New exact and approximation methods for time-dependent non-Hermitian quantum systems

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Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgement, the work presented is entirely my own.

Abstract

The focus of this thesis is new methods, approximate and exact, in the areas of time-dependent Hermitian and non-Hermitian quantum mechanics.

By utilising the Lewis-Riesenfeld method of invariants we first present an approach which makes use of time-independent approximations such as standard time-independent perturbation theory and WKB theory to provide solutions to the time-dependent Schrödinger equation [3]. The validity of the method is illustrated in its application to the study of two exactly solvable Hermitian systems, the time-dependent harmonic oscillator with Stark term and the Goldman-Krivchenko potential with a time-dependent perturbation.

Our focus then shifts to non-Hermitian systems where we present the first exact solution to the time-dependent Dyson equation for the time-dependent anharmonic quartic oscillator [4] demonstrating that it is spectrally equivalent to a time-dependent double well potential.

To aid in the construction of time-dependent Dyson maps and metrics we employ point transformations connecting time-dependent non-Hermitian systems with stationary Hermitian ones [5] to compute exact invariants. Here we study the time-dependent Swanson model and the time-dependent harmonic oscillator with complex linear potential. The approach is further applied to the time-dependent anharmonic quartic oscillator for which we present a second solution to the time-dependent Dyson equation.

A perturbative scheme for finding the time-dependent Dyson map and metric is then proposed and applied to determine exact solutions for a pair of weakly coupled two dimensional time-dependent harmonic oscillators with non-Hermitian coupling in space and momenta and the strongly coupled time-dependent anharmonic oscillator [6]. We also consider two-dimensional time-dependent harmonic oscillators where the non-Hermitian coupling is just in momenta.

Finally we explore a procedure which allows for systematic production of an infinite series of time-dependent Dyson maps governed by the symmetries of the Lewis-Riesenfeld invariants for time-dependent non-Hermitian Hamiltonians and their equivalent Hermitian Hamiltonians [7]. We find an infinite number of solutions for the aforementioned harmonic oscillators with non-Hermitian space and momenta coupling as well as the time-dependent anharmonic oscillator.

Chapter 1

Introduction

The area of non-Hermitian physics, distinct from the study of dissipative systems in which complex Hamiltonians have been used since 1928 [8–10], has become widely popular in the last 20 or so years since the publication of the seminal paper in 1998 by Bender and Boettcher [1]. Being a relatively new field its expansion into vast areas of research, both theoretical and experimental, is impressive. Notably its application in classical optics [11–19], in particular on the experimental side [20–23], has lead to its naming in 2015 as of the top 10 physics discoveries of the last 10 years [24]. The connection between classical optics and non-Hermitian quantum mechanics is realised through the paraxial approximation which allows for the comparison of the time-dependent Schrödinger equation and the Helmholtz equation where the refractive index $n(x)$, now being complex, takes the role of the potential. The imaginary component of $n(x)$ represents the gain or loss, which when balanced in a certain way ($n^*(-x) = n(x)$) results in real propagation constants. Other areas which non-Hermiticity has expanded into include acoustics [25, 26], quantum field theory [27–30], supersymmetry [31–34], topological systems [35, 36] and electronic circuits [37, 38] to name but a few. See articles [39, 40] for a more comprehensive list of fields and applications of non-Hermitian physics.

The expansion of non-Hermiticity into an array of fields can be traced back to the publication of the paper by Bender and Boettcher [1] which presented the now well established fact that a Hamiltonian need not be Hermitian to have real eigenvalues. This however was not the first time a non-Hermitian system was thought to have a real spectrum, with Bender and Boettcher being motivated by the work of Bessis and Zinn-Justin on the Hamiltonian $H = p^2 + ix^3$ [41]. Prior to this there had been additional mathematical realisations [42–50], however it was in [1] that everything was drawn together and presented.

By numerically studying Hamiltonians of the type

$$H = p^2 - (ix)^N, \qquad N \in \mathbb{R} > 0, \tag{1.1}$$

Bender and Boettcher [1] demonstrated that the energies were real for $N \geq 2$ as can be seen in figure 1.1. This remarkable feature was attributed to the fact that the

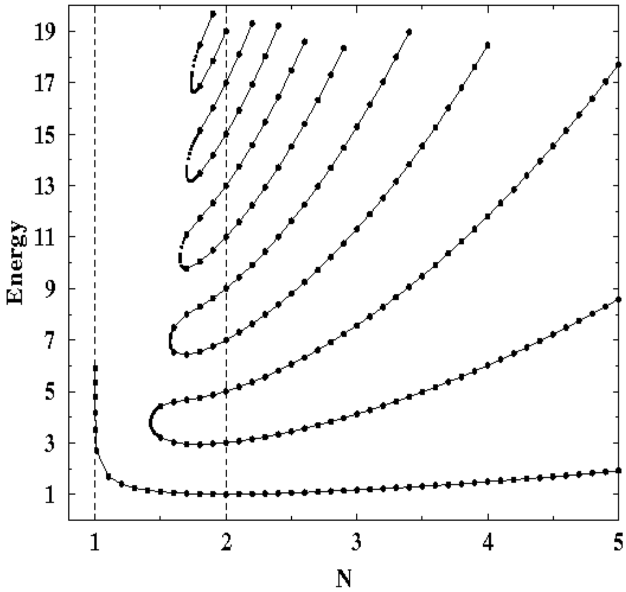

Figure 1.1: Energy spectrum for the Hamiltonian (1.1) for different values of N taken from [1].

Hamiltonians (1.1) possessed an anti-linear symmetry under which it was invariant. The typical example found throughout the literature of an anti-linear symmetry which leads to real eigenvalues is parity-time ($\mathcal{PT}$) reversal symmetry. The actions of the parity and time reversal operators on the position, momentum and imaginary unit i are[1]

$$\mathcal{P}: \quad x \to -x, \quad p \to -p, \quad i \to i, \tag{1.2}$$

$$\mathcal{T}: \quad x \to x, \quad p \to -p, \quad i \to -i, \tag{1.3}$$

such that the combined action is

$$\mathcal{PT}: \quad x \to -x, \quad p \to p, \quad i \to -i. \tag{1.4}$$

[1]Note that this is just one example of the action of $\mathcal{PT}$-symmetry, it can manifest itself differently especially when considering higher dimensional systems, see for example [51].

A Hamiltonian being $\mathcal{PT}$-symmetric however is not enough of a requirement to ensure real eigenvalues. In figure 1.1 we see that for $N < 2$ the energy levels coalesce at what is known as the exceptional point and become complex conjugate pairs. This happens because the $\mathcal{PT}$-symmetry is spontaneously broken, meaning that the eigenstates are no longer simultaneous eigenstates of the $\mathcal{PT}$ operator and Hamiltonian because the $\mathcal{PT}$ operator is anti-linear [52]. For $N \geq 2$, the spectrum is entirely real and the region is referred to as the unbroken phase. We can use an argument presented by Wigner [53] to explain this. In the unbroken regime, we have that the Hamiltonian is invariant under $\mathcal{PT}$ and that the wavefunctions are simultaneous eigenstates of the Hamiltonian H with eigenvalue ε, and the $\mathcal{PT}$ operator

$$[H, \mathcal{PT}] = 0 \quad \text{and} \quad \mathcal{PT}\phi = \phi. \tag{1.5}$$

Given that the $\mathcal{PT}$ operator is anti-linear

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi, \quad \lambda, \mu \in \mathbb{C}, \tag{1.6}$$

we simply show

$$\varepsilon\phi = H\phi = H\mathcal{PT}\phi = \mathcal{PT}H\phi = \mathcal{PT}\varepsilon\phi = \varepsilon^*\mathcal{PT}\phi = \varepsilon^*\phi, \tag{1.7}$$

and hence the eigenvalues $\varepsilon = \varepsilon^*$ are real. In the spontaneously broken regime the second requirement for real eigenvalues in (1.5) is broken, we instead have $\mathcal{PT}\phi \neq \phi$ or $\mathcal{PT}\phi_1 = \phi_2$. The Hamiltonian is still $\mathcal{PT}$-symmetric and for the two eigenstates ϕ_1 and ϕ_2 it satisfies the equations

$$H\phi_1 = \varepsilon_1\phi_1 \quad \text{and} \quad H\phi_2 = \varepsilon_2\phi_2. \tag{1.8}$$

We may now write

$$\begin{aligned} \mathcal{PT}H\phi_1 &= \mathcal{PT}\varepsilon_1\phi_1 \\ \Rightarrow H\mathcal{PT}\phi_1 &= \varepsilon_1^*\mathcal{PT}\phi_1 \\ \Rightarrow H\phi_2 &= \varepsilon_1^*\phi_2 \qquad (1.9) \\ \therefore \varepsilon_2 &= \varepsilon_1^*, \qquad (1.10) \end{aligned}$$

demonstrating that the eigenvalues occur in complex conjugate pairs. A rigorous proof of the reality of the spectrum for the Hamiltonians with unbroken $\mathcal{PT}$-symmetry can be found in [54] which utilised an equivalence between ordinary differential equations and integrable models called the ODE/IM correspondence [55–58].

A large body of work followed the initial publication of Bender and Boettcher [1] with much of it focusing on how to underpin the mathematics of a non-Hermitian quantum theory. One of the postulates of quantum mechanics states that a physical system must have associated with it a Hilbert space of state vectors and that within this space there is an inner product that has a positive norm. For Hermitian systems this inner product is defined as

$$\langle\phi|\psi\rangle := \int \phi^*(x)\psi(x)dx. \tag{1.11}$$

For non-Hermitian $\mathcal{PT}$-symmetric systems an intuitive choice for the inner product could be

$$\langle\phi|\psi\rangle^{\mathcal{PT}} := \int [\phi(x)]^{\mathcal{PT}}\psi(x)dx = \int \phi(-x)^*\psi(x)dx, \tag{1.12}$$

this would however be incorrect as the norm of the state would not always be positive. The $\mathcal{CPT}$-inner product was introduced by Bender, Brody and Jones [59] to remedy this. Defined as

$$\langle\phi|\psi\rangle^{\mathcal{CPT}} := \int [\phi(x)]^{\mathcal{CPT}}\psi(x)dx \quad \text{where} \quad [\phi(x)]^{\mathcal{CPT}} = \int \mathcal{C}(x,y)\phi^*(-y)dy, \tag{1.13}$$

The $\mathcal{C}$-operator multiplies states with negative norm by -1 resulting in a positive definite inner product $\langle\phi_n|\phi_m\rangle = \delta_{nm}$.

Computation of the $\mathcal{C}$-operator is in general a difficult process. For finite dimensional systems this can be done exactly, however approximate techniques [60–62] are usually required as knowledge of the complete set of eigenfunctions are required for its calculation as can been seen from its representation in position space

$$\mathcal{C}(x,y) = \sum_n \phi_n(x)\phi_n(y). \tag{1.14}$$

Alternatively the algebraic properties of $\mathcal{C}$ can be utilised [27], namely that $\mathcal{C}$ commutes with both the Hamiltonian and the $\mathcal{PT}$-operator and is also a reflection operator whose square is the identity

$$[\mathcal{C}, H] = 0, \quad [\mathcal{C}, \mathcal{PT}] = 0, \quad \mathcal{C}^2 = \mathbb{I}. \tag{1.15}$$

A well defined inner-product in the context of non-Hermitian systems can be guaranteed in a different way if we instead use the notion of quasi/pseudo Hermiticity. There had been early considerations predating $\mathcal{PT}$-symmetry on quasi-Hermiticity [43, 63], Mostafazadeh [64–67] developed this idea further while investigating pseudo-Hermitian Hamiltonians satisfying

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger = \rho H \rho^{-1}, \quad \rho = \eta^\dagger \eta, \tag{1.16}$$

where h and H are Hermitian and non-Hermitian Hamiltonians respectively, η is an operator commonly referred to as the Dyson map [68] and ρ, the metric. The distinction between quasi and pseudo Hermiticity is with regards to ρ. For quasi-Hermiticity ρ is positive but may not be invertible and for pseudo-Hermiticity ρ is invertible but not necessarily positive. When solving the equations (1.16) for a concrete system quasi/pseudo Hermiticity is usually assumed.

With the pseudo-Hermitian formulation the spectrum is evidently real as the Hermitian and non-Hermitian Hamiltonian are related through a similarity transformation. The solutions to the two time-independent Schrödinger equations, $h\phi = \varepsilon\phi$ and $H\psi = \varepsilon\psi$, are related through

$$\phi = \eta\psi, \tag{1.17}$$

such that the inner product definition becomes

$$\left\langle \psi | \psi' \right\rangle_\rho := \left\langle \psi | \rho \psi' \right\rangle = \left\langle \phi | \phi' \right\rangle. \tag{1.18}$$

With respect to this new metric ρ, H is Hermitian

$$\left\langle \psi | H \psi' \right\rangle_\rho = \left\langle \psi | \rho H \psi' \right\rangle = \left\langle \phi | h^\dagger \phi' \right\rangle = \left\langle H\psi | \rho \psi' \right\rangle = \left\langle H\psi | \psi' \right\rangle_\rho. \tag{1.19}$$

Obtaining solutions for η and ρ in equation (1.16) can be a cumbersome procedure, more details will be given on this in chapter 2 where we additionally generalise these equations to the time-dependent scenario.

An exciting area within $\mathcal{PT}$-symmetric quantum mechanics is the study of systems in the spontaneously broken $\mathcal{PT}$ regime. For time-independent Hamiltonians the metric ρ and Dyson map η (1.16) become ill-defined, the eigenvalues are complex the wavefunctions are no longer eigenstates of the $\mathcal{PT}$ operator. As discussed above, without the metric the inner product cannot be constructed and we do not have unitary time evolution and hence we have an inconsistent quantum mechanical framework. Fring and Frith [2, 51, 69, 70] have recently demonstrated that by introducing an explicit time-dependence into the parameters of the Hamiltonian and/or the Dyson map and metric, the spontaneously broken $\mathcal{PT}$ regime becomes physically meaningful with real energy expectation values. Further to this, physical quantities such as the entropy exhibit new effects which vary with the $\mathcal{PT}$-symmetry [71–73] (spontaneously broken, unbroken or at the exceptional point).

To study time-dependent non-Hermitian quantum systems one needs to obtain a time-dependent Dyson map and metric operator. This is a more involved process when compared with the time-independent scenario as the governing equations (1.16) are altered by them gaining additional time-derivative terms. There are existing approaches aimed at solving these time-dependent equations, in this thesis we will explore these methods further and propose new ones.

1.1 Outline

The organisation of this thesis is as follows:

In **chapter 2** we will cover in more detail time-dependent non-Hermitian quantum systems with specific focus on existing solution procedures for the Dyson map $\eta(t)$ and metric $\rho(t)$ in both the time-independent and time-dependent scenario. In subsequent chapters we will be proposing new methods, exact and approximate, for determining these quantities and so highlighting the advantages and disadvantages of previous methods will allow for the motivation of the main body of this thesis.

Chapter 3 will present the first new method we propose which has its applicability in both Hermitian and non-Hermitian time-dependent quantum systems. By utilising the Lewis-Riesenfeld method of invariants we determine a way to apply standard time-independent approximations such as time-independent perturbation theory and WKB theory to time-dependent quantum systems [3]. The effectiveness of the approach will be demonstrated by its application to the time-dependent harmonic oscillator with Stark term and the time-dependent Goldman-Krivchenko

Hamiltonian.

The time-dependent unstable anharmonic quartic oscillator will be studied in **Chapter 4**. We shall briefly cover its time-independent counterpart and how perturbation theory for the time-independent Dyson map was employed by Jones and Mateo [74] in 2006 to determine an exact metric for the model. We will then present the first exact solution for the time-dependent Dyson map [4].

In **chapter 5** we shall present the first application of point transformations to time-dependent non-Hermitian quantum systems [5]. Here the point transformations have been employed to aid in the construction of Lewis-Riesenfeld invariants which ultimately result in a simpler equation to solve for the time-dependent Dyson map. We shall apply this method to several time-dependent non-Hermitian quantum systems including the time-dependent Swanson model, the time-dependent Harmonic oscillator with complex linear term and generalised time-dependent Bender-Boettcher potentials. For the latter case we employ the method with the main aim of producing time-dependent non-Hermitian invariants, we then restrict ourselves to the anharmonic oscillator for which we determine a second exact time-dependent Dyson map.

A time-dependent version of the time-independent perturbation theory for the Dyson map shall be presented in **chapter 6** [6]. By studying weakly coupled two dimensional harmonic oscillators with non-Hermitian coupling and the strongly coupled anharmonic oscillator we are able to compute exact Dyson maps for both systems with the perturbative approach. For the former system we additionally explore the broken $\mathcal{PT}$ regime for the oscillators with a non-Hermitian coupling in space and momenta. Six inequivalent Dyson maps are obtained for this which lead to different physical behaviour as demonstrated by the energy expectation values.

Chapter 7 will explore a scheme which allows for the construction of an infinite series of time-dependent Dyson maps from two seed maps [7] governed by symmetries of the non-Hermitian and equivalent Hermitian Hamiltonians. We return first to the two dimensional harmonic oscillators with non-Hermitian coupling in space and momenta and utilise the existing six Dyson maps to illustrate the procedure. Here we demonstrate both possibilities, i.e.e when the approach breaks down and when it succeeds. The time-dependent unstable anharmonic oscillator is additionally returned to where we are able to determine the infinite series resulting in an infinite number of spectrally equivalent time-dependent double wells potentials.

In **chapter 8** the conclusions and outlook will be presented. A comparison of the existing approaches and new methods for obtaining the Dyson map/metric will also be given.

Chapter 2

Time-dependent non-Hermitian quantum systems

In this chapter we shall be focusing on time-dependent non-Hermitian quantum systems. An overview of the key equations, namely the time-dependent Dyson equation, the time-dependent quasi-Hermiticity relation and Lewis-Riesenfeld invariants, will be given. With the primary concern of this thesis being new approaches to finding solutions to these equations, we shall also discuss the relevant existing literature to highlight the challenges faced when finding solutions. The 'mending' of the spontaneously broken $\mathcal{PT}$ regime by the introduction of time will also be discussed.

2.1 Key equations and features

In the last 15 years there has be a shift in focus in the area of non-Hermitian quantum mechanics from the time-independent to the time-dependent [2, 51, 69, 75–88]. This came with unique difficulties associated with the fact that for time-dependent non-Hermitian systems the Hamiltonian loses its dual nature of being the generator of unitary time evolution and being the observable energy operator [2, 51, 76, 77]. This is because for time-dependent non-Hermitian quantum systems with time-dependent metrics, the Hamiltonian is no longer quasi-Hermitian, one instead has to define a new energy operator, which does not satisfy a time-dependent Schrödinger equation but reduces to the Hamiltonian only in the absence of time. Specific equations defining this quantity shall be given below.

The starting point for the study of time-dependent non-Hermitian quantum systems are the two time-dependent Schroödinger equations (TDSEs)

$$h(t)\left|\Phi(t)\right\rangle = i\partial_t \left|\Phi(t)\right\rangle \quad \text{and} \quad H(t)\left|\Psi(t)\right\rangle = i\partial_t \left|\Psi(t)\right\rangle, \tag{2.1}$$

where $h(t) = h^\dagger(t)$ is Hermitian and $H(t) \neq H^\dagger(t)$ is non-Hermitian. We may relate the solutions to the two TDSEs via a time-dependent Dyson map $\eta(t)$ [68]

$$\left|\Phi(t)\right\rangle = \eta(t)\left|\Psi(t)\right\rangle, \tag{2.2}$$

where $\eta(t)$ is invertible. Upon substitution into the TDSE one retrieves what it is commonly referred to as the time-dependent Dyson equation (TDDE)

$$h(t) = \eta(t)H(t)\eta(t)^{-1} + i\partial_t\eta(t)\eta(t)^{-1}. \tag{2.3}$$

As $h(t)$ is Hermitian, we may eliminate it from the TDDE by taking the complex conjugate to obtain the time-dependent quasi-Hermiticity (TDQH) relation

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\partial_t\rho(t), \tag{2.4}$$

where $\rho(t) = \eta^\dagger(t)\eta(t)$ is interpreted as the time-dependent metric. This equation gets its name from a paper by Dieudonné [63] in which quasi-Hermitian operators obey the relation $H^\dagger\rho = \rho H$, equation (2.4) is the time-dependent generalisation. The time-dependent metric operator preserves the time-dependent probability densities

$$\left\langle \Phi(t)|\tilde{\Phi}(t)\right\rangle = \left\langle \Psi(t)|\rho(t)\tilde{\Psi}(t)\right\rangle =: \left\langle \Psi(t)|\tilde{\Psi}(t)\right\rangle_\rho. \tag{2.5}$$

By taking the time-derivative of this equation we see that the left hand side vanishes due to the Hermiticity of $h(t)$, the right side vanishes if equation (2.4) holds, this therefore justifies the interpretation of $\rho(t)$ as the metric as it ensure unitary time evolution.

As mentioned above, the time-dependent non-Hermitian Hamiltonian is not the observable energy operator. This can be seen when looking at how observables in the Hermitian, $o(t)$, and non-Hermitian, $\mathcal{O}(t)$, regime are related to one another

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t). \tag{2.6}$$

That is through a similarity transformation and therefore the observables are quasi-Hermitian

$$\mathcal{O}^{\dagger}(t) = \rho(t)\mathcal{O}(t)\rho^{-1}(t). \tag{2.7}$$

Given that the Hermitian Hamiltonian $h(t)$ is observable, the presence of the time-derivative terms in both the TDDE (2.3) and the TDQH relation (2.4) results in $H(t)$ not being observable. Instead a new energy operator is defined as

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\eta^{-1}(t)\partial_t\eta(t), \tag{2.8}$$

which for a time-independent Dyson map reduces to $H(t)$. It is important to stress here that while $\tilde{H}(t)$ is the observable energy operator, it is not a Hamiltonian in the sense that it does not satisfy the original TDSE and therefore does not govern the time evolution of the system. We can demand $\tilde{H}\left|\tilde{\Psi}\right\rangle = i\partial_t\left|\tilde{\Psi}\right\rangle$, however this would be a new system.

What these key equations demonstrate is that when studying time-dependent non-Hermitian quantum systems the essential quantities that need to be determined are the metric $\rho(t)$ and the Dyson map $\eta(t)$. To do so one has the option to either solve the TDDE (2.3) for the Dyson map and then construct the metric through $\rho = \eta^{\dagger}\eta$, or solve the TDQH relation (2.4) for the metric and subsequently obtain the Dyson map. While there are many difficulties associated with solving these relations for specific systems, as will be detailed in section 2.2, there is one more key equation which allows for the determination of the metric and Dyson map. This key equation comes from considering the Lewis-Riesenfeld method of invariants [89].

The Lewis-Riesenfeld method of invariants [89] is an approach which allows one to construct the exact solution to the TDSE for a time-dependent system. The key features of the approach involve obtaining an invariant, $I(t)$, which satisfies the equation

$$\frac{dI(t)}{dt} = \partial_t I(t) - i\left[I(t), H(t)\right] = 0, \tag{2.9}$$

from which the time-dependent eigenstates can be constructed through

$$I(t)\left|\phi(t)\right\rangle = \lambda\left|\phi(t)\right\rangle, \qquad \dot{\lambda} = 0, \tag{2.10}$$

where the eigenvalues are time-independent. From here the full solution to the TDSE is computed through

$$|\Psi(t)\rangle = e^{i\alpha(t)} |\phi(t)\rangle , \quad \text{where} \quad \dot{\alpha}(t) = \langle\phi(t)|i\partial_t - H(t)|\phi(t)\rangle . \tag{2.11}$$

The utility of the invariants for time-dependent non-Hermitian quantum systems comes from how we can relate the invariant for a Hermitian and non-Hermitian system. If invariants can be constructed for the Hermitian system $h(t)$ and the non-Hermitian system $H(t)$,

$$\partial_t I_{\mathcal{H}}(t) = i\left[I_{\mathcal{H}}(t), \mathcal{H}(t)\right], \quad \text{for} \quad \mathcal{H} = h, H, \tag{2.12}$$

it can be shown that they are related via [87, 88]

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t). \tag{2.13}$$

A proof of this relation, the time-independence of the eigenvalues λ and that the eigenvectors $|\phi(t)\rangle$ satisfy the TDSE can be found in Appendix A.

We now have three key equations (2.3), (2.4) and (2.13) which can be used to determine the time-dependent Dyson map and metric. We shall now move on to how these equations have been solved in the existing literature and the limitations of each approach.

2.2 Comparison of solution procedures

It is instructive to briefly discuss how to obtain the Dyson map and metric when these quantities as well as the Hamiltonian are time-independent. By taking the Hamiltonians, Dyson map and metric to be time-independent $\{h(t), H(t), \rho(t), \eta(t)\} \rightarrow \{h, H, \eta, \rho\}$ equations (2.3) and (2.4) reduce to

$$h = \eta H \eta^{-1} \quad \text{and} \quad H^\dagger = \rho H \rho^{-1}, \tag{2.14}$$

which we will refer to as the time-independent Dyson equation (TIDE) and the time-independent quasi-Hermiticity (TIQH) relation. To solve for either η or ρ one must solve a similarity transformation which at first may seem straightforward yet there are very few known exact solutions and the process can be rather difficult [74, 90–98]. The Dyson map or metric is usually an exponential of a sum of operators for which an Ansatz needs to be made. The adjoint action of it on the non-Hermitian

Hamiltonian is computed through the Baker-Campbell-Hausdorff relation

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B] + \frac{1}{3!}[A, [A, [A, B]..., \tag{2.15}$$

which results in a set of simultaneous equations of non-Hermitian terms which need to be eliminated. Determining a correct Ansatz for a particular system can be very difficult especially when not all systems possess an exact solution. Take for example the complex cubic potential $V = ix^3$, here the metric operator is only known perturbatively [59, 99]. In fact, due to the difficulty in obtaining exact Dyson maps perturbative methods have been used to find exact solutions such as for the anharmonic quartic oscillator $V = -x^4$ [74]. A more detailed discussion on how to employ perturbation theory to determine the Dyson map/metric will be given in Chapter 4 when we study the time-dependent anharmonic quartic oscillator.

It is worth emphasising here that the Dyson map and metric are not unique as noted in [43, 91, 100]. This has been explored in more detail for the Swanson Hamiltonian [90, 93]. More interesting however is that this non-uniqueness of the metric/Dyson map can be attributed to the symmetry operators for the Hamiltonian [96]. Consider a non-Hermitian Hamiltonian H with two non-equivalent metric operators ρ and $\tilde{\rho}$ satisfying the TIQH relation (2.14)

$$H^\dagger = \rho H \rho^{-1} \quad \text{and} \quad H^\dagger = \tilde{\rho} H \tilde{\rho}^{-1}. \tag{2.16}$$

Defining $s := \rho^{-1}\tilde{\rho}$ we see that s is a symmetry operator for the system, $[H, s] = 0$. Similarly, if we consider the TIDE (2.14) and two non-equivalent Hermitian Hamiltonians

$$h = \eta H \eta^{-1} \quad \text{and} \quad \tilde{h} = \tilde{\eta} H \tilde{\eta}^{-1}, \tag{2.17}$$

and by defining $A := \tilde{\eta}\eta^{-1}$ such that

$$S := A^\dagger A \quad \text{and} \quad \tilde{S} := AA^\dagger, \tag{2.18}$$

we see that S is a symmetry operator for h and $\tilde{S}$ is symmetry operator of $\tilde{h}$

$$[h, S] = 0 \quad \text{and} \quad [\tilde{h}, \tilde{S}] = 0. \tag{2.19}$$

While the symmetries here result from two inequivalent Dyson maps we will show in Chapter 7 that for time-dependent systems that the symmetries are transferred to Lewis-Riesenfeld invariants [89] and exploit them to compute an infinite series of time-dependent Dyson maps. Even for an infinite series of Dyson maps, or just the two considered here for the time-independent scenario, one can always make the Dyson map/metric unique by specifying one more observable in the system [43].

Another interesting situation to consider before we discuss fully time-dependent Hamiltonians and metric operators/Dyson maps is that which is referred to as the metric picture [76]. The time-dependence here is included in the metric rather than in the observable as in the Heisenberg picture or the states as in the Schrödinger picture. While the non-Hermitian Hamiltonian is time-independent in this scenario the full TDDE (2.3) or (2.4) still needs to be solved.

We now move on to discussing the fully time-dependent case. In the previous section we highlighted the three key equations that can be used to find the Dyson map and metric, that being the TDDE (2.3), the TDQH relation (2.4) and the similarity transformation associated with the Lewis-Riesenfeld invariants [89] for the Hermitian and non-Hermitian system (2.13). In recent years there have been many papers aimed at finding exact solutions to these equations for different models [2, 51, 69–72, 75–77, 87, 88, 101], however, these are far and few between when compared with the time-independent case.

We will start by looking at the TDDE (2.3) and TDQH relation (2.4) together. These equations differ from their time-independent counterparts (2.14) by the additional time-derivative terms. These time-derivative terms complicate the choice of Ansatz for the Dyson map/metric in two ways. Firstly instead of solving simultaneous equations we end up with the more complicated coupled differential equations which need to be solved. Additionally we have to factorise out our Ansatz so that it is not contained in a singular exponential but instead a product of exponentials. This is because in general it is difficult to calculate for an exponential $\partial_t \exp[A(t) + B(t) + C(t)]$ with non-vanishing commutators $[A(t), B(t)] \neq 0$ and $[B(t), C(t)] \neq 0$.

Solving the TDDE (2.3) for the Dyson map and then constructing the metric or solving the TDQH relation (2.4) for the metric and subsequently obtaining the Dyson map are equivalent approaches. However, as argued in [70, 76] the latter is usually more difficult, even when the Dyson map is Hermitian as we still need to take the square root of $\rho(t) = \eta(t)^2$. Contrary to this, when solving the TDDE, given that the Dyson map is not unique and need not be Hermitian there is a problem around choosing a useful Ansatz. Once constructed though it is much easier to then obtain the metric.

The third method for solving for the Dyson map/metric is to utilise Lewis-Riesenfeld invariants together with the equation (2.13). We immediately see that the invariants are related through a similarity transformation much like the TISE (2.14). This means we bypass the need for solving coupled differential equations which makes the approach simpler [51, 87]. This is dependent on being able to obtain the invariants in the first place which requires an additional Ansatz for its form. This in turn increases the number of steps required to obtain the Dyson map. Another benefit of the invariant approach, however, is that once the invariant was obtained it, it is then much simpler to solve for the eigenfunctions as the eigenvalues as time-independent (2.10).

Overall each of the approaches has a number of advantages and disadvantages which are summarised in table 2.1. Which method is used to solve for the Dyson map/metric is usually model dependent. For example the two-dimensional harmonic oscillator with complex coupling [51] the use of Lewis-Riesenfeld invariants were shown to be beneficial as the problem of solving a time-dependent differential equation was reduced to a similarity transformation. Contrary to this, for the 2 level matrix models studied in Chapter 2 of [70] there was no clear advantage.

Approach	Advantages	Disadvantages
TDQH (2.4)	1. Does not involve h 2. Clearer structure for ρ (Hermitian)	1. $\rho \to \eta$ more difficult 2. Coupled differential equations
TDDE (2.3)	1. $\eta \to \rho$ easier 2. Less restriction on Ansatz	1. η can be Hermitian or non-Hermitian: no clear structure 2. Coupled differential equations.
Lewis-Riesenfeld invariants (2.13)	1. Similarity transformation is easier to solve. 2. Easier to solve for eigenfunctions	1. Increased number of steps 2. Ansatz for the invariant

Table 2.1: Summary of comparison of solution procedures for $\eta(t)$ and $\rho(t)$.

2.3 The spontaneously broken $\mathcal{PT}$ regime

In chapter 1 we briefly touched upon an interesting feature of time-dependent non-Hermitian quantum system in that they give physical meaning to the spontaneously broken $\mathcal{PT}$ regime [2, 51, 70, 71]. For the time-independent scenario this corresponds to the wave functions being $\mathcal{PT}$-symmetrically broken resulting in the eigenvalues being complex conjugate pairs. By introducing an explicit time-dependence into the non-Hermitian Hamiltonian the expectation values of the energy operator $\tilde{H}(t)$ become real.

In [2] the authors demonstrated this remarkable feature for the first time for a two-level spin model with the Hamiltonian

$$H = -\frac{1}{2}\left[w\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right], \qquad \omega, \lambda, \kappa \in \mathbb{R}, \tag{2.20}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices and the eignevalues are given by

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}. \tag{2.21}$$

For $|\lambda| < |\kappa|$ the eigenvalues $E_{\pm}$ will be complex conjugate pairs, a characteristic of the broken $\mathcal{PT}$ regime. By making the parameters time-dependent $\lambda \to \alpha\kappa(t)$, $\kappa \to \kappa(t)$, determining a time-dependent Dyson map $\eta(t)$ and wavefunctions $\psi_{\pm}(t)$, the expectation values of the energy operator were shown to be real provided $\alpha \neq 1$ as depicted in figure 2.1 taken from [2]. The reality of the spectrum depicted in figure 2.1 in the spontaneously broken $\mathcal{PT}$ regime ($|\alpha| < 1$) is attributed to the fact that the energy operator is different from the original time-dependent non-Hermitian Hamiltonian which is unobservable. The authors identified a new PT operator for the energy operator $\tilde{H}$ which explained this behaviour. Note also the difference in behaviour of the expectation values in figure 2.1 on either side of the exceptional point $\alpha = 1$. This behaviour and "mending" of the spontaneously broken $\mathcal{PT}$-regime has been identified in several systems now [2, 51, 70, 71]. We shall also further explore the phenomenon for some of the systems we consider in this thesis.

The energy operator is not the only physical quantity which exhibits peculiar behaviour in the broken $\mathcal{PT}$ regime, recent research has demonstrated that the entropy does as well [71–73]. The Von Neumann entropy as a particular example has been shown to exhibit three different types of behaviour associated to the $\mathcal{PT}$-

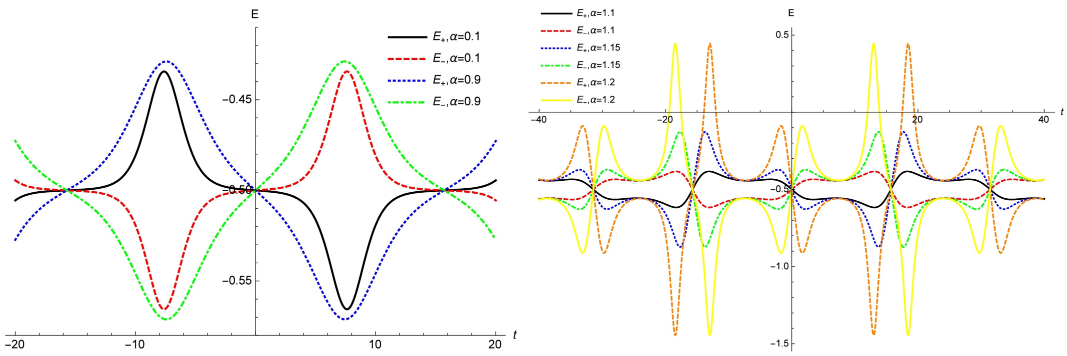

Figure 2.1: The expectation values of the energy operator $\tilde{E}_{\pm}(t) = \left\langle \psi_{\pm}(t) | \tilde{H}(t) \eta^2 \psi_{\pm}(t) \right\rangle$ for different values of α taken from [2].

symmetry of the system [71]. For the spontaneously broken regime the entropy decayed to a non-zero value in finite time. At the exceptional point point the entropy decayed to zero and in the unbroken $\mathcal{PT}$ regime there was rapid decay of the entropy to zero which subsequently revived, the entropy then continued to oscillate in this manner for all times.

Chapter 3

Time-independent approximations for Lewis-Riesenfeld invariants

In this chapter we will explore the possibility of modifying the Lewis-Riesenfeld method of invariants developed originally to find exact solutions for time-dependent quantum mechanical systems for the situation in which an exact invariant can be constructed, but the subsequently resulting time-independent eigenvalue system is not solvable exactly. Following [3] we propose to carry out this step in an approximate fashion, such as employing standard time-independent perturbation theory or the WKB approximation, and subsequently feeding the resulting approximated expressions back into the time-dependent scheme. We illustrate the quality of this approach by contrasting an exactly solvable solution to one obtained with a perturbatively carried out second step for two types of explicitly time-dependent optical potentials.

3.1 Motivation

In 1970 Askhin discovered that small particles can be trapped by using radiation pressure from continuous lasers [102]. Since then various types of optical traps have been designed [103] with wide ranging applications. Such applications include the trapping of particles [104], atoms [105] and molecules [106] as well as viruses and bacteria [107, 108]. To obtain a general understanding of how to trap particles within optical traps we can study the TDSE for an explicitly time-dependent Hamiltonian. The particles are trapped within optical potentials whose generic form may be written in a factorised way as $V(x,t) = \kappa(t)V(x)$. Here $V(x)$ would describe the shape of the trap and $\kappa(t)$ the time-dependent modulation [109]. Note that while many optical potentials can be factorised in this way, there do exist optical potentials for which this is not true such as optical lattices [110]. However in this chapter we shall only be concerning ourselves with factorisable optical potentials.

To obtain solutions to the TDSE for the types of potentials we are investigating is not an easy process. In general, one usually has to resort to approximation methods as there are very few known exact solutions to the TDSE. These approximation methods include the adiabatic and sudden approximation [111] which are carried out on the level of the time evolution operator. Additionally, there exists less general methods which have been developed to find approximate solutions to the TDSE, an example of which is the strong field approximation [112–115]. Here the systems studied have potentials of the form $V(x) + xE(t)$, where the Stark term, involving a laser field $E(t)$, is dominating or in comparable strength to the potential $V(x)$. In these scenarios the approximation scheme is a mixture of perturbative expansions based on the Du-Hamel formula also carried out on the level of the time-evolution operator [116–118]. The two perturbative expansions, one in $V(x)$ and one in $E(t)$, are mixed and then terminated after the first iteration.

Aside from the various approximation methods that exist to solve the TDSE there are exact approaches such as the Lewis-Riesenfeld method of invariants [89]. This approach has had many successes and has been used to find the exact solution for the harmonic oscillator with time-dependent mass and frequency in one [119] and two dimensions [120, 121], the damped harmonic oscillator [122], a time-dependent Coulomb potential [123], a Davydov-Chaban Hamiltonian in presence of time-dependent potential [124], a Bohr Hamiltonian with a time-dependent potential [125], time-dependent Hamiltonians given in terms of linear combinations of SU(1,1) and SU(2) generators [126–128], in the inverse construction of time-dependent Hamiltonian [129, 130], for systems on noncommutative spaces in time-dependent backgrounds [131], time-dependent non-Hermitian Hamiltonian systems [51, 75, 101, 132] and other specific systems.

For the factorisable optical potentials we shall be investigating it is possible to construct an exact Lewis-Riesenfeld invariant, as we shall demonstrate. In most cases however we will not be able to complete the Lewis-Riesenfeld method in its entirety. Following [3] we propose that one should not abandon the approach as we have massively simplified the problem by being able to construct an invariant. We have transformed the system from a time-dependent first order differential equation to a time-independent eigenvalue equation. Even if we cannot solve the time-independent eigenvalue problem exactly we recognise the fact that time-independent approximations are usually much easier to implement then time-dependent ones. We therefore propose using time-independent perturbation theory or WKB theory to solve the time-independent eigenvalue problem. In the following sections we shall demonstrate how to implement this idea within the Lewis-Riesenfeld method of invariants and then apply this directly to two different types of optical potentials.

3.2 An approximate Lewis-Riesenfeld method of invariants

We start by recalling the key steps of the method of invariants, first introduced in chapter 2, and then describe how they can be modified in an appropriate fashion. The scheme was introduced originally by Lewis and Riesenfeld [89], for the purpose of solving the TDSE

$$i\hbar\partial_t |\psi_n\rangle = H(t) |\psi_n\rangle , \tag{3.1}$$

for the time-dependent or dressed states $|\psi_n\rangle$ associated to the explicitly time-dependent Hamiltonian $H(t)$.

The Lewis-Riesenfeld method of invariants is made up of three main stages: The initial step in this approach consists of constructing a time-dependent invariant $I(t)$ from the evolution equation

$$\frac{dI(t)}{dt} = \partial_t I(t) + \frac{1}{i\hbar}[I(t), H(t)] = 0. \tag{3.2}$$

Often this step can be completed and an exact form for the invariant $I(t)$ can be found. In the next step one needs to solve the corresponding eigenvalue system of the invariant $I(t)$

$$I(t) |\phi_n\rangle = \lambda_n |\phi_n\rangle , \tag{3.3}$$

for time-independent eigenvalues λ_n and for the time-dependent states $|\phi_n\rangle$. Provided the Hamiltonian $H(t)$ is Hermitian, also the invariant $I(t)$ is Hermitian and therefore the eigenvalues λ_n are guaranteed to be real, see appendix A for a proof of this. The virtue of this equation, compared to the TDSE in (3.1), is that one has reduced the original evolutionary problem in form of a first order differential equation to an eigenvalue equation in which t simply plays the role of a parameter as λ_n has become time-independent. Hence one just needs to solve a time-independent eigenvalue problem. To complete this step the system in (3.3) needs to be solvable. It is this requirement one can weaken and employ *time-independent* approximation methods to complete step two.

The final third step relates the eigenstates in (3.3) with the complete solution of the TDSE. It was shown in [89] that the states

$$|\psi_n\rangle = e^{i\alpha_n(t)} |\phi_n\rangle , \tag{3.4}$$

satisfy the TDSE (3.1) provided that the real function $\alpha(t)$ in (3.4) obeys

$$\frac{d\alpha(t)}{dt} = \frac{1}{\hbar} \langle\phi_n| i\hbar\partial_t - H(t) |\phi_n\rangle , \tag{3.5}$$

see appendix A for further details on this. Since all the quantities on the right hand side of (3.5) have been obtained in the previous steps, one is left with a simple integration in time to determine the phase $\alpha(t)$. These key equations serve mainly for reference purposes and we refer the reader to Appendix A for more details.

For many systems we might succeed in carrying out the first step in the procedure and construct an explicit expression for the invariant $I(t)$. However, the process stalls often in the second step and for most Hamiltonians the eigenvalue equation for the invariants $I(t)$ in (3.3) can not be solved exactly. The two methods we are proposing ro use are standard time-independent perturbation theory and WKB theory.

3.2.1 Time-independent perturbation theory

The first approximation which we propose utilising to modify the Lewis-Riesenfeld method of invariants is standard time-independent perturbation theory. We are splitting the invariant as

$$I(t) = I_0(t) + \epsilon I_p(t), \tag{3.6}$$

and consider the eigenvalue equation for the full invariant and the unperturbed one separately

$$I(t)\left|\phi_n\right\rangle = \lambda_n \left|\phi_n\right\rangle, \qquad \text{and} \qquad I_0(t)\left|\phi_n^{(0)}\right\rangle = \lambda_n^{(0)}\left|\phi_n^{(0)}\right\rangle. \tag{3.7}$$

Assuming that within the perturbation term a small parameter $\epsilon \ll 1$ can be identified, we expand the eigenvalues and the eigenfunctions of the unperturbed invariant as

$$\lambda_n = \lambda_n^{(0)} + \epsilon\lambda_n^{(1)} + \epsilon^2\lambda_n^{(2)} + \mathcal{O}(\epsilon^3), \quad \text{and} \quad \left|\phi_n\right\rangle = \left|\phi_n^{(0)}\right\rangle + \epsilon\left|\phi_n^{(1)}\right\rangle + \epsilon^2\left|\phi_n^{(2)}\right\rangle + \mathcal{O}(\epsilon^3), \tag{3.8}$$

with $\lambda_n^{(k)} = 1/k!\, d\lambda_n/d\epsilon^k\big|_{\epsilon=0}$, $\phi_n^{(k)}\rangle = 1/k!\, d\phi_n/d\epsilon^k\big|_{\epsilon=0}$. The first order corrections to the eigenvalues and eigenstates of the invariants are then computed in the standard fashion for nondegenerate systems to

$$\lambda_n^{(1)} = \left\langle \phi_n^{(0)}\right| I_p \left|\phi_n^{(0)}\right\rangle, \quad \text{and} \quad \left|\phi_n^{(1)}\right\rangle = \sum\nolimits_{k\neq n} \frac{\left\langle \phi_k^{(0)}\right| I_p \left|\phi_n^{(0)}\right\rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}}\left|\phi_k^{(0)}\right\rangle, \tag{3.9}$$

respectively. For orthonormal functions ϕ_n, we obtain further constraints on the normalization of contributions in the series

$$\begin{aligned} 1 = \left\langle \phi_n \middle| \phi_n\right\rangle = \left\langle \phi_n^{(0)} \middle| \phi_n^{(0)}\right\rangle &+ \epsilon\left(\left\langle \phi_n^{(0)} \middle| \phi_n^{(1)}\right\rangle + \left\langle \phi_n^{(1)} \middle| \phi_n^{(0)}\right\rangle\right) \\ &+ \epsilon^2\left(\left\langle \phi_n^{(2)} \middle| \phi_n^{(1)}\right\rangle + \left\langle \phi_n^{(1)} \middle| \phi_n^{(1)}\right\rangle + \left\langle \phi_n^{(0)} \middle| \phi_n^{(2)}\right\rangle\right) + \dots \end{aligned} \tag{3.10}$$

Thus if the zero order wavefunction is normalized to $1 = \left\langle \phi_n^{(0)} \middle| \phi_n^{(0)}\right\rangle$, we require the higher order wave functions to satisfy the additional constraints

$$\sum_{k=0}^{\ell} \left\langle \phi_n^{(\ell-k)} \middle| \phi_n^{(k)}\right\rangle = 0. \tag{3.11}$$

Next we can use these expressions to obtain an approximate solution to the TDSE. Denoting $\left|\phi_n\right\rangle^{(1)} := \left|\phi_n^{(0)}\right\rangle + \epsilon\left|\phi_n^{(1)}\right\rangle$ we obtain

$$\left|\psi_n\right\rangle^{(1)} = e^{i\alpha_n^{(1)}(t)}\left|\phi_n\right\rangle^{(1)}, \quad \text{and} \quad \alpha^{(1)}(t) = \frac{1}{\hbar}\int dt\, {}^{(1)}\!\left\langle\phi_n\right| i\hbar\partial_t - H(t)\left|\phi_n\right\rangle^{(1)}. \tag{3.12}$$

3.2.2 WKB theory

An additional time-independent approximation which we will use to modify the Lewis-Riesenfeld method of invariants is the semi-classical WKB[1] approximation named after Wentzel [134], Kramers [135] and Brillouin [136]. This method is commonly used to find approximate solutions to the one dimensional time-independent Schrödinger equation given by

$$-\frac{d^2\psi}{d\xi^2} + \frac{p(\xi)^2}{\hbar^2}\psi(\xi) = 0 \qquad \text{where} \qquad p(\xi) = \sqrt{2m(E - V(\xi)}, \tag{3.13}$$

where ξ is the position, $p(\xi)$ is the momentum, m is the mass, E is the energy and the potential $V(\xi)$ is slowly varying. More details on how derive the WKB wave functions can be found in Appendix B.

We may solve the eigenvalue equation (3.3) by using the WKB approximation $\left|\phi_n^{\mathrm{WKB}}\right\rangle$ and compute the Lewis-Riesenfeld phase using that expression

$$\left|\psi_n^{\mathrm{WKB}}\right\rangle = e^{i\alpha_n^{\mathrm{WKB}}(t)}\left|\phi_n^{\mathrm{WKB}}\right\rangle, \quad \text{and} \quad \alpha^{\mathrm{WKB}}(t) = \frac{1}{\hbar}\int dt \left\langle\phi_n^{\mathrm{WKB}}\right| i\hbar\partial_t - H(t)\left|\phi_n^{\mathrm{WKB}}\right\rangle. \tag{3.14}$$

Assuming that the invariant $I(t)$ can be cast into the same form as a time-independent Hamiltonian, with a standard kinetic energy term and a potential $V(\xi)$, the WKB approximation to first order in $\hbar$ denoted by $\hat{\phi}$, see e.g. [137], for the eigenvalue equation (3.3) reads

$$\hat{\phi}^{\mathrm{WKB}}(\xi) = \frac{A}{\sqrt{p(\xi)}}e^{\frac{i}{\hbar}\int^{\xi}p(z)dz} + \frac{B}{\sqrt{p(\xi)}}e^{-\frac{i}{\hbar}\int^{\xi}p(z)dz} \tag{3.15}$$

in the classically allowed region, $\lambda > V(\xi)$ and

$$\hat{\phi}^{\mathrm{WKB}}(\xi) = \frac{C}{\sqrt{q(\xi)}}e^{\frac{1}{\hbar}\int^{\xi}q(z)dz} + \frac{D}{\sqrt{q(\xi)}}e^{-\frac{1}{\hbar}\int^{\xi}q(z)dz} \tag{3.16}$$

in the classically forbidden region $\lambda < V(\xi)$, where

$$p(\xi) := \sqrt{2[\lambda - V(\xi)]} \quad \text{and} \quad q(\xi) := \sqrt{2[V(\xi) - \lambda]}. \tag{3.17}$$

The constants A, B, C, D need to be determined by the appropriate asymptotic

[1]The method is also referred to as the JWKB or WKBJ method to reflect Jeffreys [133] work on the approximate solutions to second order differential equations which came prior to the Schrödinger equation.

WKB matching and normalisation conditions as explained in detail in appendix B.

3.3 Time-dependent potentials with a Stark term

We first demonstrate how to solve the TDSE (3.1) for the one-dimensional Stark Hamiltonian involving a time-dependent potential $V(x,t)$

$$H(t) = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + V(x,t) + xE(t). \tag{3.18}$$

In order to cover optical potentials of the form $V(x,t)$ in our treatment, we are slightly more general than in the standard Stark Hamiltonian where the potential is just depending on x and allow for an explicit time-dependence in the potential $V(x,t)$ as well as in an electric or laser field $E(t)$. At first we assume that the potential factorizes as $V(x,t) = \kappa(t)V(x)$. When the laser field term involving $E(t)$ dominates the potential term and $\kappa(t)$ =const several well known and successful approaches have been developed. For instance, the strong field approximation is a mixture of perturbative expansions based on the Du-Hamel formula carried out on the level of the time-evolution operator [112–115].

In our proposed approach we assume that the first step in the Lewis and Riesenfeld approach can be carried out and resort to an approximation in form of perturbation theory in the second step.

3.3.1 Construction of time-independent invariants

In order to carry out the first step in the Lewis-Riesenfeld approach to solve time-dependent systems we need to construct the invariant $I(t)$ by solving equation (3.2) for a given Hamiltonian, (3.18) in our case. For this purpose one usually makes an Ansatz by assuming the invariant to be of a similar form as the Hamiltonian

$$I(t) = \frac{1}{2}\left[\alpha(t)p^2 + \beta(t)V(x) + \gamma(t)x + \delta(t)\{x,p\} + \varepsilon(t)x^2\right]. \tag{3.19}$$

In our case it involves five unknown time-dependent coefficient functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varepsilon(t)$. As ususal we denote the anti-commutator by $\{A,B\} := AB+BA$. The substitution of (3.19) into (3.2) then yields the following first order coupled differential equations as constraints

$$\dot{\alpha} = -2\frac{\delta}{m}, \quad \gamma = 2m\alpha E, \quad \dot{\gamma} = 2\delta E, \quad \dot{\delta} = m\alpha\omega^2 - \frac{\varepsilon}{m}, \quad \dot{\varepsilon} = 2m\delta\omega^2, \tag{3.20}$$

$$\beta = m\alpha\kappa, \quad \dot{\beta} = \delta\kappa x\partial_x(\ln V). \tag{3.21}$$

Remarkably, despite being overdetermined, this system can be solved consistently. We note that the equations in (3.20) and (3.21) almost decouple entirely from each other, being only related by δ. We solve (3.20) first by parameterizing $\alpha(t) = \sigma^2(t)$ and integrating twice

$$\alpha = \sigma^2, \quad \gamma = 2m\sigma^2 E(t), \quad \delta = -m\sigma\dot{\sigma}, \quad \varepsilon = m^2\dot{\sigma}^2 + m^2\frac{\tau}{\sigma^2}. \tag{3.22}$$

The auxiliary quantity σ has to satisfy the nonlinear Ermakov-Pinney (EP) [138, 139] equation

$$\ddot{\sigma} + \omega^2\sigma = \frac{\tau}{\sigma^3}, \tag{3.23}$$

and in addition the electric field has to be parameterised by the solution of the EP-equation σ as

$$E(t) = \frac{c}{\sigma^3}. \tag{3.24}$$

The constants $c, \tau \in \mathbb{R}$ result from the integrations. We take here $\tau > 0$. Using the expression for δ from (3.22), we may now also solve the set of equations in (3.21), obtaining

$$\beta_p = m\sigma^2\kappa_p, \qquad V_p = c_p x^p, \qquad \kappa_p = \frac{\tilde{c}_p}{\sigma^{2+p}}, \tag{3.25}$$

with real integration constants $c_p, \tilde{c}_p$ and $p \in \mathbb{R}$. This means that we can not choose the electric field $E(t)$ in our Hamiltonian and the potential $V(x,t)$ entirely a priori and independently from each other. Notice that we may extend the analysis by allowing the constants $c_p, \tilde{c}_p$ to be complex, hence opening up the treatment to include non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians [1, 140, 141].

First we notice that the only time-independent potential is obtained for $p = -2$, so that the potential part in $H(t)$ becomes the solvable Goldman-Krivchenko potential [142]. Crucially, the constraining equations involving the potential (3.21) decouple from the remaining ones and since these equations are linear we may solve for potentials that factorize termwise when expanded, that is $V(x,t) = \sum_p \kappa_p(t)V_p(x)$.

For instance, for a time-dependent Gaussian potential of the form

$$V_{\text{Gauss}}(x,t) = A(t)\left(e^{-\lambda(t)x^2} - 1\right) = \sum_{n=1}^{\infty} \kappa_n(t) V_n(x),$$

we obtain

$$V_{2n} = x^{2n}, \quad \kappa_n = \frac{(-1)^n}{n!}\frac{1}{\sigma^{2+2n}}, \tag{3.26}$$

where we have to restrict $A(t) = \lambda(t) = \sigma^{-2}$. For another widely used potential, the soft Coulomb potential [143] of the form

$$V_{\text{sCoulomb}}(x,t) = A(t)\frac{1}{\sqrt{x^2 + k^2 a^2(t)}} = \sum_{n=1}^{\infty} \kappa_n(t) V_n(x),$$

with k taken to be a real constant, we obtain

$$V_{2n} = x^{2n}, \quad \kappa_n = \frac{(-1)^n (2n)!}{(2n)!!(2n)!!}\frac{1}{\sigma^{2+2n} k^{1+2n}}, \tag{3.27}$$

where we have to restrict $A(t) = 1/a(t) = \sigma^{-1}$.

As mentioned, besides the potential, also the electric field is not entirely unconstrained as they are mutually related via the EP-function σ. However, as we shall demonstrate the solutions of the EP-equation are such that it will still allow for a large class of interesting fields, notably periodic, to be investigated in an exactly solvable manner. It was found by Pinney [139] that the solutions to (3.23) are

$$\sigma = \sqrt{u_1^2 + \tau\frac{u_2^2}{W^2}}, \tag{3.28}$$

where u_1, u_2 are the two linearly independent solutions of the equation

$$\ddot{u} + \omega^2 u = 0, \tag{3.29}$$

and $W = u_1\dot{u}_2 - \dot{u}_1 u_2$ is the corresponding Wronskian. Thus taking the two solutions of (3.29) to be $u_1 = A\sin(\omega t)$ and $u_2 = B\cos(\omega t)$ with A, $B \in \mathbb{R}$, the solution to the EP-equation (3.28) acquires the form

$$\sigma(t) = \frac{1}{\sqrt{2}A\omega}\sqrt{\tau + A^4\omega^2 + (\tau - A^4\omega^2)\cos(2\omega t)}. \tag{3.30}$$

The function $\sigma(t)$ is regular since $\tau > 0$. Therefore the electric field follows to be

$$E(t) = \frac{2\sqrt{2}\omega^3 E_0}{[\omega^2 + \tau + (\omega^2 - \tau)\cos^2(\omega t)]^{3/2}}, \tag{3.31}$$

where we have chosen the constants $c = E_0$ and $A = \sqrt{\tau}/\omega$ such that $E(0) = E_0$. We note that $\omega = \sqrt{\tau}$ is a special point at which $\sigma(t) \to 1$ and also the field becomes time-independent $E(t) \to E_0$.

Assembling everything we have completed the first step in the Lewis-Riesenfeld construction procedure. The invariant acquires the form

$$I(t) = \frac{\sigma^2}{2}p^2 + \frac{m^2}{2}\left(\dot{\sigma}^2 + \frac{\tau}{\sigma^2}\right)x^2 + m\sum\nolimits_p c_p\tilde{c}_p\left(\frac{x}{\sigma}\right)^p - \frac{1}{2}m\sigma\dot{\sigma}\{x,p\} + m\sigma^2 E(t)x, \tag{3.32}$$

with $\sigma(t)$ given by (3.30) and free constants τ, m, ω, c_p, $\tilde{c}_p$ and E_0.

The second step, that is to solve the eigenvalue equation (3.3), can not be carried out exactly for all invariants $I(t)$ of the form in (3.32). We therefore resort to a perturbative approach as outlined in the previous section.

3.3.2 Testing the semi-exact solutions

Exact computation

A good indication about the quality of the perturbation theory and the WKB approximation layed out above can be obtained by comparing both approximations to an exact expression. For most cases this is of course not possible, but taking in (3.18) the potential for instance to be $V(x,t) = \kappa(t)x^2$, $\kappa(t) = 2c_\kappa/\sigma^4$, we obtain an exactly solvable system that can serve as a benchmark. In this case the expression (3.32) for the invariant simply becomes

$$I(t) = \frac{1}{2}\left[\alpha p^2 + (2\beta + \varepsilon)x^2 + \delta\{x,p\} + \gamma x\right], \tag{3.33}$$

with α, β, γ, δ, ε as specified in (3.22). The eigenvalue equation is simplified further when eliminating the anticommutator term $\{x,p\}$ by means of a unitarity tranformation $U = \exp(i\delta x^2/2\alpha)$ and the subsequent introduction of the new variable $\xi := x/\sigma$. We compute

$$\hat{I} = UIU^{-1} = -\frac{1}{2}\partial_\xi^2 + \left(\frac{\tau}{2}m^2 + mc_\kappa\right)\xi^2 + mE_0\xi. \tag{3.34}$$

The eigenvalue equation for the transformed, and in this case time-independent, invariant $\hat{I}\chi(\xi) = \lambda\chi(\xi)$ is then solved by

$$\chi(\xi) = c_1 D_{\mu_+}\left[\sqrt{2}m^{1/4}\frac{(E_0 + 2c_\kappa\xi + m\tau\xi)}{(2c_\kappa + m\tau)^{3/4}}\right] + c_2 D_{\mu_-}\left[i\sqrt{2}m^{1/4}\frac{(E_0 + 2c_\kappa\xi + m\tau\xi)}{(2c_\kappa + m\tau)^{3/4}}\right], \tag{3.35}$$

where $\mu_\pm = \pm(E_0^2 m + 4c_\kappa\lambda)/\sqrt{m}(2c_\kappa + m\tau)^{3/2} - 1/2$ and $D_\nu(z)$ denotes the parabolic cylinder function, see (C.3), (C.20) for a relation to the more familiar Hermite polynomials. Demanding that the eigenfunctions vanish asymptotically, i.e. $\lim_{\xi\to\pm\infty} \chi(\xi) = 0$, imposes the constraint $\mu_\pm = n \in \mathbb{N}_0$ and thus quantizes the eigenvalues $\lambda \to \lambda_n$. We discard the solution related to D_{μ_-}, as its corresponding eigenvalues are not bounded from below. Hence, we are left with the eigenfuctions and eigenvalues

$$\chi_n(\xi) = c_1 D_n\left[\sqrt{2}m^{1/4}\frac{(E_0 + 2c_\kappa\xi + m\tau\xi)}{(2c_\kappa + m\tau)^{3/4}}\right], \tag{3.36}$$

$$\lambda_n = \left(n + \frac{1}{2}\right)\sqrt{2mc_\kappa + m^2\tau} - \frac{mE_0^2}{4c_\kappa + 2m\tau}. \tag{3.37}$$

The eigenvalues are indeed time-independent as we expect in the context of the Lewis-Riesenfeld approach (3.3). Assembling the above and using the orthonormality property of the parabolic cylinder function $\int_{-\infty}^{\infty} D_n(x)D_m(x)dx = n!\sqrt{2\pi}\delta_{nm}$, we obtain the normalized eigenfunction $\phi_n = U^{-1}\chi_n$

$$\phi_n(x) = N_n D_n\left[a + bx\right] e^{im\dot{\sigma}x^2/2\sigma}, \tag{3.38}$$

for the operator I in (3.33) with

$$a = \frac{\sqrt{2}m^{1/4}E_0}{(2c_\kappa + m\tau)^{3/4}}, \qquad b = \frac{\sqrt{2}m^{1/4}(2c_\kappa + m\tau)^{1/4}}{\sigma}. \tag{3.39}$$

$$N_n = \frac{m^{1/8}(2c_\kappa + m\tau)^{1/8}}{\sqrt{\sigma n!\sqrt{\pi}}}, \tag{3.40}$$

Finally we compute the phase $\alpha_n(t)$ in (3.4) by means of (3.5). The right hand side yields

$$\langle\phi_n|\, i\partial_t - H(t)\,|\phi_n\rangle = -\frac{\lambda_n}{m\sigma^2} \tag{3.41}$$

so that phase becomes

$$\alpha_n(t) = -\frac{1}{m\sqrt{\tau}}\lambda_n \arctan\left[\frac{\sqrt{\tau}\tan(\omega t)}{\omega}\right], \tag{3.42}$$

where we have made us of the integrals [2]. We notice that for $\omega \to \sqrt{\tau}$ this simply reduces to $\alpha_n(t) \to -\lambda_n/m$ and the Hamiltonian becomes time-independent, so that this choice simply describes the time-independent Schrödinger equation.

Perturbative computation

Next we treat the term $V_p(x,t) = \kappa(t)x^2$ in the Hamiltonian as a perturbation, so that we may view the system as being in the strong field approximation. Accordingly we split up the invariant (3.33) as $I(t) = I_0(t) + \epsilon I_p(t)$ with

$$I_0(t) = \frac{1}{2}\left[\alpha p^2 + \varepsilon x^2 + \delta\{x,p\} + \gamma x\right], \qquad I_p(t) = \frac{m}{\sigma^2}x^2, \tag{3.45}$$

and the small expansion parameter is identified as $\epsilon \equiv c_\kappa$. First we compute the correction to the eigenvalue of the invariant. Solving the eigenvalue equation (3.7) and computing the expectation values in (3.9) we obtain

$$\lambda_n^{(0)} = \left(n + \frac{1}{2}\right) m\sqrt{\tau} - \frac{E_0^2}{2\tau}, \quad \text{and} \quad \lambda_n^{(1)} = \frac{1}{\sqrt{\tau}}\left(n + \frac{1}{2}\right) + \frac{E_0^2}{m\tau^2}. \tag{3.46}$$

As we expect, $\lambda_n^{(0)} + c_\kappa \lambda_n^{(1)}$ is precisely λ_n in (3.37) expanded up to first order in c_κ. Next we use (3.9) to compute the corrections to the wavefunctions. There are only four terms contributing in the infinite sum. We compute

$$\begin{aligned}\left|\phi_n^{(1)}\right\rangle &= \frac{1}{4m\tau}\left[\sqrt{n(n-1)}\left|\phi_{n-2}^{(0)}\right\rangle - \sqrt{(n+1)(n+2)}\left|\phi_{n+2}^{(0)}\right\rangle\right] \\ &\quad + \frac{E_0\sqrt{2}}{m^{3/2}\tau^{7/4}}\left[\sqrt{n+1}\left|\phi_{n+1}^{(0)}\right\rangle - \sqrt{n}\left|\phi_{n-1}^{(0)}\right\rangle\right].\end{aligned} \tag{3.47}$$

Finally we evaluate the perturbed expression for the phase $\alpha_n^{(1)}(t)$ using equation

[2] We used here the integrals

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2n} D_{2s+\delta}(x) D_{2r+\bar{\delta}}(x)dx &= (-1)^s 2^{s+r-n+\frac{1+\delta-\bar{\delta}}{2})}\sqrt{\pi}\Gamma\left(\frac{1}{2}+s+\delta\right)\Gamma\left(2n+1+\delta\right) \\ &\quad \times {}_3\tilde{F}_2\left(-s, n+1, n+\frac{1}{2}+\delta; \frac{1}{2}+\delta, n-r+1+\frac{\delta-\bar{\delta}}{2}; 1\right)\end{aligned} \tag{3.43}$$

for $n, s, r \in \mathbb{N}_0$ and $(\delta, \bar{\delta}) = (0,1), (0,0), (1,1)$. The function ${}_3\tilde{F}_2\left(a,b,c;d,f;z\right)$ is the regularized hypergeometric function defined as

$${}_3\tilde{F}_2\left(a,b,c;d,f;z\right) = \frac{1}{\Gamma\left(d\right)\Gamma\left(f\right)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{(d)_k(f)_k}\frac{z^k}{k!}, \tag{3.44}$$

with $(a)_k = \Gamma\left(a+k\right)/\Gamma\left(a\right)$ denoting the Pochhammer symbol. A derivation of this integral can be found in Appendix C.

(3.5). Up to first order we find

$$\alpha_n^{(1)}(t) = -\frac{\lambda_n^{(0)} + c_\kappa \lambda_n^{(1)}}{m\sqrt{\tau}} \arctan\left[\frac{\sqrt{\tau}\tan(\omega t)}{\omega}\right]. \tag{3.48}$$

Notice that for $\omega \to \sqrt{\tau}$ this simply reduces to $\alpha_n^{(1)}(t) = -t(\lambda_n^{(0)} + c_\kappa \lambda_n^{(1)})/m$. We have now obtained the full perturbative solution to the TDSE as $|\psi_n\rangle^{(1)}$ as defined in (3.12).

WKB computation

We start by determining the classical turning points $\xi_\pm$ from the condition $\lambda = V(\xi)$. We find

$$\xi_\pm = -\frac{E_0 m \pm \sqrt{m}\sqrt{mE_0^2 + 2\lambda(m\tau + 2c_\kappa)}}{m(m\tau + 2c_\kappa)}, \tag{3.49}$$

so that the WKB quantisation condition

$$\int_{\xi_-}^{\xi_+} \sqrt{2(\lambda_n - V(\xi))}d\xi = \pi\hbar\left(n + \frac{1}{2}\right), \tag{3.50}$$

yields the *exact* time-independent eigenvalues

$$\lambda_n = \left(n + \frac{1}{2}\right)\hbar\sqrt{m}(2c_\kappa + m\tau)^{1/2} - \frac{mE_0^2}{2(2c_\kappa + m\tau)}, \tag{3.51}$$

as found above in (3.37). Next we specify WKB wavefunction further. Keeping in the classically forbidden regions $\xi \in (-\infty, \xi_-)$ and $\xi \in (\xi_+, \infty)$ only the asymptotically decaying parts in (3.15) and (3.16), the corresponding WKB wavefunction are

$$\hat{\phi}_-(\xi) = \frac{C_3(-1)^n}{\sqrt{q(\xi)}} \exp\left[-\frac{1}{\hbar}\int_{\xi}^{\xi_-} q(z)dz\right], \tag{3.52}$$

and

$$\hat{\phi}_+(\xi) = \frac{C_3}{\sqrt{q(\xi)}} \exp\left[-\frac{1}{\hbar}\int_{\xi_+}^{\xi} q(z)dz\right], \tag{3.53}$$

respectively. At this point C_3 is the only undetermined constant left. Carrying out the appropriate WKB matching we obtain for the classically allowed region $\xi \in (\xi_-, \xi_+)$ the wavefunction

$$\hat{\phi}_b(\xi) = \frac{2C_3(-1)^n}{\sqrt{p(\xi)}} \cos\left[\frac{1}{\hbar}\int_{\xi_-}^{\xi} p(z)dz - \frac{\pi}{4}\right]. \tag{3.54}$$

We may compute these expressions by using the explicit expressions for the functions $q(\xi)$ and $p(\xi)$. To do so we use the same abbreviated constants a and b as defined in (3.39), that convert the potential, eigenvalues and turning points into more compact forms

$$\begin{aligned} V(\xi) &= \frac{b^4}{8}\xi^2 + a\frac{b^3}{4}\xi, \quad \lambda_n = \frac{1}{2}b^2\left(n+\frac{1}{2}\right)\hbar - \frac{a^2b^2}{8}, \\ \xi_\pm &= \frac{\pm\sqrt{2}\sqrt{2b^2 n\hbar + b^2\hbar} - ab}{b^2}. \end{aligned} \tag{3.55}$$

After a lengthy computation we obtain the WKB wavefunction in the different regions as

$$\begin{aligned} \hat{\phi}_\pm(\xi) &= \frac{F_n C_3 (\pm 1)^n \left[a + b\xi + \sqrt{q(\xi)}\right]^{\pm\left(n+\frac{1}{2}\right)} e^{\mp\frac{1}{4}\sqrt{q(\xi)}(a+b\xi)}}{\sqrt[4]{b^2 q(\xi)}}, \\ \hat{\phi}_b(\xi) &= \frac{C_3(-1)^n \cos\left[\left(n+\frac{1}{2}\right)\arctan\left(\frac{a+b\xi}{\sqrt{p(\xi)}}\right) + \frac{1}{4}\sqrt{p(\xi)}(a+b\xi) + \frac{\pi n}{2}\right]}{\sqrt[4]{b^2 p(\xi)}}, \end{aligned} \tag{3.56}$$

where

$$F_n = 2^{\mp\frac{n}{2}+\frac{1}{2}\mp\frac{1}{4}}\left[\pm\frac{\sqrt{b^2(2n+1)}}{b}\right]^{\mp\left(n+\frac{1}{2}\right)}. \tag{3.57}$$

The last remaining constant C_3 may be fixed by the normalization condition. Converting from the $\hat{I}$ eigenvalue equation back to the I eigenvalue equation with $\phi = U^{-1}\hat{\phi}$ and the variable ξ to x/σ, the normalisation condition amounts to

$$\int_{-\infty}^{x_-} \phi_-^*(x)\phi_-(x)dx + \int_{x_-}^{x_+} \phi_b^*(x)\phi_b(x)dx + \int_{x_+}^{\infty} \phi_+^*(x)\phi_+(x)dx = 1. \tag{3.58}$$

Evaluating the integrals in (3.58) using numerical integration in Mathematica, we find the n independent constant

$$C_3 \approx \frac{b}{2\sqrt{\pi\sigma(t)}}. \tag{3.59}$$

Having found the WKB eigenfunction ϕ^{WKB}, we can now compute the integrant in (3.14) that yields the WKB approximated Lewis-Riesenfeld phase

$$\alpha_n^{\text{WKB}}(t) \approx -\frac{1}{m\sqrt{\tau}}\lambda_n \arctan\left[\frac{\sqrt{\tau}\tan(\omega t)}{\omega}\right]. \tag{3.60}$$

Thus we have now obtained a WKB approximated solution ψ_n^{WKB} to the time-dependent Schrödinger equation as specified in (3.14). Let us now compare these three solutions.

WKB versus pertubation theory versus exact solution

In order to obtain an idea about the quality of these approximations let us compute some physical quantities in an exact and perturbative manner and subequently compare them. For the exact case we find the expectation values for the momentum, position and their squares as

$$\langle\psi_n| x |\psi_n\rangle = -\frac{E_0\sigma}{2c_\kappa + m\tau}, \quad \langle\psi_n| p |\psi_n\rangle = -\frac{mE_0\dot{\sigma}}{2c_\kappa + m\tau}, \tag{3.61}$$

$$\langle\psi_n| x^2 |\psi_n\rangle = \frac{E_0^2\sigma^2}{(2c_\kappa + m\tau)^2} + \frac{(2n+1)\sigma^2}{2\sqrt{m}\sqrt{2c_\kappa + m\tau}}, \tag{3.62}$$

$$\langle\psi_n| p^2 |\psi_n\rangle = \frac{m^2E_0^2\dot{\sigma}^2}{(2c_\kappa + m\tau)^2} + \frac{(2n+1)m^{1/2}}{2\sigma^2\sqrt{2c_\kappa + m\tau}}\left(2c_\kappa + m\tau + m\sigma^2\dot{\sigma}^2\right) \tag{3.63}$$

such that the uncertainty relation becomes

$$\Delta x\Delta p = (n + \frac{1}{2})\sqrt{1 + \frac{m(\tau - \omega^2)^2\sin^2(2\omega t)}{4\omega^2(2c_\kappa + m\tau)}}, \tag{3.64}$$

where as usual the squared uncertainty is defined as the squared standard deviation $\Delta A^2 := \langle\psi_n| A^2 |\psi_n\rangle - \langle\psi_n| A |\psi_n\rangle^2$ for $A = x, p$. Since the square root is always greater or equal to 1 as $c_\kappa, m, \tau > 0$, the bound in the uncertainty relation $\Delta x\Delta p \geq 1/2$ is always respected.

From the perturbed solution $|\psi_n\rangle^{(1)}$ we find

$$\langle\psi_n| x |\psi_n\rangle^{(1)} = -\frac{E_0\sigma}{m\tau} + c_k\frac{2E_0\sigma}{m^2\tau^2}, \tag{3.65}$$

$$\langle\psi_n| p |\psi_n\rangle^{(1)} = -\frac{E_0\dot{\sigma}}{\tau} + c_k\frac{2E_0\dot{\sigma}}{m\tau^2}, \tag{3.66}$$

$$\langle\psi_n| x^2 |\psi_n\rangle^{(1)} = \frac{2E_0^2 + (2n+1)m\tau^{3/2}}{2m^2\tau^2}\sigma^2 - c_k\frac{8E_0^2 + (2n+1)m\tau^{3/2}}{2m^3\tau^3}\sigma^2, \tag{3.67}$$

$$\langle\psi_n| p^2 |\psi_n\rangle^{(1)} = \frac{E_0^2\dot{\sigma}^2}{\tau^2} + \frac{m\sqrt{\tau}(2n+1)}{2\sigma^2} + \frac{m(2n+1)\dot{\sigma}^2}{2\sqrt{\tau}} \times c_k\left(\frac{n+1/2}{\sqrt{\tau}\sigma^2} - \frac{4E_0^2\dot{\sigma}^2}{m\tau^3} - \frac{(2n+1)\dot{\sigma}^2}{2\tau^{3/2}}\right), \tag{3.68}$$

and therefore

$$\Delta x \Delta p^{(1)} = \left(n + \frac{1}{2}\right)\left[\sqrt{1 + \frac{\sigma^2 \dot{\sigma}^2}{\tau}} - c_k \left(\frac{\sigma^2 \dot{\sigma}^2}{m\tau^{3/2}\sqrt{\tau + \sigma^2 \dot{\sigma}^2}}\right)\right]. \tag{3.69}$$

These expressions coincide with the exact expressions expanded up to order one in c_k. In figure 3.1 we compare the time-dependent expectation values for x, x^2, p, p^2 computed in an exact way with those computed in a perturbative fashion. In general the agreement is very good for small values of c_k. Overall the agreement is increasing for large values of n as well as m and for ω approaching $\sqrt{\tau}$.

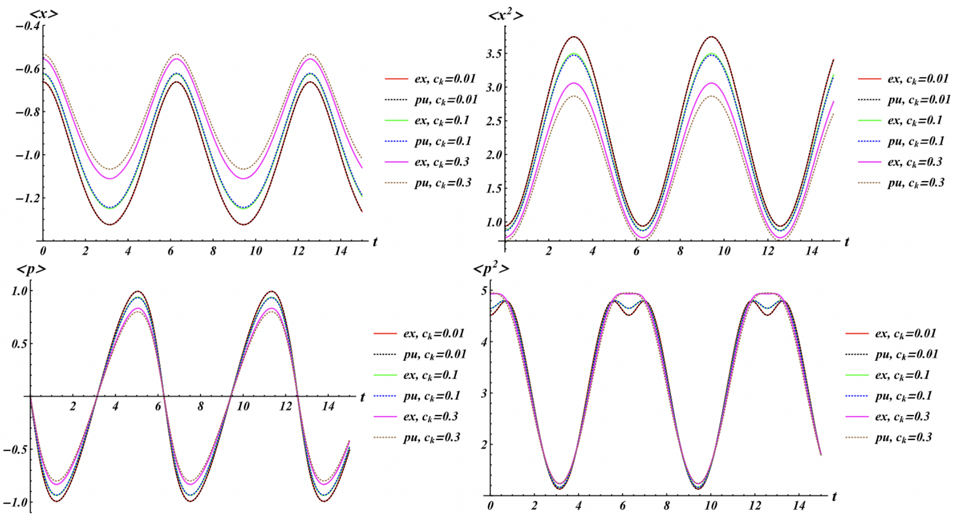

Figure 3.1: Exact versus perturbative expectation values for x, x^2, p, p^2 for $E_0 = 2$, $\omega = 1/2$, $\tau = 1$, $m = 3$ and $n = 1$ for different values of the expansion parameter c_k.

A further useful quantity to compute that illustrates the quality of the perturbative approach is the autocorrelation function

$$A_n(t) := |\langle \psi_n(t) \, |\psi_n(0)\rangle| \,. \tag{3.70}$$

Unlike the expectation values for position, momenta and their squares the autocorrelation function also captures the influence of the time-dependent phase $\alpha(t)$. We depict this function in figure 3.2. In this case the overall agreement decreases for larger values of n.

Next we compare directly the wave functions obtained three alternative ways. Figure 3.3 shows an extremely good agreement between the WKB approximation and the exact solution, except near the turning points $\xi_\pm$ where the WKB approximation is singular. The pertubative solution is in very good agreement with the

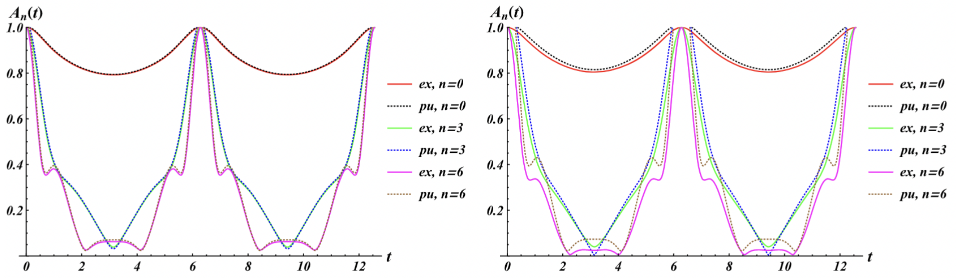

Figure 3.2: Exact versus perturbative autocorrelation function for $E_0 = 2$, $\omega = 1/2$, $\tau = 1$, $m = 3$, different values for n with $c_k = 0.1$ in the left panel and $c_k = 0.3$ in the right panel.

exact solution for small values of c_κ, as expected. With increasing values of c_κ the perturbative solution starts to deviate stronger in the negative regime for ξ and large values of n.

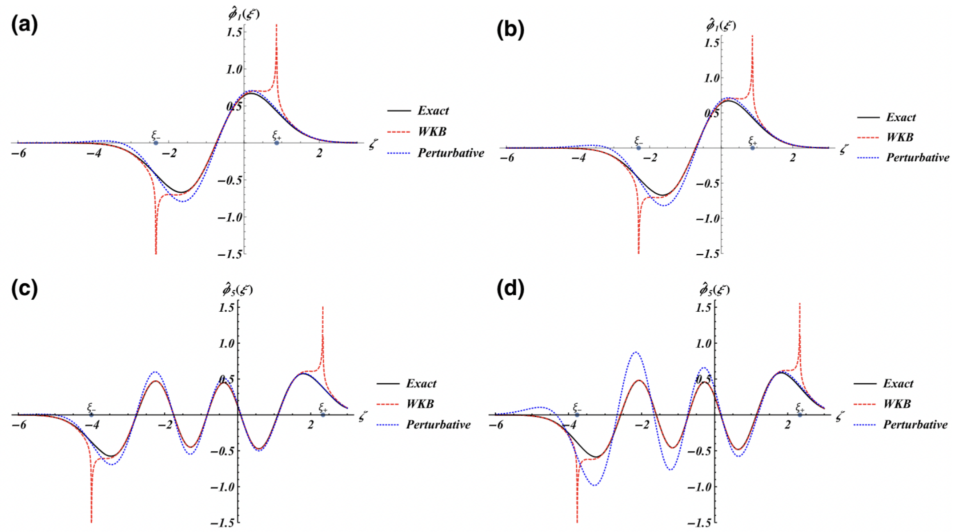

Figure 3.3: Exact versus WKB and perturbative solutions to the time-independent eigenvalue equation (3.3) for the invariant $\hat{I}$ with $\hbar = 1$, $E_0 = 1$, $m = 1$, $n = 1$, $\tau = 1$ and $c_\kappa = 0.18$ in panels (a), (c), $c_\kappa = 0.2$ in panels (b), (d). $\xi_\pm$ are the classical turning points (3.49).

Let us next see how these properties are inherited in the time-dependent system. Figure 3.4 displays the real part of the full time-dependent wave function. We observe the oscillation of the turning points with time that enter through the function $\sigma(t)$. As in the time-independent case, extremely good agreement between the WKB approximation and the exact solution, except near the turning points $x_\pm$. The perturbative solution slightly overshoots at the maxima and minima, especially in the negative time regime. The discrepancy becomes worse for larger values of n, which we do not show here.

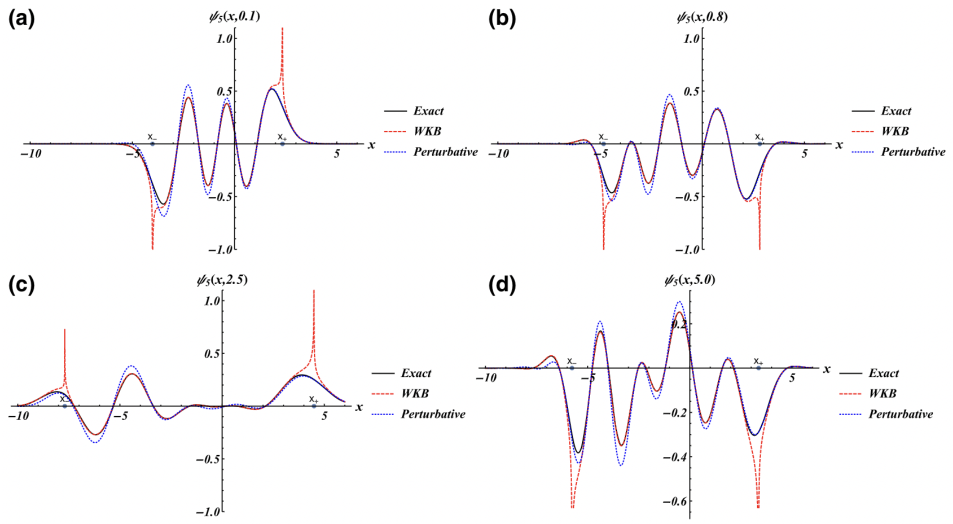

Figure 3.4: Exact versus WKB and perturbative solutions to the time-dependent Schrödinger equation (3.1) at different times with $\hbar = 1$, $E_0 = 1$, $m = 1$, $n = 5$, $\tau = 1$, $\omega = 0.5$ and $c_\kappa = 0.1$. The time-dependent classical turning points $\xi_\pm$ are indicated.

3.4 Goldman-Krivchenko potential with time-dependent perturbation

In trying to identify solvable systems we have seen in equation (3.25) that the value $p = -2$ is special as in that case the potential becomes the time-independent Goldman-Krivchenko potential [142], being a particular spiked harmonic oscillator [144, 145]. This potential may serve also as a benchmark for which we can solve the eigenvalue equation exactly and campare it to the perturbative solution. Hence we take this potential as our unperturbed system and perturb it by dropping the Stark term and replacing it by $x^2 E(t)$. Thus we consider the time-dependent Hamiltonian

$$H(t) = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \frac{m\Omega^2}{2}\frac{1}{x^2} + x^2 E(t). \tag{3.71}$$

It follows from above that the invariant for this system is

$$I(t) = \frac{1}{2}\left[\sigma^2 p^2 - m\sigma\dot{\sigma}\{x,p\} + \left(m^2\dot{\sigma}^2 + \frac{\tau m^2 + 2mE_0}{\sigma^2}\right)x^2 + m^2\sigma^2\Omega^2\frac{1}{x^2}\right], \tag{3.72}$$

with constraint $E(t) = E_0/\sigma^4$ and σ satisfying the EP-equation (3.23). Also in this case we may complete the remaining steps in the Lewis-Riesenfeld approach and hence compare the exact and the perturbative solution. We identify $E_0 \ll 1$ as the expansion parameter in the perturbative series.

3.4.1 Testing the approximate solution

Exact computation

Using the same similarity and variable transformation as in (3.34) we obtain the time-independent invariant

$$\hat{I} = UIU^{-1} = -\frac{1}{2}\partial_\xi^2 + \frac{1}{2}\left(\tau m^2 + 2mE_0\right)\xi^2 + \frac{m^2\Omega^2}{2}\frac{1}{\xi^2}, \tag{3.73}$$

where $U = \exp(i\delta x^2/2\alpha)$ and we have changed variables through $\xi := x/\sigma$. We solve the eigenvalue equation $\hat{I}\Xi(\xi) = \lambda\Xi(\xi)$ exactly obtaining the solution

$$\Xi(\xi) = \xi^{(1+b/2)}e^{-\frac{a}{2}\xi^2}\left[c_1 L_{\nu_-}^{b/2}\left(a\xi^2\right) + c_2 U\left(\nu_+, 1+b/2, \frac{m\sqrt{\tau}\xi^2}{\hbar}\right)\right], \tag{3.74}$$

where $L_\nu^\mu(z)$ denotes the generalized Laguerre polynomials, $U(\nu,\mu,z)$ the confluent hypergeometric function [146] and $\nu_\pm := \pm\left(2+b-2\lambda/a\right)/4$, $a = \sqrt{\tau m^2 + 2mE_0}$, $b := \sqrt{1+4m^2\Omega^2}$. Demanding again that the eigenfunctions vanish asymptotically, i.e. $\lim_{\zeta\to\pm\infty}\Xi(\xi) = 0$, imposes $\nu_\pm = n \in \mathbb{N}_0$ and thus quantizes λ. We discard the solution related to U, as its corresponding eigenvalues are not bounded from below, leading to the eigenfuctions and eigenvalues

$$\Xi_n(\xi) = c_1\xi^{(1+b/2)}e^{-\frac{a}{2}\xi^2}L_n^{b/2}\left(a\xi^2\right), \qquad \lambda_n = a(2n+1+b/2). \tag{3.75}$$

Assembling everything we obtain for the operator $I(t)$ the normalized eigenfunction from $U^{-1}\Xi_n(x/\sigma)$ as

$$\phi_n(x) = F_n\left(\frac{a}{\sigma^2}\right)^{(2+b)/4} x^{(1+b)/2}e^{-\frac{a}{2\sigma^2}x^2}L_n^{b/2}\left(\frac{ax^2}{\sigma^2}\right)e^{im\dot{\sigma}x^2/2\sigma\hbar}. \tag{3.76}$$

where

$$F_n = \sqrt{\frac{2n!}{[1-(-1)^b]\Gamma\left(1+n+b/2\right)}}, \tag{3.77}$$

when $\mathbf{Re}(b) > -2$ and $\mathbf{Re}(a/\sigma^2) > 0$. This completes the second step in the Lewis-Riesenfeld approach. In the third and last step we determine the phase α by means of (3.5). The right hand side is computed once more to

$$\langle\phi_n|\, i\partial_t - H(t)\,|\phi_n\rangle = -\frac{\lambda_n}{m\sigma^2}, \tag{3.78}$$

so that the phase acquires the same form as in the previous example

$$\alpha_n(t) = -\frac{1}{m\sqrt{\tau}}\lambda_n \arctan\left[\frac{\sqrt{\tau}\tan(\omega t)}{\omega}\right]. \tag{3.79}$$

Let us now compare these expressions with those obtained in the perturbative computation.

Perturbative computation

We treat the term $V_p(x,t) = x^2 E(t)$ with $E(t) = E_0/\sigma^4$ and $E_0 \ll 1$ in the Hamiltonian as a perturbation. Accordingly we split up the invariant (3.33) as $I(t) = I_0(t) + E_0 I_p(t)$ with

$$I_0(t) = \frac{1}{2}\left[\sigma^2 p^2 - m\sigma\dot{\sigma}\{x,p\} + \left(m^2\dot{\sigma}^2 + \frac{\tau m^2}{\sigma^2}\right)x^2 + m^2\sigma^2\Omega^2\frac{1}{x^2}\right], \tag{3.80}$$

$$I_p(t) = \frac{m}{\sigma^2}x^2. \tag{3.81}$$

The zeroth order wavefunction $\left|\phi_n^{(0)}\right\rangle$ is simply $\phi_n(x)$ in (3.76) with $E_0 = 0$. From (3.7) and (3.9) we compute first two terms in the perturbative series for the eigenvalues

$$\lambda_n^{(0)} = \left(2n + 1 + \sqrt{1 + 4m^2\Omega^2}/2\right) m\sqrt{\tau}, \tag{3.82}$$

$$\lambda_n^{(1)} = \left\langle \phi_n^{(0)}\right| I_p(t) \left|\phi_n^{(0)}\right\rangle = (2n+1)\sqrt{\frac{1+4m^2\Omega^2}{\tau}}. \tag{3.83}$$

As expected the eigenvalues are time-independent and $\lambda_n^{(0)} + E_0\lambda_n^{(1)}$ corresponds to (3.75) expanded to first order in E_0. Next we need to compute the infinite sum in (3.9) to determine the corrections to the wavefunctions. In this case there are only two terms contributing in the infinite sum. We compute

$$\left|\phi_n^{(1)}\right\rangle = \frac{1}{2m\tau}\left[\sqrt{(n+1)(n+1+b/2)}\left|\phi_{n+1}^{(0)}\right\rangle - \sqrt{n(n+b/2)}\left|\phi_{n-1}^{(0)}\right\rangle\right]. \tag{3.84}$$

In the last step we compute the perturbed expression for the phase $\alpha_n^{(1)}(t)$ using equation (3.5). Once more we find up to first order

$$\alpha_n^{(1)}(t) = -\frac{\lambda_n^{(0)} + c_\kappa\lambda_n^{(1)}}{m\sqrt{\tau}} \arctan\left[\frac{\sqrt{\tau}\tan(\omega t)}{\omega}\right], \tag{3.85}$$

so that we have obtained the full perturbative solution to the TDSE as $|\psi_n\rangle^{(1)}$ as defined in (3.12).

Exact versus perturbative solutions

As previously, we compute several physical quantities to compare the exact and the perturbative solution. The momentum, position, squared momentum and squared position expectation values are computed to

$$\langle\psi_n|\, x\,|\psi_n\rangle = 0,\ \ \langle\psi_n|\, x^2\,|\psi_n\rangle = \frac{(2n+\ell+3/2)\sigma^2}{\sqrt{\tau m^2+2mE_0}},\ \ \langle\psi_n|\, p\,|\psi_n\rangle = 0, \tag{3.86}$$

$$\langle\psi_n|\, p^2\,|\psi_n\rangle = \frac{(4n+2)\ell+2n+3/2}{2\ell+1}\frac{\sqrt{\tau m^2+2mE_0}}{\sigma^2} + \frac{(2n+\ell+3/2)m\dot{\sigma}^2}{\sqrt{\tau m^2+2mE_0}}.$$

In order to achieve convergence we had to impose the additional constraint $b = 2\ell+1$ with $\ell \in \mathbb{N}_0$, this ensures that the integrals involved in the determination of the above expectation values are real and finite. The uncertainty relation becomes

$$\Delta x\Delta p = \sqrt{\frac{(2n+\ell+3/2)[(4n+2)\ell+2n+3/2]}{2\ell+1} + \frac{m(2n+\ell+3/2)^2\sigma^2\dot{\sigma}^2}{(\tau m+2E_0)^2}}, \tag{3.87}$$

with the lower bound $\Delta x\Delta p \geq 1/2$ always well respected.

Using the perturbed solutions (3.84) and (3.85) we compute

$$\langle\psi_n|\, x\,|\psi_n\rangle^{(1)} = 0,\quad \langle\psi_n|\, x^2\,|\psi_n\rangle^{(1)} = (2n+\ell+3/2)(m\tau - E_0)\tfrac{\sigma^2}{\tau m^2},$$

$$\langle\psi_n|\, p\,|\psi_n\rangle^{(1)} = 0,\quad \langle\psi_n|\, p^2\,|\psi_n\rangle^{(1)} = \tfrac{(4n+2)\ell+2n+3/2}{2\ell+1}\tfrac{(m\tau+E_0)}{\sqrt{\tau}\sigma^2} + \tfrac{(2n+\ell+3/2)(\tau m-E_0)\dot{\sigma}^2}{\tau^{3/2}}. \tag{3.88}$$

The approximated uncertainty relation results to

$$\Delta x\Delta p^{(1)} = \sqrt{\frac{(2n+\ell+3/2)[(4n+2)\ell+2n+3/2]}{2\ell+1}(1-\frac{E_0^2}{m^2\tau^2})}. \tag{3.89}$$

We are now in the position to compare the exact and the pertubative solution. In figure 3.5 we compare the time-dependent expectation values for x^2 and p^2 computed in an exact way with those computed in a perturbative fashion. As in the previous example, the agreement is very good for small values of the expansion parameter, E_0 in this case. Overall the agreement is increasing for large values of n as well as m and for ω approaching $\sqrt{\tau}$.

As in the previous example we also compute the autocorrelation function (3.70) as it captures well the effect from the time-dependent phase $\alpha(t)$. We depict this

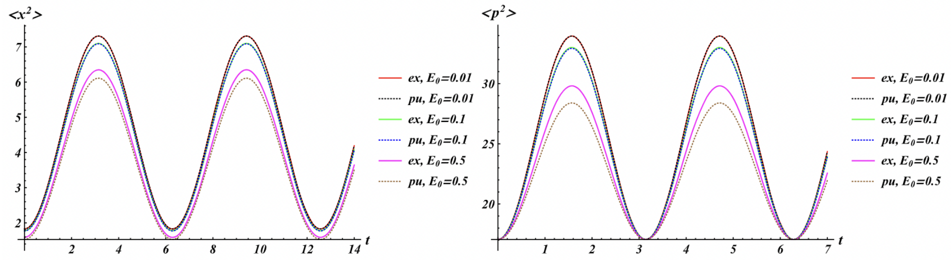

Figure 3.5: Exact versus perturbative expectation values for x^2, p^2, for $\omega = 1/2$, $\tau = 1$, $\ell = 2$, $m = 3$ and $n = 1$ for different values of the expansion parameter E_0.

function in figure 3.6. Once more, the overall agreement decreases for larger values of n.

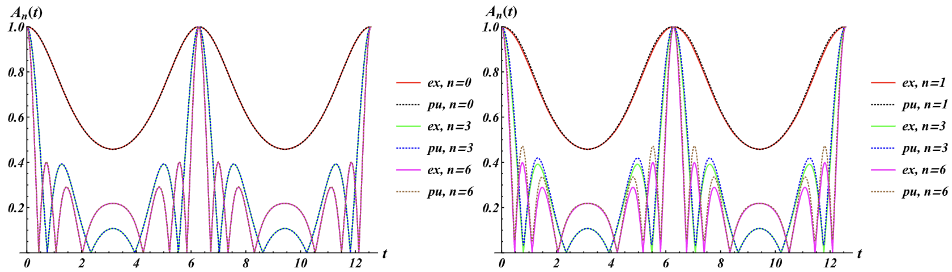

Figure 3.6: Exact versus perturbative autocorrelation function for $\omega = 1/2$, $\tau = 1$, $m = 3$, $\ell = 2$ and different values for n with $E_0 = 0.1$ in the left panel and $E_0 = 0.5$ in the right panel.

3.5 Conclusions

We have explored the possibilty of a modified approximated Lewis and Riesenfeld method by solving the time-independent eigenvalue equation in the second step by means of standard time-independent perturbation theory. We have tested the quality of this approach for two classes of optical potentials by comparing the exact solutions obtained from the completely exact solution of the Lewis-Riesenfeld approach to the approximated ones, the perturbative approach and the WKB approximation. We computed some standard expectation values and the autocorrelation functions in two alternative ways. For the pertubative approach we found in general good agreement which is naturally improved in quality for smaller values of the expansion parameters. The WKB approximation is not limited to these small parameters and only deviates significantly at the turning points.

Our semi-exactly solvable approach significantly widens the scope of the Lewis-Riesenfeld method and allows to tackle more complicated physical situations that are not possible to treat when insisting on full exact solvabilty. The validity of either approach is governed by the validity of the time-independent pertubation theory and the WKB approximation for which explicit expressions can be found in the standard literature, which then need to be adjusted to the particular potentials.

Chapter 4

Time-dependent unstable anharmonic quartic oscillator

In this chapter we construct a time-dependent double well potential as an exact spectral equivalent to the explicitly time-dependent negative quartic oscillator with a time-dependent mass term as first shown in [4]. For completeness and to establish a benchmark for comparison we shall first discuss the time-independent unstable anharmonic quartic oscillator and the techniques within non-Hermitian quantum mechanics that were used to determine the time-independent Dyson map. We shall then proceed with its time-dependent counterpart. Defining the unstable anharmonic oscillator Hamiltonian on a contour in the lower-half complex plane, the resulting time-dependent non-Hermitian Hamiltonian is first mapped by an exact solution of the time-dependent Dyson equation to a time-dependent Hermitian Hamiltonian defined on the real axis. When unitary transformed, scaled and Fourier transformed we obtain a time-dependent double well potential bounded from below. All transformations are carried out non-perturbatively so that all Hamiltonians in this process are spectrally exactly equivalent in the sense that they have identical instantaneous energy eigenvalue spectra.

4.1 Anharmonic Oscillators

Anharmonic oscillators have a wide range of applications in quantum mechanics as they describe for instance delocalization and decoherence of quantum states, e.g. [147]. They also occur naturally in relativistic models, e.g. [148]. From a mathematical point of view their nonlinear nature make them ideal testing grounds for various approximation methods, such as perturbative approaches [149]. Based on a perturbative expansion of the energy eigenvalues it was shown in [42] that the quartic anharmonic oscillator with mass term is spectrally equivalent to a double well potential with linear symmetry breaking. The first hint about the fact that even the unstable quartic anharmonic oscillator posses a well defined bounded real spectrum, despite being unbounded from below on the real axis, was proved in [150, 151], where it was proven that its energy eigenvalues series is Borel summable. The spectral equivalence between an unstable anharmonic oscillator and a complex double well potential was then proven directly by Buslaev and Grecchi in [152].

4.2 Time-independent unstable anharmonic oscillator

The time-independent unstable anharmonic quartic oscillator given by

$$H = p_x^2 - gx^4, \quad g \in \mathbb{R}, \tag{4.1}$$

where $p_x = -i\partial_x$, was first shown numerically to have a real and positive spectra by Bender and Boettcher [1] as part of a treatment of a general series of $\mathcal{PT}$-symmetric potentials given by $x^2(ix)^\varepsilon$, which were all shown to have a real spectra provided $\varepsilon \geq 0$. Jones and Mateo [74] later showed that the Hamiltonian (4.1) was spectrally equivalent to

$$h = \frac{p_x^4}{64g} - \frac{1}{2}p_x + 16gx^2. \tag{4.2}$$

In order to do this, techniques which have been developed within the area of non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics [140, 141] had to be used. We shall present a brief overview of two of the techniques in the following subsections.

4.2.1 Stokes wedges and choice of contour

The first of these techniques is associated with the fact that the Schrödinger eigenvalue problem has to be defined by sectors within complex plane known as Stokes wedges [137]. Within these wedges the wave function, $\psi(x)$, will vanish exponentially as $|x| \to \infty$. Along the centre of these wedges, known as the anti-Stokes line $\psi(x)$ will decay most rapidly, whereas on the Stokes lines which bound the wedges $\psi(x)$ is oscillatory. The locations of the Stokes wedges for different $\mathcal{PT}$-symmetric potentials and in particular the $-x^4$ potential is the main concern of this subsection.

$\mathcal{PT}$-symmetric Hamiltonians such as

$$\hat{H} = \hat{p}^2 + \hat{z}^2(i\hat{z})^\varepsilon \qquad \text{for} \qquad \varepsilon \geq 0, \tag{4.3}$$

can be considered as a complex deformation of the harmonic oscillator, where we treat z as complex. The associated Sturm Liouville eigenvalue problem with this Hamiltonian is written as

$$-\psi''(z) + z^2(iz)^\varepsilon \psi(x) = E\psi(z), \tag{4.4}$$

where we have taken $\hat{z} \to z$, $\hat{p} \to -i\frac{d}{dz}$. To determine the location of the Stokes wedges for this eigenvalue problem we must employ the WKB approximation such that we can determine the asymptotic behaviour of $\psi(z)$ for large $|z|$ [137]. A more detailed discussion of this can be found in [141], here we just present the location of the Stokes wedges as depicted in figure 4.1 for different values of ε. The locations of the Stokes lines which bound the wedges are

$$\theta_{upper,right} = \frac{\pi(2-\varepsilon)}{8+2\varepsilon} \qquad \text{and} \qquad \theta_{lower,right} = -\frac{\pi(2+\varepsilon)}{8+2\epsilon}, \tag{4.5}$$

for the right wedge and

$$\theta_{upper,left} = -\pi - \frac{\pi(2-\varepsilon)}{8+2\varepsilon} \qquad \text{and} \qquad \theta_{lower,left} = -\pi + -\frac{\pi(2+\varepsilon)}{8+2\epsilon}, \tag{4.6}$$

for the left wedge. The width of the Stokes wedges, Δ, can also be determined

$$\Delta = \theta_{upper,right} - \theta_{lover,right} = \theta_{upper,left} - \theta_{lower,left} = \frac{2\pi}{4+\varepsilon}. \tag{4.7}$$

In figure 4.1 we see that as $\varepsilon \to \infty$ the width of the wedges, $\Delta \to 0$. The wedges themselves also approach the negative imaginary axis. For the normal harmonic oscillator, which corresponds to $\varepsilon = 0$, the center of the wedges are the real line. We also see that for $\varepsilon > 2$ the wedges no longer contain the real line and that for $\varepsilon = 2$, which corresponds to the negative quartic potential, the real axis is the upper stokes line for the wedges. In order to deal with the negative quartic potential we therefore have to map the wedge back onto the real axis with the choice of a suitable

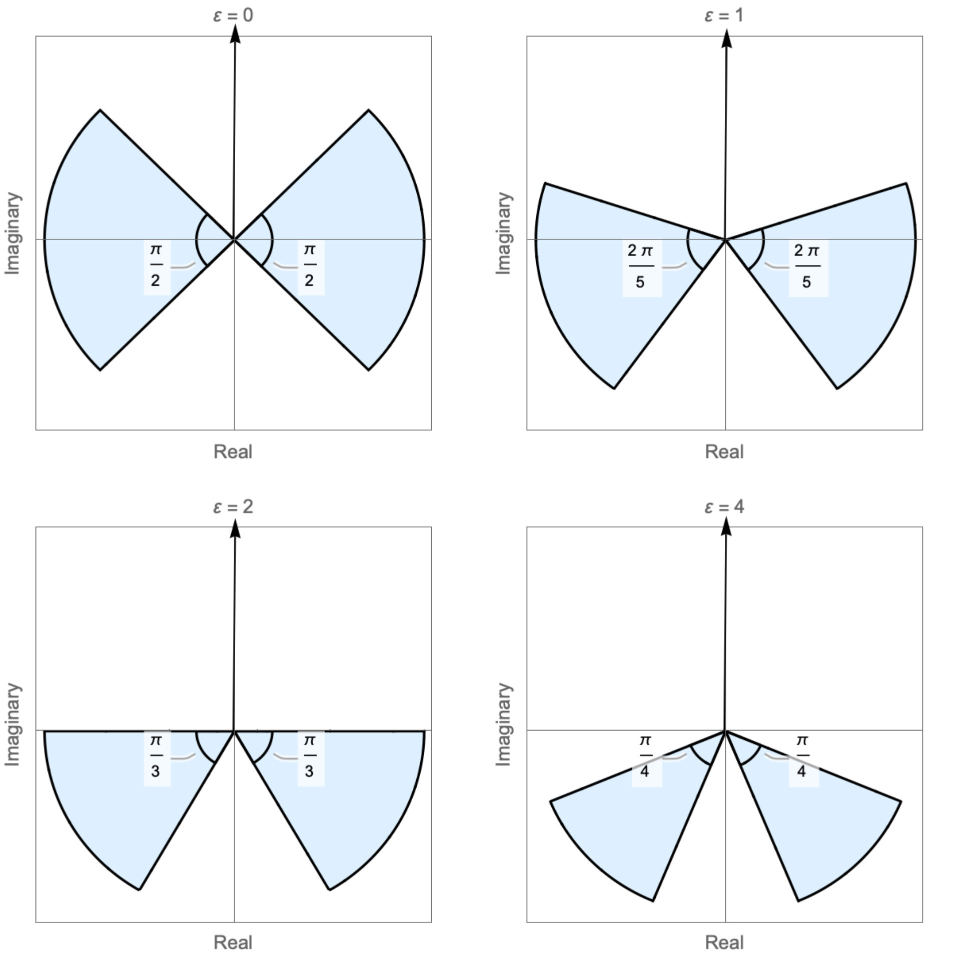

Figure 4.1: Stokes sector for the Sturm Liouville eigenvalue problem (4.4) for $\varepsilon = 0, 1, 2, 4$. The angular opening of each sector is marked and the bold black arrow is a logarithmic branch cut on the positive imaginary axis from $x = 0$ to $x = i\infty$.

parametrisation [153]. Jones and Mateo [74] found a suitable contour given by

$$z = -2i\sqrt{1 + ix}, \tag{4.8}$$

which mapped (4.1) to the new Hamiltonian

$$H = p^2 - \frac{1}{2}p + a(x^2 - 1) - 2iax + \frac{1}{2}i\left\{x, p^2\right\}, \tag{4.9}$$

where $a = 16g$. We see here that by defining the negative quartic potential on the correct contour within the complex plane we have taken a Hermitian Hamiltonian and revealed that it is manifestly non-Hermitian.

4.2.2 Perturbation theory

Another important technique employed by Jones and Mateo [74] to calculate the

Dyson map for the negative quartic potential after it had been mapped the non-Hermitian Hamiltonian given by (4.9) is perturbation theory for determining the time-independent metric and Dyson map [27, 78, 154]. We shall recall here the perturbative method.

We start by separating the non-Hermitian Hamiltonian into its real and imaginary part as

$$H = h_0 + i\epsilon h_1, \qquad \text{with } h_0^\dagger = h_0, \quad h_1^\dagger = h_1, \tag{4.10}$$

where a real parameter ϵ has been extracted from the imaginary part. Assuming here for simplicity that the Dyson map is Hermitian and of the form $\eta = e^{q/2}$, the metric operator just becomes $\rho = \eta^\dagger \eta = \eta^2 = e^q$. Making use of the Baker-Campbell-Hausdorff formula (2.15) one can then write the similarity transformation in equation (2.14) as

$$H^\dagger = \eta^2 H \eta^{-2} = H + [q, H] + \frac{1}{2!}[q,[q,H]] + \frac{1}{3!}[q,[q,[q,H]]] + ... \tag{4.11}$$

Using the decomposition (4.10) for the non-Hermitian Hamiltonian H this becomes

$$i[q, h_0] + \frac{i}{2}[q,[q,h_0]] + \frac{i}{3!}[q,[q,[q,h_0]]] + ... = \epsilon\left(2h_1 + [q,h_1] + \frac{1}{2}[q,[q,h_1]] + ...\right). \tag{4.12}$$

Expanding q further as a power series in ϵ as

$$q = \sum_{n=1}^{\infty} \epsilon^n q_n, \tag{4.13}$$

one can read off the coefficients of ϵ^n order by order upon substituting (4.13) into (4.12). One finds that $[h_0, q_2] = 0$, so that with the choice $q_2 = 0$ all even powers in (4.13) vanish. The first three nonvanishing equations are

$$[h_0, q_1] = 2ih_1, \tag{4.14}$$

$$[h_0, q_3] = \frac{i}{6}[q_1,[q_1,h_1]], \tag{4.15}$$

$$[h_0, q_5] = \frac{i}{6}\left([q_1,[q_3,h_1]] + [q_3,[q_1,h_1]] - \frac{1}{60}[q_1,[q_1,[q_1,[q_1,h_1]]]]\right). \tag{4.16}$$

Crucially, these equations provide a constructive scheme and can be solved recursively order by order for q_1, q_2, ...

Utilising this scheme Jones and Mateo [74] were able to construct a Dyson map for the Hamiltonian (4.9) which mapped it to the Hermitian Hamiltonian (4.2). This Dyson map is given by

$$\eta = \exp\left[-\frac{p^3}{3a} + 2p\right]. \tag{4.17}$$

We wish to now extend this analysis to the time-dependent regime. Whilst in this chapter we will not be using a perturbative approach to determine the time-dependent Dyson map, in chapter 5 a time-dependent counterpart to the perturbation theory in this section is laid out and subsequently applied to the time-dependent unstable anharmonic oscillator.

4.3 Time-dependent unstable anharmonic oscillator

The Hamiltonian we investigate here is similar to the one in equation (4.1), but with time-dependent coefficient functions and an additional mass term

$$H(z,t) = p^2 + \frac{m(t)}{4}z^2 - \frac{g(t)}{16}z^4, \qquad m \in \mathbb{R}, \;\; g \in \mathbb{R}^+. \tag{4.18}$$

Defining $H(z,t)$ now on the same contour in the lower-half complex plane $z = -2i\sqrt{1+ix}$ as suggested by Jones and Mateo [74], it is mapped into the non-Hermitian Hamiltonian

$$H(x,t) = p^2 - \frac{1}{2}p + \frac{i}{2}\{x,p^2\} - m(t)(1+ix) + g(t)(x-i)^2, \tag{4.19}$$

with $\{\cdot,\cdot\}$ denoting the anti-commutator. Next we attempt to solve the time-dependent Dyson equation (2.3) to find a Hermitian counterpart h.

4.3.1 Dyson map

We start by making following general Ansatz for the Dyson map

$$\eta(t) = e^{\alpha(t)x} e^{\beta(t)p^3 + i\gamma(t)p^2 + i\delta(t)p}, \qquad \alpha,\beta,\gamma,\delta \in \mathbb{R}, \tag{4.20}$$

we use the Baker-Campbell-Hausdorff formula (2.15) to compute the adjoint action of $\eta(t)$ on all terms appearing in $H(x,t)$

$$\eta x \eta^{-1} = x + \delta + 6\alpha\beta p + 2\gamma p + 3i\alpha^2\beta + 2i\alpha\gamma - 3i\beta p^2 \tag{4.21}$$

$$\begin{aligned}
\eta p\eta^{-1} &= p + i\alpha, && (4.22)\\
\eta x^2\eta^{-1} &= x^2 - 9\beta^2 p^4 - 12i\beta\,(3\alpha\beta+\gamma)\,p^3 && (4.23)\\
&\quad + \left(54\alpha^2\beta^2 + 36\alpha\beta\gamma + 4\gamma^2 - 6i\beta\delta\right)p^2 \\
&\quad +4(3\alpha\beta+\gamma)\left[\delta + i\alpha(3\alpha\beta+2\gamma)\right]p + 2\left(\delta + 3i\alpha^2\beta + 2i\alpha\gamma\right)x \\
&\quad +(6\alpha\beta+2\gamma)\{x,p\} - 3i\beta\left\{x,p^2\right\} - (3\alpha^2\beta + 2\alpha\gamma - i\delta)^2, \\
\eta p^2\eta^{-1} &= p^2 - \alpha^2 + 2i\alpha p, && (4.24)\\
\eta\{x,p^2\}\eta^{-1} &= \{x,p^2\} - 6i\beta p^4 + (24\alpha\beta+4\gamma)p^3 + \left(36i\alpha^2\beta + 12i\alpha\gamma + 2\delta\right)p^2 && (4.25)\\
&\quad +4\left(i\alpha\delta - 6\alpha^3\beta - 3\alpha^2\gamma\right)p - 2i\alpha^2\left(3\alpha^2\beta + 2\alpha\gamma - i\delta\right) \\
&\quad -2\alpha^2 x + 4i\alpha\{x,p\}.
\end{aligned}$$

The gauge like terms in (2.3) and (2.8) are calculated to

$$\begin{aligned}
i\dot\eta\eta^{-1} &= ix\dot\alpha + i\dot\beta p^3 - \left(3\dot\beta\alpha + \dot\gamma\right)p^2 - \left(3i\dot\beta\alpha^2 + 2i\dot\gamma\alpha + \dot\delta\right)p && (4.26)\\
&\quad +\dot\beta\alpha^3 + \dot\gamma\alpha^2 - i\dot\delta\alpha \\
i\eta^{-1}\dot\eta &= ix\dot\alpha + i\dot\beta p^3 - (3\dot\alpha\beta + \dot\gamma)p^2 - (2i\gamma\dot\alpha + \dot\delta)p - i\delta\dot\alpha, && (4.27)
\end{aligned}$$

where as commonly used we abbreviate partial derivatives with respect to t by an overdot. Using the expressions in (4.21)-(4.26) for the evaluation of (2.3) and demanding the right hand side to be Hermitian yields the following constraints for the coefficient functions in the Dyson map

$$\alpha = \frac{\dot g}{6g}, \quad \beta = \frac{1}{6g}, \quad \gamma = \frac{12g^3 + 6mg^2 + \dot g^2 - g\ddot g}{4\dot g g^2}, \quad \delta = c_1\frac{g}{\dot g} - \frac{g\ln g}{2\dot g}, \qquad (4.28)$$

with $c_1 \in \mathbb{R}$ being an integration constant. Moreover, the time-dependent coefficient functions in the Hamiltonian (4.18) must be related by the third order differential equation

$$9g^2\left(\dddot g - 6g\dot m\right) + 36g\dot g\,(gm - \ddot g) + 28\dot g^3 = 0. \qquad (4.29)$$

Integrating once and introducing a new parameterisation function $\sigma(t)$, we solve this equation by

$$g = \frac{1}{4\sigma^3}, \qquad \text{and} \qquad m = \frac{4c_2 + \dot\sigma^2 - 2\sigma\ddot\sigma}{4\sigma^2}, \qquad (4.30)$$

with $c_2 \in \mathbb{R}$ denoting the integration constant corresponding to the only integration we have carried out. The time-dependent Hermitian Hamiltonian in equation (2.3) then results to

$$h(x,t) = \sigma^3 p^4 + f_{pp}(t)p^2 + f_x(t)x + f_p(t)p + f_{xp}(t)\{x,p\} + f_{xx}(t)x^2 + C(t). \quad (4.31)$$

with

$$\begin{aligned}
f_{pp} &= \frac{\sigma\left\{\sigma\left[2\left(\sigma\left(\dot{\sigma}^2-4c_2\right)-2\right)\ddot{\sigma}+16c_2^2+\dot{\sigma}^4\right]+16c_2\right\}+4}{4\sigma\dot{\sigma}^2}, \\
f_p &= \frac{2c_1\left[\sigma\left(4c_2+\dot{\sigma}^2-2\sigma\ddot{\sigma}\right)+2\right]+\ln\left(4\sigma^3\right)}{12\sigma\dot{\sigma}^2}, \quad f_x = -\frac{2c_1+\ln\left(4\sigma^3\right)}{12\sigma^2\dot{\sigma}}, \\
f_{xp} &= \frac{\left(\sigma\left(\dot{\sigma}^2-4c_2\right)-2\right)}{4\sigma^2\dot{\sigma}}, \quad f_{xx} = \frac{1}{4\sigma^3}, \\
C &= \frac{\left(2c_1+\ln\left(4\sigma^3\right)\right)^2+36\dot{\sigma}^2\left(4c_2^2+\ddot{\sigma}\right)}{144\sigma\dot{\sigma}^2}+\frac{1}{8}\left(\dot{\sigma}^2-4c_2\right)\ddot{\sigma}-\frac{\dot{\sigma}^2}{4\sigma^2}.
\end{aligned}$$

We may choose to set $c_1 = c_2 = 0$ and reintroduce the original time-dependent coefficient functions $g(t)$, $m(t)$ so that the Hamiltonian simplifies to

$$\begin{aligned}
h(x,t) &= \frac{p^4}{4g}+\left(\frac{18g^2(2g+m)}{\dot{g}^2}+\frac{\dot{g}^2}{72g^3}-\frac{2g+m}{4g}\right)p^2-\frac{3\left(g^2m+g^3\right)\ln g}{\dot{g}^2}p \\
&\quad +\frac{g^2\ln(g)}{\dot{g}}x + \left(\frac{\dot{g}}{12g}-\frac{6g^2}{\dot{g}}\right)\{x,p\}+gx^2 \\
&\quad +\frac{1296g^8\ln^2 g+\dot{g}^6-36\dot{g}^4g^2(2g+m)}{5184g^5\dot{g}^2}-\frac{m}{2}. \qquad (4.32)
\end{aligned}$$

Notice that $\sigma(t)$ can be any function, but the coefficient functions $g(t)$ and $m(t)$ must be related by (4.29) that is (4.30).

The massless case for $m(t) = 0$ is more restrictive and leads to $\sigma(t)$ being a second order polynomial $\sigma(t) = \kappa_0 + \kappa_1 t + \kappa_2 t^2$ with real constants κ_i. This case is consistently recovered from (4.30) with the choice $c_2 = \kappa_1\kappa_3 - \kappa_2^2/4$. The solution found for the time-independent case in [74], would be obtained from (4.20) in the limits $\alpha \to 0$, $\beta \to 1/6g$, $\gamma \to 0$, $\delta \to i$ and $m \to 0$. While this limit obviously exists for α and β, the constraints for γ and δ are different from those reported in (4.28). In fact, setting $\delta(t) \to i\delta(t)$ enforces g to be time-independent and there is no time-dependent solution corresponding to that choice. The energy operator $\tilde{H}$ defined in (2.8) is obtained directly by adding $H(x,t)$ in (4.19) and the gauge-like term in (4.27).

4.3.2 Time-dependent double wells potential

Let us now eliminate the terms in $h(x,t)$ proportionate to x and $\{x,p\}$ by means of a unitary transformation

$$U = e^{-i\frac{f_{xp}}{2f_{xx}}p^2 - i\frac{f_x}{2f_{xx}}p}, \tag{4.33}$$

which leads to the unitary transformed Hamiltonian

$$\hat{h}(x,t) = \sigma^3 p^4 + \left(f_{pp} - \frac{f_{xp}^2}{f_{xx}}\right)p^2 + \left(f_p - \frac{f_x f_{xp}}{f_{xx}}\right)p + f_{xx}x^2 + C - \frac{f_x^2}{4f_{xx}}. \tag{4.34}$$

Similarly as in the time-independent case [74], we may scale this Hamiltonian, albeit now with a time-dependent function, $x \to (f_{xx})^{-1/2}x$. Subsequently we Fourier transform $\hat{h}(x,t)$ so that it is viewed in momentum space. In this way we obtain a spectrally equivalent Hamiltonian with a time-dependent potential

$$\begin{aligned} \tilde{h}(y,t) &= p_y^2 + \sigma^3 f_{xx}^2 y^4 + \left(f_{xx}f_{pp} - f_{xp}^2\right)y^2 + \left(\sqrt{f_{xx}}f_p - \frac{f_x f_{xp}}{\sqrt{f_{xx}}}\right)y \qquad (4.35) \\ &\quad + C - \frac{f_x^2}{4f_{xx}}, \\ &= \frac{g}{4}y^2\left(y^2 + \frac{\dot{g}^2}{36g^3} + \frac{72g^2m}{\dot{g}^2} - \frac{m}{g} + 2\right) + \frac{\left(36g^2m + \dot{g}^2\right)\sqrt{g}\ln g}{12\dot{g}^2}y \qquad (4.36) \\ &\quad + \frac{\dot{g}^4}{5184g^5} - \frac{\dot{g}^2m}{144g^3} - \frac{\dot{g}^2}{72g^2} - \frac{m}{2}, \end{aligned}$$

where for simplicity we have set $c_1 = c_2 = 0$ in (4.36). The potential in $\tilde{h}(y,t)$ is a double well that is bounded from below. We illustrate this for a specific choice of $\sigma(t)$, that is $g(t)$ and $m(t)$, in figure 4.1.

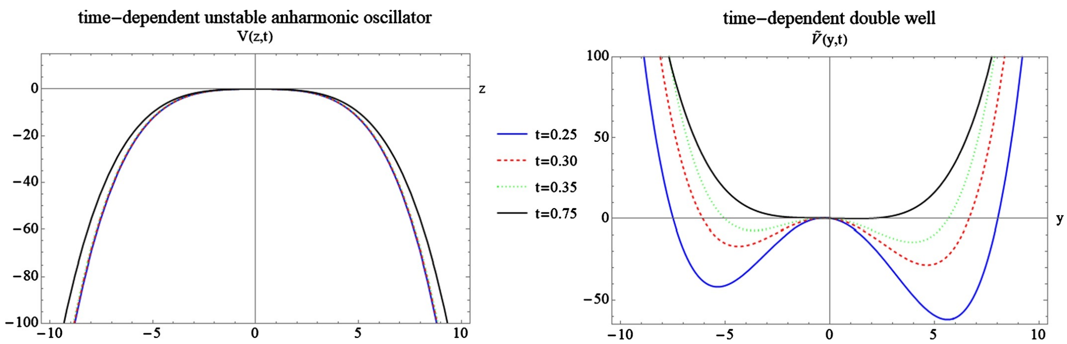

Figure 4.2: Spectrally equivalent time-dependent anharmonic oscillator potential $V(z,t)$ in (4.18) and time-dependent double well potential $\tilde{V}(y,t)$ in (4.36) for $\sigma(t) = \cosh t$, $g(t) = 1/4\cosh^3 t$, $m(t) = (\tanh^2 t - 2)/4$ at different values of time.

4.4 Conclusions

We have proven the remarkable fact that the time-dependent unstable anharmonic oscillator is spectrally equivalent to a time-dependent double well potential that is bounded from below. The transformations we carried out are summarized as follows:

$$H(z,t) \overset{z\to x}{\to} H(x,t) \overset{\text{Dyson}}{\to} h(x,t) \overset{\text{unitary transform}}{\to} \hat{h}(x,t) \overset{\text{Fourier}}{\to} \tilde{h}(y,t).$$

We have first transformed the time-dependent anharmonic oscillator $H(z,t)$ from a complex contour in a Stokes wedge to the real axis $H(x,t)$. The resulting non-Hermitian Hamiltonian $H(x,t)$ was then mapped by means of a time-dependent Dyson map $\eta(t)$ to a time-dependent Hermitian Hamiltonian $h(x,t)$. It turned out that the Dyson map can not be obtained by simply introducing time-dependence into the known solution for the time-independent case [74], but it required to complexify one of the constants and the inclusion of two additional factors. In order to obtain a potential Hamiltonian we have unitarily transformed $h(x,t)$ into a spectrally equivalent Hamiltonian $\hat{h}(x,t)$, which when Fourier transformed leads to $\tilde{h}(y,t)$ that involved a time-dependent double well potential.

In the following three chapters we shall be revisiting this model in detail. We will first compute the Lewis-Riesenfeld invariants by constructing a point transformation between the time-independent quartic oscillator and the time-dependent quartic oscillator, in doing so we obtain a second solution for the Dyson map. We will then apply time-dependent perturbation theory to determine its metric. We show that as in the time-independent case, a perturbative approach leads to an exact solution for the Dyson map. Finally we use the two Dyson maps to determine a symmetry operator for the Lewis-Riesenfeld invariants which allows for the subsequent computation of an infinite series of Dyson maps.

Chapter 5

Point transformations and exactly solvable time-dependent non-Hermitian quantum systems

In this chapter we demonstrate that complex point transformations can be used to construct non-Hermitian first integrals, time-dependent Dyson maps and metric operators for non-Hermitian quantum systems. Initially we identify a point transformation as a map from an exactly solvable time-independent system to an explicitly time-dependent non-Hermitian Hamiltonian system. Subsequently we employ the point transformation to construct the non-Hermitian time-dependent invariant for the latter system. Exploiting the fact that this invariant is pseudo-Hermitian, we construct a corresponding Dyson map as the adjoint action from a non-Hermitian to a Hermitian invariant, thus obtaining solutions to the time-dependent Dyson and time-dependent quasi-Hermiticity equation together with solutions to the corresponding time-dependent Schrödinger equation.

5.1 Introduction

One of the most convenient approaches to constructing the time-dependent Dyson map for a non-Hermitian system involves the construction of time-dependent invariants [89]. As detailed in chapter 2, and argued in [51, 87, 88, 132], this is because one has transformed the equation that needs to be solved from a time-dependent differential equation, the TDDE (2.3), to a simpler similarity transformation between a non-Hermitian and Hermitian invariant (see Appendix A for more details). While the difficulty of the problem has been reduced, the number of steps required to obtain a Dyson map has been increased. In addition to this it is not always straightforward to construct an invariant for a particular system. One usually makes an ansatz for the form of the invariant which is not guaranteed to be correct. Having a more concrete approach to constructing invariants would therefore be beneficial for constructing Dyson maps within time-dependent non-Hermitian quantum mechanics as well as in the standard Hermitian regime.

To aid in the construction of time-dependent invariants we propose using time-dependent canonical transformations known as point transformations [155]. The initial use of point transformations was in classical mechanics, this was extended to the quantum regime by DeWitt [156, 157] who utilised point transformations to settle the ambiguity problem of operator ordering. Point transformations have also been employed to construct maps between a simple exactly solvable model and a more complicated system, known as the quantum Arnold transformation [158], which was then subsequently applied to the Caldirola-Kanai oscillator [159, 160]. The main selling point of the approach, however, is that point transformations preserve conserved quantities [155]. This was exploited by Zelaya and Rosas-Ortiz [161] who recently demonstrated that point transformations may be used to compute time-dependent invariants or first integrals for Hermitian Hamiltonian systems. Extending this to the non-Hermitian regime would therefore reduce the difficulty associated with determining a Dyson map as an ansatz for the non-Hermitian invariant would no longer be required. In the subsequent sections we shall demonstrate that this can be achieved for several non-Hermitian systems including the time-dependent Swanson Hamiltonian [162], the time-dependent harmonic oscillator with complex linear term and the non-Hermitian unstable anharmonic quartic oscillator.

5.2 Invariants and Dyson maps from point transformations

The main purpose of this section is to present an alternative approach to finding $\rho(t)$ and $\eta(t)$ by exploiting *point transformations* and first integrals. Consider a non-Hermitian explicitly time-dependent Hamiltonian $H(x,t) \neq H^{\dagger}(x,t)$ satisfying the TDSE

$$H(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t). \tag{5.1}$$

As a starting point one assumes that there exists an exactly solvable time-independent reference Hamiltonian $H_0(\chi)$ satisfying the TDSE

$$H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau), \tag{5.2}$$

with χ denoting the coordinate and τ the time in this system. One may then relate (5.2) to the first TDSE (5.1) by means of a complex point transformation Γ defined as

$$\Gamma : H_0\text{-TDSE} \to H\text{-TDSE};\ [\chi,\tau,\psi(\chi,\tau)] \mapsto [x,t,\phi(x,t)]\,. \tag{5.3}$$

Here ψ and ϕ are understood to be implicit functions of χ,τ and x,t, respectively, defined by the equations (5.1) and (5.2). The variables χ, τ, ψ are treated in general as

$$\chi = P(x,t,\phi), \qquad \tau = Q(x,t,\phi), \qquad \psi = R(x,t,\phi), \tag{5.4}$$

where P, Q, R are functions of the independent variables x, t, ϕ. In practice, one may relax some of the (x,t,ϕ)-dependences of the functions P, Q, R or is even forced to do so for concrete systems.

Having identified the point transformation Γ on the level of the TDSEs one may subsequently apply it exclusively to the time-independent Hamiltonian $H_0(\chi)$ as

$$\Gamma : H_0(\chi) \to I_H(x,t). \tag{5.5}$$

Since real point transformations preserve conserved quantities [155], and $I_H(x,t)$ acquired a time-dependence via the point transformation Γ, it is suggestive to assume that also complex point transformations have this property and that $I_H(x,t)$ is actually the time-dependent conserved Lewis-Riesenfeld [89] invariant for the non-Hermitian time-dependent Hamiltonian $H(x,t)$ in (5.1) satisfying

$$i\hbar\frac{dI_H}{dt} = i\hbar\partial_t I_H + [I_H, H] = 0. \tag{5.6}$$

Since H is non-Hermitian, also its first integral, the invariant I_H, must be non-Hermitian, which is evident from (5.6).

As argued in [51, 87, 88, 132] one may map this non-Hermitian invariant I_H to a Hermitian invariant I_h by means of a time-dependent similarity transformation $\eta(t)$ as

$$\eta(t)I_H(t)\eta^{-1}(t) = I_h(t). \tag{5.7}$$

Remarkably the map $\eta(t)$ is indeed the Dyson map solving the first equation in (2.3) and the Hermitian operator I_h is the Lewis-Riesenfeld invariant for the Hermitian time-dependent Hamiltonian $h(t)$, identified in (2.3), satisfying

$$i\hbar\frac{dI_h}{dt} = i\hbar\partial_t I_h + [I_h, h] = 0. \tag{5.8}$$

In summary, we have a four step method that leads not only to the solutions $\phi(x,t)$ of the TDSE (5.1), but also an explicit expression for the metric operator. The first step consists of selecting a suitable time-independent reference Hamiltonian $H_0(\chi)$ and point transform its corresponding TDSE (5.2). In the second step we fix the free parameters by matching the transformed TDSE with a TDSE for a non-Hermitian target Hamiltonian $H(t)$ (5.1), hence identifying the point transformation Γ by means of (5.3). In the third step we obtain the invariant $I_H(t)$ by acting with Γ on the time-independent reference Hamiltonian $H_0(\chi)$ and in the fourth step we construct the Dyson map η as a similarity transformation by means of (5.7). In case the TDSE for $H_0(\chi)$ is solvable we obtain by construction also the solutions to the original TDSE for $H(x,t)$. Here our main focus is on the construction of η and ρ. Let us now demonstrate how this four step strategy is carried out for a concrete time-dependent non-Hermitian Hamiltonian.

5.3 Point transforming exactly solvable reference Hamiltonians

One of the simplest choices for an exactly solvable reference Hamiltonian $H_0(\chi)$ one can make is to take the time-independent Hermitian harmonic oscillator

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \qquad m, \omega \in \mathbb{R}. \tag{5.9}$$

First we identify the point transformation of $H_0(\chi)$ in general terms. Expressing the momentum operator P in the position representation $P = -i\hbar\partial_\chi$, we act with the point transformation Γ on the TDSE (5.2). Simplifying the general functional dependence as stated in (5.4) to

$$\chi = \chi(x,t), \qquad \tau = \tau(t), \qquad \psi = A(x,t)\phi(x,t), \tag{5.10}$$

we convert all partial derivatives in the TDSE from the (χ, τ) to the (x,t)-variables obtaining the point transformed differential equation

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0, \tag{5.11}$$

with

$$B_0(x,t) = -i\hbar\frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(2\frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x}\right), \tag{5.12}$$

$$V_0(x,t) = \frac{1}{2}m\tau_t\chi^2\omega^2 - i\hbar\left(\frac{A_t}{A} - \frac{A_x\chi_t}{A\chi_x}\right) - \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(\frac{A_{xx}}{A} - \frac{A_x\chi_{xx}}{A\chi_x}\right). \tag{5.13}$$

This form of equation (5.11) was previously derived in [161], more details of this computation can be found in Appendix D. However, we allow for a major difference by admitting the potential V_0 of the target Hamiltonian to be complex. The first two assumptions in (5.10) on the functional dependence when compared to the most general dependence $\chi(x,t,\phi)$, $\tau(x,t,\phi)$ are made for convenience to simplify the calculation. The last factorization property of ψ in (5.10) is already using an assumption made on the structure of the target differential equation. Since the TDSE is a linear equation in the fields it does not contain a ϕ_x^2 term so that $\psi_{\phi\phi} = 0$. Hence the linear dependence in ϕ.

Since the reference Hamiltonian is a choice, we shall explore here some further simple options

$$H_0^{(1)}(\chi) = \frac{P^2}{2m} \tag{5.14}$$

$$H_0^{(2)}(\chi) = H_0(\chi) + a\chi, \qquad a \in \mathbb{R}, \tag{5.15}$$

$$H_0^{(3)}(\chi) = H_0(\chi) + ib\chi, \qquad b \in \mathbb{R}, \tag{5.16}$$

$$H_0^{(4)}(\chi) = H_0(\chi) + a\{\chi, P\}. \tag{5.17}$$

We note that the reference Hamiltonian does not have to be Hermitian. Then the general point transformed differential equation (5.11) associated with each these reference Hamiltonians remains the same, yet the explicit forms of $B_0(x,t)$ (5.12) and $V_0(x,t)$ (5.13) differ. For the choices (5.14)-(5.17) we obtain

$$B_1(x,t) = B_0(x,t), \qquad V_1(x,t) = V_0(x,t) - \frac{1}{2}m\omega^2\chi^2\tau_t, \tag{5.18}$$

$$B_2(x,t) = B_0(x,t), \qquad V_2(x,t) = V_0(x,t) + a\chi\tau_t, \tag{5.19}$$

$$B_3(x,t) = B_0(x,t), \qquad V_3(x,t) = V_0(x,t) + ib\chi\tau_t, \tag{5.20}$$

$$B_4(x,t) = B_0(x,t) + \frac{2ia\hbar\chi\tau_t}{\chi_x}, \qquad V_4(x,t) = V_0(x,t) - \frac{2ia\chi\hbar A_x\tau_t}{A\chi_x} - ia\hbar\tau_t. \tag{5.21}$$

In order to proceed to the second step in the procedure we need to select a target Hamiltonian.

5.4 The time-dependent Swanson model as the target Hamiltonian

As a concrete example for a target Hamiltonian we consider here a prototype non-Hermitian Hamiltonian system, the time-dependent version of the Swanson Hamiltonian [162]. In its standard formulation in terms of bosonic creation a and annihilation operators $a^\dagger$, the time-dependent version may be written in the form

$$\tilde{H}_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \tilde{\alpha}(t)a^2 + \tilde{\beta}(t)\left(a^\dagger\right)^2, \quad \omega(t), \tilde{\alpha}(t), \tilde{\beta}(t) \in \mathbb{R} \tag{5.22}$$

which is clearly non-Hermitian when $\tilde{\alpha} \neq \tilde{\beta}^*$. Dyson maps for the time-independent and time-dependent version were found in [90] and [86], respectively. In order to apply the point transformations it is more convenient to convert the Hamiltonian into coordinate and momentum variables x, p, which is easily achieved. Using the standard representations $a = (x + ip)/2$ and $a^\dagger = (x - ip)/2$ we obtain

$$\begin{aligned}\tilde{H}_S(t) = &\frac{1}{2}\left[\omega(t) - \tilde{\alpha}(t) - \tilde{\beta}(t)\right]p^2 + \frac{1}{2}\left[\omega(t) + \tilde{\alpha}(t) + \tilde{\beta}(t)\right]x^2 \\ &+ \frac{i}{2}\left[\tilde{\alpha}(t) - \tilde{\beta}(t)\right]\{x,p\} + \frac{\omega(t)}{2}.\end{aligned} \tag{5.23}$$

Expressing the time-dependent functions $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$, $\omega(t)$ in terms of new time-dependent functions $\alpha(t)$, $\Omega(t)$ and $M(t)$ as

$$\tilde{\alpha}(t) = \frac{M(t)\Omega(t)^2}{4} - \frac{1}{4M(t)} + \alpha(t), \quad \tilde{\beta}(t) = \frac{M(t)\Omega(t)^2}{4} - \frac{1}{4M(t)} - \alpha(t),$$
$$\omega(t) = \frac{M(t)\Omega(t)^2}{2} + \frac{1}{2M(t)}, \tag{5.24}$$

the Hamiltonian is converted into the simpler form

$$H_S(x,t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2}\Omega(t)^2x^2 + i\alpha(t)\{x,p\}, \tag{5.25}$$

with $M, \Omega \in \mathbb{R}$, $\alpha \in \mathbb{C}$, which is clearly still non-Hermitian for $\alpha \neq 0$. The Swanson Hamiltonian is $\mathcal{PT}$-symmetric for $\mathcal{PT}: x \to -x$, $p \to p$, $i \to -i$ and all time-dependent coefficient functions transforming as $\mathcal{PT}: M, \Omega, \alpha \to M, \Omega, \alpha$. Taking $\alpha = \alpha_R - i\alpha_I$[1] which is complex, this requires $\mathcal{PT}: \alpha_R \to \alpha_R$, $\alpha_I \to -\alpha_I$. We notice here that the option $\alpha \in \mathbb{C}$, rather than $\alpha \in \mathbb{R}$, does not exist in the time-independent case when one wishes to maintain the $\mathcal{PT}$-symmetry of the Hamiltonian.

We will explore here two versions of this target Hamiltonian, in one we keep the mass time-independent by setting the time-dependent coefficient in the kinetic energy term to a constant, $M(t) \to m$, and in the other option we take the mass term to be generically time-dependent [119, 163]. Let us now identify the point transformation Γ according to (5.3) for the specified pairs of Hamiltonians.

5.4.1 Point transformation $\Gamma_0^S : H_0(\chi) \to H_S(x,t)$, time-independent mass

Having specified the target Hamiltonian as $H_S(x,t)$ with m, we express the time-dependent Schrödinger equation (5.2) in the position representation as

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}m\Omega(t)x^2\phi = 0. \tag{5.26}$$

With $H_0(\chi)$ as reference Hamiltonian, the direct comparison with (5.11) leads to the three constraints

$$\frac{\tau_t}{\chi_x^2} = 1, \quad B_0(x,t) = -2\hbar\alpha(t)x, \quad V_0(x,t) = \frac{1}{2}m\Omega(t)x^2 + \hbar\alpha(t). \tag{5.27}$$

[1]We have corrected here a minor typo in the manuscript [5] from $\alpha = \alpha_R + i\alpha_I \to \alpha = \alpha_R - i\alpha_I$.

Apart from being a complex equation, the first constraint in (5.27) is the same as the one found in [161], where it was solved by

$$\tau(t) = \int^t \frac{ds}{\sigma^2(s)}, \qquad \text{and} \qquad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)}, \tag{5.28}$$

with now complex functions $\gamma(t)$ and $\sigma(t)$. Using these expressions in the second constraint in (5.27) yields the equation

$$i\frac{\hbar}{m}\frac{A_x}{A} + \gamma_t + 2i\alpha x - (x+\gamma)\frac{\sigma_t}{\sigma} = 0, \tag{5.29}$$

which may be solved by

$$A(x,t) = \exp\left\{\frac{im}{\hbar}\left[\left(\gamma_t - \gamma\frac{\sigma_t}{\sigma}\right)tx + \left(it\alpha - \frac{\sigma_t}{2\sigma}\right)x^2 + \delta(t)\right]\right\}, \tag{5.30}$$

where $\delta(t)$ is a complex valued function corresponding to the integration constant in the x integration. Proceeding with these expressions to the third constraint in (5.27) yields

$$\begin{aligned} &-i\hbar\frac{\sigma_t}{2\sigma} - \frac{m}{2}\left(2\gamma\gamma_t\frac{\sigma_t}{\sigma} + \gamma_t^2 + \gamma^2\frac{\sigma_t^2}{\sigma^2} - \frac{\omega^2\gamma^2}{\sigma^4} - 2m\delta_t\right) \\ &+\frac{m\gamma}{\sigma}\left[\sigma_{tt} - \frac{\gamma_{tt}}{\gamma}\sigma - \frac{\omega^2}{\sigma^3}\right]x + \frac{m}{2\sigma}\left[\sigma_{tt} - \left(2i\alpha_t - 4\alpha^2 - \Omega\right)\sigma - \frac{\omega^2}{\sigma^3}\right]x^2 = 0. \end{aligned} \tag{5.31}$$

The x-independent term in (5.31) vanishes for

$$\delta(t) = \frac{\gamma}{2\sigma}(\sigma\gamma_t - \gamma\sigma_t) - \frac{i\hbar}{2m}\log\sigma \tag{5.32}$$

Furthermore, we recognize that the square brackets of the coefficient functions for the x and x^2 dependent terms amount both to the ubiquitous Ermakov-Pinney equation [138, 139] with the constraint

$$\frac{\gamma_{tt}}{\gamma} = 2i\alpha_t - 4\alpha^2 - \Omega := \kappa(t), \tag{5.33}$$

respectively. The general solution to this version of the Ermakov-Pinney (EP) equation, as given by the coefficient functions, can be constructed in terms of the two fundamental solutions $u(t)$ and $v(t)$ to the equations $\ddot{u} + \kappa(t)u = 0$, $\ddot{v} + \kappa(t)v = 0$ as

$$\sigma(t) = \left(Au^2 + Bv^2 + 2Cuv\right)^{1/2}, \tag{5.34}$$

where the constants A, B, C are constrained as $C^2 = AB - \omega^2/W$ with Wronskian $W = u\dot{v} - v\dot{u}$. Given that $\kappa(t)$ is now complex, the time τ and the coordinate χ inevitably become complex, unless we take $\alpha_t = 0$. As we see from (5.24) the latter option still keeps all the coefficients time-dependent although in a somewhat more restricted form.

5.4.2 Point transformation $\hat{\Gamma}_0^S : H_0(\chi) \to H_S(x,t)$, time-dependent mass

Let us now switch on the time-dependence in the mass so that we have to compare the transformed equation (5.11) with

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega(t)^2x^2\phi = 0, \tag{5.35}$$

instead of (5.26). The direct comparison then changes the three constraints (5.27) into

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x,t) = -2\hbar\alpha(t)x, \quad V(x,t) = \frac{1}{2}M(t)\Omega(t)x^2 + \hbar\alpha(t). \tag{5.36}$$

Thus, the first constraint in (5.36) differs now from the one found in [161] as a result of the introduction of an explicit time-dependent mass. As we show next, this change from a time-independent to a time-dependent mass permits us to keep the time τ and the coordinate χ to be real for more generic time-dependent coefficient functions. Taking a general form for the mass as

$$M(t) = m\sigma(t)^n, \tag{5.37}$$

allows us to easily distinguish between the time-independent and time-dependent cases, with the former obtained for $n = 0$. The first constraint in (5.36) is now solved by

$$\tau(t) = \int^t \sigma(y)^r dy \qquad \text{and} \qquad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}, \tag{5.38}$$

where we identify $n = -r - 2s$. Using these expressions in the second constraint in (5.36) yields the equation

$$\sigma^{r+2s}\frac{h}{m}\frac{A_x}{A} - i\gamma_t + is(x+\gamma)\frac{\sigma_t}{\sigma} + 2\alpha x = 0, \tag{5.39}$$

which may be solved by

$$A(x,t) = \exp\left\{\frac{im\sigma^{-1-r-2s}}{\hbar}\left[\left(\sigma\gamma_t - s\gamma\sigma_t\right)x + \left(i\alpha\sigma - \frac{1}{2}s\sigma_t\right)x^2 + \delta(t)\right]\right\} \quad (5.40)$$

where $\delta(t)$ is a complex valued function corresponding to the integration constant in the x integration. Proceeding with these expressions to the third constraint in (5.36) yields

$$\begin{aligned} &- i\hbar\frac{q\sigma^{1+r+2s}\sigma_t}{2} - m\left\{\gamma\omega^2\sigma^{2r+2} - \sigma[r+2s]\gamma_t\sigma_t + \gamma s\left[(r+s+1)\sigma_t^2 - \sigma\sigma_{tt}\right]\right\}x \\ &+ \sigma^2\gamma_{tt} + \frac{1}{2}m\left\{\sigma\left[\sigma\left(4\alpha^2 - \omega^2\sigma^{2r} + \Omega^2\right) + s\sigma_{tt}\right] - s[r+s+1]\sigma_t^2\right\}x^2 \\ &+ \frac{1}{2}m\left\{\left[2i\alpha\sigma(r+2s)\sigma_t - 2i\sigma^2\alpha_t\right]x^2 + 2(1+r+2s)\delta\sigma_t - 2\sigma\delta_t + \sigma^2\gamma_t^2\right\} \\ &+ \frac{m}{2}\left[-2s\gamma\sigma\gamma_t\sigma_t + \gamma^2\left(s^2\sigma_t^2 - \omega^2\sigma^{2+2s}\right)\right] = 0. \end{aligned} \quad (5.41)$$

The x-independent term in (5.41) vanishes for

$$\delta(t) = \frac{\gamma}{2}\left(\sigma\gamma_t - s\gamma\sigma_t\right) + \sigma^{1+r+2s}\left(c_1 - \frac{is\hbar}{2m}\log\sigma\right), \quad (5.42)$$

where c_1 is a constant. The term proportional to x^2 in (5.41) is a non-linear second order differential equation in σ. To ensure that σ is real, hence our space-time is real, we set the imaginary term to be equal too zero

$$\alpha_R[(r+2s)\sigma_t - 4\sigma\alpha_I] - \sigma(\alpha_R)_t = 0. \quad (5.43)$$

This equation is satisfied for

$$\alpha_I = \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_R}\right). \quad (5.44)$$

We notice from here that since $\alpha_I \propto \partial_t$ it does indeed transform as $\alpha_I \to -\alpha_I$ under $\mathcal{PT}$ as is required for H_S to be $\mathcal{PT}$-symmetric. The terms proportional to x^2 and x vanish for

$$\sigma_{tt} = \sigma\left(\frac{2\alpha_R\left(2\Omega^2\alpha_R + 8\alpha_R^3 + (\alpha_R)_{\rm tt}\right) - 3\left(\alpha_R\right)_t^2}{2r\alpha_R^2}\right) + \frac{\left(\frac{r}{2}+1\right)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}, \quad (5.45)$$

and

$$\gamma_{tt} = \frac{\gamma}{2r}\left(\frac{s\left(16\alpha_R^4 - 3\left(\alpha_R\right)_t^2 + 2\alpha_R(\alpha_R)_{\rm tt}\right)}{\alpha_R^2} + 4s\Omega^2\right) + \frac{(r+2s)\gamma_t\sigma_t}{\sigma}$$
$$- \frac{\gamma}{2r}\frac{(r+2s)\left(2\omega^2\sigma^{2r+2} + rs\sigma_t^2\right)}{\sigma^2}, \tag{5.46}$$

respectively. These equations can be reduced to solvable ones for specific choices of r, s, α_R, α_I and γ. We discuss now some special choices.

$\underline{\boldsymbol{\alpha_{\mathbf{I}} = 0}}$

Setting now $\alpha_I = 0$, we may solve directly for α_R in (5.44). We obtain

$$\alpha_R = c_2\sigma^{r+2s}. \tag{5.47}$$

Taking the mass to be time-independent and hence α to be time-independent by setting $r = -2s$ and $s = 1$, equations (5.45) and (5.46) reduce to

$$\sigma_{tt} = -4c_2^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2 \qquad \text{and} \qquad \gamma_{tt} = -\gamma\left(4c_2^2 + \Omega^2\right), \tag{5.48}$$

respectively. Both of these equations are solvable, with the first being the nonlinear Ermakov-Pinney equation [138, 139], solved as stated above and the second is harmonic oscillator solved by

$$\gamma = \kappa_1\gamma_1 + \kappa_2\gamma_2, \tag{5.49}$$

where γ_1, γ_2 are the two linearly independent solutions (depending here on the choice of Ω) and κ_1, κ_2 are constants.

Another interesting choice is to take $r = -s - 1$ with $s = -1$, in doing so we end up with

$$\sigma_{tt} = \frac{4c_2^2}{\sigma^3} - \sigma\omega^2 + \sigma\Omega^2 \qquad \text{and} \qquad \gamma_{tt} = -\gamma\left(\frac{4c_2^2}{\sigma^4} + \Omega^2\right) - \frac{2\gamma_t\sigma_t}{\sigma} \tag{5.50}$$

where again the first equation is a version of the non-linear EP equation. However, now the Ermakov-Pinney equation is real without any restrictions on $\alpha(t)$, so that also the time τ and the coordinate χ are real. The second equation is a damped harmonic oscillator equation, which we may solve explicitly or simply take the integration constant γ to be zero.

$\underline{\gamma = \mathbf{0}}$

Opting now for the second choice we set $\gamma = 0$ and parametrize

$$\alpha_R = \sigma^{-2-r}, \tag{5.51}$$

equation (5.45) reduces to

$$\sigma_{tt} = \frac{-\omega^2\sigma^{2r+1} + 4\sigma^{-2r-3} + \sigma\Omega^2}{r+1}, \tag{5.52}$$

with α now being genuinely complex

$$\alpha = \alpha_R - i\frac{r+s+1}{2}\partial_t \ln(\sigma). \tag{5.53}$$

Choosing $r = 0$ or $r = -2$ results in equation (5.52) being the respective EP equations given by

$$\sigma_{\rm tt} = \frac{4}{\sigma^3} + \sigma\left(\Omega^2 - \omega^2\right) \quad \text{or} \quad \sigma_{\rm tt} = \frac{\omega^2}{\sigma^3} - \sigma\left(\Omega^2 + 4\right). \tag{5.54}$$

As we have taken $\gamma = 0$ we do not need to pick a concrete value for s.

When setting $r = -2$ we do not need to choose a concrete form for α_R as in this case equation (5.45) reduces to the Ermakov-Pinney equation

$$\sigma_{tt} = \frac{\omega^2}{\sigma^3} - f\sigma, \quad \text{with} \quad f = 4\alpha_R^2 - \frac{3\,(\alpha_R)_t^2}{4\alpha_R^2} + \frac{(\alpha_R)_{\rm tt}}{2\alpha_R} + \Omega^2. \tag{5.55}$$

$\underline{\gamma \neq \mathbf{0}}$

When $\gamma \neq 0$, we still have the same parametrization of α_R and choices for r as in the previous section but we now have to restrict s so that equation (5.45) is solvable. For instance, when $r = -2$, if we choose $s = 1$, we have

$$\gamma_{tt} = -\gamma(4 + \Omega^2), \tag{5.56}$$

which is solvable by (5.49).

5.4.3 Point transformations $\hat{\Gamma}^S_{1,2,4} : H_0^{(1,2,4)}(\chi) \to H_S(x,t)$, time-dependent mass

Let us now explore the point transformations that result when changing the reference Hamiltonian, but keeping the target Hamiltonian to be $H_S(x,t)$ with time-dependent mass. Considering now the second constraint in (5.27) together with (5.18)-(5.21) we can identify the fields $A_i(x,t)$ for the the reference Hamiltonians (6.34)-(5.17). Solving the constraints we find

$$A_1(x,t) = A_2(x,t) = A(x,t), \qquad A_4(x,t) = A(x,t) \exp\left[-\frac{iamx(x+2\gamma)}{\hbar \sigma^{2s}} \right], \quad (5.57)$$

such that the $A_i(x,t)$ are identical for the same $B_i(x,t)$. Solving next the third constraint in (5.27) for (6.34)-(5.17) we notice that we always require (5.44) to hold in order to ensure that space-time remains real. In contrast, the other time-dependent functional coefficient δ and the constraining equations for σ and γ vary for each reference Hamiltonians. We obtain

$$\begin{aligned}
H_0^{(1)}: &\quad \delta_0^{(1)} = \delta, \quad \sigma_{tt}^{(1)} = \sigma_{tt} + \frac{2\omega^2\sigma^{1+2r}}{r}, \quad \gamma_{tt}^{(1)} = \gamma_{tt} + \frac{(r+2s)\omega^2\gamma\sigma^{2r}}{r}, \\
H_0^{(2)}: &\quad \delta_0^{(2)} = \delta - \sigma^{1+r+2s}\frac{a}{2m}\int^t \gamma\sigma^{r-s}, \quad \sigma^{(2)} = \sigma, \quad \gamma_{tt}^{(2)} = \gamma_{tt} - \frac{a\sigma^{2r+s}}{m}, \\
H_0^{(4)}: &\quad \delta_0^{(4)} = \delta + 2a\sigma^{1+r+2s}\int^t \gamma\sigma^{-1-2s}(s\gamma\sigma_t - \sigma\gamma_t), \quad \sigma_{tt}^{(4)} = \sigma_{tt} + \frac{8a^2\sigma^{1+2r}}{r}, \\
&\quad \gamma_{tt}^{(4)} = \gamma_{tt} + \frac{4a^2(r+2s)\gamma\sigma^{2r}}{r}.
\end{aligned}$$

Here we understand that σ_{tt} and γ_{tt} are to be replaced by the right hand sides of equations (5.45) and (5.46), respectively.

5.4.4 Non-Hermitian invariants from Γ_i^S

Having constructed the various point transformations Γ_i^j that relate the TDSEs (5.1) and (5.2) for $H^j(x,t)$ and $H_0^i(\chi)$, respectively, we proceed to the third step in our scheme and employ the point transformations now to act on $H_0^i(\chi)$ exclusively, as specified in (5.5). In this way we obtain directly the invariant I_H for the non-Hermitian Hamiltonian H.

Non-Hermitian invariant from Γ_0^S, time-independent mass

Acting with Γ_0^S, as constructed in section 5.4.1, on $H_0(\chi)$ we obtain the invariant

$$
\begin{aligned}
I_H(x,t) \quad = \quad & \frac{\sigma^2}{2m}p^2 + m\left(\frac{\gamma\omega^2}{\sigma^2} + 2i\alpha(\sigma^2\gamma_t - \gamma\sigma\sigma_t) - \sigma\sigma_t\gamma_t + \gamma\sigma_t^2\right)x \qquad (5.58)\\
& + \frac{1}{2}\sigma\left[2i\alpha\sigma - \sigma_t\right]\{x,p\} + \frac{m}{2}\left[(\sigma_t - 2i\alpha\sigma)^2 + \frac{\omega^2}{\sigma^2}\right]x^2\\
& + \sigma\left(\sigma\gamma_t - \gamma\sigma_t\right)p + \frac{m}{2}\left(\frac{\gamma^2\omega^2}{\sigma^2} + \gamma^2\sigma_t^2 + \sigma^2\gamma_t^2 - 2\gamma\gamma_t\sigma\sigma_t\right).
\end{aligned}
$$

We verified that the expression for I_H in (5.58) does indeed satisfy the Lewis-Riesenfeld equation (3.2). Thus $I_H(x,t)$ is the non-Hermitian invariant or first integral for the non-Hermitian Hamiltonian $H(x,t)$. We stress that the invariant has been obtained by a direct calculation and did not involve any assumption or guess work on the general form of the invariant, which one usually has to make when solving (3.2) directly.

Non-Hermitian invariant from $\hat{\Gamma}_0^S$, time-dependent mass

Similarly acting with $\hat{\Gamma}_0^S$, as constructed in section 5.4.2, on $H_0(\chi)$ we obtain the invariant

$$
\begin{aligned}
\hat{I}_H(x,t) = & \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - s\gamma\sigma^{-r-1}\sigma_t)p + \frac{4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t}{4\alpha_R\sigma^{r+1}}\{x,p\}\\
& + \frac{4m\omega^2\alpha_R^2\sigma^{2r+2} - m(4\sigma\alpha_R^2 - ir\alpha_R\sigma_t + i\sigma(\alpha_R)_t)^2}{8\alpha_R^2\sigma^{2(r+s+1)}}x^2\\
& + \frac{2\gamma m\omega^2\alpha_R\sigma^{2r+2} + m(\sigma\gamma_t - s\gamma\sigma_t)(4i\sigma\alpha_R^2 + r\alpha_R\sigma_t\sigma(\alpha_R)_t)}{2\alpha_R\sigma^{2(r+s+1)}}x\\
& + \frac{1}{2}m\sigma^{-2(r+s+1)}[\gamma^2\omega^2\sigma^{2r+2} + (\sigma\gamma_t - s\gamma\sigma_t)^2]. \qquad (5.59)
\end{aligned}
$$

Once more we convince ourselves that $\hat{I}_H(x,t)$ does indeed satisfy (3.2).

Non-Hermitian invariant from $\hat{\Gamma}_{1,2,4}^S$, time-dependent mass

The action of $\hat{\Gamma}_{1,2,4}^S$ from section 5.4.3 on $H_0^{(1,2,4)}(\chi)$ yields the invariants

$$
\begin{aligned}
\hat{I}_H^{(1)}(x,t) = & \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - s\gamma\sigma^{-r-1}\sigma_t)p + \frac{4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t}{4\alpha_R\sigma^{r+1}}\{x,p\}\\
& - \frac{m(4\sigma\alpha_R^2 - ir\alpha_R\sigma_t + i\sigma(\alpha_R)_t)^2}{8\alpha_R^2\sigma^{2(r+s+1)}}x^2
\end{aligned}
$$

$$+\frac{m(\sigma\gamma_t - s\gamma\sigma_t)(4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t)}{2\alpha_R\sigma^{2(r+s+1)}}x$$
$$+\frac{1}{2}m\sigma^{-2(r+s+1)}(\sigma\gamma_t - s\gamma\sigma_t)^2, \tag{5.60}$$

$$\begin{aligned}\hat{I}_H^{(2)}(x,t) =& \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - s\gamma\sigma^{-r-1}\sigma_t)p + \frac{4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t}{4\alpha_R\sigma^{r+1}}\{x,p\}\\ &+\frac{4m\omega^2\alpha_R^2\sigma^{2r+2} - m(4\sigma\alpha_R^2 - ir\alpha_R\sigma_t + i\sigma(\alpha_R)_t)^2}{8\alpha_R^2\sigma^{2(r+s+1)}}x^2\\ &+\frac{2(a\sigma^s + \gamma m\omega^2)\alpha_R\sigma^{2r+2} + m(\sigma\gamma_t - s\gamma\sigma_t)(4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t)}{2\alpha_R\sigma^{2(r+s+1)}}x\\ &+\frac{1}{2}\sigma^{-2(r+s+1)}\left[\gamma\sigma^{2r+2}\left(2a\sigma^s + \gamma m\omega^2\right) + m\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2\right],\end{aligned} \tag{5.61}$$

and

$$\begin{aligned}\hat{I}_H^{(4)}(x,t) =& \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - s\gamma\sigma^{-r-1}\sigma_t)p + \frac{4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t}{4\alpha_R\sigma^{r+1}}\{x,p\}\\ &+\frac{-4m(4a^2-\omega^2)\alpha_R^2\sigma^{2r+2} - m(4\sigma\alpha_R^2 - ir\alpha_R\sigma_t + i\sigma(\alpha_R)_t)^2}{8\alpha_R^2\sigma^{2(r+s+1)}}x^2\\ &+\frac{-2\gamma m(4a^2-\omega^2)\alpha_R\sigma^{2r+2} + m(\sigma\gamma_t - s\gamma\sigma_t)(4i\sigma\alpha_R^2 + r\alpha_R\sigma_t - \sigma(\alpha_R)_t)}{2\alpha_R\sigma^{2(r+s+1)}}x\\ &+\frac{1}{2}\sigma^{-2(r+s+1)}\left[m\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2 - m\gamma^2(4a^2-\omega^2)\sigma^{2r+2}\right].\end{aligned} \tag{5.62}$$

Let us now compare the invariants obtained. First of all we notice that all our invariants can be brought into the form

$$I_H = a_r p^2 + b_r p + (c_r + ic_i)\{x,p\} + (d_r + id_i)x^2 + (e_r + ie_i)x + f_r, \tag{5.63}$$

where we abbreviated the complex time-dependent coefficient functions in I_H and separate them into real and imaginary parts by denoting $x = x_r + ix_i$ with $x_r, x_i \in \mathbb{R}$, $x \in \{a,b,c,d,e,f\}$. When written in this form we notice a very peculiar property that for all of our invariants the time-dependent coefficient functions are related to each other as

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m\alpha_R\sigma^{-r-2s}. \tag{5.64}$$

As we will see in the next subsection this property is responsible for the fact that all invariants lead to same Dyson map. Notice that when using the conventions as in (5.63) for the Hamiltonian $H_S(x,t)$ and using the same parameterization for $M(t)$ and $\alpha(t)$, the last relation also holds for the coefficients in the Hamiltonian. We also note that if we were to take $a \to ia$ in $H_0^{(4)}(\chi)$, the associated invariant would still posses the same properties as a only appears squared in it. When comparing the expressions for the invariants $I_H^{(i)}$ one needs to keep in mind that the constraining equations also change with the index i.

5.4.5 Dyson maps and metric operators

We may now carry out the last step in our scheme and construct a Dyson map by acting adjointly on the invariants I_H. We can verify that the Dyson map constructed in [86] does indeed map I_H to a Hermitian invariant. Alternatively, when utilizing the property (5.64) we may identify the time-dependent Dyson map as

$$\eta = \exp\left(-m\alpha_R \sigma^{-r-2s} x^2\right), \tag{5.65}$$

with the associated Hermitian invariant being given by

$$I_h = a_r p^2 + b_r p + c_r\{x, p\} + \left(d_r + 4m^2 a_r \alpha_R^2 \sigma^{-2r-4s}\right) x^2 + e_r x + f_r. \tag{5.66}$$

The corresponding Hermitian Hamiltonian is computed to be

$$h = \frac{\sigma^{r+2s}}{2m} p^2 + \left(2m\alpha_R^2 \sigma^{-r-2s} + \frac{1}{2} m\sigma^{-r-2s}\Omega^2\right) x^2 + \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_R}\right)\{x, p\}, \tag{5.67}$$

which is an extended version of the time-dependent harmonic oscillator with time-dependent mass. For the special choice $\alpha_R = \sigma^{r+2s}$ the coefficient function $\alpha(t)$ becomes real, the Dyson map becomes time-independent and h reduces to the time-dependent harmonic oscillator. The metric operator is constructed to be

$$\rho = \eta^\dagger \eta = \exp(-2m\alpha_R \sigma^{-r-2s} x^2). \tag{5.68}$$

5.5 The time-dependent harmonic oscillator with complex linear term as the target Hamiltonian

To further illustrate the method and demonstrate the importance of the choice of $H_0(\chi)$ we shall be next considering the time-dependent harmonic oscillator with a time-dependent complex linear term

$$H_{CL}(x,t) = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)\Omega(t)^2x^2 + i\beta(t)x, \qquad M,\Omega,\beta \in \mathbb{R} \tag{5.69}$$

which has been previously studied in [87, 164]. As a reference Hamiltonian we take now $H_0^{(3)}(\chi)$ as defined in (5.16). We have also considered $H_0(\chi)$ as a reference Hamiltonian which leads to a point transformation that renders space-time to be complex.

5.5.1 Point transformation Γ_3^{CL} from $H_0^{(3)}(\chi)$ to $H_{CL}(x,t)$

We have already identified the equations for $B_3(x,t)$ and $V_3(x,t)$ for the reference Hamiltonian $H_0^{(3)}(\chi)$ in (5.20). Comparing now with the time-dependent Schrödinger equation for the target Hamiltonian (5.2) in the position representation, we find the three constraints

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \qquad B(x,t) = 0, \qquad V(x,t) = \frac{1}{2}M(t)\Omega(t)^2x^2 + i\beta(t)x. \tag{5.70}$$

The first constraint in (5.70) is solved in the same way as in section 5.4.2, i.e. by equations (5.38), together with (5.37). Substituting these expressions into the second constraint in (5.70) and then solving for the field $A(x,t)$ yields

$$A(x,t) = \exp\left\{\frac{im\sigma^{-1-r-2s}}{\hbar}\left[(\sigma\gamma_t - s\gamma\sigma_t)\,x - \frac{1}{2}s\sigma_t x^2 + \delta(t)\right]\right\}, \tag{5.71}$$

where $\delta(t)$ is a complex time-dependent function associated with the integration carried out. Next we use all of our expressions obtained in the third constraint in (5.70), obtaining

$$\begin{aligned}
&- m\left[\omega^2\sigma^{2r+2} + s(r+s+1)\sigma_t^2 - \sigma\left(s\sigma_{tt} + \sigma\Omega^2\right)\right]x^2 \\
&+ 2i\sigma^{r+2}\left(-b\sigma^{r+s} + i\gamma m\omega^2\sigma^r + \beta\sigma^{2s}\right)x - \gamma\sigma^{2r+2}\left(\gamma m\omega^2 + 2ib\sigma^s\right) \\
&+ 2m\left[\sigma(r+2s)\gamma_t\sigma_t + \gamma s\left(\sigma\sigma_{tt} - (r+s+1)\sigma_t^2\right) - \sigma^2\gamma_{tt}\right]x - ihs\sigma_t\sigma^{r+2s+1} \\
&+ m\left\{2\sigma_t\left[\delta(r+2s+1) - \gamma s\sigma\gamma_t\right] + \gamma^2s^2\sigma_t^2 + \sigma\left[\sigma\gamma_t^2 - 2\delta_t\right]\right\} = 0.
\end{aligned} \tag{5.72}$$

Firstly we notice that the x-dependent term in (5.72) contains an imaginary term which would result in space-time becoming complex. However, when setting

$$\beta = b\sigma^{r-s}, \tag{5.73}$$

the imaginary term vanishes and space-time remains real. Secondly we find that the x-independent terms in (5.72) vanishes for

$$\delta(t) = \frac{\gamma}{2}\left(\sigma\gamma_t - s\gamma\sigma_t\right) + \sigma^{1+r+2s}\left(c_1 - \frac{is\hbar}{2m}\log\sigma - i\int^t \frac{b\gamma\sigma^{r-s}}{m}\right). \tag{5.74}$$

Finally, the remaining terms proportional to x^2 and x result in the two second order auxiliary differential equations

$$\sigma_{tt} = \frac{\omega^2\sigma^{2r+2} - \sigma^2\Omega^2}{s\sigma} + \frac{(r+s+1)\sigma_t^2}{\sigma} \qquad \text{and} \qquad \gamma_{tt} = \frac{(r+2s)\gamma_t\sigma_t}{\sigma} - \gamma\Omega^2, \tag{5.75}$$

respectively. As discussed in the previous section there are different choices of r and s for which these equations reduce into ones with known solutions. As before, we shall not pick concrete values for r and s so we keep the derivation of the invariant and subsequent Dyson map as general as possible.

5.5.2 Non-Hermitian invariant from Γ_3^{CL}

Acting with Γ_3^{CL}, as constructed in the previous section on $H_0^{(3)}(\chi)$ we obtain the invariant

$$\begin{aligned} I_H(x,t) = &\frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t)p - \frac{1}{2}s\sigma^{-r-1}\sigma_t\{x,p\} \\ &+ \frac{1}{2}m\sigma^{-2(r+s+1)}\left(\omega^2\sigma^{2r+2} + s^2\sigma_t^2\right)x^2 \\ &\sigma^{-2(r+s+1)}\left[ms\sigma_t\left(\gamma s\sigma_t - \sigma\gamma_t\right) + \sigma^{2r+2}\left(\gamma m\omega^2 + ib\sigma^s\right)\right]x \\ &+ \frac{1}{2}\sigma^{-2(r+s+1)}\left[m\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2 + \gamma\sigma^{2r+2}\left(\gamma m\omega^2 + 2ib\sigma^s\right)\right]. \end{aligned} \tag{5.76}$$

We have verified that this expression does indeed satisfy the Lewis-Riesenfeld equation (3.2).

5.5.3 Time-dependent Dyson map and metric operator

To determine the time-dependent Dyson map associated with the non-Hermitian invariant (5.76) we use the following abbreviated version of the invariant

$$I_H = a_r p^2 + b_r p + c_r\{x,p\} + d_r x^2 + (e_r + ie_i)x + f_r + if_i, \tag{5.77}$$

using the same conventions as in (5.63).

Making now the general Ansatz for the Dyson map

$$\eta(t) = e^{\epsilon(t)p} e^{\lambda(t)x}, \qquad \epsilon, \lambda \in \mathbb{R}, \tag{5.78}$$

we compute the adjoint action of the Dyson map on all the operators that appear in the non-Hermitian invariant. We find that (5.78) maps $I_H(x,t)$ indeed to a Hermitian counterpart when the following constraints are satisfied

$$\epsilon = \frac{a_r f_i}{a_r e_r - b_r c_r}, \qquad \lambda = \frac{c_r \epsilon}{a_r}, \qquad e_i = \frac{2(c_r^2 - a_r d_r) f_i}{b_r c_r - a_r e_r}. \tag{5.79}$$

The time-dependent functions from above do indeed satisfy these equations and when using the explicit expressions for time-dependent coefficient functions from (5.76) the time-dependent Dyson map results to

$$\eta(t) = \exp\left(\frac{b\sigma^s}{m\omega^2} p\right) \exp\left(-\frac{bs\sigma^{-1-r-s}\sigma_t}{\omega^2} x\right), \tag{5.80}$$

with σ to be determined by the auxiliary equation (5.75). The corresponding Hermitian invariant is computed to

$$\begin{aligned} I_h(x,t) = {} & \frac{\sigma^{2s}}{2m} p^2 + (\sigma^{-r}\gamma_t - \gamma s \sigma^{-r-1}\sigma_t)p - \frac{1}{2} s\sigma^{-r-1}\sigma_t \{x,p\} \\ & + \frac{1}{2} m \sigma^{-2(r+s+1)} \left(\omega^2 \sigma^{2r+2} + s^2 \sigma_t^2\right) x^2 \qquad (5.81) \\ & + m\sigma^{-2(r+s+1)} \left[\gamma \omega^2 \sigma^{2r+2} + s\sigma_t \left(\gamma s \sigma_t - \sigma \gamma_t\right)\right] x \\ & + \frac{\frac{b^2 + \gamma^2 m^2 \omega^4 \sigma^{-2s}}{\omega^2} + m^2 \sigma^{-2(r+s+1)} \left(\gamma^2 s^2 \sigma_t^2 + \sigma^2 \gamma_t^2\right)}{2m}. \qquad (5.82) \end{aligned}$$

Finally we use the Dyson map (5.78) in the time-dependent Dyson equation (2.3) to compute the corresponding Hermitian Hamiltonian as

$$h(t) = \frac{\sigma^{r+2s}}{2m} p^2 + \frac{1}{2} m \sigma^{-r-2s} \Omega^2 x^2 + \frac{b^2 \sigma^{-r-2} \left(\sigma^2 \Omega^2 - s^2 \sigma_t^2\right)}{2m\omega^4}, \tag{5.83}$$

which is a time-dependent harmonic oscillator with a time-dependent free term.

5.6 Construction of wavefunctions and energy spectra

Now that we have constructed the point transformations between the reference and target Hamiltonians, and computed a metric for the models we considered, it is instructive to find the wavefunctions.

5.6.1 The time-dependent Swanson model

Wavefunctions of the reference Hamiltonian

For the purpose of determining the wavefunctions for the time-dependent Swanson Hamiltonian we choose to use the time-independent harmonic oscillator (5.9) as our reference Hamiltonian. The full solution to the time-dependent Schrödinger equation is found through separation of variables

$$H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_t\psi(\chi,\tau) \quad \text{where} \quad \psi(\chi,\tau) = e^{-i\frac{E_n\tau}{\hbar}}\theta(\chi) \tag{5.84}$$

with $\xi(\chi)$ satisfying the time-independent Schrödinger equation

$$H_0(\chi)\theta(\chi) = E_n\theta(\chi). \tag{5.85}$$

The normalised wavefunctions are given by

$$\theta(\chi) = \sqrt{\frac{1}{2^n n!}\sqrt{\frac{m\omega}{\pi\hbar}}}e^{-\frac{m\omega}{2\hbar}\chi^2}H_n\left(\sqrt{\frac{m\omega}{\hbar}}\chi\right), \tag{5.86}$$

where $H_n(x)$ denote the Hermite polynomials.

Wavefunctions of the target Hamiltonian

We many now find the solution to the TDSE for the time-dependent Swanson Hamiltonian (5.25) by utilising the point transformation. By nature of its construction the wavefunctions for the target and reference Hamiltonian, ϕ and ψ, respectively are related via

$$\psi(\chi,\tau) = A(x,t)\phi(x,t), \tag{5.87}$$

such that

$$\phi(x,t) = A^{-1}(x,t)\psi(\chi(x,t),\tau(t)) \tag{5.88}$$

where $A(x,t)$ is given by equation (5.40), and $\chi(x,t)$ and $\tau(t)$ by (5.38). We have verified that (5.88) does indeed satisfy the TDSE for the Hamiltonian (5.25).

We may additionally construct the wavefunctions for the Hermitian Hamiltonian (5.67) satisfying the TDSE

$$h(t)\Theta(x,t) = i\hbar\partial_t\Theta(x,t). \tag{5.89}$$

through

$$\Theta(x,t) = \eta(t)\phi(x,t) = \eta(t)A^{-1}(x,t)\psi(\chi(x,t),\tau(t)), \tag{5.90}$$

where $\eta(t)$ is given by equation (5.65).

Energy spectra

We proceed now to determining the time-dependent expectation values of the energy operator $\tilde{H}(t)$ (2.8). Setting $\gamma(t) = 0$ and $\hbar = 1$ we obtain

$$\begin{aligned} E_n(t) = \langle\Theta|h(t)|\Theta\rangle = \frac{(2n+1)\sigma^{-r-2}}{16\omega\alpha_R^2} & \left\{2\sigma(r+2s)\alpha_R\sigma_t(\alpha_R)_t + 16\sigma^2\alpha_R^4 - \sigma^2(\alpha_R^2)_t \right. \\ & \left. + \alpha_R^2\left[4\omega^2\sigma^{2r+2} - r(r+4s)\sigma_t^2 + 4\sigma^2\Omega^2\right]\right\}. \end{aligned} \tag{5.91}$$

These expectation values are guaranteed to be real as $\sigma, \alpha_R, r, s, \Omega \in \mathbb{R}$.

We shall now present some plots of the instantaneous energy spectra (5.91) for different choices of α_R. Firstly, for the choice $r = -2$ such that the auxiliary equation σ satisfies is reduced to (5.55). Choosing $f(t) = \Omega^2(t)$ we solved the differential equation for $\alpha_R(t)$ and obtain

$$\alpha_R(t) = \frac{4k_1}{64 + k_1^2(t + k_2^2)}. \tag{5.92}$$

Taking the frequency to be time-independent $\Omega(t) = \Omega_0$ we obtain the solution to the Ermakov-Pinney equation as (5.55)

$$\sigma(t) = \sqrt{A_1 q_1^2(t) + A_2 q_1(t) q_2(t) + A_3 q_2(t)^2}, \tag{5.93}$$

where $q_1(t)$ and $q_2(t)$ are the two linearly independent solutions to the homogeneous equation

$$q''(t) + \Omega_0^2 q(t) = 0, \tag{5.94}$$

given by $q_1(t) = \cos(\Omega_0 t)$ and $q_2(t) = \sin(\Omega_0 t)$. The constants A_1, A_2 and A_3 are constrained via

$$A_2^2 - 4A_1A_3 = -\frac{\omega^2}{W_0} \tag{5.95}$$

where $W_0 = q_1\dot{q}_2 - \dot{q}_1 q_2 =$ constant is the Wronkskian. A plot of the real and finite energy spectra for different values of n can be found in figure 5.1.

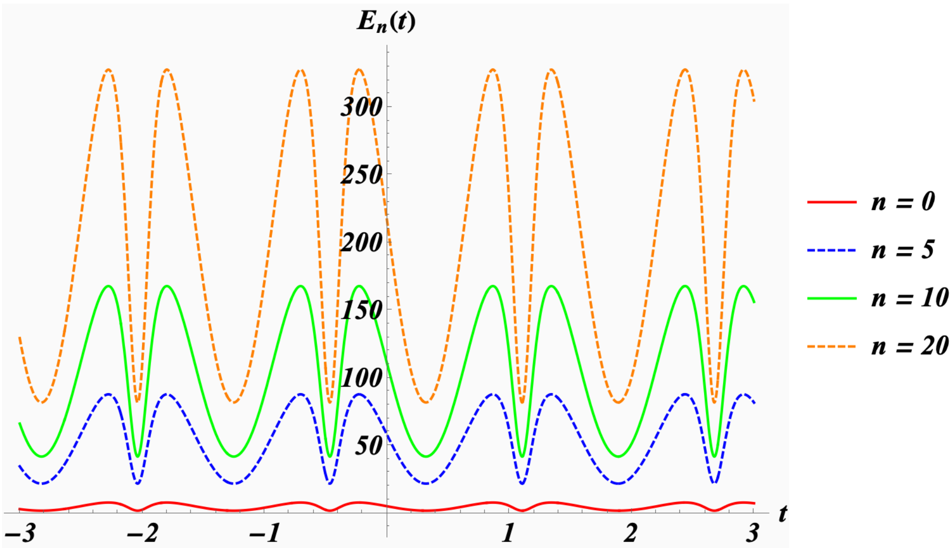

Figure 5.1: The instantaneous energy expectation values (5.91) for $r = -2$, α_R given by equation (5.92) and $A_1 = 2$, $A_3 = 1.2$, $\omega = 1.6$, $\Omega_0 = 2$, $k_1 = 0.1$, $k_2 = 0.3$ and $s = 4$.

Another interesting choice to consider is that of the time-independent mass which corresponds to $\alpha_R = c_2$ with $r = -2s$ and $s = 1$. In this case the auxiliary equation for σ is given by equation (5.48). Taking this time the frequency to be time-dependent and given by $\Omega(t) = \sin(\Omega_0 t)$, the solution to equation (5.48) is given by equation (5.93) where q_1 and q_2 are the two linearly independent solutions to the equation

$$q''(t) + [4c_2^2 + \sin^2(\Omega_0 t)]q(t) = 0, \tag{5.96}$$

given by

$$q_1 = C\left(\frac{8c_2^2+1}{2\Omega_0^2}, \frac{1}{4\Omega_0^2}, \Omega_0 t\right) \quad \text{and} \quad q_2 = S\left(\frac{8c_2^2+1}{2\Omega_0^2}, \frac{1}{4\Omega_0^2}, \Omega_0 t\right), \tag{5.97}$$

where C and S denote the even and odd Mathieu functions respectively. The constants A_1, A_2 and A_3 are again constrained by (5.95). Figure 5.2 contains a plot of the energy expectation values (5.91) for these time-dependences.

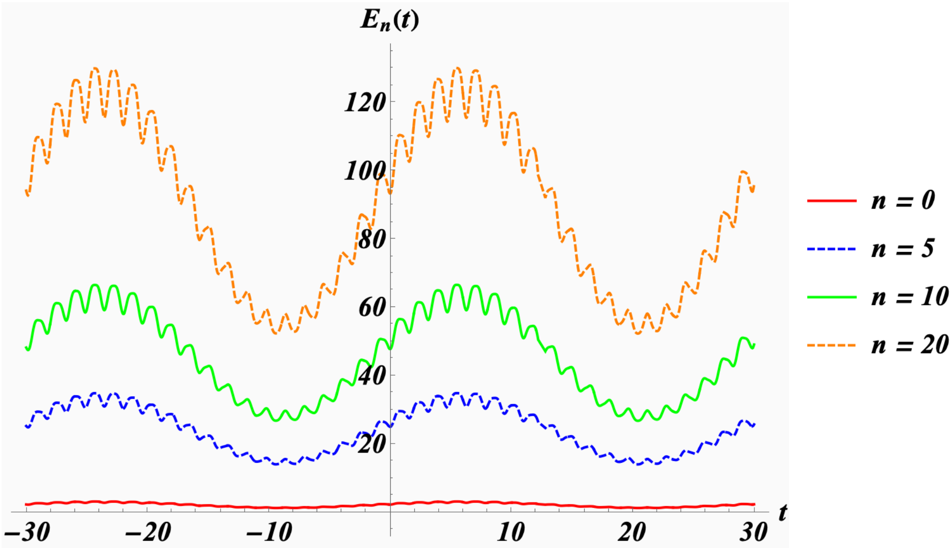

Figure 5.2: The instantaneous energy expectation values (5.91) for $\alpha_R = c_2\sigma^{r+2s}$ with $r = -2s$ and $s = 1$ with $A_1 = 2$, $A_3 = 1.2$, $\omega = 1.6$, $\Omega_0 = 2$ and $c_2 = 1$.

5.6.2 Harmonic oscillator with complex linear term

Wavefunctions of the reference Hamiltonian

For the harmonic oscillator with complex linear term the reference Hamiltonian $H_0^{(3)}(\chi)$ (5.16) is non-Hermitian, we therefore have to employ the time-independent Dyson equation (2.14) to determine the wavefunctions. This is readily done and we obtain

$$\eta_0(\chi) = e^{\frac{b}{m\omega^2}P}, \tag{5.98}$$

as a time-independent Dyson map with

$$h_0^{(3)}(\chi) = \eta_0(\chi)H_0^{(3)}(\chi)\eta_0^{-1}(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2 + \frac{b^2}{2m\omega^2}, \tag{5.99}$$

being the corresponding Hermitian Hamiltonian satisfying the time-independent Schrödinger equation

$$h_0^{(3)}(\chi)\vartheta(\chi) = E_n\vartheta(\chi). \tag{5.100}$$

We determine the wavefunctions and eigenvalues to be given by

$$\vartheta(\chi) = \sqrt{\frac{1}{2^n n!}\sqrt{\frac{m\omega}{\pi\hbar}}}e^{-\frac{m\omega}{2\hbar}\chi^2}H_n\left(\sqrt{\frac{m\omega}{\hbar}}\chi\right), \tag{5.101}$$

and

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{b^2}{2m\omega^2}, \tag{5.102}$$

respectively. The wavefunctions for the reference Hamiltonian $H_0^{(3)}(\chi)$ satisfying the time-dependent Schrödinger equation

$$H_0^{(3)}(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau) \tag{5.103}$$

are then constructed as

$$\psi(\chi,\tau) = e^{-i\frac{E_n\tau}{\hbar}}\eta_0^{-1}(\chi)\vartheta(\chi). \tag{5.104}$$

Wavefunctions of the target Hamiltonian

As before, the wavefunctions $\phi(x,t)$ of the target Hamiltonian (5.69) are related to those of the reference Hamiltonian through

$$\phi(x,t) = A^{-1}(x,t)\psi(\chi(x,t),\tau(t)) \tag{5.105}$$

with $A(x,t)$ being given by (5.71) and $\chi(x,t)$ and $\tau(t)$ by (5.38). Interestingly here is that the wavefunctions for the reference Hamiltonian involved the time-independent Dyson map $\eta_0(\chi)$. Under the point transformation Γ_3^{CL} this transforms as

$$\eta_0(\chi) \overset{\Gamma_3^{CL}}{\Longleftrightarrow} \eta_0(x,t) = e^{\frac{b\sigma^s}{m\omega^2}p}e^{-\frac{bs\sigma^{-1-r-s}\sigma_t}{\omega^2}x}e^{-i\frac{b^2s\sigma^{-1-r}\sigma_t}{2m\omega^4}}. \tag{5.106}$$

We can now determine the wavefunctions for the Hermitian Hamiltonian satisfying

$$h(x,t)\Theta(x,t) = i\hbar\partial_t\Theta(x,t), \tag{5.107}$$

where $\Theta(x,t) = \eta(x,t)\phi(x,t)$ with $\eta(x,t) = \eta(t)$ in (5.80). Note here that

$$\eta_0^{-1}(x,t)\eta(x,t) = e^{i\frac{b^2s\sigma^{-1-r}\sigma_t}{2m\omega^4}}, \tag{5.108}$$

such that we may write the full wavefunction for $h(t)$ as

$$\Theta(x,t) = e^{i\frac{b^2s\sigma^{-1-r}\sigma_t}{2m\omega^4}}e^{-i\frac{E_n\tau(t)}{\hbar}}A^{-1}(x,t)\vartheta(\chi(x,t),\tau(t)), \tag{5.109}$$

which we have verified does satisfy (5.107).

Energy spectra

We now move on to calculating the time-dependent expectation values of the energy operator $\tilde{H}(t)$ (2.8). For simplicity we take $\gamma(t) = 0$ and $\hbar = 1$ and obtain

$$E_n(t) = \langle\Theta|h(t)|\Theta\rangle = \frac{\sigma^{-2-r}}{4m\omega^4}\left\{m(2n+1)\omega^3\left[\sigma^2\left(\omega^2\sigma^{2r}+\Omega^2\right)+s^2\sigma_t^2\right]\right. \\ \left. + 2b^2\left(\sigma^2\Omega^2 - s^2\sigma_t^2\right)\right\}, \tag{5.110}$$

where $h(t)$ is given by (5.83). $E_n(t)$ is guaranteed to be real as $\sigma, \Omega, \omega, b, m, s, r \in \mathbb{R}$.

For completeness we include here a plot of energies $E_n(t)$ for a particular time-dependence. First taking $r = -s-1$ with $s = 1$ the auxiliary equation for σ (5.75) reduces to

$$\sigma_{tt} + \Omega^2\sigma = \frac{\omega^2}{\sigma^3}. \tag{5.111}$$

For $\Omega = \cos(\Omega_0 t)$ the solution to this equation is given by (5.93) with q_1 and q_2 being the two linearly independent solutions to the equation

$$q''(t) + \cos^2(\Omega_0 t)q(t) = 0, \tag{5.112}$$

given by

$$q_1 = C\left(\frac{1}{2\Omega_0^2}, -\frac{1}{4\Omega_0^2}, \Omega_0 t\right) \quad \text{and} \quad q_2 = S\left(\frac{1}{2\Omega_0^2}, -\frac{1}{4\Omega_0^2}, \Omega_0 t\right), \tag{5.113}$$

where C and S denote again the even and odd Mathieu functions. Figure 5.3 contains

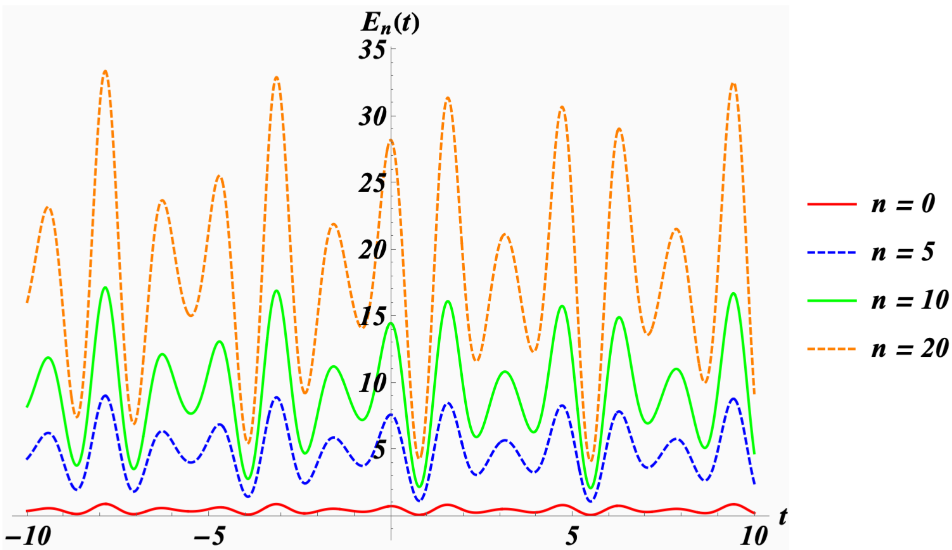

Figure 5.3: The instantaneous energy expectation values (5.110) for with $r = -s-1$ and $s = 1$ with $A_1 = 3$, $A_3 = 2.5$, $\omega = 1.6$, $\Omega_0 = 2$, $m = 1.5$ and $b = 0.7$.

a plot of the energies (5.110) for this time-dependence.

5.6.3 From the time-independent to the time-dependent Dyson map

For the harmonic oscillator with complex linear term we needed to construct the time-independent Dyson map $\eta_0(\chi)$. We notice that when point transformed, and so it becomes time-dependent, this is an additional Dyson map for the time-dependent target Hamiltonian corresponding to the Hermitian Hamiltonian

$$h(x,t) = \frac{\sigma^{r+2s}}{2m}p^2 + \frac{1}{2}m\sigma^{-r-2s}\Omega^2 x^2 + \frac{b^2\sigma^r}{2m\omega^2}, \tag{5.114}$$

satisfying

$$h(x,t)\Xi(x,t) = i\hbar\partial_t \Xi(x,t). \tag{5.115}$$

Further to this, we also find that by using the constructed point transformation on the time-dependent Schrödinger equation for $h_0(\chi)$

$$h_0(\chi)\xi(\chi,\tau) = i\hbar\partial_\tau \xi(\chi,\tau), \tag{5.116}$$

we obtain the corresponding equation for $h(x,t)$. These relationships for the harmonic oscillator with complex coupling are summarised in figure 5.4. For this particular system after having constructed the point transformation Γ_3^{CL} between the reference and target Hamiltonian, as well as the time-independent Dyson map for the reference Hamiltonian, we were able to use the point transformation to construct the time-dependent Dyson map and equivalent Hermitian Hamiltonian for

$$\begin{array}{ccc} & h_0(\chi) = \eta_0(\chi)H_0(\chi)\eta_0^{-1}(\chi), & \\ H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau), & \eta_0(\chi), & h_0(\chi)\xi(\chi,\tau) = i\hbar\partial_t\xi(\chi,\tau), \\ \updownarrow \Gamma_3^{CL} & \updownarrow \Gamma_3^{CL} & \updownarrow \Gamma_3^{CL} \\ H(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t), & \eta(x,t), & h(x,t)\Xi(x,t) = i\hbar\partial_t\Xi(x,t), \\ & h(x,t) = \eta(x,t)H(x,t)\eta^{-1}(x,t) + i\partial_t\eta(x,t)\eta^{-1}(x,t) & \end{array}$$

Figure 5.4: Schematic representation of how the point transformation Γ_3^{CL} can be used to construct time-dependent Dyson maps and time-dependent Hermitian Hamiltonians from their time-independent counterparts for the time-dependent Harmonic oscillator with complex coupling.

the target Hamiltonian. This has completely bypassed the time-dependent Dyson equation as well as the need for the construction of the invariants. It remains an open question as to whether one can formulate general criteria for which systems this can be applied to.

5.7 Point transformations between Bender-Boettcher Hamiltonians

For the purpose of computing a non-Hermitian invariant associated with a time-dependent non-Hermitian Hamiltonian, the reference Hamiltonian need not be Hermitian, as demonstrated in the previous section, or exactly solvable. We shall now construct point transformations utilising the Bender-Boettcher potentials to demonstrate this.

5.7.1 The reference and target Hamiltonians

We shall be considering the time-independent harmonic oscillator with a general Bender-Boettcher potential

$$H^{(n)}_{0,BB}(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2 + \lambda\chi^2(i\chi)^n, \tag{5.117}$$

as the reference Hamiltonian. The time-dependent counterpart

$$H^{(n)}_{BB}(x,t) = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)\Omega(t)^2x^2 + \Lambda(t)x^2(ix)^n, \tag{5.118}$$

will be the target Hamiltonian. Generally neither of these Hamiltonians are exactly solvable.

The general form of the point transformed differential equation (5.11) associated with this new reference Hamiltonian (5.117) and the explicit form of $B_0(x,t)$ (5.12) remains the same yet the explicit form of $V_0(x,t)$ (5.13) differs

$$V_{BB}(x,t) = V_0(x,t) + \lambda\chi^2(i\chi)^n\tau_t. \tag{5.119}$$

5.7.2 Point transformations Γ^{BB}_n from $H^{(n)}_{0,BB}(\chi)$ to $H^{(n)}_{BB}(x,t)$

We compare the point transformed differential equation (5.11) for the reference

Hamiltonian (5.117) with the TDSE associated with the target Hamiltonian (5.118)

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - \frac{1}{2}M(t)\Omega(t)^2x^2\phi - \Lambda(t)x^2(ix)^n\phi = 0. \tag{5.120}$$

This comparison leads to the following constraints

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B_{BB}(t) = 0, \quad V_{BB}(t) = \frac{1}{2}M(t)\Omega(t)^2x^2 + \Lambda(t)x^2(ix)^n. \tag{5.121}$$

The first two constraints are solved in the same way as in (5.5.1), that is by equations (5.38), (5.37) and (5.40) yet we chose to take $\gamma(t) = 0$ for simplicity such that $A(x,t)$ is now given by

$$A(x,t) = \exp\left[\frac{im\sigma^{-1-r-2s}}{\hbar}\left(\delta - \frac{1}{2}s\sigma_t x^2\right)\right], \tag{5.122}$$

where $\delta(t)$ is again an integration constant that needs to be determined. Using all of the expressions in third constraint in (5.121) yields

$$\begin{aligned}&\frac{\sigma^{2-r-2s}}{2}\left\{\sigma\left[ms\sigma_{tt} - \sigma(2\lambda\sigma^{2r}(ix\sigma^{-s})^n + m\sigma^{2r}\omega^2 - m\Omega^2 - 2(ix)^n\sigma^{r+2s}\Lambda)\right]\right\}x^2\\ &+\frac{\sigma^{-2-r-2s}}{2}\left\{\sigma_t[2m(1+r+2s)\delta - i\hbar\sigma^{1+r+2s}] - 2m\sigma\delta_t\right\}\\ &-\frac{ms(1+r+s)\sigma^{2-r-2s}\sigma_t^2}{2}x^2 = 0.\end{aligned} \tag{5.123}$$

We first notice that the non-Hermitian terms can be eliminated by setting

$$\Lambda = \lambda\sigma^{r-2s-ns}. \tag{5.124}$$

The x-independent terms vanish for

$$\delta(t) = \sigma^{1+r+2s}\left(c_1 - \frac{is\hbar}{2m}\log\sigma\right). \tag{5.125}$$

The remaining terms which are proportional to x^2 vanish with σ satisfying the auxiliary equation

$$\sigma_{tt} = \frac{\omega^2\sigma^{2r+2} - \Omega^2\sigma^2}{s\sigma} + \frac{(r+s+1)\sigma_t^2}{\sigma}. \tag{5.126}$$

As detailed in previous sections there are several choices of r and s for which the auxiliary equation reduces to one with known solutions.

5.7.3 Non-Hermitian invariants from Γ_n^{BB}

If we now act with the point transformations Γ_n^{BB} on the reference Hamiltonians (5.117) we obtain the invariants

$$I_{H,BB}^{(n)} = \frac{\sigma^{2s}}{2m}p^2 - \frac{1}{2}s\sigma^{-r-1}\sigma_t\{x,p\} + \frac{1}{2}\sigma^{-2(1+r+s)}\left\{\sigma^{2+2r}\left[m\omega^2 + 2\lambda\left(ix\sigma^{-s}\right)^n\right] + ms^2\sigma_t^2\right\}x^2 \tag{5.127}$$

We have verified that this expressions does satisfy the Lewis-Riesenfeld equation (3.2) for the Hamiltonians (5.118).

5.7.4 The time-dependent anharmonic quartic oscillator

An interesting case to consider here now is the anharmonic oscillator. Taking $r = -2$, $s = 1$ (time-independent mass), $\Omega(t) = 0$ and $n = 2$ the target Hamiltonian (5.118) and corresponding non-Hermitian invariant (5.127) take the form

$$H_{BB}^{(2)} = \frac{p^2}{2m} - \frac{\lambda}{\sigma^6}x^4, \tag{5.128}$$

and

$$I_{H,BB}^{(2)} = \frac{\sigma^2}{2m}p^2 - \frac{\sigma\sigma_t}{2}\{x,p\} + \frac{1}{2}\left(\frac{m\omega^2}{\sigma^2} + m\sigma_t^2\right)x^2 - \frac{\lambda}{\sigma^4}x^4, \tag{5.129}$$

respectively. The auxiliary equation (5.126) reduces to

$$\sigma_{tt} = \frac{\omega^2}{\sigma^3}, \tag{5.130}$$

which has solutions

$$\sigma = \pm\frac{\sqrt{(\omega^2 + c_2^2c_3^2) + c_2^2t^2 + 2c_2^2c_3t}}{\sqrt{c_2}}, \tag{5.131}$$

where c_2, c_3 are constants of integration. This solution for the time-dependence of the system directly aligns with the one recovered when determining the Dyson map for the time-dependent anharmonic oscillator without mass term in Chapter 4 (4.30) if we identify

$$\kappa_0 = \omega^2 + c_2 c_3^2, \quad \kappa_1 = 2c_2 c_3, \quad \kappa_2 = c_2, \quad \lambda = \frac{1}{64}, \quad m = \frac{1}{2}. \tag{5.132}$$

We can define the invariant (5.129) in the lower half complex plane by using the contour $x \to -2i\sqrt{1+ix}$ which was used by Jones and Mateo [74] to define the anharmonic oscillator in the lower half complex plane

$$I_{LH}^{(2)} = \sigma^2 p^2 + \left(2i\sigma\sigma_t - \frac{1}{2}\sigma^2\right) p + \frac{1}{4\sigma^4}x^2 - \frac{i}{2\sigma^4}\left(1 + 2\omega^2\sigma^2 + 2\sigma^4\sigma_t^2\right) x$$
$$- \sigma\sigma_t\{x,p\} + \frac{1}{2}i\sigma^2\{x,p^2\} - \frac{1}{4\sigma^4} - \frac{\omega^2}{\sigma^2} - \frac{1}{2}i\sigma\sigma_t - \sigma_t^2. \tag{5.133}$$

This expression is indeed an invariant for the Hamiltonian (5.128) when it has also been defined on the lower half complex plane using the contour $x \to -2i\sqrt{1+ix}$ and has set λ and m according to (5.132).

We have verified that the Dyson map detailed by equations (4.20) and (4.28) does indeed map the invariant (5.133) to the Hermitian invariant for the Hamiltonian (4.32) with $m = 0$. More interestingly, we are now able to find an additional Dyson map for the time-dependent anharmonic oscillator given that it is much easier to solve for time-dependent Dyson maps on the level of the invariant. We find that

$$\eta(t) = e^{\alpha(t)p^3} e^{\beta(t)x} e^{\gamma(t)p + i\delta(t)p}, \tag{5.134}$$

where

$$\alpha = \frac{2\sigma^6}{3}, \quad \beta = -\frac{\sigma_t}{\sigma}, \quad \gamma = -1 - 2\omega^2\sigma^2, \quad \delta = \frac{[c_4 - \frac{1}{2}\ln(\sigma)]\sigma}{\sigma_t} \tag{5.135}$$

is a new Dyson map which when adjointly acting on (5.133) produces the Hermitian invariant

$$I_h^{(2)} = \eta I_{LH}^{(2)} \eta^{-1} = \sigma^8 p^4 - 2\omega^2\sigma^4 p^2 - \frac{1}{2}\sigma^2 p + \frac{1}{4\sigma^4}x^2 + \frac{\delta}{2\sigma^4}x + \frac{\delta}{4\sigma^4} + \omega^4. \tag{5.136}$$

We now use the Dyson map (5.134) in the TDDE (2.3) to compute the Hermitian Hamiltonian

$$h = \sigma^6 p^4 - 2\omega^2\sigma^2 p^2 + \frac{(2c_4 - \ln(\sigma))\left(\sigma^2\sigma_t^2 + \omega^2\right)}{2\sigma^2\sigma_t^2} p + \frac{1}{4\sigma^6} x^2$$
$$- \frac{\ln(\sigma) - 2c_1}{4\sigma^5\sigma_t} x + \frac{\sigma_t}{\sigma}\{x, p\} - \frac{\omega^4}{\sigma^2} + \frac{(\ln(\sigma) - 2c_1)^2}{16\sigma^4\sigma_t^2} - 2\omega^2\sigma_t^2, \tag{5.137}$$

which differs from the previously found solution in [4]. The point transformation, whilst increasing the number of steps taken to determine the time-dependent Dyson map have drastically reduced the difficulty in obtaining one.

5.8 Conclusions

We have demonstrated that point transformations can be utilized to construct non-Hermitian invariants for non-Hermitian Hamiltonians. In turn these invariants may then be used to construct Dyson maps simply in form of similarity transformations, which automatically satisfy the time-dependent Dyson equation (2.3). Thus we have bypassed solving this more complicated equation directly. When starting from an exactly solvable reference Hamiltonian the scheme yields also the solution for the TDSE of the target Hamiltonian. By construction the solutions only form an orthonormal system when equipped with a metric operator that is obtained trivially from the constructed Dyson map. We have shown that several different reference Hamiltonians may lead to the same Dyson map. In addition to this we have shown that for the purpose of producing an invariant for a time-dependent non-Hermitian system that the reference Hamiltonian need not be exactly solvable.

Additionally, for the harmonic oscillator with complex coupling we have demonstrated a remarkable feature of the point transformation in that it allowed for the construction of a time-dependent Dyson map from a time-independent one. Further to this, we were also able to employ the point transformation to construct the time-dependent Hermitian counterpart to the non-Hermitian Hamiltonian. This bypassed the need for the time-dependent Dyson equation and the invariants.

Finally we applied the approach to the more complicated model of the time-dependent anharmonic quartic oscillator. Once we had obtained the invariant for the non-Hermitian system we defined it on a contour in the lower complex plane. We were then able to more easily determine a new Dyson map for this system. We now have two different Dyson maps for the anharmonic quartic oscillator which both have the same auxiliary equation. In Chapter 7 we will again revisit this model to determine an infinite series of Dyson maps from the two solutions we have already found.

Chapter 6

Time-dependent perturbation theory for the metric

In this chapter we propose a perturbative approach to determine the time-dependent Dyson map and the metric operator associated with time-dependent non-Hermitian Hamiltonians. We will apply this method to a pair of explicitly time-dependent two dimensional harmonic oscillators that are weakly coupled to each other in a $\mathcal{PT}$-symmetric fashion. The non-Hermitian couplings we consider will be of the type $i(xy + p_x p_y)$ and $ip_x p_y$. The former of these models can be described by an algebra comprised of four generators with the latter requiring a more complicated ten dimensional algebra. We will then consider the strongly coupled explicitly time-dependent negative quartic anharmonic oscillator potential for which we have already come across twice in this thesis in chapters 4 and 5. We demonstrate that once the perturbative Ansatz is set up the coupled differential equations resulting order by order may be solved recursively in a constructive manner, thus bypassing the need for making any guess for the Dyson map or the metric operator. Exploring the ambiguities in the solutions of the order by order differential equations naturally leads to a whole set of inequivalent solutions for the Dyson maps and metric operators implying different physical behaviour as demonstrated for the expectation values of the time-dependent energy operator.

6.1 Motivation

Both in the time-independent and time-dependent scenario the metric ρ is the key quantity that needs to be determined when studying non-Hermitian systems. If this quantity is not defined then one cannot construct a positive definite inner product, calculate observables, or root the non-Hermitian theory in a well defined Hilbert space [1, 140, 141, 165]. The metric is therefore required for a physical interpretation of a non-Hermitian system and so having a systematic way to determine it is vital.

If a non-Hermitian Hamiltonian has no explicit time-dependence, then one would usually solve either the time-independent quasi-Hermiticity relation directly for ρ or the time-independent Dyson equation for η and construct the metric through $\rho = \eta^\dagger \eta$ (2.14). For many models this approach has led to the Dyson map being exactly known, see for e.g. [90, 93, 94, 96]. The solvability of these models however is rare trait and perturbative approaches need to be applied instead. Consider for example the non-Hermitian time-independent complex cubic potential $V = ix^3$: no exact solution for the metric is known and up to now it is in fact only calculated perturbatively [59, 99]. This perturbative approach has additional benefits for when there does exist an exact solution. The systematic method allowed Jones and Mateo [74] to calculate the exact metric for unstable anharmonic oscillator as we discussed in chapter 4.

For a time-dependent non-Hermitian system one would similarly solve either the time-dependent quasi-Hermiticity relation (2.4) or time-dependent Dyson equation (2.3) directly for the metric $\rho(t)$ and Dyson map $\eta(t)$, respectively. Alternatively, one may utilize the Lewis-Riesenfeld method of invariants [89] to reduce the problem of solving for the Dyson map to one of a similarity transformation [6, 51, 87, 88], which has been demonstrated in chapter 5. As in the time-independent case there are many exact solutions for the metric and Dyson map [2, 4, 6, 69, 75–77, 86, 166] which usually rely on making inspired guesses for the Ansatz. In contrast, the powerful feature of the time-independent perturbative approach mentioned above is that it is entirely constructive and may be solved order by order. No such perturbative approach has been developed in the time-dependent regime. We therefore propose in this chapter a method to determine the time-dependent metric for time-dependent non-Hermitian systems using perturbation theory.

6.2 Time-dependent perturbation theory

In chapter 4 section 4.2.2 we introduced time-independent perturbation theory for determining the time-independent Dyson map. In this section we shall propose a similar procedure as in the time-independent case, however, we solve the time-dependent quasi-Hermiticity relation in (2.4) for $\rho(t)$ rather than the time-independent Dyson equation for $\eta(t)$. We separate the Hamiltonian as

$$H(t) = h_0(t) + i\epsilon h_1(t), \qquad \text{with } h_0(t) = h_0^\dagger(t),\ h_1(t) = h_1^\dagger(t), \tag{6.1}$$

with $\epsilon \ll 1$ being a time-independent expansion parameter. By comparing with the time-independent case let us now motivate our Ansatz for the perturbative version of the time-dependent Dyson map. First we note that the operators $\check{q}_n$ in (4.13) might consist of a sum of operators with different amounts of terms at each order. Thus they may be expanded further at each order in terms of operators $\tilde{q}_i^{(n)}$ as $\check{q}_n \rightarrow 2\sum_{i=1}^{N_n} \tilde{\gamma}_i^{(n)}(t)\tilde{q}_i^{(n)}$ with real coefficient functions that become now time-dependent $\gamma_i^{(n)}(t)$. The factor 2 is introduced for convenience and will be useful below. The upper limit of the sum N_n takes into account that we may need different amounts of operators at each order in ϵ. Then with the introduction of time, the operator q in (4.13) is replaced by

$$q(t) = 2\sum_{n=1}^{\infty}\sum_{i=1}^{N_n} \epsilon^n \tilde{\gamma}_i^{(n)}(t)\tilde{q}_i^{(n)}. \tag{6.2}$$

This version is highly unsuitable for the time-dependent case as we have to compute $\partial_t\eta(t)$ or $\partial_t\rho(t)$ in equations (2.3) and (2.4). In general this calculation is complicated for expressions of the form $e^{\tilde{A}(t)+\tilde{B}(t)+\tilde{C}(t)+\ldots}$ with non-vanishing commutators $[\tilde{A}(t), \tilde{B}(t)]$, $[\tilde{A}(t), \tilde{C}(t)]$, ... We therefore factorize the exponential with a sum in its argument into a product of exponentials $e^{A(t)}e^{B(t)}e^{C(t)}\ldots$. The explicit relations between the operators $\tilde{A}$, $\tilde{B}$, $\tilde{C}$,... and the A, B, C,... are usually very complicated, see for instance equations (6) and (7) in reference [77]. Assuming now in addition that at each order the operators $\tilde{q}_i^{(n)}$ belong to the same closed algebra with generators q_i, for $i = 1, \ldots, j$, we can simply convert (6.2) into

$$q(t) = 2\sum_{i=1}^{j}\sum_{n=1}^{k} \epsilon^n \gamma_i^{(n)}(t) q_i, \tag{6.3}$$

where we also swapped the two sums and terminated the second sum at some finite limit k. We can now factorize the Dyson map as

$$\eta(t) = e^{q(t)/2} = \prod_{i=1}^{j} \exp\left(\sum_{n=1}^{k} \epsilon^n \gamma_i^{(n)}(t) q_i\right) = \prod_{i=1}^{j}\prod_{n=1}^{k} \exp\left(\epsilon^n \gamma_i^{(n)}(t) q_i\right). \qquad (6.4)$$

The product in (6.4) is understood to be ordered $\prod_{i=1}^{j} a_i = a_1 a_2 \ldots a_j$. The precise relations between the $\gamma_i^{(n)}(t)$ and the $\tilde{\gamma}_i^{(n)}(t)$ are left unspecified, but these would only be relevant if one takes the expression in (6.2) as a starting point. Instead one may simply view the factorized Ansatz (6.4) as more fundamental. The limits j, k and the generators q_i may be pre-selected leaving the time-dependent coefficient functions $\gamma_i^{(n)}(t)$ as the unknown quantities that need to be determined. Taking the generators to be Hermitian $q_i = q_i^\dagger$, the metric acquires the form

$$\rho(t) = \eta(t)^\dagger \eta(t) = \prod_{i=j}^{1}\left[\prod_{n=k}^{1} \exp\left(\epsilon^n \gamma_i^{(n)} q_i\right)\right] \prod_{i=1}^{j}\left[\prod_{n=1}^{k} \exp\left(\epsilon^n \gamma_i^{(n)} q_i\right)\right], \qquad (6.5)$$

where $\prod_{i=j}^{1}$ denotes the reverse ordered product, that is $\prod_{i=j}^{1} a_i = a_j a_{j-1} \ldots a_1$. For $k = 1$ the relevant terms in the metric are therefore identified to be

$$\rho^{(1)}(t) = \left[\prod_{i=j}^{1} \exp\left(\epsilon \gamma_i^{(1)} q_i\right)\right]\left[\prod_{i=1}^{j} \exp\left(\epsilon \gamma_i^{(1)} q_i\right)\right]. \qquad (6.6)$$

Upon substituting this expression into the time-dependent quasi-Hermiticity relation in (2.4), and expanding up to first order in ϵ we obtain the first order differential equation

$$ih_1 + \sum_{i=1}^{j}\left(\gamma_i^{(1)}[q_i, h_0] + i\dot{\gamma}_i^{(1)} q_i\right) = 0. \qquad (6.7)$$

We observe from this equation that we can multiply the Dyson map by a factor involving a time-independent phase that commutes with the Hermitian part of the Hamiltonian. This is analogous to time-independent first order equation (4.14), which can be retrieved from (6.7) by setting the time-derivative terms to zero with $j = 1$ and $\gamma_1^{(1)} = 1/2$.

To second order the relevant metric results to

$$\rho^{(2)}(t) = \prod_{i=j}^{1}\left[\prod_{l=2}^{1} \exp(\epsilon^l \gamma_i^{(l)} q_i)\right] \prod_{i=1}^{j}\left[\prod_{l=1}^{2} \exp(\epsilon^l \gamma_i^{(l)} q_i)\right], \qquad (6.8)$$

where this time we have only kept terms up to order ϵ^2 in the argument of the exponential function. We substitute this into the time-dependent quasi-Hermiticity relation in (2.4), and only keep terms that are proportional to ϵ^2, obtaining

$$\sum_{i=1}^{j}\left(2\sum_{r=1,\neq i}^{j}\left(\gamma_i^{(1)}\gamma_r^{(1)}[q_r,[q_i,h_0]]+i\dot{\gamma}_i^{(1)}\gamma_r^{(1)}[q_r,q_i]\right)+(\gamma_i^{(1)})^2[q_i,[q_i,h_0]]\right)$$
$$+2\sum_{i=1}^{j}\left(\gamma_i^{(2)}[q_i,h_0]+i\gamma_i^{(1)}[q_i^1,h_1]+\frac{1}{2!}(\gamma_i^{(1)})^2[q_i,[q_i,h_0]]+i\dot{\gamma}_i^{(2)}q_i\right)=0 \qquad (6.9)$$

The equations resulting from higher order in ϵ can be derived in a closely related fashion. Similar to the time-independent case, these equations can be solved recursively. In contrast, we find here that the even ordered equations are also important, as will be demonstrated below.

Some remarks are needed with regards to the Ansatz made for the perturbative series. First of all we assumed here that $\eta(t)$ is Hermitian in (6.5), which is not necessary and in fact implies that we are missing some of the solutions as we shall see below. The second point to notice is that we have not made any assumptions about the operators in the exponentials, which are in turn determined by (6.7), (6.8) and the corresponding higher order equations. Nonetheless, we made some assumptions about the form of the products in (6.4) as explained and motivated above. We also need to make an assumption about the limits in the product. Let us now demonstrate for a concrete example that the recursive solutions of the order by order equations (6.7), (6.8), ... do indeed lead to meaningful solutions of the time-dependent quasi-Hermiticity relation in (2.4). As it clear from the above equations, the solutions procedure for the time-dependent case is much more involved than in the time-independent case. However, the above and especially the examples below demonstrate that one may indeed solve the equations recursively order by order.

6.3 Time-dependent coupled non-Hermitian harmonic oscillators

Throughout this thesis we will be performing an extensive analysis on two dimensional time-dependent harmonic oscillators with complex coupling. In particular we will be focusing on harmonic oscillators where the non-Hermitian couplings are $i(xy+p_xp_y)$ and ip_xp_y. The Hamiltonians are given by

$$H_1(t) = \frac{a(t)}{2}\left(p_x^2 + x^2\right) + \frac{b(t)}{2}\left(p_y^2 + y^2\right) + i\frac{\lambda(t)}{2}\left(xy + p_x p_y\right) + \frac{\mu(t)}{2}\left(xp_y - yp_x\right), \tag{6.10}$$

and

$$H_2(t) = \frac{a(t)}{2}\left(p_x^2 + x^2\right) + \frac{b(t)}{2}\left(p_y^2 + y^2\right) + i\Lambda(t)p_x p_y, \tag{6.11}$$

respectively, and involve the time-dependent coefficient functions $a(t), b(t), \lambda(t), \mu(t), \Lambda(t) \in \mathbb{R}$. These non-Hermitian Hamiltonians are symmetric with respect to two different $\mathcal{PT}$-transformations, $[\mathcal{PT}_\pm, H_{1,2}] = 0$, where the antilinear maps are given by $\mathcal{PT}_\pm : x \to \pm x, y \to \mp y, p_x \to \mp p_x, p_y \to \pm p_y, i \to -i$. The Hamiltonian (6.10) generalizes the system previously studied in [51] for $\mu = 0$ and $a = b$.

The Hamiltonians (6.10) and (6.11) can be re-expressed in terms of ten Hermitian generators given by

$$K_\pm^z = \frac{1}{2}(p_z^2 \pm z^2), \quad K_0^z = \frac{1}{2}\{z, P_z\}, \quad J_\pm = \frac{1}{2}(xp_y \pm yp_x), \quad I_\pm = \frac{1}{2}(xy \pm p_x p_y), \tag{6.12}$$

where $z = x, y$. As laid out in [51, 69], we then obtain a closed algebra with non-vanishing commutation relations

$$[K_0^x, K_\pm^x] = 2iK_\mp^x, \quad [K_0^y, K_\pm^y] = 2iK_\mp^y, \quad [K_+^x, K_-^x] = 2i, \tag{6.13}$$
$$[K_+^y, K_-^y] = 2iK_0^y, \quad [K_0^x, J_\pm] = -iJ_\mp, \quad [K_0^y, J_\pm] = iJ_\mp, \tag{6.14}$$
$$[K_0^x, I_\pm] = -iI_\mp, \quad [K_0^y, I_\pm] = -iI_\mp \quad [K_\pm^x, J_+] = \pm iI_\mp, \tag{6.15}$$
$$[K_\pm^y, J_+] = \pm iI_\mp, \quad [K_\pm^x, J_-] = \mp iI_\pm, \quad [K_\pm^y, J_-] = \pm iI_\pm, \tag{6.16}$$
$$[K_\pm^x, I_+] = \pm iJ_\mp, \quad [K_\pm^y, I_+] = -iJ_\mp, \quad [K_\pm^x, I_-] = \mp iJ_\pm, \tag{6.17}$$
$$[K_\pm^y, I_-] = -iJ_\pm, \tag{6.18}$$
$$[J_+, J_-] = \frac{i}{2}(K_0^x - K_0^y), \quad [I_+, I_-] = -\frac{i}{2}(K_0^x + K_0^y), \tag{6.19}$$
$$[J_+, I_\pm] = \pm\frac{i}{2}(K_\mp^x + K_\mp^y), \quad [J_-, I_\pm] = \pm\frac{i}{2}(K_\mp^x - K_\mp^y). \tag{6.20}$$

The $\mathcal{PT}$-symmetry that leaves this algebra invariant manifests itself as

$$\mathcal{PT}_\pm : \quad K_0^{x,y} \to -K_0^{x,y}, \quad K_\pm^{x,y} \to K_\pm^{x,y}, \quad I_\pm \to -I_\pm, \quad J_\pm \to J_\pm, \quad i \to -i, \tag{6.21}$$

and the Hamiltonians (6.10) and (6.11) are now re-expressed as

$$H_1(t) = a(t)K_+^x + b(t)K_+^y + i\lambda(t)I_+ + \mu(t)J_-, \tag{6.22}$$

and

$$H_2(t) = a(t)K_+^x + b(t)K_+^y + i\Lambda(t)\left(I_+ - I_-\right), \tag{6.23}$$

respectively.

In the following two sections we shall be using the time-dependent perturbation theory laid out above to determine metrics and hence the Dyson map for the oscillators with an $i(xy + p_x p_y)$ and $ip_x p_y$ coupling.

6.4 $i(xy + p_x p_y)$ coupled oscillators

As we only require a subalgebra of (6.13)-(6.20) for this model we simplify our notation here by rewriting the Hamiltonian (6.22) as

$$H_1(t) = a(t)K_1 + b(t)K_2 + i\lambda(t)K_3 + \mu(t)K_4, \tag{6.24}$$

where we have identified $K_1 = K_+^x, K_2 = K_+^y, K_3 = I_+, K_4 = J_-$. These four generators form a closed subalgebra on their own satisfying the commutation relations

$$\begin{aligned} &[K_1, K_2] = 0, && [K_1, K_3] = iK_4, && [K_1, K_4] = -iK_3, \\ &[K_2, K_3] = -iK_4, && [K_2, K_4] = iK_3, && [K_3, K_4] = i(K_1 - K_2)/2. \end{aligned} \tag{6.25}$$

Denoting $c(t) := a(t) - b(t)$, we shall be considering the three different cases for $H_1(t)$, characterized as:

$$\text{case 1}: \quad c(t) = 0 \quad \text{and} \quad \mu(t) = 0, \tag{6.26}$$

$$\text{case 2}: \quad c(t) \neq 0 \quad \text{and} \quad \mu(t) = 0, \tag{6.27}$$

$$\text{case 3}: \quad c(t) = 0 \quad \text{and} \quad \mu(t) \neq 0. \tag{6.28}$$

The first order perturbation equation (6.7) that needs to be satisfied has many different types of solutions for each of these cases. Therefore we shall present the different solutions in separate sections below. We will also discuss the possibility of $\eta^\dagger \neq \eta$ captured by letting some of the coefficient functions $\gamma_i^{(l)}$ to be purely imaginary.

As noticed in [2, 51] and discussed in chapter 2, an interesting feature of the explicitly time-dependent systems is that the spontaneously broken regime of the time-independent system becomes physical. To see whether this is also the case here we briefly discuss the time-independent version of the Hamiltonian (6.24) with $\dot{a} = \dot{b} = \dot{\lambda} = \dot{\mu} = 0$ in order to create a benchmark for the $\mathcal{PT}$-broken and $\mathcal{PT}$-symmetric regions in the parameter space. Taking the Dyson map to be of the form

$$\eta = \exp(\theta K_4), \qquad \text{with} \quad \theta = \operatorname{arctanh}\left(-\frac{\lambda}{c}\right), \tag{6.29}$$

and acting adjointly on H leads to the Hermitian Hamiltonian

$$h = \eta H_1 \eta^{-1} = \frac{1}{2}\left(a+b\right)\left(K_1+K_2\right) + \frac{1}{2}\sqrt{c^2-\lambda^2}(K_1-K_2) + \mu K_4, \tag{6.30}$$

with eigenvalues

$$E_{n,m} = \frac{1}{2}(1+n+m)(a+b) + \frac{1}{2}(n-m)\sqrt{c^2-\lambda^2}\sqrt{1+\frac{\mu^2}{c^2-\lambda^2}}. \tag{6.31}$$

We notice for the cases 1 and 3, that is when $c = 0$, the Dyson map is ill-defined and also the eigenvalues are complex so that these two cases are always in the spontaneously broken $\mathcal{PT}$-regime. For case 2 we identify a $\mathcal{PT}$-symmetric regime when $|\lambda| < |c|$ and a spontaneously broken regime otherwise. Let us now demonstrate that the spontaneously broken $\mathcal{PT}$-regimes can become physical when an explicit time-dependence is introduced. We need to treat the cases 1 and 2 separately from the case 3, as we find that the perturbative expansions for the metric have no common overlap.

6.4.1 Metric and Dyson maps with $\mu(t) = 0$, cases 1 and 2

We will now show how the above perturbative equations can be solved systematically order by order in ϵ. We treat here the non-Hermitian term as a small perturbation and set $\lambda(t) \to \epsilon\lambda(t)$ with $\epsilon \ll 1$. When succeeding in constructing a complete infinite series we may set ϵ back to 1 depending on the convergence properties. Focusing at first on the cases 1 and 2 with $\mu(t) = 0$, the first order equation (6.7) for the Hamiltonian $H_1(t)$ in (6.24) becomes

$$i\lambda(t)K_3 + \sum_{i=1}^{j} \left(\gamma_i^{(1)} \left[q_i, a(t)K_1 + b(t)K_2 \right] + i\dot{\gamma}_i^{(1)} q_i \right) = 0. \tag{6.32}$$

When compared to the corresponding time-independent equation (4.14), we notice that besides having to satisfy the commutative structure, the coefficient functions are not just a set of functions of the parameters in the model, but correspond now to a system of coupled differential equations. As our algebra is four dimensional we have now the options to take the limit in (6.32) as $j \in \{1, 2, 3, 4\}$ with corresponding generators $q_i \in \{K_1, K_2, K_3, K_4\}$. Taking now at first $j = 4$ with $q_1 = K_4$, $q_2 = K_3$, $q_3 = K_1$ and $q_4 = K_2$, the first order equation becomes

$$i\left(\lambda + c\gamma_1^{(1)} + \dot{\gamma}_2^{(1)}\right) K_3 + i\left(\dot{\gamma}_1^{(1)} - c\gamma_2^{(1)}\right) K_4 + i\dot{\gamma}_3^{(1)} K_1 + i\dot{\gamma}_4^{(1)} K_2 = 0. \tag{6.33}$$

Thus setting the coefficients of all K_i in (6.33) to zero, we obtain two coupled first order equations for $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$. Moreover, we conclude that $\gamma_3^{(1)}$ and $\gamma_4^{(1)}$ are time-independent. As our goal is to find a time-dependent metric and Dyson map we set them both to zero $\gamma_3^{(1)} = \gamma_4^{(1)} = 0$. Having now fixed $j = 2$ and the corresponding $q_1 = K_4$, $q_2 = K_3$, we can simply evaluate the higher order equations obtaining the constraints by setting the coefficient functions to zero. The first equation contains the key foundational structure for the entire series. Note that here however, the ordering of $q_1 = K_4$ and $q_2 = K_3$ is unimportant, the ordering only has an impact on the higher order equations, which we shall also demonstrate.

6.4.2 Hermitian η with $q_1 = K_4$ and $q_2 = K_3$

Keeping now the choice of the q_i as indicated above, we derive the differential equations to be satisfied at each order in ϵ. The first five orders of the equations to be satisfied for the $\gamma_1^{(l)}(t)$ are

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = c\gamma_2^{(1)}, \tag{6.34}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = c\gamma_2^{(2)}, \tag{6.35}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = c\left[\frac{1}{6}\left(\gamma_2^{(1)}\right)^3 + \gamma_2^{(3)}\right], \tag{6.36}$$

$$\epsilon^4: \quad \dot{\gamma}_1^{(4)} = c\left[\frac{1}{2}\left(\gamma_2^{(1)}\right)^2 \gamma_2^{(2)} + \gamma_2^{(4)}\right], \tag{6.37}$$

$$\epsilon^5: \quad \dot{\gamma}_1^{(5)} = c\left[\frac{1}{120}\left(\gamma_2^{(1)}\right)^5 + \frac{1}{2}\gamma_2^{(1)}\left(\gamma_2^{(2)}\right)^2 + \frac{1}{2}\left(\gamma_2^{(1)}\right)^2 \gamma_2^{(3)} + \gamma_2^{(5)}\right]. \tag{6.38}$$

For $\gamma_2(t)$ we obtain the first order differential equations

$$\epsilon^1: \quad \dot{\gamma}_2^{(1)} = -c\gamma_1^{(1)} - \lambda, \tag{6.39}$$

$$\epsilon^2: \quad \dot{\gamma}_2^{(2)} = -c\gamma_1^{(2)}, \tag{6.40}$$

$$\epsilon^3: \quad \dot{\gamma}_2^{(3)} = c\left[\frac{1}{3}\left(\gamma_1^{(1)}\right)^3 - \gamma_1^{(3)} - \frac{1}{2}\gamma_1^{(1)}\left(\gamma_2^{(1)}\right)^2\right], \tag{6.41}$$

$$\epsilon^4: \quad \dot{\gamma}_2^{(4)} = c\left[\left(\gamma_1^{(1)}\right)^2\gamma_1^{(2)} - \gamma_1^{(4)} - \frac{1}{2}\gamma_1^{(2)}\left(\gamma_2^{(1)}\right)^2 - \gamma_1^{(1)}\gamma_2^{(1)}\gamma_2^{(2)}\right], \tag{6.42}$$

$$\begin{aligned}\epsilon^5: \quad \dot{\gamma}_2^{(5)} = c\Bigg[&\gamma_1^{(1)}\left(\gamma_1^{(2)}\right)^2 - \frac{2}{15}\left(\gamma_1^{(1)}\right)^5 + \left(\gamma_1^{(1)}\right)^2\gamma_1^{(3)} - \gamma_1^{(5)} \\ &+\frac{1}{6}\left(\gamma_1^{(1)}\right)^3\left(\gamma_2^{(1)}\right)^2 - \frac{1}{24}\gamma_1^{(1)}\left(\gamma_2^{(1)}\right)^4 - \frac{1}{2}\gamma_1^{(3)}\left(\gamma_2^{(1)}\right)^2 \\ &-\gamma_1^{(2)}\gamma_2^{(1)}\gamma_2^{(2)} - \frac{1}{2}\gamma_1^{(1)}\left(\gamma_2^{(2)}\right)^2 - \gamma_1^{(1)}\gamma_2^{(1)}\gamma_2^{(3)}\Bigg]\end{aligned} \tag{6.43}$$

These equations reveal the underlying structure that distinguishes the different cases. Whilst the equations look rather complex, they contain all the information that can be used to obtain the solutions up to fifth order that can even be extrapolated to the exact solutions.

Perturbation theory to the exact Dyson map and Hermitian Hamiltonians

We shall now demonstrate how to use these equations to obtain the Dyson map and hence the metric. Proceeding similarly as for the first order equation (6.33), we may solve the set of equations (6.34)-(6.38), (6.39)-(6.43) recursively order by order to obtain the explicit expressions for the coefficient functions $\gamma_1^{(i)}(t)$ and $\gamma_2^{(i)}(t)$, $i = 1, 2, \ldots$ We will not report these expressions here. In the next step we extrapolate from the first terms by trying to identify a combination of standard functions whose Taylor expansion matches the first terms in the perturbative series.

For case 1, when $c(t) = 0$, we notice from (6.33) that also $\dot{\gamma}_1^{(1)} = 0$ when requiring Hermiticity of h. As the Hermitian part of the Hamiltonian $H(t)$ is given by $h_0(t) = a(t)(K_1 + K_2)$, we now have $[h_0(t), K_i] = 0$ so that all of the generators in this algebra commute with $h_0(t)$. As a consequence of this we observe that all orders of the perturbation equations disappear except for one. This is also seen by setting $c = 0$ in (6.34)-(6.43) so that the only relevant equation left is

$$\dot{\gamma}_2^{(1)}(t) = -\lambda(t). \tag{6.44}$$

Hence, we easily obtain the exact solution

$$\gamma_1^{(1)}(t) = \gamma_1(t) = k_1, \qquad \gamma_2^{(1)}(t) = \gamma_2(t) = -\int^t \lambda(s)ds + k_2,$$

with two integration constants k_1, k_2.

For case 2, when $c(t) \neq 0$, all of the right hand sides of the differential equations are proportional to $c(t)$, except for the one for $\dot{\gamma}_2^{(1)}(t)$ in (6.39). Assuming $\lambda(t)$ to be a real multiple of $c(t)$ the equations become fully integrable and we are able to solve the equations order by order, even leading to an exact solution. Keeping for instance terms up to fifth order we obtain

$$[\dot{\gamma}_1(t)]^{[5]} = \sum_{i=1}^{5} \epsilon^i \dot{\gamma}_1^{(i)}(t) = c(t) \left[\epsilon \sinh\left(\sum_{i=1}^{5} \epsilon^i \gamma_2^{(i)}(t)\right)\right]^{[5]} = c(t)\left\{\epsilon \sinh\left[\gamma_2(t)\right]\right\}^{[5]}, \tag{6.45}$$

and

$$\begin{aligned}[\dot{\gamma}_2(t)]^{[5]} &= \sum_{i=1}^{5} \epsilon^i \dot{\gamma}_2^{(i)}(t) \\ &= -\lambda(t) - c(t)\left\{\epsilon\left[\cosh\left(\sum_{i=1}^{5} \epsilon^i \gamma_2^{(i)}(t)\right)\right]\left[\tanh\left(\sum_{i=1}^{5} \epsilon^i \gamma_1^{(i)}(t)\right)\right]\right\}^{[5]} \\ &= -\lambda(t) - c(t)\left(\epsilon \cosh[\gamma_2(t)] \tanh[\gamma_1(t)]\right)^{[5]}. \end{aligned} \tag{6.46}$$

Here the superscript [5] means we only retain terms up to order 5 in ϵ. In fact, we have verified the validity of the closed form to eleventh order, by extending and solving the sets of equations (6.34)-(6.38) and (6.39)-(6.43).

Assuming now the expressions in (6.45) and (6.46) to be exact, we may set $\epsilon = 1$ and subsequently solve them for $\gamma_1(t)$ and $\gamma_2(t)$. Letting $\lambda(t)$ be any real multiple of $c(t)$, that is

$$c(t) = p\lambda(t) \qquad \text{where} \qquad p \in \mathbb{R}, \tag{6.47}$$

we are able to solve the relevant equations exactly and express γ_2 as a function of γ_1 as

$$\gamma_2(\gamma_1) = \pm \operatorname{arccosh}\left\{-\frac{1}{2}\operatorname{sech}(\gamma_1)\left[k_1 + \frac{2}{p}\sinh(\gamma_1)\right]\right\}, \tag{6.48}$$

with k_1 being an integration constant. Relation (6.48) is obtained by integrating $\dot{\gamma}_2/\dot{\gamma}_1 = \partial\gamma_2/\partial\gamma_1$ with respect to γ_1. Parameterizing $\gamma_1(t)$ by a new function $\chi(t)$ as

$$\gamma_1 = \operatorname{arcsinh}(\chi), \tag{6.49}$$

the two differential equations for $\dot{\gamma}_1(t)$ and $\dot{\gamma}_2(t)$ can be converted into the linear second order equation entirely in χ

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} + (p^2-1)\lambda^2\chi = k_1\frac{p}{2}\lambda^2. \tag{6.50}$$

We solve equation (6.50) by

$$\chi(t) = \frac{e^{-2\sqrt{1-p^2}\left(k_2-\frac{1}{2}\int^t \lambda(s)ds\right)}}{4(1-p^2)}\left[\left(e^{2\sqrt{1-p^2}\left(k_2-\frac{1}{2}\int^t \lambda(s)ds\right)} - pk_1\right)^2 + (k_1^2-4)\left(1-p^2\right)\right]. \tag{6.51}$$

Notice that in fact we are solving the two first order equations for $\dot{\gamma}_1(t)$ and $\dot{\gamma}_2(t)$, so that there are only two integration constants and no additional linear independent solution for the second order equation (6.50). We have to impose here $|p| < 1$ to ensure the reality of χ and hence γ_2, γ_1.

Having obtained an exact Dyson map, we can invoke the time-dependent Dyson equation (2.3) and compute the Hermitian counterparts to $H_1(t)$, which consists of two decoupled harmonic oscillators in both cases 1 and 2

$$h(t) = f_+(t)K_1 + f_-(t)K_2. \tag{6.52}$$

For case 1 we find $f_\pm(t) = a$ and for case 2 we obtain

$$f_\pm(t) = b + \frac{p\lambda}{2} \mp \frac{\lambda(2\chi + pk_1)}{4(1+\chi^2)}. \tag{6.53}$$

We may also compute real time-dependent energy expectation values from these expressions as will be shown below.

In the following three subsections we shall be repeating this process to determine five further Dyson maps, one Hermitian and four non-Hermitian. While the procedure for the computation of each map is similar, there are technicalities associated with obtaining the parametrizations of the time-dependent coefficient functions $\gamma_i(t)$'s and the auxiliary equations. For the non-Hermitian Dyson maps we also have to modify the Ansatz in (6.4). In section 6.4.6 the reader can find a summary of all relevant information for the six Dyson maps we consider here.

6.4.3 Hermitian η with $q_1 = K_3$ and $q_2 = K_4$

As mentioned above, the order in which we take our $q_i's$ is important and only manifests in the higher order equations. Therefore we now take $q_1 = K_3$ and $q_2 = K_4$ and derive the perturbative equations. For $\gamma_1(t)$ we find that the first four orders of the equations that need to be satisfied are

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = -c\gamma_2^{(1)} - \lambda, \tag{6.54}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = -c\gamma_2^{(2)}, \tag{6.55}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = -c\left[\frac{1}{6}\left(\gamma_2^{(1)}\right)^3 + \gamma_2^{(3)}\right] - \frac{1}{2}\left(\gamma_2^{(1)}\right)^2\lambda, \tag{6.56}$$

$$\epsilon^4: \quad \dot{\gamma}_1^{(4)} = -c\left[\frac{1}{2}\left(\gamma_2^{(1)}\right)^2\gamma_2^{(2)} + \gamma_2^{(4)}\right] - \gamma_2^{(1)}\gamma_2^{(2)}\lambda. \tag{6.57}$$

The first order differential equations for $\gamma_2(t)$ are

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = c\gamma_1^{(2)}, \tag{6.58}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = c\gamma_1^{(2)}, \tag{6.59}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = -c\left[\frac{1}{3}\left(\gamma_1^{(1)}\right)^3 - \gamma_1^{(3)} - \frac{1}{2}\gamma_1^{(1)}\left(\gamma_2^{(1)}\right)^2\right] + \gamma_1^{(1)}\gamma_2^{(1)}\lambda, \tag{6.60}$$

$$\epsilon^4: \quad \dot{\gamma}_1^{(4)} = -c\left[\left(\gamma_1^{(1)}\right)^2\gamma_1^{(2)} - \gamma_1^{(2)} - \frac{1}{2}\gamma_1^{(2)}\left(\gamma_2^{(1)}\right)^2 - \gamma_1^{(1)}\gamma_2^{(1)}\gamma_2^{(2)}\right] + \lambda\left[\gamma_1^{(2)}\gamma_2^{(1)} + \gamma_1^{(1)}\gamma_2^{(2)}\right]. \tag{6.61}$$

We can now extrapolate this information to find another exact solution for the time-dependent Dyson map.

Perturbation theory to the exact Dyson map and Hermitian Hamiltonians

We may solve the equations (6.54)-(6.57) and (6.58)-(6.61) recursively order by order and match them to a Taylor expansion of a combination of standard functions as we did in the previous section. In doing this we obtain

$$\dot{\gamma}_1 = -\lambda\cosh(\gamma_2) - c\sinh(\gamma_2), \quad \text{and} \quad \dot{\gamma}_2 = \left[c\cosh(\gamma_2) + \lambda\sinh(\gamma_2)\right]\tanh(\gamma_1), \tag{6.62}$$

which are exact. We shall now solve these equations for each case.

<u>For case 1</u> we have that $c(t) = 0$. We may solve for γ_1 in terms of γ_2 and obtain

$$\gamma_2(\gamma_1) = \text{arcsinh}[k_1 \, \text{sech}(\gamma_1)], \tag{6.63}$$

where k_1 is an integration constant. By letting

$$\gamma_1 = \text{arccosh}(\chi), \tag{6.64}$$

we may convert the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ into the dissipative Ermakov-Pinney equation [138, 139]

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2 \chi = \frac{k_1^2 \lambda^2}{\chi^3}. \tag{6.65}$$

The solution to this equation is given by

$$\chi(t) = \left[1 + (1 + k_1^2) \sinh^2 \left(k_2 - \int^t \lambda(s) ds\right)\right]^{1/2}. \tag{6.66}$$

<u>For case 2</u> we must restrict $c(t) = \lambda(t)$ for the equations to become solvable. We again express γ_1 in terms γ_2

$$\gamma_2(\gamma_1) = \ln \left[\frac{k_2}{\cosh(\gamma_1)}\right], \tag{6.67}$$

here k_2 is an integration constant. We again let γ_1 to be given by equation (6.64) which then converts the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ into

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} = \frac{k_2^2 \lambda^2}{\chi^3}, \tag{6.68}$$

which has solution

$$\chi(t) = \left[1 + \left(k_2 - k_3 \int^t \lambda(s) ds\right)^{1/2}\right]^2. \tag{6.69}$$

The resulting Hermitian Hamiltonians obtained by substituting the Dyson maps described by these equations into the TDDE are of the general form (6.52) where the time-dependent functions are given by

$$f_\pm(t) = b \pm \frac{\lambda k_1}{2\chi^2}, \tag{6.70}$$

for case 1 and

$$f_\pm(t) = b + \frac{\lambda}{2} \pm \frac{\lambda k_2}{2\chi^2}, \tag{6.71}$$

for case 2.

6.4.4 Non-Hermitian η with $q_1 = K_4$ and $q_2 = K_1, K_2$

Making now the choice $q_1 = K_4$, $q_2 = K_1, K_2$ the perturbative expansion yields $\dot{\gamma}_1^{(l)} = \dot{\gamma}_2^{(l)} = 0$, so that the entire metric becomes time-independent. However, η does not have to be Hermitian as assumed in the Ansatz (6.4). Thus allowing $\gamma_i^{(l)} \in \mathbb{C}$ in general, we now modify the Ansatz to $\gamma_1^{(\ell)} \in \mathbb{R}$, $\ell = 1, 2, \ldots$, $\gamma_2^{(\ell)} \in i\mathbb{R}$, $\ell = 0, 1, 2, \ldots$, $\gamma_3^{(\ell)} = \gamma_4^{(\ell)} = 0$. The perturbative constraints up to order ϵ^3 then read

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = \pm\lambda \sin\left(\gamma_2^{(0)}\right), \tag{6.72}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = \pm\lambda\gamma_2^{(1)} \cos\left(\gamma_2^{(0)}\right), \tag{6.73}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = \pm\lambda\gamma_2^{(2)} \cos\left(\gamma_2^{(0)}\right) \mp \frac{1}{2}\lambda\left(\gamma_2^{(1)}\right)^2 \sin\left(\gamma_2^{(0)}\right), \tag{6.74}$$

and for $\gamma_2(t)$ we obtain

$$\epsilon^1: \quad \dot{\gamma}_2^{(0)} = \pm c \pm \lambda\frac{\cos\left(\gamma_2^{(0)}\right)}{\gamma_1^{(1)}}, \tag{6.75}$$

$$\epsilon^2: \quad \dot{\gamma}_2^{(1)} = \mp\frac{\lambda}{\gamma_1^{(1)}}\left[\frac{\gamma_1^{(2)}}{\gamma_1^{(1)}}\cos\left(\gamma_2^{(0)}\right) + \gamma_2^{(1)}\sin\left(\gamma_2^{(0)}\right)\right], \tag{6.76}$$

$$\epsilon^3: \quad \dot{\gamma}_2^{(2)} = \pm\frac{\lambda}{\gamma_1^{(1)}}\left\{\left[\frac{\left(\gamma_1^{(1)}\right)^2}{3} + \left(\frac{\gamma_1^{(2)}}{\gamma_1^{(1)}}\right)^2 - \frac{\gamma_1^{(3)}}{\gamma_1^{(1)}} - \frac{\gamma_2^{(1)}}{2}\right]\cos\left(\gamma_2^{(0)}\right) + \left[\frac{\gamma_1^{(2)}\gamma_2^{(1)}}{\gamma_1^{(1)}} - \gamma_2^{(2)}\right]\sin\left(\gamma_2^{(0)}\right)\right\}, \tag{6.77}$$

where the upper sign solution is taken for $q_2 = K_1$ and the lower sign for $q_2 = K_2$.

Perturbation theory to the exact Dyson map and Hermitian Hamiltonians

Once again we may solve these equations order by order for the coefficient functions $\gamma_i^{(\ell)}$ and subsequently try to extrapolate the series to all orders. We find the exact constraining equations for $\gamma_1(t)$ and $\gamma_2(t)$ by demanding the non-Hermitian terms in $h(t)$ to vanish

$$\dot{\gamma}_1 = \pm\lambda \sin\left(\gamma_2\right), \qquad \text{and} \qquad \dot{\gamma}_2 = \pm c \pm \lambda\cos\left(\gamma_2\right)\coth(\gamma_1).$$

We may now solve these equations separately in each case.

For case 1 we can solve for γ_1 in terms of γ_2 obtaining

$$\gamma_1(\gamma_2) = \text{arcsinh}\left[k_1 \sec(\gamma_2)\right], \tag{6.78}$$

with integration constant k_1. By letting

$$\gamma_2 = \mp \arctan(\chi), \tag{6.79}$$

the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are converted into the linear second order differential equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2\chi = 0. \tag{6.80}$$

We observe that the auxiliary equation (6.50) reduces to equation (6.80) in the limit $p \to 0$ which also holds for the solution (6.51). We have two constants of integration left after having carried out the limit.

For case 2 we set $c(t) = p\lambda(t)$ as then the equations become solvable. In this case it is more convenient to express γ_2 in terms of γ_1

$$\gamma_2(\gamma_1) = \arccos\left[-p\coth(\gamma_1) - i\frac{1}{2}k_1 \,\text{cosech}(\gamma_1)\right], \tag{6.81}$$

where k_1 is an integration constant that we set to 0 to ensure the reality of γ_2. Letting

$$\gamma_1 = \text{arccosh}\left(\chi\right), \tag{6.82}$$

the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are converted into the linear second order differential equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} + (p^2 - 1)\lambda^2\chi = 0. \tag{6.83}$$

We note that equations (6.83) is obtained from (6.50) in the limit $k_1 \to 0$, which also holds for the solution (6.51). As we have already chosen one of the integration constants, there is only one left in this case, i.e. k_2.

After imposing the constraints, the remaining Hermitian part of the Hamiltonian is of the same general form as the one reported in (6.52), albeit with different forms for the coefficient functions

$$f_\pm(t) = b - \frac{\lambda(\pm 1 + \sqrt{1 + (1+\chi^2)k_1^2})}{2(1+\chi^2)k_1}, \tag{6.84}$$

in case 1 and

$$f_{\pm}(t) = b + \frac{p\lambda\chi}{2(\chi \mp 1)}, \tag{6.85}$$

in case 2, respectively for $q_2 = K_1$. For $q_2 = K_2$ we have

$$f_{\pm}(t) = b + \frac{\lambda(\mp 1 + \sqrt{1 + (1 + \chi^2)k_1^2})}{2(1 + \chi^2)k_1}, \tag{6.86}$$

in case 1 and

$$f_{\pm}(t) = b + p\lambda - \frac{p\lambda\chi}{2(\chi \pm 1)}, \tag{6.87}$$

in case 2.

6.4.5 Non-Hermitian η with $q_1 = K_3$ and $q_2 = K_1$ or $q_2 = K_2$

We now make the choice $q_1 = K_3$, $q_2 = K_1, K_2$ and again modify our Ansatz to $\gamma_1^{(\ell)} \in \mathbb{R}$, $\ell = 1, 2, \ldots$, $\gamma_2^{(\ell)} \in i\mathbb{R}$, $\ell = 0, 1, 2, \ldots$, $\gamma_3^{(\ell)} = \gamma_4^{(\ell)} = 0$. The perturbative constraints up to order ϵ^3 then read

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = -\lambda \cos\left(\gamma_2^{(0)}\right), \tag{6.88}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = \lambda\gamma_2^{(1)} \sin\left(\gamma_2^{(0)}\right), \tag{6.89}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = \lambda\gamma_2^{(2)} \sin\left(\gamma_2^{(0)}\right) + \frac{1}{2}\lambda\left(\gamma_2^{(1)}\right)^2 \cos\left(\gamma_2^{(0)}\right), \tag{6.90}$$

and for $\gamma_2(t)$ we obtain

$$\epsilon^1: \quad \dot{\gamma}_2^{(0)} = \pm c + \lambda\frac{\sin\left(\gamma_2^{(0)}\right)}{\gamma_1^{(1)}}, \tag{6.91}$$

$$\epsilon^2: \quad \dot{\gamma}_2^{(1)} = -\frac{\lambda}{\gamma_1^{(1)}}\left[\frac{\gamma_1^{(2)}}{\gamma_1^{(1)}} \sin\left(\gamma_2^{(0)}\right) - \gamma_2^{(1)} \cos\left(\gamma_2^{(0)}\right)\right], \tag{6.92}$$

$$\begin{aligned}\epsilon^3: \quad \dot{\gamma}_2^{(2)} = \frac{\lambda}{\gamma_1^{(1)}}&\left\{\left[\frac{\left(\gamma_1^{(1)}\right)^2}{3} + \left(\frac{\gamma_1^{(2)}}{\gamma_1^{(1)}}\right)^2 - \frac{\gamma_1^{(3)}}{\gamma_1^{(1)}} - \frac{\gamma_2^{(1)}}{2}\right] \sin\left(\gamma_2^{(0)}\right)\right. \\ &\left. - \left[\frac{\gamma_1^{(2)}\gamma_2^{(1)}}{\gamma_1^{(1)}} - \gamma_2^{(2)}\right] \cos\left(\gamma_2^{(0)}\right)\right\},\end{aligned} \tag{6.93}$$

where the upper sign solution is taken for $q_2 = K_1$ and the lower sign for $q_2 = K_2$.

Perturbation theory to the exact Dyson map and Hermitian Hamiltonians

By solving the equations order by order and extrapolating to all orders we find that the exact constraining equations for $\gamma_1(t)$ and $\gamma_2(t)$ to be given by

$$\ddot{\gamma}_1 = -\lambda\cos(\gamma_2), \quad \text{and} \quad \dot{\gamma}_2 = \pm c + \lambda\sin(\gamma_2)\coth(\gamma_1). \tag{6.94}$$

We now proceed to solve these equations separately for the two cases.

<u>For case 1</u> we solve first for γ_2 in terms of γ_1 and obtain

$$\gamma_1(\gamma_2) = \text{arcsinh}[k_1\csc(\gamma_2)], \tag{6.95}$$

where k_2 is an integration constant. By letting

$$\gamma_2 = \text{arccot}(\chi), \tag{6.96}$$

we find that the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ may be converted into the linear second order differential equation given by (6.80). The solution to this differential equation is given by equation (6.51) in the limit $p \to 0$.

<u>For case 2</u> we must set $c(t) = p\lambda(t)$ to ensure that the equations are solvable. This time we solve for γ_1 in terms of γ_2 and obtain

$$\gamma_2(\gamma_1) = \arcsin\left\{\frac{1}{2}\left[k_2 \mp 2p\cosh(\gamma_1)\right]\text{cosech}(\gamma_1)\right\}, \tag{6.97}$$

where k_2 is a constant of integration. By letting

$$\gamma_1 = \text{arccosh}(\chi), \tag{6.98}$$

we find that the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ may be converted into equation (6.50) which has solution (6.51).

We may now determine the resulting Hermitian Hamiltonians by substituting the Dyson maps into the TDDE (2.3). For all the maps the form of the Hermitian Hamiltonian is given by (6.52). For $q_2 = K_1$ we have that the time-dependent coefficient functions are given by

$$f_\pm(t) = b + \frac{\lambda[\mp 1 - \sqrt{1 + k_1^2(1+\chi^2)}]}{2k_1(1+\chi^2)}, \tag{6.99}$$

for case 1 and

$$f_\pm(t) = b + \frac{\lambda(2p\chi - k_1)}{4(\chi \mp 1)}, \tag{6.100}$$

for case 2. For $q_2 = K_2$ we have

$$f_\pm(t) = b + \frac{\lambda[\pm 1 - \sqrt{1 + k_2^2(1+\chi^2)}]}{2k_2(1+\chi^2)}, \tag{6.101}$$

for case 1 and

$$f_\pm(t) = b - \frac{\lambda(2p\chi + k_2)}{4(\chi \pm 1)}, \tag{6.102}$$

for case 2.

6.4.6 Summary of exact Dyson maps, auxiliary equations and Hermitian Hamiltonians

Having made a distinction in the setup of the perturbative treatment between Hermitian and non-Hermitian Dyson maps, it has been possible to find six unique exact Dyson maps and hence metrics for both cases 1 and 2. While we have already presented the procedure to find all these maps we here present them in a more compact form for ease of reference. We will also be referring back to this section in subsequent chapters when we present a method of determining an infinite series of Dyson maps. In table 6.1 you will find all Dyson maps we have derived using the perturbative method, all the Dyson maps have the general form

$$\eta(t) = \exp[\gamma_1(t)q_1]\exp[\gamma_2(t)q_2], \tag{6.103}$$

which $q_1, q_1, \gamma_1(t)$ and $\gamma_2(t)$ given in table 6.1.

q_1, q_2	$\dot{\gamma}_1(t)$	$\dot{\gamma}_2(t)$
K_4, K_3	$c\sinh(\gamma_2)$	$-c\cosh(\gamma_2)\tanh(\gamma_1) - \lambda$
K_3, K_4	$-\lambda\cosh(\gamma_2) - c\sinh(\gamma_2)$	$[c\cosh(\gamma_2) + \lambda\sinh(\gamma_2)]\tanh(\gamma_1)$
$K_4, iK_{1,2}$	$\pm\lambda\sin(\gamma_2)$	$\pm c \pm \lambda\cos(\gamma_2)\coth(\gamma_1)$
$K_3, iK_{1,2}$	$-\lambda\cos(\gamma_2)$	$\pm c + \lambda\sin(\gamma_2)\coth(\gamma_1)$

Table 6.1: Coupled first order differential equation constraints on the time-dependent coefficient functions γ_1 and γ_2 in the Dyson map η, for different choices of q_1 and q_2.

All presented solutions and cases are new, except for the Hermitian case with $q_1 = K_3$, $q_2 = K_4$, $c = 0$ which reproduces a solution found in [51], with the difference that the Dyson map we are considering here are missing the two factors involving the time-independent K_1 and K_2 terms. The parameterization of $\gamma_{1,2}$ in terms of a new function, that we always denote as $\chi(t)$, are not obvious and differ

q_1, q_2	constraint	$\gamma_1(\chi)$	$\gamma_2(\chi)$	constraint
K_4, K_3	$c = 0$	*	*	*
K_4, K_3	$c = p\lambda$	$\text{arcsinh}\,(\chi)$	$\text{arccosh}\left(-\frac{k_1+2p\chi}{2\sqrt{1+\chi^2}}\right)$	$-\frac{k_1+2p\chi}{2\sqrt{1+\chi^2}} \leq 1$
K_3, K_4	$c = 0$	$\text{arccosh}\,(\chi)$	$\text{arcsinh}\left(\frac{k_1}{\chi}\right)$	$\chi > 1$
K_3, K_4	$c = \lambda$	$\text{arccosh}\,(\chi)$	$\ln\left(\frac{k_1}{\chi}\right)$	$\chi > 1$
$K_4, iK_{1,2}$	$c = 0$	$\text{arcsinh}\left(k_1\sqrt{1+\chi^2}\right)$	$\mp\arctan(\chi)$	*
$K_4, iK_{1,2}$	$c = p\lambda$	$\text{arccosh}\,(\chi)$	$\arccos\left(-\frac{p\chi}{\sqrt{\chi^2-1}}\right)$	$\chi > 1$
$K_3, iK_{1,2}$	$c = 0$	$\text{arcsinh}\left(k_1\sqrt{1+\chi^2}\right)$	$\text{arccot}(\chi)$	*
$K_3, iK_{1,2}$	$c = p\lambda$	$\text{arccosh}\,(\chi)$	$\arcsin\left(\frac{k_1\mp 2p\chi}{2\sqrt{\chi^2-1}}\right)$	$\chi > 1$

Table 6.2: Parameterisation of γ_1 and γ_2 in terms of the auxiliary function χ with additional constraint on $c(t)$ for different choices of q_1 and q_2. The constraints in the last column result from the parameterization. A $*$ indicates no constraint.

for the multitude of maps we consider and are therefore presented in table 6.2. We may only solve these equations upon imposing an additional restriction on the time-dependent functions in the Hamiltonian, which are also reported in table 6.2.

We derived a total of five different auxiliary equations for the maps we found. As discussed in the previous subsections, combining the equations for the constraints on γ_1 and γ_2 leads to a set of second order auxiliary equations that we present in table 6.3.

q_1, q_2	constraint	auxiliary equation
$K_4,\ K_3$	$c = 0$	none
K_4, K_3 $K_3, iK_{1,2}$	$c = p\lambda$	$\text{Aux}_1:\quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\,\dot{\chi} - (1-p^2)\lambda^2\chi = k_1\frac{p}{2}\lambda^2$
$K_{3,4},\ iK_{1,2}$	$c = 0$	$\text{Aux}_2:\quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\,\dot{\chi} - \lambda^2\chi = 0$
$K_4,\ iK_{1,2}$	$c = p\lambda$	$\text{Aux}_3:\quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\,\dot{\chi} - (1-p^2)\lambda^2\chi = 0$
$K_3,\ K_4$	$c = 0$	$\text{Aux}_4:\quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\,\dot{\chi} - \lambda^2\chi = k_1^2\lambda^2\frac{1}{\chi^3}$
$K_3,\ K_4$	$c = \lambda$	$\text{Aux}_5:\quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\,\dot{\chi} = k_1^2\lambda^2\frac{1}{\chi^3}$

Table 6.3: Auxiliary equations to be satisfied by quantities in the parameterisation of the functions γ_1 and γ_2 together with the additional constraint on $c(t)$ for different choices of q_1 and q_2.

Solutions to the auxiliary equations As the last step we disentangle the parameterisations for γ_1 and γ_2 by solving the auxiliary equations for χ. We have encountered one case with no restrictions at all, three types of linear second order equations and two versions of the nonlinear Ermakov-Pinney (EP) equation [138, 139].

We already reported the solutions to the linear equations referred to as Aux_1 in table 6.3 in (6.51), from which we obtain the solution to Aux_2 in the limit $p \to 0$ and Aux_3 in limit $k_1 \to 0$. The solutions to Aux_4 and Aux_5 are given by equations (6.66) and (6.69) respectively. With time-dependent harmonic oscillators we usually find that the time-dependence is governed by an Ermakov-Pinney equation, we will demonstrate that this is still the case when we compute the wavefunctions associated with all the Hermitian Hamiltonians.

Finally we turn to the resulting Hermitian Hamiltonian $h(t)$ that is always of the general form of two uncoupled harmonic oscillators (6.52) with different time-dependent coefficient functions $f_\pm(t)$ as reported in table 6.4.

q_1, q_2	constraint	$f_\pm(t)$	η
K_4, K_3	$c = 0$	a	η_1
K_4, K_3	$c = p\lambda$	$b + \frac{p\lambda}{2} \mp \frac{\lambda(2\chi+pk_1)}{4(1+\chi^2)}$	η_1
K_3, K_4	$c = 0$	$b \pm \frac{\lambda k_1}{2\chi^2}$	η_2
K_3, K_4	$c = \lambda$	$b + \frac{\lambda}{2} \pm \frac{\lambda k_1}{2\chi^2}$	η_2
K_4, iK_1	$c = 0$	$b - \frac{\lambda(\pm 1+\sqrt{1+(1+\chi^2)k_1^2})}{2(1+\chi^2)k_1}$	η_3
K_4, iK_1	$c = p\lambda$	$b + \frac{p\lambda\chi}{2(\chi\mp 1)}$	η_3
K_4, iK_2	$c = 0$	$b + \frac{\lambda(\mp 1+\sqrt{1+(1+\chi^2)k_1^2})}{2k_1(1+\chi^2)}$	η_4
K_4, iK_2	$c = p\lambda$	$b + p\lambda - \frac{p\lambda\chi}{2(\chi\pm 1)}$	η_4
K_3, iK_1	$c = 0$	$b + \frac{\lambda\left[\mp 1-\sqrt{1+k_1^2(1+\chi^2)}\right]}{2k_1(\chi^2+1)}$	η_5
K_3, iK_1	$c = p\lambda$	$b + \frac{\lambda(2p\chi-k_1)}{4(\chi\mp 1)}$	η_5
K_3, iK_2	$c = 0$	$b + \frac{\lambda\left[\pm 1-\sqrt{1+k_1^2(1+\chi^2)}\right]}{2k_1(1+\chi^2)}$	η_6
K_3, iK_2	$c = p\lambda$	$b - \frac{\lambda(2p\chi+k_1)}{4(\chi\pm 1)}$	η_6

Table 6.4: Time-dependent coefficient in the Hermitian Hamiltonian $h(t) = f_+(t)K_1 + f_-(t)K_2$ together with the additional constraint on $c(t)$ for different choices of q_1 and q_2. In the last column we report a short notation for the Dyson maps of the particular cases that we shall use below for convenience.

6.4.7 Time-dependent eigenfunctions, energies and $\mathcal{PT}$-symmetry breaking

Next we present the expectation values for the time-dependent energy operator $\tilde{H}(t)$ as defined in equation (2.8). Since each of the Hermitian Hamiltonians constructed from any of the similarity transformations simply consists of two uncoupled harmonic oscillators (6.52) with different time-dependent coefficient functions, we can easily construct the total wavefunction as a product of the wavefunctions for a harmonic oscillator with real time-dependent mass and frequency of the form $\tilde{h}(t) = f(t)/2(p_x^2 + x^2)$. The latter problem was solved originally in [119]. Adapting to our notation and including a normalization constant, found in [51], the time-dependent wavefunction is given by

$$\tilde{\phi}_n(x,t) = \frac{e^{i\alpha_n(t)}}{\sqrt{2^n n! \sqrt{\pi}\chi(t)}} \exp\left[\left(\frac{i}{f(t)}\frac{\dot{\chi}(t)}{\chi(t)} - \frac{1}{\chi^2(t)}\right)\frac{x^2}{2}\right] H_n\left[\frac{x}{\chi(t)}\right], \tag{6.104}$$

where $H_n[x]$ denotes the n-th Hermite polynomial in x and the phase is given by

$$\alpha_n(t) = -\left(n + \frac{1}{2}\right)\int_0^t \frac{f(s)}{\chi^2(s)} ds. \tag{6.105}$$

The auxiliary function $\chi(t)$ is constrained by the dissipative Ermakov-Pinney equation of the form

$$\ddot{\chi} - \frac{\dot{f}}{f}\dot{\chi} + f^2\chi = \frac{f^2}{\chi^3}. \tag{6.106}$$

Interestingly this is equation Aux_4 in table 6.3 with $\lambda \to if$, $k_1^2 = i$. However, the solution (6.66) to Aux_4 reduces to 1 for these parameter choices. Instead, equation (6.106) is solved by

$$\chi(t) = \sqrt{\sqrt{1+\kappa^2} + c\cos\left[2\int^t f(s)ds\right]}, \tag{6.107}$$

with integration constant κ. The expectation value of K_1 is then computed to

$$\left\langle \tilde{\phi}_n(x,t)\right| K_1 \left|\tilde{\phi}_m(x,t)\right\rangle = \left(n + \frac{1}{2}\right)\sqrt{1+\kappa^2}\delta_{n,m}. \tag{6.108}$$

Hence, the solution to the full time-dependent Schrödinger equation for the Hermitian Hamiltonian $h(t)$ in (6.52) is simply the product of the two wavefunctions in (6.104)

$$\Psi_h^{n,m}(x,y,t) = \tilde{\phi}_n^{f_+}(x,t)\tilde{\phi}_m^{f_-}(y,t), \tag{6.109}$$

from which we calculate the instantaneous energy expectation values

$$E^{n,m}(t) = \left\langle \Psi_h^{n,m}(t)\right| h(t) \left|\Psi_h^{n,m}(t)\right\rangle = \sum_{i=-,+} f_i(t)\left(n + \frac{1}{2}\right)\sqrt{1+\kappa_i^2}. \tag{6.110}$$

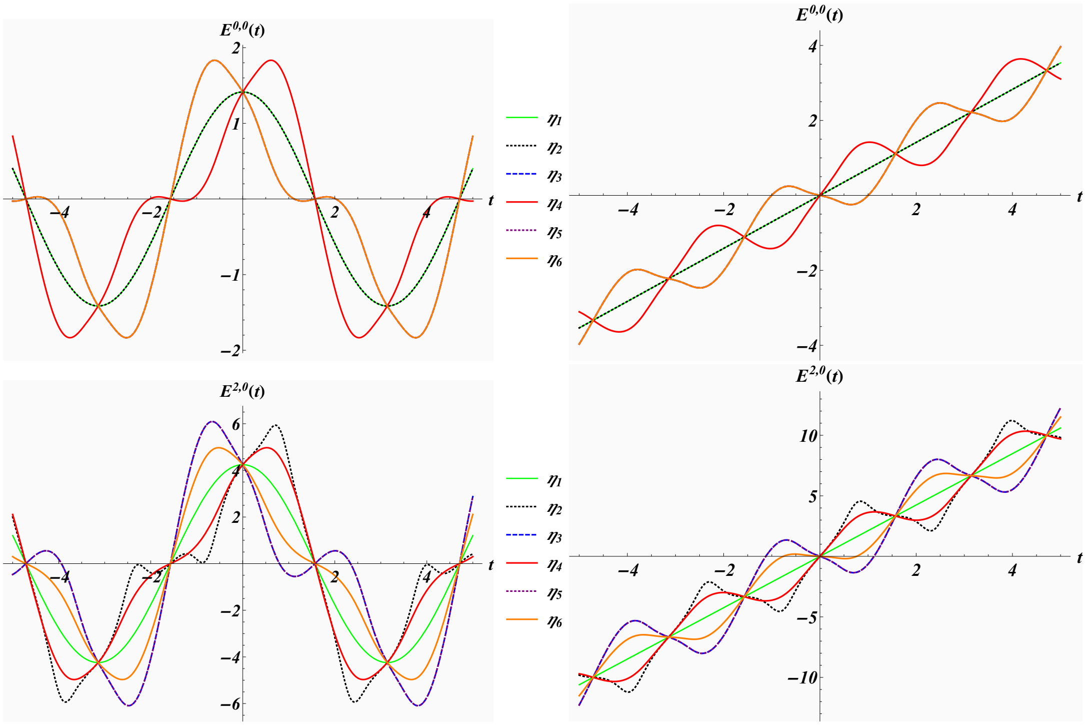

Figure 6.1: The instantaneous energy spectra (6.110) associated with the six Dyson maps for $\lambda(t) = \sin(2t)$ for case 1 with $\kappa_+ = \kappa_- = 1$, $k_1 = 2$. In panels (a), (c) we have $a(t) = \cos(t)$ and in panels (a), (c) we that $a(t) = t/2$.

These expectation values are real provided $f_\pm(t), \kappa_\pm \in \mathbb{R}$. For case 1 this is simply guaranteed by taking the parameter and time-dependent functions to be real. For case 2 we can not freely choose and have to respect the constraints resulting as a consequence of the parameterization as reported in table 6.2. As the auxiliary function $\chi(t)$ must be real, the additional constraint $|p| < 1$ results from the form of the solution (6.51), together with $k_1, k_2 \in \mathbb{R}$. For concrete choices of the time-dependent coefficient functions we can now directly evaluate the expressions for $E_i^{n,m}(t)$ corresponding to the Dyson maps $\eta_i(t)$ explicitly by computing the auxiliary functions $\chi(t)$ and the functions $f_i(t)$. The Dyson map η_2 leads to somewhat different behaviour. This is understood by the fact that it can only be constructed at $c = 0$ and at what would be the exceptional point in the time-independent scenario $c = \lambda$. Hence also the energies exhibit slightly different characteristics. Taking the above mentioned constraints into account there are large regions in the parameter space for which all of the energies $E_i^{n,m}(t)$ are real and hence physical. We illustrate the behaviour of these energies for each of the Dyson maps in figues 6.1 and 6.2 for some concrete choices. First of all we observe from figure 6.1 the crucial feature that the instantaneous energy is real and finite. Secondly we note that despite sharing

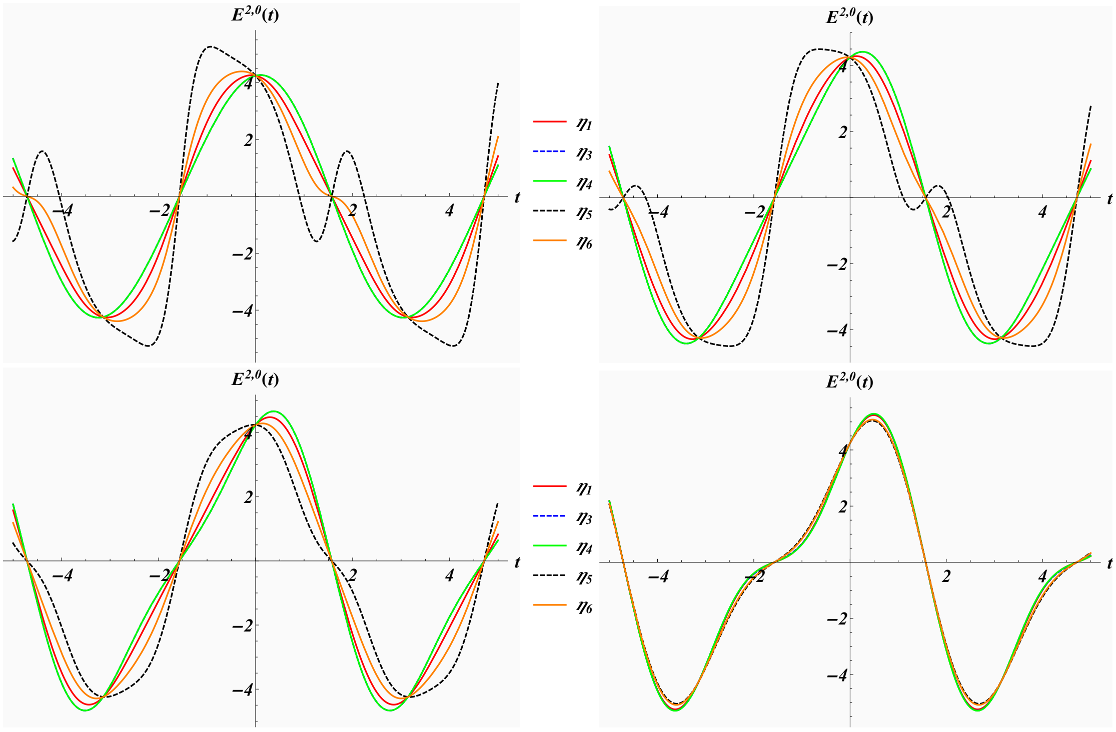

Figure 6.2: The instantaneous energy spectra (6.110) associated with five Dyson maps for $\lambda(t) = \sin(2t)$, $a(t) = \cos(t)$ for case 2 with $\kappa_+ = \kappa_- = 1$, $k_1 = 2.5$, $k_2 = 1$. We have $p = -0.1$, $p = -0.3$, $p = -0.5$, $p = -0.9$ in panels (a), (b), (c), (d), respectively.

the same non-Hermitian Hamiltonian, the theories related to different Dyson maps can lead to quite different physical behaviour in the energy. Similar to the time-independent scenario, this is the known fact that the Hamiltonian alone does not define a unique definite physical system, but to define the physics one also needs to specify the metric, i.e. the Dyson map. We note that some of the energies can become degenerate, $E_1^{n,n} = E_2^{n,n}$, which can however split when $n \neq m$. As is also expected from the explicit expressions, the differences are more amplified the larger $|n - m|$. In case 2, when we have non vanishing values of the parameter p, these effects are even more amplified as can be seen in figure 6.2. We notice a strong sensitivity with regard to p.

The constraints resulting from the parameterization, $|p| < 1$, imply that we are in the regime with spontaneously broken $\mathcal{PT}$-symmetry when compared to the time-independent case. Therefore, we observe the same phenomenon that was first noted in [2, 51], namely that the introduction of a time-dependence into the metric will mend the spontaneously broken $\mathcal{PT}$-regime so that it becomes physically meaningful. In this case this manifests itself by the fact that the instantaneous energy is real.

6.4.8 Metric and Dyson maps with $\mu(t) \neq 0$, case 3

Finally we also discuss the case 3 by including a Hermitian coupling term into the Hamiltonian in addition to the non-Hermitian one. This case turns out to be more complicated to solve, but may also be tackled successfully by our perturbative method. Keeping the expression (6.6) as our Ansatz for the perturbative expansion for the metric we obtain the same first order equation (6.7), but now involving

$$h_0(t) = a(t)\left(K_1 + K_2\right) + \mu(t) K_4 \qquad \text{and} \qquad h_1(t) = \lambda(t) K_3. \tag{6.111}$$

Since all generators of the algebra commute with $K_1 + K_2$ the only nontrivial contribution in the commutator of that relation results from the term involving K_4 in h_0. Taking now

$$q_1 = K_1, \qquad q_2 = K_2, \qquad q_3 = K_3, \tag{6.112}$$

leads to the following first order equations for the time-dependent coefficient functions

$$\dot{\gamma}_1^{(1)}(t) = -\frac{1}{2}\mu(t)\gamma_3^{(1)}(t), \tag{6.113}$$

$$\dot{\gamma}_2^{(1)}(t) = \frac{1}{2}\mu(t)\gamma_3^{(1)}(t), \tag{6.114}$$

$$\dot{\gamma}_3^{(1)}(t) = \mu(t)\left[\gamma_1^{(1)}(t) - \gamma_2^{(1)}(t)\right] - \lambda(t). \tag{6.115}$$

We see immediately that $\gamma_2^{(1)}(t) = -\gamma_1^{(1)}(t) + C$, where C is a constant. We take $C = 0$ which then also simplifies equations (6.115).

Proceeding now in the same manner as in the previous cases by extrapolation to the full series, we find that the following two equations need to be satisfied

$$\dot{\gamma}_1(t) = -\frac{1}{2}\sinh[\gamma_3(t)]\mu(t) \qquad \text{and} \qquad \dot{\gamma}_3(t) = \cosh[\gamma_3(t)]\tanh[2\gamma_1(t)]\mu(t) - \lambda(t). \tag{6.116}$$

Letting $\lambda = p\mu$, we can express γ_3 as a function of γ_1

$$\gamma_3(\gamma_1) = \pm\,\text{arccosh}\left[p\tanh(2\gamma_1) - \frac{k_1}{2}\,\text{sech}(2\gamma_1)\right]. \tag{6.117}$$

Setting

$$\gamma_1 = \frac{1}{2}\,\text{arcsinh}(\chi), \tag{6.118}$$

the two first order equations (6.116) are converted into the linear second order auxiliary equation (6.50) with $\lambda \to \mu$. The resulting Hermitian Hamiltonian consists now not only of two decoupled harmonic oscillators, but also contains an additional Hermitian term in form of K_4

$$h(t) = a(t)\,(K_1 + K_2) - \frac{k_1 + 2p\chi(t)}{2\,[1+\chi(t)^2]}\mu(t)K_4. \tag{6.119}$$

As in the previous two cases, we may also construct a non-Hermitian solution for the Dyson map by means of the perturbative approach. From the first order equation we observe that also $q_3 = iK_4$ with q_1 and q_2 as in (6.112) leads to a solution. Extrapolating to all orders yields now the two equations

$$\dot{\gamma}_1(t) = -\frac{1}{2}\sin[\gamma_3(t)]\lambda(t) \qquad \text{and} \qquad \dot{\gamma}_3(t) = \mu(t) - \cos[\gamma_3(t)]\coth[2\gamma_1(t)]\lambda(t). \tag{6.120}$$

As before we must restrict $\lambda(t) = p\mu(t)$ so that we may solve for γ_3 in terms of γ_1

$$\gamma_3(\gamma_1) = \pm\arccos\left\{\frac{[2 - ik_2 + 2\cosh(2\gamma_1)]\,\mathrm{cosech}(2\gamma_1)}{2p}\right\}. \tag{6.121}$$

We set here $k_2 = 0$ in order to obtain a real solution. Letting now

$$\gamma_1 = \frac{1}{2}\operatorname{arccosh}(\chi), \tag{6.122}$$

the two first order equations (6.120) are now converted into the linear second order auxiliary equation (6.50) with $\lambda \to \mu$ and $k_1 \to 0$. Similarly as the resulting Hamiltonian for the Hermitian Dyson map the resulting Hermitian Hamiltonian contain a K_4 besides the two uncoupled harmonic oscillators

$$h(t) = a(t)\,(K_1 + K_2) + \frac{\mu(t)}{\chi(t) - 1}K_4. \tag{6.123}$$

The generator K_4 can be identified with the standard angular momentum operator L_z and can be eliminated from $h(t)$ in (6.119) and (6.123) by means of a unitary transformation, see for instance [167]. Subsequently the eigenfunctions and expectation values of the resulting system of two uncoupled harmonic oscillators can be obtained similarly as for the cases 1 and 2 presented in detail in the previous section.

6.5 ip_xp_y coupled oscillators

We shall now seek solutions for the metric using perturbation theory for the Hamiltonian $H_2(t)$ given by equation (6.23). As we did for the oscillators with the $i(xy + p_xp_y)$ coupling we shall briefly consider the time-independent scenario, that being when $\dot{a} = \dot{b} = \dot{\Lambda} = 0$, which cannot be found in the literature. In this case the TDDE (2.14) becomes time-independent and reduces to a similarity transformation which is solved by

$$\eta = \exp(\theta_1 J_-)\exp(\theta_2 J_+), \tag{6.124}$$

where

$$\theta_1 = -\operatorname{arctanh}\left[\frac{\Lambda}{(a-b)\sqrt{1+\frac{\Lambda^2}{(a+b)^2}}}\right], \quad \text{and} \quad \theta_2 = -\arctan\left[\frac{\Lambda}{a+b}\right] \tag{6.125}$$

The corresponding Hermitian Hamiltonian is now decoupled and given by

$$\begin{aligned} h = {} & \frac{1}{2}(a+b)\sqrt{1+\frac{\Lambda^2}{(a+b)^2}}\left(K_+^x + K_+^y\right) + \frac{1}{2}\frac{\sqrt{(a-b)^2 - 4ab\Lambda^2}}{\sqrt{(a+b)^2+\Lambda^2}}\left(K_+^x - K_+^y\right) \\ & + \frac{1}{2}\frac{\Lambda^2}{(a+b)\sqrt{1+\frac{\Lambda^2}{(a+b)^2}}}\left(K_-^x + K_-^y\right). \end{aligned} \tag{6.126}$$

We see from this that the mapping is only valid, that is we have a $\mathcal{PT}$-symmetric regime, when $(a-b)^2 > 4ab\Lambda^2$. For $a = \pm b$ the Dyson map becomes ill-defined and and we are always in the spontaneously broken $\mathcal{PT}$-regime as we were in the previous section. We shall now demonstrate once again that by introducing an explicit time-dependence into the parameters a, b, Λ we can make the broken $\mathcal{PT}$-regime physical.

6.5.1 Metrics and Dyson maps

We shall now show how to solve the perturbative equations systematically to determine a Dyson map. We once again treat the non-Hermitian term as a perturbation and set $\Lambda(t) \to \epsilon\Lambda(t)$ where $\epsilon \ll 1$. The first order equation (6.7) for the Hamiltonian (6.23) becomes

$$i\Lambda(t)(I_+ - I_-) + \sum_{i=1}^{j}\left(\gamma_i^{(1)}[q_i, a(t)K_+^x + b(t)K_+^y] + i\dot{\gamma}_i^{(1)}q_i\right) = 0 \tag{6.127}$$

The underlying algebra for this problem is ten dimensional, we therefore have the options to take the limit in (6.127) as $j \in \{1,2,3,4,5,6,7,8,9,10\}$ with corresponding generators $q_i \in \{K_+^x, K_+^y, I_+, J_-, I_-, J_+, K_-^x, K_-^y, K_0^y, K_0^x\}$ obeying the relations (6.13)-(6.20). To illustrate all the possible solutions for the metric and Dyson map we take $j = 10$ with $q_1 = K_+^x, q_2 = K_+^y, q_3 = I_+, q_4 = J_-, q_5 = I_-, q_6 = J_+, q_7 = K_-^x, q_8 = K_-^y, q_9 = K_0^y$ and $q_{10} = K_0^x$, the first order equation becomes

$$\begin{aligned} &i\dot{\gamma}_1^{(1)} K_+^x + i\dot{\gamma}_2^{(1)} K_+^y + i\left(\Lambda + \dot{\gamma}_3^{(1)} + c\gamma_4^{(1)}\right) I_+ + i\left(\dot{\gamma}_4^{(1)} - c\gamma_3^{(1)}\right) J_- \\ &+ i\left(\dot{\gamma}_5^{(1)} - \Lambda - d\gamma_6^{(1)}\right) I_- + i\left(\dot{\gamma}_6^{(1)} + d\gamma_5^{(1)}\right) J_+ + i\left(\dot{\gamma}_7^{(1)} + 2a\gamma_{10}^{(1)}\right) K_-^x \\ &+ i\left(\dot{\gamma}_8^{(1)} + 2b\gamma_9^{(1)}\right) K_-^y + i\left(\dot{\gamma}_9^{(1)} - 2b\gamma_8^{(1)}\right) K_0^y + i\left(\dot{\gamma}_{10}^{(1)} - 2a\gamma_7^{(1)}\right) K_0^x = 0, \end{aligned} \tag{6.128}$$

where $c = a - b$ and $d = a + b$. By setting all the coefficients of the generators to zero obtain a set of 8 coupled differential equations. We in fact have two subsets of coupled differential equations, one involving $\gamma_3^{(1)}, \gamma_4^{(1)}, \gamma_5^{(1)}$ and $\gamma_6^{(1)}$ and the other set involving $\gamma_7^{(1)}, \gamma_8^{(1)}, \gamma_9^{(1)}$ and $\gamma_{10}^{(1)}$. The latter set of coupled differential equations do not involve Λ and so we set all the time-dependences to zero along with $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$. Interestingly we can identify different maps made up of just two or three generators depending upon whether $c = 0$. For example, by setting $c = 0$ we see that the equation (6.128) can be satisfied by only keeping the $q_3 = I_+$ and $q_6 = J_+$ terms in the Ansatz. Alternatively, with $c \neq 0$ we see that if we only keep terms $q_4 = J_-$, $q_5 = I_-$ and $q_6 = J_+$ then equation (6.128) is also satisfied. We shall be exploring the former of these options in the subsequent sections.

6.5.2 Hermitian η, $q_1 = I_+$, $q_2 = J_+$, and $c = 0$

Taking now $q_1 = I_+$ and $q_2 = J_+$ we may derive the perturbative equations up to sixth order in ϵ. Interestingly we have to actually modify how we construct our perturbative series. In the previous section when using two generators in the Ansatz it led to two sets of perturbative differential equations. For this scenario we end up with three sets, the first two being a set of coupled differential equations with the third relating a and Λ via a series. We therefore write our $\Lambda \to \epsilon\Lambda$ instead as

$$\Lambda \to \sum_{i=1}^{j} \epsilon^j \Lambda^{(j)}. \tag{6.129}$$

This allows us to look for a series solutions relation between a and Λ.

For $\gamma_1(t)$ we now obtain as the first five perturbative equations

$$\epsilon^1: \quad \dot{\gamma}_1^{(1)} = -\Lambda^{(1)}, \tag{6.130}$$

$$\epsilon^2: \quad \dot{\gamma}_1^{(2)} = -\Lambda^{(2)}, \tag{6.131}$$

$$\epsilon^3: \quad \dot{\gamma}_1^{(3)} = \frac{\left(\Lambda^{(1)}\right)^3}{8a^2} - \Lambda^{(3)}, \tag{6.132}$$

$$\epsilon^4: \quad \dot{\gamma}_1^{(4)} = \frac{3\left(\Lambda^{(1)}\right)^2\Lambda^{(2)}}{8a^2} - \Lambda^{(4)}, \tag{6.133}$$

$$\epsilon^5: \quad \dot{\gamma}_1^{(5)} = -\frac{3\left(\Lambda^{(1)}\right)^5}{128a^4} + \frac{3\Lambda^{(1)}\left(\Lambda^{(2)}\right)^2}{8a^2} + \frac{3\left(\Lambda^{(1)}\right)^2\Lambda^{(3)}}{8a^2} - \Lambda^{(5)}. \tag{6.134}$$

For $\gamma_2(t)$ we have

$$\epsilon^1: \quad \dot{\gamma}_2^{(1)} = 0, \tag{6.135}$$

$$\epsilon^2: \quad \dot{\gamma}_2^{(2)} = 0, \tag{6.136}$$

$$\epsilon^3: \quad \dot{\gamma}_2^{(3)} = -\frac{\gamma_1^{(1)}\left(\Lambda^{(1)}\right)^2}{2a}, \tag{6.137}$$

$$\epsilon^4: \quad \dot{\gamma}_2^{(4)} = -\frac{\gamma_1^{(2)}\left(\Lambda^{(1)}\right)^2}{2a} - \frac{\gamma_1^{(1)}\Lambda^{(1)}\Lambda^{(2)}}{a}, \tag{6.138}$$

$$\epsilon^5: \quad \dot{\gamma}_2^{(5)} = -\frac{\gamma_1^{(3)}\left(\Lambda^{(1)}\right)^2}{2a} + \frac{\left(\gamma_1^{(1)}\right)^3\left(\Lambda^{(1)}\right)^2}{6a} + \frac{\gamma_1^{(1)}\left(\Lambda^{(1)}\right)^4}{16a^3} - \frac{\gamma_1^{(2)}\Lambda^{(1)}\Lambda^{(2)}}{a} - \frac{\gamma_1^{(1)}\left(\Lambda^{(2)}\right)^2}{2a} - \frac{\gamma_1^{(1)}\Lambda^{(1)}\Lambda^{(3)}}{a}. \tag{6.139}$$

The third set of equations we derive relate a to the $\Lambda^{(i)}$'s via the $\gamma_2^{(i)}$'s, we obtain

$$\epsilon^1: \quad \gamma_2^{(1)} = -\frac{\Lambda^{(1)}}{2a}, \tag{6.140}$$

$$\epsilon^2: \quad \gamma_2^{(2)} = -\frac{\Lambda^{(2)}}{2a}, \tag{6.141}$$

$$\epsilon^3: \quad \gamma_2^{(3)} = \frac{\left(\Lambda^{(1)}\right)^3}{24a^3} - \frac{\Lambda^{(3)}}{2a}, \tag{6.142}$$

$$\epsilon^4: \quad \gamma_2^{(4)} = \frac{\left(\Lambda^{(1)}\right)^2\Lambda^{(2)}}{8a^3} - \frac{\Lambda^{(4)}}{2a}, \tag{6.143}$$

$$\epsilon^5: \quad \gamma_2^{(5)} = -\frac{\left(\Lambda^{(1)}\right)^5}{160a^5} + \frac{\Lambda^{(1)}\left(\Lambda^{(2)}\right)^2}{8a^3} + \frac{\left(\Lambda^{(1)}\right)^2\Lambda^{(3)}}{8a^3} - \frac{\Lambda^{(5)}}{2a}. \tag{6.144}$$

Now that we have obtained our three sets of perturbative equations we can match them to some Taylor expansions of standard functions to obtain an exact solution for the Dyson map.

Perturbation theory to the exact Dyson map and Hermitian Hamiltonian

We find that by extrapolating the equations (6.130) - (6.134), (6.135) - (6.139) and (6.140) - (6.144) to all orders and matching with standard functions, we find the following closed and exact forms for the time-dependences in the Dyson map

$$\dot{\gamma}_1 = -\Lambda\cos(\gamma_2), \quad \dot{\gamma}_2 = \sin(\gamma_2)\tanh(\gamma_1)\Lambda, \quad \gamma_2 = -\arctan\left(\frac{\Lambda}{2a}\right). \tag{6.145}$$

Given that we already have a concrete form for γ_2 it is instructive to use the equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ to solve for γ_1 in terms of γ_2. In doing so we obtain

$$\gamma_1(\gamma_2) = \text{arccosh}\left[-\frac{1}{2}ic_1\csc(\gamma_2)\right]. \tag{6.146}$$

Choosing now $c_1 = -2i$ and substituting the equation for γ_2 we get

$$\gamma_1 = \text{arccosh}\left[\sqrt{1+\frac{4a^2}{\Lambda^2}}\right], \tag{6.147}$$

which given that a, Λ are real, γ_1 is also always real. We now have a concrete form of γ_1 and γ_2 yet we still need to find the relation between a and Λ. Upon substitution of γ_1 and γ_2 into either the equation for $\dot{\gamma}_1$ or $\dot{\gamma}_2$ we pull out the following differential equation that needs to be satisfied

$$\Lambda\dot{a} + a(\Lambda^2 - \dot{\Lambda}) = 0. \tag{6.148}$$

By letting $a = \frac{\Lambda}{2f}$ this equation is converted to the more familiar

$$\dot{f} = \Lambda f, \tag{6.149}$$

which has solution

$$f = c\exp\left[\int^t \Lambda(s)ds\right], \tag{6.150}$$

where c is a constant.

Now that we have our Dyson map we may substitute it into the TDDE (2.3) to obtain the Hermitian Hamiltonian

$$h = \frac{\sqrt{1+f^2}\Lambda}{2f}\left(K_+^x + K_+^y\right) + \frac{f^2\Lambda}{2(1+f^2)}\left(K_-^x + K_-^y\right), \tag{6.151}$$

which is a decoupled two-dimensional time-dependent harmonic oscillator.

6.5.3 Hermitian η, $q_1 = J_+$, $q_2 = I_+$, and $c = 0$

We shall now take $q_1 = J_+$ and $q_2 = I_+$ and once again expand Λ as in (6.129). The perturbative equations for $\dot{\gamma}_1$ and $\dot{\gamma}_2$ read

$$\epsilon^n : \quad \dot{\gamma}_1^{(n)} = 0 \quad \text{and} \quad \dot{\gamma}_2^{(n)} = -\Lambda^{(n)}. \tag{6.152}$$

The first four equations which relate a and Λ are given by

$$\epsilon^1 : \quad \gamma_1^{(1)} = -\frac{\Lambda^{(1)}}{2a}, \tag{6.153}$$

$$\epsilon^2 : \quad \gamma_1^{(2)} = -\frac{\Lambda^{(2)}}{2a}, \tag{6.154}$$

$$\epsilon^3 : \quad \gamma_1^{(3)} = -\frac{\left(\gamma_2^{(1)}\right)^2 \Lambda^{(1)}}{4a} + \frac{\left(\Lambda^{(1)}\right)^3}{24a^3} - \frac{\Lambda^{(2)}}{2a}, \tag{6.155}$$

$$\epsilon^4 : \quad \gamma_1^{(4)} = -\frac{\gamma_2^{(2)}\gamma_2^{(1)}\Lambda^{(1)}}{2a} - \frac{\left(\gamma_2^{(1)}\right)^2 \Lambda^{(2)}}{4a} + \frac{\left(\Lambda^{(1)}\right)^2 \Lambda^{(2)}}{8a^3} - \frac{\Lambda^{(4)}}{2a}. \tag{6.156}$$

We see here that the perturbative equations are much simpler to solve and we may read off directly the form of γ_1 and γ_2. The equations (6.153)-(6.156) are slightly more involved.

From perturbation theory to the exact Dyson map and Hermitian Hamiltonian

We immediately see from equation (6.152) that we have

$$\gamma_1 = -c \qquad \text{and} \qquad \gamma_2 = -\int^t \Lambda(s)ds, \tag{6.157}$$

where c is a constant (we have taken a minus sign here for presentation purposes later). By extrapolating and matching equations (6.153) - (6.156) to the Taylor expansion of standard functions we obtain the following relationship between a and Λ

$$a = -\frac{1}{2}\cot(\gamma_1)\cosh(\gamma_2)\Lambda = \frac{1}{2}\cot(c)\cosh\left[\int^t \Lambda(s)ds\right]\Lambda. \tag{6.158}$$

We have now obtained what is another exact Dyson map, however it is for a different non-Hermitian Hamiltonian as a is given by a different time-dependent function.

We may now substitute this Dyson map into the TDDE (2.3) to obtain the following Hermitian Hamiltonian

$$h = \frac{1}{2}\csc(c)\cosh\left[\int^t \Lambda(s)ds\right]\Lambda\left(K_+^x + K_+^y\right) + \frac{1}{2}\sinh\left[\int^t \Lambda(s)ds\right]\Lambda\left(K_0^x + K_0^y\right), \tag{6.159}$$

which is a Hermitian version of Swanson type [162] and decoupled.

6.5.4 Summary

We have demonstrated that we may use time-dependent perturbation theory to obtain solutions for the Dyson map for the non-Hermitian Hamiltonian described by (6.23). For both solutions we have that $a = b$ such that in the time-independent scenario the Dyson map given by equations (6.124) and (6.125) is ill-defined and the system is always in the spontaneously broken $\mathcal{PT}$-regime. By allowing the parameters a and Λ to be explicitly time-dependent we were able construct two time-dependent Dyson maps in this regime. Technically these maps are for different non-Hermitian Hamiltonians, the relationships between a and Λ are different for each map. However they both exist for $a = b$ and lead to either two-dimensional time-dependent uncoupled harmonic oscillators (6.151) or two-dimensional Hermitian version of the time-dependent Swanson type oscillators (6.159).

6.6 Anharmonic oscillator - revisited

In this section we discuss an example for which the previous versions of the perturbative expressions for the metric or the Dyson map do not however lead to any solution. In fact, as we will demonstrate one does not only have to change the Ansatz, but one also needs to rescale the Hamiltonian in order to introduce the perturbative parameter in the right terms and treat the non-Hermitian part as a strong rather than a weak perturbation.

Unstable anharmonic oscillators have been the testing ground for perturbative methods for nonlinear systems for more than fifty years [42, 150–152, 168]. As discussed in Chpater 4 it is only fairly recently that an exact solution for the time-independent unstable anharmonic quartic oscillator was found by Jones and Mateo [74]. They used ideas from non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics [140, 141] and applied a perturbative approach that turned out to be exact. Recently we [4] also solved the explicitly time-dependent version of this model in an exact manner (see Chapters 4 and 5). These exact solutions found in [4] will serve here as a benchmark for our perturbative approach, so that we consider the same Hamiltonian, but with the time-dependent mass term set to zero

$$H(z,t) = p^2 - \frac{g(t)}{16} z^4, \qquad g \in \mathbb{R}^+. \tag{6.160}$$

Defining $H(z,t)$ on the contour $z = -2i\sqrt{1+ix}$ as proposed in [74], it is mapped into the non-Hermitian Hamiltonian

$$H(x,t) = p^2 - \frac{1}{2}p + \frac{i}{2}\{x, p^2\} + g(t)(x-i)^2, \tag{6.161}$$

where $\{\cdot,\cdot\}$ denotes as usual the anti-commutator. As mentioned using our previous versions for the perturbative Ansatz does not lead to a solvable first order equation or a recursive system. Instead we change our Ansatz to

$$\rho(t) = \eta(t)^\dagger \eta(t) = \prod_{i=j}^{1} \left[\prod_{l=k}^{1} \exp\left(\epsilon^{-l} (\gamma_i^{(l)})^\dagger q_i \right) \right] \prod_{i=1}^{j} \left[\prod_{l=1}^{k} \exp\left(\epsilon^{-l} (\gamma_i^{(l)}) q_i \right) \right]. \tag{6.162}$$

As we are expanding in ϵ^{-1} we assume here that perturbation parameter, $\epsilon \gg 1$, is large. The reason for this is that in addition we also need to scale the Hamiltonian (6.161) as $x \to \epsilon x$. Separating now into a Hermitian and non-Hermitian term, $h_0(t)$ and $h_p(t)$, respectively, we have

$$h_0(t) = p^2 - \frac{1}{2}p + \epsilon^2 g(t) x^2 - g(t), \qquad \text{and} \qquad h_p(t) = -2i\epsilon g(t) x + \frac{1}{2} i\epsilon \{x, p^2\}. \tag{6.163}$$

Thus instead of adding a small non-Hermitian perturbation to the Hermitian part, we have perturbed by a large term and also scaled up the harmonic oscillator term. Our Hamiltonian acquires therefore the following generic form

$$H(t) = h_1(t) + \epsilon^2 h_2(t) + i\epsilon h_3(t), \tag{6.164}$$

which together with the Ansatz (6.162) leads to the new first order equation

$$2ih_3(t) + \sum_{i=1}^{j} \left[\left((\gamma_i^{(1)} + (\gamma_i^{(1)})^\dagger \right) [q_i, h_2(t)] \right] = 0. \tag{6.165}$$

From this equation we can see that if any of the time-dependent coefficient functions $\gamma_i^{(1)}$'s are purely imaginary, then their contributions vanishes at this order and if they are real we simply acquire a factor of 2. For any time-dependent coefficient functions we will therefore modify the Ansatz such that the summation changes from $\sum_{i=1}^{j}$ to $\sum_{i=0}^{j}$. This version of the Ansatz leads to a recursive system that can be solved systematically order by order. In our example for the Hamiltonian (6.161) we identify

$$h_3(t) = h_p(t) \qquad \text{and} \qquad h_2(t) = g(t)x^2, \tag{6.166}$$

and may satisfy the lowest order equation with the choice

$$q_1 = x, \qquad q_2 = p^2, \qquad q_3 = p^2, \qquad q_4 = p, \tag{6.167}$$

where for q_3 and q_4 we are taking their time-dependent coefficient functions to be purely imaginary. In doing so we end up with following equations that need to be satisfied

$$\gamma_2^{(1)} = \frac{1}{6g}, \qquad \text{and} \qquad \gamma_3^{(0)} = \frac{1}{2\gamma_1^{(1)}}. \tag{6.168}$$

At order ϵ^0 we read off the constraining equations

$$\gamma_2^{(2)} = 0 \qquad \text{and} \qquad \gamma_1^{(2)} = -2\left(\gamma_1^{(1)}\right)^2 \gamma_3^{(1)}. \tag{6.169}$$

Continuing to order ϵ^{-1} we find the constraints

$$\gamma_1^{(1)} = \frac{\dot{g}}{6g}, \;\; \gamma_1^{(3)} = -\frac{\gamma_3^{(2)}\dot{g}^2}{18g^2} + \frac{\dot{g}^3}{72g^4} + \frac{\left(\gamma_3^{(1)}\right)^2 \dot{g}^3}{54g^3} - \frac{\dot{g}\ddot{g}}{72g^3}, \;\; \dot{\gamma}_4^{(0)} + \gamma_4^{(0)}\left(\frac{\ddot{g}}{\dot{g}} - \frac{\dot{g}}{g}\right) = -\frac{1}{3}. \tag{6.170}$$

The last equation is solved to

$$\gamma_4^{(0)} = \frac{c_1 g}{\dot{g}} - \frac{g \ln g}{2\dot{g}}. \tag{6.171}$$

At order ϵ^{-2} we obtain $\gamma_3^{(1)} = 0$, and therefore with (6.169) we have $\gamma_1^{(2)} = 0$. At order ϵ^{-3} we obtain

$$\gamma_3^{(2)} = \frac{\dot{g}^2 - g\ddot{g}}{4g^2\dot{g}}, \tag{6.172}$$

which implies with (6.170) that $\gamma_1^{(3)} = 0$. Some features hold for all remaining orders in ε. We have $\gamma_2^{(i)} = 0$ for all $i \geq 2$. We also find that at every order ϵ^{-n}, where

$n \geq 2$ the differential equation

$$\frac{\gamma_4^{(n-1)}\dot{g}^2}{3g^2} + \frac{\dot{g}\dot{\gamma}_4^{(n-1)}}{3g} + \frac{\gamma_4^{(n-1)}\ddot{g}}{3g} = 0, \tag{6.173}$$

occurs, which is solved by

$$\gamma_4^{(n-1)} = \frac{c_{n-1}g}{\dot{g}}. \tag{6.174}$$

Another equation that appears at all orders ϵ^{-n} for $n \geq 2$ is given by

$$\gamma_1^{(n+2)} = -\frac{\gamma_3^{(n+1)}\dot{g}^2}{18g^2}. \tag{6.175}$$

This is solved at all orders if we have

$$\gamma_1^{(n+2)} = 0 \qquad \text{and} \qquad \gamma_3^{(n+1)} = 0, \tag{6.176}$$

for $n \geq 2$. When eliminating the γs from these equations we are left with a differential equation entirely in g given by

$$\frac{14\dot{g}^3}{9g^2} + \frac{2\dot{g}\ddot{g}}{g} - \frac{\dddot{g}}{2} = 0. \tag{6.177}$$

Parameterizing $g = \frac{1}{2}\sigma^{-3}$ this equation reduces to

$$\sigma^2 \dddot{\sigma} = 0 \tag{6.178}$$

which is easily solved by $\sigma(t) = c_1 + c_2 t + c_3 t^2$.

Assembling all our results we extrapolate to all orders, i.e. an exact solution. Setting therefore $\varepsilon = 1$ gives the time-dependent Dyson map of the form

$$\eta(t) = \exp[\gamma_1(t)x]\exp[\gamma_2(t)p^3 + i\gamma_3(t)p^2 + i\gamma_4(t)p], \tag{6.179}$$

with

$$\gamma_1 = \frac{\dot{g}}{6g}, \quad \gamma_2 = \frac{1}{6g}, \quad \gamma_3 = \frac{12g^3 + \dot{g}^2 - g\ddot{g}}{4\dot{g}g^2}, \quad \gamma_4 = \frac{g}{\dot{g}}\left(c_1 - \frac{\log g}{2}\right), \tag{6.180}$$

which is in precise agreement with the Dyson map we previously found in [4] and presented in Chapter 4.

6.7 Conclusions

In this chapter we have demonstrated how to set up a perturbative approach that allows to construct the metric operator and the Dyson map in a recursive manner order by order in a perturbative parameter that may be very small or very large. Unlike the time-independent perturbation theory, whose formulation is fairly canonical, the time-dependent version allows for many more variants. We found four different types of perturbative expansions. The Ansatz (6.4) is the most natural one when the Dyson map is assumed to be Hermitian and needs to be slightly modified when one allows η to be non-Hermitian. In both of these versions the non-Hermitian term was treated as a small perturbation. For the ip_xp_y oscillators we found that we also needed to consider a series expansion for the time-dependence of the non-Hermitian term. For the anharmonic oscillator we demonstrated that this approach can not be applied universally and has to be altered for some models for which one needs to treat the non-Hermitian term and parts of the Hermitian term as large perturbations. Consequently the perturbative expansion needs to be in the inverse of the large perturbative parameter.

When compared to the time-independent scenario, all our approaches have in common that the order-by-order equations do not just determine the commutative structure of the q_is, but computations are more involved as in addition one needs to solve coupled sets of differential equations for the time-dependent coefficient functions, which is also possible order by order. Moreover, we observed that the key structure is already determined by the lowest order equation.

Although the main emphasis in this chapter is on the perturbation theory, with regard to the specific example studied we found many new Dyson maps for the $i(xy + p_xp_y)$ coupled non-Hermitian harmonic oscillator. We saw that these different maps lead to different types of physical behaviour, as shown explicitly for the time-dependent energy expectation values. When compared to the time-independent case, all our solutions are only valid in what would be the spontaneously broken $\mathcal{PT}$-regime, except for one example that is defined on what would be the exceptional point. So similar to the effect observed in [2, 51], this regime becomes physically meaningful in the time-dependent setting. However, unlike as in some of the previously studied systems, one can not crossover to the $\mathcal{PT}$-regime and is confined to the broken phase. It remains an open issue to formulate general criteria that characterize precisely when this possibility occurs for time-dependent systems and when not.

We shall be revisiting the $i(xy + p_x p_y)$ coupled oscillators in chapter 7 where we demonstrate that we can use the solutions we found for the Dyson maps to construct an infinite series of Dyson map all with differing physical behaviours.

Chapter 7

Infinite series of Dyson maps

In this chapter we propose and explore a scheme that leads to an infinite series of time-dependent Dyson maps which associate different Hermitian Hamiltonians to a uniquely specified time-dependent non-Hermitian Hamiltonian [7]. We identify the underlying symmetries responsible for this feature respected by various Lewis-Riesenfeld invariants. The latter are used to facilitate the explicit construction of the Dyson maps and metric operators. We shall consider two concrete examples, a two-dimensional system of oscillators that are coupled to each other in a non-Hermitian $\mathcal{PT}$-symmetrical fashion and the time-dependent anharmonic oscillator. The former of these systems allows us to demonstrate the full working of the scheme and how it can break down.

7.1 Introduction

Throughout this thesis we have encountered different time-dependent non-Hermitian systems for which we have been able to determine two or more nonequivalent metrics leading to different physical behaviour. This non-uniqueness of the metric has been established for nearly 30 years [43] and is attributed to the fact that the usual starting point when studying these non-Hermitian systems is to consider only fixing one observable, e.g. the Hamiltonian H. To render the metric unique one must specify at least one further observable such as the position x.

For time-independent non-Hermitian Hamiltonians H, who can be related to a Hermitian Hamiltonian, h, through the time-independent Dyson equation (2.14) it has been shown that the ambiguity in the metric operator is associated with the symmetries of the equivalent Hermitian Hamiltonians [96]. For many known models explicit solutions, including some of their ambiguities, have been constructed [74, 90, 93, 94, 96].

When the non-Hermitian Hamiltonian is explicitly time-dependent it is no longer related to a Hermitian counterpart via a similarity transformation and instead related through the TDDE (2.3). In this scenario we therefore expect that the symmetries responsible for the ambiguity of the metric are no longer associated with the Hermitian Hamiltonians. We shall in fact demonstrate that this is indeed thhe case and that the governing symmetries are those of the Lewis-Riesenfeld invariants [89] of the Hermitian Hamiltonians. We shall exploit these symmetries to propose and explore a scheme that leads to an infinite series of Dyson maps constructed from two seed maps, hence an infinite series of equivalent Hermitian Hamiltonians, albeit with different physics as we shall demonstrate.

7.2 Infinite symmetries and series of Dyson maps from two seeds

Our starting point is an explicitly non-Hermitian time-dependent Hamiltonian $H \neq H^\dagger$ satisfying the TDSE $H(x,t)\psi(x,t) = i\hbar\partial_t\psi(x,t)$. We further assume that we have two different time-dependent Dyson maps, $\eta(t)$ and $\tilde{\eta}(t)$, satisfying the time-dependent Dyson equations (TDDE)

$$h = \eta H \eta^{-1} + i\hbar\partial_t\eta\eta^{-1}, \qquad \text{and} \qquad \tilde{h} = \tilde{\eta} H \tilde{\eta}^{-1} + i\hbar\partial_t\tilde{\eta}\tilde{\eta}^{-1}, \tag{7.1}$$

involving two different time-dependent Hermitian Hamiltonians $h = h^\dagger$, $\tilde{h} = \tilde{h}^\dagger$ that also obey their respective TDSEs $h(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t)$ and $\tilde{h}(x,t)\tilde{\phi}(x,t) = i\hbar\partial_t\tilde{\phi}(x,t)$. The wavefunctions are related as $\phi = \eta\psi$, $\tilde{\phi} = \tilde{\eta}\psi$ and therefore $\tilde{\phi} = A\phi$, where we employed the first of the operators

$$A := \tilde{\eta}\eta^{-1} \qquad \text{and} \qquad \tilde{A} := \eta^{-1}\tilde{\eta}. \tag{7.2}$$

The operator $\tilde{A}$ is defined for later purposes. Next we eliminate the Hamiltonian H from the two equations in (7.1), such that the two Hermitian Hamiltonians are seen to be related as

$$\tilde{h} = AhA^{-1} + i\hbar\partial_t AA^{-1}. \tag{7.3}$$

As argued and shown for concrete examples in [5, 51, 87, 88, 132] and throughout this thesis, once the Dyson maps are known one may relate the respective Lewis-Riesenfeld invariants $I_{\mathcal{H}}$, with $\mathcal{H} = H, h, \tilde{h}$, satisfying [89]

$$i\hbar \frac{dI_{\mathcal{H}}}{dt} = i\hbar \partial_t I_{\mathcal{H}} + [I_{\mathcal{H}}, \mathcal{H}] = 0, \tag{7.4}$$

simply by means of similarity transformations as

$$I_h = \eta I_H \eta^{-1}, \qquad I_{\tilde{h}} = \tilde{\eta} I_H \tilde{\eta}^{-1}, \quad \Rightarrow I_{\tilde{h}} = A I_h A^{-1}. \tag{7.5}$$

Each of the invariants satisfies an eigenvalue equation with time-independent eigenvalues and eigenfunctions that are simply related by a phase factor to the wavefunctions satisfying the respective TDSE. Exploiting the Hermiticity of the invariants I_h and $I_{\tilde{h}}$, the latter relation in (7.5) implies that the operators

$$S := A^\dagger A \qquad \text{and} \qquad \tilde{S} := A A^\dagger \tag{7.6}$$

are symmetries for the invariants I_h and $I_{\tilde{h}}$, respectively, with

$$[I_h, S] = 0 \qquad \text{and} \qquad \left[I_{\tilde{h}}, \tilde{S} \right] = 0. \tag{7.7}$$

Thus S and $\tilde{S}$ also satisfy the Lewis-Riesenfeld equations for the Hermitian h-Hamiltonian system and the $\tilde{h}$-Hamiltonian system

$$i\hbar \frac{dS}{dt} = i\hbar \partial_t S + [S, h] = 0, \qquad i\hbar \frac{d\tilde{S}}{dt} = i\hbar \partial_t \tilde{S} + \left[\tilde{S}, \tilde{h} \right] = 0. \tag{7.8}$$

In turn this means that

$$I_h' = I_h + S, \qquad \text{and} \qquad I_{\tilde{h}}' = I_{\tilde{h}} + \tilde{S} \tag{7.9}$$

are new invariants for the Hamiltonians h and $\tilde{h}$, respectively.

Another symmetry with an interesting consequence is an $\tilde{A}$-symmetry, see (7.2), of the non-Hermitian invariant I_H, as it implies that the two invariants related to the Hermitian systems are identical

$$\left[I_H, \tilde{A} \right] = 0 \qquad \Leftrightarrow \qquad I_h = I_{\tilde{h}}, \tag{7.10}$$

and in turn, the equality of two invariants associated to different Hermitian Hamiltonians implies an $\tilde{A}$-symmetry of the non-Hermitian invariant I_H. This is easily established by making use of the pseudo-Hermiticity relations for the invariants (7.5).

7.2.1 Iteration of two Dyson maps

While certain symmetries of the invariants imply the presence of two inequivalent Dyson maps and vice versa, we will now construct further time-dependent Dyson maps, say $\check{\eta}$ or $\hat{\eta}$, from two given ones, say η and $\tilde{\eta}$. We start by constructing a third Dyson map making use of either of the following statements:

(S1) If and only if the adjoint action of A on the invariant $I_{\tilde{h}}$, $AI_{\tilde{h}}A^{-1}$, is Hermitian then $\check{h} = A\tilde{h}A^{-1} + i\hbar\partial_t AA^{-1}$ is a new Hamiltonian that is related to the non-Hermitian Hamiltonian H by the time-dependent Dyson equation $\check{h} = \check{\eta}H\check{\eta}^{-1} + i\hbar\partial_t\check{\eta}\check{\eta}^{-1}$ with $\check{\eta} := \tilde{\eta}\eta^{-1}\tilde{\eta}$.

(S2) If and only if the inverse adjoint action of A on the invariant I_h, $A^{-1}I_hA$ is Hermitian then $\hat{h} = A^{-1}hA - i\hbar A^{-1}\partial_t A$ is a new Hamiltonian that is related to the non-Hermitian Hamiltonian H by the time-dependent Dyson equation $\hat{h} = \hat{\eta}H\hat{\eta}^{-1} + i\hbar\partial_t\hat{\eta}\hat{\eta}^{-1}$ with $\hat{\eta} := \eta\tilde{\eta}^{-1}\eta$.

At first we prove (S1) in reverse: Assuming that $\check{\eta} := \tilde{\eta}\eta^{-1}\tilde{\eta}$ is a new time-dependent Dyson map that maps the non-Hermitian Hamiltonian H to a Hermitian one, the TDDE $\check{h} = \check{\eta}H\check{\eta}^{-1} + i\hbar\partial_t\check{\eta}\check{\eta}^{-1}$ holds by definition. Replacing now H in this equation by means of the first equation in (2.3) and using the definition (7.2) for A, equation $\check{h} = A\tilde{h}A^{-1} + i\hbar\partial_t AA^{-1}$ follows directly. In turn this implies that the adjoint action of A on $I_{\tilde{h}}$ yields the Lewis-Riesenfeld invariant $I_{\check{h}}$. Since $I_{\check{h}}$ is Hermitian, so is $AI_{\tilde{h}}A^{-1}$. The direct statement is shown by checking whether $AI_{\tilde{h}}A^{-1}$ is Hermitian and then reversing the steps in the previous argument. Similarly we may prove (S2).

Thus for practical purposes when given the two time-dependent Dyson maps η, $\tilde{\eta}$ and the invariants $I_{\tilde{h}}$, I_h we can simply check whether $AI_{\tilde{h}}A^{-1}$ and/or $A^{-1}I_hA$ are Hermitian and subsequently deduce the form of the new Dyson map. Alternatively one may of course also assume the given forms for $\check{\eta}$ and $\hat{\eta}$ with a subsequent check of whether the right hand sides of the corresponding Dyson equations are Hermitian, thus defining new Hermitian Hamiltonians.

Having now obtained two new time-dependent Dyson maps, we may include them into the set of the two starting Dyson maps to construct yet more and more maps by iteration. We summarize the first step as outlines above, i.e. the constuction of $\breve{\eta} =: \eta_3$ and $\hat{\eta} =: \eta_4$ from the seed maps η and $\tilde{\eta}$, as

$$\eta, \tilde{\eta} \quad \begin{array}{l} \nearrow \quad \eta_3 = \tilde{\eta}\eta^{-1}\tilde{\eta} = A\tilde{\eta} \\ \\ \searrow \quad \eta_4 = \eta\tilde{\eta}^{-1}\eta = A^{-1}\eta \end{array} \quad . \tag{7.11}$$

Replacing now in the next step the seed maps by new maps obtained in the previous step we obtain, up to the Hermiticity check,

$$\eta, \eta_3 \quad \begin{array}{l} \nearrow \quad \eta_5 = \tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta} = A^3\tilde{\eta} \\ \\ \searrow \quad \eta_6 = \eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta = A^{-2}\eta \end{array} \quad , \tag{7.12}$$

$$\eta, \eta_4 \quad \begin{array}{l} \nearrow \quad \eta_6 = \eta\tilde{\eta}^{-1}\eta\eta^{-1}\eta\tilde{\eta}^{-1}\eta = \eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta \\ \\ \searrow \quad \tilde{\eta} = \eta\eta^{-1}\tilde{\eta}\eta^{-1}\eta \end{array} \quad , \tag{7.13}$$

$$\tilde{\eta}, \eta_3 \quad \begin{array}{l} \nearrow \quad \eta = \tilde{\eta}\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\tilde{\eta} \\ \\ \searrow \quad \eta_7 = \tilde{\eta}\eta^{-1}\tilde{\eta}\tilde{\eta}^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta} = \tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta} = A^2\tilde{\eta} \end{array} \quad , \tag{7.14}$$

$$\tilde{\eta}, \eta_4 \quad \begin{array}{l} \nearrow \quad \eta_7 = \tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta} \\ \\ \searrow \quad \eta_8 = \eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta = A^{-3}\eta \end{array} \quad , \tag{7.15}$$

$$\eta_3, \eta_4 \quad \begin{array}{l} \nearrow \quad \eta_9 = \eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta\tilde{\eta}^{-1}\eta = A^{-4}\eta \\ \\ \searrow \quad \eta_{10} = \tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta}\eta^{-1}\tilde{\eta} = A^4\tilde{\eta} \end{array} \quad . \tag{7.16}$$

Continuing in this manner we obtain a series of Dyson maps of the general form

$$\eta^{(n)} := A^n\eta, \qquad \tilde{\eta}^{(n)} := A^{-n}\tilde{\eta}, \qquad \text{with } n \in \mathbb{Z}. \tag{7.17}$$

When combined in the way described above we only obtain new maps of the same form

$$\tilde{\eta}^{(n)},\tilde{\eta}^{(m)} \quad \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \quad \begin{array}{c} \tilde{\eta}^{(2m-n)} \\ \\ \tilde{\eta}^{(2n-m)} \end{array} \quad , \qquad \tilde{\eta}^{(n)},\eta^{(m)} \quad \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \quad \begin{array}{c} \eta^{(2m-n-1)} \\ \\ \tilde{\eta}^{(2n-m+1)} \end{array} \quad , \tag{7.18}$$

$$\eta^{(n)},\tilde{\eta}^{(m)} \quad \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \quad \begin{array}{c} \tilde{\eta}^{(2m-n+1)} \\ \\ \eta^{(2n-m-1)} \end{array} \quad , \qquad \eta^{(n)},\eta^{(m)} \quad \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \quad \begin{array}{c} \eta^{(2m-n)} \\ \\ \eta^{(2n-m)} \end{array} \quad . \tag{7.19}$$

As discussed above, for the iteration to proceed we need to verify at each step the Hermiticity of the right hand side of the time-dependent Dyson equation or the adjointly mapped invariants. Thus we require the relevant A-operators involving the new maps

$$\tilde{\eta}^{(m)}\left(\tilde{\eta}^{(n)}\right)^{-1} = A^{m-n}, \qquad \eta^{(m)}\left(\tilde{\eta}^{(n)}\right)^{-1} = A^{m-n-1}, \tag{7.20}$$

$$\tilde{\eta}^{(m)}\left(\eta^{(n)}\right)^{-1} = A^{m-n+1}, \quad \eta^{(m)}\left(\eta^{(n)}\right)^{-1} = A^{m-n}. \tag{7.21}$$

Naturally we may repeat the symmetry arguments from the previous section using the newly constructed Dyson maps, thus obtaining an infinite set of symmetry operators, provided that the Hermiticity property holds at each of the iterative steps.

7.3 Two dimensional $\mathcal{PT}$-symmetrically coupled oscillators

The first system we shall consider to demonstrate the working of the above scheme will be the two-dimensional time-dependent oscillators which are coupled in a $\mathcal{PT}$-symmetric fashion (6.24). We performed an extensive analysis on these oscillators in Chapter 6 resulting in the construction of six Dyson maps for both cases 1 and 2. The amount of Dyson maps found makes this system an ideal testing ground to potentially determine the aforementioned infinite series of Dyson maps.

7.3.1 Six seed Dyson maps

We shall consider here case 1, that is when $c(t) = 0$ and $\mu(t) = 0$ for the Hamiltonian

(6.24). In this case the six Dyson maps determined were all of the form

$$\eta(t) = \exp[\gamma_1(t)q_1]\exp[\gamma_2(t)q_2], \tag{7.22}$$

where q_1, q_2 were operators part of the subalgebra $\{K_1, K_2, K_3, K_4\}$ which satisfy the commutations relations (6.25) and $\gamma_1(t), \gamma_2(t)$ are time-dependent functions constrained by two coupled first order differential equations. In section 6.4.6 a summary of the Dyson maps found, the auxiliary equations which the time-dependent functions satisfy and the resulting Hermitian Hamiltonians can be found. Specifically in table 6.1 we report the first order differential equations which $\gamma_1(t)$ and $\gamma_2(t)$ satisfy. In table 6.2 we report the parametrizations of γ_1 and γ_2 in terms of the auxiliary function χ which satisfies an auxiliary equation that can be found in table 6.3. The resulting Hermitian Hamiltonians for each of the Dyson maps were of the form

$$h(t) = f_+(t)K_1 + f_-(t)K_2, \tag{7.23}$$

and details of the time-dependent functions $f_\pm(t)$ for each of the maps are specified table 6.4.

We present here a summary of this relevant information for each of the Dyson maps in table 7.1. The auxiliary functions x and χ governing the time-dependence

η_i	q_1, q_2	γ_1	γ_2	$f_\pm(t)$
η_1	K_4, K_3	*	*	a
η_2	K_3, K_4	$\operatorname{arccosh}(\chi)$	$\operatorname{arcsinh}\left(\frac{k}{\chi}\right)$	$a \pm \frac{k\lambda}{2\chi^2}$
η_3	K_4, iK_1	$\operatorname{arcsinh}\left(k_3\sqrt{1+x^2}\right)$	$-\arctan(x)$	$a - \frac{\lambda(\pm 1+\sqrt{1+(1+x^2)k_3^2})}{2(1+x^2)k_3}$
η_4	K_4, iK_2	$\operatorname{arcsinh}\left(k_4\sqrt{1+x^2}\right)$	$\arctan(x)$	$a + \frac{\lambda(\mp 1+\sqrt{1+(1+x^2)k_4^2})}{2(1+x^2)k_4}$
η_5	K_3, iK_1	$\operatorname{arcsinh}\left(k_5\sqrt{1+x^2}\right)$	$\operatorname{arccot}(x)$	$a + \frac{\lambda(\pm 1+\sqrt{1+(1+x^2)k_5^2})}{2(1+x^2)k_5}$
η_6	K_3, iK_2	$\operatorname{arcsinh}\left(k_6\sqrt{1+x^2}\right)$	$\operatorname{arccot}(x)$	$a - \frac{\lambda(\mp 1+\sqrt{1+(1+x^2)k_6^2})}{2(1+x^2)k_6}$

Table 7.1: Inequivalent Dyson maps η_i with specific operators q_1, q_2 in the factorisation (7.22), and parametrisations for γ_1, γ_2 in terms of the auxiliary functions χ or x_i together with the time-dependent functions $f_\pm(t)$ in $h(t)$. For η_2 we demand that $\chi > 1$ so that the Dyson map is well-defined.

were found to be

$$\mathrm{Aux}_1: \quad \ddot{x}_i - \frac{\dot{\lambda}}{\lambda}\dot{x}_i - \lambda^2 x_i = 0 \qquad \text{and} \qquad \mathrm{Aux}_2: \quad \ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2\chi = k^2\frac{\lambda^2}{\chi^3}, \tag{7.24}$$

where $i = 3, 4, 5, 6$. The second equation in (7.24) is the ubiquitous Ermakov-Pinney equation [138, 139].

Note that the parametrization of γ_2 in table 7.1 for Dyson maps $\eta_3 - \eta_6$ could be positive or negative and we would still obtain Aux_1 as the auxiliary equation. What would differ however is the first order equations resulting from combining the differential equations for $\gamma_1(t)$ and $\gamma_2(t)$ as detailed in table 6.1. To ensure that the first order equation is the same for all the maps we have chosen here these specific parametrisations for γ_2 so that it will be easier to combine the Dyson maps to create new ones.

7.3.2 Relation between auxiliary equations

To carry out the discussion as set out in section 7.2 we have the somewhat unappealing feature that various seed Dyson maps are governed by different types of auxiliary equations. Here we comment briefly on a feature previously not commented on in previous chapters, and show that with a different parametrisation also η_2 is in fact constrained by the linear second order equation in (7.24). To demonstrate that this can be achieved we briefly recall how to solve the TDDE for η_2, but with a different parametrisation.

Assuming for this purpose the Dyson map η_2 to be of the form (7.22) with $\gamma_1(t), \gamma_2(t)$ unknown and $q_1 = K_3$, $q_2 = K_4$, we substitute η_2 into the TDDE and find that $h_2(t)$ is indeed made to be Hermitian if the following coupled first order differential equations are satisfied

$$\dot{\gamma}_1 = -\lambda \cosh \gamma_2 \qquad \text{and} \qquad \dot{\gamma}_2 = \lambda \tanh \gamma_1 \sinh \gamma_2. \tag{7.25}$$

These equations match the ones obtained through perturbation theory in section 6.4.3. To solve these equations for γ_1 and γ_2 we notice first that we can eliminate λ and dt from the above equations to give

$$\frac{d\gamma_2}{d\gamma_1} = -\tanh \gamma_1 \tanh \gamma_2(\gamma_1), \tag{7.26}$$

which we solve to

$$\gamma_2 = \operatorname{arcsinh}(c \operatorname{sech} \gamma_1), \tag{7.27}$$

with c being an integration constant. As reported in chapter 6 in [6], we parametrised

$\gamma_1 = \operatorname{arccosh} \chi$ which lead to the Ermakov-Pinney equation Aux$_2$ as auxiliary equation. When instead we define $\gamma_1 = \operatorname{arccosh} \sqrt{1+x_2^2}$ and let $c = -1/k_2$, we find that the central equation to be satisfied is now also Aux$_1$, similarly as for the other cases. We have now a new way of writing the Dyson map η_2 so that all of the maps found are governed by the same central equation, with

$$\eta_2 : \quad \gamma_1 = \operatorname{arccosh} \sqrt{1+x_2^2}, \quad \gamma_2 = \operatorname{arcsinh} \left(-\frac{1}{k_2\sqrt{1+x_2^2}} \right), \quad f_\pm = \mp \frac{\lambda}{2k_2(1+x_2^2)}. \tag{7.28}$$

Thus we have found a way to convert the nonlinear dissipative Ermakov-Pinney equation given by Aux$_2$ to the linear second order differential equation Aux$_1$ by the relation

$$\chi = \sqrt{1+x_i^2} \qquad \text{with} \quad k = -\frac{1}{k_i}. \tag{7.29}$$

This appears to be somewhat miraculous, but one needs to stress here that this is only possible when employing also the first order equations resulting from (7.25) for the respective variables, i.e.

$$\dot{x}_2 = -\frac{\lambda\sqrt{1+k_2^2(1+x_2^2)}}{k_2}, \tag{7.30}$$

for the new parametrisation. Notice also that the constraint imposed on $\chi > 1$ is automatically satisfied with the new parametrisation.

7.3.3 Construction of invariants

As outlined in section 7.2, in order to construct new Dyson maps we must first calculate invariants for each of the Hermitian Hamiltonians associated to each of the seed Dyson maps $\eta_i, i = 2, \ldots, 6$, or solve the TDDE (2.3). Using the corresponding expressions for h_i we solve equation (3.2) and find the invariant

$$I_{h_i}(t) = c_1K_1 + c_2K_2 + c_3 \cos\left[c_4 - \int^t f_{+-}^i(s)ds\right] K_3 - c_3 \sin\left[c_4 - \int^t f_{+-}^i(s)ds\right] K_4 \tag{7.31}$$

where $f_{+-}^i := f_+^i - f_-^i$ is the difference between the time-dependent functions in (7.68) occurring in the Hermitian Hamiltonian, and the c_1, c_2, c_3, c_4 are real constants. Notice that in all cases the difference takes on the same form up to an overall sign

$$f_{+-}^{2,3,4}(t) = -\frac{\lambda}{k_i(1+x_i^2)} = -f_{+-}^{5,6}(t), \tag{7.32}$$

such that the corresponding Hermitian invariants are identical.

While the cases $i = 2, \ldots, 6$ have been unified, the case $i = 1$ still stands out as in this case $f_+ = f_-$, so that the invariant in (7.31) is rendered time-independent. We therefore need to construct an additional invariant for $h_1(t)$. To achieve that we need to enlarge the algebra by six additional elements such that the ten-dimensional algebra is now given by the generators in (6.12). These ten generators satisfy the commutation relations detailed in equations (6.13) - (6.20).

Assuming now that the invariant is also spanned by these generators, we found as another solution to (3.2) a universal solution for all six cases

$$I_{h_i}(t) = \alpha_+(K_+^x+K_-^x)+\beta_+(K_+^x-K_-^x)+\alpha_-(K_+^y+K_-^y)+\beta_-(K_+^y-K_-^y)+\delta_+K_0^x+\delta_-K_0^y, \tag{7.33}$$

for $i = 1, \ldots, 6$. The time-dependent functions are constrained by

$$\alpha_\pm(t) = \rho_\pm(t)^2, \qquad \beta_\pm(t) = \frac{1}{\rho_\pm(t)^2} + \frac{\dot\rho_\pm(t)^2}{f_\pm(t)^2}, \qquad \delta_\pm(t) = -\frac{2\rho_\pm(t)\dot\rho_\pm(t)}{f_\pm(t)}, \tag{7.34}$$

where the auxiliary functions $\rho_\pm$ satisfy the dissipative Ermakov-Pinney equation

$$\ddot\rho_\pm - \frac{\dot f_\pm}{f_\pm}\dot\rho_\pm + f_\pm^2\rho_\pm = \frac{f_\pm^2}{\rho_\pm^3}. \tag{7.35}$$

We will exploit these ambiguities and use which ever invariant is most useful in a certain context. Noting that the invariant in (7.31) is much simpler than the one in (7.33), we shall be using it below for the Hermitian Hamiltonians h_i associated with the Dyson maps η_i, $i = 2, \ldots, 6$. In turn we shall use the invariant (7.33) only for the Hermitian Hamiltonian h_1 associated with η_1, for which it simplifies further due to the relation $f_+ = f_-$ that implies $\rho_+ = \rho_-$.

We also construct the non-Hermitian invariant for the non-Hermitian Hamiltonian H_1 in (6.24) for case 1 by directly solving equation (3.2). We find

$$I_{H_1} = C_1(t)K_1 + C_2(t)K_2 + C_3(t)K_3 + iC_4(t)K_4, \tag{7.36}$$

with

$$C_1 = \frac{c_1}{2} + c_3 \cosh\left(c_4 - \int^t \lambda(s)ds\right), \qquad (7.37)$$

$$C_2 = \frac{c_1}{2} - c_3 \cosh\left(c_4 - \int^t \lambda(s)ds\right), \quad C_3 = c_2, \qquad (7.38)$$

$$C_4 = 2c_3 \sinh\left(c_4 - \int^t \lambda(s)ds\right). \qquad (7.39)$$

Using the equations in (7.5) we may relate the various invariants up to the stated ambiguities. We have verified that the inverse adjoint actions of η and $\tilde{\eta}$ on I_{H_1} in (7.36) are indeed invariants for I_h and $I_{\tilde{h}}$, respectively, albeit different from the invariants in (7.31) and (7.33) up the aforementioned ambiguities.

For our non-Hermitian Hamiltonian $H_1(t)$ in (6.24) for case 1, we have now a number of seed Dyson maps η_i at hand together with their associated Hermitian Hamiltonians h_i and their respective Lewis-Riesenfeld invariants I_{h_i}. Thus we can now carry out the scheme laid out in section 7.2 and construct an infinite series of Dyson maps from two of these seed maps. We will not present here all thirty possibilities that may result from these six maps as there is considerable overlap in the solution procedure as well as the resulting Hermitian Hamiltonians. Instead we select various examples that exhibit different types of features including an example for which the mechanism breaks down.

7.3.4 Seed maps $\eta = \eta_3$ and $\tilde{\eta} = \eta_4$ - unitary operator A

The central operator to compute first is A as defined in (7.2). We start with a simple example for which some of its factors commute. Taking $\eta = \eta_3$, $\tilde{\eta} = \eta_4$ as specified in table 1 and setting $k_3 = k_4 = k$, $x_3 = x_4 = x$ we obtain

$$A = \eta_4 \eta_3^{-1} = e^{\operatorname{arcsinh}\left(k\sqrt{1+x^2}\right)K_4} e^{i\arctan(x)(K_1+K_2)} e^{-\operatorname{arcsinh}\left(k\sqrt{1+x^2}\right)K_4} \qquad (7.40)$$

$$= e^{i\arctan(x)(K_1+K_2)}. \qquad (7.41)$$

The last equality results from the fact that $[K_1 + K_2, K_4] = 0$. According to the statement (S1) in section 7.2, we need to guarantee next that $AI_{h_4}A^{-1}$ is Hermitian. For the case at hand this is easily seen to be the case as A is a unitary operator and I_{h_4} is Hermitian. Thus, according to (S1) a new Dyson map is given by

$$\eta^{(1)} = A\eta_4 = e^{i\arctan(x)(K_1+K_2)} e^{\operatorname{arcsinh}(k\sqrt{1+x^2})K_4} e^{i\arctan(x)K_2}$$
$$= e^{\operatorname{arcsinh}(k\sqrt{1+x^2})K_4} e^{i\arctan(x)(K_1+2K_2)}, \qquad (7.42)$$

which in turn is not unitary. Next we compute the associated Hermitian Hamiltonian from the TDDE (2.3) simply by substituting into the right hand side all the known quantities

$$h^{(1)} = \left[a + \frac{\lambda\left(3\sqrt{1+k^2(1+x^2)}-1\right)}{2k(1+x^2)}\right] K_1 + \left[a + \frac{\lambda\left(3\sqrt{1+k^2(1+x^2)}+1\right)}{2k(1+x^2)}\right] K_2. \qquad (7.43)$$

Using next the relation (7.17) it is now straightforward to calculate the infinite series of Dyson maps. At each step the Hermiticity of the adjoint action of the higher order A operators, as defined in (7.20), (7.21), on the Hermitian invariants is guaranteed by the fact that also any power of A is a unitary operator. We find

$$\eta^{(n)} = A^n\eta_4 = e^{\operatorname{arcsinh}(k\sqrt{1+x^2})K_4} e^{i\arctan(x)[K_1+(n+1)K_2]}, \qquad (7.44)$$

with corresponding infinite series of Hermitian Hamiltonians

$$h^{(n)} = h^{(1)} + \frac{(n-1)\lambda\sqrt{1+k^2(1+x^2)}}{k(1+x^2)}(K_1+K_2). \qquad (7.45)$$

In a similar fashion we use the second relation in (7.17) to obtain the new Dyson maps

$$\tilde{\eta}^{(n)} = A^n\eta_3 = e^{\operatorname{arcsinh}(k\sqrt{1+x^2})K_4} e^{-i\arctan(x)[(n+1)K_1+K_2]}, \qquad (7.46)$$

with corresponding Hermitian Hamiltonians

$$\tilde{h}^{(n)} = \left(\frac{\lambda}{2k(1+x^2)}\right)(K_2-K_1) + \left[a - \frac{(2n+1)\lambda\sqrt{1+k^2(1+x^2)}}{2k(1+x^2)}\right](K_1+K_2). \qquad (7.47)$$

Since A is a unitary operator the symmetry operator, as defined in (7.6) is simply the unit operator, i.e. $S = S^\dagger = \mathbb{I}$. Moreover the unitarity of A also implies that the relation between the two Hermitian Hamiltonians (7.3) simply becomes a non-Abelian gauge symmetry between two Hermitian Hamiltonians. In this case the metric operators do not to change in the iteration process

$$\rho^{(n)} = \eta^{(n)\dagger}\eta^{(n)} = \eta_4^\dagger\eta_4 = \rho_4, \qquad \text{and} \qquad \tilde{\rho}^{(n)} = \tilde{\eta}^{(n)\dagger}\tilde{\eta}^{(n)} = \eta_3^\dagger\eta_3 = \rho_3. \tag{7.48}$$

7.3.5 Seed maps $\eta = \eta_2$ and $\tilde{\eta} = \eta_3$ - nonunitary operator A

Once again we start with the construction of the operator A

$$\begin{aligned} A :&= \eta_3\eta_2^{-1} = \\ &= e^{\operatorname{arcsinh}\left(k\sqrt{1+x^2}\right)K_4} e^{-i\arctan(x)K_1} e^{\operatorname{arcsinh}\left(\frac{1}{k\sqrt{1+x^2}}\right)K_4} e^{-\operatorname{arccosh}\left(\sqrt{1+x^2}\right)K_3}, \end{aligned} \tag{7.49}$$

where we have used $\eta = \eta_2$, $\tilde{\eta} = \eta_3$ as defined in table 1 and set $k_2 = k_3 = k$, such that $x_2 = x_3 = x$. According to (S1) we need to determine again whether the quantity $AI_{h_3}A^{-1}$ is Hermitian in order to proceed. A lengthy computation can be avoided here by noting that the Hermitian invariants I_h for h_2 and h_3 are identical. Thus we have

$$AI_{h_3}A^{-1} = \eta_3\eta_2^{-1}I_{h_2}\eta_2\eta_3^{-1} = \eta_3 I_H \eta_3^{-1} = I_{h_3} = I_{h_3}^\dagger = \left(AI_{h_3}A^{-1}\right)^\dagger, \tag{7.50}$$

Consequently (S1) is implying that

$$\eta^{(1)} = A\eta_3, \tag{7.51}$$

constitutes a new Dyson map. With the help of the TDDE (2.3) we determine the corresponding Hermitian Hamitonian to

$$h^{(1)} = \left[a - \frac{\lambda\left(1+2\sqrt{1+k^2(1+x^2)}\right)}{2k(1+x^2)}\right]K_1 + \left[a - \frac{\lambda\left(2\sqrt{1+k^2(1+x^2)}-1\right)}{2k(1+x^2)}\right]K_2. \tag{7.52}$$

As previously, we use the relation (7.17) to calculate the infinite series of Dyson maps. With

$$\eta^{(n)} = A^n\eta_3, \qquad \text{and} \quad \tilde{\eta}^{(n)} = A^{-n}\eta_2, \tag{7.53}$$

we can use relation (7.50) repeatedly to ensure that at each level the adjoint action of the higher order As on the Hermitian invariants is Hermitian. Using the TDDE (2.3) for the new maps we obtain the Hermitian Hamiltonians

$$h^{(n)} = \left(\frac{\lambda}{2k(1+x^2)}\right)(K_2 - K_1) + \left[a - \frac{(n+1)\lambda\sqrt{1+k^2(1+x^2)}}{2k(1+x^2)}\right](K_1 + K_2), \tag{7.54}$$

from the first map in (7.53) and

$$h^{(n)} = h^{(n)} = \left(\frac{\lambda}{2k(1+x^2)}\right)(K_2 - K_1) + \left[a - \frac{n\lambda\sqrt{1+k^2(1+x^2)}}{2k(1+x^2)}\right](K_1 + K_2), \tag{7.55}$$

from the second.

We may now also compute the symmetry operators for I_{h_2} and I_{h_3}. The symmetry operator is readily written down as

$$S := A^\dagger A = e^{-\operatorname{arccosh}\left(\sqrt{1+x^2}\right)K_3} e^{\operatorname{arcsinh}\left(\frac{1}{k\sqrt{1+x^2}}\right)K_4} e^{-i\arctan(x)K_1} e^{\operatorname{arcsinh}\left(k\sqrt{1+x^2}\right)K_4}$$
$$\times e^{\operatorname{arcsinh}\left(k\sqrt{1+x^2}\right)K_4} e^{-i\arctan(x)K_1} e^{\operatorname{arcsinh}\left(\frac{1}{k\sqrt{1+x^2}}\right)K_4} e^{-\operatorname{arccosh}\left(\sqrt{1+x^2}\right)K_3}.$$

Thus we may now explicitly verify the symmetry relation (7.17), best calculated in the form $SI_{h_2}S^{-1} = I_{h_2}$. Similarly, the symmetry operator for I_{h_3} should be given by $\tilde{S}$, which is indeed the case as we verified explicitly.

7.3.6 Seed maps $\eta = \eta_1$ and $\tilde{\eta} = \eta_2, \eta_3, \eta_4$ - breakdown of the iteration

From the previous two examples one might get the impression that the iteration procedure can always be carried out with any two seed Dyson maps. However, this is not the case when the Hermiticity condition does not hold. To verify this we relied in the previous section on the fact that the invariants for the two Hermitian Hamiltonians resulting from the seed maps were identical. This is not the case when involving η_1 as a seed map and any of the other five maps, as can be seen from (7.33) when comparing the functions $f_\pm$. Thus in this case the Hermiticity condition needs to be verified more explicitly.

Let us now carry out the calculation for the seed Dyson maps chosen to be $\eta = \eta_1$ and $\tilde{\eta} = \eta_2$. We start from the expression for A

$$A := \eta_2\eta_1^{-1} = e^{\operatorname{arccosh}\left(\sqrt{1+x_2^2}\right)K_3} e^{-\operatorname{arcsinh}\left(\frac{1}{k_2\sqrt{1+x_2^2}}\right)K_4} e^{\int^s \lambda(s)ds K_3} \tag{7.56}$$

where η_2 is defined as in (7.28) and η_1 as in table 1. Next we compute to quantity $AI_{h_2}A^{-1}$ where I_{h_2} is given by (7.33). After a lengthy calculation we find that this quantity is non-Hermitian and given by

$$
\begin{aligned}
AI_{h_2}A^{-1} = \quad & \left[\frac{1}{2}\Gamma_{+}^{++} + \frac{\kappa}{2k_2}\right] K_1 \left[\frac{1}{2}\Gamma_{+}^{++} - \frac{\kappa}{2k_2}\right] K_2 + i\frac{\cosh(g)}{k_2\Delta\sqrt{1+x_2^2}}\Gamma_{-}^{+-}K_3 \\
& + i\frac{\kappa(1+x_2^2)+\cosh(g)\Delta}{k_2x_2\sqrt{1+x_2^2}}\Gamma_{-}^{+-}K_4 \\
& + i\left\{\frac{(1+k_2^2)(\delta_- + \delta_+)}{k_2^2x_2}\cosh(g)K_5 - \frac{1}{k_2^2x_2}\left[k_2x_2\Gamma_{-}^{-+} - \kappa\left(\delta_- + \delta_+\right)\Delta\right]\right\} K_5 \\
& + i\left\{\frac{(1+k_2^2)\Gamma_{+}^{--}}{k_2^2x_2}\cosh(g) + \frac{1}{k_2^2x_2}\left[k_2x_2(\delta_- - \delta_+) + \kappa\Gamma_{+}^{--}\Delta\right]\right\} K_6 \\
& - \frac{\left(\delta_- + \delta_+\right)\Delta - \Gamma_{+}^{--}\left(\left(k_2^2+1\right)x_2 - k_2^2x_2\right)}{2k_2^2x_2\sqrt{x_2^2+1}}\cosh(g)(K_7+K_8) \\
& + \left[\kappa\frac{\delta_- + \delta_+ - x_2\Gamma_{+}^{--}\Delta}{2k_2^2x_2\sqrt{x_2^2+1}} + \frac{\left(\delta_+ - \delta_-\right)x_2 - \Gamma_{-}^{+-}\Delta}{2k_2\sqrt{x_2^2+1}}\right] K_7 \\
& + \left[\kappa\frac{-\delta_- - \delta_+ - x_2\Gamma_{+}^{--}\Delta}{2k_2^2x_2\sqrt{x_2^2+1}} + \frac{\Gamma_{-}^{+-}\Delta + \left(\delta_+ - \delta_-\right)x_2}{2k_2\sqrt{x_2^2+1}}\right] K_8 \\
& + \frac{\Gamma_{+}^{--}\Delta - \left(\delta_- + \delta_+\right)\left(\left(k_2^2+1\right)x_2 - k_2^2x_2\right)}{2k_2^2x_2\sqrt{x_2^2+1}}\cosh(g)((K_9+K_{10}) \\
& + \left[\kappa\frac{\Gamma_{+}^{--} - \left(\delta_- + \delta_+\right)x_2\Delta}{2k_2^2x_2\sqrt{x_2^2+1}} + \frac{\left(\delta_- - \delta_+\right)\Delta + x_2\Gamma_{-}^{+-}}{2k_2\sqrt{x_2^2+1}}\right] K_9 \\
& + \left[-\kappa\frac{\Gamma_{+}^{--} + \left(\delta_- + \delta_+\right)x_2\Delta}{2k_2^2x_2\sqrt{x_2^2+1}} + \frac{\left(\delta_+ - \delta_-\right)\Delta + x_2\Gamma_{-}^{+-}}{2k_2\sqrt{x_2^2+1}}\right] K_{10},
\end{aligned} \tag{7.57}
$$

where we introduced the abbreviations

$$
\Gamma_{\delta_1}^{\delta_2\delta_3} := \alpha_- + \delta_1\alpha_+ + \delta_2\beta_- + \delta_3\beta_+ \quad \Delta := \sqrt{1+k_2^2(1+x_2^2)} \quad g := \int^t \lambda(s)ds, \tag{7.58}
$$

with $\delta_i = \pm 1$, $i = 1, 2, 3$. We simplified here our expressions using the identity

$$
\sinh(g) = \frac{\kappa + \cosh(g)\sqrt{1+k_2^2(1+x_2^2)}}{k_2x_2}, \tag{7.59}
$$

which is verified using the first order constraint (7.30). As the invariant is non-Hermitian, the iteration process breaks down and by (S1) we deduce that $A\eta_2$ is not a Dyson map. We have also carried out the equivalent calculation for the seed map choices $\eta = \eta_1$ and $\tilde{\eta} = \eta_3$ and η_4, reaching the same conclusion.

7.4 The time-dependent anharmonic quartic oscillator

Another system for which we have found multiple Dyson maps is the time-dependent anharmonic quartic oscillator [4]

$$H(z,t) = p_z^2 - \frac{1}{64\sigma^6(t)} z^4, \tag{7.60}$$

which, when defined on a contour in the lower-half complex plane $z = -2i\sqrt{1+ix}$ as suggested by Jones and Mateo [74], is mapped the to the non-Hermitian Hamiltonian

$$H(x,t) = p^2 - \frac{1}{2}p + \frac{i}{2}\{x, p^2\} + \frac{1}{4\sigma^6(t)}(x-i)^2. \tag{7.61}$$

7.4.1 Two seed Dyson maps

We present here in a compact and unified form the two solutions for the Dyson maps we have already found along with the corresponding Hermitian Hamiltonians. The Dyson maps are given by

$$\eta = \exp(\alpha x)\exp(\beta p^3 + i\gamma p^2 + i\delta p), \tag{7.62}$$

and

$$\tilde{\eta} = \exp(\beta p^3)\exp(\alpha x)\exp(\tilde{\gamma} p + i\delta p), \tag{7.63}$$

where

$$\alpha = -\frac{\dot{\sigma}}{\sigma}, \quad \beta = \frac{2\sigma^6}{3}, \quad \delta = \frac{[c - \frac{1}{2}\ln(\sigma)]\sigma}{\dot{\sigma}}, \tag{7.64}$$

$$\gamma = \frac{2\sigma^5\dot{\sigma}^2 - 2\omega^2\sigma^3 - \sigma}{2\dot{\sigma}}, \quad \tilde{\gamma} = -1 - 2\omega^2\sigma^2 \tag{7.65}$$

and σ satisfies the auxiliary equation

$$\dot{\sigma} = \frac{\omega^2}{\sigma^3}, \tag{7.66}$$

which has solution

$$\sigma = \frac{\sqrt{c_1{}^2 t^2 + 2c_1{}^2 c_2 t + \omega^2 + c_1{}^2 c_2{}^2}}{\sqrt{c_1}}. \tag{7.67}$$

where c_1, c_2 are real valued constants. The first of these maps, η, was found by making a suitable anstatz on the level of the TDDE (see Chapter 4 for more details) and when substituted into the TDDE results in the Hermitian Hamiltonian[1]

$$h = \sigma^6 p^4 + \frac{1}{4}\left(8\sigma^4\dot{\sigma}^2 + \frac{1+2\omega^2\sigma^2}{\sigma^4\dot{\sigma}^2} - 2\right)p^2 + \frac{\ln(\sigma)-2c}{4\sigma^4\dot{\sigma}^2}p + \frac{1}{4\sigma^6}x^2 \tag{7.68}$$

$$- \frac{\ln(\sigma)-2c}{4\sigma^5\dot{\sigma}}x + \frac{2\sigma^4\dot{\sigma}^2 - 2\omega^2\sigma^2 - 1}{4\sigma^5\dot{\sigma}}\{x,p\} + \frac{(\ln(\sigma)-2c)^2}{16\sigma^4\dot{\sigma}^2} + \frac{2\sigma^6\dot{\sigma}^4 - \sigma^2\dot{\sigma}^2 + \omega^2}{2\sigma^4}. \tag{7.69}$$

The second Dyson map, $\tilde{\eta}$, was found by utilising point transformations to construct an invariant, I_H, for the Hamiltonian (7.61) which was then subsequently employed to determine a Dyson map through the similarity transformation $I_{\tilde{h}} = \tilde{\eta} I_H \tilde{\eta}^{-1}$. Substituting the Dyson map into the TDDE led to the new and different Hermitian Hamiltonian

$$\tilde{h} = \sigma^6 p^4 - 2\omega^2\sigma^2 p^2 + \frac{(2c - \ln(\sigma))\left(\sigma^2\dot{\sigma}^2 + \omega^2\right)}{2\sigma^2\sigma^2}p + \frac{1}{4\sigma^6}x^2 - \frac{\ln(\sigma)-2c}{4\sigma^5\dot{\sigma}}x + \frac{\dot{\sigma}}{\sigma}\{x,p\} - \frac{\omega^4}{\sigma^2} + \frac{(\ln(\sigma)-2c)^2}{16\sigma^4\dot{\sigma}^2} - 2\omega^2\dot{\sigma}^2. \tag{7.70}$$

7.4.2 Construction of invariants

To carry out the procedure laid out in section 7.2 we must now construct invariants for the Hamiltonians (7.68) and (7.70). We start by constructing an invariant for the non-Hermitian Hamiltonian (7.61). Fortunately we have already obtained this invariants in Chapter 5 by utilising point transformations. We reproduce equation (5.133) here for convenience,

$$I_H = \sigma^2 p^2 + \left(2i\sigma\dot{\sigma} - \frac{1}{2}\sigma^2\right)p + \frac{1}{4\sigma^4}x^2 - \frac{i}{2\sigma^4}\left(1 + 2\omega^2\sigma^2 + 2\sigma^4\dot{\sigma}^2\right)x - \sigma\dot{\sigma}\{x,p\} + \frac{1}{2}i\sigma^2\{x,p^2\} - \frac{1}{4\sigma^4} - \frac{\omega^2}{\sigma^2} - \frac{1}{2}i\sigma\dot{\sigma} - \dot{\sigma}^2. \tag{7.71}$$

[1] The appearance of the Dyson map and Hermitian Hamiltonian differs from that presented in Chapter 4. It is in fact the same Dyson map yet with the change $g = \frac{1}{4\sigma^6}$ instead of $g = \frac{1}{4\sigma^3}$ so that it coincides with the map $\tilde{\eta}$ found using point transformations in Chapter 5. The auxiliary equations also differ yet their solutions can be made equivalent, more details on this are in Chapter 5 section 5.6.4.

The Hermitian invariant associated with the Dyson map η is then obtained via

$$I_h = \eta I_H \eta^{-1} = \sigma^8 p^4 + \left(\sigma^2 + \frac{\sigma^2\omega^4 + \frac{1}{4\sigma^2} + \omega^2}{\dot{\sigma}^2} + \sigma^6\dot{\sigma}^2\right)p^2 + \frac{1}{4\sigma^4}x^2 \quad (7.72)$$
$$- \left(\frac{\sigma^2}{2} + \frac{2\delta\sigma^2\omega^2 + \delta}{2\sigma^3\dot{\sigma}} + \delta\sigma\dot{\sigma}\right)p + \frac{\delta}{2\sigma^4}x - \left(\frac{2\sigma^2\omega^2 + 2\sigma^4\dot{\sigma}^2 + 1}{4\sigma^3\dot{\sigma}}\right)\{x,p\}$$
$$+ \omega^4 + \frac{\delta^2}{4\sigma^4},$$

and the Hermitian invariant for $\tilde{\eta}$ is given by

$$I_{\tilde{h}} = \tilde{\eta} I_H \tilde{\eta}^{-1} = \sigma^8 p^4 - 2\omega^2\sigma^4 p^2 - \frac{1}{2}\sigma^2 p + \frac{1}{4\sigma^4}x^2 + \frac{\delta}{2\sigma^4}x + \frac{\delta}{4\sigma^4} + \omega^4. \quad (7.73)$$

The invariant I_h is considerably more involved than $I_{\tilde{h}}$.

We may now employ the Dyson maps and invariants to construct an infinite series of Dyson maps for the time-dependent anharmonic quartic oscillator.

7.4.3 Infinite series of Dyson maps from η and $\tilde{\eta}$

We start by constructing the operator A as in (7.2)

$$A := \tilde{\eta}\eta^{-1} = \exp(\beta p^3)\exp(\alpha x)\exp(\tilde{\gamma} p - i\gamma p^2 - \beta p^3)\exp(-\alpha x). \quad (7.74)$$

We may simplify this expression by utilising the braiding relation

$$e^{\hat{A}} f(\hat{B}) e^{-\hat{A}} = f\left(e^{\hat{A}}\hat{B}e^{-\hat{A}}\right), \quad \text{if} \quad f(\hat{B}) = \sum_n C_n \hat{B}^n, \quad (7.75)$$

meaning that $f(\hat{B})$ can be expressed as a power series[2]. Letting $\hat{A} = \alpha x$ and $\hat{B} = \tilde{\gamma} p - i\gamma p^2 - \beta p^3$ we may write

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = -\beta p^3 - i(3\alpha\beta + \gamma)p^2 + (3\alpha^2\beta + 2\alpha\gamma + \tilde{\gamma})p + i\alpha[\alpha(\alpha\beta + \gamma) + \tilde{\gamma}], \quad (7.76)$$

such that (7.74) simplifies too

[2] A simple proof of this identity is as follows

$$e^{\hat{A}} f(\hat{B}) e^{-\hat{A}} = \sum_n C_n e^{\hat{A}}\hat{B}^n e^{-\hat{A}} = \sum_n C_n \left(e^{\hat{A}}\hat{B}e^{-\hat{A}}\right)^n = f\left(e^{\hat{A}}\hat{B}e^{-\hat{A}}\right),$$

as

$$e^{\hat{A}}\hat{B}^n e^{-\hat{A}} = e^{\hat{A}}\hat{B}e^{-\hat{A}}e^{\hat{A}}\hat{B}e^{-\hat{A}}e^{\hat{A}}\hat{B}e^{-\hat{A}}...\, e^{\hat{A}}\hat{B}e^{-\hat{A}} = \left(e^{\hat{A}}\hat{B}e^{-\hat{A}}\right)^n.$$

$$A = \exp\left\{(3\alpha^2\beta + 2\alpha\gamma + \tilde{\gamma})p - i(3\alpha\beta + \gamma)p^2 + i\alpha[\alpha(\alpha\beta + \gamma) + \tilde{\gamma}]\right\}. \tag{7.77}$$

Following (S1) we now need to determine whether the quantity $AI_{\tilde{h}}A^{-1}$ is Hermitian. Several lengthy calculations may be avoided in the construction of the infinite series of Dyson maps here by noting the adjoint action of A on each of the variables in $I_{\tilde{h}}$. Firstly, given that A is a function of p and p^2 we know that the adjoint action on p, p^2 and p^4 produces no new terms. For the final three terms we obtain

$$AxA^{-1} = x + \frac{\sigma\left(2\sigma^2\omega^2 + 1\right)}{\dot{\sigma}}p + 2\sigma^5\dot{\sigma}p, \tag{7.78}$$

$$Ax^2A^{-1} = x^2 + \frac{\left(2\sigma^3\omega^2 + \sigma + 2\sigma^5\dot{\sigma}^2\right)^2}{\dot{\sigma}^2}p^2 + \left(\frac{2\sigma^3\omega^2 + \sigma}{\dot{\sigma}} + 2\sigma^5\dot{\sigma}\right)\{x,p\}, \tag{7.79}$$

$$A\{x,p\}A^{-1} = \{x,p\} + \left(\frac{2\left(2\sigma^3\omega^2 + \sigma\right)}{\dot{\sigma}} + 4\sigma^5\dot{\sigma}\right)p^2, \tag{7.80}$$

which are all Hermitian. Therefore we have that $AI_{\tilde{h}}A^{-1} = \tilde{I}^{(1)}$ is Hermitian and (S1) implies that

$$\tilde{\eta}^{(1)} = A\tilde{\eta} \tag{7.81}$$

is indeed a new Dyson map. Substituting $\tilde{\eta}^{(1)}$ into the TDDE (2.3) produces the corresponding Hermitian Hamiltonian

$$\tilde{h}^{(1)} = \sigma^6 p^4 + \tilde{f}^{(1)}_{p^2}p^2 + \tilde{f}^{(1)}_{p}p + \frac{1}{4\sigma^6}x^2 + \frac{2c - \ln(\sigma)}{4\sigma^5\dot{\sigma}}x + \tilde{f}^{(1)}_{xp}\{x,p\} + \tilde{f}^{(1)}_{f}, \tag{7.82}$$

where

$$\tilde{f}^{(1)}_{p^2} = \frac{5}{2} + \frac{8\sigma^4\omega^4 + 6\sigma^2\omega^2 + 1}{4\sigma^4\dot{\sigma}^2}, \quad \tilde{f}^{(1)}_{xp} = \frac{2\sigma^2\omega^2 + 6\sigma^4\dot{\sigma}^2 + 1}{4\sigma^5\dot{\sigma}}, \tag{7.83}$$

$$\tilde{f}^{(1)}_{p} = \frac{(2c - \ln(\sigma))\left(4\sigma^2\omega^2 + 4\sigma^4\dot{\sigma}^2 + 1\right)}{4\sigma^4\dot{\sigma}^2}, \tag{7.84}$$

$$\tilde{f}^{(1)}_{f} = \frac{(\ln(\sigma) - 2c)^2 - 8\dot{\sigma}^2\left(4\sigma^2\omega^4 + \sigma^2\left(8\sigma^2\omega^2 - 1\right)\dot{\sigma}^2 + 2\sigma^6\dot{\sigma}^4 + \omega^2\right)}{16\sigma^4\dot{\sigma}^2}. \tag{7.85}$$

We now extend this analysis to determine an infinite series of Dyson maps. Firstly we note that the quantity $A^nI_{\tilde{h}}A^{-n} = \tilde{I}^{(n)}$ is always Hermitian as the adjoint action of A on all of the terms in $I_{\tilde{h}}$ is Hermitian and produces no new terms as demonstrated by equations (7.78) - (7.80) and the fact that A is only a function of

p and p^2. From (S1) we therefore know that

$$\tilde{\eta}^{(n)} = A^n \tilde{\eta} \tag{7.86}$$

constitutes an infinite series of Dyson maps. We have managed to determine the corresponding infinite series of Hermitian Hamiltonians by finding patterns in the time-dependent coefficient functions of $p^2, p, \{x,p\}$ and the free term. We have that

$$\tilde{h}^{(n)} = \sigma^6 p^4 + \tilde{f}_{p^2}^{(n)} p^2 + \tilde{f}_p^{(n)} p + \tilde{f}_{xx} x^2 + \tilde{f}_x x + \tilde{f}_{xp}^{(n)} \{x,p\} + \tilde{f}_f^{(n)}, \tag{7.87}$$

where

$$\begin{aligned}
\tilde{f}_{p^2}^{(n)} &= \frac{3n}{2} + n^2 + 2\left(n^2-1\right)\sigma^2\omega^2 + (n-1)n\sigma^4\dot{\sigma}^2 \qquad (7.88)\\
&\quad + \frac{n\left[4(n+1)\sigma^4\omega^4 + 2(2n+1)\sigma^2\omega^2 + n\right]}{4\sigma^4\dot{\sigma}^2},\\
\tilde{f}_p^{(n)} &= \frac{[2c-\ln(\sigma)]\left[2(n+1)\sigma^2\omega^2 + 2(n+1)\sigma^4\dot{\sigma}^2 + n\right]}{4\sigma^4\dot{\sigma}^2}, \qquad (7.89)\\
\tilde{f}_{xp}^{(n)} &= \frac{2n\sigma^2\omega^2 + 2(n+2)\sigma^4\dot{\sigma}^2 + n}{4\sigma^5\dot{\sigma}}, \qquad (7.90)\\
\tilde{f}_f^{(n)} &= \frac{[\ln(\sigma)-2c]^2}{16\sigma^4\dot{\sigma}^2} - \frac{2(n+1)\sigma^2\omega^4 + 2n\sigma^6\dot{\sigma}^4 + n\omega^2}{2\sigma^4} \qquad (7.91)\\
&\quad - \frac{\dot{\sigma}^2\left[4(n+1)\sigma^4\omega^2 - n\sigma^2\right]}{2\sigma^4},\\
\tilde{f}_x &= \frac{2c-\ln(\sigma)}{4\sigma^5\dot{\sigma}}, \qquad \tilde{f}_{xx} = \frac{1}{4\sigma^6}, \qquad (7.92)
\end{aligned}$$

In a similar fashion we may also derive the other infinite series of Dyson maps given by

$$\eta^{(n)} = A^{-n}\eta, \tag{7.93}$$

which is associated with the Hermitian Hamiltonians

$$h^{(n)} = \sigma^6 p^4 + f_{p^2}^{(n)} p^2 + f_p^{(n)} p + f_{xx} x^2 + f_x x + f_{xp}^{(n)} \{x,p\} + f_f^{(n)}, \tag{7.94}$$

where

$$f_p^{(n)} = \frac{[\ln(\sigma)-2c]\left(2n\sigma^2\omega^2 + 2n\sigma^4\dot{\sigma}^2 + n + 1\right)}{4\sigma^4\dot{\sigma}^2}, \tag{7.95}$$

$$f_{p^2}^{(n)} = n^2 + 2(n+2)n\sigma^2\omega^2 + \frac{n}{2} - \frac{1}{2} + (n+1)(n+2)\sigma^4\dot{\sigma}^2 + \frac{(n+1)\left(2\sigma^2\omega^2+1\right)\left(2n\sigma^2\omega^2+n+1\right)}{4\sigma^4\dot{\sigma}^2} \tag{7.96}$$

$$f_{xp}^{(n)} = -\frac{(n+1)\left(2\sigma^2\omega^2+1\right)}{4\sigma^5\dot{\sigma}} - \frac{(n-1)\dot{\sigma}}{2\sigma}, \tag{7.97}$$

$$f_f^{(n)} = \frac{(\ln(\sigma)-2c)^2}{16\sigma^4\dot{\sigma}^2} + \frac{\omega^2\left(2n\sigma^2\omega^2+n+1\right)+\sigma^2\dot{\sigma}^2\left[n\left(4\sigma^2\omega^2-1\right)-1\right]}{2\sigma^4} + (n+1)\sigma^2\dot{\sigma}^4, \tag{7.98}$$

$$f_x = \frac{2c-\ln(\sigma)}{4\sigma^5\dot{\sigma}}, \qquad f_{xx} = \frac{1}{4\sigma^6}, \tag{7.99}$$

It is straightforward to compute the symmetry operators for $I_{\tilde{h}}$ and I_h. We obtain

$$S := A^\dagger A = \exp[2(6\alpha^2\beta + 2\alpha\gamma + \tilde{\gamma})p]. \tag{7.100}$$

We have verified that $SI_{\tilde{h}}S^{-1} = I_{\tilde{h}}$ and $SI_hS^{-1} = I_h$.

7.4.4 Comparison of infinite spectrally equivalent double wells

We shall now directly compare the time-dependent unstable anharmonic oscillator potential in (7.60) with the infinite number of spectrally equivalent potentials in (7.86)-(7.91) and (7.93)-(7.98).

The Hamiltonians (7.86) and (7.93) can be written as

$$H^{(n)} = \sigma^6p^4 + F_{p^2}^{(n)}p^2 + F_p^{(n)}p + F_{xx}x^2 + F_x x + F_{xp}^{(n)}\{x,p\} + F_f^{(n)}. \tag{7.101}$$

where

$$F_{p^2}^{(n)} = f_{p^2}^{(n)}, \tilde{f}_{p^2}^{(n)}, \quad F_p^{(n)} = f_p^{(n)}, \tilde{f}_p^{(n)}, \quad F_{xx} = f_{xx}, \tilde{f}_{xx}, \quad F_x = f_x, \tilde{f}_x, \tag{7.102}$$

$$F_{xp}^{(n)} = f_{xp}^{(n)}, \tilde{f}_{xp}^{(n)}, \quad F_f^{(n)} = f_f^{(n)}, \tilde{f}_f^{(n)}. \tag{7.103}$$

We now need to eliminate the terms proportional to x and $\{x,p\}$. We achieve this with the following unitary transformation

$$U = e^{-i\frac{F_{xp}^{(n)}}{2F_{xx}}p^2 - i\frac{F_x}{F_{xx}}p}, \tag{7.104}$$

where the unitary transformed Hamiltonians are given by

$$\hat{H}^{(n)} = \sigma^6 p^4 + \left[F^{(n)}_{p^2} - \frac{\left(F^{(n)}_{xp}\right)^2}{F_{xx}} \right] p^2 + \left[F^{(n)}_{p} - \frac{F_x F^{(n)}_{xp}}{F_{xx}} \right] p + F_{xx}x^2 + F^{(n)}_{f} - \frac{F_x^2}{4F_{xx}}. \tag{7.105}$$

Next we scale these Hamiltonians with a time-dependent function $x \to (\tilde{F}_{xx})^{-1/2}x$ and subsequently Fourier transform them so that they are viewed in momentum space. We obtain the spectrally equivalent Hamiltonians with time-dependent potentials as

$$\begin{aligned}\bar{H}^{(n)}(y,t) = p_y^2 + \sigma^6 \left(F_{xx}\right)^2 y^4 + \left[F_{xx}F^{(n)}_{p^2} - \left(F^{(n)}_{xp}\right)^2 \right] y^2 + F^{(n)}_{f} \\ + \left[\sqrt{F_{xx}}F^{(n)}_{p} - \frac{F_x \tilde{f}^{(n)}_{xp}}{\sqrt{F_{xx}}} \right] y - \frac{F_x^2}{4F_{xx}}.\end{aligned} \tag{7.106}$$

The time-dependent double wells potentials associated with the Dyson maps $\tilde{\eta}^{(n)}$ are

$$\begin{aligned}\bar{\tilde{V}}^{(n)}(y,t) = \frac{y^4}{4} + \frac{\left[4c\sigma\omega^2 - 2\sigma\omega^2\ln(\sigma)\right]}{8\sigma^6\dot{\sigma}^2} y + \frac{n\omega^2\left(2\sigma^2\omega^2+1\right)}{8\sigma^8\dot{\sigma}^2} y^2 \\ + \frac{\sigma^2\dot{\sigma}^2\left[n - 4(n+1)\sigma^2\omega^2\right] - 2(3n+1)\sigma^6\dot{\sigma}^4}{8\sigma^8\dot{\sigma}^2} y^2 + \frac{\omega^2\left(2n\sigma^2\omega^2+n+1\right)}{2\sigma^4} \\ + \frac{\sigma^2\dot{\sigma}^2\left[n\left(4\sigma^2\omega^2-1\right)-1\right] + 2(n+1)\sigma^6\dot{\sigma}^4}{2\sigma^4}.\end{aligned} \tag{7.107}$$

Plots of these double wells potentials for different values of n and t can be found in figure 7.1. The time-dependent double wells potentials for the Dyson map $\eta^{(n)}$ are given by

$$\begin{aligned}\bar{V}^{(n)}(y,t) = \frac{y^4}{4} - \frac{\left[2c - \ln(\sigma)\right]\left[2\sigma^4\dot{\sigma}^2 - 1 - 2\sigma^2\omega^2\right]}{8\sigma^7\dot{\sigma}^2} y + \frac{2(5n+1)\sigma^6\dot{\sigma}^4}{8\sigma^8\dot{\sigma}^2} y^2 \\ + \frac{\sigma^2\dot{\sigma}^2\left[4(2n+1)\sigma^2\omega^2+n+1\right] - \left[(n+1)\omega^2\left(2\sigma^2\omega^2+1\right)\right]}{8\sigma^8\dot{\sigma}^2} y^2 \\ + \frac{\omega^2\left(2n\sigma^2\omega^2+n+1\right) + \sigma^2\dot{\sigma}^2\left(n\left(4\sigma^2\omega^2-1\right)-1\right) + 2(n+1)\sigma^6\dot{\sigma}^4}{2\sigma^4}.\end{aligned} \tag{7.108}$$

These potentials are plotted in figure 7.2 for different values of n and t. We have determined an infinite number of time-dependent double wells which are spectrally equivalent to the time-dependent unstable anharmonic oscillator. The two infinite series of Dyson maps do in fact lead to potentials with different characteristics for the same parameter values.

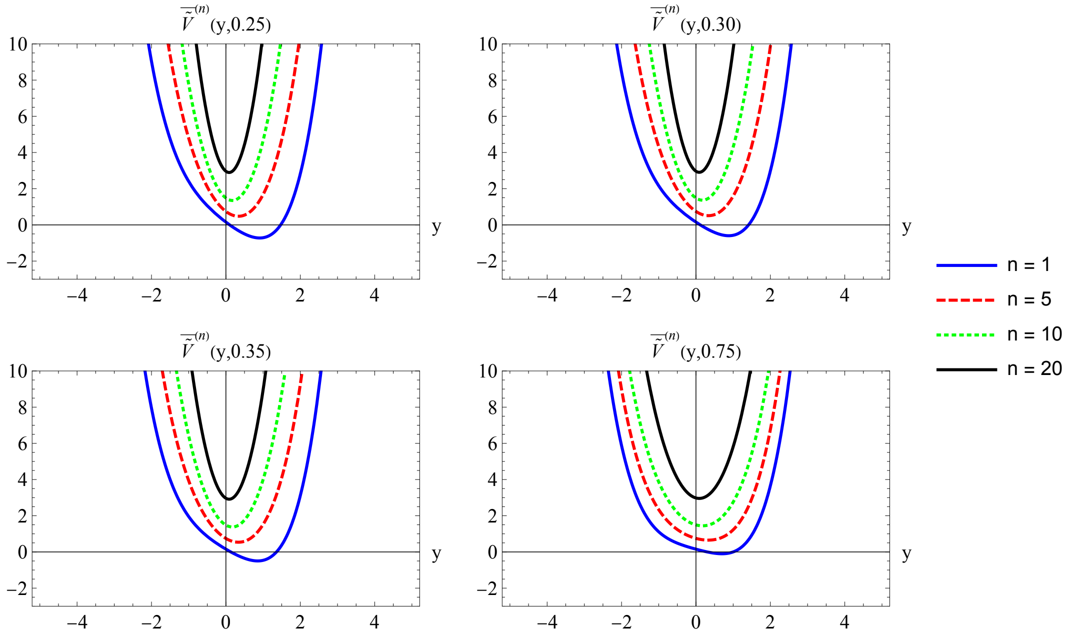

Figure 7.1: The time-dependent double wells potentials in 7.107 for $c = 0$, $c_1 = 0.1$, $c_2 = 0.5$ and $\omega. = 1.2$ at different times and different values of n.

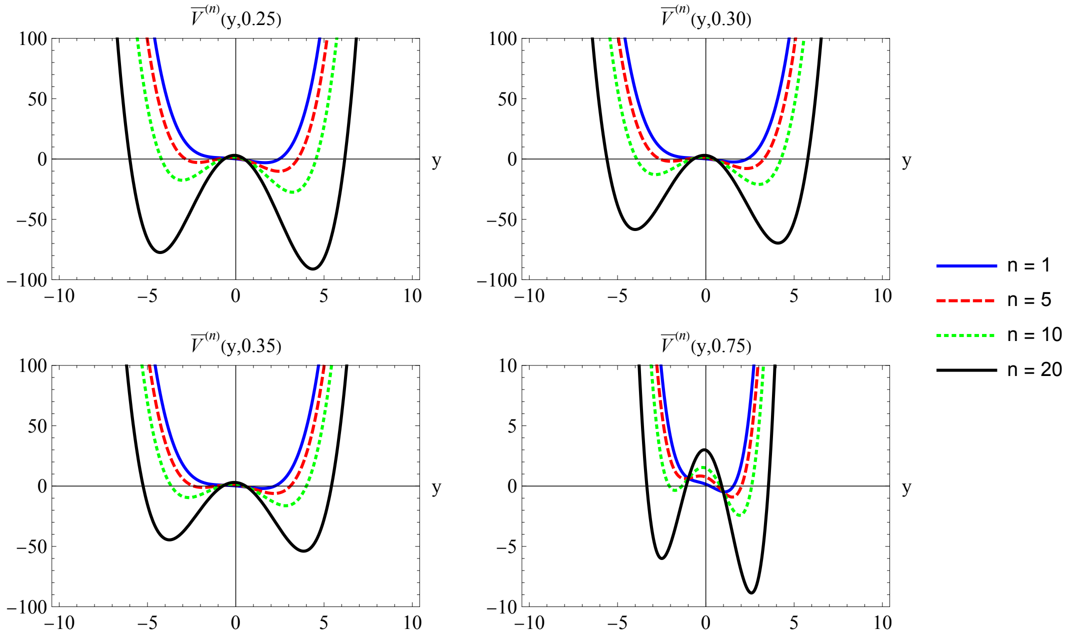

Figure 7.2: The time-dependent double wells potentials in 7.108 for $c = 0$, $c_1 = 0.1$, $c_2 = 0.5$ and $\omega. = 1.2$ at different times and different values of n.

7.5 Conclusions

We have proposed a scheme that allows to compute new time-dependent Dyson maps from two seed maps in a iterative fashion for a given non-Hermitian time-dependent Hamiltonian. As argued in general in section 7.2, in principle the iteration process might continue indefinitely, thus leading to an infinite series of time-dependent Dyson maps including their associated Hermitian Hamiltonians. The symmetry operators S of the Lewis-Riesenfeld invariants govern this behaviour. Thus when the symmetry is broken also the iteration procedure breaks down. We carried out the procedure in detail for a two-dimensional system of harmonic oscillators that are coupled to each other in a non-Hermitian, but $\mathcal{PT}$-symmetrical, fashion and the time-dependent anharmonic oscillator. For the former we have presented in detail three examples that exhibit different types of behaviours, but we have verified that similar results are obtained when starting from different sets of seed functions. For the latter we started with only two seed maps yet were able to construct the infinite series of Dyson maps. We have focused in our analysis mainly on the relations between the various Hamiltonians and their corresponding invariants, but having obtained the Dyson maps, and therefore the metric operators, it is straightforward to extend the considerations to the associated wave functions and inner product structures on the physical Hilbert space.

Chapter 8

Conclusions and outlook

8.1 Conclusions

The focus of this thesis has been to provide new methods, both exact and approximate, which aid in the study of time-dependent non-Hermitian quantum systems. Of the four new methods presented one is associated with approximate solutions to the TDSE based upon on the Lewis-Riesenfeld method of invariants with the remaining being concerned with solution procedures for the time-dependent Dyson map $\eta(t)$ and metric $\rho(t)$. To demonstrate the validity of these approaches we have studied a wide array of both Hermitian and non-Hermitian time-dependent systems. For the non-Hermitian case we additionally explored the spontaneously broken $\mathcal{PT}$-regime which would have been discarded as unphysical for time-independent systems.

The first method we presented allows for the construction of approximate solutions to the TDSE by utilising time-independent approximation methods such as time-independent perturbation theory of WKB theory. These time-independent techniques can be utilised in conjunction with the Lewis-Riesenfeld method of invariants as the eigenvalue equation which the invariant satisfies has time-independent eigenvalues. By studying two different systems with factorisable optical potentials, for which an exact solution can be constructed, we demonstrated by calculating expecation values and the autocorrelation function that both time-independent methods produced a fairly accurate solution to the TDSE. For the time-independent perturbation theory the agreement between exact and approximate solutions naturally improved for decreasing expansion parameter and for the WKB approximation we found the main discrepancies around the turning points, where the WKB wavefunction blows up. Overall, by not insisting on full exact solvability of the Lewis-Riesenfeld method of invariants will allow for the study of more complicated time-dependent systems.

In chapters 5, 6, and 7 we were concerned with finding new ways to determine the Dyson map $\eta(t)$ and metric $\rho(t)$. In chapter 2 we covered the existing approaches highlighting the limitations for which some we wanted to overcome. For example, a clear disadvantage in utilising the Lewis-Riesenfeld method of invariants in the calculation of $\eta(t)$ and $\rho(t)$ was the reliance on an Ansatz for the time-dependent non-Hermitian Hamiltonian $H(t)$. We overcame this by proposing the use of point transformations in chapter 5 and demonstrating for the first time that they can be used for the construction of non-Hermitian invariants for non-Hermitian Hamiltonians. We also showed the flexibility of the approach by connecting time-dependent non-Hermitian systems with exactly solvable time-independent Hermitian and non-Hermitian systems as well as non-exactly solvable time-independent non-Hermitian systems. While the approach suffers the same disadvantage as utilising Lewis-Riesenfeld invariants of increased number steps to obtain $\eta(t)$ and $\rho(t)$, it is imbued with the same advantages with the addition of bypassing a need for an Anstaz for the invariant.

In chapter 6 we considered a perturbative method for the computation of $\eta(t)$ and $\rho(t)$. By applying this to a system of two-dimensional time-dependent harmonic oscillators with a non-Hermitian $i(p_x p_y + xy)$ coupling as well as the time-dependent unstable anharmonic oscillator we were able to obtain exact solutions. For the former system, by treating the non-Hermitian term as a small perturbation, we were able to identify six unique Dyson map solutions. For the latter we treated the perturbation as strong and recovered the exact solution determined in [4]. For both cases we saw that we needed to modify the Ansatz for the metric in the perturbation theory if the Dyson map was non-Hermitian. Further to this, we additionally explored two-dimensional time-dependent harmonic oscillators where the non-Hermitian coupling was $ip_x p_y$. This system has first been studied in this thesis and we were able to identify two exact solutions for the Dyson map where we again had to modify the perturbative procedure to search for a series solution relating the time-dependent parameters in the Hamiltonian. The study of these three systems has highlighted that while the perturbative procedure allows for the construction of exact and multiple solutions for the Dyson map, it cannot be applied universally. The Ansatz for the form of the metric appears to be model dependent.

In addition to obtaining several solutions for the $i(p_x p_y + xy)$ and $ip_x p_y$ oscillators, we also explored the broken $\mathcal{PT}$-regime for both systems. All Dyson maps constructed were valid in the broken $\mathcal{PT}$-regime bar one for the $i(p_x p_y + xy)$ oscillators, which was valid only at the exceptional point. This is similar to the effects which were observed in [2, 51, 70, 71], yet here our solutions are restricted to a particular parameter space. We cannot cross from the unbroken to the broken $\mathcal{PT}$-regime.

Motivated by the number of Dyson maps we had found for the $i(p_x p_y + xy)$ oscillators, in chapter 7 we explored an approach which allowed for the construction of an infinite series of Dyson maps by exploiting the symmetries of the Lewis-Riesenfeld invariants for the non-Hermitian and associated Hermitian Hamiltonians. If the symmetry was broken, then as demonstrated the iteration procedure could not be completed. Therefore, even with two Dyson maps for a particular system there is no guarantee that you can construct an infinite series of Dyson maps and equivalent Hermitian Hamiltonians.

Throughout this thesis we also presented a new extensive analysis on the time-dependent anharmonic oscillator. Previously we had obtained one exact solution for the Dyson map [4] by solving the TDDE (2.3) with an Ansatz. By constructing a point transformation between general Bender-Boetcher Hamiltonians and subsequently selecting the concrete potential as $-x^4$ we were able to construct a time-dependent non-Hermitian invariant for the system. From there, after defining the invariant on the correct contour as first done in [74], we were then able to more easily identify a second time-dependent Dyson map, which when combined with the first, as outlined in chapter 7, was a symmetry operator for the invariant. This then allowed for the construction of the infinite series of Dyson maps and Hermitian Hamiltonians which, after being unitary and Fourier transformed, contained an infinite number of spectrally equivalent time-dependent double wells terms.

Table 8.1 contains a summary of methods which can be used to obtain $\eta(t)$ and $\rho(t)$ including advantages and disadvantages. The first three approaches were outlined in chapter 2 and the final three correspond to the new works presented in this thesis.

Approach	Advantages	Disadvantages
TDQH (2.4)	1. Doesn't involve h 2. Clearer structure for ρ (Hermitian)	1. $\rho \to \eta$ more difficult 2. Coupled differential equations
TDDE (2.3)	1. $\eta \to \rho$ easier 2. Less restriction on Ansatz	1. η can be Hermitian or non-Hermitian: no clear structure 2. Coupled differential equations.
Lewis-Riesenfeld invariants (2.13)	1. Similarity transformation is easier to solve. 2. Easier to solve for eigenfunctions	1. Increased number of steps 2. Ansatz for the invariant
Point transformations (Chapter 5)	1. Advantages of invariant approach 2. No Ansatz for invariant	1. Increased number of steps
Perturbation theory (Chapter 6)	1. Obtain exact solutions 2. Identify several solutions	1. Can not be applied universally
Infinite Series (Chapter 7)	1. Infinite solutions	1. Two starting η's 2. No guarantee series can constructed

Table 8.1: Summary of comparison of existing and new solution procedures for $\eta(t)$ and $\rho(t)$.

8.2 Outlook

With the work presented here being concerned with new methods in the time-dependent quantum mechanics there are naturally many follow on questions.

Firstly, with regard to using time-independent approximations to find solutions to the TDSE. A logical next step would be to apply the procedure to more complicated systems, for example those with a Gaussian potential, we demonstrated how to construct an invariant for such a model in chapter 3. The time-dependent anharmonic oscillator now has an infinite number of spectrally equivalent Hermitian Hamiltonians and corresponding invariants, this approach could be utilised to study the spectra and eigenfunctions. There could also be systems for which an exact invariant cannot be constructed, weakening the first step of the approach to deal with approximate invariants would open up the procedure to deal with more complex physical phenomena.

We only considered here point transformations between one dimensional Hamiltonians. Extending the approach to higher dimensional systems would be of interest in both the Hermitian and non-Hermitian regime. More complicated choices of both reference and target Hamiltonian is also worth study, and if the resulting invariant is not exactly solvable we can utilise the approximate method outlined in chapter 3 to obtain eigenfunctions.

A model for which the metric is only known perturbatively in the time-independent scenario is the complex cubic $V = ix^3$ potential. Using the perturbative approach outlined in chapter 6 to study the time-dependent version of this model is of great interest. There are issues here however associated with how to set up the perturbative scheme correctly, as we demonstrated it cannot be applied universally. Determining criteria for when to use the different types of Ansatz for the metric would therefore also be beneficial.

As for all the other methods considered, it would be a natural next step to carry out the infinite series scheme for more concrete models. Other questions however arise from having an infinite number of solutions for the Dyson map/metric. We know from chapter 6 that different solutions for the metric lead to different physical behaviour as exhibited in the instantaneous energy spectra for the $i(p_x p_y + xy)$ oscillators. Do we observe similar effects for other physical quantities such as the entropy?

Another open question which is related more to the broader field of non-Hermitian quantum mechanics rather than just the methods is how we move from the time-independent to the time-dependent regime. As can be seen for the time-dependent anharmonic oscillator in chapter 4, we cannot recover the solution for the time-independent Dyson map from the time-dependent one. Exploring the connection between the time-independent and the time-dependent could also give a generic argument as to why the broken $\mathcal{PT}$-regime is 'mended' with time.

To conclude, this thesis has provided an array of new methods in the areas of time-dependent Hermitian and non-Hermitian quantum systems. There now exists a multitude of approaches, both approximate and exact, which can be utilised to determine the metric and Dyson map, the application of which will hopefully increase our understanding of phenomena related to time-dependent non-Hermitian quantum mechanics.

Appendix A

Lewis-Riesenfeld Invariants

In this appendix we shall be providing further details on the Lewis-Riesenfeld method of invariants [89]. We shall demonstrate that the eigenavalues of a Lewis-Riesenfeld invariants are indeed time-independent, that the eigenvectors of the invariants do satisfy the TDSE, and also how to relate a Hermitian and non-Hermitian invariant via a similarity transformation involving the time-dependent Dyson map .

A.1 Time-independent eigenvalues

To demonstrate that the eigenvalues, λ_n, are indeed time-independent we start with eigenvalue equation which the invariant $I(t)$ satisfies (2.10). Taking the scalar product of this equation with the bra $\langle\phi_n|$, and assuming orthonormal eigenvectors $\langle\phi_n|\phi_m\rangle = \delta_{nm}$, we may write the time-derivative of the eigenvalues as

$$\begin{aligned}
\partial_t\lambda_n &= \frac{\partial}{\partial t}\langle\phi_n|I|\phi_n\rangle, \\
&= \left(\partial_t\langle\phi_n|\right) I|\phi_n\rangle + \langle\phi_n|\partial_t I|\phi_n\rangle + \langle\phi_n|I\left(\partial_t|\phi_n\rangle\right), \\
&= \lambda_n\left(\partial_t\langle\phi_n|\right)|\phi_n\rangle + \frac{i}{\hbar}\langle\phi_n|\left[I,H\right]|\phi_n\rangle + \lambda_n\langle\phi_n|\left(\partial_t|\phi_n\rangle\right), \\
&= \lambda_n\partial_t\left(\langle\phi_n|\phi_n\rangle\right) - \frac{i}{\hbar}\langle\phi_n|H\lambda_n - \lambda_n H|\phi_n\rangle, \\
&= 0, \qquad\qquad (A.1)
\end{aligned}$$

where we have used the fact that the eigenvalues are real guaranteed by the fact that $I^\dagger = I$.

A.2 Eigenvectors satisfy the TDSE

To show that the eigenvectors in the eigenvalue equation (2.10) do satisfy the TDSE we again start with the eigenvalue equation. Taking the time-derivative we obtain

$$\begin{aligned}\lambda_n|\dot{\phi}_n\rangle &= \dot{I}|\phi_n\rangle + I|\dot{\phi}_n\rangle, \\ &= -\frac{i}{\hbar}\left[H, I\right]|\phi_n\rangle + I|\dot{\phi}_n\rangle, \\ &= -\lambda_n\frac{i}{\hbar}H|\phi_n\rangle + \frac{i}{\hbar}IH|\phi_n\rangle + I|\dot{\phi}_n\rangle. \end{aligned} \tag{A.2}$$

If we now take the scalar product with the bra $\langle\phi_m|$ we obtain

$$\lambda_n\langle\phi_m|\dot{\phi}_n\rangle = -\lambda_n\frac{i}{\hbar}\langle\phi_m|H|\phi_n\rangle + \lambda_m\frac{i}{\hbar}\langle\phi_m|H|\phi_n\rangle + \lambda_m\langle\phi_m|\dot{\phi}_n\rangle. \tag{A.3}$$

We may re-arrange this to obtain

$$i\hbar(\lambda_n - \lambda_m)\langle\phi_m|\dot{\phi}_n\rangle = (\lambda_n - \lambda_m)\langle\phi_m|H|\phi_n\rangle. \tag{A.4}$$

For this equation to hold for both $\lambda_n = \lambda_m$ and $\lambda_n \neq \lambda_m$ we must have that

$$i\hbar\langle\phi_m|\dot{\phi}_n\rangle = \langle\phi_m|H|\phi_n\rangle, \tag{A.5}$$

and therefore $|\phi_n\rangle$ must satisfy the TDSE.

We may also derive the time-dependent phase function given by (3.5). To do this we consider the time-dependent states $|\psi_n\rangle = e^{i\alpha_n(t)}|\phi_n\rangle$ and take the time derivative

$$|\dot{\psi}_n\rangle = i\dot{\alpha}_n e^{i\alpha_n}|\phi_n\rangle + e^{i\alpha_n}|\dot{\phi}_n\rangle. \tag{A.6}$$

As $|\dot{\psi}_n\rangle = -\frac{i}{\hbar}e^{i\alpha_n}H|\phi_n\rangle$ we have that

$$\dot{\alpha}_n = \frac{1}{\hbar}\langle\phi_n|i\hbar\partial_t - H|\phi_n\rangle. \tag{A.7}$$

Therefore the phase $\alpha_n(t)$ is given by equation (2.11).

A.3 Similarity transform between Hermitian and non-Hermitian invariants

We now wish to prove that the relation

$$I_h = \eta I_H \eta^{-1}, \tag{A.8}$$

holds. To do so we start with the evolution equation (2.12)

$$\frac{dI_h}{dt} = \partial_t I_h - i\left[I_h, h\right] = 0, \tag{A.9}$$

and substiute the expression (A.8) into the right hand side obtaining

$$\partial_t \left(\eta I_H \eta^{-1}\right) - i\left[\eta I_H \eta^{-1}, h\right] = 0. \tag{A.10}$$

The first term in this expression is expanded as

$$\begin{aligned}\partial_t \left(\eta I_H \eta^{-1}\right) &= \left(\partial_t \eta\right) I_H \eta^{-1} + \eta \left(\partial_t I_H\right) \eta^{-1} + \eta I_H \left(\partial_t \eta^{-1}\right), \\ &= \left(\partial_t \eta\right) I_H \eta^{-1} + i\eta[I_H, H]\eta^{-1} + \eta I_H \left(\partial_t \eta^{-1}\right)\end{aligned} \tag{A.11}$$

the second term requires a little more attention. We first substitute the TDDE (2.3), the similarity transformation (A.8) and expand the addition term in the commutator giving

$$\begin{aligned}[I_h, h] &= \left[\eta I_H \eta^{-1}, \eta H \eta^{-1} + i(\partial_t \eta)\eta^{-1}\right] \\ &= \left[\eta I_H \eta^{-1}, \eta H \eta^{-1}\right] + i\left[\eta I_H \eta^{-1}, (\partial_t \eta)\eta^{-1}\right].\end{aligned} \tag{A.12}$$

To simplify this expression we make repeated use of the following commutator identity

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC. \tag{A.13}$$

and the following equation for the derivative of the inverse of the Dyson map

$$(\partial_t \eta^{-1}) = -\eta^{-1}(\partial_t \eta)\eta^{-1}. \tag{A.14}$$

The first term in (A.12) simplifies to

$$
\begin{aligned}
[\eta I_H\eta^{-1}, \eta H\eta^{-1}] &= \eta I_H[\eta^{-1}, \eta H\eta^{-1}] + \eta[I_H, \eta H\eta^{-1}]\eta^{-1} + [\eta, \eta H\eta^{-1}]I_H\eta^{-1} \\
&= \eta I_H \left(\eta^{-1}\eta H\eta^{-1} - \eta H\eta^{-1}\eta^{-1}\right) + \eta \left(I_H\eta H\eta^{-1} - \eta H\eta^{-1}I_H\right)\eta^{-1} \\
&\qquad + \left(\eta\eta H\eta^{-1} - \eta H\eta^{-1}\eta\right) I_H\eta^{-1} \\
&= \eta I_H H\eta^{-1} - \eta H I_H\eta^{-1} \\
&= \eta[I_H, H]\eta^{-1}, \qquad \text{(A.15)}
\end{aligned}
$$

and the second term in (A.12) reduces to

$$
\begin{aligned}
[\eta I_H\eta^{-1}, (\partial_t\eta)\eta^{-1}] &= \eta I_H[\eta^{-1}, (\partial_t\eta)\eta^{-1}] + \eta[I_H, (\partial_t\eta)\eta^{-1}]\eta^{-1} + [\eta, (\partial_t\eta)\eta^{-1}]I_H\eta^{-1} \\
&= \eta I_H \left(\eta^{-1}(\partial_t\eta)\eta^{-1} - (\partial_t\eta)\eta^{-1}\eta^{-1}\right) \\
&\qquad + \eta \left(I_H(\partial_t\eta)\eta^{-1} - (\partial_t\eta)\eta^{-1}I_H\right)\eta^{-1} \\
&\qquad + \left(\eta(\partial_t\eta)\eta^{-1} - (\partial_t\eta)\eta^{-1}\eta\right) I_H\eta^{-1}, \\
&= \eta I_H\eta^{-1}(\partial_t\eta)\eta^{-1} - (\partial_t\eta)I_H\eta^{-1}, \\
&= -\eta I_H(\partial_t\eta^{-1}) - (\partial_t\eta)I_H\eta^{-1}. \qquad \text{(A.16)}
\end{aligned}
$$

Combining (A.15) with (A.16) we have shown that

$$
[I_h, h] = \eta[I_H, H]\eta^{-1} - i\eta I_H(\partial_t\eta^{-1}) - i(\partial_t\eta)I_H\eta^{-1}, \qquad \text{(A.17)}
$$

which when substituted into (A.10) with (A.11) verifies that the expression (A.8) holds.

Appendix B

WKB theory

In this appendix we shall demonstrate how to derive the WKB wave functions following [169], the connection formulae and the quantization condition for the one-dimensional TISE.

Our starting point is the one dimensional TISE

$$-\frac{d^2\psi}{dx^2} + \frac{p(x)^2}{\hbar^2}\psi(x) = 0 \qquad \text{where} \qquad p(x) = \sqrt{2m(E - V(x)}. \tag{B.1}$$

If $V(x)$ were constant then the solution to this equation provided $E > V$ would be plane waves $\psi(x) = Ae^{\pm ipx/\hbar}$. Motivated by this solution we assume that when $V(x)$ is not constant but instead slowly varying the wave functions takes the form

$$\psi(x) = e^{i\frac{S(x)}{\hbar}}. \tag{B.2}$$

We substitute this into the Schrödinger equation (B.1) to obtain

$$i\hbar S''(x) - S'(x)^2 + p(x)^2 = 0. \tag{B.3}$$

Expanding $S(x)$ now as a power series in $\hbar$

$$S(x) = \sum_{n=0}^{\infty} \hbar^n S_n(x), \tag{B.4}$$

substituting into (5.41) and equating powers of $\hbar$, we may pull out the zeroth and first order equations which need to be satisfied

$$S_0'(x)^2 = p(x)^2 \qquad \text{and} \qquad S_0''(x) = -2iS_0'(x)S_1'(x). \tag{B.5}$$

The respective solutions to these equations are

$$S_0(x) = \pm \int^x p(y)dy \qquad \text{and} \qquad \ln S_0'(x) = -2iS_1(x) + c, \tag{B.6}$$

where c is a constant. If we now approximate $S(x)$ by only keeping terms up to first order in $\hbar$ meaning

$$S(x) \approx S_0(x) + \hbar S_1(x), \tag{B.7}$$

we obtain the WKB wave function

$$\psi^{WKB}(x) = \frac{A}{\sqrt{p(x)}} e^{i\int^x p(y)dy} + \frac{B}{\sqrt{p(x)}} e^{-i\int^x p(y)dy}, \tag{B.8}$$

where A and B are constants to be determined. We have taken a superposition of the two possible linearly independent states.

The wave function given by (B.8) is actually only valid in the "classically allowed" region where $E > V(x)$. We can obtain the wave function in what is known as the "classically forbidden" region where $E < V(x)$. In this case the momentum $p(x)$ becomes imaginary and the wave function is given by

$$\psi^{WKB}(x) = \frac{C}{\sqrt{q(x)}} e^{\int^x q(y)dy} + \frac{D}{\sqrt{q(x)}} e^{-\int^x q(y)dy}, \tag{B.9}$$

where

$$q(x) = \sqrt{2m(V(x) - E)}. \tag{B.10}$$

.

There is a difference in behaviour of the WKB wave functions in the two regions. In the classically allowed region the solution is sinusoidal and in the forbidden region the wave function decays exponentially. The transition between these two regions of differing behaviour happens at the turning point. These turning points are defined through $V(x_i) = E$, with x_i being the location of these turning points. The WKB solutions break down here as the amplitude of the wave functions diverge. We therefore need a way to connect the two wave functions on either side of the turning points with a "patching" wave function.

Following Griffiths [169] we shall now demonstrate how to "patch" the two wave functions either side of a turning point we shall consider a potential $V(x)$ where the classical turning point is located at $x = 0$. To the left of the turning point is the classically allowed regions and therefore the solutions are sinusoidal and the WKB wave function is given by equation (B.8). To the right of the turning point will be the classically forbidden region where the solution is exponentially decaying the WKB wave function is given by (B.9). In the patching region which is in the vicinity of the turning point $x = 0$ we shall approximate the potential by a straight line

$$V(x) \approx E + V'(0)x. \tag{B.11}$$

We now need to solve the Schrödinger equation (B.1) for the linearized potential for the patching wave function ψ_p

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} + (E + V'(0)x)\psi_p = \psi_p. \tag{B.12}$$

By setting

$$\alpha = \left[\frac{2m}{\hbar^2}V'(0)\right]^{1/3} \tag{B.13}$$

and changing variables through $z = \alpha x$ we may recast equation (B.12) as the Airy equation

$$\frac{d^2\psi_p}{dz^2} = z\psi_p. \tag{B.14}$$

The general solution to the Airy equation is

$$\psi_p = aAi(z) + bBi(z), \tag{B.15}$$

where $Ai(z)$ and $Bi(z)$ are the Airy functions, a plot of these functions is given in figure 3.1 displaying the aforementioned sinusoidal and dacying or increasing behaviour on either side of the turning point. The integral representation of these functions are

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3} + sz\right) ds, \tag{B.16}$$

$$Bi(z) = \frac{1}{\pi}\int_0^\infty \left[\sin\left(\frac{s^3}{3} + sz\right) + e^{-\frac{s^3}{3}+sz}\right] ds. \tag{B.17}$$

We however will only be concerned with the asymptotic forms of the Airy functions given by

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-\frac{2}{3}z^{3/2}} \qquad \text{and} \qquad Bi(z) \sim \frac{1}{\sqrt{\pi}z^{1/4}}e^{\frac{2}{3}z^{2/3}}, \tag{B.18}$$

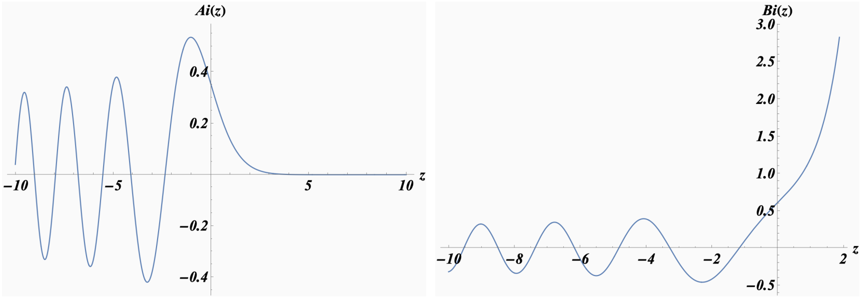

Figure B.1: Graphs of the Airy functions $Ai(z)$ and $Bi(z)$ respectively.

when $z \gg 0$. When $z \ll 0$ we have

$$\begin{aligned} Ai(z) &\sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] \\ &= \frac{1}{\sqrt{\pi}(-z)^{1/4}} \left[e^{i\frac{2}{3}(-z)^{\frac{3}{2}}} e^{i\frac{\pi}{4}} - e^{-i\frac{2}{3}(-z)^{\frac{3}{2}}} e^{-i\frac{\pi}{4}}\right], \end{aligned} \tag{B.19}$$

and

$$\begin{aligned} Bi(z) &\sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] \\ &= \frac{1}{\sqrt{\pi}(-z)^{1/4}} \left[e^{i\frac{2}{3}(-z)^{\frac{3}{2}}} e^{i\frac{\pi}{4}} + e^{-i\frac{2}{3}(-z)^{\frac{3}{2}}} e^{-i\frac{\pi}{4}}\right]. \end{aligned} \tag{B.20}$$

We now need to determine the WKB wave function in the vicinity of the turning point. We shall start to the left of the turning point and assume that the slope is positive ($V'(x) > 0$). In this region where $x < 0$ the momentum will be given by

$$p(x) = \sqrt{2m(E - V(x))} \sim \hbar\alpha^{3/2}\sqrt{-x}, \tag{B.21}$$

therefore the WKB wave function is

$$\psi_l = \frac{A}{\sqrt{\hbar}(-\alpha^3 x)^4} e^{-i\frac{2}{3}(-\alpha x)^{\frac{3}{2}}} + \frac{B}{\sqrt{\hbar}(-\alpha^3 x)^4} e^{i\frac{2}{3}(-\alpha x)^{\frac{3}{2}}}. \tag{B.22}$$

Similarly to the right of the turning point where $x > 0$ and we are in the classically forbidden region the WKB wave function can now be written as

$$\psi_r(x) = \frac{C}{\sqrt{\hbar}(\alpha^3 x)^4} e^{\frac{2}{3}(\alpha x)^{\frac{3}{2}}} + \frac{D}{\sqrt{\hbar}(\alpha^3 x)^4} e^{-\frac{2}{3}(\alpha x)^{\frac{3}{2}}}. \tag{B.23}$$

We will now patch these WKB wave functions to correct forms of the asymptotic Airy functions. In the left region, the WKB wave function (B.22) re-written in terms of z as

$$\psi_l(z) = \frac{A}{\sqrt{\hbar\alpha}(-z)^{\frac{1}{4}}} e^{-i\frac{2}{3}(-z)^{\frac{3}{2}}} + \frac{B}{\sqrt{\hbar\alpha}(-z)^{\frac{1}{4}}} e^{i\frac{2}{3}(-z)^{\frac{3}{2}}}, \tag{B.24}$$

will match with the $z \ll 0$ form of the Airy functions. Setting $\psi_l(z) = aAi(z) + bBi(z)$ and solving for a and b we get

$$a = \sqrt{\frac{\pi}{\hbar\alpha}} \left(Ae^{-i\frac{\pi}{4}} + Be^{i\frac{\pi}{4}} \right) \quad \text{and} \quad b = \sqrt{\frac{\pi}{\hbar\alpha}} \left(Ae^{i\frac{\pi}{4}} + Be^{-i\frac{\pi}{4}} \right). \tag{B.25}$$

Similarly in the right region we may set $\psi_r(z) = aAi(z) + bBi(z)$ yet we now use the asymptotic forms of the Airy functions for $z \gg 0$. In doing so we obtain a and b in terms of C and D

$$a = \sqrt{\frac{4\pi}{\hbar\alpha}} D \qquad \text{and} \qquad b = \sqrt{\frac{\pi}{\hbar\alpha}} C. \tag{B.26}$$

Setting equations (B.25) and (B.26) equal to one another allows us to now write down the connection formulae for a right hand barrier ($V'(x) > 0$)

$$D = \frac{1}{2}(Ae^{-i\frac{\pi}{4}} + Be^{i\frac{\pi}{4}}), \qquad C = \frac{1}{2}(Ae^{i\frac{\pi}{4}} + Be^{-i\frac{\pi}{4}}), \tag{B.27}$$

$$A = \frac{1}{2}(De^{i\frac{\pi}{4}} + Ce^{-i\frac{\pi}{4}}), \qquad B = \frac{1}{2}(De^{-i\frac{\pi}{4}} + Ce^{i\frac{\pi}{4}}). \tag{B.28}$$

The equivalent formulae for a left hand barrier ($V'(x) < 0$) can be derived in a similar fashion.

We can summarise the connection formulae as

$$\frac{2}{\sqrt{p(x)}} \cos\left[\int_x^a p(y)dy/\hbar - \frac{\pi}{4}\right] \quad \leftarrow \quad \frac{1}{\sqrt{q(x)}} \exp\left[-\int_a^x q(y)dy/\hbar\right], \tag{B.29}$$

and

$$-\frac{1}{\sqrt{p(x)}} \sin\left[\int_x^a p(y)dy/\hbar - \frac{\pi}{4}\right] \quad \rightarrow \quad \frac{1}{\sqrt{q(x)}} \exp\left[\int_a^x q(y)dy/\hbar\right], \tag{B.30}$$

for a right hand barrier. For a left hand barrier we have

$$\frac{1}{\sqrt{q(x)}} \exp\left[-\int_x^a q(y)dy/\hbar\right] \quad \rightarrow \quad \frac{2}{\sqrt{p(x)}} \cos\left[\int_a^x p(y)dy/\hbar - \frac{\pi}{4}\right], \tag{B.31}$$

and

$$\frac{1}{\sqrt{q(x)}} \exp\left[\int_x^a q(y)dy/\hbar\right] \quad \leftarrow \quad -\frac{1}{\sqrt{p(x)}} \sin\left[\int_a^x p(y)dy/\hbar - \frac{\pi}{4}\right]. \tag{B.32}$$

We can now use the derived matching conditions of the WKB wave functions to derive the quantization condition. To do this we shall consider bound states in a quantum well as depicted for a generic potential $V(x)$ in figure B.2. The classical

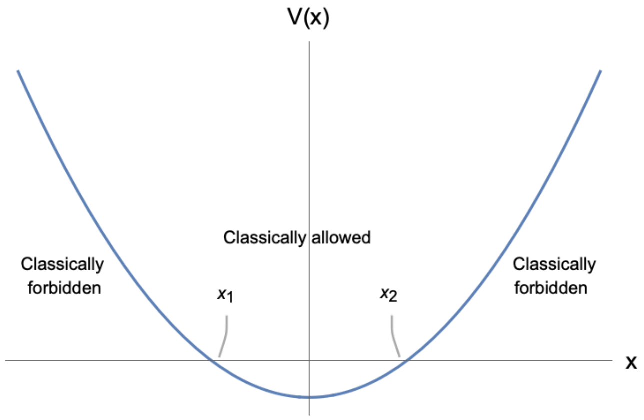

Figure B.2: Schematic representation of a potential well $V(x)$ with classical turning points x_1 and x_2.

turning points determined from the condition $V(x) = E$ are labelled x_1 and x_2. The well will be defined by three regions. Region 1 will be a classically forbidden region. Region 2 will be the classically allowed region in between the turning point x_1 and x_2. The third region will again be a classically forbidden region. At x_1 there is is a left hand barrier ($V'(x) < 0$) and at x_2 there is a right hand barrier ($V'(x) > 0$). For region 1 we start by neglecting the part of the wave function that would blow up at $-\infty$, we find the WKB wave function in the region to be

$$\psi_1(x) = \frac{C_1}{\sqrt{q(x)}} \exp\left[-\int_x^{x_1} q(y)dy/\hbar\right], \tag{B.33}$$

with C_1 being a constant. Similarly in region 3 we have

$$\psi_3(x) = \frac{C_3}{\sqrt{q(x)}} \exp\left[-\int_{x_2}^{x} q(y)dy/\hbar\right]. \tag{B.34}$$

By using the connection formulae given by equations (B.31) and (B.29) we obtain two results for the wave function in region 2 given by

$$\psi_2(x) = \frac{2C_1}{\sqrt{p(x)}} \cos\left[\int_{x_1}^{x} p(y)dy/\hbar - \frac{\pi}{4}\right], \tag{B.35}$$

and

$$\psi_2(x) = \frac{2C_3}{\sqrt{p(x)}} \cos\left[\int_{x}^{x_2} p(y)dy/\hbar - \frac{\pi}{4}\right]. \tag{B.36}$$

These wave functions have to be equal. Therefore if we use the fact that

$$\int_{x}^{x_2} dy = \int_{x_1}^{x_2} dy - \int_{x_1}^{x} dy, \tag{B.37}$$

we can rewrite equation (B.36) as

$$\psi_2(x) = \frac{2C_3}{\sqrt{p(x)}} \cos\left[\int_{x_1}^{x_2} p(x')dx'/\hbar - \int_{x_1}^{x} p(x')dx'/\hbar - \frac{\pi}{4}\right]. \tag{B.38}$$

If we now let

$$A = \int_{x_1}^{x_2} p(x')dx'/\hbar \qquad \text{and} \qquad B = \int_{x_1}^{x} p(x')dx'/\hbar. \tag{B.39}$$

We can write

$$\begin{aligned} \cos\left[A - \left(B + \frac{\pi}{4}\right)\right] &= \sin\left[A - \left(B - \frac{\pi}{4}\right)\right] \\ &= \sin[A]\cos\left[B - \frac{\pi}{4}\right] - \cos[A]\sin\left[B - \frac{\pi}{4}\right]. \end{aligned} \tag{B.40}$$

We need this expression to be proportional to $\cos[B - \pi/4]$ so that it can match correctly with (B.35). This leads to the condition that $\cos[A] = 0$, this is achieved if $A = (n + 1/2)\pi$ and leads to the quantization condition:

$$\int_{x_1}^{x_2} p(x)dx = \left(n + \frac{1}{2}\right)\pi\hbar. \tag{B.41}$$

Furthermore, we have the requirement that $\sin[A] = (-1)^n$, and hence $C_1 = (-1)^n C_3$.

Appendix C

Integral derivation

In this appendix we shall derive the integral given by equation (3.43) in the main body of this thesis. To do so we shall be considering the following integral

$$\int_{-\infty}^{\infty} x^{2n} D_S(x) D_R(x) dx, \tag{C.1}$$

and consider three separate cases.

C.1 S and R are even

For S and R both even we shall consider

$$\int_{0}^{\infty} x^{2n} D_{2s}(x) D_{2s}(x) dx. \tag{C.2}$$

The parabolic cylinder functions are related to the physicist's Hermite polynomials [170] through

$$D_{2s}(x) = 2^{-s} e^{-x^2/4} H_{2s}\left(\frac{x}{\sqrt{2}}\right), \tag{C.3}$$

which allows us to rewrite our integral as

$$2^{-(s+r)} \int_{0}^{\infty} x^{2n} e^{-x^2/2} H_{2s}\left(\frac{x}{\sqrt{2}}\right) H_{2r}\left(\frac{x}{\sqrt{2}}\right) dx. \tag{C.4}$$

The Hermite polynomials are related to the Laguerre polynomials [170] through

$$H_{2s}(x) = (-1)^s 2^{2s} s! L_s^{(-1/2)}(x^2), \tag{C.5}$$

and so our integral becomes

$$(-1)^{2+r}2^{s+r}s!r!\int_0^\infty x^{2n}e^{-x^2/2}L_s^{(-1/2)}\left(\frac{x^2}{2}\right)L_r^{(-1/2)}\left(\frac{x^2}{2}\right)dx. \tag{C.6}$$

If we change variables to $y = x^2/2$ our integral reduces to

$$(-1)^{s+r}2^{s+r+n-1/2}s!r!\int_0^\infty y^{n-1/2}e^{-y}L_s^{(-1/2)}(y)\,L_r^{(-1/2)}(y)\,dy, \tag{C.7}$$

We notice that this integral is of a similar form to one derived by Mavromatis [171], which is given by

$$\int_0^\infty x^\mu e^{-x}L_m^{(\alpha)}(x)L_n^{(\beta)}(x)dx = \binom{m+\alpha}{m}\binom{n+\beta-\mu-1}{n}\Gamma[\mu+1]{}_3F_2\left(-m,\mu+1,\mu-\beta+1;\alpha+1,\mu-\beta-n+1;1\right) \tag{C.8}$$

where $\mathbf{Re}(\mu) > -1$ and $n, m \in \mathbb{N}$, and the function ${}_3F_2(a,b,c;d,f;z)$ is the hypergeometric function defined as

$${}_3F_2(a,b,c;d,f;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{(d)_k(f)_k}\frac{z^k}{k!}, \tag{C.9}$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ denoting the Pochhammer symbol. Using this we can therefore write down a form for the integral given by (C.1) when s and r are both even:

$$\int_{-\infty}^\infty x^{2n}D_{2s}(x)D_{2r}(x)dx = (-1)^{s+r}2^{1/2+s+r+n}s!r!\binom{s-1/2}{s}\binom{r-n-1}{r}\Gamma\left[n+\frac{1}{2}\right] \times {}_3F_2\left(-s,n+\frac{1}{2},n+1;\frac{1}{2},n-r+1;1\right). \tag{C.10}$$

Unfortunately this result is only valid for when $n \geq r$. To overcome this we first write the hypergeometric function in equation (C.10) as power series using equation (C.9) noting that the series will terminate as s:

$${}_3F_2\left(-s,n+\frac{1}{2},n+1;\frac{1}{2},n-r+1;1\right) = \sum_{k=0}^{s}\frac{(-s)_k(n+1/2)_k(n+1)_k}{(1/2)_k(n-r+1)_k}\frac{1}{k!}. \tag{C.11}$$

We can also write the binomial involving r in equation (C.10) in terms of a Pochhammer symbol

$$\binom{r-n-1}{r} = \frac{(-1)^r}{r!}(n-r+1)_r, \tag{C.12}$$

By multiplying this by equation (C.11) and noting that

$$\frac{(n-r+1)_r(n+1)_k}{(n-r+1)_k} = (k-r+1+n)_r, \tag{C.13}$$

we obtain

$$\begin{aligned}
&\binom{r-n-1}{r}{}_3F_2\left(-s, n+\frac{1}{2}, n+1; \frac{1}{2}, n-r+1; 1\right) = \\
&\qquad = \frac{(-1)^r}{r!}\sum_{k=0}^{s}\frac{(k-r+1+n)_r(-s)_k(n+1/2)_k}{k!(1/2)_k}, \\
&\qquad = \frac{(-1)^r}{r!}\sqrt{\pi}\Gamma[1+n]{}_3\tilde{F}_2\left(-s, n+1, n+\frac{1}{2}; \frac{1}{2}, n+1-r; 1\right),
\end{aligned} \tag{C.14}$$

where the function ${}_3\tilde{F}_2\,(a,b,c;d,f;z)$ is the regularized hypergeometric function defined by

$${}_3\tilde{F}_2\,(a,b,c;d,f;z) = \frac{1}{\Gamma[d]\Gamma[f]}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{(d)_k(f)_k}\frac{z^k}{k!}. \tag{C.15}$$

Furthermore, by writing

$$\binom{s-1/2}{s} = \frac{\Gamma[s+1/2]}{\sqrt{\pi}s!}, \tag{C.16}$$

as well as

$$\Gamma\left[n+\frac{1}{2}\right]\Gamma[1+n] = 2^{-2n}\sqrt{2\pi}\Gamma[1+2n], \tag{C.17}$$

we obtain our final form for equation (C.10)

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^{2n}D_{2s}(x)D_{2r}(x)dx = \\
&(-1)^s 2^{1/2+s+r-n}\sqrt{\pi}\Gamma[1+2n]\Gamma\left[s+\frac{1}{2}\right]{}_3\tilde{F}_2\left(-s, n+1, n+\frac{1}{2}; \frac{1}{2}, n+1-r; 1\right),
\end{aligned} \tag{C.18}$$

which is valid for all $n > -1$ and $s, r \in \mathbb{N}$.

C.2 S and R are odd

For the case where S and R are both odd we shall consider the following integral

$$\int_0^\infty x^{2n} D_{2s+1}(x) D_{2r+1}(x) dx. \tag{C.19}$$

We shall follow the same procedure as before. First we write the parabolic cylinder functions in terms of the physicist's Hermite polynomials

$$D_{2s+1}(x) = 2^{-(s+1/2)} e^{-x^2/4} H_{2s+1}\left(\frac{x}{\sqrt{2}}\right), \tag{C.20}$$

which results in our integral being

$$2^{-(1+s+r)} \int_0^\infty x^{2n} e^{-x^2/2} H_{2s+1}\left(\frac{x}{\sqrt{2}}\right) H_{2r+1}\left(\frac{x}{\sqrt{2}}\right) dx. \tag{C.21}$$

We now write the Hermite polynomials in terms of Laguerre polynomials, which for odd S and R are given by

$$H_{2s+1}(x) = (-1)^s 2^{2s+1} s! x L_s^{(1/2)}(x^2). \tag{C.22}$$

Our integral now reads

$$(-1)^{s+r} 2^{(s+r)} s! r! \int_0^\infty x^{2n+2} e^{-x^2/2} L_s^{)1/2)}(x^2) L_r^{(1/2)}(x^2) dx. \tag{C.23}$$

By substituting $y = x^2/2$, the integral reduces to a familiar form given by

$$(-1)^{s+r} 2^{s+r+n+1/2} s! r! \int_0^\infty y^{n+1/2} e^{-y} L_s^{(1/2)}(y) L_r^{(1/2)}(y) dy, \tag{C.24}$$

Using the formula derived by Mavromatis [171] given in equation (C.8) we obtain

$$\begin{aligned}\int_{-\infty}^\infty x^{2n} D_{2s+1}(x) D_{2r+1}(x) dx &= (-1)^{s+r} 2^{s+r+n+3/2} s! r! \binom{s+1/2}{s} \binom{r-n-1}{r} \\ &\times \Gamma\left[n+\frac{3}{2}\right] {}_3F_2\left(-s, n+\frac{3}{2}, n+1; \frac{3}{2}, n-r+1; 1\right).\end{aligned} \tag{C.25}$$

We now need to modify this formula to ensure that it is valid for all combinations of n and r. We first write the hypergeometric function as a power series using equation

(C.9)

$$ {}_3F_2\left(-s, n+\frac{3}{2}, n+1; \frac{3}{2}, n-r+1; 1\right) = \sum_{k=0}^{s} \frac{(-s)_k (n+3/2)_k (n+1)_k}{(3/2)_k (n-r+1)_k} \frac{1}{k!}. \quad \text{(C.26)} $$

We then write the binomial involving r in equation (C.25) in terms of Pochhammer symbols using equation (C.12), and then utilise the relation given in equation (C.13) to obtain

$$ \begin{aligned} &\binom{r-n-1}{r} {}_3F_2\left(-s, n+\frac{3}{2}, n+1; \frac{3}{2}, n-r+1; 1\right) = \\ &\qquad = \frac{(-1)^r}{r!} \sum_{k=0}^{s} \frac{(k-r+1+n)_r (-s)_k (n+3/2)_k}{k!(3/2)_k}, \\ &\qquad = \frac{1}{2}\frac{(-1)^r}{r!}\sqrt{\pi}\Gamma[1+n] {}_3\tilde{F}_2\left(-s, n+\frac{3}{2}, n+1; \frac{3}{2}, n-r+1; 1\right), \quad \text{(C.27)} \end{aligned} $$

where ${}_3\tilde{F}_2\,(a, b, c; d, f; z)$ is the regularized hypergeometric function defined by equation (C.15). If we combine this with the fact that we can write

$$ \binom{s+1/2}{s} = \frac{2\Gamma\left[s+\frac{3}{2}\right]}{s!\sqrt{\pi}}, \quad \text{(C.28)} $$

and

$$ \Gamma[1+n]\Gamma\left[n+\frac{3}{2}\right] = 2^{-1-2n}\sqrt{\pi}\Gamma[2+2n], \quad \text{(C.29)} $$

we obtain our final form for equation (C.25):

$$ \begin{aligned} &\int_{-\infty}^{\infty} x^{2n} D_{2s+1}(x) D_{2r+1}(x) dx = \\ &(-1)^s 2^{s+r-n+3/2}\sqrt{\pi}\Gamma[2+2n]\Gamma\left[s+\frac{3}{2}\right] {}_3\tilde{F}_2\left(-s, n+\frac{3}{2}, n+1; \frac{3}{2}, n-r+1; 1\right), \end{aligned} \quad \text{(C.30)} $$

which is valid for all $n > -1$ and $s, r \in \mathbb{N}$.

C.3 S is even and R is odd

For the final case we have that S is even and R is odd, and therefore we shall consider the following integral

$$ \int_{0}^{\infty} x^{2n} D_{2s}(x) D_{2r+1}(x) dx. \quad \text{(C.31)} $$

Using equations (C.3) and (C.20) we can write this integral in terms of Hermite polynomials

$$2^{-(s+r+1/2)} \int_0^\infty x^{2n} e^{-x^2/2} H_{2s}\left(\frac{x}{\sqrt{2}}\right) H_{2r+1}\left(\frac{x}{\sqrt{2}}\right) dx. \tag{C.32}$$

If we now use equations (C.5) and (C.22) we can write this integral in terms of Laguerre polynomials

$$(-1)^{s+r} 2^{s+r} s! r! \int_0^\infty x^{2n+1} e^{-x^2/2} L_s^{(-1/2)}\left(\frac{x^2}{2}\right) L_r^{(1/2)}\left(\frac{x^2}{2}\right) dx, \tag{C.33}$$

If we change variables to $y = x^2/2$ we obtain

$$(-1)^{s+r} 2^{s+r+n} s! r! \int_0^\infty y^n e^{-y} L_s^{(-1/2)}(y) L_r^{(1/2)}(y) dy. \tag{C.34}$$

If we now use equation (C.8) we obtain

$$\int_{-\infty}^{\infty} x^{2n} D_{2s}(x) D_{2r+1}(x) dx = (-1)^{s+r} 2^{r+s+n+1} s! r! \binom{s-1/2}{s} \binom{r-n-1/2}{r}$$
$$\times \Gamma[n+1]\, {}_3F_2\left(-s, n+1, n+\frac{1}{2}; \frac{1}{2}, n-r+1+\frac{1}{2}; 1\right). \tag{C.35}$$

For this equation we now wish to obtain a form that is similar to equations (C.18) and (C.30). In order to do this we proceed in the same way as before by first writing the hypergeometric function in terms of a power series using equation (C.9) yielding

$${}_3F_2\left(-s, n+1, n+\frac{1}{2}; \frac{1}{2}, n-r+1+\frac{1}{2}; 1\right) = \sum_{k=0}^{s} \frac{(-s)_k (n+1)_k (n+1/2)_k}{(1/2)_k (n-r+1/2)_k} \frac{1}{k!}. \tag{C.36}$$

We now need to write the binomial involving r in equation (C.35) in terms of Pochhammer symbols to obtain

$$\binom{r-n-1/2}{r} = \frac{(-1)^r}{r!}(n-r+1/2)_r. \tag{C.37}$$

If we now multiply this by equation (C.36) and use the the following relationship involving Pochammer symbols

$$\frac{(n-r+1/2)_r (n+1/2)_k}{n-r+1/2)_k} = ((k-r+1/2+n)_r, \tag{C.38}$$

we obtain

$$\binom{r-n-1/2}{r}{}_3F_2\left(-s,n+1,n+\frac{1}{2};\frac{1}{2},n-r+1+\frac{1}{2};1\right)=$$
$$=\frac{(-1)^r}{r!}\sum_{k=0}^{s}\frac{(k-r+1/2+n)_r(-s)_k(n+1)_k}{k!(1/2)_k}$$
$$=\frac{(-1)^r}{r!}\sqrt{\pi}\Gamma\left[n+\frac{1}{2}\right]{}_3\tilde{F}_2\left(-s,n+1,n+\frac{1}{2};\frac{1}{2},n-r+1+\frac{1}{2};1\right) \tag{C.39}$$

Combining this with

$$\binom{s-1/2}{s}=\frac{\Gamma\left[s+\frac{1}{2}\right]}{s!\sqrt{\pi}}, \tag{C.40}$$

and

$$\Gamma[n+1]\Gamma\left[n+\frac{1}{2}\right]=2^{-2n}\sqrt{\pi}\Gamma[1+2n], \tag{C.41}$$

we obtain

$$\int_{-\infty}^{\infty}x^{2n}D_{2s}(x)D_{2r+1}(x)dx=$$
$$(-1)^s2^{r+s+1-n}\sqrt{\pi}\Gamma[1+2n]\Gamma\left[s+\frac{1}{2}\right]{}_3\tilde{F}_2\left(-s,n+1,n+\frac{1}{2};\frac{1}{2},n-r+1+\frac{1}{2};1\right), \tag{C.42}$$

which is valid for $n>-1$ and $s,r\in\mathbb{N}$.

Finally we can now combine the final formulae for all 3 cases given by equations (C.18), (C.30) and (C.42) into a universal form given by

$$\int_{-\infty}^{\infty}x^{2n}D_{2s+\delta}(x)D_{2r+\bar{\delta}}(x)dx=(-1)^s2^{s+r-n+\frac{1+\delta-\bar{\delta}}{2}}\sqrt{\pi}\Gamma\left[s+\frac{1}{2}+\delta\right]$$
$$\times\Gamma[2n+1+\delta]{}_3\tilde{F}_2\left(-s,n+1,n+\frac{1}{2}+\delta;\frac{1}{2}+\delta,n-r+1+\frac{\delta-\bar{\delta}}{2};1\right) \tag{C.43}$$

for $n,r,s\in\mathbb{N}_0$ and $(\delta,\bar{\delta})=(0,1),(0,0),(1,1)$. Our formula generalises the result of Mavromatis [171].

Appendix D

Point transformations

In this appendix we shall demonstrate how to derive equations (5.11), (5.12) and (5.13) by utilising point transformations [155, 161].

We wish to determine the point transformation which maps the Schrödinger equation (5.2) to (5.1) for $H_0(\chi)$ being the time-independent harmonic oscillator (5.9). The wave functions ψ and ϕ are implicit functions of (χ, τ) and (x, t) respectively and the functional dependence of χ, τ and ψ is

$$\chi = \chi(x,t), \quad \tau = \tau(x,t), \quad \psi = R(x,t,\phi(x,t)). \tag{D.1}$$

We compute the total derivatives of χ and τ with respect to x and t

$$\frac{d\psi}{dx} = \psi_\chi \chi_x + \psi_\tau \tau_x = R_\phi \phi_x + R_x, \tag{D.2}$$
$$\frac{d\psi}{dt} = \psi_\chi \chi_t + \psi_\tau \tau_t = R_\phi \phi_t + R_t. \tag{D.3}$$

We may solve this system of equations for the unknown functions ψ_χ and ψ_τ

$$\psi_\chi = \frac{1}{J}\left(\tau_t R_\phi \phi_x - \tau_x R_\phi \phi_t + \tau_t R_x - \tau_x R_t\right), \tag{D.4}$$
$$\psi_\tau = \frac{1}{J}\left(\chi_x R_\phi \phi_t - \chi_t R_\phi \phi_x + \chi_x R_t - \chi_t R_x\right), \tag{D.5}$$

here $J = \chi_x \tau_t - \chi_t \tau_x$ is the Jacobian. As there is a momentum squared term we also compute the derivative

$$
\begin{aligned}
\frac{d^2\psi}{dx^2} &= \psi_{\chi,\chi}\chi_x^2 + \psi_\chi\chi_{x,x} + 2\psi_{\chi,\tau}\chi_x\tau_x + \psi_\tau\tau_{x,x} + \psi_{\tau,\tau}\tau_x^2 \\
&= R_{\phi,\phi}\phi_x^2 + R_\phi\phi_{x,x} + 2R_{\phi,x}\phi_x + R_{x,x}.
\end{aligned} \tag{D.6}
$$

Present in (D.6) is the nonlinear term $R_{\phi,\phi}\phi_x^2$, there are no nonlinear terms present in the TDSEs we are considering and therefore to eliminate this we factorise the wavefunction as

$$
\psi = A(x,t)\phi, \tag{D.7}
$$

such that $R_{\phi,\phi} = 0$. Similarly, in (D.6) we notice a term proportional to a double derivative in the time co-ordinate for the reference system $\psi_{\tau,\tau}\tau_x^2$, there are again no such terms present in the TDSE so we opt to take

$$
\tau = \tau(t), \tag{D.8}
$$

such that $\tau_x = 0$. We may now solve (D.6) for $\psi_{\chi,\chi}$ and simplify expressions (D.4) and (D.5) to obtain

$$
\psi_{\chi,\chi} = \frac{1}{\chi_x^2}\left[\left(A_{x,x} - \frac{\chi_{x,x}}{\chi_x}\right)\phi + \left(2A_x - \frac{A\chi_{x,x}}{\chi_x}\right)\phi_x + A\phi_{x,x}\right], \tag{D.9}
$$

$$
\psi_\chi = \frac{1}{\chi_x}\left(A_x\phi + A\phi_x\right), \tag{D.10}
$$

$$
\psi_\tau = \frac{1}{\tau_t}\left[\left(A_t - \frac{A_x\chi_t}{\chi_x}\right)\phi - \frac{A\chi_t}{\chi_x}\phi_x + A\phi_t\right]. \tag{D.11}
$$

Substituting these into the TDSE for time-independent harmonic oscillator (5.9) we obtain the point transformed TDSE

$$
i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{x,x} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0, \tag{D.12}
$$

where

$$
B_0(x,t) = -i\hbar\frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(2\frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x}\right), \tag{D.13}
$$

$$
V_0(x,t) = \frac{1}{2}m\tau_t\chi^2\omega^2 - i\hbar\left(\frac{A_t}{A} - \frac{A_x\chi_t}{A\chi_x}\right) - \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(\frac{A_{xx}}{A} - \frac{A_x\chi_{xx}}{A\chi_x}\right). \tag{D.14}
$$

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