STRUCTURAL PROPERTIES OF LINEAR SYSTEMS

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DECLARATION

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MATHEMATICAL NOTATIONS

— R, C: fields of real, complex numbers

— $R(s)$: field of rational functions in the variable s with real coefficients

— $R[s]$: ring of polynomials in s with real coefficients

— $R_{pr}(s)$: ring of proper rational functions

— $\mathcal{F}$: denotes a general field, or ring

— $\mathcal{F}^{p\times k}$: set of matrices with $p \times k$ dimensions and elements over $\mathcal{F}$, thus $R^{p\times k}(s)$, $R^{p\times k}[s]$,... denote the corresponding set of matrices with elements over $R(s)$, $R[s]$,...

— $R_{\mathcal{P}}(s)$: ring of proper rational function which have no poles in a symmetric set of the complex plane Ω, which excludes at least one point of the real axis.

— $\mathcal{V}$: denotes a finite dimensional vector space over some field $\mathcal{F}$ (usual cases the real vector spaces (R-vector spaces), rational vector spaces ($R(s)$-vector spaces).

— $\dim\mathcal{V}$: denotes the dimension of a vector space.

— $\mathcal{F}^n$: set of all n-dimensional vectors (n-tuples) of elements of $\mathcal{F}$. $R^n, C^n, R^n(s), ...$: n-dimensional vector spaces over $\mathcal{F}$.

— If $\mathcal{V}$ is a subspace of $R^n, (R^n(s))$, the $\underline{v} \in \mathcal{V}$ denotes a vector of $R^n(R^n(s))$ that belongs to $\mathcal{V}$. If $\dim\mathcal{V} = d$ and $\{\underline{v}_1, ..., \underline{v}_d\}$ is a basis of $\mathcal{V}$, then $V = [\underline{v}_1, ..., \underline{v}_d] \in R^{n\times d}$ denotes a basis matrix of $\mathcal{V}$.

— If $H \in \mathcal{F}^{p\times k}, \mathcal{F}$ a field, then $\rho_{\mathcal{F}}(H)$ denotes the rank of H over $\mathcal{F}$, $\mathcal{N}_r\{H\}$ the right null space and $\mathcal{N}_l(H)$ the left null space of H.

— Z denotes the set of integers, Z^+ the positive integers and Z_0^+ the nonnegative integers ($Z^+ \cup \{0\}$).

— If $n \in Z^+$, then $< n >= \{1, 2, ..., n\}$ and if a property holds for $i \in< n >$, that implies that it is true for all $i = 1, 2, ..., n$.

—$H \in \mathcal{F}^{p\times p}, |H|$ denotes the determinant of H.

— State space description:

$$S(A, B, C, D) : \begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{u}, & A \in R^{n\times n}, B \in R^{n\times l}, C \in R^{m\times n}, D \in R^{m\times l} \\ \underline{y} = C\underline{x} + D\underline{u} & \underline{x} \in R^n, \underline{u} \in R^l, \underline{y} \in R^m \end{cases}$$

Assumptions: $\rho(B) = l, \rho(C) = m$

N: left annihilator of B, $(NB = 0, \rho(N) = n - l, N \in R^{(n-l)\times n})$

$B^\dagger$: left inverse of B $(B^\dagger B = I_l, \rho(B^+) = l, B^\dagger \in R^{l\times n})$

M: right annihilator of C, $(CM = 0, \rho(M) = n - m, M \in R^{n\times(n-m)})$

$L \in R^{l\times n}$: state feedback, $Q \in R^{n\times m}$: output injection

$F \in R^{l\times m}$: output feedback, $K \in R^{l\times m}$: squaring down

$T \in R^{n \times n}, |T| \neq 0$: state coordinate transformation

$R \in R^{l \times l}, |R| \neq 0$: input coordinate transformation

$P \in R^{m \times m} |P| \neq 0$: output coordinate transformation

$\sigma(A)$: spectrum of A (eigenvalues of A including multiplicities)

λ: eigenvalue of $A, \underline{u}_\lambda$: eigenvector of A for λ eigenvalue

$J(A)$: Jordan canonical block of A.

$A = UJ(A)V$, Jordan decomposition of A.

f.e.d.: finite elementary divisors.

i.e.d.: infinite elementary divisors.

c.m.i.: column minimal indices.

r.m.i.: row minimal indices.

GR: Grassman Representative.

— Transfer function description

$G(s) = C(sI - A)^{-1}B + D \in R^{m \times l}(s)$: transfer function matrix

$r = \rho_{R(s)}\{G(s)\}$: normal rank of $G(s)$

$t(s) \in R[s], \partial[t]$: degree of $t(s)$.

— $G(s) = N_r(s)D_r(s)^{-1} = D_l(s)^{-1}N_l(s)$

$N_r(s) \in R^{m \times l}[s], D_r(s) \in R^{l \times l}[s]$: Right Matrix Fraction Description (R.M.F.D.)

$N_l(s) \in R^{m \times l}[s], D_l(s) \in R^{m \times m}[s]$: Left Matrix Fraction Description (L.M.F.D.)

—$T(s) \in R^{p \times k}[s] : S(T)$: Smith normal form of $T(s)$

$$S(T) = \left[\begin{array}{ccc|c} f_1(s) & & & 0 \\ & \ddots & & \\ & & f_r(s) & \\ \hline & 0 & & 0 \end{array}\right] \qquad r = \rho_{R(s)}\{T(s)\}$$

$f_i(s)$: invariant polynomials of $T(s)$

—$T(s) \in R^{p \times k}(s) : M(T)$: Smith-McMillan form of $T(s)$

$$M(T) = \left[\begin{array}{ccc|c} \frac{\epsilon_1(s)}{\psi_1(s)} & & & 0 \\ & \ddots & & \\ & & \frac{\epsilon_r(s)}{\psi_r(s)} & \\ \hline & 0 & & 0 \end{array}\right] \qquad r = \rho_{R(s)}\{T(s)\}$$

$\epsilon_i(s)$: invariant zero polynomials, $\epsilon_1(s)/\epsilon_2(s)/\cdots/\epsilon_r(s)$

$\psi_i(s)$: invariant pole polynomials, $\psi_r(s)/\psi_{r-1}(s)/\cdots/\psi_1(s)$, $(/)$ divides

$z(s) = \Pi_{i=1}^{r}\epsilon_i(s)$: zero polynomial of $T(s)$

$p(s) = \Pi_{i=1}^{r}\psi_i(s)$: pole polynomial of $P(s)$

$\delta_M(T)$: McMillan degree of $T(s)$

—$R_u^{m\times m}[s]$: set of $m \times mR[s]$-unimodular matrices

—$R_{bpr}^{m\times m}(s)$: Set of $m \times mR_{pr}(s)$-unimodular matrices, biproper ($U(s) \in R_{pr}^{m\times m}(s)$, then $U^{-1}(s) \in R_{pr}^{m\times m}(s)$).

— $T(s) \in R^{p\times k}[s] : T(s) = s^d T_d + \cdots + sT_1 + T_0, T_i \in R^{p\times k}, T_d \neq 0, d = \partial_s[T]$: scalar degree of $T(s)$

$\delta = \partial_m[T]$: matrix degree of $T(s)$ (maximal degree amongst the maximal order minors of $T(s)$)

— $t(s) = n(s)/d(s) \in R(s)$

$\delta_\infty(t) = \partial[d] - \partial[n]$: valuation at infinity of $t(s)$

— $T(s) \in R^{p\times k}(s), M_\infty(T)$: Smith-McMillan form at infinity of $T(s)$

$$M_\infty(T) = \left[\begin{array}{ccc|c} s^{q_1} & & & 0 \\ & \ddots & & \\ & & s^{q_r} & \\ \hline & 0 & & 0 \end{array}\right] \qquad r = \rho_{R(s)}\{T(s)\}, q_1 \geq q_2 \geq \cdots \geq q_r$$

$q_i > 0$: orders of infinite poles, $q_i < 0$; $|q_i|$: orders of infinite zeros

$\delta_M^\infty(T) = \sum q_i : q_i$: McMillan degree at infinity of $T(s)$

$\nu(T) = \delta_M(T) + \delta_M^\infty(T)$: extended McMillan degree of $T(s)$.

— $G \in C^{m\times l}, G = Y\sum U^*$: singular value decomposition (SVD)

$Y \in C^{m\times m}, U \in C^{l\times l}$ unitary matrices

$\sum = \text{p-diag}\{\sigma_1, ..., \sigma_r\} \in R^{m\times l}, r = \min(m, l), \sigma_1 \geq \cdots \geq \sigma_r$

p-diag: (pseudo-diagonal) i.e. $\left[\begin{array}{cccc} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ \hline & & 0 & \end{array}\right]$, or $\left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_r & \end{array}\right]$

$\sum(G)$: set of singular values of G

$\bar{\sigma}(G)$: maximal singular value, $\underline{\sigma}(G)$: minimal singular value

Columns of Y, U: left, right singular vectors of G.

$A \in \mathcal{F}^{p\times k}$:

$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_t \end{bmatrix}, A_i \in \mathcal{F}^{p_i\times k_i}, A = \text{b-diag}\{A_1, ..., |A_t\}$ (block-diagonal) $A = \text{diag}\{A_1, ..., A_t\}$, if $A_i \in \mathcal{F}^{p_i\times p_i}$

ABSTRACT

The aim of the thesis is to examine a number of properties related to the set of invariants of linear dynamical systems under different types of transformations and for both state space and transfer function representations. The general objectives have been to classify the invariants according to their generic, non-generic nature, establish links between different types of invariants and system properties and explore the nature and formation of invariants on interconnected systems. More specifically, state space invariants have been classified according to genericity, non-genericity using generic properties of matrix pencils and generic values of controllability, observability indices have been worked out using the generic properties of piecewise Arithmetic Progression Sequence. Similar classification results have been obtained for transfer function models. Some new results on the importance of Plucker invariants have been derived and new tests for controllability, observability were obtained in terms of the rank properties of appropriate controllability-, observability Plucker matrices. Finally, the relationship between the invariants of composite system structure and the invariants of the subsystems has been examined. In particular, it has been shown that under general assumptions on the system structure composition, the zeros of the composite system as well as controllability-, observability indices and decoupling zeros are simply the aggregate of the corresponding subsystem invariants. The effect of loss of input, output channels on the nature of the invariants of the composite system has also been examined.

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Chapter 1

INTRODUCTION

System structure characterises the potential of a system for feedback control; however, the relationships between the values of system structure indicators and limits of achievable performance are not always very clear. The development of system structure is the byproduct of actions taken in the different stages of an engineering design; the mechanism of formation of system structure during the process synthesis and instrumentation stages of engineering design, are not however well understood. Establishing the links between achievable performance and structural properties, as well as structure formation through the stages of engineering design, are long term goals of certain new directions in control theory [Kar.,3] [Kar.,4].

The subject of control theory and design is well developed in the case of linear systems and when the system model (state space of transfer function) has a fixed structure (as far as inputs, outputs) and well defined parameters. Two of the main goals of control theory have been: (i) to determine the characterisation of the solvability conditions of different control problems in terms of the values of appropriate system invariants, (ii) to establish the links between system structural characteristics and the achievable limits of performance under compensation (relationships between performance indicators and system structure). Most of the work so far is in area (i) and has been within the frame of the structural approaches (algebraic, geometric, algebrogeometric etc.), whereas, the second area has been considered within the framework of frequency response as well as state space design methods. The importance of the results in those two areas is that they allow the classification of system models according to their potential for an easy or difficult nature of the control problem; the classification criteria which are used are defined by the type, values of the structural characteristics. The greater the difficulty of the control problem, the larger of the cost of the control action and this can be considerable; thus, the task of designing systems with good control characteristics is a very important problem. This, however, is not a traditional task for control theory, which always assumes a given fixed model. This non-standard task for control theory may be addressed only by the structural approaches which deal with the tools, as well as the criteria of this appropriate system synthesis. The origin of this new role of the control theory goes back to the work of [Ros.,1] [MacF. & Kar.,1] [Kar. & Kou.,1] and more recently to the work in [Kar., & Gia.,1] [Kar.,3] [Kar.,4], where a rather general formulation of the different structure assignment issues is given.

The need for exercising some control and possibly directing the process of building up system models has been recently addressed within the [ESPRIT II Project], EPIC, (see [Kar.,3]) where an attempt to integrate the various tools and issues of the defferent stages of the engineering design has been made. In fact, it has been recognised that the system model on which the control design is performed is not something given, or fixed, but the result of an evolutionary mechanism which unfolds as we go through the design stages on choosing subsystems, interconnecting them, define manipulated and controlled variables and finally deciding about the structure of the control scheme [Kar.,3] [Kar.,4]; the above system design steps, shape the final model characteristics. Understanding this structure evolution mechanism and try to direct it towards the generation of models with "good" control structural characteristics is a major task which is posed for the structural approaches. Taking into account that some of the early design stages are characterised by inaccurate and simplified models, it is essential that the structural approaches develop also along the direction of the uncertain parameters and possible variable dynamic complexity models. The two main tasks of the structural approaches are thus, (i) Development of all aspects which may allow the generation of system model synthesis procedures with the system structure as an important criterion. (ii) Expansion of their concepts and tools into the area of parameter uncertainty and possibly variable complexity models.

The various approaches of the theory of linear systems which deal with the structural system aspects are divided into two broad categories: The state space and the transfer function approaches. The state space approach is well developed and suited for the study of systems properties such as redundancy, minimality, controllability, observability etc. within the state space framework a variety of techniques has been developed for the solution of the design problems and feedback compensation. The structural characteristics (invariants and indicators) are well developed within this framework and the computational tools are those of standard numerical linear algebra.

Two main different roads have emerged within the transfer function description of a linear system: the algebraic approaches and the frequency response approaches. The first family of approaches treats the system as an operator between rational vector spaces and the basic tools are algebraic (polynomial matrix theory, integral matrices, theory of rings); the second, views the system as a map between spaces of periodic signals and thus its tools are those of complex analysis.

There has been a considerable interest in the structural properties of multivariable linear systems in the last thirty years. Several researchers have investigated structural properties and this has led to the introduction of different types of invariants under various transformation groups. The theory of a singular pencil of matrices has been instrumental for the state space aspect of the theory. [Kal.,2] [Ros.,1] etc. has shown how Kronecker's theory of singular pencils plays a fundamental role in determining the invariants of the pair (A,B) under input-space, state-space and state feedback transformations. Pursuing this train of thought, [Tho.,1] applied Kronecker's theory of singular pencils to multivariable systems in state-space form and obtained a canonical form under operations of strict equivalence of pencils. This work was based on special operations and transformations of the system and involved apart from coordinate transformations, state feedback and output injection. Using the geometric approach, [Mor.,1] has provided a deeper understanding of the formal results of pencil theory by explaining the nature of these results in geometric terms. The matrix pencil approach was further developed by ([Kar.,1] [Jaf. & Kar., 1] [Kar. & MacB.,1]) by establishing a unifying pencil and transformation group treatment of invariants and canonical forms and establishing the links with geometric theory by providing a complete characterisation of invariant spaces, in terms of matrix pencils and Kronecker invariants.

The concept of a zero of a multivariable system has been investigated dynamically, algebraically as well as in relation to the geometric properties of a system. [Ros.,1] apparently was the first to define the zeros of a transfer function matrix as the set of zeros of the numerator polynomials in the Smith-McMillan form of the transfer function matrix of a system. An integrated dynamic, state space approach to the zero characterisation was given by [MacF. & Kar.,1], whereas a thorough study of the state space zero structure was given by [Kar. & Kou.,1] using the mechnism of matrix pencils. The extension of the theory to the case of infinite poles and zeros has taken also a lot of attention by researchers such as [Kai,1] [Pug. & Rat.,1] [Var. Lim. & Kar.,1] etc.

Recent works by [Won.,1] [Kim., 1] [Wil. & Hes.,1] have shown the importance of generic and non-generic properties of a system in control theory. [She. & Pear] have extended the results of Lin on structural controllability of single-input linear systems to multi-input linear systems. An interesting byproduct of this extension is in the application of generic analysis to the determination of the rank of a structured matrix. For unstructured generic systems some further extensions of this work are considered here such as that of computing the generic values of minimal indices; these invariants are important for solvability conditions of exact control synthesis problems and their determination is an issue, which is considered here.

[Wil. & Hes.,1] [Bro. & Byr., 1] have shown the importance of generic conditions in establishing the necessary conditions for generic pole placement. [Kar., & Gia.,1] [Kar., & Gia.,2] have established a new exterior algebra and algebraic geometry based framework, which on one hand introduces new system invariants in terms of the Grassmann vectors and Plucker matrices and on the other hand allows the derivation of an integrated framework for study of solvability, as well as computation of solutions of problems described as frequency assignment. One of the goals of this thesis is to study further the properties of invariants of the exterior algebra framework by investigating some of their generic properties, provide computational procedures for finding their generic values and establish certain links between controllability indices, system properties and Grassmann type of invariants. Some of the most fundamental concepts characterising the coupling of internal mechanism to its environment are those of controllability and observability [Kal.,1]. Both controllability, observability properties express the interaction of internal mechanism with the environment represented by the inputs, outputs. Thus controllability, observability properties are shaped at the instrumentation stage of the process design. Controllability, observability are concepts essential for state feedback design. Two more indicators, playing a key role in state space design are the controllability-, observability-Plucker matrices. The importance of such matrices has been demonstrated in [Kar., & Gia.,2] where the notions of controllability-, observability-Plucker matrices are introduced. In this thesis a new criterion for controllability and observability respectively is established in terms of the rank properties of the corresponding

Plucker matrices. Plucker matrices have been recently used in the investigation of frequency assignment problems in linear multivariable systems. The approach relies on the notion of the canonical polynomial Grassmann representative of a rational vector space and on the associated Plucker matrix, which has been defined by [Kar., & Gia.,3] [Kar., & Gia.,2]. Within this framework, the frequency assignment problem is reduced to the study of a linear problem, defined on the Plucker matrix and to a standard problem of decomposability of multivectors.

One additional objectives of this thesis is to determine the controllability, observability and zeros of composite systems, which are formed by interconnection of several multivariable subsystems. It is shown that the controllability and observability of composite systems are related to the controllability and observability of their subsystems and the effect of total loss of set of inputs, outputs, on these properties under sensor, actuator failure is examined.

The thesis is structured as follows: In Chapter 2, a comprehensive introduction to the fundamental algebraic tools, which are relevant in the study of structural properties of the system is given. The specific objective of this thesis is to provide a short review of descriptions, basic concepts and tools from polynomial and rational matrix theory, which will be used as background material for the following chapters. Clearly, the aim here is to provide a set of definitions, which establishes some common terminology and it is supplemented with references, where the material is treated in an appropriate way.

In Chapter 3, we first discuss the effect of transformations of the fundamental system properties and the theory of invariants for state space models, as well as transfer function models and where possible indicate their desirable values.

Chapter 4 is mainly concerned with the study of the generic values of invariants. It is shown that the generic set of column minimal indices, $I_c(F,G)$, and row minimal indices, $I_e(F,G)$, may be deduced from the properties of a generic piecewise Arithmetic Progression Sequence defined on the ordered pair (F,G) [Kar. & Kal.1]. Furthermore, the generic values of the rest of transfer function and state space invariants is also worked out there.

In Chapter 5, rank properties of controllability, observability Plucker matrices are discussed and a necessary and sufficient condition for the Plucker matrix P_A of a least degree matrix $A(s)$ to have full rank is examined, as well as the generic values of this rank. It is shown that system controllability, observability is equivalent to the full rank properties of the corresponding Plucker matrices.

In Chapter 6, we are concerned with the structural properties of the composite systems. Input-state, state-output restriction pencil is used in derivation of most of the results. The general case of input-state, state-output restricted pencil of composite and aggregate systems with full inputs and output is considered first and the results are then extended to the case where one or more inputs are lost. It is assumed that the transfer function of each subsystem remains unchanged after the connection and the system represented by its transfer function matrix is controllable and observable. It is shown that the controllability, observability and zero structure properties of composite system under full input, output structure are simply given as aggregates of corresponding properties of subsystems. Finally, it was shown that the controllability indices and input decoupling zeros (observability indices and output decoupling zeros) of the complete composite system under total loss of subsystem inputs (outputs) are given as the union of those defined by the subsystems.

Chapter 2

BACKGROUND FROM LINEAR SYSTEM AND MATHEMATICS

2.1 Introduction

Modern control theory and design uses concepts and tools from almost every single branch of mathematics. The aim of this section is to introduce some terminology and define the basic mathematical concepts and tools, which are essential for the presentation of the system concepts in the following sections. For rigorous definitions we refer to the references here, we try to emphasise the relevance to applications and the computational aspects of the concepts and tools. The following topics are considered as essential:

i) Basic definitions from abstract algebra.

ii) Basic tools from matrix theory.

iii) Basic concepts and tools from polynomial and rational matrix theory.

iv) Basic concepts from matrix pencil theory

It should be emphasised that this section serves as basic terminology and does not aspire to be an introduction to mathematics for control theory.

2.2 Basic definitions from abstract algebra

In the study of properties of systems a number of concepts from abstract algebra [Macl., & Bir.,1] are essential. We distinguish:

2.2.1 Equivalence Relations, Equivalence Classes, Invariants

On a given system representation different types of transformation may be applied which results in a family of system representations. The resulting family may be described as an equivalence class. The notion of equivalence class is central in parametrising families of systems and has important implications in the study of system structure, identification and control. The crucial mathematical concepts are defined below.

If A is a set, then a relation R on A is a subset of $A \times A$ (Cartesian product, set of ordered pairs $(x, y), x, y \in A$). A relation R is called an equivalence relation (ER), if it is "reflexive" ($(x, x) \in R, \forall x \in A$), "symmetric" $(x_1, x_2) \in R$ implies $(x_2, x_1) \in R$, and "transitive" ($(x_1, x_2) \in R$ and $(x_2, x_3) \in R$ implies $(x_1, x_3) \in R$). If R is an equivalence relation, $x \in A$, then $R(x) = \{y : y \in A \text{ and } (x, y) \in R\}$ denotes the R-equivalence class of x (R-EC), i.e. all $y \in A$ which are equivalent to x in the sense defined by R. The set of all R-equivalences classes $R(x), x \in A$ is called the quotient set of A modulo R and it is denoted by A/R. If R is an ER, then the family of all R-equivalence classes forms a partition of A, i.e. A may be expressed as the union of disjoined classes and represented in diagrammatic terms as:

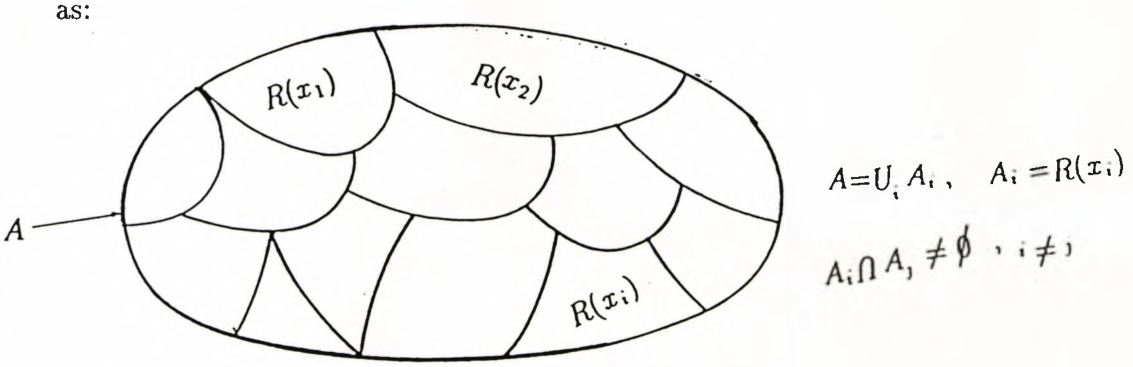


Figure 2.1 Partitioning of a set

Every $R(x_i) = A_i$ R-EC may be represented by one element, say x'_i, and called the representative of A_i and the set of all representatives τ_1 of the $R(x_i)$ R-EC is called a system of distinct representatives (R-SDR). If A, P are two sets, R an ER, then function $f : A \rightarrow P$ is called an invariant of R when $(x, y) \in R$ implies $f(x) = f(y)$.

An invariant f of R is called complete, when $f(x) = f(y)$ implies $(x, y) \in R$. A complete invariant defines a one to one correspondence between the R-ECs $R(x)$ and the image of f. By specialising the complete invariant f such that $P \subset A$, we define a canonical element $C \in R(x)$ which uniquely characterises $R(s)$ and it is referred to as canonical form.

Example (2.2.1): Let A be the set of $n \times n$ real matrices $A(A \in R^{n \times n})$. For $A_1, A_2 \in A$, we define an ER by

$$A_1 E A_2 \text{ if } A_1 = P A_2 Q, P, Q \in R^{n \times n}, |P|, |Q| \neq 0 \tag{2.1}$$

and $E(A) = \{PAQ, \text{ all } P, Q \in R^{n \times n}, |P|, |Q| \neq 0\}$ is the E-EC. If ρ is the rank function defined on $A(\rho : A \rightarrow \{0, 1, ..., n\})$, then $\rho(A)$ is a complete invariant of $E(A)$ and the set of matrices of A

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, I_n \tag{2.2}$$

where I_k is the $k \times k$ identity matrix is the set of all possible canonical forms for the quotient set A/E. The set in (2.2) defines an E-SDR, and the partitioning of A is Figure (2.1) contains $n + 1$ E-EC's each one characterised by a given rank $r \in \{0, 1, ..., n\}$.

2.2.2 Groups of Transformations

For systems, representations, compensation transformations generate equivalence classes. The formal way of representing such transformations is in terms of groups. The equivalence class is then generated by the action of the group on the system. In simple terms, a group is a set A with an internal composition rule (binary operation) $*$ which has the properties associativity, existence of a identity element, and existence of inverse for every element.

Example (2.2.2): (canonical example): Consider the set

$$H = \left\{ h = (P, Q) : P \in R^{p\times p}, Q \in R^{k\times k}, |P| = 0, |Q| = 0 \right\} \tag{2.3}$$

with a composition rule * defined by:

$$h_1 * h_2 = (P_1, Q_1) * (P_2, Q_2) = (P_1 P_2, Q_2 Q_1) \tag{2.4}$$

The set H with the * operation is a group with (I_p, I_k) identity and will be denoted, in short by H and referred to as the matrix transformation group. If A is the set of $p \times k$ real matrices, then for every $A \in R^{p\times k}$, the action of H on A is defined by every

$$h \in H, hoA = (P, Q)oA = PAQ \tag{2.5}$$

These concepts will be used to define formally the different equivalence classes of systems obtained under special group of the H-type.

2.2.3 Rings in Control Theory

Modern control theory is using representations based either on polynomials, proper rational functions, or proper and stable rational functions. The properties of such sets are similar and they are described in abstract terms by those of Euclidean rings. The essence of such mathematical structure is described by the properties of polynomials under addition and multiplication. The term Euclidean, refers to the existence of division in the sense of division of polynomials. A central concept in Euclidean rings is the notion of the degree function. For polynomials, this function is well known, whereas for the other important rings, it will be explained in the examples. Another key concept in any ring is that of the unit, that is those elements which have a multiplicative inverse. If every element of the ring (apart from zero) has an inverse, then the ring is called a field. Typical examples of fields are the real, complex numbers (R,C) and the rational functions (R(s)). A proper treatment of these important algebraic concepts may be found in [MacL.& Bir.,1] and with a control application flavour in [Vid,1], [Vard., & Kar.,1].

Example (2.2.3): (Rational functions): The set of rational functions $R(s)(t(s) = n(s)/d(s) \in R(s), n(s), d(s) \in R[s])$ is a field. For every $t(s) = n(s)/d(s) \in R(s)$ the function defined by

$$\delta_\infty(t(s)) = q_\infty = \partial[d(s)] - \partial[n(s)] \text{ if } t(s) \neq 0, \delta_\infty(t(s)) = \infty \text{ if } t(s) = 0 \tag{2.6}$$

is known as valuation and if $q_\infty > 0$ we say that $t(s)$ has a zero at $s = \infty$ of order q_∞, if $q_\infty < 0$, we say that $t(s)$ has a pole at $s = \infty$ of order $|q_\infty|$ and if $q_\infty = 0$, it is called biproper. The set of all proper rational functions $R_{pr}(s)$ is characterised by the property that $\delta_\infty(t) \geq 0$, all $t(s) \in R_{pr}(s)$ and it is a Euclidean ring with degree $\gamma_\infty(t)$ expresses the total number of zeros at $s = \infty$ of $t(s) \in R_{pr}(s)$. The units of $R_{pr}(s)$ are the biproper functions and the units of $R[s]$ are the constants. If $t(s) = n(s)/d(s) \in R_{pr}(s)$, then it has no poles at $s = \infty$ and the finite poles are given by the zeros of $d(s)$; $t(s)$ may have zeros at $s = \infty$, and their number is defined by $\gamma_\infty(t)$, whereas the finite zeros are given by the zeros of $n(s)$.

If $t(s) = n(s)/d(s) \in R_P(s)(P = \Omega \cup \{\infty\}, \Omega = C^+$ for instance) i.e. $t(s)$ is proper and $d(s)$ has no zeros in Ω, then

$$\gamma_P(t) \equiv \gamma_\infty + \{\text{total number of zeros in } \Omega \text{ of } n(s)\} \tag{2.7}$$

is defined as the P-degree of $t(s)$. $R_P(s)$ is a Euclidean degree with $\gamma_P(\cdot)$ degree and its units are the biproper rational functions which have no poles and no zeros in Ω.

Two polynomials $t_1(s), t_2(s) \in R[s]$ are said to be coprime if they have no common zeros. Similarly, $t_1(s), t_2(s) \in R_P(s)$ are said to be coprime if they have no common zeros in $P = \Omega \cup \{\infty\}$ (They might have common zeros in Ω^c (complement of Ω) however).

2.3 Basic definitions and Properties of Matrices

In the following, we shall refer to $C, R, R(s)$ fields by $\mathcal{F}$, whenever a definition, property is valid on either of them. We introduce here some basic definitions and properties.

2.3.1 Basic definitions

If $A \in \mathcal{F}^{m \times n}$, then $\rho(A) = r \leq \min(m, n)$ denotes the rank of A over $\mathcal{F}$ and the numbers $n_r(A) = n - r, n_l(A) = m - r$, denote the right-, left-nullity of A. If $n_r(A) = 0$, the A is called right regular and if $n_l(A) = 0$, it is called left regular. With A, we may associate the vector spaces

$$\mathcal{R}_A^c \equiv c.sp\{A\}, \mathcal{R}_A^r \equiv r.sp\{A\} \tag{2.8}$$

$$\mathcal{N}_A^r \equiv \mathcal{N}_r(A) = \{\underline{x} : A\underline{x} = \underline{0}\}, \mathcal{N}_A^l \equiv \mathcal{N}_l(A) = \{\underline{y}^t : \underline{y}^t A = \underline{0}^t\} \tag{2.9}$$

where $\mathcal{R}_A^c, \mathcal{R}_A^r$ are the column-, row- spaces of A and $\mathcal{N}_A^r, \mathcal{N}_A^l$ are the right-, left-null spaces of A. These spaces are vector spaces over $\mathcal{F}$ and $\dim \mathcal{R}_A^c = \dim \mathcal{R}_A^r = r$, $\dim \mathcal{N}_A^r = n_r(A)$, $\dim \mathcal{N}_A^l = n_l(A)$. For every A, there exist $Q \in \mathcal{F}^{m \times m}, R \in \mathcal{F}^{n \times n}$, $|Q|, |R| \neq 0$, such that

$$QAR = \left[\begin{array}{c|c} I_r & O_{r,n-r} \\ \hline O_{m-r,r} & O_{m-r,n-r} \end{array}\right] \tag{2.10}$$

where Q, R may be defined as products of elementary row, column transformations. For the special cases $n_r(A) = 0, n_l(A) = 0$, the above becomes

$$QA = \left[\begin{array}{c} A_l^{\dagger} \\ \hline A_l^{\perp} \end{array}\right] A = \left[\begin{array}{c} I_n \\ \hline 0 \end{array}\right] \quad \text{if } n_r(A) = 0 \tag{2.11}$$

$$AR = A\left[\begin{array}{cc} A_r^{\dagger}, & A_r^{\perp} \end{array}\right] = [I_m, 0], \text{ if } n_l(A) = 0 \tag{2.12}$$

and $A_l^{\dagger}, A_r^{\dagger}$ are referred to as left-, right inverse ($A_l^{\dagger} A = I_n$, $AA_r^{\dagger} = I_m$) and $A_l^{\perp}, A_r^{\perp}$ as left-, right-annihilators (the rows of $A_l^{\perp}$ define a basis for $\mathcal{N}_A^l$ and the columns of $A_A^{\perp}$ a basis for $\mathcal{N}_A^r$). Computing the above matrices may be achieved by singular value decomposition (SVD), if $\mathcal{F} = R$ or C. A matrix A is called block diagonal and denoted by $A \equiv$block diag$\{A_i\}$ if

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} \tag{2.13}$$

If $A = (a_{ij}) \in \mathcal{F}^{m \times n}$, then A is called diagonal and write $A \equiv$ diag $\{a_1, ..., a_k\}$, if $a_{ij} = 0$, all $i \neq j$ and $a_{ii} = a_i$ for $i = 1, 2, ..., k, k = \min\{m, n\}$. A matrix is sparse, if it has relatively few nonzero entries. A matrix A is called structured, if it has a number of fixed zero elements, but the nonzero elements may take generic values and they are called the generic elements. The nonzero-zero structure of such matrices may be conveniently displayed by letting "$\times$" denote an arbitrary nonzero scalar as shown below

$$A = \begin{bmatrix} 0 & \times & 0 & \times & \times \\ \times & 0 & 0 & 0 & \times \\ 0 & \times & 0 & 0 & \times \\ \times & 0 & 0 & \times & 0 \end{bmatrix}$$

2.3.2 Jordan Canonical form of a square constant matrix

Let $\mathcal{F}$ be either R, or C and $A \in \mathcal{F}^{n \times n}$. The polynomial

$$\phi(\lambda) = |\lambda I - A| = (\lambda - \lambda_1)^{p_1} \dots (\lambda - \lambda_k)^{p_k},$$

with $\lambda_i \neq \lambda_j, i = 1, 2, \dots, k$ is known as the characteristic polynomial and its roots as the eigenvalues of A. The set of distinct values of the eigenvalues, $\phi(A) \equiv \{\lambda_i, i \in \underline{k}\}$ will be referred to as the root range and the multiplicity p_i of λ_i in $\phi(\lambda)$ as its algebraic multiplicity. The number $q_i \equiv n_r(\lambda_i I - A)$ is defined as the geometric multiplicity of λ_i. For every $\lambda_i \in \phi(A), p_i \geq q_i$ and if $p_i = q_i$ for all $\lambda_i \in \phi(A)$, then A is called simple, or cyclic; otherwise, i.e. for at least a $\lambda_i \in \phi(A), p_i > q_i$, then it is called nonsimple, noncyclic. Clearly, $p_1 + \dots + p_k = n$.

For every $A \in \mathcal{F}^{n \times n} (\mathcal{F} = R$, or $C)$ there exists $U \in C^{n \times n}, |U| \neq 0$ such that

$$A = U \text{bl.diag}\ \{J(\lambda_1); \dots; J(\lambda_k)\}\, U^{-1} = UJ(A)U^{-1}, \lambda_i \in \phi(A) \tag{2.14}$$

$$J(\lambda_i) \equiv \text{bl.diag}\ \{J_{\nu_{i,1}}(\lambda_i); \dots; J_{\nu_{i,q_i}}(\lambda_i)\}, \nu_{i,1} + \dots + \nu_{i,q_i} = p_i \tag{2.15}$$

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in C^{k \times k} \tag{2.16}$$

The pair (U, U^{-1}) are known as similarity transformations, $J(A)$ as the Jordan canonical form and $J_k(\lambda)$ as an elementary Jordan block. $J(A)$ is unique up to permutations of the diagonal blocks. The ordered set of dimensions of the Jordan blocks for $\lambda_i \in \phi(A), \rho(A, \lambda_i) \equiv \{\nu_{i,1} \leq \dots \leq \nu_{i,q_i}\}$ is defined as the λ_i-Segre' characteristic of A or equivalently, the ordered set of degrees of elementary divisors at λ of the corresponding matrix $(\lambda_i I - A)$.

2.3.3 Exterior product of vectors and compound matrices [Mar. & Min,1]

We shall denote by $Q_{k,n}$ the set of strictly increasing, lexicographically ordered sequences of k integers chosen from $\{1, 2, \dots, n\}$. Let $A \in \mathcal{F}^{m \times n}$ (where $\mathcal{F}$ is R, C, or $R(s)$), $1 \leq r \leq \min(m, n)$. The r-th compound matrix of A is the $\binom{m}{r} \times \binom{n}{r}$

matrix whose entries are the minors $|A[\alpha/\beta]|$, that is the determinants of the submatrices defined by the rows corresponding to $\alpha \in Q_{r,m}$ and columns corresponding to $\beta \in Q_{r,n}$. This matrix is denoted by $C_r(A)$. For example, if $A \in \mathcal{F}^{3\times 3}$ and $r = 2$, then

$$C_2(A) = \begin{bmatrix} A^{1,2}_{1,2} & A^{1,2}_{1,3} & A^{1,2}_{2,3} \\ A^{1,3}_{1,2} & A^{1,3}_{1,3} & A^{1,3}_{2,3} \\ A^{2,3}_{1,2} & A^{2,3}_{1,3} & A^{2,3}_{2,3} \end{bmatrix}$$

If $A \in \mathcal{F}^{m\times m}$, $\underline{\alpha}_i, i \in \underline{n}$ denote the columns and $\underline{\alpha}^t_j, j \in \underline{m}$ the rows of A, then we define [Mar,Min,1]:

i) If $m \leq n, C_m(A) \equiv \underline{\alpha}^t_1 \wedge \cdots \wedge \underline{\alpha}^t_m$ is an $1 \times \begin{pmatrix} n \\ m \end{pmatrix}$ row vector and it is called the <u>exterior</u>, or <u>Grassman Product</u> of the rows of A.

ii) If $m \geq n, C_n(A) \equiv \underline{\alpha}_1 \wedge \cdots \wedge \underline{\alpha}_n$ is an $\begin{pmatrix} m \\ n \end{pmatrix}$-column vector and it is called the exterior, or Grassman product of the columns of A.

Binet-Cauchy Theorem:

Suppose $A \in M_{m,p}(\mathcal{F}), B \in M_{p,n}(\mathcal{F})$ and $C = AB \in M_{m,n}(\mathcal{F})$. If $1 \leq r \leq \min(m,n,p), \alpha \in Q_{r,m}, \beta \in Q_{r,n}$, then

$$\det\left\{C^{\alpha]}_{\beta]}\right\} = C^{\alpha}_{\beta} = \sum_{\omega \in Q_{r,p}} \det\left\{A^{\alpha]}_{\omega]}\right\} \cdot \det\left\{B^{\omega]}_{\beta]}\right\} \tag{2.17}$$

For the special case of $A \in M_{n,m}, B \in M_{m,n}$, then the determinant of the product is given in terms of the minors

$$|C| = \det C^{\alpha]}_{\beta]} = C^{1,\ldots,m}_{1,\ldots,m} = \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq m} A^{1,2,\ldots,m}_{k_1,k_2,\ldots,k_m} B^{k_1,k_2,\ldots,k_m}_{1,2,\ldots,m} \tag{2.18}$$

or, the determinant of C is the sum of the products of all possible minors of the maximal (m-th) order of A into the corresponding minors of the same order of B.

2.4 Polynomial and Rational Matrices

Polynomial and rational matrices appear throughout linear systems theory and some of the basic definitions and notations are summarised here.

2.4.1 Basic definitions and notation

For a $P(s) \in R^{m\times n}(s)$ the rank over $R(s), \rho(P(s)) = r \leq \min(m,n)$ is referred to as normal rank, whereas the rank over $C, \rho_C(P(\lambda)) = r_\lambda$, for some $\lambda \in C$, is called the local rank at $s = \lambda$. The tools for investigating rank properties are the Smith-McMillan forms. If $r = \min(m,n)$, then $P(s)$ is said to be nondegenerate, otherwise, $r < \min(m,n)$, it will be called degenerate.

If $\underline{x}(s) = [x_1(s), ..., x_m(s)]^t \in R^m[s]$, then $\partial[\underline{x}(s)] = \max\{\partial[x_i(s)], i \in \underline{m}\}$ is defined as the degree of $\underline{x}(s)$. If $P(s) = P_d s^d + \cdots + sP_1 + P_0 \in R^{m\times n}[s], P_i \in R^{m\times n}, P_d \neq 0$, then $d \equiv \partial_s[P(s)]$ is defined as the scalar degree of $P(s)$.

If $P(s) = [\underline{p}_1(s), ..., \underline{p}_n(s)] \in R^{m\times n}[s]$, $m \geq n$ and $\rho(P(s)) = r = n$, then the set of indices $\mathcal{I}_P = \{\delta_i : \delta_i = \partial[\underline{p}_i(s), i \in \underline{n}\}$ is defined as the set of column degrees and $c_P \equiv \sum_{i=1}^{n} \delta_i$ as the column complexity of $P(s)$ (row degrees and row complexity are defined in a similar manner). The $\begin{pmatrix} m \\ n \end{pmatrix}$ polynomial vector $C_n(P(s)) = \underline{p}_1(s) \wedge \cdots \wedge \underline{p}_n(s) = \underline{p}(s)\wedge$ is called the Grassman vector of $P(s)$ and $\partial[\underline{p}(s)\wedge] \equiv \partial[P(s)]$ is referred to as the matrix degree, or simple degree of $P(s)$. If $\underline{p}_i(s) = \underline{p}_{i,h} s^{\delta_i} + \cdots + \underline{p}_{i,0}$, then we may write

$$P(s) = [\underline{p}_{1,h}, ..., \underline{p}_{n,h}] \text{ diag } \{s^{\delta_1}, ..., s^{\delta_n}\} + \underline{\hat{P}}(s) \tag{2.19}$$

where the columns of $\underline{\hat{P}}(s)$ have degrees less than δ_i. The matrix $P_h = [\underline{p}_{1,h}, ..., \underline{p}_{n,h}] \equiv [P(s)]_h \in R^{m\times n}$ is referred to as the high column coefficient matrix of $P(s)$ and if $\rho(P_h) = n$ then $P(s)$ is called column reduced. (high row coefficient matrix, and row reducedness is defined similarly).

If $U(s) \in \mathcal{K}^{m\times m}$, where $\mathcal{K}$ is either $R, R[s], R_{pr}(s)$, or $R_p(s)$, and $|U(s)|$ is a unit of $\mathcal{K}$, then it will be called an $(m,\mathcal{K})$-unimodular and will designate it as $U(s) \in U(m,\mathcal{K})$. Such matrices are products of elementary transformations over $\mathcal{K}$. If $P(s), P'(s) \in R^{m\times n}(s)$ and

$$P'(s) = L(s)P(s)R(s) \tag{2.20}$$

where $L(s) \in U(m,\mathcal{K}), R(s) \in U(n,\mathcal{K})$, then they are said to be $\mathcal{K}$-equivalent and this is denoted by $P(s)E_{\mathcal{K}}P'(s)$. If $\mathcal{K} = R$, then they are called strict equivalent. If $P(s)E_{\mathcal{K}}P'(s)$ and $L(s) = I_m$, or $R(s) = I_n$, then they are called right-, left-equivalent and this is denoted by $P(s)E^r_{\mathcal{K}}P'(s), P(s)E^l_{\mathcal{K}}P'(s)$ respectively. $E_{\mathcal{K}}, E^r_{\mathcal{K}}, E^l_{\mathcal{K}}$ are equivalence relations and the corresponding equivalence classes of $P(s)$ are denoted by $E_{\mathcal{K}}(P), E^r_{\mathcal{K}}(P), E^l_{\mathcal{K}}(P)$ respectively.

2.4.2 Smith, Smith-McMillan forms

For a matrix $P(s) \in R^{m\times n}(s)$ canonical forms may be defined under left, or right equivalence over $\mathcal{K}$ (where $\mathcal{K}$ is either $R[s]$, $R_{pr}(s)$, or $R_p(s)$). Such forms are referred to as Hermite forms, if $P(s)$ is defined over $\mathcal{K}$ and Hermite-McMillan form if $P(s)$ is a general rational. Under $\mathcal{K}$-equivalence we define respectively the Smith forms (if $P(s)$ is from $\mathcal{K}$) and Smith-McMillan forms (if $P(s)$ is general rational). The Smith, Smith-McMillan forms are central in the study of structure of linear systems and they are described next.

(a) Smith form over $R[s]$ [Gan,1]

Let $P(s) \in R^{m\times n}[s], \rho(P(s)) = r \leq \min(m,n)$. There exist $L(s) \in U(m, R[s])$, $R(s) \in U(n, R[s])$ such that

$$L(s)P(s)R(s) = S_P(s) = \left[\begin{array}{c|c} S_P^*(s) & 0 \\ \hline \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{array}\right] \begin{array}{l} \} \; r \\ \} \; m-r \end{array} \tag{2.21}$$

$$S_P^*(s) = \operatorname{diag}\{f_1(s), ..., f_r(s)\}, f_i(s) \in R[s] \tag{2.22}$$

$S_P(s)$ is called the Smith form and the monic polynomials $f_i(s), i \in \underline{r}$ are the invariant polynomials of $P(s)$ and satisfy the division property $f_i(s)/f_{i+1}(s) \forall i = 1, 2, ..., r-1, (f_i(s)$ divides $f_{i+1}(s))$. The set of $f_i(s)$ may be defined by the Smith algorithm. Thus, let $d_0(s) = 1, d_i(s)$ be the monic greatest common divisor (GCD) of all $i \times i$ order minors, $i = 0, 1, 2, ..., r$. Then $d_i(s)/d_{i+1}(s), i = 0, 1, 2, ..., r-1$ (divisibility) property and

$$f_i(s) = d_i(s)/d_{i-1}(s), i = 1, 2, ..., r \tag{2.23}$$

The above formula describes the Smith algorithm. The polynomial $z_P(s) \equiv \Pi_{i=1}^{r} f_i(s)$ is called the zero polynomial of $P(s)$. If $z_P(s)$ is factorised into irreducible factors over C as

$$z_P(s) = (s-z_1)^{\tau_1} ... (s-z_\mu)^{\tau_\mu}, z_i \in C, z_i \neq z_j \tag{2.24}$$

then the set $\phi_P = \{z_i, i \in \underline{\mu}\}$ is called the root range, z_i a zero of $P(s)$ and τ_i the algebraic multiplicity of z_i. The zeros are the frequencies for which $P(z_i)$ loses rank below its normal rank r and the number $v_i = n_r(P(z_i)) + r - n$ is defined as the geometric multiplicity of z_i. Generally, $v_i \leq \tau_i$ and if equality holds, the zero is called Simple. The matrix $P(s)$ is called simple, if all zeros are simple, otherwise, it is called nonsimple. By factorising each of the $f_i(s)$ into irreducible factors over C and collecting all terms corresponding to the zero z_i, we define the set of elementary divisor for z_i, $D_{P,z_i} \equiv \{(s-z_i)^{\bar{q}_{ik}}, k = 1, 2, ..., v_i\}$, where v_i is the geometric multiplicity and $\sum_{k=1}^{\nu_i} \bar{q}_{ik} = \tau_i$.

The Smith form is defined for matrices over $P_{pr}(s)$ and $R_P(s)$ and with the appropriate modifications their form is similar.

(b) Smith-McMillan form over $R[s]$ [Kai,1]

If $P(s) \in R^{m\times n}(s), \rho(P(s)) = r \leq \min(m,n)$ and $d(s)$ is the least common multiple (LCM) of the denominators of the elements of $P(s)$, then $P(s) = d(s)^{-1}N(s)$, where $N(s) \in R^{m\times n}[s]$. If

$$L(s)N(s)R(s) = S_N(s) \tag{2.25}$$

is the Smith reduction of $N(s)$, then the Smith-McMillan form of $P(s)$ is defined by

$$M_P(s) = \frac{1}{d(s)} S_N(s) \in R^{m\times n}(s) \tag{2.26}$$

where in $M_P(s)$ all possible numerator-denominator cancellations are assumed to have been carried out. Thus,

$$L(s)P(s)R(s) = M_P(s) = \left[\begin{array}{c|c} M_P^*(s) & 0 \\ \hline \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{array}\right] \begin{array}{l} \} \ r \\ \} \ m-r \end{array} \tag{2.27}$$

$$\tag{2.28}$$

$$M_P^*(s) = \ \operatorname{diag}\{\epsilon_i(s)/\psi_i(s), i \in \mathrm{r}\} \tag{2.29}$$

The sets of $\{\epsilon_i(s), i \in \tilde{r}\}$, $\{\psi_i(s), i \in \tilde{r}\}$ are the elementary zero- pole-polynomials of $P(s)$ and satisfy the divisibility properties

$$\epsilon_1(s)/\epsilon_2(s)/.../\epsilon_r(s), \psi_r(s)/\psi_{r-1}(s)/.../\psi_1(s) \tag{2.30}$$

The polynomials $z_P(s) \equiv \Pi_{i=1}^{r}\epsilon_i(s), p_P(s) \equiv \Pi_{i=1}^{r}\psi_i(s)$ are defined as the zero, pole polynomials of $P(s)$ and $\partial[p_P(s)] \equiv \delta_M$ is defined as McMillan degree of $P(s)$. The Smith-McMillan form over $R_p(s)$ is of similar structure (with the appropriate changes) and it is described in [Vard,Kar,1]. The Smith-McMillan form is the natural tool to describe the rank properties (local and normal rank) for all $s \in C$; however, does not provide information for the rank of $P(s)$ at $s = \infty$.

(C) Smith-McMillan form at $s = \infty$ [Vard,Lim.,Kar.,1]

The structure and rank properties of a rational matrix at $s = \infty$ is defined by the Smith-McMillan form at $s = \infty$. Thus, let $P(s) \in R^{m \times n}(s), \rho(P(s)) = r$. There exist $L(s) \in U(m, R_{pr}(s)), R(s) \in U(n, R_{pr}(s))$ such that

$$L(s)P(s)R(s) = M_P^{\infty}(s) = \left[\begin{array}{c|c} M_P^{*\infty} & 0 \\ \hline \underbrace{0}_{r} & \underbrace{0}_{n-r} \end{array}\right] \begin{array}{l} \} \ r \\ \} \ m-r \end{array} \tag{2.31}$$

$$M_P^{\infty *}(s) = \operatorname{diag}\{s^{q_1}, ..., s^{q_r}\}, q_1 \geq \cdots \geq q_r \tag{2.32}$$

$M_P^{\infty}(s)$ is uniquely defined by $P(s)$ and it is called the Smith-McMillan form at $s = \infty$ of $P(s)$. If Π_∞ is the number of q_i's with $q_i > 0$, then we say that $P(s)$ has Π_∞ poles at infinity, each one of order $q_i > 0$. If η_∞ is the number of q_i's with $q_i < 0$, then we say that $P(s)$ has η_∞ zeros at infinity, each one of order $|q_i|$. The number $\delta_M^{\infty}(P) \equiv \sum_{i=1}^{\Pi_\infty} q_i, q_i > 0$, is defined as the McMillan degree at infinity of $P(s)$. If $P(s)$ is proper, then it has no poles at infinity and $M_P^{\infty} = S_P^{\infty}(s)$ is the Smith form at $s = \infty$ describing the infinite zero structure of $P(s)$. In this thesis, the term McMillan degree refers to the total number of finite poles and McMillan degree at ∞ refers to the total number of poles at infinity. We shall refer to the sum of those two numbers as the extended McMillan degree.

$M_P^{\infty}(s)$ may be defined from the standard Smith-McMillan form of $P(1/w)$ [Verg., 1] [Pug. & Rat.,2] at $w = 0$. Alternatively, the q_i's may be computed by the valuation algorithm. Thus, if $\epsilon_i \equiv$ least valuation among the valuations of all $i \times i$ minors of $P(s), i = 1, 2, ..., r$ then

$$q_i = \epsilon_{i-1} - \epsilon_i, i = 1, 2, ..., r, \epsilon_0 = 0 \tag{2.33}$$

A matrix $P(s)$ which is a column, or row block of some $R_{pr}(s)$-unimodular matrix has no poles and no zeros at $s = \infty$ and it is called left-, or right-biproper.

$M_P^{\infty}(s)$ is one of the local canonical forms of $P(s)$ that reveals its structure at a single point, here at $s = \infty$. By considering the $\widetilde{P}(w) \equiv P(1/w)$ rational matrix, then $M_P^{\infty}(s), M_{\widetilde{P}}^{\infty}$ defines by a mere substitution of $s = w$, the $M_P^0(s)$, which is the local Smith-McMillan form at $s = 0$; $M_P^0(s)$ reveals the structure of $P(s)$ at $s = 0$, The final and initial asymptotes of the Bode plots of $P(s)$ together with the valuation algorithm may be used to compute $M_P^{\infty}(s), M_P^0(s)$ from experimental data. In fact, for the case $M_P^{\infty}(s)$, the final asymptotes of the Bode diagrams of the elements of the transfer functions define the valuations at ∞ of the corresponding elements, then the valuation algorithm described before together with some genericity arguments may be used to compute all possible order valuations of the given matrix; such a procedure has been described in [Var. Lim. & Kar.,1]. An example illustrating this procedure is given below.

Example (2.4.1): Let

$$T(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & s^3 & \frac{s^2}{s+1} \\ \frac{s+2}{s^2+0.2s+1} & \frac{1}{s^3} & \frac{1}{(s+2)^2} \end{bmatrix}$$

then the least valuation ϵ_1 among the valuations of all first order minors (i.e. elements) of $T(s)$ is:

$$\epsilon_1 = \min\{2, -3, -1, 1, 3, 2\} = -3$$

The least valuation ϵ_2 among all second order minors is:

$$\epsilon_2 = \min\{-2, 0, -1\} = -2 \text{ and } \epsilon_0 = 0$$

therefore

$$\begin{aligned} q_1 &= \epsilon_0 - \epsilon_1 = 0 - (-3) = 3 \\ q_2 &= \epsilon_1 - \epsilon_2 = -3 - (-2) = -1 \end{aligned}$$

and so $T(s)$ has one pole at infinity of order 3 and one zero at infinity of order one.

Thus, the Bode magnitude diagrams corresponding to above example are shown in the Figure and the valuation matrix $V = \{\delta(t_{ij}(s))\}$ is given by $V = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 3 & 2 \end{bmatrix}$.

From V it follows that

$$\xi_1(T) = -3 \quad \text{and} \quad \xi_2(T) = \min\{2+3,\ 1-3,\ 2+2,\ 1-1,\ 2-3,\ 3-1\} = -2$$

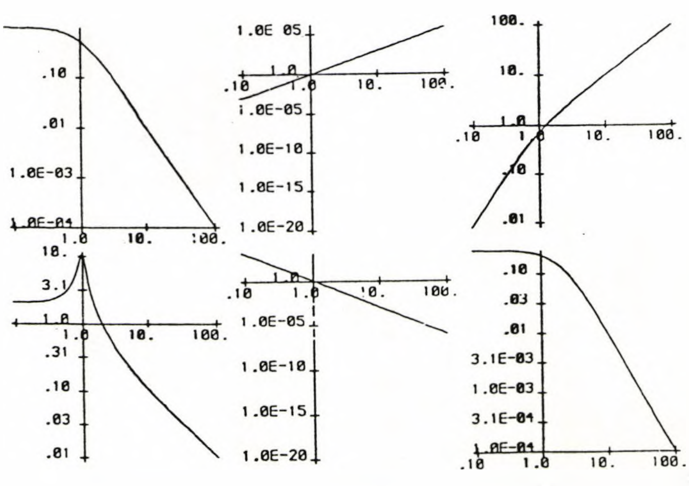


Figure 2.1: Bode magnitude array for Example (2.4.1)

For the case $M_P^o(s)$ the analysis is dual to the previously described one. In fact now we are using the initial asymptotic slopes at zero.

If we are given the transfer function, we are using the valuation algorithm however, even from experimently derived frequency responses we can always computae the inital and final slopes from the corresponding Bode plot.

2.4.3 Matrix, Divisors and Coprimeness of Polynomial Matrices

The solvability conditions of many control problems are reduced to a test of coprimeness of polynomial, or rational matrices (from a specific ring of importance to control). The basic definitions and tests are given here for the case of polynomial matrices [Kai,1]. The corresponding extensions to the case of matrices over $R_{pr}(s)$, or $R_p(s)$ are along similar lines and may be found in [Vard.,Kar.1,2].

Let $P(s) \in R^{m\times n}[s], \rho(P(s)) = n$. A matrix $R(s) \in R^{n\times n}[s]$ such that

$$P(s) = P'(s)R(s), P'(s) \in R^{m\times n}[s] \tag{2.34}$$

is called a right matrix divisor (RMD) of $P(s)$. If $\bar{R}(s)$ is any other RMD and

$$R(s) = W(s)\bar{R}(s), W(s) \in R^{n\times n}[s] \tag{2.35}$$

then $R(s)$ is called right greatest matrix divisor (RGMD) of $P(s)$. If $\rho(P(s)) = m$, the notions of left matrix divisors (LMD) and left greatest matrix divisor (LGMD) are defined similarly. A matrix $P(s) \in R^{m\times n}[s]$, $\rho(P(s)) = n$ is called right irreducible, if all RMDs are $R(s)$-unimodular. Nonunimodular RMDs contain a subset of the zeros of the original matrix. A matrix $P(s)$ is least degree, if it has no zeros, i.e. $\rho_C(P(\lambda)) = n, \forall\lambda \in C$, or equivalently $S_P(s) = [I_n, 0]^t$. A left irreducible matrix is defined in a similar manner. A matrix $P(s) \in R^{m\times n}[s]$, $\rho(P(s)) = n$ (or m), is called a minimal basis, if it is right (left) irreducible and column (row) reduced. Minimal bases have no finite and no infinite zeros.

If $P_r \equiv \{P_i(s) \in R^{k_i\times m}[s], i \in \tilde{\nu}\}$ is a set of matrices, then the matrix

$$T_P^r(s) \equiv \begin{bmatrix} P_1(s) \\ \vdots \\ P_\nu(s) \end{bmatrix} \in R^{k\times m}[s], k = \sum_{i=1}^{\nu} k_i \tag{2.36}$$

is called a matrix representative of P_r; P_r is right regular, if $\rho(T_P^r(s)) = m$. If P_r is right regular, then a right common matrix divisor (RCMD) and a right greatest common matrix divisor (RGCMD) of P_r is defined as a RMD and a RGMD of $T_P^r(s)$ respectively. The set P_r is called right coprime (RC), if it is right regular and $T_P^r(s)$ is right irreducible. For a set of matrices with the same number of rows, a matrix representative, left regularity property, left common matrix divisors, (LCMD), left greatest common matrix divisor (LGCMD) and left coprimeness are defined in a similar manner. The above definitions for matrices over $R[s]$ have their counterparts for matrices over $R_{pr}(s), R_P(s)$ with the appropriate changes. (see [Vard.,Kar.1,2]).

2.4.4 Matrix Pencils [Gan,1]

If $F, G \in R^{m \times n}$, then the polynomial matrix $L(s) = sF - G$ is called a matrix pencil. Polynomial matrices of this type play an important role in the study of state space models and matrix computations. Their theory is richer than that of general polynomial matrices and the basic concepts are summarised below.

The family of $m \times n$ pencils is denoted by $\mathcal{L}_{m,n}(s)$. $L_1(s) = sF_1 - G_1, L_2(s) = sF_2 - G_2 \in \mathcal{L}_{m,n}(s)$ are said to be strict equivalent, if there exist $R \in U(m, R), Q \in U(n, R)$ such that $L_2(s) = RL_1(s)Q$. If $m = n$ and $|sF - G| \not\equiv 0$ then the pencil is called regular, otherwise it is called singular. If $sF - \hat{s}G$ is the homogeneous pencil and $f_i(s, \hat{s}), i \in \tilde{r}, r = \rho(sF - G)$, are the homogeneous invariant polynomials (defined by Smith algorithm), then elementary divisors (ed) of the type $\hat{s}^q$ are referred to as infinite e.d. (∞-ed) and those of the type $(s - \alpha\hat{s})^p$ as finite ed (f-ed). The link between the ∞-e.d. (divisors of the type $\hat{s}^q$) and the orders of ∞-zeros of $sF - G$, as these are defined by the Smith-McMillan form at ∞, has been established in [Vard. & Kar.,4] and it is referred to as "plus one" property. In fact, for every ∞-e.d. $\hat{s}^q, q \geq 2$, there is an ∞-zero of order $q - 1$, whereas linear elementary divisor of the type $\hat{s}$ do not correspond to infinite zeros. If the pencil is singular, at least one of the following equations has a solution for polynomial vectors $\underline{x}(s), \underline{y}(s)^t$

$$(sF - G)\underline{x}(s) = \underline{0}, \qquad \underline{y}(s)^t(sF - G) = \underline{0}^t \tag{2.37}$$

If $\{\underline{x}_i(s), i \in \tilde{\mu}\}, \{\underline{y}_j(s)^t, j \in \tilde{\nu}\}$ are minimal polynomial bases for $\mathcal{N}_r\{sF - G\}, \mathcal{N}_l\{sF - G\}$ respectively (in the sense defined in section (2.4.3)) and $\{\epsilon_i, i \in \tilde{\mu}\}, \{\eta_j, j \in \tilde{\nu}\}$ denote the corresponding degrees, then the set of ϵ_i are known column minimal indices (c.m.i.) and the set of η_j as row minimal indices (r.m.i.) of the pencil. The set of f-ed, ∞-ed, cmi, rmi, uniquely characterise the strict equivalence class of $sF - G, E_R(F,G)$, and there exists a canonical form, the Kronecker canonical form defined by some appropriate pair (R,Q) by $R(sF-G)Q = sF_k - G_k$, of the type

$$sF_k - G_k = \text{bl.diag}\{0_{g,h}; ...; L_\epsilon(s); ...; \hat{L}_\eta(s); ...; sH_q - I_q; ...; sI_p - J_p(\alpha); ...\} \quad (2.38)$$

where $O_{g,h}$ is a zero block defined by the g zero rmi, h zero cmi, $L_\epsilon(s), \hat{L}_\eta(s)$ are blocks associated with nonzero ϵ cmi, η rmi, $sH_q - I_q$ a block associated with the $\hat{s}^q\infty$-ed and $sI_p - J_p(\alpha)$ a block associated with the f-ed $(s-\alpha)^p$. The structure of these blocks is defined below:

$$\begin{aligned} L_\epsilon(s) &= s\,[I_\epsilon, \underline{0}] - [\underline{0}, I_\epsilon] \qquad \epsilon \times (\epsilon+1) \text{ block} \\ \tilde{L}_\eta(s) &= s\begin{bmatrix} I_n \\ \hline \underline{0}^t \end{bmatrix} - \begin{bmatrix} \underline{0}^t \\ \hline I_\eta \end{bmatrix} \qquad (\eta+1)\times\eta \text{ block} \end{aligned} \quad (2.39)$$

$$sH_q - I_q = s\left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{q-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] - I_q \qquad q\times q \text{ block}$$

$$sI_p - J_p(\alpha) = sI_p - \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & \cdots & 0 & \alpha \end{bmatrix} \qquad p\times p \text{ block} \quad (2.40)$$

2.4.5 Matrix fraction description

Consider a linear time invariant multivariable system giving rise to a transfer function matrix $G(s) \in R^{m\times l}(s)$, $\text{rank}_{R(s)}G(s) = \min\{m,l\}$. It is then well known that $G(s)$ can always be factored (in a non-unique way) as

$$G(s) = D_l^{-1}(s)N_l(s) = N_r(s)D_r^{-1}(s) \quad (2.41)$$

where $N_l(s), N_r(s) \in R^{m\times l}[s], D_l(s) \in R^{m\times m}[s], D_r(s) \in R^{l\times l}[s]$ with $\det D_l(s)$, $\det D_r(s) \neq 0$. The pair $(D_r(s), N_r(s))$ $(D_l(s), N_l(s))$ is called a right (left) matrix fraction description (MFD) of the transfer function matrix $G(s)$.

The above definitions, show that matrix fraction description provide a natural generalisation of the scalar rational function representation of single input-single output systems, though in the multivariable case we have to distinguish between right and left descriptions.

However, we have to note that there is a certain duality in these descriptions. Furthermore, we shall often omit the subscript r.

2.4.6 The Grassman representative of a vector space and the Plucker Co-ordinate [Kar.,5][Kar.&Gia,3]

Let $\mathcal{V}$ be an $m-$dimensional subspace of an n-dimensional vector space $\mathcal{U}$ over a field F. The map $h : \mathcal{V} \to \mathcal{U}$ defined by $h(x) = \underline{x}, x \in \mathcal{V}$ is linear and there is a unique homomorphism $\hat{h} : \wedge\mathcal{V} \to \wedge\mathcal{U}$ associated with h. Note that $\wedge$ denotes the wedge or exterior product and $\wedge^m$ denotes the m-th exterior power. [Mar.,1] See also Section (2.3.3). Since $\dim\mathcal{V} = m, \wedge^m\mathcal{V}$ is a one-dimensional space and it is mapped by $\hat{h}$ onto a one-dimensional subspace of $\wedge^m\mathcal{U}$. Thus if $V = \{\underline{v}_i, i = 1, ..., m\}$ is a basis of $\mathcal{V}$ then $\wedge^m\mathcal{V}$ is spanned by the element $\underline{v}_1 \wedge ... \wedge \underline{v}_m$ and $\hat{h}$ maps this element onto

$$\hat{h}(\underline{v}_1 \wedge ... \wedge \underline{v}_m) = \hat{h}(\underline{v}_1) \wedge ... \wedge \hat{h}(\underline{v}_m) = \underline{v}_1 \wedge ... \wedge \underline{v}_m \tag{2.42}$$

in $\wedge^m\mathcal{U}$. The vectors $\underline{v}_i, i = 1, 2, ..., m$ are linearly independent and so $\underline{v}_1 \wedge ... \wedge \underline{v}_m$ is a non-zero element of $\wedge^m\mathcal{U}$. In fact the injection map $h : \mathcal{V} \to \mathcal{U}$ defined by $h(x) = \underline{x}, \underline{x} \in \mathcal{V}$ induces an injection map $\wedge^m h : \wedge^m\mathcal{V} \to \wedge^m\mathcal{U}$ defined by $\wedge^m h(\underline{x}\wedge) = \underline{x}\wedge$, $\underline{x}\wedge \in \wedge^m\mathcal{U}$. The vector $\{\underline{v}_1, ..., \underline{v}_m\}$ spans a one-dimensional subspace of $\wedge^m\mathcal{U}$ which depends only on $\mathcal{U}$. Now let $U = \{\underline{u}_j, j = 1, ..., n\}$ be basis of $\mathcal{U}$, then using matrix representation we have the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d] & U \arrow[d] \\ F^m \arrow[r, "A = H_U^V"] & F^n \end{tikzcd}$$

where $A = H_U^V$ is the matrix representation of h with respect to V and U. In fact if

$$h(\underline{v}_i) = \underline{v}_i = \sum_{j=1}^{n} a_{ij}\underline{u}_j \tag{2.43}$$

then $A = H_U^V$ is the matrix

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

The column span of A is a subspace of F^n and it is the representation of $h(V)$ with respect to the basis V, U. The representation of $\underline{v}_1 \wedge ... \wedge \underline{v}_m$ with respect to the basis $\wedge^m \mathcal{V}, \wedge^m \mathcal{U}$ is defined by the commutative diagram and thus

$$C_m\left[H_V^{U'}\right] = C_m\left[H_U^V\right] C_m\left[Q_{V'}^V\right] = C_m\left[H_U^V\right] q, q = \det Q_U^V \in F - \{0\} \tag{2.44}$$

The two vectors $\underline{t} = \underline{v}_1 \wedge ... \wedge \underline{v}_m, \underline{t}' = \underline{v}_1' \wedge ... \wedge \underline{v}_m'$ are related by

$$t' = [..., \underline{u}_w \wedge, ...] \begin{bmatrix} \vdots \\ c_w \\ \vdots \end{bmatrix} = [..., u_w \wedge, ...] \begin{bmatrix} \vdots \\ c_w \\ \vdots \end{bmatrix} q = q\underline{t} \text{ or } c_w = ac'_w, \quad w \in Q_{m,n} \tag{2.45}$$

The above derivation indicates that co-ordinate transformations (change of bases) defines another co-ordinate transformation in the exterior product spaces which expresses the Binet-Cauchy Theorem described in Section (2.3.3) and manifested by (2.45).

<u>Example (2.4.2)</u>: Let $\mathcal{V}$ be a vector space defined by the columns of the matrix $V = \begin{pmatrix} 1 & -2 \\ 3 & -5 \\ 7 & -9 \end{pmatrix}$.

In this case we have $C_2(V) = C_2\begin{pmatrix} 1 & -2 \\ 3 & -5 \\ 7 & -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}$.

If $V' = \begin{pmatrix} 3 & 2 \\ 8 & 5 \\ 16 & 9 \end{pmatrix}$ is another basis for $\mathcal{V}$. In this case we have $C_2(V') =$

$$C_2 \begin{pmatrix} 3 & 2 \\ 8 & 5 \\ 16 & 9 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ -8 \end{pmatrix}.$$

By Binet-Cauchy Theorem

$$C_2(V') = C_2(V)q \text{ where } q = \det Q$$

i.e.

$$C_2 \begin{pmatrix} 3 & 2 \\ 8 & 5 \\ 16 & 9 \end{pmatrix} = C_2 \begin{pmatrix} 1 & -2 \\ 3 & -5 \\ 7 & -9 \end{pmatrix} \cdot \det Q \text{ where } Q = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Definition (2.4.1): The scalars c_w of equation (2.45) are called Plucker Co-ordinates of the subspace $\mathcal{V}$ relative to the bases V of $\mathcal{V}$ and U of $\mathcal{U}$.

□

Equation (2.45) shows that any two sets of Plucker Co-ordinate of $\mathcal{V}$, which correspond to two different bases of $\mathcal{V}$, with respect to the fixed base U of $\mathcal{U}$ differ by a non-zero scalar factor. Hence the ratios of c'_w's are the same as the corresponding ratios of c_w's ($c_{w1} = qc'_{w_1}, c_{w2} = qc'_{w_2}$ and so $c_{w1}/c_{w2} = c'_{w_1}/c'_{w_2}$). Therefore the ratios are uniquely determined by V.

Sometimes, the ratios of the c'_w, rather than the c_w themselves, are called the Plucker Coordinate of V.

Consider now the vector space $F^{\sigma+1}$ of $(\sigma+1)$-tuples $\underline{x} = (x_0, x_1, ..., x_\sigma), x_i \in F$. Let us call two such vectors $\underline{x}$ and $\underline{y}$ equivalent if they are both non-zero and if $\underline{x} = q\underline{y}$ for some $q \in F - \{0\}$. This equivalence relation splits the non-zero vectors in $F^{\sigma+1}$ into equivalence classes, and clearly each equivalence class consists of all the non-zero elements in a one-dimensional subspace of $F^{\sigma+1}$. Thus the equivalence classes are in one-to-one correspondence with the lines through the origin of $F^{\sigma+1}$.

Definition (2.4.2): The set of all equivalence classes of non-zero vectors in $F^{\sigma+1}$ as defined above, is called the projective space of dimension σ over F denoted by $P^\sigma(F)$. Each equivalence class defines a point of this projective space. If Q is any point in $P^\sigma(F)$ and if $\underline{x} = (x_0, ..., x_\sigma)$ is any vector of the equivalence class which defines Q, then the x_i's are called homogeneous co-ordinates of Q.

□

If we set $\sigma = \begin{pmatrix} n \\ m \end{pmatrix} - 1 = \dim \wedge^m \mathcal{U} - 1$, then we can easily see that the Plucker co-ordinates of V, enumerated in lexicographic order, may be considered as the homogeneous co-ordinates of a point in $P^\sigma(F)$. However, every point in $P^\sigma(F)$ does not represent an m-dimensional subspace of $\mathcal{U}$. Elements of $\wedge^m \mathcal{V}$ of the type $q\underline{v}_1 \wedge \ldots \wedge q\underline{v}_m$ where $\underline{v}_1, \ldots, \underline{v}_m$ are linearly independent vectors of $\mathcal{V}$ and $q \in F - \{0\}$ are called simple or decomposable m-vectors. Decomposable multivectors uniquely define m-dimensional subspaces of $\mathcal{U}$ and it is shown below.

Proposition (2.4.1): Let $\mathcal{U}$ be an n-dimensional vector space over F and let $\underline{y}\wedge = \underline{y}_1 \wedge \ldots \wedge \underline{y}_m, \underline{z}\wedge = \underline{z}_1 \wedge \ldots \wedge \underline{z}_m$ be two decomposable non-zero elements of $\wedge^m \mathcal{U}$ and let us denote by $\mathcal{V}_y =$ span $\{\underline{y}_1, \ldots, \underline{y}_m\}$ and $\mathcal{V}_z =$ span $\{\underline{z}_1, \ldots, \underline{z}_m\}$ the subspaces of $\mathcal{U}$ defined by $\underline{y}\wedge$ and $\underline{z}\wedge$ respectively. Necessary and sufficient condition for $\mathcal{V}_y = \mathcal{V}_z$ is

$$\underline{y}\wedge = \underline{y}_1 \wedge \ldots \wedge \underline{y}_m = q\underline{z}_1 \wedge \ldots \wedge q\underline{z}_m \wedge, q \in F - \{0\} \tag{2.46}$$

□

Definition (2.4.3): Let $\mathcal{U}$ be a vector space over a field F with $\dim \mathcal{U} = n$. The Grassmannian $G(m,n)$ is defined as the set of m-dimensional subspaces $\mathcal{V}$ of $\mathcal{U}$; $G(m,n)$ actually admits the structure of an manifold which is known as the Grassmann manifold.

□

Definition (2.4.4): If $\mathcal{V}$ is any m-dimensional subspace of the n-dimensional vector space $\mathcal{U}$, then any non-zero decomposable multivector $\underline{v} \wedge \ldots \wedge \underline{v}_m$ with $\underline{v}_i \in \mathcal{V}, i = 1, 2, \ldots, m$ is called a Grassman representative of $\mathcal{V}$. We have already seen that all the Grassmann representatives differ only by non-zero scalar factors so that we shall denote any one of them simply by $g(\mathcal{V})$.

□

What we have constructed so far is a well defined mapping $g : G(m, U) \to P^\sigma(F)$, by associating to every $v \in G(m,n)$ Plucker co-ordinates $(\ldots, c_w, \ldots), w \in Q_{m,n}$. This mapping is known as the Plucker Embedding of $G(m,n)$ in the projective space $P^\sigma(F)$. The Plucker image of $G(m, \mathcal{U})$ in $P^\sigma(F)$ is called the Grassmann variety in $P^\sigma(F)$.

2.5 Conclusion

In this chapter, the basic tools from Mathematics and Systems which are essential for the development of the subsequent chapters have been examined. The proper cover of the relevant issues is given in the listed references.

Chapter 3

REVIEW OF INVARIANTS OF LINEAR SYSTEMS

3.1 Introduction

On a given system we may apply different types of transformations, some of them corresponding to a change of representation and some others having a compensation, or feedback interpretation. The theory of system invariants is important for control theory and design since they describe structural characteristics which remain unaffected under the transformation and thus indirectly are related to the limits of compensation. Here we try to summarise the basic invariants and where possible indicate their desirable values. This chapter is structured as follows: We first discuss the effect of transformations of the fundamental system properties. Then we discuss the theory of invariants for state space models and finally for transfer function models. The topic on system invariants and canonical forms is quite extensive. Here we attempt to summarise the basic aspects of the theory.

3.2 System Transformations and Fundamental System Properties [Won.,1] [Kar. & MacB.,1] [Kar., & Gia.,1]

On a state space model $S(A,B,C,D)$ we may apply co-ordinate and feedback transformations. Thus, we consider the following cases:

a) System properties under co-ordinate transformations;

b) System properties under co-ordinate transformations and feedback.

Coordinate transformations are of the type $\underline{x} = Q\underline{x}', \underline{y} = T\underline{y}', \underline{u} = R\underline{u}'$ where Q, T, R are square nonsingular matrices. The effect of these transformations on system properties is summarised below.

Result (3.1): If Q, T, R are state, output, input coordinate transformations, then

i) The characteristic polynomial $\phi(\lambda) = |\lambda I - A|$, the eigenvalues and associated Segre characteristic, are invariant under all (Q, T, R) transformations.

ii) The controllability, observability are invariant under all (Q, T, R) transformations.

iii) The transfer function matrix and Markov parameters are invariant under all Q transformations.

□

Thus, controllability properties may be inferred from any model obtained from $S(A,B,C,D)$ and Q transformations. The eigenvectors, however, are functions of Q and their description changes with the changing of Q. The eigenframe is important when we deal with a coordinate frame characterising physical states.

Under the feedback transformations L, K, F expressing state, output feedback and output injection respectively illustrated by the diagram (3.1) we have:

Result (3.2): If L, K, F denote state-, output-feedback and output injection respectively, then:

i) Controllability and stabilisability are invariant under all L, K.

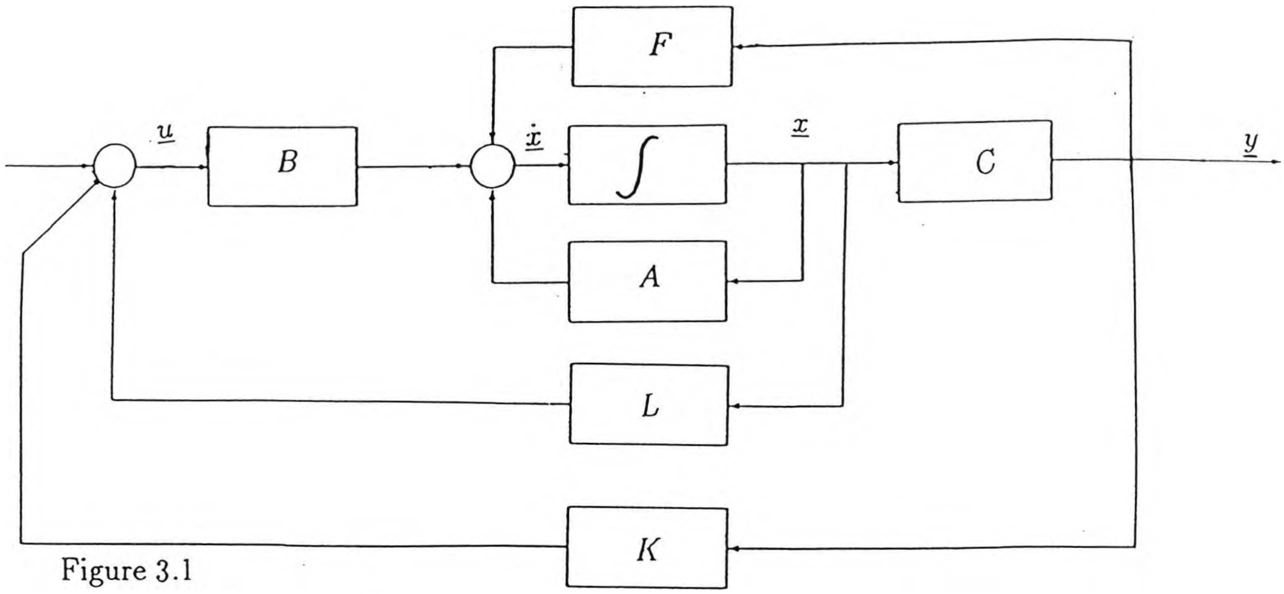


Figure 3.1

ii) Controllability is invariant under all F, iff the system has no zeros. For any system, controllability is invariant under a generic F.

iii) If the system is stabilisable, then stabilisability is invariant under all F, iff the system has no right half plane zeros. For any stabilisable system, stabilisability is invariant under a generic F.

iv) Observability and detectability are invariant under all F, K.

v) Observability is invariant under all L, iff the system has no zeros. For any system, observability is invariant under a generic L.

vi) If the system is detectable, then detectability is invariant under all L, iff the system has no right half plane zeros. For any detectable system, detectability is invariant under a generic L.

□

The presence of finite zeros implies that for certain families of output injection we loose controllability and for certain families of state feedback we loose observability [Sha. & Kar.1]. The presence of right half plane zeros has corresponding implications to loss of stabilisability, detectability under certain families of output injections, output feedback correspondingly.

More general types of transformations, which preserve the transfer function and certain properties of PMDs are discussed in [Ros.,1] [Kai,1] [Pug. Hay. & Fre.,1]

[Pug. Hay. & Wal.,1]. Since such topics are not relevant for our present topic, they will not be considered here.

The notion of coordinate transformations for state space models, has its equivalent in the transfer function matrix context, which is that of unimodular matrices. According to what sort of fractional description we consider for $G(s)$, we have the K-unimodular matrices $U(m, K)$ where m denotes the dimension of the square matrix elements from K (K is $R[s]$, $R_{pr}(s)$, or $R_p(s)$ and if $Q \in U(m, K)$, then $|Q|$ is a unit of K). The role of coordinate transformations in the system representation is emphasised by the following result.

Result (3.3): Let $G(s) \in R_{pr}^{m\times l}(s)$. Then,

i) [Kal.,1] The minimal state space models $S_i, i = 1, 2$ have the same transfer function $G(s)$, iff they are related by a state coordinate transformation,

ii) [Ros.,1] [Kai,1] The left,right K-coprime MFD pairs $(A_{1i}, B_{1i}), (B_{2i}, A_{2i}), i = 1, 2$ (K is $R[s]$, $R_{pr}(s)$ or $R_p(s)$) have the same transfer function $G(s)$, iff

$$[A_{12}, B_{12}] = L[A_{11}, B_{11}], L \in U(m, K) \tag{3.1}$$

$$\begin{bmatrix} B_{22} \\ A_{22} \end{bmatrix} = \begin{bmatrix} B_{21} \\ A_{21} \end{bmatrix} R, R \in U(m, K) \tag{3.2}$$

□

3.3 State space invariants

On state space models we may apply different types of representation, compensation transformations and thus a variety of invariants and canonical forms are defined. Summarising the most fundamental types of state space invariants, is the aim of this section. Central to the definition and computation of most of the invariants is the theory of Kronecker invariants (and associated canonical form) of matrix pencils [Gan.,1]. It is worth pointing out that the complexity of invariants and associated canonical form increases as the complexity of the transformations which are involved in the definition of the equivalence classes decreases. The presentation thus follows a path of increased complexity.

The most general types of transformations that may be applied on the $S(A, B, C, D)$ system are those defined by Q, T, R state, output, input coordinate transformations, state feedback L and output injection F. Based on Q, T, R, L, F transformations, we may define the following ordered sets of transformations [Kar.,7]:

$$\mathcal{H}_k = \{H_k : H_k = (Q, T, R; L, F)\} \tag{3.3}$$

$$\mathcal{H}_B^r = \{H_B^r : H_B^r = (Q, R; L)\}, \mathcal{H}_B^l = \{H_B^l : H_B^l = (Q, T; F)\} \tag{3.4}$$

$$\mathcal{H}_C = \{H_C : H_C = (Q, T, R; 0, 0) = (Q, T, R)\} \tag{3.5}$$

$$\mathcal{H}_C^{is} = \{H_c^{is} : H_C^{is} = (Q, 0, R) = (Q, R)\} \tag{3.6}$$

$$\mathcal{H}_C^{os} = \{H_C^{os} : H_C^{os} = (Q, T, 0) = (Q, T)\} \tag{3.7}$$

$$\mathcal{H}_C^s = \{H_C^s : H_C^s(Q, 0, 0) = (Q)\} \tag{3.8}$$

These transformation form groups (under a standard composition rule described in chapter 2); $\mathcal{H}_k, \mathcal{H}_B^r, \mathcal{H}_B^l$ will be referred to as the Kronecker, right-,left-Brunovsky groups and $\mathcal{H}_C, \mathcal{H}_C^{is}, \mathcal{H}_C^{os}, \mathcal{H}_C^s$ as general-, input-state-, state-output-, state-coordinate groups respectively. The action of these groups on the system may be expressed as action on pencils associated with the corresponding type of system which is considered. Thus,

i) Action of $\mathcal{H}_k, \mathcal{H}_C$ on $S(A, B, C, D)$ is defined by:

$$\begin{bmatrix} sI - A' & -B' \\ -C' & -D' \end{bmatrix} = \begin{bmatrix} Q^{-1} & F \\ 0 & T \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} Q & 0 \\ L & R \end{bmatrix} \tag{3.9}$$

ii) Action of $\mathcal{H}_B^r, \mathcal{H}_C^{is}$ on $S(A, B)$ is defined by:

$$\begin{bmatrix} sI - A', & -B' \end{bmatrix} = Q^{-1} \begin{bmatrix} sI - A, & -B \end{bmatrix} \begin{bmatrix} Q & 0 \\ L & R \end{bmatrix} \tag{3.10}$$

iii) Action of $\mathcal{H}_B^l, \mathcal{H}_C^{os}$ on $S(A, C)$ is defined by: $sI - A' = Q^{-1}(sI - A)$,

$$\begin{bmatrix} sI - A' \\ -C' \end{bmatrix} = \begin{bmatrix} Q^{-1} & F \\ 0 & T \end{bmatrix} \begin{bmatrix} sI - A \\ -C \end{bmatrix} Q \tag{3.11}$$

iii) Action of $\mathcal{H}_C^s$ on $S(A)$ is defined by: $sI - A' = Q^{-1}(sI - A)$.

We consider next the types of invariants and canonical forms that may be defined on state space models under the different groups. We distinguish those involving coordinate transformations only and then those also involving feedback.

(I) Invariants and canonical forms under coordinate transformations

(a) State coordinate transformation on $S(A)$ [Tur. & Ait.,1]

For the system $S(A) : \dot{\underline{x}} = A\underline{x}$, coordinate transformations are known also as similarity transformations. The structure of eigenvalues defines the invariants and canonical form [Gan.,1].

Result (3.4): If $\phi(A)$ is the root range of A, and $S(A, \lambda) = \{v_1 < ... < v_q\}$ is the Segre characteristic for every $\lambda \in \phi(A)$, then the set $\{\phi(A); S(A, \lambda) \text{ all } \lambda \in \phi(A)\}$ is a complete invariant for similarity equivalence on $S(A)$. The corresponding canonical form is the Jordan canonical form,

$$J(A) = \text{ diag } \{...; J(\lambda); ...\}, J(\lambda) = \text{ diag } \{J_{v_1}(\lambda); ...; J_{v_q}(\lambda)\} \tag{3.12}$$

where $J_k(\lambda) = \lambda I_k + H_k$ is a typical $k \times k$ λ-Jordan block.

The invariants and canonical form may be computed algebraically by use of the Smith form of $sI - A$ (computation of set of e.d.'s), or by alternative means based on sequences of numbers [Kar.,2]. The maximum of the geometric multiplicities of eigenvalues,is denoted by μ and referred to as the Segre index.

Remark (3.1): The similarity invariants define the nature of elementary motions of $S(A)$ and characterise stability properties. For eigenvalues on the imaginary axis it is essential to compute the corresponding Segre characteristics, since this defines the difference between Lyapunov stability and instability. The Segre index μ (max of q for all eigenvalues) defines the minimum number of inputs, outputs which are needed for controllability, observability, when inputs and outputs are selected.

If $v(\lambda)$ is the maximal degree elementary divisor at λ, then $\bar{n} = \sum v(\lambda)$ defines the degree of the minimal polynomial of A. Alternative canonical forms, such as those of the companion type may be found in [Gan.,1].

(b) State, input coordinate transformations on $S(A, B)$

We examine here the basic invariants and canonical forms under state coordinate transformations and then extend them to include state and input coordinate transformations. The canonical forms have implications for identification and state space design. The emphasis is put on invariants, whereas for a proper surveying of canonical forms we refer to references. Throughout this section we assume that $S(A, B)$ has n state, l inputs and $\rho(B) = l$.

For the pair (A,B) we define the sequence of matrices

$$Q_{c,k} = [B, AB, ..., A^k B], k = 0, 1, 2, ... \tag{3.13}$$

where $Q_{c,n-1} = Q_c$ is the controllability matrix and $\rho(Q_{c,k}) < \rho(Q_{c,k+1})$.

Definition (3.1): [Kai,1] The smallest integer μ for which $\rho(Q_{c,m}) = \rho(Q_{c,\mu+v})$ is defined as the controllability index of $S(A,B)$. If we assume that the linearly independent columns of Q_c in order from left to right have been found and rearrange these independent columns as

$$\underline{b}_1, A\underline{b}_1, ..., A^{\mu_1 - 1}\underline{b}_1; ...; \underline{b}_l, A\underline{b}_l, ..., A^{\mu_l - 1}\underline{b}_l \tag{3.14}$$

then set of indices $\{\mu_i, i \in l\}$ are called the controllability indices of $S(A,B)$.

□

In order to state the next result on controllability indices, we have to introduce some tools from the matrix pencil characterisation of system properties [Kar. & Kou.,1] [Kar. & MacB.,1].

Definition (3.2): Let $S(A,B,C)$ be a linear system, $A \in R^{n\times n}, B \in R^{n\times l}, C \in R^{m\times n}$, rank $(B) = l$, rank $(C) = m, N \in R^{(n-l)\times n}$ be a basis matrix for $\mathcal{N}_l(B)$, (left annihilator of B) and $M \in R^{n\times(n-m)}$ be a basis matrix for $\mathcal{N}_r(C)$ (right annihilator of C). We may define the following restricted pencils:

i) The input-state restriction pencil $R(s) = sN - NA \in R^{(n-l)\times n}[s]$.

ii) The state-output restriction pencil $P(s) = sM - MA \in R^{n\times(n-m)}[s]$.

iii) The zero pencil $Z(s) = sNM - NAM \in R^{(n-l)\times(n-m)}[s]$.

□

Remark (3.2): [Kar. & Kou.,1] [Kar. & MacB.,1] The pencils $R(s), P(s)$ completely characterise the controllability, observability and feedback invariant properties of $S(A,B,C)$, where as $Z(s)$ defines completely the zero structure of the system. The characterisation of the above properties is in terms of the Kronecker invariants [Gan.,1] of the corresponding pencils.

□

Some important properties of these indices are summarised by the following result [Kai,1], [Che,1], [Kar. & MacB.,1]. Note $\mu_i > l$, for all $i = 1, ..., l$ and the zero value appears, only when $\rho(B) < l$ (which is not considered here).

Result (3.5): [Kar., & Gia.,1] For the set $I_c = \{\mu_i, i \in \tilde{l}\}$ of controllability indices of $S(A, B)$ the following hold true:

i) The controllability index $\mu = \max\{\mu_1, \mu_2, ..., \mu_l\}$

ii) If $\bar{n}$ is the degree of the minimal polynomial of A, then

$$n/l \leq \mu \leq \min(\bar{n}, n - l + 1) \leq n - l + 1 \tag{3.15}$$

iii) $\mu_1 + \mu_2 + \cdots + \mu_l < n$ and equality holds iff the system is controllable. Furthermore, $\sum_{i=l} \mu_i = n_c$ is the dimension of the controllable space of the system and $n - n_c$ defines the total number of uncontrollable modes.

iv) The controllability indices are invariant under state, input coordinate transformations and state feedback.

v) The set I_c is the same with the set of column minimal indices of the pencil $P_c(s) = [sI - A, -B]$.

vi) The set $I_c = \{\mu_i, i \in \tilde{l}\}$ defines the set of column minimal indices $\{\tilde{\mu}_i\}$ of the pencil $R(s) = sN - NA$ by the rule $\tilde{\mu}_i = \mu_i - 1, i = 1, 2, ..., l$.

vii) If $G(s) = N(s)D(s)^{-1}$ is any $R(s)$-right coprime MFD with $D(s)$ column reduced and $S(A, B, C)$ is a minimal realisation of $G(s)$ (assume $G(s)$ strictly proper), then the column degrees of $D(s)$ define the controllability indices of $S(A, B)$.

□

Remark (3.3): The set of controllability indices and the set of f.e.d. of $P_c(s)$ pencil are invariant under $\mathcal{H}_C^{is}$ group, but they are not complete. That is, more invariants are needed to define a complete set.

□

Defining a complete set of invariants for $S(A, B)$ under $\mathcal{H}_C^{s}, \mathcal{H}_C^{is}$ groups is related to the theory of canonical forms under coordinate transformations which is extensively treated in [Kai,1]. The Popov canonical form [Pop.,1], is a unique form under similarity for $S(A, B)$. Such a canonical from contains all additional information about the new invariants, which are now a set of real numbers. We may illustrate the structure of this canonical from in terms of an example. Thus consider a controllable system with $n = 5, \mu_1 = 2, \mu_2 = 3, l = 2$.

The Popov canonical form has the following general structure where $\times$ denote uniquely defined constants.

$$A_c = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ \times & \times & 0 & \times & \times \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \times & 0 & \times & \times & \times \end{array}\right] \qquad B_c = \left[\begin{array}{cc} 0 & 0 \\ 1 & \times \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \tag{3.16}$$

The subsystems defined by diagonal blocks are intercoupled and this coupling is defined by the crate diagram [Kai,1] and it is an invariant. This unique canonical form is a useful tool in system identification. If input coordinate transformations are also used, then the canonical form has an identical A matrix, A_c, but the $\times$s in B_c are eliminated. Different types of "pseudo canonical" forms (not unique) exist in the literature [Kai,1], which once more demonstrate the minimal indices structure of the pair, but their nonzero elements are not all of them uniquely defined.

(c) <u>State-output coordinate transformations on $S(A, C)$</u>

Note that the definitions and results presented for (A, B) pairs have their equivalents for the case of (A, C) pairs by using "transposed duality" arguments, that is (A^t, C^t) is first seen as a state, input pair and by transposition and use of the changes:

Controllability to observability, right MFD to left MFD, input to output etc. all definitions and results may be stated for state, output pairs (A, C). The set of observability indices is denoted by $I_o = \{\theta_i, i \in \widetilde{m}\}$, where $\widetilde{m}$ is the number of outputs and θ denotes the observability index which now satisfies the following inequality.

$$n/m \leq \theta \leq \min(\bar{n}, n - m + 1) \leq n - m + 1 \tag{3.17}$$

where $\bar{n}$ is the degree of the minimal polynomial.

(d) <u>State-coordinate transformations of $S(A, B, C)$</u>

For systems $S(A, B, C)$, the theory of invariants and canonical forms is richer than that of $S(A, B), S(A, C)$ systems, since both aspects of the above two subsystems are involved. The set of controllability, observability indices are invariants, as well as the sets of input, output decoupling zeros and finite, infinite zeros. Note that the additional invariants, which will be defined under the Kronecker group $\mathcal{H}_k$, are also invariant under $\mathcal{H}_C^s$, since $\mathcal{H}_C^s$ is a subgroup of $\mathcal{H}_k$. The canonical forms, which have been defined in the literature do not always demonstrate the structure of all of these invariants.

If Q is a transformation that brings (A, B) to Popov form (A_c, B_c) defined before, then the output $C_c = CQ$ is uniquely defined and (A_c, B_c, C_c) is an input based canonical form [Kar., & Gia.,1]. Similarly, if Q' is a transformation that brings (A, C) to the corresponding Popov form (A_o, C_o), then $B_o = Q^{-1}B$ is uniquely defined and (A_o, B_o, C_o) is an output based canonical form [Kar., & Gia.,1]. The Popov canonical forms $(A_c, B_c, C_c), (A_o, B_o, C_o)$ are related to the realisation of transfer functions based on canonical right, left MFDs, that is those which are in a "echelon type form" [Kai,1]. Alternative canonical forms, based on the idea of balancing the controllability and observability Grammians have been defined [Obe. & McF,1]; such forms are more robust to model parameter uncertainties and play a key role in model reduction.

The canonical forms and invariants under coordinate transformations are important in system parametrisation, identification and model reduction. They are also useful as convenient form for studying state space design problems; however, some considerable numerical effort (and associated numerical difficulties) may be associated with their derivation. For the purposes of this project, the invariants and canonical forms that involve feedback seem to be more relevant, since they indicate what is the "best" that can be achieved under feedback.

(II) Invariants and canonical forms under coordinate transformations and feedback

The types of feedback used are state feedback and output injection; the case of output feedback is considered briefly at the end, since it is still an active area in control theory. The transformations $\mathcal{H}_B^r, \mathcal{H}_B^l, \mathcal{H}_k$ contain as subgroups the $\mathcal{H}_C^{is}, \mathcal{H}_C^{os}, \mathcal{H}_C$; thus, a number of coordinate transformations invariants are not preserved under the more general groups, which are considered now.

a) Coordinate transformations and state feedback on $S(A, B)$

Under the action of $\mathcal{H}_B^r$ group (input, state coordinate transformations and state feedback) on $S(A, B)$ systems, we obtain an equivalence class of systems $\mathcal{E}_B(A, B)$, referred to as the Brunovsky orbit of $S(A,B)$. If $\mathcal{I}_c = \{\mu_i, i \in \tilde{l}\}$ is the set of controllability indices, or equivalently c.m.i. of $P_c(s)$, $\Phi_{ID} = \{\lambda : \rho[\lambda I - A, -B] < n\}$, and $\mathcal{D}_{ID} = \{(s-\lambda_i)^{\tau_i}, \lambda \in \Phi_{ID}, , i = 1, ..., k\}$ [Kal.,2] is the set of f.e.d. of $P_c(s)$ (defining the structure of input decoupling zeros) then we may summarise the properties of $\mathcal{E}_B(A,B)$ as follows [Bru.1], [Kal.,2], [Kar. & MacB.,1].

Result (3.6): For the Brunovsky orbit $\mathcal{E}_B(A,B)$, the following hold true:

i) The sets $\mathcal{I}_c, \mathcal{D}_{ID}$ are complete and independent invariants of $\mathcal{E}_B(A,B)$

ii) There is a uniquely defined canonical form, the generalised Brunovsky form, $S(A_B, B_B)$, which in pencil form is described by

$$\begin{aligned} P_C^B(s) &= [sI - A_B, -B_B] \\ &= \begin{bmatrix} sI - A_C & 0 & -B_C \\ 0 & sI - A_{ID} & 0 \end{bmatrix} \end{aligned} \tag{3.18}$$

where $A_{ID} = \text{diag}\ \{J_{\tau_i}(\lambda_i), i = 1, 2, ..., k\}, J_{\tau_i}(\lambda_i)$ is the Jordan block associated with $(s-\lambda_i)^{\tau_i}, A_C = \text{diag}\ \{H_j : j = \mu_1, ..., \mu_l\}, H_j$ is the $j \times j$ standard nilpotent matrix and $B_C = \text{bl-diag}\{\underline{w}_j, j = \mu_1, ...\mu_l\}, \underline{w}_j = [0, ...0, 1]^t \in R^j$.

□

$S(A_C, B_C)$ is the controllable subsystem and if $S(A,B)$ is controllable, then $sI - A_{ID}$ is not present in (3.18). Controllability indices and the structure and values of decoupling zeros are the only invariants under $\mathcal{H}_B^r$.

Remark (3.4): Controllability indices are essential for identification and study of control theory problems such as: Assignment of Jordan forms by state feedback [Ros.,1], structure and parametrisation of controllability subspaces [Won.,1] etc. It seems, that the most relevant for our present task is the value of the controllability index μ.

□

b) Coordinate transformations and output injection on $S(A,C)$

The results in the previous section have their duals for the Brunovsky orbit $\mathcal{E}_B(A,C)$, obtained from $S(A,C)$ under $\mathcal{H}_B^l$. The essence of the duality is that defined by transposition. The set of observability indices $\mathcal{I}_o$ and set of f.e.d. of

$P_o(s), \mathcal{D}_{OD}$, defining the structure of output decoupling zeros are complete invariants and the corresponding canonical form (A_B, B_B) by transposition [Mor.,1], [Tho.,1].

c) Coordinate transformations, state feedback and output injection on $S(A, B, C, D)$: Kronecker invariants and canonical form

For the $S(A, B, C, D)$ state space model with transfer function $G(s)$, (n: states, l: inputs, m; outputs) the action of the Kronecker group $\mathcal{H}_k$ on S produces an equivalence class $\mathcal{E}_k(A, B, C, D)$ which will be referred to as the Kronecker orbit of $S(A, B, C, D)$. The natural tool to represent $S(A, B, C, D)$ is the system matrix pencil $P(s)((n+m) \times (n+l))$

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \tag{3.19}$$

We assume $\rho(P(s)) = r, \rho(G(s)) = \rho$ (over $R(s)$) and $\rho([C, D]) = m, \rho([B^t, D^t]) = l$.

The pencil $P(s)$ is characterised by Kronecker invariants [Gan.,1], which are defined below:

Definition (3.3): For the system $S(A, B, C, D)$, described by $P(s)$, we define:

i) $\mathcal{D}_z = \{(s - z_i)^{\tau_i}, i \in \tilde{\Pi}\}$ the set of f.e.d., which define the finite zero structure of $S(A, B, C, D)$; the number $n_f = \sum_{i=1}^{\Pi} \tau_i$ is called the finite zero order of the system.

ii) $\mathcal{D}_\infty = \{\hat{s}^{q_i} : 1 = q_1 = \cdots = q_\delta < q_{\delta+1} \leq \cdots \leq q_\sigma\}$ the set of ∞-ed, which define the infinite zero structure of $S(A, B, C, D)$; ∞-ed of the $\hat{s}$ type are called linear infinite zero divisors (lizd) and those of the $\hat{s}^q, q > 1$, are called nonlinear infinite zero divisor (n-lizd). The number $n_\infty = \sum_{i=1}^{\sigma}(q_i - 1)$ is defined as the infinite zero order of the system.

iii) $\mathcal{I}_r = \{\epsilon_i : 0 < \epsilon_1 \leq - \cdots \leq \epsilon_p\}, \mathcal{I}_l = \{\eta_i : 0 < \eta_1 \leq \cdots \leq \eta_t\}$ are the sets of c.m.i., r.m.i. respectively of $P(s)$ and they are called the right-, left-indices of the system. The numbers $n_r = \sum_{i=1}^{p} \epsilon_i, , n_l = \sum_{i=1}^{t} \eta_i$ are called the right-, left-order respectively of the system.

□

Remark (3.5): The finite and infinite zero structure is characterised in physical terms by frequency transmission problems. The right, left indices are associated with the blocking of families of signals which are not necessarily of the simple exponential type [Kar. & Kou.,1].

□

The importance of the $\mathcal{D}_z, \mathcal{D}_\infty, \mathcal{I}_r, \mathcal{I}_l$ sets defined on $S(A, B, C, D)$ is described below [Mor.,1], [Tho.,1], [Kar. & MacB.,1].

Result (3.7): For the Kronecker orbit $\mathcal{E}_k(A, B, C, D)$ the following hold true:

i) The set $\{\mathcal{D}_z; \mathcal{C}_\infty; \mathcal{I}_r; \mathcal{I}_l\}$ defined by $S(A, B, C, D)$ is a complete and independent invariant.

ii) There is a uniquely defined canonical form, the Kronecker canonical form $S(A_k, B_k, C_k, D_k)$, which in pencil form is described by

$$P_k = \begin{bmatrix} sI - A_k & -B_k \\ -C_k & -D_k \end{bmatrix} \tag{3.20}$$

$$= \left[\begin{array}{cccc|ccc} sI - A_\epsilon & 0 & 0 & 0 & -B_\epsilon & 0 & 0 \\ 0 & sI - A_\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sI - A_\infty & 0 & 0 & -B_\infty & 0 \\ 0 & 0 & 0 & sI - A_f & 0 & 0 & 0 \\ \hline 0 & -C_\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C_\infty & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_\delta \end{array}\right]$$

where

$$\begin{aligned} A_\epsilon &= \text{diag}\ \{A_j : j = \epsilon_1, ..., \epsilon_p\}, n_r \times n_r, A_\eta = \text{diag}\ \{A_j : j = \eta_1, ..., \eta_t\}, n_l \times n_l \\ A_\infty &= \text{diag}\ \{A_j : j = f_1, ..., f_{\sigma-\delta}, f_i = q_i - 1, i = \delta + 1, ..., \delta\}, n_\infty \times n_\infty \\ A_f &= \text{diag}\ \{J_{\tau_i}(z_i) : i \in \tilde{\Pi}\}, n_f \times n_f \end{aligned}$$

where $J_{\tau_i}(z_i)$ are Jordan blocks characterising $(s - z_i)^{\tau_i}$, $A_j = H_j$, is the $j \times j$ standard nilpotent matrix and

$$\begin{aligned} C_\eta &= \text{bl.diag}\ \{\underline{v}_j^t : j = \eta_1, ..., \eta_t\}, C_\eta \in R^{t \times n_l} \\ C_\infty &= \text{bl.diag}\ \{\underline{v}_j^t : j = f_1, ..., f_{\sigma-\delta}\}, C_\infty \in R^{(\sigma-\delta) \times n_\infty} \\ B_\infty &= \text{bl.diag}\ \{\underline{w}_j : j = f_1, ..., f_{\sigma-\delta}\}, B_\infty \in R^{n_\infty \times (\sigma-\delta)} \\ B_\epsilon &= \text{bl.diag}\ \{\underline{w}_j : j = \epsilon_1, ..., \epsilon_p\}, B_\epsilon \in R^{n_r \times p} \end{aligned}$$

where $\underline{v}_j^t = [1, 0, ..., 0]$, $1 \times j$ and $\underline{w}_j = [0, ..., 0, 1]^t$ is a $j \times 1$ vector.

iii) If r, ρ are the ranks of $P(s), G(s)$ respectively, then the following relationships hold true amongst the numbers of the invariants

a) $r = n + \rho, p = l - \rho, t = m - \rho, n = n_f + n_\infty + n_r + n_l$

b) $\sigma = \rho$ and $\delta = \rho(D)$

c) There are zero c.m.i., zero r.m.i., iff $[B^t, D^t]^t, [C, D]$ are rank deficient respectively.

iv) The transfer function matrix of $S(A_k, B_k, C_k, D_k)$ is

$$G_k(s) = C_k(sI - A_k)^{-1}B_k + D_k = \begin{bmatrix} M_\infty^*(s) & 0_{\sigma,p} \\ 0_{t,\sigma} & 0_{t,p} \end{bmatrix} \tag{3.21}$$

$$M_\infty^*(s) = \text{ diag } \{s^{1-q_1}, ..., s^{1-q_\mu}\} \tag{3.22}$$

where $G_k(s)$ is the Smith form at $s = \infty$ of $G(s)$ [Var. Lim. & Kar.,1].

□

The above summary of results demonstrates the structure of state space models under the most general types of transformations that may be applied on state space models. The results may be simplified for strictly proper systems in the obvious manner. The importance of the result is that it establishes the numbers and relationships between different invariants which enter to the solvability condition of many control problems.

Remark (3.6): The number of divisors at infinity of $P(s)$ is equal to the rank of $G(s)$. There exists a number of linear divisors at infinity equal to the rank of D; for strictly proper systems, all divisors at infinity are nonlinear, ie. $q_i \geq 2$. The orders of infinite zeros are defined by $f_i = q_i - 1$, where q_i are the degrees of nonlinear divisors at infinity. The f_i define the generic asymptotic root locus pattern and terminal Nyquist phases. If $\rho(D) = \rho(G(s))$, then $G(s)$ has no infinite zeros, or equivalently all q_i's are equal to 1. For strictly proper, square systems with $\rho = m = l$, all orders of infinite zeros of $G(s)$, f_i, are equal to 1 iff $\rho(CB) = m = l$, higher order of infinite zeros emerge $\rho(CB) < m = l$.

□

Remark (3.7): The Kronecker form $S(A_k, B_k, C_k, D_k)$ is maximally uncontrollable and unobservable and the dimension of the minimal system is defined by the infinite zero order. State feedback and output injection are equivalent to post-,pre-multiplication of transfer function by $R_{pr}(s)$-unimodular matrices; the special element of $\mathcal{H}_k$ that reduce S to its Kronecker form, is equivalent to a pair of $R_{pr}(s)$-unimodular matrices which reduce $G(s)$ to its Smith form at $s = \infty$ of $G(s)$.

□

Remark (3.8): For right regular systems $(\rho = l), n_r = 0$ (no right indices) and for left regular systems $(\rho = m), n_l = 0$, (no left indices). For left-right regular systems $(\rho = m = l)$ (square nondegenerate systems), $n_r = n_l = 0$ and

$$n_f + n_\infty = n \tag{3.23}$$

which shows that total number of finite and infinite zeros is equal to the dimension of the state space. For such systems, the total number of finite zeros satisfies the conditions:

i) $D \neq 0 : n_f \leq n$ and $n_f = n$ iff $\rho(D) = m = l$

ii) $D = 0 : n_f \leq n - m = n - l$ and equality holds, iff $\rho(CB) = m = l$.

□

For strictly proper square systems, the number $n - m = n - l$ defines an upper bound on the total number of finite zeros. The right and left indices are related to the synthesis problems such as squaring down, model matching etc. Their relationships to transfer function invariants will be discussed later on.

Remark (3.9): The infinite zeros of $P(s)$ and $Z(s)$ (zero pencil) are the same, if $q_i', i \in \tilde{\tau}$ are the degrees of the divisors at $s = \infty$, with $q_i' \geq 3$ of $P(s)$, then the degrees of restricted zero divisors of $Z(s)$ are $q_i' - 2, i \in \tilde{\tau}$.

□

d) Invariants under constant output feedback

The problem of finding invariants and canonical forms under constant output feedback is a problem related to output feedback pole assignment and stabilisation. Concerning the problem of finding invariants and canonical forms under this feedback, there has been only very recently some progress [Yan.,1],[Fur. & Hel.,1]. The types of invariants, which are defined are in terms of Bezoutians and, at the moment, they are only in a rather abstract form which cannot be easily translated on state space parameters. This topic will not be examined here, since the results are not in a form that can be exploited for the present task.

3.4 Transfer function invariants

With a transfer function matrix $G(s) \in R^{m\times l}(s)$ we may always associate the $\mathcal{K}$-coprime MFDs, $G = A_1^{-1}B_1 = B_2A_2^{-1}$, where $\mathcal{K}$ is $R[s], R_{pr}(s)$ of $R_p(s)$ and with them we associate the left-,right-MFD matrices [Kar., & Gia.,1]

$$T_l = [A_1, B_1] \in \mathcal{K}^{m\times(m+l)}, T_r = \begin{bmatrix} B_2 \\ A_2 \end{bmatrix} \in \mathcal{K}^{(m+l)\times l} \tag{3.24}$$

on G, T_l, T_r we may apply different types of transformations which are based on the ring $\mathcal{K}$ which is used to describe fractionally a rational function. These transformations are defined by the $\mathcal{K}$-unimodular matrices and the basic tools are those defined by the Smith, Smith-McMillan forms, as well as those Hermite, Hermite McMillan forms. The results are summarised next and their significance for the structure of linear systems is also discussed. Throughout this section it is assumed that $\rho(G) = r \leq \min(m, l)$ and that $\mathcal{K}$ is any Euclidian ring such that $R(s)$ may be expressed as the field of the fractions of $\mathcal{K}$. For control theory applications $\mathcal{K}$ is $R[s], R_{pr}(s), R_p(s)$ or $R_o(s)$ (the rational function which have no poles at $s = 0$).

a) Smith McMillan forms over $\mathcal{K}$

If L, R are $\mathcal{K}$-unimodular matrices ($L \in U(m, \mathcal{K}), R \in U(l, \mathcal{K})$), then the natural equivalence $\mathcal{E}_{\mathcal{K}}(G)$ is defined by pre-, post multiplication of G by L, R and LGR is the general element of the orbit (equivalence class) $\mathcal{E}_{\mathcal{K}}(G)$, If $G \notin \mathcal{K}^{m\times l}$, a canonical form and invariants is defined by the Smith-McMillan form over $\mathcal{K}$ [Kai,1], [Ros.,1], [Vard. & Kar.,1], [Var. Lim. & Kar.,1].

Result (3.8): The orbit $\mathcal{E}_{\mathcal{K}}(G)$ is characterised by a canonical form $M_G^{\mathcal{K}}$, the Smith-McMillan form over $\mathcal{K}$, where

$$M_G^{\mathcal{K}} = \begin{bmatrix} M_G^{*\mathcal{K}} & 0 \\ \underbrace{0}_{r} & \underbrace{0}_{l-r} \end{bmatrix} \begin{matrix} \} \ r \\ \} \ m-r \end{matrix} \tag{3.25}$$

$$M_G^{*\mathcal{K}} = \text{diag}\ \{\epsilon_i/\psi_i, i \in \tilde{r}\} \tag{3.26}$$

where $\{\epsilon_i, \psi_i\}$ are $\mathcal{K}$-coprime element of $\mathcal{K}$, uniquely defined (modulo $\mathcal{K}$-units) and satisfy the divisibility properties

$$\epsilon_1/\epsilon_2/\ldots/\epsilon_r, \psi_r/\psi_{r-1}/\ldots/\psi_1 \tag{3.27}$$

The $\epsilon_i, \psi_i, i \in \tilde{r}$ are the elementary $\mathcal{K}$-zero, pole-functions of G and together with r define a complete and independent set of invariants under $\mathcal{E}_{\mathcal{K}}$-equivalence. □

For the various rings $\mathcal{K}$ of interest we have:

Remark (3.10): The Smith-McMillan form over the different rings $\mathcal{K}$ reveal the following information about the system:

i) For $\mathcal{K} = R[s]$ indicates the zero, pole structure of G over C. The polynomials ϵ_i, ψ_i define the finite zeros, poles of G and $\partial(\Pi_{i=1}^{r}\psi_i)$ the finite McMillan degree δ_m of G. This canonical form does not reveal any information about the structure of G at $s = \infty$.

ii) For $\mathcal{K} = R_{\mathcal{P}}(s), \mathcal{P} = \Omega \cup \{\infty\}$ indicates the zero, pole structure of G over $\mathcal{P}$. The proper and Ω-stable functions ϵ_i, ψ_i define the zeros, poles of G. This canonical form does not reveal anything about the structure of G in the region Ω^c (the complement of Ω with respect to C).

iii) For $\mathcal{K} = R_{pr}(s)$ indicates the zero, pole structure of G at infinity only, but nothing about the structure of G over C. The ϵ_i, ψ_i are the proper rational functions of the type $(1/s)^q, q > 0$ indicating the orders of infinite zeros, poles of G.

iv) For $\mathcal{K} = R_o(s)$ indicates the zero, pole structure of G at $s = 0$ only but nothing about the structure of G over $C - \{0\}$, or $s = \infty$. The ϵ_i, ψ_i are polynomials of the type $s^p, p \geq 0$ indicating zero, pole type of G at $s = 0$.

□

Smith-McMillan forms reveal the basic pole, zero structure over different subsets of $C \cup \{\infty\}$; the standard rule for analysis is the form over $R[s]$, whereas that over $R_p(s)$ is essential for studies of stabilisation in the generalised Ω-sense. The Smith-McMillan forms over $R_{pr}(s)$, or $R_o(s)$ are local, since they reveal the structure at $\{\infty\}$, or $\{0\}$ respectively; The first is important for characterisation of properness and infinity zero structure, whereas the second is essential for the study of steady-state tracking, disturbance rejection. Local Smith-McMillan forms may be defined by simple tests from the elements of $G(s)$ [Var. Lim. & Kar.,1].

b) Smith form over $\mathcal{K}$

If $G \in \mathcal{K}^{m\times l}$, then the Smith-McMillan form is reduced to the $\mathcal{K}$-Smith form, which is defined as in (3.25-3.27) with the only difference that all the ψ_i's are 1; that is G has no poles over $\mathcal{K}$, but only possibly zeros. Smith forms are essential tools for $\mathcal{K}$-coprimeness tests and thus they are involved in the characterisation of irreducible $\mathcal{K}$-MFDs, as well as solvability of matrix equations over $\mathcal{K}$.

c) Rational vector spaces and transfer function matrix invariants

Under $\mathcal{E}_{\mathcal{K}}$ type of equivalence the column, row spaces of a transfer function change. A richer set of invariants, which is directly related to pre-, post-compensation of transfer functions, is defined under left-, or right $\mathcal{K}$-unimodular equivalence. If $G \in R^{m\times l}, L \in U(m,\mathcal{K}), R \in U(l,\mathcal{K})$ then G and $G' = GR$ are $\mathcal{K}$-right equivalent and is denoted by $G\mathcal{E}^r_{\mathcal{K}}G'$, and G and $G'' = LG$ are $\mathcal{K}$-left equivalent and is denoted by $G\mathcal{E}^l_{\mathcal{K}}G''$; the corresponding equivalence classes, orbits are denoted by $\mathcal{E}^r_{\mathcal{K}}(G), \mathcal{E}^l_{\mathcal{K}}(G)$.

Definition (3.4): Let $G \in R^{m\times l}(s), \rho(G) = r \leq \min(m,l)$ and let $G = A_1^{-1}B_1 = B_2A_2^{-1}$ be $\mathcal{K}$-coprime left, right MFDs ($\mathcal{K} = R[s], R_p(s), R_{pr}(s)$). With the given G and for T_l, T_r defined as in (3.24) we define:

i) $\mathcal{X}_{c,G} = col.sp_{R(s)}\{G\}, \mathcal{X}_{r,G} = row.sp_{R(s)}\{G\}$ as the $R(s)$-column, row-vector space of G respectively and $\mathcal{N}_{r,G} = \mathcal{N}_r\{G\}, \mathcal{N}_{l,G} = \mathcal{N}_l\{G\}$ as the $R(s)$-right-, left-null space of G correspondingly.

ii) $\mathcal{Y}_{l,G} = row.sp_{R(s)}\{T_l\}, \mathcal{Y}_{l,G} = col.sp_{R(s)}\{T_r\}$ as the $R(s)$-composite left-, right-space of G respectively, where T_l, T_r defined by (3.24).

iii) $\mathcal{M}^{\mathcal{K}}_{c,G} = col.sp_{\mathcal{K}}\{B_2\}, \mathcal{M}^{\mathcal{K}}_{r,G} = row.sp_{\mathcal{K}}\{B_1\}$ as the $\mathcal{K}$-column-, row- module of G respectively and $\mathcal{T}^{\mathcal{K}}_{l,G} = row.sp_{\mathcal{K}}\{T_l\}, \mathcal{T}^{\mathcal{K}}_{r,G} = col.sp_{\mathcal{K}}\{T_r\}$ as the $\mathcal{K}$-composite-left-, right-module of G correspondingly.

iv) $\mathcal{M}^{*\mathcal{K}}_{c,G}$ is the set of all $\underline{x} \in \mathcal{K}^m$ vectors which are in $\mathcal{X}_{c,G}$ and $\mathcal{M}^{*\mathcal{K}}_{r,G}$ is the set of all $\underline{y} \in \mathcal{K}^l$ vectors such that $\underline{y}^t \in \mathcal{X}_{r,G}$.

□

For any rational transfer function matrix the following general invariants may be established [Kai,1], [Vard. & Kar.,1], [Ros.,1] etc.

Result (3.9): For all rings $\mathcal{K}$: $R[s], R_p(s), R_{pr}(s)$ the following properties hold true:

i) $\mathcal{X}_{c,G}, \mathcal{N}_{l,G}$ are invariants of $\mathcal{E}^r_{\mathcal{K}}(G)$ and $\mathcal{X}_{r,G}, \mathcal{N}_{r,G}$ are invariants of $\mathcal{E}^l_{\mathcal{K}}(G)$; these properties also hold true for $\mathcal{K} = R(s)$.

ii) For all $\mathcal{K}$-MFDs, not necessarily coprime, $\mathcal{Y}_{l,G}$ is invariant for the left MFDs and $\mathcal{Y}_{r,G}$ is invariant for the right MFDs.

iii) $\mathcal{M}^{\mathcal{K}}_{l,G}$ is invariant of $\mathcal{E}^r_{\mathcal{K}}(G)$ and $\mathcal{M}^{\mathcal{K}}_{r,G}$ is invariant of $\mathcal{E}^l_{\mathcal{K}}(G)$.

iv) $\mathcal{T}^{\mathcal{K}}_{l,G}, \mathcal{T}^{\mathcal{K}}_{r,G}$ are complete invariants for all left, right $\mathcal{K}$-coprime MFDs respectively.

v) $\mathcal{M}^{*\mathcal{K}}_{c,G}$ are maximal $\mathcal{K}$-modules which have the following properties

a) If $\rho(G) = l, G$ may be factorised as

$$G = B_r Z_r D_r^{-1}, B_r \in \mathcal{K}^{m\times l}, Z_r, D_r \in \mathcal{K}^{l\times l} \tag{3.28}$$

where $(B_r Z_r, D_r)$ is $\mathcal{K}$-right coprime, B_r is $\mathcal{K}$-right-irreducible and $col.sp_{\mathcal{K}}(B_r) = \mathcal{M}^{*\mathcal{K}}_{c,G}$; furthermore, $\mathcal{M}^{*\mathcal{K}}_{c,G}$ is invariant for any $GQ, Q \in U(l, R(s))$.

b) If $\rho(G) = m, G$ may be factorised as

$$G = D_r^{-1} Z_l B_l^{-1}, B_l \in \mathcal{K}^{m\times l}, Z_l, D_l \in \mathcal{K}^{m\times m} \tag{3.29}$$

where $(B_l, Z_l D_l)$ is $\mathcal{K}$-left coprime, B_l is $\mathcal{K}$-left-irreducible and $row.sp_{\mathcal{K}}(B_l) = \mathcal{M}^{*\mathcal{K}}_{r,G}$; furthermore, $\mathcal{M}^{*\mathcal{K}}_{r,G}$ is invariant for any $PG, P \in U(m, R(s))$.

□

The above summary of results clearly indicates that the theory of transfer function invariants is related to the theory of invariants of rational vector spaces and $\mathcal{K}$-modules contained in them [For.,1], [Kai,1], [Vard. & Kar.,2]. This theory is quite rich and becomes rather concrete, in terms of the theory of minimal bases [For.,1], [Vard. & Kar.,2], or equivalently by using tools from exterior algebra [Kar., & Gia.,3]. The types of invariants defined for each of the $\mathcal{K}$-rings are essentially the same; only the interpretation depends on the nature of $\mathcal{K}$. Thus, we may consider throughout the rest of the section the standard case $\mathcal{K} = R[s]$. Some useful interpretations of the mathematical result (3.9) are given next.

Remark (3.11):

i) Pre-, post-compensation of G by a square full rank rational compensator leaves invariant the rational vector spaces $\mathcal{X}_{r,G}, \mathcal{X}_{c,G}$ respectively; thus, $\mathcal{X}_{r,G}, \mathcal{X}_{c,G}$ are not spaces characterising a single transfer function, but a family of transfer functions.

ii) The rational vector spaces $\mathcal{Y}_{l,G}, \mathcal{Y}_{r,G}$ characterise all left-, right-MFDs of G and thus are common to all state space models that have a common transfer function; these spaces are "personal" space of G. The modules $\mathcal{T}_{l,G}, \mathcal{T}_{r,G}$ (defined for $\mathcal{K} = R[s]$) characterise all left-, right-$R[s]$-coprime MFDs and thus they are invariants of all minimal realisations of G.

iii) The modules $\mathcal{M}^{\mathcal{K}}_{c.G}, \mathcal{M}^{\mathcal{K}}_{r.G}$ for $\mathcal{K} = R_{pr}(s), R_p(s)$ define invariants under post-,pre-multiplication respectively by proper, proper and stable square rational transfer functions.

□

For a rational vector space $\mathcal{X}, \mathcal{X} \in R^n(s)$, with $\dim\mathcal{X} = p$, the theory of concrete invariants, based on the polynomial interpretation, has three alternative directions:

i) Minimal degree $R[s]$-bases

ii) $R[s]$-Hermite forms.

iii) Plucker matrices

and they are defined below for the general $\mathcal{X}$ and then specialised to the rational vector spaces with a transfer function G.

Definition (3.5): Let $X(s) \in R^{n\times p}[s]$ be a polynomial basis matrix of $\mathcal{X}$, i.e. $\rho(X) = p, col.sp_{R(s)}\{X\} = \mathcal{X}$, and let $X(s) = [..., \underline{x}_i(s), ...], \underline{x}_i(s) \in R^n[s], \partial[x_i(s)] = \delta_i, i \in \tilde{p}$.

i) [For.,1] $X(s)$ will be called an <u>$R[s]$-minimal basis</u> ($R[s]$-MB) if it is right irreducible (no finite zeros) and column reduced (full rank high column coefficient matrix). It is called an <u>ordered</u>-$R[s]$-MB if $\delta_i < \delta_{i+1}, i \in \tilde{p}$. The set $\mathcal{I}_X = \{\delta_i, i \in \tilde{p} : \delta_i < \delta_{i+1}\}$ is called the <u>Forney dynamical indices</u> (FDI) of $X(s)$ and $\delta_F = \sum_{i=1}^{p} \delta_i$ the <u>Forney dynamical order</u> (FDO) of $X(s)$.

ii) [Kar., & Gia.,2]. The polynomial multivector $\underline{g}(X) = \underline{x}_1(s) \wedge .. \wedge \underline{x}_p(s) = C_p(X) \in R[s]^\nu, \nu = \begin{pmatrix} n \\ p \end{pmatrix}$, is defined as an <u>$R[s]$-Grassman representative</u> ($R[s]$-GR) of $\mathcal{X}$. If $X(s)$ is right irreducible, then $\underline{g}(X)$ is called a <u>canonical</u> $R[s]$-GR. If $\underline{g}(X)$ is canonical and $\partial[\underline{g}(X)] = \delta$, then it may be expressed as

$$\underline{g}(X) = P_\delta \underline{e}_\delta(s), \underline{e}_\delta(s) = \begin{bmatrix} 1, & s, & \cdots, & s^\delta \end{bmatrix}^t, P_\delta \in R^{\nu \times (\delta+1)} \tag{3.30}$$

and P_δ is called a <u>Plucker matrix</u> (PM) of $\mathcal{X}$.

□

<u>Result (3.10)</u>: [For.,1] For any rational vector space $\mathcal{X}$ the following properties hold true:

i) All $R[s]$-MBs of $\mathcal{X}$ define the same $R[s]$-module $\mathcal{M}^*$, which is a maximal Noetherian module [Mar.,1].

ii) All ordered-$R[s]$-MB have the same set $\mathcal{I}_X$ of FDIs and thus $\mathcal{I}_X$ and δ_F are invariants of $\mathcal{X}$.

iii) There exist a uniquely defined $R[s]$-MB, the <u>echelon type basis</u>, the elements of which uniquely characterise $\mathcal{X}$.

iv) If $X(s) \in R(s)^{n \times p}$ is any rational basis of $\mathcal{X}$, then it may be factorised as

$$X(s) = N(s)Z(s)D(s)^{-1} \tag{3.31}$$

where $N(s)$ is an $R[s]$-MB, $Z(s), D(s)$ are $p \times p$ polynomial matrices defining the finite zeros, poles respectively of $X(s)$.

A dual statement of the result for row rational spaces is obvious. The above result may be expressed with respect to the $R_{pr}(s), R_p(s)$ rings [Vard. & Kar.,2]; the essential conceptual difference is that the minimal degree basis, then becomes a minimal McMillan degree basis. Minimal bases are essential tools in the study of

"Minimal design problems" that is problems where a minimal McMillan degree complexity solution are sought. Note that a general $R[s]$-MB does not define a complete invariant of $\mathcal{X}$; echelon type $R[s]$-MB define a complete, basis free invariant, but their construction is extremely elaborate (see [Kai,1]). For most of the applications the set $\mathcal{I}_X$, which characterises $\mathcal{X}$ and thus denoted by $\mathcal{I}_{\mathcal{X}}$, and δ_F are the essential invariants. In the study of determinantal assignment problem (DAP) (Pole, zero assignment) alternative forms of complete invariants are most suitable, the Plucker matrices [Kar., & Gia.,2].

For a transfer function matrix $G, G \in R_{pr}(s)^{m\times l}$, with $\rho(G) = \min(m, l)$ we have the rational vector spaces $\mathcal{X}_{c,G}, \mathcal{X}_{r,G}, \mathcal{N}_{r,G}, \mathcal{N}_{l,G}$ which are "not personal spaces" of G, as well as the "personal spaces" $\mathcal{Y}_{l,G}$ and $\mathcal{Y}_{r,G}$. The Forney dynamic indices and Forney order of these subspaces will be denoted by $\mathcal{I}(A), \delta_F(A)$, where A is the corresponding space. A number of properties of the above invariants are summarised below [Kai,1] etc.

Result (3.11): For the family of rational vector spaces associated with the transfer function matrix $G, G \in R_{pr}(s)^{m\times l}, \rho(G) = \min(m, l)$, we have the following properties:

i) $\mathcal{I}(\mathcal{Y}_{l,G})$ defines the observability indices and $\mathcal{I}(\mathcal{Y}_{r,G})$ the controllability indices of any realisation of G; Furthermore, $\delta_F(\mathcal{Y}_{r,G}) = \delta_F(\mathcal{Y}_{l,G}) = \delta_M(G)$, the McMillan degree of G.

ii) If $m > l$, then $\mathcal{N}_{r,G} = 0, \delta_F(\mathcal{X}_{c,G}) = \delta_F(\mathcal{N}_{l,G})$, and $\mathcal{I}(\mathcal{X}_{r,G}) = \{0, ..., 0\}$, that is the identity matrix I_l, is an $R[s]$-MB of $\mathcal{X}_{r,G}$.

iii) If $m < l$, then $\mathcal{N}_{l,G} = 0, \delta_F(\mathcal{X}_{r,G}) = \delta_F(\mathcal{N}_{r,G})$ and $\mathcal{I}(\mathcal{X}_{c,G}) = \{0, ..., 0\}$ that is the identity matrix I_m, is an $R[s]$-MB of $\mathcal{X}_{c,G}$.

iv) If $m = l, \mathcal{N}_{l,G} = 0, \mathcal{N}_{r,G} = 0, \mathcal{I}(\mathcal{X}_{c,G}) = \{0, ..., 0\}, \mathcal{I}(\mathcal{X}_{r,G}) = \{0, ..., 0\}$, that is the identity matrix I_m is an $R[s]$-MB of $\mathcal{X}_{c,G}, \mathcal{X}_{r,G}$.

□

The nontrivial set $\mathcal{I}(\mathcal{X}_{c,G})$, when $m \geq l$, or $\mathcal{I}(\mathcal{X}_{r,G})$, when $m \leq l$, will be referred to as external dynamical indices (EDI) of G and are invariant under square full rank post-, pre-compensation. These indices are important in the study of compensation, as well as squaring down of systems.

An alternative, complete set of transfer function invariants (to that defined by the echelon type bases), which is useful in the study of DAP problem is defined by the following result [Kar., & Gia.,2].

Result (3.12): For and rational vector space $\mathcal{X}$ with X_1, X_2 basis matrices the following properties hold true:

i) If $\underline{g}(X_1), \underline{g}(X_2)$ are any two $R[s]$-GRs, of $\mathcal{X}$, then $\underline{g}(X_1) = \underline{g}(X_2)t$ where $t \in R(s)$.

ii) A canonical $R[s]$-GR is coprime (has no zeros) and uniquely characterises $\mathcal{X}$ (modulo $c \in R, c \neq 0$); furthermore, $\partial[\underline{g}(X)] = \delta_F(\mathcal{X})$.

iii) A Plucker matrix is a complete invariant (modulo $c \in R, c \neq 0$) of $\mathcal{X}$.

□

A canonical-$R[s]$-GR, or equivalently a Plucker matrix is a complete invariant of $\mathcal{X}$ and in this sense is equivalent to the echelon type minimal bases of $\mathcal{X}$. The relationship between the parameters in the echelon form and the coefficient in the Plucker matrix is not known yet and this is an open question. In fact, all least degree bases give rise to Plucker matrices differing only by a scalar [Kar., & Gia.,3]. For the basic subspaces associated with G, the corresponding Plucker matrices are defined below [Kar., & Gia.,2]:

i) If $m > l, P_c(G)$ is the $\begin{pmatrix} m \\ l \end{pmatrix} \times (\delta_{F,C} + 1)$ is the Plucker matrix of $\mathcal{X}_{c,G}$, where $\delta_{F,C} = \delta_F(\mathcal{X}_{c,G})$; the Plucker matrix of $\mathcal{X}_{r,G}$ is $P_r(G) = 1$.

ii) If $m < l, P_r(G)$ is the $(\delta_{F,r} + 1) \times \begin{pmatrix} l \\ m \end{pmatrix}$ is the Plucker matrix of $\mathcal{X}_{r,G}$, where $\delta_{F,r} = \delta_F(\mathcal{X}_{r,G})$; the Plucker matrix of $\mathcal{X}_{c.G}$ is $P_c(G) = 1$.

iii) If $m = l, P_r(G) = 1, P_c(G) = 1$ are the Plucker matrices of $\mathcal{X}_{r,G}, \mathcal{X}_{c,G}$.

iv) $P(T_l), P(T_r)$ are the $(n+1) \times \begin{pmatrix} m+l \\ m \end{pmatrix}, \begin{pmatrix} m+l \\ l \end{pmatrix} \times (n+1)$

Plucker matrices of $\mathcal{Y}_{l,G}, \mathcal{Y}_{r,G}$ respectively, where $n = \delta_M(G)$.

Plucker matrices associated with the basic matrix pencils may also be defined, as it has been shown in previous section. The matrices $P_c(G), P_l(G)$ are essential in the study of zero assignment problems by "squaring down", whereas $P(T_l), P(T_r)$ are crucial in the study of the pole assignment by constant or dynamic output feedback. The Plucker matrices have special properties, which are not fully explored; for control system design, their significance lies in that their column, or row space defines the family of pole, zero polynomials coefficient vectors. Their significance will become clear in later chapters.

d) Hermite, Hermite-McMillan forms and invariants

With a transfer function matrix we associate rational vector spaces, as well as $\mathcal{K}$-modules. The notions of $\mathcal{K}$-right-, left-equivalence defined before is intimately related to compensation theory under special types of compensator; thus, if $\mathcal{K} = R_{pr}(s)$, or $R_p(s)$, then the corresponding equivalence classes are systems obtained under proper, proper and stable pre-, or post-compensation. The theory of right-, left-equivalence produces types of invariants based on the modules contained in a rational vector space. We distinguish the following cases:

i) Transfer functions (matrices) with elements from a given ring $\mathcal{K}$.

ii) Transfer functions (matrices) with elements rational functions, i.e. fractions of elements of $\mathcal{K}$.

The first case is related to the theory of Hermite forms, whereas the second to the case of Hermite-McMillan forms. In the following, by $\mathcal{K}$ we mean either of the cases $R[s], R_{pr}(s), R_p(s)$.

i) Hermite forms: We consider matrices $G \in \mathcal{K}^{m\times l}$, assume $\rho(G) = m (m \leq l)$ and consider the case of $\mathcal{K}$-left equivalence, $\mathcal{E}^l_{\mathcal{K}}$. The case of $\mathcal{K}$-right equivalence as well as the case where $\rho(G) < \min(m, l)$ may be found in references [Mar.,1].

Result (3.13): For a matrix G with the above properties there exists $L \in U(m, l)$ such that

$$LG = H_G^{l,\mathcal{K}} = \begin{bmatrix} 0 & \cdots & 0 & \times & \cdots & \times & \cdots & \times & \cdots & \times \\ 0 & \cdots & 0 & 0 & \cdots & \times & \cdots & \times & \cdots & \times \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \times & \cdots & \times \end{bmatrix} \quad (3.32)$$

where $H_G^{l,\mathcal{K}}$ is called the $\mathcal{K}$-Hermite row form of G and its elements associated with the ρ_i rows $i \in \widetilde{m}$ and γ_j columns, $j \in \widetilde{l}$ satisfy the conditions ($\partial[\cdot]$ denote the degree in $\mathcal{K}$)P

a) $i \in \widetilde{m}$, the ρ_i row has a leading nonzero $\mathcal{K}$-monic element h_{in_i} (leading entry) such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m \leq n$.

b) $i \in \widetilde{m}$, then

i) if $h_{in_i} = 1, h_{jn_i} = 0, j < i$.

ii) if $h_{in_i} \neq 1, \partial[h_{jn_i}] < \partial[h_{in_i}], j < i$ such that $h_{jn_i} \neq 0$

c) $\forall \gamma_j$ row s.t. $j < n_1, \gamma_j$ is zero $\forall \gamma_j$ s.t. $n_i < j < n_{i+1}$ with $i \in \widetilde{m-1}$, then the last $m-i$ entries of γ_j are zero.

$\square$

The Hermite form $H_G^{l,\mathcal{K}}$ is unique (modulo diagonal scaling by units) and its elements define a complete set of invariants of $\mathcal{E}_{\mathcal{K}}^l(G)$.

A similar result may be stated for $\mathcal{K}$-right equivalence and the corresponding <u>$\mathcal{K}$-Hermite column form</u> of G is denoted by $H_G^{r,\mathcal{K}}$. The set of indices $J = \{n_i, i \in \widetilde{m}\}$ are defined as <u>Hermite indices</u> (row, or column) and they are also invariants. The elements of $H_G^{r,\mathcal{K}}, H_G^{l,\mathcal{K}}$ uniquely define the column, row modules of G and thus may also be used to provide invariants equivalent to those of the echelon type basis, if G is right, left irreducible. For $\mathcal{K} = R_{pr}(s)$ the corresponding Hermite forms are related to the "system interactor" [Wol.,1], which in turn defines the decouplability properties of the system.

ii) <u>Hermite McMillan forms</u>: For a general rational matrix $G \in R(s)^{m \times l}$ of full rank we may define canonical forms under $\mathcal{K}$-left, right-equivalence as follows: Let every element of G be expressed as coprime fraction of elements of $\mathcal{K}$ and let d be the least common multiple of the denominators of the elements of G. Then we may write

$$G = \frac{1}{d} N, N \in \mathcal{K}^{m \times l} \tag{3.33}$$

If $H_N^{l,\mathcal{K}}$ is the $\mathcal{K}$-row-Hermite form of N we may write

$$H_N^{l,k} = LN, L \in U(m, \mathcal{K}) \tag{3.34}$$

and thus

$$H_G^{l,k} = \frac{1}{d} H_N^{l,k} = LG \tag{3.35}$$

is defined as the <u>$\mathcal{K}$-row-Hermite-McMillan form</u> of G, where in $H_G^{l,\mathcal{K}}$ all possible numerator-denominator cancellations are assumed to have been carried out.

Result (3.14): The $\mathcal{K}$-row-Hermite-McMillan form of G, $H_G^{l,\mathcal{K}}$ is a complete invariant of $\mathcal{E}_{\mathcal{K}}^l(G)$.

□

Note that the structure of $H_G^{l,\mathcal{K}}$ is similar to that of $\mathcal{K}$-row Hermite form, i.e. "upper staircase", but its elements are rational functions. The corresponding structure and result for the $\mathcal{K}$-column-Hermite-McMillan form is similar. For $\mathcal{K} = R_{pr}(s)$, or $R_p(s)$, the corresponding Hermite-McMillan forms are essential in defining the limits of pre-, post-compensation under proper, or proper and stable compensators. Questions such as "what is nature of simplest possible Nyquist diagrams that may be obtained under proper or proper and stable compensators", may be answered in terms of the invariants of such forms. However, the potential of such forms for control synthesis/design has not been fully explored. Some of the invariants of transfer functions under left, right $\mathcal{K}$-equivalence are summarised below. [Vard. & Kar.,2].

Result (3.15): Let $G \in R_{pr}(s)^{m\times l}$ and assume that $\rho(G) = l(m \geq l)$.

i) The set of Forney dynamical indices $\mathcal{I}(\mathcal{X}_{c,G})$ and Forney order δ_F of $\mathcal{X}_{c,G}$ are invariants of $\mathcal{E}_{\mathcal{K}}^r(G)$ for all $\mathcal{K} : R[s], R_{pr}(s), R_p(s)$,

ii) The set of finite zeros and poles of G, together with their corresponding multiplicities are invariants of $\mathcal{E}_{\mathcal{K}}^r(G)$ for $\mathcal{K} = R[s]$; the infinite zeros, poles are not necessarily invariant under this equivalence.

iii) The set of infinite zeros and poles of G, together with their corresponding multiplicities are invariants of $\mathcal{E}_{\mathcal{K}}^r(G)$ for $\mathcal{K} = R_{pr}(s)$; the finite zeros, poles are not necessarily invariant under this equivalence.

iv) The set of zeros and poles of G, together with their corresponding multiplicities, in $\mathcal{P} = \Omega \cup \{\infty\}$ are invariants of $\mathcal{E}_{\mathcal{K}}^r(G)$ for $\mathcal{K} = R_p(s)$; the poles and zeros of G in Ω^c (the complement of Ω with respect to C) are not necessarily invariant under this equivalence.

v) If $\delta_M, \delta_F, z_\infty, z_F$ are the McMillan degree, Forney order of $\mathcal{X}_{c,G}$, total number of infinite, finite zeros of G, then

$$\delta_M = z_\infty + z_F + \delta_F \tag{3.36}$$

□

Parts i) to iv) of the above result may also be stated for a general $G(s)$ (not necessarily proper) since they are based on the properties of Smith-McMillan form in the region $P = \Omega \cup \{\infty\}$ [Vard. & Kar.,1].

The last relationship indicates that under all types of compensation which preserve δ_F, the difference between McMillan degree and total number of zeros remains constant. Since for square systems $\delta_F = 0$, (3.36) also indicates that for square systems, the McMillan degree is equal to the total number of zeros. The Forney order plays a crucial role under squaring down [Kar., & Gia.,1], since it indicates the total number of newly created zeros under squaring down. A result similar to that stated for right equivalence, may be stated for left equivalence.

3.5 Conclusions

The aim of this chapter was to survey the fundamental linear system invariants which emerge as tools for control system design. The functional relationships between system model parameters and invariants are not always simple and explicit. This imposes severe difficulties in the effort to develop systematic procedures for assigning values for all invariants. The material in this chapter provides the background on which some of the following chapters will build upon.

Chapter 4

GENERICITY, GENERIC VALUES OF STATE SPACE AND TRANSFER FUNCTION MATRIX INVARIANTS

4.1 Introduction

The definition of the set of invariants of a singular pencils $sF - G \in R^{m \times n}[s]$ under strict equivalence (S.E.) relies on the algebraic notions of Smith form and minimal bases [For.,1]. A related topic, that is the study of properties of a whole family of models having fixed certain fundamental parameters (such as number of inputs, states, outputs, McMillan degree), but with the rest of the parameters taking generic values, is examined here. In this chapter we examine the generic values and properties of state space and transfer function matrix invariants.

For the generic singular pencil $sF - G$, the right characteristic sequence (r.c.s.) of (F, G) [Kar. & Kal.1], $C_r(F, G)$ is completely defined by the generic set $\mathcal{I}_c(F, G)$ and it satisfies the Arithmetic Progression Property (A.P.P.) for all indices apart from a finite number of them [Kar. & Kal.1], which are referred to as singular points. It is due to this property that the characteristic sequence $C_r(F, G)$ is referred to as a Piecewise Arithmetic Progression Sequence. In this chapter, it will be shown that the generic set of column minimal indices (c.m.i.), $\mathcal{I}_c(F, G)$, and the row minimal indices (r.m.i.), $\mathcal{I}_r(F,G)$, may be deduced from the properties of a generic piecewise Arithmetic Progression Sequence defined on the ordered pair (F,G). The singular points of $C_r(F,G)$ define the distinct values of generic (c.m.i.), whereas the gaps at the singular points define the multiplicity of the corresponding generic (c.m.i.). The number theoretic characterisation of $\mathcal{I}_c(F,G), \mathcal{I}_r(F,G)$ also provides a computational procedure, which is independent from the minimal bases algebraic definition and it is based on rank test of Toeplitz matrices defined on (F,G). The results on generic properties of matrix pencils are then used to characterise the generic values of state space invariants. Finally the characterisation of generic values and properties of transfer function matrix invariants is considered.

4.2 Basic definition on Genericity [Won.,1]

Let $A, B, ...$ be matrices with elements in R and suppose $\Pi(A, B, ...)$ is some property which may be asserted about them. In applications where $A, B, ...$ represent data of a physical problem, it is often important to know various topological features of Π. For instance, if Π is true at a nominal parameter set $P = (A_o, B_o, ...)$ it may be desirable or natural that Π be true at points P in a neighbourhood of P_o, corresponding to small derivations of the parameters from their nominal values.

Most of the properties of interest to us will turn out to hold true for all sets of parameter values except possibly those which correspond to points P which lie on some algebraic hypersurface in a suitable parameter space, and which are thus, in an intuitive sense, atypical. To make this idea precise, we borrow some terminology from algebraic geometry. Consider a point which lies on some algebraic hypersurface in a suitable parameter space $P = (P_1, ..., P_N) \in R^N$ and consider polynomials $\phi(\lambda_1, ..., \lambda_n)$ with coefficients in R. A variety $V \subset R^N$ is defined to be the locus of common zeros of a finite number of polynomials $\phi_1, ..., \phi_k$:

$$V = \{P : \phi_i(P_1, ..., P_N) = 0, i \in k\}$$

V is <u>proper</u> if $V \neq R^N$ and <u>non-trivial</u> if $V \neq \emptyset$. A property Π is mearly a function $\Pi : R^N \to \{0,1\}$, where $\Pi(P) = 1$ (or 0) means Π hold (or fails) at P. Let V be a proper variety, we shall say that Π is <u>generic relative</u> to V provided $\Pi(P) = 0$ only for points $P \in V$; and that Π is <u>generic</u> provided such a V exists. If Π is generic, we sometimes write $\Pi = 1(g)$.

If we assign to R^N the usual Euclidean topology, then, a property Π is said to be well-posed at P if Π holds throughout some neighbourhood of P in R^N. By extension a "Problem" which is parametrized by data in R^N will be called well-posed at the data point P, if it is solvable for all data points P' is some neighbourhood of P. If V is any variety in R^N, it is clear from the continuity of its defining polynomials, that V is a closed subset of R^N. Thus, if Π is generic relative to V (so that V is proper), then Π is well-posed at every point in the complement, V^c. Let $P_o \in V$, with V nontrivial and proper. It is clear that every neighbourhood of P_o contains points $P \in V^c$; otherwise, each defining polynomial ϕ of V vanishes identically in some neighbourhood of P_o, hence vanishes on R^N, and therefore, $V = R^N$, in contradiction to the assumption that V is proper. Thus, if Π is generic relative to V and if Π fails at P_o, Π can be made to hold if P_o is shifted by a suitable perturbation, which can be chosen arbitrarily small. We conclude that the set of points P where a generic property is well-posed, is both open and dense in R^N; furthermore, it can be shown that its complement has zero Lebesgue measure.

As a primitive illustration of these ideas, let $C \in R^{m\times n}, y \in R^{m\times 1}$ and consider assertion: there exists $x \in R^{n\times 1}$ such that $Cx = y$. Say that $P := (C, y)$ has property Π (i.e. $\Pi(P) = 1$), if and only if our assertion is true. By listing the elements of C and y in some arbitrary order, regard P as a data point in $R^N, N = mn + m$. Now $\Pi(P) = 1$ if and only if $y \in ImC$ i.e.

$$\text{rank}[C, y] = \text{ rank } C \tag{4.1}$$

It follows easily that Π is well-posed at P if and only if rank $C = m$, and Π is generic if and only if $m \leq n$. To verify these statements note first that (4.1) fails only if

$$\text{rank } C = d(ImC) < d(y) = m \tag{4.2}$$

If $\mathcal{V}$ is a vector space, then $d(\mathcal{V})$ denotes the dimension of the vector space $\mathcal{V}$.

But (4.2) implies that all $m \times m$ minors of C vanish: Let $V \subset R^N$ be the variety so determined. If $m \leq n, V$ is clearly proper, hence Π is generic, as claimed. On the other hand, if $m \geq n + 1$, (4.1) holds only if all $(n + 1) \times (n + 1)$ minors of $[C, y]$ vanish. The variety W so defined is proper, and $\Pi(P) = 0$ for $P \in W^c$, hence Π can not be generic. Finally, if rank $C = m$ at P then (equivalently) at least one $m \times m$ minor of C is nonzero at P,hence nonzero in a neighbourhood of P, so Π is well-posed at P. Conversely, if rank $C < m$ at P then a suitable $\tilde{y}$, with $|\tilde{y} - y|$

arbitrarily small, will make rank $[C, \widetilde{y}] =$ rank $C + 1$ namely, if $\widetilde{P} := (C, \widetilde{y})$, then $\Pi(\widetilde{P}) = 0$, hence Π is not well-posed at P. The study of the generic invariants of pencils dominates most of this chapter, and this is given next.

4.3 Generic Invariants of Pencils and State Space Theory

Pencils are characterised by finite elementary divisors, infinite elementary divisors, column minimal indices and row minimal indices [Gan.,1]. Regular pencils are characterised by finite e.d. and infinite e.d. Given a regular pencil $sF - G$ we may find its elementary divisors as we have shown in chapter 2. We shall now consider a singular pencil of matrices $sF - G$ of dimension $m \times n, m < n$ where F and G are generic meaning that F and G have full rank.

Proposition (4.1): Let $(F, G) \in R^{m \times n} \times R^{m \times n}$. Then generically

$$\text{rank}_{R(s)}\{sF - G\} = \min(m, n) = m$$

Proof: We start by forming the m-th compound matrix of the pencil $sF - G$. By setting the m-th compound matrix $C_m(sF - G) = 0$ this implies that all the maximal order minors which are polynomials in s with coefficients from R and they are identically zero. If V is any variety subset of the parameter space containing all the coefficients satisfying above equations then it is clear from the continuity of its defining polynomials that V is a closed subset of parameter space. Now we have to prove that there is at least one point in the whole parameter space which does not belong to the solution space (set of points of the parameter space which satisfy the above equations). As we are looking at the generic case, there must exist a point in the whole parameter space which dose not belong to the solution space, otherwise, the pencil looses rank and is non-generic case. So the pencil has full rank, i.e. m.

To make the above proof more clear, consider

$$sF - G = \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & 0 & s+1 \end{bmatrix} \quad \text{where } m = 3, n = 4$$

showing that $sF - G$ is a full rank pencil and V is a subset of a proper variety.

□

Proposition (4.2): If $sF - G \in R^{m\times n}[s]$ is generic, then:

(i) If $m < n$, $sF - G$ is characterised only by c.m.i.

(i) If $m > n$, $sF - G$ is characterised only by r.m.i.

(i) If $m = n$, $sF - G$ is characterised only by distinct finite elementary divisors.

Proof: If $m < n$, then given that $\rho_{R(s)}\{sF - G\} = m$, we have that

$$\begin{aligned} \mathcal{N}_l(F,G) = \{0\} &\quad\Leftrightarrow\quad \text{There is no r.m.i.} \\ \mathcal{N}_r(F,G) \neq \{0\} &\quad\Leftrightarrow\quad \text{There exists c.m.i.} \end{aligned}$$

For the pencil $sF - G$ to have i.e.d., F should loose rank; however F has generically full rank so there exist no i.e.d.

For the pencil $sF - G$ to have f.e.d, we have to find the maximal order minors which they turn out to be a set of generic polynomials and these polynomials are generically coprime i.e. they have no common divisors; so generically there exist no f.e.d. The proof for the other cases is similar.

□

A generic singular pencil $sF - G$ with $m < n$ is characterised by a unique element $sF_k - G_k$. This unique element is the Kronecker canonical form defined by the set of c.m.i. as:

$$sF_k - G_k = \text{quasi-diag}\{0_\rho : L_{\epsilon_{\rho+1}}(s), ..., L_{\epsilon_\mu}(s)\}$$

where 0_ρ is the block parameterised by the zero c.m.i, $L_{\epsilon_i}(s)$ is the block corresponding to the non-zero c.m.i.,

$$L_\epsilon(s) = \underbrace{\begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s & -1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s & -1 \end{bmatrix}}_{\epsilon+1} \left.\vphantom{\begin{bmatrix} s\\0\\0\\ \vdots\\0 \end{bmatrix}}\right\} \epsilon \tag{4.3}$$

where $\epsilon = \epsilon_i, i = \rho + 1, ..., \mu$.

For all $\epsilon > 0$, the associated block $L_\epsilon(s)$ has the form $L_\epsilon(s) = sL_\epsilon - \hat{L}_\epsilon$, where $L_\epsilon = [I_\epsilon|0], \hat{L}_\epsilon = [0|I_\epsilon] \in R^{\epsilon\times(\epsilon+1)}$ and $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_\rho = 0 \leq \epsilon_{\rho+1} \leq \epsilon_{\rho+2} \leq \cdots \leq \epsilon_\mu$ and we have

$$m = \sum_{j=\rho+1}^{\mu} \epsilon_j = \sum_{i=1}^{\mu} \epsilon_i \tag{4.4}$$

$$n = \rho + \sum_{j=\rho+1}^{\mu} (\epsilon_j + 1) = \sum_{i=1}^{\mu} (\epsilon_i + 1) \tag{4.5}$$

So $\mu = (n - m)$.

4.4 Generic values of Invariants of Singular Pencils

i) Toeplitz characterisation of c.m.i. (r.m.i.) [Kar. & Kal.1]

Let $sF - G \in R^{m\times n}[s]$ be a singular pencil with $\rho_{R(s)}\{sF - G\} = r \leq \min(m, n)$, $\mathcal{D}(F, G)$ be the set of elementary divisors (e.d.) (finite and infinite), $\mathcal{I}_c(F, G)$ be the set of column minimal indices (c.m.i.) of the pencil $(sF - G)$ and $\mathcal{I}_r(F, G)$ the set of row minimal indices (r.m.i) of the pencil $(sF - G)$ [For.,1] [Gan.,1]. The strict equivalence (S.E.) notion defined on $sF - G$ pencils will be denoted by $\mathcal{E}$ and the strict equivalence class, or orbit, of $sF - G$ by $\mathcal{E}(F, G)$. Two ordered pairs $(F, G), (F', G') \in R^{m\times n} \times R^{m\times n}$ will be called S.E., $(F, G)\mathcal{E}(F', G')$, if the corresponding pencils are S.E., i.e. $(sF - G)\mathcal{E}(sF' - G')$. $\{\mathcal{D}(F, G), \mathcal{I}_r(F, G), \mathcal{I}_c(F, G)\}$ define a complete set of invariants for $\mathcal{E}(F, G)$ and the corresponding Kronecker canonical form will be denoted by S.

The singularity of $sF - G$ implies linear dependence (over $R(s)$) amongst its columns and/or its rows; thus, there exist polynomial vectors $\underline{x}(s) \in R^n[s], \underline{y}(s) \in R^m[s]$ such that at least one of the following conditions are satisfied

$$(sF - G)\underline{x}(s) = \underline{0}, \quad \Leftrightarrow \quad \underline{x}(s) \in \mathcal{N}_r\{sF - G\} \equiv \mathcal{X}_r(F, G) \tag{4.6}$$

$$\underline{y}^t(s)(sF - G) = \underline{0}^t, \quad \Leftrightarrow \quad \underline{y}^t(s) \in \mathcal{N}_l\{sF - G\} \equiv \mathcal{X}_l(F, G) \tag{4.7}$$

$\mathcal{X}_r(F, G), \mathcal{X}_l(F, G)$ are rational vector spaces (over $R(s)$) and the sets $\mathcal{I}_c(F, G), \mathcal{I}_r(F, G)$ are the Forney dynamical indices [For.,1] characterising the polynomial minimal bases of $\mathcal{X}_r(F, G), \mathcal{X}_l(F, G)$ respectively. We shall consider the case where F, G are generic and with $m < n$ there is only c.m.i. With the pair $(F, G) \in R^{m\times n} \times R^{m\times n}$ we associate the following sequence of matrices:

$$T_r(F, G) = \left\{T_k(F, G) \in R^{(k+1)m\times kn}, k = 1, 2, ...,\right\}$$

where

$$T_1 = \begin{bmatrix} F \\ -G \end{bmatrix}, T_2(F,G) = \begin{bmatrix} F & 0 \\ -G & F \\ 0 & -G \end{bmatrix}, \cdots,$$

$$T_k(F,G) = \underbrace{\begin{bmatrix} F & 0 & \cdots & 0 & 0 \\ -G & F & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -G & F \\ 0 & 0 & \cdots & 0 & -G \end{bmatrix}}_{\text{k blocks}} \left.\vphantom{\begin{bmatrix} F \\ -G \\ \vdots \\ 0 \\ 0 \end{bmatrix}}\right\} k+1 \text{ blocks} \tag{4.8}$$

$T_r(F,G)$ will be called the right Toeplitz sequence and $T_k(F,G)$ the k-th right Toeplitz matrix of (F,G). The sequences

$$\begin{aligned} P_r(F,G) &\equiv \left\{\mathcal{N}_r^k : \mathcal{N}_r^k = \mathcal{N}_r\{T_k(F,G)\}, k = 1,2,...\right\} \\ C_r(F,G) &\equiv \left\{\theta_k : k = -1,0,1,..., \text{ where } \theta_{-1} = \theta_0 = 0 \text{ and } \theta_k = \dim\mathcal{N}_r^k, k \geq 1\right\} \end{aligned} \tag{4.9}$$

will be referred to as the right characteristic spaces sequence (r.c.s.s) and right characteristic sequence (r.c.s) of (F,G) respectively; the real space $\mathcal{N}_r^k$ will be called the k-th right characteristic space of (F,G). $C_r(F,G)$ will be called neutral and shall be denoted by $C_r(F,G) = \{0\}$, if for all $k, \theta_k = 0$.

ii) The Piecewise Arithmetic Progression Sequence

The sequence $C_r(F,G)$ is completely defined by the set $\mathcal{I}_c(F,G)$ and this is explained bwlow.

Lemma (4.1): [Kar. & Kal.1] The $C_r(F,G)$ sequence of a general pencil $sF-G$ is characterised for every $k = 0,1,2,...$ by the property

$$\theta_k \leq (\theta_{k+1} + \theta_{k-1})/2, \theta_{-1} = \theta_0 = 0 \tag{4.10}$$

In particular, we have that:

i) Strict inequality holds, if and only if $k = \epsilon$, where ϵ is a c.m.i.

ii) Equality holds, if and only if k is not the value of c.m.i.

□

The set of integers is partitioned by the set $\{0 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_\mu\}$. For all k in the range $[\epsilon_j + 1, ..., \epsilon_{j+1} - 1]$ the Arithmetic Progression Property (A.P.P.) $\theta_k = (\theta_{k+1} + \theta_{k-1})/2$ holds true; this relationship, however, can not be extended in the range of values $[..., \epsilon_j - 1]$ or $[\epsilon_{j+1} + 1, ...]$, since for $k = \epsilon_j$ and $k = \epsilon_{j+1}, \theta_{\epsilon_j} < (\theta_{\epsilon_j - 1} + \theta_{\epsilon_j + 1})/2$ and $\theta_{\epsilon_{j+1}} < (\theta_{\epsilon_{j+1}-1} + \theta_{\epsilon_{j+1}+1})/2$. Therefore, $C_r(F,G)$ satisfies the (A.P.P) in the range $[\epsilon_j + 1, ..., \epsilon_{j+1} - 1]$, but violates the (A.P.P) at the boundary values $\epsilon_j, \epsilon_j + 1$. Because of this property, $C_r(F,G)$ will be referred to as a Piecewise Arithmetic Progression Sequence (P.A.P.S). The integer k for which the (A.P.P) breaks down will be called the singular points of $C_r(F,G)$. A measure of deviation from the (A.P.P) is the number

$$\delta_k = (\theta_{k+1} - \theta_k) - (\theta_k - \theta_{k-1}) = \theta_{k+1} + \theta_{k-1} - 2\theta_k \tag{4.11}$$

which will be called the gap of $C_r(F,G)$ at k. If k is singular, then $\delta_k > 0$, otherwise $\delta_k = 0$.

iii) Generic Rank of Toeplitz Matrices and the Generic P.A.P.s

The above results hold true as long as the rank of Toeplitz matrices are known. In this section, we find first the generic rank of Toeplitz matrices; we consider the k-th right Toeplitz matrix

$$T_k(F,G) = \underbrace{\begin{bmatrix} F & 0 & \cdots & 0 & 0 \\ -G & F & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -G & F \\ 0 & 0 & \cdots & 0 & -G \end{bmatrix}}_{\text{k blocks}} \left.\vphantom{\begin{bmatrix} F \\ F \\ \vdots \\ F \\ F \end{bmatrix}}\right\} k+1 \text{ blocks}$$

Proposition (4.3): The k-th right Toeplitz matrix $T_k(F,G)$ is rank equivalent to

$$\left[\begin{array}{c|c} I_{km} & \underline{0} \\ \hline \underline{0} & -A^{k-1}B, ..., -A^2B, -AB, -B \end{array}\right]$$

for some appropriate pair of matrices (A,B) where $A \in R^{m \times m}, B \in R^{m \times (n-m)}$; furthermore if F,G are generic, then so are A and B.

Proof: Since F,G are generic, there exists a coordinate framework such that we may write $F = [I_m, 0]$ and $G = [A, B]$. To get the reduced form of $T_k(F,G)$, we then start from the matrix

$$\begin{bmatrix} I_m & 0 & 0 & 0 & & & \\ -A & -B & I_m & 0 & & & 0 \\ 0 & 0 & -A & -B & & & \\ & & & & \ddots & & \\ 0 & & & & & I_m & 0 \\ & & & & & -A & -B \end{bmatrix}$$

which is the same as $T_k(F, G)$ with the only difference that we have substituted for F and G. First, we try to eliminate the A's; the way to do this, for example the A in position (2,1), we multiply first row by A and add it to the second row block. Having eliminated the A in position (2,1), multiply second row by A and add it to the third row. By doing this, we can eliminate all the A's, and we produce AB's (starting from third row, position (3,2)) going down to $A^{k-1}B$. Now, we should eliminate all B's, AB's. The way to do this is again, for example, the B in position (2,2), first multiply third column by B and add it to the second column, or the AB in position (3,2)m multiply fifth column by AB and add it to the 2^{nd} column and so on. By continuing with these eliminations, we finally get a matrix in the form

$$\left[\begin{array}{c|c} I_{km} & \underline{0} \\ \hline \underline{0} & -A^{k-1}B, -A^{k-2}B, ..., -A^2B, -AB, -B \end{array}\right]$$

□

<u>**Lemma (4.2)**</u>: For the pair (A, B) above, the matrix $[B, AB, ..., A^{m-1}B]$ has generically rank m, if A, B are generic matrices.

<u>Proof</u>: We start by forming the m-th compound matrix of the matrix $[B, AB, ..., A^{m-1}B]$. By setting the m-th compound matrix to zero implies that all the maximal order minors of the matrix $[B, AB, ..., A^{m-1}B]$ are identically zero. Let V be the variety in R^N containing all the solutions which satisfy the resulting equations. Now we have to show that V is proper. In order to do this, we have to show that there is at least one point in the whole parameter space, which does not belong to the solution space (i.e. the space which solution to th equations above are simultaneously zero and is a subset of the parameter space). As we are dealing with generic case, there must exist a point in the whole parameter space which does not belong to the solution space, that is we have to show that there is a full rank matrix $[B, AB, ..., A^{m-1}B]$; however, we can always construct a controllable system. So the matrix $[B, AB, ..., A^{m-1}B]$ has rank m.

$\square$

From the above we have:

Proposition (4.4): The generic rank of the k-th right Toeplitz matrix of (F, G) is $km + m = (k+1)m$.

$\square$

iv) Generic Values of Minimal Indices

The values of minimal indices for unstructured generic systems are important elements in the solvability conditions of exact control synthesis problems and they are examined here using the previous results. We first note:

Remark (4.1): For a generic square pencil, the finite zeros are distinct and they appear in pairs of complex conjugate zeros. Furthermore, such pencils generically have no i.e.d.

$\square$

For a generic $m \times n$ pencil with $m < n$ the only invariants are those defined by column minimal indices. The following result provides the characterisation of the generic values of c.m.i. of pencils and thus provide tools for characterising the generic values of the controllability, observability indices of linear systems.

Proposition (4.5): Let $sF - G$ be an $m \times n, m < n$, generic pencil and let $\underline{k}$ be the integer

$$\underline{k} = \min\left\{k : k > \frac{m}{n-m}\right\} \tag{4.12}$$

then, the smallest minimal index is $\epsilon_1 = \underline{k} - 1$ and has multiplicity $\rho_1 = \underline{k}(n-m) - m$.

Proof: Let $\underline{k}$ be the smallest minimal index such that $\underline{k} > \frac{m}{n-m}$, the sequence $C_r(F, G)$ is defined by

$$\begin{aligned} \theta_{-1} = 0, \theta_0 = 0, \theta_1 = 0, ..., \theta_{\underline{k}-2} = 0, \theta_{\underline{k}-1} = 0, \theta_{\underline{k}} = \underline{k}(n-m) - m, \\ \theta_{\underline{k}+1} = (\underline{k}+1)(n-m) - m, \theta_{\underline{k}+2} = (\underline{k}+2)(n-m) - m, ... \end{aligned} \tag{4.13}$$

By computing the gap for $\underline{k} - 1$ we have

$$\delta_{\underline{k}-1} = \theta_{\underline{k}} + \theta_{\underline{k}-2} - 2\theta_{\underline{k}-1} = \underline{k}(n-m) - m + 0 = \underline{k}(n-m) - m \neq 0$$

from which and Lemma (4.1) the result follows.

□

We may now state the following theorem:

Theorem (4.1): Let $F, G \in R^{m \times n}, m < n$, and assume that (F, G) is generic (that is the matrices F, G are generic). For the pencil $sF - G$ the following holds true:

i) $sF - G$ has only c.m.i, that is, it has no finite, infinite-ed and no r.m.i.

ii) If $\underline{k} = \min\{k \in Z^+ : k > m/(n-m)\}$, then the set $\mathcal{I}_c(F, G)$ of c.m.i. of $sF - G$ is defined by:

a) If $\underline{k} = \frac{n}{n-m}$ then $\mathcal{I}_c(F, G) = \{(\epsilon_1, \rho_1) : \epsilon_1 = \underline{k} - 1, \rho_1 = n - m\}$; that is it has one cmi ϵ_1 with multiplicity ρ_1.

b) If $\underline{k} \neq \frac{n}{n-m}$ then $\mathcal{I}_c(F, G) = \{(\epsilon_1, \rho_1), (\epsilon_2, \rho_2) : \epsilon_1 = \underline{k} - 1, \rho_1 = \underline{k}(n-m) - m, \epsilon_2 = \underline{k}, \rho_2 = n - \underline{k}(n-m)\}$.

Proof: From proposition (4.5) we have that

$$\epsilon_1 = \underline{k} - 1, \rho_1 = \underline{k}(n-m) - m > 0 \tag{4.14}$$

i) If $\underline{k} = \frac{n}{n-m}$, then $\rho_1 = \frac{n}{n-m}(n-m) - m = n - m$ and since there are $n - m$ indices ϵ_1 the search for more singular points of the sequence stops, this establishes part a.

ii) The pair (ϵ_1, ρ_1) has already defined. For $k = \underline{k}$ the corresponding gap is given by:

$$\begin{aligned} \delta_{\underline{k}} &= \theta_{\underline{k}+1} + \theta_{\underline{k}-1} - 2\theta_{\underline{k}} = (\underline{k}+1)(n-m) - m + 0 - 2\underline{k}(n-m) + 2m \\ &= \underline{k}(n-m) + (n-m) - m - 2\underline{k}(n-m) + 2m = n - \underline{k}(n-m) \end{aligned} \tag{4.15}$$

Since $\underline{k} \neq \frac{n}{n-m}$, clearly $\delta_{\underline{k}} \neq 0$. Furthermore, since $\delta_i \geq 0$ and $\delta_i \neq 0$, it follows that

$$\epsilon_2 = \underline{k}, \rho_2 = n - \underline{k}(n-m) > 0$$

Note that since

$$\rho_1 + \rho_2 = \underline{k}(n-m) - m + n - \underline{k}(n-m) = n - m \tag{4.16}$$

and $\rho_1 > 0 \rightarrow \rho_2 < n - m$. Since $\rho_1 + \rho_2 = n - m$, the search for more singular points stops, since all of them have been defined.

$\square$

Example (4.4.1): If $n = 50, m = 48$, then $\frac{m}{n-m} = \frac{48}{2} = 24$ and $\underline{k} = 25$. However, $\underline{k} = 25 = \frac{n}{n-m} = \frac{50}{2} = 25$ and thus, there is only one value of minimal indices.

$\square$

A similar result may be stated for pencils with $m > n$ and this defines the generic values of r.m.i.s. From the above and its dual we have:

Corollary (4.1): For the generic system $S(A, B, C)$ with n states, l inputs, and m outputs the generic values of controllability indices $\mathcal{I}_c$ and observability indices $\mathcal{I}_o$ are defined by:

a) If $\underline{k}$ is the smallest integer such that $k > n/l$, then

i) If $\underline{k} = \frac{n}{l} + 1$, then $\mathcal{I}_c = \{(\mu_1, \rho_1) : \mu_1 = n/l, \rho_1 = l\}$

ii) If $\underline{k} \neq \frac{n}{l} + 1$, then $\mathcal{I}_c = \{(\mu_1, \rho_1), (\mu_2, \rho_2) : \mu_1 = k - 1, \rho_1 = kl - n, \mu_2 = k, \rho_2 = n + l - kl\}$. where μ_i are distinct values of controllability indices and ρ_i are the corresponding multiplicities.

b) If $\underline{k}$ is the smallest integer such that $k > n/m$, then

i) If $\underline{k} = \frac{n}{m} + 1$, then $\mathcal{I}_o = \{(\theta_1, \sigma_1) : \theta_1 = n/m, \sigma_1 = m\}$

ii) If $\underline{k} \neq \frac{n}{m} + 1$, then $\mathcal{I}_o = \{(\theta_1, \sigma_1), (\theta_2, \sigma_2) : \theta_1 = k - 1, \sigma_1 = km - n, \theta_2 = k, \sigma_2 = n + m - km\}$. where θ_i are distinct values of observability indices and σ_i are the corresponding multiplicities.

$\square$

This result establishes the generic values of the controllability and observability indices.

Remark (4.2): The generic value of the controllability, observability indices μ, θ are defined by:

a) μ is the smallest integer for which $\mu \geq n/l$.

b) θ is the smallest integer for which $\theta \geq n/m$.

$\square$

The result on the generic values of c.m.i., r.m.i. may be used to find the generic values of c.m.i, r.m.i of the pencil

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & 0 \end{bmatrix}$$

Corollary (4.2): If $S(A, B, C)$ is generic, then for the following three cases we have:

For $m < l$, if $\underline{k}$ is the smallest integer such that $k > \frac{m+n}{l-m}$, then

i) If $\underline{k} = \frac{n+l}{l-m}$, then $\mathcal{I}_c(P) = \{(\tilde{\epsilon}_1, \tilde{\rho}_1) : \tilde{\epsilon}_1 = k - 1, \tilde{\rho}_1 = l - m\}$, that is, it has one c.m.i. $\tilde{\epsilon}_1$ with multiplicity $\tilde{\rho}_1$.

ii) If $\underline{k} \neq \frac{n+l}{l-m}$, then $\mathcal{I}_c(P) = (\tilde{\epsilon}_1, \tilde{\rho}_1), (\tilde{\epsilon}_2, \tilde{\rho}_2) : \tilde{\epsilon}_1 = k-1, \tilde{\rho}_1 = k(l-m)-(m+n), \tilde{\epsilon}_2 = k, \tilde{\rho}_2 = n + l(1-k) + km\}$; that is we have the values $\tilde{\epsilon}_1, \tilde{\epsilon}_2$ with corresponding multiplicities $\tilde{\rho}_1, \tilde{\rho}_2$.

For $m > l$, if $\underline{k}$ is the smallest integer such that $k > \frac{m+n}{m-l}$, then

i) If $\underline{k} = \frac{n+l}{m-l}$, then $\mathcal{I}_r(P) = \{(\tilde{\eta}_1, \tilde{\sigma}_1) : \tilde{\eta}_1 = k - 1, \tilde{\sigma}_1 = m - l\}$, that is, it has one r.m.i. $\tilde{\eta}_1$ with multiplicity $\tilde{\sigma}_1$.

ii) If $\underline{k} \neq \{\frac{n+l}{m-l}\}$, then $\mathcal{I}_r(P) = \{(\tilde{\eta}_1, \tilde{\sigma}_1), (\tilde{\eta}_2, \tilde{\sigma}_2) : \tilde{\eta}_1 = k - 1, \tilde{\sigma}_1 = k(m - l) - (l + n), \tilde{\eta}_2 = k, \tilde{\sigma}_2 = n + m(1 - k) + kl\}$; that is we have two values $\tilde{\eta}_1, \tilde{\eta}_2$ with corresponding multiplicities $\tilde{\sigma}_1, \tilde{\sigma}_2$.

For $m = l$, it has only distinct f.e.d.

□

We illustrate the above result by means of the following example.

Example (4.4.2):

i) For $m < l$, if $n = 24, m = 48, l = 50$, then $\frac{m+n}{l-m} = 36$ and $\underline{k} = 37$. However, $\underline{k} = \frac{n+l}{l-m} = 37$ and thus there is one value of minimal indices.

ii) For $m > l$, if $n = 28, m = 50, l = 24$, then $\frac{m+n}{m-l} = 3$ and $\underline{k} = 4$. However, $\underline{k} = \frac{n+l}{m-l} = 2$ and thus $\underline{k} \neq \frac{n+l}{m-l}$ and we have two values of minimal indices.

4.5 Generic Properties of Proper Transfer Function Matrices

In the study of generic properties of transfer function matrices we examine the structural properties of them, as defined by the Smith-McMillan forms, over $R[s]$ and at infinity and for a generic transfer function. Note that the infinite poles and zeros of a rational matrix may be defined as follows [Var. Lim. & Kar.,1]:

Definition (4.1): Let $G(s) \in R^{m\times l}(s), \rho_{R(s)}\{G(s)\} = r$ and let $M_\infty^*(s) = diag\{s^{q_i}, i \in \tilde{r}, q_1 \geq \cdots \geq q_r\}$ be the essential part of the Smith McMillan form at ∞ of $G(s)$.

i) If π_∞ is the number of q_i's with $q_i > 0, i \in \tilde{r}$, then we say that $G(s)$ has π_∞ poles at infinity, each one of order $q_i > 0$,

ii) If ϵ_∞ is the number of q_i's with $q_i < 0, i \in \tilde{r}$, then we say that $G(s)$ has ϵ_∞ zeros at infinity, each one of order $|q_i|$,

iii) The number $\delta_M^\infty(G) = \sum_{j=1}^{\pi_\infty} \bar{q}_i$, where $\bar{q}_i$ are the positive q_i's is defined as the McMillan degree at infinity of $G(s)$.

□

Remark (4.3): Let $G(s) \in R^{m\times l}(s), \rho_{R(s)}\{G(s)\} = r$. If ϵ_∞ is the number of infinite zeros of $G(s)$, then $\rho\{G(\infty)\} = r - \epsilon_\infty$.

□

If $G(s) \in R_{pr}^{m\times l}(s)$, then $G(s)$ has no poles at $s = \infty$, but it may have zeros at $s = \infty$. Now we consider whether there exists any zeros if the elements of $G(s)$ are generic.

Theorem (4.2): Let $G(s) \in R^{m\times l}(s), \rho_{R(s)}\{G(s)\} = r, m \neq l$, if $G(s)$ is generic, then $G(s)$ has no finite zeros.

Proof: Write $G(s)$ in terms of Coprime Matrix Fraction Description (c-MFD), $G(s) = N(s)D^{-1}(s)$ where $(N(s), D(s))$ are coprime MFD's. Consider now the r-th compound $C_r(N(s))$. The zero polynomial is defined as the greatest common divisor of the entries of $C_r(N(s))$. The elements of $C_r(N(s))$ are generic polynomials which means that they are coprime. Thus $N(s)$ has no finite zeros and so $G(s)$ has also no zeros.

$\square$

By showing that proper generic transfer function $G(s)$ has no finite zeros, we can state the following theorem,

Theorem (4.3): Let $G(s) = N(s)D^{-1}(s) = \widetilde{N}(s)\widetilde{D}^{-1}(s) \in R_{pr}^{m\times l}(s)$, where $m \neq l$, if $G(s)$ is proper generic then $G(s)$ has no infinite poles and no infinite zeros.

Proof: From remark (4.3), if $G(s)$ is proper generic then it has no poles at infinity. To prove that $G(s)$ has no zeros at infinity, recall that the infinite zero structure of $G(s)$ is given be the finite zero structure of $G(1/w)$ which generically from previous result has no zeros at $w = 0$. Thus $G(s)$ has no zeros at $s = \infty$.

$\square$

We examine next the relationships between finite poles, zeros and infinite poles, zeros of a proper generic rational matrix $G(s) \in R_{pr}^{m\times l}(s), \rho_{R(s)}\{G(s)\} = r$, with a rational vector space $R_G^c \equiv colsp_{R(s)}\{G(s)\}$ which clearly has dimension $\dim R_G^c = r$. The set $\mathcal{P}$ of all polynomial vectors which are contained in R_G^c is an $R[s]$-Module and its definition is considered next.

Definition (4.2): Let $G(s) \in R^{m\times l}(s)$ and $\rho_{R(s)}\{G(s)\} = l$, we may write

$$G(s) = N(s) \cdot \frac{1}{d(s)} \tag{4.17}$$

where $d(s)$ is the l.c.m of the denominators of $G(s)$. Clearly $\rho_{R(s)}\{N(s)\} = l$ and thus $N(s)$ is a polynomial bases of R_G^c; thus, every rational vector space has a polynomial bases. Let $N(s) = [n_1(s), ..., n_l(s)]$ and define the set

$$\mathcal{M}_N \equiv col.sp_{R[s]}\{N(s)\} = \{\underline{x}(s) : \underline{x}(s) = \sum_{i=1}^{l} t_i(s)\underline{n}_i(s), t_i(s) \in R[s]\} \tag{4.18}$$

$\square$

The set $\mathcal{M}_N$ under the standard operations of addition and scalar multiplication by elements of $R[s]$, is a finitely generated module, that is, it is generated by a finite number of polynomial vectors. [Ros.,1] [For.,1] [Kai,1].

Definition (4.3): Let $N(s) \in R^{m\times l}[s], \rho_{R(s)}\{N(s)\} = l$ and let $R_N^c = sp_{R(s)}\{N(s)\}$, then,

i) $N(s)$ is called a least degree basis (l.d.b) of R_N^c, if it has no finite zeros.

ii) $N(s)$ is called a minimal basis of R_N^c if it is a l.d.b and it is column reduced [Kai,1], that is its high coefficient column matrix has full rank.

□

Definition (4.4): Let $N(s) = [\underline{n}_1(s), ..., \underline{n}_l(s)] \in R^{m \times l}[s]$, be any minimal basis matrix of the l-dimensional $R(s)$-vector space $\mathcal{X}$. Let $\delta_i = \partial_s[\underline{n}_i(s)]$ and assume the columns are ordered by $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_l$. The set of indices $I_{\mathcal{X}} = \{\delta_i : \delta_1 \leq \delta_2 \leq \cdots \leq \delta_l\}$ is an invariant of $\mathcal{X}$ and they are referred to as Forney dynamical indices; the minimal degree $\delta_F = \sum_{i=1}^{l} \delta_i$, of the maximal module $\mathcal{M}_{\mathcal{X}}^* = \mathcal{M}_N \equiv colsp_{R[s]}\{N(s)\}$ is called the Forney dynamical order of $\mathcal{X}$.

□

Remark (4.4): [Vard. & Kar.,3] Let ϵ_i be the least valuation among all $i \times i$ minors of $G(s), i = 1, 2, ..., r$ and $q_i, i \in \tilde{r}$ the invariants of the Smith-McMillan form at infinity, and $p_\infty(G), z_\infty(G)$ be the number of poles and zeros at infinity with orders taken into account. Then for all j,

$$\epsilon_j = \sum_{i=1}^{j} q_i, j = 1, 2, ..., r \tag{4.19}$$

and for $j = r$, we have

$$\delta_\infty(G) = \epsilon_r = -\sum_{i=1}^{r} q_i = z_\infty(G) - p_\infty(G) \tag{4.20}$$

where $\delta_\infty(G)$ is the valuation at infinity of $G(s)$.

□

Theorem (4.4): Let $G(s) = N(s)D^{-1}(s) \in R^{m \times l}(s)$ be any R-MFD, i.e. $N(s) \in R^{m \times l}[s], D(s) \in R^{l \times l}[s]$ and let $\rho_{R(s)}\{G(s)\} = l$. Then

$$\delta_\infty(G) = \partial_m(D) - \partial_m(N) \tag{4.21}$$

where $\delta_\infty(G)$ is the valuation at infinity of $G(s)$.

Furthermore, if $N(s), D(s)$ is any R-CMFD pair, then $\partial_m(D) \equiv \delta_M(G)$, the McMillan degree, and (4.21) implies

$$\delta_\infty(G) = \delta_M(G) - \partial_m(N) \tag{4.22}$$

$\square$

Consider now a $G(s) \in R^{m \times l}(s), \rho_{R(s)}\{G(s)\} = l$. If $z_f(G), p_f(G)$ denote the total number of finite zeros, poles of $G(s)$ respectively and $z_\infty(G), p_\infty(G) = \delta_M^\infty(G)$ are the total number of infinite zeros, poles with multiplicities accounted for, we have the following relationship.

Corollary (4.3): Let $G(s) \in R^{m \times l}(s), \rho_{R(s)}\{G(s)\} = l$. Then

$$\delta_M(G) + \delta_M^\infty(G) = z_\infty(G) + z_f(G) + \delta_F \tag{4.23}$$

$\square$

Condition (4.23) expresses the important property that for a general full rank, rational matrix, the total number of finite and infinite poles is equal to the total number of finite and infinite zeros, plus the Forney dynamical order δ_F of R_G^c. For the case where $G(s)$ is proper generic, which means that $G(s)$ has no finite zeros and no infinite zeros, poles the relation (4.23) becomes:

Corollary (4.4): Let $G(s) \in R^{m \times l}(s)$ be a proper generic, $\rho_{R(s)}\{G(s)\} = l$. Then

$$\delta_M(G) = \delta_F \tag{4.24}$$

$\square$

Condition (4.24) implies that the total number of finite poles is equal to the Forney dynamical order of R_G^c, for a generic system.

Corollary (4.5): Let $G(s) \in R^{m \times l}(s)$, $\rho_{R(s)}\{G(s)\} = l$ with $\delta_M = n$ and $G(\infty) = D$, then

i) If $m = l$, then all Forney dynamical indices and the Forney order are zero. Furthermore, if $D \neq 0$, then we have n distinct finite zeros and no infinite zeros, whereas if it is strictly proper ($D = 0$), then we have $n - m$ distinct finite zeros and m first order infinite zeros.

ii) If $m \neq l$, we have no finite zeros; furthermore,

- **a)** If $D \neq 0$, we have no infinite zeros and the generic value of the Forney order $\delta_F = n$.

b) If $D = 0$, we have $\min(m, l)$ first order infinite zeros and the generic value of the Forney order is $\delta_F = n - \min(m, l)$.

Proof: This result follows from the Kronecker canonical decomposition of system matrix pencil [Kar. & MacB.,1] [Kar. & Kou.,1].

□

4.6 Conclusion

In this chapter, it was seen that nonsquare generic $m \times n$ pencils are characterised only by c.m.i if $m < n$, or r.m.i. if $m > n$ and if they are square then they are characterised only by distinct f.e.d. The generic form of the right, left characteristic sequence has been worked out and this has led to the determination of the generic values of c.m.i., r.m.i. of singular pencils. These results were then used to determine the generic values of c.m.i., r.m.i. type invariants of state space models.

For transfer function models, the generic types of invariants have been defined and some relationships between Forney invariants and McMillan degree was defined.

Chapter 5

COMPUTATION AND RANK PROPERTIES OF PLUCKER MATRICES

5.1 Introduction

Recent work [Kar., & Gia.,2] has demonstrated the importance of controllability, observability Plucker matrices in control theory and a new criterion for controllability, observability respectively was given in terms of the corresponding Plucker matrices. The Plucker matrices $P(A, B), P(A, C)$ characterise the pairs $(A, B), (A, C)$ respectively; thus, it is expected that system controllability, observability should be connected with the properties of $P(A, B), P(A, C)$ respectively.

This chapter is mainly structured on the computation and rank properties of Plucker matrices. The structure of the Plucker matrix P_A of a least degree matrix $A(S)$ (i.e. has coprime rows) and computation of P_A from the original data in terms of the Grassman vector, of the structure matrix $S(s)$, of $A(s)$ is also given here. Then, necessary and sufficient conditions for P_A to have full rank are examined, as well as the generic value of the rank of such matrices. The Plucker matrices are important invariants which characterise the solvability of pole, zero Determinental Assignment Problems [Kar., & Gia.,1] [Kar., & Gia.,2] [Gia. & Kar.,1]. Thus, the computation of Plucker matrices and the investigation of their rank are integral parts of the solvability of DAP, as well as computation of its solutions.

5.2 Relationships Between Controllability, Observability and Corresponding Plucker Matrices

Our aim here is to establish the rank properties of Plucker matrices associated with a given system. Let $T(s) = [sI - A, -B] = s[I_n, 0] - [A, B]$ be a matrix pencil of dimension $n \times (n+l)$. This pencil has l column minimal indices and finite elementary divisors for any (A, B). If (A, B) controllable, then it has no f.e.d.

The Plucker matrix of $T(s)$ plays a key role in the study of the pole assignment by state feedback problem and it is defined by [Kar., & Gia.,2]

$$\begin{aligned} C_n(T(s)) = C_n([sI - A, -B)] &= g(A, B) \\ &= e_n(s)P(A, B), e_n(s) = [1, s, ..., s^n] \end{aligned} \quad (5.1)$$

where $g(A, B) \in R^{1\times\binom{n+l}{n}}$ is defined as Grassman vector of $T(s)$ with degree n and $P(A, B) \in R^{(n+1)\times\binom{n+l}{n}}$ is the Plucker matrix of $T(s)$. The problem considered here is to investigate the relationship between rank $P(A, B)$ and system controllability matrix. In the following we will use the result:

Lemma (5.1): (Sylvester-Frank) [Mar. & Min,1] Let $A \in R^{n\times n}$ be an $n \times n$ real matrix then

$$|C_r(A)| = |A|^{\binom{n-1}{r-1}} \quad (5.2)$$

□

Lemma (5.2): (Brunovsky-Transformation) [Bru.1] Consider the set of all matrix pairs $(A, B) \in R^{n\times n} \times R^{n\times m}$ and the following transformation group $\mathcal{H}_B^r$, i.e.

$$\mathcal{H}_B^r : \begin{cases} R^{n\times n} \times R^{n\times m} \Longrightarrow R^{n\times n} \times R^{n\times m} \\ (A, B) \Longrightarrow (Q(A + BL)Q^{-1}, QBR^{-1}) \end{cases} \quad (5.3)$$

with

$$Q^{-1} : R^n \rightarrow R^n \text{ state coordinate transform}$$

$$R : R^m \rightarrow R^m \text{ input coordinate transform}$$
$$L : R^n \rightarrow R^m \text{ state feedback transform}$$

Every Brunovsky transformation $\mathcal{H}_B^T$ is an element of a transformation group represented by a triple (Q, L, R) and its action on the pair (A, B) defines an orbit.

□

Brunovsky type of transformations corresponding to column space transformations on $P(A, B)$ and thus leave the rank $P(A, B)$ invariant. This is stated below:

Proposition (5.1): Rank $P(A, B)$ is invariant under the Brunovsky group $\mathcal{H}_B^T$.

□

Proof: If

$$Q[sI - A, -B] \underbrace{\begin{bmatrix} Q^{-1} & 0 \\ L & R \end{bmatrix}}_{\equiv U} = [sI - A', -B'] = T'(s) \tag{5.4}$$

where

Q^{-1}	is state coordinate transform
L	is state feedback transform
R	is input coordinate transform

then, by the Binet-Cauchy theorem [Mar. & Min,1] we have

$$C_n(T'(s)) = |Q|\underline{g}(A, B)C_n(U) = \underline{g}(A', B') \tag{5.5}$$

thus

$$P(A', B') = |Q| \cdot P(A, B)C_n(U) \tag{5.6}$$

Since $U \in R^{(n+l)\times(n+l)}$, $|U| \neq 0$, then $C_n(U)$ is nonsingular (see Lemma (5.1)). Thus (5.6) expresses equivalence and does not affect the rank.

□

Remark (5.1): We may use any special form under $\mathcal{H}_B^T$ and thus any system in the orbit to investigate the rank properties of $P(A, B)$.

□

The above result suggests that in the study of the rank of $P(A, B)$ we may use any canonical form of $T(s)$. In the following we investigate the relationship between controllability and rank of the corresponding Plucker matrix.

Lemma (5.2): If the controllable pair $(A, \underline{b})$ is in the controllable canonical form i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5.7}$$

then the Grassman vector and the corresponding Plucker matrix have the form:

$$\underline{g}(A, b) = \left[|sI - A|, s^{n-1}, -s^{n-2}, ..., (-1)^{n-2}s, (-1)^{n-1}\right] \tag{5.8}$$

and thus

$$P(A, \underline{b}) = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 & (-1)^{n-1} \\ a_1 & 0 & 0 & \cdots & (-1)^{n-2} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-2} & 0 & -1 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{5.9}$$

Proof: We can see that the first term in $\underline{g}(A, \underline{b})$ is the maximal degree minor which is always $|sI - A|$. We note that for all other minors apart from the first, the last column of $[sI - A, -b]$ has to be included and thus all such minors are elements of the exterior product of the submatrix of $[sI - A, -b]$ which is obtained by deleting the last row. Mind that these elements have to be appropriately arranged, according to the original lexicographic ordering. This submatrix is of order $(n + 1) \times (n + 1)$ and is of the form

$$\left[\begin{array}{cccccc|c} s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s & -1 & 0 \end{array}\right]$$

and thus from (5.1) and the special structure of the above matrix we have $g(A, B) =$

$e_n(s)P(A,B)$ where

$$P(A,B) = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 & (-1)^{n-1} \\ a_1 & 0 & 0 & \cdots & (-1)^{n-2} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-2} & 0 & -1 & \cdots & 0 & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

clearly from the above structure we have that controllability implies that

$$\text{rank}(P(A,B)) = n+1$$

□

Remark (5.2): For a general pair (A,B) with

$$\underline{g}(A,B) = [|sI-A|, \phi_1(s), \cdots, \phi_\rho(s)] \quad \rho = \begin{pmatrix} n+l \\ n \end{pmatrix} \Rightarrow \partial[\phi_i(s)] < n, i \in \tilde{\rho} \quad (5.10)$$

□

Thus in $\underline{g}(A,B)$, maximal order minors with the potential to have maximal rank are those which always contain the first column corresponding to $|sI-A|$; that is $(n+1)$ elements (polynomials) of $\underline{g}(A,B)$ which defines a $(n+1)\times(n+1)$ submatrix in $P(A,B)$. This submatrix may be nonsingular if $|sI-A|$ is one of $(n+1)$-polynomials, otherwise it is always singular. From $[sI-A,-B]$ by selecting a column of B the submatrix $[sI-A,-b]$ defines a subset of $(n+1)$-coordinates of $\underline{g}(A,B)$ which has the potential to have a full rank Plucker matrix if $(A,\underline{b})$ is controllable.

Theorem (5.1): (A,B) is controllable, if and only if

$$\text{rank}(P(A,B)) = n+1$$

Proof: Controllability of (A,B) implies $\exists L$ (state-feedback) such that

$$[sI-A,-B]\begin{bmatrix} I_n & 0 \\ L & I_l \end{bmatrix} = [sI-A',-B], \text{ with } (A',B) \text{ cyclic}$$

If (A',B) is cyclic, there exist a $u \in R^l$ such that (A',Bu) is controllable [Che,1].

We may now construct an input coordinate transformation R: $R = [\underline{u}, R']$ with $Bu = b'$, i.e. (A', b') is controllable that is

$$[sI - A', -B]\begin{bmatrix} I_n & 0 \\ 0 & [u, R'] \end{bmatrix} = \left[sI - A', [b', \widetilde{B}]\right]$$

There always exist a state coordinate transformation which transforms (A', b') into the controllable canonical form $(\widetilde{A}, \widetilde{b})$. So we have transformed a controllable pair (A,B) and thus $T(s)$ into a form $[sI - \widetilde{A}; \widetilde{b}; \widetilde{B}']$. Using Lemma (5.2) and Proposition (5.1) that rank $P(A, B)$ is invariant under Brunovsky transformation, then the result is established, since $[sI - \widetilde{A}, \widetilde{b}]$ is in the controllable canonical form. The opposite is proved by contradiction.

$\square$

<u>**Corollay (5.1)**</u>: For a general pair (A, B) we have that

$$\mathrm{rank}(P(A, B)) = r + 1$$

where r is the dimension of the controllable subspace.

<u>Proof</u>: For the sake of simplicity, assume the case of distinct input decoupling zeros. Uncontrollability implies that there exist (Q, L, R) that would bring $[sI - A, -B]$ directly into the extended Brunovsky canonical form [Kar. & MacB.,1] defined below:

$$\left[\begin{array}{c|c|c} \overbrace{sI - A_\epsilon}^{n_\epsilon} & \overbrace{0}^{n_f} & \overbrace{-B_\epsilon}^{p} \\ \hline 0 & sI - A_f & 0 \end{array}\right] \begin{array}{l} \} \; n_\epsilon \\ \} \; n_f \end{array} = [sI - A_B, -B_B]$$

where A_ϵ =diag $\{A_{\epsilon_1}, ..., A_{\epsilon_p}\}$ and B_ϵ = block diag $\{\underline{w}_{\epsilon_1}, ..., \underline{w}_{\epsilon_p}\}$

$$A_{\epsilon_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; A_f = \mathrm{J}, \; n = n_\epsilon + n_f, \; n_\epsilon = \textstyle\sum_{i=1}^{p} \epsilon_i, \; A_B = \left[\begin{array}{c|c} A_\epsilon & 0 \\ \hline 0 & A_f \end{array}\right],$$

$$B_B = \left[\begin{array}{c} B_\epsilon \\ \hline 0 \end{array}\right], \; w_{\epsilon_i} = \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right\} \epsilon_i$$

is a generalisation of the Brunovsky canonical form for controllable system, to systems that need not be completely controllable [Kar. & MacB.,1].

Let $\phi(s) = |sI - A_f|$; we have to show that $\phi(s)$ is the greatest common divisor of $\underline{g}(A_B, B_B)$. To calculate $\underline{g}(A_B, B_B)$, we have to take every column of $sI - A_f$, since otherwise, the corresponding minor is identically zero and the number of zero minors is equal to n_f. By doing so, we see that all the minors have $|sI - A_f|$ as a common factor which shows that $|sI - A_f|$ is a g.c.d. of $\underline{g}(A_B, B_B)$. By contradiction argument it then follows that $|sI - A_f|$ is a greatest common divisor (gcd). If $s = \lambda$ is a root of $\phi(s)$, then

$$[1, \lambda, ..., \lambda^{n-1}]P(A_B, B_B) = 0 \Leftrightarrow \mathcal{N}(P(A_B, B_B)) \neq \{0\} \Leftrightarrow$$
$$\text{rank}(P(A_B, B_B)) < n + 1$$

In fact, $\text{rank}(P(A_B, B_B)) = r + 1$ where r is the dimension of the controllable subspace.

5.3 Computation of Plucker Matrices of General Rational Vector Spaces

The aim of the following sections is to discuss the systematic computation and then to investigate the rank properties of Plucker matrices of least degree polynomial matrices. An essential part in this study, is defining the structure of the Grassman vector which leads us to the computation of the Plucker matrix of the least degree matrix from the original data. A general procedure how to compute the Plucker matrix step by step is given.

Definition (5.1): Let $A(s) \in R^{m \times p}[s], \rho_{R(s)}\{A(s)\} = p, m \geq p$ and let $A = [\underline{a}_1(s), ..., \underline{a}_p(s)]$. The Plucker matrix of $A(s)$ is defined by

$$\begin{aligned} C_p(A(s)) &= \underline{a}_1(s) \wedge ... \wedge \underline{a}_p(s) = \underline{g}(A(s)) \\ &= P_A \underline{e}_\delta(s), \underline{e}_\delta(s) = [1, s, ..., s^\delta]^t \end{aligned} \tag{5.11}$$

where $\delta = \partial\{\underline{g}(A(s))\}$ and $P_A \in R^{\binom{m}{p} \times (\delta+1)}$ is defined as the Plucker matrix of $A(s)$.

$\square$

The problems examined here are first the computation of P_A from the original data and then the investigation of its rank properties.

Remark (5.3): If $A(s)$ has zeros (finite), then for each one of them $z, \underline{g}(A(z)) = \underline{0} = P_A \underline{e}_\delta(z)$ and thus $\mathcal{N}_r(P_A) \neq \{0\}$. We distinguish the following two cases:

i) $\begin{pmatrix} m \\ p \end{pmatrix} < (\delta+1)$: in this case, $P_A \underline{e}_\delta(z) = 0$ may be satisfied with $\rho(P_A)$ either $= \begin{pmatrix} m \\ p \end{pmatrix}$, or $< \begin{pmatrix} m \\ p \end{pmatrix}$ and nothing can be inferred for the rank of P_A.

ii) $\begin{pmatrix} m \\ p \end{pmatrix} > (\delta+1)$: in this case, $P_A \underline{e}_\delta(z) = 0$ implies that $\rho(P_A) < (\delta+1)$.

$\square$

Note that $\underline{g}(A(s))$ is an invariant of $\mathcal{X} \equiv col.sp_{R(s)}\{A(s)\}$ modulo $R[s]$ [Kar., & Gia.,1], and thus we can consider the Plucker matrices which correspond to matrices $A(s)$ having no zeros. Furthermore, since by unimodular equivalence we can make such bases, column reduced; such transformation simply multiply the Plucker matrix by a constant. In the following we shall assume that $A(s)$ is a minimal basis matrix which is ordered according to ascending degrees i.e., if

$$A(s) = [..., \underline{a}_i(s), ...], \partial[\underline{a}_i(s)] = \delta_i, \text{ then } 0 \le \delta_1 \le \cdots \le \delta_p.$$

Under these assumptions we may write:

$$\underline{a}_i(s) = A_i \underline{e}_{\delta_i}(s), A_i \in R^{m \times (\delta_i+1)}, i \in \tilde{p} \tag{5.12}$$

and thus

$$A(s) = \underbrace{[A_1, ..., A_p]}_{\equiv T_A} \underbrace{\begin{bmatrix} \underline{e}_{\delta_1}(s) & & \underline{0} \\ & \ddots & \\ \underline{0} & & \underline{e}_{\delta_p}(s) \end{bmatrix}}_{\equiv S(s)} \tag{5.13}$$

where $T_A \in R^{m\times q}, q = \sum_{i=1}^{p}(\delta_i + 1)$ is the <u>coefficient matrix</u> of $A(s)$ and $S(s)$ is the <u>structure matrix</u> of $A(s)$ defined by the <u>index</u> $\mathcal{I} = \{\delta_i : 0 \leq \delta_1 \leq \cdots \leq \delta_p\}$.

By combining equations (5.11) and (5.13) we have:

$$C_p(A(s)) = C_p(T_A)C_p(S(s)) = P_A\underline{e}_\delta(s) \tag{5.14}$$

Note that

$$C_p(S(s)) = \underline{g}(S(s)) = P_S\underline{e}_\delta(s) = \underline{g}_S(s) \tag{5.15}$$

where $\delta = \sum_{i=1}^{p} \delta_i$ and $P_S \in R^{\binom{q}{p}\times(\delta+1)}$. Thus

$$P_A = C_p(T_A)P_S \tag{5.16}$$

The problem here is to define the structure of $\underline{g}(S(s))$ and thus of P_S. The above problem is central to our study, since the structure of P_S will define which part of $C_p(T_A)$ is essential for the structure of P_A. This problem is also related to the computation of P_A from the original data. Note that $S(s)$ is defined explicitly as:

$$S(s) = \begin{bmatrix} 1 & & & \\ s & & & \\ \vdots & & & \\ s^{\delta_1} & & & \underline{0} \\ & 1 & & \\ & s & & \\ & \vdots & & \\ & s^{\delta_2} & & \\ & & \ddots & \\ \underline{0} & & & 1 \\ & & & s \\ & & & \vdots \\ & & & s^{\delta_p} \end{bmatrix} \in R^{q\times p}[s], q = \sum_{i=1}^{p}(\delta_i + 1) \tag{5.17}$$

Some of the basic properties of the $\underline{g}(S(s)) \equiv \underline{g}_S(s)$ Grassman vector are examined next. We first note that

$$\underline{g}_S(s) \in R^{\nu}[s], \nu = \binom{q}{p}, \partial[\underline{g}_S(s)] = \delta = \sum_{i=1}^{p} \delta_i$$

and every entry in $\underline{g}_S(s)$ is parametrised by a sequence $w = (i_1, i_2, ..., i_p) \in Q_{p,q}$ and thus $\underline{g}_S(s)$ may be denoted by

$$\underline{g}_S(s) = \begin{bmatrix} \vdots \\ g_w(s) \\ \vdots \end{bmatrix}, w = (i_1, i_2, ..., i_p) \in Q_{p,q}$$

$g_w(s)$ are referred to as Plucker coordinates of $S(s)$

Central Issues: In the computation of $S(s)$ we are concerned with the following issues:

- Define those $w \in Q_{p,q}$ such that $g_w(s) \equiv 0$;
- Define the form of nonzero $g_w(s)$ and their corresponding location (in terms of w).

□

We introduce first some notation and definitions:

Definition (5.2): Given the set of ordered integers $0 \le \delta_1 \le \cdots \le \delta_p$, $q = \sum_{i=1}^{p}(\delta_i + 1)$ the interval of integers $[1, ..., q]$ is partitioned into intervals as shown below:

$$\underbrace{1, 2, ..., \delta_1 + 1}_{\Delta_1}; \underbrace{\delta_1 + 2, ..., \delta_1 + \delta_2 + 2}_{\Delta_2}; ...; \underbrace{\delta_1 + \cdots + \delta_{p-1} + p, ..., q}_{\Delta_p} \tag{5.18}$$

For each integer $k \in \{1, ..., q\}$ we associate two parameters, its index $\equiv \mathcal{V}(k)$ indicating the interval where it belongs to and its Stathm $\equiv \sigma(k)$ indicating the relative order in the interval. Thus, if

$$\sum_{j=1}^{i-1}(\delta_j + 1) < k \le \sum_{j=1}^{i}(\delta_j + 1)$$

$k \in \Delta_i$ and this may be denoted by $\mathcal{V}(k) \equiv \Delta_i$; if $k \in \Delta_i$, then its Stathm $\sigma(k)$ is defined by

$$\sigma(k) = k - \sum_{j=1}^{i-1}(\delta_j + 1) - 1 \tag{5.19}$$

□

Using the above notation we may state the following result.

Proposition (5.4): Let $S(s)$ be the structure matrix defined by the set of indices $0 \leq \delta_1 \leq \cdots \leq \delta_p$. Then the Plucker coordinates $g_w(s), w = (i_1, i_2, ..., i_p) \in Q_{p,q}$ of $g_S(s)$ have the following properties:

i) $g_w(s) \not\equiv 0$, if and only if for the sequence $w = (i_1, i_2, ..., i_p) \in Q_{p,q}, i_1 \in \Delta_1, i_2 \in \Delta_2, ..., i_p \in \Delta_p$.

ii) $g_w(s) \equiv 0$, if at least two indices in w are taken from the same interval.

iii) If $i_1 \in \Delta_1, i_2 \in \Delta_2, ..., i_p \in \Delta_p$, then

$$g_w(s) = s^{\sigma(w)}, \sigma(w) = \sigma(i_1) + \cdots + \sigma(i_p) \tag{5.20}$$

Proof: Note that if $S[w]$ is any $p{\times}p$ submatrix of $S(s)$ where $w = (i_1, i_2, ..., i_p) \in Q_{p,q}$, then the following properties are deduced by inspection of the structure of $S(s)$,

1) Every row of $S(s)$ contains only one non-zero element of the type $s^\beta, \beta = 0, 1, ..., \delta_i$;

2) If a column of $S(s)$ contains more than one non-zero elements, then from observation (1), it follows that there is at least one zero column in $S[w]$ and thus $|S[w]| \equiv 0$.

3) The condition that $S[w]$ has a column with more than one nonzero elements is equivalent to that at least two indices from w are taken from the same interval.

4) From (1), it follows that the presence of an identically zero column is equivalent to that there is another column in $S[w]$ with at least two nonzero elements.

5) If every column in $S[w]$ contains only one nonzero element then the submatrix $S[w]$ is diagonal and has the form

$$S[w] = \begin{bmatrix} s^{\sigma(i_1)} & & & 0 \\ & s^{\sigma(i_2)} & & \\ & & \ddots & \\ 0 & & & s^{\sigma(i_p)} \end{bmatrix} \tag{5.21}$$

For the following reasons:

- Since $i_1 < i_2 < \cdots < i_p$ and $i_1 \in \Delta_1$ (otherwise we have two indices from the same interval and thus an entirely zero column) the nonzero element in the first column is on the first row, i.e. in (1,1) position and by inspection, has a value $s^{\sigma(i_1)}$.

- For the same reasons as before, $i_2 \in \Delta_2$, and since every row has only one nonzero element, it can not be on the first row. Since i_2 defines the second row and all nonzero elements are on the second column, it follows that the second element associated with i_2 is in the (2,2) position of $S[w]$ and has obviously value $s^{\sigma(i_2)}$.

The results generalise and the structure of $S[w]$ as in (5.21) is clearly established. The above arguments establish the result.

□

The above result suggests a method for computing the family of sequences from $Q_{p,q}$ which have nonzero Plucker coordinates, as well as the form of these elements. We define the following:

Definition (5.3): A sequence $w = (i_1, i_2, ..., i_p) \in Q_{p,q}$ for which $i_1 \in \Delta_1, i_2 \in \Delta_2, ..., i_p \in \Delta_p$ is called nonsingular; otherwise, i.e. if more than one indices are taken from the same interval, then w is called singular.

□

Remark (5.4): The set of singular and nonsingular sequences are complementary subsets of $Q_{p,q}$, i.e. by defining one we define the other as the complementary set with respect to $Q_{p,q}$.

□

Remark (5.5): The singular sequences define the zero Plucker coordinates, whereas the nonsingular ones are the nonzero elements.

□

The set of nonsingular sequences may be computed by constructing the following table:

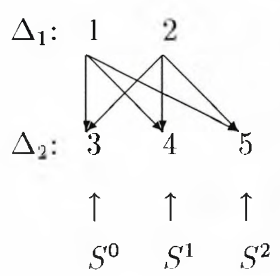
Structured Table

$$
\begin{array}{lllll}
\Delta_1 & \rightarrow & 1, & 2, & \cdots, \quad \delta_1 + 1 \\
\Delta_2 & \rightarrow & \delta_1 + 2, & \delta_1 + 3, & \cdots, \quad \delta_1 + \delta_2 + 2 \\
\vdots & & \vdots & \vdots & \qquad \vdots \\
\Delta_p & \rightarrow & \delta_1 + \cdots + \delta_{p-1} + p, & \delta_1 + \cdots + \delta_{p-1} + p + p + 1, & \cdots, \quad \delta_1 + \delta_2 + \cdots + \delta_p + p \\
 & & \uparrow & \uparrow & \qquad \uparrow \\
 & & S^0 & S^1 & \qquad S^{\delta_p}
\end{array}
$$

which indicates the index sets and the Stathm of each element in the set. $S^0, S^1, ..., S^{\delta_p}$ are the columns of the structure matrix $S(s)$ which indicates the set of nonsingular sequences where nonsingular sequences define the nonzero Plucker coordinates.

From the definition of the nonsingular sequences we have that: The set $\Omega_{p,q}(\delta_1, ..., \delta_p)$ of nonsingular sequences of $Q_{p,q}$ may be defined from the above table as paths passing once through the elements of each of the Δ_i index sets. To demonstrate the construction of $\Omega_{p,q}(\delta_1, ..., \delta_q)$ we give the following example.

<u>Example (5.1)</u>: Let $\delta_1 = 1, \delta_2 = 2$ then from (5.17) we get $p = 2, q = 5$



$$
\begin{array}{lll}
(1,3) & \rightarrow & 1 \\
(1,4) & \rightarrow & s^1 \\
(1,5) & \rightarrow & s^2 \\
(2,3) & \rightarrow & s^1 \\
(2,4) & \rightarrow & s^2 \\
(2,5) & \rightarrow & s^3
\end{array}
$$

This shows that the set $\Omega_{2,5}(\delta_1, \delta_2)$ of nonsingular sequences of $Q_{2,5}$ may be defined from the table as path passing once through the elements of each of the Δ_1, Δ_2 index set, which gives the result for nonzero Plucker coordinates.

and thus

$$P_S = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ S^0 & S^1 & S^2 & S^3 \end{array}\right] \begin{array}{cc} (1,2) & \rightarrow \\ (1,3) & \rightarrow \\ (1,4) & \rightarrow \\ (1,5) & \rightarrow \\ (2,3) & \rightarrow \\ (2,4) & \rightarrow \\ (2,5) & \rightarrow \\ (3,4) & \rightarrow \\ (3,5) & \rightarrow \\ (4,5) & \rightarrow \end{array} \begin{bmatrix} 0 \\ 1 \\ s \\ s^2 \\ s \\ s^2 \\ s^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{g}(S(s))$$

<u>Example (5.2)</u>: Let $\delta_1 = 1, \delta_2 = 2, \delta_3 = 2 \rightarrow p = 3, q = 8$, constructing the structured table

$$\left.\begin{array}{ccccc} \delta_1 = 1 & \rightarrow & 1, & 2 & \\ \delta_2 = 2 & \rightarrow & 3, & 4, & 5 \\ \delta_3 = 2 & \rightarrow & 6, & 7, & 8 \\ & & \uparrow & \uparrow & \uparrow \\ & & S^0 & S^1 & S^2 \end{array}\right\} \Rightarrow$$

$$\begin{array}{c|cccccc|}
(1,3,6) & 1 & 0 & 0 & 0 & 0 & 0 \\
(1,3,7) & 0 & 1 & 0 & 0 & 0 & 0 \\
(1,3,8) & 0 & 0 & 1 & 0 & 0 & 0 \\
\cline{2-7}
(1,4,6) & 0 & 1 & 0 & 0 & 0 & 0 \\
(1,4,7) & 0 & 0 & 1 & 0 & 0 & 0 \\
(1,4,8) & 0 & 0 & 0 & 1 & 0 & 0 \\
\cline{2-7}
(1,5,6) & 0 & 0 & 1 & 0 & 0 & 0 \\
(1,5,7) & 0 & 0 & 0 & 1 & 0 & 0 \\
(1,5,8) & 0 & 0 & 0 & 0 & 1 & 0 \\
\cline{2-7}
(2,3,6) & 0 & 1 & 0 & 0 & 0 & 0 \\
(2,3,7) & 0 & 0 & 1 & 0 & 0 & 0 \\
(2,3,8) & 0 & 0 & 0 & 1 & 0 & 0 \\
\cline{2-7}
(2,4,6) & 0 & 0 & 1 & 0 & 0 & 0 \\
(2,4,7) & 0 & 0 & 0 & 1 & 0 & 0 \\
(2,4,8) & 0 & 0 & 0 & 0 & 1 & 0 \\
\cline{2-7}
(2,5,6) & 0 & 0 & 0 & 1 & 0 & 0 \\
(2,5,7) & 0 & 0 & 0 & 0 & 1 & 0 \\
(2,5,8) & 0 & 0 & 0 & 0 & 0 & 1 \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 & S^0 & S^1 & S^2 & S^3 & S^4 & S^5
\end{array}$$

The above matrix P_S has been computed from the structured table. Note this is the essential part of P_S after deleting the zero rows.

We continue with the rest of the structured table to show the computation of the nonzero elements of P_S, which is a path passing once through the elements of each of $\Delta_1, \Delta_2, \Delta_3$ index sets.

$$(1,3) \quad \left.\begin{array}{lcl} (1,3,6) & \to & 1 \\ (1,3,7) & \to & s \\ (1,3,8) & \to & s^2 \end{array}\right\} \underline{e}_{\delta_3}(s)$$

$$(1,4) \quad \left.\begin{array}{lcl} (1,4,6) & \to & s \\ (1,4,7) & \to & s^2 \\ (1,4,8) & \to & s^3 \end{array}\right\} s\underline{e}_{\delta_3}(s)$$

$$(1,5) \quad \left.\begin{array}{lcl} (1,5,6) & \to & s^2 \\ (1,5,7) & \to & s^3 \\ (1,5,8) & \to & s^4 \end{array}\right\} s^{\delta_2}\underline{e}_{\delta_3}(s)$$

$$(2,3) \quad \left.\begin{array}{lcl} (2,3,6) & \to & s \\ (2,3,7) & \to & s^2 \\ (2,3,8) & \to & s^3 \end{array}\right\} s^{\delta_1}\underline{e}_{\delta_3}(s)$$

$$(2,4) \quad \left.\begin{array}{lcl} (2,4,6) & \to & s^2 \\ (2,4,7) & \to & s^3 \\ (2,4,8) & \to & s^4 \end{array}\right\} s^{\delta_1+1}\underline{e}_{\delta_3}(s)$$

$$(2,5) \quad \left.\begin{array}{lcl} (2,5,6) & \to & s^3 \\ (2,5,7) & \to & s^4 \\ (2,5,8) & \to & s^5 \end{array}\right\} s^{\delta_1+\delta_2}\underline{e}_{\delta_3}(s)$$

Example (5.3): $\delta_1 = 1, \delta_2 = 1, \delta_3 = 2, \delta_4 = 2 \Rightarrow p = 4, q = 10$

$$\left.\begin{array}{lclll}\delta_1 & \rightarrow & 1 & 2 & \\ \delta_2 & \rightarrow & 3 & 4 & \\ \delta_3 & \rightarrow & 5 & 6 & 7 \\ \delta_4 & \rightarrow & 8 & 9 & 10\end{array}\right\} \Rightarrow$$

To compute the nonzero elements of P_S we continue as follows:

$$(1,3) \begin{cases} \rightarrow (1,3,5) \rightarrow \left.\begin{array}{l}(1,3,5,8)\\(1,3,5,9)\\(1,3,5,10)\end{array}\right\} \underline{e}_{\delta_4}(s) \\ \rightarrow (1,3,6) \rightarrow \left.\begin{array}{l}(1,3,6,8)\\(1,3,6,9)\\(1,3,6,10)\end{array}\right\} s\,\underline{e}_{\delta_4}(s) \\ \rightarrow (1,3,7) \rightarrow \left.\begin{array}{l}(1,3,7,8)\\(1,3,7,9)\\(1,3,7,10)\end{array}\right\} s^{\delta_3}\underline{e}_{\delta_4}(s) \end{cases}$$

$$(1,4) \begin{cases} \rightarrow (1,4,5) \rightarrow \left.\begin{array}{l}(1,4,5,8)\\(1,4,5,9)\\(1,4,5,10)\end{array}\right\} s^{\delta_2}\underline{e}_{\delta_4}(s) \\ \rightarrow (1,4,6) \rightarrow \left.\begin{array}{l}(1,4,6,8)\\(1,4,6,9)\\(1,4,6,10)\end{array}\right\} s^{\delta_2+1}\underline{e}_{\delta_4}(s) \\ \rightarrow (1,4,7) \rightarrow \left.\begin{array}{l}(1,4,7,8)\\(1,4,7,9)\\(1,4,7,10)\end{array}\right\} s^{\delta_2+\delta_3}\underline{e}_{\delta_4}(s) \end{cases}$$

$$(2,3) \begin{cases} \rightarrow (2,3,5) \rightarrow \left.\begin{array}{l}(2,3,5,8)\\(2,3,5,9)\\(2,3,5,10)\end{array}\right\} s^{\delta_1}\underline{e}_{\delta_4}(s) \\ \rightarrow (2,3,6) \rightarrow \left.\begin{array}{l}(2,3,6,8)\\(2,3,6,9)\\(2,3,6,10)\end{array}\right\} s^{\delta_1+1}\underline{e}_{\delta_4}(s) \\ \rightarrow (2,3,7) \rightarrow \left.\begin{array}{l}(2,3,7,8)\\(2,3,7,9)\\(2,3,7,10)\end{array}\right\} s^{\delta_1+\delta_3}\underline{e}_{\delta_4}(s) \end{cases}$$

$$
\begin{tikzcd}
 & & (2,4,5,8) & \\
 & (2,4,5) \arrow[ur] \arrow[r] \arrow[dr] & (2,4,5,9) & s^{\delta_1+\delta_2}\underline{e}_{\delta_4}(s) \\
 & & (2,4,5,10) & \\
 & & (2,4,6,8) & \\
(2,4) \arrow[uuur] \arrow[r] \arrow[dddr] & (2,4,6) \arrow[ur] \arrow[r] \arrow[dr] & (2,4,6,9) & s^{\delta_1+\delta_2+1}\underline{e}_{\delta_4}(s) \\
 & & (2,4,6,10) & \\
 & & (2,4,7,8) & \\
 & (2,4,7) \arrow[ur] \arrow[r] \arrow[dr] & (2,4,7,9) & s^{\delta_1++\delta_2+\delta_3}\underline{e}_{\delta_4}(s) \\
 & & (2,4,7,10) &
\end{tikzcd}
$$

The generation of the final subvector of $g(s)$ corresponding to nonzero entries may be achieved by observing the generation of the various elements.

$\square$

Note that the structured table may be seen as a table of Stathms as shown below:

$$
\left.\begin{array}{cclll}
\{\Delta_1\} & \rightarrow & 0 & 1 & \\
\{\Delta_2\} & \rightarrow & 0 & 1 & \\
\{\Delta_3\} & \rightarrow & 0 & 1 & 2 \\
\{\Delta_4\} & \rightarrow & 0 & 1 & 2
\end{array}\right\}
$$

Furthermore each nonsingular sequence $w = (i_1, ..., i_p) \in Q_{p,q}$ may be represented by a vector, where each coordinate denotes the Stathm of the corresponding integer in $\{\Delta_i\}$, i.e. to w we associate

$$
w(i_1, i_2, ..., i_p) \rightarrow \begin{bmatrix} \sigma(i_1) \\ \sigma(i_2) \\ \vdots \\ \sigma(i_p) \end{bmatrix}
$$

and this is called the <u>Stathm representation</u> of $w = (i_1, ..., i_p)$. The generation of sequences for example (5.3) is illustrated below:

<u>Example (5.3)</u>: We continue with the last example and through it we demonstrate the systematic computation of the coordinates of $\underline{g}_S(s)$. The steps illustrated below are of general nature and may be used for any $S(s)$ structure matrix.

Step 1: Consider the composition of Δ_1, Δ_2 intervals, i.e.

$$\begin{array}{lcll} \Delta_1 & : & 1 & 2 \\ \Delta_2 & : & 3 & 4 \\ & & \uparrow & \uparrow \\ & & S^0 & S^1 \end{array} \qquad \text{or} \qquad \begin{array}{lll} \sigma\{\Delta_1\} & :0 & 1 \\ \sigma\{\Delta_2\} & :0 & 1 \end{array}$$

$\{\Delta_1, \Delta_2\} \equiv$ Composition of Δ_1, Δ_2 is defined by
$\{\Delta_1, \Delta_2\} = \{(1,3),(1,4),(2,3),(2,4)\}$

The Stathm representation of $\{\Delta_1, \Delta_2\}$ is

$$\sum\{\Delta_1, \Delta_2\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Note: There is an one to one mapping between $\{\Delta_1, \Delta_2\}$, and $\sum\{\Delta_1, \Delta_2\}$.

Step 2: Consider the composition of $\Delta_1, \Delta_2, \Delta_3$, this is defined as:

$$\{\Delta_1, \Delta_2, \Delta_3\} = \left\{ w^{(3)} : w^{(3)} = (w^{(2)}, i_3), \forall w^{(2)} \in \{\Delta_1, \Delta_2\}, i_3 \in \Delta_3 \right\}$$

i.e.

$$\begin{array}{ccc} & (1,3) & \\ & \downarrow & \\ & (5,6,7) & \\ \swarrow & \downarrow & \searrow \\ (1,3,5) & (1,3,6) & (1,3,7) \end{array} \qquad \begin{array}{ccc} & (1,4) & \\ & \downarrow & \\ & (5,6,7) & \\ \swarrow & \downarrow & \searrow \\ (1,4,5) & (1,4,6) & (1,4,7) \end{array}$$

$$\begin{array}{ccc} & (2,3) & \\ & \downarrow & \\ & (5,6,7) & \\ \swarrow & \downarrow & \searrow \\ (2,3,5) & (2,3,6) & (2,3,7) \end{array} \qquad \begin{array}{ccc} & (2,4) & \\ & \downarrow & \\ & (5,6,7) & \\ \swarrow & \downarrow & \searrow \\ (2,4,5) & (2,4,6) & (2,4,7) \end{array}$$

and

$$\begin{aligned} \{\Delta_1, \Delta_2, \Delta_3\} &= \{(1,3,5),(1,3,6),(1,3,7),(1,4,5),(1,4,6),(1,4,7),(2,3,5), \\ &\quad ,(2,3,6),(2,3,7),(2,4,5),(2,4,6),(2,4,7)\} \end{aligned}$$

The Stathm set representation may be defined in a similar manner by:

$$\begin{tikzcd}
& \begin{pmatrix}0\\0\end{pmatrix} \arrow[d] & \\
& (0,1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\2\end{pmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \begin{pmatrix}0\\1\end{pmatrix} \arrow[d] & \\
& (0,1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\1\end{pmatrix} & \begin{pmatrix}0\\1\\2\end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}
& \begin{pmatrix}1\\0\end{pmatrix} \arrow[d] & \\
& (0,1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\2\end{pmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \begin{pmatrix}1\\1\end{pmatrix} \arrow[d] & \\
& (0,1,2) \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{pmatrix}1\\1\\0\end{pmatrix} & \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\1\\2\end{pmatrix}
\end{tikzcd}$$

$= \sum\{\Delta_1, \Delta_2, \Delta_3\}$

Note that from the above diagram we can construct the final vector as follows:

If $\sigma_{x_1,\ldots,x_k} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \equiv \sum_{i=1}^{k} x_i$ is the Stathm of the corresponding element, then the final vector representation of the reduced Grassman vector may be defined as follows:

Let

$$\begin{aligned}
\phi_{x_1,x_2,\ldots,x_{p-1}} &\equiv s^{\sigma_{x_1,\ldots,x_{p-1}}} \\
f_{x_1,x_2,\ldots,x_{p-1}} &\equiv \phi_{x_1,\ldots,x_{p-1}} \cdot \underline{e}_{\delta_p}(s) \in R^{(\delta_p+1)}[s]
\end{aligned}$$

Thus we have for this specific example:

$$\begin{array}{lcl} f_{0,0,0} & \rightarrow & \underline{e}_2(s) \\ f_{0,0,1} & \rightarrow & s\,\underline{e}_2(s) \\ f_{0,0,2} & \rightarrow & s^2\,\underline{e}_2(s) \\ f_{0,1,0} & \rightarrow & s\,\underline{e}_2(s) \\ f_{0,1,1} & \rightarrow & s^2\,\underline{e}_2(s) \\ f_{0,1,2} & \rightarrow & s^3\,\underline{e}_2(s) \\ f_{1,0,0} & \rightarrow & s\,\underline{e}_2(s) \end{array} \qquad \begin{array}{lcl} f_{1,0,1} & \rightarrow & s^2\,\underline{e}_2(s) \\ f_{1,0,2} & \rightarrow & s^3\,\underline{e}_2(s) \\ f_{1,1,0} & \rightarrow & s^2\,\underline{e}_2(s) \\ f_{1,1,1} & \rightarrow & s^3\,\underline{e}_2(s) \\ f_{1,1,2} & \rightarrow & s^4\,\underline{e}_2(s) \end{array}$$

The above examples indicate how we can generalise the computational procedure. This is described below:

General Procedure

The Stathm table may be used for the definition of the nonsingular sequences and the form of the reduced Grassman representative. For the computations we follow the steps:

Step 1. From the indices $\delta_1, ..., \delta_p$ define the Stathm table:

STATHM TABLE

$$\begin{array}{rcllll} \{\Delta_1\} & \rightarrow & 0, & 1, & \cdots, & \delta_1 \\ \{\Delta_2\} & \rightarrow & 0, & 1, & \cdots, & \delta_2 \\ & & \vdots & \vdots & & \vdots \\ \{\Delta_{p-1}\} & \rightarrow & 0, & 1, & \cdots, & \delta_{p-1} \\ \{\Delta_p\} & \rightarrow & 0, & 1, & \cdots, & \delta_p \end{array}$$

Step 2: Define the composition set $\{\Delta_1, \Delta_2, ..., \Delta_{p-1}\}$ using the recursive relations:

$$\begin{aligned} \{\Delta_1\} &= \{w_{i(1)} : w_{i(1)} = (i(1) - 1), i(1) \in \widetilde{\delta_1 + 1}\} \\ &= \{(0), (1), ..., (\delta_1)\} \text{ ordered lexicographically} \\ \{\Delta_1, \Delta_2\} &= \{w_{i(1)i(2)} : w_{i(1)i(2)} = (w_{i(1)} : i(2) - 1), \forall w_{i(1)} \in \{\Delta_1\}, i(2) \in \widetilde{\delta_2 + 1}\} \\ &= \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 1 \\ \delta_2 \end{pmatrix}; ...; \begin{pmatrix} \delta_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right\} \\ & \qquad \text{ordered lexicographically} \end{aligned}$$

$$\{\Delta_1, \Delta_2, ..., \Delta_k\} \equiv \{w_{i(1)...i(k-1)i(k)} : w_{i(1)...i(k-1)i(k)} = (w_{i(1)...i(k-1)}, i(k) - 1), \forall w_{i(1),...,i(k-1)} \in \{\delta_1, ..., \Delta_{k-1}\} \text{ and } i(k) \in \widetilde{\delta_k + 1}\}$$

Step 3: Having defined the composition $\{\Delta_1, \Delta_2, ..., \Delta_{p-1}\}$ which is the lexicographically ordered set, then for every $w_{i(1),...i(p-1)} \in \{\Delta_1, ..., \Delta_{p-1}\}$ we define the corresponding part of the reduced GR (Grassman Representative) as:

If $w_{i(1),...,i(p-1)} = \{x_1, ..., x_{p-1}\}$ then

$$f_{x_1,x_2,...,x_{p-1}} = s^{x_1+\cdots+x_{p-1}} \underline{e}_{\delta_p}(s) \tag{5.22}$$

Step 4: The reduced GR may then be expressed as the vector

$$\begin{bmatrix} \vdots \\ \hline f_{x_1,x_2,...,x_{p-1}} \\ \hline \vdots \end{bmatrix} \}\delta_p + 1, (x_1, x_2, ..., x_{p-1}) \in \{\Delta_1, \Delta_2, ..., \Delta_{p-1}\} \tag{5.23}$$

Thus the element corresponding to the $w = (x_1, x_2, ..., x_{p-1}, x_p)$ sequence of Stathm is

$$g_w(s) = s^{x_1+x_2+...+x_{p-1}+x_p} \text{ in the } (x_1, x_2, ..., x_{p-1}, x_p) \text{ position} \tag{5.24}$$

□

Proposition (5.5): if $w = (x_1, x_2, ..., x_{p-1}, x_p) \in \{\Delta_1, ..., \Delta_p\}$ is a Stathm representation, then

$$g_{w'}(s) = s^{\,x_1+\cdots+x_p} \tag{5.25}$$

The proof of the above is a straightforward consequence of the previous analysis.

□

Remark (5.6): The Grassman vector $\underline{g}_S(s)$ is an $\begin{pmatrix} q \\ p \end{pmatrix}$-dimensional vector, where $q = \sum_{i=1}^{p}(\delta_i + 1)$. The reduced Grassman vector defined by the nonsingular sequences, $\underline{g}_S^{r}(s)$ is a τ-dimensional vector, where

$$\tau = \Pi_{i=1}^{p}(\delta_i + 1) < \begin{pmatrix} \sum_{i=1}^{p}(\delta_i + 1) \\ p \end{pmatrix}$$

The value of τ follows from the definition of the nonsingular sequences.

□

Example (5.4): Note $\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{(n-k)!k!}$

i) $\delta_1 = 1, \delta_2 = 2 \rightarrow p = 2, q = 5$

$$\underbrace{\left(\begin{array}{c} q \\ p \end{array}\right) = \left(\begin{array}{c} 5 \\ 2 \end{array}\right) = 10}_{\text{total dimension}}, \qquad \underbrace{\tau = 2 \times 3 = 6}_{\text{reduced dimension}}$$

ii) $\delta_1 = 1, \delta_2 = 2, \delta_3 = 2 \rightarrow p = 3, q = 8$

$$\underbrace{\left(\begin{array}{c} q \\ p \end{array}\right) = \left(\begin{array}{c} 8 \\ 3 \end{array}\right) = 56}_{\text{total dimension}}, \qquad \underbrace{\tau = 2 \times 3 \times 3 = 18}_{\text{reduced dimension}}$$

iii) $\delta_1 = 1, \delta_2 = 1, \delta_3 = 2, \delta_4 = 2 \rightarrow p = 4, q = 10$

$$\underbrace{\left(\begin{array}{c} q \\ p \end{array}\right) = \left(\begin{array}{c} 10 \\ 4 \end{array}\right) = 210}_{\text{total dimension}}, \qquad \underbrace{\tau = 2 \times 2 \times 3 \times 2 = 24}_{\text{reduced dimension}}$$

Remark (5.7): It is obvious from the examples that there is a considerable reduction in the computation effort by using a procedure based on the nonsingular sequences.

□

We may now return to some of the original questions which have been asked. Thus the rank question of P_S Plucker matrix is considered first.

Proposition (5.6): For any set $\{\delta_i, i \in \tilde{p}\}$ the Plucker matrix P_S of $S(s)$ has always full rank.

Proof: From the construction of $\underline{g}^r_S(s)$ it is obvious that there exist a subvector of $\underline{g}^r_S(s)$ having the form $\underline{e}_\delta(s) = [1, s, ..., s^\delta]^t, \delta = \sum_{i=1}^p \delta_i$. To the vector $\underline{e}_\delta(s)$, there corresponds a minimal order minor of P_S which is $I_{\delta+1}$ and this proves the full rank property of P_S.

□

An example indicating the form of the Plucker matrix associated with $S(s)$ is given below.

Example (5.5): Let $\delta_1 = 1, \delta_2 = 1, \delta_3 = 2$. The Plucker matrix associated with $\underline{g}_S(s)$ is defined below:

$$P_S^T = \begin{array}{c} (0,0,0) \rightarrow \\ (0,0,1) \rightarrow \\ (0,0,2) \rightarrow \\ (0,1,0) \rightarrow \\ (0,1,1) \rightarrow \\ (0,1,2) \rightarrow \\ (0,2,0) \rightarrow \\ (0,2,1) \rightarrow \\ (0,2,2) \rightarrow \\ (1,0,0) \rightarrow \\ (1,0,1) \rightarrow \\ (1,0,2) \rightarrow \\ (1,1,0) \rightarrow \\ (1,1,1) \rightarrow \\ (1,1,2) \rightarrow \\ (1,2,0) \rightarrow \\ (1,2,1) \rightarrow \\ (1,2,2) \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow S^0 \\ \rightarrow S^1 \\ \rightarrow S^2 \\ \rightarrow S^1 \\ \rightarrow S^2 \\ \rightarrow S^3 \\ \rightarrow S^2 \\ \rightarrow S^3 \\ \rightarrow S^4 \\ \rightarrow S^1 \\ \rightarrow S^2 \\ \rightarrow S^3 \\ \rightarrow S^2 \\ \rightarrow S^3 \\ \rightarrow S^4 \\ \rightarrow S^3 \\ \rightarrow S^4 \\ \rightarrow S^5 \end{array}$$

The set $\{\Delta_1, \Delta_2, ..., \Delta_p\}$ defines immediately the structure of the reduced P_S, P_S^T.

Grouping according to Powers:

S^0 : $(0,0,0)$

S^1 : $(0,0,1)$, $(0,1,0)$, $(1,0,0)$

S^2 : $(0,0,2)$, $(0,1,1)$, $(0,2,0)$, $(1,0,1)$, $(1,1,0)$

S^3 : $(0,1,2)$, $(0,2,1)$, $(1,0,2)$, $(1,1,1)$, $(1,2,0)$

S^4 : $(0,2,2)$, $(1,1,2)$, $(1,2,1)$

S^5 : $(1,2,2)$

□

Summary

The Plucker matrix P_A is expressed by (5.16) as:

$$P_A = C_p(T_A)P_S \tag{5.26}$$

where

$$T_A = [A_1, ..., A_p] \in R^{m \times q}, \qquad q = \sum_{i=1}^{p}(\delta_i + 1)$$

$$P_S \in R^{\binom{q}{p} \times (\delta+1)}, \qquad \delta = \sum_{i=1}^{p} \delta_i$$

Note:

i) We denote by $\Delta = \{\delta_1, ..., \delta_p\}$ and $\{\Delta\}$ the nonsingular sequences;

ii) P_S has a number of zero rows, each one of them parametrised by a singular sequence $w \in Q_{p,q}$. The submatrix of P_S obtained by deleting the zero rows without changing the relative position of the rows characterised by the nonsingular sequences. It is defined as the <u>Δ-Reduced Plucker Structure Matrix</u> and it is denoted by P_S^{Δ}. Clearly, $P_S^{\tau} \in R^{\tau \times (\delta+1)}$, where $\tau = \Pi_{i=1}^{p}(\delta_i + 1)$.

iii) The submatrix of $C_p(T_A)$ obtained by deleting all columns associated with the nonsingular sequences, but retaining their relative ordering, is called the <u>Δ-reduced compound</u> and denoted by $C_p^{\Delta}(T_A)$. Clearly

$$P_A = C_p^{\Delta}(T_A)P_S^{\Delta} \tag{5.27}$$

iv) The nonsingular sequences $\{\Delta\} = \{\Delta_1, ..., \Delta_p\}$ are defined by the composition of the Stathm structure diagram

$$\begin{array}{lllllllll}
\{\Delta_1\} & \rightarrow & 0, & 1, & \cdots, & \delta_1 & & & \\
 & & & & & \vdots & & & \\
\{\Delta_2\} & \rightarrow & 0, & 1, & \cdots & \cdots & \cdots & \delta_2 & \\
\vdots & & \vdots & \vdots & & & & \vdots & \\
\{\Delta_p\} & \rightarrow & 0, & 1, & \cdots & \cdots & \cdots & \cdots & \delta_p
\end{array}$$

and each sequence $w = (x_1, x_2, ..., x_p), x_i \in [0, \delta_i]$ uniquely defines the $g_w(s)$ Plucker coordinate as $s^{x_1+\cdots+x_p}$ and thus the corresponding row in P_S^{Δ}.

v) All sequences with a given Stathm contribute in the definition of the corresponding column in P_A. In fact,

$$\Omega_k \equiv \{w = (x_1, ..., x_p) : w \in \{\Delta\} \text{ and } \sigma(w) = x_1 + \cdots + x_p = k\} \tag{5.28}$$

and

$$C_A^{\Delta}(T_A) = [\cdots, \underline{t}_w, \cdots] \tag{5.29}$$

$$P_A = [\underline{p}_0, \underline{p}_1, \cdots, \underline{p}_k, \cdots, \underline{p}_\delta] \tag{5.30}$$

then from the structure of P_S^{Δ} it follows that

$$\underline{p}_k = \sum_{w \in \Omega_k} \underline{t}_w \tag{5.31}$$

$\square$

Theorem (5.2): Let $\Omega_k = \{w\}$ be the subset of the nonsingular sequences of $Q_{p,q}$ with a given Stathm $k, k = 0, 1, ..., \delta = \sum_{i=1}^{p} \delta_i$. Furthermore, let $\underline{t}_w$ be the columns of $C_p(T_A)$ corresponding to the $\omega \in \Omega_k$ sequences. The k-th column of the P_A Plucker matrix is defined by

$$\underline{p}_k = \sum_{w \in \Omega_k} \underline{t}_w, k = 0, 1, 2, ..., \delta \tag{5.32}$$

$\square$

The proof follows from the above analysis. It is clear, that in defining P_A, it is essential to have an efficient procedure algorithm for computing the Ω_k set. An important issue in these computations is the following problem:

Problem: Given the Stathm structured table,

STATHM STRUCTURED TABLE

$$\begin{array}{lcccccccc}
\{\Delta_1\} & \to & 0, & 1, & \cdots, & & \delta_1 - 1, & \delta_1 \\
\{\Delta_2\} & \to & 0, & 1, & \cdots, & & \delta_2 - 1, & \delta_2 \\
\vdots & & & & & & & \\
\{\Delta_p\} & \to & 0, & 1, & \cdots & \cdots & \cdots\ \delta_p - 1, & \delta_p
\end{array}$$

define the set of all paths $(x_1, x_2, ..., x_p)$ (p-tuples), where $x_i \in [0, 1, ..., \delta_i]$ such that

$$x_1 + x_2 + \cdots + x_p = k, k \text{ fixed} \tag{5.33}$$

$\square$

This is a problem of partitioning the k-integer into integers taken from the Δ_i intervals. This problem is referred to as a k-path problem in the $\{\Delta_1, ..., \Delta_p\}$ table, and can be solved as follows:

Solution of the k-path problem

Let $\{\delta_1, \delta_2, ..., \delta_p, 0 \leq \delta_1 \leq \cdots \leq \delta_p\} \equiv \Delta$ be a set of integers, and k some fixed integer, $k \leq \delta = \sum_{i=1}^{p} \delta_i$. The set of sequences

$$\Omega_k \equiv \{w = (x_1, x_2, ..., x_p) : w \in \{\Delta\} \text{ and } \sigma(w) = x_1 + \cdots + x_p = k\}$$

may be constructed by a step by step procedure as described below. We first define:

Definition (5.4): If k is an integer from $[0, 1, ..., \delta]$, then any ordered p-tuple $\{x_1, x_2, ..., x_p\}, x_i \in [0, 1, ..., \delta]$ such that $x_1 + \cdots + x_p = k$ is called an oriented p-partition of k. If ordering in the partition $\{x_1, x_2, ..., x_p\}$ is not of importance, then it is like referring to the family of all permutations of $\{x_1, x_2, ..., x_p\}$ and any such representative is denoted by $< x_1, x_2, ..., x_p >$ and referred to as an apolar p-partition of k.

□

Example (5.4): The construction of Ω_k will be illustrated by an example where $\delta_1 = 1, \delta_2 = 2, \delta_3 = 2$. We follow the steps:

Step 1: Given k, define all apolar p-partitions of k. i.e. if $k = 4$, then the set of 3-partition is

$$< 4, 0, 0 >, < 3, 1, 0 >, < 2, 2, 0 >, < 2, 1, 1 >$$

and its construction is summarised by the table:

4	0	0
3	1	0
2	2	0
2	1	1

Step 2: From the set of apolar p-partitions define the subset with the property that if $< x_1, x_2, ..., x_p >$ is the apolar partition, then $x_i \leq \max\{\delta_i, i \in \tilde{p}\}$, i.e. for our example with $\max\{\delta_i\} = 2$ the subset with the above property is:

$$< 2, 2, 0 >, \qquad < 2, 1, 1 >$$

This subset is called the Δ-apolar partition set of k.

Step 3: From the set define the permutation of each apolar partition, i.e.

$$< 2,2,0 > \quad \rightarrow \quad \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$< 2,1,1 > \quad \rightarrow \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Note that since the integers in any $< x_1, x_2, ..., x_p >$ may repeat themselves, the set of p-oriented partitions is less than $p!$. in number for any p-apolar partition.

The set defined above will be referred to as the Δ-oriented partition set of k.

Step 4: From the Δ-oriented partition set of k define Ω_k as the subset with the property that if

$$\{x_1, x_2, ..., x_p\} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}, \text{ then } x_i \in [0, 1, ..., \delta_i]$$

i.e. for our example with $\delta_1 = 1, \delta_2 = 2, \delta_3 = 2$

$$\Omega_4 = \left\{ \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

where as the rest i.e. $(2, 0, 2)$ fail the criteria since

$$x_1 = 2, < \delta_1 = 1$$

$\square$

Example (5.6): Let $\delta_1 = 1, \delta_2 = 2, \delta_3 = 2$.

1) $k = 0 : \Omega_0 = \{(0, 0, 0)\}$ obviously

2) $k = 1 :< 1, 0, 0 >$ the only apolar partition

$$\Delta\text{-oriented set : } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

all acceptable and thus

$$\Omega_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$$

3) $k = 2 :< 2,0,0 >, < 1,1,0 >$ the only apolar partitions

$$\Delta\text{-oriented set} : \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

all acceptable except $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and thus

$$\Omega_2 = \{(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$$

4). $k = 3 :< 3,0,0 >, < 2,1,0 >, < 1,1,1 >, < 2,1,0 >, < 1,1,1 >$ the only apolar partitions acceptable

$$\Delta\text{-oriented set} : \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are not acceptable (see step 2) and thus

$$\Omega_3 = \{(1,1,1),(1,0,2),(1,2,0),(0,2,1),(0,1,2)\}$$

$k = 4 : \Omega_4 = \{(0,2,2),(1,2,1),(1,1,2),\}$ defined before.

$k = 5 :< 5,0,0 >, < 4,1,0 >, < 3,2,0 >, < 3,1,1 >, < 2,2,1 >$ are not acceptable except $< 2,2,1 >$ since $\max\{\delta_i\} = 2$ so $< 2,2,1 >$ the only apolar partition.

$$\Delta\text{-oriented set} : \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

the only acceptable one is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$,

$$\Omega_5 = \{(1,2,2)\}$$

5.4 Rank Properties of Plucker Matrices

The results so far, indicate how we can relate the Plucker matrix to the original data defined on the polynomial matrix. These results provide the means for discussing the rank properties of Plucker matrices which is considered next: For the $A(s) \in R^{m \times n}[s]$ we may write:

$$P_A = C_A^{\Delta}(T_A) \cdot P_S^{\Delta} = C_p(T_A)P_S \tag{5.34}$$

where

$$\begin{aligned} P_S^{\Delta} \in R^{\tau \times (\delta+1)}, &\qquad G_A \equiv C_p^{\Delta}(T_A) \in R^{\binom{m}{p} \times \tau} \\ \rho(P_S^{\Delta}) = (\delta+1) \leq \tau &= \Pi_{i=1}^{p}(\delta_i + 1), \delta = \sum_{i=1}^{p} \delta_i \end{aligned}$$

Note that $\tau = \delta + 1$ only when $\delta_1 = \delta_2 = \cdots = \delta_p = 0$; otherwise $\tau > \delta + 1$. We shall denote by

$$\mathcal{P}_S^{\Delta} \equiv col.sp\{P_S^{\Delta}\}, \qquad \mathcal{N}_S^{\Delta} \equiv \mathcal{N}_l\{P_S^{\Delta}\} \tag{5.35}$$

$$\mathcal{N}_A^{\Delta} \equiv \mathcal{N}_r\{G_A\}, \qquad \mathcal{R}_A^{\Delta} \equiv row.sp\{G_A\} \tag{5.36}$$

with this notation, we may state the following results:

Proposition (5.7): If $A(s)$ is a minimal basis such that $T_A \in R^{m \times m}$, i.e. $m = \sum(\delta_i + 1)$, then if $\rho(T_A) = m$, we also have that $\rho(P_A) = \delta + 1$.

Proof: $P_A = C_p(T_A)P_S$ and $\rho(T_A) = m$, implies $C_p(T_A)$ has full rank. Thus $\rho(P_A) = \rho(P_S) = \delta + 1$.

□

Consider now the rank properties of P_A. By inspection of dimensions as below:

$$\binom{m}{p}\left\{\overbrace{\boxed{C_P(T_A)}}^{\tau}\overbrace{\boxed{P_S}}^{\delta+1}\right\}\tau = \left.\overbrace{\boxed{P_A}}^{\delta+1}\right\}\binom{m}{p}$$

the following result is derived:

Proposition (5.8): If $\begin{pmatrix} m \\ p \end{pmatrix} < (\delta+1)$ then

$$\rho(P_A) < \begin{pmatrix} m \\ p \end{pmatrix} \tag{5.37}$$

or equivalently $\mathcal{N}_l(P_A) \neq \{0\}$, implies that

$$\mathcal{N}_l\{G_A\} \neq \{0\} \Leftrightarrow \rho(G_A) < \begin{pmatrix} m \\ p \end{pmatrix} \tag{5.38}$$

and/or

$$\mathcal{R}_A^{\Delta} \cap \mathcal{N}_S^{\Delta} \neq \{0\} \tag{5.39}$$

Proof: Note that

$$\rho(P_A) < \begin{pmatrix} m \\ p \end{pmatrix} \Leftrightarrow \exists \underline{x}^t \neq 0 : \underline{x}^t P_A = 0$$

or

$$\underline{x}^t G_A P_A^{\Delta} = 0$$

If we define $\underline{y}^t = \underline{x}^t G_A$, then loss of rank is equivalent to either of the following conditions:

1. $\underline{y}^t = 0 \Leftrightarrow \underline{x}^t G_A = 0$.
2. $\underline{y}^t \neq 0$ and $\underline{y}^t P_S^{\Delta} = 0$

For each of the above we have:

1. $\underline{x}^t G_A = 0$: This implies that $\mathcal{N}_l\{G_A\} \neq \{0\}$. However, since $\begin{pmatrix} m \\ p \end{pmatrix} < (\delta+1) \leq \tau, \mathcal{N}_l\{G_A\} \neq \{0\}$, iff $\rho(G_A) < \begin{pmatrix} m \\ p \end{pmatrix}$.
2. $\underline{y}^t \neq 0, \underline{y}^t P_S^{\Delta} = 0$: Since $\underline{y} \in R_A^{\Delta} = \text{row-sp}\{G_A\}$, this condition implies that $R_A^{\Delta} \cap \mathcal{N}_S^{\Delta} \neq \{\underline{0}\}$ this leads to the result.

Either of the above conditions implies $\mathcal{N}_l(P_A) \neq \{0\}$, or equivalently $\rho(P_A) < \begin{pmatrix} m \\ p \end{pmatrix}$.

$\square$

Proposition (5.9): If $\begin{pmatrix} m \\ p \end{pmatrix} \geq (\delta+1)$, then $\rho(P_A) < (\delta+1)$ if and only if

$$\mathcal{N}_A^\Delta \cap P_S^\Delta \neq \{0\}$$

A necessary condition for $\mathcal{N}_A^\Delta$ to be $\neq \{0\}$ in the case where $(\delta+1) \leq \tau \leq \begin{pmatrix} m \\ p \end{pmatrix}$ is that $\rho(G_A) < \tau$.

Proof: We write the above as

$$\rho(P_A) < (\delta+1) \Leftrightarrow \exists \underline{x} \neq \underline{0} : \underline{P}_A \underline{x} = \underline{0} \text{ or } G_A P_S^\Delta \underline{x} = \underline{0}.$$

If $\underline{y} = P_S^\Delta \underline{x}$, then since P_S^Δ has full column rank, it follows that $\underline{y} \neq \underline{0}$ and

$$G_A \underline{y} = 0, \underline{y} = P_S^\Delta \underline{x} \tag{5.40}$$

The above condition implies that

$$\underline{y} \in \mathcal{N}_r\{G_A\}, \underline{y} \in P_S^\Delta \tag{5.41}$$

or that

$$\mathcal{N}_A^\Delta \cap P_S^\Delta \neq \{0\} \tag{5.42}$$

A necessary condition for (5.42) to be satisfied is that $\mathcal{N}_r\{G_A\} = \mathcal{N}_A^\Delta \neq \{0\}$, we distinguish the following cases:

a) $(\delta+1) \leq \tau \leq \begin{pmatrix} m \\ p \end{pmatrix}$.

b) $(\delta+1) \leq \begin{pmatrix} m \\ p \end{pmatrix} < \tau$.

For each of the above cases we have:

a) $(\delta+1) \leq \tau \leq \begin{pmatrix} m \\ p \end{pmatrix}$

In this case $\mathcal{N}_r\{G_A\} \neq \{0\}$ implies $\rho(G_A) < \tau$ i.e. G_A looses rank (necessary condition).

b) $(\delta+1) \leq \begin{pmatrix} m \\ p \end{pmatrix} < \tau$.

In this case, $\mathcal{N}_r\{G_A\}$ is always $\neq \{0\}$ and condition (5.42) is necessary and sufficient.

□

The above analysis allows the reduction of study of rank of P_A to properties of G_A, P_S^Δ matrices and associated spaces.

Proposition (5.10): For the Plucker matrix P_A the following properties hold true:

i) If $\begin{pmatrix} m \\ p \end{pmatrix} < (\delta+1)$, then

$$\rho(G_A) = \begin{pmatrix} m \\ p \end{pmatrix} \text{ and } \mathcal{R}_A^\Delta \cap \mathcal{N}_S^\Delta = \{0\} \tag{5.43}$$

are both necessary and sufficient conditions for $\rho(P_A) = \begin{pmatrix} m \\ p \end{pmatrix}$.

ii) $(\delta+1) \leq \tau \leq \begin{pmatrix} m \\ p \end{pmatrix}$ then

$$\rho(G_A) = \tau \tag{5.44}$$

is necessary and sufficient condition for $\rho(P_A) = \delta + 1$.

iii) If $(\delta+1) \leq \begin{pmatrix} m \\ p \end{pmatrix} \leq \tau$, then

$$\mathcal{N}_A^\Delta \cap \mathcal{P}_S^\Delta = \{0\} \tag{5.45}$$

is necessary and sufficient condition for $\rho(P_A) = \delta + 1$.

This result follows from the analysis preceding the statement of Propositions (5.8) and (5.9).

Now we consider the generic rank properties of a Plucker matrix. For a matrix $A(s)$, such that

$$A(s) \in R^{m \times p}[s], \rho_{R(s)}\{A(s)\} = p \text{ and let } A(s) = [\underline{a}_1(s), ..., \underline{a}_p(s)]$$

its Plucker matrix is defined by

$$C_p(A(s)) = P_A e_\delta(s), e_\delta(s) = [1, s, ..., s^\delta]^t$$

where $P_A \in R^{\binom{m}{p} \times (\delta+1)}$ is defined as the Plucker matrix of $A(s)$. The problem here is to investigate the condition such that P_A has generically full rank.

From equation (5.31) we know $\underline{p}_k = \sum_{w \in \Omega_k} \underline{t}_w$, if $\underline{t}_w$ are the columns of $C_p(T_A)$ corresponding to the $w \in \Omega_k$ sequences, then the k-th column of the P_A Plucker matrix is defined by

$$\underline{p}_k = \sum_{w \in \Omega_k} \underline{t}_w, k = 0, 1, ..., \delta \tag{5.46}$$

Corollary (5.1): The Plucker matrix P_A has generically full rank.

Proof: According to the dimensions, we start by forming either $\binom{m}{p}$-th or $(\delta+1)$-th compound matrix of P_A. By setting the chosen compound matrix to zero implying that all the maximal order minors of the matrix P_A are identically zero. Let V be the variety in R^N containing all the solutions which satisfy the resulting equations. Now we have to show that V is proper. In order to do this, we say there is at least one point in the whole parameter space which does not belong to the solution space (i.e. the space which solution to the equations above are simultaneously zero and is a subset of the parameter space). As we are dealing with generic case, there must exist a point in the whole parameter space which does not belong to the solution space. That is we have to show that there is a full rank matrix P_A; however, we can always construct a matrix. So the matrix P_A has generically full rank.

$\square$

5.5 Conclusion

In this chapter, it has been proved that the controllability Plucker matrix $P(A, B)$ has rank equal to $(n+1)$ if and only if (A, B) is controllable and equal to $(r+1)$ for a general pair (A, B) where r is the dimension of the controllable subspace. A systematic procedure for the computation of Plucker matrices of least degree polynomial matrices has been introduced and their rank properties have been investigated. The results have allowed the derivation of a result showing that the Plucker matrices associated with least degree polynomial matrices have generically full rank.

The results here provide a link of the controllability, observability properties to the rank of appropriate Plucker matrices, establish a computational procedure for determining the Plucker matrices and establish some interesting rank properties of them.

Chapter 6

STRUCTURAL PROPERTIES OF GENERAL INTERCONNECTED SYSTEMS

6.1 Introduction

The theory of structural properties of composite system has attracted a lot of attention in recent years [Ros.,1] [Ros. & Pug., 1] [Gil.,1] [Cal. & Des.,1] [Pug. & Kaf., 1] [Kai,1] etc.. In [Gil.,1], the controllability and observability of composite systems are related to the controllability and observability of their subsystems. [Ros. & Pug., 1] have shown that the closed-loop system will always be strictly proper (whenever it is defined) for the case of proper $G(s)$ and have investigated the decoupling zeros of a composite system which are known [Ros.,1] to be invariant under strict system equivalence applied to the composite system matrix. More recently [Pug. & Rat.,1] and [Pug.,1] have provided necessary and sufficent conditions for closed-loop properness in the case of a general open-loop $G(s)$ and a simple sufficient condition for composite system properness has been derived in [Pug. & Kaf., 1].

In forming composite systems by interconnecting a family of subsystems, the two important ingredients are the subsystem models and the nature of the interconnection topology (the interconnection graph). An ideal interconnection scheme is that characterised by the completeness assumption [Kar.,8], [Cal. & Des.,1]; this assumption allows the characterisation of a number of important system properties, such as controllability, observability etc. as the aggregates of the corresponding properties of subsystems [Kar.,8]. The main objective of this chapter is to provide a formulation and appropriate tools for the study of the effect of diversion from the completeness assumption, as well as characterise the effects of these diversions on the formation of properties such as controllability, observability, zero structure formation. The diversion from completeness is studied by assuming first that, the composite system under study is the result of failures, or structural changes on a complete composite system, where these changes are due to loss of inputs, outputs either total, or partial at the subsystem level. The study of controllability, observability and zero structure properties under such failure conditions aim at establishing the values and nature of the associated Kronecker invariants with these properties (minimal indices, elementary divisors) rather than simply testing for such properties. An essential part of the investigation is to locate the part of the interconnection graph, which becomes essential under the above failure conditions.

The main idea behind the work here is to try to relate the structural aspects of the composite system in terms of the structural aspects of the subsystems and the nature of the interconnections. The present approach relies on the use of the restricted pencils [Kar. & MacB.,1] [Kar. & Kou.,1] for the composite system; this analysis leads to the computation of the restricted pencils of the composite systems which are expressed in a simple way in terms of the restriction pencils of the subsystems. Some basic assumptions in dealing with composite system are that the transfer function of each subsystem provides a representation for the subsystem, that is each subsystem is both controllable and observable.

The problem we shall examine here are related to the effect of reducing the number of actuating variables $\underline{u}_i$ (inputs) and/or measurement variables $\underline{y}_i$ (outputs) at subsystem level, on the resulting Kronecker invariant structure of the centralised (simple) and complete composite system. It will be shown that, controllability, observability, zero structure properties of complete composite systems are simply given as aggregates (direct sum) of corresponding properties of subsystems.

It will also be shown that, the problems of input-, output- and input-output reduction on the Kronecker structure of a centralised system are equivalent to matrix pencil augmentation problems by row-, column- and row and column pencils [Kar. & Vaf.,1]. Such problems deal with issues of Kronecker structure evolution assignment under the operations of matrix pencil augmentation. We shall investigate the effect of the partial, or total loss of inputs, or outputs on the basic system properties, such as controllability, observability, zeros as well as on the composite system structure. It is assumed that the original composite satisfies the completeness assumption and thus our interest here is to qualify the effect of deviation from completeness by loss of inputs, or outputs on the resulting composite system, which will be demonstrated that for every deviation from comleteness, the study of controllability, observability may be reduced to subproblems in a structural sense and controllability indices, observability indices, input decoupling zeros and output decoupling zeros of the complete composite system are given as aggregates of those defined by the subsystems. We summarise first some results on the role of pencils in the characterisation of the system properties.

6.2 Matrix pencils and structural properties

In this section some of the basic definitions and properties of matrix pencils arising in linear system theory which we shall use later on in this chapter to describe the structural properties of composite systems are summarised.

6.2.1 Input-state pencil

The pencil $[sI - A, -B]$ is known as input-state, or controllability pencil [Ros.,1] [MacF. & Kar.,1] and the invariants of $[sI - A, -B]$ are very closely associated with the controllability properties of the system. A system is uncontrollable iff there exist finite elementary divisors in $[sI - A, -B]$. This implies the existence of a non-zero constant vector $\underline{v}^t$ and a frequency s_0 such that

$$\underline{v}^t[s_0 I - A, -B] = 0 \tag{6.1}$$

Let N be a maximal rank left annihilator of B (a basis matrix for $\mathcal{N}_l(B)$) and $B^\dagger$ be a left inverse of B, i.e.

$$NB = 0, \qquad B^\dagger B = I_l \tag{6.2}$$

we may now write $\underline{v}^t = [\underline{v}_1^t, \underline{v}_2^t]\begin{bmatrix} N \\ \hline B^\dagger \end{bmatrix}$ where $\begin{bmatrix} N \\ \hline B^\dagger \end{bmatrix}$ is a full rank matrix and thus (6.1) becomes

$$[\underline{v}_1^t, \underline{v}_2^t]\begin{bmatrix} N \\ \hline B^\dagger \end{bmatrix}[s_0 I - A, -B] = 0$$

$\Leftrightarrow$

$$\underline{v}_1^t(s_0N - NA) + \underline{v}_2^t B^\dagger(s_0I - A) = 0$$

$$\underline{v}_2^t = 0$$

$\Leftrightarrow$

$$\underline{v}_1^t(s_0N - NA) = 0 \tag{6.3}$$

The last equation implies that there exist finite elementary divisors of the pencil $sN - NA$ iff the system $S(A, B)$ is uncontrollable. The pencil $sN - NA$ is known as the restricted input-state pencil [Kar.,1] [Kar.,7]. It can be proved [Kar. & MacB.,1] that a controllable system yields an input-state pencil characterised only by a set of column minimal indices $\{\epsilon_i + 1 = \sigma_{l-i+1}, i \in l\}$, where σ_k denotes the controllability indices of the pair (A, B) and ϵ_i are the c.m.i. of the restricted input-state pencil $sN - NA$. For uncontrollable systems the canonical form of $sN - NA$ contains additional blocks to those corresponding to the column minimal indices; these new blocks correspond to finite elementary divisors, which in turn define the input decoupling zeros of the system [Ros.,1]. The pencils$[sI - A, -B]$and $sN - NA$ have the same f.e.d., but their c.m.i. are related by the "plus one" property described above.

It was shown that if T is the coordinate transformation bringing the pair (A, B) in the Luenberger controllable companion form [Kar.,1], then a mere multiplication of $sN - NA$ on the right by T^{-1} brings the pencil in the Kronecker canonical form. The transformation T^{-1} belongs to the class of strict equivalent transformations [Gan.,1] and, as such, does not affect the Kronecker invariants. Another important set of transformations on the pair (A, B), is the set of state/output feedback transformations; the input-state pencil that corresponds to a closed loop pair $(A - BL, B)$ is $sN - N(A - BL) = sN - NA$, since $NB = 0$, and thus we are led to the following theorem:

Theorem (6.1): [Kar.,1] The input-state pencil $sN - NA$ corresponding to the pair (A, B) and its Kronecker canonical form are invariant under state feedback.

$\square$

6.2.2 State output pencil

In the previous section we found another pencil with reduced dimensions than $[sI-A,-B]$ which characterised the equivalence class of the systems $S(A,B)$ under feedback. In this section we shall repeat the analysis for the $S(A,C)$ pair using the concepts of observability or rather unobservability, instead of those of controllability. Note that a system is unobservable iff there exists finite elementary divisors of the pencil $[sI-A^t,-C^t]^t$ [Ros.,1] which is referred to as the state-output pencil. This implies the existence of a non-zero vector $\underline{u}$ and a frequency $s_0 \in C$ such that

$$\begin{bmatrix} s_0 I - A \\ -C \end{bmatrix} \underline{u} = 0 \tag{6.4}$$

Let M be a maximal rank right annihilator of C (i.e. a basis matrix for $\mathcal{N}_r(C)$) and $C^\dagger$ be a right inverse of C, i.e.

$$CM = 0, \qquad CC^\dagger = I_m \tag{6.5}$$

we may always write

$$\underline{u} = [M|C^\dagger] \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} \tag{6.6}$$

then the system (6.4) may be expressed as

$$\begin{bmatrix} s_0 I_n - A \\ -C \end{bmatrix} [M|C^\dagger] \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} s_0 M - AM & s_0 C^\dagger - AC^\dagger \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} = 0$$

which leads to $\underline{u}_2 = 0$ and thus

$$(s_0 M - AM)\underline{u}_1 = 0 \tag{6.7}$$

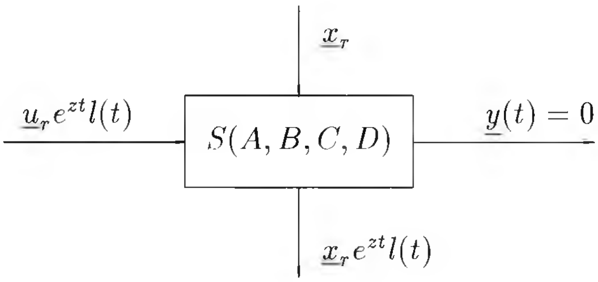
Condition (6.7) implies that there exist finite elementary divisors of the pencil $sM - AM$ iff the system $S(A,C)$ is unobservable. The pencil $sM - AM$ is known as the restricted state-output pencil [Kar. & MacB.,1] [Kar.,7].

It can be proved [Kar. & MacB.,1] that observable systems yield an state-output pencil characterised only by a set of row minimal indices $\{\eta_i + 1 = \rho_{m-i+1}, i \in m\}$, where ρ_k denotes the observability indices of the pair (A,C) and η_i are the r.m.i.

of $sM - AM$. For unobservable systems the canonical form of $sM - AM$ contains additional blocks to those corresponding to the row minimal indices; these new blocks correspond to finite elementary divisors, which in turn define the output decoupling zeros of the system. Once more the state-output and restricted state-output pencils have the same f.e.d. and their r.m.i. are characterised by the "plus one" property described above. It was shown [Kar. & MacB.,1] that if V is the coordinate transformation bringing the pair (A, C) to the Luenberger observable form then a mere multiplication of $sM - AM$ on the right by V^{-1} brings the pencil to its Kronecker canonical form. The transformation V^{-1} belongs to the class of strict equivalence transformations and, as such, preserves the Kronecker canonical form.

6.3 Zero pencil

The concept of a zero is strongly connected with the physical problem of a system $S(A, B, C, D)$ whose output response remains identically zero even though the system input and states are themselves non-zero. This situation may be represented diagrammatically as



The condition for the solution of this problem are expressed in the following theorem:

<u>**Theorem (6.2)**</u>: [MacF. & Kar.,1] For a proper system $S(A, B, C, D)$ for which the number of inputs l is less than or equal to the number of outputs m a necessary and sufficient condition for an input

$$\underline{u}(t) = \underline{u}_r exp(zt) 1(t)$$

to yield a rectilinear motion in the state space of the form

$$\underline{x}(t) = \underline{x}_r exp(zt)1(t)$$

and to be such that

$$\underline{y}(t) \equiv 0 \text{ for } t > 0$$

is that

$$P(z)\left[\frac{\underline{x}_r}{\underline{u}_r}\right] = 0, P(z) = \begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} \tag{6.8}$$

□

From (6.8) we have that the solutions in z give the values of the complex variable s for which $P(s)$ loses column rank. This is only possible for values of s which coincide with the finite elementary divisors of $P(s)$. The frequencies z define the set of finite invariant zeros. The vector solutions $\underline{x}_r, \underline{u}_r$ that correspond to the finite invariant zeros are called the state and input zero directions [MacF. & Kar.,1].

The finite zeros and zero directions are related to the finite elementary divisors on the system matrix pencil $P(s)$. In the study of the properties of zeros and zero directions a simpler form than $P(s)$ has also been used [Kar. & Kou.,1]. A new pencil is derived of reduced dimensions, which simplifies the study of the zero behaviour, since it is restricted only to the properties of state; this pencil is known as the zero pencil and may be defined from the conditions characterising the output zeroing problem for a strictly proper system as shown next. We should first note that condition (6.8) for strictly proper systems implies:

$$(zI - A)\underline{x}_r = B\underline{u}_r \tag{6.9}$$

$$C\underline{x}_r = 0 \tag{6.10}$$

The last equation implies that $\underline{x}_r \in kerC$, so that

$$\underline{x}_r = M\underline{v}_r \tag{6.11}$$

where M is a basis matrix representation of ker$C$ and $\underline{v}_r$ is an appropriate constant vector. Substitution of equation (6.11) into (6.8) and premultiplication by the full rank transformation $\left[\frac{N}{B^\dagger}\right]$, where N is a full rank left annihilator of B and $B^\dagger$ is a left inverse of B gives

$$(zNM - NAM)\underline{v}_r = 0 \tag{6.12}$$

$$\underline{u}_r = B^\dagger(zI - A)\underline{x}_r \tag{6.13}$$

since equations (6.11), (6.12) and (6.13) are equivalent to (6.8). These conditions lead to the definition of frequencies z and vectors $\underline{u}_r$ and $\underline{x}_r$, which are the zeros and the zero directions of the system. The matrix pencil $sNM - NAM$ is known as the zero pencil [Kar. & Kou.,1].and its structure characterises the zero structure of the system, which is also the structure that remains invariant under the general set of state space transformations. These transformations are those of the Kronecker group which involves state feedback, output injection, and state, input, output coordinate transformations [Mor.,1]. Under these transformations the system $S(A, B, C, D)$ may be reduced to a canonical form, $S_k(A_k, B_k, C_k, D_k)$ known as the Kronecker canonical form [Mor.,1] [Tho.,1] [Kar. & MacB.,1]. The relationship between the Kronecker canonical form $S_k(A_k, B_k, C_k, D_k)$ and the Kronecker form of the zero pencil is established by the following result [Kar. & MacB.,1]:

Theorem (6.3): Let $S(A, B, C)$ be a strictly proper linear system with the following set of invariants, defined by the system matrix pencil $P(s)$.

i) $(s - s_i)^{\tau_i}, i = 1, ..., r$ finite elementary divisors

ii) $\hat{s}^{q_i}, i = 1, ..., \mu, 0 < q_1 \leq \cdots \leq q_\mu$ infinite elementary divisors

iii) $0 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_p$ column minimal indices

iv) $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_t$ row minimal indices.

Then,

(a) If B, C have full rank and $G(s)$ has full normal rank, then $\epsilon_1 > 0, \eta_1 > 0$ and $q_1 \geq 2$.

(b) Let $S_k(A_k, B_k, C_k)$ be the Kronecker canonical form [Kar. & MacB.,1] of the system $S(A, B, C)$. The zero pencil $Z_k(s)$ computed on the system $S_k(A_k, B_k, C_k)$ is in Kronecker canonical form with blocks corresponding to the infinite elementary divisors and the row minimal indices rearranged. The invariants of $Z_k(s)$ are related to the invariants of $S(A, B, C)$ in the following manner:

1) The finite elementary divisors of $Z_k(s)$ are equal to the finite elementary divisors of $S(A, B, C)$.

2) The infinite elementary divisors of $z_k(s), \hat{s}^{\widetilde{q_i}}$ are defined by $\widetilde{q_i} = q_i - 2, i = 1, ..., \mu$

3) The column and row minimal indices of $z_k(s)$ are defined by

$$\begin{aligned} \tilde{\epsilon}_i &= \epsilon_i - 1, i = 1, 2, ..., p \\ \tilde{\eta}_j &= \eta_j - 1, j = 1, 2, ..., t \end{aligned}$$

□

The result of this section are used in the following section to establish links between the structural properties of composite systems and those of the subsystems.

6.4 Composite System: The Equivalent Feedback Configuration

A process is always synthesised by connecting subprocesses (subsystems) and the two fundamental ingredients of the composite system model are:

(i) The topology (graph) of system interconnections $\mathcal{F}$.

(ii) The family $\mathcal{T}$ of subsystem models.

The aim here is to investigate the links between the structural aspects of the composite system, the structural aspects of the subsystems and the interconnection graph $\mathcal{F}$. This problem is of immense importance, especially in the early stages of designing systems by interconnecting subprocesses, since it has important implications on the synthesis of composite structures with desirable control structure characteristics [Ros. & Pug., 1].

It is assumed that each subsystem $\sum_i$ is represented by a proper rational transfer function matrix $G_k(s) \in R^{q_k \times p_k}(s)$, that is the subsystems S_i are both controllable and observable or more generally stabilisable and detectable [Won.,1]. Furthermore, in forming composite structures we assume that there are no loading effects, that is each subsystem transfer function remains unchanged after the connections [Che.,2]. Note that the assumption that the subsytems are completely characterised by their transfer functions, does not imply that the composite systems are completely characterised by their trasfer functions. In forming the composite sytem structure we make the following assumptions [Cal. & Des.,1]:

__**Interconnection Assumptions**__: For each subsystem $\sum_k(G_k \in R_{pr}^{q_k \times p_k}(s)), k = 1, 2, ..., \mu$ we have the interconnection structure shown in Figure (6.1): to __each subsystem__ $\sum_k$ with __output__ $\underline{y}_k$ and __input__ $\underline{e}_k$ we associate a __summing node__ with the following characteristics:

(i) Its outputs is the subsystem inputs $\underline{e}_k$ i.e. $\underline{e}_k = [\underline{e}_1^t, ..., \underline{e}_\mu^t]^t$.

(ii) Its inputs at the subsystem level are:

- **(a)** an exogenous input $\underline{u}_k$ (always assignable, or disturbance signal);
- **(b)** other inputs, which are feedback of the form $F_{kj}\underline{y}_j, j = 1, 2, ..., \mu$ where $F_{kj} \in R_{pr}^{p_k \times q_k}$ denotes a proper dynamic matrix from $\underline{y}_j$ to the k-th summing node (very frequently F_{kj} may be real and some of them may be zero).

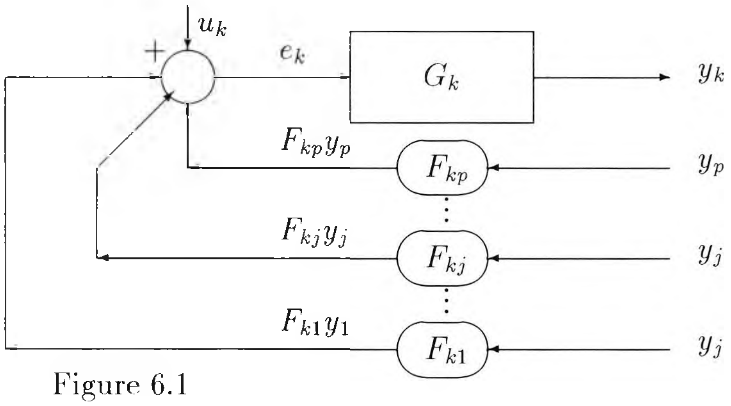


Figure 6.1

An interconnected system satisfying the above assumptions will be called a __complete__ composite system and shall be denoted by $\sum_c(\mathcal{T}, \mathcal{F})$. The implications of the above assumptions are that the subsystems $\sum_k, k = 1, 2, ..., \mu$ are interconnected according to the equations

$$\underline{e}_k = \underline{u}_k + \sum_{j=1}^{p} F_{kj}\underline{y}_j, \underline{y}_k = G_k(s)\underline{e}_k \tag{6.14}$$

where $\underline{e}_k, \underline{u}_k, \underline{y}_k$ denote the Laplace transforms of the corresponding vector signals. If n_i denotes the McMillan degree of $G_i(s)$, then by aggregation we may define the global quantities

$$\begin{aligned} q &= \sum_{k=1}^{\mu} q_k, p = \sum_{k=1}^{\mu} p_k, \\ n &= \sum_{k=1}^{\mu} n_k \end{aligned} \tag{6.15}$$

$$
\begin{aligned}
\underline{u} &= [\underline{u}_1^t, ..., \underline{u}_\mu^t]^t \in R^p \\
\underline{y} &= [\underline{y}_1^t, ..., \underline{y}_\mu^t]^t \in R^q \\
\underline{e} &= [\underline{e}_1^t, ..., \underline{e}_\mu^t]^t \in R^p \\
\underline{x} &= [\underline{x}_1^t, ..., \underline{x}_\mu^t]^t \in R^n \\
F &= [F_{kj}]_{k,j\in\tilde{\mu}} \in R^{p\times q}, G = \text{block-diag}\{G_k(s), k \in \tilde{\mu}\} \in R_{pr}^{q\times p}(s)
\end{aligned} \tag{6.16}
$$

where $\underline{x}_k$ denotes the state vecotr of the $S_k(A_k, B_k, C_k, D_k)$ minimal realisation of $G_k(s)$. Using the aggregate expressions we may express (6.14) as

$$\underline{e} = \underline{u} + F\underline{y}, \underline{y} = G(s)\underline{e} \tag{6.17}$$

which describes the feedback system shown in Figure (6.1).

The above representation of composite systems (as a feedback configuration) has important implications for the present work:

(i) It provides a systematic method for representing composite systems (with implications on the transition from process configurations to process transfer functions).

(ii) It allows the formulation of the process synthesis problem (interconnection of subprocesses) as a feedback design problem.

The matrix F is a representation of the topology $\mathcal{F}$ of the interconnections and will be called the interconnection matrix of $\sum_c$; the aggregate system is denoted by S_a and it is representd by the aggregate transfer function $G(s)$ with the composite system $\sum_c$ we define the following two transfer functions

$$H_{eu}(s) : \underline{u} \to \underline{e}, H_{eu}(s) = (I - FG(s))^{-1} \in R^{p\times p}(s) \tag{6.18}$$

$$H_{yu}(s) : \underline{u} \to \underline{y}, H_{yu}(s) = G(s)(I - FG(s))^{-1} \in R^{q\times p}(s) \tag{6.19}$$

where

$$H_{eu}(s) = I - FH_{yu}(s) \tag{6.20}$$

The composite sytem will be called well-formed, if all transfer functions are well defined and will be called well-posed, if all closed-loop transfer functions are well defined and proper.

Remark (6.1): [Vid,1] The system $\sum_c$ is well formed, iff $|I - FG(s)| \neq 0$and it is well-posed, iff $|I - FG(\infty)| \neq 0$.

□

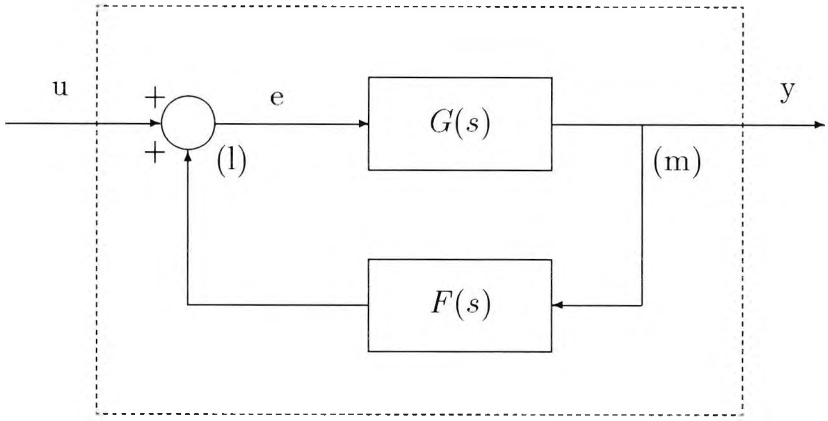
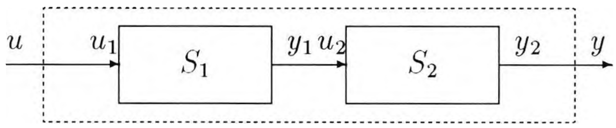
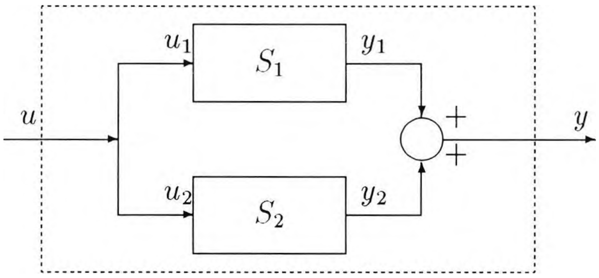


Figure 6.2: Interconnected System $\sum$: The Feedback System Obtained after Aggregation

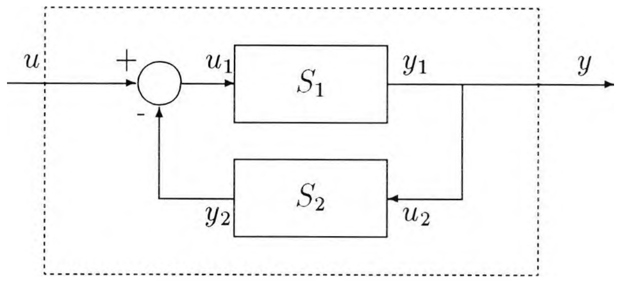
We consider proper systems $S_i(A_i, B_i, C_i, D_i)$ with transfer function matrices $G_i(s) = C_i(sI - A_i)^{-1}B_i + D_i, i = 1, 2, ...$ An interconnected system consisting of a number of subsystems S_i will be denoted by $\sum_c$. The basic interconnection schemes for two systems are shown below.



$\sum 1$: Cascade or Tandem Connection



$\sum 2$: Parallel Connection



$\sum 3$: Feedback Connection

Figure 6.3

The composite systems described above are defined by the composite state space descriptions, and whether the composite transfer functions describe these systems depends on the relationships between poles and zeros of the subsystems [Che,1], [Kai,1] etc. Note that the above connections are well posed under the following conditions:

a) Tandem connection: Always

b) Parallel connection: If $G_1(s) \neq -G_2(s)$

c) Feedback connection: If $|I + G_1(\infty)G_2(\infty)| = |I + D_1D_2| \neq 0$

For two systems S_1, S_2 which are completely characterised by their proper transfer function matrices $G_1(s), G_2(s)$, any composite well posed connection of S_1 and S_2 is completely characterised by its composite transfer matrix $G_{12}(s)$, if and only if [Che,1]

$$\delta_m(G_{12}(s)) = \delta_m(G_1(s)) + \delta_m(G_2(s)) \tag{6.21}$$

For the different types of connections described in above, the above condition for the representation of the composite system by its composite transfer function matrix may become more explicit as conditions for coprimeness of the polynomial matrices defined by the $R[s]$-irreducible MFDs of $G_1(s)$ and $G_2(s)$ (see [Che,1], [Kai,1] etc.) For the simple case of single-input, single-output (SISO) systems S_i which are completely characterised by their proper rational functions $g_i(s), i = 1, 2$ we have the following:

a) The <u>tandem connection</u> of S_1 and S_2 is completely characterised by $g_{12}(s) = g_2(s)g_1(s)$ if and only if there is no pole-zero cancellation between $g_1(s)$ and $g_2(s)$.

b) The parallel connection of S_1 and S_2 completely characterised by $g_{12}(s) = g_1(s) + g_2(s)$, if and only if $g_1(s)$ and $g_2(s)$ do not have any pole in common.

c) The feedback connection of S_1 and S_2 is completely characterised by $g_{12}(s) = (1 + g_1(s)g_2(s))^{-1}g_1(s)$, if and only if there is no pole of $g_2(s)$ cancelled by any zero of $g_1(s)$.

The problem of representation of composite systems by the composite transfer function is always related to controllability and observability of the composite system. The feedback configuration of Fig. (6.3) does not always have these two properties. Controllability and observability of a system, always depends on the selection of the inputs and outputs. An enlarged feedback configuration, denoted in Fig. (6.4), has always the property of controllability and observability for the composite input vector $[r_1^t, r_2^t]^t$, and output vector $[y_1^t, y_2^t]^t$ and will be called the complete feedback configuration. Such configuration will be used again in the discussion of the general control design problem and it is well-posed if $|I + G_1(\infty)G_2(\infty)| \neq 0$. For such a configuration we may define

$$\begin{bmatrix} \underline{u}_1(s) \\ \underline{u}_2(s) \end{bmatrix} = H(s) \begin{bmatrix} \underline{r}_1(s) \\ \underline{r}_2(s) \end{bmatrix}, \text{ where } H(s) = \begin{bmatrix} I_1 & G_2(s) \\ -G_1(s) & I_2 \end{bmatrix}^{-1} \tag{6.22}$$

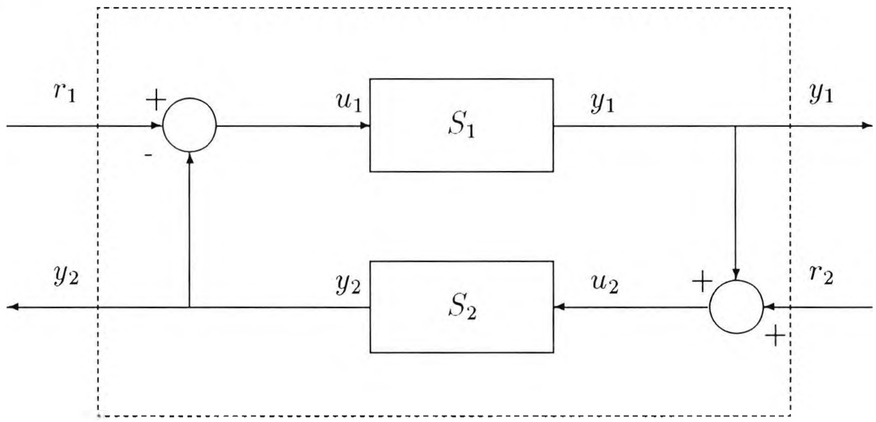


Fig. 6.4: $\sum_{12}$: Complete feedback configuration

and $H(s)$ exists under the well posedness assumption and it is known as error transfer function (other transfer functions may also be defined). If $\sum_{12}^{CF}$ denotes the composite state space equations and assume that $G_i(s)$ are complete representations of S_i, then $H(s)$ completely describes the $\sum_{12}^{CF}$ composite system [Che,1], [Vid,1].

The feedback configuration above is a natural representation of general interconnected systems [Cal. & Des.,1]. Thus assume that the interconnected system $\sum$

is obtained by coupling p subsystems, S_k, each one of them described completely by their proper transfer functions $G_k(s)$, i.e. $G_k(s) \in R_{pr}(s)^{q_k \times p_k}$. For example, consider the interconnection shown in Figure (6.5). (In the following, we work in the s-domain (Laplace transforms) and thus we omit (s)).

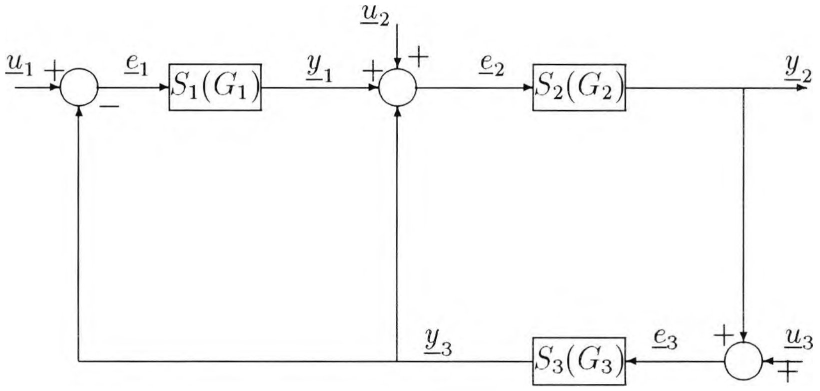


Fig. 6.5

For the example of Figure (6.5), we have

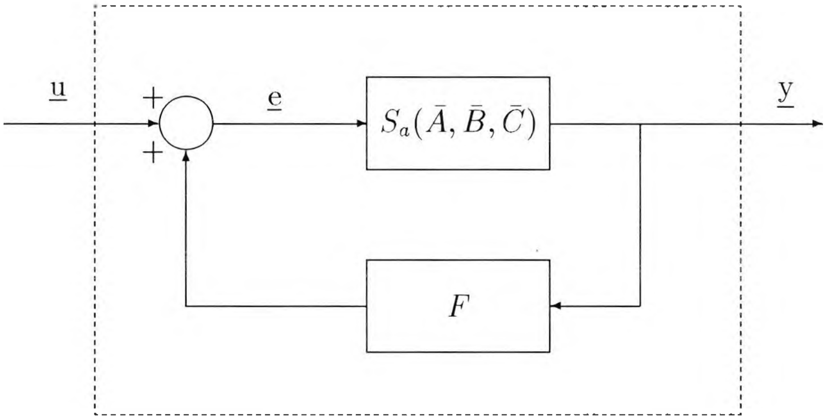
$$G = \begin{array}{cc} \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0, \\ 0 & 0 & G_3 \end{bmatrix} & \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \\ \begin{array}{ccc} l_1 & l_2 & l_3 \end{array} & \end{array} \qquad F = \begin{array}{cc} \begin{bmatrix} 0 & 0 & -I \\ I & 0 & -I \\ 0 & I & 0 \end{bmatrix} & \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} \\ \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} & \end{array}$$

6.5 General State Space Description of Complete Composite System

The composite system may be represented as a feedback structure as shown in Figure (6.2). Under the assumption of well-posedness all transfer functions H_{yu}, H_{eu} are proper. The state space form description of the composite system is considered next. Note that for the sake of simplicity, we consider only strictly proper systems. If the system equations are defined by

$$S(A,B,C): \begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{e}, A \in R^{n \times n}, B \in R^{n \times l} \\ \underline{y} = C\underline{x}, \quad C \in R^{m \times n} \end{cases} \tag{6.23}$$

or a composite system $\sum_c(S_a(\bar{A},\bar{B},\bar{C});F)$, where $S_a(\bar{A},\bar{B},\bar{C}) = \{S_i(A_i,B_i,C_i), i \in \tilde{\mu}\}$ is the aggregate system and F is the interconnection matrix expressing the graph structure of the composite system. It is also assumed that $\sum_c$ is a complete composite system [Kar.,8] and thus it is describe by the feedback configuration



where

$$S_a(\bar{A},\bar{B},\bar{C}) : \begin{cases} \dot{\underline{x}} = \bar{A}\underline{x} + \bar{B}\underline{e} \\ \underline{y} = \bar{C}\underline{x} \end{cases} \tag{6.24}$$

with

$$\begin{aligned} \bar{A} &= \text{block-diag}\{A_i, i \in \tilde{\mu}\}, A_i \in R^{n_i \times n_i} \\ \bar{B} &= \text{block-diag}\{B_i, i \in \tilde{\mu}\}, B_i \in R^{n_i \times l_i} \\ \bar{C} &= \text{block-diag}\{C_i, i \in \tilde{\mu}\}, C_i \in R^{m_i \times n_i} \end{aligned}$$

describing the aggregate system with vectors

$$\underline{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \underline{e}_\mu \end{bmatrix}, \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_\mu \end{bmatrix}, \underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_\mu \end{bmatrix}, \underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_\mu \end{bmatrix} \tag{6.25}$$

where the interconnection is expressed by

$$\underline{e} = \underline{u} + F\underline{y} \tag{6.26}$$

where $F = [F_{kj}]_{k,j \in \tilde{\mu}}$ expresses the structure of the interconnecting graph. From the above (6.24) and (6.26)) we have:

$$\underline{e} = \underline{u} + F\underline{y} = \underline{u} + F\bar{C}\underline{x}$$

and $\dot{\underline{x}} = \bar{A}\underline{x} + \bar{B}\underline{e} = \bar{A}\underline{x} + \bar{B}(\underline{u} + F\bar{C}\underline{x}) = (\bar{A} + \bar{B}F\bar{C})\underline{x} + \bar{B}\underline{u}$ we may summarise as follows:

Proposition (6.1): The composite system state equations of the complete system are given by

$$S_c(A_c, B_c, C_c) : \begin{cases} \underline{\dot{x}} = A_c\underline{x} + B_c\underline{e} \\ \underline{y} = C_c\underline{x} \end{cases} \tag{6.27}$$

where

$$A_c = \bar{A} + \bar{B}F\bar{C} \tag{6.28}$$

$C_c = \bar{C}, B_c = \bar{B}$ and $\bar{A}, \bar{B}$ and $\bar{C}$ are the state space parameters describing the aggregate model.

□

The composite system is called complete [Kar.,8] (see interconnection assumption) when for each subsystem the number of independent variables in $\underline{u}_i$ is equal to the number of interconnection variables in $\underline{e}_i$. We assume as before that the component subsystems $S_i(A_i, B_i, C_i)$ are both controllable and observable (or stabilisable and detectable) for all $i = 1, 2, ..., \mu$.

The problem we consider next is the investigation of the controllability properties of the composite system with full inputs at the subsystem level.

6.5.1 Input-state Restriction Pencil of Composite and Aggregate Systems with Full Inputs

Consider the complete composite system described by equation (6.27)and let $\bar{N}$ be a left annihilator of $\bar{B}$, where

$$\bar{N} = \begin{bmatrix} N_1 & & & 0 \\ & N_2 & & \\ & & \ddots & \\ 0 & & & N_\mu \end{bmatrix} \tag{6.29}$$

and N_i are left annihilators for B_i. The input state Restriction Pencil is $s\bar{N} - \bar{N}A_c$ and may be expressed as shown below. We first note that the pencil

$$s\bar{N} - \bar{N}\bar{A} = \begin{bmatrix} sN_1 - N_1A_1 & & & 0 \\ & sN_2 - N_2A_2 & & \\ & & \ddots & \\ 0 & & & sN_\mu - N_\mu A_\mu \end{bmatrix} \tag{6.30}$$

is the input-state Restriction pencil of the aggregate system without the interconnection. Given that $\bar{N}\bar{B} = 0$, then the restriction pencil of the composite system is:

$$s\bar{N} - \bar{N}A_c = s\bar{N} - \bar{N}(\bar{A} + \bar{B}F\bar{C}) = s\bar{N} - \bar{N}\bar{A} = \begin{bmatrix} sN_1 - N_1A_1 & & & 0 \\ & sN_2 - N_2A_2 & & \\ & & \ddots & \\ 0 & & & sN_\mu - N_\mu A_\mu \end{bmatrix} \tag{6.31}$$

the above leads to the following result.

Theorem (6.4): If $S_i(A_i, B_i), i = 1, ..., \mu$ are controllable and the composite system is well formed, then the composite system with full inputs (as those present in the subsystems) is also controllable.

Proof: Since the input-state restriction pencil of the composite system is the direct sum of the input-state restriction pencils of the subsystems, controllability properties are expressed as the aggregates of the corresponding properties defined on the subsystems.

□

Corollary (6.1): Uncontrollability of a subsystem results in uncontrollability of the composite system. Furthermore, the dimension of the controllable space of the composite system is the sum of the dimensions of the controllable subspaces of subsystems.

□

We consider next the observability properties of the composite system with full outputs at the subsystem level.

6.5.2 State-Output Restriction Pencil of Composite and Aggregate Systems with Full Outputs

Consider the complete composite system described by equation (6.27) and let $\bar{M}$ be a right annihilator of $\bar{C}$, where

$$\bar{M} = \begin{bmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_\mu \end{bmatrix} \tag{6.32}$$

where M_i are right annihilators for C_i. The state-output restriction pencil is $s\bar{M} - A_c\bar{M}$ and may be expressed as shown below. We first not that the pencil

$$s\bar{M} - \bar{A}\bar{M} = \begin{bmatrix} sM_1 - A_1M_1 & & & 0 \\ & sM_2 - A_2M_2 & & \\ & & \ddots & \\ 0 & & & sM_\mu - A_\mu M_\mu \end{bmatrix} \tag{6.33}$$

is the state-output restriction pencil of the aggregate system without the interconnection. Given that $\bar{C}\bar{M} = 0$, then the restriction pencil of the composite system is:

$$\begin{aligned} s\bar{M} - A_c\bar{M} &= s\bar{M} - (\bar{A} + \bar{B}F\bar{C})\bar{M} = s\bar{M} - \bar{A}\bar{M} = \\ &= \begin{bmatrix} sM_1 - A_1M_1 & & & 0 \\ & sM_2 - A_2M_2 & & \\ & & \ddots & \\ 0 & & & sM_\mu - A_\mu M_\mu \end{bmatrix} \end{aligned} \tag{6.34}$$

The above leads to the following result.

Theorem (6.5): If $S_i(A_i, C_i), i = 1, ..., \mu$ are observable and the composite system is well formed, then the composite system with full outputs (as those present in the subsystems) is also observable.

Proof: Since the state-output restriction pencil of the composite system is the direct sum of the state-output restriction pencils of the subsystems, observability properties are expressed as the aggregates of the corresponding properties defined on the subsystems.

$\square$

Corollary (6.2): Unobservability of a subsystem results in a unobservable composite system. Furthermore, the dimension of the observable space of the composite system is the sum of the dimensions of the observable subspaces of subsystems.

$\square$

We consider next the zero properties of the composite system with full inputs and outputs at the subsystem level.

6.5.3 The Zero Pencil of Composite and Aggregate Systems with Full Inputs and Outputs

Consider the composite system described by equation (6.27) and let $\bar{N}, \bar{M}$ be left and right annihilators of $\bar{B}, \bar{C}$ respectively, where $\bar{N}, \bar{M}$ are as in (6.29) and (6.32).

The zero pencil is $s\bar{N}\bar{M} - \bar{N}A_c\bar{M}$ and may be expressed as shown below. We first note that the pencil

$$s\bar{N}\bar{M} - \bar{N}\bar{A}\bar{M} =$$

$$= \begin{bmatrix} sN_1 - N_1A_1 & & & & 0 \\ & \ddots & & & \\ & & sN_i - N_iA_i & & \\ 0 & & & \ddots & \\ & & & & sN_\mu - N_\mu A_\mu \end{bmatrix} \begin{bmatrix} M_1 & & & & 0 \\ & \ddots & & & \\ & & M_i & & \\ 0 & & & \ddots & \\ & & & & M_\mu \end{bmatrix}$$

$$= \begin{bmatrix} sN_1M_1 - N_1A_1M_1 & & & & 0 \\ & \ddots & & & \\ & & sN_iM_i - N_iA_iM_i & & \\ 0 & & & \ddots & \\ & & & & sN_\mu M_\mu - N_\mu A_\mu M_\mu \end{bmatrix} \tag{6.35}$$

Theorem (6.6): The zero pencil of the composite system is the direct sum of the zero pencils of the subsystems, and thus the zero properties are expressed as the aggregate of the corresponding properties defined on the subsystems.

$\square$

The study of the zero pencil expressed in the form of equation (6.35) affords all the necessary insight for the analysis of the zero structure of the composite system.

Now we investigate the loss of inputs, outputs and the effect on the Kronecker structure of the resulting linear systems.

6.6 Loss of Inputs, Outputs and the Effect on the Kronecker Structure of the Resulting Linear Systems

The problems which we will examine here are related to the effect of reducing the number of actuating variables $\underline{u}_i$ and/or measurement variables $\underline{y}_i$ at subsystem level, on the resulting Kronecker invariant structure of the system; this problem arises either to failure conditions in actuators, sensors, or due to that not all possible actuation variables, sensor variables are used. These problems may be referred to as input-, output-projection problems and they will be examined for both the case of simple system described by (6.23) and for the case of the composite sytem described by (6.27), (6.28). The problems discussd above may be formally defined as:

Definition (6.1): Given the system $S(A, B, C)$ as in (6.23) we define:

i) If $R \in R^{l \times p}, \rho(R) = p < l$, then $S'(A', B', C')$ is called R-input reduced system if

$$A' = A, B' = BR, C' = C \tag{6.36}$$

ii) If $P \in R^{q \times m}, \rho(P) = q < m$, then $S'(A', B', C')$ is called P-output reduced system if

$$A' = A, B' = B, C' = PC \tag{6.37}$$

iii) If $R \in R^{l \times p}, \rho(R) = p < l, P \in R^{q \times m}, \rho(P) = q < m$ then $S('A', B', C')$ is called (R,P)-input output reduced system, if

$$A' = A, B' = BR, C' = PC \tag{6.38}$$

□

The above definition for a simple system $S(A, B, C)$ may be extended to the case of complete composite systems $S_c(A_c, B_c, C_c)$ as follows:

Definition (6.2): Given the complete composite system $S_c(A_c, B_c, C_c)$ as in (6.27), (6.28) we define:

i) If $R_i \in R^{l_i \times p_i}, \rho(R_i) = p_i < l_i$, then $S'(A', B', C')$ is called <u>R_i-input reduced system</u> if

$$A' = A_c, B' = B_c \cdot \text{block-diag}\{I_{l_1}; ...; R_i; ...; I_{l_\mu}\}, C' = C_c \tag{6.39}$$

and will be called <u>totally i-th input reduced system</u> if

$$A'' = A_c, C'' = C_c, B'' = \begin{bmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_{i-1} & & & 0 \\ & & & 0 & \updownarrow n_i & \\ & & & B_{i+1} & & \\ 0 & & & & \ddots & \\ & & & & & N_\mu \end{bmatrix} \tag{6.40}$$

ii) If $P_i \in R^{q_i \times m_i}, \rho(P_i) = q_i < m_i$, then $S'(A', B', C')$ is called <u>P_i-output reduced system</u> if

$$A' = A_c, B' = B_c, C' = \text{block-diag}\{I_{m_1}; ...; P_i; ...; I_{m_\mu}\}C_c \tag{6.41}$$

and will be called <u>totally i-th output reduced system</u>, if

$$A'' = A_c, B'' = B_c, C'' = \begin{bmatrix} C_1 & & & & & \\ & \ddots & & & & \\ & & C_{i-1} & & & 0 \\ & & & \underset{\overset{\longleftrightarrow}{i}}{0} & C_{i+1} & \\ 0 & & & & \ddots & \\ & & & & & C_\mu \end{bmatrix} \tag{6.42}$$

iii) If $R_i \in R^{l_i \times p_i}, \rho(R_i) = p_i < l_i, P_i \in R^{q_i \times m_i}, \rho(P_i) = q_i < m_i$ then $S'(A', B', C')$ is called <u>(R_i, P_i)-input output reduced system</u>, if it is R_i-input reduced and P_i-output reduced. Finally, $S'(A', B', C')$ will be called <u>totally (i,j)-reduced system</u>, if it is totally i-th input reduced and totally j-th output reduced.

□

The input, output reduction of $S(A, B, C)$ or $S_c(A_c, B_c, C_c)$ express either failure conditions, or selection of subsets of potential inputs, outputs and represent deviation from the completeness. In the following we shall examine the following problems.

i) Effect of input, output reduction on the Kronecker structure of a centralised system $S(A, B, C)$.

ii) Effect of input, output reduction on the resulting system properties and Kronecker structure of a complete composite system.

The latter problem aims at qualifying the deviation from completeness in composite system structures.

6.7 Input, Output Reduction Problems for a Centralised System

We consider the system described by (6.23)

$$S(A,B,C): \begin{cases} \underline{\dot{x}} = A\underline{x} + B\underline{e}, A \in R^{n\times n}, B \in R^{n\times l} \\ \underline{y} = C\underline{x}, \quad C \in R^{m\times n} \end{cases} \tag{6.43}$$

and let $R \in R^{l\times p}, \rho(R) = p < l, P \in R^{q\times m}, \rho(P) = q < m$ be input-,output-reduction matrices. Input-output reduction by (R, P) implies that we define new inputs, outputs by

$$\underline{e} = R\underline{u}\,, \underline{u} \in R^p, \underline{z} = P\underline{y}, \underline{z} \in R^q \tag{6.44}$$

and the (R, P)-reduced system is $S(A, BR, PC)$. We consider next the structure of the basic pencil associated with the $S(A, BR, PC)$ system. We first note

Remark (6.2): Let $B^\dagger \in R^{l\times n}, N \in R^{(n-l)\times n}$ be a pair of left inverse and left annihilator of B and let $R^\perp$ and $R^\dagger$ be a corresponding pair of annihilator and inverse for R. A pair of a left inverse and left annihilator of $B' = BR$ is defined by $(N', B^{\dagger'})$ where

$$\begin{bmatrix} N' \\ B^{\dagger'} \end{bmatrix} = \begin{bmatrix} N \\ R^\perp B^\dagger \\ R^\dagger B^\dagger \end{bmatrix} \text{ and } \begin{bmatrix} N \\ R^\perp B^\dagger \\ R^\dagger B^\dagger \end{bmatrix} BR = \begin{bmatrix} 0 \\ I_p \end{bmatrix} \tag{6.45}$$

□

Remark (6.3): Let $C^{\dagger} \in R^{n \times m}, M \in R^{n \times (n-m)}$ be a pair of right inverse and right annihilator of C and let $P^{\dagger}$ and $P^{\perp}$ be a corresponding pair of inverse and annihilator inverse for P. A pair of a right inverse and left annihilator of $C' = PC$ is defined by $(M', C^{\dagger'})$ where

$$\begin{bmatrix} M' & C^{\dagger'} \end{bmatrix} = \begin{bmatrix} M; C^{\dagger}P^{\perp} & C^{\dagger}P^{\dagger} \end{bmatrix} \text{ and } PC \begin{bmatrix} M; C^{\dagger}P^{\perp} & C^{\dagger}P^{\dagger} \end{bmatrix} = \begin{bmatrix} 0 & I_q \end{bmatrix} \tag{6.46}$$

□

As a result of the above two remarks we have the following result regarding the restriction pencils of the reduced systems:

Proposition (6.2): For the (R, P)-reduced system $S(A, BR, PC)$ the input-state, state-output and zero pencils respectively are given by:

$$sN' - N'A = \begin{bmatrix} sN - NA \\ sR^{\perp}B^{\dagger} - R^{\perp}B^{\dagger}A \end{bmatrix} = \begin{bmatrix} sN - NA \\ R^{\perp}(sB^{\dagger} - B^{\dagger}A) \end{bmatrix} \tag{6.47}$$

$$sM' - AM' = \left[sM - AM; sC^{\dagger}P^{\perp} - AC^{\dagger}P^{\perp}\right] = \left[sM - AM; (sC^{\dagger} - AC^{\dagger})P^{\perp}\right]$$

$$sN'M' - N'AM' = \begin{bmatrix} sNM - NAM & (sN - NA)C^{\dagger}P^{\perp} \\ R^{\perp}(sB^{\dagger} - B^{\dagger}A)M & R^{\perp}(sB^{\dagger} - B^{\dagger}A)C^{\dagger}P^{\perp} \end{bmatrix} \tag{6.48}$$

The above result readily follows from the definition of the pencils and remarks (6.2) and (6.3). Expressions from (6.47) to (6.48) demonstrate that the problems of input-, output- and input-output reduction are equivalent to matrix pencils augmentation problems by row-, column and row and column pencils. Such problems deal with issues of Kronecker structure evolution assignment under the operations of matrix pencil augmentation and they have also emerged within the framework of studying the cover problem of the geometric theory [Kar. & Vaf.,1]. The study of the matrix pencil augmentation problems is considered in detail in [Kar. & Vaf.,1].

6.8 Input, Output Reduction Problems for a Composite System

The aim of the present study is to investigate the effect of the partial, or total loss of inputs, or outputs on the basic system properties, such as controllability, observability, zeros, on composite system structures. It is assumed that the original composite system satisfies the completeness assumption and thus our interest here is to qualify the effect of deviations from completeness on the resulting composite system. Some basic results on the complete composite system are considered first and then we examine the effects of deviating from completeness for the composite system.

6.8.1 Basic Properties of Complete Composite Systems

If $S_i(A_i, B_i, C_i)$ denotes the i-th subsystem of the F-connected composite system $\sum_c(S_a; F)$ we may define the pencils

$$R_i(s) = sN_i - N_iA_i \tag{6.49}$$

$$P_i(s) = sM_i - A_iM_i \tag{6.50}$$

$$Z_i(s) = sN_iM_i - N_iA_iM_i \tag{6.51}$$

where $R_i(s), P_i(s), Z_i(s)$ denote the input-state-, state-output restriction and zero pencils respectively. The completeness assumption, implies the following result establishing the relationship between the aggregate and complete composite system respectively.

Theorem (6.7): Let $S_a(\bar{A}, \bar{B}, \bar{C})$ be the aggregate and $S_c(A_c, B_c, C_c)$ the corresponding complete composite system $\sum_c(S_a; F)$, where F is the interconnection graph matrix. For the system pencils of the composite and aggregate systems we have the following relationships:

$$R_a(s) = R_c(s) = \text{ block-diag } \{...; R_i(s); i \in \tilde{\mu}\} \tag{6.52}$$

$$P_a(s) = P_c(s) = \text{ block-diag } \{...; P_i(s); i \in \tilde{\mu}\} \tag{6.53}$$

$$Z_a(s) = Z_c(s) = \text{ block-diag } \{...; Z_i(s); i \in \tilde{\mu}\} \tag{6.54}$$

Proof:

From the definition, the B_c, C_c matrices are the same with those of the aggregate system $\bar{B}, \bar{C}$ and thus the corresponding annihilators are

$$N_c = \bar{N} = \text{ block-diag } \{...; N_i; ...\} \tag{6.55}$$

$$M_c = \bar{M} = \text{ block-diag } \{...; M_i; ...\}$$

and thus

$$R_c(s) \;=\; sN_c - N_cA_c = s\bar{N} - \bar{N}(\bar{A} + \bar{B}F\bar{C}) = \tag{6.56}$$

$$= \quad s\bar{N} - \bar{N}\bar{A} = R_a(c) = \text{ block-diag } \{...; R_i(s); i \in \tilde{\mu}\}$$

Since by definition $\bar{N}\bar{B} = 0$.

Similarly, we have

$$\begin{aligned} P_c(s) &= sM_c - A_cM_c = s\bar{M} - (\bar{A} + \bar{B}F\bar{C})\bar{M} = \\ &= s\bar{M} - \bar{A}\bar{M} = P_a(c) = \text{ block-diag } \{...; P_i(s); i \in \tilde{\mu}\} \end{aligned} \tag{6.57}$$

Since by definition $\bar{C}\bar{M} = 0$. The last part follows from (6.55) and (6.56) i.e.

$$\begin{aligned} &Z_c(s) = R_c(s)M_c = R_a(s)\bar{M} = s\bar{N}\bar{M} - \bar{N}\bar{A}\bar{M} \\ &\text{block-diag } \{...; Z_i(s); i \in \tilde{\mu}\} \end{aligned} \tag{6.58}$$

□

From the above result we have a number of important corollaries, which follow directly from the block diagonal form of the pencil $R_c(s), P_c(s), Z_c(s)$.

Corollary (6.3): The controllability, observability and zero properties of the complete composite system S_c which are defined by the Kronecker structure of the $R_c(s), P_c(s), Z_c(s)$ pencils, are aggregates of the corresponding subsystem properties and independent from the interconnection structure F. In particular:

i) If $\mathcal{I}_c(S), \mathcal{D}_{id}(S)$ denote the sets of controllability indices and input-decoupling ed of a system S, then for the complete composite system we have:

$$\mathcal{I}_c(S_c) = \cup_{i=1}^{\mu}\mathcal{I}_c(S_i) \tag{6.59}$$

$$\mathcal{D}_{id}(S_c) = \cup_{i=1}^{\mu}\mathcal{D}_{id}(S_i) \tag{6.60}$$

ii) If $\mathcal{I}_o(S), \mathcal{D}_{od}(S)$ denote the sets of observability indices and output decoupling ed of a system S, then for the complete composite system we have:

$$\mathcal{I}_o(S_c) = \cup_{i=1}^{\mu}\mathcal{I}_o(S_i) \tag{6.61}$$

$$\mathcal{D}_{od}(S_c) = \cup_{i=1}^{\mu}\mathcal{D}_{od}(S_i) \tag{6.62}$$

iii) If $\mathcal{I}_r^z(S), \mathcal{I}_l^z(S), \mathcal{D}_f^z(S), \mathcal{D}_\infty^z(S)$ are the sets of right, left output nulling indices, finite zero ed, infinite zero ed of a system S, then for the complete composite system we have:

$$\mathcal{I}_r^z(S_c) = \cup_{i=1}^{\mu}\mathcal{I}_r^z(S_i), \mathcal{I}_l^z(S_c) = \cup_{i=1}^{\mu}\mathcal{I}_l^z(S_i) \tag{6.63}$$

$$\mathcal{D}_f^z(S_c) = \cup_{i=1}^{\mu}\mathcal{D}_f^z(S_i), \mathcal{D}_\infty^z(S_c) = \cup_{i=1}^{\mu}\mathcal{D}_\infty^z(S_i) \tag{6.64}$$

Proof:

It is known [Kar. & MacB.,1] that the pencils $R(s), P(s)$ and $Z(s)$ define respectively the sets of controllability indices/input decoupling ed, observability indices/output decoupling ed and right, left output nulling indices, finite zero ed, infinite zero ed. From the block diagonal structure of the pencils $R_c(s), P_c(s)$ and $Z_c(s)$ established by Theorem (6.7), the result follows that the corresponding sets are expressed as unions of the invariants defined on the subsystem level.

□

The above results demonstrate that the completeness assumption is very strong and in fact it makes the specific structure of the interconnection graph rather redundant. The effect of the interconnection graph will become clear under the loss of input, output conditions, expressing deviation from the completeness assumption, which are considered next.

6.8.2 Input-reduction at Subsystem Level

We consider the case of an R_i-input reduced system and then of a totally i-th input reduced system. In those two cases the coresponding B matrices are

$$B' = \begin{bmatrix} B_1 & & & & 0 \\ & \ddots & & & \\ & & B_i R_i & & \\ & & & \ddots & \\ 0 & & & & B_\mu \end{bmatrix}, B'' = \begin{bmatrix} B_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & B_{i-1} & & & & \\ & & & 0 & & & \\ & & & & B_{i+1} & & \\ & & & & & \ddots & \\ 0 & & & & & & B_\mu \end{bmatrix} \tag{6.65}$$

using the results of Section (6.7) we have that the corresponding left annihilators are

$$N' = \begin{bmatrix} N_1 & & & 0 \\ & \ddots & & \\ & & \begin{matrix} N_i \\ R_i^{\perp} B_i^{\dagger} \end{matrix} & \\ 0 & & & \ddots & \\ & & & & N_\mu \end{bmatrix} \tag{6.66}$$

$$N_i'' = \begin{bmatrix} N_1 & & & & & & \\ & \ddots & & & & & \\ & & N_{i-1} & & & 0 & \\ & & & I_{n_i} & & & \\ & & & & N_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & N_\mu \end{bmatrix} \tag{6.67}$$

Using the above two expressions we may investigate the controllability properties of the resulting input reduced system by computing the restriction pencils for the two cases.

R_i-Input Reduction

The restriction pencil for the R_i-input reduction is

$$R(R_i; s) = sN' - N'\bar{A} - N'\bar{B}F\bar{C} \tag{6.68}$$

From (6.66) and the form of $\bar{A}$ we have

$$sN' - N'\bar{A} = \begin{bmatrix} R_1(s) & & & & \\ & \ddots & & & 0 \\ & & R_i(s) & & \\ & & Q(R_i, s) & & \\ & 0 & & \ddots & \\ & & & & R_\mu(s) \end{bmatrix} \tag{6.69}$$

where

$$Q(R_i, s) = sR_i^\perp B_i^\dagger - R_i^\perp B_i^\dagger A_i \tag{6.70}$$

Given that

$$N'\bar{B} = \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 0 & \\ & & R_i^\perp & \\ & 0 & & \ddots \\ & & & & 0 \end{bmatrix} \tag{6.71}$$

It follows that

$$N'\bar{B}F\bar{C} = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & 0 & 0 \\ -R_i^{\perp}F_{i1}C_1 & -R_i^{\perp}F_{ii}C_i & -R_i^{\perp}F_{i\mu}C_\mu \\ & & \ddots \\ 0 & & 0 \end{bmatrix} \tag{6.72}$$

Thus we are led to the following result:

Proposition (6.3): If $\sum_c(S_a; F)$ is a complete composite system, then the input restriction pencil of the R_i-input reduced system is

$$R(R_i; s) = \begin{bmatrix} R_1(s) & & 0 \\ \vdots & \ddots & \vdots \\ 0 & R_i(s) & 0 \\ -R_i^{\perp}F_{i1}C_1 & Q(R_i, s) - R_i^{\perp}F_{ii}C_i & -R_i^{\perp}F_{i\mu}C_\mu \\ & & \\ 0 & & R_\mu(s) \end{bmatrix} \tag{6.73}$$

The above result is an extension of Proposition (6.2) stated for centralised systems to the case of composite systems and indicates that R_i-input reduction corresponds to a row pencil augmentation. In particular, if we partition the interconnectioin matrix F according to the natural partitioning implied by the partitioning of the $\underline{e}, \underline{y}$ vectors in (6.25) i.e.

$$F = \begin{array}{c} \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1i} & \cdots & F_{1\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ F_{i1} & F_{i2} & \cdots & F_{ii} & \cdots & F_{i\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ F_{\mu 1} & F_{\mu 2} & \cdots & F_{\mu i} & \cdots & F_{\mu\mu} \end{bmatrix} \end{array} \begin{array}{c} \updownarrow l_1 \\ \\ \updownarrow l_i \\ \\ \updownarrow l_\mu \end{array} \tag{6.74}$$

and by denoting

$$L = F\bar{C} \;\; = \begin{bmatrix} F_{11}C_1 & \cdots & F_{1i}C_i & \cdots & F_{1\mu}C_\mu \\ \vdots & & \vdots & & \vdots \\ F_{i1}C_1 & \cdots & F_{ii}C_i & \cdots & F_{i\mu}C_\mu \\ \vdots & & \vdots & & \vdots \\ F_{\mu 1}C_1 & \cdots & F_{\mu i}C_i & \cdots & F_{\mu\mu}C_\mu \end{bmatrix} \tag{6.75}$$

$$= \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1i} & \cdots & L_{1\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ L_{i1} & L_{i2} & \cdots & L_{ii} & \cdots & L_{i\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ L_{\mu 1} & L_{\mu 2} & \cdots & L_{\mu i} & \cdots & L_{\mu\mu} \end{bmatrix} = \begin{bmatrix} L_1 \\ \vdots \\ L_i \\ \vdots \\ L_\mu \end{bmatrix} \tag{6.76}$$

we may state:

Remark (6.4): For every R_i-input reduction problem, only the i-th row block of the state interconnection matrix $L = F\bar{C}$ affects the controllability properties of the corresponding composite system. In fact, the corresponding input-state restriction pencil may be expressed as

$$R(R_i, s) = \left[\begin{array}{ccc} R_1(s) & & \\ & \ddots & 0 \\ 0 & R_i(s) & \\ & & \ddots \\ & & R_\mu(s) \\ \hline 0 & Q(R_i, s) & 0 \end{array}\right] - \left[\begin{array}{c} \\ \\ 0 \\ \\ \hline R_i^{\perp} L_i \end{array}\right] \tag{6.77}$$

□

It is worth pointing out that the system graph enter now as a perturbation on the block diagonal structure and this perturbation is expressed by $R_i^{\perp} L_i$ on the added pencil row block. The effect of the $R_i^{\perp} L_i$ partitioned structure on aggregate controllability properties will be examined later on.

We consider now the boundary case where we have a total loss of system inputs at the i-th subsystem. In this case, N'' is given by (6.67) and the corresponding input-state restriction pencil is

$$R_{i)}(s) = sN_i'' - N_i''\bar{A} - N_i''\bar{B}F\bar{C} \tag{6.78}$$

Note that

$$sN_i'' - N_i''\bar{A} = \begin{bmatrix} R_1(s) & & & & \\ & \ddots & & & 0 \\ & & sI - A_i & & \\ & & & \ddots & \\ & 0 & & & R_\mu(s) \end{bmatrix} \tag{6.79}$$

and that

$$N_i'' \bar{B} F \bar{C} = N_i'' \bar{B} L \tag{6.80}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & B_i & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} L = \begin{bmatrix} & 0 & \\ & & \\ B_i L_{i1} & B_i L_{ii} & B_i L_{i\mu} \\ & & \\ & 0 & \end{bmatrix} \tag{6.81}$$

where L is partitioned as before. From the above two conditions we are led to the following result.

Proposition (6.4): If $\sum_c(S_a; F)$ is a complete composite system, then the input-state restriction pencil of the i-th totally input-reduced system is

$$R_{i)}(s) = \left[\begin{array}{ccccc} R_1(s) & & & & \\ & \ddots & & 0 & \\ & & R_i(s) & & \\ & 0 & & \ddots & \\ & & & & R_\mu(s) \\ \hline 0 & & sB_i^\dagger - B_i^\dagger A_i & & 0 \end{array}\right] - \left[\begin{array}{c} \\ \\ 0 \\ \\ \\ \hline L_i \end{array}\right] \tag{6.82}$$

or

$$R_{i)}(s) = \begin{bmatrix} R_1(s) & & & & \\ & \ddots & & 0 & \\ -B_i L_{i1} & & sI - A_i - B_i L_{ii} & & -B_i L_{i\mu} \\ & 0 & & \ddots & \\ & & & & R_\mu(s) \end{bmatrix} \tag{6.83}$$

Proof:

By (6.79) and (6.81) condition (6.83) follows immediately. If $B_i^\dagger, N_i$ is a pair of a left inverse of B_i and a left annihilator, then by multiplication by

$$\begin{bmatrix} I & & & \\ & \ddots & 0 & \\ & & N_i & \\ & & B_i^\dagger & \\ & 0 & & \ddots & \\ & & & & I \end{bmatrix} = \Theta, |\Theta| \neq 0$$

of the $R_{i)}(s)$ pencil, we obtain an equivalent pencil $R'_{i)}(s)$ which is also an input-state restriction pencil (corresponding to a different left annihilator). By rearrangements of the blocks we obtain expression (6.82).

□

Once more, total loss of inputs at the subsystem level corresponds to augmentation of the corresponding pencil by a row block and subsequent perturbation away from the block diagonal structure by the term L_i, the i-th block of the state interconnection matrix.

6.8.3 Output-reduction at Subsystem Level

We consider the case of an P_i-output reduced system and then of a totally i-th output reduced system. In those two cases the corresponding C matrices are

$$C' = \begin{bmatrix} C_1 & & & & \\ & \ddots & & 0 & \\ & & P_i C_i & & \\ & 0 & & \ddots & \\ & & & & C_\mu \end{bmatrix}, C'' = \begin{bmatrix} C_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & C_{i-1} & & & & \\ & & & 0 & C_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & C_\mu \end{bmatrix} \tag{6.84}$$

Using the result of Section (6.7) we have that the corresponding right annihilators are

$$M' = \begin{bmatrix} M_1 & & & & \\ & \ddots & & 0 & \\ & & M_i & C_i^\dagger P_i^\perp & \\ & 0 & & & \ddots & \\ & & & & & M_\mu \end{bmatrix} \tag{6.85}$$

$$M_i'' = \begin{bmatrix} M_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & M_{i-1} & & & & \\ & & & I_{m_i} & & & \\ & & & & M_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & M_\mu \end{bmatrix} \tag{6.86}$$

Using the above two expressions we may investigate the observability properties of the resulting output-reduced system by computing the state-output restriction pencils for the two cases.

P_i-output reduction

The state-output restriction pencil for the P_i-output reduction is

$$P(P_i; s) = sM' - \bar{A}M' - \bar{B}F\bar{C}M' \tag{6.87}$$

from (6.84) and the form of $\bar{A}$ we have

$$sM' - \bar{A}M' = \begin{bmatrix} P_1(s) & & & & \\ & \ddots & & & 0 \\ & & P_i(s) & E(P_i, s) & \\ & 0 & & & \ddots \\ & & & & & P_\mu(s) \end{bmatrix} \tag{6.88}$$

where

$$E(P_i, s) = sC_i^\dagger P_i^\perp - A_i C_i^\dagger P_i^\perp \tag{6.89}$$

Given that

$$\bar{C}M' = \begin{bmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & 0 & P_i^\perp & \\ & 0 & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} \tag{6.90}$$

It follows that

$$\bar{B}F\bar{C}M' = \begin{bmatrix} 0 & -B_1F_{1i}P_i^\perp & 0 \\ \vdots & \vdots & \vdots \\ & -B_iF_{ii}P_i^\perp & \\ \vdots & \vdots & \vdots \\ 0 & -B_\mu F_{\mu i}P_i^\perp & 0 \end{bmatrix} \tag{6.91}$$

Thus we are led to the following result:

Proposition (6.5): If $\sum_c(S_a; F)$ is a complete composite system, then the state-output restriction pencil of the P_i-output reduced system is

$$P(P_i, s) = \begin{bmatrix} P_1(s) & 0 & -B_1 F_{1i} P_i^{\perp} & 0 \\ \vdots & & \vdots & \vdots \\ 0 & P_i(s) & E(P_i, s) - B_i F_{ii} P_i^{\perp} & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & -B_\mu F_{\mu i} P_i^{\perp} & P_\mu(s) \end{bmatrix} \tag{6.92}$$

The above result is an extension of Proposition (6.2) stated for centralised systems to the case of composite systems and indicates that P_i-output reduction corresponds to a column pencil augmentation. In particular, if we partition the interconnectioin matrix F according to the natural partitioning implied by the partitioning of the $\underline{e}, \underline{y}$ vectors in (6.25) i.e.

$$F = \begin{array}{c} \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1i} & \cdots & F_{1\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ F_{i1} & F_{i2} & \cdots & F_{ii} & \cdots & F_{i\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ F_{\mu 1} & F_{\mu 2} & \cdots & F_{\mu i} & \cdots & F_{\mu\mu} \end{bmatrix} \end{array} \begin{array}{c} \updownarrow l_1 \\ \\ \updownarrow l_i \\ \\ \updownarrow l_\mu \end{array} \tag{6.93}$$

and by denoting

$$K = \bar{B}F = \begin{bmatrix} B_1 F_{11} & \cdots & B_i F_{1i} & \cdots & B_\mu F_{1\mu} \\ \vdots & & \vdots & & \vdots \\ B_1 F_{i1} & \cdots & B_i F_{ii} & \cdots & B_\mu F_{i\mu} \\ \vdots & & \vdots & & \vdots \\ B_1 F_{\mu 1} & \cdots & B_i F_{\mu i} & \cdots & B_\mu F_{\mu\mu} \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1i} & \cdots & K_{1\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ K_{i1} & K_{i2} & \cdots & K_{ii} & \cdots & K_{i\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ K_{\mu 1} & K_{\mu 2} & \cdots & K_{\mu i} & \cdots & K_{\mu\mu} \end{bmatrix} = \begin{bmatrix} K_1 \\ \vdots \\ K_i \\ \vdots \\ K_\mu \end{bmatrix}$$

we may state:

Remark (6.5): For every P_i-output reduction problem, only the i-th column block of the state interconnection matrix $K = \bar{B}F$ affects the observability properties of the corresponding composite system. In fact, the corresponding state-output restriction pencil may be expressed as

$$R(P_i, s) = \left[\begin{array}{ccccc|c} P_1(s) & & & & & 0 \\ & \ddots & & & & \\ & & P_i(s) & & & E(P_i, s) \\ 0 & & & \ddots & & \\ & & & & P_\mu(s) & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & K_i P_i^{\perp} \end{array}\right] \tag{6.94}$$

□

It is worth pointing out that the system graph enter now as a perturbation on the block diagonal structure and this perturbation is expressed by $K_i P_i^{\perp}$ on the added pencil column block. The effect of the $K_i P_i^{\perp}$ partitioned structure on aggregate observability properties will be examined later on.

We consider now the boundary case where we have a total loss of system outputs at the i-th subsystem. In this case, M'' is given by (6.86) and the corresponding state-output restriction pencil is

$$P_{i)}(s) = sM_i'' - \bar{A}M_i'' - \bar{B}F\bar{C}M_i'' \tag{6.95}$$

Note that

$$sM_i'' - \bar{A}M_i'' = \begin{bmatrix} P_1(s) & & & & \\ & \ddots & & & 0 \\ & & sI - A_i & & \\ & & & \ddots & \\ & 0 & & & P_\mu(s) \end{bmatrix} \tag{6.96}$$

and that

$$\begin{aligned} \bar{B}F\bar{C}M_i'' &= K\bar{C}M_i'' \\ &= K \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & C_i & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & K_{i1}C_i & \\ & \vdots & \\ 0 & K_{ii}C_i & 0 \\ & \vdots & \\ & K_{\mu i}C_i & \end{bmatrix} \end{aligned}$$

where K is partitioned as before. From the above two conditions we are led to the following result.

Proposition (6.6): If $\sum_c(S_a; F)$ is a complete composite system, then the state-output restriction pencil of the i-th totally output-reduced system is

$$P_{i)}(s) = \left[\begin{array}{ccccc|c} P_1(s) & & & & & 0 \\ & \ddots & & 0 & & 0 \\ & & P_i(s) & & & sC_i^\dagger - A_iC_i^\dagger \\ & 0 & & \ddots & & 0 \\ & & & & P_\mu(s) & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & \\ 0 & \\ 0 & K_i \\ 0 & \\ 0 & \end{array}\right] \tag{6.97}$$

or

$$P_{i)}(s) = \begin{bmatrix} P_1(s) & & -K_{1i}C_i & & \\ & \ddots & & & 0 \\ & & sI - A_i - K_{ii}C_i & & \\ 0 & & & \ddots & \\ & & -K_{\mu i}C_i & & P_\mu(s) \end{bmatrix} \tag{6.98}$$

Proof:

By (6.96) and (6.97) condition (6.98) follows immediately. If $C_i^\dagger, M_i$ is a pair of a right inverse of C_i and a right annihilator, then by multiplication by

$$\begin{bmatrix} I & & & & \\ & \ddots & & 0 & \\ & & M_i & C_i^\dagger & \\ & 0 & & \ddots & \\ & & & & I \end{bmatrix} = \Gamma, |\Gamma| \neq 0$$

of the $P_{i)}(s)$ pencil, we obtain an equivalent pencil $P'_{i)}(s)$ which is also an state-output restriction pencil (corresponding to a different right annihilator). By rearrangements of the blocks we obtain expression (6.97).

□

Once more, total loss of outputs at the subsystem level corresponds to augmentation of the corresponding pencil by a column block and subsequent perturbation away from the block diagonal structure by the term K_i, the i-th block of the state interconnection matrix.

6.8.4 Input-Output Reduction at Subsystem Level

We consider the case of an (R_i, P_j)-input-output reduced system and then of a totally (i,j)-input-output reduced system. In those two cases the corresponding B and C matrices are as in (6.65) and (6.84). Using the result of Section (6.7) together with the corresponding left annihilators (N', N'') as in (6.71) (6.67) and right annihilators (M', M'') as in (6.85), (6.86), we may investigate the zero properties of the resulting input-output reduced system by computing the zero pencil for the two cases.

(R_i, P_j)**-input-output Reduction**:

The zero pencil for the (R_i, P_j)-input-output reduction is

$$\begin{aligned}
Z(R_i, P_j; s) &= [R(R_i; s)]M' \text{ or} \\
&= N'[P(P_i; s)] \\
&= \begin{bmatrix} sN_iM_i - N_iA_iM_i & (sN_i - N_iA_i)C_i^\dagger P_i^\perp \\ R_i^\perp(sB_i^\dagger - B_i^\dagger A_i)M_i & R_i^\perp(sB_i^\dagger - B_i^\dagger A_i)C_i^\dagger P_i^\perp \end{bmatrix} \\
&= \begin{bmatrix} Z_i(s) & R_i(s)C_i^\dagger P_i^\perp \\ Q(R_i, s)M_i & Q(R_i, s)C_i^\dagger P_i^\perp \end{bmatrix}
\end{aligned} \tag{6.99}$$

where $Q(R_i, s)$ has been defined by equation (6.70). Expression (6.99) demonstrates that the problem of input-output reduction is equavalent to matrix pencils augmentation problem by row and column pencils. Such problem deal with issues of Kronecker structure evolution assignment under the operations on matrix pencil augmentation.

We consider now the boundary case where we have a total loss of system inputs and outputs at the i-th subsystem. In this case, the zero pencil of (i,j)-th totally input-output reduced system is

$$Z_{i,j)}(s) = \begin{bmatrix} Z_1(s) & & & & \\ & \ddots & & & 0 \\ & & sI - A_i & & \\ & 0 & & \ddots & \\ & & & & Z_\mu(s) \end{bmatrix} \tag{6.100}$$

Once more, total loss of inputs and outputs at the subsystem level corresponds to augmentation of the corresponding pencil by a row and column block.

Using the expressions (6.99) and (6.100), we can state the following general result for the zero pencil of the complete composite system.

Theorem (6.8): Let $\sum_c(S_a; F)$ be any complete composite system and $S_i(A_i, B_i, C_i)$ denote the i-th subsystem and let $R_i(s) = sN_i - N_iA_i, P_i(s) = sM_i - A_iM_i, Z_i(s) = sN_iM_i - N_iA_iM_i, \bar{Q}_i(s) = sI - A_i, i = 1, s, ..., \mu$ be the pencils associated with the i-th subsystem. Assuming that total loss of inputs, and/or outputs may take at the subsystem level, then for the resulting composite system $S_c(A_c, B_c, C_c)$ the zero pencil $Z_c(S)$ may be expressed as

$$Z_c(s) = \text{ block-diag } \{X_1(s); ...; X_i(s); ...; X_\mu(s)\} \tag{6.101}$$

where $X_i(s)$ block is associated with the i-th subsystem and it is of the following type:

i) If all inputs and outputs of the i-th subsystem are present, then $X_i(s) = Z_i(s)$.

ii) If all inputs are lost at the i-th subsystem, then $X_i(s) = P_i(s)$.

iii) If all outputs are lost at the i-th subsystem, then $X_i(s) = R_i(s)$.

iv) If all inputs and all outputs are lost at the i-th subsystem, then $X_i(s) = \bar{Q}_i(s)$.

□

The cases considered here can provide some useful tools for investigating the controllability, observability and zero properties of a complete composite system under loss of inputs and/or outputs may take at the subsystem level, which are considered next.

6.9 Controllability, Observability, Connectivity under Total Loss of Subsystem Input, Outputs

It has been shown that under total loss of subsystem inputs at the i-th subsystem, the resulting input restriction pencil may be expressed as

$$R_{i)}(s) = \begin{bmatrix} R_1(s) & & & & \\ & \ddots & & 0 & \\ -B_iL_{i1} & & sI - A_i - B_iL_{ii} & & -B_iL_{i\mu} \\ & 0 & & \ddots & \\ & & & & R_\mu(s) \end{bmatrix} \tag{6.102}$$

and similarly the state-output restriction pencil for total loss of outputs at the i-th subsystem may be expressed as

$$P_{i)}(s) = \begin{bmatrix} P_1(s) & & -K_{1i}C_i & & \\ & \ddots & & 0 & \\ & & sI - A_i - L_{ii}C_i & & \\ & 0 & & \ddots & \\ & & -K_{\mu 1}C_i & & P_\mu(s) \end{bmatrix} \tag{6.103}$$

where the K matrix (output injection) is defined from (6.74) by

$$K = \bar{B}F = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1i} & \cdots & K_{1\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ K_{i1} & K_{i2} & \cdots & K_{ii} & \cdots & K_{i\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ K_{\mu 1} & K_{\mu 2} & \cdots & K_{\mu i} & \cdots & K_{\mu\mu} \end{bmatrix} = \begin{bmatrix} K_1; & ...; & K_i; & ...; & K_\mu \end{bmatrix} \tag{6.104}$$

There is clearly a transposed duality between the $R_{i)}(s), P_{i)}(s)$ pencils [Kar., & Hay.,1] (i.e. column minimal indices become row minimal indices and vice-versa, whereas the elementary divisors become the same) and thus we may restrict ourselves to the study of $R_{i)}(s)$. All definitions and procedures given for controllability ($R_{i)}(s)$) may then be interpreted for observability.

We note that both $R_{i)}(s), P_{i)}(s)$ are structured matrices, that is they have zero blocks in fixed locations, whereas the i-th block of rows

$$\begin{bmatrix} & 0 & \\ -B_i L_{i1} & -B_i L_{ii} & -B_i L_{i\mu} \\ & 0 & \end{bmatrix} = Q_i(s) \tag{6.105}$$

acts as a perturbation on the block-diag$\{ R_1(s); ...; sI - A_i; ...; R_\mu(s)\}$ and the i-th block of columns

$$\begin{bmatrix} & -K_{1i}C_i & \\ & \vdots & \\ 0 & -K_{ii}C_i & 0 \\ & \vdots & \\ & -K_{\mu i}C_i & \end{bmatrix} = T_i(s) \tag{6.106}$$

acts as a perturbation on the block-diag$\{P_1(s); ...; sI - A_i; ...; P_\mu(s)\}$. It is worth pointing out that the F_{ij} interconnection gain blocks are not always non zero and a number of them may be identically zero due to the nature of the interconnection graph.

For the given interconnection matrix F we may give the following definition.

Definition (6.3): For the interconnection matrix F defined in partitioned form as in (6.74) we define as:

(i) The i-th row characteristic of F as the set of indices $\mathcal{C}_r(i) \equiv \{j_1 < j_2 < \cdots < j_\nu, \nu \leq \mu\}$ for which

$$F_{ij_1} = F_{ij_2} = \cdots = F_{ij_\nu} = 0 \tag{6.107}$$

(ii) The i-th column characteristic of F as the set of indices $\mathcal{C}_c(i) \equiv \{k_1 < k_2 < \cdots < k_\sigma, \sigma \leq \mu\}$ for which

$$F_{k_1 i} = F_{k_2 i} = \cdots = F_{k_\sigma i} = 0 \tag{6.108}$$

□

We shall denote by $\mathcal{C}'_r(i) \equiv \{j'_1 < j'_2 < \cdots < j'_{\nu'}, \nu' \leq \mu\}, \mathcal{C}'_c(i) \equiv \{k'_1 < k'_2 < \cdots < k'_{\sigma'}, \sigma' \leq \mu\}$ the complementary sets of indices of $\mathcal{C}_r(i), \mathcal{C}_c(i)$ respectively in $\{1, 2, ..., \mu\}$. The significance of the above definition is demonstrated by the following result:

Theorem (6.9): Consider the complete composite sytem $S_c(A_c, B_c, C_c)$ defined as in (6.27), (6.28) and let F be the interconnection matrix which is naturally partitioned as in (6.74). If $S_c(A_c, B_c^{i)}, C_c), S_c(A_c, B_c, C_c^{i)})$ are the subsystems obtained by total loss of inputs at i-th subsystem, total loss of outputs at i-th subsystem respectively, then the following properties hold true:

(i) If $\mathcal{C}_r(i), \mathcal{C}'_r(i)$ are the i-th row characteristics and its complement respectively, then the state input restriction pencil of $S_c(A_c, B_c^{i)}, C_c)$ may be expressed as:

$$R_{i)}(s) = \begin{bmatrix} R_{j_1}(s) & & & \\ & \ddots & & 0 \\ & & R_{j_\nu}(s) & \\ 0 & & & R'_i(s) \end{bmatrix} \tag{6.109}$$

where

$$R'_i(s) = \begin{bmatrix} R_{j'_1}(s) & & & & \\ & \ddots & & & \\ -B_i L_{ij'_1} & & sI - A_i - B_i L_{ii} & & -B_i L_{ij'_{\nu'}} \\ & & & \ddots & \\ & & & & R_{j'_{\nu'}}(s) \end{bmatrix} \tag{6.110}$$

(ii) If $\mathcal{C}_c(i), \mathcal{C}'_c(i)$ are the i-th column characteristics and its complement respectively, then the state-output restriction pencil of $S_c(A_c, B_c, C_c^{i)})$ may be expressed as:

$$P_{i)}(s) = \begin{bmatrix} P_{k_1}(s) & & & \\ & \ddots & & \\ & & P_{k_\sigma}(s) & \\ & & & P'_i(s) \end{bmatrix} \tag{6.111}$$

where

$$P'_i(s) = \begin{bmatrix} P_{k'_1}(s) & & -K_{k'_1 i}C_i & & \\ & \ddots & & & \\ & & sI - A_i - K_{ii}C_i & & \\ & & & \ddots & \\ & & -K_{k'_{\sigma'} i}C_i & & P_{k'_{\sigma'}}(s) \end{bmatrix} \tag{6.112}$$

Proof:

The result is proved for Part (i) and Part (ii) follows along similar lines. From (6.102) we note that for all indices in $\mathcal{C}_r(i) = \{j_1 < j_2 < \cdots < j_\nu, \nu \leq \mu\}$ the column blocks corresponding to those indices have zeros above and below the $R_{j_l}(s)$

blocks as well as to the right and left of them; thus, by successive column and row permutations and starting from the $R_j(s)$ block we can transform $R_{i)}(s)$ in (6.102) to

$$R'_{i}(s) = \left[\begin{array}{c|c} R_{j'_1}(s) & \\ \hline & \\ & R''_{i)}(s) \end{array}\right] \tag{6.113}$$

Note that for the rest of the indices in $\mathcal{C}_r(i)$ the property that to the left and right of R_{j_l}, as well as above and below to have zeros is preserved in $R''_{i)}(s)$, and thus by repeating successively the above first step we get part (i) of the result. Part (ii) follows along similar lines.

□

The submatrices $R'_i(s), P'_i(s)$ cannot be blocked diagonalised anymore; to the $R'_i(s), P'_i(s)$ pencils there corresponds subsystems $S'_i(A'_i, B'_i, C'_i)$ and $\bar{S}_i(\bar{A}_i, \bar{B}_i, \bar{C}_i)$ respectively. These subsystems may be readily constructed and referred to as <u>i-th input irreducible</u>, <u>i-th output irreducible</u> subsystems respectively. From the above result we have:

Corollary (6.4): For a system $S_c(A_c, B_c^{i)}, C_c)$ obtained from the complete system under total loss of i-th subsystem inputs, the controllability properties are given as aggregates of those defined by the subsystems $S_{k^\cdot}(A_{k^\cdot}, B_{k^\cdot}, C_{k^\cdot})$ where $k^\cdot = j_1, ..., j_\nu$, $\mathcal{C}_r(i) = \{j_1 < \cdots < j_\nu\}$ is the i-th row characteristic and of the i-th input irreducible subsystem $S'_i(A'_i, B'_i, C'_i)$.

□

Note that by controllability properties we refer to those associated with controllability indices and input-decoupling zeros. A similar result may be stated for total loss of outputs at i-th subsystem as shown below.

Corollary (6.5): For a system $S_c(A_c, B_c, C_c^{i)})$ obtained from the complete system under total loss of i-th subsystem outputs, the observability properties are given as aggregates of those defined by the subsystems $S_{k^-}(A_{k^-}, B_{k^-}, C_{k^-})$ where $k^- = k_1, ..., k_\sigma$, $\mathcal{C}_c(i) = \{k_1 < \cdots < k_\sigma\}$ is the i-th column characteristic and of the i-th output irreducible subsystem $\bar{S}_i(\bar{A}_i, \bar{B}_i, \bar{C}_i)$.

□

Note that by observability properties we refer to those associated with observability indices and output-decoupling zeros.

Remark (6.6): If $\mathcal{C}_r(i) = \{\emptyset\}(\mathcal{C}_c(i) = \{\emptyset\})$, then in the decomposition (6.109) (6.111) the diagonal part is non-existent and the system may be referred to as strongly connected as far as i-th input (i-th output). If $\mathcal{C}_r(i) = \{1, ..., \mu\}$, then only the diagonal part in (6.109) is present and the system is referred to as weakly connected as far as i-th input.

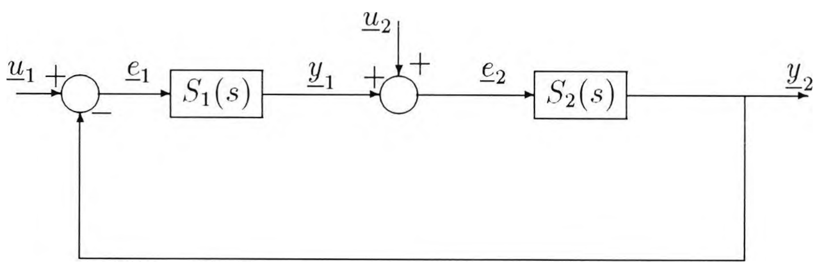
□

The above analysis demonstrates that for every deviation from completeness (by loss of inputs, or outputs), the study of controllability, observability may be reduced to subproblems in a structural sense. For each of the input-, output-irreducible subsystems a proper investigation has to be carried out using the various known controllability, observability tests.

We illustrate the above statements by means of the following examples.

Example (6.1): COMPOSITE STRUCTURE (I)

The following figure shows the block diagram of a composite system with two subsystems.



The system equations derived from the above figure are:

$$\begin{cases} \underline{e}_1 = \underline{u}_1 - \underline{y}_2 \\ \underline{e}_2 = \underline{u}_2 + \underline{y}_1 \end{cases} \Rightarrow \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}_{\equiv F} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \tag{6.114}$$

$$\left\{\begin{array}{l} \begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{\equiv\bar{A}} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}}_{\bar{B}} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \\ \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}}_{\equiv\bar{C}} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \end{array}\right. \tag{6.115}$$

Thus, the composite state matrix equations are

$$\begin{aligned} \bar{A} + \bar{B}F\bar{C} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} 0 & -B_1 \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} 0 & -B_1C_2 \\ B_2C_1 & 0 \end{bmatrix} \end{aligned}$$

or

$$\bar{A} + \bar{B}F\bar{C} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \tag{6.116}$$

Therefore the composite system description can be written as:

$$\left\{\begin{array}{l} \begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix}}_{\equiv A_c} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}}_{B_c} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} \\ \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}}_{\equiv C_c} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \end{array}\right. \tag{6.117}$$

Consider next the restriction pencils of the composite system. In this example, three input-output cases are considered. First, the full input, full output case is given. We first note that given the system, we associate the input state restriction pencil, state output restriction pencil, zero pencil as shown below:

$$\left\{\begin{array}{lll} S_1(A_1, B_1, C_1): & \to & R_1(s) = sN_1 - N_1A_1, P_1(s) = sM_1 - A_1M_1, \\ & & Z_1(s) = sN_1M_1 - N_1A_1M_1 \\ S_2(A_2, B_2, C_2): & \to & R_2(s) = sN_2 - N_2A_2, P_2(s) = sM_2 - A_2M_2, \\ & & Z_2(s) = sN_2M_2 - N_2A_2M_2 \end{array}\right. \tag{6.118}$$

Consider the composite system described above and let N and M be left and right annihilators of B and C such that

$$NB = 0 \text{ where } N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, CM = 0 \text{ where } M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \tag{6.119}$$

To study the controllability properties of the system with full input, full output, the input-state restriction pencil is derived as follows:

$$R(s) = \text{ block-diag}\{R_1(s); R_2(s)\} \tag{6.120}$$

For an insight into the observability properties, the state output restriction pencil is given be:

$$P(s) = \text{ block-diag}\{P_1(s); P_2(s)\} \tag{6.121}$$

To investigate the zero properties of the system, from either of the above pencils the zero pencil can be derived:

$$Z(s) = \text{ block-diag}\{Z_1(s); Z_2(s)\} \tag{6.122}$$

we are therefore led to the following proposition.

Proposition (6.7): The controllability, observability, zero structure properties of the composite system under full input, output structure are simply given as aggregates (direct sum) of corresponding properties of two subsystems.

□

We consider next the case where the total loss of subsystem input structure has occurred. Assume that $\underline{u}_2 = 0$ without loss of generality. This leads to the following reduced composite system description:

$$\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ 0 \end{bmatrix} \tag{6.123}$$

or

$$\dot{\underline{x}} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \underline{u}_1, \underline{y} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \underline{x} \tag{6.124}$$

Let

$$N_2'' = \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} \tag{6.125}$$

To study the controllability properties of the system when there is a loss of input, the input state restriction pencil is derived as follows:

$$
\begin{aligned}
sN_2'' - N_2''A &= \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A_1 & B_1C_2 \\ -B_2C_1 & sI - A_2 \end{bmatrix} \\
&= \begin{bmatrix} sN_1 - N_1A_1 & 0 \\ -B_2C_1 & sI - A_2 \end{bmatrix}
\end{aligned} \tag{6.126}
$$

This is a strongly connected system and is of the form of expression (6.110). By Corollary (6.4), the controllability properties are given as those defined by the 2nd-input irreducible system $S'(A', B_2')$, where $A' = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}$, $B_2' = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$.

□

To investigate the zero properties of the system when there is a loss of input, we derive the zero pencil as follows:

$$
sN_2''M - N_2''AM = \text{ block-diag } \{Z_1(s); P_2(s)\} \tag{6.127}
$$

which verifies the general result previously derived.

Last, let us assume that the first output is measured only. This leads to the total loss of subsystem output, and the reduced composite system description is given by:

$$
\begin{cases} \dot{\underline{x}} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \underline{u} \\ \underline{y} = [C_1, 0]\underline{x} \end{cases} \tag{6.128}
$$

Let

$$
M_2'' = \begin{bmatrix} M_1 & 0 \\ 0 & I \end{bmatrix} \tag{6.129}
$$

To investigate the observability properties, when there is a loss of output, we may define the state-output restriction pencil as follows:

$$
\begin{aligned}
sM_2'' - AM_2'' &= \begin{bmatrix} sI - A_1 & B_1C_2 \\ -B_2C_1 & sI - A_2 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} sM_1 - A_1M_1 & B_1C_2 \\ 0 & sI - A_2 \end{bmatrix}
\end{aligned} \tag{6.130}
$$

This is a strongly connected system and is of the form of expression (6.112). By Corollary (6.5), the observability properties are given as those defined by the 2nd-output irreducible system $S'(A', B_2')$ where $A' = \begin{bmatrix} A_1 & -B_1C_2 \\ 0 & A_2 \end{bmatrix}$, $C_2' = [C_1, 0]$

□

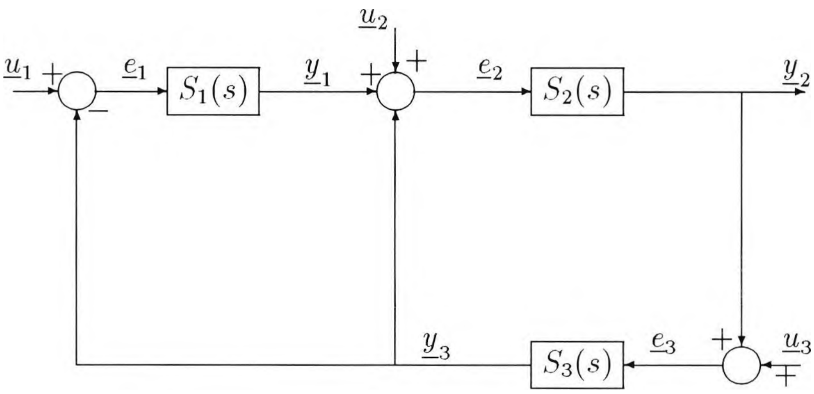
To obtain the zero properties of the system when there is a loss of output, we defined the zero pencil as follows:

$$sNM_2'' - NAM_2'' = \text{ block-diag } \{Z_1(s); R_2(s)\} \tag{6.131}$$

The above verifies the previously worked out result.

Example (6.2): COMPOSITE STRUCTURE (II)

The following figure shows the block diagram of a composite system with three subsystems.



The system equations as derived from the above figure are:

$$\left\{ \begin{array}{l} \underline{e}_1 = \underline{u}_1 - \underline{y}_3 \\ \underline{e}_2 = \underline{u}_2 + \underline{y}_1 - \underline{y}_3 \\ \underline{e}_3 = \underline{u}_3 + \underline{y}_2 \end{array} \right] \Rightarrow \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -I \\ I & 0 & -I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \end{bmatrix} \tag{6.132}$$

This leads to the aggregate system equations

$$\left\{\begin{array}{l}\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\underline{x}}_3\end{bmatrix}=\underbrace{\begin{bmatrix}A_1&0&0\\0&A_2&0\\0&0&A_3\end{bmatrix}}_{\equiv\bar{A}}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\end{bmatrix}+\underbrace{\begin{bmatrix}B_1&0&0\\0&B_2&0\\0&0&B_3\end{bmatrix}}_{\equiv\bar{B}}\begin{bmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{bmatrix}\\ \begin{bmatrix}\underline{y}_1\\ \underline{y}_2\\ \underline{y}_3\end{bmatrix}\underbrace{\begin{bmatrix}C_1&0&0\\0&C_2&0\\0&0&C_3\end{bmatrix}}_{\equiv\bar{C}}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\end{bmatrix}\end{array}\right. \tag{6.133}$$

Thus, the composite state matrix equations are:

$$\bar{A}+\bar{B}F\bar{C}=\begin{bmatrix}A_1&0&0\\0&A_2&0\\0&0&A_3\end{bmatrix}+\begin{bmatrix}B_1&0&0\\0&B_2&0\\0&0&B_3\end{bmatrix}\begin{bmatrix}0&0&-I\\I&0&-I\\0&I&0\end{bmatrix}\begin{bmatrix}C_1&0&0\\0&C_2&0\\0&0&C_3\end{bmatrix}$$

or

$$\bar{A}+\bar{B}F\bar{C}=\begin{bmatrix}A_1&0&-B_1C_3\\B_2C_1&A_2&-B_2C_3\\0&B_3C_2&A_3\end{bmatrix} \tag{6.134}$$

Therefore the composite system description can be written as:

$$\left\{\begin{array}{l}\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\underline{x}}_3\end{bmatrix}=\underbrace{\begin{bmatrix}A_1&0&-B_1C_3\\B_2C_1&A_2&-B_2C_3\\0&B_3C_2&A_3\end{bmatrix}}_{\equiv A_c}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\end{bmatrix}+\underbrace{\begin{bmatrix}B_1&0&0\\0&B_2&0\\0&0&B_3\end{bmatrix}}_{\equiv B_c}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\end{bmatrix}\\ \begin{bmatrix}\underline{y}_1\\ \underline{y}_2\\ \underline{y}_3\end{bmatrix}\underbrace{\begin{bmatrix}C_1&0&0\\0&C_2&0\\0&0&C_3\end{bmatrix}}_{\equiv C_c}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\end{bmatrix}\end{array}\right. \tag{6.135}$$

Consider next the restriction pencil of the composite system. In this example, three different cases are considered. First, the full input, full output case is given. We first note that given the system, we associate the input-state restriction pencil, state-output restriction pencil, zero pencil as shown below for each subsystem:

$$\begin{cases} S_1(A_1,B_1,C_1): & \rightarrow \quad R_1(s)=sN_1-N_1A_1, P_1(s)=sM_1-A_1M_1, \\ & \qquad Z_1(s)=sN_1M_1-N_1A_1M_1 \\ S_2(A_2,B_2,C_2): & \rightarrow \quad R_2(s)=sN_2-N_2A_2, P_2(s)=sM_2-A_2M_2, \\ & \qquad Z_2(s)=sN_2M_2-N_2A_2M_2 \\ S_3(A_3,B_3,C_3): & \rightarrow \quad R_3(s)=sN_3-N_3A_3, P_3(s)=sM_3-A_3M_3, \\ & \qquad Z_3(s)=sN_3M_3-N_3A_3M_3 \end{cases}$$

Consider the composite system described by the above figure and let N and M be left and right annihilators of B and C respectively such that

$$NB=0 \text{ where } N=\begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix}, CM=0 \text{ where } M=\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}$$

To study the controllability properties of the system with full input, full output, the input-state restriction pencil is derived as follows:

$$R(s)= \text{ block-diag } \{R_1(s); R_2(s); R_3(s)\} \tag{6.136}$$

To investigate the observability properties of the system, the state-output restriction pencil is given by:

$$P(s)= \text{ block-diag } \{P_1(s); P_2(s); P_3(s)\} \tag{6.137}$$

To obtain the zero properties of the system, with full input, full output, the zero pencil is defined by:

$$Z(s)= \text{ block-diag } \{Z_1(s); Z_2(s); Z_3(s)\} \tag{6.138}$$

The above verifies the previously given general results. Consider next the case where total loss of subsystem input structure has occurred. Assume that $\underline{u}_1=0$ (without loss of generality). This leads to the following reduced composite system description:

$$\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \\ \dot{\underline{x}}_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & -B_1C_3 \\ B_2C_1 & A_2 & -B_2C_3 \\ 0 & B_3C_2 & A_3 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix} + \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix} \begin{bmatrix} 0 \\ \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} \tag{6.139}$$

or

$$\underline{\dot{x}} = \begin{bmatrix} A_1 & 0 & -B_1C_3 \\ B_2C_1 & A_2 & -B_2C_3 \\ 0 & B_3C_2 & A_3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} \begin{bmatrix} \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} \tag{6.140}$$

Let

$$N''_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \tag{6.141}$$

To investigate the controllability properties when there is a total loss of subsystem input, we may define the input-state restriction pencil as follows:

$$\begin{aligned} sN''_1 - N''_1A &= \begin{bmatrix} I & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{bmatrix} sI - A_1 & 0 & -B_1C_3 \\ B_2C_1 & sI - A_2 & -B_2C_3 \\ 0 & B_3C_2 & sI - A_3 \end{bmatrix} \\ &= \begin{bmatrix} sI - A_1 & 0 & -B_1C_3 \\ 0 & sN_2 - N_2A_2 & 0 \\ 0 & 0 & sN_3 - N_3A_3 \end{bmatrix} \end{aligned} \tag{6.142}$$

This is a weakly connected system so by using Theorem (6.9), by successive row and column operations we can transform (6.142) to two subsystems, i.e.

$$\left[\begin{array}{c|cc} sN_2 - N_2A_2 & 0 & 0 \\ \hline 0 & sN_3 - N_3A_3 & 0 \\ 0 & -B_1C_3 & sI - A_1 \end{array}\right]$$

where the block

$$\begin{bmatrix} sN_3 - N_3A_3 & 0 \\ -B_1C_3 & sI - A_1 \end{bmatrix}$$

is a strongly connected system and is of the form as expression (6.110). By Corollary (6.4), the controllability properties are given as aggrtegate of those defined by the subsystems $S_2(A_2, B_2)$ and $S'(A', B_1')$ where $A' = \begin{bmatrix} A_3 & 0 \\ B_1C_3 & A_1 \end{bmatrix}$, $B_1' = \begin{bmatrix} B_3 \\ 0 \end{bmatrix}$.

□

To investigate the zero properties of the system, when there is a loss of input, we derive the zero pencil as follows:

$$sN''_1M_1 - N''_1AM_1 = \text{block-diag}\ \{P_1(s); Z_2(s); Z_3(s)\} \tag{6.143}$$

which once more verifies the previously stated general result.

Last, let us assume that the first and second outputs are measured (again without loss of generality). This leads to the total loss of subsystem output, and the reduced composite system description is given by:

$$\begin{cases} \dot{\underline{x}} = \begin{bmatrix} A_1 & 0 & -B_1C_3 \\ B_2C_1 & A_2 & -B_2C_3 \\ 0 & B_3C_2 & A_3 \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix} \underline{u} \\ \underline{y} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} \underline{x} \end{cases} \tag{6.144}$$

let

$$M_3'' = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{6.145}$$

To investigate the observability properties, when there is a loss of output, we may define the state-output restriction pencil as follows:

$$\begin{aligned} sM_3'' - AM_3'' &= \begin{bmatrix} sI - A_1 & 0 & B_1C_3 \\ -B_2C_1 & sI - A_2 & B_2C_3 \\ 0 & -B_3C_2 & sI - A_3 \end{bmatrix} \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} sM_1 - A_1M_1 & 0 & B_1C_3 \\ 0 & sM_2 - A_2M_2 & B_2C_3 \\ 0 & 0 & sI - A_3 \end{bmatrix} \end{aligned} \tag{6.146}$$

Using Theorem (6.9), by successive row and column operations we can transform (6.130) to

$$\begin{bmatrix} sM_1 - A_1M_1 & B_1C_3 & 0 \\ 0 & sI - A_3 & 0 \\ 0 & B_2C_3 & sM_2 - A_2M_2 \end{bmatrix}$$

which is a stronly connected system and is of the form of expression (6.112). By Corollary (6.5), the observability properties are given as those defined by the 3rd-output irreducible system $S'(A', C_3')$, where $A' = \begin{bmatrix} A_1 & -B_1C_3 & 0 \\ 0 & A_3 & 0 \\ 0 & -B_2C_3 & A_2 \end{bmatrix}$,

$C_3' = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}$.

□

To obtain the zero properties of the system when there is a loss of output, we define the zero pencil as

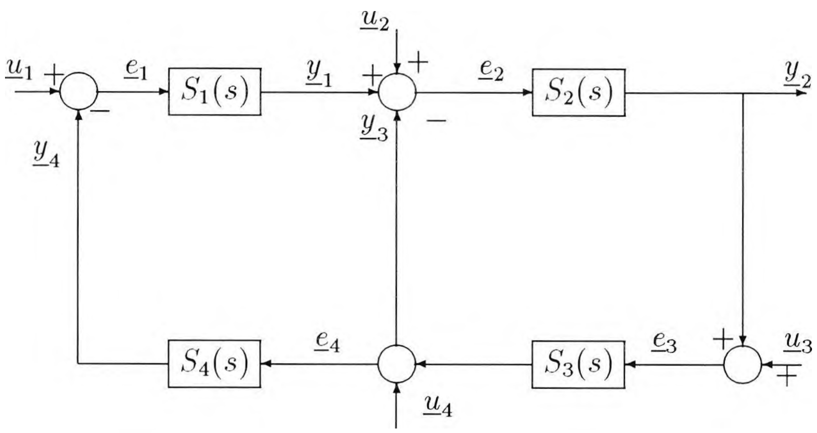
$$sNM_3'' - NAM_3'' = \text{ block-diag } \{Z_1(s); Z_2(s); R_3(s)\} \tag{6.147}$$

The above verifies the previously derived result.

Let us now consider the following example which extends the composite system by a further subsystem.

Example (6.3): COMPOSITE STRUCTURE (III)

The following figure shows the block diagram of a composite system with four subsystems.



The system equations as derived from the above figure are:

$$\left\{\begin{array}{l} \underline{e}_1 = \underline{u}_1 - \underline{y}_4 \\ \underline{e}_2 = \underline{u}_2 + \underline{y}_1 - \underline{y}_3 \\ \underline{e}_3 = \underline{u}_3 + \underline{y}_2 \\ \underline{e}_4 = \underline{u}_4 + \underline{y}_3 \end{array}\right. \Rightarrow \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \\ \underline{e}_4 \end{bmatrix} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \\ \underline{u}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -I \\ I & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{bmatrix} \tag{6.148}$$

This leads to the following aggregate system equation:

$$\left\{\begin{array}{l}
\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \\ \dot{\underline{x}}_3 \\ \dot{\underline{x}}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}}_{\equiv \bar{A}} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}}_{\equiv \bar{B}} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \\ \underline{e}_4 \end{bmatrix} \\
\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{bmatrix} \underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\equiv \bar{C}} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \end{bmatrix}
\end{array}\right. \tag{6.149}$$

Thus, the composite state matrix equation is:

$$\begin{aligned}
\bar{A} + \bar{B}F\bar{C} \;=\; & \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \\
& + \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -I \\ I & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}
\end{aligned}$$

or

$$\bar{A} + \bar{B}F\bar{C} = \begin{bmatrix} A_1 & 0 & 0 & -B_1C_4 \\ B_2C_1 & A_2 & B_2C_3 & 0 \\ 0 & B_3C_2 & A_3 & 0 \\ 0 & 0 & B_4C_3 & A_4 \end{bmatrix} \tag{6.150}$$

Therefore, the composite system description can be written as:

$$\left\{\begin{array}{l}\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\underline{x}}_3\\ \dot{\underline{x}}_4\end{bmatrix}=\underbrace{\begin{bmatrix}A_1 & 0 & 0 & -B_1C_4\\ B_2C_1 & A_2 & B_2C_3 & 0\\ 0 & B_3C_2 & A_3 & 0\\ 0 & 0 & B_4C_3 & A_4\end{bmatrix}}_{\equiv A_c}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ \underline{x}_4\end{bmatrix}+\underbrace{\begin{bmatrix}B_1 & 0 & 0 & 0\\ 0 & B_2 & 0 & 0\\ 0 & 0 & B_3 & 0\\ 0 & 0 & 0 & B_4\end{bmatrix}}_{\equiv B_c}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\\ \underline{u}_4\end{bmatrix}\\ \begin{bmatrix}\underline{y}_1\\ \underline{y}_2\\ \underline{y}_3\\ \underline{y}_4\end{bmatrix}=\underbrace{\begin{bmatrix}C_1 & 0 & 0 & 0\\ 0 & C_2 & 0 & 0\\ 0 & 0 & C_3 & 0\\ 0 & 0 & 0 & C_4\end{bmatrix}}_{\equiv C_c}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ \underline{x}_4\end{bmatrix}\end{array}\right. \tag{6.151}$$

Consider next the restriction pencils of the composite system. In this example, two input-output cases are considered. We first note that given the system we associate input-state restriction pencil, state-output restriction pencil and zero pencil as shown below:

$$S_i(A_i, B_i, C_i): \quad \rightarrow \quad \begin{array}{l} R_i(s) = sN_i - N_iA_i,\, P_i(s) = sM_i - A_iM_i, \\ Z_i(s) = sN_iM_i - N_iA_iM_i \end{array} \quad , i = 1, 2, 3, 4$$

The total loss of subsystem input structure is occurred now. Assume that $u_1 = 0$, without loss of generality. This leads to the following reduced composite system description.

$$\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\underline{x}}_3\\ \dot{\underline{x}}_4\end{bmatrix}=\begin{bmatrix}A_1 & 0 & 0 & -B_1C_4\\ B_2C_1 & A_2 & B_2C_3 & 0\\ 0 & B_3C_2 & A_3 & 0\\ 0 & 0 & B_4C_3 & A_4\end{bmatrix}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ \underline{x}_4\end{bmatrix}+\begin{bmatrix}B_1 & 0 & 0 & 0\\ 0 & B_2 & 0 & 0\\ 0 & 0 & B_3 & 0\\ 0 & 0 & 0 & B_4\end{bmatrix}\begin{bmatrix}0\\ \underline{u}_2\\ \underline{u}_3\\ \underline{u}_4\end{bmatrix} \tag{6.152}$$

or

$$\dot{\underline{x}}=\begin{bmatrix}A_1 & 0 & 0 & -B_1C_4\\ B_2C_1 & A_2 & B_2C_3 & 0\\ 0 & B_3C_2 & A_3 & 0\\ 0 & 0 & B_4C_3 & A_4\end{bmatrix}\underline{x}+\begin{bmatrix}0 & 0 & 0\\ B_2 & 0 & 0\\ 0 & B_3 & 0\\ 0 & 0 & B_4\end{bmatrix}\begin{bmatrix}0\\ \underline{u}_2\\ \underline{u}_3\\ \underline{u}_4\end{bmatrix} \tag{6.153}$$

It is clear that a left annihilator is defined by

$$N_1'' = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & N_3 & 0 \\ 0 & 0 & 0 & N_4 \end{bmatrix} \tag{6.154}$$

To investigate the controllability properties of the system, when there is a loss of input, we may define the input-state restriction pencil as follows:

$$sN''_1 - N''_1 A = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & N_3 & 0 \\ 0 & 0 & 0 & N_4 \end{bmatrix} \begin{bmatrix} sI - A_1 & 0 & 0 & -B_1C_4 \\ B_2C_1 & sI - A_2 & B_2C_3 & 0 \\ 0 & B_3C_2 & sI - A_3 & 0 \\ 0 & 0 & B_4C_3 & sI - A_4 \end{bmatrix}$$

$$= \begin{bmatrix} sI - A_1 & 0 & 0 & -B_1C_4 \\ 0 & sN_2 - N_2A_2 & 0 & 0 \\ 0 & 0 & sN_3 - N_3A_3 & 0 \\ 0 & 0 & 0 & sN_4 - N_4A_4 \end{bmatrix} \tag{6.155}$$

This is a weakly connected system so by using Theorem (6.9), by successive row and column operations we can transform (6.155) to two subsystems, i.e.,

$$\left[\begin{array}{cc|cc} sN_2 - N_2A_2 & 0 & 0 & 0 \\ 0 & sN_3 - N_3A_3 & 0 & 0 \\ \hline 0 & 0 & sI - A_1 & -B_1C_4 \\ 0 & 0 & 0 & sN_4 - N_4A_4 \end{array}\right]$$

where the block

$$\begin{bmatrix} sI - A_1 & -B_1C_4 \\ 0 & sN_4 - N_4A_4 \end{bmatrix}$$

is a strongly connected system and is of the form of expression (6.110). By Corollary (6.4), the controllability properties are given as aggregate of those defined by the subsystems $S_2(A_2, B_2), S_3(A_3, B_3)$ and $S'(A', B_1')$, where $A' = \begin{bmatrix} A_1 & B_1C_4 \\ 0 & A_4 \end{bmatrix}$, $B_1' = \begin{bmatrix} 0 \\ B_4 \end{bmatrix}$.

□

Consider next when there is a total loss of subsystem input and output. Assume that $\underline{u}_1 = 0$ and first output is not measured. To investigate the zero properties of the system, from matrix (6.142), the zero pencil can be derived:

$$sN_1''M_1'' - N_1''AM_1'' = \text{ block-diag } \{\bar{Q}_1(s); Z_2(s); Z_3(s); Z_4(s)\} \qquad (6.156)$$

where $\bar{Q}_1(s) = sI - A_1$.

□

This once more verifies the previously stated general result.

The above three examples demonstrate that when there is partial loss of inputs, or outputs then the interconnection structure plays a crucial role in defining the controllability, observability properties of the resulted system. For each of the input-, output-irreducible subsystems a proper investigation has to be carried out using the various known controllability, observability tests.

6.10 Conclusion

The input-state (state-output) restriction pencil was used for studying the effect of the structure F on controllability (observability) and zero structure of the complete composite system, when total loss of subsystem input (output) occurs as well as location of the formed input (output) decoupling zeros. It was shown that, controllability, observability and zero structure properties of complete composite system under full input, output structure are simply given as aggregates of corresponding properites of the subsystems. It was also shown that, the problems of input-, output- and input-output reduction on the Kronecker structure of a centralised system are equivalent to matrix pencil augmentation problems by row-, column- and row-column pencils. It was proved that for every deviation from completeness by loss of inputs, or outputs on the resulting composite system, the study of controllability, observability may be reduced to a number of subproblems in a structural sense where the system properties are given as aggregates of those defined by the subsytems. For each of the subsytems a proper investigation has to be carried out using various known controllability, observability tests.

Chapter 7

CONCLUSION

In the study of properties of linear systems, as well as the analysis and design of control systems, a variety of approaches has been developed so far. The classification of the different approaches is based on the model that the approach uses, as well as the tools which are developed.

The state space and transfer function descriptions are only two extremes of a whole spectrum of possible descriptions of finite dimensional linear systems. The notion of invariants plays a key role in the characterisation of solvability conditions of different control problems for both the state space and the transfer function based approaches which are considered here. The general aim of the thesis has been to explore further the properties and role of invariants by establishing links between them and system properties, characterisation of their generic values and finally looking at the role of them in composite systems.

The study of properties of a whole family of models having fixed certain fundamental parameters (such as number of inputs, outputs, McMillan degree), but with the rest of parameters taking generic values has been examined here. This topic has been treated both in the state space setup using matrix pencils, as well as with transfer function models. There is still a number of difficulties to deal with unstructured models such as Forney indices, etc., in fact, there is a need for further work on the level of development of concepts and theoretical tools. An important obstacle in the development of some of these concepts and tools has been the lack of numerically efficient tools for algebraic computations, as well as the extension of some of the algebraic notions and approaches to uncertain model cases. Extension of this characterisation to structured (in terms of a given graph) generic system is an important area which needs further consideration.

The importance of Plucker matrices has been shown in [Kar., & Gia.,2], where tests for controllability, observability in terms of the corresponding Plucker matrices have been given. A new different proof for controllability, observability of the system was given here in terms of the rank of the appropriate controllability-, observability Plucker matrices. It has been shown that system controllability, observability is equivalent to the full rank properties of the corresponding Plucker matrices. Further work is needed in linking the Plucker matrices related to transfer function descriptions with the system properties and values of other types of invariants.

The relationship of the structural properties of interconnected systems with those at the subsystem level have been examined and in particular those related to controllability, observability and zeros. The controllability, observability properties of a centralised system as well as composite system under partial, or total loss of subsystem inputs, outputs have been investigated and it has been shown that controllability, observability properties are invariant under composition if the subsystems are controllable or observable respectively. It was also shown that, for every deviation from completeness by loss of inputs, or outputs, the study of system properties may be reduced to a number of subproblems in a structural sense, where the aggregates of those subproblems define the overall system properties.

There is a need to expand the scope by addressing issues of the control theory beyond the present fixed model assumptions. There are a number of issues that have been touched here but need further consideration such as: i) Tackling issues related to system representations and transformations in a unifying manner. ii) Development of matrix pencil approach for state space analysis and synthesis. ii) Development of results characterising system properties in terms of types and values of system invariants, using any structural approaches. Further developments are needed, however, if the algebraic-geometric methods are to be transformed to a synthesis tool. Important issues here are: i) to determine the characterisation of the solvability conditions of different control problems in terms of the values of appropriate system invariants. ii) to establish links between system structural characteristics and achievable limits of performance under compensation (relationships between performance indicators and system structure).

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