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The Copula Derived from the SAHARA Utility Function

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Abstract: A new Archimedean copula family is presented that was derived from the SAHARA utility function introduced in the economic literature in 2011. Its properties are discussed, and its flexibility and versatility are demonstrated. It is left tail decreasing or right tail increasing, but unlike mainstream Archimedean families, not necessarily stochastically increasing at the same time. It is shown that the family fits very well to a dataset of previously studied coupled lives in the literature.

Keywords: copula; Archimedean generator; dependence; coupled lives

1. Introduction

Archimedean copulas, which in two dimensions are of the form $C_\psi(u_1, u_2) = \psi\big(\psi^{-1}(u_1) + \psi^{-1}(u_2)\big)$ where ψ is the one-dimensional generator, have become a popular mode of modelling dependence in both finance and insurance. Several ways of constructing copula families are given in Chapter 3 of Joe (2015). The interpretation of the generator as the Williamson transform of a radial random variable has given rise to new Archimedean families; see McNeil and Nešlehová (2009, 2010). Archimedean copulas are a flexible class due to the ease with which new Archimedean copulas with an enriched parameter space can be constructed from existing ones using transformations. For the bivariate case, five of such transformations, namely, left composition, right composition, scaling, exponentiation, and the linear combination of the (inverse) generator, were introduced in the literature by Genest et al. (1998). They were reviewed by Michiels and De Schepper (2012), and in more detail in Michiels and De Schepper (2009), with the focus on the so-called λ function (which is the ratio of the inverse generator to its derivative). For the latter, see also Michiels et al. (2011).

In the literature, several generalized families were constructed that contain the Archimedean class as a special case, e.g., the Archimax family in Capéraà et al. (2000) (which includes extreme value copulas as another special case). The background risk model where one random variable ("systemic risk") acts multiplicatively on a series of other random variables ("idiosyncratic risks") is the basis of generalization of Archimedean copulas in several ways, as demonstrated in Côté and Genest (2019) and Marri and Moutanabbir (2022).

Commonly used families typically feature a generator being completely monotonic and thereby the Laplace transform of a mixing random variable. Common examples include Clayton and Frank (as far as dependence is positive), Gumbel–Hougaard, and Joe. This subfamily of Archimedean copulas, also known as shared frailty models, has the advantage of being valid in any dimension. Recent applications regarding shared frailty models involve the aforementioned background risk model; see (Albrecher et al. 2011; Furman et al. 2021; Sarabia et al. 2018).

The Gumbel–Hougaard copula is a good fit to the well-known dataset of loss vs. Allocated Loss Adjustment Expenses (ALAE), which is the object of statistical inference in several publications, starting with Genest et al. (1998). For extensive analysis, consult Joe (2015). Probably there are several other case studies of dependence in insurance where the use of shared frailty models is appropriate.

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Shared frailty dependency models have the property of being conditionally increasing (CI) and multivariate totally positive of order 2 (MTP_2), as shown in Müller and Scarsini (2005). In the specific case of two dimensions, this is known as TP_2, implying stochastically increasing (SI), which in turn implies both left tail decreasing (LTD) and right tail increasing (RTI). If the two involved marginal random variables are remaining lifetimes, LTD/RTI implies that the hazard rate of the one upon hazard (e.g., death or default) of the other goes up. The stronger condition SI implies that the hazard rate of the one given the hazard of the other (e.g., death or default) at time t decreases as t goes up. This was noted in Spreeuw (2006). The underlying assumption of SI may work out well in reliability theory. Consider two printers available to office staff. When one fails, the other one, pending the repair of the first, is used more intensively and thereby exposed to greater strain. It is sensible to assume that the longer the first printer is out of order, the higher the failure rate of the remaining machine would be.

For coupled lives, however, it is not so clear-cut. On the one hand, the event of the death of one life usually triggers an elevated mortality of the surviving life, so the assumption of LTD/RTI seems sensible. In addition, such lives are exposed to common risks due to permanently living together and due to a similar background ("birds of a feather flock together", the so called long-term dependence according to Hougaard (2000)). On the other hand, however, there is also the phenomenon of the event of the one life dying leading to the mortality of the remaining life temporarily going up, which is the so-called broken-heart syndrome (short-term dependence, also attributed to Hougaard (2000)). Similar nonstandard features may apply to other cases in insurance (and finance as well). In short, there is a case for constructing copula families allowing for flexibility in terms of type of dependence, such as LTD/RTI but not necessarily SI.

In this paper, we introduce a new Archimedean copula family that is based on a link between Archimedean generators and utility functions; see Spreeuw (2010) for more details. Unlike mainstream copulas, this family has the property of being LTD/RTI, but not necessarily SI, the latter being clearly indicated by the sign of one of the parameters.

The outline of the paper is as follows. Section 2 gives the basic definitions of Archimedean copula, and the dependence concepts of LTD/RTI and SI. Section 3 introduces the new family and analyzes its basic properties. Section 4 fits the new Archimedean family to the section of censored remaining lifetime data of coupled lives, as in Luciano et al. (2008). Section 5 sets out a conclusion.

2. Basic Definitions

Define ψ as the generator of a 2-dimensional Archimedean copula, being strictly continuous, strictly decreasing, convex, with $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$. The copula itself is then specified as

$$C_\psi(u_1, u_2) = \psi\Big(\psi^{-1}(u_1) + \psi^{-1}(u_2)\Big), \tag{1}$$

where u_1, u_2 each take values between 0 and 1.

Next are definitions of the tail concepts of left tail decreasing (LTD), right tail increasing (RTI) and stochastically increasing (SI), based on two random variables X and Y, and their copula C. They can all be found in Chapter 5 of Nelsen (2006).

Definition 1. *Y is LTD in X (notation LTD$(Y|X)$) $\Leftrightarrow$ $\Pr[Y \le y | X \le x]$ is nonincreasing in x for all y. For an exchangeable copula C (i.e., $C(u,v) = C(v,u)$ for $0 \le u, v \le 1$) of random variables X and Y, LTD$(Y|X)$ and LTD$(X|Y)$ are equivalent.*

Definition 2. *Y is RTI in X (notation RTI$(Y|X)$) $\Leftrightarrow$ $\Pr[Y > y | X > x]$ is nondecreasing in x for all y. For an exchangeable copula C (i.e., $C(u,v) = C(v,u)$ for $0 \le u, v \le 1$) of random variables X and Y, RTI$(Y|X)$ and RTI$(X|Y)$ are equivalent.*

Definition 3. *Y is SI in X (notation $SI(Y|X)$) $\Leftrightarrow \Pr[Y \le y|X = x]$ is nonincreasing in x for all y.*

The following propositions are from Avérous and Dortet-Bernadet (2004). The second one was originally shown in Capéraà and Genest (1993).

Proposition 1. *If C is Archimedean with generator ψ, $LTD(Y|X)$ or $LTD(X|Y)$ if and only if ψ is logconvex. Likewise, if $\hat{C}$, the rotated copula (also known as survival copula) of C (so $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ for $0 \le u, v \le 1$) is Archimedean with generator $\widehat{\psi}$ is $RTI(Y|X)$ or $RTI(X|Y)$ if and only if $\widehat{\psi}$ is logconvex.*

Proposition 2. *If C is Archimedean with differentiable generator ψ, $SI(Y|X)$ or $SI(X|Y)$ if and only if $-\psi'$ is logconvex.*

3. SAHARA Family

The SAHARA copula family is derived from the Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) utility function introduced in Chen et al. (2011). This utility function is specified below.

$$\varphi_{\theta,\epsilon}(s) = \begin{cases} -\frac{1}{(1+1/\theta)^2-1}\left(s-\epsilon+\sqrt{\delta^2+(s-\epsilon)^2}\right)^{-(1+1/\theta)} \cdot\left(s-\epsilon+(1+1/\theta)\sqrt{\delta^2+(s-\epsilon)^2}\right) & \theta \neq 0 \\ \frac{1}{2}\ln\left((s-\epsilon)+\sqrt{\delta^2+(s-\epsilon)^2}\right) + \frac{1}{2}\delta^{-2}(s-\epsilon)\left(\sqrt{\delta^2+(s-\epsilon)^2}-(s-\epsilon)\right) & \theta = 0. \end{cases}, \quad \theta \in (-\infty,-1)\cup(0,\infty), \delta > 0, \epsilon \in \mathbb{R}. \tag{2}$$

As shown in Spreeuw (2010), a strict Archimedean generator can be obtained from a utility function φ if $\varphi(\infty) = \lim_{s\longrightarrow\infty} \varphi(s) < \infty$. For SAHARA, this is the case when $\theta > 0$. Then, applying the formula $\psi_{\theta,\epsilon}(s) = \{\varphi_{\theta,\epsilon}(\infty) - \varphi_{\theta,\epsilon}(s)\}\{\varphi_{\theta,\epsilon}(\infty) - \varphi_{\theta,\epsilon}(0)\}^{-1}$ leads to the Archimedean generator

$$\psi_{\theta,\epsilon}(s) = \left(\frac{s-\epsilon+\sqrt{\delta^2+(s-\epsilon)^2}}{-\epsilon+\sqrt{\delta^2+\epsilon^2}}\right)^{-(1+1/\theta)} \left(\frac{s-\epsilon+(1+1/\theta)\sqrt{\delta^2+(s-\epsilon)^2}}{-\epsilon+(1+1/\theta)\sqrt{\delta^2+\epsilon^2}}\right). \tag{3}$$

Remark 1. *This approach of obtaining an Archimedean generator from a utility function is not to be confused with the method of obtaining the* inverse *of an Archimedean generator from a utility function. For the latter, consult Spreeuw (2014).*

The SAHARA utility function was inspired by nonmonotone risk aversion coefficient

$$AR_\varphi(s) = \frac{1+1/\theta}{\sqrt{(s-\epsilon)^2+\delta^2}},$$

which, unlike common utility functions, is not monotone in its argument. It is rather increasing for $s < \epsilon$ attaining a maximum for $s = \epsilon$, and decreasing for $s > \epsilon$. The SAHARA utility function found applications in both finance and insurance (Bernard et al. 2021; Bernard and Kwak 2016; Brachetta and Schmidli 2020; Chen et al. 2021; Chen and Vellekoop 2017; Li and Ma 2018; Schumacher 2018). As shown in Spreeuw (2010), a risk aversion monotone decreasing (increasing) on the positive real line in general implies that the corresponding Archimedean copula is stochastic increasing (stochastic decreasing) in two dimensions. The former property applies to the vast majority of commonly applied copula families (including all those whose generator is a completely monotonic function). For $\epsilon > 0$ the copula is neither stochastic increasing nor stochastic decreasing. To the best of our knowledge, there are hardly any copula families that share this property.

For $\epsilon < 0$, the condition imposed on δ can be somewhat relaxed to $\delta \geq 0$, $\delta = 0$ corresponding to the Clayton copula with parameter θ. Due to scaling, for some real-valued $\delta^* > 0$ and ϵ^*, $(\delta, \epsilon) = (\delta^*, \epsilon^*)$ gives exactly the same Archimedean copula as $(\delta, \epsilon) = \left(1, \epsilon^*/\delta^*\right)$. To see this, we divide in (3) both numerator and denominator by $\delta > 0$. This gives:

$$\psi_{\theta,\epsilon}(s) = \left(\frac{\frac{s}{\delta} - \frac{\epsilon}{\delta} + \sqrt{1 + \left(\frac{s}{\delta} - \frac{\epsilon}{\delta}\right)^2}}{-\frac{\epsilon}{\delta} + \sqrt{1 + \left(\frac{\epsilon}{\delta}\right)^2}}\right)^{-(1+1/\theta)} \left(\frac{\frac{s}{\delta} - \frac{\epsilon}{\delta} + (1 + 1/\theta)\sqrt{1 + \left(\frac{s}{\delta} - \frac{\epsilon}{\delta}\right)^2}}{-\frac{\epsilon}{\delta} + (1 + 1/\theta)\sqrt{1 + \left(\frac{\epsilon}{\delta}\right)^2}}\right).$$

Now, ϵ being real-valued implies that ϵ/δ is real-valued as well. In addition, it is well-known that a generator is only defined up to a multiple constant. In other words, for $\beta > 0$, $\psi_{\theta,\epsilon}(s)$ and $\psi_{\theta,\epsilon}(\beta\, s)$ generate the same Archimedean copula. $\delta^* \longrightarrow^+ 0$ is equivalent to $\left|\epsilon^*/\delta^*\right| \longrightarrow \infty$. Hence, without loss of generality, we take $\delta = 1$ from now on bearing in mind that, for $\epsilon \longrightarrow -\infty$, the Clayton copula with parameter θ is obtained as a limiting case. For $\epsilon \longrightarrow \infty$, the Clayton copula is again obtained as a limiting case, although now with parameter $-\frac{\theta}{2\theta+1}$.

It can be numerically shown that this copula (a) for a fixed θ decreases in concordance with increasing ϵ; (b) for negative fixed ϵ, it increases in concordance with increasing θ; and (c) for a positive fixed ϵ, it first decreases and then increases in concordance in terms of θ. For any finite ϵ, $\theta \downarrow 0$ and $\theta \longrightarrow \infty$ lead to independence and comonotonicity, respectively.

According to Theorem 4.3 of Joe (1997), p. 91, for Archimedean copulas with a strict generator, the population version of Kendall's tau can be written as:

$$\tau = 1 - 4 \int_{s=0}^{\infty} s \left(\frac{d}{ds}\psi_{\theta,\epsilon}(s)\right)^2 ds.$$

Using software system *Wolfram Mathematica* (see Wolfram Research Inc. (2017)) this gives the expression:

$$\tau = 1 - \frac{(2\theta+1)\left((\theta+2)(3\theta+2) + 4(\theta+2)\epsilon^4 + 12(\theta+1)\epsilon^2 + 2(\theta+4)\epsilon\sqrt{\epsilon^2+1} + 4(\theta+2)\epsilon^3\sqrt{\epsilon^2+1}\right)}{(\theta+2)(3\theta+2)\left(\theta + \epsilon\left(\sqrt{\epsilon^2+1} + \epsilon\right) + 1\right)^2},$$

for $\epsilon = 0$ considerably reducing to $\tau = \{\theta/(\theta+1)\}^2$. Taking the limit for $\epsilon \to -\infty$, keeping θ constant, gives $\tau \to \theta/(\theta+2)$. This is the well-known formula of Kendall's tau for the Clayton copula, and therefore not very surprising. Taking the limit $\epsilon \to \infty$, keeping θ constant, gives $\tau \to -\theta/(3\theta+2)$. This implies that, for increasing θ, the range of values taken by τ increases. So, the lowest possible value of τ for this family is $\lim_{\theta \longrightarrow \infty} -\theta/(3\theta+2) = -1/3$. For fixed nonpositive ϵ, τ is monotone increasing in θ from 0 to 1. For fixed and finite positive ϵ, τ as a function of θ first decreases until a certain negative minimum that is greater than $-1/3$, and increases afterwards. The greater the value of ϵ is, the greater the value of θ for which the minimum is reached and the smaller the minimal value. The difference between Kendall's tau for $\epsilon = 0$ and $\epsilon \to -\infty$ is

$$\frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2 = \frac{\theta}{(\theta+2)(\theta+1)^2},$$

which is zero for $\theta = 0$, increasing until $\theta = \left(\sqrt{5} - 1\right)/2 \approx 0.618$ (which for $\epsilon = 0$ and $\epsilon \to -\infty$ gives values of Kendall's tau of 0.236 and 0.146, respectively), then decreasing for increasing θ and ultimately vanishing. There is no upper tail dependence, while the lower tail dependence coefficient is $2^{-1/\theta}$.

Another interesting feature of this family concerns the conditional survival copula, that is, if the copula of the conditional joint survival function $\overline{H}(\mathbf{x}|\mathbf{y}) = \Pr[X_1 > x_1, X_2 > x_2 \,|X_1 > y_1, X_2 > y_2]$. If the joint survival function $\overline{H}(\mathbf{x})$ has an Archimedean copula with generator $\psi(s), s \geq 0$, the conditional joint survival function $\overline{H}(\mathbf{x}|\mathbf{y})$ also has an Archimedean copula with generator

$$\psi_y(s) = \psi(s+t)/\psi(t), \tag{4}$$

where $t = \psi^{-1}(\overline{H}(\mathbf{y}))$. (Usually the conditional copula is rather given in terms of the inverse generator $\psi_y^{-1}(s) = \psi^{-1}(s\overline{H}(\mathbf{y})) - \psi^{-1}(\overline{H}(\mathbf{y}))$, see (Charpentier 2003; Spreeuw 2006; Sungur 2002)) Applying (4) to the SAHARA family gives

$$\psi_{y,\theta,\epsilon}(s) = \left(\frac{s+t-\epsilon+\sqrt{\delta^2+(s+t-\epsilon)^2}}{t-\epsilon+\sqrt{\delta^2+(t-\epsilon)^2}}\right)^{-(1+1/\theta)} \left(\frac{s+t-\epsilon+(1+1/\theta)\sqrt{\delta^2+(s+t-\epsilon)^2}}{t-\epsilon+(1+1/\theta)\sqrt{\delta^2+(t-\epsilon)^2}}\right), \tag{5}$$

so the conditional copula is again SAHARA with parameters θ and $\epsilon - t$. It follows that dependence increases over time, and the copula converges to Clayton with parameter θ. Again, this is unlike most other copula families where the limiting dependence is either none (independence) or perfect positive.

Some scatterplots follow in Figures 1–3, for Kendall's tau fixed at 0.25 and varying values for ϵ and θ. As ϵ went up, we encountered on the one hand increasingly positive dependence in the bottom left part, and increasingly negative dependence in the top right half. Such families could be considered when data feature strongly positive dependence for small values, and weakly positive, no, or even negative dependence for large values.

The SAHARA copula is clearly flexible and versatile. A drawback is that the inverse of the generator is not available in closed form, like some families introduced in McNeil and Nešlehová (2010), Hua and Joe (2011) and Hua (2015), rendering computations more complicated.

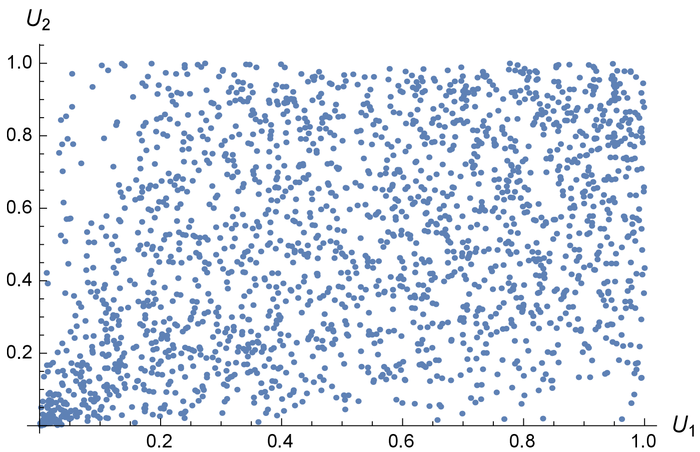

Figure 1. Scatterplot of the SAHARA copula for $\theta = 1$ and $\epsilon = 0$; $\tau = 0.25$.

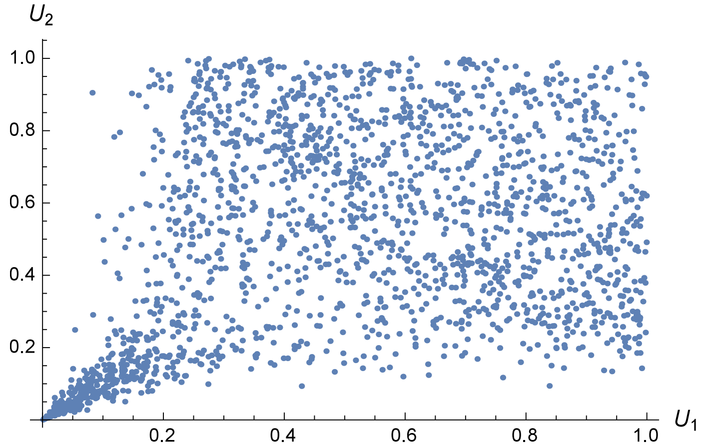

Figure 2. Scatterplot of the SAHARA copula for $\theta = 4.5464$ and $\epsilon = 2$; $\tau = 0.25$.

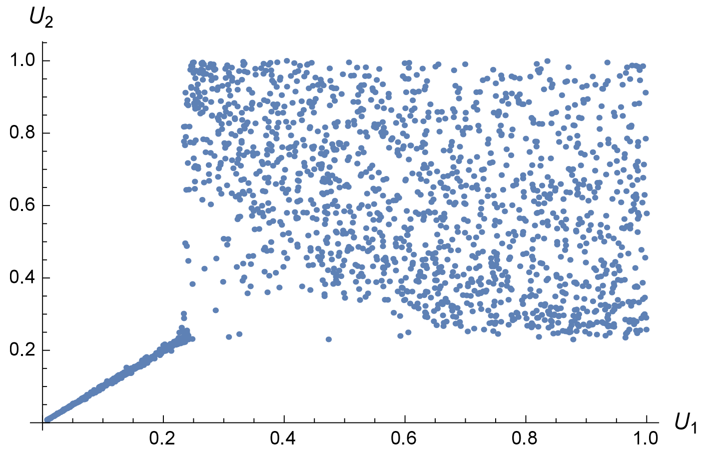

Figure 3. Scatterplot of the SAHARA copula for $\theta = 69.11$ and $\epsilon = 10$; $\tau = 0.25$.

4. Application

For the numerical application in this section, we use the example about modelling dependence of coupled lives in Luciano et al. (2008) and Spreeuw (2014). The two publications used different data, although they both concern samples specified as generations from the same large dataset of annuitants from a Canadian insurer. In this section, copula families are fitted into the data from Luciano et al. (2008) rather than those from Spreeuw (2010). We follow the same procedure of modelling and calibration as that in Luciano et al. (2008) and Spreeuw (2014). Some elaboration on deriving the empirical generator is in order to render this paper self-contained.

The joint survival function of two remaining lifetimes T_x^m (male, age x at the start of the observation) and T_y^f (female, age y at the start of the observation) is given in terms of a survival copula C_{xy} as

$$S_{xy}(s,t) = C_{xy}(S_x^m(s), S_y^f(t)).$$

In this setup, the lives are coupled at the time when they are observed (rather than at birth, as in, e.g., Frees et al. (1996)), just like in Carriere (2000). Using a modified version of the procedure by Wang and Wells (2000), the performance of a candidate Archimedean copula is judged on the basis of distance between the empirical Kendall function, denoted by $\widehat{K}_{n(xy)}$, and the theoretical Kendall function, denoted by $K_{\psi_{\widehat{\mathbf{A}}}^{-1}(xy)}(v)$, where $\psi_{\widehat{\mathbf{A}}}^{-1}$ is the inverse generator of the copula concerned, with $\widehat{\mathbf{A}}$ being the parameter vector estimate minimizing the distance between $\widehat{K}_{n(xy)}$ and $K_{\psi_{\widehat{\mathbf{A}}}^{-1}(xy)}(v)$. For single parameter copulas,

A = θ, while for families with two parameters, $\mathbf{A} = \{\theta, \epsilon\}$. The distance or error is defined under the L^2 norm (so, in the usual quadratic sense). Therefore,

$$error\left(\psi_{\widehat{\mathbf{A}}}^{-1}(xy)\right) = \int_{\xi}^{1} \left(K_{\psi_{\widehat{\mathbf{A}}}^{-1}(xy)}(v) - \widehat{K}_{n(xy)}(v)\right)^2 dv,$$

with

$$\widehat{\mathbf{A}} = \arg\min_{\mathbf{A}} \int_{\xi}^{1} \left(K_{\psi_{\mathbf{A}}^{-1}(xy)}(v) - \widehat{K}_{n(xy)}(v)\right)^2 dv.$$

Given that the data were right censored, the lower bound ξ was greater than zero. In this example, it is taken to be the smallest value for which $\widehat{K}_{n(xy)}$ is positive:

$$\xi = \min\left\{\nu : \widehat{K}_{n(xy)}(v) > 0\right\}.$$

The empirical Kendall function, denoted by $\widehat{K}_{n(xy)}$, was derived from Dabrowska's nonparametric estimator of the joint survival function (see Dabrowska (1988)). Given that the data were right censored, with many observations being doubly censored, $\widehat{K}_{n(xy)}$ is zero between 0 and a certain value $\xi_1 > 0$, at which point it jumps. In this case, $\xi_1 = 0.23$. The pseudo-maximum likelihood (PML) procedure uses as input rescaled Kaplan–Meier estimates of the marginal survival functions in order to accommodate censoring.

Luciano et al. (2008) fit the data to Clayton, Gumbel-Hougaard, Frank, entry 20 of Table 4.2 in Nelsen (2006) ("4.2.20 Nelsen") and the so-called Special copula. For convenience, the last two are listed below with their generators:

1. 4.2.20 Nelsen: $\psi_\theta(t) = \{\log[e+t]\}^{-\frac{1}{\theta}}, \quad \theta > 0.$
2. Special: $\psi_\theta(t) = \left(\frac{-t+\sqrt{t^2+4}}{2}\right)^{\frac{1}{\theta}}, \quad \theta > 0.$

Luciano et al. (2008) concluded that 4.2.20 Nelsen fit the data best. In this section, we compare its performance with that of SAHARA and the best contender of common two-parameter families from Joe (1997, 2015), i.e., BB2. Its generator is:

$$\psi_\theta(t) = \left\{1 + \frac{\log[1+t]}{\epsilon}\right\}^{-\frac{1}{\theta}}, \quad \theta, \epsilon > 0.$$

The 4.2.20 Nelsen is a special case of BB2 arising for $\epsilon = 1$.

Results are given in Table 1. The positive estimate for ϵ indicates the absence of SI.

Table 1. Results for several copula families.

Copula	Parameter Estimates	Error $\varphi_{\widehat{\theta}}^{[-1]}(xy)$
4.2.20 Nelsen	$\widehat{\theta} = 1.005$	0.720
BB2 Joe (1997)	$\widehat{\theta} = 1.469$; $\hat{\epsilon} = 0.383$	0.667
SAHARA	$\widehat{\theta} = 0.204$; $\hat{\epsilon} = 0.914$	0.293

As in Luciano et al. (2008), we performed a graphical comparison between the theoretical and the empirical K functions through transformation $\lambda(w) = w - K(w)$ for $\xi_1 \leq w \leq 1$. The result can be found in Figure 4. SAHARA achieved a significant improvement to the fit compared to the other families.

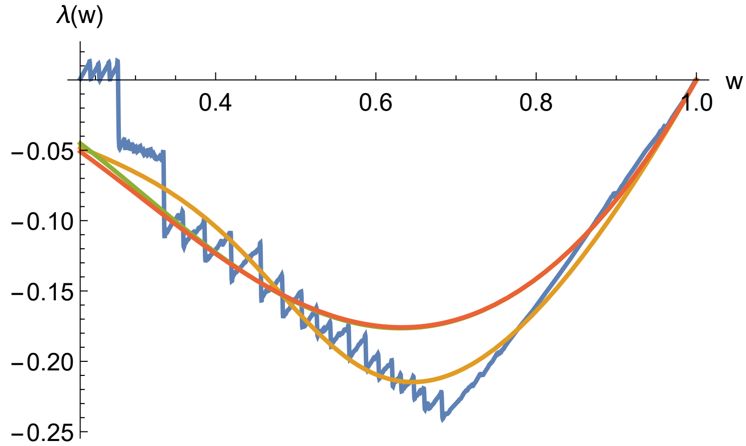

Figure 4. Graphical comparison between theoretical $\lambda(w) = w - K(w)$ for SAHARA (yellow), BB2 (green) and 4.2.20 Nelsen (orange) and empirical one (blue).

Now consider the notion of SI in more detail. For two random variables X_1 and X_2, X_2 being SI in X_1 is equivalent to $\Pr[X_2 > x_2 | X_1 = x_1]$ being nondecreasing in x_1 for all x_2. Related to this is the notion of long-term dependence as introduced in Hougaard (2000). If we define $\mu^m\left(t \middle| T_y^f = t_y\right)$ as the conditional force of mortality of life (x) at duration t given $T_y^f = t_y < t$ ((y) dies at duration t_y), then the dependence between T_x^m and T_y^f is of the long-term type if $\mu^m\left(t \middle| T_y^f = t_y\right)$ is constant or decreasing as a function of t_y, while dependence is short-term if $\mu^m\left(t \middle| T_y^f = t_y\right)$ is increasing as a function of t_y. To understand this, it is important to that, as indicated before, for Archimedean copulas, stochastic increasing (SI) is equivalent to $-\psi'$ being logconvex. Spreeuw (2006) showed that this property of the generator also applies to long-term dependence, implying that SI and long-term dependence are actually equivalent. On the other hand, however, for the SAHARA family, we have:

$$\frac{\partial\left\{\ln\left[-\psi'_{\theta,\epsilon}(s)\right]\right\}}{\partial s} = -\frac{1 + 1/\theta}{\sqrt{(s-\epsilon)^2 + 1}},$$

which is monotone increasing in s across the board for negative ϵ. For positive values of ϵ, however, the expression decreases in s for $0 \leq s < \epsilon$, so the SI property does not hold. The positive parameter estimate for ϵ suggests that short-term rather than long-term dependence may prevail between the coupled lives. To investigate this further, we analysed the data in the same vein as in Spreeuw and Owadally (2013), who devise an augmented Markov model to allow for short-term dependence for the entire dataset. Results are reported in Table 2.

In this table, e denotes the time in which an integer number of years that have elapsed since the death of the partner. So, e.g., $e = 0$ concerns the lives that were bereaved less than a year ago. For each possible value of e (noting that each life was observed for 5 years or less), we calculate number of deaths reported, the risk exposure, and the overall mortality rate being the ratio of the values in the second and third column. So, for instance, the risk exposure of lives who lost their partner less than a year ago is 604.87, and there were 69 lives that died within one year after their partner. Now, long-term dependence implies that the mortality rate in the last column should be increasing as a function of e, but the results in Table 2 show that this is not the case, and that short-term dependence may be present even though the aggregate mortality rate for $e = 4$ is higher than for e equal to 1, 2 or 3.

Ideally, one such table should be shown for each gender. However, due to the small number of observed deaths in the dataset, in particular for higher values of e caused by heavy censoring, males and females were combined. Results in Table 2 should thus be interpreted as an indication of possible short-term dependence, rather than firm evidence.

Table 2. Mortality for all couples, with e denoting the number of years since partner's death.

	Deaths	Exposure	Mortality
Partner dead			
$e = 0$	69	604.87	0.114075
$e = 1$	17	428.44	0.039679
$e = 2$	9	277.76	0.032403
$e = 3$	4	155.08	0.025590
$e = 4$	3	49.67	0.060395
Partner alive	751	34,631.45	0.021685

5. Conclusions

In this paper, we introduced a new Archimedean copula family derived from the SAHARA utility function. With SAHARA utility first increasing to a maximum and subsequently decreasing, the corresponding copula family allows for stochastically increasing (SI) and non-SI at the same time, depending on the sign of one of the parameters. As the numerical application shows, this family could fit the mortality data of coupled lives well. The parameter estimates suggest the possible existence of short-term dependence, i.e., the mortality of bereaved lives increases on bereavement but diminishes later.

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Data Availability Statement: Anyone who would like to use the data should consult Professor Edward W. (Jed) Frees of University of Wisconsin-Madison, USA, e-mail: jfrees@bus.wisc.edu, or Professor Emiliano Valdez of University of Connecticut, USA, e-mail: emiliano.valdez@uconn.edu.

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Conflicts of Interest: The author declares no conflict of interest.

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