivatives. In the spinor formalism there is a similar relation when such commutators are applied to spinors

$$\begin{aligned}&(\varepsilon_{A'B'}\nabla_{C'(A}\nabla_{B)}{}^{C'} + \varepsilon_{AB}\nabla_{C(A'}\nabla_{B')}{}^{C})\kappa^D \\ &= (\varepsilon_{A'B'}\Psi_{ABD}{}^{C} + \Lambda\varepsilon_{A'B'}(\varepsilon_{AD}\varepsilon_B{}^C + \varepsilon_A{}^C\varepsilon_{BD}) + \varepsilon_{AB}\Phi_{A'B'D}{}^{C})\kappa^D,\end{aligned} \tag{5.28}$$

where $\kappa^D$ is an arbitrary spinor. Let us set $\kappa^D = \varepsilon_{\mathbf{D}}{}^D$ and take components of (5.28). Using (5.23), we then obtain the NP equations in the GHP formalism [89]

$$þ\rho - ð'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa + \Phi_{00}, \tag{5.29a}$$

$$þ\sigma - ð\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \tag{5.29b}$$

$$þ\tau - þ'\kappa = (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Psi_1 + \Phi_{01}, \tag{5.29c}$$

$$ð\rho - ð'\sigma = (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \Psi_1 + \Phi_{01}, \tag{5.29d}$$

$$ð\tau - þ'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \Phi_{02}, \tag{5.29e}$$

$$þ'\rho - ð'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 - 2\Lambda. \tag{5.29f}$$

Applying the prime operation to (5.29), we obtain six more relations. Equations (5.29) and their primed partners are equivalent to the Ricci identities of the tensor calculus [96].
The remaining equations of the NP formalism take the form of commutator expressions. To derive them, consider the following commutator of intrinsic derivatives acting on an arbitrary scalar $f$

$$[\nabla_{\mathbf{AB'}}, \nabla_{\mathbf{CD'}}]\, f = \Big(\nabla_{\mathbf{AB'}}(\varepsilon_{\mathbf{C}}{}^{Q}\varepsilon_{\mathbf{D'}}{}^{Q'}) - \nabla_{\mathbf{CD'}}(\varepsilon_{\mathbf{A}}{}^{Q}\varepsilon_{\mathbf{B'}}{}^{Q'})\Big)\, \nabla_{QQ'} f. \tag{5.30}$$

Taking components of (5.30) and using (5.23), the following commutator expressions may be derived

$$
\begin{aligned}
þþ' - þ'þ &= (\bar{\tau} - \tau')ð + (\tau - \bar{\tau}')ð' - p(\kappa\kappa' - \tau\tau' + \Psi_2 + \Phi_{11} - \Lambda) \\
&\quad - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2 + \Phi_{11} - \Lambda), && (5.31a) \\
ðð' - ð'ð &= (\bar{\rho}' - \rho')þ + (\rho - \bar{\rho})þ' + p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda) \\
&\quad - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2 - \Phi_{11} - \Lambda), && (5.31b) \\
þð - ðþ &= \bar{\rho}ð + \sigma ð' - \bar{\tau}'þ - \kappa þ' - p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}' + \Phi_{01}), && (5.31c) \\
þ'ð' - ð'þ' &= \bar{\rho}'ð' + \sigma'ð - \bar{\tau}þ' - \kappa'þ + p(\rho\kappa' - \tau\sigma' + \Psi_3) + q(\bar{\sigma}\bar{\kappa}' - \bar{\rho}'\bar{\tau} + \Phi_{21}), && (5.31d) \\
þð' - ð'þ &= \rho ð' + \bar{\sigma}ð - \tau'þ - \bar{\kappa}þ' - q(\bar{\rho}'\bar{\kappa} - \bar{\tau}'\bar{\sigma} + \bar{\Psi}_1) - p(\sigma'\kappa - \rho\tau' + \Phi_{10}), && (5.31e) \\
þ'ð - ðþ' &= \rho'ð + \bar{\sigma}'ð' - \tau þ' - \bar{\kappa}'þ + q(\bar{\rho}\bar{\kappa}' - \bar{\tau}\bar{\sigma}' + \bar{\Psi}_3) + p(\sigma\kappa' - \rho'\tau + \Phi_{12}). && (5.31f)
\end{aligned}
$$

## 5.4 Spinor form of the Bianchi identity

The Riemann tensor satisfies an important differential identity called the *Bianchi identity*

$$\nabla_{[a} R_{bc]de} = 0, \tag{5.32}$$

which is equivalent to [59]

$$\nabla^{a\,*} R_{abcd} = 0, \tag{5.33}$$

where we define the (left) Hodge dual operation to act on the first and second indices as follows

$${}^{*}R_{abcd} = \tfrac{1}{2} e_{ab}{}^{pq} R_{pqcd}. \tag{5.34}$$

Substituting (4.9) into (5.33) and using (5.34) gives a differential relation between the Weyl tensor, Ricci tensor and Ricci scalar

$$\nabla^a C_{abcd} = \nabla_{[c} R_{d]b} + \tfrac{1}{6} g_{b[c} \nabla_{d]} R. \tag{5.35}$$

Taking the trace of (5.35) gives

$$\nabla^a R_{ab} - \tfrac{1}{2} \nabla_b R = 0. \tag{5.36}$$

The spinor equivalent of (5.35) is

$$\nabla^A_{B'} \Psi_{ABCD} = \nabla^{A'}_B \Phi_{CDA'B'} - 2\varepsilon_{B(C} \nabla_{D)B'} \Lambda. \tag{5.37}$$

In terms of the GHP formalism, the components of (5.37) are given by the following six equations

$$\begin{aligned}
&\text{þ}\Psi_1 - \eth'\Psi_0 - \text{þ}\Phi_{01} + \eth\Phi_{00} \\
&\quad = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 + \bar{\tau}'\Phi_{00} - 2\bar{\rho}\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \bar{\kappa}\Phi_{02}, && \text{(5.38a)} \\
&\text{þ}\Psi_2 - \eth'\Psi_1 - \eth'\Phi_{01} + \text{þ}'\Phi_{00} + 2\text{þ}\Lambda \\
&\quad = \sigma'\Psi_0 - 2\tau'\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + 2\bar{\rho}'\Phi_{00} - 2\bar{\tau}\Phi_{01} - 2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02}, && \text{(5.38b)} \\
&\text{þ}\Psi_3 - \eth'\Psi_2 - \text{þ}\Phi_{21} + \eth\Phi_{20} - 2\eth'\Lambda \\
&\quad = 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_3 - \kappa\Psi_4 - 2\rho'\Phi_{10} + 2\tau'\Phi_{11} + \bar{\tau}'\Phi_{20} - 2\bar{\rho}\Phi_{21} + \bar{\kappa}\Phi_{22}, && \text{(5.38c)} \\
&\text{þ}\Psi_4 - \eth'\Psi_3 - \eth'\Phi_{21} + \text{þ}'\Phi_{20} \\
&\quad = 3\sigma'\Psi_2 - 4\tau'\Psi_3 + \rho\Psi_4 - 2\kappa'\Phi_{10} + 2\sigma'\Phi_{11} + \bar{\rho}'\Phi_{20} - 2\bar{\tau}\Phi_{21} + \bar{\sigma}\Phi_{22}, && \text{(5.38d)} \\
&\text{þ}\Phi_{12} + \text{þ}'\Phi_{01} - \eth\Phi_{11} - \eth'\Phi_{02} + 3\eth\Lambda \\
&\quad = \left(\rho' + 2\bar{\rho}'\right)\Phi_{01} + \left(2\rho + \bar{\rho}\right)\Phi_{12} - \left(\tau' + \bar{\tau}\right)\Phi_{02} - 2\left(\tau + \bar{\tau}'\right)\Phi_{11} \\
&\quad - \bar{\kappa}'\Phi_{00} - \kappa\Phi_{22} + \sigma\Phi_{21} + \bar{\sigma}'\Phi_{10}, && \text{(5.38e)} \\
&\text{þ}\Phi_{11} + \text{þ}'\Phi_{00} - \eth\Phi_{10} - \eth'\Phi_{01} + 3\text{þ}\Lambda \\
&\quad = \left(\rho' + \bar{\rho}'\right)\Phi_{00} + 2\left(\rho + \bar{\rho}\right)\Phi_{11} - \left(\tau' + 2\bar{\tau}\right)\Phi_{01} - \left(2\tau + \bar{\tau}'\right)\Phi_{10} \\
&\quad - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} + \sigma\Phi_{20} + \bar{\sigma}\Phi_{02}, && \text{(5.38f)}
\end{aligned}$$

and their corresponding primed relations.

## 5.5 Calculating spin-coefficients using Cartan's calculus

The spin-coefficients can be calculated efficiently using Cartan's differential calculus [89]. First we take the exterior derivative of the dual basis one-forms

$$\mathrm{d}\mathbf{l} = l_{[\mathbf{b},\mathbf{a}]}e^{\mathbf{ma}}e^{\mathbf{nb}}(\mathbf{e_m} \wedge \mathbf{e_n}), \tag{5.39}$$

$$\mathrm{d}\mathbf{n} = n_{[\mathbf{b},\mathbf{a}]}e^{\mathbf{ma}}e^{\mathbf{nb}}(\mathbf{e_m} \wedge \mathbf{e_n}), \tag{5.40}$$

$$\mathrm{d}\mathbf{m} = m_{[\mathbf{b},\mathbf{a}]}e^{\mathbf{ma}}e^{\mathbf{nb}}(\mathbf{e_m} \wedge \mathbf{e_n}), \tag{5.41}$$

where a comma denotes partial differentiation and

$$\mathbf{e}_1 = \mathbf{l}, \mathbf{e}_2 = \mathbf{n}, \mathbf{e}_3 = \mathbf{m}, \mathbf{e}_4 = \bar{\mathbf{m}}, \tag{5.42}$$

which gives

$$\begin{aligned}
\mathrm{d}\mathbf{l} &= l_{[\mathbf{b},\mathbf{a}]}\left(e^{1\mathbf{a}}e^{2\mathbf{b}}(\mathbf{l}\wedge\mathbf{n}) + e^{1\mathbf{a}}e^{3\mathbf{b}}(\mathbf{l}\wedge\mathbf{m}) + e^{1\mathbf{a}}e^{4\mathbf{b}}(\mathbf{l}\wedge\bar{\mathbf{m}})\right) \\
&\quad + l_{[\mathbf{b},\mathbf{a}]}\left(e^{2\mathbf{a}}e^{3\mathbf{b}}(\mathbf{n}\wedge\mathbf{m}) + e^{2\mathbf{a}}e^{4\mathbf{b}}(\mathbf{n}\wedge\bar{\mathbf{m}}) + e^{3\mathbf{a}}e^{4\mathbf{b}}(\mathbf{m}\wedge\bar{\mathbf{m}})\right), \qquad (5.43a) \\
\mathrm{d}\mathbf{n} &= n_{[\mathbf{b},\mathbf{a}]}\left(e^{1\mathbf{a}}e^{2\mathbf{b}}(\mathbf{l}\wedge\mathbf{n}) + e^{1\mathbf{a}}e^{3\mathbf{b}}(\mathbf{l}\wedge\mathbf{m}) + e^{1\mathbf{a}}e^{4\mathbf{b}}(\mathbf{l}\wedge\bar{\mathbf{m}})\right) \\
&\quad + n_{[\mathbf{b},\mathbf{a}]}\left(e^{2\mathbf{a}}e^{3\mathbf{b}}(\mathbf{n}\wedge\mathbf{m}) + e^{2\mathbf{a}}e^{4\mathbf{b}}(\mathbf{n}\wedge\bar{\mathbf{m}}) + e^{3\mathbf{a}}e^{4\mathbf{b}}(\mathbf{m}\wedge\bar{\mathbf{m}})\right), \qquad (5.43b) \\
\mathrm{d}\mathbf{m} &= m_{[\mathbf{b},\mathbf{a}]}\left(e^{1\mathbf{a}}e^{2\mathbf{b}}(\mathbf{l}\wedge\mathbf{n}) + e^{1\mathbf{a}}e^{3\mathbf{b}}(\mathbf{l}\wedge\mathbf{m}) + e^{1\mathbf{a}}e^{4\mathbf{b}}(\mathbf{l}\wedge\bar{\mathbf{m}})\right) \\
&\quad + m_{[\mathbf{b},\mathbf{a}]}\left(e^{2\mathbf{a}}e^{3\mathbf{b}}(\mathbf{n}\wedge\mathbf{m}) + e^{2\mathbf{a}}e^{4\mathbf{b}}(\mathbf{n}\wedge\bar{\mathbf{m}}) + e^{3\mathbf{a}}e^{4\mathbf{b}}(\mathbf{m}\wedge\bar{\mathbf{m}})\right). \qquad (5.43c)
\end{aligned}$$

The spin-coefficients are defined as follows

$$\begin{aligned}
2\mathrm{d}\mathbf{l} &= (\gamma' + \bar{\gamma}')(\mathbf{l}\wedge\mathbf{n}) + (\bar{\beta} - \beta' - \bar{\tau})(\mathbf{l}\wedge\mathbf{m}) + (\beta - \bar{\beta}' - \tau)(\mathbf{l}\wedge\bar{\mathbf{m}}) \\
&\quad - \bar{\kappa}(\mathbf{n}\wedge\mathbf{m}) - \kappa(\mathbf{n}\wedge\bar{\mathbf{m}}) + (\rho - \bar{\rho})(\mathbf{m}\wedge\bar{\mathbf{m}}), \qquad (5.44a) \\
2\mathrm{d}\mathbf{n} &= -(\gamma + \bar{\gamma})(\mathbf{l}\wedge\mathbf{n}) - \kappa'(\mathbf{l}\wedge\mathbf{m}) - \bar{\kappa}'(\mathbf{l}\wedge\bar{\mathbf{m}}) \\
&\quad + (-\tau' + \beta' - \bar{\beta})(\mathbf{n}\wedge\mathbf{m}) + (-\bar{\tau}' + \bar{\beta}' - \beta)(\mathbf{n}\wedge\bar{\mathbf{m}}) - (\rho' - \bar{\rho}')(\mathbf{m}\wedge\bar{\mathbf{m}}), \qquad (5.44b) \\
2\mathrm{d}\mathbf{m} &= -(\tau - \bar{\tau}')(\mathbf{l}\wedge\mathbf{n}) + (\gamma - \bar{\gamma} - \bar{\rho}')(\mathbf{l}\wedge\mathbf{m}) - \bar{\sigma}'(\mathbf{l}\wedge\bar{\mathbf{m}}) \\
&\quad + (-\gamma' + \bar{\gamma}' - \rho)(\mathbf{n}\wedge\mathbf{m}) - \sigma(\mathbf{n}\wedge\bar{\mathbf{m}}) + (\beta + \bar{\beta}')(\mathbf{m}\wedge\bar{\mathbf{m}}). \qquad (5.44c)
\end{aligned}$$

Comparing (5.44) with (5.43) we obtain

$$\begin{array}{lll}
\gamma' + \bar{\gamma}' = l_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}n^{\mathbf{b}}, & \beta - \bar{\beta}' - \tau = l_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}m^{\mathbf{b}}, & \bar{\beta} - \beta' - \bar{\tau} = l_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}\bar{m}^{\mathbf{b}}, \\
-\bar{\kappa} = l_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}m^{\mathbf{b}}, & -\bar{\kappa} = l_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}\bar{m}^{\mathbf{b}}, & \rho - \bar{\rho} = l_{[\mathbf{b},\mathbf{a}]}m^{\mathbf{a}}\bar{m}^{\mathbf{b}}, \\
-\gamma - \bar{\gamma} = n_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}n^{\mathbf{b}}, & -\kappa' = n_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}m^{\mathbf{b}}, & -\bar{\kappa}' = n_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}\bar{m}^{\mathbf{b}}, \\
-\tau' + \beta' - \bar{\beta} = n_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}m^{\mathbf{b}}, & -\bar{\tau}' + \bar{\beta}' - \beta = n_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}\bar{m}^{\mathbf{b}}, & -\rho' + \bar{\rho}' = n_{[\mathbf{b},\mathbf{a}]}m^{\mathbf{a}}\bar{m}^{\mathbf{b}}, \\
-\tau + \bar{\tau}' = m_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}n^{\mathbf{b}}, & \gamma - \bar{\gamma} - \bar{\rho}' = m_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}m^{\mathbf{b}}, & -\bar{\sigma}' = m_{[\mathbf{b},\mathbf{a}]}l^{\mathbf{a}}\bar{m}^{\mathbf{b}}, \\
-\gamma' + \bar{\gamma}' - \rho = m_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}m^{\mathbf{b}}, & -\sigma = m_{[\mathbf{b},\mathbf{a}]}n^{\mathbf{a}}\bar{m}^{\mathbf{b}}, & \beta + \bar{\beta}' = m_{[\mathbf{b},\mathbf{a}]}m^{\mathbf{a}}\bar{m}^{\mathbf{b}}.
\end{array} \qquad (5.45)$$

Equations (5.45) contain only 24 independent real equations, thus they enable us to solve uniquely for the twelve complex spin-coefficients in terms of the null tetrad [32]. All the formulae hitherto are well-known and can be found in the literature [107, 89, 100, 52, 74].

# Chapter 6

# The Bach equations

In the hope to describe more varied phenomena, the theory of General Relativity has been generalised in various different ways. To name just a few, including torsion in the connection, scalar-tensor theories and the inclusion of higher-order curvature terms in the Lagrangian, see [25] for an overview. An example of the last was proposed by Bach in 1921 [8], where he introduced a set of field equations now called the Bach equations. These equations can be derived from a Lagrangian equal to the square of the Weyl tensor. The corresponding action is conformally invariant, hence the Bach equations are also conformally invariant. Conformal gravity theories are defined by field equations that determine only the conformal structure of the spacetime manifold. Hence, the Bach equations represent an early example of such a theory. In the previous chapter, we noted some different mathematical formalisms which can be used to solve the field equations, including coordinate or tetrad based approaches, Cartan's calculus of differential forms. However, our focus has been primarily on the spinor calculus in its compacted GHP spin-coefficient form. In the present chapter, we apply this formalism to the Bach equations. Although each formalism makes the same physical predictions, they differ in the ease of their calculations. In particular, the spin-coefficient formalism has led to many exact solutions, which would have been otherwise difficult to find [98]. As a simple application of the formalism, we reconstruct two well-known solutions: the PP-wave spacetime and the static spherically-symmetric spacetime.

Among the most important differences between the Bach equations and the Einstein field equations (EFE) are that they are fourth-order differential equations in the metric, as opposed to the second-order EFE. Furthermore, they are conformally invariant, a consequence of this invariance is that the conformal scale factor is left undetermined by them. The theory is, however, different to Weyl's [110] conformal theory in that the spacetime geometry remains Riemannian and therefore the covariant derivative of the metric is zero (metric compatibility (4.3)).

An important remark, from a physical standpoint, is that every spacetime locally conformal to an Einstein space (vanishing of traceless Ricci tensor) is a solution of the Bach equations. Therefore, the physically important Schwarzschild, Kerr, gravitational wave and most Friedmann-Lemaitre-Robertson-Walker cosmological spacetimes are also solutions of the Bach equations.

Whilst this is an attractive feature of the Bach equations, it is far from sufficient to argue that they are therefore physically relevant. With this is mind, we wish to emphasise not the physicality of the Bach equations, but the efficiency of the spin-coefficient formalism in solving them.

The main new achievement of this chapter is the translation of the Bach equations into compacted spin-coefficient form [43]. We suggest that this formulation may be used as an efficient alternative to tensor methods for solving the Bach equations. Arguments in support of this suggestion are the following. Firstly, the spin-coefficient formalism deals entirely with scalar quantities, which are easily manipulated and may take the form of explicit functions. Moreover, because the spin coefficients are complex quantities, the number of them are reduced to 12, instead of the 24 required in a orthonormal tetrad formalism, or the 40 Christoffel symbols used in coordinate based approaches. As a consequence, fewer terms arise in the calculation. Another advantage of the spin-coefficient formalism is that the Weyl tensor (or spinor) has an economic representation as just five complex Weyl scalars. Since the Weyl tensor (or spinor) is the primary field variable in the Bach equations, they have a similarly economic representation in this formalism. Lastly, it has been shown that the GHP formalism incorporates conformal transformations in a straightforward way [89], which should prove useful in studying conformally invariant equations such as the Bach equations. In support of the idea that the GHP formalism is particularly well suited to solving the Bach equations, we give two straightforward applications of the formalism by solving the Bach equations for a PP-wave spacetime [39, 41, 63] and a static spherically symmetric spacetime [41, 65], where the general solution for the curvature spinors are obtained in explicit form.

The tensor form of the Bach equations may be derived from the following action

$$S = \int C_{abcd}C^{abcd}\sqrt{g}\,\mathrm{d}^4x, \tag{6.1}$$

where $C_{abcd}$ is the Weyl tensor, introduced earlier, which represents the trace-free part of the Riemann tensor (4.9). They are equivalent to the vanishing of the Bach tensor, which is defined as follows [96, 57]

$$B_{ab} = (\nabla^c\nabla^d - \tfrac{1}{2}R^{cd})C_{acbd}, \tag{6.2}$$

where $\nabla_a$ is the covariant derivative and $R_{cd}$ is the Ricci tensor, cf. (4.7). The Bach tensor satisfies the following relations

$$B_{ab} = B_{ba}, \quad g^{ab}B_{ab} = 0, \quad \nabla^a B_{ab} = 0, \quad B_{ab} = \bar{B}_{ab}. \tag{6.3}$$

such that it is symmetric, traceless, divergence-free and real. In the last relation we have used an over-bar to denote the complex conjugate of a tensor, cf. (2.7). Furthermore, under the conformal rescaling

$$g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab}, \tag{6.4}$$

we have

$$\hat{B}_{ab} = B_{ab}, \tag{6.5}$$

such that it is conformally invariant and therefore so too are the Bach equations. The Bach spinor is defined as follows[1] [96, 57, 90]

$$B_{ABA'B'} = 2(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD}, \tag{6.6}$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$ is the (complex) Weyl spinor and $\Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \bar{\Phi}_{ABC'D'}$ is the (real, traceless) Ricci spinor. We have also denoted the complex conjugation of a spinor with an over-bar, see (2.7). The Bach spinor has the same symmetries as the Ricci spinor

$$B_{ABA'B'} = B_{(AB)(A'B')}, \quad \bar{B}_{ABC'D'} = B_{ABC'D'}. \tag{6.7}$$

## 6.1 Bach equations in terms of spin-coefficients

With the help of (5.23) the translation of (6.6) into spin-coefficient form is straightforward. According to (5.7) and its complex conjugate relation we first take components of (6.6) as follows

$$\tfrac{1}{2}B_{\mathbf{ABA'B'}} = -\bar{\varepsilon}_{\mathbf{B'}}{}^{B'}\varepsilon_{\mathbf{A}}{}^{A}\varepsilon_{\mathbf{B}}{}^{B}\varepsilon_{\mathbf{C}}{}^{C}\nabla^{\mathbf{C}}_{\mathbf{A'}}\nabla^{D}_{B'}\Psi_{ABCD} - \Phi^{\mathbf{CD}}_{\mathbf{A'B'}}\Psi_{\mathbf{ABCD}}. \tag{6.8}$$

Writing out the individual terms of the summation in the derivative term of (6.8), we find terms of the form of the left hand side of (5.23) with $\Psi_{ABCD}$ playing the role of $\xi_{A\ldots D\ldots G'\ldots K'}$. Substituting the right hand side of (5.23) with the corresponding values of $r, r', t, t'$ we obtain an expression in terms of the derivative operators and spin-coefficients of the GHP formalism.

[1]The ordering of unprimed in relation to primed indices is inconsequential, therefore, it is not necessary to stagger up and down indices of different types.

The Bach tensor in spin-coefficient form is then given as follows[2]

$$\begin{aligned}\tfrac{1}{2}B_{01} = {}& (\text{þ} - 3\rho)[(\text{þ}' - 2\rho')\Psi_1 - (\eth - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3] \\ & + (\eth' - \tau')[(\eth - 4\tau)\Psi_1 - (\text{þ}' - \rho')\Psi_0 + 3\sigma\Psi_2] \\ & + 2\kappa[(\text{þ}' - 3\rho')\Psi_2 - (\eth - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4] \\ & + \bar{\tau}'[(\eth' - 2\tau')\Psi_1 - (\text{þ} - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3] \\ & + \bar{\rho}'[(\text{þ} - 4\rho)\Psi_1 - (\eth' - \tau')\Psi_0 + 3\kappa\Psi_2] - \Phi_{21}\Psi_0 + 2\Phi_{11}\Psi_1 - \Phi_{01}\Psi_2,\end{aligned} \tag{6.9a}$$

$$\begin{aligned}\tfrac{1}{2}B_{21} = {}& (\text{þ}' - 3\rho')[(\text{þ} - 2\rho)\Psi_3 - (\eth' - 3\tau')\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1] \\ & + (\eth - \tau)[(\eth' - 4\tau')\Psi_3 - (\text{þ} - \rho)\Psi_4 + 3\sigma'\Psi_2] \\ & + 2\kappa'[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0] \\ & + \bar{\tau}[(\eth - 2\tau)\Psi_3 - (\text{þ}' - 3\rho')\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1] \\ & + \bar{\rho}[(\text{þ}' - 4\rho')\Psi_3 - (\eth - \tau)\Psi_4 + 3\kappa'\Psi_2] - \Phi_{01}\Psi_4 + 2\Phi_{11}\Psi_3 - \Phi_{21}\Psi_2,\end{aligned} \tag{6.9b}$$

$$\begin{aligned}\tfrac{1}{2}B_{10} = {}& (\text{þ} - 2\rho)[(\eth' - 3\tau')\Psi_2 - (\text{þ} - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4] \\ & + (\eth' - 2\tau')[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\Psi_1 - \sigma'\Psi_0 + 2\kappa\Psi_3] \\ & + \kappa[(\eth' - 4\tau')\Psi_3 - (\text{þ} - \rho)\Psi_4 + 3\sigma'\Psi_2] \\ & + \bar{\kappa}[(\text{þ}' - 3\rho')\Psi_2 - (\eth - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4] \\ & + \bar{\sigma}[(\eth - 3\tau)\Psi_2 - (\text{þ}' - 2\rho')\Psi_1 - \kappa'\Psi_0 + 2\sigma\Psi_3] - \Phi_{20}\Psi_1 + 2\Phi_{10}\Psi_2 - \Phi_{00}\Psi_3,\end{aligned} \tag{6.10a}$$

$$\begin{aligned}\tfrac{1}{2}B_{12} = {}& (\text{þ}' - 2\rho')[(\eth - 3\tau)\Psi_2 - (\text{þ}' - 2\rho')\Psi_1 + 2\sigma\Psi_3 - \kappa'\Psi_0] \\ & + (\eth - 2\tau)[(\text{þ}' - 3\rho')\Psi_2 - (\eth - 2\tau)\Psi_3 - \sigma\Psi_4 + 2\kappa'\Psi_1] \\ & + \kappa'[(\eth - 4\tau)\Psi_1 - (\text{þ}' - \rho')\Psi_0 + 3\sigma\Psi_2] \\ & + \bar{\kappa}'[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0] \\ & + \bar{\sigma}'[(\eth' - 3\tau')\Psi_2 - (\text{þ} - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4] - \Phi_{02}\Psi_3 + 2\Phi_{12}\Psi_2 - \Phi_{22}\Psi_1,\end{aligned} \tag{6.10b}$$

[2]We adhere to the convention that a differential operator acts only on the symbol (or bracketed expression) which immediately follows it-unless this is also a differential operator.

$$
\begin{aligned}
\tfrac{1}{2}B_{00} = {} & (þ - 3\rho)[(ð' - 2\tau')\Psi_1 - (þ - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3] \\
& + (ð' - \tau')[(þ - 4\rho)\Psi_1 - (ð' - \tau')\Psi_0 + 3\kappa\Psi_2] \\
& + 2\kappa[(ð' - 3\tau')\Psi_2 - (þ - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4] \\
& + \bar{\kappa}[(þ' - 2\rho')\Psi_1 - (ð - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3] \\
& + \bar{\sigma}[(ð - 4\tau)\Psi_1 - (þ' - \rho')\Psi_0 + 3\sigma\Psi_2] - \Phi_{20}\Psi_0 + 2\Phi_{10}\Psi_1 - \Phi_{00}\Psi_2,
\end{aligned} \tag{6.11a}
$$

$$
\begin{aligned}
\tfrac{1}{2}B_{02} = {} & (ð - 3\tau)[(þ' - 2\rho')\Psi_1 - (ð - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3] \\
& + (þ' - \rho')[(ð - 4\tau)\Psi_1 - (þ' - \rho')\Psi_0 + 3\sigma\Psi_2] \\
& + 2\sigma[(þ' - 3\rho')\Psi_2 - (ð - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4] \\
& + \bar{\sigma}'[(ð' - 2\tau')\Psi_1 - (þ - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3] \\
& + \bar{\kappa}'[(þ - 4\rho)\Psi_1 - (ð' - \tau')\Psi_0 + 3\kappa\Psi_2] - \Phi_{22}\Psi_0 + 2\Phi_{12}\Psi_1 - \Phi_{02}\Psi_2,
\end{aligned} \tag{6.11b}
$$

$$
\begin{aligned}
\tfrac{1}{2}B_{20} = {} & (ð' - 3\tau')[(þ - 2\rho)\Psi_3 - (ð' - 3\tau')\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1] \\
& + (þ - \rho)[(ð' - 4\tau')\Psi_3 - (þ - \rho)\Psi_4 + 3\sigma'\Psi_2] \\
& + 2\sigma'[(þ - 3\rho)\Psi_2 - (ð' - 2\tau')\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0] \\
& + \bar{\sigma}[(ð - 2\tau)\Psi_3 - (þ' - 3\rho')\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1] \\
& + \bar{\kappa}[(þ' - 4\rho')\Psi_3 - (ð - \tau)\Psi_4 + 3\kappa'\Psi_2] - \Phi_{00}\Psi_4 + 2\Phi_{10}\Psi_3 - \Phi_{20}\Psi_2,
\end{aligned} \tag{6.11c}
$$

$$
\begin{aligned}
\tfrac{1}{2}B_{22} = {} & (þ' - 3\rho')[(ð - 2\tau)\Psi_3 - (þ' - 3\rho')\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1] \\
& + (ð - \tau)[(þ' - 4\rho')\Psi_3 - (ð - \tau)\Psi_4 + 3\kappa'\Psi_2] \\
& + 2\kappa'[(ð - 3\tau)\Psi_2 - (þ' - 2\rho')\Psi_1 + 2\sigma\Psi_3 - \kappa'\Psi_0] \\
& + \bar{\kappa}'[(þ - 2\rho)\Psi_3 - (ð' - 3\tau')\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1] \\
& + \bar{\sigma}'[(ð' - 4\tau')\Psi_3 - (þ - \rho)\Psi_4 + 3\sigma'\Psi_2] - \Phi_{02}\Psi_4 + 2\Phi_{12}\Psi_3 - \Phi_{22}\Psi_2,
\end{aligned} \tag{6.11d}
$$

$$
\begin{aligned}
\tfrac{1}{2}B_{11} = {} & (þ - 2\rho)[(þ' - 3\rho')\Psi_2 - (ð - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4] \\
& + (ð' - 2\tau')[(ð - 3\tau)\Psi_2 - (þ' - 2\rho')\Psi_1 - \kappa'\Psi_0 + 2\sigma\Psi_3] \\
& + \kappa[(þ' - 4\rho')\Psi_3 - (ð - \tau)\Psi_4 + 3\kappa'\Psi_2] \\
& + \sigma'[(ð - 4\tau)\Psi_1 - (þ' - \rho')\Psi_0 + 3\sigma\Psi_2] \\
& + \bar{\rho}'[(þ - 3\rho)\Psi_2 - (ð' - 2\tau')\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0] - \Phi_{21}\Psi_1 + 2\Phi_{11}\Psi_2 - \Phi_{01}\Psi_3.
\end{aligned} \tag{6.12}
$$

Setting equations (6.9), (6.10), (6.11) and (6.12) to zero gives the Bach equations in spin-coefficient form. The Bach equations in the GHP formalism are interchanged by the prime and asterisk operations respectively in the following way, cf. (5.2)

$$
B_{rs} \mapsto B_{tu} \quad 0 \leftrightarrow 2 \quad 1 \leftrightarrow 1, \tag{6.13}
$$

$$
\begin{aligned}
B_{01} &\leftrightarrow -B_{01},\\
B_{21} &\leftrightarrow -B_{21},\\
B_{10} &\leftrightarrow B_{12},\\
B_{00} &\leftrightarrow B_{02},\\
B_{22} &\leftrightarrow B_{20},\\
B_{11} &\leftrightarrow -B_{11}.
\end{aligned} \tag{6.14}
$$

Therefore, the Bach equations split into the four groups (6.9), (6.10), (6.11) and (6.12) which transform only amongst themselves under the combined action of (6.13) and (6.14).

## 6.2 Applications of the Bach equations in spin-coefficient form

### 6.2.1 PP-wave spacetime

As a straightforward example, we solve the Bach equations for a metric corresponding to a PP-wave spacetime [3], which may be represented in a coordinate chart $(u, v, x, y)$ by the following fundamental form

$$
g_{\mathbf{ab}}\mathrm{d}x^{\mathbf{a}}\mathrm{d}x^{\mathbf{b}} = 2H\mathrm{d}u^2 + (\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v\,\mathrm{d}u) - \mathrm{d}x^2 - \mathrm{d}y^2, \tag{6.15}
$$

where $H = H(u, x, y)$. Substituting (6.15) into (5.2) and using (5.3) we find that the components of the null tetrad $e_{\mathbf{m}}{}^{\mathbf{a}}$ are given by the following matrix

$$
e_{\mathbf{m}}{}^{\mathbf{a}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & -H & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}, \tag{6.16}
$$

where

$$
e_{\mathbf{ma}} = \begin{pmatrix} l_{\mathbf{a}} \\ n_{\mathbf{a}} \\ m_{\mathbf{a}} \\ \bar{m}_{\mathbf{a}} \end{pmatrix}, \tag{6.17}
$$

[3]A PP-wave was defined in [39] to be any Lorentzian manifold which admits a covariantly constant null vector field. Furthermore, the authors showed that a metric for such a spacetime can always be written in the form (6.15), which was investigated earlier, see [19].

and

$$l_{\mathbf{a}} = \frac{1}{\sqrt{2}}(2, 0, 0, 0), \tag{6.18a}$$

$$n_{\mathbf{a}} = \frac{1}{\sqrt{2}}(H, 1, 0, 0), \tag{6.18b}$$

$$m_{\mathbf{a}} = \frac{1}{\sqrt{2}}(0, 0, -1, -i), \tag{6.18c}$$

$$\bar{m}_{\mathbf{a}} = \frac{1}{\sqrt{2}}(0, 0, -1, i). \tag{6.18d}$$

Substituting (6.18) into (5.45) we obtain the following non-zero spin-coefficient for a PP-wave spacetime

$$\sqrt{2}\kappa' = -\partial H, \tag{6.19}$$

where we have introduced the following complex coordinates

$$z = x + iy, \quad \bar{z} = x - iy, \tag{6.20}$$

and the following notation for the partial derivatives

$$\partial = \partial_z = \tfrac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} = \partial_{\bar{z}} = \tfrac{1}{2}(\partial_x + i\partial_y), \tag{6.21}$$

familiar from complex manifold theory [109]. All other spin-coefficients are zero. Substituting (6.19) into the NP equations (5.29), we find the following non-zero curvature spinor components

$$\Psi_4 = -\delta'\kappa' = \partial^2 H, \tag{6.22}$$

$$\Phi_{22} = -\delta\kappa' = \bar{\partial}\partial H, \tag{6.23}$$

all other curvature spinor components are zero. Substituting (6.23) and (6.22) into (6.9), (6.11), (6.12) and (6.10) we find that all the Bach equations are trivially satisfied except for (6.11d) which gives

$$\eth^2\Psi_4 = \delta^2\Psi_4 = 2\bar{\partial}^2\Psi_4 = 0. \tag{6.24}$$

The general solution to (6.24) is

$$\Psi_4(u, z, \bar{z}) = \bar{z}\psi_1(u, z) + \psi_2(u, z), \tag{6.25}$$

where $\psi_1(u, z)$ and $\psi_2(u, z)$ are arbitrary complex functions. From (6.22) and (6.23) we find

$$\bar{\partial}\Psi_4 = \partial\Phi_{22}, \tag{6.26}$$

which is the only non-trivial Bianchi identity remaining for a PP-wave spacetime, see (5.38). Substituting (6.25) into (6.26) we find the general solution for the Ricci spinor

$$\Phi_{22}(u,z,\bar{z}) = \phi(u,z) + \bar{\phi}(u,\bar{z}), \tag{6.27}$$

where

$$\psi_1 = \partial\phi. \tag{6.28}$$

The function $H$ can be obtained implicitly by integrating twice either (6.25) or (6.27) according to (6.22) or (6.23) respectively. The explicit solutions (6.25) and (6.27) have not been found in the literature, however, they follow as a special case of the examples considered in [63, 41]. As mentioned in the introduction to this chapter, it is well known that an Einstein space, defined by the condition $\Phi_{ab} = 0$, solves the Bach equations. In this particular case, from (6.27) and (6.28) we see that $\psi_1 = 0$. Since in this case the Ricci tensor is zero, only the Weyl curvature remains, hence this solution corresponds to a pure gravity wave.
In order to gain a better insight into the physical role played by the curvature components (6.22) and (6.23), we consider the following (different) special case of a *plane wave* [81], which is a PP-wave where the function $H$ depends on $z$ and $\bar{z}$ in the following way

$$H(u,z,\bar{z}) = \frac{1}{2}\begin{pmatrix} z, & \bar{z}\end{pmatrix}\begin{pmatrix} H_{zz} & H_{z\bar{z}} \\ \bar{H}_{z\bar{z}} & \bar{H}_{zz}\end{pmatrix}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}, \tag{6.29}$$

where $H_{zz} = H_{zz}(u)$ and $H_{z\bar{z}} = H_{z\bar{z}}(u)$ are arbitrary functions of $u$. Substituting (6.29) into (6.22) and (6.23) we find

$$\Phi_{22} = H_{z\bar{z}} + \bar{H}_{z\bar{z}}, \tag{6.30}$$

$$\Psi_4 = H_{zz}, \tag{6.31}$$

and in this case (6.24) is trivially satisfied since $\Psi_4$ depends only on $u$. The modulus and argument of $\Psi_4$ correspond respectively to the amplitude and polarization of the gravitational plane wave. The square root of $\Phi_{22}$ corresponds to the electromagnetic part of the wave, the polarization of the electromagnetic part, whilst still an arbitrary function of $u$, does not contribute to the curvature [81].

### 6.2.2 Static spherically-symmetric spacetime

For our second example we consider a static spherically symmetric spacetime, i.e. the conditions under which the Schwarzschild solution is the unique solution of the EFE. The solution to the Bach equations under these conditions was found in [65], where the authors show that a static spherically symmetric spacetime may be represented in a coordinate chart $(t,r,\theta,\phi)$ by the following fundamental form

$$g_{\mathbf{ab}}\mathrm{d}x^{\mathbf{a}}\mathrm{d}x^{\mathbf{b}} = \frac{p^2}{r^2}\left(A^2\mathrm{d}t^2 - A^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{6.32}$$

where $p = p(r)$ and $A = A(r)$. Since the Bach equations (6.6) are conformally invariant we may make the following conformal rescaling

$$g_{\mathbf{ab}} \mapsto \hat{g}_{\mathbf{ab}} = \frac{r^2}{p^2}g_{\mathbf{ab}}, \tag{6.33}$$

such that our metric is transformed to

$$\hat{g}_{\mathbf{ab}}\mathrm{d}x^{\mathbf{a}}\mathrm{d}x^{\mathbf{b}} = A^2\mathrm{d}t^2 - A^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{6.34}$$

All quantities (e.g. spin-coefficients and spinor components) will now refer to the conformally rescaled metric (6.34), however, we omit the hats on these quantities. Substituting (6.34) into (5.2) we find

$$e_{\mathbf{m}}{}^{\mathbf{a}} = \frac{1}{\sqrt{2}}\begin{pmatrix} A^{-1} & -A & 0 & 0 \\ A^{-1} & A & 0 & 0 \\ 0 & 0 & -r^{-1} & -ir^{-1}\csc\theta \\ 0 & 0 & -r^{-1} & ir^{-1}\csc\theta \end{pmatrix}, \tag{6.35}$$

such that

$$l_{\mathbf{a}} = \frac{1}{\sqrt{2}}(A, A^{-1}, 0, 0), \tag{6.36a}$$

$$n_{\mathbf{a}} = \frac{1}{\sqrt{2}}(A, -A^{-1}, 0, 0), \tag{6.36b}$$

$$m_{\mathbf{a}} = \frac{1}{\sqrt{2}}(0, 0, r, ir\sin\theta), \tag{6.36c}$$

$$\bar{m}_{\mathbf{a}} = \frac{1}{\sqrt{2}}(0, 0, r, -ir\sin\theta). \tag{6.36d}$$

Substituting (6.35) and (6.36) into (5.43) and (5.44) we calculate the following spin-coefficients [35]

$$\rho = -\rho' = \frac{A}{\sqrt{2}r}, \tag{6.37a}$$

$$\gamma = -\gamma' = -\frac{\dot{A}}{2\sqrt{2}}, \tag{6.37b}$$

$$\beta = \beta' = \frac{\cot\theta}{2\sqrt{2}r}, \tag{6.37c}$$

where we have denoted differentiation with respect to the radial coordinate $r$, not the time, with an over-dot because the prime is already in use. All other spin-coefficients are zero. Substituting

(6.37) into (5.29) and (5.31) we find that the NP equations and commutator expressions reduce to the following

$$\begin{aligned} -2r\rho\dot{\gamma} &= -4\gamma^2 + \Psi_2 + \Phi_{11} - \Lambda, && (6.38a)\\ 4\gamma\rho &= \Psi_2 + 2\Lambda, && (6.38b)\\ \frac{1}{2r^2} &= \rho^2 - \Psi_2 + \Phi_{11} + \Lambda, && (6.38c) \end{aligned}$$

where $\bar{\Psi}_2 = \Psi_2$ and all other curvature spinor components are zero. From (6.37) and (6.38) we find that all the Bach equations are trivially satisfied except for (6.11a) and (6.12). The former is given by

$$(þ - 3\rho)\left[(þ - 3\rho)\Psi_2\right] = 0. \tag{6.39}$$

Expanding (6.39) and using $þ\rho = \rho^2$ and $þ\Psi_2 = -r\rho\dot{\Psi}_2$, which follow from (5.29), and from (5.21), (6.36) and (6.37) respectively, we find that it reduces to

$$\rho^2\left[r^2\ddot{\Psi}_2 + 6(r\dot{\Psi}_2 + \Psi_2)\right] = 0. \tag{6.40}$$

If $\rho = 0$ then from (6.37a) and (6.32) we see that this solution trivially corresponds to a singular metric tensor. Therefore, we must have

$$r^2\ddot{\Psi}_2 + 6(r\dot{\Psi}_2 + \Psi_2) = 0, \tag{6.41}$$

from which we obtain the general solution

$$\Psi_2(r) = \frac{c_1}{r^2} + \frac{c_2}{r^3}, \tag{6.42}$$

where $c_1$ and $c_2$ are constants of integration. Equation (6.12) is given by the following

$$(þ - 2\rho)((þ' - 3\rho')\Psi_2) + \rho'(þ - 3\rho)\Psi_2 + 2\Phi_{11}\Psi_2 = 0. \tag{6.43}$$

Using (6.39), we may further simply (6.43)[4]

$$(2\gamma\rho + \rho^2)\left(r\dot{\Psi}_2 + 3\Psi_2\right) + \Phi_{11}\Psi_2 = 0. \tag{6.44}$$

Equations (6.44) and (6.38) are the only ones remaining that we need to solve. To achieve this, we make use of the Bianchi identities (5.38b) and (5.38f), which in our example are given by

$$\begin{aligned} (þ - 3\rho)\Psi_2 - 2\rho\Phi_{11} + 2þ\Lambda &= 0, && (6.45)\\ (þ - 4\rho)\Phi_{11} + 3þ\Lambda &= 0, && (6.46) \end{aligned}$$

[4]Note that $þþ'\Psi_2 = -þ^2\Psi_2 - 4\gammaþ\Psi_2$ and $þ\rho' = -þ\rho - 4\gamma\rho$.

which simplify to the following

$$\rho \left( r\dot{\Psi}_2 + 3\Psi_2 + 2\Phi_{11} + 2r\dot{\Lambda} \right) = 0, \tag{6.47}$$

$$\rho \left( r\dot{\Phi}_{11} + 4\Phi_{11} + 3r\dot{\Lambda} \right) = 0. \tag{6.48}$$

Assuming $\rho \neq 0$, we then substitute (6.42) and (6.48) into (6.47) to find

$$\Phi_{11} = -\frac{3c_1}{2r^2} + \frac{c_3}{r}, \tag{6.49}$$

$$\Lambda = -\frac{c_1}{2r^2} + \frac{c_3}{r} - \lambda, \tag{6.50}$$

where $c_3, \lambda$ are constants of integration. Substituting (6.42), (6.49) and (6.50) into (6.38c), we obtain $\rho^2$ and hence from (6.37a) we find $A^2$ to be

$$A^2 = 1 + 6c_1 + \frac{2c_2}{r} - 4c_3 r + 2\lambda r^2. \tag{6.51}$$

We find $\gamma$ from (6.37b) and substitute it, along with (6.42), (6.49) and (6.51), into (6.44) to obtain the following relation between our constants of integration

$$3{c_1}^2 + c_1 + 2c_2c_3 = 0. \tag{6.52}$$

The third term of (6.51) is present in the *Schwarzschild* solution where $c_2$ would be equal to (minus) the Newtonian mass [29]. The fifth term is the cosmological constant term, which is present in the de Sitter spacetimes. The second and fourth term, however, are peculiar to the solution of the Bach equations, (cf. [65] where $c_3$ is proportional to their constant $\gamma$). The spherically symmetric solution to Einstein's vacuum equations is the well-known Schwarzschild solution. Due to a theorem of Birkhoff, which states that any spherically symmetric solution of the Einstein vacuum equations is locally equivalent to the Schwarzschild spacetime [49], one may wonder whether the same is true of the spherically symmetric solution to the Bach equations. In fact, the analogous theorem is true [94]. Therefore, as in the case of the Schwarzschild solution, the assumption that our metric be static is not necessary.

Since there exist very few exact solutions to the Bach equations in the literature, we hope that our translation of the Bach equations into spin-coefficient form will be useful for finding new exact solutions that would be otherwise difficult to obtain. This hope is sustained by the fact that the spin-coefficient formalism has already proven to be a powerful method for finding exact solutions to the EFE [98].

In order to show the efficiency of the formalism applied to the Bach equations, we chose two straightforward examples, the plane-fronted wave spacetime and a static spherically symmetric spacetime. In comparison to coordinate based tensor methods, the calculations involved are shorter. For example, in a standard approach, one would start with the metric, from which the 40 Christoffel symbols are derived, from these the Riemann tensor follows and its trace and trace-free parts give the Ricci and Weyl tensors. These are then substituted into the Bach equations yielding fourth-order partial differential equations in the metric components. On the other hand, starting from the Bach equations in spin-coefficient form, the steps are far fewer, as shown in our examples. Furthermore, the equations are never more than second-order, since one solves directly for the curvature components as opposed to the metric components. In more complicated problems we expect the improved efficiency to increase.

# Chapter 7

# Conformal Einstein space

After the spherically symmetric solution to the Bach equations was found in [65], it was noted in [66] that the solution was conformally related to the de Sitter-Schwarzschild solution (also known as the Kottler [98] or Weyl-Trefftz metric [41]) of the Einstein equations with cosmological constant. Centrally symmetric solutions of the Bach equations and their relation to solutions of the Einstein equations were investigated earlier in [41]. In this chapter, we look at this relationship in more detail, which leads us to consider *conformal Einstein spaces*. An Einstein space is defined as a spacetime with zero traceless Ricci tensor. Spacetimes which satisfy Einstein's vacuum equations with a cosmological constant are Einstein spaces. From both a physical and a mathematical point of view, Einstein spaces represent an interesting class of spacetimes. In the context of conformally invariant theories of gravity, conformal Einstein spaces, which are spacetimes related to an Einstein space by a conformal transformation, play an analogous role. In [57], the necessary and sufficient conditions were given for a spacetime to be a conformal Einstein space. The proof requires that the complex invariant

$$J = \Psi_{ABCD}\Psi^{CD}{}_{EF}\Psi^{EFAB} \tag{7.1}$$

of the Weyl spinor is non-zero. The conditions are equivalent to the vanishing of the Bach tensor (6.2) (or equivalently the Bach spinor (6.6)) and the Eastwood-Dighton (ED) tensor [13], which we introduce presently.

## 7.1 Eastwood-Dighton tensor in terms of spin coefficients

The ED tensor arose during an investigation into *local twistors* in twistor theory [37]. It was shown that conformally invariant spinors, such as the ED, Weyl and Bach spinors, derive naturally from the curvature twistor, defined via local twistor transport. In [44], it was shown that the local twistor connection coincides with the normal conformal Cartan connection, a well-studied object in conformal geometry [56].

The ED tensor is given in terms of the Weyl tensor as

$$E_{abc} = \Psi_{ABCD} \nabla^{DD'} \bar{\Psi}_{A'B'C'D'} - \bar{\Psi}_{A'B'C'D'} \nabla^{DD'} \Psi_{ABCD}, \tag{7.2}$$

where

$$E_{abc} = E_{(abc)}, \qquad E^{a}{}_{ab} = 0, \qquad \hat{E}_{abc} = E_{abc}, \qquad \bar{E}_{abc} = -E_{abc}, \tag{7.3}$$

such that (7.2) is symmetric, trace-free and conformally invariant, as is the Bach tensor, c.f (6.3). However, the last condition in (7.3) shows that the ED spinor is pure imaginary, in contrast to the Bach tensor, which is real. In terms of the GHP formalism, the components of (7.2) are given by the following sixteen equations,

$$
\begin{aligned}
E_{00} =&\Psi_0 \left[(\text{þ}' - \bar\rho')\,\bar\Psi_0 - (\text{ð}' - 4\bar\tau)\,\bar\Psi_1 - 3\bar\sigma\bar\Psi_2\right]\\
&+ \Psi_1 \left[(\text{þ} - 4\bar\rho)\,\bar\Psi_1 - (\text{ð} - \bar\tau')\,\bar\Psi_0 + 3\bar\kappa\bar\Psi_2\right] - c.c., && (7.4a)\\
E_{33} =&\Psi_4 \left[(\text{þ} - \bar\rho)\,\bar\Psi_4 - (\text{ð} - 4\bar\tau')\,\bar\Psi_3 - 3\bar\sigma'\bar\Psi_2\right]\\
&+ \Psi_3 \left[(\text{þ}' - 4\bar\rho')\,\bar\Psi_3 - (\text{ð}' - \bar\tau)\,\bar\Psi_4 + 3\bar\kappa'\bar\Psi_2\right] - c.c., && (7.4b)\\
E_{01} =&\Psi_0 \left[(\text{þ}' - 2\bar\rho')\,\bar\Psi_1 - (\text{ð}' - 3\bar\tau)\,\bar\Psi_2 + \bar\kappa'\bar\Psi_0 - 2\bar\sigma\bar\Psi_3\right]\\
&+ \Psi_1 \left[(\text{þ} - 3\bar\rho)\,\bar\Psi_2 - (\text{ð} - 2\bar\tau')\,\bar\Psi_1 - \bar\sigma'\bar\Psi_0 + 2\bar\kappa\bar\Psi_3\right] - c.c., && (7.4c)\\
E_{32} =&\Psi_4 \left[(\text{þ} - 2\bar\rho)\,\bar\Psi_3 - (\text{ð} - 3\bar\tau')\,\bar\Psi_2 + \bar\kappa\bar\Psi_4 - 2\bar\sigma'\bar\Psi_1\right]\\
&+ \Psi_3 \left[(\text{þ}' - 3\bar\rho')\,\bar\Psi_2 - (\text{ð}' - 2\bar\tau)\,\bar\Psi_3 - \bar\sigma\bar\Psi_4 + 2\bar\kappa'\bar\Psi_1\right] - c.c., && (7.4d)\\
E_{02} =&\Psi_0 \left[(\text{þ}' - 3\bar\rho')\,\bar\Psi_2 - (\text{ð}' - 2\bar\tau)\,\bar\Psi_3 + 2\bar\kappa'\bar\Psi_1 - \bar\sigma\bar\Psi_4\right]\\
&+ \Psi_1 \left[(\text{þ} - 2\bar\rho)\,\bar\Psi_3 - (\text{ð} - 3\bar\tau')\,\bar\Psi_2 - 2\bar\sigma'\bar\Psi_1 + \bar\kappa\bar\Psi_4\right] - c.c., && (7.4e)\\
E_{31} =&\Psi_4 \left[(\text{þ} - 3\bar\rho)\,\bar\Psi_2 - (\text{ð} - 2\bar\tau')\,\bar\Psi_1 + 2\bar\kappa\bar\Psi_3 - \bar\sigma'\bar\Psi_0\right]\\
&+ \Psi_3 \left[(\text{þ}' - 2\bar\rho')\,\bar\Psi_1 - (\text{ð}' - 3\bar\tau)\,\bar\Psi_2 - 2\bar\sigma\bar\Psi_3 + \bar\kappa'\bar\Psi_0\right] - c.c., && (7.4f)\\
E_{03} =&\Psi_0 \left[(\text{þ}' - 4\bar\rho')\,\bar\Psi_3 - (\text{ð}' - \bar\tau)\,\bar\Psi_4 + 3\bar\kappa'\bar\Psi_2\right]\\
&+ \Psi_1 \left[(\text{þ} - \bar\rho)\,\bar\Psi_4 - (\text{ð} - 4\bar\tau')\,\bar\Psi_3 - 3\bar\sigma'\bar\Psi_2\right] - c.c., && (7.4g)\\
E_{30} =&\Psi_4 \left[(\text{þ} - 4\bar\rho)\,\bar\Psi_1 - (\text{ð} - \bar\tau')\,\bar\Psi_0 + 3\bar\kappa\bar\Psi_2\right]\\
&+ \Psi_3 \left[(\text{þ}' - \bar\rho')\,\bar\Psi_0 - (\text{ð}' - 4\bar\tau)\,\bar\Psi_1 - 3\bar\sigma\bar\Psi_2\right] - c.c., && (7.4h)\\
E_{10} =&\Psi_1 \left[(\text{þ}' - \bar\rho')\,\bar\Psi_0 - (\text{ð}' - 4\bar\tau)\,\bar\Psi_1 - 3\bar\sigma\bar\Psi_2\right]\\
&+ \Psi_2 \left[(\text{þ} - 4\bar\rho)\,\bar\Psi_1 - (\text{ð} - \bar\tau')\,\bar\Psi_0 + 3\bar\kappa\bar\Psi_2\right] - c.c., && (7.4i)\\
E_{23} =&\Psi_3 \left[(\text{þ} - \bar\rho)\,\bar\Psi_4 - (\text{ð} - 4\bar\tau')\,\bar\Psi_3 - 3\bar\sigma'\bar\Psi_2\right]\\
&+ \Psi_2 \left[(\text{þ}' - 4\bar\rho')\,\bar\Psi_3 - (\text{ð}' - \bar\tau)\,\bar\Psi_4 + 3\bar\kappa'\bar\Psi_2\right] - c.c., && (7.4j)\\
E_{11} =&\Psi_1 \left[(\text{þ}' - 2\bar\rho')\,\bar\Psi_1 - (\text{ð}' - 3\bar\tau)\,\bar\Psi_2 + \bar\kappa'\bar\Psi_0 - 2\bar\sigma\bar\Psi_3\right]\\
&+ \Psi_2 \left[(\text{þ} - 3\bar\rho)\,\bar\Psi_2 - (\text{ð} - 2\bar\tau')\,\bar\Psi_1 - \bar\sigma'\bar\Psi_0 + 2\bar\kappa\bar\Psi_3\right] - c.c., && (7.4k)\\
E_{22} =&\Psi_3 \left[(\text{þ} - 2\bar\rho)\,\bar\Psi_3 - (\text{ð} - 3\bar\tau')\,\bar\Psi_2 + \bar\kappa\bar\Psi_4 - 2\bar\sigma'\bar\Psi_1\right]\\
&+ \Psi_2 \left[(\text{þ}' - 3\bar\rho')\,\bar\Psi_2 - (\text{ð}' - 2\bar\tau)\,\bar\Psi_3 - \bar\sigma\bar\Psi_4 + 2\bar\kappa'\bar\Psi_1\right] - c.c., && (7.4l)\\
E_{12} =&\Psi_1 \left[(\text{þ}' - 3\bar\rho')\,\bar\Psi_2 - (\text{ð}' - 2\bar\tau)\,\bar\Psi_3 + 2\bar\kappa'\bar\Psi_1 - \bar\sigma\bar\Psi_4\right]\\
&+ \Psi_2 \left[(\text{þ} - 2\bar\rho)\,\bar\Psi_3 - (\text{ð} - 3\bar\tau')\,\bar\Psi_2 - 2\bar\sigma'\bar\Psi_1 + \bar\kappa\bar\Psi_4\right] - c.c., && (7.4m)\\
E_{21} =&\Psi_3 \left[(\text{þ} - 3\bar\rho)\,\bar\Psi_2 - (\text{ð} - 2\bar\tau')\,\bar\Psi_1 + 2\bar\kappa\bar\Psi_3 - \bar\sigma'\bar\Psi_0\right]\\
&+ \Psi_2 \left[(\text{þ}' - 2\bar\rho')\,\bar\Psi_1 - (\text{ð}' - 3\bar\tau)\,\bar\Psi_2 - 2\bar\sigma\bar\Psi_3 + \bar\kappa'\bar\Psi_0\right] - c.c., && (7.4n)\\
E_{13} =&\Psi_1 \left[(\text{þ}' - 4\bar\rho')\,\bar\Psi_3 - (\text{ð}' - \bar\tau)\,\bar\Psi_4 + 3\bar\kappa'\bar\Psi_2\right]\\
&+ \Psi_2 \left[(\text{þ} - \bar\rho)\,\bar\Psi_4 - (\text{ð} - 4\bar\tau')\,\bar\Psi_3 - 3\bar\sigma'\bar\Psi_2\right] - c.c., && (7.4o)\\
E_{20} =&\Psi_3 \left[(\text{þ} - 4\bar\rho)\,\bar\Psi_1 - (\text{ð} - \bar\tau')\,\bar\Psi_0 + 3\bar\kappa\bar\Psi_2\right]\\
&+ \Psi_2 \left[(\text{þ}' - \bar\rho')\,\bar\Psi_0 - (\text{ð}' - 4\bar\tau)\,\bar\Psi_1 - 3\bar\sigma\bar\Psi_2\right] - c.c., && (7.4p)
\end{aligned}
$$

where *c.c.* stands for complex conjugate. Note that the components can be divided into pairs, where each member is transformed into the other by the prime operation.

## 7.2 Conformal C-space

In [102], a *C-space* was defined as a spacetime satisfying the following condition

$$\nabla^{DD'}\Psi_{ABCD} = 0. \tag{7.5}$$

From (7.2) it follows that all C-spaces have vanishing ED tensors. Furthermore, because (7.2) is conformally invariant, this is true for all spaces which are conformal transformations of C-spaces, such spaces are referred to as *conformal C-spaces*. After a conformal transformation, $\hat{g}_{ab} = \Omega^2 g_{ab}$, (7.5) transforms as

$$\hat{\nabla}^{D}_{D'}\hat{\Psi}_{ABCD} = \Omega^{-1}\left(\nabla^{D}_{D'}\Psi_{ABCD} + \Upsilon^{D}_{D'}\Psi_{ABCD}\right) = 0, \tag{7.6}$$

where

$$\Upsilon^{DE'} = \Omega^{-1}\nabla^{DE'}\Omega, \tag{7.7}$$

and $\bar{\Upsilon}^{DE'} = \Upsilon^{DE'}$ since $\bar{\Omega} = \Omega$. Substituting (7.7) into (7.6) and rearranging yields

$$\nabla^{DE'}\left(\Omega^{-1}\Psi_{ABCD}\right) = 0. \tag{7.8}$$

Setting (7.2) to zero and contracting with $\bar{\Psi}^{A'B'C'D}$ yields

$$\bar{\Psi}^{A'B'C'E'}\Psi_{ABCD}\nabla^{DD'}\bar{\Psi}_{A'B'C'D'} - \bar{\Psi}^{A'B'C'E'}\bar{\Psi}_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD} = 0. \tag{7.9}$$

The Weyl spinor satisfies the following identity

$$\bar{\Psi}^{A'B'C'E'}\bar{\Psi}_{A'B'C'D'} = \frac{\bar{I}}{2}\varepsilon_{D'}{}^{E'}, \qquad \bar{I} = \bar{\Psi}^{A'B'C'D'}\bar{\Psi}_{A'B'C'D'}. \tag{7.10}$$

Rearranging (7.9) and using (7.10) yields

$$\nabla^{DE'}\Psi_{ABCD} + V^{DE'}\Psi_{ABCD} = 0, \tag{7.11}$$

where $V^{DE'}$ is defined as

$$V^{DE'} = -\frac{2}{\bar{I}}\bar{\Psi}^{A'B'C'E'}\nabla^{DD'}\bar{\Psi}_{A'B'C'D'}. \tag{7.12}$$

In [57], it was shown that if $J \neq 0$, it follows from (7.11) that (7.12) is equal to a gradient

$$V^{DE'} = V^{-1}\nabla^{DE'}V. \tag{7.13}$$

As before, substituting (7.13) into (7.11) and rearranging yields

$$\nabla^{DE'}\left(V^{-1}\Psi_{ABCD}\right) = 0. \tag{7.14}$$

In terms of the GHP formalism, the components of (7.12) are given by the following four equations

$$\begin{aligned}
\frac{\bar{I}}{2}V_{00'} = &-\bar{\Psi}_0\left[\left(ð - 4\bar{\tau}'\right)\bar{\Psi}_3 - \left(þ - \bar{\rho}\right)\bar{\Psi}_4 + 3\bar{\sigma}'\bar{\Psi}_2\right] \\
&+ 3\bar{\Psi}_1\left[\left(ð - 3\bar{\tau}'\right)\bar{\Psi}_2 - \left(þ - 2\bar{\rho}\right)\bar{\Psi}_3 + 2\bar{\sigma}'\bar{\Psi}_1 - \bar{\kappa}\bar{\Psi}_4\right] \\
&- 3\bar{\Psi}_2\left[\left(ð - 2\bar{\tau}'\right)\bar{\Psi}_1 - \left(þ - 3\bar{\rho}\right)\bar{\Psi}_2 + \bar{\sigma}'\bar{\Psi}_0 - 2\bar{\kappa}\bar{\Psi}_3\right] \\
&+ \bar{\Psi}_3\left[\left(ð - \bar{\tau}'\right)\bar{\Psi}_0 - \left(þ - 4\bar{\rho}\right)\bar{\Psi}_1 - 3\bar{\kappa}\bar{\Psi}_2\right], \qquad (7.15a)\\
\frac{\bar{I}}{2}V_{11'} = &-\bar{\Psi}_4\left[\left(ð' - 4\bar{\tau}\right)\bar{\Psi}_1 - \left(þ' - \bar{\rho}'\right)\bar{\Psi}_0 + 3\bar{\sigma}\bar{\Psi}_2\right] \\
&+ 3\bar{\Psi}_3\left[\left(ð' - 3\bar{\tau}\right)\bar{\Psi}_2 - \left(þ' - 2\bar{\rho}'\right)\bar{\Psi}_1 + 2\bar{\sigma}\bar{\Psi}_3 - \bar{\kappa}'\bar{\Psi}_0\right] \\
&- 3\bar{\Psi}_2\left[\left(ð' - 2\bar{\tau}\right)\bar{\Psi}_3 - \left(þ' - 3\bar{\rho}'\right)\bar{\Psi}_2 + \bar{\sigma}\bar{\Psi}_4 - 2\bar{\kappa}'\bar{\Psi}_1\right] \\
&+ \bar{\Psi}_1\left[\left(ð' - \bar{\tau}\right)\bar{\Psi}_4 - \left(þ' - 4\bar{\rho}'\right)\bar{\Psi}_3 - 3\bar{\kappa}'\bar{\Psi}_2\right], \qquad (7.15b)\\
\frac{\bar{I}}{2}V_{01'} = &-\bar{\Psi}_1\left[\left(ð - 4\bar{\tau}'\right)\bar{\Psi}_3 - \left(þ - \bar{\rho}\right)\bar{\Psi}_4 + 3\bar{\sigma}'\bar{\Psi}_2\right] \\
&+ 3\bar{\Psi}_2\left[\left(ð - 3\bar{\tau}'\right)\bar{\Psi}_2 - \left(þ - 2\bar{\rho}\right)\bar{\Psi}_3 + 2\bar{\sigma}'\bar{\Psi}_1 - \bar{\kappa}\bar{\Psi}_4\right] \\
&- 3\bar{\Psi}_3\left[\left(ð - 2\bar{\tau}'\right)\bar{\Psi}_1 - \left(þ - 3\bar{\rho}\right)\bar{\Psi}_2 + \bar{\sigma}'\bar{\Psi}_0 - 2\bar{\kappa}\bar{\Psi}_3\right] \\
&+ \bar{\Psi}_4\left[\left(ð - \bar{\tau}'\right)\bar{\Psi}_0 - \left(þ - 4\bar{\rho}\right)\bar{\Psi}_1 - 3\bar{\kappa}\bar{\Psi}_2\right], \qquad (7.15c)\\
\frac{\bar{I}}{2}V_{10'} = &-\bar{\Psi}_3\left[\left(ð' - 4\bar{\tau}\right)\bar{\Psi}_1 - \left(þ' - \bar{\rho}'\right)\bar{\Psi}_0 + 3\bar{\sigma}\bar{\Psi}_2\right] \\
&+ 3\bar{\Psi}_2\left[\left(ð' - 3\bar{\tau}\right)\bar{\Psi}_2 - \left(þ' - 2\bar{\rho}'\right)\bar{\Psi}_1 + 2\bar{\sigma}\bar{\Psi}_3 - \bar{\kappa}'\bar{\Psi}_0\right] \\
&- 3\bar{\Psi}_1\left[\left(ð' - 2\bar{\tau}\right)\bar{\Psi}_3 - \left(þ' - 3\bar{\rho}'\right)\bar{\Psi}_2 + \bar{\sigma}\bar{\Psi}_4 - 2\bar{\kappa}'\bar{\Psi}_1\right] \\
&+ \bar{\Psi}_0\left[\left(ð' - \bar{\tau}\right)\bar{\Psi}_4 - \left(þ' - 4\bar{\rho}'\right)\bar{\Psi}_3 - 3\bar{\kappa}'\bar{\Psi}_2\right]. \qquad (7.15d)
\end{aligned}$$

We have shown that when (7.6) holds, the space is conformal to a C-space, with the corresponding conformal factor $V$.

Imposing the condition that the Bach tensor must vanish, reduces the class from conformal C-spaces to a subset called *conformal Einstein spaces*. Alternatively, we can first find a solution to the Bach equations, then check if it is conformally Einstein by substituting the solution into (7.2) to see whether it vanishes. If it does, the spacetime can be transformed into an Einstein space with conformal factor, $V$, calculated from the Weyl spinor according to (7.12) (equivalently (7.15)) and (7.13).

## 7.3 Conformal factor for static spherically-symmetric spacetime

In section 6.2.2, we found two solutions to the Bach equations. According to the algebraic classification of the Weyl spinor (Petrov types) [90], the PP-wave spacetime is type $\{4\}$ (or $\{N\}$ for null), meaning all of its principal null directions coincide. In this case the complex invariant (7.1) vanishes, hence the proof in [57] does not apply. However, the spherically symmetric solution is type $\{22\}$ (or $\{D\}$ for doubly degenerate), where the principal null directions occur in two pairs. In this case $J \neq 0$, so the proof does apply. From (7.15), we find $V_{01'} = 0$, $V_{10'} = 0$ and

$$\begin{aligned} V_{00'} = V_{11'} &= \frac{6}{\bar{I}} \bar{\Psi}_2 \left( þ - 3\bar{\rho} \right) \bar{\Psi}_2 \\ &= \Psi_2^{-1} D \Psi_2 + 3\rho \\ &= -\rho \left( r \Psi_2^{-1} \dot{\Psi}_2 + 3 \right), \end{aligned} \tag{7.16}$$

where we have used $\bar{I} = 6\bar{\Psi}_2{}^2$, $\bar{\Psi}_2 = \Psi_2$ and $þ\Psi_2 = D\Psi_2 = -r\rho\dot{\Psi}_2$. On the other hand, taking the 00 component of (7.13) with indices lowered gives

$$V_{00'} = V^{-1} D V = -r\rho V^{-1} \dot{V}. \tag{7.17}$$

Equating (7.16) and (7.17) gives

$$V^{-1} \dot{V} = \Psi_2^{-1} \dot{\Psi}_2 + \frac{3}{r}, \tag{7.18}$$

for $\rho \neq 0$. Integrating (7.18) yields

$$V = \frac{r^3}{r_0} \Psi_2, \tag{7.19}$$

where $r_0$ is a constant of integration. From (7.14), it follows that $V^{-1}\Psi_2$ will satisfy the Einstein vacuum equations. Indeed, rearranging (7.19) gives

$$V^{-1} \Psi_2 = \frac{r_0}{r^3}, \tag{7.20}$$

which represents the non-zero Weyl component of the well-known Schwarzschild solution with $r_0 = -M$, where $M$ is the Newtonian mass. Accordingly, we can find $V$ by substituting (6.42) into (7.19) with $r_0 = c_2$, which gives

$$V = 1 + \frac{c_1 r}{c_2}. \tag{7.21}$$

Soon after the spherically symmetric solution to the Bach equations was found, it was noted that the solution was conformally related to a solution of the Einstein field equations [66]. This particular point was discussed further in [95], where the conformal factor was given explicitly and agrees with (7.21). Because our calculation uses the curvature directly, as opposed to the metric, our method represents a complementary approach. Moreover, because the Weyl scalar $\Psi_2$ is conformally weighted and invariant under transformations of the spin-basis [74], it allows the correct conformal factor to be identified straightforwardly as in (7.21).

We note that the generalisation to the axis-symmetric case, considered in [66], follows rapidly from the relationship between the Schwarzschild and Kerr solutions, which is manifest in the spin-coefficient formalism [70]. The Weyl spinor for the Kerr spacetime is calculated from the Schwarzschild solution by formally replacing $r \to r - ia\cos\theta$, where, $r, \theta$ are *oblate spheroidal* coordinates[92]. In [104], this transformation was shown to be a particular case of a more general transformation, which sends solutions to other solutions for a large sub-class of the *Kerr-Schild* metrics [1]. The same transformation can be used to find the axis-symmetric solution to the Bach equations, which will be similarly conformally related to the Kerr solution [66].

# Chapter 8

# Lanczos potential theory

In this chapter, we introduce an approach to solving the Einstein field equations known as *Lanczos potential theory* [58, 103, 10, 72], see [76] for a historical review. Our interest in Lanczos potential theory is due to the importance of the Lanczos tensor (spinor) in its role as a potential for the Weyl tensor (spinor). The Weyl spinor has especial significance in any conformal gravity theory, due to its conformal invariance, therefore, one expects the Lanczos spinor also to be significant.

The two-spinor formalism has proved itself to be very useful when analysing the Weyl-Lanczos equations, which are of central importance to Lanczos potential theory [64, 114, 105, 53, 22]. Furthermore, due to the considerable simplifications which occur, the spin-coefficient formalism has provided much success in the search for explicit solutions [4, 74]. We will apply this formalism to calculate the Lanczos spinor for the two solutions found in section (6.2).

## 8.1 Weyl-Lanczos equations

In [58], whilst analysing the self-dual part of the Riemann tensor in four-dimensional spacetime, Lanczos discovered a tensor, $H_{abc}$ of third-order. It was later shown in [103] that this tensor generates the Weyl tensor differentially. The corresponding set of equations are now referred to as the WL equations

$$\begin{aligned} C_{abcd} =& \nabla_d H_{abc} + \nabla_b H_{cda} + \nabla_c H_{bad} + \nabla_a H_{dcb} \\ &+ \nabla_e H^e{}_{ac} g_{bd} + \nabla_e H^e{}_{bd} g_{ac} - \nabla_e H^e{}_{ad} g_{bc} - \nabla_e H^e{}_{bc} g_{ad}, \end{aligned} \tag{8.1}$$

where $H_{abc}$ is the Lanczos tensor, skew-symmetric in $ab$. Equations (8.1) are obtained only after imposing the following gauge conditions

$${}^*H^{ab}{}_a = 0, \qquad H_{ab}{}^b = 0, \qquad \nabla_c H_{ab}{}^c = 0. \tag{8.2}$$

Equations (8.1) show that the Weyl tensor can be generated by differentiation from a potential-like equation in the same way that the Maxwell-Faraday tensor is obtained by differentiating the electromagnetic potential in Maxwell's theory.
The WL equations (8.1) are given in the spinor formulation in [74]. When (8.2) are applied they take the following simple form

$$\Psi_{ABCD} = 2\nabla_D^{D'} H_{ABCD'}, \tag{8.3}$$

where $H_{ABCD'}$ is the Lanczos spinor, symmetric in $AB$. The gauge conditions in spinor form are given by

$$H_{AB}{}^{B}{}_{C'} = 0, \qquad \nabla^{CC'} H_{ABCC'} = 0. \tag{8.4}$$

The first condition makes $H_{ABCD'}$ a symmetric spinor

$$H_{ABCC'} = H_{(ABC)C'}. \tag{8.5}$$

## 8.2 Weyl-Lanczos equations in terms of spin-coefficients

Using (5.23), we can translate the WL equations (8.3) into the GHP formalism

$$\tfrac{1}{2}\Psi_0 = (\eth - \bar{\tau}')H_{00} - (\text{þ} - \bar{\rho})H_{01} + 3\sigma H_{10} - 3\kappa H_{11}, \tag{8.6a}$$
$$\tfrac{1}{2}\Psi_1 = (\text{þ}' - \bar{\rho}')H_{00} - (\eth' - \bar{\tau})H_{01} + 3\tau H_{11} - 3\rho H_{11}, \tag{8.6b}$$
$$\tfrac{1}{2}\Psi_1 = (\eth - \bar{\tau}')H_{10} - (\text{þ} - \bar{\rho})H_{11} + \rho' H_{00} - \tau' H_{01} + 2\sigma H_{20} - 2\kappa H_{21}, \tag{8.6c}$$
$$\tfrac{1}{2}\Psi_2 = (\text{þ}' - \bar{\rho}')H_{10} - (\eth' - \bar{\tau})H_{11} - 2\rho H_{21} + 2\tau H_{20} - \sigma' H_{01} + \kappa' H_{00}, \tag{8.6d}$$
$$\tfrac{1}{2}\Psi_2 = (\eth - \bar{\tau}')H_{20} - (\text{þ} - \bar{\rho})H_{21} + 2\rho' H_{10} - 2\tau' H_{11} + \sigma H_{30} - \kappa H_{31}, \tag{8.6e}$$
$$\tfrac{1}{2}\Psi_3 = (\text{þ}' - \bar{\rho}')H_{20} - (\eth' - \bar{\tau})H_{21} - \rho H_{31} + \tau H_{30} - 2\sigma' H_{11} + 2\kappa' H_{10}, \tag{8.6f}$$
$$\tfrac{1}{2}\Psi_3 = (\eth - \bar{\tau}')H_{30} - (\text{þ} - \bar{\rho})H_{31} - 3\tau' H_{21} + 3\rho' H_{20}, \tag{8.6g}$$
$$\tfrac{1}{2}\Psi_4 = (\text{þ}' - \bar{\rho}')H_{30} - (\eth' - \bar{\tau})H_{31} - 3\sigma' H_{21} + 3\kappa' H_{20}, \tag{8.6h}$$

where we have labelled the components of the Lanczos spinor as

$$\begin{aligned}
H_{00} &= H_{ABCC'} o^A o^B o^C o^{C'}, \\
H_{10} &= H_{ABCC'} o^A o^B \iota^C o^{C'}, \\
H_{20} &= H_{ABCC'} o^A \iota^B \iota^C o^{C'}, \\
H_{30} &= H_{ABCC'} \iota^A \iota^B \iota^C o^{C'}, \\
H_{01} &= H_{ABCC'} o^A o^B o^C \iota^{C'}, \\
H_{11} &= H_{ABCC'} o^A o^B \iota^C \iota^{C'}, \\
H_{21} &= H_{ABCC'} o^A \iota^B \iota^C \iota^{C'}, \\
H_{31} &= H_{ABCC'} \iota^A \iota^B \iota^C \iota^{C'},
\end{aligned} \tag{8.7}$$

according to (5.11). This notation follows naturally from the translation formulae (5.23), although it is different to that used in [74]. According to (5.17), the Lanczos spinor components are weighted scalars of the following types

$$\begin{aligned} &H_{00} \quad (3,1), \\ &H_{10} \quad (1,1), \\ &H_{20} \quad (-1,1), \\ &H_{30} \quad (-3,1), \\ &H_{01} \quad (3,-1), \\ &H_{11} \quad (1,-1), \\ &H_{21} \quad (-1,-1), \\ &H_{31} \quad (-3,-1). \end{aligned} \tag{8.8}$$

The WL equations in the GHP formalism are interchanged by the prime and asterisk operations respectively in the following way,

$$H_{rs} \mapsto -H_{tu} \quad \begin{array}{ll} r \leftrightarrow t & s \leftrightarrow u \\ 0 \leftrightarrow 3 & 0 \leftrightarrow 1 \\ 1 \leftrightarrow 2 & \end{array}, \tag{8.9}$$

$$\begin{aligned} H_{00} &\leftrightarrow H_{01}, \\ H_{10} &\leftrightarrow H_{11}, \\ H_{20} &\leftrightarrow H_{21}, \\ H_{30} &\leftrightarrow H_{31}, \\ H_{01} &\leftrightarrow -H_{00}, \\ H_{11} &\leftrightarrow -H_{10}, \\ H_{21} &\leftrightarrow -H_{20}, \\ H_{31} &\leftrightarrow -H_{30}. \end{aligned} \tag{8.10}$$

## 8.3 Applications of the Weyl-Lanczos equations in spin-coefficient form

In section 6.2, we found the non-zero Weyl spinor components for two solutions to the Bach equations. The first was the PP-wave spacetime, which is in an example of a Petrov type $N$ spacetime, where the only non-zero Weyl scalar is $\Psi_4$. For a general type $N$ spacetime, comparing (8.6) with the NP field equations (5.29) suggests that the Lanczos coefficients can be given in terms of the spin coefficients [74]. Specialising to the PP-wave spacetime, the only non-zero spin coefficient is $\kappa'$, hence the one non-trivial equation from (8.6) is

$$\tfrac{1}{2}\Psi_4 = -\eth' H_{31}. \tag{8.11}$$

Comparing (8.11) to (6.22), we see that the only non-zero component of the Lanczos spinor is

$$H_{31} = \tfrac{1}{2}\kappa', \tag{8.12}$$

where the explicit solution follows from (6.25) by integration.
The second solution we found was that for a static spherically symmetric spacetime, which is an example of a Petrov type $D$ spacetime. For type $D$ spacetimes, it has been suggested that the following conditions on the Lanczos coefficients hold [74, 73]

$$\begin{aligned} H_{31} &= H_{00}, \\ H_{21} &= H_{10}, \\ H_{11} &= H_{20}, \\ H_{01} &= H_{30}, \end{aligned} \tag{8.13}$$

which can be understood as relating each Lanczos coefficient to the negative of its primed counterpart, in accordance with (8.9) and (6.37). Furthermore, for a static spherically symmetric spacetime, we may assume that $H_{rs} = H_{rs}(r, \theta)$. Substituting (8.13) into (8.6), we find

$$(\delta - 2\beta)H_{00} - (D - 2\gamma - \rho)H_{30} = 0, \tag{8.14a}$$
$$-(D + 4\gamma - \rho)H_{00} - (\delta + 4\beta)H_{30} = 3\rho H_{20}, \tag{8.14b}$$
$$\delta H_{10} - (D - \rho)H_{20} = \rho H_{00}, \tag{8.14c}$$
$$-(D + 2\gamma - \rho)H_{10} - (\delta + 2\beta)H_{20} = 2\rho H_{10} + \tfrac{1}{2}\Psi_2, \tag{8.14d}$$
$$(\delta + 2\beta)H_{20} - (D + 2\gamma - \rho)H_{10} = 2\rho H_{10} + \tfrac{1}{2}\Psi_2, \tag{8.14e}$$
$$-(D - \rho)H_{20} - \delta H_{10} = \rho H_{00}, \tag{8.14f}$$
$$(\delta + 4\beta)H_{30} - (D + 4\gamma - \rho)H_{00} = 3\rho H_{20}, \tag{8.14g}$$
$$-(D - 2\gamma - \rho)H_{30} - (\delta - 2\beta)H_{00} = 0. \tag{8.14h}$$

Taking linear combinations of (8.14) gives the following equivalent set of equations

$$(D - 2\gamma - \rho)H_{30} = 0, \tag{8.15a}$$
$$-(D + 4\gamma - \rho)H_{00} = 3\rho H_{20}, \tag{8.15b}$$
$$\delta H_{10} = 0, \tag{8.15c}$$
$$(\delta + 2\beta)H_{20} = 0, \tag{8.15d}$$
$$-(D + 2\gamma + \rho)H_{10} = \tfrac{1}{2}\Psi_2, \tag{8.15e}$$
$$-(D - \rho)H_{20} = \rho H_{00}, \tag{8.15f}$$
$$(\delta + 4\beta)H_{30} = 0, \tag{8.15g}$$
$$(\delta - 2\beta)H_{00} = 0, \tag{8.15h}$$

which can be solved to yield the following the Lanczos coefficients

$$\begin{aligned}
H_{00} &= H_{31} = 0, \\
H_{10} &= H_{21} = -\tfrac{2}{3}\gamma, \\
H_{20} &= H_{11} = 0, \\
H_{30} &= H_{01} = \rho \csc^2\theta.
\end{aligned} \tag{8.16}$$

The explicit solutions then follow from (6.37a), (6.37b) and (6.51).
We see from (8.12) and (8.16), that the non-zero Lanczos coefficients are proportional to the spin-coefficients $\kappa'$, $\gamma$ and $\rho$ respectively. This is true also in the analogous solutions to Einstein's field equations [74]. For the spherically symmetric case, the mathematical steps are identical to those in the Schwarzschild spacetime [75]. However, the explicit form of the Lanczos coefficients as functions of spacetime is different. Moreover, in accordance with the previous chapter, we find that the explicit solution corresponding to (8.16) is conformally related to the corresponding solution for the de Sitter-Schwarzschild spacetime with conformal factor given by (7.21).

# Chapter 9

# Duality rotations and helicity

In this chapter, we consider a second set of conformally invariant field equations called the *massless free-field equations*. We restrict our considerations to conformally flat spacetime, due to the algebraic consistency relations which exist for the massless field equations in conformally curved space [20, 89]. In contrast to the Bach equations, where comparatively few exact solutions are known, the general solution to the massless free-field equations, for arbitrary spin, is given by means of a contour integral expression involving arbitrary analytic functions [85]. A more detailed description is given in terms of twistor theory [84], where it was shown that the functions are more correctly described as elements of *first sheaf cohomology groups* [90, 52]. In light of this, instead of applying the two-spinor formalism to finding exact solutions, as we did for the Bach equations, we follow a different line of enquiry for the massless free-field equation. The motivation stems from a certain conserved quantity for the Maxwell equations, given in terms of a three-surface integral over a conserved current density [24]. There it was referred to as the 'screw action', on account of its equivalence to the expression for helicity when expanded into Fourier components. In [21], Noether's theorem was used to show that the corresponding symmetry was that of a duality transformation. Although both helicity and duality are central notions in spinor theory and even more so in twistor theory, no reference to a field conservation law for helicity has been found in the spinor-twistor literature. Instead, the connection between the helicity constant arising in particle theory and that arising in the description of massless fields is described via 'twistor first quantization' [78, 90]. That is, via a quantum description regarding the field as a one-particle state. One the other hand, while there exist other types of contour integrals which produce a twistor expression for conserved currents, the conserved quantities explicitly described are electric charge, energy-momentum and angular momentum [79, 78]. Therefore, it appeared that a study of the helicity constant in field theory and its relation to duality rotations using the two-spinor formalism would be worthwhile. At a minimum, it would be a new derivation of results derived previously using tensor methods.

## 9.1 Massless free-field equations

The massless free-field equations are given as follows

$$\nabla^{A_1 A'_1} \phi_{A_1 \ldots A_n} = 0, \tag{9.1}$$

where $\phi_{A_1 \ldots A_n}$ is totally symmetric and the spin $s$ of the field is equal to half the number of unprimed indices $s = \frac{n}{2}$. Taking the complex conjugate of (9.1) yields

$$\nabla^{A_1 A'_1} \bar{\phi}_{A'_1 \ldots A'_n} = 0, \tag{9.2}$$

where $\phi_{A'_1 \ldots A'_n}$ is again totally symmetric, also describes a massless field of spin $s$, equal to half the number of primed indices. Equations (9.1) and (9.2) are referred to as the anti- and self dual cases, respectively. This terminology corresponds to how they transform under the Hodge dual operation

$$*\phi_{A_1 \ldots A_n} = -i\phi_{A_1 \ldots A_n}, \tag{9.3}$$
$$*\bar{\phi}_{A'_1 \ldots A'_n} = i\bar{\phi}_{A'_1 \ldots A'_n}. \tag{9.4}$$

Physically important specific cases of (9.1) and (9.2) include the Weyl equation $n = 1$, the source-free Maxwell equations $n = 2$ and the linearised gravity equations $n = 4$ (weak-field Einstein vacuum equations). The corresponding case for $n = 0$ would be the wave equation

$$\nabla^{AA'} \nabla_{AA'} \phi = 0. \tag{9.5}$$

Assuming the spinors have a conformal weight of $-1$, such that

$$\hat{\phi}_{A_1 \ldots A_n} = \Omega^{-1} \phi_{A_1 \ldots A_n}, \tag{9.6}$$
$$\hat{\bar{\phi}}_{A_{1'} \ldots A_{n'}} = \Omega^{-1} \bar{\phi}_{A_{1'} \ldots A_{n'}}, \tag{9.7}$$

we can use (4.24) to show that (9.1) and (9.2) are invariant under the conformal rescaling (4.20), that is [89]

$$\hat{\nabla}^{A_1 A'_1} \hat{\phi}_{A_1 \ldots A_n} = \Omega^{-3} \nabla^{A_1 A'_1} \phi_{A_1 \ldots A_n} = 0, \tag{9.8}$$
$$\hat{\nabla}^{A_1 A'_1} \hat{\bar{\phi}}_{A'_1 \ldots A'_n} = \Omega^{-3} \nabla^{A_1 A'_1} \bar{\phi}_{A'_1 \ldots A'_n} = 0. \tag{9.9}$$

## 9.2 The helicity operator

### 9.2.1 Spin 1 Maxwell

In [36] the conserved quantity corresponding to the duality rotation,

$$\begin{aligned} F_{ab} &\rightarrow \cos\theta F_{ab} + \sin\theta {}^{*}F_{ab}\,, \\ {}^{*}F_{ab} &\rightarrow \cos\theta {}^{*}F_{ab} - \sin\theta F_{ab}, \end{aligned} \tag{9.10}$$

was derived. It was shown that the infinitesimal form of the transformation (9.10) changes the action by a total derivative, hence Noether's theorem applies. The corresponding Noether charge was calculated in both the second order Lagrangian and the first order Hamiltonian formulations. Written in terms of the Maxwell tensor, its dual and their respective potentials, the constant is

$$H_1 = \int_\Sigma \left(A^{b*}F_{ab} - C^b F_{ab}\right) \mathrm{d}^3x^a, \tag{9.11}$$

where

$$F_{ab} = \nabla_{[a}A_{b]}, \qquad {}^{*}F_{ab} = \nabla_{[a}C_{b]}. \tag{9.12}$$

Whilst the current appearing in the integrand of (9.11) depends on the choice of gauge (due to the presence of the gauge-dependent potentials), with suitable boundary conditions the integral is in fact gauge-invariant [2]. By expanding into Fourier components, equation (9.11) gives the difference between the number of right and left circularly polarised waves, furthermore, it agrees exactly with the *Stokes parameter V* [55, 16]. In particle physics, this would represent the difference between the number of right and left circularly polarised photons, known as the helicity [24, 21, 2, 12]. Helicity is in fact a general concept, which exists in many areas of physics. In particle physics, it is represented as the projection of the spin angular momentum in the direction of the linear momentum. In field theory, it has an analogous representation as the projection of the spin angular momentum in the direction of wave propagation. In every case, it is a pseudoscalar quantity related to the angular momentum or vorticity. For instance, in fluid mechanics it is used as a measure of the degree of knottedness of vortex lines [68], and in (magneto)hydrodynamics the first term of (9.11) measures the winding of magnetic lines of force, thereby characterising the topological configuration of these vortex lines [93].
Noether's theorem provides the link between symmetry transformations of spacetime and corresponding conserved quantities, like energy-momentum-angular-momentum. According to [12, 23, 18], the corresponding transformation for helicity is considered to be the duality rotation, cf. (9.10) in the case of the Maxwell field strength. A complementary method exhibiting this relationship makes use of the Pauli-Lubanski (PL) vector

$$S_a = \tfrac{1}{2} e_{abcd} p^b M^{cd}, \tag{9.13}$$

where $p^b$ is the momentum and $M^{cd}$ the angular momentum. In the case of zero rest mass particles and massless fields, (9.13) is a null vector and is proportional to the momentum [79]

$$S^a = sp^a, \tag{9.14}$$

where the proportionality constant $s$ is equal to the helicity. Using tensor methods, the relationship between the duality operator and the helicity was derived in [17, 6]. Using two-spinors, the spin-one-half case (Weyl equation) was derived in [46]. We will first give the analogous calculation for spin-one and spin-two and then give the general results for a spin $s$ massless field. The action of a Poincaré generator on a tensor or spinor field is defined by the Lie derivative of the field with respect to the associated Killing vector. According to (9.13) and (9.14), in order to derive the infinitesimal transformation corresponding to the helicity, we need to define the infinitesimal transformation corresponding to the momenta and angular-momenta. These are given as follows

$$\mathcal{L}_k \phi^{MN} = k^b \nabla_b \phi^{MN}, \tag{9.15}$$

where $k^b$ is constant Killing vector, and

$$\mathcal{L}_l \phi^{MN} = L^c{}_d x^d \nabla_c \phi^{MN} - L^{(MD'}{}_{XD'} \phi^{N)X}, \tag{9.16}$$

where

$$l^c = L^c{}_d x^d, \tag{9.17}$$

is a Killing vector and $L_{cd}$ is antisymmetric and constant. There product gives

$$\begin{aligned}
\mathcal{L}_k \mathcal{L}_l \phi^{MN} &= k^b \nabla_b \left( L^c{}_d x^d \nabla_c \phi^{MN} - L^{(MD'}{}_{XD'} \phi^{N)X} \right) \\
&= k^b \left( L^c{}_b \nabla_c \phi^{MN} + L^c{}_d x^d \nabla_b \nabla_c \phi^{MN} - L^{(MD'}{}_{XD'} \nabla_b \phi^{N)X} \right) \\
&= k^b L^{cd} \left( g_{bd} \nabla_c \phi^{MN} + x_d \nabla_b \nabla_c \phi^{MN} + \varepsilon_{C'D'} \varepsilon_C{}^{(M} \nabla_b \phi^{N)}_D \right).
\end{aligned} \tag{9.18}$$

In accordance with (9.13) we replace $k^b L^{cd}$ with $u_a e^{abcd}$

$$\begin{aligned}
u_a S^a \phi^{MN} &= u_a e^{abcd} \left( g_{bd} \nabla_c \phi^{MN} + x_d \nabla_b \nabla_c \phi^{MN} + \varepsilon_{C'D'} \varepsilon_C{}^{(M} \nabla_b \phi^{N)}_D \right) \\
&= u_a e^{abcd} \varepsilon_{C'D'} \varepsilon_C{}^{(M} \nabla_b \phi^{N)}{}_D \\
&= i u_{AA'} \left( \varepsilon^{AC} \varepsilon^{BD} \varepsilon^{A'D'} \varepsilon^{B'C'} - \varepsilon^{AD} \varepsilon^{BC} \varepsilon^{A'C'} \varepsilon^{B'D'} \right) \varepsilon_{C'D'} \varepsilon_C{}^{(M} \nabla_b \phi^{N)}_D \\
&= -i u^{B'(M} \nabla_{BB'} \phi^{N)B} - i u^{B'}_A \nabla^{(M}_{B'} \phi^{N)A} \\
&= i u_a \nabla^a \phi^{MN},
\end{aligned} \tag{9.19}$$

where the first two terms in the first line are symmetric in $b, d$, and hence vanish when contracted with $e^{abcd}$. In the third equality we substituted the spinor equivalent of the alternating tensor

$$e^{abcd} = i\varepsilon^{AC}\varepsilon^{BD}\varepsilon^{A'D'}\varepsilon^{B'C'} - i\varepsilon^{AD}\varepsilon^{BC}\varepsilon^{A'C'}\varepsilon^{B'D'}. \tag{9.20}$$

In the last equality we have used Maxwell's equations such that, in the fourth line, the first term vanishes and the second term is totally symmetric in $M, N$ and $A$.
Comparing (9.19) with (9.14) and (9.15) we find that

$$\hat{s}\phi_{MN} = i\phi_{MN}, \tag{9.21}$$

where we distinguish the helicity operator with a hat over the symbol $s$. According to (9.21), for a spin-1 Maxwell spinor the infinitesimal transformation corresponding to the helicity is simply multiplication by the complex unit $i$. As emphasised in [17], in the context of particle physics, the helicity operator is almost always represented via momentum eigenstates [91]. But as the PL vector method has shown, this decomposition is not strictly necessary. In the standard way, we can obtain the finite transformation from the infinitesimal one by exponentiating

$$\phi_{MN} \to e^{i\theta}\phi_{MN}. \tag{9.22}$$

The Maxwell spinor defines the anti-self-dual part of the Maxwell tensor

$$\phi_{AB}\varepsilon_{A'B'} = \frac{1}{2}\left(F_{ab} + i^*F_{ab}\right). \tag{9.23}$$

Substituting (9.22) into (9.23) and taking the real imaginary parts yields the duality rotations (9.10). In the case of spin-one, we have shown that the finite transformation (9.22) is precisely the duality rotation (9.10). This agrees with the interpretation that the generator of duality rotations, via Noether's theorem, corresponds to the conserved quantity helicity [12, 23, 18]. In the next section we give the analogous calculation for spin-two.

### 9.2.2 Spin 2 Linear Gravity

Since the derivation of the action of the helicity operator for spin-two linear gravity is analogous to that of spin-one, in [17, 6] the final result was given without the explicit calculation. We include the calculation here in terms of two-spinors. The tensor quantity analogous to the Maxwell field is the first order tensor $K_{abcd}$ which, in the absence of sources, is defined as [89]

$$K_{abcd} = \lim\left(u^{-1}C_{abcd}\right), \tag{9.24}$$

where $u$ is a parameter such that when $u = 0$ we recover Minkowski spacetime, that is $C_{abcd} = 0$. However, in the limit $u \to 0$, the limit (9.24) is well defined. This procedure is equivalent to the more common description of linearised Einstein theory in terms of a perturbation potential field $h_{ab}$, where

$$g_{ab}(u) = g_{ab} + uh_{ab} + O(u^2). \tag{9.25}$$

In terms of the first order Weyl tensor, the corresponding duality rotation takes the following form

$$\begin{aligned} K_{abcd} &\to \cos\theta K_{abcd} + \sin\theta^* K_{abcd}, \\ {}^*K_{abcd} &\to \cos\theta^* K_{abcd} - \sin\theta K_{abcd}. \end{aligned} \tag{9.26}$$

As in (9.18), we take the product of the Lie derivatives corresponding to the linear and angular momentum and apply the result to the first order Weyl spinor $\phi_{ABCD}$,

$$\begin{aligned} \mathcal{L}_k\mathcal{L}_l\phi^{KLMN} &= k^b\nabla_b\left(L^c{}_d x^d\nabla_c\phi^{KLMN} - 2L^{(KD'}{}_{XD'}\phi^{LMN)X}\right) \\ &= k^b\left(L^c{}_b\nabla_c\phi^{KLMN} + L^c{}_d x^d\nabla_b\nabla_c\phi^{KLMN} - 2L^{(KD'}{}_{XD'}\nabla_b\phi^{LMN)X}\right) \\ &= k^b L^{cd}\left(g_{bd}\nabla_c\phi^{KLMN} + x_d\nabla_b\nabla_c\phi^{KLMN} + 2\varepsilon_{C'D'}\varepsilon_C{}^{(K}\nabla_b\phi_D^{LMN)}\right). \end{aligned} \tag{9.27}$$

In accordance with (9.13) we replace $k^b L^{cd}$ with $u_a e^{abcd}$,

$$\begin{aligned} u_a S^a\phi^{KLMN} &= u_a e^{abcd}\left(g_{bd}\nabla_c\phi^{KLMN} + x_d\nabla_b\nabla_c\phi^{KLMN} + 2\varepsilon_{C'D'}\varepsilon_C{}^{(K}\nabla_b\phi_D^{LMN)}\right) \\ &= 2u_a e^{abcd}\epsilon_{C'D'}\varepsilon_C{}^{(K}\nabla_b\phi^{LMN)}{}_D \\ &= 2iu_{AA'}\left(\epsilon^{AC}\epsilon^{BD}\epsilon^{A'D'}\epsilon^{B'C'} - \epsilon^{AD}\epsilon^{BC}\epsilon^{A'C'}\epsilon^{B'D'}\right)\epsilon_{C'D'}\varepsilon_C{}^{(K}\nabla_b\phi^{LMN)}{}_D \\ &= -2iu^{B'(K}\nabla_{BB'}\phi^{LMN)B} - 2iu_A^{B'}\nabla_{B'}^{(K}\phi^{LMN)A} \\ &= 2iu_a\nabla^a\phi^{KLMN}, \end{aligned} \tag{9.28}$$

where the first two terms in the first line are symmetric in $b, d$ and hence vanish when contracted with $e^{abcd}$. In the third equality we have substituted (9.20). In the last equality we have used the spin-two massless free-field equations such that, in the fourth line, the first term vanishes and the second term is totally symmetric in $K, L, M, N$ and $A$.
Comparing (9.19) with (9.14) and (9.15) we find that

$$\hat{s}\phi_{KLMN} = 2i\phi_{KLMN}, \tag{9.29}$$

such that for a anti-self-dual spin-2 Weyl spinor the infinitesimal transformation corresponding to the helicity is simply multiplication by $2i$. In the standard way, we can obtain the finite transformation from the infinitesimal one by exponentiating

$$\phi_{KLMN} \to e^{2i\theta}\phi_{KLMN}. \tag{9.30}$$

The first-order Weyl spinor defines the anti-self-dual part of the first-order Weyl tensor

$$\phi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} = \frac{1}{2}\left(K_{abcd} + i^*K_{abcd}\right). \tag{9.31}$$

Substituting (9.30) into (9.31) and taking the real imaginary parts does not produce precisely the duality rotation (9.26) because of the factor two, which necessarily appears due to the first-order Weyl spinor having four indices, in contrast to the Maxwell spinor, which has two indices.

#### Spin $s$ massless fields

The previous cases of spin-one and spin-two massless fields easily generalise to spin $s$ massless fields

$$\hat{s}\phi_{A_1\ldots A_n} = \tfrac{n}{2}i\phi_{A_1\ldots A_n}, \tag{9.32}$$

such that the infinitesimal transformation corresponding to the helicity is simply multiplication by $\frac{n}{2}i$. In the standard way, we can obtain the finite transformation from the infinitesimal one by exponentiating

$$\phi_{A_1\ldots A_n} \to e^{\frac{ni}{2}\theta}\phi_{A_1\ldots A_n}. \tag{9.33}$$

An advantage of the two-spinor formalism is that the spin-one-half case (Weyl equation) is included in (9.32) on the same footing as the other spin values. When acting on spinors with integral spin (even number of indices), the essential part of the helicity operator is the duality operator, which agrees with known results using tensor methods [17, 6]. Nevertheless, the multiplying factor $s = \frac{n}{2}$, indicates that for arbitrary spin values, the correlation between helicity and duality rotations is not as direct as in the spin-one case, where the helicity transformation is precisely that of a duality rotation. The transformation which corresponds precisely to the helicity constant, in the sense of Noether's theorem, is given later using twistor theory and standard methods from symplectic geometry. Moreover, the origin of the multiplying factor can also be better understood in that context. Therefore, in the next section we introduce the necessary concepts from symplectic geometry, which will also allow us to derive the helicity constant for spin $s$ fields.

## 9.3 Symplectic two-form

In the previous chapter we derived the transformation corresponding to the helicity constant. In this section we make use of this transformation to derive the helicity constant. In order to achieve this, we give a brief introduction to the *covariant phase space* construction for field theories, restricting ourselves to the concepts required to derive conserved quantities. The general theory is given in [31, 112] and a concise summary in [7]. The advantage of the covariant phase construction over non-covariant constructions is that is does not require a

preferred instant of time. It achieves this by constructing a phase space isomorphic to the space of dynamically allowed histories. In non-covariant constructions, this is achieved by fixing an instant of time. But this is not necessary for our purposes. For there exists a symplectic two-form on the phase space which is Lie-dragged by the dynamical vector field, and is therefore independent of the choice of initial instant. We can use the symplectic two-form to derive an expression for conserved quantities. In general, they are obtained as the generators of canonical transformations corresponding to symmetries of the system. Using tensor methods, the authors in [40] derived the expression for the helicity constant for the Maxwell spin-one case. Using spinor methods we derive the same expression for spin-s massless fields.
We assume that the dynamics are specified by a Lagrangian density

$$L(\phi, \nabla\phi), \tag{9.34}$$

which is an invariant function of the fields and their first covariant derivative. The field equations are obtained by requiring that the variation of the action integral

$$S = \int_V L(\phi, \nabla\phi)\epsilon, \tag{9.35}$$

where $\epsilon$ is the space-time volume element, should vanish under any variation $X^\alpha$ of $\phi^\alpha$ which vanishes on the boundary of $V$. The variation of the action is

$$dS = \int_V \left(\frac{\partial L}{\partial\phi^\alpha} - \nabla_a \frac{\partial L}{\partial\phi^\alpha_a} X^\alpha\right)\epsilon + \int_\Sigma \frac{\partial L}{\partial\phi^\alpha_a} X^\alpha \mathrm{d}\sigma_a, \tag{9.36}$$

where $\phi^\alpha_a = \nabla_a\phi^\alpha$ and $d\sigma_a = n_a d\sigma$, $d\sigma$ is volume element and $n^a$ the unit normal on the boundary $\Sigma = \partial V$. The second term defines a potential one-form $\theta$ which in general depends on $\Sigma$

$$\theta(X) = \int_\Sigma \frac{\partial L}{\partial\phi^\alpha_a} X^\alpha \mathrm{d}\sigma_a, \tag{9.37}$$

and resembles the analogous expression $\theta = p_a dq^a$ from particle mechanics. Taking the exterior derivative, we obtain

$$\omega(X, Y) = \frac{1}{2}\int_\Sigma \omega^a \mathrm{d}\sigma_a \tag{9.38}$$

where $X$ and $Y$ are solutions of the linearised equations. The symplectic current $\omega^a$ is defined as

$$\omega^a = \frac{\partial^2 L}{\partial\phi^\beta\partial\phi^\alpha_a}\left(Y^\alpha X^\beta - X^\alpha Y^\beta\right) + \frac{\partial^2 L}{\partial\phi^\beta_b\partial\phi^\alpha_a}\left(Y^\alpha\nabla_b X^\beta - X^\alpha\nabla_b Y^\beta\right). \tag{9.39}$$

Due to the linearised field equations $\nabla_a\omega^a = 0$, therefore $\omega(X, Y)$ is independent of $\Sigma$ assuming the linearised fields tend to zero fast enough at spatial infinity. Furthermore, because $\omega$ is the exterior derivative a one-form, it is closed $d\omega = 0$ and represents the (pre)symplectic two-form. In general, the two-form will be degenerate; motions corresponding to gauge transformations give rise to directions in which $\omega$ vanishes. The gauge freedom is removed by factoring out the integral manifold of the degenerate directions resulting in a reduced phase space with non-degenerate symplectic two-form. Let $X$ be an infinitesimal canonical transformation, by definition we then have $\mathcal{L}_X\omega = 0$. Furthermore, if there exists a potential $\theta$ that is also Lie dragged by $X$, then we have

$$\mathcal{L}_X\theta \equiv i_X\omega + d(\theta(X)) = 0, \tag{9.40}$$

where $i_X$ is the interior product such that

$$i_X\omega = \omega(X, \cdot), \tag{9.41}$$

and $H = \theta(X)$ is the unique Hamiltonian (up to a constant) which generates the canonical transformation $X$. In symplectic geometry, $\theta(X)$ is referred to as the moment map. When a symmetry group exists, each element of the corresponding Lie algebra gives rise to a vector field $X$ which preserves the two-form $\omega$. For a given trajectory in the phase space, the conserved quantities $\theta(X)$ map elements of the Lie algebra linearly to constants $H$, that is, they are elements of the dual of the Lie algebra. For the Poincaré group, the constants include the energy-momentum-angular-momentum, whence the name moment map.

## 9.4 The helicity expression for integer spin massless fields

For a spin $s$ anti-self-dual field, the canonical transformation corresponding to the helicity constant was given in (9.32), such that the infinitesimal helicity transformation for a spin-s anti-self-dual fields is simply multiplication by $is$, whereas for self-dual fields it is multiplication by $-is$. We denote the corresponding 'helicity Hamiltonian' as $H_s$, which can be identified up to a constant by the following relation (9.40)

$$i_s\omega = \omega(\hat{s}, \cdot) = -dH_s. \tag{9.42}$$

We will consider the helicity expression for different values of integer spin. Firstly, a scalar field satisfying the wave equation (9.5) has zero indices, $n = 0$. Therefore, according to (9.32) it has zero helicity, as expected. Let us then start with the spin-one case. Substituting the Maxwell Lagrangian into (9.38) and (9.39), we find the corresponding symplectic form [112]

$$\omega(\delta\phi, \delta\phi') = \int_\Sigma \left(\delta\phi_{AB}\delta\bar{\eta}'^{B}_{A'} - \delta\phi'_{AB}\delta\bar{\eta}^{B}_{A'} + c.c.\right) \mathrm{d}\sigma^{AA'}, \tag{9.43}$$

where *c.c.* stands for complex conjugate and $\delta\phi$ and $\delta\phi'$ are solutions of the linearised equations. Some authors would write simply $\phi$ when the field equations are linear to begin with, as is the case here. We do not follow this convention however, instead we represent our variations always as $\delta\phi$, where they were previously written as $X$ and $Y$. The $\delta$ satisfies the properties of the exterior derivative on the space of solutions [34]. According to (9.42), we calculate the helicity by using the interior product of the phase space vector, corresponding to the infinitesimal helicity transformation (9.32), with the symplectic two-form,

$$\begin{aligned}\omega(\hat{s}\phi, \delta\phi) &= \int_\Sigma \left(i\phi_{AB}\delta\bar{\eta}^B_{A'} - \delta\phi_{AB}(-i)\bar{\eta}^B_{A'} + c.c.\right) \mathrm{d}\sigma^{AA'} \\ &= i\delta \int_\Sigma \left(\phi_{AB}\bar{\eta}^B_{A'} - \bar{\phi}_{A'B'}\eta^{B'}_A\right) \mathrm{d}\sigma^{AA'}. \end{aligned} \tag{9.44}$$

Comparing (9.44) with (9.42) we find

$$H_1(\phi) = i \int_\Sigma \left(\phi_{AB}\bar{\eta}^B_{A'} - \bar{\phi}_{A'B'}\eta^{B'}_A\right) \mathrm{d}\sigma^{AA'}, \tag{9.45}$$

where the potential satisfies the following equation[79][1]

$$\nabla^{AA'}\eta^{B'}_A = 0, \tag{9.46}$$

and the Maxwell potential gives rise to the Maxwell spinor

$$\phi_{AB} = \nabla_{AA'}\eta^{A'}_B, \tag{9.47}$$

which is symmetric due to (9.46). The Maxwell spinor with unprimed indices represents the complex anti-self-dual part of the Maxwell tensor

$$\phi_{AB}\varepsilon_{A'B'} = \frac{1}{2}\left(F_{ab} + i^*F_{ab}\right), \tag{9.48}$$

whereas the spinor with primed indices represents the complex self-dual part

$$\bar{\phi}_{A'B'}\varepsilon_{AB} = \frac{1}{2}\left(F_{ab} - i^*F_{ab}\right). \tag{9.49}$$

The complex potential $\eta^{AA'}$ can also be written in terms of its real and imaginary parts

$$\eta^{AA'} = \frac{1}{2}\left(A^a - iC^a\right), \tag{9.50}$$

and

$$\bar{\eta}^{AA'} = \frac{1}{2}\left(A^a + iC^a\right), \tag{9.51}$$

which represent the usual Maxwell potential and the potential for the dual tensor, respectively [14]. Substituting eqs. (9.48) to (9.51) into (9.45) recovers (9.11). Our result is precisely the spinor form of the tensor expression given in [18]. The tensor form is written as the imagi-

[1]In [3] the symmetric part of (9.47) is called an adjoint equation, which is essential to their general classification method of conservation laws.

nary part of a complex quantity, which is the contraction of the right hand sides of (9.48) and (9.51).

In the case of spin-two linearised gravity we have the following symplectic form

$$\omega(\delta\phi,\delta\phi') = \int_\Sigma \left(\delta\phi_{ABCD}\delta\bar{\eta}'^{BCD}_{A'} - \delta\phi'_{ABCD}\delta\bar{\eta}^{BCD}_{A'} + c.c.\right) \mathrm{d}\sigma^{AA'}. \tag{9.52}$$

We calculate the helicity as follows

$$\begin{aligned}\omega(\hat{s}\phi,\delta\phi) &= \int_\Sigma \left(2i\phi_{ABCD}\delta\bar{\eta}^{BCD}_{A'} - \delta\phi_{ABCD}(-2i)\bar{\eta}^{BCD}_{A'} + c.c.\right) \mathrm{d}\sigma^{AA'} \\ &= 2i\delta\int_\Sigma \left(\phi_{ABCD}\bar{\eta}^{BCD}_{A'} - \bar{\phi}_{A'B'C'D'}\eta^{B'C'D'}_{A}\right) \mathrm{d}\sigma^{AA'}.\end{aligned} \tag{9.53}$$

Comparing (9.53) with (9.42) we find

$$H_2(\phi) = 2i\int_\Sigma \left(\phi_{ABCD}\bar{\eta}^{BCD}_{A'} - \bar{\phi}_{A'B'C'D'}\eta^{B'C'D'}_{A}\right) \mathrm{d}\sigma^{AA'}. \tag{9.54}$$

The potential chain is given as [88]

$$\begin{aligned}\nabla_{BB'}\eta^{B'C'D'}_{A} &= \chi^{C'D'}_{AB}, \\ \nabla_{CC'}\chi^{C'D'}_{AB} &= \gamma^{D'}_{ABC}, \\ \nabla_{DD'}\gamma^{D'}_{ABC} &= \phi_{ABCD},\end{aligned} \tag{9.55}$$

where all fields are totally symmetric in their primed and unprimed indices separately. Furthermore, they satisfy the following

$$\begin{aligned}\nabla^{AA'}\eta^{B'C'D'}_{A} &= 0, \\ \nabla^{AA'}\chi^{C'D'}_{AB} &= 0, \\ \nabla^{AA'}\gamma^{D'}_{ABC} &= 0, \\ \nabla^{AA'}\phi_{ABCD} &= 0.\end{aligned} \tag{9.56}$$

The first-order Weyl spinor with unprimed indices represents the complex anti-self-dual part of the first-order Weyl tensor

$$\phi_{ABCD}\varepsilon_{A'B'}\varepsilon_{'D'} = \frac{1}{2}\left(K_{abcd} + i^{*}K_{abcd}\right), \tag{9.57}$$

whereas the spinor with primed indices represents the complex self-dual part

$$\bar{\phi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} = \frac{1}{2}\left(K_{abcd} - i^{*}K_{abcd}\right). \tag{9.58}$$

Our expression (9.54) looks seemingly different to that given in [11], but this is only due to it being presented in different variables. For fields tending to zero at infinity, we can integrate (9.54) by parts using (9.55) [88],

$$\int \phi_{ABCD}\bar{\eta}^{BCD}_{A'} d\sigma^{AA'} = -\int \gamma^{D'}_{ABC}\bar{\chi}^{BC}_{A'D'}\mathrm{d}\sigma^{AA'}. \tag{9.59}$$

The potential $\chi^{C'D'}_{AB}$ represents the complex linearised metric, its real and imaginary parts give the two potentials used in [11], which are the usual linear metric potential and the potential for the dual Weyl tensor. The $\gamma^{D'}_{ABC}$ represent the linearised spin-coefficients (or Christoffel symbols) which, in accordance with (8.3) and (9.55), resemble a linearised version of the Lanczos spinor. In [50], yet another set of variables, referred to as *superpotentials*, is used to represent the helicity constant. These superpotentials are examples of Hertz type potentials [82, 99].
For integer $s$, the general expression for the helicity is given as

$$H_s(\phi) = is\int_\Sigma \left(\phi_{A_1A_2\ldots A_n}{}^{(1)}\bar{\eta}^{A_2\ldots A_n}_{A'_1} - \bar{\phi}_{A_1'A_2'\ldots A_n'}{}^{(1)}\eta^{A'_2\ldots A'_n}_{A_1}\right)\mathrm{d}\sigma^{A_1A_1'}, \tag{9.60}$$

where $s = \frac{n}{2}$. The corresponding spin $s$ potential chain is given as [78, 79]

$$\nabla_{A_{r+1}A'_{r+1}}{}^{(r)}\eta^{A'_{r+1}A'_{r+2}\ldots A'_{n-r}}_{A_1\ldots A_r} = {}^{(r+1)}\eta^{A'_{r+2}\ldots A'_{n-r}}_{A_1\ldots A_{r+1}}, \tag{9.61}$$

where $r \leqslant n$ such that the last field in the recursive chain is

$${}^{(n)}\eta_{A_1\ldots A_n} = \phi_{A_1\ldots A_n}. \tag{9.62}$$

Every field in the chain satisfies

$$\nabla^{A_rA'_r}{}^{(r)}\eta^{A'_{r+1}\ldots A'_{n-r}}_{A_1\ldots A_r} = 0, \tag{9.63}$$

when $n = r$ we recover the massless field equations (9.1). In order to have the correct dimension of helicity (or action), our helicity expressions eqs. (9.45), (9.54) and (9.60) should have Planck's constant $\hbar$ reinstated. This is done via (9.32), as in [6]. However, in contrast to the derivation given in [6], we have not introduced a positive/negative frequency decomposition of the fields. As is well known, this splitting is used in quantum field theory in order to construct a Hilbert space of states, with a corresponding positive definite Hermitian scalar product [91]. This can be introduced via an (almost) complex structure, which we discuss next, since it has relevance also for duality and hence helicity.

## 9.5 Complex structure

One approach to defining a complex manifold $M$ is to define an extra structure on a smooth real manifold. This extra structure is referred to as a complex structure [30]. A complex structure $J$ on a real vector space $V$ can be defined by a real tensor $J$ (linear map) satisfying [52]

$$J^2 = -1. \tag{9.64}$$

An *almost complex structure* $M$ is a tensor field of complex structures on the tangent bundle $TM$. It is called integrable if a certain tensor constructed from $J$, called the Nijenhuis tensor, is zero. If $J$ is integrable then $M$ is a complex manifold with almost complex structure $J$ [69]. In the complexified vector space $V_{\mathbb{C}}$, $J$ has two eigenspaces with the eigenvalues $\pm i$. Therefore, from (9.32) we see that the infinitesimal transformation corresponding to the helicity is a constant multiple of an (almost) complex structure, in this case the hodge duality operator. In four dimensions, the corresponding eigenspaces of the duality operator are two-forms that are either self-dual or anti-self-dual. Moreover, since these can be represented respectively by unprimed and primed symmetric spinors, the corresponding complex vector space can be thought of as spin-space.
A different complex structure, which is employed in quantum field theory, gives a decomposition of the field into positive and negative frequencies as follows

$$J(\phi) = i\phi^+ - i\phi^-, \tag{9.65}$$

where $\phi$ is a spinor or tensor field (indices suppressed) and $\phi^{\pm}$ represent the positive negative frequency parts of $\phi$. In quantum field theory, the vector space $V$ of real solutions $\phi$ to the field equations is made into a complex vector space via (9.65), where multiplication is defined as

$$(a + ib).\phi := a\phi + bJ.\phi, \tag{9.66}$$

for real $a$ and $b$. The complex structure $J$ is *compatible* with the symplectic structure, which means it gives rise to a Hermitian scalar product

$$\langle\phi|\phi'\rangle = \omega(\phi, J\phi) + i\omega(\phi, \phi'), \tag{9.67}$$

which is positive definite so that $(V, \langle|\rangle)$ is a Hilbert space. The complex structure defined by (9.65) makes the action of the Poincaré group on the Hilbert space unitary. The helicity operator on the Hilbert space is then given as follows [6]

$$\hat{S} = -sJ.D, \tag{9.68}$$

where $J$ provides the positive/negative frequency splitting and $D$ is the Hodge duality operator. The corresponding helicity is defined as

$$H_S(\phi) = \langle \phi | \hat{S} \phi \rangle \,. \tag{9.69}$$

Substituting (9.68) and (9.65) into (9.67), we find that (9.69), where $\phi$ is regarded as an element on the one-particle Hilbert space, agrees with the helicity expression (9.60) given earlier. Therefore, we have two equivalent expressions interpreted respectively as the helicity of a one-particle state and of the corresponding classical field. The same equivalence, but for the energy, was postulated in [5] in order to single out the unique complex structure (9.65). We note that when $\phi$ is regarded as an element on the one-particle Hilbert space, with the corresponding positive/negative frequency decomposition (9.65), the corresponding helicity is positive if it has positive frequency, whereas it is negative if it has negative frequency. Therefore, as emphasised in [6], for a quantum field that has both positive- and negative-frequency components, there is no direct relationship between duality and helicity.

## 9.6 Conserved quantities in particle mechanics and field theory

From (9.32), we have shown that the infinitesimal transformation corresponding to the helicity constant is essentially the duality rotation acting as an (almost) complex structure. However, it also included multiplication by a half-integer, representing the spin of the massless field. In [11], when comparing the action of the duality operator in the spin-one and spin-two cases, the author correctly noted that an extra factor of two needs to be included in the latter case due to the different values of the spin. However, because only the duality transformation was considered, the multiplying factor representing the spin was missing, and so it was necessary to put this factor in by hand. In contrast, using the PL vector method, the half-integer representing the spin value appears automatically from the Lie derivatives of the rotations. Although the PL vector has its origins in particle mechanics, we have shown it to be useful in deriving a field conservation law also. The basic reason being its relation to symmetry transformations of spacetime and the essential identity between how conservation laws in particle mechanics and field theory arise. In this section, we describe this process in general terms.

It is in the presence of a Killing vector or conformal Killing vector, that there is a close similarity in the way in which conserved quantities arise in particle mechanics and in the continuous case of fields. For test particles, where we assume the particle does not back-react on the spacetime, we assume they move on geodesics, which means the particle's momentum is parallelly propagated

$$p^a \nabla_a p^b = 0, \tag{9.70}$$

where $p^a$ is the particle's momentum, which is tangent to the particle's world line. If a (conformal) Killing vector $\xi^b$ exists, then we have

$$p^a \nabla_a \left(\xi_b p^b\right) = \xi_b p^a \nabla_a p^b + p^a p^b \nabla_{(a}\xi_{b)}. \tag{9.71}$$

The first term on the right vanishes due to (9.70). The second term will vanish if $\xi^b$ is a Killing vector due to Killing's equation $\nabla_{(a}\xi_{b)} = 0$. If it is a (proper) conformal Killing vector then $\nabla_{(a}\xi_{b)} \propto g_{ab}$, so that we must also have $p^a$ null. In either case we obtain a conserved quantity $Q$ along the particle's worldline

$$Q = \xi_b p^b. \tag{9.72}$$

Furthermore, due to the linearity of $p^a$ in (9.72), if the sum of the momenta is conserved in collisions between particles, then the sum of the quantities $Q$ will also be conserved. We can interpret this as having a conservation law for the system as a whole in the sense that the total flux of $Q$ across the boundary of a region of spacetime must be zero. In this case we can expect a close similarity with the continuous case of fields. The quantity (9.72) may be generalised to the following non-linear expression

$$Q = \xi_{a\ldots d} p^a \ldots p^d, \tag{9.73}$$

where $\xi_{a\ldots d} = \xi_{(a\ldots d)}$ is referred to as a Killing tensor, which satisfies

$$\nabla_{(e}\xi_{a\ldots d)} = 0. \tag{9.74}$$

If (9.74) holds, then (9.73) is conserved along the particle's worldline. However, due to the non-linearity in the momenta, the total $Q$ will not be conserved in particle collisions. We may also generalise (9.74) in the following way

$$\nabla_{(e}\xi_{ab\ldots d)} = g_{(ea}\eta_{b\ldots d)} \tag{9.75}$$

for some $\eta_{b\ldots d}$. In this case, we still get a conserved quantity if $p^a$ is null. Equation (9.75) states that the symmetric trace-free part of $\nabla_{(e}\xi_{ab\ldots d)}$ is zero. A symmetric trace-free tensor is represented in terms of a spinor which is symmetric in all its primed indices and unprimed indices respectively[89]. Therefore, the spinor translation of (9.75) is

$$\nabla^{(E}_{(E'}\xi^{A\ldots D)}_{A'\ldots D')} = 0, \tag{9.76}$$

where without loss of generality we can also assume symmetry in both sets of indices

$$\xi^{A\ldots D}_{A'\ldots D'} = \xi^{(A\ldots D)}_{(A'\ldots D')}, \tag{9.77}$$

where it is referred to as a *Killing spinor*. Equation (9.76) can incorporate cases with unequal number of primed and unprimed indices, where the use of spinors is now essential. In this case, we obtain a complex quantity $Q$ given by

$$Q = \xi^{A\ldots DF'\ldots K'}\bar{\pi}_A \ldots \bar{\pi}_D \pi_{F'} \ldots \pi_{K'}, \tag{9.78}$$

where $p^a = \bar{\pi}^A \pi^{A'}$ is the future-null momentum. The quantity Q will be conserved if

$$p^a \nabla_a \pi^{B'} = 0, \tag{9.79}$$

which says that the flag plane of $\pi^{B'}$ must be parallelly propagated in addition to the flag-pole $\bar{\pi}^B \pi^{B'}$. When the number of primed and unprimed indices are equal, the Killing spinor $\xi^{A\ldots DA'\ldots D'}$ is Hermitian and can be represented as the tensor $\xi^{a\ldots d}$, in which case we get a real $Q$ according to (9.73). In the case of just one spinor index, the expression (9.78) is linear in either $\pi^{A'}$ or $\bar{\pi}^A$. Therefore, in analogy to the case of linear momentum, we can expect a corresponding conservation law for the whole system. Accordingly, this conservation law should be evident in the case of continuous fields, which we turn to now.

In the case of a continuous field, the symmetric energy-momentum tensor $T_{ab} = T_{ba}$ takes the role of the momentum. For suppose the continuity equation holds

$$\nabla^a T_{ab} = 0. \tag{9.80}$$

Then we have

$$\nabla^a \left(\xi^b T_{ab}\right) = \xi^b \nabla^a T_{ab} + T_{ab} \nabla^{(a} \xi^{b)}. \tag{9.81}$$

The first term vanishes due to (9.80). If $\xi^b$ is a Killing vector the second term vanishes. Else, if it is a (proper) conformal Killing vector then we must also have $T_{ab}$ traceless. In either case we obtain a conserved current $J^a$, where

$$J_a = \xi^b T_{ab}, \tag{9.82}$$

and

$$\nabla_a J^a = 0. \tag{9.83}$$

This shows the similarity, particularly for currents linear in energy-momentum, to the particle case. Moreover, further currents exist, in addition to the helicity and the energy-momentum-stress. In [3], a complete classification was given of all locally constructed conserved currents associated with the massless free field equations (9.1). These include generalised *zilch currents* [61], and less well studied *chiral currents*, which have odd parity under a duality rotation [3].

In field theory, we showed that conserved quantities arise from a conserved current according to

(9.83). The link between a conserved current and its corresponding conserved quantity, often referred to as a generalised charge, is provided by the fundamental theorem of exterior calculus, which in different dimensions is more commonly referred to as the divergence theorem of Gauss, or the Kelvin-Stokes curl theorem. In differential form notation, the theorem is as follows

$$\int_V \mathrm{d}^*\boldsymbol{J} = \int_{\partial V} {}^*\boldsymbol{J}, \tag{9.84}$$

where ${}^*\boldsymbol{J}$ is a $p$-form and $V$ is a compact $p+1$-dimensional oriented volume whose boundary $\partial V$ is $p$-dimensional and is also oriented and compact. In four-dimensional spacetime, the conserved current $J^a$ in (9.83) is dual to the 3-form ${}^*\boldsymbol{J}$. In differential form notation, (9.83) says that the corresponding 3-form is closed

$$\mathrm{d}^*\boldsymbol{J} = 0. \tag{9.85}$$

Substituting (9.85) into (9.84) gives

$$\int_{\partial V} {}^*\boldsymbol{J} = 0. \tag{9.86}$$

In accordance with $J^a$ being a current vector, ${}^*\boldsymbol{J}$ describes the flow (or flux) of the conserved quantity (generalised charge) across the boundary $\partial V$. Hence, (9.86) states that the total flowing into the volume $V$ is equal to the total flowing out, which expresses the conservation of generalised charge.

Recall that the massless free-field equations (9.1) are conformally invariant if the spinor field has conformal weight equal to $-1$, cf.(9.6). If the conformal weight of the potential field ${}^{(1)}\bar{\eta}_{A'_1}^{A_2 \dots A_n}$ in (9.60) is also equal to $-1$, then the conformal weight of the one-form current in the integrand of (9.60) is equal to $-2$. Raising the tensor index yields a vector current of conformal weight $-4$. Since the alternating tensor has a conformal weight of 4, dualising produces a 3-form of conformal weight 0, hence

$$\widehat{{}^*\boldsymbol{J}} = {}^*\boldsymbol{J}. \tag{9.87}$$

From (9.87), it follows that the helicity charge

$$H_s = \int_\Sigma {}^*\boldsymbol{J}, \tag{9.88}$$

has conformal weight zero, that is, it is conformally invariant. This would then agree with the fact that the helicity operator and the scalar product can be constructed from functions of *twistors*, which are conformally invariant objects. In the next section, we describe some relevant details of twistor theory [90, 52], including the representation of the helicity operator.

## 9.7 The helicity operator in twistor theory

We started this chapter with the goal of showing exactly how the duality transformation corresponds to the helicity constant in the way that spacetime translations and rotations correspond to energy-momentum and angular momentum (where Lorentz boosts would correspond to uniform motion in a straight line of the mass centre). From the perspective of Hamiltonian mechanics, quantities that correspond to each other in this way are canonically conjugate variables. Since the duality rotation changes the phase of the spinor, phase and helicity are also conjugate variables. The momentum and angular momentum variables are invariant with respect to phase changes, hence so is the symplectic two-form

$$\omega = dp_a \wedge dx^a, \tag{9.89}$$

which describes a particle with zero spin. The generalisation to particles with non-zero spin was given in [97]. An elegant representation, particularly for massless particles, is given in terms of twistors [33, 113]. Moreover, when the basic phase space variables are given in terms of twistors, the canonical relationship between phase and helicity is manifest. Many definitions of a twistor exist [78]. For our purposes, they may be thought of as the reduced (Weyl) spinors for the pseudo-orthogonal group, $O(2,4)$, [90]. Thus in the same way that spin vectors form a complex two-dimensional vector space (spin-space) in which $SL(2,\mathbb{C})$ acts, the twistors form a four-dimensional vector space in which $SU(2,2)$ acts. Accordingly we have the following group isomorphisms

$$SL(2,\mathbb{C}) \to O_+(1,3), \tag{9.90}$$

$$SU(2,2) \to O(2,4) \to C(1,3), \tag{9.91}$$

where each map is a 2-1 isomorphism and $C(1,3)$ is the conformal group. Furthermore, in analogy to how a two-spinor can be thought of as a square root of a (future-pointing) null vector, a twistor can be thought of as a square root of the energy-momentum-angular-momentum structure of a zero-rest-mass particle [86]. Earlier we introduced concepts from symplectic geometry in order to give an expression for conserved quantities. The main idea is that conserved quantities arise as observables which generate canonical transformations on phase space, cf. (9.40). It was first shown in [33] that the twistor symplectic structure for null geodesics is identical to the standard symplectic structure on the phase space (cotangent bundle) over spacetime. Therefore, we can describe how the helicity constant arises in twistor theory according to the symplectic methods already introduced. In term of the twistor phase space coordinates $(Z^a, \bar{Z}_a)$, the potential one-form is given by

$$\theta = \tfrac{i}{2}\left(Z^a d\bar{Z}_a - \bar{Z}_a dZ^a\right). \tag{9.92}$$

Taking the exterior derivative of (9.92) gives the symplectic two-form,

$$\omega = idZ^a \wedge d\bar{Z}_a, \tag{9.93}$$

which is closed $d\omega = 0$. The vector field corresponding to the helicity Hamiltonian is given by

$$X_s = \frac{i}{2}\left(Z^a \frac{\partial}{\partial Z^a} - \bar{Z}_a \frac{\partial}{\partial \bar{Z}_a}\right). \tag{9.94}$$

Therefore, according to (9.40) the helicity constant is given as

$$\begin{aligned} H_s(Z, \bar{Z}) = i_s\theta &= \tfrac{i}{2}\left(i_s(Z^a d\bar{Z}_a) - i_s(\bar{Z}_a dZ^a)\right) \\ &= \tfrac{1}{2} Z^a \bar{Z}_a. \end{aligned} \tag{9.95}$$

The infinitesimal transformation on the phase space coordinates $(Z^a, \bar{Z}_a)$ is

$$\delta Z^a = iZ^a, \qquad \delta\bar{Z}_a = -i\bar{Z}_a, \tag{9.96}$$

exponentiating gives the corresponding finite transformation

$$Z^a \to e^{i\theta} Z^a, \qquad \bar{Z}_a \to e^{-i\theta}\bar{Z}_a. \tag{9.97}$$

Equations (9.96) and (9.97) should be compared to their respective analogues in the case a spin-s spinor field (9.32) and (9.33). The finite transformation (9.97) is precisely that which corresponds to the helicity constant in the sense of Noether's theorem. It is simply a phase transformation of the twistor coordinates, similar in form to the duality rotations of a spinor field (9.33). Furthermore, in twistor theory, there exist contour integral formulae giving the general solution to the spin-s massless field equations in terms of twistors [84], which thereby provide the link between the transformations (9.96) for twistors and (9.32) for fields.

Twistors can also be expressed, relative to a spacetime origin, in terms of a pair of two-spinors $\omega^A$ and $\pi_{A'}$ as follows

$$Z^a = \left(\omega^A, \pi_{A'}\right), \qquad \bar{Z}_a = \left(\bar{\pi}_A, \bar{\omega}^{A'}\right). \tag{9.98}$$

The two-spinors $\omega^A$ and $\pi_{A'}$ are related to the momentum and angular momentum in the following way

$$p_a = \pi_{A'}\bar{\pi}_A, \tag{9.99}$$

$$M_{ab} = i\omega_{(A}\bar{\pi}_{B)}\varepsilon_{A'B'} - i\bar{\omega}_{(A'}\pi_{B')}\varepsilon_{AB}, \tag{9.100}$$

where $p_a$ is necessarily future-null. The helicity can be represented in terms of $\omega^A$ and $\pi_{A'}$ by substituting (9.98) into (9.95). Alternatively, the same result follows by substituting (9.99) and (9.100) into (9.13) [90],

$$S_a = {}^*M_{ab}p^b = \left(\omega_{(A}\bar{\pi}_{B)}\varepsilon_{A'B'} + \bar{\omega}_{(A'}\pi_{B')}\varepsilon_{AB}\right)\bar{\pi}^B\pi^{B'}$$
$$= \frac{1}{2}\left(\omega^B\bar{\pi}_B + \bar{\omega}^{B'}\pi_{B'}\right)\bar{\pi}_A\pi_{A'}. \quad (9.101)$$

Comparing (9.101) with (9.14) and (9.99), we can write

$$H_s = s = \frac{1}{2}\left(\omega^A\bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}\right). \quad (9.102)$$

In terms of two-spinors, the vector field (9.94) is

$$X_s = \frac{i}{2}\left(\omega^A\frac{\partial}{\partial\omega^A} + \pi_{A'}\frac{\partial}{\partial\pi_{A'}} - \bar{\omega}^{A'}\frac{\partial}{\partial\bar{\omega}^{A'}} - \bar{\pi}_A\frac{\partial}{\partial\bar{\pi}_A}\right). \quad (9.103)$$

The corresponding infinitesimal transformations of the two-spinors are

$$\delta\omega^A = i\omega^A, \qquad \delta\pi_{A'} = i\pi_{A'}, \qquad \delta\bar{\omega}^{A'} = -i\bar{\omega}^{A'}, \qquad \delta\bar{\pi}_A = -i\bar{\pi}_A, \quad (9.104)$$

and their finite counterparts

$$\omega^A \to e^{i\theta}\omega^A, \qquad \pi_{A'} \to e^{i\theta}\pi_{A'}, \qquad \bar{\omega}^{A'} \to e^{-i\theta}\bar{\omega}^{A'}, \qquad \bar{\pi}_A \to e^{-i\theta}\bar{\pi}_A. \quad (9.105)$$

Comparing (9.105) with (3.2), we see that the finite helicity transformation can be understood as a rotation of the flag plane of the spinors $\omega^A$ and $\pi_{A'}$. In order that (9.100) transform correctly under translations as momenta and angular momenta, the twistor two-spinor parts must satisfy

$$\omega^A = \tilde{\omega}^A - ix^{AA'}\pi_{A'}, \quad (9.106)$$
$$\pi_{A'} = \tilde{\pi}_{A'}, \quad (9.107)$$

where $\tilde{\omega}$ and $\tilde{\pi}$ are constant spinor fields whose values coincide with $\omega$ and $\pi$ respectively at the origin. From (9.106) and (9.107), the following equations follow

$$\nabla^{(A}_{A'}\omega^{B)} = 0, \quad (9.108)$$

$$\nabla_{BA'}\omega^C = -i\epsilon_B{}^C\pi_{A'}, \quad (9.109)$$

$$\nabla_{AA'}\pi_B = 0, \quad (9.110)$$

where (9.108) is referred to as the *twistor equation*. In conformally flat space, where the Weyl tensor is zero, (9.108) also implies (9.106) and (9.107). If $\omega^A$ is assumed to have a conformal weight of zero, $\hat{\omega}^A = \omega^A$, then (9.108) is conformally invariant [90]. The solutions to the twistor equation (9.108) constitute a four-dimensional vector space over the complex numbers, and give a useful definition of twistor space. Due to the conformal invariance of (9.108), twistors are conformally invariant objects. It follows that expressions constructed from twistors, such as the helicity (9.95) and the scalar product [51, 48], are also conformally invariant, in agreement with the conformal invariance of the helicity integral (9.88), cf. (9.87).

Comparing (9.108) with the spinor conformal Killing equation (9.76), we see that the twistor equation is just a particular case. Therefore, $\omega^A$ may be considered to be a one-index Killing spinor. Furthermore, from (9.102), we see that the helicity constant is linear in the momentum. In accordance with our earlier discussion, we therefore expect to find a corresponding conservation law for the entire system, in the form of a conserved current vector. Our earlier derivation of the conserved helicity integrals suggest that the corresponding conserved current, for a spin $s$ massless field, is given by the integrand of our helicity integral (9.60).

It was noted in the introduction to this chapter, that the connection between the helicity constant arising in particle theory (9.95) and that arising in field theory (9.32) is usually described via 'twistor first quantization', i.e., via a quantum description regarding the field as a one-particle state. On the other hand, according to the discussion on complex structure in section 9.5 and conserved quantities in section 9.6, the connection can also be described classically in terms of (conformal) Killing vectors or Killing spinors. Consider the familiar example of the energy, the conserved quantity in particle theory is represented as

$$E = t^a p_a, \tag{9.111}$$

where, according to (9.71), $E$ is constant along the particle's worldline, $t^a$ is the Killing vector corresponding to a time translation and $p_a$ is the particle momentum, cf. (9.72). The corresponding result in field theory is

$$E = \omega(\mathcal{L}_t\phi, \phi) = \int_\Sigma t^a T_{ab} \mathrm{d}\sigma^b, \tag{9.112}$$

where $\phi$ is any scalar, tensor or spinor field and $T_{ab}$ is its corresponding energy-momentum tensor, conserved according to (9.80). In the case of helicity, the analogue of (9.111) is given by (9.95) or equivalently (9.102). The analogue of (9.112) is the helicity integral (9.60). Curiously, on first sight, there is no mention of a Killing spinor in (9.60). However, according to (9.102), the multiplying factor $s$ in (9.60) is actually a function of $\omega^A$ and $\pi_{A'}$, but it is covariantly constant. Therefore, according to (9.102), we could rewrite (9.60) as

$$\begin{aligned} H_s(\phi) &= \omega(\hat{s}\phi, \phi) \\ &= \frac{i}{2}\int_\Sigma \left(\omega^A \bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}\right)\left(\phi_{A_1 A_2 \ldots A_n}{}^{(1)}\bar{\eta}_{A'_1}^{A_2\ldots A_n} - \bar{\phi}_{A_{1'} A_{2'} \ldots A_{n'}}{}^{(1)}\eta_{A_1}^{A'_2\ldots A'_n}\right)\mathrm{d}\sigma^{A_1 A_1{}'}. \end{aligned} \tag{9.113}$$

The multiplying factor $s$ is now interpreted as playing an analogous role to the Killing vector components $t^a$ in (9.112). The obvious difference being that (9.102) is a (pseudo-) *scalar* quantity. According to twistor theory, it forms the trace part of a Hermitian twistor. The corresponding trace-free part contains fifteen parameters, providing the twistor representation of a conformal Killing vector corresponding to the fifteen continuous symmetries of the conformal group [90, 52]. Therefore, as is the case for energy-momentum and angular momentum, the connection between the conserved helicity constant in particle and field theory can be understood via the concept of a Killing spinor, according to the general theory described in section 9.6.
According to twistor theory, each spinor can be represented as a function (more correctly an element of first sheaf cohomology) on twistor space. The vector field corresponding to the helicity (9.94) is essentially determined by the Euler homogeneity operator [90], such that the homogeneity degree of the twistor function determines the helicity via an eigenvalue equation, in accordance with Euler's homogenous function theorem. The description in terms of helicity eigenstates is the standard one and is equivalent to the discussion given above.

# Chapter 10

# Summary and Outlook

In this thesis, we have investigated two sets of conformally invariant equations using the two-spinor formalism. First we looked at the Bach equations, where the focus was on finding exact solutions. Our main original contribution was the translation of the spinor equations into compacted spin-coefficient form. As an application of the formalism, we reconstructed two previously known exact solutions, namely the PP-wave spacetime and the static spherically-symmetric spacetime. Since the latter was found to be conformally related to the Schwarzschild-de Sitter solution, it was natural to look at the conditions under which a general solution of the Bach equations was conformal to a solution of the Einstein field equations. For a certain class of spacetimes, which excludes type $N$ spacetimes such as the PP-wave, the necessary and sufficient conditions are the vanishing of the Bach and Eastwood-Dighton tensors. For the spherically symmetric solution, we applied the conditions in order to find the conformal factor for the Weyl scalar, which transforms a solution of the Bach equations to a solution of the Einstein equations. We applied the spin-coefficient formalism also to the Weyl-Lanczos equations in order to find the Lanczos coefficients for the PP-wave and spherically symmetric spacetimes. In both cases, we showed that they were proportional to the spin coefficients.

Further extensions of this work could include the search for new exact solutions of the Bach equations. As in the case of the EFE, the NP spin-coefficient formalism will be a useful calculus. In comparison to the large number of known exact solutions to the EFE, the number of known solutions to the Bach equations is relatively few, particularly solutions which are not conformal Einstein spaces. Part of the reason is the increased difficulty of solving non-linear higher-order derivative equations. As is the case for the EFE, computer algebra packages can play a useful role [62]. In particular, when solving equations using the NP formalism, spinor packages such as [47] can be used.

The second set of equations we investigated were the massless free-field equations for spin $s$. In conformally flat spacetime, the general solution can be represented by means of a contour integral expression involving arbitrary analytic functions. Therefore, in contrast to our investigation of the Bach equations, where the focus was on exact solutions, our approach was to investigate the relationship between duality rotations and a certain conserved quantity, in-

terpreted physically as the helicity. Using concepts from symplectic geometry, such as the symplectic two-form, we derived an expression for the helicity of an integer spin $s$ field in terms of a three-surface integral over a conserved current density. In general, the conserved current is a real four-vector, constructed from the spin $s$ field and a certain potential from its corresponding potential chain. Moreover, the conformal invariance of the helicity integral expression follows from the conformal invariance of the massless free-field equations.
Further extensions of this work could be to translate the helicity integral expressions into twistor form, as was done in [78] for the electric charge, mass-energy and angular momentum. According to the symplectic approach, the result should agree with the moment map (9.40), where the vector field is (9.94), i.e. the twistor helicity Hamiltonian vector field. Moreover, the symplectic product is the standard one [88], closely related to the scalar product (9.67), given in terms of twistor functions. Another extension concerns the generalisation of conservation laws to curved spacetimes. In the introduction to section 9, we noted that in conformally curved spacetime, algebraic consistency relations exist for the massless field equations. Similar conditions hold also for the potential equations (9.63) and the twistor equation (9.108). Furthermore, the (conformal) Killing equation has non-zero solutions only when the spacetime possesses continuous symmetries. In spite of these difficulties, much progress has been made in describing conservation laws in curved spacetime. Especially for conserved quantities, such as the electric charge or mass-energy-angular-momentum, that permit a *quasi-local* description [87, 101]. In such approaches, the conserved quantities are expressed as integrals over closed spacelike two-surfaces. For a particular class of two-surfaces, so called *non-contorted*, the quasi-local mass construction produces many appealing results [106]. However, in the *contorted* case, modifications are required [52]. For such cases, the main obstacle is that the standard definition of the twistor norm (9.102), interpreted physically as the helicity, is not constant. For the contorted cases, various alternatives for the twistor norm have been suggested. For example, in [89] three proposals are put forward: an averaging procedure over the two-sphere, the inclusion of the Gaussian curvature of the two-sphere, the inclusion of the determinant of any four linearly independent solutions to (the tangential parts of) the twistor equation (9.108). Due to the physical interpretation of the norm as helicity, the discussion in section 9.7 suggests an alternative tentative proposal along the lines of (9.113), but generalised so as to apply to curved spacetime. It would be interesting to investigate whether this line of inquiry could lead to a modification of the twistor norm suitable for contorted surfaces in curved spacetimes.

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