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The Implicit McMillan Degree and Network Descriptions

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The paper addresses the problem of evaluating the *Implicit McMillan degree* δm of $W^{-1}(s)$, where $W^{-1}(s)$ denotes the transfer function of a passive *RLC* electrical network [1]. The *Implicit McMillan degree* δm specifies the minimum number of dynamic elements needed to completely characterize the passive *RLC* network, i.e. an electrical network that contains only passive elements (capacitors, inductors and resistors), and associates it with the rank properties of the passive element matrices. A fact that in the circuit literature is intuitively accepted but not rigorously proved is that this degree must be equal to the minimum number of independent dynamical elements in the network (Livada 2017), (Leventides et al. 2014). In this paper, we investigate this finding, showing that the maximum possible *Implicit McMillan degree* δm of such networks is given by $rankL + rankC$, and that this value is reached when certain necessary and sufficient conditions are satisfied [a].

Keywords: systems theory, McMillan degree, passive RLC networks, network theory, matrix pencils

[a]We should note that throughout this paper we consider only passive *RLC* networks that contain all three types of passive elements, i.e. inductors, capacitors and resistors. The results established do not apply in the case of networks containing only capacitors and inductors. This is a generic case and the results obtained are valid under certain conditions.

1. Introduction

Classical network theory (E. A. Guillemin 1957), (Shearer et al. 1971), (Seshu & Reed 1961), (Vlach & Singhal 1994) introduces for a large family of systems an integral- differential description, in terms of the impedance and admittance models, which in turn provide an implicit system description. In this description the natural topology of the network introduced by the R, L, and C structural matrices is explicitly described. These type of networks may be considered within the systems theory setting (Karcanias 2008), modelled as a set of integro-differential equations relating the basic variables of the network i.e. the vectors of currents and voltage sources (Seshu & Reed 1961), (Vlach & Singhal 1994). In the frequency domain these equations are transformed to the so called loop or impedance model where a rational matrix of the form $W(s) = s\mathbf{L} + \mathbf{R} + s^{-1}\mathbf{C}$ plays the role of a generalised transfer function. The properties of $W(s)$ are central to the study of the network from the system theory point of view. In this work we address the problem of associating the McMillan degree of $W^{-1}(s)$ with the matrices **R,L,C** of the network elements. The McMillan degree defines the minimum number of dynamic elements needed to describe the network fully. A result which is intuitively known but not rigorously proven in

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the circuit literature (E. A. Guillemin 1957), (Van Valkenburg 1960), (Seshu & Reed 1961) is that this degree has to be equal to the minimum independent number of capacitors and inductors in the circuit. The main theoretic tools which were used for the derivation of the following results are given in the framework of compound matrices and exterior algebra (Marcus 1973). Here we examine rigorously this question proving that the maximum possible McMillan degree of such networks is given by $rank(\mathbf{L}) + rank(\mathbf{C})$ and this value is attained provided some regularity (or independence) conditions are valid for the network. These conditions are necessary and sufficient, i.e. optimal, and they are expressed in various forms that are all testable. The first set of conditions are of determinantal type and relate the highest and lowest order coefficients of s in the expansion of the determinant $det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$ to the matrices **L,R,C**. The second set of conditions relates the property of these coefficients to be non-zero with some rank properties of matrices related to the three fundamental matrices **R,L,C**. These conditions imply some regularity properties for the network similar to the ones considered intuitively in the literature. This result and analysis may be used as a starting point for a more general study of this type of networks in terms of its algebraic properties such as the study of the McMillan form of $W(s)$ the nature of zero elementary divisors or the structure of the pole divisors. Such considerations will make possible the use of system and control theoretic tools in RLC network theory and will facilitate the definition and solution of new analysis and design problems.

1.1 *Problem Statement*

In this paper we consider the dynamic properties of a passive *RLC* network, i.e. an electrical network that contains only passive elements (capacitors, inductors and resistors), as described by its impedance model (E. A. Guillemin 1957), (Shearer et al. 1971).
RLC networks are of great importance both from a theoretical and applied perspective. Although, these type of networks have wide applicability, i.e. in filters, oscillators circuits or variable tuned circuits, the paper performs a theoretical analysis related to the system aspects of certain descriptions used within *RLC* topologies. This work has been motivated by the "System Re-Engineering problem", aspects of which have been addressed in (Karcanias 2008), (Livada 2017).
In the impedance model model the variables are selected such that the vertex law is automatically satisfied. Solving these equations involves the selection of internal independent loops, the definition of loop currents and the transformation of current sources to equivalent voltage sources. If we denote by $(I_{s1}, I_{2s}, ..., I_{sq})$ the set of the Laplace transforms of the loop currents and by $(u_{s1}, ..., u_{sq})$ the set of Laplace transforms of equivalent voltage sources, then the impedance model (Shearer et al. 1971) is defined by: $Z(s)$, where $z_{ii}(s)$ is the sum of impedances in loop and $z_{ij}(s)$ is the sum of impedances common between loops i and j. These equations can be written in short as:

$$Z(s)I_s(s) = u_s(s) \tag{1.1}$$

This is referred to as the impedance model and the symmetric matrix $Z(s)$ is referred to as the network impedance matrix. Equivalently, the admittance model is described in short as:

$$Y(s)v(s) = I_s(s) \tag{1.2}$$

and the symmetric matrix $Y(s)$ is referred to as the network admittance matrix. This general modelling approach for passive network provides a description of networks in terms of symmetric integral-differential operators, the impedance and admittance models which are described in a general way by the unified operator $W(s)$ (Livada 2017):

$$W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} \tag{1.3}$$

where for the case of impedance we have that $\mathbf{L}$ is the matrix whose entries are functions of the inductances, $\mathbf{C}$ is the matrix of capacitances whose entries are functions of the capacitances and finally $\mathbf{R}$ is the matrix whose entries are functions of the network's resistances. The operator $W(s)$ is thus a unifying description of the $Y(s)$ and $Z(s)$ matrices and its properties will be considered next [1]. The $W(s)$ matrix is symmetric and the structure of L, C, R matrices characterizes the associated network topology (Shearer et al. 1971), (Livada 2017). Such matrices have a structure and properties that underpin the development of system theoretic framework based on network models. The operator $W(s)$ describes the dynamics of the network and of special interest are the properties of its zeros (Livada 2017), (Leventides & Karcanias 2009). Furthermore, this integral - differential operator, defined by $W(s)$, introduces a new implicit, i.e. no inputs system description:

$$(p\mathbf{L} + p^{-1}\mathbf{C} + \mathbf{R})\xi = 0$$

where ξ can be seen as an internal vector. Such a description has no inputs and no outputs but as a rational matrix, $W(s)$, has a McMillan degree which is linked to a notion of minimality of the implicit description. The McMillan degree (Antsaklis & Michel 1997) of the system may be computed via various methods, i.e. by determining the Smith-McMillan form or via exterior algebra and a determinantal treatment of the problem. The main purpose of the paper is, for an *RLC* network that is described by the general operator:

$$W(s) = Z(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R}$$

find a relationship between the McMillan degree of the network and the rank of the matrices of the dynamical elements. The McMillan degree of the system may be computed in terms of the transfer function of the network, which is described by the $W^{-1}(s)$ operator. The main purpose of this paper is to compute an upper bound of this degree in terms of the elements of the network, to derive testable conditions and to interpret the results.

We should note that throughout this paper we consider only passive *RLC* networks that contain all three types of passive elements, i.e. inductors, capacitors and resistors. The results established do not apply in the case of networks containing only capacitors and inductors. This is a generic case and the results obtained are valid under certain conditions.These conditions are related with the matrices **L,R,C**, i.e. we assume that:

- $\mathbf{L}$ is a random matrix of prespecified rank p with no specific structure,
- $\mathbf{C}$ is a random matrix of prespecified rank q with no specific structure, and
- $\mathbf{R}$ is a random matrix

Hence one can select specific examples where the Implicit McMillan degree and the McMillan degree are not equal.

2. Implicit McMillan Degree and its Calculation

In this section we establish a relationship between the $W(s)$ operator that describes a general *RLC* network, i.e. an electrical network that contains only passive elements (capacitors, inductors and resistors) and the *Implicit McMillan degree* of this network. By considering a generic case of such a network, i.e.

[1] it should be noted here that this unifying description cannot be used as a description for the driving-point impedance of one-port networks containing multiple inductors and capacitors (Hughes 2018)

consisting of all three types of passive elements, we calculate an upper bound for this degree and we establish links between the Implicit McMillan Degree and the ranks of the matrices of the dynamical elements (i.e. inductors and capacitors) (Leventides et al. 2014), (Livada 2017).

The Implicit McMillan degree δ_m calculated below may be a lower bound of the McMillan degree of $W^{-1}(s)$. In the generic case where the matrices **L,R,C** are those defined above and the fraction in equation (2.1), i.e. $\frac{\varepsilon_1(s)\cdots\varepsilon_n(s)}{\psi_1(s)\cdots\psi_n(s)}$, is coprime, these two degrees are equal. Although, there are some special cases of electrical circuits where δ_m is less than the McMillan degree.
The following Theorem establishes a link between the McMillan degree [2] of a general *RLC* network and its general operator $W(s)$. Furthermore, a formula for the computation of the *Implicit McMillan degree* is stated (Leventides et al. 2014), (Livada 2017).

DEFINITION 2.1 Let a passive electrical *RLC* network consisting only of inductors, capacitors and resistors. The network's cardinality is denoted by n and is equal to the number of independent loops of the network (Alexander & Sadiku 1989). A loop is said to be independent if it contains at least one branch which is not a part of any other independent loop. Independent loops or paths result in independent sets of equations.

THEOREM 2.1 Let $W^{-1}(s)$ be the transfer function of an *RLC* network (Livada 2017), where $W(s) = s\mathbf{L}+s^{-1}\mathbf{C}+\mathbf{R}$ and $W(s)$ non-singular. Then, the *Implicit McMillan degree* of $W^{-1}(s)$ is given by:

$$\delta_m = n_{\max} - \min(n_{\min}, n)$$

where $n_{\max}$ and $n_{\min}$ are the maximum and minimum degrees of s in the expansion of the determinant:

$$\det(s^2\mathbf{L}+s\mathbf{R}+\mathbf{C})$$

and n denotes the cardinality of the network.

Proof.
The Smith-McMillan form (Berger et al. 2019, Schrader & Sain 1988, Karcanias 2009) of $W^{-1}(s)$ is described by the following equation:

$$W^{-1}(s) = V_1(s)\begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & & \\ & \ddots & \\ & & \frac{\varepsilon_n(s)}{\psi_n(s)} \end{bmatrix} V_2(s) \tag{2.1}$$

where: $V_1(s), V_2(s)$ unimodular, $\varepsilon_i/\varepsilon_{i+1}$, ψ_i/ψ_{i-1} and ε_i, ψ_i coprime polynomials. Computing the determinants at both sides of (2.1) we get:

$$\frac{s^n}{\det(s^2\mathbf{L}+s\mathbf{R}+\mathbf{C})} = \frac{\varepsilon_1(s)\cdots\varepsilon_n(s)}{\psi_1(s)\cdots\psi_n(s)} \tag{2.2}$$

The *Implicit McMillan degree* of $W^{-1}(s)$ is given by the degree of the polynomial:

$$p(s) = \psi_1(s)\cdots\psi_n(s)$$

[2]The McMillan degree of a transfer-function matrix is the total number of poles in the diagonal elements of the matrix in its McMillan form. This number determines the order of any minimal state-space realization of the transfer-function matrix or the minimal order of coprime matrix-fraction models.

The polynomial $p(s)$ can be taken from the left hand part of (2.2) as the polynomial remaining from $\det(s^2\mathbf{L}+s\mathbf{R}+\mathbf{C})$ after the maximum possible cancellations of the powers of s in the corresponding left hand part ratio of (2.2). If we let:

$$\det(s^2\mathbf{L}+s\mathbf{R}+\mathbf{C}) = \alpha_{n_{\max}} s^{n_{\max}} + \alpha_{n_{\max-1}} s^{n_{\max-1}} + \cdots + \alpha_{n_{\min}} s^{n_{\min}}$$

then the maximum possible term of s that can be canceled is $s^{\min(n_{\min},n)}$, therefore:

$$p(s) = \psi_1(s)\cdot\psi_2(s)\ldots\psi_n(s) = \alpha_{n_{\max}} s^{n_{\max}-min(n_{\min},n)} + \cdots + \alpha_{n_{\min}} s^{n_{\min}-\min(n_{\min},n)}$$

and hence the degree of $p(s)$ is $n_{\max}-min(n_{\min},n)$, which is the Implicit McMillan degree of $W^{-1}(s)$. □

The next theorem establishes an upper bound for the degree of the determinant of the polynomial $\mathbf{Z_a}(\mathbf{s}) = s^2\mathbf{L}+s\mathbf{R}+\mathbf{C}$ relatively to the ranks of the matrices of the dynamical elements, i.e. $\mathbf{L},\mathbf{C}$ (Leventides et al. 2014), (Livada 2017).

THEOREM 2.2 Let $\mathbf{Z_a}(\mathbf{s}) = s^2\mathbf{L}+s\mathbf{R}+\mathbf{C}$ with $rank(\mathbf{L}) = p$, $rank(\mathbf{C}) = q$, let n denote the cardinality of network and let the polynomial $\det[\mathbf{Z_a}(\mathbf{s})] = \alpha s^{n_2} + \cdots + \beta s^{n_1}$ with the powers in descending order. Then: $n_2 - \min(n,n_1) \leqslant p+q$, when $n \geqslant n_1$ and $n_2 - \min(n,n_1) \leqslant p$, when $n < n_1$. Additionally, the maximum value for $n_2 - \min(n,n_1)$, which is $p+q$ is obtained when $n_2 = n+p$ and $n_1 = n-q$.

Proof. Developing the determinant $det[\mathbf{Z_a}(\mathbf{s})]$ we can get it as sums of determinants taking f_1 rows from $s^2\mathbf{L}$, f_2 rows from $s\mathbf{R}$ and the remaining rows from $\mathbf{C}$. In this case, the polynomial part of this term will be: $s^{2f_1+f_2}$. Furthermore, we have the following constraints for f_1, f_2:

(i) $f_1, f_2 \geqslant 0$

(ii) $f_1+f_2 \leqslant n$

(iii) $f_1 \leqslant p$ (if we select more rows of L than its rank, the coefficient of $s^{2f_1+f_2}$ will be zero).

(iv) $n-f_1-f_2 \leqslant q$ (for similar reasons as in (iii)).

Now as: $f_1 \leqslant p$, $f_1+f_2 \leqslant n$ we get $2f_1+f_2 \leqslant n+p$, with the equality achieved when both $f_1 = p$ and $f_1+f_2 = n$ i.e. when: $f_1 = p$ and $f_2 = n+p$ (we can also see that all constraints are satisfied). Hence,

$$\max(2f_1+f_2) = n+p \tag{2.3}$$

This maximum value is attained exactly when $f_1 = p$ and $f_2 = n-p$. Additionally, selecting f_3 rows from $s\mathbf{R}$ and f_4 rows from $\mathbf{C}$, the degree for n_1 is: $2(n-f_3-f_4)+f_3$ and we have to minimize:

$$\min 2(n-f_3-f_4)+f_3 \tag{2.4}$$

subject to the following constraints for f_3 and f_4:

(i) $f_3, f_4 \geqslant 0$

(ii) $f_3+f_4 \leqslant n$

(iii) $f_4 \leqslant q$.

The solution to this problem is: $f_3+f_4=n$, $f_4=q$, thus $f_3=n-q$ and the minimum degree is $(\min 2(n-f_3-f_4)+f_3)$: $n-q$. Hence, for the McMillan degree $\delta_m = n_{\max}-\min(n_{\min},n) = n_2-\min(n_1,n)$ we distinguish the following two cases:

Case 1: When $n_1 \leqslant n$, then $\delta_m = n_2-n_1$. To maximize δ_m we have to maximize n_2 and minimize n_1. Thus, $\delta_{m_{\max}} = n+p-(n-q) = p+q$.

Case 2: When $n_1 > n$, then $\delta_m = n_2-n$. To maximize δ_m we have to maximize n_2, which is $n_2 = n+p$ and $\delta_{m_{\max}} = n+p-n = p$.

Hence, taking into account the two cases, the maximum possible Implicit McMillan degree is:

$$\delta_{m_{\max}} = p+q$$

when $n_2 = n+p$ and $n_1 = n-q$. □

3. Necessary and Sufficient Conditions satisfied by the Implicit McMillan Degree

In this section we investigate the necessary and sufficient conditions for determining the *Implicit McMillan degree* of an *RLC* network (Livada 2017), (Leventides et al. 2014).

The first Theorem provides a formula for the maximum and minimum coefficients of the determinant of the matrix representation of the circuit (i.e. $\mathbf{Z_a}(\mathbf{s}) = s^2\mathbf{L}+s\mathbf{R}+\mathbf{C}$).

THEOREM 3.1 Let $\mathbf{Z_a}(\mathbf{s}) = s^2\mathbf{L}+s\mathbf{R}+\mathbf{C}$ the matrix representation of a *RLC* circuit. Let $k_{\max}$, $k_{\min}$, $\mathrm{n}_{\max}$, $n_{\min}$ be the maximum and minimum coefficients and degrees of the determinant $\det[\mathbf{Z_a}(\mathbf{s})]$ respectively. Let us also define as $rank(\mathbf{L}) = p$, $rank(\mathbf{C}) = q$ which implies that

$$C_p(\mathbf{L}) = \alpha_1 \cdot \alpha_2^T \ , \ \alpha_1, \alpha_2 \in \mathbb{R}^{\binom{n}{p}\times 1}$$

and that

$$C_q(\mathbf{C}) = \beta_1 \cdot \beta_2^T \ , \ \beta_1, \beta_2 \in \mathbb{R}^{\binom{n}{q}\times 1}$$

where $C_p(\mathbf{L})$ and $C_q(\mathbf{C})$ denote the p-th and q-th compound matrices of $\mathbf{L}$ and $\mathbf{C}$ respectively (Gantmacher & Krein 2002). Then the following hold true:

(i) When $p < n$ then: $n_{\max} \leqslant n+p$ and $n_{\max}$ takes the maximum possible value $n+p$ if and only if

$$k_{\max} = tr(C_p(\mathbf{L}) \cdot Adj_p(\mathbf{R})) = \alpha_2^T \cdot Adj_p(\mathbf{R}) \cdot \alpha_1 \neq 0.$$

where Adj_p denotes the p-th adjugate of $\mathbf{R}$ (Gantmacher & Krein 2002). In the case where $n = p$ then:

$$k_{\max} = \det(\mathbf{L}) \neq 0.$$

(ii) When $q < n$ then: $n_{\min} \geqslant n-q$ and $n_{\min}$ takes the minimum possible value $n-q$ if and only if

$$k_{\min} = tr(C_q(\mathbf{C}) \cdot Adj_q(\mathbf{R})) = \beta_2^T \cdot Adj_q(\mathbf{R}) \cdot \beta_1 \neq 0.$$

where Adj_q denotes the q-th adjugate of $\mathbf{R}$ (Gantmacher & Krein 2002). Particularly, when $n = q$ then:

$$k_{\min} = \det(\mathbf{C}) \neq 0.$$

Proof. We denote by l_i, r_i, c_i the columns of the matrices **L**, **R**, **C** respectively. The $\det[\mathbf{Z_a}(\mathbf{s})]$ is the sum of the terms:

$$(-1)^{\sigma} \cdot \underbrace{l_{i_1} \wedge l_{i_2} \wedge \cdots \wedge l_{i_{f_1}}}_{f_1 \text{ from } \mathbf{L}} \wedge \underbrace{r_{j_1} \wedge r_{j_2} \wedge \cdots \wedge r_{j_{f_2}}}_{f_2 \text{ from } \mathbf{R}} \wedge \underbrace{c_{m_1} \wedge c_{m_2} \wedge \cdots \wedge c_{m_{n-f_1-f_2}}}_{n-f_1-f_2 \text{ from } \mathbf{C}} \cdot s^{2f_1+f_2} \tag{3.1}$$

where "$\wedge$" denotes the wedge (or exterior) product (Bourbaki 1989).

(a) To determine the maximum possible degree of the polynomial $\det[\mathbf{Z_a}(\mathbf{s})]$ we have to solve the *integer-programming problem*:

$$\begin{gathered} \max\ n = 2f_1 + f_2 \\ \text{s.t.} \\ f_1, f_2 \geqslant 0,\ f_1 + f_2 \leqslant n,\ f_1 \leqslant p, n - f_1 - f_2 \leqslant q \end{gathered}$$

This has the obvious solution: $f_1 = p,\ f_2 = n - p$ and $n_{\max} = 2p + n - p = n + p$ i.e. take p columns from **L** and $n - p$ columns from **R**. In this case:

$$k_{\max} = \sum_{\omega \in Q_n^p} A_{\omega}$$

where A_{ω} are all $n \times n$ determinants of matrices formed by p rows from **L** and $n-p$ complementary rows from **R**. For a given selection of columns of **L**: $\omega = (i_1, i_2, \cdots, i_p) \in Q_n^p$ the *Laplace Expansion Theorem* (Meyer 2000) gives:

$$A_{\omega} = \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega,\beta} \cdot Adj_p(\mathbf{R})_{\beta,\omega}$$

Therefore,

$$k_{\max} = \sum A_{\omega} = \sum_{\omega \in Q_n^p} \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega,\beta} \cdot Adj_p(\mathbf{R})_{\beta,\omega} = tr(C_p(\mathbf{L}) \cdot Adj_p(\mathbf{R}))$$

Since, **L** has rank p we have: $C_p(\mathbf{L}) = \alpha_1 \cdot \alpha_2^T$. Thus,

$$k_{\max} = tr(C_p(\mathbf{L}) \cdot Adj_p(\mathbf{R})) = \alpha_2^T \cdot Adj_p(\mathbf{R})) \cdot \alpha_1$$

(b) The minimum degree can be determined by solving the following *integer-programming problem*:

$$\begin{gathered} \min 2(n - f_3 - f_4) + f_3 \\ \text{s.t.} \\ f_3 + f_4 \leqslant n,\ f_3, f_4 \geqslant 0,\ f_4 \leqslant q \end{gathered}$$

which has the obvious solution: $f_3 + f_4 = n$, $f_4 = q$ and thus, $f_3 = n - q$. In this case:

$$\min 2(n - f_3 - f_4) + f_3 = 2(n - n + q - q) + n - q = n - q$$

Then,

$$k_{\min} = \sum_{\omega \in Q_n^q} B_{\omega}$$

where B_ω are all the $n \times n$ determinants of matrices formed by q rows of $\mathbf{C}$ and $n-q$ rows of $\mathbf{R}$. For $\omega = (i_1, i_2, ..., i_q) \in Q_n^q$ using the *Laplace Expansion Theorem* (Meyer 2000) we have:

$$B_\omega = \sum_{\beta \in Q_n^q} C_q(\mathbf{C})_{\omega,\beta} \cdot Adj_q(\mathbf{R})_{\beta,\omega}$$

Therefore,

$$k_{\min} = \sum B_\omega = \sum_{\omega \in Q_n^q} \sum_{\beta \in Q_n^q} C_q(\mathbf{C})_{\omega,\beta} \cdot Adj_q(\mathbf{R})_{\beta,\omega} = tr(C_q(\mathbf{C}) \cdot Adj_q(\mathbf{R}))$$

Since, $\mathbf{C}$ has rank q we have: $C_q(\mathbf{C}) = \beta_1 \cdot \beta_2^T$, proving that:

$$k_{\min} = tr(C_q(\mathbf{C}) \cdot Adj_q(\mathbf{R})) = \beta_2^t \cdot Adj_q(\mathbf{R})) \cdot \beta_1$$

□

The necessary conditions for the maximum and minimum coefficients k_{n+p} and k_{n-q} respectively to be non zero are given in the following proposition (Livada 2017), (Leventides et al. 2014).

PROPOSITION 3.1 (1) A *necessary condition* for $k_{n+p} \neq 0$, is that both matrices $\begin{bmatrix} \mathbf{L} & \mathbf{R} \end{bmatrix}$, $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$ have full rank.

(2) A *necessary condition* for $k_{n-q} \neq 0$, is that both matrices $\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}$, $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}$ have full rank.

Proof.

(1) As the coefficient of k_{n+p} is the sum of certain $n \times n$ minors of $\begin{bmatrix} \mathbf{L} & \mathbf{R} \end{bmatrix}$ or $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$, if these matrices are not full rank all these minors have to be zero and therefore k_{n+p} must be zero.

(2) Similar to (1).

We should note here that if $\begin{bmatrix} \mathbf{L} & \mathbf{R} \end{bmatrix}$ has full rank then $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$ will have full rank as well. Similar results apply for the matrices $\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}$. □

PROPOSITION 3.2 Let $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, $\mathbf{L}' \in \mathbb{R}^{n \times p}$, $\mathbf{L}'' \in \mathbb{R}^{p \times n}$ and $p < n$. Then:

$$C_p(\mathbf{L}'') \cdot Adj_p(\mathbf{R}) \cdot C_p(\mathbf{L}') = (-1)^p \cdot \begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix}$$

where $|\cdot|$ stands for determinant.

Proof. Developing $A = \begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix}$ with respect to the last p rows we get:

$$A = \sum_{\omega} (-1)^{n+1+n+2+...+n+p+j_1+j_2+...+j_p} \cdot \left|L''_\omega\right| \cdot \left|R_\omega \mathbf{L}'\right| \tag{3.2}$$

where $\omega = (j_1, j_2, ..., j_p) \in Q_n^p$, L''_ω are the entries of $C_p(\mathbf{L}'')$ and R_ω is the part of $\mathbf{R}$ with $j_1, j_2, \ldots, j_p$ columns excluded, then expanding

$$\begin{vmatrix} R_\omega & \mathbf{L}' \end{vmatrix}$$

with respect to its last p columns (i.e. $\mathbf{L}'$) we get:

$$\left|R_\omega \mathbf{L}'\right| = \sum_\beta (-1)^{n-p+1+n-p+2+\ldots+n+f_1+f_2+\ldots+f_p} \cdot |R_\omega| \cdot \left|L'_\beta\right| \tag{3.3}$$

where $\beta = (f_1, f_2, ..., f_p) \in Q_n^p$, L'_β are the entries of $C_p(\mathbf{L}')$ and R_ω is the part of $\mathbf{R}$ with the ω rows and β columns excluded. Substituting (5.8) into (5.7) we get:

$$\begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix} = (-1)^{n+1+\ldots+n_p+n-p+1+\ldots+n} \cdot \sum_{\omega,\beta \in Q_n^p} (-1)^{j_1+j_2+\ldots+j_p+f_1+f_2+\ldots+f_p} \cdot |L_\omega| \left|R_{\omega,\beta}\right| \left|L'_\beta\right| =$$

$$= (-1)^p \cdot C_p(\mathbf{L}') \cdot Adj_p(\mathbf{R}) \cdot C_p(\mathbf{L}'')$$

□

Equivalently, the following Corollary is established:

COROLLARY 3.1 Let $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$, $\mathbf{C}' \in \mathbb{R}^{n\times q}$, $\mathbf{C}'' \in \mathbb{R}^{q\times n}$ and $q < n$. Then:

$$C_q(\mathbf{C}'') \cdot Adj_q(\mathbf{R}) \cdot C_q(\mathbf{C}') = (-1)^q \cdot \begin{vmatrix} \mathbf{R} & \mathbf{C}' \\ \mathbf{C}'' & \mathbf{0} \end{vmatrix}$$

The next Theorem provides a description for the maximum coefficient of the determinant with respect to the rank properties of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of an *RLC* network (Livada 2017), (Leventides et al. 2014).

THEOREM 3.2 (i) If $p < n$ then:

$$k_{n+p} = C_p(\mathbf{L}'') \cdot Adj_p(\mathbf{R}) \cdot C_p(\mathbf{L}') \neq 0 \text{ (where } Adj_n(\mathbf{R}) = 1)$$
$$\text{if and only if } rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{L})$$

(ii) If $p = n$ then: $\det(\mathbf{L}) \neq 0$ if and only if $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{L})$

Proof. Let $p = rank(\mathbf{L})$. Moreover,

$$rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \leqslant rank(\mathbf{L}) + rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \end{bmatrix}\right) = n + p$$

Therefore, for

$$rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + p$$

there must be

$$C_{n+p}\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \neq \underline{0}.$$

Taking into account the identity:

$$\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix}$$

by the Binet-Cauchy theorem (Marcus & Minc 1964) we have:

$$C_{n+p}\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right) \cdot C_{n+p}\left(\begin{bmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix}\right) \cdot C_{n+p}\left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix}\right).$$

Hence,

$$C_{n+p} = \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \neq \underline{0} \text{ if and only if } \det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right) \neq 0.$$

Since $k_{n+p} = (-1)^p \cdot \det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right)$ (Proposition 3.2), we have that: $k_{n+p} \neq 0$ if and only if $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n+p$. □

The next Corollary expresses a similar result as Theorem 3.2 for the minimum coefficient of the determinant with respect to the rank properties of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of an *RLC* network (Livada 2017), (Leventides et al. 2014).

COROLLARY 3.2 (i) If $q < n$ then: $k_{n-q} \neq 0$ if and only if $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{C})$.

(ii) If $q = n$ then: $\det(\mathbf{C}) \neq 0$ if and only if $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{C})$

COROLLARY 3.3 Let δ_m be the Implicit McMillan degree of $W^{-1}(s) = \left(s\mathbf{L} + \mathbf{R} + {}^1\!/_s\mathbf{C}\right)^{-1}$. Then the following are equivalent:

(a) $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$.

(b) $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{L})$ and $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) = n + rank(\mathbf{C})$.

COROLLARY 3.4 Necessary conditions for $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$ are:

(a) $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \end{bmatrix}\right) = n$.

(b) $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}\right) = n$.

(c) $rank(\mathbf{R}) \geqslant n - \min(rank(\mathbf{L}), \mathrm{rank}(\mathbf{C}))$.

4. A Graph Theoretic Interpretation of the Necessary and Sufficient Conditions

In this section we analyze a graph systematic approach of the necessary and sufficient conditions that were established in Section 3. We emphasize mostly in implementing this conditions in terms of the *graph incidence matrices* of the $\mathbf{L}, \mathbf{R}, \mathbf{C}$ matrices of the network. Such an approach will provide a more clear result on the link between the *Implicit McMillan degree* δ_m and the topology of the *RLC* network. Firstly, we will introduce the notion of an *incidence matrix* of a graph or a network, which is crucial for the development of this graph approach.

DEFINITION 4.1 An incidence matrix $G^T \in \mathbb{R}^{m\times n}$ is a matrix with $i,i=1,\dots,m$ rows and $j,j=1,\dots,n$ columns. Each row of the matrix corresponds to an element of the network, i.e. capacitor, inductance, resistor and each column corresponds to a loop or node of the given *RLC* network. Hence, an entry G_{ij} in the matrix is:

a. 1 if element i is present in loop / node j and thecurrent I_{ij} flows through the element i in the clockwise direction.

b. -1 if element i is present in loop / node j and thecurrent I_{ij} flows through the element i in the counter clockwise direction.

c. 0 if element i is not present in loop j.

The following remark provides a description of the $\mathbf{L},\mathbf{R},\mathbf{C}$ matrices of an *RLC* network in terms of the associated incidence matrices defined in Definition 4.1.

REMARK 4.1 Each one of the matrices $\mathbf{L},\mathbf{R},\mathbf{C}$ can be decomposed into corresponding dyads as:

$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} R_i \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} L_i \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \frac{1}{C_i} \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

with entries: 1 if element i is present in loop / node j and the current I_{ij} flows through the element i in the clockwise direction, -1 if element i is present in loop / node j and thecurrent I_{ij} flows through the element i in the counter clockwise direction, or 0 if element i is not present in loop j. If all elements $R_i, L_i, \frac{1}{C_i}$ are gathered and the matrices $\mathbf{R}$,$\mathbf{L}$,$\mathbf{C}$ are formed accordingly then we have the following representation.

If G^T denotes the incidence matrix (Gross et al. 2013) for the matrices $\mathbf{R}$,$\mathbf{L}$,$\mathbf{C}$ then these matrices can be represented by:

$$\begin{aligned} \mathbf{R} &= G_R \cdot D_R \cdot G_R^T \\ \mathbf{L} &= G_L \cdot D_L \cdot G_L^T \\ \mathbf{C} &= G_C \cdot D_C \cdot G_C^T \end{aligned} \tag{4.1}$$

where D_C, D_R, D_L represent the diagonal matrices with entries the capacitances, resistances and inductances respectively in a given network and G_C^T, G_R^T, G_L^T denote the incidence matrices of $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{L}$ matrices respectively.

The next two theorems provide equivalent expressions for the maximum and minimum coefficients $k_{\max}$ and $k_{\min}$ (as were developed in Section 3) respectively not to be zero.

THEOREM 4.1 Let $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, $\mathbf{L}' \in \mathbb{R}^{n\times p}$ and $\mathbf{L}'' \in \mathbb{R}^{p\times n}$. If $\mathbf{L}''$ is decomposed as $\mathbf{L}'' = D_L \cdot G_L^T$ and $\mathbf{L}' = G_L$ then by Theorem 3.1, Proposition 3.2 and equation (4.1) we have that:

If G_L^T and G_R^T are non square incidence matrices (Gross et al. 2013) then an equivalent expression for $k_{\max} \neq 0$ is:

$$C_p(G_L^T) \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

where $J_{n,p}$ appropriate matrix (Price 1947),(Nambiar & Sreevalsan 2001): with entries 1 and -1 such that (4.4) holds true.

Proof. We know from Theorem 3.1 that $k_{\max} \neq 0$ if and only if

$$C_p(\mathbf{L}'') \cdot Adj_p(\mathbf{R}) \cdot C_p(\mathbf{L}') \neq \underline{0} \tag{4.2}$$

Let's denote by $\mathbf{L}'' = D_L \cdot G_L^T$ and by $\mathbf{L}' = G_L$ then using that $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$ and developing equation (4.2) we will have that:

$$\begin{array}{l} \det D_L \cdot C_p(G_L^T) \cdot Adj_p(\overbrace{G_R D_R G_R^T}^{n\times n}) \cdot C_p(G_L) = \\ \det D_L \cdot C_p(G_L^T) \left[J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot {J_{n,p}}^T\right] \cdot C_p(G_L) = \\ \det D_L \cdot C_p(G_L^T) \left[J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot {J_{n,p}}^T\right] \cdot C_p(G_L) \end{array} \tag{4.3}$$

<u>Note:</u> In equation (4.3) the p-th adjugate $Adj_p(B)$ of an $n \times n$ matrix $\mathbf{B}$ can be decomposed as (Price 1947),(Nambiar & Sreevalsan 2001):

$$Adj_p(\mathbf{B}) = \left(J_{n,p} \cdot C_{n-p}(\mathbf{B}) \cdot J_{n,p}^T\right) \tag{4.4}$$

Using for equation (4.3) the Binet-Cauchy theorem (Marcus & Minc 1964) we have that:

$$\begin{array}{l} \det D_L \cdot C_p(G_L)^T \left[J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot {J_{n,p}}^T\right] \cdot C_p(G_L) = \\ \det D_L \cdot C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \cdot C_{n-p}(D_R) \cdot C_{n-p}(G_R)^T \cdot J_{n,p}^T \cdot C_p(G_L) \end{array}$$

Thus, for non-square matrices G_R^T,G_L^T the equivalent expression for $k_{\max} \neq 0$ is:

$$C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

□

REMARK 4.2 If G_R^T,G_L^T are square matrices, then the equivalent expression for $k_{\max} \neq 0$ is:

$$C_p(G_L)^T \cdot Adj_p(G_R^T) \neq \underline{0}$$

THEOREM 4.2 Let $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$ $\mathbf{C}' \in \mathbb{R}^{n\times q}, \mathbf{C}'' \in \mathbb{R}^{q\times n}$. If $\mathbf{C}''$ is decomposed as $\mathbf{C}'' = D_C \cdot G_C^T$ and $\mathbf{C}' = G_C$ then by Theorems 4.1 and 3.1 and Proposition 3.2 we have that:

If G_C^T and G_R^T are non square incidence matrices (Gross et al. 2013), then an equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C^T) \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

where $J_{n,p}$ appropriate matrix (Price 1947),(Nambiar & Sreevalsan 2001): with entries 1 and -1 such that (4.4) holds true.

Proof. We know from Theorem 3.1 that $k_{\min} \neq 0$ if and only if

$$C_q(\mathbf{C}'') \cdot Adj_q(\mathbf{R}) \cdot C_q(\mathbf{C}') \neq \underline{0} \tag{4.5}$$

Let's denote by $\mathbf{C}'' = D_C \cdot G_C^T$ and by $\mathbf{C}' = G_C$ then using that $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$ and developing equation (4.5) we will have that:

$$\begin{aligned} &\det D_C \cdot C_q(G_C^T) \cdot Adj_q(\overbrace{G_R D_R G_R^T}^{n\times n}) \cdot C_q(G_C) = \\ &= \det D_C \cdot C_q(G_C^T)\left[J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}{}^T\right]^T \cdot C_q(G_C) = \\ &= \det D_C \cdot C_q(G_C^T)\left[J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}{}^T\right] \cdot C_q(G_C) \end{aligned} \tag{4.6}$$

Using for equation (4.6) the Binet-Cauchy theorem (Marcus & Minc 1964) we have that:

$$\begin{aligned} &= \det D_C \cdot C_q(G_C)^T\left[J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}{}^T\right] \cdot C_q(G_C) \\ &= \det D_C \cdot C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \cdot C_{n-q}(D_R) \cdot C_{n-q}(G_R)^T \cdot J_{n-q}^T \cdot C_q(G_C) \end{aligned}$$

Hence, for non-square matrices G_R^T, G_C^T the equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

□

REMARK 4.3 For square matrices G_R^T, G_C^T the equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C)^T \cdot Adj_q(G_R^T) \neq \underline{0}$$

The next Theorem shows under which conditions the maximum and minimum coefficients $k_{\max}$ and $k_{\min}$ are non zero.

THEOREM 4.3 For a given network represented by the matrices $\mathbf{R},\mathbf{L},\mathbf{C}$ and the associated incidence matrices (Gross et al. 2013) G_L^T, G_R^T, G_C^T then:

1. The minimum coefficient of the Implicit McMillan degree is non-zero, i.e. $k_{\min} \neq 0$, if and only if

   $$C_q(G_C^T) \cdot Adj_q(G_R^T) \neq \underline{0}$$

   where G_C^T and G_R^T are square incidence matrices (Gross et al. 2013) or

   $$C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

   where G_R^T, G_C^T are non square matrices. Equivalently, at least one determinant formed by q rows of G_C^T and $(n-q)$ rows from G_R^T is non-zero.

2. The maximum coefficient of the Implicit McMillan degree is non-zero, i.e. $k_{\max} \neq 0$, if and only if

   $$C_p(G_L^T) \cdot Adj_p(G_R^T) \neq \underline{0}$$

   where G_R^T, G_L^T are square incidence matrices (Gross et al. 2013) or

   $$C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

   where G_R^T, G_L^T non square matrices. Equivalently, at least one determinant formed by p rows of G_L^T and $(n-p)$ rows from G_R^T is non-zero.

Finally, the following corollary expresses the necessary conditions for the Implicit McMillan degree δ_m to achieve the upper bound. The necessary and sufficient conditions for this are presented in Remark 4.4.

COROLLARY 4.1 For a given network represented by the matrices **R**,**L**,**C** **necessary conditions** for $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$ are:

- $rank \begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} = n$

- $rank \begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} = n$

REMARK 4.4 The implicit McMillan degree of a network satisfies $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$ if and only if both of the following two conditions hold:

1. If there is a set of linearly independent rows formed by $(n-q)$ rows of the incidence matrix of **R** and q rows of the incidence matrix of **C**.

2. If there is a set of linearly independent rows formed by $(n-p)$ rows of the incidence matrix of **R** and p rows of the incidence matrix of **L**.

5. Network Pencil $P(s)$ and its Link to the Implicit McMillan degree of the Network

In this section we try to establish an expression for the maximum possible Implicit McMillan degree δ_m of an *RLC* network using the associated loop network pencil $P(s)$ (Livada 2017):

$$P(s) = s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} s\mathbf{L}+\mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = s\mathbf{F} + \mathbf{G} \tag{5.1}$$

As mentioned in the previous sections the maximum possible Implicit McMillan degree of an *RLC* network is given by:

$$\delta_m = n_{\max} - \min(n_{\min}, n)$$

where n is the cardinality of the network and $n_{\max}$, $n_{\min}$ are the maximum and minimum powers of s in the expansion of the determinant $\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$.

We can reformulate the above determinantal expression in terms of matrix pencils as:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = \det \begin{bmatrix} s\mathbf{L}+\mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = \det\left(s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \right) \tag{5.2}$$

To determine the maximum value of s in this determinantal expression, i.e. $s^{n_{\max}}$, which is s^{n+p} (Theorem 3.1), we need to select all the last n rows from $s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, p rows from $s \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}$ and $n-p$ complementary rows from $\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}$.

Hence,

$$A_{\omega} = \begin{vmatrix} l_1 & 0 \\ l_2 & 0 \\ \vdots & \vdots \\ l_p & 0 \\ r_{p+1} & c_{p+1} \\ \vdots & \vdots \\ r_n & c_n \\ \mathbf{0} & \mathbf{I} \end{vmatrix} = \begin{vmatrix} l_1 \\ l_2 \\ \vdots \\ r_{p+1} \\ \vdots \\ r_n \end{vmatrix} \tag{5.3}$$

and the coefficient of s^{n+p}, i.e $k_{\max}$ is $k_{n+p} = \sum\limits_{\omega} A_{\omega}$, where ω stands for different selections of $l_1, l_2, \ldots l_p$, and $|\cdot|$ stands for determinant. To continue, we can use the same procedure as in Section 3.

Equivalently, to determine the minimum power of s in the expansion of the determinant (5.2), we need to consider the following (Livada 2017):

$$\det \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = \det(s\mathbf{R} + \mathbf{C}) \tag{5.4}$$

Then, we will select q rows from $\mathbf{C}$ and $n-q$ complementary rows from $\mathbf{R}$. Now, the minimum coefficient $k_{\min}$ of s^{n-q} will be given by $k_{\min} = \sum\limits_{\omega} B_{\omega}$, where ω stands for q different selections of the rows of $\mathbf{C}$. To continue, we can use the same procedure as in Section 3.

6. Examples

In this section we will demonstrate the use of previous theorems and test the necessary and sufficient conditions in the following examples (Livada 2017), (Leventides et al. 2014).

EXAMPLE 6.1 First, let us investigate an RLC network with $n = 4$ loops, 2 inductors and 1 capacitor arranged as shown in Figure 6.1. The operator $\mathbf{Z_a}(s) = s \cdot W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ is given by the following matrices:

The autonomous network of the figure can be represented by the following symmetric matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$:

$$\mathbf{L} = \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \\ -L_1 & -L_2 & L_1 + L_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.1}$$

$$\mathbf{R} = \begin{bmatrix} R_1 & 0 & 0 & -R_1 \\ 0 & R_2 & 0 & -R_2 \\ 0 & 0 & R_3 & 0 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 \end{bmatrix} \tag{6.2}$$

$$\mathbf{C} = \begin{bmatrix} {C_1}^{-1} & -{C_1}^{-1} & 0 & 0 \\ -{C_1}^{-1} & {C_1}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.3}$$

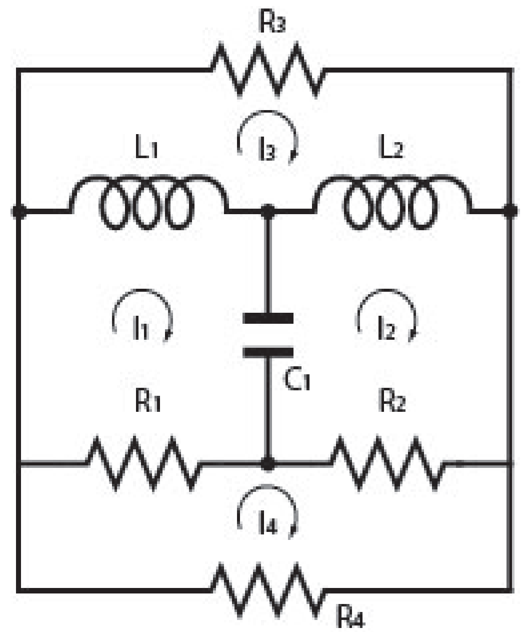

FIG. 1. RLC autonomous network with $n = 4$, $p = 2$, $q = 1$

By inspection:

$$rank(\mathbf{L}) = p = 2$$

and

$$rank(\mathbf{C}) = q = 1$$

.

Using the formulas derived from Theorems 2.1, 2.2, 3.1 we may find the minimum and maximum coefficients of the determinant of the $\mathbf{Z_a}$ operator. For these coefficients we need to compute:

(I) $C_p(\mathbf{L}) = C_2(\mathbf{L})$, because $p = 2$.

(II) $C_q(\mathbf{C}) = C_1(\mathbf{C}) = C$, because $q = 1$.

(III) $Adj_q(\mathbf{R}) = Adj_1(\mathbf{R})$ and $Adj_p(\mathbf{R}) = Adj_2(\mathbf{R})$.

Thus, we have:

$$C_2(\mathbf{L}) = \begin{bmatrix} L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \\ -L_1L_2 & L_1L_2 & 0 & -L_1L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\alpha_1} \cdot \underbrace{\begin{bmatrix} L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \end{bmatrix}}_{\alpha_2^T} \tag{6.4}$$

$$C_1(\mathbf{C}) = \mathbf{C} = \begin{bmatrix} {C_1}^{-1} & -{C_1}^{-1} & 0 & 0 \\ -{C_1}^{-1} & {C_1}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{\beta_1}} \cdot \underbrace{\begin{bmatrix} {C_1}^{-1} & -{C_1}^{-1} & 0 & 0 \end{bmatrix}}_{\boldsymbol{\beta_2^T}} \tag{6.5}$$

Finally, for the *compound adjugates* of R we have that:

$$Adj_1(\mathbf{R}) = \begin{bmatrix} R_2R_3(R_1+R_4) & R_1R_2R_3 & 0 & R_1R_2R_3 \\ R_1R_2R_3 & R_1R_3(R_2+R_4) & 0 & R_1R_2R_3 \\ 0 & 0 & R_1R_2R_4 & 0 \\ R_1R_2R_3 & R_1R_2R_3 & 0 & R_1R_2R_3 \end{bmatrix}$$

$$Adj_2(\mathbf{R}) = \begin{bmatrix} R_3(R_1+R_2+R_4) & 0 & R_2R_3 & 0 & -R_1R_3 & 0 \\ 0 & R_2(R_1+R_4) & 0 & R_1R_2 & 0 & -R_1R_2 \\ R_2R_3 & 0 & R_2R_3 & 0 & 0 & 0 \\ 0 & R_1R_2 & 0 & R_1(R_2+R_4) & 0 & -R_1R_2 \\ -R_1R_3 & 0 & 0 & 0 & R_1R_3 & 0 \\ 0 & -R_1R_2 & 0 & -R_1R_2 & 0 & R_1R_2 \end{bmatrix} \tag{6.6}$$

Hence, for the maximum and minimum coefficients using the following formulas:

$$k_{\max} = \alpha_2^T \cdot Adj_p(\mathbf{R}) \cdot \boldsymbol{\alpha_1}$$

and

$$k_{\min} = \beta_2^T \cdot Adj_q(\mathbf{R}) \cdot \boldsymbol{\beta_1}$$

we finally find that:

$$k_{\min} = {C_1}^{-1}(R_1+R_2)R_3R_4$$

$$\begin{aligned} k_{\max} &= L_1L_2(R_3R_4 + R_1(R_3+R_4) + R_2(R_3+R_4)) \\ &= L_1L_2R_3R_4 + L_1L_2R_1R_3 + L_1L_2R_1R_4 + L_1L_2R_2R_3 + L_1L_2R_2R_4 \end{aligned}$$

and by subtracting their corresponding degrees n_{max}, n_{min} we get the Implicit McMillan degree: $\delta_m = 3$.

Alternatively, we may use the composite matrices as denoted in Proposition 3.2 and Corollary 3.1:

$$(-1)^q \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} \tag{6.7}$$

$$(-1)^p \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} \tag{6.8}$$

Firstly, we need to decompose matrix $\mathbf{C}$ from 6.3 to its corresponding dyads, $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$, as indicated below, where $\mathbf{C}' \in \mathbb{R}^{\mathbf{4\times 1}}$ and $\mathbf{C}'' \in \mathbb{R}^{\mathbf{1\times 4}}$. Then, $\mathbf{C}$ can be written as:

$$\mathbf{C} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} {C_1}^{-1} & -{C_1}^{-1} & 0 & 0 \end{bmatrix}$$

Hence, the composite matrix which used to calculate the minimum coefficient of the $det(\mathbf{Z_a})$ operator, k_{min}, is expressed as:

$$(-1)^q \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} = (-1) \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 \\ 0 & R_2 & 0 & -R_2 & -1 \\ 0 & 0 & R_3 & 0 & 0 \\ -R_1 & -R_2 & 0 & R_1+R_2+R_4 & 0 \\ {C_1}^{-1} & -{C_1}^{-1} & 0 & 0 & 0 \end{vmatrix} \tag{6.9}$$

Similarly, we need to decompose matrix $\mathbf{L}$ from 6.1 to its corresponding dyads, $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, where $\mathbf{L}' \in \mathbb{R}^{\mathbf{4\times 2}}$ and $\mathbf{L}'' \in \mathbb{R}^{\mathbf{2\times 4}}$. Then, $\mathbf{L}$ can be written as:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \end{bmatrix} \tag{6.10}$$

and the composite matrix which is used to calculate the highest coefficient k_{max} is expressed as:

$$(-1)^p \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} = (-1)^2 \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 & 0 \\ 0 & R_2 & 0 & -R_2 & 0 & 1 \\ 0 & 0 & R_3 & 0 & -1 & -1 \\ -R_1 & -R_2 & 0 & R_1+R_2+R_4 & 0 & 0 \\ L_1 & 0 & -L_1 & 0 & 0 & 0 \\ 0 & L_2 & -L_2 & 0 & 0 & 0 \end{vmatrix} \tag{6.11}$$

Therefore, by computing the determinants of the composite matrices above we derive the minimum coefficient as:

$$k_{\min} = {C_1}^{-1}(R_1+R_2)R_3R_4 \tag{6.12}$$

and the maximum coefficient:

$$k_{\max} = L_1L_2R_3R_4 + L_1L_2R_1R_3 + L_1L_2R_1R_4 + L_1L_2R_2R_3 + L_1L_2R_2R_4 \tag{6.13}$$

exactly same as before. Thus, it is verified that both computational methods produces the same results, i.e. Implicit McMillan degree $\delta_m = 3$.

Applying the *Graph Systematic Approach* discussed in section 5 and using the formulation derived in remark 4.1 we can express each one of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of the network as:

Matrix of capacitors $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} C_1{}^{-1} & -C_1{}^{-1} & 0 & 0 \\ -C_1{}^{-1} & C_1{}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C_1{}^{-1} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{G_C} \cdot C_1{}^{-1} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}}_{G_C^T}$$

Matrix of inductances $\mathbf{L}$:

$$\mathbf{L} = \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \\ -L_1 & -L_2 & L_1 + L_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L_1 \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + L_2 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \cdot L_1 \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot L_2 \cdot \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}}_{G_L} \cdot \overbrace{\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}}^{D_L} \cdot \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{G_L^T}$$

Matrix of resistors **R:**

$$\mathbf{R} = \begin{bmatrix} R_1 & 0 & 0 & -R_1 \\ 0 & R_2 & 0 & -R_2 \\ 0 & 0 & R_3 & 0 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 \end{bmatrix} = R_1 \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} +$$

$$+ R_2 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + R_3 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R_4 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{G_R} \cdot \overbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix}}^{D_R} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{G_R^T}$$

Next, we will test whether the necessary and sufficient conditions derived in Corollary 4.1 and Remark 4.4 for the Implicit McMillan degree of the network are met. Hence, the following composite matrices need to be formulated:

$$\text{a. } \begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 0 & -1 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ 1 & 0 & 0 & -1 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the first matrix we can choose $q = 1$ lines from G_C^T and $(n-q) = (4-1) = 3$ lines from G_R^T (these lines are demonstrated above in bold letters) that are linearly independent. Similarly, from the last composite matrix we can choose $p = 2$ lines from G_L^T and $(n-p) = (4-2) = 2$ lines from G_R^T (in bold) that are linearly independent with each other.

Thus, we conclude that the necessary and sufficient conditions for the Implicit McMillan degree are satisfied in this particular example.

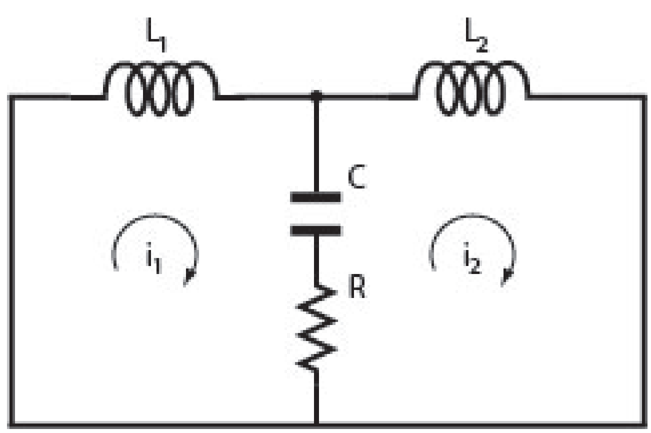

FIG. 2. RLC autonomous network with $n = 2$, $p = 2$, $q = 1$

EXAMPLE 6.2 Now, lets examine a peculiar *RLC* network with $n = 2$ loops, 2 inductors, 1 capacitor and 1 resistance arranged as shown in Figure 6. The operator $\mathbf{Z_a}(s) = s^2\mathbf{L} + \mathbf{sR} + \mathbf{C}$ for the *RLC* network is:

$$\mathbf{Z_a}(s) = s^2 \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} + s \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} + \begin{bmatrix} C^{-1} & -C^{-1} \\ -C^{-1} & C^{-1} \end{bmatrix}$$

In this example, if we use the previous results, we expect the Implicit McMillan degree of the system to be equal with the number of dynamical elements (i.e. inductors and capacitors). So, $\delta_m = 3$. Then, we compute as previously the maximum and minimum coefficients and their corresponding degrees

$$k_{\max} = L_1 L_2 \cdot s^4$$

and $k_{\min} = C^{-1}(L_1 + L_2) \cdot s^2$. As we can see, $\delta_\mu = k_{max} - k_{min} = 4 - 2 = 2 \neq 3$ as we expected.

This is because the the necessary and sufficient conditions are not valid in this case.

Applying the *Graph Systematic Approach* discussed in Section 5 and using the formulation derived in Remark 4.1 we can express each one of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of the network as:

Matrix of capacitors $\mathbf{C}$:

$$\begin{bmatrix} C^{-1} & -C^{-1} \\ -C^{-1} & C^{-1} \end{bmatrix} = C^{-1} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{G_C} \cdot \overbrace{C^{-1}}^{D_C} \cdot \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{G_C^T}$$

Matrix of inductances $\mathbf{L}$:

$$\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = L_1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + L_2 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot L_1 \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot L_2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G_L} \cdot \overbrace{\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}}^{D_L} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G_L^T}$$

Matrix of resistors **R:**

$$\begin{bmatrix} R & -R \\ -R & R \end{bmatrix} = R \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{G_R} \cdot \overbrace{R}^{D_R} \cdot \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{G_R^T}$$

To determine whether the necessary and sufficient conditions derived in Corollary 4.1 and Remark 4.4 for the Implicit McMillan degree of the network are met, we need to formulate the following composite matrices:

a. $\begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

From the first matrix we can choose $p = 2$ lines from G_L^T and $(n-p) = (2-2) = 0$ lines from G_R^T (these lines are demonstrated above in bold letters) that are linearly independent. In contrast, from the last composite matrix we cannot choose $q = 1$ lines from G_C^T and $(n-q) = (2-1) = 1$ lines from G_R^T that are linearly independent with each other.

Thus, we conclude that the necessary and sufficient conditions for the Implicit McMillan degree are not met in this particular example.

7. Concluding remarks

The purpose of this paper was to develop a framework with which *RLC* networks could be treated as control systems with a generalised transfer function $W^{-1}(s)$. For a general *RLC* network described by the *Implicit network operator* $W(s)$ the Implicit McMillan degree δ_m was calculated, which expresses the maximum number of independent dynamical elements of the system. We established an upper bound for this degree, which is $\delta_m = rank(\mathbf{L}) + rank(\mathbf{C})$ and this is attained when certain regularity conditions for *RLC* networks are met (Livada 2017), (Leventides et al. 2014). Three different types of regularity conditions were established, i.e. determinantal, rank and graph theoretic. Furthermore, this framework was reformulated by introducing matrix pencils theory and some results were established, using the associated loop pencil of the network $P(s)$. Finally, two applications demonstrating these regularity conditions were developed.

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