A structural model of debt pricing with creditor-determined liquidation

Max Bruche[a], Hassan Naqvi[b,c,*]

[a] *CEMFI, Casado del Alisal 5, 28014 Madrid, Spain.*
[b] *Department of Finance, School of Business, Mochtar Riady Building, 15 Kent Ridge Drive, Singapore 119245. Tel: +65-65165552, Fax: +65-67792083.*
[c] *National University of Singapore Risk Management Institute*

Abstract

This paper develops a continuous time asset pricing model of debt and equity in a framework where equityholders decide when to default but creditors decide when to liquidate. This framework is relevant for environments where creditors exert a significant influence on the timing of liquidation, such as those of countries with creditor-friendly bankruptcy regimes, or in the case of secured debt. The interaction between the decisions of equityholders and creditors introduces an agency problem whereby equityholders default too early and creditors subsequently liquidate too early. Our model allows us to assess quantitatively how this problem affects the timing of default and liquidation, optimal capital structure, and spreads.

Keywords: defaultable debt pricing, creditor induced liquidation, premature liquidation
JEL: G12, G13, G32, G33

*Corresponding author
Email addresses: bruche@cemfi.es (Max Bruche), naqvi@nus.edu.sg (Hassan Naqvi)

Preprint submitted to Elsevier *January 6, 2010*

1. Introduction

In much of the continuous-time debt pricing literature, it has typically been assumed that default is tantamount to liquidation. In the models of Leland (1994) and Leland and Toft (1996), for example, equityholders choose a boundary at which they will default, knowing that once they do so, the firm is immediately liquidated.

In practice most of the companies which default go into a period of reorganization and may or may not be liquidated.[1] Using US data, Gilson et al. (1990) find that in their sample only about 5% of the bankruptcies in Chapter 11 are converted into Chapter 7 liquidations. Using data on distressed UK companies, Franks and Sussman (2005) find no evidence of automatic liquidation upon default.

Recently, some debt pricing models have therefore attempted to separate the notions of default and liquidation. For instance, in the model by François and Morellec (2004), default is triggered by the equityholders but the company is liquidated only once the consecutive time spent in default exceeds an *exogenous* grace period. Similarly, in the model of Moraux (2002) liquidation occurs when the cumulative time spent in default exceeds an exogenous grace period, and in the model of Galai et al. (2007), liquidation occurs when consecutive time in distress, weighted by distress severity exceeds an exogenous threshold.[2] In all of these papers, default is determined by equityholders but liquidation is triggered by an exogenous criterion which is meant to capture the Court's behavior under Chapter 11.

This paper contributes to this literature by considering an alternative, *endogenous* mechanism for liquidation, which is more representative of creditor-friendly bankruptcy regimes or secured debt: Once equityholders have defaulted, debt covenants are triggered, which give *debtholders* the option to liquidate the firm. In our setup, debtholders do not liquidate the firm immediately upon default but are willing to accept reduced coupon payments (i.e. some default on coupon payments) in the hope that the firm's fundamentals will improve. If the firm's fundamentals deteriorate, however, debtholders eventually liquidate. The point at which equityholders default furthermore affects the incentives to liquidate; the earlier equityholders default, the lower the continuation value to debtholders, and hence the earlier they will want to liquidate. Equityholders, when choosing when to default, take into account how their default decision affects the liquidation decision of debtholders.

This produces very different implications for the timing of liquidation—as opposed to models with an exogenous grace period, the time of liquidation can be very soon after default if firm fundamentals deteriorate sufficiently, or very long after default if firm fundamentals are bad but stable for a long time. In our model, the time between the initial default and subsequent liquidation will be random, and not related to a grace period.

Also, it implies that debtholders have incentives to liquidate too early, and equityholders have incentives to default too early, which affects debt values and optimal capital

[1] The time spent in the reorganization period varies immensely. Franks and Torous (1989) report that in their sample, firms on average spend a period of 4 years in Chapter 11.

[2] In a slightly more general version of this approach, Broadie et al. (2007) posit that a firm is liquidated either once the consecutive time spent in default exceeds an exogenous grace period, or once the asset value hits a boundary chosen by equityholders.

structures. Firstly, debtholders typically get all of the liquidation value but only a part of the upside to all claimants if the firm's situation improves. They therefore have incentives to liquidate early, decreasing firm value. Secondly, equityholders anticipate that debtholders will not liquidate immediately upon default. They therefore have incentives to default earlier than they would otherwise. If there is a cost associated with default/ financial distress, this cost is then more likely to be incurred, decreasing firm value. Both effects produce an agency cost, whose size is also affected by the interaction between the two decisions. We show in a numerical example how this can lead to lower optimal leverage and higher spreads.

Relative to the above-mentioned literature that considers liquidation as a result of exogenous grace periods, our model is a better representation of an environment where debtholders have a strong influence over the timing of liquidation, like in the case of secured debt, or creditor-friendly bankruptcy regimes. For instance, in the UK most banks hold a floating charge over the assets of a company and if the firm is unable to meet its obligations then the banks have the power to appoint an administrative receiver who then supervises the running of the firm and who has the power to put the company into liquidation.[3] The UK Bankruptcy Code closely resembles the South African "judicial management" and the Australian "official management" and gives relatively more powers to the creditors vis-a-vis the US Chapter 11.

Our model is also related to the work of Mella-Barral (1999). In his setup, debtholders also do not liquidate immediately upon default but are willing to accept reduced coupon payments (i.e. some default on coupon payments) in the hope that the firm's fundamentals will improve. However, in his model whenever a default occurs new debt contracts are drawn up until the firm is eventually liquidated. In contrast, in our model, although debt contracts are temporarily violated when equityholders default on coupon payments, they are not renegotiated or replaced. (This is consistent with the US Trust Indenture Act of 1939 which prohibits firms from permanently changing the 'core' terms of bond indentures, which include the principal amount, the interest rate and the stated maturity, unless all the creditors agree unanimously.[4])

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 reports the analytical solutions for the values of equity, debt, and the firm, taking the default and bankruptcy boundaries as given. Section 4 characterizes and discusses the optimal default and liquidation boundaries chosen by equityholders and debtholders respectively. Section 5 studies the implications of the model in the context of a numerical example. Section 6 concludes.

[3]There is at least anecdotal evidence that as a result of debtholders being able to enforce liquidation in the UK, there are instances of premature liquidation. For example, Woolridge (1987) reports that floating charge holders "apply themselves ruthlessly to the realization of assets to satisfy the charge. . . in some cases with scant regard for the future of the company", or Hart (1995, p. 168) argues that "the bank may decide against keeping a good company going because it does not see the upside potential."

[4]If the firm goes to Chapter 11, then these core terms can be altered if a two-thirds majority by value and a simple majority by number is reached within each class of creditors. However unanimity is required outside of Chapter 11.

2. The Model

A firm has access to a production process which produces a stochastic cash flow x with two inputs: the human capital of a penniless manager, and a (physical) productive asset. Financiers set up a firm to become equityholders: they purchase the productive asset, hire a manager,[5] and finance part of the initial outlay by issuing perpetual debt[6] with a promised coupon payment of $b > 0$ per period, and part of it with their own wealth.[7] All potential managers have many outside job opportunities all paying a flow of $a > 0$, and therefore the manager needs to be paid a salary flow of a per instant of time. The manager's human capital can be interpreted as a technical skill which is valuable to the firm and without which the firm cannot be run. Hence if the manager's wage income is not paid, she leaves the firm and the firm produces no cash flow.

We suppose that capital markets are frictionless and that there are no informational asymmetries between agents. Interest rates are non-stochastic and flat at r.[8]

Once a firm is operational, equityholders can decide whether or not to default on coupon payments. When equityholders do not default on coupon payments, they receive residual cash flows gross of taxes of $x - (a + b)$, on which they are liable to pay taxes at a tax rate of τ per period, where $\tau \in (0, 1)$. Thus, the residual cash flows to equityholders net of taxes are given by $(1 - \tau)(x - (a + b))$.[9] The payoff to the equityholders might be negative if the fundamental cash flow x is too low to cover both coupon payments and the salary of the manager.

When equityholders default on coupon payments, the fundamental cash flow is reduced to θx, where $\theta \in (0, 1)$ reflects the cost of financial distress. In default, debtholders receive the residual cash flows $\theta x - a$, where $\theta x - a < b$ whilst equityholders receive nothing. Once a default occurs, debt covenants are triggered which gives debtholders the right to liquidate. This put option to liquidate may or may not be exercised by the debtholders.[10]

---

[5] We do not model principal-agent problems between the equityholders and the manager in this paper. Recently some continuous time pricing models have focused on managerial related agency problems. See, for instance, DeMarzo and Sannikov (2006) and Morellec and Smith (2007).

[6] For simplicity we assume that debt is perpetual. For an analysis of the effect of maturity on debt pricing, see, for instance, Dumitrescu (2007) and Leland and Toft (1996).

[7] As is commonly supposed in the literature we assume that the equityholders do not hold any debt in the firm. Realdon (2007) develops a structural valuation model when the equityholders are also debtholders and shows that equityholders expropriate other debtholders by repaying their own credit before bankruptcy.

[8] Longstaff and Schwartz (1995) develop a simple model of debt pricing in which they drop the usual assumption of non-stochastic interest rates. However, like most other papers in the literature the focus of this paper is on default risk and hence we assume that interest rates are constant.

[9] In the model, cash flows to debtholders are not taxed; τ should therefore be interpreted as a net tax advantage to debt in the sense of Miller (1977).

[10] Note that given this setup the equityholders will not have an incentive to default on coupon payments as long as the cash flows are sufficient to service debt. This is in contrast to the strategic debt servicing models of Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997). In their models, equityholders default even at high cash flows to the extent that the payoff to the debtholders is just sufficient such that they do not have an incentive to liquidate the firm. This strategic debt service will not happen here because we require that equityholders cannot pay themselves a dividend when defaulting.

As argued by Jensen (1986), one motivation for issuing debt is to commit the managers of the firm to pay out future cash flows. However for such an objective to be effective there has to be some punishment

In the event of liquidation the manager is fired, and the physical capital is sold for total proceeds of K. We assume that $0 < K < \frac{b}{r}$, and that absolute priority is respected in liquidation, such that all of the liquidation proceeds go to debtholders. Since the liquidation payoff to debtholders is less than $\frac{b}{r}$, debt will be risky.

Although for the main part of the paper we treat K as constant, the analysis can be generalized to a situation in which K is a function of x, as long as this function satisfies some conditions. Basically, these conditions boil down to requiring that in the case of an *unlevered* firm, there can be situations in which the going-concern value is so low that the firm is better off being liquidated and the remaining assets diverted to other uses, i.e. liquidation is optimal for a low enough firm value. It turns out that if this holds for an *unlevered* firm, it also holds for a *levered* firm. For our purposes, since we want to be able to discuss optimal versus suboptimal liquidation boundaries, we impose these conditions. Since a constant K is the simplest example for which these conditions are satisfied, this is the case which we treat in the main text. A full discussion of the case where K is a function of x is relegated to Appendix D.

The cash flows x follow a geometric Brownian motion under the pricing measure defined by the money market account, i.e.

$$dx(t) = \mu x(t)dt + \sigma x(t)d\tilde{W}(t), \tag{1}$$

where the parameters μ and σ represent the drift and volatility terms respectively and $d\tilde{W}(t)$ is the increment of a Brownian motion under the pricing measure.

Since debt and equity are perpetual claims, the optimal decision to default and the optimal decision to liquidate can be expressed in terms of optimally chosen *constant* boundaries for the cash flow at which equityholders default and debtholders liquidate. We let $\hat{x}$ denote the critical boundary at which equityholders choose to default, and $\bar{x}$ denote the critical boundary at which debtholders choose to liquidate. Equityholders will choose $\hat{x}$ to maximize the value of equity. They choose first, taking into account the effect that their choice of $\hat{x}$ has on the choice of liquidation boundary $\bar{x}$ by the debtholders (i.e. they act as Stackelberg leaders). Debtholders choose $\bar{x}$ to maximize the value of debt, taking $\hat{x}$ as given.

3. Valuation

In this section, we discuss the value of equity, debt, and the firm.

3.1. The value of equity

Equity is a perpetual claim to the cash flows $(1 - \tau)(x - a - b)$ outside of default, and 0 in default, until the firm is liquidated. The liquidation payoff to equity is zero. A standard pricing argument (requiring discounted gains processes to be martingales under the pricing measure $\mathbb{Q}$) produces a pricing ordinary differential equation (ODE).

---

if the commitment is not fulfilled. This punishment can take various forms. In our model it is the shift of the control rights that gives force to this commitment. Hart and Moore (1998) show that such a characteristic of a debt contract which gives the lender the right to seize and liquidate the project upon default is useful in persuading entrepreneurs to pay out cash flows rather than diverting them.

The equity value $E(x)$ solves the following ODE:

$$\frac{1}{2}\sigma^2 x^2 E''(x) + \mu x E'(x) - rE(x) + e(x) = 0 \tag{2}$$

where

$$e(x) = \begin{cases} (1-\tau)(x-(a+b)) & \text{if } x \geq \hat{x} \quad \text{Region 1: no-default} \\ 0 & \text{if } x < \hat{x} \quad \text{Region 2: default} \end{cases} \tag{3}$$

Let $E_1(x)$ be the value of equity in the no-default region, and $E_2(x)$ be the value of equity in the default region. Then the value of equity will obey the following boundary conditions:

$$\lim_{x\to\infty} E_1(x) = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a+b}{r}\right) \tag{4}$$

$$E_1(\hat{x}) = E_2(\hat{x}) \tag{5}$$

$$E_1'(\hat{x}) = E_2'(\hat{x}) \tag{6}$$

$$E_2(\bar{x}) = 0 \tag{7}$$

Eqns. (5) and (7) are obvious value-matching conditions. Eqn. (4) rules out bubbles, and states that for very large cash flows, the firm should essentially be default-risk free. Eqn. (6) states the value function should not change abruptly when the flow payoff function changes. Dixit (1993) shows that this is a necessary condition for no arbitrage. (See Karatzas and Shreve (1988) for a more rigorous discussion.)

Solving the differential equation (2) subject to the above boundary conditions we obtain the following result.

Proposition 1. *The value of equity in the no-default region is given by*

$$E_1(x;\hat{x},\bar{x}) = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a+b}{r}\right) + \left[E_2(\hat{x};\hat{x},\bar{x}) - (1-\tau)\left(\frac{\hat{x}}{r-\mu} - \frac{a+b}{r}\right)\right]\left(\frac{x}{\hat{x}}\right)^{-\gamma} \tag{8}$$

while the value of equity in the default region is given by

$$E_2(x;\hat{x},\bar{x}) = (1-\tau)Z(\hat{x})\left(\frac{x}{\hat{x}}\right)^{\delta} + \left[0 - (1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right]\left(\frac{x}{\bar{x}}\right)^{-\gamma} \tag{9}$$

where

$$Z(\hat{x}) = \frac{(1+\gamma)}{\delta+\gamma}\frac{\hat{x}}{r-\mu} - \frac{\gamma}{\delta+\gamma}\frac{a+b}{r}.$$

The powers δ and γ are the positive and negative roots respectively of the characteristic quadratic equation $\xi(\xi-1)\sigma^2/2 + \xi\mu - r = 0$. Finally, for $x < \bar{x}$ the value of equity is 0.

Proof. See Appendix A. □

The value of equity in the no-default region, E_1, can be interpreted as the value of receiving the cash flows net of coupons and taxes (the first term of (8)), plus the (negative) value of a claim that swaps this against E_2 when the firm enters the default region. The first term in (9) is the value of receiving positive cash flows in the event that the firm exits the default region. The second term swaps this value against 0 when $\bar{x}$ is hit and the firm is liquidated.

The solution of the equity function from Proposition 1 is illustrated in Figure 1. Note that the value of equity is often positive even in the default region even though the economic payoff to the equityholders in this region is zero. This is because as long as the firm is not liquidated, there is always a positive probability that the firm might exit default and hence the equity holders might be able to earn economic profits in the future.

3.2. The value of debt

The debt value solves the following ODE

$$\frac{1}{2}\sigma^2 x^2 B''(x) + \mu x B'(x) - rB(x) + p(x) = 0 \tag{10}$$

where

$$p(x) = \begin{cases} b & \text{for } x \in [\hat{x}, \infty) \quad \text{Region 1: no-default} \\ \theta x - a & \text{for } x \in [\bar{x}, \hat{x}) \quad \text{Region 2: default} \end{cases} \tag{11}$$

The liquidation payoff to debtholders is K.

Let B_1 be the value of debt in Region 1, and let B_2 be the value of debt in Region 2. Then the value of corporate debt will satisfy the differential equation (10) subject to the following boundary conditions

$$\lim_{x\to\infty} B_1(x) = \frac{b}{r} \tag{12}$$

$$B_1(\hat{x}) = B_2(\hat{x}) \tag{13}$$

$$B_1'(\hat{x}) = B_2'(\hat{x}) \tag{14}$$

$$B_2(\bar{x}) = K \tag{15}$$

As before, (13) and (15) are obvious value-matching conditions. Eqn. (12) rules out bubbles, and states that for very large cash flows, debt should essentially be default-risk free. Analogous to the equity pricing case, (14) rules out arbitrage opportunities.

Solving the differential equation (10) given the boundary conditions just described yields the following result.

Proposition 2. *The value of debt in the no-default region is given by*

$$B_1(x; \hat{x}, \bar{x}) = \frac{b}{r} + \left(B_2(\hat{x}; \hat{x}, \bar{x}) - \frac{b}{r}\right)\left(\frac{x}{\hat{x}}\right)^{-\gamma} \tag{16}$$

while the value of debt in the default region is given by

$$B_2(x; \hat{x}, \bar{x}) = \frac{\theta x}{r-\mu} - \frac{a}{r} - Z(\theta\hat{x})\left(\frac{x}{\hat{x}}\right)^{\delta} + \left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} - Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{x}{\bar{x}}\right)^{-\gamma}. \tag{17}$$

Finally, for $x < \bar{x}$ the value of debt is K.

Proof. See Appendix A. □

As before, these formulas can be interpreted in terms of claims that swap values for other values when default and liquidation boundaries are hit.

The solution from Proposition 2 is depicted in Figure 1. Note that as is the usual case, the value of debt is a concave function of the state variable in good states of the world. However, in the default region the value of debt is a convex function of the state variable reflecting the fact that the bondholders are in effect the residual claimants as long as the firm remains in default.

3.3. The market value of the levered firm

Given the market price of equity and the market price of debt we can calculate the market value of the levered firm, V, as the sum of the value of all outstanding claims.[11] We define V_1 as being equal to the market value of the levered firm in the no-default region while V_2 denotes the market value of the levered firm in default region. Since $V_1 = E_1 + B_1$ and $V_2 = E_2 + B_2$ we have the following corollary to Propositions 1 and 2.

Corollary 1. *The value of the levered firm in the no-default region is given by*

$$V_1(x;\hat{x}),\bar{x}) = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a}{r}\right) + \tau\frac{b}{r} + \left[V_2(\hat{x};\hat{x},\bar{x}) - \left((1-\tau)\left(\frac{\hat{x}}{r-\mu} - \frac{a}{r}\right) + \tau\frac{b}{r}\right)\right]\left(\frac{x}{\hat{x}}\right)^{-\gamma} \tag{18}$$

while the value of the levered firm in the default region is given by

$$V_2(x;\hat{x},\bar{x}) = \frac{\theta x}{r-\mu} - \frac{a}{r} + \left(Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x})\right)\left(\frac{x}{\hat{x}}\right)^{\delta} + \left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} + \left(Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x})\right)\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{x}{\bar{x}}\right)^{-\gamma}. \tag{19}$$

In the no-default region, the market value of the levered firm is the post-tax value of cash flows (net of manager salary), plus the value of the debt tax shield. The third term represents the value of a claim that swaps this against the market value of the levered firm in the default region, when x hits $\hat{x}$.

In the default region, the market value of the levered firm is the value of receiving the now untaxed cash flows, but subject to distress cost θ, net of manager salary (first two terms), plus the value associated with receiving the higher cash flows (not subject to distress cost) when x rises above $\hat{x}$ (term in $(Z(\hat{x}) - Z(\theta\hat{x}))\left(\frac{x}{\hat{x}}\right)^{\delta}$), minus the value associated with having to pay taxes again once x rises above $\hat{x}$ (second term). The fourth term represents the value of a claim that swaps this against K, when x hits $\bar{x}$.

The valuation function of the levered firm is depicted in Figure 1.

[11]Note that the market value of the levered firm is not equal to the market value of an unlevered firm because of taxes, potentially different liquidation boundaries, and financial distress costs, i.e. a Modigliani-Miller style capital structure irrelevance result will *not* hold here.

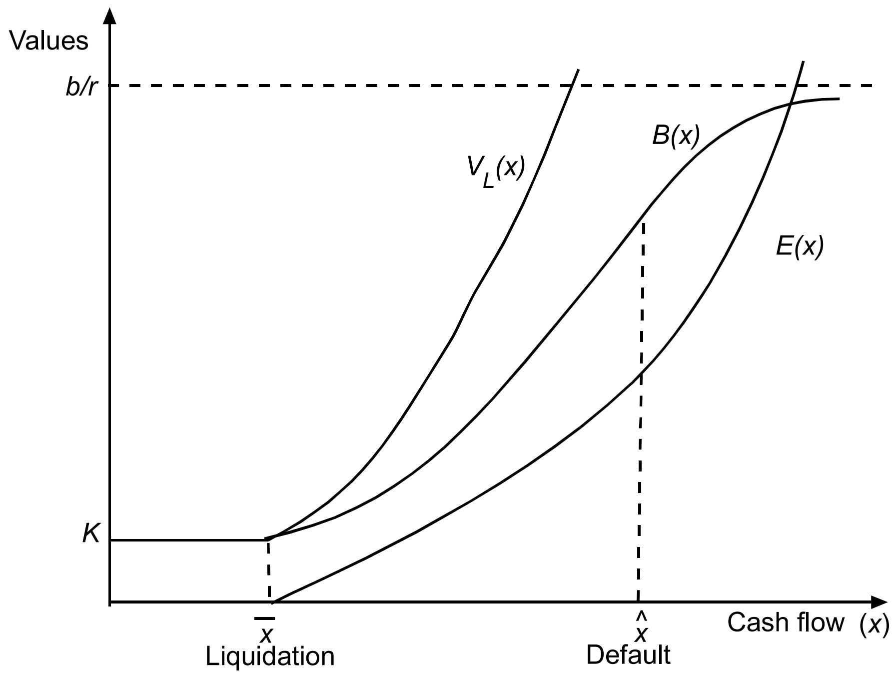

Figure 1: Values of debt, equity, and the firm
Values of debt ($B(x)$, equity ($E(x)$), and the firm ($V(x)$) as a function of cash flow x.

4. Optimal default and liquidation

In this section, we discuss the optimal default and liquidation thresholds chosen by equityholders and debtholders respectively. A natural point of comparison will turn out to be the default boundary at which equityholders would choose to default if default implied immediate liquidation as in Leland (1994).

4.1. Optimal default boundary chosen by equityholders

Suppose that the liquidation boundary $\bar{x}$ is below the default boundary $\hat{x}$, i.e. default does not imply immediate liquidation. It can then be shown that equityholders will inject some cash to delay default, but less so than in the case in which default implies immediate liquidation.

Proposition 3. *If default does not imply immediate liquidation, the default boundary $\hat{x}^*$ that maximizes the value of equity satisfies the following inequality*

$$x^{\dagger} < \hat{x}^* < a + b, \tag{20}$$

where $x^{\dagger}$ is the default boundary that maximizes the equity value for the case where default leads to immediate liquidation.

Proof. See Appendix C.2. □

The optimal default boundary $\hat{x}^*$ can be characterized via the appropriate first order condition associated with maximizing the value of equity:

$$\frac{\partial E(x;\hat{x},\bar{x})}{\partial \hat{x}} + \frac{\partial E(x;\hat{x},\bar{x})}{\partial \bar{x}} \frac{d\bar{x}}{d\hat{x}} = 0. \tag{21}$$

The first term in the above equation measures the *direct* effect of the choice of default boundary $\hat{x}$ on the value of equity while the second term measures the *indirect* effect of the choice of default boundary $\hat{x}$ on the liquidation boundary. The choice of default threshold affects the continuation value of debt and hence indirectly influences the debtholders' liquidation decision. Since equityholders are the Stackelberg leaders they take into account the indirect effect of default timing on the liquidation threshold when maximizing the value of equity.

Note that the default boundary directly affects equity value via its effect on cash flows: Choosing a default boundary above $a + b$ means giving up the positive cash flows $x - (a + b)$ for some x, because equityholders cannot receive dividends while the firm is in default. On the other hand, choosing a default boundary below $a + b$ means having to make positive injections $a + b - x$ for some x. If we only consider these direct effects on cash flows, the optimal default point would be given by $\hat{x} = a + b$.

Furthermore, the default boundary indirectly affects equity value via affecting the liquidation boundary chosen by debtholders, $\bar{x}_B^*$. (As it turns out, increasing the default boundary $\hat{x}$ increases the optimal liquidation boundary $\bar{x}_B^*$ chosen by debtholders, see Appendix C.1). This is because increasing the default boundary decreases the continuation value to debtholders and vice versa.

If equityholders could indirectly choose the liquidation boundary $\bar{x}$, which boundary would they choose if they ignored the direct effect on the value of equity via cash flows? This depends on whether the chosen *default* boundary $\hat{x}$ is above or below $x^\dagger$given that the equity value at $x^\dagger$ smooth pastes to zero.[12]

If $\hat{x}$ is above $x^\dagger$, then at the default boundary $\hat{x}$, the equity value is positive. This implies that the continuation value to equityholders is always positive in the default region. Since the liquidation payoff to equityholders is zero, they would therefore prefer a liquidation boundary as low as possible, i.e. the equity value would be decreasing in the liquidation boundary.

On the other hand, if $\hat{x}$ is below $x^\dagger$, then at the default boundary $\hat{x}$, the equity value is negative. This implies that the continuation value of equityholders is negative in the default region. Since the liquidation payoff to equityholders is zero, they would therefore prefer a liquidation boundary as high as possible, i.e. the equity value would be increasing in the liquidation boundary.

Hence, if equityholders only consider the *indirect* effect of the default boundary on the value of equity via the *liquidation* boundary, they would choose a *default* boundary of $\hat{x} = x^\dagger$, as this would maximize the value of equity.

[12]To see this, note first that $x^\dagger$ is the default boundary that would be chosen if default implied immediate liquidation, via a first order condition, or equivalently, a smooth-pasting condition. Alternatively, it can be derived as the $\hat{x}$ that sets $E_2(x;\hat{x},\bar{x})$ to zero—since if the boundary payoff at $\hat{x}$ to equity holders is zero, condition (6) then imposes smooth pasting to zero. This implies that $E_2(\hat{x};\hat{x},\bar{x})$ is positive or negative depending on whether $\hat{x}$ is above or below $x^\dagger$.

Given that there exists both a direct and an indirect effect, it follows that the optimal default threshold would lie between $x^\dagger$ and $a + b$. In other words the optimal default threshold trades off the direct effect of the default boundary on equity via cash flows with the indirect effect on the value of equity via the liquidation boundary.

Intuitively, if default leads to instantaneous liquidation then the default decision is irreversible and hence it is optimal for equityholders to keep injecting liquidity until equity is worthless. However, if the default decision is not irreversible and furthermore if debtholders do not liquidate the firm immediately upon default but are willing to accept residual cash flows which are less than the promised coupon payments, then this gives an incentive to equityholders to default earlier.

4.2. Optimal liquidation boundary chosen by debtholders

It can be shown that the optimal *liquidation* boundary $\bar{x}_B^*$ is below the default boundary $\hat{x}$, i.e. debtholders will tolerate some default before liquidating. The key assumption that drives is result is that the liquidation payoff to debtholders K is less than $\frac{b}{r}$ where the latter represents the value of receiving the coupon payments b forever.

An intuitive argument for why this is the case is as follows: The liquidation payoff to debtholders is K, so they will optimally liquidate once their continuation value falls to K. Since $\frac{b}{r}$ is the value of receiving the coupon b forever with probability 1, the continuation value can never be below $\frac{b}{r}$ unless debtholders tolerate some amount of default before liquidating. If $K < \frac{b}{r}$, this necessarily implies that at the time of liquidation, the continuation value of debtholders is below $\frac{b}{r}$, and that debtholders therefore must have tolerated some default before liquidation. This result is formally summarized in Proposition 4.

Proposition 4. *For any given default boundary $\hat{x}$, if $K < \frac{b}{r}$, the optimal liquidation boundary $\bar{x}_B^*$ that maximizes the value of debt falls below the default boundary $\hat{x}$. The optimal liquidation boundary $\bar{x}_B^*$ is unique. It is given via an implicit function $f_B(\bar{x}_B^*) \equiv 0$.*

Proof. See Appendix C.1. □

4.3. Discussion

Unlike Leland (1994) where default is synonymous with liquidation, our model implies that default does not lead to immediate liquidation. Liquidation in our model occurs only when firm fundamentals fall sufficiently following a default by the equityholders. In a setup where default implies instantaneous liquidation, equityholders would have an incentive to inject relatively more liquidity compared to our setup in order to prevent liquidation.

As discussed in the introduction, recently, a number of papers including that of e.g. François and Morellec (2004) have modelled default in a setup where default does not imply immediate liquidation. As is the case in our model, in these models, equityholders inject less cash vis-a-vis the case of Leland (1994) and hence default occurs earlier. However, in these models the liquidation decision is determined exogenously.[13] For instance,

[13]In the case of Broadie et al. (2007) liquidation may occur either because too much time is spent in Chapter 11 bankruptcy or simply because because the firm value reaches the liquidation value.

in François and Morellec (2004) the firm is liquidated if the time spent in default (an observation period or grace period) exceeds an exogenously specified number of days. Thus liquidation happens with a certain exogenous probability once the equityholders default on the coupon payments. In contrast, in our model liquidation can happen at *any* time after default, provided that the cash flow of the firm falls sufficiently.

Furthermore, in the model of François and Morellec (2004), an increase in the anticipated length of the grace period reduces the probability of liquidation and hence equityholders' incentives to default earlier increases. However, in our model there is a positive relation between the default threshold and the liquidation boundary. As discussed earlier, this is the case because an earlier default reduces the probability of exiting default and thus lowers the continuation value of the debtholders. This in turn gives the debtholders an incentive to liquidate earlier.[14]

Finally, it can be shown that the debtholders in our model have an incentive to liquidate the firm prematurely relative to the liquidation boundary that maximizes firm value, taking the default boundary as given. Since $\hat{x}^* > x^\dagger$, the equity value is positive in the default region. However, the debtholders do not internalize the positive continuation value of equity when choosing their liquidation timing. Debtholders realize that even if the firm was restored to good health their upside would be limited by the level of the coupon b. Hence the debtholders' put option to liquidate is less valuable compared to that of a stakeholder which had residual rights to the cash flows associated with all claims on the firm in the no-default region. Debtholders therefore have an incentive to liquidate prematurely. This is formally stated in the following Proposition.

Proposition 5. *For a given $\hat{x}$, there exists an optimal liquidation boundary $\bar{x}_V^*$ that maximizes* the value of the levered firm *that satisfies the following inequality*

$$0 < \bar{x}_V^* < \bar{x}_B^* < \hat{x},$$

as long as $\hat{x} > x^\dagger$. The optimal $\bar{x}_V^*$ is given via an implicit function $f_V(\bar{x}_V^*) \equiv 0$.

Proof. See Appendix C.3. □

5. Numerical analysis

We next study the numerical implications that our model has for the default and liquidation decisions, optimal capital structure and spreads. Regarding the choice of parametric values we follow François and Morellec (2004) as far as is feasible. We set the riskless interest rate to $r = 6\%$, and the net tax advantage of debt to $\tau = 20\%$. Unless stated otherwise, we set the salary cost $a = 1$, the coupon $b = 5$, and the liquidation value $K = 30$. It can be shown that in our model, $r - \mu$, plays the same role as the payout rate, and that the post-tax value of cash flows net of salary costs, $(1-\tau)\left(\frac{x}{r-\mu} - \frac{a}{r}\right)$ can be interpreted as the going concern asset value. Since François and Morellec (2004) set the

[14]Notice that in the case of François and Morellec (2004) an exogenous probability of liquidation determines the timing of default decision by the equityholders. In our model, equityholders when contemplating default take into account the fact their default timing will affect the liquidation timing of debtholders.

payout rate to 5% and the current asset value to 100, we therefore set $\mu = 1\%$, and the current cash flow $x = 7.08$. François and Morellec (2004) choose an asset value volatility of 20% in their base scenario; for simplicity, we set $\sigma = 20\%$ (which corresponds to a slightly higher volatility of the going concern asset value of around 23%). Finally, we set $\theta = 0.7$, which amounts to a 12.5% reduction of (post-tax) cash flows in financial distress.

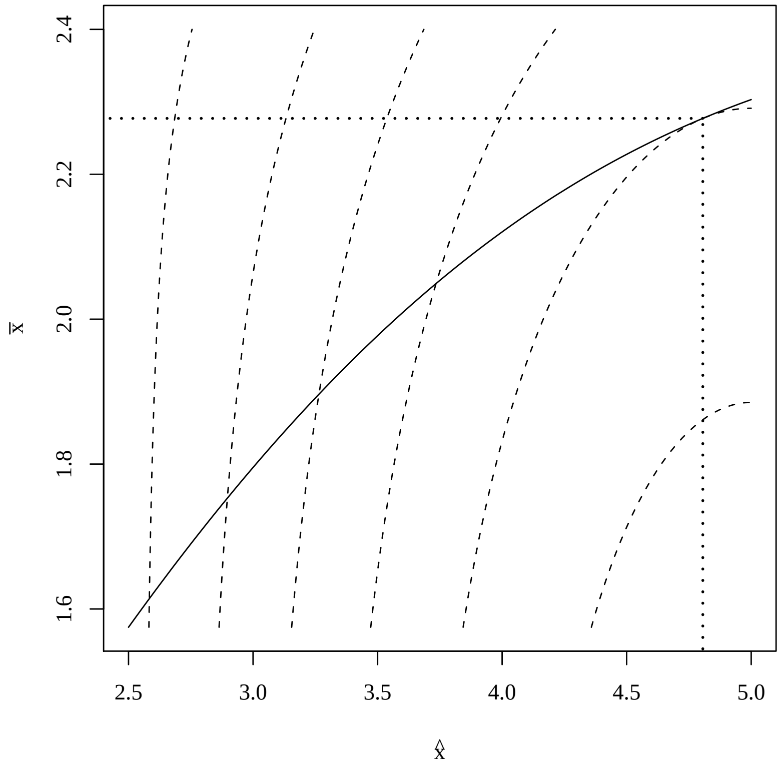

Figure 2: Optimal default decision
The solid line is the optimal $\bar{x}_B^*$ as a function of $\hat{x}$, as chosen by debtholders. The dashed lines are iso-equity-value curves. The tangency point, indicated by dotted lines, is the equity-value maximizing choice of $\hat{x}$ for shareholders, given that $\bar{x}_B^*$ is determined by debtholders.

In Figure 2 we illustrate how the optimal default and liquidation boundaries are determined. On the horizontal axis, we have different values for the default boundary $\hat{x}$ while on the vertical axis, we have different values for the liquidation boundary $\bar{x}$. The solid line is the liquidation boundary $\bar{x}_B^*(\hat{x})$ chosen by debtholders, taking the default boundary $\hat{x}$ as given. Note that the solid line is upward sloping implying a positive relationship between default and liquidation thresholds. (In contrast, as discussed earlier, in the case of François and Morellec (2004) the exogenous probability of liquidation determines the default threshold chosen by equityholders.) The dashed lines depict combinations of the default and liquidation boundaries for which the value of equity is constant (iso-equity-value lines). Equity value is increasing for lower liquidation boundaries, and increasing for default boundaries closer to $a+b=5$ (where $b=4$ in this example). The equityholders, acting as Stackelberg leaders, choose a default boundary $\hat{x}$ (which implies a choice of the liquidation boundary $\bar{x}_B^*$ by the debtholders) that maximizes the value of equity; i.e. they pick the point on $\bar{x}_B^*(\hat{x})$ at which the iso-equity-value lines are just tangent to the solid line. For the given numerical example, this point is at $\hat{x}^*=4.81$, which implies a liquidation boundary of $\bar{x}_B^*=2.28$, as indicated in the figure.

Numerically, it is clear here that for the given parameters, the direct effect of $\hat{x}$ on the value of equity (via cash flows) is more important than the indirect effect (via changing the liquidation boundary $\bar{x}_B^*$), producing a default boundary close to $a+b$ rather than close to $x^\dagger$. Here, because default does not imply immediate liquidation (in fact, the distance between $\hat{x}^*$ and $\bar{x}_B^*$ is quite large), equityholders care more about not injecting extra cash relative to liquidation timing.

For the given default and liquidation boundaries, leverage (defined as the proportion of debt value in total firm value) is around 49.72%. The spread, defined as $b/B-r$, is around 141 bp. The risk-neutral probability of hitting the default boundary within one year is around 5.8%, while the market recovery, defined as the value of debt at default as a fraction of face value (b/r) is around 66%. The risk-neutral probability of hitting the liquidation boundary is virtually nil, reflecting the very large distance between the default and the liquidation boundaries.[15][16]

This raises the question as to what determines the distance between the default and liquidation boundaries. A key determining feature here is the relationship between K, the liquidation value, and b/r, the face value of debt. When b/r is small in relation to K, debtholders have a strong incentive to liquidate early, which implies that equityholders have a strong incentive to delay default, and the distance will be small. Conversely, when b/r is large in relation to K, debtholders have a weak incentive to liquidate early, which implies that equityholders have a weak incentive to delay default, and the distance will be large. Taking into account both these effects we expect the distance between default and liquidation to be decreasing in $K-b/r$ and vice versa. This is illustrated in Figure 3, where we have kept K fixed at 30, but vary b.

As expected it can be seen from the Figure that the distance between the default boundary (solid line) and the liquidation boundary (dashed line) is small when b is small, tending to zero as $b\rightarrow rK=1.8$. In the limit when $K=b/r$, debtholders liquidate immediately upon default. This implies that equityholders will inject a lot of cash and liquidate only when the cash flow reaches $x^\dagger=2.5$. Conversely, for large coupons of $b=5$, debtholders have only weak incentives to liquidate early. Since they liquidate late anyway, the default boundary has only a very weak indirect effect via the liquidation boundary on the value of equity. The direct effect dominates, and equityholders choose to default very soon after the cash flow becomes negative (for x very close to $a+b=6$).

In between the extreme values, raising b has two effects on the boundaries. Firstly, increasing b decreases the incentives of debtholders to liquidate early. This makes it easier for equityholders to default earlier. But if they default earlier, this in turn tends

[15] The default and liquidation probabilities can be worked out via standard formulas.

[16] Note that this tendency to generate large differences between the probabilities of default and liquidation is not limited to our model but seems to be relatively general in models that separate the notions of default and liquidation. For example Broadie et al. (2007) report a similar finding.

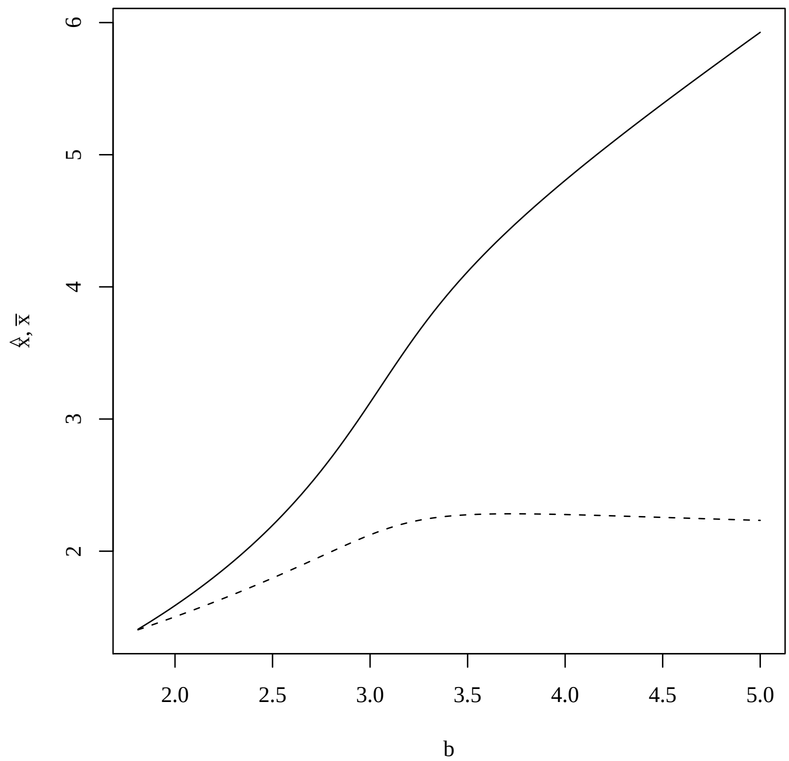

Figure 3: Default and liquidation boundaries as a function of b
The solid line is $\hat{x}^*$, and the dashed line is $\bar{x}_B^*$.

to raise the incentives of debtholders to liquidate early, unless the distance between the boundaries is already very large. This in turn tends to increase the incentives of debtholders to liquidate early. These overall effects are captured by Figure 3.

We next examine the issue of optimal capital structure in the ex ante sense of Leland (1994), i.e. if an entrepreneur wants to exit the firm what should be the optimal mix of debt and equity that he should choose such that it maximizes the value of the proceeds. In other words, which value of b maximizes firm value?

In Figure 4, we plot the firm value in our model (solid line), and compare this to the firm value in the case where default implies immediate liquidation (dashed line). We can see that for small values of b, when the distance between the default and liquidation boundaries is very small, the lines overlap. For large values of b, when the distance between default and liquidation is large, the solid line is much lower than the dashed line. In other words, for large b the firm value is much lower compared to the case where default and liquidation are synonymous. This is the case because an extra cost of financial distress is incurred in the states corresponding to the default region. Obviously, when default implies immediate liquidation, this cost is not incurred, leading to higher

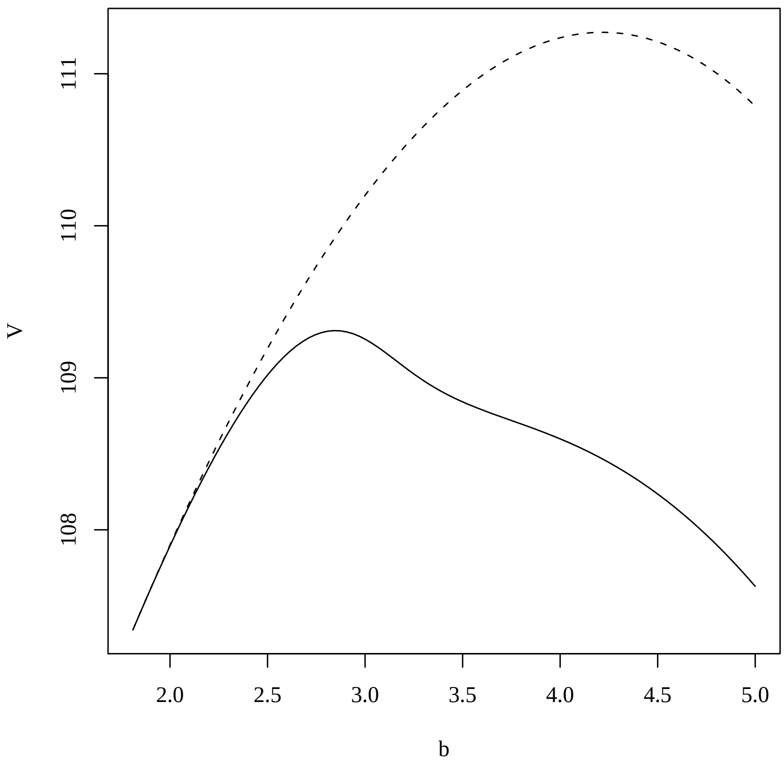

Figure 4: Optimal capital structure
The solid line is the firm value V in our model, the dashed line is the firm value V under the assumption that default implies immediate liquidation.

firm values.

Given the agency cost of debt as well as the cost of financial distress the optimal amount of debt is much lower relative to the case in Leland (1994). The optimal leverage implied by our model is 40.24% but it is 55.07% for the case where default is tantamount to liquidation as in Leland (1994). The extra costs associated with debt have an adverse effect on the value of debt and this explains why the spreads implied by our model are higher compared to the case where default leads to instantaneous liquidation as can be seen in Figure 5.

6. Conclusion

In much of the continuous-time debt pricing literature, it has been assumed that default is tantamount to liquidation, e.g. in the paper of Leland (1994). Noting that in practice this is not the case, several more recent papers have separated the notions of default and liquidation, by defining liquidation to result when a firm spends an ex-

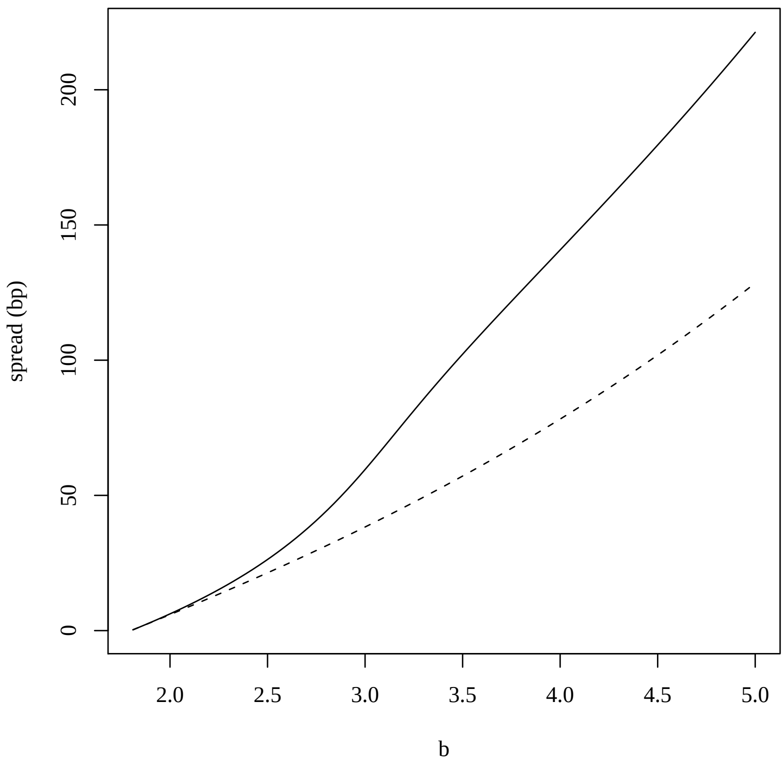

Figure 5: Spreads
The solid line is the spread in our model, the dashed line is the spread under the assumption that default implies immediate liquidation.

ogenously specified time in default. These papers argue that this feature captures the essence of Chapter 11.

In this paper, we consider an alternative mechanism that allows a separation between the notions of default and liquidation: we allow *debtholders* to liquidate the firm, as long as equityholders are defaulting on coupon payments, while equity holders decide when to default, taking into account the effect of their actions on the debtholders' decision to liquidate. This assumption is more representative of environments in which debtholders have a strong influence over the timing of liquidation, like in the case of secured debt, or creditor-friendly bankruptcy regimes such as the one in the UK.

In our setup, debtholders do not liquidate immediately upon default. Nevertheless, debtholders have incentives to liquidate too early in the sense that they do not take into account the full continuation value to all claimants when making their decision. Since debtholders do not liquidate immediately upon default, shareholders can get away with some default, and default earlier relative to the case where default implies immediate liquidation as in the model of Leland (1994). In our model, there are deadweight costs associated with default, implying that this early default reduces overall firm value. Both effects produce an agency cost of debt, which can lead to lower optimal leverage and higher spreads, as shown in a numerical example.

Acknowledgements

We thank Ron Anderson, Mujtaba Mian, Hyun Song Shin and Abhijeet Singh for valuable comments, as well as seminar and conference participants at EEA-ESEM, FIRS, FMG at LSE, Said Business School, and HKUST. Bruche gratefully acknowledges financial support of the Spanish Ministry of Science and Innovation through the Consolider-Ingenio 2010 Project "Consolidating Economics". Naqvi gratefully acknowledges financial support by NUS via Research Grant #R-315-000-060-112, and thanks GAM in collaboration with the FMG at LSE for generously sponsoring the Gilbert de Botton Award for excellence in finance research of which an earlier version of this paper was a beneficiary.

Appendix

Appendix A. Valuation

Proof of proposition 1. It is easy to verify that the general solution of the ODE (2) for the two regions is given by

$$E_1(x) = A_1 x^{\delta} + A_2 x^{-\gamma} + (1-\tau)\left(\frac{x}{r-\mu} - \frac{a+b}{r}\right) \tag{A.1}$$

$$E_2(x) = A_3 x^{\delta} + A_4 x^{-\gamma} \tag{A.2}$$

where A_1, A_2, A_3 and A_4 are integration constants. As $x \to \infty$, x^{δ} explodes. Thus given the no bubbles condition (4), A_1 must be zero. We know that the value of equity is zero when x falls below $\bar{x}$. To determine the values of E_1 and E_2 note that we have three unknowns A_2, A_3 and A_4 and three equations given by the boundary conditions (5), (6) and (7). Solving for the three unknowns, yields the solutions in Eqns. (8) and (9). □

Proof of Proposition 2. It is easy to verify that the general solution of the ODE (10) for the two regions is given

$$B_1(x) = N_1 x^{\delta} + N_2 x^{-\gamma} + \frac{b}{r} \tag{A.3}$$

$$B_2(x) = N_3 x^{\delta} + N_4 x^{-\gamma} + \frac{\theta x}{r-\mu} - \frac{a}{r}. \tag{A.4}$$

where N_1, N_2, N_3 and N_4 are integration constants. As $x \to \infty$, x^{δ} explodes. Thus given the no bubbles condition (12), N_1 must be zero. We know that the value of debt is K when x falls below $\bar{x}$. To determine the values of B_1 and B_2 note that we have three unknowns N_2, N_3, N_4 and three equations given by the boundary conditions (13), (14) and (15). Solving for the integration constants in terms of $\bar{x}$, yields the solutions in Eqns. (16) and (17). □

Appendix B. Some intermediate results

We present some lemmas here which allow the proofs in the subsequent sections to be expressed in a more succinct way.

Lemma 1.

$$(\delta-1)(1+\gamma)r - \delta\gamma(r-\mu) = 0. \tag{B.1}$$

Proof. This follows from the quadratic equation that δ and $-\gamma$ solve. To see this, rewrite the above equation as

$$(1-\delta+\gamma)r = \delta\gamma\mu, \tag{B.2}$$

and insert the definitions of $\delta, -\gamma$:

$$\{\delta, -\gamma\} = \frac{-\mu + \frac{1}{2}\sigma^2}{\sigma^2} \pm \frac{\sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{B.3}$$

□

Lemma 2.

$$\hat{x}Z' - \delta Z(\hat{x}) \begin{cases} > 0 & \textit{for } \hat{x} < a+b \\ = 0 & \textit{for } \hat{x} = a+b \\ < 0 & \textit{for } \hat{x} > a+b \end{cases} \tag{B.4}$$

where $Z' = \frac{dZ(y)}{dy} = \frac{1+\gamma}{\delta+\gamma}\frac{1}{r-\mu}$.

Proof. Using the definition of $Z(\hat{x})$, we can rewrite the expression as

$$-\frac{1}{\delta+\gamma}\left\{(\delta-1)(1+\gamma)\frac{\hat{x}}{r-\mu} - \delta\gamma\frac{a+b}{r}\right\} \tag{B.5}$$

If $\hat{x} = a+b$, the expression will be zero, by Lemma 1. If $\hat{x} > a+b$, the expression in curly brackets will be positive, and the entire expression will be negative, since $(\delta-1)(1+\gamma) > 0, r-\mu > 0)$, and by Lemma 1. Similarly If $\hat{x} < a+b$, the expression in curly brackets will be negative, and the whole expression will be positive. □

Lemma 3.

$$Z(y) \begin{cases} > 0 & \textit{for } y > x^\dagger, \\ = 0 & \textit{for } y = x^\dagger, \\ < 0 & \textit{for } y < x^\dagger, \end{cases} \tag{B.6}$$

where $x^\dagger = \frac{\gamma}{1+\gamma}\frac{r-\mu}{r}(a+b)$*. Also,* $x^\dagger < a+b$*.*

Proof. Solve $Z(x^\dagger) \equiv 0$ for $x^\dagger$, and note that $Z(y)$ is increasing in y to obtain the first statement.. To see that $x^\dagger < a+b$, note that $\frac{\gamma}{1+\gamma}\frac{r-\mu}{r} < 1$ can be rearranged to produce $-\gamma\mu < r$. But since $-\gamma$ solves

$$-\mu\gamma - r + \frac{1}{2}\sigma^2\gamma(\gamma+1) = 0, \tag{B.7}$$

and because $\sigma^2, \gamma > 0$, we have that $-\mu\gamma < r$, and hence that $x^\dagger < a+b$. □

Appendix C. Optimal boundaries

Appendix C.1. Debt-value maximizing liquidation boundary

The liquidation boundary $\bar{x}_B^*$ that maximizes the value of debt can be worked out either via direct optimization or via a smooth-pasting condition. Here, we work out this boundary via direct optimization.

$B_1(x;\hat{x},\bar{x})$ does not depend on $\bar{x}$ directly, but only via $B_2(\hat{x};\hat{x},\bar{x})$. We can furthermore see that $B_1(x;\hat{x},\bar{x})$ depends positively on $B_2(\hat{x};\hat{x},\bar{x})$. Therefore, picking a $\bar{x}$ that maximizes $B_2(\hat{x};\hat{x},\bar{x})$ is equivalent to picking the $\bar{x}$ that maximizes $B_1(x;\hat{x},\bar{x})$.

$B_2(\hat{x};\hat{x},\bar{x})$ is given by

$$B_2(\hat{x};\hat{x},\bar{x}) = \frac{\theta\hat{x}}{r-\mu} - \frac{a}{r} - Z(\theta\hat{x}) + \left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} - Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^\delta\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma}. \tag{C.1}$$

The first derivative of $B_2(\hat{x};\hat{x},\bar{x})$ w.r.t. $\bar{x}$ is

$$\begin{aligned}\frac{\partial B_2(\hat{x})}{\partial \bar{x}} = \frac{1}{\bar{x}}\left[0 - \left(\frac{\theta\bar{x}}{r-\mu} - \delta Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma} \\ + \gamma\frac{1}{\bar{x}}\left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} - Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma}.\end{aligned} \tag{C.2}$$

The sign of this derivative will be equal to the sign of

$$f_B(\bar{x}) = \underbrace{\gamma K}_{P(\bar{x})} + \underbrace{\gamma\frac{a}{r} - (1+\gamma)\frac{\theta\bar{x}}{r-\mu}}_{Q(\bar{x})} + \underbrace{(\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_B(\bar{x})}. \tag{C.3}$$

A $\bar{x}$ that produces a local *maximum* for B will be given by a root of this function, where the function needs to cross the $\bar{x}$-axis from *above.*

There might be several local maxima. A set of sufficient conditions for single-crossing from above, below $\hat{x}$, ensuring a single optimal $\bar{x}^*_B < \hat{x}$, are : (1) $f_B(0) > 0$, (2) $f_B(\hat{x}) < 0$, and (3) $f'_B(\bar{x}) < 0$ for $\bar{x} \in [0, \hat{x})$. This just says that (1) for low enough cash flows, you do want to liquidate, (2) for high enough cash flows ($x \geq \hat{x}$), you do not want to liquidate, and (3) the higher the cash flow, the less likely you are to want to liquidate.

It is obvious that conditions (1) and (2) are satisfied by a constant $K < \frac{b}{r}$. To see that condition (3) is satisfied, take derivatives of $f'_B(\bar{x})$:

$$f'_B(\bar{x}) = \underbrace{0}_{P'(\bar{x})} + \underbrace{\left(-(1+\gamma)\frac{\theta\bar{x}}{r-\mu}\frac{1}{\bar{x}}\right)}_{Q'(\bar{x})} + \underbrace{\delta(\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\frac{1}{\bar{x}}}_{R'_B(\bar{x})}. \tag{C.4}$$

We can now show that $f'_B(\bar{x}) = Q'(\bar{x}) + R'_B(\bar{x}) < 0$ in the relevant interval. Note initially that $Q'(0) + R'_B(0) < 0$. We note that at the other endpoint $\bar{x} = \hat{x}$, $Q'(\hat{x}) + R'_B(\hat{x}) \leq 0$ at $\bar{x} = \hat{x}$, with equality only for $\theta = 1$, by Lemma 2.

$Q(\bar{x}) + R_B(\bar{x})$ is either strictly concave, or strictly convex in $[0, \hat{x})$: $Q(\bar{x})$ is linear. If $Z(\theta\hat{x}) < 0$, then $R_B(\bar{x})$ is strictly decreasing and concave in $[0, \hat{x})$, if $Z(\theta\hat{x}) > 0$, then $R_B(\bar{x})$ is strictly increasing and convex in $[0, \hat{x})$.

Since $Q'(\bar{x})+R'_B(\bar{x})$ is negative at $\bar{x} = 0$, and non-positive at $\bar{x} = \hat{x}$, and $Q(\bar{x})+R_B(\bar{x})$ is either strictly convex or concave, $Q(\bar{x})+R_B(\bar{x})$ must be monotonic decreasing in $[0, \hat{x})$.

Effect of $\hat{x}$ on $\bar{x}^*_B$. We showed that $f_B(\bar{x})$ can be written as $P(\bar{x})+Q(\bar{x})+R_B(\bar{x})$, where $\hat{x}$ only appears in $R_B(\bar{x})$, and $R_B(\bar{x})$ is given by

$$R_B(\bar{x}) = (\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}. \tag{C.5}$$

Taking partial derivatives w.r.t. $\hat{x}$, we see that

$$\frac{\partial R_B}{\partial \hat{x}} = \frac{1}{\hat{x}}(\delta+\gamma)\left(\theta\hat{x}Z' - \delta Z(\theta\hat{x})\right)\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}. \tag{C.6}$$

By Lemma 2, this will be positive for $\theta\hat{x} < a+b$, zero for $\theta\hat{x} = a+b$, and negative for $\theta\hat{x} > a+b$. We can restrict attention to values of $\hat{x} \leq \frac{a+b}{\theta}$: If $\hat{x} > \frac{a+b}{\theta}$, this implies that there are some levels of the cash flow $x < \hat{x}$ in the default region such that the residual cash flow that $\theta x - a$ debtholders obtain in default is *larger* than the promised coupon payment b. In this case, the firm is clearly *not* defaulting on promised coupon payments, and an $\hat{x} > \frac{a+b}{\theta}$ is therefore meaningless.

Since $\hat{x}$ enters $f_B(\bar{x})$ only via $R_B(\bar{x})$, and since $\frac{\partial f_B(\bar{x})}{\partial \bar{x}} < 0$ in the relevant interval, we obtain that

$$\frac{d\bar{x}_B^*}{d\hat{x}} \begin{cases} > 0 \text{ for } \hat{x} < \frac{a+b}{\theta}, \\ = 0 \text{ for } \hat{x} = \frac{a+b}{\theta}, \end{cases} \tag{C.7}$$

via the implicit function theorem.

Appendix C.2. Equity-value maximizing default boundary $\hat{x}^*$

In this subsection, we characterize $\hat{x}^*$ via consideration of the total derivative of E w.r.t. $\hat{x}$,

$$\frac{dE(x;\hat{x},\bar{x})}{d\hat{x}} = \frac{\partial E(x;\hat{x},\bar{x})}{\partial \hat{x}} + \frac{\partial E(x;\hat{x},\bar{x})}{\partial \bar{x}} \frac{d\bar{x}_B^*(\hat{x})}{d\hat{x}}. \tag{C.8}$$

We initially consider the benchmark case of Leland (1994), in which default equals liquidation, i.e. $\bar{x} = \hat{x}$. In this case, the total derivative above reduces to

$$\frac{dE(x;\hat{x},\hat{x})}{d\hat{x}}. \tag{C.9}$$

Setting this to zero produces a first order condition, or (equivalently) a smooth-pasting condition.

We then consider the more general case where $\bar{x} < \hat{x}$. We first derive the partial derivative of E w.r.t. $\hat{x}$, and show that it is positive for $\hat{x} < a+b$, zero for $\hat{x} = a+b$, and negative for $\hat{x} > a+b$. We then show that the partial derivative of E w.r.t $\bar{x}$ is positive for $\hat{x} > x^\dagger = \frac{\gamma}{1+\gamma}\frac{r-\mu}{r}(a+b)$, zero for $\hat{x} = x^\dagger$, and negative for $\hat{x} < x^\dagger$. Combining this with the results on the derivative of $\bar{x}_B^*$ w.r.t. $\hat{x}$ (see Appendix C.1), this will allow showing that $x^\dagger < \hat{x}^* < a+b$.

Appendix C.2.1. The Leland (1994) benchmark ($\bar{x} = \hat{x}$)

Suppose the firm is immediately liquidated when default occurs ($\bar{x} = \hat{x}$) This means that the equity pricing function now is

$$E(x;\hat{x},\hat{x}) = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a+b}{r}\right) + \left[0 - (1-\tau)\left(\frac{\hat{x}}{r-\mu} - \frac{a+b}{r}\right)\right]\left(\frac{x}{\hat{x}}\right)^{-\gamma}, \tag{C.10}$$

as can be shown by solving the corresponding ODE with the appropriate boundary conditions, or taking appropriate limits of the expressions for the equity pricing functions as previously derived.

Let $x^\dagger$ denote the default boundary, then

$$x^\dagger = \frac{\gamma}{1+\gamma}\frac{r-\mu}{r}(a+b). \tag{C.11}$$

By Lemma 3, we know that $x^\dagger < a+b$.[17]

[17] The term $(1-\tau)\frac{x}{r-\mu}$ here plays the role of a post-tax "asset value" as in Leland (1994). In terms of the "asset value" the default boundary could be written as $(1-\tau)\frac{x^\dagger}{r-\mu} = \frac{\gamma}{1+\gamma}(1-\tau)\frac{a+b}{r}$, which is closer to the original formulation in Leland (1994).

Appendix C.2.2. Partial derivative of E w.r.t. $\hat{x}$

Now suppose that $\bar{x} < \hat{x}$. We show that both $\partial E_1(x;\hat{x},\bar{x})/\partial\hat{x}$ and $\partial E_2(x;\hat{x},\bar{x})/\partial\hat{x}$ are both positive for $\hat{x} < a+b$, both zero for $\hat{x} = a+b$, and both negative for $\hat{x} > a+b$, $\forall x,\bar{x}$.

With some algebra, it can be shown that

$$\frac{\partial E_1(x;\hat{x},\bar{x})}{\partial\hat{x}} = \frac{1}{\hat{x}}(1-\tau)\left\{\hat{x}Z' - \delta Z(\hat{x})\right\}\left(1-\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta+\gamma}\right)\left(\frac{x}{\hat{x}}\right)^{-\gamma}, \tag{C.12}$$

where we note that all terms apart from the term in curly brackets are positive.

With some more algebra, it can be shown that

$$\frac{\partial E_2(x;\hat{x})}{\partial\hat{x}} = \frac{1}{\hat{x}}(1-\tau)\left\{\hat{x}Z' - \delta Z(\hat{x})\right\}\left(\left(\frac{x}{\hat{x}}\right)^{\delta} - \left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\left(\frac{x}{\bar{x}}\right)^{-\gamma}\right), \tag{C.13}$$

where we note again that all terms apart from the term in curly brackets are positive.

We can see that the sign of $\partial E_1(x)/\partial\hat{x}$ and the sign of $\partial E_2(x)/\partial\hat{x}$ is therefore the same as the sign of $\hat{x}Z' - \delta Z(\hat{x})$. By Lemma 2, we now can see that the $\hat{x}$ that sets the partial derivative of $E(x;\hat{x},\bar{x})$ w.r.t. $\hat{x}$ equal to zero is $\hat{x} = a+b$, for any $\bar{x} < a+b$.

Appendix C.2.3. Partial derivative of E w.r.t. $\bar{x}$

We can see that E_1 depends positively on E_2, and that $\bar{x}$ only affects E_2. The sign of the effect of $\bar{x}$ on E_1 is therefore identically to the sign of the effect of $\bar{x}$ on E_2. We have

$$\frac{\partial E_2(x;\hat{x},\bar{x})}{\partial\bar{x}} = -(\delta+\gamma)(1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\left(\frac{x}{\bar{x}}\right)^{-\gamma}\frac{1}{\bar{x}}.$$

By Lemma 3, the sign of this derivative is positive when $\hat{x} < x^{\dagger}$, zero when $\hat{x} = x^{\dagger}$, and negative when $\hat{x} > x^{\dagger}$.

Appendix C.2.4. First- and second-order-condition

For the total derivative

$$\frac{dE(x;\hat{x},\bar{x})}{d\hat{x}} = \frac{\partial E(x;\hat{x},\bar{x})}{\partial\hat{x}} + \frac{\partial E(x;\hat{x},\bar{x})}{\partial\bar{x}}\frac{d\bar{x}^*_B(\hat{x})}{d\hat{x}}, \tag{C.14}$$

we know that the first term on the RHS is positive for $\hat{x} < a+b$ and negative for $\hat{x} > a+b$.

Now note that $x^{\dagger} < a+b < \frac{a+b}{\theta}$ by Lemma 3 and the fact that $0 < \theta < 1$. For the second term, due to the properties of the $\partial E/\partial\bar{x}$ and $d\bar{x}^*_B/\hat{x}$, we know that this is positive for $\hat{x} < x^{\dagger}$, when $\partial E/\partial\bar{x} > 0$ is positive and $d\bar{x}^*_B/d\hat{x} > 0$. We know that it is negative for $x^{\dagger} < \hat{x} < \frac{a+b}{\theta}$, when $\partial E/\partial\bar{x} < 0$, and $d\bar{x}^*_B/d\hat{x} > 0$. We know that it is zero for $\hat{x} = \frac{a+b}{\theta}$, when $d\bar{x}^*_B/d\hat{x} = 0$. We can ignore the situation where $\hat{x} > \frac{a+b}{\theta}$, because such an $\hat{x}$ is meaningless (see section Appendix C.1).

This allows making a statement about the sign of the total derivative:

	$\frac{\partial E(x;\hat{x},\bar{x})}{\partial\hat{x}}$	$\frac{\partial E(x,\hat{x},\bar{x})}{\partial\bar{x}}\frac{d\bar{x}^*_B}{\hat{x}}$	$\frac{dE(x;\hat{x},\bar{x})}{d\hat{x}}$
$\hat{x} < x^{\dagger}$	> 0	> 0	> 0
$x^{\dagger} < \hat{x} < a+b$	> 0	< 0	?
$a+b < \hat{x} < \frac{a+b}{\theta}$	< 0	< 0	< 0
$\hat{x} = \frac{a+b}{\theta}$	< 0	0	< 0

(C.15)

By continuity, there must be an $\hat{x}^*$ between $x^\dagger$ and $a+b$ at which the total derivative crosses the $\hat{x}$-axis from above. At this $\hat{x}^*$, the value of E will be maximized.

Appendix C.3. Levered-firm-value maximizing liquidation boundary

$V_1(x;\hat{x},\bar{x})$ does not depend on $\bar{x}$ directly, but only via $V_2(\hat{x};\hat{x},\bar{x})$. We can furthermore see that $V_1(x;\hat{x},\bar{x})$ depends positively on $V_2(\hat{x};\hat{x},\bar{x})$. Therefore, picking a $\bar{x}$ that maximizes $V_2(\hat{x};\hat{x},\bar{x})$ is equivalent to picking the $\bar{x}$ that maximizes $V_1(x;\hat{x},\bar{x})$.

$V_2(\hat{x};\hat{x},\bar{x})$ is given by

$$\begin{aligned} V_2(\hat{x};\hat{x},\bar{x}) &= \frac{\theta\hat{x}}{r-\mu} - \frac{a}{r} + Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x}) \\ &\quad + \left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} + \left(Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x})\right)\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma}. \end{aligned} \tag{C.16}$$

The first derivative of $V_2(\hat{x};\hat{x},\bar{x})$ w.r.t. $\bar{x}$ is

$$\begin{aligned} \frac{\partial B_2(\hat{x})}{\partial \bar{x}} &= \frac{1}{\bar{x}}\left[0 - \left(\frac{\theta\bar{x}}{r-\mu} + \delta\left(Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x})\right)\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma} \\ &\quad + \gamma\frac{1}{\bar{x}}\left[K - \left(\frac{\theta\bar{x}}{r-\mu} - \frac{a}{r} + \left(Z(\hat{x}) - Z(\theta\hat{x}) - \tau Z(\hat{x})\right)\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}\right)\right]\left(\frac{\hat{x}}{\bar{x}}\right)^{-\gamma}. \end{aligned} \tag{C.17}$$

The sign of this derivative will be equal to the sign of

$$\begin{aligned} f_V(\bar{x}) &= \underbrace{\gamma K}_{P(\bar{x})} + \underbrace{\left(\gamma\frac{a}{r} - (1+\gamma)\frac{\theta\bar{x}}{r-\mu}\right)}_{Q(\bar{x})} \\ &\quad + \underbrace{(\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_B(\bar{x})} - \underbrace{(\delta+\gamma)(1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_V(\bar{x})} \\ &= f_B(\bar{x}) - \underbrace{(\delta+\gamma)(1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_V(\bar{x})}. \end{aligned} \tag{C.18}$$

Note that $R_V(0) = 0$, that $R_V(\bar{x})$ will be positive, convex, and increasing if $Z(\hat{x}) > 0$, and negative, concave, and decreasing if $Z(\hat{x}) < 0$. Since we are subtracting $R_V(\bar{x})$, this implies that the root of $f_V(\bar{x})$, $\bar{x}_V^*$, will be smaller than the root of $f_B(\bar{x})$, $\bar{x}_B^*$, iff $Z(\hat{x}) > 0$, because this will ensure that $f_V(\bar{x}) < f_B(\bar{x})$ for the relevant range of $\bar{x}$. By Lemma 3, $Z(\hat{x}) > 0$ exactly when $\hat{x} > x^\dagger$.

Appendix D. Liquidation payoff K a function of cash flow x

We first discuss why we need to diverge from the typical assumption that the liquidation payoff is a fraction of the "asset value" of the firm. (This assumption is made e.g. by Leland (1994), and repeated in much of the literature.) Even though we cannot adopt this assumption, we can allow K to be a function of the cash flow x, but with some restrictions on this function. We subsequently discuss these restrictions.

Comparison to a standard assumption. A standard assumption in the literature, see e.g. Leland (1994), is that the liquidation payoff is a fraction of the "asset value" of the firm. We show here that in the context of this assumption, it is difficult to talk meaningfully about *optimal* liquidation. Consider an unlevered firm, with a (traded) post-tax asset value ω, a fraction β of which is paid out as a flow to the owner. Suppose the evolution of ω under $\mathbb{Q}$ is described by the following SDE:

$$d\omega(t) = (r-\beta)\omega(t)dt + \sigma\omega(t)d\tilde{W}(t) \tag{D.1}$$

(ω plays the role that $(1-\tau)\frac{x}{r-\mu}$ plays in our context.) Suppose that the firm is liquidated once ω hits $\bar{\omega}$, and that the liquidation payoff is $(1-\lambda)\bar{\omega}$, where $0<\lambda<1$. What is the value $v(\omega)$ of the unlevered firm?

The pricing ODE will be

$$\frac{1}{2}\sigma^2\omega^2 v''(\omega) + (r-\beta)\omega v'(\omega) - rv(\omega) + \beta\omega = 0, \tag{D.2}$$

with boundary conditions

$$\lim_{\omega\to\infty} v(\omega) = \omega \tag{D.3}$$

$$v(\bar{\omega}) = (1-\lambda)\bar{\omega}. \tag{D.4}$$

The solution is

$$v(\omega;\bar{\omega}) = \omega + [(1-\lambda)\bar{\omega} - \bar{\omega}]\left(\frac{\omega}{\bar{\omega}}\right)^{-\gamma} = \omega - \lambda\left(\frac{\omega}{\bar{\omega}}\right)^{-\gamma}. \tag{D.5}$$

This is essentially the equation derived by Leland (1994), except that here, the coupon payment set to zero (this is an unlevered firm).

What is the optimal liquidation boundary in this case? By direct differentiation, it can be seen that the first derivative of v w.r.t. $\bar{\omega}$ is

$$-(1+\gamma)\lambda\left(\frac{\omega}{\bar{\omega}}\right)^{-\gamma}. \tag{D.6}$$

Note that since $\lambda>0$, this is always negative. This means that it is never optimal to liquidate (this is of course also obvious by direct inspection of the pricing function). Clearly, with this type of corner solution, it is hard to discuss optimal liquidation.

K as a function of cash flow x. Let the liquidation payoff to debt be a function $K(\bar{x})$ of the cash flow at liquidation, $\bar{x}$. This means that the last boundary condition in the derivation of the prices needs to be modified in the obvious way. All resulting pricing formulas are essentially the same, with the constant K now replaced with the function $K(\bar{x})$ everywhere.

We can repeat the derivations above to obtain new implicit functions $f_B(\bar{x})$, $f_V(\bar{x})$ that determine the optimal liquidation boundaries. We have

$$f_B(\bar{x}) = \underbrace{\bar{x}K'(\bar{x}) + \gamma K(\bar{x})}_{P(\bar{x})} + \underbrace{\gamma\frac{a}{r} - (1+\gamma)\frac{\theta\bar{x}}{r-\mu}}_{Q(\bar{x})} + \underbrace{(\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_B(\bar{x})}, \tag{D.7}$$

and

$$
\begin{aligned}
f_V(\bar{x}) = \underbrace{\bar{x}K'(\bar{x}) + \gamma K(\bar{x})}_{P(\bar{x})} + \underbrace{\left(\gamma\frac{a}{r} - (1+\gamma)\frac{\theta\bar{x}}{r-\mu}\right)}_{Q(\bar{x})} \\
+ \underbrace{(\delta+\gamma)Z(\theta\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_B(\bar{x})} - \underbrace{(\delta+\gamma)(1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_V(\bar{x})} \\
= f_B(\bar{x}) - \underbrace{(\delta+\gamma)(1-\tau)Z(\hat{x})\left(\frac{\bar{x}}{\hat{x}}\right)^{\delta}}_{R_V(\bar{x})}.
\end{aligned}
\tag{D.8}
$$

There might be several local maxima, each associated with a root of these functions where the function crosses the $\bar{x}$-axis from above. E.g. in the case of $f_B(\bar{x})$, a set of sufficient conditions for single-crossing from above, below $\hat{x}$, ensuring a single optimal $\bar{x}_B^*$ still is: (1) $f_B(0) > 0$, (2) $f_B(\hat{x}) < 0$, and (3) $f_B'(\bar{x}) < 0$ for $\bar{x} \in [0, \hat{x})$. This just says that (1) for low enough cash flows, you do want to liquidate, (2) for high enough cash flows ($x \geq \hat{x}$), you do not want to liquidate, and (3) the higher the cash flow, the less likely you are to want to liquidate. These conditions implicitly put restrictions on the first and second derivatives of the function $K(\bar{x})$.

We now show how these conditions relate to similar conditions that one would have to impose for an optimal liquidation boundary $\bar{x}_U^*$ to exist for an *unlevered* firm. Let U denote the value of the unlevered firm. The holder of the unlevered firm receives cash flows $(1-\tau)(x-a)$ until the firm is liquidated at cash flow $\bar{x}$, when she receives $K(\bar{x})$. The pricing ODE is

$$\frac{1}{2}\sigma^2 x^2 U''(x) + \mu x U'(x) - rU(x) + (1-\tau)(x-a) = 0, \tag{D.9}$$

and the boundary conditions are

$$\lim_{x\to\infty} U = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a}{r}\right) \tag{D.10}$$

$$U(\bar{x}) = K(\bar{x}). \tag{D.11}$$

Solving, one obtains

$$U(x;\bar{x}) = (1-\tau)\left(\frac{x}{r-\mu} - \frac{a}{r}\right) + \left[K(\bar{x}) - \left(\frac{\bar{x}}{r-\mu} - \frac{a}{r}\right)\right]\left(\frac{x}{\bar{x}}\right)^{-\gamma}. \tag{D.12}$$

By direct differentiation, we find that the sign of $\partial U/\partial \bar{x}$ is equal to the sign of

$$f_U(\bar{x}) = \underbrace{\bar{x}K'(\bar{x}) + \gamma K(\bar{x})}_{P(\bar{x})} + \underbrace{(1-\tau)\left(\gamma\frac{a}{r} - (1+\gamma)\frac{\bar{x}}{r-\mu}\right)}_{Q_U(\bar{x})}. \tag{D.13}$$

A $\bar{x}_U^*$ that produces a local *maximum* for U will be given by a root of this function, where the function needs to cross the $\bar{x}$-axis from *above*.

Comparing $f_U(\bar{x})$ with $f_V(\bar{x})$ and $f_B(\bar{x})$, we can see that they are very similar. Reasonable conditions on $K(\bar{x})$ that ensure the existence and uniqueness of an optimal liquidation boundary $\bar{x}_U^*$ that maximizes the value of the unlevered firm *imply* that (1) and (3) are going to be satisfied for $f_B(\bar{x})$, and hence also $f_V(\bar{x})$. An additional assumption on $K(\bar{x})$ is then still needed to ensure that $\bar{x}_B < \hat{x}$. This would be the analogue of the assumption $K < \frac{b}{r}$, i.e. that debt is risky.

Anderson, R., Sundaresan, S., 1996. Design and valuation of debt contracts. Review of Financial Studies 9 (1), 37–68.
Broadie, M., Chernov, M., Sundaresan, S., June 2007. Optimal debt and equity values in the presence of chapter 7 and chapter 11. Journal of Finance 62 (3), 1341–1377.
DeMarzo, P. M., Sannikov, Y., December 2006. Optimal security design and dynamic capital structure in a continuous-time agency model. Journal of Finance 61 (6), 2681–2724.
Dixit, A. K., 1993. The art of smooth pasting. Vol. 55 of Fundamentals of Pure and Applied Economics. Harwood Academic Publishers, Reading, UK.
Dumitrescu, A., 2007. Valuation of defaultable bonds and debt restructuring. Journal of Corporate Finance 13 (1), 94–111.
François, P., Morellec, E., 2004. Capital structure and asset prices: Some effects of bankruptcy procedures. Journal of Business 77 (2), 387–411.
Franks, J. R., Sussman, O., 2005. Financial distress and bank restructuring of small to medium size UK companies. Review of Finance 9 (1), 65–96.
Franks, J. R., Torous, W. N., July 1989. An empirical investigation of U.S. firms in reorganization. Journal of Finance 44 (3), 747–769.
Galai, D., Raviv, A., Wiener, Z., 2007. Liquidation triggers and the valuation of equity and debt. Journal of Banking & Finance 31 (12), 3604–3620.
Gilson, S. C., John, K., Lang, L. H. P., 1990. Troubled debt restructurings: An empirical study of private reorganization of firms in default. Journal of Financial Economics 27 (2), 315–353.
Hart, O. D., 1995. Firms, Contracts, and Financial Structure. Clarendon Lectures in Economics. Oxford University Press, Oxford, UK.
Hart, O. D., Moore, J. H., February 1998. Default and renegotiation: a dynamic model of debt. Quarterly Journal of Economics 113 (1), 1–41.
Jensen, M. C., 1986. Agency costs of free cash flow, corporate finance, and takeovers. American Economic Review 76 (2), 55–77.
Karatzas, I., Shreve, S. E., 1988. Brownian Motion and Stochastic Calculus. Springer-Verlag.
Leland, H., Toft, K., 1996. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 51 (3), 987–1019, papers and Proceedings of the Fifty-Sixth Annual Meeting of the American Finance Association, San Francisco California, January 5-7, 1996.
Leland, H. E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49 (4), 1213–1252.
Longstaff, F. A., Schwartz, E. S., 1995. A simple approach to valuing risky debt. Journal of Finance 50 (3), 789–821.
Mella-Barral, P., 1999. The dynamics of default and debt reorganization. Review of Financial Studies 12 (3), 535–578.
Mella-Barral, P., Perraudin, W., 1997. Strategic debt service. Journal of Finance 52 (2), 531–556.
Miller, M. H., May 1977. Debt and taxes. Journal of Finance 32 (2), 261–275.
Moraux, F., November 2002. Valuing corporate liabilities when the default threshold is not an absorbing barrier, working Paper.
Morellec, E., Smith, C. W., 2007. Agency conflicts and risk management. Review of Finance 11 (1), 1–23.
Realdon, M., 2007. Valuation of the firm's liabilities when equity holders are also creditors. Journal of Business Finance & Accounting 34 (5-6), 950–975.
Woolridge, F. H., 1987. Administration Procedure. Jordans, Bristol, UK.