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The Crossings of Boundaries by Vector Gaussian Processes with Applications to Problems in Reliability

by

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A thesis submitted to The City University for the degree of Doctor of Philosophy

November, 1992.

CONTENTS

Tables and Illustrations

Acknowledgements

I would like to thank Prof. Henry Wynn for his continued support and encouragement and to thank his predecessor as my supervisor, Dr. Alan Winterbottom. My thanks are due also to Robert Matthews of the City of London Polytechnic for all his help with computer software during the writing-up. My belated thanks to Gordon Belcher of Marconi Avionics who was responsible for introducing me to the problem of nuisance disconnects, and last but not least, to my wife Sheila for her steadfast and understanding support.

Declaration

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Abstract

The problem of 'nuisance disconnects' in high integrity redundant systems is shown to be mathematically equivalent to the problem of the crossings of the boundary of a region by a vector stochastic process. A number of other engineering situations are similarly modelled by such multivariate crossing processes.

Let X(t) be a vector valued stationary Gaussian process, having continuous sample paths a.s., and let U be the number of exits, in the interval (0,1] , from a region Γ having a boundary $\partial\Gamma$ consisting of a finite number of regular elements. We prove the formula of Belyaev for the factorial moments of U under conditions on the process similar to those of Ylvisaker(1966). We further show that these conditions are sufficient to guarantee $E(U) < \infty$.

The validity of the formula of Belyaev does not imply the existence of the moments of all orders. We show that, for a two- dimensional Gaussian process, the variance of U is finite if

$$\int_0^\delta \| \theta''(t) \| t^{-1}\, dt < \infty$$

for some $\delta > 0$, where $\theta''(t) = R''(t) - R''(0)$ and $R''(t)$ is the second derivative of the covariance matrix of the process.

It is well known that the duration of an exceedence above a high level, by a stationary Gaussian process, has an asymptotic Rayleigh distribution. In chapter 4, we show that, for the two dimensional processes of the present study, the Rayleigh distribution is but one of three asymptotic distributions possible for the duration of an exceedence above a large boundary.

In the final chapters we comment on the problem of 'nuisance disconnects' in the light of the theoretical developments of the previous chapters. A discussion of the relation of our work to that of earlier authors and of possible avenues for future research is also included.

1 INTRODUCTION

1.1 Nuisance Disconnects

My interest in boundary and level crossing problems was stimulated by the engineering problem of 'nuisance disconnects' which occurs in the high integrity control systems found in modern aircraft and in nuclear reactors. Many 'fly-by-wire' aircraft are intrnsically unstable and depend on the control system for dynamic stability. It is therefore vital that the integrity of the control system is not compromised by the failure of a single component.

To seek to maintain the integrity of a control system subject to one or more faults, parallel redundant lanes are introduced into the system. Triplex or quadruplex redundant systems are typical of flight control systems [Gill, 1977; Ahern, et al.,1976] allowing up to two-failure survival. The introduction of redundant circuits of itself, only increases the chance of a component failure.

To improve the chances of survival of the control system, we must be able to detect faulty lanes and disconnect them, leaving the working lanes to operate with signals uncorrupted by the faulty lane or lanes. This function is performed by a 'voter-monitor ' which compares the signals in the working lanes and decides according to some rule or criterion whether a lane is faulty and to be disconnected. The other function of the voter-monitor is to output a consolidated signal by combining or averaging the inputs from the lanes that are currently deemed to be working.

The disconnection of a lane will induce a sudden, though transitory, change in the consolidated output from the monitor. This so called transient, if large enough, could have a catastrophic effect on the system.

The size of transients can be made small by using a criterion with narrow tolerances in the voter-monitor. However, if there is noise in the lane signals, as there always is in practice, then the probability, of a disconnection occuring without a fault in any of the lanes, will become unacceptably large. The event of a disconnection occuring without a fault in any of the lanes, is a so called 'nuisance disconnect'.

1.2 Example

For definiteness, we assume a triplex system with a voter-monitor which outputs the arithmetic mean of the signals from the working lanes. Such a system is illustrated in figure 1.1, where three sensors independently measure a particular flight parameter and the outputs of the sensors are fed into a voter-monitor.

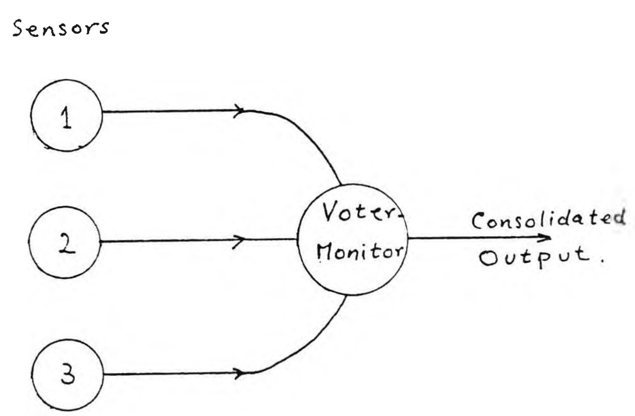

figure 1.1 Model of a redundant system with voter-monitor

In practice, the output from each sensor might be fed into three monitors, providing three independently obtained consolidated outputs for input to computers or actuators.

Assuming all three lanes are deemed to be working, the monitor will disconnect a lane if the magnitudes of the disparities of that lane with the other two is greater than or equal to some tolerance level u. Thus if the signal levels in the three lanes are X_1, X_2, X_3, the disparities can be denoted by $s_1 = X_2 - X_3$, $s_2 = X_3 - X_1$, and $s_3 = X_1 - X_2$. Suppose, for example, that lane 3 is about to be disconnected, then $\min\{ |s_1|, |s_2| \} = u$ and $|s_3| < u$. In other words, a lane is disconnected if the median of the magnitudes $|s_1|$, $|s_2|$, $|s_3|$ attains and subsequently exceeds the level u. What we have described here is, of course, only one of many possible decision criteria.

Since $s_1 + s_2 + s_3 = 0$ identically, we can represent the disparities by a point P in a two-dimensional space. In fact, we represent s_1, s_2, s_3 as the orthogonal projections of the position vector **x**, of P in the disparity space, on three axes at 120 degrees to each other. In such a representation, our disconnection criterion corresponds to a region containing the origin and having the star-shaped boundary shown in figure 1.2 .

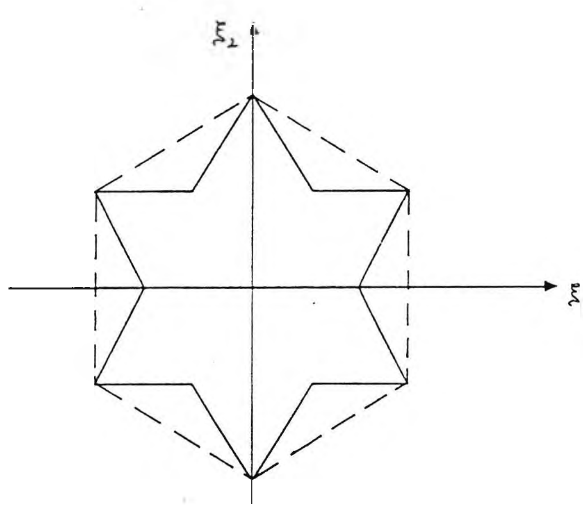

figure 1.2 The star-shaped boundary in disparity space

The state of the monitor, or more precisely of the three lanes being monitored, is represented by a point ξ in the two-dimensional disparity space which generates a stochastic process $\xi(t)$ over time. While the stochastic process $\xi(t)$ remains within the star-shaped region of figure 1.2, the monitor assumes the lanes are all working and will not disconnect them. However, once the process $\xi(t)$ crosses the star-shaped boundary, one of the three lanes is disconnected on the assumption that it has failed. Which lane is disconnected depends on which part of the boundary is crossed.

Once a lane has been disconnected, two things happen. The monitoring regime must change to provide a consolidated output from the remaining two lanes instead of the original three. This change in the consolidation regime introduces a transient in the system corresponding to the sudden change in output from the monitor.

By way of example, suppose the new output is taken to be the mean of the two working lanes and suppose lane 1 is the disconnected lane. Just prior to the disconnection the monitor output was $(X_1 + X_2 + X_3)/3$ and just after the disconnection, the output becomes $(X_2 + X_3)/2$, giving rise to a transient

$$t_1 = (X_2 + X_3)/2 - (X_1 + X_2 + X_3)/3 = (s_2 - s_3)/6.$$

At disconnection, one of $| s_2 |$ and $| s_3 |$ will equal u and the other will be greater than u. Further, s_2 and s_3 will be of opposite signs, since $s_1 + s_2 + s_3 = 0$ and $| s_1 | < u$. It follows that $| s_2 - s_3 | = 2u + | s_1 |$ and hence that $u/3 < | t_1 | < u/2$. In general, a disconnection will lead to a transient that will vary in magnitude between one third and one half u, depending on how far the crossing point is from the origin.

Transients, if large, can have a disastrous effect on a system. A regime which gives the smallest transients for a given tolerance level will yield transients of constant magnitude. Using means to consolidate the input signals, such a regime would cause us to disconnect a lane as soon as its signal exceeds the mean signal of all three lanes by more than the tolerance u. That is, we would disconnect lane 1, when

$$| X_1 - (X_1 + X_2 + X_3)/3 | = | 2X_1 - X_2 - X_3 |/3 = u ,$$

with $| X_j - (X_1 + X_2 + X_3)/3 | < u$, for $j = 2, 3$. The coresponding boundary in disparity space, shown by the dashed hexagonal line in figure 1.2, would give transients of constant magnitude $u/2$.

1.3 Model Assumptions

We suppose the signal in lane i consists of a stimulus $Z(t)$ corrupted by an additive noise process or measurement error $W_i(t)$, thus

$$X_i(t) = Z(t) + W_i(t) \quad , \qquad (1.3.1)$$

for $i = 1, 2, 3$. Here $\mathbf{W}(t) = \{ W_1, W_2, W_3 \}$ is a stationary vector process, independent of $Z(t)$, which we will assume to be Gaussian whenever convenient. However, we do not assume the components $W_i(t)$ to be independently nor identically distributed. For the most part, we assume the measurement process to be unbiased, ie $E(W_i(t)) = 0$, for $i = 1, 2, 3$, although in chapter 5 we will consider the effect of a measurement bias in one of the lanes.

With these assumptions, the disparities are independent of the stimulus. The problem of 'nuisance disconnects' reduces to the study of the exits of a multivariate stationary stochastic process from a simply-connected region of R^2.

We introduce coordinates into the disparity space by

$$\begin{aligned} \xi_1 &= (-2X_1 + X_2 + X_3)/\sqrt{3} \\ &= (-2W_1 + W_2 + W_3)/\sqrt{3} \end{aligned} \qquad (1.3.2)$$

$$\begin{aligned} \xi_2 &= - X_2 + X_3 \\ &= - W_2 + W_3 \end{aligned} \qquad (1.3.3)$$

Thus if $L(t) = E[\mathbf{W}(0)\,\mathbf{W}(t)^T] = [\, l_{ij}\,]$ is the covariance matrix of $\mathbf{W}(t)$, then the covariance matrix of $\xi(t)$ is given by

$$R(t) = E[\,\xi(0)\,\xi(t)^T\,]$$
$$= \begin{bmatrix} -2/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1 & 1 \end{bmatrix} L \begin{bmatrix} -2/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1 & 1 \end{bmatrix}^T$$

On carrying out the matrix multiplication, the elements of R(t) become:

$$R_{11} = \tfrac{4}{3}\, l_{11} + \tfrac{1}{3}\, l_{22} + \tfrac{1}{3}\, l_{33} - \tfrac{4}{3}\, l_{12} - \tfrac{4}{3}\, l_{13} + \tfrac{2}{3}\, l_{23}$$
$$R_{12} = -\frac{1}{\sqrt{3}}\, l_{22} + \frac{1}{\sqrt{3}}\, l_{33} + \frac{2}{\sqrt{3}}\,(\, l_{12} - l_{13}\,) \qquad (1.3.4)$$
$$R_{22} = l_{22} + l_{33} - 2\, l_{23} \quad .$$

making full use of the symmetry of L.

1.4 Applications

A widely discussed application of the crossings of a vector stochastic process occurs in civil and mechanical engineering (Lindgren (1980a), Veneziano et.al. (1977)). In this application, $\mathbf{X}(t)$ is a vector representing a random load on some structure. The structure remains safe while $\mathbf{X}(t)$ lies inside a connected region S of the state space. The probability that the structure remains intact throughout the interval [0, T], $P\{\, \mathbf{X}(t) \in S \,;\ \text{for all } t \in [0, T]\,\}$, is directly related to the distribution of time to the first exit from S, by the process $\mathbf{X}(t)$.

The types of region that have been used in practice, [Lindgren (1980a)], are

(a) $S = \{\, (x_1, x_2) \in R^2 :\ x_1{}^2 + x_2{}^2 \le r \,\}$

(b) $S = \{\, (x_1, x_2, x_3) \in R^3 :\ x_1 + \max(\,|\, x_2 \,|, |\, x_3 \,|\,) \le r \,\}$

(c) $S = \{\, (x_1, x_2) \in R^2 :\ \max(\,|\, a\, x_1 + a'\, x_2 |, |\, b\, x_1 + b'\, x_2 |\,) \le r \,\}$

The regions of examples (b) and (c) are similar in type to the region of our application to nuisance disconnects, which we give, as example (d), for comparison .

(d) $S = \{(x_1, x_2) \in R^2 : \text{med}(\,|x_2|, |\sqrt{\tfrac{3}{2}}x_1 + \tfrac{1}{2}x_2|, |\sqrt{\tfrac{3}{2}}x_1 - \tfrac{1}{2}x_2|\,) \leq u\}$,

where med(a, b, c) denotes the median of a, b, c .

Example (d) along with examples (b) and (c) yield polyhedral shaped regions. Note that in all cases, with the aid of the max and med functions, we have been able to express the S-regions in the form $f(\mathbf{x}) \leq u$, or equivalently in the form $f(\mathbf{x}) \leq 0$. Thus, formally at least, the problem of the crossings of a boundary by a vector stochastic process $\mathbf{X}$ is reduced to that of the level crossings of the univariate process $Y = f(\mathbf{X})$. For this and other reasons, we give a brief review of level crossings in the next section and follow it with a review of the much sparser literature on boundary crossings. In the last section, we outline the scope of the thesis and describe the contents of the succeeding chapters.

1.5 Level Crossings

The best reference to the early work on level crossings is the seminal book, Stationary and Related Stochastic Processes, by Cramer and Leadbetter (1967). Another useful source of references to the early literature is the review article by Blake and Lindsey (1973). In this section we review the three crossing problems directly relevent to the present thesis, the moments of the number of crossings in an interval, the existence of the variance of the number of crossings and the duration of exceedences above a high level.

Throughout this section we assume $X(t)$ to be a real-valued stationary Gaussian process having zero mean and continuous sample paths with probability one, for $t \in [0,1]$. The process is also assumed to have a covariance function $r(t)$ and a spectral distribution function $F(\lambda)$, in the real sense, such that

$$r(t) = E[X(s+t)X(s)] = \int_0^\infty \cos \lambda t \ \ dF(\lambda) \ . \tag{1.5.1}$$

We define spectral moments of even order 2k, $k = 0, 1, \ldots,$ by

$$\lambda_{2k} = \int_0^\infty \lambda^{2k} \ \ dF(\lambda) \ . \tag{1.5.2}$$

It is well known (Cramer and Leadbetter (1967)) , that λ_{2k} is finite if and only if r(t) has a finite derivative of order 2k at the origin. In this case $r^{(2k)}(t)$ is continuous and

$$\lambda_{2k} = (-1)^k \, r^{(2k)}(0) \ .$$

Let S be the set of continuous functions f(t) on [0,1] for which $f(t) \neq 0$ when $t = k2^{-n}$, $k = 0, \ldots, 2^n$, $n = 1, 2, \ldots$. For $f(t) \in S$, we define N(f) to be the number of zeros of f(t) in [0,1] and T(f) to be the number of tangential zeros. A tangential zero occurs at t_0 if $f(t_0) = 0$ and there is a neighbourhood of t_0 on which f has a constant sign. A crossing zero occurs at $t_0 \in [0,1]$ provided every neighbourhood of t_0 contains points t_1 and t_2 such that $f(t_1)f(t_2) < 0$. Clearly, if a zero is not tangential it is a crossing, hence N(f) $=$ T(f) + C(f) , where C(f) is the number of crossing zeros in the interval [0,1].

In this taxonomy of zeros introduced by Ylvisaker (1965) and Cramer and Leadbetter (1967), crossing zeros has the subcategories 'up-crossings' and 'down-crossings'. An up-crossing of zero is said to occur at t_0 if there exists $\varepsilon > 0$ such that, $f(t) < 0$ for $t_0 - \varepsilon < t < t_0$ and $f(t) > 0$ for $t_0 < t < t_0 + \varepsilon$, a down-crossing is similarly defined. Isolated crossing zeros must be either up or down-crossings, thus if $C < \infty$ then $C = U + D$, where U, D denote the numbers of up and down-crossings in [0,1], respectively.

Since X(t) has a continuous distribution for $t \in [0,1]$, it follows that $X(t) \in S$ a.s. and that N, T, C etc. are defined for X(t) a.s. More generally, we define N_u, T_u, C_u, . . . for the crossings and tangencies of a level u by X(t). In this notation, the number of zeros, tangencies and zero crossings in [0,1] are denoted N_0, T_0, C_0, respectively.

Bulinskya (1961) and Ylvisaker (1965) have shown that $T = 0$ a.s. under a wide range of conditions. Thus $N = C$ a.s. and hence, in particular, $E(N_u) = E(C_u)$. The formula for the expected number of crossings of the level u in [0,1],

$$E(C_u) = \frac{1}{\pi}\left(\frac{\lambda_2}{\lambda_0}\right)^{1/2} e^{-u^2/(2\lambda_0)} \tag{1.5.3}$$

was first obtained by Rice (1945) under the assumption of a discrete spectrum. The formula (1.5.3) has been derived by a number of authors under successively weaker conditions [see for example Ivanov (1960), Bulinskya (1961)]. The best result for stationary Gaussian processes was obtained by Ito (1964) and Ylvisaker (1965) who showed that (1.5.3) held for processes with continuous sample paths if $\lambda_2 < \infty$ and that $E(C_u) = \infty$, if $\lambda_2 = \infty$.

The earlier authors had assumed a sample function derivative for the process and used an integral method due to Kac (1943) to count the number of crossing zeros. Both Ito and Ylvisaker utilised a new counting method based on the use of indicator random variables.

Approximate X(t) by a polygonal process $X_n(t)$ tied to X(t) at points $t = t_i = i\,2^{-n}$, for $i = 0, 1, \ldots, 2^n$ and let C_n be the number of zero crossings by $X_n(t)$ in the interval [0,1]. If C_{ni} is the indicator of the event

$X[(i-1)\,2^{-n}]\,X[\,i\,2^{-n}\,] < 0$, it can be shown that $C_n = \Sigma_i\, C_{ni}$ is an increasing sequence and $C_n \to C$, as $n \to \infty$, a.s.

Similarly we can write for up-crossings of zero $U_n = \Sigma\, U_{ni}$, where U_{ni} is the indicator of the event $X[(i-1)\,2^{-n}] < 0 < X(i\,2^{-n})$. As before, U_n forms an increasing sequence a.s. However it is no longer true that $U_n \uparrow U$, as $n \to \infty$, without qualification. Unlike the crossing situation, $U_{ni} = 1$ does not invariably imply an up-crossing in the interval $(\,(i-1)\,2^{-n}\,,\ i\,2^{-n}\,)$. This will only be true, with probability one, if the crossings are all isolated, ie if $C < \infty$.

This method for counting up-crossings is used by Cramer and Leadbetter (1965) and Ylvisaker (1966) in deriving a formula for the factorial moments of the number of up-crossings. Cramer and Leadbetter obtain the fornula for the factorial moment M_k, of order k, of the number of up-crossings of a level, under the assumption of a continuous sample path derivative and the existence of a joint density $p_t\,(x_1, \ldots, x_k; y_1, \ldots, y_k)$ of the variables $X(t_1), \ldots, X(t_k), X'(t_1), \ldots, X'(t_k)$ for distinct $t_1, \ldots, t_k$. In our terminology their formula becomes

$$\begin{aligned} M_k &= E\left[U(U-1)\cdots(U-k+1)\right] \\ &= \int_0^1 \cdots \int_0^1 dt_1 \cdots dt_k \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_k\, p_t(u\,;\,y)\, dy_1 \cdots dy_k\,, \end{aligned} \qquad (1.5.4)$$

where $p_t(u\,;\,y) = p_t(u, \ldots, u\,;\,y_1, \ldots, y_k)$. Ylvisaker (1966) has proved that the condition of a continuous sample path derivative can be replaced by the existence of the second spectral moment λ_2, in which case (1.5.4) will hold whether M_k is finite or not.

Both Cramer and Leadbetter (1965) and Ylvisaker (1966) appear to assume incorrectly that U_n tends monotonely to U, as $n \to \infty$, without qualification.

Since the conditions of both authors guarantee $E(C) < \infty$, it follows that $C < \infty$ a.s. and hence their assumptions are justified.

Just prior to the publication of Cramer and Leadbetter (1965) , Leadbetter and Cryer (1965) had obtained a formula for the variance of the number of zeros of a stationary Gaussian process and had shown that the variance would be finite if, for some $\delta > 0$,

$$\int_0^\delta \frac{\lambda_2 + r''(t)}{t}\, dt < \infty \,. \tag{1.5.5}$$

This sufficient condition was later shown to be necessary by Geman (1972) . Belyaev (1966) and Miroshin (1973) have given sufficient conditions for the existence of the moments of the number of crossings of a level by a non-stationary Gaussian process. For stationary Gaussian processes, Cuzick (1975) has obtained necessary and sufficient conditions for the moments of the number of zero crossings to be finite. If a stationary Gaussian process has a covariance function

$$r(t) = 1 - \frac{1}{2}t^2 + \frac{1}{6}C\,|t|^3 + o(|t|^3) \quad , \tag{1.5.6}$$

for small t , its crossing moments of all orders are finite.

Marcus (1977) has given formulae for the generalised moments of the numbers of crossings by a real valued stochastic process with abssolutely continuous sample paths.The extremely general conditions given for the validity of the moment formulae are so close to the result that verifying the conditions would, in many cases, be as difficult as giving a direct proof of the formulae. Marcus has applied his conditions to a class of non-stationary Gaussian processes, which satisfy a bound on the joint densities proved by Cramer and Leadbetter (1967) for stationary Gaussian processes, to obtain sufficient conditions for the validity of moment formulae for non-stationary Gaussian processes.

As we have already noted, in many applications the quantity of principal interest is the distribution of the time to the first crossing. Unfortunately the problem of finding the distribution of time to first crossing or the distributions of times between crossings has proved intractable, except in one or two isolated instances. In consequence interest has focused on asymptotic results for crossings of high levels.

It is well known that the crossings of a level by a stationary Gaussian process form a regular stationary point process, if $\lambda_2 < \infty$ [Cramer and Leadbetter (1967) p 201]. Volkonskii and Rozanov (1959, 1961) have proved, under a rather complicated mixing condition, that the up-crossings of a level u form an asymptotic Poisson process, as $u \rightarrow \infty$. The mixing condition has been successively weakened and simplified by Cramer (1966) and Berman (1971). The latter showed that the result holds if the covariance function satisfies the condition $r(t) \log t \rightarrow 0$, as $t \rightarrow \infty$.

Another form of asymptotic result, more pertinent to the present discussion, relates to the distribution of the duration of an excursion above a high level. Let Δ denote the time from an up-crossing of a level u to the following dowm-crossing. If the process X(t) is stationary and ergodic, then the mean duration $E(\Delta) = \theta = \frac{1}{\mu} P\{ X(0) > u \}$, where μ is the mean up-crossing rate. If F(t) is the distribution function of Δ / θ, Rice (1958) has shown that

$$\lim_{u \rightarrow \infty} F(t) = 1 - e^{-(\pi/4) t^2} \qquad (1.5.7)$$

ie the duration, measured in units of its mean, has an asymptotic Rayleigh distribution, as $u \rightarrow \infty$.

Kac and Slepian (1959) have pointed to the fact that F(.), in (1.5.7), is a conditional distribution and as such can depend critically on the way the conditioning is carried out. Using ergodic arguments, they have shown that the conditioning should be applied in what they call the 'horizontal window" sense.

Cramer and Leadbetter (1967) following the work of Volkonskii and Rozanov (1961) have given a rigorous proof of (1.5.7) assuming two conditions on the covariance function r(t) :

(i) $$r(t) = 1 - \frac{\lambda_2}{2!}t^2 + \frac{\lambda_4}{4!}t^4 + o(t^4) \quad , \text{as } t \to 0 . \tag{1.5.8}$$

(ii) $$r(t) = O(t^{-\alpha}) \quad , \text{for some } \alpha > 0, \text{ as } t \to \infty . \tag{1.5.9}$$

Belyaev and Nosko (1969) have obtained (1.5.7) under a slightly weaker condition on r(t) , as $t \to 0$, than (1.5.8). They have also anticipated our result in chapter 4, by showing that the durations of excursions, of a bivariate stationary Gaussian process above a large circular boundary, have the asymptotic distribution of (1.5.7).

1.6 Vector Processes

Let $\mathbf{X}(t)$ be a vector stochastic process taking values in R^p and having continuous sample paths. In this section we consider the exits of a process $\mathbf{X}(t)$ from a bounded, simply connected, region $\Gamma \subset R^p$. At an exit, we assume the process crosses the boundary $\partial\Gamma$ and enters a region $\Gamma' \subset R^p$, the complement of $\Gamma \cup \partial\Gamma$. In what follows, we will often refer to Γ as the admissible or safe region, since, in applications, exits from Γ correspond to failures of one kind or another.

It is natural to regard exits and entrances as crossings of the boundary $\partial\Gamma$ and to talk of 'boundary crossings' by analogy with 'level crossings' in the univariate case. Unfortunately this terminology leads to some ambiguity. The term 'boundary crossing' is often used in connection with sequential analysis, which is concerned with a univariate discrete parameter stochastic process exceeding a level which is a function of the parameter. In the present work 'boundary crossing' will always have the former connotation of exits from, or entrances to, a region by a multivariate continuous parameter stochastic process.

Belyaev (1968) gave, without proof, an integral formula for the factorial moments of the number of exits by a vector process under rather complicated conditions on the process. Contrary to Belyaev's assertion, his condition (1) on the process is not satisfied by stationary Gaussian processes having continuously differentiable sample paths, as we show in the appendix .

Suppose $g(\mathbf{x})$ is a real function on R^p such that $\Gamma = \{ \mathbf{x} \in R^p : g(\mathbf{x}) \leq 0 \}$ and hence $\partial\Gamma = \{ \mathbf{x} \in R^p : g(\mathbf{x}) = 0 \}$. As we have previously observed, the boundary crossings of $\partial\Gamma$ by the vector process $\mathbf{X}(t)$ are equivalent to the zero crossings of the univariate process $Y = g[\mathbf{X}(t)]$. Assuming $g(.)$ is continuously differentiable in the neighbourhood of $\partial\Gamma$ with the possible exception of a finite number of points of $\partial\Gamma$, Lindgren (1980b) applies the method of Marcus (1977) to extend a result of Belyaev (1968) concerning the expected number of exits across the boundary $\partial\Gamma$.

A number of authors including Belyaev (1968) , Belyaev and Nosko (1969) , Bolotin (1971) , Hasofer (1974) and Veneziano et. al. (1977), have calculated the mean crossing rate for Gaussian processes for a variety of smooth and polyhedral boundaries. Formulae for asymptotic crossing rates of stationary vector Gaussian processes for large boundaries have been obtained by Breitung (1988).

Sharpe (1978), Lindgren (1980a) and Aronowich and Adler (1985, 1986) have developed the level crossing theory of χ^2 processes.If $\mathbf{X}(t)$ is a stationary vector Gaussian process with independent components $X_1(t), \ldots, X_p(t)$ having zero means and unit variances, then $Y(t) = X_1^2(t) + \ldots + X_p^2(t)$ is said to be a χ^2 process. Clearly the up-crossings of level u by $Y(t)$ are equivalent to the exits of $\mathbf{X}$ from a p-sphere of radius $\sqrt{u}$. Using these two different approaches Sharpe (1978) and Lindgren (1980a) obtain the following expression for the expected number of up-crossings of the level u by a χ^2 process

$$E(U) = \left(\frac{\lambda_2}{\pi}\right)^{1/2} (u/2)^{(p-1)/2} e^{-u/2} / \Gamma(p/2)$$

where λ_2 is the second spectral moment of the component Gaussian processes.

1.7 Scope of Thesis

In chapter 2 we give a proof of Belyaev's formula, for the factorial moments of the number of exits by a vector Gaussian process, under very general conditions. There is no proof in the literature. Belyaev (1968) originally gave the formula without proof and under very different conditions. The approach we adopt to the proof is to describe the boundary $\partial\Gamma$ of the admissible region explicitly and attempt to generalise the approach of Ylvisaker (1966) and Cramer and Leadbetter (1965).

In chapter 3 we explore the conditions for the existence of the variance of the number of exits and obtain sufficient conditions similar to those obtained by Leadbetter and Cryer (1965) in the univariate case.

Chapter 4 is devoted to the problem of the asymptotic distribution of the duration of an excursion outside a large boundary. Three types of asymptotic distribution are obtained, one of which is the Rayleigh distribution familiar from the level crossings of Gaussian processes.

By way of example, in chapter 5 we apply some of the results of the earlier chapters to the problem of nuisance disconnects, described in sections 1.1 to 1.3 . In particular, we compute the expected number of exits and the mean duration of an excursion for a number of different admissible regions and discuss their relevance to some questions concerning nuisance disconnections. This chapter also contains a summary of the results obtained in the earlier chapters, with some discussion of the place of the results in relation to the published work and of possible future developments.

2 FACTORIAL MOMENTS

2.1 Preliminaries

Let $\mathbf{X}(t) = \{X_1(t), \ldots, X_p(t)\}$, $0 \le t \le 1$, be a real vector-valued stationary Gaussian process, having continuous sample paths with probability one. In general, we assume that $\mathbf{X}(t)$ has mean zero and a covariance matrix R(t), having a continuous second derivative at the origin. The latter condition is equivalent to the existence of a finite second order spectral moment matrix λ_2 , and ensures the existence of a mean square derivative $\mathbf{X}'(t)$, for $t \in [0,1]$ [Cramer and Leadbetter (1967)].

In this chapter we study the crossings of the boundary $\partial\Gamma$ of a simply connected open subset $\Gamma \subset R^p$, by the process $\mathbf{X}(t)$, $0 \le t \le 1$. The boundary $\partial\Gamma$ is assumed throughout to consist of a finite number of regular elements of finite extent. Within each regular element the coordinates of the points are assumed to be continuously differentiable functions of p -1 parameters $a_1, \ldots, a_{p-1}$. Further we assume that $\partial\Gamma$ has no double points, ie on crossing $\partial\Gamma$ from Γ we enter a subset Γ' which is the complement of $\Gamma \cup \partial\Gamma$ in R^p . We start our study by giving a careful definition of the different types of crossing of $\partial\Gamma$.

Let G be the class of continuous functions f(t) on [0,1] , taking values in R^p , such that $f(t_{ni}) \notin \partial\Gamma$, for all $t_{ni} = i\, 2^{-n}$, $i = 0, 1, \ldots, 2^n$; $n = 1, 2, \ldots$

For $f(t) \in G$, we define the following types of crossing:

(i) The function f(t) is said to exit from Γ at t_0 , if there exists $\varepsilon > 0$, such that $f(t) \in \Gamma$ for $t_0 - \varepsilon < t < t_0$, and $f(t) \in \Gamma'$ for $t_0 < t < t_0 + \varepsilon$. Denote by U the number of exits by f in [0,1].

(ii) The function f(t) is said to enter Γ at t_0, if there exists $\varepsilon > 0$, such that $f(t) \in \Gamma'$ for $t_0 - \varepsilon < t < t_0$, and $f(t) \in \Gamma$ for $t_0 < t < t_0 + \varepsilon$. Denote by D the number of entrances by f in [0,1].

(iii) The function f(t) is said to have a crossing of $\partial\Gamma$ at t_0, if, in each neighbourhood of t_0, there exists t_1 and t_2 such that $f(t_1) \in \Gamma$ and $f(t_2) \in \Gamma'$. Denote by C the number of crossings in [0,1].

Since exits and entrances are clearly crossings, $C \geq U+D$. In fact if t_0 is an isolated crossing of $\partial\Gamma$ then it must be either an exit or an entrance, hence, if C is finite, we must have $C = U+D$.

Assuming $\mathbf{X}(t)$ has a continuous density for all $t \in [0,1]$, as would be the case if R(0) is non-singular, it follows that $\mathbf{X}(t) \in G$ a.s. Thus the above definitions and relations will apply to $\mathbf{X}(t)$ except on an exceptional set having probability zero.

Let U_{ni} be the indicator of the event $\{\mathbf{X}(t_{n\,i-1}) \in \Gamma;\ \mathbf{X}(t_{ni}) \in \Gamma'\}$ and D_{ni} the indicator of the event $\{\mathbf{X}(t_{n\,i-1}) \in \Gamma';\ \mathbf{X}(t_{ni}) \in \Gamma\}$. Write $C_{ni} = U_{ni} + D_{ni}$, $U_n = \Sigma\, U_{ni}$ and $C_n = \Sigma\, C_{ni}$, where the sum in each case is over $i = 1, 2, \ldots, 2^n$ for $n = 1, 2, \ldots$. In the next lemma we show that, in certain circumstances, the random variables U_n and C_n approximate U and C, respectively, as n tends to infinity.

Lemma 2.1 As $n \to \infty$, $C_n \uparrow C$ a.s. and, if C is finite with probability one, then $U_n \uparrow U$ a.s.

Proof. For $\mathbf{X}(t) \in G$, $U_{ni} = 1$ implies $U_{n+1\,2i-1} + U_{n+1\,2i} = 1$, similarly $C_{ni} = 1$ implies $C_{n+1\,2i-1} + C_{n+1\,2i} = 1$. Since, if $\mathbf{X}(t_{n\,i-1}) \in \Gamma$ and $\mathbf{X}(t_{ni}) \in \Gamma'$, then, at the mid-point $t_{n+1\,2i-1}$, either $\mathbf{X} \in \Gamma$ or $\mathbf{X} \in$

Γ' , and one of $U_{n+1\ 2i-1}$ and $U_{n+1\ 2i}$ is zero and the other unity. It follows that the sequences $\{U_n\}$ and $\{C_n\}$ are non-decreasing a.s. Since, by continuity, $\mathbf{X}(t_1) \in \Gamma$ and $\mathbf{X}(t_2) \in \Gamma'$ implies $\mathbf{X}(t)$ has a crossing between t_1 and t_2 , we must have $C_n \leq C$ a.s.

Suppose $C < \infty$, then the crossings can be separately contained in disjoint intervals $I_1, \ldots, I_C$. Hence there exists a positive integer n_o such that $C_n = C$, for $n \geq n_o$, and therefore $\lim_{n\to\infty} C_n = C$. Further, let $I_1, \ldots, I_U$ be the subset of the C disjoint intervals which contain the exits among the C crossings. For n large enough, the intervals $[t_{n\ i-1}, t_{ni}]$ containing an exit will lie entirely within the corresponding I_j , thus there exists n_o , such that $U_n = U$, for $n \geq n_o$. Therefore $\lim_{n\to\infty} U_n = U$ and the second part of the lemma is proved.

Now suppose that $\lim_{n\to\infty} C_n < \infty$ for some sample path in G. Since $\{C_n\}$ is non-decreasing, $C_n = m$, for $n \geq n_o$, for some positive integers m, n_o . Thus, for $n \geq n_o$, if $\mathbf{X}(t_{n\ i-1}) \in \Gamma$ and $\mathbf{X}(t_{ni}) \in \Gamma$, then $\mathbf{X}(t_{n+k\ j}) \in \Gamma$, for all $t_{n+k\ j} \in [t_{n\ i-1}, t_{ni}]$, $k = 1, 2, \ldots$, since C_n must remain unchanged as n increases. Therefore, by continuity, $\mathbf{X}(t) \in \Gamma \cup \partial\Gamma$ for $t \in [t_{n\ i-1}, t_{ni}]$ and there can be no crossings between $t_{n\ i-1}$ and t_{ni} . Similarly, if $\mathbf{X}(t_{n\ i-1}) \in \Gamma'$ and $\mathbf{X}(t_{ni}) \in \Gamma'$, for $n \geq n_o$, it will follow that $\mathbf{X}(t) \in \Gamma \cup \partial\Gamma$, for $t \in [t_{n\ i-1}, t_{ni}]$, and the interval will contain no crossings. These considerations show that there is just one crossing in each of the intervals $[t_{n_o\ i-1}, t_{n_o\ i}]$ that contribute to C_{n_o} and hence that $\mathbf{X}(t)$ has at most $m = \lim_{n\to\infty} C_n$ crossings. The lemma now follows since, if $C = \infty$, then $\lim_{n\to\infty} C_n = \infty$, for the opposite conclusion would lead to a contradiction.

It is worth noting that, had we defined U_n and C_n to be the number of exits or crossings, respectively, of the straight line process tied to $\mathbf{X}(t)$ at $t = t_{ni}$,

$i = 0, 1, \ldots, 2^n$, the sequences $\{U_n\}$ and $\{C_n\}$ would not have been monotone.

The lemma we have just proved also holds for level crossings if C and U are interpreted as the numbers of crossings and up-crossings, respectively. In the context of level crossings, Ylvisaker (1966) and Cramer and Leadbetter (1965) erroneously assume $U_n \uparrow U$ a.s. , without the extra condition $C < \infty$. However, their main results remain valid, since $E(C) < \infty$ for the class of stationary Gaussian processes considered by them, and in consequence $C < \infty$ a.s.

Clearly the quantities U_n and C_n are measurable on the sample space and hence random variables. The lemma gives conditions under which $C_n \to C$ and $U_n \to U$ on a set of measure one. If the probability measure is complete, then C and U are measurable in case $C_n \to C$ and $U_n \to U$.

2.2 The Expected Number of Crossings

Following lemma 2.1, the monotone convergence theorem implies

$$E(C_n) = \sum_{i=1}^{2^n} P\{C_{ni} = 1\} \uparrow E(C) , \tag{2.2.1}$$

as $n \to \infty$.

For $0 < t \leq 1$ and $n = 1, 2, \ldots$ let $i(t)$ be the integer between 1 and 2^n such that $(i - 1)/2^n < t \leq i/2^n$ and define $\phi_n(t)$ by

$$\phi_n(t) = 2^n P\{C_{ni} = 1\} , \tag{2.2.2}$$

where $i = i(t)$. Thus (2.2.1) can be rewritten

$$E(C_n) = \int_0^1 \phi_n(t)\, dt \uparrow E(C) \tag{2.2.3}$$

as $n \to \infty$.

Now

$$P\{C_{ni} = 1\} = P\{U_{ni} = 1\} + P\{D_{ni} = 1\} \qquad (2.2.4)$$

and

$$P\{U_{ni} = 1\} = P\{\mathbf{X}(t_{n\,i-1}) \in \Gamma ; \mathbf{X}(t_{ni}) \in \Gamma'\}$$
$$= P\{\mathbf{X}_i - 2^{-n}\mathbf{Y}_i \in \Gamma \;;\; \mathbf{X}_i \in \Gamma'\} ,$$

where we have written $\mathbf{X}_i = \mathbf{X}(t_{ni})$ and $\mathbf{Y}_i = 2^n(\mathbf{X}_i - \mathbf{X}_{i-1})$.

For $0 < t \leq 1$, and $i = i(t)$, let $p_{nt}(\mathbf{x} ; \mathbf{y})$ denote the probability density function of $\mathbf{X}_i$ and $\mathbf{Y}_i$. Then we can write

$$P\{U_{ni} = 1\} = \int\int_{A_n(\mathbf{y})} p_{nt}(\mathbf{x} ; \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} , \qquad (2.2.5)$$

where $A_n(\mathbf{y})$ is the subset of $\mathbf{x}$ for which $\mathbf{x} - 2^{-n}\mathbf{y} \in \Gamma$ and $\mathbf{x} \in \Gamma'$.

If $\mathbf{x} \in A_n(\mathbf{y})$, the line segment $\mathbf{x} - \theta\, 2^{-n}\mathbf{y}$, $0 \leq \theta \leq 1$, must cross $\partial\Gamma$ at least once. For the sake of uniqueness, suppose $\theta = \beta$ corresponds to the first crossing in going from $\mathbf{x} - 2^{-n}\mathbf{y}$ to $\mathbf{x}$. Suppose this crossing occurs at the point $\mathbf{x}(a) = \mathbf{x}(a_1, \ldots, a_{p-1})$ of $\partial\Gamma$, so that we have

$$\mathbf{x} = \mathbf{x}(a) + \beta 2^{-n}\mathbf{y} \qquad (2.2.6)$$

and $\mathbf{x} - \theta\, 2^{-n}\mathbf{y} \in \Gamma$ for $\beta < \theta \leq 1$. Since the crossing at $\mathbf{x}(\mathbf{a})$ is from Γ to Γ', if $\mathbf{a}$ is a regular point of $\partial\Gamma$, we must have $\nu^T.\mathbf{y} \geq 0$, where ν is the unit outward drawn normal to $\partial\Gamma$ at $\mathbf{x}(\mathbf{a})$.

Equation (2.2.6), for fixed $\mathbf{y}$, defines a one-one transformation from $(a_1, \ldots, a_{p-1}, \beta)$ to $\mathbf{x}$ which is differentiable if $\mathbf{a}$ corresponds to a regular point of $\partial\Gamma$.

The Jacobian of the transformation is

$$\frac{\partial(\mathbf{x})}{\partial(\mathbf{a},\beta)} = 2^{-n}\nu^T.\mathbf{y}\ \mathrm{Det}\left[\frac{\partial\mathbf{x}(\mathbf{a})}{\partial a_1}, \cdots, \frac{\partial\mathbf{x}(\mathbf{a})}{\partial a_{p-1}}, \nu\right]$$

since the first p -1 columns of the determinant span the tangent hyperplane of $\partial\Gamma$ at $\mathbf{x}(\mathbf{a})$. Substituting for $\mathbf{x}$ in terms of $\mathbf{a},\beta$ in the integral of (2.2.5), we get

$$P\{U_{ni} = 1\} = \int d\mathbf{y} \int_{\partial\Gamma} \int_{B_n} p_{nt}\left[\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y}\ ;\ \mathbf{y}\right] 2^{-n} \nu^T.\mathbf{y}\ d\beta\ d\mathbf{w} \quad , \qquad (2.2.7)$$

where $B_n = \{\beta : \mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y} \in \Gamma'$ and $\mathbf{x}(\mathbf{a}) - (\theta - \beta)\, 2^{-n}\, \mathbf{y} \in \Gamma$, for all θ, with $\beta < \theta \leq 1\}$. In deriving (2.2.7) we have used the well known result

$$d\mathbf{w} = \left| \operatorname{Det}\left[\frac{\partial \mathbf{x}(\mathbf{a})}{\partial a_1}, \ldots, \frac{\partial \mathbf{x}(\mathbf{a})}{\partial a_{p-1}}, \nu\right] \right| da_1 \cdots da_{p-1}$$

for the surface element $d\mathbf{w}$ of $\partial\Gamma$ at $\mathbf{x}(\mathbf{a})$. By a similar arguement we would get

$$P\{D_{ni} = 1\} = \int d\mathbf{y} \int_{\partial\Gamma} \int_{B_n} p_{nt}\left[\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y}\ ;\ \mathbf{y}\right] 2^{-n} \left|\nu^T.\mathbf{y}\right| d\beta\ d\mathbf{w} \quad , \qquad (2.2.8)$$

where the definition of B_n differs from that in (2.2.7), in particular B_n is empty if $\nu^T.\mathbf{y} > 0$. Substituting from (2.2.7) and (2.2.8) into (2.2.2) and using (2.2.4), we obtain

$$\phi_n(t) = \int \int_{\partial\Gamma} \int_{B_n} p_{nt}\left[\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y}\ ;\ \mathbf{y}\right] \left|\nu^T.\mathbf{y}\right| d\beta\ d\mathbf{w}\ d\mathbf{y} \quad , \qquad (2.2.9)$$

The definition of B_n in (2.2.9) depends on the sign of $\nu^T.\mathbf{y}$; if $\nu^T.\mathbf{y} > 0$, B_n is defined as in (2.2.7), if $\nu^T.\mathbf{y} < 0$, then B_n is the subset of $\beta \in (0,1)$ such that $\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y} \in \Gamma$ and $\mathbf{x}(\mathbf{a}) - (\theta - \beta)\, 2^{-n}\, \mathbf{y} \in \Gamma'$, for all θ, with $\beta < \theta \leq 1$, and, if $\nu^T.\mathbf{y} = 0$, we put $B_n = \emptyset$, since in this case the integrand is zero. If $\mathbf{x}(\mathbf{a})$ is a regular point of $\partial\Gamma$, and $\nu^T.\mathbf{y} \neq 0$, we can find an integer $n_o > 0$ such that the line segment from $\mathbf{x} - 2^{-n}\, \mathbf{y}$ to $\mathbf{x} + 2^{-n}\, \mathbf{y}$ contains no point of $\partial\Gamma$ other than $\mathbf{x}(\mathbf{a})$. Hence $B_n = (0,1)$ for all $n \geq n_o$.

The covariance matrix of $\mathbf{X}_i$ and $\mathbf{Y}_i$ is

$$\Lambda_n = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} ,$$

where the p x p sub-matrices A, B, C are given by

$$A = E(\mathbf{X}_i \mathbf{X}_i^{\,T}) = R(0)$$

$$B = E(\mathbf{X}_i \mathbf{Y}_i^{\,T}) = 2^n [R(0) - R(2^{-n})] \tag{2.2.10}$$

$$C = E(\mathbf{Y}_i \mathbf{Y}_i^{\,T}) = 2^{2n} [2R(0) - R(- 2^{-n}) - R(2^{-n})] .$$

It is well known [Cramer (1940)] that the covariance matrix

$$R(t) = E\left[\mathbf{X}(s + t) \mathbf{X}^T(s)\right] \tag{2.2.11}$$

of the continuous stationary process X(t) has a spectral representation

$$R(t) = \int_0^\infty \cos \lambda t \, dF(\lambda) + \int_0^\infty \sin \lambda t \, dG(\lambda) \quad , \tag{2.2.12}$$

where the elements of $F(\lambda)$ and $G(\lambda)$ are functions of bounded variation and $\Delta H = \Delta F + i\,\Delta G$ is a non-negative definite Hermitean matrix. In particular, the diagonal element $F_{r\,r}(\lambda)$ is a non-decreasing function of λ, and

$$(\Delta F_{r\,s})^2 + (\Delta G_{r\,s})^2 \leq \Delta F_{r\,r} \Delta F_{s\,s} \ , \tag{2.2.13}$$

for $r \neq s$, $r,s = 1, 2, \ldots, p$.

In order to consider the limiting behaviour in (2.2.9) we need the following lemma.

Lemma 2.2 If the matrix of second order spectral moments

$$\lambda_2 = \int_0^\infty \lambda^2 \, dF(\lambda) \tag{2.2.14}$$

exists and is finite, then R(t) is twice continuously differentiable and as $h \to 0$

$$\frac{R(t + h) - R(t)}{h} \to R'(t) \quad ,$$

and

$$\frac{2R(t) - R(t - h) - R(t + h)}{h^2} \to - R''(t)$$

uniformly in t .

Proof. Note ΔG is skew-symmetric and for $r \neq s$, by (2.2.13), it follows that $|\Delta G_{rs}| \leq \frac{1}{2}(\Delta F_{rr} + \Delta F_{ss})$, and hence

$$\left|\int_0^\infty \lambda^2 \, dG_{rs}\right| \leq \int_0^\infty \lambda^2 |dG_{rs}| \leq \frac{1}{2}\int_0^\infty \lambda^2 (dF_{rr} + dF_{ss})$$

Thus by (2.2.14) it follows that $\int_0^\infty \lambda^2 \, dG$ exists.

From (2.2.12) we obtain

$$\frac{R(t+h) - R(t)}{h} = \int_0^\infty \frac{\cos \lambda(t+h) - \cos \lambda t}{h} dF(\lambda) + \\ + \int_0^\infty \frac{\sin \lambda(t+h) - \sin \lambda t}{h} dG(\lambda). \qquad (2.2.15)$$

For $\lambda > 0$, we readily show that the two integrands of the right hand side of (2.2.15) are bounded by 2λ. Let $\acute{I}(\lambda) = \sum_1^p F_{rr}(\lambda)$, then $\acute{I}(\lambda)$ is a non-decreasing function of λ and $\int_0^\infty \lambda^2 \, d\acute{I}(\lambda)) < \infty$, by (2.2.14), hence $\int_0^\infty \lambda \, d\acute{I}(\lambda) < \infty$. It follows from the above considerations that the right hand member of (2.2.15) tends, element by element, to

$$-\int_0^\infty \lambda \sin \lambda t \, dF + \int_0^\infty \lambda \cos \lambda t \, dG = R'(t), \qquad (2.2.16)$$

as $h \to 0$, by dominated convergence. Further, the derivative is continuous in t, since each component integral is dominated by $\int_0^\infty \lambda \, d\acute{I}(\lambda)$.

Again from (2.2.12) we find

$$\frac{2R(t) - R(t-h) - R(t+h)}{h^2} = \int_0^{\infty} \cos \lambda t \left(\frac{\sin \frac{\lambda h}{2}}{\frac{\lambda h}{2}} \right)^2 \lambda^2 \, dF + \int_0^{\infty} \sin \lambda t \left(\frac{\sin \frac{\lambda h}{2}}{\frac{\lambda h}{2}} \right)^2 \lambda^2 \, dG \;. \tag{2.2.17}$$

Since each integral is dominated by $\int_0^{\infty} \lambda^2 \, d\acute{I}(\lambda)$, the right hand member of (2.2.16) tends to

$$\int_0^{\infty} \cos \lambda t \; \lambda^2 \, dF + \int_0^{\infty} \sin \lambda t \; \lambda^2 \, dG = -R''(t) \;, \tag{2.2.18}$$

element by element, as $h \to 0$, and the continuity of $R''(t)$ follows from the continuity of $\cos \lambda t$ and $\sin \lambda t$ by dominated convergence.

To demonstrate that the convergence of (2.2.15) to (2.2.16) is uniform in t , we consider the (r, s) element of $[R(t+h) - R(t)]/h - R'(t)$. This element is dominated by the integral $\int_0^{\infty} 6\lambda \, d\acute{I}$, which in view of the above is finite.

Hence, given $\varepsilon > 0$, we can choose $\lambda_0 > 0$ such that $\int_{\lambda_0}^{\infty} 6\lambda \, d\acute{I} \leq \varepsilon/2$. Further, given $\eta > 0$, we can find $\delta > 0$ such that $|(\sin x)/x - 1| < \eta$ and $|(\cos x - 1)/x| < \eta$, if $|x| < \delta$. Thus, for $|\lambda_0 h| < \delta$, we readily show

$$\begin{aligned} \left| \left[\frac{R(t+h) - R(t)}{h} - R'(t) \right]_{rs} \right| &\leq \sqrt{2}\eta \int_0^{\lambda_0} \lambda \, |dF_{rs}| + \sqrt{2}\eta \int_0^{\lambda_0} \lambda \, |dG_{rs}| + \int_{\lambda_0}^{\infty} 6\lambda \, d\acute{I} \\ &\leq 2\sqrt{2}\eta \int_0^{\infty} \lambda \, d\acute{I} + \frac{\varepsilon}{2} \;. \end{aligned} \tag{2.2.19}$$

Since $\int_0^\infty \lambda \, d\acute{I}(\lambda)$ is finite, we can choose η so that the right hand member of (2.2.19) is less than or equal to ε, for all t, and the uniform convergence is proved.

Using the finiteness of $\int_0^\infty \lambda^2 \, d\acute{I}(\lambda)$ we can similarly prove the uniformity of the convergence of (2.2.17) to (2.2.18).

Under the conditions of lemma 2.2,

$$\Lambda_n \to \Lambda = \begin{bmatrix} R(0) & -R'(0) \\ -R'(0) & -R''(0) \end{bmatrix},$$

as $n \to \infty$. Thus, if Λ is non-singular, then, for some integer $n_0 > 0$, Λ_n must be non-singular, for all $n \geq n_0$, and hence, as $n \to \infty$, $p_{nt}(\mathbf{x} ; \mathbf{y}) \to p_t(\mathbf{x} ; \mathbf{y})$; the joint density function of $\mathbf{X}(t)$ and its mean square derivative $\mathbf{X}'(t)$.

Before considering the limit, as $n \to \infty$, in (2.2.9), we need a bound for the integrand. We start by deriving a bound for $p_{nt}(\mathbf{x} ; \mathbf{y})$ using a modification of an argument due to Cramer and Leadbetter (1967). The density $p_{nt}(\mathbf{x} ; \mathbf{y})$ is multivariate normal with covariance matrix Λ_n and zero means and hence does not depend on t. Consequently we write $p_n(.)$ for $p_{nt}(.)$, and $p(.)$ for $p_t(.)$.

Lemma 2.3 If the matrix of second order spectral moments is finite and Λ is non-singular, then there exists a positive constant K and a positive integer n_0 such that

$$p_n(\mathbf{x} ; \mathbf{y}) \leq K \exp\left\{-\frac{1}{4}\mathbf{y}^T C^{-1}\mathbf{y}\right\},$$

for all x and $n \geq n_0$, where $C = -R''(0)$.

Proof. Now $p_n(\mathbf{x} ; \mathbf{y})$ can be written $p_n(\mathbf{x} \mid \mathbf{y}) \, p_n(\mathbf{y})$, where

$$p_n(\mathbf{x} \mid \mathbf{y}) = (2\pi)^{-p/2} |L_n|^{1/2} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T L_n (\mathbf{x} - \mathbf{m})\right\}$$

is the conditional probability density function of X given Y . The p x p matrix L_n consists of the first p rows and columns of Λ_n^{-1} and is non-negative definite. Thus , for n large enough for Λ_n to be non-singular, we have

$$p_n(\mathbf{x} ; \mathbf{y}) \leq (2\pi)^{-p/2} |L_n|^{1/2} p_n(\mathbf{y}) = (2\pi)^{-p} |\Lambda_n|^{-1/2} \exp\left\{-\frac{1}{2}\mathbf{y}^T C_n^{-1} \mathbf{y}\right\}$$

for all $\mathbf{x}, \mathbf{y}$.

By lemma 2.2, $C_n \to C = -R''(0)$, as $n \to \infty$. Consider the ratio $\mathbf{y}^T C_n^{-1} \mathbf{y} / \mathbf{y}^T C^{-1} \mathbf{y}$. This is bounded below by the smallest eigenvalue of $C\,C_n^{-1}$. Since $C\,C_n^{-1} \to I$, the identity matrix of order p, as $n \to \infty$, the eigenvalue will tend to unity . Hence, for all n larger than some integer $n_o > 0$, $\mathbf{y}^T C_n^{-1} \mathbf{y} / \mathbf{y}^T C^{-1} \mathbf{y} \geq 1/2$.

Therefore

$$\begin{aligned} p_n(\mathbf{x} ; \mathbf{y}) &\leq (2\pi)^{-p} |\Lambda_n|^{-1/2} \exp\{-\tfrac{1}{2} \mathbf{y}^T C_n^{-1} \mathbf{y}\} \\ &\leq (2\pi)^{-p} |\Lambda_n|^{-1/2} \exp\{-\tfrac{1}{4} \mathbf{y}^T C^{-1} \mathbf{y}\} \\ &\leq K \exp\{-\tfrac{1}{4} \mathbf{y}^T C^{-1} \mathbf{y}\} \quad , \end{aligned}$$

for $n \geq n_o$ and all $\mathbf{x}, \mathbf{y}$.

Theorem 2.1 Let $\mathbf{X}(t)$ be a p-variate stationary Gaussian process having continuous sample paths, with probability one, covariance matrix $R(t)$ and finite second order spectral moment matrix λ_2 .

If the 2p x 2p matrix

$$\Lambda = \begin{bmatrix} R(0) & -R'(0) \\ -R'(0)^T & -R''(0) \end{bmatrix} \qquad (2.2.20)$$

is non-singular, then for ϕ_n defined by (2.2.2),

$$\phi_n(t) \to \phi(t) = \int_{\partial\Gamma} \int p(\mathbf{x} ; \mathbf{y}) \, |\nu^T \mathbf{y}| \, d\mathbf{y} \, dw$$

and $E(C) = \int_0^1 \phi(t)\, dt < \infty$.

Proof. By lemma 2.2, $\Lambda_n \rightarrow \Lambda$ and $p_n(\mathbf{x} ; \mathbf{y}) \rightarrow p(\mathbf{x} ; \mathbf{y})$ as $n \rightarrow \infty$. By lemma 2.3, there exists $n_0 > 0$ such that

$$p_n(\mathbf{x} ; \mathbf{y}) \le K \exp\{ -\tfrac{1}{4} \mathbf{y}^T C^{-1} \mathbf{y} \}$$

for $n \ge n_0$. Hence, from (2.2.9),

$$\phi_n(t) \le \int_{\partial\Gamma} \int \int_0^1 p_n(\mathbf{x} + \beta 2^{-n} \mathbf{y} ; \mathbf{y}) |\mathbf{v}^T.\mathbf{y}| \, d\beta \, d\mathbf{y} \, d\mathbf{w}$$

$$\le \int_{\partial\Gamma} \int |\mathbf{y}| K \exp\left\{ -\tfrac{1}{4} \mathbf{y}^T C^{-1} \mathbf{y} \right\} d\mathbf{y} \, d\mathbf{w} < \infty$$

since $\partial\Gamma$ has finite extent.

Since, as $n \rightarrow \infty$, $B_n \rightarrow (0,1)$ and $p_n(\mathbf{x}(\mathbf{a}) + \beta 2^{-n} \mathbf{y} ; \mathbf{y}) \rightarrow p(\mathbf{x}(\mathbf{a}) ; \mathbf{y})$ then, by dominated convergence, $\phi_n(t) \rightarrow \phi(t)$, where

$$\phi(t) = \int_{\partial\Gamma} \int p(\mathbf{x}(\mathbf{a}) ; \mathbf{y}) |\mathbf{v}^T.\mathbf{y}| \, d\mathbf{y} \, d\mathbf{w} \; .$$

Thus

$$E(C) = \lim_{n \rightarrow \infty} \int_0^1 \phi_n(t) \, dt = \int_0^1 \phi(t) \, dt$$

is finite.

Under the conditions of theorem 2.1, it therefore follows that $C < \infty$, with probability one.

2.3 Factorial Moments

It follows from lemma 2.1 that, if $C < \infty$ with probability one, for any positive integer k, $U_n (U_n - 1) \ldots (U_n - k + 1) \uparrow U (U - 1) \ldots (U - k + 1)$ a.s., and by monotone convergence, the kth factorial moment

$$M_k = E[U(U-1) \cdots (U-k+1)] = \lim_{n \rightarrow \infty} E[U_n(U_n - 1) \cdots (U_n - k + 1)] \; .$$

As can readily be shown [Cramer and Leadbetter (1965)]

$$U_n(U_n - 1) \cdots (U_n - k + 1) = \Sigma' \, U_{ni_1} U_{ni_2} \cdots U_{ni_k} \; ,$$

where the summation is over all sets of k distinct integers from 1 to 2^n . On taking expectations, we have

$$E[U_n(U_n - 1)\cdots(U_n - k + 1)] = \Sigma' P\{U_{ni_1} U_{ni_2}\cdots U_{ni_k} = 1\} \quad . \qquad (2.3.1)$$

Define functions f_n , $n = 1, 2, \ldots$, on $Q = (0,1]^k$ by

$$f_n(\mathbf{t}) = \begin{cases} 2^{nk} P\{U_{ni_1} U_{ni_2}\cdots U_{ni_k} = 1\} & \text{if } i_1\cdots i_k \text{ distinct} \\ 0 & \text{if } i_1\cdots i_k \text{ are not distinct}, \end{cases} \qquad (2.3.2)$$

where $\mathbf{t} = (t_1, \ldots, t_k) \in Q$ and $i_r = i(t_r)$. Not only does $f_n(\mathbf{t}) = 0$ if $i_r = i_s$ for $r \neq s$, but also if $i_r = i_s + 1$, for some $r \neq s$, since $U_{ni}\, U_{n\ i+1} = 0$ a.s., as follows from the definition of the indicator random variables U_{ni} .

Combining (2.3.1) and (2.3.2), we can write

$$E[U_n(U_n - 1)\cdots(U_n - k + 1)] = \int_Q f_n(\mathbf{t})\, d\mu \qquad (2.3.3)$$

where μ denotes Lesbegue measure on Q. Futhermore, by Fatou's lemma, we have

$$M_k = \lim_{n \to \infty} \int_Q f_n(\mathbf{t})\, d\mu \geq \int_Q \liminf f_n(\mathbf{t})\, d\mu \quad . \qquad (2.3.4)$$

Thus, if $f_n(\mathbf{t})$ converges pointwise to a limit function $f(\mathbf{t})$, almost everywhere on Q, then from (2.3.4) we will have

$$\int_Q f(\mathbf{t})\, d\mu \leq M_k \quad . \qquad (2.3.5)$$

In order to show that a limit function $f(\mathbf{t})$ does exist for certain stationary Gaussian processes we need the following variant on a theorem of Ylvisaker (1966).

Theorem 2.2 If for each $\varepsilon > o$, there exists an integrable function $g_\varepsilon(.)$ on the subset $A_\varepsilon = \{\mathbf{t} = (t_1, \dots, t_k) : | t_i - t_j | \geq \varepsilon,\ i \neq j\}$ of Q, such that, for all n, $f_n(\mathbf{t}) \leq g_\varepsilon(\mathbf{t}), \mathbf{t} \in A_\varepsilon$, and, if $C \leq \infty$ a.s., then there exists $f(\mathbf{t})$ on Q such that $f_n \to f$ a.e. on Q and

$$E[U(U-1)\cdots(U-k+1)] = \int_Q f\, d\mu \quad ,$$

whether finite or not.

Proof. Let $C = \{\mathbf{t} : (i_j - 1)\, 2^{-n} < t_j \leq i_j\, 2^{-n},\ j = 1, \dots, k\}$ for some choice of $i_1, \dots, i_k$. If $i_1, \dots, i_k$ are distinct integers then from (2.3.2)

$$f_n(\mathbf{t}) = 2^{nk}\, P\{U_{ni_1} U_{ni_2} \cdots U_{ni_k} = 1\} \quad ,$$

and

$$\int_C f_n\, d\mu = P\{U_{ni_1} U_{ni_2} \cdots U_{ni_k} = 1\} \quad . \qquad (2.3.6)$$

It follows from the definition of U_{ni}, that $U_{ni} = 1$ implies $U_{n+1\ 2i-1} + U_{n+1\ 2i} = 1$ and $U_{n+1\ 2i-1} U_{n+1\ 2i} = 0$ a.s. Therefore

$$\begin{aligned} P\{U_{ni_1} U_{ni_2} \cdots U_{ni_k} = 1\} &\leq P\{\prod_{j=1}^{k} (U_{n+1\ 2i_j-1} + U_{n+1\ 2i_j}) = 1\} \\ &= P\{\textstyle\sum^* U_{n+1\ r_1} \cdots U_{n+1\ r_k} = 1\} \\ &= \textstyle\sum^* P\{U_{n+1\ r_1} \cdots U_{n+1\ r_k} = 1\} \quad . \qquad (2.3.7) \end{aligned}$$

where $\sum^*$ denotes summation over the 2^k terms obtained by setting $r_j = 2\,i_j - 1$ or $2\,i_j$, $j = 1, \dots, k$. The last equality follows from the fact that the 2^k events $\{U_{n+1\ r_1} \cdots U_{n+1\ r_k} = 1\}$ are mutually exclusive.

Now

$$P\{U_{n+1\ r_1} \cdots U_{n+1\ r_k} = 1\} = \int_{C_r} f_{n+1}\, d\mu \quad ,$$

where $C_r = \{\mathbf{t} : (r_j - 1)\, 2^{-n-1} < t_j \leq r_j\, 2^{-n-1},\ j = 1, \dots, k\}$. Thus

$$\Sigma^* P\{ U_{n+1\, r_1} \cdots U_{n+1\, r_k} = 1 \} = \Sigma^* \int_{C_r} f_{n+1}\, d\mu = \int_{\cup^* C_r} f_{n+1}\, d\mu = \int_C f_{n+1}\, d\mu$$

and from (2.3.6) and (2.3.7), we have

$$\int_C f_n\, d\mu \le \int_C f_{n+1}\, d\mu \quad . \tag{2.3.8}$$

Note , if $i_1, \ldots, i_k$ are not distinct then $f_n = 0$ and (2.3.8) holds trivially.

Now let $H_n = \cup^* C$, where the union is taken over all subsets ($i_1, \ldots, i_k$) for which $| i_r - i_s | > 1$ for all $r \neq s$. In other words H_n is the set of all $\mathbf{t} \in Q$ such that , for the given n, $| i(t_r) - i(t_s) | > 1$, for all $r \neq s$. As Ylvisaker (1966) observes, the sequence $\{f_n\}$ forms a submartingale on Q . Similarly the sequence $\{ f_{n+r}\, I(H_n) \}$, where $I(H_n)$ is the indicator function of H_n , is a submartingale in $r \ge 0$, for each fixed n . Since $H_n \subset A_{2^{-n}}$, the sequence $\{f_{n+r}\, I(H_n)\}$ is dominated by the integrable function $g_{2^{-n}}$, hence, by the martingale convergence theorem, there exists an integrable function f such that $f_{n+r} \to f$ a.e. on H_n , as $r \to \infty$. As $n \to \infty$, $\mu(Q - H_n) \to 0$, hence $f_n \to f$ a.e. on Q and

$$\int_Q f_n\, d\mu = \int_{H_n} f_n\, d\mu \le \int_{H_n} f_{n+r}\, d\mu \to \int_{H_n} f\, d\mu \le \int_Q f\, d\mu$$

using the submartingale property of $\{f_n\}$ and the fact that f is non-negative.

Thus we have proved that

$$E[U(U-1)\cdots(U-k+1)] = \lim_{n \to \infty} \int_Q f_n\, d\mu \le \int_Q f\, d\mu \ , \tag{2.3.9}$$

and the result follows from (2.3.5), since $C < \infty$.

If we assume the matrix of second order spectral moments λ_2 is finite, $\mathbf{X}(t)$ can be shown to posess a mean square derivative $\mathbf{X}'(t)$. For $\mathbf{t} \in Q$ with $t_i \neq t_j$, $i \neq j$, the joint distribution of $\mathbf{X}(t_1), \ldots, \mathbf{X}(t_k), \mathbf{X}'(t_1), \ldots, \mathbf{X}'(t_k)$ is multivariate normal with zero means and covariance matrix Σ_t . If Σ_t is non-singular, the joint distribution will have a probability density function which we denote by $p_t(\mathbf{x}\,;\mathbf{y}) = p_t(\mathbf{x}_1,\ldots, \qquad \mathbf{x}_k;\ \mathbf{y}_1,\ldots,\mathbf{y}_k)$.

For $\mathbf{t}\in H_n$, let Σ_{nt} be the covariance matrix of $\mathbf{X}_{i_1},\ldots,\mathbf{X}_{i_k};\mathbf{Y}_{i_1},\ldots \mathbf{Y}_{i_k}$, where we have written $\mathbf{X}_i = \mathbf{X}(i\,2^{-n})$, $\mathbf{Y}_i = 2^n(\mathbf{X}_i - \mathbf{X}_{i-1})$ and $i_r = i(t_r)$, for $r = 1,\ldots,k$. If the joint density of the 2k variables exists, we denote it by $p_{nt}(\mathbf{x}\,;\mathbf{y}) = p_{nt}(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathbf{y}_1,\ldots,\mathbf{y}_k)$.

If $\mathbf{t}\in A_\varepsilon$, for some $\varepsilon > 0$, by lemma 2.2, $\Sigma_{nt}\to\Sigma_t$, as $n\to\infty$, uniformly in $\mathbf{t}$. It follows that if Σ_t is non-singular on A_ε, then there exists $n_o > 0$ such that Σ_{nt} is non-singular on A_ε, if $n \ge n_o$.

Theorem 2.3 Let $\mathbf{X}(t)$, $0\le t\le 1$, be a stationary Gaussian process satisfying the conditions of theorem 2.1. If in addition, for some integer $k > 1$ and all $\varepsilon > 0$, Σ_t is non-singular on A_ε, then the factorial moment of U of order k, is given by the following equality whether it is finite or not.

$$M_k = \int_0^1\!\!\cdots\!\int_0^1 dt_1\cdots dt_k \int_{\partial\Gamma}\!\cdots\!\int_{\partial\Gamma} dw_1\cdots dw_k \int\!\cdots\!\int p_t(x,y)\prod_{i=1}^{k}(v_i^T y_i)^+ dy_1\cdots dy_k\ ,$$

where, for $i = 1, 2, \ldots, k$, v_i is the outward drawn normal and dw_i the surface element of $\partial\Gamma$ at $\mathbf{x}_i$.

Proof. From (2.3.2) and the subsequent comments, $f_n(\mathbf{t}) = 0$ unless $\mathbf{t}\in H_n$ in which case

$$f_n(\mathbf{t}) = 2^{nk}\, P\{U_{ni_1} U_{ni_2}\cdots U_{ni_k} = 1\}$$

$$= 2^{nk}\int\!\cdots\!\int d\mathbf{y}_1\cdots d\mathbf{y}_k \int_{A_n(y_k)}\!\cdots\!\int_{A_n(y_1)} p_{nt}(\mathbf{x}_1,\ldots,\mathbf{x}_k;\ \mathbf{y}_1,\ldots,\mathbf{y}_k)\, d\mathbf{x}_1\cdots d\mathbf{x}_k \qquad (2.3.10)$$

where $A_n(\mathbf{y})$ is the subset of $\mathbf{x}$ for which $\mathbf{x} - 2^{-n}\mathbf{y}\in\Gamma$ and $\mathbf{x}\in\Gamma'$.

Following the arguments leading to (2.2.6) and (2.2.7) we may substitute $\mathbf{a}_i$, β_i for $\mathbf{x}_i$ in the right hand side of (2.3.10) to obtain

$$f_n(\mathbf{t}) = \int \cdots \int dy_1 \cdots dy_k \int_{\partial\Gamma} \cdots \int_{\partial\Gamma} \int_{B_n} \cdots \int_{B_n} p_{nt}\left[x(a) + \beta 2^{-n} y \,;\, y\right] \prod_{i=1}^{k} (v_i^T y_i) \, d\beta \, dw \,, \tag{2.3.11}$$

where we have written $x(a) + \beta\, 2^{-n}\, y$ for $\mathbf{x}(\mathbf{a}_1) + \beta_1\, 2^{-n}\, \mathbf{y}_1, \ldots,$ $\mathbf{x}(\mathbf{a}_k) + \beta_k\, 2^{-n}\, \mathbf{y}_k$ and y for $\mathbf{y}_1, \ldots, \mathbf{y}_k$.

Since, given $\varepsilon > 0$, $\Sigma_{\mathbf{t}}$ is non-singular on A_ε, it follows from the remarks preceding the theorem that there exists n_0 such that $\Sigma_{n\mathbf{t}}$ is non-singular on A_ε for $n \geq n_0$. Thus, for n large enough, p_{nt} exists and formulae (2.3.10) and (2.3.11) are valid for $\mathbf{t} \in A_\varepsilon$.

Writing

$$\Sigma_t = \begin{bmatrix} A_t & B_t \\ B_t^T & C_t \end{bmatrix},$$

where C_t is the covariance matrix of $\mathbf{X}'(t_1), \ldots, \mathbf{X}'(t_k)$, we can employ the arguments of lemma 2.3 to obtain the bound

$$p_{nt}(\mathbf{x}\,;\,\mathbf{y}) = K \exp\{-\tfrac{1}{4}\, \mathbf{y}^T C_t^{-1}\, \mathbf{y}\} \,, \tag{2.3.12}$$

for $\mathbf{t} \in A_\varepsilon$ and $n \geq n_0$, for some $n_0 > 0$. The inequality (2.3.12) holds for all $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_k)$. In the second member of (2.3.12) the kp-dimensional vector $\mathbf{y}$ is defined by $\mathbf{y}^T = [\mathbf{y}_1^T, \ldots, \mathbf{y}_k^T]$.

We now return to (2.3.11) and consider the limit of $f_n(\mathbf{t})$, as $n \to \infty$. For fixed $\mathbf{y}_1, \ldots, \mathbf{y}_k\,;\, \mathbf{a}_1, \ldots, \mathbf{a}_k\,;\, \beta_1, \ldots, \beta_k$ we can show that

$$p_{nt}(\,\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n}\, \mathbf{y}\,;\, \mathbf{y}) \to p_t(\,\mathbf{x}(\mathbf{a})\,;\, \mathbf{y})$$

as $n \to \infty$. Also, if $v_i^T.y_i > 0$, then the corresponding B_n in the right hand of (2.3.11) will tend to the interval $(0,1)$, whereas if $v_i^T.y_i < 0$ then $B_n = \emptyset$.

By (2.3.12) , the integrand in (2.3.11) is dominated by the integrable function $K \exp \{-\frac{1}{4} y^T C_t^{-1} y\} |y_1| \ldots |y_k|$. Hence there exists a function f(**t**) such that $f_n(\mathbf{t}) \to f(\mathbf{t})$, $\mathbf{t} \in A_\varepsilon$, and

$$f(\mathbf{t}) = \int \cdots \int dy_1 \cdots dy_k \int_{\partial\Gamma} \cdots \int_{\partial\Gamma} p_t[x(a); y] \prod_i (v_i^T y)^+ \, dw_1 \cdots dw_k \qquad (2.3.13)$$

a.e. on Q .

For $\mathbf{t} \in A_\varepsilon$, we have $f_n(\mathbf{t}) \le g_\varepsilon(\mathbf{t})$, for all $n \ge n_0$, where the bounding function is defined by

$$g_\varepsilon(\mathbf{t}) = \int \cdots \int dy_1 \cdots dy_k \int_{\partial\Gamma} \cdots \int_{\partial\Gamma} K\exp\{-\frac{1}{4} y^T C_t^{-1} y\} \prod_i |y_i| \, dw_1 \cdots dw_k \quad . (2.3.14)$$

As $\Sigma_\mathbf{t}$ is non-singular on the closure of A_ε, so is C_t . It follows that $g^*_\varepsilon(\mathbf{t})$, the continuous extension of $g_\varepsilon(\mathbf{t})$, is bounded on the closure of A_ε , since it is compact , and hence $g_\varepsilon(\mathbf{t})$ is integrable on A_ε . The result now follows from theorem 2.2 .

It only remains to prove that $p_{nt}(\mathbf{x}(\mathbf{a}) + \beta\, 2^{-n} \mathbf{y} \, ; \mathbf{y})$ tends to $p_t(\mathbf{x}(\mathbf{a}) \, ; \mathbf{y})$, as $n \to \infty$, if **y**, **a**, β are held fixed . Since $\Sigma_\mathbf{t}$ is non-singular for $\mathbf{t} \in A_\varepsilon$, it follows from lemma 2.2 that $\Sigma_{n\mathbf{t}} \to \Sigma_\mathbf{t}$, and hence $p_{nt}(\mathbf{x} \, ; \mathbf{y})$ tends to $p_t(\mathbf{x} \, ; \mathbf{y})$ as $n \to \infty$. The required limit will be assured if we can show that the sequence $\{p_{nt}(\mathbf{x} \, ; \mathbf{y})\}$ is equicontinuous as functions of $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$, ie given $\varepsilon > 0$, we can find $\delta > 0$ such that $| p_{nt}(\mathbf{x} \, ; \mathbf{y}) - p_{nt}(\mathbf{x}' ; \mathbf{y}) | \le \varepsilon$ if $| \mathbf{x} - \mathbf{x}' | \le \delta$, for all n .

The normal density $p_{nt}(\,.\,)$ is given by

$$p_{nt}(\mathbf{z}) = (2\pi)^{-kp} |\Sigma_{nt}^{-1}|^{1/2} \exp\{-\frac{1}{2} \mathbf{z}^T \Sigma_{nt}^{-1} \mathbf{z}\} \quad ,$$

for $\mathbf{t} \in A_\varepsilon$, for some $\varepsilon > 0$. In what follows, we drop the reference to **t** in the notation as it remains constant.

Consider

$$|p_n(\mathbf{z}) - p_n(\mathbf{z}')| \leq (2\pi)^{-kp} |\Sigma_n^{-1}|^{1/2} \left| \exp\{\tfrac{1}{2}\mathbf{z}'^T\Sigma_n^{-1}\mathbf{z}' - \tfrac{1}{2}\mathbf{z}^T\Sigma_n^{-1}\mathbf{z}\} - 1 \right| \quad . \quad (2.3.15)$$

Since $\Sigma_n \rightarrow \Sigma$ non-singular , therefore there exists K > 0 such that $|\Sigma_n^{-1}|^{1/2} \leq K$ for some integer $n_o > 0$. Now

$$\mathbf{z}'^T\Sigma_n^{-1}\mathbf{z}' - \mathbf{z}^T\Sigma_n^{-1}\mathbf{z} = (\mathbf{z}' - \mathbf{z})^T\Sigma_n^{-1}(\mathbf{z}' - \mathbf{z}) + 2\,\mathbf{z}^T\Sigma_n^{-1}(\mathbf{z}' - \mathbf{z})$$

since Σ_n is symmetric, and hence

$$\left| \mathbf{z}'^T\Sigma_n^{-1}\mathbf{z}' - \mathbf{z}^T\Sigma_n^{-1}\mathbf{z} \right| \leq |\mathbf{z}' - \mathbf{z}|^2 \|\Sigma_n^{-1}\| + 2|\mathbf{z}|.|\mathbf{z}' - \mathbf{z}|.\|\Sigma_n^{-1}\| . \quad (2.3.16)$$

As $\Sigma_n^{-1} \rightarrow \Sigma^{-1}$, as $n \rightarrow \infty$, we can find L > 0 such that $\| \Sigma_n^{-1} \| \leq L$, for all n larger than some $n_o > 0$. The inequality (2.3.16) now gives

$$\left| \mathbf{z}'^T\Sigma_n^{-1}\mathbf{z}' - \mathbf{z}^T\Sigma_n^{-1}\mathbf{z} \right| \leq (2|\mathbf{z}|\, L + |\mathbf{z}' - \mathbf{z}|\, L) |\mathbf{z}' - \mathbf{z}|$$

and from (2.3.15), since $|e^x - 1| \leq |x|$ for $|x| \leq 1$ and , we find

$$|p_n(\mathbf{z}) - p_n(\mathbf{z}')| \leq (2\pi)^{-pk} K (2|\mathbf{z}| + |\mathbf{z}' - \mathbf{z}|) L |\mathbf{z}' - \mathbf{z}| \quad . \quad (2.3.17)$$

For fixed **z**, given $\varepsilon > 0$, by (2.3.17) we can find $\delta > 0$, independent of n, such that $| p_n(\mathbf{z}) - p_n(\mathbf{z}') | \leq \varepsilon$, if $| \mathbf{z} - \mathbf{z}' | \leq \delta$, which proves the equicontinuity of $\{p_{nt}(\mathbf{z})\}$.

3. SUFFICIENT CONDITIONS FOR THE VARIANCE OF U

3.1 Preliminaries

The formula for the kth factorial moment of the number of exits, U, from a simply -connected region Γ has been obtained under general conditions which do not guarantee the finiteness of the moment for $k > 1$. In this chapter we obtain sufficient conditions for M_2 to be finite, for a bivariate stationary Gaussian process on the interval $(0, T]$.

Throughout this chapter, let $\mathbf{X}(t)$, $0 \leq t \leq T$, be a real bivariate stationary Gaussian process with zero mean, having continuous sample paths with probability one. We further assume that $\mathbf{X}(t)$ has a non-degenerate distribution for $0 \leq t \leq T$ and a finite matrix λ_2 of second order spectral moments. It follows that the covariance matrix $R(t)$ has a continuous second derivative at the origin and, for small t ,

$$R(t) = \lambda_0 + \lambda_1 t - \tfrac{1}{2}\lambda_2 t^2 + \theta(t) \quad , \qquad (3.1.1)$$

where the coefficients λ_0 , λ_2 are symmetric 2 x 2 matrices, while λ_1 is a skew-symmetric 2 x 2 matrix . The residual term $\theta(t)$ satisfies $\theta(0) = \theta'(0) = 0$ and $\| \theta'(t) \|$ tends to zero, as $t \rightarrow 0$. Here, as in the previous chapter, $\| \theta \|$ denotes the Euclidean norm $\sup \{\mathbf{n}^T \theta^T \theta \, \mathbf{n}\}^{1/2}$, where the supremum is taken over all unit vectors $\mathbf{n}$. As previously noted, the finiteness of $\lambda_2 = - R''(0)$ implies the existence of a mean square derivative $\mathbf{X}'(t)$. We assume that the covariance matrix

$$\begin{bmatrix} R(0) & R'(0) \\ - R'(0) & - R''(0) \end{bmatrix}$$

of $\mathbf{X}(t)$, $\mathbf{X}'(t)$ is non-singular .

If , at some point in time, $\mathbf{X}$ is contained in a simply-connected two-dimensional region Γ, then, at some subsequent time, the process may cross the boundary $\partial\Gamma$ and exit from Γ . In this chapter U denotes the number of exits of $\mathbf{X}$ from Γ in the interval (0, T] .

We further assume that the boundary $\partial\Gamma$ has no double points and consists of a finite number of regular arcs of finite extent . A tangent τ exists at all regular points of $\partial\Gamma$, where it has a continuous derivative $\frac{d\tau}{ds}$ with respect to arc length s . At a vertex of $\partial\Gamma$, where two regular arcs meet, τ assumes limiting values τ^- from below and τ^+ from above . Finally, we assume $\partial\Gamma$ has no cusps, ie if ϕ is the angle measured from τ^- to τ^+ in an anticlockwise sense, we assume $-\pi < \phi < \pi$.

It follows from the work of the previous chapter, that E(C) , and hence E(U), is finite if $\lambda_2 < \infty$. Further, the second factorial moment of U is given by

$$M_2(T) = E\,[U(U-1)]$$

$$= \int_0^T\!\!\int_0^T dt_1 dt_2 \int_{\partial\Gamma}\int_{\partial\Gamma} ds_1 ds_2 \int\!\!\int p_t(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_2)\,(\nu_1^T\mathbf{y}_1)^+(\nu_2^T\mathbf{y}_2)^+ d\mathbf{y}_1 d\mathbf{y}_2\ , \qquad (3.1.2)$$

where $\mathbf{x}_i = \mathbf{x}(s_i)$, $i = 1, 2$. In (3.1.2) we assume the covariance matrix of $\mathbf{X}(t_1)$, $\mathbf{X}(t_2)$, $\mathbf{X}'(t_1)$, $\mathbf{X}'(t_2)$, denoted by

$$\Lambda = \begin{bmatrix} R(0) & R(t) & R'(0) & R'(t) \\ R(t)^T & R(0) & -R'(t) & R'(0) \\ -R'(0) & -R'(t) & -R''(0) & -R''(t) \\ R'(t)^T & -R'(0) & -R''(t)^T & -R''(0) \end{bmatrix}, \qquad (3.1.3)$$

is non-singular, for $t = t_2 - t_1 \neq 0$.

Since the process is stationary, the density $p_t(.) = p_{t_1 t_2}(.)$ depends on t_1 , t_2 only through the difference t . Integrating over the tangential components of $\mathbf{y}_1$, $\mathbf{y}_2$ and writing y_1 , y_2 for the normal velocities $\nu_1^T.\mathbf{y}_1$, $\nu_2^T.\mathbf{y}_2$, respectively, (3.1.2) becomes

$$M_2(T) = 2\int_0^T (T - t)\, f(t)\, dt \quad , \tag{3.1.4}$$

where

$$f(t) = \int_{\partial\Gamma}\int_{\partial\Gamma} p_t(\mathbf{x}_1,\mathbf{x}_2)\, h_t(s_1,s_2)\, ds_1 ds_2 \quad , \tag{3.1.5}$$

and

$$h_t(s_1,s_2) = \int_0^\infty\int_0^\infty p_t(y_1,y_2 \mid \mathbf{x}_1,\mathbf{x}_2)\, y_1 y_2\, dy_1 dy_2 \quad . \tag{3.1.6}$$

In (3.1.6), $p_t(y_1, y_2 \mid \mathbf{x}_1, \mathbf{x}_2)$ is the density of the normal velocities, given $\mathbf{X}(t_1) = \mathbf{x}_1$, $\mathbf{X}(t_2) = \mathbf{x}_2$. This conditional distribution is $N(\mathbf{m}, \Sigma)$, with

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}^T B^T A^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{3.1.7}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}^T (C - B^T A^{-1} B) \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix} , \tag{3.1.8}$$

where the 4 x 4 matrices A, B, C are defined by the partition

$$\Lambda = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} . \tag{3.1.9}$$

Expression (3.1.5) is obtained from (3.1.4) by writing the joint density $p_t(\mathbf{x}_1, \mathbf{x}_2 ; y_1, y_2)$, in terms of the conditional density $p_t(y_1, y_2 \mid \mathbf{x}_1, \mathbf{x}_2)$ and the marginal density $p_t(\mathbf{x}_1, \mathbf{x}_2)$ of $\mathbf{X}(t_1), \mathbf{X}(t_2)$.

Theorem 3.1 Suppose the process and boundary $\partial\Gamma$ are as described above. If the curvature $\left| \frac{d\tau}{ds} \right|$ is bounded on the regular arcs of $\partial\Gamma$, then a sufficient condition for the finiteness of the second moment $M_2 = E[U(U - 1)]$ is

$$\int_0^\delta \frac{\| \theta''(t) \|}{t}\, dt < \infty \quad ,$$

for some $\delta > 0$.

Proof. Write $\rho = \sigma_{12} / (\sigma_{11} \sigma_{22})^{1/2}$, and substitute

$z_1 = (y_1 - m_1) \sigma_{11}^{-1/2}$, $z_2 = (y_2 - m_2) \sigma_{22}^{-1/2}$ into the integral in

(3.1.6) to obtain

$$h_t = \int_{-\mu_2}^{\infty} \int_{-\mu_1}^{\infty} \phi(z_1,z_2 ; \rho) (\mu_1 + z_1) (\mu_2 + z_2) (\sigma_{11} \sigma_{22})^{1/2} dz_1dz_2 \quad , \qquad (3.1.10)$$

where $\mu_i = m_i \sigma_{ii}^{-1/2}$ (i = 1, 2) and

$$\phi(z_1,z_2 ; \rho) = \frac{(1 - \rho^2)^{-1/2}}{2\pi} \exp\left\{ - \frac{z_1^2 + z_2^2 - 2\rho z_1z_2}{2(1 - \rho^2)} \right\}$$

is the standardised bivariate normal density .

Differentiating (3.1.10) with respect to ρ and employing the well known result $\frac{\partial\phi}{\partial\rho} = \frac{\partial^2 \phi}{\partial z_1 \partial z_2}$, [Cramer and Leadbetter (1967) , p. 26] , we obtain, on

integrating by parts with respect to z_1 and z_2 ,

$$\frac{\partial h_t}{\partial\rho} = (\sigma_{11}\sigma_{22})^{1/2} \Phi(z_1,z_2 ; \rho) \quad , \qquad (3.1.11)$$

where $\Phi(z_1, z_2 ; \rho)$ is the bivariate normal distribution function .

When $\rho = 0$, $h_t = (\sigma_{11} \sigma_{22})^{1/2} \psi(m_1) \psi(m_2)$ with

$$\psi(\mu) = \int_{-\mu}^{\infty} (2\pi)^{-1/2} e^{-\frac{1}{2} z^2} (\mu + z) dz = \mu\Phi(\mu) + \phi(\mu) \; , \qquad (3.1.12)$$

where $\phi(.)$ and $\Phi(.)$ are the density and distribution functions, respectively, of the standard normal distribution . It follows that we can write

$$h_t = (\sigma_{11}\sigma_{22})^{1/2} \psi(\mu_1)\psi(\mu_2) + (\sigma_{11}\sigma_{22})^{1/2} \int_0^{\rho} \Phi(\mu_1,\mu_2 ; r) \, dr$$

and, on substituting for $\psi(.)$ using (3.1.12), we obtain

$$h_t = m_1 m_2\, \Phi(\mu_1)\Phi(\mu_2) + m_1 \sigma_{22}^{1/2}\, \Phi(\mu_1)\phi(\mu_2) + \sigma_{11}^{1/2} m_2\, \phi(\mu_1)\Phi(\mu_2) + + (\sigma_{11}\sigma_{22})^{1/2} \left[\phi(\mu_1)\phi(\mu_2) + \int_0^{\rho} \Phi(\mu_1,\mu_2\,;\, r)\, dr \right] . \tag{3.1.13}$$

Since L is non-singular if $t_1 \neq t_2$, the matrices A and Σ are non-singular for $t \neq 0$. Thus the density $p_t\,(\mathbf{x}_1, \mathbf{x}_2)$ will be uniformly bounded for $t \geq \delta$, for any $\delta > 0$. Furthermore, the right-hand side of (3.1.13) is a polynomial in m_1 and m_2 with coefficients which are bounded for $\delta \leq t \leq T$. The non-singularity of A also implies that m_1 and m_2 are bounded linear functions of $\mathbf{x}_1$ and $\mathbf{x}_2$. From these observations we deduce that

$$f(t\,) = \int_{\partial\Gamma}\int_{\partial\Gamma} p_t(\mathbf{x}_1,\mathbf{x}_2)\; h_t(s_1,s_2)\; ds_1 ds_2$$

is finite for $\delta \leq t \leq T$, and hence the integral in (3.1.4), for t in the range δ to T, is finite. Clearly if M_2 (T) is to be infinite, it must be from the behaviour of f(t) for small t .

3.2 The Behaviour of f(t) for $0 \leq t \leq \delta$

We consider the contribution to f(t) from three regions of the domain of intergration, $\partial\Gamma \otimes \partial\Gamma$, in (3.1.5). For some $\varepsilon > 0$, consider the division of the domain of intergration over s_1, s_2 into the following three regions:

(a) s_2 outside $[\, s_1 - \varepsilon \,,\, s_1 + \varepsilon \,]$ and all s_1 ;

(b) $s_2 \in [\, s_1 - \varepsilon \,,\, s_1 + \varepsilon \,]$ and all s_1 such that $[\, s_1 - \varepsilon \,,\, s_1 + \varepsilon \,]$ contains no vertex of $\partial\Gamma$;

(c) $s_2 \in [\, s_1 - \varepsilon \,,\, s_1 + \varepsilon \,]$ and all s_1 such that $[\, s_1 - \varepsilon \,,\, s_1 + \varepsilon \,]$ contains a vertex.

Let Δ_ε (s) be the subject of $\partial\Gamma$ outside the closed interval $[\, s - \varepsilon \,,\; s + \varepsilon \,]$.

3.3 Region (a)

With our assumptions about the vertices of $\partial\Gamma$, if ε is small enough then $|\,\mathbf{x}(s_1) - \mathbf{x}(s_2)\,| \geq \varepsilon / C$ for some $C > 1$. Writing the joint density in (3.1.5) in terms of the conditional density of $\mathbf{X}(t_2)$, given $\mathbf{X}(t_1) = \mathbf{x}_1$, the contribution to f(t) of the region is

$$f_a(t) = \int_{\partial\Gamma} \int_{\Delta_\varepsilon(s_1)} p(\mathbf{x}_1)\, p_t(\mathbf{x}_2 \,|\, \mathbf{x}_1)\, h_t(s_1,s_2)\, ds_2 ds_1 \ . \qquad (3.3.1)$$

Conditional on $\mathbf{X}(0)) = \mathbf{x}_1$, we find [see appendix A], $\mathbf{X}(t_1) \sim N[\, R(t)^T R(0)^{-1} \mathbf{x}_1 \,,\, S(t)\,]$, where

$$\begin{aligned} S(t) &= R(0) - R(t)^T R(0)^{-1} R(t) \qquad (3.3.2) \\ &= \kappa_2 t^2 - (\theta + \theta^T) \end{aligned}$$

for small t, where $\kappa_2 = \lambda_2 + \lambda_1 \lambda_o^{-1} \lambda_1$. Thus, for small t, we have

$$p_t(\mathbf{x}_2 \,|\, \mathbf{x}_1) \cong \frac{|\kappa_2|^{-1/2}}{2\pi} t^{-2} \exp\{ -\frac{1}{2} (\mathbf{x}_2 - \mathbf{x}_1)^T \frac{1}{t^2} \kappa_2^{-1} (\mathbf{x}_2 - \mathbf{x}_1) \} \qquad (3.3.3)$$

Lemma 3.1 If $|\,\mathbf{x}_1 - \mathbf{x}_2\,| \geq \varepsilon'$, for some $\varepsilon' > 0$, then, as $t \to 0$,

$$h_t \leq \frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{t^2} [\, 1 + o(1)\,] \ . \qquad (3.3.4)$$

Proof After a deal of straightforward though tedious algebra [see appendix], we find, for small t ,

$$C - B^T A^{-1} B = \begin{bmatrix} \frac{\theta' + \theta'^T}{t} - \frac{\theta + \theta^T}{t^2} \,, & -\theta'' + \frac{2\theta'}{t} - \frac{\theta + \theta^T}{t^2} \\ -\theta'' + \frac{2\theta'}{t} - \frac{\theta + \theta^T}{t^2} \,, & \frac{\theta' + \theta'^T}{t} - \frac{\theta + \theta^T}{t^2} \end{bmatrix} + o(t) \qquad (3.3.5)$$

and $\quad B^T A^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} =$

$$\left[\begin{array}{l} \left\{1 + \left(-\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2}\right)\kappa_2^{-1} + O(t)\right\}v + \left\{\left(-\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2}\right)\kappa_2^{-1}\lambda_1\lambda_0^{-1} + O(t)\right\}x_1 \\ \\ \left\{1 + \left(-\frac{\theta'^T}{t} + \frac{\theta + \theta^T}{t^2}\right)\kappa_2^{-1} + O(t)\right\}v + \left\{\left(-\frac{\theta'^T}{t} + \frac{\theta + \theta^T}{t^2}\right)\kappa_2^{-1}\lambda_1\lambda_0^{-1} + O(t)\right\}x_1 \end{array}\right. \tag{3.3.6}$$

where we have written $\mathbf{v} = (\mathbf{x}_2 - \mathbf{x}_1) / t$.

Since $|\mathbf{x}_1|$ is bounded on $\partial\Gamma$, substituting (3.3.6) into (3.1.7) we find

$$| m_i | \leq \frac{| \mathbf{x}_2 - \mathbf{x}_1 |}{t} [1 + o(1)] \quad , \quad i = 1, 2 \quad ,$$

as $t \rightarrow 0$, with $0 \leq t \leq \delta$. Since, from (3.3.5) and (3.1.8), $\sigma_{ii} = o(1)$ as $t \rightarrow 0$, the result follows from (3.1.13).

Thus from (3.3.3) and (3.3.4), for δ small enough and $0 \leq t \leq \delta$,

$$p_t(\mathbf{x}_2 | \mathbf{x}_1) h_t \leq K \exp\left\{ - \frac{(\mathbf{x}_2 - \mathbf{x}_1)^T \kappa_2^{-1}(\mathbf{x}_2 - \mathbf{x}_1)}{2t^2} \right\} \frac{|\mathbf{x}_2 - \mathbf{x}_1|^2}{t^4} \tag{3.3.7}$$

for positive constant K. Since $|\mathbf{x}_2 - \mathbf{x}_1| \geq \varepsilon / C$ on region (a) , a simple maximisation gives

$$p_t(\mathbf{x}_2 | \mathbf{x}_1) h_t \leq K \exp\left\{ - \frac{\varepsilon^2}{2\mu C^2} t^{-2} \right\} t^{-4} \quad ,$$

where μ is the largest eigenvalue of κ_2 and K is a positive constant , not necessarily the same as in (3.3.7). Hence $p_t (\mathbf{x}_2 | \mathbf{x}_1) h_t$ tends to zero, as $t \rightarrow 0$, uniformly on region (a) . Since $p(\mathbf{x}_1)$ is bounded on $\partial\Gamma$, and $\partial\Gamma$ has finite length , it follows , for fixed ε, that $f_a (t)$ in (3.3.1) tends to zero , as $t \rightarrow 0$.

3.4 Region (b)

For fixed s_1 , substituting $s_2 = s_1 + t s$ we get

$$\int_{s_1 - \varepsilon}^{s_1 + \varepsilon} p_t(\mathbf{x}_2 | \mathbf{x}_1) h_t \, ds_2 = \int_{-\varepsilon/t}^{\varepsilon/t} t^{-1} p_t(\mathbf{v}(st) | \mathbf{x}_1) h_t \, ds \quad , \tag{3.4.1}$$

where $p_t (\mathbf{v} | \mathbf{x}_1)$ is the conditional density of $\mathbf{v} = (\mathbf{x}_2 - \mathbf{x}_1) / t$. For

$s_2 \in [\, s_1 - \varepsilon , s_1 + \varepsilon \,]$, $\mathbf{x}(s)$ posesses a continuous second derivative at $s = s_1$ and $s = s_2$. On expanding $\mathbf{x}(s_2)$ about s_1 , we get, on rearrangement ,

$$\mathbf{v}(st) = s\,\tau_1 + \frac{1}{2}s^2 t\frac{d\tau_1}{ds_1} + o(s^2 t) \; .$$

Multiplying through by the unit normal ν_1^T we obtain , to first order in t ,

$$\nu_1^T\mathbf{v}(st) = \frac{s^2}{2}t\,\nu_1^T\frac{d\tau_1}{ds_1} = -\frac{s^2 t}{2}c_1 \; , \tag{3.4.2}$$

where $\tau_1 = \tau(s_1)$ etc. , and c_1 is the curvature of $\partial\Gamma$ at s_1 . Similarly , writing $s_1 = s_2 - s\,t$ and expanding about s_2 , we find to the same degree of approximation

$$\nu_2^T\mathbf{v}(st) = -\frac{s^2}{2}t\,\dot{\nu}_2^T\frac{d\tau_2}{ds_2} = \frac{s^2 t}{2}c_2 \; , \tag{3.4.3}$$

where c_2 is the curvature of $\partial\Gamma$ at s_2 .

Substituting (3.3.6), (3.4.2) and (3.4.3) into (3.1.7) we find, since $|\,\mathbf{x}_1\,|$ is bounded on $\partial\Gamma$ and $|\,\mathbf{v}(st)\,| \leq s$,

$$\left.\begin{aligned} m_1 &= \nu_1^T\left(-\frac{\theta'}{t} + \frac{\theta+\theta^T}{t^2}\right)\kappa_2^{-1}(\mathbf{v} + \lambda_1\lambda_0^{-1}\mathbf{x}_1) + a_1 O(t) \\ m_2 &= \nu_2^T\left(-\frac{\theta'^T}{t} + \frac{\theta+\theta^T}{t^2}\right)\kappa_2^{-1}(\mathbf{v} + \lambda_1\lambda_0^{-1}\mathbf{x}_1) + a_2 O(t) \end{aligned}\right\} , \tag{3.4.4}$$

where a_1 , a_2 are functions of s, s_1, t bounded by a quadratic function of s, for all s_1 and $t \leq \delta$. In deriving (3.4.4), we have assumed c_1 , c_2 bounded on $\partial\Gamma$. Substituting for m_1, m_2 from (3.4.4) into (3.1.13) we obtain

$$\begin{aligned} h_t = A(\sigma_{11}\sigma_{22})^{1/2} &+ B\sigma_{11}^{1/2}\nu_2^T\left(-\frac{\theta'^T}{t} + \frac{\theta+\theta^T}{t^2}\right)\mathbf{b} + B\sigma_{22}^{1/2}\nu_1^T\left(-\frac{\theta'}{t} + \frac{\theta+\theta^T}{t^2}\right)\mathbf{b} + \\ &+ C\,\nu_2^T\left(-\frac{\theta'^T}{t} + \frac{\theta+\theta^T}{t^2}\right)\mathbf{b}\,\nu_1^T\left(-\frac{\theta'}{t} + \frac{\theta+\theta^T}{t^2}\right)\mathbf{b} + \eta t \; , \end{aligned} \tag{3.4.5}$$

where A, B, C are bounded functions of s, s_1, t for $t \leq \delta$, and $\mathbf{b} = \kappa_2^{-1}(\,\mathbf{v}(st) + \lambda_1\,\lambda_0^{-1}\mathbf{x}_1\,)$. The residual η is a function of s, s_1, t which is bounded by a quartic in s, uniformly in s_1, and $t \leq \delta$, and $\eta \to 0$, as $t \to 0$

We now turn to approximating the conditional density $p_t(\mathbf{v} \mid \mathbf{x}_1)$ which occurs in (3.4.1) .

Lemma 3.2 Let K, ε be positive constants such that $K\varepsilon \leq 1/2$ and $\left| \frac{d\tau}{ds} \right| \leq$ K on the regular arcs of $\partial\Gamma$. Then, if δ is small enough,

$$p_t(\mathbf{v}(st) \mid \mathbf{x}_1) \leq C \exp\left\{ -\frac{s^2}{16\mu} \right\},$$

for $0 \leq t \leq \delta$, $|st| \leq \varepsilon$ and $\mathbf{x}_1 \in \partial\Gamma$, where μ is the largest eigenvalue of κ_2 and C is a positive constant .

Proof. It follows from (3.3.2) that, conditional on $\mathbf{X}(0) = \mathbf{x}_1$,

$$t^{-1}[\mathbf{X}(t) - \mathbf{X}(0)] \sim N[t^{-1}(R(t)^T R(0)^{-1} - 1)\mathbf{x}_1,\ t^{-2} S(t)]$$

and from (3.1.1), for small t,

$$t^{-1}(R(t)^T R(0)^{-1} - 1)\mathbf{x}_1 = (-\lambda_1\lambda_0^{-1} - \frac{1}{2}\lambda_2\lambda_0^{-1} t + \theta^T\lambda_0^{-1}t^{-1})\mathbf{x}_1 .$$

Hence, for small t, we can write

$$p_t(\mathbf{v} \mid \mathbf{x}_1) = \frac{1}{2\pi}\left|t^{-2}S(t)\right|^{-1/2}\exp\left\{-\frac{1}{2}\mathbf{u}^T[t^{-2}S]^{-1}\mathbf{u}\right\}, \tag{3.4.6}$$

where

$$\mathbf{u} = \mathbf{v} + (\lambda_1\lambda_0^{-1} + \frac{1}{2}\lambda_2\lambda_0^{-1} t - \theta^T\lambda_0^{-1}t^{-1})\mathbf{x}_1 . \tag{3.4.7}$$

Since the matrix A, defined by (3.1.9), is non-singular for $t > 0$,

$$t^{-2} S(t) = \kappa_2 - t^{-2}(\theta + \theta^T) + O(t)$$

is non-singular and tends to κ_2 as $t \to 0$.Thus, for $0 \leq t \leq \delta$ and $\delta > 0$ small enough, we have

$$\mathbf{u}^T[t^{-2}S]^{-1}\mathbf{u} \geq \frac{1}{2}\mathbf{u}^T\kappa_2^{-1}\mathbf{u} \geq \frac{|\mathbf{u}|^2}{2\mu}$$

for all $\mathbf{u}$. In view of the above inequality and (3.26), we find

$$p_t(\mathbf{v} \mid \mathbf{x}_1) \;\geq\; C' \exp\left\{ - \frac{|\mathbf{u}|^2}{4\mu} \right\}, \tag{3.4.8}$$

for some positive constant C'.

If s_1 and $s_2 = s_1 + st$, with $|st| \leq \varepsilon$, are two points of a regular arc of $\partial\Gamma$, then by the mean value theorem

$$|\mathbf{x}(s_2) - \mathbf{x}(s_1) - (s_2 - s_1)\,\tau_1| = \left| \frac{1}{2}(s_2 - s_1)^2 \left(\frac{d^2\mathbf{x}}{ds^2}\right)_{s = s_3} \right| \leq \frac{1}{2}(s_2 - s_1)^2 K,$$

where τ_1 is the tangent at s_1, and s_3 lies between s_1 and s_2. Substituting for s_2 and dividing by $|st|$, we obtain $\left|\frac{\mathbf{v}}{s} - \tau_1\right| \leq \frac{1}{2}|st|\,K \leq \frac{1}{2}\varepsilon K \leq \frac{1}{4}$, from which it follows that $|\mathbf{v}| \geq \frac{3}{4}|s|$. Thus, from (3.4.7), $|\mathbf{u}| = |\mathbf{v} + \alpha| \geq |\mathbf{v}| - |\alpha| \geq \frac{3}{4}|s| - |\alpha|$, where we have written

$$\alpha = (\lambda_1\lambda_0^{-1} + \frac{1}{2}\lambda_2\lambda_0^{-1}t - \theta^T\lambda_0^{-1}t^{-1})\mathbf{x}_1 = \lambda_1\lambda_0^{-1}\mathbf{x}_1 + O(t),$$

and therefore

$$|\mathbf{u}|^2 \geq \begin{cases} (\frac{3}{4}|s| - |\alpha|)^2 & \text{for } |s| \geq \frac{4}{3}|\alpha| \\ 0 & \text{for } |s| < \frac{4}{3}|\alpha|. \end{cases}$$

Since $|\mathbf{x}_1|$ is bounded under the conditions of the lemma, $|\alpha|$ is bounded for δ sufficiently small and hence we can find a positive constant a for which

$$|\mathbf{u}|^2 \geq \frac{1}{4}|s|^2 - a^2.$$

The lemma now follows on combining the above inequality with (3.4.8).

From (3.1.5), (3.4.1) and lemma 3.2, the contribution of region (b) to f(t), for $0 \leq t \leq \delta$, can be written

$$f_b(t) = \int_{\Delta_\varepsilon} \int_{-\varepsilon/t}^{\varepsilon/t} t^{-1}\, p_t(\mathbf{v}(st) \mid \mathbf{x}_1)\, p(\mathbf{x}_1)\, h_t \, ds\, ds_1$$

$$\leq \int_{\Delta_\varepsilon} \int_{-\varepsilon/t}^{\varepsilon/t} t^{-1}\, C \exp\left\{ -\frac{s^2}{16\mu} \right\} p(\mathbf{x}_1)\, h_t \, ds\, ds_1 \tag{3.4.9}$$

where Δ_ε is the subset of s such that the closed interval $[\, s - \varepsilon\, , s + \varepsilon\,]$ does not contain a vertex of $\partial\Gamma$.

Now (3.4.5) gives

$$h_t \leq A(\sigma_{11}\sigma_{22})^{1/2} + B\sigma_{11}^{1/2}\, |\mathbf{b}| \left\| -\frac{\theta'^T}{t} + \frac{\theta + \theta^T}{t^2} \right\| + \\ + B\sigma_{22}^{1/2}\, |\mathbf{b}| \left\| -\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2} \right\| + C|\mathbf{b}|^2 \left\| -\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2} \right\|^2 + |\eta|\, t\ , \tag{3.4.10}$$

and

$$|\, \mathbf{b}\, | \;\leq\; \|\, \kappa_2^{-1} \|\, [\, |\, s\, | + |\, \lambda_1 \lambda_0^{-1} \mathbf{x}_1\, |\,]\ .$$

Since $p(\, \mathbf{x}_1\,)$ and $|\, \mathbf{x}_1\, |$ are bounded on $\partial\Gamma$, substituting (3.4.10) into the second member of (3.4.9) , we find

$$f_b(t) \leq A\, (\sigma_{11}\sigma_{22})^{1/2}\, t^{-1} + B\, (\sigma_{11}^{1/2} + \sigma_{22}^{1/2}) \left\| -\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2} \right\| t^{-1} + \\ + C \left\| -\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2} \right\|^2 t^{-1} + \eta\ , \tag{3.4.11}$$

where A, B, C and η are positive constants .

Let $w(t) = \|\, \theta''(t)\, \|$, which following (3.1.1) is continuous and o(1), as $t \to 0$. It is an elementary matter to prove [see appendix A, section 3] that the integrals $\int_0^\delta \frac{\|\, \theta'(t)\, \|}{t^2}\, dt$ and $\int_0^\delta \frac{\|\, \theta(t)\, \|}{t^3}\, dt$ are bounded by $\int_0^\delta \frac{w(t)}{t}\, dt$, which is finite under the conditions of the theorem .

Since, for small t, using (3.1.8) and (3.3.5), we have

$$\sigma_{11} = v_1^T \left[\frac{\theta' + \theta'^T}{t} - \frac{\theta + \theta^T}{t^2} \right] v_1 + o(t) ,$$

with a similar expression for σ_{22}, it follows that

$$\int_0^\delta \frac{\sigma_{11}}{t} dt \leq 2\int_0^\delta \frac{\| \theta' \|}{t^2} dt + 2\int_0^\delta \frac{\| \theta \|}{t^3} dt + \int_0^\delta o(1)\, dt .$$

Hence $\int_0^\delta \frac{\sigma_{11}}{t} dt$, and similarly $\int_0^\delta \frac{\sigma_{22}}{t} dt$, are finite.

Applying the Cauchy - Schwartz inequality, we get

$$\int_0^\delta \frac{(\sigma_{11}\sigma_{22})^{1/2}}{t} dt \leq \left\{ \int_0^\delta \frac{\sigma_{11}}{t} dt \int_0^\delta \frac{\sigma_{22}}{t} dt \right\}^{1/2}$$

and hence the first member of the inequality is finite . Using arguments similar to the preceding, we can show that $\int_0^\delta \left\| -\frac{\theta'}{t^2} + \frac{\theta + \theta^T}{t^3} \right\| dt$ is bounded by $\int_0^\delta 3\frac{w(t)}{t} dt$ and hence finite . Bearing in mind that $\sigma_{11}{}^{1/2}$, $\sigma_{22}{}^{1/2}$ and $\left\| -\frac{\theta'}{t} + \frac{\theta + \theta^T}{t^2} \right\|$ are bounded on $[0, \delta]$, it follows from (3.4.11) and the succeeding remarks that $\int_0^\delta f_b(t)\, dt$ is finite .

3.5 Region (c)

We may suppose, without loss of generality, s = 0 at the vertex $\mathbf{x}_0$. The contribution of the region to f(t) may be written

$$\begin{aligned} f_c(t) &= \int_{-\varepsilon}^{\varepsilon} \int_{s_1 - \varepsilon}^{s_1 + \varepsilon} p(\mathbf{x}_1)\, p(\mathbf{x}_2 \mid \mathbf{x}_1)\, h_t\, ds_2\, ds_1 \\ &= I_{--} + I_{-+} + I_{+-} + I_{++} , \end{aligned} \tag{3.5.1}$$

where the four summands correspond to the partition of the range of integration into four disjoint regions according to the signs of the two variables of integration s_1 and s_2 . Thus I_{-+} is the integral over $s_2 \in [0, s_1 + \varepsilon]$, for all

$s_1 \in [-\varepsilon, 0]$. The integrals I_{--} and I_{++} may be subsumed under region (b), so we concentrate on obtaining bounds for I_{-+} and I_{+-} .

Now consider I_{-+} , where $s_1 < 0$ and $s_2 > 0$. To first order in s

$$x_1 = x_0 + s_1 \tau^-$$

$$x_2 = x_0 + s_2 \tau^+$$

and therefore

$$v = \frac{1}{t}(x_2 - x_1) = \frac{s_2}{t}\tau^+ - \frac{s_1}{t}\tau^- \quad .$$

Thus we have

$$v_1^T.v = (\nu^-)^T v = \frac{s_2}{t}(\nu^-)^T\tau^+ = -\frac{s_2}{t}\sin\phi \ , \tag{3.5.2}$$

and

$$v_2^T.v = (\nu^+)^T v = \frac{s_1}{t}(\nu^+)^T\tau^- = -\frac{s_1}{t}\sin\phi \ , \tag{3.5.3}$$

where ϕ is the angle between τ^- and τ^+, as shown in figure 3.1 for an external vertex .

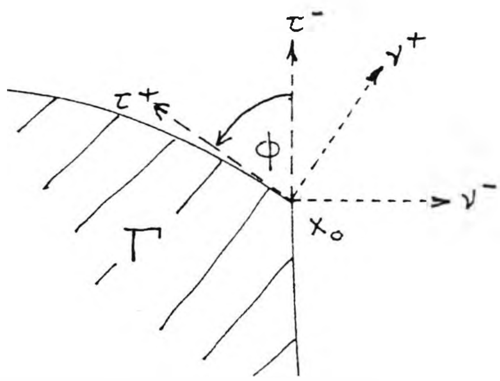

figure 3.1 The angle ϕ at an external vertex of $\partial\Gamma$.

From (3.1.13) it simply follows that h_t satisfies

$$h_t \le (\sigma_{11}\sigma_{22})^{1/2}\{\psi(\mu_1)\psi(\mu_2) + 1\} \ , \tag{3.5.4}$$

with $\psi(\mu)$ defined as in (3.1.12) . For small t , from (3.1.7) and (3.3.6), we have

$$m_i = v_i^T v + o(1) \ , \qquad i = 1, 2$$

and, on substituting from (3.5.2) and (3.5.3), we get , approximately,

$$m_1 = -\frac{s_2}{t}\sin\phi \ , \qquad m_2 = -\frac{s_1}{t}\sin\phi \ . \tag{3.5.5}$$

Hence m_1 and m_2 have opposite signs . For the sake of definiteness, we suppose $\phi > 0$, as would be the case for an external vertex, in which case $m_1 < 0$ and $m_2 > 0$, for small t .

It follows immediately from (3.1.12) that

$$\left.\begin{array}{ll} \psi(\mu) \le \mu + \phi(0) = \mu + (2\pi)^{-1/2} & \text{if } \mu > 0 \\ \text{and } \ \psi(\mu) \le \phi(\mu) \le (2\pi)^{-1/2} & \text{if } \mu < 0. \end{array}\right\} \tag{3.5.6}$$

Since $\mu_1 < 0$ and $\mu_2 > 0$, (3.5.4) and (3.5.6) give

$$\begin{aligned} h_t &\le (\sigma_{11}\sigma_{22})^{1/2} \{[\ \mu_2 + (2\pi)^{-1/2}\ ]\ (2\pi)^{-1/2} + 1\ \} \\ &= (2\pi)^{-1/2}\,\sigma_{11}^{1/2} m_2 + \ \{(2\pi)^{-1} + 1\}(\sigma_{11}\sigma_{22})^{1/2} \end{aligned} \tag{3.5.7}$$

where we have employed the definition of μ_2 following equation (3.1.10)

Introducing the conditional density $p_t(\mathbf{v} \mid \mathbf{x}_1) = t^2\, p_t(\mathbf{x}_2 \mid \mathbf{x}_1)$ into the integral

$$I_{-+} = \int_{-\varepsilon}^{0} \int_{0}^{s_1+\varepsilon} p(\mathbf{x}_1)\ p(\mathbf{x}_2 \mid \mathbf{x}_1)\ h_t\ ds_2\ ds_1$$

and making the substitutions $s_1 = t\,w_1$, $s_2 = t\,w_2$, we get

$$I_{-+} = \int_{-\varepsilon/t}^{0} \int_{0}^{w_1+\varepsilon/t} p(\mathbf{x}_1)\ p_t(\mathbf{v} \mid \mathbf{x}_1)\ h_t\ dw_2\ dw_1 \ , \tag{3.5.8}$$

with $\mathbf{v} = w_2\,\tau^+ - w_1\,\tau^-$. Since $p(\mathbf{x}_1)$ is bounded on $\partial\Gamma$, (3.5.7) and (3.5.8) give, for small t ,

$$I_{-+} \le K \int_{-\varepsilon/t}^{0} \int_{0}^{w_1+\varepsilon/t} p_t(\mathbf{v} \mid \mathbf{x}_1) \left[(2\pi)^{-1/2}\,\sigma_{11}^{1/2} m_2 + \left(1 + \frac{1}{2\pi}\right)(\sigma_{11}\sigma_{22})^{1/2}\right] dw_2\ dw_1$$

for some positive constant K and, since the integrand is positive,

$$I_{-+} \leq K \int_{-\varepsilon/t}^{0} \int_{0}^{w_1 + \varepsilon/t} p_t(\mathbf{v} \mid \mathbf{x}_1) \left[-(2\pi)^{-1/2} \sigma_{11}^{1/2} w_1 \sin\phi + \left(1 + \frac{1}{2\pi}\right) (\sigma_{11}\sigma_{22})^{1/2} \right] dw_2 dw_1 \tag{3.5.9}$$

on substituting for m_2 from (3.5.5) .

As $t \to 0$, $p_t(\mathbf{v} \mid \mathbf{x}_1)$ tends to a non-singular bivariate normal density . Thus, for small t , $p_t(\mathbf{v} \mid \mathbf{x}_1)$ is proportional to a non-singular density for w_1, w_2 and consequently the integrals in the second member of (3.5.9) are finite . Hence, there exists $\delta > 0$, such that, for $0 \leq t \leq \delta$,

$$I_{-+} \leq A\,\sigma_{11}^{1/2} + B(\sigma_{11}\sigma_{22})^{1/2} \tag{3.5.10}$$

for positive constants A and B . Clearly a similar inequality holds for I_{+-} .

Since $\sigma_{11}{}^{1/2} = o(1) = \sigma_{22}{}^{1/2}$, as $t \to 0$, it follows from (3.5.10) and the remarks that follow (3.5.1) that $\int_0^\delta f_c(t)\,dt$ is finite .

Combining the results for the three regions, we have shown that, under the conditions of theorem 3.1, $\int_0^\delta f(t)\,dt$ is finite and the theorem follows

4. THE DURATION OF AN EXCURSION

4.1 Preliminaries

In this chapter we consider the excursions of a Gaussian process outside a large two-dimensional star-shaped region. Let $D = \{ D_L ; L > 0 \}$ be a family of boundaries of similar two-dimensional star-shaped regions Γ_L , indexed by a parameter L proportional to the length of D_L . A ray drawn from the origin meets each boundary in just one point .Thus, if $\mathbf{x}(s, L)$ is the point on D_L with arc-length s, we can write $\mathbf{x}(s, L) = L\mathbf{x}(s /L, 1)$and it follows that $u = s / L$ and L provide a coordinate system in the plane .

As in the previous chapters, we assume $D_L = \partial\Gamma_L$ to consist of a finite number of regular arcs of finite length . A tangent τ exists at all interior points of the regular arcs,where it has a continuous derivative with respect to arc length s . At a vertex of D_L, where two regular arcs meet, τ assumes limiting values τ_- from below and τ_+ from above the vertex . Further we assume D_L has no cusps .

Thoughout this chapter $\mathbf{X}(t)$ is a two-dimensional stationary Gaussian process with mean zero and covariance matrix R(t) which satisfies :

(i) $$R(t) = \lambda_o + \lambda_1 t - \frac{1}{2}\lambda_2 t^2 + \frac{1}{3!}\lambda_3 t^3 + \frac{1}{4!}\lambda_4 t^4 + \theta , \quad (4.1.1)$$

where $\theta(0) = \theta'(0) = \theta''(0) = \theta'''(0) = 0$ and

$\theta^{(iv)}(0) = o(1)$, for small t ; and

(ii) $$\| R(t) \| = O(t^{-\alpha}), \quad \alpha > 0 , \quad (4.1.2)$$

for large t .

This last condition ensures that the spectrum of $\mathbf{X}(t)$ is everywhere continuous and hence the process is ergodic .

As follows from chapter 2 , the existence of λ_2 will ensure that the mean number of exits, μ, in a unit interval is finite . Further, by a lemma of R.L. Dobrushin, it follows that the sequence of crossings of D_L form a regular stationary stream of events [Cramer and Leadbetter (1967) , p. 201] . As an extension of the notation of chapter 1, we write $U(t_1, t_2)$ for the number of exits from Γ in the interval $(t_1, t_2]$ and $D(t_1, t_2)$ for the number of entrances into Γ in the same interval . Since the stream of events is regular, we have in particular $P\{ U(0, t) \geq 1 \} = \mu t + o(t)$, for small t .

Given an exit at t = 0, we consider the distribution of the time to re-entrance into Γ_L, as $L \to \infty$. The probability of re-entrance before time t, conditional on an exit at t = 0 , in the 'horizontal window' sense of Kac and Slepian (1959), is

$$\begin{aligned} F_1(t) &= 1 - \lim_{\varepsilon \to 0} \frac{P\{ U(-\varepsilon, 0) \geq 1 ; D(0, t) = 0 \}}{P\{ U(-\varepsilon, 0) \geq 1 \}} \\ &= 1 - \frac{1}{\mu} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P\{ U(-\varepsilon, 0) \geq 1 ; D(0, t) = 0 \} \quad , \qquad (4.1.3) \end{aligned}$$

on using $P\{ U(-\varepsilon, 0) \geq 1 \} = \mu\varepsilon + o(\varepsilon)$. For an ergodic process, it can be shown [Cramer and Leadbetter (1967)] that $F_1(t)$ is a proper distribution function .

The mean duration θ of an excursion is given by

$$\theta = \int_0^\infty t \, dF_1(t) = \frac{1}{\mu} P\{\mathbf{X} \in \Gamma' \} \quad , \qquad (4.1.4)$$

where Γ' is the set of points exterior to D .

Using θ as a scaling unit, we look for the limit of $F_1(\theta t)$, as $L \to \infty$. The following lemma provides a justification of our approach to finding the asymptotic distribution of the duration of an excursion .

Lemma 4.1 If the limit function

$$G(\tau) = \lim_{L\to\infty} \frac{1}{\mu} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E[\, U(-\varepsilon, 0)\, D(0, \theta\tau)\,]$$

exists for $\tau \geq 0$ and is a distribution function, then $\lim_{L\to\infty} F_1(\theta\tau) = G(\tau)$.

Proof. The proof derives from the work of Volkonskii and Rozanov (1961) . From (4.1.3) we have, for $t_1 < t_2$,

$$F_1(t_2) - F_1(t_1) = \frac{1}{\mu} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} P\{\, U(-\varepsilon, 0) \geq 1 \,;\, D(0, t_1) = 0 \,;\, D(t_1, t_2) \geq 1 \,\}$$

$$\leq \frac{1}{\mu} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} P\{\, U(-\varepsilon, 0) \geq 1 \,;\, D(t_1, t_2) \geq 1 \,\}$$

$$\leq \frac{1}{\mu} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E[\, U(-\varepsilon, 0)\, D(t_1, t_2)\,] \ . \tag{4.1.5}$$

Writing $H(t, L) = F_1(\theta t)$ to show the dependence on L explicity, since $F_1(0) = 0$, we obtain from (4.1.5)

$$H(t, L) \leq \frac{1}{\mu} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E[\, U(-\varepsilon, 0)\, D(0, \theta t)\,] \ .$$

Hence ,

$$\limsup_{L\to\infty} H(t, L) \leq G(t) \ . \tag{4.1.6}$$

Now assume that $\liminf H(t, L) = G(t) - 4c$, for some $t > 0$ and some $c > 0$. Hence we can find an unbounded sequence $\{a_n\}$ such that

$$H(t, a_n) < G(t) - 2c \ , \tag{4.1.7}$$

for all n . Since, from (4.1.5), for any $s > t$,

$$\limsup_{L \to \infty} [\, H(s, L) - H(t, L)\,] \le G(s) - G(t) \;,$$

it follows that

$$H(s, a_n) - H(t, a_n) < G(s) - G(t) + c \;,$$

for all sufficiently large n . Utilising (4.1.7) we find

$$H(s, a_n) < G(s) - c < 1 - c \tag{4.1.8}$$

for any $s > t$, since G(.) is a distribution function by assumption, only supposing n to be sufficiently large . However , $H(s, a_n)$ is a distribution function with unit mean, hence $H(s, a_n) > 1 - 1/s$, for all $s > 0$.

If $s > 1/c$, this implies a contradiction of (4.1.8) , and hence of the original assumption that lim inf $H(t, L) < G(t)$. Thus, we have proved that lim inf $H(t, L) \ge G(t)$ for all t , and the lemma follows from (4.1.6) .

Using arguments similar to those used in chapter 2 for the factorial moments, we obtain, for $0 < \tau_1 < \tau_2$,

$$E\,[\, U(-\varepsilon, 0)\, D(\tau_1, \tau_2)\,] = \int_{-\varepsilon}^{0} dt_1 \int_{\tau_1}^{\tau_2} q(t_2 - t_1)\, dt_2 \;, \tag{4.1.9}$$

where

$$q(t) = \int\int_{D\,D}\int\int p_t(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)\,(\nu_1^T \mathbf{y}_1)^+ (\nu_2^T \mathbf{y}_2)^-\, d\mathbf{y}_1 d\mathbf{y}_2\, ds_1 ds_2 \;.$$

In the integrand, the function $p_t(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$ is the multivariate normal density of $\mathbf{X}(0)$, $\mathbf{X}(t)$, $\mathbf{X}'(0)$, $\mathbf{X}'(t)$ at $\mathbf{X}(0) = \mathbf{x}_1 = \mathbf{x}(s_1)$, $\mathbf{X}(t) = \mathbf{x}_2 = \mathbf{x}(s_2)$; ν_1 , ν_2 are unit outward drawn normals to D_L at s_1 and s_2 , respectively, and $(.)^+$, $(.)^-$ denote positive and negative parts .

We assume that the distribution of $\mathbf{X}(0)$, $\mathbf{X}(t)$, $\mathbf{X}'(0)$, and $\mathbf{X}'(t)$ is non-singular, for $t > 0$, and that $p_t(.)$ is a bounded continuous function of t. It follows that $q(t)$ is a continuous function for $t > 0$.

Thus we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E[U(-\varepsilon, 0) D(\tau_1, \tau_2)] = \int_{\tau_1}^{\tau_2} q(t)\, dt ,$$

and (4.1.5) becomes

$$F_1(\tau_2) - F_1(\tau_1) \leq \frac{1}{\mu} \int_{\tau_1}^{\tau_2} q(t)\, dt , \qquad (4.1.10)$$

Since $F_1(t)$ is continuous at $t = 0$ [Cramer and Leadbetter (1967) , § 11.5] and the integral in the second member of (4.1.10) is ultimately increasing as $\tau_1 \to 0$, the inequality will hold when $\tau_1 = 0$. To make the dependence on L explicit, we write

$$G(\tau, L) = \frac{1}{\mu} \int_0^{\theta\tau} q(t)\, dt , \qquad (4.1.11)$$

Before we turn to consideration of the limit of $G(\tau, L)$, as L tends to infinity, we first obtain asymptotic expressions for μ and θ.

4.2 Asymptotic Expressions for μ and θ.

As we show in chapter 2, for a stationary Gaussian process, $\mu = E[U(0, 1)]$ is given by

$$\mu = \int_D [\sigma_n \phi(\beta_n/\sigma_n) + \beta_n \Phi(\beta_n/\sigma_n)]\, p[\mathbf{x}(s)]\, ds , \qquad (4.2.1)$$

where $\sigma_n^2 = \nu^T (\lambda_2 + \lambda_1\lambda_o^{-1}\lambda_1)\nu$, $\beta_n = -\nu^T \lambda_1\lambda_o^{-1}\mathbf{x}$ and where $\phi(.)$ and $\Phi(.)$ are the density and distribution functions, respectively, of the standard normal distribution. The other function in the integrand, $p[\mathbf{x}] =$

$(2\pi)^{-1} | \lambda_o |^{-1/2} e^{-Q/2}$, where $Q = \mathbf{x}^T \lambda_o^{-1} \mathbf{x}$, is the marginal density of $\mathbf{X}(0)$.

For the remainder of this chapter, we will suppose that Q has a unique minimum on D_L at $\mathbf{x} = \mathbf{x}_o$, where without loss of generality we assume $s = 0$.

Theorem 4.1 As $L \to \infty$, μ satisfies the following limits .

(i) If $\mathbf{x}_o$ is a regular point of D_L ,

$$\frac{\mu}{p[\mathbf{x}_0]} \to \left(\frac{\sigma_0^2}{k^2} + p^2 \right)^{1/2} ,$$

where $k^2 = \frac{1}{2} (\frac{d^2 Q}{ds^2})_o$, $p = \tau^T \lambda_1 \nu$; the suffix zero indicating the quantities are to be evaluated at $s = 0$.

(ii) If $\mathbf{x}_o$ is a vertex of D_L and $\lambda_1 \neq 0$,

$$\frac{\mu}{p[\mathbf{x}_0]} \to | p | .$$

(iii) If $\mathbf{x}_o$ is a vertex of D_L and $\lambda_1 = 0$,

$$\frac{L \mu}{p[\mathbf{x}_0]} \to \frac{1}{\sqrt{2\pi}} \left(\frac{\sigma_+}{k^+} + \frac{\sigma_-}{k^-} \right) ,$$

where $k^+ = \frac{1}{L} \tau_+^T \lambda_o^{-1} \mathbf{x}_o$, $k^- = -\frac{1}{L} \tau_-^T \lambda_o^{-1} \mathbf{x}_o$ are positive constants and σ_+ , σ_- denote σ_n evaluated with $\nu = \nu_+$ and $\nu = \nu_-$, respectively. The vectors ν_+ and ν_- are the limiting unit outward drawn normals corresponding to τ_+ and τ_- .

Proof. For any given $\delta > 0$, there exists $f > 0$ such that $[Q(\mathbf{x}) - Q(\mathbf{x}_o)] / L^2 \geq f^2$ for all $\mathbf{x}(s) \in D_L$ such that $| s | \geq L\delta$. To see this, write $Q_L(s) = Q[\mathbf{x}(s, L)]$, then $Q_L(s) = Q[\mathbf{x}(s, L)] = Q[L\mathbf{x}(u, 1)] = L^2 Q[\mathbf{x}(u, 1)] = L^2 Q_1(u)$, with $u = s / L$, as follows from the similarity of the boundaries D_L. Since Q_L has a unique minimum at $s = 0$, therefore Q_1 has a unique minimum at $u = 0$, and hence $[Q_L(s) - Q_L(0)] / L^2 = Q_1(u) - Q_1(0) > 0$, for $| u | > \delta > 0$.

Let us define $f^2 = \inf_{|u|>\delta} [\, Q_1(u) - Q_1(0) \,] > 0$, from which we obtain $[\, Q(\mathbf{x}) - Q(\mathbf{x}_0) \,] / L^2 \geq f^2$. Since σ_n is bounded and $\beta_n = O(L)$, as $L \to \infty$, the integrand of (4.2.1) , for $|s| \geq L\delta$, is bounded by

$$KL\, p[\mathbf{x}(s)] = KL\, p[\mathbf{x}_0] \exp\{-\tfrac{1}{2}[\, Q(\mathbf{x}) - Q(\mathbf{x}_0) \,]\} \leq KL\, p[\mathbf{x}_0]\, e^{-\frac{1}{2}f^2L^2} ,$$

for some positive constant K , as $L \to \infty$. Thus, since the perimeter of D_L = O(L),

$$\int_{|s|\geq L\delta} [\sigma_n \phi(\beta_n/\sigma_n) + \beta_n \Phi(\beta_n/\sigma_n)]\, p[\mathbf{x}(s)]\, ds$$

$$\leq p[x_0]\, K L\, e^{-\frac{1}{2}f^2L^2} \int_{|s|\geq L\delta} ds$$

$$\leq p[\mathbf{x}_0]\, K'L^2 e^{-\frac{1}{2}f^2L^2} ,$$

where K' is a positive constant . Thus, from (4.2.1) , we can write

$$\mu = \int_{-L\delta}^{L\delta} [\sigma_n \phi(\beta_n/\sigma_n) + \beta_n \Phi(\beta_n/\sigma_n)]\, p[\mathbf{x}(s)]\, ds + + p[\mathbf{x}_0]\, O\left(L^2 e^{-\frac{1}{2}f^2L^2}\right) \tag{4.2.2}$$

(i) At $\mathbf{x}_0$, $\dfrac{dQ}{ds} = 2\tau_0^T \lambda_0^{-1}\mathbf{x}_0 = 0$, hence $\beta_n = 0$, since $\lambda_1 \nu = p\,\tau$, where p is a parameter of the bivariate process related to the circulation of the velocity field .

For $\delta > 0$ small enough, $[\, Q(\mathbf{x}) - Q(\mathbf{x}_0) \,] / s^2$ is bounded away from zero and $|\beta_n / s|$ is bounded for $-L\delta < s < L\delta$. Thus

$$\sigma_n \phi(\beta_n / \sigma_n) + \beta_n \Phi(\beta_n / \sigma_n)$$

is dominated by a linear function of s, and

$$p[x(s)] \leq p[\mathbf{x}_0]\, e^{-\frac{1}{2}a s^2} ,$$

for some positive constant a . It follows that the integrand in (4.2.2) is dominated by an integrable function of s .

Further, as $L \to \infty$, for fixed s, $[\,Q(\mathbf{x}) - Q(\mathbf{x}_o)\,] / s^2 \to \frac{1}{2}\left(\frac{d^2Q}{ds^2}\right)_o = k^2$,

and $\beta_n / s \to b = \left(\frac{d\beta_n}{ds}\right)_o$, where the derivatives are evaluated at s = 0.

Dividing (4.2.2) throughout by $p[\mathbf{x}_o]$ and letting $L \to \infty$, we find

$$\frac{\mu}{p[\mathbf{x}_0]} \to \int_{-\infty}^{\infty} [\, \sigma_0\, \phi(bs/\sigma_0) + bs\, \Phi(bs/\sigma_0)\,]\, e^{-\frac{1}{2}k^2 s^2} ds ,$$

where s_o is evaluated at s = 0. The result (i) follows on evaluating the integral

(ii) In this case the derivative $\frac{dQ}{ds}$ does not exist at $\mathbf{x}_o$. However, since Q is a minimum at $\mathbf{x}_o$, we have $\left(\frac{dQ}{ds}\right)_{o+} = 2\tau_+^T \lambda_o^{-1}\mathbf{x}_o > 0$ and $\left(\frac{dQ}{ds}\right)_{o-} = 2\tau_-^T \lambda_o^{-1}\mathbf{x}_o < 0$ for the limiting values of the derivative. Thus $k^+ = \frac{1}{L}\tau_+^T \lambda_o^{-1}\mathbf{x}_o$ and $k^- = -\frac{1}{L}\tau_-^T \lambda_o^{-1}\mathbf{x}_o$ are positive constants independent of L.

We can choose $\delta > 0$, so that $[\,Q(\mathbf{x}) - Q(\mathbf{x}_o)\,] / (\,|s|\,L\,)$ is bounded away from zero for $|s| < L\,\delta$. As $L \to \infty$, for fixed s,

$$\frac{Q(\mathbf{x}) - Q(\mathbf{x}_0)}{L\,|s|} \to \begin{cases} 2k^+ & \text{for } s > 0 \\ 2k^- & \text{for } s < 0. \end{cases} \tag{4.2.3}$$

Since, using the definition of Q_L,

$$\frac{Q(\mathbf{x}) - Q(\mathbf{x}_0)}{L\,|s|} = \frac{Q_1(s/L) - Q_1(0)}{s/L} ,$$

allowing L to tend to infinity with s fixed, the last member will tend to $\left(\frac{dQ_1}{du}\right)_{o+}$ or $-\left(\frac{dQ_1}{du}\right)_{o-}$ according as s > 0 or s < 0, respectively.

On division by $p[\mathbf{x}_o]$, the integral in (4.2.2) can be written

$$I_L = \int_{-L\delta}^{L\delta} g(s)\, f(s)\, ds , \tag{4.2.4}$$

where

$$g(s) = \sigma_n \phi(\beta_n / \sigma_n) + \beta_n \Phi(\beta_n / \sigma_n) \quad ,$$

and

$$f(s) = \frac{p[\mathbf{x}(s)]}{p[\mathbf{x}_0]} = \exp\left\{ -\frac{1}{2}[Q(\mathbf{x}) - Q(\mathbf{x}_0)] \right\} .$$

On introducing a new variable of integration w = Ls in (4.2.4) , we obtain

$$I_L = \int_{-L^2\delta}^{L^2\delta} \frac{1}{L} g\left(\frac{w}{L}\right) f\left(\frac{w}{L}\right) dw \quad . \tag{4.2.5}$$

Now consider

$$\frac{\beta_n}{L} = -\frac{1}{L} \nu(s)^T \lambda_1 \lambda_0^{-1} \mathbf{x}(s, L) = -\nu(s)^T \lambda_1 \lambda_0^{-1} \mathbf{x}(u, 1) \quad ,$$

where $u = s / L$. Hence β_n / L is bounded as a function of s , uniformly in L .

Writing $s = w / L$, $u = w / L^2$, in the above and, letting L tend to infinity with w fixed, we find

$$\frac{\beta_n}{L} \rightarrow \begin{cases} p\, \tau_+^T \lambda_0^{-1} \mathbf{x}(0, 1) = p\, k^+ , & \text{if } w > 0 \\ p\, \tau_-^T \lambda_0^{-1} \mathbf{x}(0, 1) = p\, k^- , & \text{if } w < 0 \end{cases} , \tag{4.2.6}$$

on using the relation $\lambda_1 \nu = p\, \tau$.

If we assume $p > 0$, then $\beta_n \rightarrow \infty$, as $L \rightarrow \infty$, if $w > 0$, and $\beta_n \rightarrow -\infty$, if $w < 0$. It follows that $\frac{1}{L} g(\frac{w}{L}) \rightarrow p\, k^+$, if $w > 0$, and $\frac{1}{L} g(\frac{w}{L}) \rightarrow 0$, if $w < 0$. Further,

$$Q[\mathbf{x}(s)] - Q(\mathbf{x}_0) = \frac{L^2[Q_1(u) - Q_1(0)]}{L\,|s|} |w| = \frac{Q_1(u) - Q_1(0)}{|u|} |w|$$

on writing $u = w / L^2$. Thus, letting L tend to infinity with w fixed, we obtain

$$Q[\mathbf{x}(s)] - Q(\mathbf{x}_0) \rightarrow \begin{cases} 2k^+ |w| \quad , & w > 0 \\ 2k^- |w| \quad , & w < 0 . \end{cases} \tag{4.2.7}$$

Since the integrand of (4.2.5) is dominated by an integrable function of the form $Ke^{-a|w|}$, on letting L tend to infinity in (4.2.5), we obtain

$$I_L \to \int_0^\infty p\, k^+ e^{-k^+ w}\, dw \;=\; p \quad .$$

If instead $p < 0$, then $\beta_n \to -\infty$, as $L \to \infty$, if $w > 0$, and $\beta_n \to \infty$, if $w < 0$. It follows that, as $L \to \infty$,

$$\frac{1}{L}\, g\left(\frac{w}{L}\right) \to \begin{cases} 0 \;, & \text{if } w > 0 \,, \\ -pk^- \;, & \text{if } w < 0 \,, \end{cases}$$

and

$$I_L \to \int_{-\infty}^{0} -p\, k^- e^{k^- w} dw \;=\; -p \quad ,$$

which proves part (ii) .

(iii) This follows from arguments similar to those used in (ii), on observing that , when $\lambda_1 = 0$, $\beta_n = 0$ and

$$g\left(\frac{w}{L}\right) \to \begin{cases} (2\pi)^{-1/2}\, \sigma_+ \;, & \text{if } w > 0 \,, \\ (2\pi)^{-1/2}\, \sigma_- \;, & \text{if } w < 0 \,, \end{cases}$$

as L tends to infinity for fixed w . This completes the proof of theorem 4.1 .

By (4.1.4) , $\theta = \frac{1}{\mu} P\{ \mathbf{X} \in \Gamma' \}$. Hence we seek an asymptotic expression for $P\{ \mathbf{X} \in \Gamma' \}$, for large L . We start by transforming the usual integral for $P\{ \mathbf{X} \in \Gamma' \}$ by the introduction of polar coordinates $u = s / L$, L in the plane.

Suppose, for the moment, that Γ' is the exterior of D_L , when $L = L_0$. Then

$$P\{\mathbf{X} \in \Gamma'\} \;=\; \int_{\Gamma'} p[\mathbf{x}]\, d\mathbf{x} \;=\; \int_{D_{L_0}} \int_{L_0}^{\infty} p[\mathbf{x}(Lu, L)]\, |v^T\mathbf{x}(u, 1)|\, L\, dL du \quad .$$

On writing $Q = L^2 q$ and integrating over L, we get

$$P\{\mathbf{X} \in \Gamma'\} \;=\; \int \frac{1}{q}\, p[\mathbf{x}(L_0 u, L_0)]\, |v^T\mathbf{x}(u, 1)|\, du \quad .$$

Dropping the suffix zero and reintroducing $s = L\,u$ as variable of integration, we obtain

$$P\{\mathbf{X} \in \Gamma'\} = \int_D \frac{1}{Q}\, p[\mathbf{x}(s)]\, v^T.\mathbf{x}\, ds \ , \tag{4.2.8}$$

where we have dropped the modulus bars from $v^T\mathbf{x}$, since $v^T\mathbf{x} \geq 0$ for a star-shaped boundary .

Lemma 4.2 If Q has a unique minimum at $\mathbf{x}_0$, where $s = 0$, we find the following limits as L tends to infinity .

(i) If $\mathbf{x}_0$ is a regular point of D_L, then

$$\frac{L\, P\{\mathbf{X} \in \Gamma'\}}{p[\mathbf{x}_0]} \rightarrow \frac{(2\pi)^{1/2}}{q_0}\, \frac{v_0^T\mathbf{x}(0,\ 1)}{k} \ .$$

(ii) If $\mathbf{x}_0$ is a vertex of D_L, then

$$\frac{L^2\, P\{\mathbf{X} \in \Gamma'\}}{p[\mathbf{x}_0]} \rightarrow -\frac{\sin\phi}{k^+k^-} \ ,$$

where k, k^+, k^- are defined in theorem 4.1, ϕ is the angle between τ_+ and τ_- measured in an anticlockwise sense, and $q_0 = \mathbf{x}(0,\ 1)^T \lambda_0^{-1}\mathbf{x}(0,\ 1)$.

Proof. Since $q_0 > 0$, the limits follow by arguments similar to those used in the proof of theorem 4.1 .

In case (i), the result follows from the limit

$$\frac{L\, v^T\mathbf{x}}{Q} = \frac{v^T\mathbf{x}}{L\, \mathbf{x}(\frac{s}{L},\ 1)^T \lambda_0^{-1}\, \mathbf{x}(\frac{s}{L},\ 1)} \rightarrow \frac{v_0^T\mathbf{x}(0,\ 1)}{\mathbf{x}(0,\ 1)^T \lambda_0^{-1}\mathbf{x}(0,\ 1)} \tag{4.2.9}$$

as L tends to infinity, with s fixed .

In case (ii) the same approach as used in theorem 4.1 yields

$$\frac{L^2\, P\{\mathbf{X} \in \Gamma'\}}{p[\mathbf{x}_0]} \rightarrow \frac{1}{q_0}\left[\frac{v_+^T\mathbf{x}(0,\ 1)}{k^+} + \frac{v_-^T\mathbf{x}(0,\ 1)}{k^-}\right] \tag{4.2.10}$$

Expressing $\mathbf{x}_0$ as a linear combination of τ_+ and τ_-, we obtain

$$q_0 = \frac{1}{\sin\phi}\left[\nu_-^T \mathbf{x}(0,1)\, k^+ + \nu_+^T \mathbf{x}(0,1)\, k^- \right] \quad ,$$

where as before ϕ is the angle between τ_- and τ_+ measured in an anticlockwise sense and hence negative at an internal vertex . Substitution of the expression for q_o into (4.2.10) gives the required result .

Theorem 4.2 As L tends to infinity, θ satisfies the following limits .

(i) If $\mathbf{x}_o$ is a regular point of D_L , then

$$L\,\theta \to \frac{(2\pi)^{1/2}}{\alpha}\left[\nu_0^T \lambda_2 \nu_0 - p^2 \kappa\, \alpha \right]^{-1/2} \quad ,$$

where $\alpha = \nu_o^T \lambda_o^{-1} \mathbf{x}(0, 1)$ and κ is the curvature of D_1 at $s = 0$.

(ii) If $\mathbf{x}_o$ is a vertex of D_L and $\lambda_1 \neq 0$, then

$$L^2\,\theta \to \frac{-\sin\phi}{k^+ k^- \,|p|} \quad .$$

(iii) If $\mathbf{x}_o$ is a vertex of D_L and $\lambda_1 = 0$, then

$$L\,\theta \to \frac{-(2\pi)^{1/2}\sin\phi}{k^-\sigma_+ + k^+\sigma_-} \quad .$$

Proof. Since $\theta = \frac{1}{\mu} P\{\, \mathbf{X} \in \Gamma' \,\}$,

$$L\,\theta = \frac{L\, P\{\mathbf{X} \in \Gamma'\}}{p[\mathbf{x}_0]} \cdot \frac{p[\mathbf{x}_0]}{\mu}$$

$$\to \frac{(2\pi)^{1/2}}{q_0}\, \frac{\nu_0^T \mathbf{x}(0,1)}{k} \left[\frac{\sigma_0^2}{k^2} + p^2 \right]^{-1/2} \tag{4.2.11}$$

by theorem 4.1 and lemma 4.2 . Since Q has a minimum at $\mathbf{x}_o$, $\lambda_o^{-1}\mathbf{x}_o$ is parallel to ν_o and we can write $q_o = \nu_o^T \mathbf{x}(0, 1) . \nu_o^T \lambda_o^{-1} \mathbf{x}(0, 1) =$ $\nu_o^T \mathbf{x}(0, 1)\, \alpha$. Further

$$k^2 = \frac{1}{2}\left(\frac{d^2Q}{ds^2} \right)_0 = \left(\tau^T \lambda_0^{-1} \tau + \left(\frac{d\tau}{ds} \right)^T \lambda_0^{-1} \mathbf{x} \right)_0$$

and $x(s, L) = L\, x(u, 1)$, where $u = s / L$, which, differentiating twice with respect to s, gives $\frac{d\tau}{ds} = \frac{1}{L}\, \mathbf{x}''(u, 1)$; the primes denoting differentation with respect to u . Since u is the arclength on D_1 , we have $\mathbf{x}''(u, 1) = -\kappa\,\nu$, where

κ is the curvature of D_1 and ν is the common normal to D_1 at $\mathbf{x}(u, 1)$, and to D_L at $\mathbf{x}(s, L)$, by similarity . Thus we can write

$$k^2 = \left(\tau^T \lambda_0^{-1} \tau - \frac{\kappa}{L} \nu^T \lambda_0^{-1} \mathbf{x} \right)_0 . \tag{4.2.12}$$

Bearing in mind $\lambda_1 \nu_0 = p\, \tau_0$, we have

$$\sigma_0^2 = \nu_0^T \lambda_2 \nu_0 - p^2 \tau_0^T \lambda_0^{-1} \tau_0 \quad ,$$

and, since $\frac{1}{L} \mathbf{x}_0 = \mathbf{x}(0, 1)$, using (4.2.12), we have

$$\sigma_0^2 + p^2 k^2 = \nu_0^T \lambda_2 \nu_0 - p^2 \kappa \alpha \quad ,$$

from which the result (i) follows . Results (ii) and (iii) follow immediately from theorem 4.1 and lemma 4.2 .

4.3 The Asymptotic Distributions of Excursions

From (4.1.9) and (4.1.11) we have

$$G(\tau, L) = \frac{1}{\mu} \int_{D_L}\!\!\int p(\mathbf{x}_1, \mathbf{y}_1)\, J_L(\tau, s_1, \mathbf{y}_1)\, (\nu_1^T \mathbf{y}_1)^+ \, d\mathbf{y}_1\, ds_1 \; , \tag{4.3.1}$$

where

$$J_L(\tau, s_1, \mathbf{y}_1) = \int_0^{\theta\tau}\!\!\int\!\!\int p_t\, (\mathbf{x}_2, \mathbf{y}_2 \mid \mathbf{x}_1, \mathbf{y}_1)\, (\nu_2^T \mathbf{y}_2)^- \, d\mathbf{y}_2\, ds_2\, dt \tag{4.3.2}$$

In the above $p(\mathbf{x}_1, \mathbf{y}_1)$ is the density of $\mathbf{X}(0)$, $\mathbf{X}'(0)$ and $p_t(\mathbf{x}_2, \mathbf{y}_2 \mid \mathbf{x}_1, \mathbf{y}_1)$ the density of $\mathbf{X}(t)$, $\mathbf{X}'(t)$, conditional on $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$.

The covariance matrix of $\mathbf{X}(0)$, $\mathbf{X}'(0)$ is

$$\Sigma_0 = \begin{bmatrix} R(0) & R'(0) \\ -R'(0) & -R''(0) \end{bmatrix} = \begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix} \tag{4.3.3}$$

and the conditional distribution is normal with mean vector :

$$\begin{bmatrix} E[\,\mathbf{X}(t) \mid \mathbf{x}_1, \mathbf{y}_1\,] \\ E[\,\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1\,] \end{bmatrix} = \begin{bmatrix} R(t) & R'(t) \\ -R'(t) & -R''(t) \end{bmatrix}^T \begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{bmatrix} \tag{4.3.4}$$

and covariance matrix :

$$\begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix} - \begin{bmatrix} R(t) & R'(t) \\ -R'(t) & -R''(t) \end{bmatrix}^T \begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} R(t) & R'(t) \\ -R'(t) & -R''(t) \end{bmatrix} . \tag{4.3.5}$$

We can use the results of appendix B, section (e), to rewrite (4.3.2) as

$$J_L(\tau, s_1, \mathbf{y}_1) = \int_0^{\theta\tau} \int p_t\,(\mathbf{x}(s_2) \mid \mathbf{x}_1, \mathbf{y}_1\,)\, j(t, s_2\,; s_1, \mathbf{y}_1)\, ds_2\, dt \tag{4.3.6}$$

where

$$j(t, s_2\,; s_1, \mathbf{y}_1) = \int p_t\,(\mathbf{y}_2 \mid \mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_1\,)\,(\mathbf{v}_2^T\mathbf{y}_2)^-\, d\mathbf{y}_2\ . \tag{4.3.7}$$

In the above integrals, $p_t\,(\mathbf{x} \mid \mathbf{x}_1, \mathbf{y}_1\,)$ is the conditional density of $\mathbf{X}(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$ and $p_t\,(\mathbf{y}_2 \mid \mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_1\,)$ the density of $\mathbf{X}'(t\,)$, conditional on $\mathbf{X}(t) = \mathbf{x}_2$, $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$. Details of the latter conditional distribution are derived in section (d) of appendix B .

Evaluating the integral in (4.3.7) , we find

$$j(t, s_2\,; s_1, \mathbf{y}_1) = -\,m\,\Phi(-\frac{m}{\sigma}) + \sigma\,\phi(-\frac{m}{\sigma})\ , \tag{4.3.8}$$

where $m = \mathbf{v}_2{}^T\mathbf{m} = \mathbf{v}_2{}^T E\,[\,\mathbf{X}'(t\,) \mid \mathbf{x}_1, \mathbf{y}_1] + \mathbf{v}_2{}^T\, b^T\, a^{-1}\mathbf{z}$,

with $\mathbf{z} = \mathbf{x}_2 - E\,[\,\mathbf{X}(t\,) \mid \mathbf{x}_1, \mathbf{y}_1]$, and $\sigma^2 = \mathbf{v}_2{}^T\,(\,c - b^T\, a^{-1} b\,)\mathbf{v}_2$,

where a, b, c are 2 x 2 matrices defined by equation (B. 14) of appendix B, in terms of the covariance matrix of the conditional distribution of $\mathbf{X}(t\,)$, $\mathbf{X}'(t\,)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$. The functions Φ (.) and ϕ (.) on the right hand side of (4.3.8) are the standard normal distribution function and density function, respectively .

The experience of the previous section suggests we will need the limit of $J_L(\tau, s_1, \mathbf{y}_1)$, as L tends to infinity, either for fixed s_1 or for $s_1 = x / L$, with x fixed. The following lemma gives sufficient conditions for the existence of such limits.

Lemma 4.3 Let $A_\tau = \{ (s_2, t) : 0 \le t \le \theta\tau \ , \ 0 \le s_2 < \| D_L \| \}$ be the domain of integration in (4.3.6), and define the set $N_\varepsilon = \{ (s_2, t) : |\mathbf{z}| \le \theta\varepsilon \}$, for $\varepsilon > 0$, where $\mathbf{z}$ is as defined following (4.3.8).

If there exist positive real numbers δ, ε, and L_o, with $\varepsilon < \frac{1}{3}\delta^2$, such that $t \ge \theta\delta$ and $v_2^T E[\, \mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1] \le -\delta$ on N_ε, for $L \ge L_o$, then $J_L(\tau, s_1, \mathbf{y}_1)$ is bounded for $L \ge L_o$.

Further, if, for $L \ge L_o$, $N_\varepsilon \subset A_\tau$, then $J_L \to 1$ as L tends to infinity, alternatively, if $N_\varepsilon \subset (A_\tau)'$, the complement of A_τ, then $J_L \to 0$, as L tends to infinity.

Proof. The distribution of $\mathbf{X}(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$ is normal with mean $E[\, \mathbf{X}(t) \mid \mathbf{x}_1, \mathbf{y}_1]$ and covariance matrix a, as follows from section (d) of appendix B. Since a is positive definite, if $\mathbf{z}$ is a two-dimensional vector, then $\mathbf{z}^T a^{-1} \mathbf{z} / |\mathbf{z}|^2 \ge 1/\alpha_1$, where α_1 is the largest eigenvalue of a. Thus we have

$$\mathbf{z}^T a^{-1} \mathbf{z} \ge |\mathbf{z}|^2 / \alpha_1 \ \ge \ |\mathbf{z}|^2 / (\mathrm{Sp}(a)) \ ,$$

where Sp(a) denotes the spur of matrix a. The result (B.15) of appendix B shows that the elements of a are $O(t^4)$, for small t, and hence $\mathrm{Sp}(a) = O(t^4)$. It follows that

$$p_t(\mathbf{x}_2 \mid \mathbf{x}_1, \mathbf{y}_1) = \frac{|a|^{-1/2}}{2\pi} \exp\{-\tfrac{1}{2}\mathbf{z}^T a^{-1} \mathbf{z}\} \le K t^{-4} \exp\{-k\,\varepsilon^2 \theta^2 / t^4\}$$

on $(N_\varepsilon)'$, for some positive constants K and k.

We now partition the range of integration in (4.3.6) using N_ε, and write $J_L = I_1 + I_2$, where I_1 denotes the integral over $A_\tau \cap (N_\varepsilon)'$. Substituting from the above into the integral of (4.3.6) and using (4.3.8), we find

$$I_1 \le \int\int_{A_\tau} Kt^{-4} \exp\{-k\,\varepsilon^2\theta^2/t^{-4}\}\,|\,j\,|\,ds_2\,dt\ , \qquad (4.3.9)$$

From the definitions following (4.3.8), we have

$$|\,m\,| \le |\,v_2^T E[\,\mathbf{X}'(t)\,|\,\mathbf{x}_1, \mathbf{y}_1]\,| + |\,v_2^T\, b^T a^{-1}\mathbf{z}\,|\ .$$

By (4.3.4), the first term on the right hand side is bounded by a linear function of L, similarly $|\,\mathbf{z}\,|$ is bounded by a linear function of L, for large L. Since, from equation (B.19) of the appendix, $b^T a^{-1} = O(t^{-1})$, it follows that $t\,|\,m\,|$ is bounded by a linear function of L. Further $\sigma^2 = o(t^2) = o(\theta^2)$, for fixed τ, by (B.20) and since $\theta \to 0$, as L tends to infinity, we have

$$|\,j(t, s_2\,;\,s_1, \mathbf{y}_1)\,| \le |\,m\,| + \frac{\sigma}{\sqrt{2\pi}} \le K'\,L\,t^{-1} \qquad (4.3.10)$$

for $L \ge L_0$, where K' is a positive constant. Substituting from (4.3.10) into (4.3.9) and carrying out the integration over t, we obtain

$$I_1 \le K_1\,L\,\|D_L\|\,\frac{\theta^{-2}}{4k\varepsilon^2}\exp\{-k\,\varepsilon^2/(\theta^2\tau^4)\}\ ,$$

for some positive constant K_1. Since $\|\,D_L\,\| = O(L)$ and $\theta \to 0$, as a power of L, it follows from the above inequality that I_1 tends to zero, as L tends to infinity.

The vector $\zeta = t^{-2}\mathbf{z}$ is a function of s_2 and t. The transformation $(s_2, t) \to \zeta$ has Jacobian

$$\frac{\partial(\zeta)}{\partial(s_2, t)} = t^{-4} v_2^T\left\{ E[\,\mathbf{X}'(t)\,|\,\mathbf{x}_1, \mathbf{y}_1\,] + \frac{2}{t}\mathbf{z}\right\}$$

which, under the conditions of the lemma, is non-zero throughout N_ε. In fact,

$$\left|\frac{\partial(\zeta)}{\partial(s_2, t)}\right| t^4 \ge \left|v_2^T E[\,\mathbf{X}'(t)\,|\,\mathbf{x}_1, \mathbf{y}_1\,]\right| + \left|\frac{2}{t}\mathbf{z}\right| \ge \delta - 2\frac{\varepsilon}{\delta} > \frac{1}{3}\delta\ , \qquad (4.3.11)$$

on N_ε.

Introducing ζ as variable of integration in the integral of (4.3.6) , we obtain

$$I_2 \leq \int \int_{N_\varepsilon} p_t\,(\mathbf{x}(s_2) \mid \mathbf{x}_1, \mathbf{y}_1)\, j\, ds_2\, dt$$

$$= \int_{|z| \leq \theta\varepsilon} \psi_t(\zeta)\, j \left| \nu_2^T E[\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1] + \frac{2}{t} \nu_2^T \mathbf{z} \right|^{-1} d\zeta \,, \qquad (4.3.12)$$

where $\psi_t(\zeta) = t^4\, p_t\,(\mathbf{x}(s_2) \mid \mathbf{x}_1, \mathbf{y}_1)$ is the density function of $N(\underline{0}\,,\ t^{-4}\, a\,)$.

Using (4.3.8) and the following definition of m, we have

$$j \leq |\,m\,| + \frac{\sigma}{\sqrt{2\pi}} \leq \left| \nu_2^T E[\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1] \right| + \left| \nu_2^T b^T a^{-1} \mathbf{z} \right| + \frac{\sigma}{\sqrt{2\pi}}\,. \qquad (4.3.13)$$

The 2 x 2 matrix b^Ta^{-1} is a function of t , determined solely by the stochastic process, which is $O(t^{-1})$ for small t , by (B.19) . Thus we can find a positive constant K, independent of L, such that

$$\| b^T a^{-1} \| \leq K / t \,,$$

and therefore, by the conditions of the lemma,

$$\left| \nu_2^T b^T a^{-1} \mathbf{z} \right| \leq \frac{K}{t}\, \theta\varepsilon \leq K \frac{\varepsilon}{\delta} \qquad (4.3.14)$$

on N_ε . Combining inequalities (4.3.11), (4.3.13) and (4.3.14), we find

$$j \left| \nu_2^T E[\,\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1\,] + \frac{2}{t} \nu_2^T \mathbf{z} \right|^{-1} \leq 3 + 3K\, \varepsilon\, \delta^{-2} + 3\delta^{-1} \frac{\sigma}{\sqrt{2\pi}} \qquad (4.3.15)$$

and therefore the first member is bounded on N_ε , for fixed τ , since $\sigma \to 0$ with θ, as L tends to infinity .

From its definition following (4.3.12), we have

$$\psi_t(\zeta) = t^4 \frac{|a|^{-1/2}}{2\pi} \exp\{-\frac{1}{2} \zeta^T t^4 a^{-1} \zeta \} \,, \qquad (4.3.16)$$

and, for small t , appendix B gives

$$a = \frac{1}{4} t^4 d + o(t^4) \qquad (B.15)$$

where the 2 x 2 matrix d is given by (B.13) . Thus, for given τ and $L \geq L_0$, for some L_0 which may depend on τ ,

$$\psi_t(\zeta) \leq \frac{4}{\pi} |d|^{-1/2} \exp\{-|\zeta|^2/\|d\|\} . \tag{4.3.17}$$

It follows , with the aid of (4.3.12) and (4.3.15), that I_2 is bounded for $L \geq L_o$ and the first part of the lemma is proved .

If $N_\varepsilon \subset (A_\tau)'$, then $N_\varepsilon \cap A_\tau = \varnothing$, therefore $I_2 = 0$ and $J_L = I_1 \to 0$, as $L \to \infty$.

On the other hand, if $N_\varepsilon \subset A_\tau$, (4.3.12) holds as an equality . Further

$$j \left| v_2^T E[\, \mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1 \,] + 2t\, v_2^T \zeta \right|^{-1} \to 1 \ ,$$

as L tends to infinity with ζ fixed, since , from (4.3.8) for $m < 0$,

$$j = |m| + o(\sigma) \ ,$$

as $\sigma \to 0$, and

$$v_2^T E[\, \mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1 \,] + 2t\, v_2^T \zeta = m + o(\theta) \ ,$$

as $\theta \to 0$, by section (d) of appendix B. Further, from (B.19) we have, $\psi_t(\zeta) \to \psi_o(\zeta)$, the density function of $N(0, \frac{1}{4} d)$, as $t \to 0$.

Under the conditions of the lemma, $\theta\delta \leq t \leq \theta\tau$ on N_ε , therefore $\{\zeta : |\zeta| \leq \frac{\varepsilon}{\theta\tau^2}\} \subset N_\varepsilon \subset \{\zeta : |\zeta| \leq \frac{\varepsilon}{\theta\delta^2}\}$ and hence $N_\varepsilon \to R^2$, as $L \to \infty$ and $\theta \to 0$. Since, by (4.3.17), ψ_t is uniformly bounded by a function integrable on R^2, thus $I_2 \to \int \psi_o \, d\zeta = 1$, as $L \to \infty$, and since $I_1 \to 0$ the lemma is proved.

The expression (4.3.1) for $G(\tau, L)$ may be rewritten

$$G(\tau, L) = \frac{1}{\mu} \int p(\mathbf{x}_1)\, g_L(\tau, s_1)\, ds_1 \ , \tag{4.3.18}$$

where

$$g_L(\tau, s_1) = \int p(\mathbf{y}_1 \mid \mathbf{x}_1)\, J_L(\tau, s_1, \mathbf{y}_1)\, (v_1^T \mathbf{y}_1)^+\, d\mathbf{y}_1 \ , \tag{4.3.19}$$

$\mathbf{x}_1 = \mathbf{x}(s_1, L)$ and where $p(\mathbf{x}_1)$ is the density function of $\mathbf{X}(0)$ and $p(\mathbf{y}_1 \mid \mathbf{x}_1)$ the density of $\mathbf{X}'(0)$, conditional on $\mathbf{X}(0) = \mathbf{x}_1$. Thus, from section (e) of appendix B, $p(\mathbf{x})$ is the density of $N(\underline{0}, \lambda_o)$ and $p(\mathbf{y} \mid \mathbf{x}_1)$ the density of $N(-\lambda_1\lambda_o^{-1}\mathbf{x}_1, \lambda_2 + \lambda_1\lambda_o^{-1}\lambda_1)$.

Assuming the conditions of lemma 4.3 are met, J_L will be bounded by some constant K and hence, from (4.3.19),

$$g_L(\tau, s_1) \le K \int p(\mathbf{y}_1 \mid \mathbf{x}_1)\, (v_1^T \mathbf{y}_1)^+ \, d\mathbf{y}_1$$

from which it follows that g_L is bounded by a linear function of L, as L tends to infinity. Utilising the arguments of the proof of theorem 4.1, we obtain

$$\mu G(\tau, L) = \int_{-\delta L}^{\delta L} p(\mathbf{x}_1)\, g_L \, ds_1 + p(\mathbf{x}_0)\, O\left(L^2 e^{-\frac{1}{2} f^2 L^2}\right) \tag{4.3.20}$$

where $\delta > 0$ and f^2 are as defined in theorem 4.1 . Thus to consider the limit of $G(\tau, L)$, as $L \to \infty$, we will need to ascertain the limit of g_L , as $L \to \infty$ with τ fixed .

Substituting $\mathbf{y}_1 = -\lambda_1\lambda_o^{-1}\mathbf{x}_1 + \eta$ into (4.3.19), we find

$$g_L(\tau, s_1) = \int p(\eta)\, J_L(\tau, s_1, \mathbf{y}_1)\, (\beta_n + v_1^T \eta)^+ \, d\eta \ , \tag{4.3.21}$$

where β_n is as defined in (4.2.1) with $v = v_1$, and where $p(\eta)$ is the density function of $N(\underline{0}, \lambda_2 + \lambda_1\lambda_o^{-1}\lambda_1)$. With the aid of lemma 3 , we now explore the limiting behaviour of $G(\tau, L)$ and g_L for each of the three cases of theorem 4.1 separately .

4.4 Case (i)

In this case $\theta = O(1/L)$, by theorem 4.2, and we look for solutions of the equations $\mathbf{z} = 0$ in which $t = O(1/L)$, with $s_1 = O(1)$, and $\mathbf{y}_1 = -\lambda_1\lambda_o^{-1}\mathbf{x}_1 + \eta$, with $\eta = O(1)$ and $v_1^T \mathbf{y}_1 > 0$.

From (B.10) of appendix B, the equation $\mathbf{z} = 0$ becomes

$$\mathbf{x}(s_2) = \mathbf{x}_1 + t\,\mathbf{y}_1 - \frac{1}{2} t^2 \lambda_2 \lambda_0^{-1} \mathbf{x}_1 + O(1/L^2) \ , \tag{4.4.1}$$

on substituting $\mathbf{y}_1 = -\lambda_1\lambda_o^{-1}\mathbf{x}_1 + \eta$ in the coefficient of t^2 , and noting that $\sigma_{22}\lambda_1\lambda_o^{-1} = \lambda_2^{-1}\lambda_1\sigma_{11}$ and $\sigma_{11} = [\lambda_o + \lambda_1\lambda_2^{-1}\lambda_1]^{-1}$ (see appendix B). Writing $s_2 = s_1 + s$, assuming s_1 and s_2 are points of the same regular arc of D_L, we have the Taylor expansion

$$\mathbf{x}(s_2) = \mathbf{x}_1 + s\,\tau_1 - \frac{1}{2} s^2 \frac{\kappa}{L} \nu_1 + o(1/L), \tag{4.4.2}$$

where κ is the curvature of D_1 at s/L and τ_1, ν_1 the unit tangent and normal, respectively, at s_1. Equating the right hand sides of (4.4.1) and (4.4.2) and resolving along τ_1 and ν_1, we get

$$s = t\,\tau_1^T\mathbf{y}_1 - \frac{1}{2} t^2 \tau_1^T \lambda_2 \lambda_0^{-1}\mathbf{x}_1 + o(1/L) \tag{4.4.3}$$

and

$$-\frac{1}{2} s^2 \frac{\kappa}{L} = t\,\nu_1^T\mathbf{y}_1 - \frac{1}{2} t^2 \nu_1^T \lambda_2 \lambda_0^{-1}\mathbf{x}_1 + o(1/L) . \tag{4.4.4}$$

Eliminating s between (4.4.3) and (4.4.4), and solving the resulting equation for t, we find

$$t = 2\,\nu_1^T\mathbf{y}_1 \left[\nu_1^T\lambda_2 \lambda_0^{-1}\mathbf{x}_1 - \frac{\kappa}{L} (\tau_1^T\lambda_1\lambda_0^{-1}\mathbf{x}_1)^2 \right]^{-1} + o(1/L) , \tag{4.4.5}$$

ignoring the trivial solution $t = 0$.

Expressing $\mathbf{x}_1$ and ν_1 as power series in s_1, up to terms of order $1/L$, in $\nu_1^T\mathbf{y}_1 = -\nu_1^T\lambda_1\lambda_o^{-1}\mathbf{x}_1 + \nu_1^T\eta$, we find

$$\nu_1^T\mathbf{y}_1 = -\nu_0^T\lambda_1\lambda_0^{-1}\mathbf{x}_0 - s_1\nu_0^T\lambda_1\lambda_0^{-1}\tau_0 - s_1\frac{\kappa}{L}\tau_0^T\lambda_1\lambda_0^{-1}\mathbf{x}_0 + \nu_0^T\eta + o(1) ,$$

where the suffix zero on $\mathbf{x}$, τ, ν corresponds to $s_1 = 0$. Recalling that $\lambda_1\nu = p\tau$ and $\nu_o^T\lambda_1\lambda_o^{-1}\mathbf{x}_o = 0$, we obtain

$$\nu_1^T\mathbf{y}_1 = s_1\, p\, k^2 + \nu_0^T\eta + o(1) , \tag{4.4.6}$$

where $k^2 = \tau_o^T\lambda_o^{-1}\tau_o - \frac{\kappa}{L}\nu_o^T\lambda_o^{-1}\mathbf{x}_o$ is defined in theorem 1. It follows that $\nu_1^T\mathbf{y}_1$ is $O(1)$, as $L \to \infty$.

Let $t = t_L$, $s = s_L$ denote the non-trivial solution of $\mathbf{z} = 0$. It is easily seen that the denominator of the right hand side of (4.4.5) is $O(L)$ and hence, in view of

(4.4.6), $t_L = O(1/L)$, as $L \to \infty$ and, from (4.4.3), we find $s_L = O(1)$. Since $\theta = O(1/L)$, $u_L = t_L/\theta = O(1)$ and tends to a limit u_∞, as $L \to \infty$.

From (5.4.5), since $\frac{1}{L}\mathbf{x}_1 = \mathbf{x}\left(\frac{s_1}{L}, 1\right)$, and $\nu_1 \to \nu_o$ etc., we obtain

$$L\, t_L \to 2\, \nu_0^T \mathbf{y}_1 \left[\alpha\, \nu_0^T \lambda_2\, \nu_0 - \kappa\, p^2 \alpha^2\right]^{-1}, \tag{4.4.7}$$

as $L \to \infty$, on writing $\alpha = \nu_o{}^T \lambda_o{}^{-1} \mathbf{x}(0, 1)$ and observing that $\lambda_o{}^{-1}\mathbf{x}_o$ is parallel to ν_o. Employing theorem 4.2 for the limit of $L\theta$, we obtain

$$u_\infty = \left(\frac{2}{\pi}\right)^{1/2} \nu_0^T \mathbf{y}_1 \left[\, \nu_0^T \lambda_2 \nu_0 - \kappa\, p^2\, \alpha\,\right]^{-1/2}. \tag{4.4.8}$$

Writing $t = t_L + \delta t$, $s = s_L + \delta s$ and expanding $\mathbf{z}$ about (t_L, s_L), where $\mathbf{z} = 0$ we get

$$\mathbf{z} = \tau\, \delta s - \mathbf{w}\, \delta t ,$$

approximately, where $\mathbf{w} = E[\, \mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1]$. Solving the vector equation for δs, δt we get , for small $\mathbf{z}$,

$$\delta t = - \frac{\nu^T \mathbf{z}}{\nu^T \mathbf{w}} , \quad \delta s = \tau^T \mathbf{z} - \frac{\tau^T \mathbf{w}}{\nu^T \mathbf{w}}\, \nu^T \mathbf{z} , \tag{4.4.9}$$

where τ, ν, $\mathbf{w}$ are to be evaluated at $t = t_L$, $s_2 = s_1 + s_L$.

Not only are we interested in $\nu_2{}^T \mathbf{w}$ at $\mathbf{z} = 0$, but more generally for $(t, s_2) \in N_\varepsilon$. Substituting $\mathbf{y}_1 = -\lambda_1 \lambda_o{}^{-1} \mathbf{x}_1 + \eta$ in the coefficient of t in (B.11) and observing $t = O(1/L)$, we obtain

$$\mathbf{w} = E\left[\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1\right] = \mathbf{y}_1 - t\, \lambda_2 \lambda_0^{-1} \mathbf{x}_1 + O(1/L) . \tag{4.4.10}$$

Writing $\nu_2 = \nu_1 + s\frac{\kappa}{L}\, \tau_1 + o(1/L)$, $t = t_L + \delta t$, $s = s_L + \delta s$, with the aid of (4.4.10), (4.4.3) and (4.4.5), we find

$$\nu_2^T \mathbf{w} = -\nu_1^T \mathbf{y}_1 - \nu_1^T \lambda_2 \lambda_0^{-1} \mathbf{x}_1\, \delta t - \frac{\kappa}{L}\, \tau_1^T \lambda_1 \lambda_0^{-1} \mathbf{x}_1\, \delta s + o(1) . \tag{4.4.11}$$

In particular, $\nu_2{}^T \mathbf{w} = -\nu_1{}^T \mathbf{y}_1 + o(1)$ at $\mathbf{z} = 0$ and, since $\nu_1{}^T \mathbf{y}_1 > 0$, $\nu_2{}^T \mathbf{w} < 0$ for large L .

On N_ε , (4.4.9) gives

$$|\delta t| \leq \frac{\theta\,\varepsilon}{|v^T \mathbf{w}|} \leq \frac{2\,\theta\,\varepsilon}{v_1^T \mathbf{y}_1}, \tag{4.4.12}$$

and

$$|\delta s| \leq \theta\varepsilon + \left|\frac{\tau^T \mathbf{w}}{v^T \mathbf{w}}\right| \theta\varepsilon \leq \frac{2|p\,\alpha|L}{v_1^T \mathbf{y}_1}\theta\varepsilon , \tag{4.4.13}$$

since $\tau^T \mathbf{w}$ is dominated by $\tau^T \mathbf{y}_1 = - p\,\alpha\,L + O(1)$ and $|v^T \mathbf{w}| >$ $(v_1{}^T \mathbf{y}_1) / 2$, for large L . Thus, since $v_1{}^T \mathbf{y}_1 = O(1)$ and $\theta = O(1 / L)$, from (4.4.11) it follows that, for large L,

$$\left| v_2^T \mathbf{w} + v_1^T \mathbf{y}_1 \right| \leq K\varepsilon \tag{4.4.14}$$

for some positive constant K .

Since $v_1{}^T \mathbf{y}_1 > 0$, from (4.4.6) we may assume $s_1\, p\, k^2 + v_o{}^T \eta > 0$. If we choose δ such that $0 < \delta < s_1\, p\, k^2 + v_o{}^T \eta$, then, by (4.4.14), we can find $L_o > 0$ and ε such that $v^T \mathbf{w} \leq -\delta$ on N_ε , for all $L \geq L_o$. For a given δ, this statement remains true if ε is replaced by a smaller positive value .

Suppose that $u_\infty < \tau$ and $0 < \delta < u_\infty$, then , since u_L tends to u_∞ , as L tends to infinity, we may choose L_o and $\varepsilon > 0$ such that $u_L + 2\varepsilon / (v_1{}^T \mathbf{y}_1) < \tau$ and $u_L - 2\varepsilon / (v_1{}^T \mathbf{y}_1) > \delta$, for $L \geq L_o$. Thus on N_ε ,

$$t = t_L + \delta t = \theta\, u_L + \delta t \leq \theta\, u_L + \frac{2\,\theta\,\varepsilon}{v_1^T \mathbf{y}_1} < \theta\,\tau ,$$

by (4.4.12), similarly we find $t > \theta\delta$. Since we may choose $\delta < s_1\, p\, k^2 + v_o{}^T \eta$ and $\delta < u_\infty$ and ensure $\varepsilon < \frac{1}{3}\delta^2$ without contradiction, the conditions of lemma 4.3 are met if $u_\infty < \tau$, with $N_\varepsilon \subset A_\tau$, for $L \geq L_o$.

Suppose now that $u_\infty > \tau$, we may choose L_o and $\varepsilon > 0$ such that $u_L - 2\varepsilon / (v_1{}^T \mathbf{y}_1) > \tau$, and hence $t > \tau\,\theta$ on N_ε , for $L \geq L_o$. Again the conditions of lemma 4.3 can be met, but with $N_\varepsilon \subset (A_\tau)'$, for $L \geq L_o$. From lemma 4.3 , it now follows that $J_L(\tau, s_1, \mathbf{y}_1)$ is bounded and

$$J_L(\tau, s_1, y_1) \to \begin{cases} 1, & \text{if } u_\infty < \tau \\ 0, & \text{if } u_\infty > \tau, \end{cases}$$

as $L \to \infty$.

From (4.4.6) $v_1^T y_1 \to s_1 p k^2 + v_0^T \eta$ and by dominated convergence, as $L \to \infty$, (4.3.21) gives

$$g_L(\tau, s_1) \to \int p(\eta)\,(s_1 p k^2 + v_0^T \eta)^+ \, d\eta \quad , \tag{4.4.15}$$

where the integration is over the region

$$u_\infty = \left(\frac{2}{\pi}\right)^{1/2} \left[v_0^T \lambda_2 v_0 - \kappa p^2 \alpha \right]^{-1/2} (s_1 p k^2 + v_0^T \eta) < \tau \ ,$$

ie. over $v_0^T \eta < - s_1 p k^2 + t / c$, where we have written

$$c = \left(\frac{2}{\pi}\right)^{1/2} [v_0^T \lambda_2 v_0 - \kappa p^2 \alpha]^{-1/2} .$$

Integrating out the tangential component of η in (4.4.15), we get

$$\begin{aligned} g_L &\to \int_{-s_1 p k^2}^{-s_1 p k^2 + \tau/c} p(\eta_0)\,(s_1 p k^2 + \eta_0)\, d\eta_0 \\ &= \int_0^{\tau/c} \frac{1}{\sigma_0} \phi\left(\frac{y - s_1 p k^2}{\sigma_0}\right) y \, dy \end{aligned} \tag{4.4.16}$$

on writing $y = s_1 p k^2 + \eta_0$, $\eta_0 = v_0^T \eta$, where $\phi(.)$ is the standard normal density function and $\sigma_0^2 = v_0^T (\lambda_2 + \lambda_1 \lambda_0^{-1} \lambda_1) v_0$.

The arguments leading to theorem 4.1 applied to (4.3.20), now give

$$\frac{\mu G(\tau, L)}{p[\mathbf{x}_0]} \to \int_{-\infty}^{\infty} e^{-\frac{1}{2} k^2 s_1^2} \int_0^{\tau/c} \frac{1}{\sigma_0} \phi\left(\frac{y - s_1 p k^2}{\sigma_0}\right) y \, dy \, ds_1 \ ,$$

as L tends to infinity. On changing the order of integration in the limit and integrating over s_1, we obtain

$$\lim_{L \to \infty} \frac{\mu G}{p[\mathbf{x}_0]} = \int_0^{\tau/c} \frac{1}{\sigma_0} \left[\frac{k^2}{\sigma_0^2} (\sigma_0^2 + p^2 k^2) \right]^{-1/2} \exp \left\{ - \frac{y^2}{2(\sigma_0^2 + p^2 k^2)} \right\} y \, dy$$

$$= \frac{1}{k} (\sigma_0^2 + p^2 k^2)^{1/2} \left\{ 1 - \exp \left(- \frac{(\tau/c)^2}{2(\sigma_0^2 + p^2 k^2)} \right) \right\}$$

on evaluating the integral . Observing that $\sigma_0^2 + p^2 k^2 = v_0^T \lambda_2 v_0 +$

$- \kappa p^2 \alpha$ and employing theorem 4.1(i) , we finally obtain

$$G(\tau) = \lim_{L \to \infty} G(\tau, L) = 1 - e^{-\pi \tau^2 /4} \quad . \tag{4.4.17}$$

Since the limit $G(\tau)$ exists and is a distribution function, it follows, from lemma 4.1, that the duration of an excursion has an asymptotic distribution, as $L \to \infty$, for which $G(\tau)$ is the distribution function .

4.5 Case (ii)

Since the mean excursion time $\theta = O(1 / L^2)$ we look for solutions of $\mathbf{z}(t, s_2) = 0$ with $t = O(1 / L^2)$, $s_1 = O(1 / L)$ and $v_1^T \mathbf{y}_1 > 0$. As in the corresponding case of theorem 4.1 , we suppose initially that $p > 0$.

For $s = O(1 / L)$, we can write

$$\mathbf{x}(s) = \begin{cases} \mathbf{x}_0 + s \, \tau_+ + o(1/L) \ , & \text{for } s > 0 \\ \mathbf{x}_0 + s \, \tau_- + o(1/L) \ , & \text{for } s < 0 \ , \end{cases} \tag{4.5.1}$$

where $\mathbf{x}_0 = \mathbf{x}(0, L)$ is the position of the vertex at which $Q(\mathbf{x})$ has a minimum on D_L . There are four cases to consider, depending on the signs of s_1 and s_2, but initially we suppose $s_1 > 0$ and $s_2 < 0$. Employing (4.5.1) , the equation $\mathbf{z} = 0$ becomes

$$\mathbf{x}_0 + s_2 \tau_- = \mathbf{x}_0 + s_1 \tau_+ + t \, \mathbf{y}_1 + o(1/L) \ , \tag{4.5.2}$$

which, on resolving along v_- and v_+ , gives

$$t = \frac{s_1 \sin\phi}{v_-^T y_1} + o(1/L^2) \tag{4.5.3}$$

and

$$s_2 = \frac{t\, v_+^T y_1}{\sin\phi} + o(1/L) , \tag{4.5.4}$$

where we have used $v_-^T \tau_+ = -\sin\phi$ and $v_+^T \tau_- = \sin\phi$. Since Q only can have a minimum at an internal vertex of D_L , where $\sin\phi < 0$, we must have $v_-^T y_1 < 0$ for (4.5.3) to give a positive value for t and $v_+^T y_1 > 0$ for (4.5.4) to give $s_2 < 0$, for large L . Further, assuming y_1 to be O(L) , we see $t = O(1/L^2)$ and $s = O(1/L)$.

We now write $s_1 = x/L$ and $y_1 = -\lambda_1 \lambda_0^{-1} x_1 + \eta$, and letting L tend to infinity with x and η fixed, (4.2.6) gives

$$\frac{1}{L} v_1^T y_1 = \frac{\beta_n + v_1^T \eta}{L} \rightarrow \begin{cases} pk^+ , & \text{if } x > 0 . \\ -pk^- , & \text{if } x < 0 . \end{cases} \tag{4.5.5}$$

Further, from (4.5.3),

$$u_L = \frac{t_L}{L} = \frac{x \sin\phi}{(-v_-^T \lambda_1 \lambda_0^{-1} x_1 + v_-^T \eta)\, L\theta} + o(1) ,$$

and letting L tend to infinity and using theorem 4.2, we get

$$u_L \rightarrow u_\infty = \frac{|p|}{p} k^+ x . \tag{4.5.6}$$

By (4.4.10) we have $v_2^T w = v_2^T y_1 + O(1/L)$, and

$$\frac{1}{L} v_2^T y_1 = -\frac{1}{L} v_2^T \lambda_1 \lambda_0^{-1} x_1 + O(1/L) \rightarrow -pk^- ,$$

since $s_2 < 0$. Thus for $p > 0$, $v_2^T w \leq -\delta$, for any $\delta > 0$, and $L \geq L_0$, for some L_0 .

Since $t = u_L \theta + \delta t$, with $|\delta t| \leq \dfrac{2\theta\varepsilon}{v_-^T y_1} = O(\varepsilon / L^3)$, on N_ε , we can find real numbers $\varepsilon > 0$ and $\delta > 0$ such that $\theta\delta < t$ on N_ε , for $L \geq L_0$. The above observations, together with lemma 4.3, show that, for $x > 0$ and $s_2 = s_L < 0$, J_L is bounded for $L \geq L_0$ and

$$J_L(\tau, \frac{x}{L}, \mathbf{y}_1) \rightarrow \begin{cases} 1 & \text{if } k^+x < \tau \\ 0 & \text{if } k^+x > \tau , \end{cases} \tag{4.5.7}$$

as $L \rightarrow \infty$, by (4.5.6) assuming $p > 0$.

Assume now that $s_1 > 0$ and $s_2 > 0$. Following the analysis of case (i) , (4.4.5) gives

$$t_L = 2\nu_1^T\mathbf{y}_1 \left[\nu_1^T\lambda_2\lambda_0^{-1}\mathbf{x}_1 - \frac{\kappa}{L}(\tau_1^T\lambda_1\lambda_0^{-1}\mathbf{x}_1)^2 \right]^{-1} + O(1/L) ,$$

which on substituting $\mathbf{y}_1 = -\lambda_1\lambda_0^{-1}\mathbf{x}_1 + \eta$ gives

$$t_L \rightarrow \frac{2pk^+}{\nu_+^T\lambda_2\lambda_0^{-1}\mathbf{x}(0,1) - \kappa p^2\{\nu_+^T\lambda_0^{-1}\mathbf{x}(0,1)\}^2}$$

as $L \rightarrow \infty$. Thus $t_L / \theta \rightarrow \infty$, for $p > 0$, and therefore $t_L > \tau\,\theta$ for $L \geq L_o$ and any given $\tau > 0$. Hence by lemma 4.3 , J_L will be bounded for $L \geq L_o$, and $s_2 < 0$ to yield the conclusion that $J_L \rightarrow 0$, as L tends to infinity .

Finally we consider the solutions of $\mathbf{z} = 0$ with $s_1 < 0$ and $s_2 > 0$. The methods used in the earlier mixed situation give in the present case,

$$t = -\frac{s_1 \sin\phi}{\nu_+^T\mathbf{y}_1} + o(1/L^2) \tag{4.5.8}$$

and

$$s_2 = -\frac{t\,\nu_-^T\mathbf{y}_1}{\sin\phi} + o(1/L) . \tag{4.5.9}$$

Substituting $\mathbf{y}_1 = -\lambda_1\lambda_0^{-1}\mathbf{x}_1 + \eta$ and $s_1 = x / L$, $x < 0$, in (4.5.8) and employing theorem 4.2, we obtain

$$\begin{aligned} u_L = \frac{t_L}{\theta} &= \frac{-x\sin\phi}{(-\nu_+^T\lambda_1\lambda_0^{-1}\mathbf{x}_1 + \nu_+^T\eta)L\theta} + o(1) \\ \rightarrow u_\infty &= \frac{|p|}{p}k^-x \quad , \end{aligned} \tag{4.5.10}$$

as $L \rightarrow \infty$. Thus, for $p > 0$, $u_\infty < 0$ when $x < 0$, and $t_L < 0$ for $L \geq L_o$, for some sufficiently large L_o .

As in the corresponding case of theorem 4.1 , we substitute $s_1 = x / L$ into the integral in (4.3.20) to obtain , on division by $p[\mathbf{x}_0]$,

$$I_L = \int_{-L^2\delta}^{L^2\delta} \frac{1}{L} g_L\left(\frac{x}{L}\right) f\left(\frac{x}{L}\right) dx \quad , \tag{4.5.11}$$

where $f(s) = \exp\{-\frac{1}{2}[\, Q(\mathbf{x}) - Q(\mathbf{x}_0)]\,\}$ and $g_L(s) = g_L(\tau, s)$ is given by (4.3.21). Thus, from (4.3.21) , we have

$$\frac{1}{L} g_L(\tau, \frac{x}{L}) = \int p(\eta)\, J_L(\tau, \frac{x}{L}, y_1)\left(\frac{\beta_n + v_1^T\eta}{L}\right)^+ d\eta \quad , \tag{4.5.12}$$

where the integrand is dominated by $p(\eta)$ multiplied by a linear function of $|\,\eta\,|$ and hence is integrable .

Letting $L \to \infty$ in (4.5.12) with $x > 0$, we obtain

$$\frac{1}{L} g_L(\tau, \frac{x}{L}) \to \begin{cases} pk^+ \int p(\eta)\, d\eta & x < \tau / k^+ \\ 0 & x > \tau / k^+ \end{cases}$$

using (4.2.6) and (4.5.7) with $p > 0$. On the other hand, letting L tend to infinity in (4.5.12) with $x < 0$, we obtain

$$\frac{1}{L} g_L(\tau, \frac{x}{L}) \to 0 \qquad x < 0 \quad ,$$

since $J_L \to 0$ for all $x < 0$, if $p > 0$.

The integrand of (4.5.11) is dominated by an integrable function of the form $Ke^{-a|x|}$, therefore, letting L tend to infinity in (4.5.11), we obtain

$$I_L \to \int_0^{\tau/k^+} p\, k^+ e^{-k^+x}\, dx = p\,(\,1 - e^{-\tau}\,) \;.$$

On refering back to (4.3.20), we have proved that

$$\frac{\mu G(\tau, L)}{p[\mathbf{x}_0]} \to p\,(\,1 - e^{-\tau}\,) \;.$$

from which, using theorem 4.1, we obtain

$$G(\tau\,) = \lim_{L \to \infty} G(\tau, L) = 1 - e^{-\tau} \quad , \quad \tau > 0 \,. \tag{4.5.13}$$

Here again $G(\tau\,)$ is a distribution function .

A similar analysis with $p < 0$ yields,

$$\frac{1}{L} g_L(\tau, \frac{x}{L}) \to \begin{cases} 0 & 0 < x \\ -p k^- & -\tau / k^- < x < 0 \\ 0 & x < -\tau / k^- \end{cases}$$

and hence

$$\frac{\mu G(\tau, L)}{p[\mathbf{x}_0]} \to \int_{-\tau/k^-}^{0} -p k^- e^{k^- x} dx = -p(1 - e^{-\tau}) .$$

Theorem 4.1 now gives

$$G(\tau) = \lim_{L \to \infty} G(\tau, L) = 1 - e^{-\tau} , \quad \tau > 0$$

and we see that, for $p \neq 0$, the duration of an exceedence has an asymptotic distribution with distribution function given by (4.5.13) , on applying lemma 4.1 .

4.6 Case (iii)

In this case $\theta = O(1 / L)$, hence we look for solutions of the equation $\mathbf{z} = 0$, with $t = O(1 / L)$ and $s_1 = O(1 / L)$. Substituting from (4.4.1) and (4.5.1), for $s_1 > 0$ and $s_2 < 0$, we obtain the equation

$$s_2\tau_- = s_1\tau_+ + t\,\mathbf{y}_1 - \frac{1}{2}t^2 \lambda_2\lambda_0^{-1}\mathbf{x}_0 + o(1/L) , \qquad (4.6.1)$$

where $\mathbf{x}_0$ is the vertex of D_L at which Q is a minimum .

Resolving (4.6.1) along ν_- and ν_+ , we get

$$0 = -s_1 \sin\phi + t\,\nu_-^T\mathbf{y}_1 - \frac{1}{2}t^2 \nu_-^T\lambda_2\lambda_0^{-1}\mathbf{x}_0 + o(1/L) , \qquad (4.6.2)$$

$$s_2 \sin\phi = + t\,\nu_+^T\mathbf{y}_1 - \frac{1}{2}t^2 \nu_+^T\lambda_2\lambda_0^{-1}\mathbf{x}_0 + o(1/L) , \qquad (4.6.3)$$

Since $\mathbf{x}_0$ is a minimum for Q, the vector $\lambda_0^{-1}\mathbf{x}_0$ lies between ν_- and ν_+, in consequence at least one of $\nu_-^T\lambda_2\lambda_0^{-1}\mathbf{x}_0$ and $\nu_+^T\lambda_2\lambda_0^{-1}\mathbf{x}_0$ must be positive . For the present discussion , we assume both are positive and write $c^- = \frac{1}{2}\theta\,\nu_-^T\lambda_2\lambda_0^{-1}\mathbf{x}_0$, $c^+ = \frac{1}{2}\theta\,\nu_+^T\lambda_2\lambda_0^{-1}\mathbf{x}_0$ which , by theorem 4.2, tend to finite limits as $L \to \infty$.

Equation (4.6.4) can be rewritten

$$y^- = \frac{s_1 \sin\phi}{t} + \frac{1}{2} t\, v_-^T \lambda_2 \lambda_0^{-1} \mathbf{x}_0 + o(1) \;, \qquad (4.6.4)$$

where we have introduced $y^- = v_-^T \mathbf{y}_1$ and $y^+ = v_+^T \mathbf{y}_1$. Since $s_1 > 0$ and $\sin\phi < 0$, the right hand side of (4.6.4) is an increasing function of t , for $t > 0$. As t runs from zero to $\theta\tau$, y^- goes from $-\infty$ to $s_1 \sin\phi / (\theta\tau) + c^- \tau$. Thus, on writing $s_1 = \theta x$, the condition $t < \theta\tau$ becomes

$$y^- < \frac{x \sin\phi}{\tau} + \tau c^- \;. \qquad (4.6.5)$$

The condition $s_2 < 0$, in view of (4.6.3), becomes

$$\frac{t}{\theta} < \frac{y^+}{c^+} + o(1) \;, \qquad (4.6.6)$$

for large L .

Combining (4.6.4), (4.6.5) and (4.6.6), we find the following conditions on $\mathbf{y}_1$ for equation (4.6.1) to have a solution with $s_2 < 0$ and $t < \theta\tau$, in the limit as L tends to infinity .

$$\left.\begin{array}{lll} \text{Either} & \dfrac{y^+}{c^+} \geq \tau \text{ and } y^- < \dfrac{x \sin\phi}{\tau} + \tau c^- & \\ \text{or} & \dfrac{y^+}{c^+} < \tau \text{ and } y^- < \dfrac{x \sin\phi}{y^+} c^+ + y^+ \dfrac{c^-}{c^+} \;. & \end{array}\right| \qquad (4.6.7)$$

We now look for solutions of $\mathbf{z} = 0$ with $s_1 > 0$ and $s_2 > 0$. Substituting from (4.4.1) and (4.5.1), we obtain the equation

$$s_2 \tau_+ = s_1 \tau_+ + t\, \mathbf{y}_1 - \frac{1}{2} t^2 \lambda_2 \lambda_0^{-1} \mathbf{x}_0 + o(1/L) \;, \qquad (4.6.8)$$

From (4.6.8) we get, on writing $s_1 = \theta x$,

$$t = \frac{2 y^+}{v_+^T \lambda_2 \lambda_0^{-1} \mathbf{x}_0} + o(1/L) \;, \qquad (4.6.9)$$

and

$$-\frac{s_2 \sin\phi}{\theta} = -x \sin\phi + \frac{y^- y^+}{c^+} - c^- \left(\frac{y^+}{c^+}\right)^2 > 0 \;, \qquad (4.6.10)$$

since $s_2 > 0$.

In the limit, assuming $y^+ > 0$, the inequality of (4.6.10) gives

$$y^- > \frac{x \sin \phi}{y^+} c^+ + y^+ \frac{c^-}{c^+} \quad , \tag{4.6.11}$$

and, in view of (4.6.9), the condition $t < \theta\tau$ becomes

$$\frac{y^+}{c^+} < \tau \quad . \tag{4.6.12}$$

Combining (4.6.7), (4.6.11) and (4.6.12), the equation $\mathbf{z} = 0$ will have a solution, for $s_1 > 0$, with $t < \theta\tau$, if $y^+ / c^+ < \tau$, or, if $y^+ / c^+ \geq \tau$ and $y^- < (x \sin \phi) / \tau + \tau c^-$, for large L .

If, for a given $x > 0$, $\mathbf{y}_1$ lies in the interior of the shaded region of figure 4.1, there will be a solution of $\mathbf{z} = 0$ with $t = t_L$ and $t_L < \theta\tau$, for $L \geq L_o$, for some L_o . If, on the other hand, $\mathbf{y}_1$ lies outside the shaded region and $y^+ > 0$, then $t_L > \theta\tau$, for $L \geq L_o$.

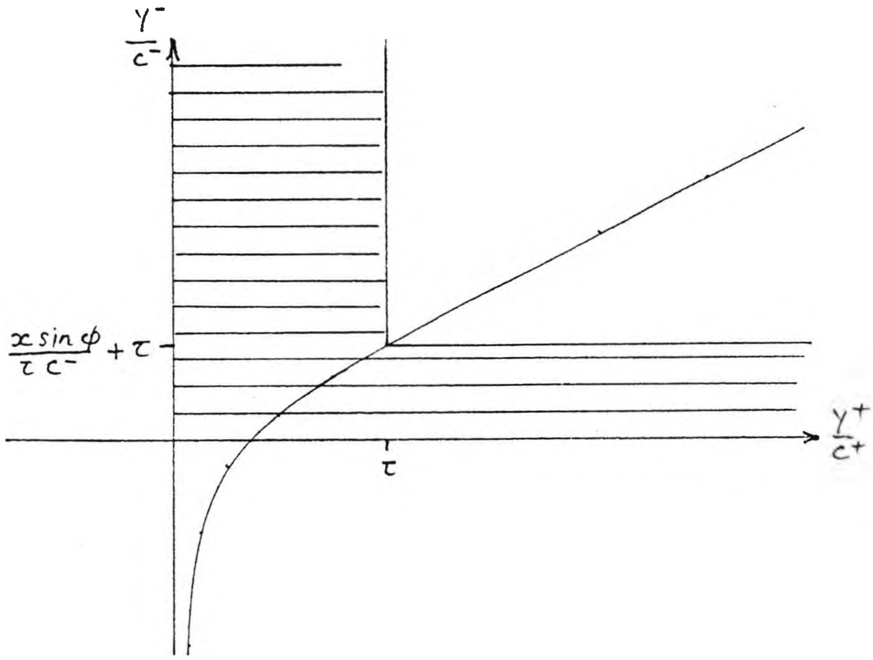

figure 4.1 Region giving a solution of $\mathbf{z} = 0$.

By (4.4.9) and (4.4.12), $t = t_L + \delta t$ on N_ε with $| \delta t | \leq \dfrac{\theta \varepsilon}{|v^T \mathbf{w}|}$. Since as we shall show $v^T \mathbf{w} < 0$ at $\mathbf{z} = 0$, we can choose δ and $\varepsilon > 0$ such that $\theta \delta < t < \theta \tau$ on N_ε, for $L \geq L_o$, if $\mathbf{y}_1$ lies inside the shaded region of figure 4.1 .

Alternatively we can choose $\varepsilon > 0$ such that $t > \theta\tau$ on N_ε, for $L \geq L_o$, if $\mathbf{y}_1$ lies outside the shaded region of figure 4.1 .

We now turn to the value of $\nu^T\mathbf{w}$ at $\mathbf{z} = 0$. Since $s_2 = O(1 / L)$, if $s_2 < 0$, we have from (4.4.10)

$$\nu^T\mathbf{w} = y^- - t_L \nu_-^T \lambda_2 \lambda_0^{-1} \mathbf{x}_0 + O(1/L) \quad ,$$

assuming $s_1 = O(1 / L)$. Substituting the unique positive root of equation (4.6.4) for t_L , we obtain

$$\nu^T\mathbf{w} = - \left\{ (y^-)^2 - 2 s_1 \sin\phi\, \nu_-^T \lambda_2 \lambda_0^{-1} \mathbf{x}_0 \right\}^{1/2} + o(1) \quad ,$$

and therefore

$$\nu^T\mathbf{w} \leq - \left\{ -4 x \sin\phi\, c^- \right\}^{1/2} \quad ,$$

for large L . In the case where $s_2 > 0$,

$$\nu^T\mathbf{w} = \nu_+^T \mathbf{y}_1 - t_L \nu_+^T \lambda_2 \lambda_0^{-1} \mathbf{x}_0 + O(1/L) \quad ,$$

which on substituting for t_L from (4.6.9) gives

$$\nu^T\mathbf{w} = - y^+ + o(1) \quad ,$$

and again $\nu^T\mathbf{w}$ will be negative for large L, if $y^+ > 0$.

Substituting $s_1 = \theta x$ into (4.3.20) and (4.3.21) and noting that $\beta_n = 0$, we obtain

$$\mu G(\tau, L) = \int_{-L\delta/\theta}^{L\delta/\theta} p(\mathbf{x}_1)\, g_L(\tau, \theta x)\, \theta\, dx + p[\mathbf{x}_0]\, O\!\left(L^2 e^{-\frac{1}{2} f^2 L^2} \right)$$

and

$$g_L(\tau, \theta x) = \int p(\mathbf{y}_1)\, J_L(\tau, \theta x, \mathbf{y}_1)\, (\nu_1^T \mathbf{y}_1)^+\, d\mathbf{y}_1 \quad , \qquad (4.6.13)$$

where we have replaced η by $\mathbf{y}_1$. Dividing the first equation by $\theta\, p[\mathbf{x}_o]$, we obtain

$$\frac{\mu G(\tau, L)}{\theta p[\mathbf{x}_0]} = \int_{-L\delta/\theta}^{L\delta/\theta} f(x)\, g_L(\tau, \theta x)\, dx + O\!\left(L^3 e^{-\frac{1}{2} f^2 L^2} \right) \qquad (4.6.14)$$

where

$$f(x) = \exp \left\{ -\frac{1}{2} \left[Q_L(\theta x) - Q_L(0) \right] \right\} . \qquad (4.6.15)$$

The previous condiderations of this section together with lemma 4.3 , ensure that $J_L(\tau, \theta x, \mathbf{y}_1) \to 1$ on the shaded region of figure 4.1 and $J_L(\tau, \theta x, \mathbf{y}_1) \to 0$ on the unshaded region, where $y^+ > 0$, if L tends to infinity with $\tau > 0$, $x > 0$ and $\mathbf{y}_1$ fixed . Letting L tend to infinity in (4.6.13) with $x > 0$, then gives

$$g_L(\tau, \theta x) \to \int p(\mathbf{y}_1)\, (\nu_+^T \mathbf{y}_1)^+\, d\mathbf{y}_1 \quad ,$$

where the domain of integration is the shaded region of figure 4.1 . Introducing y^+, y^- as variables of integration we can write the limit

$$g_L(\tau, \theta x) \to \int_0^{\tau c^+} \int_{-\infty}^{\infty} p(\mathbf{y}_1)\, y^+ \frac{dy^- dy^+}{|\sin \phi|} + \int_{\tau c^+}^{\infty} \int_{-\infty}^{\frac{x \sin \phi}{\tau} + \tau c^-} p(\mathbf{y}_1)\, y^+ \frac{dy^- dy^+}{|\sin \phi|} \quad ,$$

where c^+, c^- assume their limiting values .

Since $p(\mathbf{y}_1)$ is the density function of $N(\underline{0}, \lambda_2)$, it is a simple matter to show that

$$\int_0^{\infty} \int_{-\infty}^{\infty} p(\mathbf{y}_1)\, y^+ \frac{dy^- dy^+}{|\sin \phi|} = \frac{\sigma_+}{\sqrt{2\pi}}$$

where $\sigma_+ = [\, \nu_+^T \lambda_2 \nu_+ \,]^{1/2}$, whence, for $x > 0$, the limit becomes

$$g_L(\tau, \theta x) \to \frac{\sigma_+}{\sqrt{2\pi}} - \int_{\tau c^+}^{\infty} \int_{\frac{x \sin \phi}{\tau} + \tau c^-}^{\infty} p(\mathbf{y}_1)\, y^+ \frac{dy^- dy^+}{|\sin \phi|} \quad . \tag{4.6.16}$$

The corresponding result for $x < 0$ can be shown to be

$$g_L(\tau, \theta x) \to \frac{\sigma_-}{\sqrt{2\pi}} - \int_{\tau c^-}^{\infty} \int_{-\frac{x \sin \phi}{\tau} + \tau c^+}^{\infty} p(\mathbf{y}_1)\, y^- \frac{dy^+ dy^-}{|\sin \phi|} \quad . \tag{4.6.17}$$

where $\sigma_- = [\, \nu_-^T \lambda_2 \nu_- \,]^{1/2}$, as L tends to infinity .

Writing $\theta x = L\,\theta x / L$, since $L\theta$ tends to a positive limit , as L tends to infinity, the theory leading to (4.2.7) will give

$$Q_L(\theta x) - Q_L(0) \to \begin{cases} 2\,\alpha^+ |x| & , \; x > 0 \\ 2\,\alpha^- |x| & , \; x < 0 \end{cases} \tag{4.6.18}$$

as $L \to \infty$ with x fixed, where we have written

$$\alpha^{\pm} = k^{\pm} \lim_{L \to \infty} L\theta = k^{\pm}\left[- \frac{(2\pi)^{1/2} \sin \phi}{k^{-}\sigma_{+} + k^{+}\sigma_{-}} \right] .$$

By theorems 4.1 and 4.2, we have

$$\frac{\theta p[x_0]}{\mu} \to - \frac{2\pi\, k^{+} k^{-} \sin \phi}{(k^{-}\sigma_{+} + k^{+}\sigma_{-})^2} ,$$

as L tends to infinity . Further, by (4.6.18), as $L \to \infty$,

$$f(x) \to e^{-\alpha^{\pm} |x|}$$

and the integrand in (4.6.14) is dominated by a function of the form $Ke^{-a|x|}$, $a > 0$. Thus allowing L to tend to infinity in (4.6.14), we find

$$\lim_{L \to \infty} G(\tau, L) \to - \frac{2\pi\, k^{+} k^{-} \sin \phi}{(k^{-}\sigma_{+} + k^{+}\sigma_{-})^2} \left\{ \int_0^{\infty} e^{-\alpha^{+}x}\, g_{+}(x)\, dx + \int_{-\infty}^{0} e^{-\alpha^{-}x}\, g_{-}(x)\, dx \right\} ,$$

where g_{+}, g_{-} denote the limit functions of (4.6.16) and (4.6.17) . After some minor simplifications involving replacing x by $-|x| \sin \phi$ as variable of integration, the limit can be written

$$G(\tau) = \lim_{L \to \infty} G(\tau, L) = 1 - \frac{2\pi\, [\, I_{+}(\tau) + I_{-}(\tau)\,]}{k^{+} k^{-} \left[\frac{\sigma_{+}}{k^{+}} + \frac{\sigma_{-}}{k^{-}} \right]^2} , \qquad (4.6.19)$$

where

$$\left.\begin{aligned} I_{+}(\tau) &= \int_0^{\infty} e^{-\alpha^{+}x} \int_{\tau c^{+}}^{\infty} \int_{\tau c^{-} - \frac{x}{\tau}}^{\infty} p(y_1)\, y^{+} \frac{dy^{-}\, dy^{+}}{|\sin \phi|}\, dx , \\ I_{-}(\tau) &= \int_0^{\infty} e^{-\alpha^{-}x} \int_{\tau c^{-}}^{\infty} \int_{\tau c^{+} - \frac{x}{\tau}}^{\infty} p(y_1)\, y^{-} \frac{dy^{+}\, dy^{-}}{|\sin \phi|}\, dx . \end{aligned}\right\} \qquad (4.6.20)$$

In the simplification the definitions of the α's have been slightly revised . For convenience we give the definitions of the various constants involved in (4.6.19) and (4.6.20) below .

$$
\left.\begin{aligned}
k^{\pm} &= \pm\,\tau_{\pm}^{T}\,\lambda_0^{-1}\mathbf{x}(0,1) \\
\alpha^{\pm} &= (2\pi)^{1/2}\,k^{\pm}[\,k^{-}\sigma_{+} + k^{+}\sigma_{-}\,]^{-1} \\
\frac{c^{+}}{\sigma_{+}} &= \left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{2}(1-\rho)\,\sigma_{-}\,\alpha^{+} \\
\frac{c^{-}}{\sigma_{-}} &= \left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{2}(1-\rho)\,\sigma_{+}\,\alpha^{-} \\
\rho &= v_{-}^{T}\lambda_2 v_{+} \,/\, (\sigma_{+}\sigma_{-}) \quad .
\end{aligned}\right\} \qquad (4.6.21)
$$

By assumption c^{+}, c^{-} are positive constants, hence from (4.6.20) $I_{+}(\tau)$ and $I_{-}(\tau)$ tend to zero, as $\tau \to \infty$, and $G(\tau)$ of (4.6.19) is a distribution function. It therefore follows from lemma 4.1, that $G(\tau)$ is the asymptotic distribution function of the length of an excursion, measured in units of the mean θ.

The combined results of this chapter serve to prove the following theorem .

Theorem 4.3. Under the conditions on the stationary Gaussian process $\mathbf{X}(t)$ and the family of boundaries D_L, $L > 0$, described in the introduction to chapter 4 , there exists a limiting distribution function $G(t) = \lim F_1(\theta t)$, for the duration of an excursion outside D_L, scaled by the mean duration $\theta = \int t\, dF_1(t)$.

If $p[\mathbf{x}]$, the density of $\mathbf{X}(0)$, has a unique maximum on D_L at $\mathbf{x}_o$, the limiting distribution function is given below depending on the status of $\mathbf{x}_o$ and on λ_1, the first derivative of the covariance matrix of $\mathbf{X}$ at the origin .

(i) If $\mathbf{x}_o$ is a regular point of D_L, then

$$G(\tau) = 1 - e^{-\pi\tau^2/4} \quad .$$

(ii) If $\mathbf{x}_o$ is a vertex of D_L and $\lambda_1 \neq 0$, then

$$G(\tau) = 1 - e^{-\tau} \quad .$$

(iii) If $\mathbf{x}_0$ is a vertex of D_L and $\lambda_1 = 0$, then

$$G(\tau) = 1 - \frac{2\pi \left[I_+(\tau) + I_-(\tau) \right]}{k^+ k^- \left| \qquad \right|^2} \quad ,$$

where the functions I_+, I_- and associated constants are defined by (4.6.20) and (4.6.21) .

5 APPLICATIONS AND CONCLUSIONS

5.1 Introduction

In preceeding chapters, we have explored a number of aspects of the crossings of boundaries by vector Gaussian processes. In chapters 3 and 4 we restricted the treatment to 2-dimensions in order to simplify the notation. The working assumptions we have made in previous chapters will be discussed later in this chapter, along with our conclusions on the scope of our results in relation to the current literature. However, before we address these matters, we will illustrate the power of our results by applying some of them to the problem of nuisance disconnects, described in chapter 1.

This is not the place for a full discussion of so technical a subject. Certainly a full discussion of the relative performance of different monitoring regimes is beyond the scope of the present study, since 'nuisance disconnects' form only one aspect of the monitor's performance. The performance of monitors subject to actual failures is completely outside the context of this report, since we assume throughout this thesis that the processes are stationary and correspondingly that the system remains working normally.However, as well as comparing different monitoring regimes in relation to the numbers of nuisance disconnects and the magnitudes of the transients, we can consider the effects, on the expected number of nuisance disconnects, of constant differences in the response of the sensors, for example the effect of bias or of an increase in measurement variance for one or more sensors.

5.2 Applications to Nuisance Disconnects

5.2.1 Computation of μ and θ for polygonal boundaries

Let $\mathbf{x}_1 = \mathbf{x}(s_1)$ and $\mathbf{x}_2 = \mathbf{x}(s_2)$, ($s_1 < s_2$) , be the vertices of a side of a polygonal boundary. The contribution of the side to $\mu = E(U) = M_1$ is given by

$$\Delta\mu = \int_{s_1}^{s_2} \int p\,(\mathbf{x}(s), \mathbf{y}\,)\,(\mathbf{n}^T\mathbf{y})^+\, d\mathbf{y}\; ds \quad ,$$

where $\mathbf{n}$ is the unit outward drawn normal to the boundary at $\mathbf{x}(s)$, on using theorem 2.3 and observing that p $(\mathbf{x}, \mathbf{y})$ is independent of t , for k = 1. Writing p $(\mathbf{x}, \mathbf{y})$ = p $(\mathbf{y} \mid \mathbf{x})$ p$(\mathbf{x})$, where p$(\mathbf{x})$ is the marginal density of $\mathbf{X}(0)$ and p $(\mathbf{y} \mid \mathbf{x})$ is the conditional density of $\mathbf{X}'(0)$, given $\mathbf{X}(0) = \mathbf{x}$ (see appendix B, section e), and integrating over $\mathbf{y}$, we obtain

$$\Delta\mu = \int_{s_1}^{s_2} \left\{ \sigma_n\, \phi\left(\frac{\beta_n}{\sigma_n}\right) + \beta_n\, \Phi\left(\frac{\beta_n}{\sigma_n}\right) \right\} p[\mathbf{x}(s)]\; ds \quad ,$$

where $\beta_n = E(\,\mathbf{n}^T\mathbf{X}'(0) \mid \mathbf{X}(0)\,) = -\,\mathbf{n}^T R'(0)\, R(0)^{-1}\mathbf{x}$,

and $\sigma_n^2 = V(\,\mathbf{n}^T\mathbf{X}'(0) \mid \mathbf{X}(0)\,) = \mathbf{n}^T(\,-\,R''(0) + R'(0)\, R(0)^{-1}R'(0)\,)\,\mathbf{n}$.

Writing $\mathbf{x} = p\,\mathbf{n} + s\,\mathbf{t}$, where $\mathbf{t} = (\,\mathbf{x}_2 - \mathbf{x}_1\,) / (\,s_1 - s_2\,)$ is the constant unit tangent of the side and p is the perpendicular distance of the side from the origin, $R(0) = \lambda_0$, $R'(0) = 0$, $-\,R''(0) = \lambda_2$ and substituting $\beta_n = 0$, $\sigma_n^2 = \mathbf{n}^T\lambda_2\,\mathbf{n}$, we obtain

$$\Delta\mu = \left[\frac{1}{2\pi} \left(\frac{\mathbf{n}^T\lambda_2\,\mathbf{n}}{\mathbf{n}^T\lambda_0\,\mathbf{n}} \right)^{1/2} \Phi\left[\frac{\mathbf{t}^T\lambda_0^{-1}\mathbf{x}}{(\mathbf{t}^T\lambda_0^{-1}\mathbf{t})^{1/2}} \right] \exp\left\{ -\,p^2 / (2\mathbf{n}^T\lambda_0\mathbf{n}) \right\} \right]_{\mathbf{x}_1}^{\mathbf{x}_2} \tag{5.2.1}$$

on integrating over s.

From equation (4.2.8) , the probability that the process lies outside the acceptance region Γ, is given by

$$P\{\, \mathbf{X}(t) \in \Gamma' \,\} = \int_{\partial\Gamma} \frac{1}{Q}\, p[\,\mathbf{x}(s)\,]\, |\mathbf{n}^T\mathbf{x}|\, ds \tag{5.2.2}$$

where $Q = \mathbf{x}^T\lambda_0^{-1}\mathbf{x}$, and $p[\mathbf{x}] = (2\pi)^{-1} \mid \lambda_0 \mid^{-1/2} e^{-\,Q/2}$. Thus we can calculate θ, the mean duration of an excursion outside Γ, from $\theta = \frac{1}{\mu}\, P\{\, \mathbf{X}(t\,) \in \Gamma' \,\}$.

The Fortran programme EXIT.FOR calculates μ and θ for polygonal boundaries $\partial\Gamma$ using (5.2.1) and (5.2.2). For a polygon with specified vertices, the programme calculates the contributions of each side of $\partial\Gamma$ to μ and $P\{ \mathbf{X}(t) \in \Gamma' \}$ and sums these contributions over the sides of the polygon. The integration in (5.2.2) is evaluated numerically using Simpson's rule, with a given number, M, of steps for each side. The standard normal distribution function $\Phi(.)$ in (5.2.1) is obtained from the NAG subroutine S15ABF. The programme contains facilities which allow us to increase or decrease the size of the boundary and to translate and rotate the boundary in the plane.

5.2.2 Comparison of monitoring regimes

In this section we compare the two monitoring regimes, introduced in chapter 1, in terms of the rate of nuisance disconnects and the level of transients. As we shall see, above a certain limit, the more we relax the tolerances the fewer the nuisance disconnects but only at the expense of larger transients, on average. Thus, in practice, one consraint on our increasing the tolerances in the monitor and thereby reducing the incidence of nuisance disconnects, is the ability of the system to withstand transients above a certain size.

As well as the 12-sided star and hexagonal acceptance regions which characterise the two monitoring regimes of chapter 1, we include the circular region of the χ^2 process as a further comparitor. We assume here that the noise processes in the three sensors are identical and independent . Thus , from (1.3.4), we may write $R(0) = I$, $R'(0) = 0$, and $R''(0) = -\omega^2 I$, where I is the two-dimensional identity matrix . To aid comparison, we have chosen the basic acceptance regions of each type, so that , in the notation of chapter 4, $L = 1$ corresponds to the boundary whose minimum distance from the origin is unity. It follows that L is the minimum distance of D_L from the origin, which has been used by Hasofer and Lind (1974) as an index of the reliability of the corresponding system. The following diagrams show graphs of E(U) against L, for the three types of regime, for $\omega = 3$.

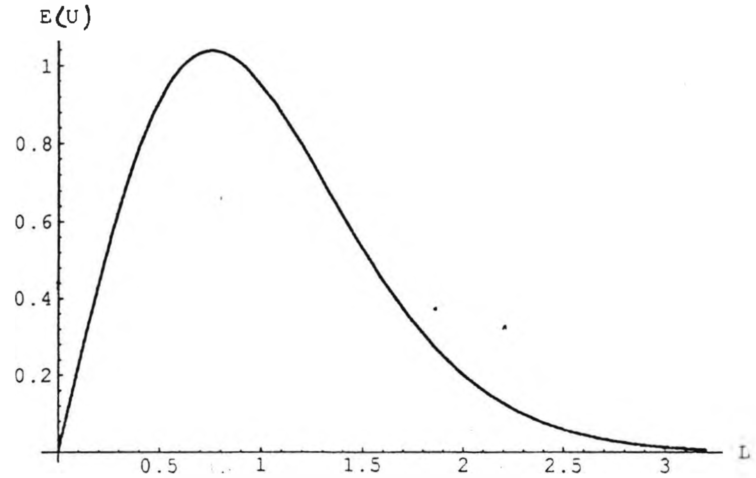

figure 5.1 Graph of E(U) v L for the 12-sided star

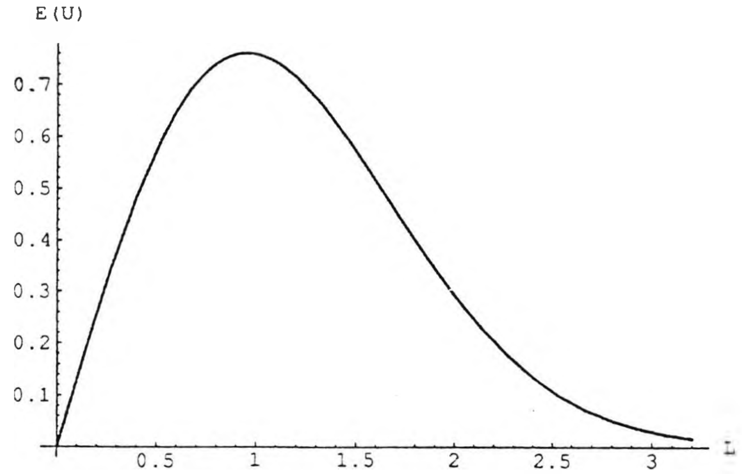

figure 5.2 Graph of E(U) v L for the Hexagon

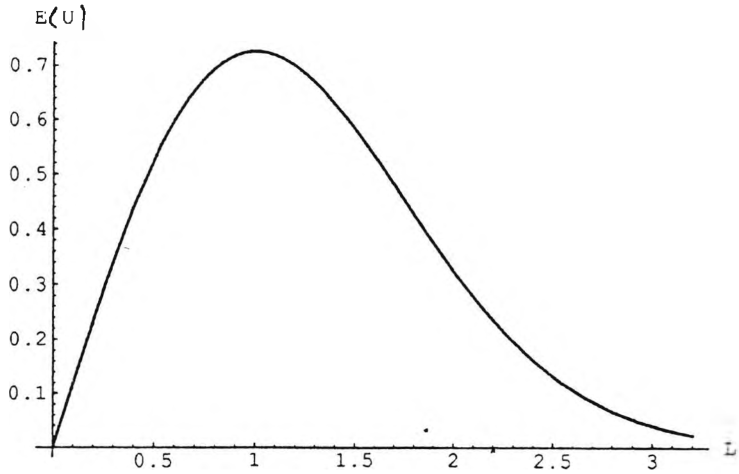

figure 5.3 Graph of E(U) v L for the Circle

The graph in figure 5.3 was obtained using the formula of Sharpe (1978), quoted in section 1.6 of this report, with $u = L^2$, $\lambda_2 = \omega^2$ and $p = 2$, which yields

$$E(U) = \frac{\omega}{\sqrt{2\pi}} L e^{-L^2/2} \quad . \tag{5.2.3}$$

What is most noticeable about the three graphs of E(U) against L is their similar general shape, in spite of the very different boundaries of the three regions. In each case E(U) increases with L, for small values of L, up to some maximum and subsequently steadily decreases as L increases beyond the maximum . The reasons for the general shape of these graphs are not difficult to see. When the region Γ is small, $P\{\mathbf{X}(t) \in \Gamma\} = O(L^2)$ and the process spends most of the time outside Γ. The process has a small probability of hitting Γ and spends little time in Γ before it exits the region. Thus, in this range, an increase in L, leading to an increase in the size of Γ, will improve the chances of the process hitting Γ and therefore cause μ to increase. On the other hand, when Γ is large, the process spends a large proportion of the time inside Γ and has difficulty in reaching the boundary . Thus, for large L, a decrease in L will improve the chances of the process leaving Γ and will cause an increase in μ .

In practical applications we may want μ to be 10^{-5} or smaller, implying a large acceptance region and L in the asymptotic region . Below we give asymptotic formulae for $\mu = E(U)$, for large L, when $R(0) = I$ and $R''(0) = -\omega^2 I$.

12-sided star

$$\mu \sim \frac{12\,\omega}{\pi\sqrt{2\pi}} L^{-1} e^{-L^2/2} \quad . \tag{5.2.4}$$

Hexagon

$$\mu \sim \frac{3\,\omega}{\pi} e^{-L^2/2} \quad . \tag{5.2.5}$$

Circle

$$\mu \sim \frac{\omega}{\sqrt{2\pi}} L\, e^{-L^2/2} \quad . \tag{5.2.6}$$

Formulae (5.2.4) and (5.2.5) follow by the application of theorem 1, chapter 4, on multiplying by 6 to take account of the 6 minima in each case [Breitung, 1988]. The formula (5.2.6) for the circle is the exact formula of Sharpe. In this case there are no isolated minima of Q on D_L, since Q is constant on the boundary, and theorem 1 of chapter 4 does not apply. We discuss the question of multiple minima and the relation of our work to the work of Breitung (1988) in section 5.3 .

For a given value of L, the transients from the 12-sided star shaped region are distributed in the range from $L / (2\sqrt{3})$ to $\sqrt{3}\, L / 4$, whereas the transients from the hexagonal region are a constant $L / (2\sqrt{3})$. If we compare the average rate of nuisance disconnects (exits) from these two regions using the asymptotic expressions (5.2.4) and (5.2.5), we see that the ratio of the expected number of nuisance disconnects, for the 12-sided star to that for the hexagon, decreases like L^{-1}, for large L. Alternatively, if we compare E(U) for the 12-sided star at L with E(U) for the hexagon at k L, for fixed $k > 1$, then the ratio of the expected rate of nuisance disconnects increases exponentially with L. In other words, the hexagonal region will generate far fewer nuisance disconnects for a transient level only a little above the minimum for the 12-sided star.

Another way of considering the trade-off between numbers of nuisance disconnects and levels of transients is to compare E(U) for regions giving the same mean size of transients. In the following table, we compute the mean transient for the 12-sided star and give the corresponding E(U), for a range of values of L. We also compute the value of L, such that the constant transient of the hexagonal region should equal the mean transient of the 12-sided star shaped region, and quote the corresponding E(U) for the hexagonal region.

Table 5.1 **Comparison of rates of nuisance disconnects for boundaries with the same mean transient**

12-sided star			Hexagon	
L	E(U)	Mean Transient	L	E(U)
1.0	0.9519042	0.34939	1.2103	0.7096878
2.0	0.2011057	0.65096	2.2550	0.1818855
3.0	0.0130972	0.92932	3.2193	0.0150767
4.0	0.0003231	1.20852	4.1864	0.0004411
5.0	0.0000030	1.48993	5.1613	0.0000047
6.0	0.0000000	1.77288	6.1414	0.0000000

For large L, it turns out that the mean transient for the 12-sided star is given by $(L + L^{-1}) / (2\sqrt{3}) + O(L^{-3})$. Hence an hexagonal region with size parameter $L + L^{-1}$ will have the same mean transient, asymptotically, as the 12-sided star of size L. Substituting for L into (5.2.5) , we find the following asymptotic expression for μ

$$\mu \sim \frac{3\omega}{\pi} e^{-1} e^{-L^2/2} \tag{5.2.7}$$

Thus the asymptotic effect of equating the mean transients is to reduce μ by a factor e^{-1}. In other words, for L large enough, the 12-sided star will have a smaller rate of nuisance disconnects than the hexagonal region with equal mean transient . Thus, although the 12-sided star will, on average, give fewer nuisance disconnects, if L is large enough, the hexagonal region will ensure the magnitude of any transients is constant , thereby eliminating the possibility of a catastrophically large transient .

5.2.3 Effect of bias in one sensor

In this section we suppose that one of the sensors of section 1.3 suffers a bias w and write, without loss of generality, $E(W_1) = w$, $E(W_2) = E(W_3) = 0$. By (1.3.2) and (1.3.3), it follows that the mean of the two-dimensional process $\mathbf{X}(t)$ becomes $(-\frac{2}{\sqrt{3}}w, 0)$, leaving the covariance structure of the process unchanged. The effect of such a shift in the mean of the process can, of course, be modelled using the zero mean process by applying an equal but opposite shift $b = \frac{2}{\sqrt{3}}w$ to the boundary.

Figures 5.4 and 5.5 show the effect of such a bias on the rate of nuisance disconnects for the 12-sided star and hexagonal boundaries, for small values of L and a range of values of the bias b.

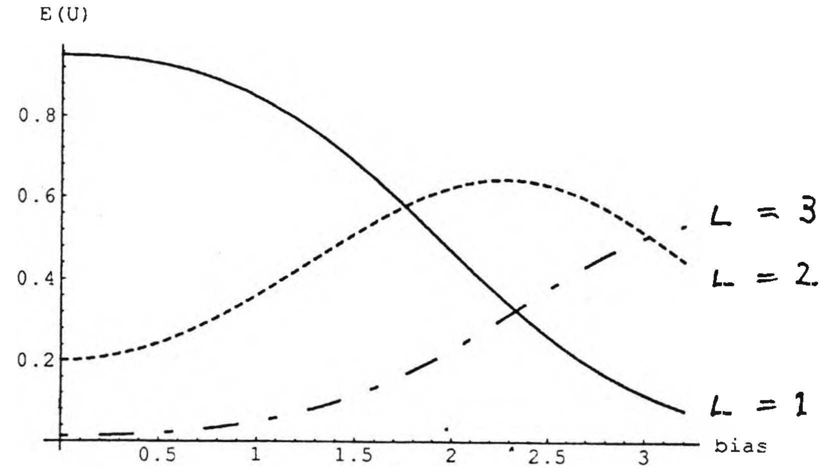

figure 5.4 Graph of E(U) v bias for the 12-sided star.

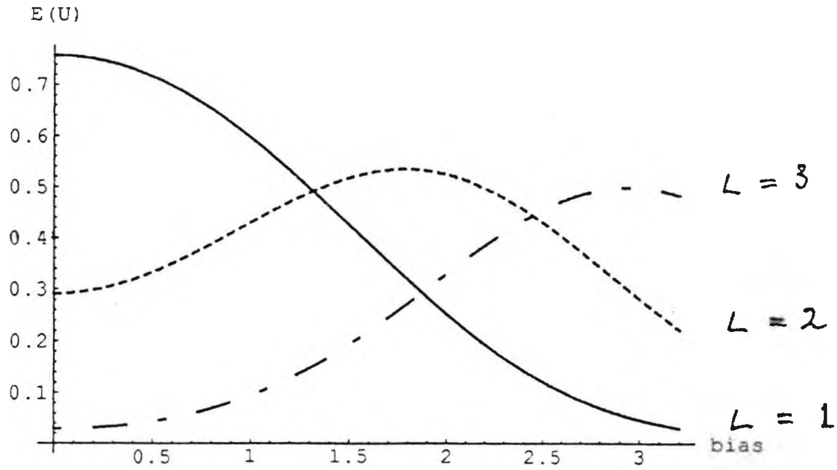

figure 5.5 Graph of E(U) v bias for the Hexagon

As we observe from the graphs of figures 5.4 and 5.5, E(U) has a stationary point at $b = 0$ for the two regions and for all values of L, as was to be expected on the grounds of symmetry. The stationary point is a local maximum when $L = 1$, but is a minimum when $L = 2, 3$. We also notice the development of a maximum at or near the point $b = L$, for the larger values of L. It is of interest to note, that Veneziano et. al. (1977) give a formula for the rate of exits from a circular region whose centre is displaced from the origin . Using this formula, we find that $b = 0$ is a maximum for E(U) if the radius $L < \sqrt{2}$, and is a minimum if $L > \sqrt{2}$.

Clearly for small regions, any gain in the number of exits through one part of the boundary, due to the increasing proximity of the origin, is more than compensated for by the loss of exits over the increasingly large part of the boundary which is receding from the origin . On the other hand, when the region is large and E(U) is small at $b = 0$, the number of exits will increase, as the origin approaches a side or vertex of the boundary. Moreover, if the origin approaches a side of a large polygonal boundary, we might expect that the rate of exits would behave like the rate of up-crossings for a univariate Gaussian process, namely

$$\mu \equiv \frac{\omega}{2\pi} e^{-\frac{1}{2}(L-b)^2}$$

for large L and $b \approx L$.

Certainly this expectation is born out for the hexagonal region. The situation regarding the 12-sided star shaped region is slightly different, in that, as we continue to displace the boundary, the origin approaches, or is approached by, an internal vertex. Estimating the asymptotic behaviour of E(U) is more difficult in this case. However, as we see from figure 5.4, the maximum, for large L, occurs after the origin has passed the vertex, ie when $b > L$.

For our present application, interest is likely to centre on the effect of bias when L is large. From what we have observed, a small bias will not increase the rate of nuisance disconnects drastically. However, it is the relative rather than the absolute increase in E(U) that is relevant and E(U) will itself be small in any application. The development of an asymptotic formula, giving E(U) for small values of b and large L, would be of considerable utility.

5.2.4 Increased measurement variance

We now suppose that the variance of sensor 1 is, for whatever reason, twice as large as that of the other two sensors whose measurement variances remain equal and unchanged. Thus we may assume that the covariance matrix of the noise process **W** , at the origin, is given by L(0) = diag (1, 0.5, 0.5) and in consequence of equations (1.3.4) we obtain R(0) = diag ($\frac{5}{3}$, 1) . Although it does not follow that the second derivative of the covariance matrix at the origin, R"(0), of necessity remains proportional to R(0), if we assume that the spectral distribution of W_1 remains unaltered when the variance is increased, then the diagonal elements of R"(0) are modified in direct proportion to those of R(0), and we put R"(0) = diag (15, 9). In the two following figures, we compare the rates of nuisance disconnects for the process with increased measurement variance and the basic process, over a range of values of L, for the 12-sided star and the hexagon.

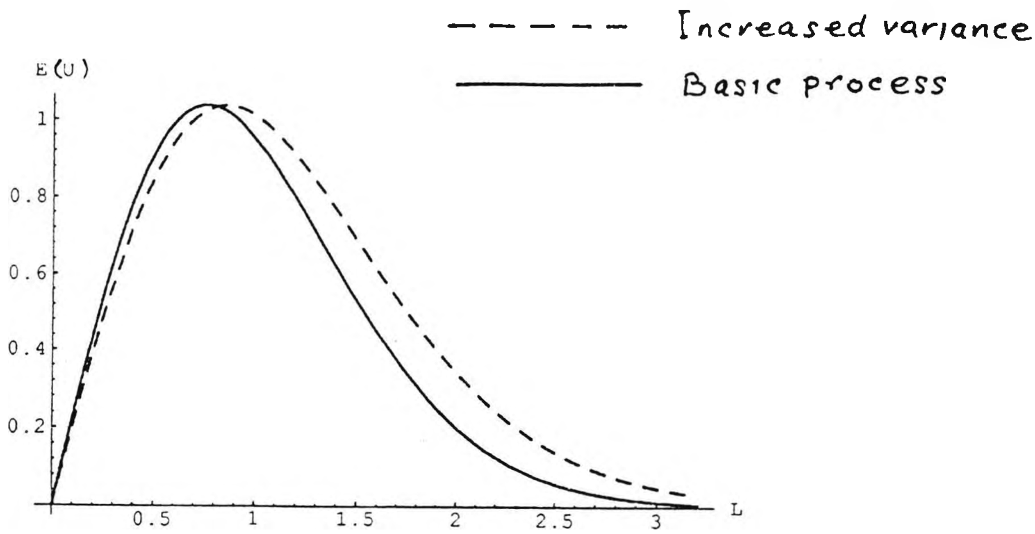

figure 5.6 The effect of increased measurement variance on E(U) ;12-sided star

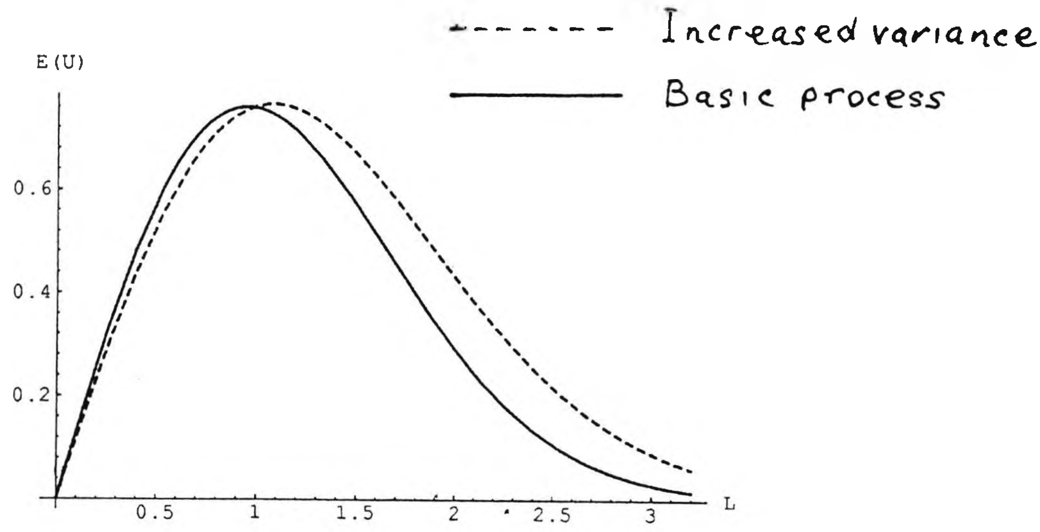

figure 5.7 The effect of increased measurement variance on E(U); Hexagon

In both figures we see that the effect of the increase in measurement variance is to displace the maxima of the graphs to the right, leading to an increase in the rate of nuisance disconnects for large L. This effect can be further explored using the asymptotic results of theorem 1, chapter 4. Below we give asymptotic expressions for the ratios of the rate of nuisance disconnects for the process with increased variance, μ_1, and that for the base process, μ_0, for our two polygonal regions.

Hexagon $$\frac{\mu_1}{\mu_0} \cong \frac{\sqrt{2}}{3} e^{L^2/5}$$

12-sided star $$\frac{\mu_1}{\mu_0} \cong \frac{\sqrt{10}}{6} e^{L^2/5}$$

For L = 3, these formulae show that , for both regions, the ratio μ_1 / μ_0 is roughly equal to 3, as we can verify from the graphs of figures 5.6 and 5.7. This disparity in the rate of nuisance disconnects clearly increases with L, thus for L = 5 the ratio is of the order of 10^2. Furthermore, the assumptions we made concerning R"(0) do not crucially affect the conclusions, since a different choice of R''(0) would only cause μ_1 to be multiplied by a constant factor ; the exponential factor in the ratios is solely determined by R(0).

5.3 Conclusions

If we are to construct a theory of the crossings of boundaries by Gaussian vector stochastic processes, we must either construct a direct theory of vector Gaussian processes or alternatively construct a theory of non-Gaussian univariate processes and their level crossings . In chapter 2 we have given a proof of Belyaev's formula for the factorial moments of the number of exits from a region, by a vector Gaussian process, under very general conditions on the process and the boundary of the region. There is no direct proof in the literature. Belyaev (1968) gave his celebrated formula without proof, under very different conditions to ours. The alternative approach has been adopted by Marcus (1977), who has given formulae for the moments of the number of crossings of levels by a non-Gaussian process, under extremely general sufficient conditions . Lindgren (1980b) has used the results of Marcus (1977) to give a formal proof of the formula for the expected number of exits .

The merit of our approach is its applicability to all vector stationary Gaussian processes for which the matrix of second order spectral moments exists. Also that it makes explicit any assumptions about the boundary $\partial\Gamma$. Our approach is also easily generalisable to non-stationary Gaussian processes and to limited classes of non-Gaussian processes . Although we assume throughout our processes are Gaussian, by different choices of boundary we can obtain results for the crossings of a wide range of univariate processes. For example, considering the exits of an isotropic Gaussian process (Veneziano et. al.

(1977)) from a circular region of radius u, is equivalent to considering the up-crossings of the level u^2 by the χ^2 process $Y = \mathbf{X}^T\mathbf{X}$. More generally, if we consider the exits from an n-dimensional region $\Gamma = \{ \mathbf{x} \in R^n ; \ g(x) < 0 \}$ by a stationary Gaussian process $\mathbf{X}(t)$, this is equivalent to studying the up-crossings of zero by the non-Gaussian process $Y = g[\mathbf{X}(t)]$.

Our sufficient condition on the covariance R(.), for the existence of the variance of the number of exits, is a natural generalisation of the sufficient condition (1.5.5) of Leadbetter and Cryer (1965). The restriction to two-dimensional regions in chapter 3 is mainly for ease of presentation, the results are easily generalisable to p-dimensions. It is worth noting, that since our sufficent condition does not depend on the shape of the boundary, the condition applies to the crossing variance of the χ^2 - process and to other processes which can be represented as a function of a multivariate Gaussian process.

Belyaev (1966) and Cuzick (1975) have given sufficient and necessary conditions for a stationary Gaussian process to posess finite crossing moments of all orders. In retrospect the result of Leadbetter and Cryer (1965) can be characterised as the application of Belyaev's general sufficient condition to the second order moments. The sufficient condition on the covariance function (1.5.6), for a real stationary Gaussian process to have finite crossing moments of all orders [Cusick (1975)], is more restrictive than the sufficient condition given by Leadbetter and Cryer (1965) for finite crossing variance. As Geman (1972) has shown Leadbetter and Cryer's sufficient condition to be necessary, we might reasonably entertain the hope that our sufficient condition will also prove to be necessary.

It is well known that the duration of an excursion, by a stationary Gaussian process above a high level, has an asymptotic Rayleigh distribution . We have shown, in chapter 4, that the duration of an excursion above a large two-dimensional boundary can exhibit a wider range of asymptotic behaviour . Although the Rayleigh distribution will always occur if all the points of the boundary are regular, should the boundary contain vertices, where two regular elements meet, it is possible for two other types of asymptotic distribution to occur .

The two other types of asymptotic distribution occur when p[**x**] has a unique maximum at a vertex, $\mathbf{x}_0$, of D_L . The nature of the distribution depends on whether the skew-symmetric matrix $\lambda_1 = 0$ or not . If $\lambda_1 \neq 0$, the asymptotic distribution is exponential with unit mean, whereas if $\lambda_1 = 0$, the asymptotic distribution is determined by the limiting distribution function of (4.6.19) and the complicated integrals of (4.6.20). The latter distribution also can be shown to have a unit mean, as expected, since we have used the mean θ to scale the duration.

In obtaining these asymptotic distributions, we have derived asymptotic expressions for $\mu = E(U)$, $P\{\mathbf{X}(t) \in \Gamma'\}$ and θ, for the three cases depending on whether the maximum $\mathbf{x}_0$ is a regular point or a vertex and whether $\lambda_1 = 0$ or $\lambda_1 \neq 0$. Table 5.2 below compares the values of μ obtained from the asymptotic formulae (5.2.4) and (5.2.5), with $\omega = 3$, and the corresponding exact values obtained from the Fortran program EXIT.FOR .

table 5.2 **Comparison of exact and asymptotic values for μ**

	12-sided Star		Hexagon	
L	Asymptotic	Exact	Asymptotic	Exact
2.0	0.3093454	0.2011057	0.3877070	0.2914731
2.5	0.0803438	0.0580430	0.1258700	0.1071262
3.0	0.0169284	0.0130972	0.0318249	0.0291750
3.5	0.0028572	0.0023216	0.0062667	0.0059953
4.0	0.0003834	0.0003231	0.0009610	0.0009409
4.5	0.0000407	0.0000353	0.0001148	0.0001137
5.0	3.41×10^{-6}	3.02×10^{-6}	1.07×10^{-5}	1.06×10^{-5}
5.5	2.24×10^{-7}	2.02×10^{-7}	7.73×10^{-7}	7.72×10^{-7}

Unlike the situation of chapter 3, the working assumptions of chapter 4 , two-dimensional regions and unique maximum of p[x] on D_L , are important to the development of the results. It is not immediately clear how the relaxation of these assumptions would affect the results of chapter 4. Even cursory consideration suffices to provide hints of the likely growth in the variety of pathological behaviour with increasing dimension. Equally the admission of multiple maxima of p[**x**] could in principle lead to complicated mixtures of limiting distributions. Elucidation of these matters must await further research.

Breitung (1988) has obtained asymptotic crossing rates for stationary Gaussian vector processes using an implicit definition of the safe region $S = \{ \mathbf{x} \in R^n \; ; \; g(\mathbf{x}) > 0 \}$ in terms of a real function g(.) on R^n . It is assumed that $\min | \mathbf{x} | = 1$, on the boundary $G = \{ \mathbf{x} \in R^n \; ; \; g(\mathbf{x}) = 0 \}$, and that this minimum distance from the origin is attained at only a finite number of points.

A sequence of surfaces similar to G is introduced by $G(\beta) = \{\mathbf{x} ; g(\beta^{-1}\mathbf{x}) = 0\}$, so that $\beta \geq 1$ is the Hasofer-Lind index of reliability, and Breitung derives an asymptotic formula for the expected number of crossings, $C(\beta)$, of $G(\beta)$ by a stationary Gaussian process $\mathbf{X}(t)$, standardised so that $R(0) = I$.

The formula given in corollary 4.3 of Breitung (1988) for the expected number of crossings of $G(\beta)$ is identical to our result of theorem 1(i), chapter 4, assuming as we do only a single minimum for Q. Of course, Breitung's result only applies in the regular case, since his differentiability conditions on g(.) preclude the existence of vertices on the boundary surfaces.

As Lindgren (1980a) and others have made clear, many safe regions which occur in practical problems do have vertices in their boundaries. Whether due to vertices, symmetry or accident, we have found many situations where Q has assumed more than one minimum on the boundary of Γ. The analysis of Breitung (1988) serves to show that the asymptotic behaviour of μ is the sum of contributions from each of the minima. We have made use of this observation in deriving asymptotic formulae for the 12-sided star and the hexagon which, for reasons of symmetry, have 6 minima each.

As we have seen already, in the case of a circular boundary, Q is constant on the boundary if the process is isotropic, and neither the theory of Breitung nor the theory of chapter 4 can provide asymptotic expressions for μ. However, it is easy to show that if Q is a minimum on an interval of D_L then

$\mu = O(L\, e^{-\frac{1}{2}L^2})$, for large L.

5.4 Future Developments

1. Generalise the results of chapter 2 to non-stationary Gaussian processes and to stationary non-Gaussian processes. An intriguing question for Gaussian

processes, is whether $E(C) = \infty$ necessarily implies that $C = \infty$, with probability one?

2. Extend the work on the variance of the number of exits to p-dimensional regions and explore the necessity of our 2-dimensional sufficient condition .

3. The work on asymptotic distributions of excursions needs extending, both to relax some of the working assumptions and to generalise the results to higher dimensional regions. With the move to multivariate processes, another form of asymptotic result becomes possible, not encountered with univariate Gaussian processes. As we have seen, as $L \rightarrow 0$, $E(U)$ becomes small and the excursions of the process inside Γ shorten in duration . Thus as well as asymptotic distributions for excursions outside large boundaries, we have now the possibility of asymptotic distributions for excursions inside small boundaries . The corresponding asymptotic distributions for the χ^2 - process have been discussed by Aronowich & Adler (1986).

Appendix A

1. Distribution of $\mathbf{X}(t)$, conditional on $\mathbf{X}(0) = \mathbf{x}_1$.

The joint distribution of $\mathbf{X}(0)$ and $\mathbf{X}(t)$ is normal with zero means and covariance matrix

$$A = \begin{bmatrix} R(0) & R(t) \\ R(t)^T & R(0) \end{bmatrix}$$

It can be shown that $\mathbf{X}(t) - R(t)^T R(0)^{-1}\mathbf{X}(0)$ has covariance $R(0) - R(t)^T R(0)^{-1} R(t)$ and is uncorrelated with $\mathbf{X}(0)$. By the normality of the distributions, it follows that the conditional distrbution of $\mathbf{X}(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, is

$$N[\, R(t)^T R(0)^{-1}\mathbf{x}_1 ,\; R(0) - R(t)^T R(0)^{-1} R(t)\,] .$$

Since, for small t ,

$$R(t) = \lambda_0 + \lambda_1 t - \frac{1}{2}\lambda_2 t^2 + \theta(t) \tag{1.1}$$

a little algebra gives

$$S(t) = R(0) - R(t)^T R(0)^{-1} R(t) = \kappa_2 t^2 - \theta - \theta^T + O(t^3) , \tag{1.2}$$

where $\kappa_2 = \lambda_2 + \lambda_1 \lambda_o^{-1} \lambda_1$.

Using the matrix identity

$$(A + B)^{-1} = A^{-1} - (A + B)^{-1} B A^{-1} \tag{1.3}$$

we find

$$\begin{aligned} S^{-1} &= \frac{1}{t^2} \kappa_2^{-1} + S^{-1} (\theta + \theta^T + O(t^3)) \frac{1}{t^2} \kappa_2^{-1} \\ &= \frac{1}{t^2} \kappa_2^{-1} (1 + o(1)) . \end{aligned}$$

Thus for small t , we have

$$p_t (\mathbf{x}_2 \mid \mathbf{x}_1) = \frac{|\kappa_2|^{-1/2}}{2\pi} t^{-2} \exp\left\{ -\frac{1}{2} (\mathbf{x}_2 - \mathbf{x}_1)^T \frac{1}{t^2} \kappa_2^{-1} (\mathbf{x}_2 - \mathbf{x}_1) \right\} (1 + o(1)) , \tag{1.4}$$

since $R(t)^T R(0)^{-1} \mathbf{x}_1 = \mathbf{x}_1 + O(t)$.

It follows from the above that, conditional on $\mathbf{X}(0) = \mathbf{x}_1$, the distribution of

$[\,\mathbf{X}(t) - \mathbf{X}(0)\,]/t$ is

$$N\left[(-\lambda_1\lambda_0^{-1} - \frac{1}{2}\lambda_2\lambda_0^{-1}t + \theta^T\lambda_0^{-1}t^{-1})\mathbf{x}_1 \;,\; \kappa_2 - \frac{\theta + \theta^T}{t^2}\right]. \tag{1.5}$$

2. <u>Conditional Distribution of $\nu_1^T\mathbf{X}'(t_1)$ and $\nu_2^T\mathbf{X}'(t_2)$, given $\mathbf{X}(t_1) = \mathbf{x}_1$, $\mathbf{X}(t_2) = \mathbf{x}_2$</u>.

The joint distribution of $\mathbf{X}(t_1)$, $\mathbf{X}(t_2)$, $\mathbf{X}'(t_1)$, $\mathbf{X}'(t_2)$ is normal with zero means and covariance matrix Λ, given by

$$\Lambda = \begin{bmatrix} R(0) & R(t) & R'(0) & R'(t) \\ R(t)^T & R(0) & -R'(t)^T & R'(0) \\ -R'(0) & -R'(t) & -R''(0) & -R''(t) \\ R'(t)^T & -R'(0) & -R''(t)^T & -R''(0) \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{2.1}$$

where $t = t_2 - t_1$, and the partition defines the 4 x 4 matrices A, B, C.

Applying the general argument of section (a), we can show that the distribution of $\mathbf{X}'(t_1)$, $\mathbf{X}'(t_2)$, conditional on $\mathbf{X}(t_1) = \mathbf{x}_1$, $\mathbf{X}(t_2) = \mathbf{x}_2$ is

$$N\left[B^TA^{-1}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} \;,\; C - B^TA^{-1}B\right].$$

Thus the mean of $\nu_1^T{\cdot}\mathbf{X}'(t_1)$ is $\nu_1^T B^T A^{-1}\mathbf{x}_1$, the mean of $\nu_2^T{\cdot}\mathbf{X}'(t_2)$ is $\nu_2^T B^T A^{-1}\mathbf{x}_2$ while the variance of $\nu_1^T \mathbf{X}'(t_1)$ is $\nu_1^T{\cdot}(C - B^T A^{-1}B)\,\nu_1$ etc. Hence, conditional on $\mathbf{X}(t_1) = \mathbf{x}_1$, $\mathbf{X}(t_2) = \mathbf{x}_2$, $\nu_1^T{\cdot}\mathbf{X}'(t_1)$ and $\nu_2^T{\cdot}\mathbf{X}'(t_2)$ are normally distributed with mean vector $\mathbf{m}$ and covariance matrix Σ, where

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}^T B^TA^{-1}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} \tag{3.2}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}^T (C - B^T A^{-1} B) \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix} . \qquad (3.3)$$

Starting from the expansion (A.1.1) for $R(t)$, we obtain expansions for m and Σ, valid for small t .

Writing the matrix A in the form

$$A = \begin{bmatrix} I & 0 \\ R(t)^T R(0)^{-1} & I \end{bmatrix} . \begin{bmatrix} R(0) & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & R(0)^{-1} R(t) \\ 0 & I \end{bmatrix}$$

we get

$$A^{-1} = \begin{bmatrix} I & -R(0)^{-1} R(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} R(0)^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -R(t)^T R(0)^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_0^{-1} + \lambda_0^{-1} R(t) S^{-1} R(t)^T \lambda_0^{-1} & -\lambda_0^{-1} R(t) S^{-1} \\ -S^{-1} R(t)^T \lambda_0^{-1} & S^{-1} \end{bmatrix} \qquad (2.4)$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{ij} are 2 x 2 sub-matrices of the above partition of A^{-1}.

Expanding in powers of t , using (A.1.1), we get

$$S = \lambda_0 - R(t)^T \lambda_0^{-1} R(t) = \kappa_2 t^2 + \frac{1}{2} \kappa_3 t^3 \cdots\cdots - (\theta + \theta^T)$$

$$= t^2 (\kappa_2 - \phi_2) \quad ,$$

where $\kappa_3 = \lambda_2 \lambda_0^{-1} \lambda_1 - \lambda_1 \lambda_0^{-1} \lambda_2$ and $\phi_2 = o(1)$, for small t . Applying the identity (A.1.3), we find

$$S^{-1} = t^{-2} \left[\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} + O(t) + O\left(\frac{\theta + \theta^T}{t^2} \right)^2 \right] \qquad (2.5)$$

which immediately provides an expansion of A_{22} .

Let $U = \lambda_o - R(t)\lambda_o^{-1}R(t)^T = t^2(\kappa_2 - \phi_1)$, where $\phi_1 = o(1)$ for small t , thus as above we find

$$U^{-1} = t^{-2}\left[\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta + \theta^T}{t^2}\kappa_2^{-1} + O(t) + O\left(\frac{\theta + \theta^T}{t^2}\right)^2\right] \quad (2.6)$$

Now it follows directly from the above definitions that

$$U\lambda_o^{-1}R(t) = R(t)\lambda_o^{-1}S$$

and hence $\quad \lambda_o^{-1}R(t)S^{-1} = U^{-1}R(t)\lambda_o^{-1}$

from which we simply obtain

$$U^{-1} = \lambda_o^{-1} + \lambda_o^{-1}R(t)S^{-1}R(t)^T\lambda_o^{-1} = A_{11} \;. \quad (2.7)$$

Further $A_{12} = A_{21}^T = -\lambda_o^{-1}R(t)S^{-1}$ and on expanding $R(t)$ and S^{-1} in powers of t , we find

$$A_{12} = t^{-2}\left[\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta + \theta^T}{t^2}\kappa_2^{-1} + O(t) + O\left(\frac{\theta + \theta^T}{t^2}\right)^2\right] \quad (2.8)$$

From (A.2.1)

$$B^T = \begin{bmatrix} -\lambda_1 & -\lambda_1 \\ -\lambda_1 & -\lambda_1 \end{bmatrix} + \begin{bmatrix} 0 & \lambda_2 t - \theta' \\ -\lambda_2 t + \theta'^T & 0 \end{bmatrix}$$

and therefore

$$B^T A^{-1} = \begin{bmatrix} -\lambda_1(A_{11} + A_{21}) & -\lambda_1(A_{12} + A_{22}) \\ -\lambda_1(A_{11} + A_{21}) & -\lambda_1(A_{12} + A_{22}) \end{bmatrix} + \begin{bmatrix} (\lambda_2 t - \theta')A_{21} & (\lambda_2 t - \theta')A_{22} \\ (-\lambda_2 t + \theta'^T)A_{11} & (-\lambda_2 t + \theta'^T)A_{12} \end{bmatrix} . \quad (2.9)$$

Expansion of $C - B^TA^{-1}B$ for small t .

Post multiplying (A.2.9) by B, we find

$$B^TA^{-1}B = \begin{bmatrix} -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22})\lambda_1 & -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22})\lambda_1 \\ -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22})\lambda_1 & -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22})\lambda_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} -\lambda_1(A_{12}+A_{22})(\lambda_2 t - \theta'^T) & -\lambda_1(A_{11}+A_{21})(-\lambda_2 t + \theta') \\ -\lambda_1(A_{12}+A_{22})(\lambda_2 t - \theta'^T) & -\lambda_1(A_{11}+A_{21})(-\lambda_2 t + \theta') \end{bmatrix} + \tag{2.10}$$

$$+ \begin{bmatrix} (\lambda_2 t - \theta')(A_{21}+A_{22})\lambda_1 & (\lambda_2 t - \theta')(A_{21}+A_{22})\lambda_1 \\ (-\lambda_2 t + \theta'^T)(A_{11}+A_{12})\lambda_1 & (-\lambda_2 t + \theta'^T)(A_{11}+A_{12})\lambda_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} (\lambda_2 t - \theta')A_{22}(\lambda_2 t - \theta'^T) & (\lambda_2 t - \theta')A_{21}(-\lambda_2 t + \theta') \\ (-\lambda_2 t - \theta'^T)A_{12}(\lambda_2 t - \theta'^T) & (-\lambda_2 t + \theta'^T)A_{11}(-\lambda_2 t + \theta') \end{bmatrix}.$$

From (A.2.4), we have

$$A_{11} + A_{21} + A_{12} + A_{22} = \lambda_0^{-1} - \lambda_0^{-1}\lambda_1\kappa_2^{-1}\lambda_1\lambda_0^{-1} + - \lambda_0^{-1}\lambda_1\kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1}\lambda_1\lambda_0^{-1} + O(t) \tag{2.11}$$

using the expression (A.2.5) for S^{-1} . Thus the common submatrices making up the first term of (A.2.10) are

$$-\lambda_1(A_{11}+A_{21}+A_{12}+A_{22})\lambda_1 = -\lambda_1\lambda_0^{-1}\lambda_1 + \lambda_1\lambda_0^{-1}\lambda_1\kappa_2^{-1}\lambda_1\lambda_0^{-1}\lambda_1 + + \lambda_1\lambda_0^{-1}\lambda_1\kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1}\lambda_1\lambda_0^{-1}\lambda_1 + O(t) \tag{2.12}$$

Using (A.2.4) and substituting from (A.2.5) and (A.2.6) for S^{-1} and U^{-1} , we obtain for the second term of (A.2.10), neglecting terms of order O(t),

$$\begin{bmatrix} \lambda_1\lambda_0^{-1}\lambda_1(\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1})\left(\lambda_2 - \frac{\theta'^T}{t}\right) & -\lambda_1\lambda_0^{-1}\lambda_1(\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1})\left(-\lambda_2 + \frac{\theta'}{t}\right) \\ \lambda_1\lambda_0^{-1}\lambda_1(\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1})\left(\lambda_2 - \frac{\theta'^T}{t}\right) & -\lambda_1\lambda_0^{-1}\lambda_1(\kappa_2^{-1} + \kappa_2^{-1}\frac{\theta+\theta^T}{t^2}\kappa_2^{-1})\left(-\lambda_2 + \frac{\theta'}{t}\right) \end{bmatrix}$$

Similarly for the third term of (A.2.10) we find

$$\begin{bmatrix} \left(\lambda_2 - \frac{\theta'}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right) \lambda_1 \lambda_0^{-1} \lambda_1 \;, & \text{ditto} \\ \left(\lambda_2 - \frac{\theta'^T}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right) \lambda_1 \lambda_0^{-1} \lambda_1 \;, & \text{ditto} \end{bmatrix} + O(t)$$

Using the expansions (A.2.5), (A.2.6) and (A.2.8), the fourth and last term of (A.2.10) becomes

$$\begin{bmatrix} \left(\lambda_2 - \frac{\theta'}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right)\left(\lambda_2 - \frac{\theta'^T}{t}\right) & \left(\lambda_2 - \frac{\theta'}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right)\left(\lambda_2 - \frac{\theta'}{t}\right) \\ \left(\lambda_2 - \frac{\theta'^T}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right)\left(\lambda_2 - \frac{\theta'^T}{t}\right) & \left(\lambda_2 - \frac{\theta'^T}{t}\right)\left(\kappa_2^{-1} + \kappa_2^{-1} \frac{\theta + \theta^T}{t^2} \kappa_2^{-1} \right)\left(\lambda_2 - \frac{\theta'}{t}\right) \end{bmatrix}$$

Collecting terms we finally obtain

$$B^T A^{-1} B = \begin{bmatrix} \lambda_2 + \frac{\theta + \theta^T}{t^2} - \frac{\theta' + \theta'^T}{t} & \lambda_2 + \frac{\theta + \theta^T}{t^2} - \frac{2\theta'}{t} \\ \lambda_2 + \frac{\theta + \theta^T}{t^2} - \frac{2\theta'^T}{t} & \lambda_2 + \frac{\theta + \theta^T}{t^2} - \frac{\theta' + \theta'^T}{t} \end{bmatrix} + O(t) \,. \quad (2.13)$$

Now (A.1.1) and (A.2.1) together with (A.2.13) give

$$C - B^T A^{-1} B = \begin{bmatrix} -\frac{\theta + \theta^T}{t^2} + \frac{\theta' + \theta'^T}{t} & -\frac{\theta + \theta^T}{t^2} + \frac{2\theta'}{t} - \theta'' \\ -\frac{\theta + \theta^T}{t^2} + \frac{2\theta'^T}{t} - \theta''^T & -\frac{\theta + \theta^T}{t^2} + \frac{\theta' + \theta'^T}{t} \end{bmatrix} + O(t) \,. (2.14)$$

<u>Expansion, for small t , of</u> $B^T A^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$

It follows by (A.1.5), that $[\, \mathbf{X}(t) - \mathbf{X}(0) \,] / t$ has finite mean and variance as t tends to zero . Hence we may assume $\mathbf{x}_1 - \mathbf{x}_2 = O(t)$, and consequently write

$$B^T A^{-1} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = B^T A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 - \mathbf{x}_1 \end{bmatrix} . \quad (2.15)$$

From (A.2.9), we now find

$$B^T A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22}) & -\lambda_1(A_{12}+A_{22}) \\ -\lambda_1(A_{11}+A_{21}+A_{12}+A_{22}) & -\lambda_1(A_{12}+A_{22}) \end{bmatrix} + \tag{2.16}$$

$$+\begin{bmatrix} (\lambda_2 t - \theta')(A_{21}+A_{22}) & (\lambda_2 t - \theta')A_{22} \\ (\lambda_2 t - \theta'^T)(A_{11}+A_{12}) & (-\lambda_2 t + \theta'^T)A_{22} \end{bmatrix}$$

On combining (A.2.15) and (A.2.16) and employing (A.2.5), (A.2.6) and (A.2.11), we find, neglecting terms of order t ,

$$B^T A^{-1}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} a\,\kappa_2^{-1}\lambda_1\lambda_0^{-1}, & 1 + a\,\kappa_2^{-1} \\ a^T\kappa_2^{-1}\lambda_1\lambda_0^{-1}, & 1 + a^T\kappa_2^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \dfrac{(\mathbf{x}_2 - \mathbf{x}_1)}{t} \end{bmatrix}, \tag{2.17}$$

where we have written $a = (\theta + \dot{\theta}^T)\, t^{-2} - \theta'\, t^{-1}$.

Expansion of Σ and $\mathbf{m}$, for small t .

From (A.2.2) and (A.2.17), we obtain directly

$$\begin{aligned} m_1 &= v_1^T a\,\kappa_2^{-1}\lambda_1\lambda_0^{-1}\mathbf{x}_1 + v_1^T(1 + a\,\kappa_2^{-1})\,(\mathbf{x}_2 - \mathbf{x}_1)/t \\ m_2 &= v_2^T a^T\kappa_2^{-1}\lambda_1\lambda_0^{-1}\mathbf{x}_1 + v_2^T(1 + a^T\kappa_2^{-1})\,(\mathbf{x}_2 - \mathbf{x}_1)/t \end{aligned} \tag{2.18}$$

for small t .

From (A.2.3) and (A.2.14), we obtain

$$\Sigma = \begin{bmatrix} v_1^T\left(-\dfrac{\theta+\theta^T}{t^2} + \dfrac{\theta'+\theta'^T}{t}\right)v_1 & v_1^T\left(-\dfrac{\theta+\theta^T}{t^2} + \dfrac{2\theta'}{t} - \theta''\right)v_2 \\ v_2^T\left(-\dfrac{\theta+\theta^T}{t^2} + \dfrac{2\theta'^T}{t} - \theta''^T\right)v_1 & v_2^T\left(-\dfrac{\theta+\theta^T}{t^2} + \dfrac{\theta'+\theta'^T}{t}\right)v_2 \end{bmatrix} \tag{2.19}$$

for small t .

The expansions (A.2.18), (A.2.19), in common with all the expansions in this section, neglect terms of the first degree in t along with powers and products of θ / t^2 , θ' / t , θ'' , and t . In expansions of the mean such as (A.2.18) we have assumed that $\mathbf{x}_2 - \mathbf{x}_1 = O(t)$.

If the expansion of $R(t)$ given in (1.1) is valid beyond the term in t^2, then $\theta = O(t^3)$, therefore θ / t^2, θ' / t, θ'' are all $O(t)$ and their powers and products will be $O(t^2)$. In chapter 3 we are mainly concerned with the situation where θ is larger than $O(t^3)$, for small t. Thus θ / t^2, θ' / t, θ'' will tend to zero more slowly than $O(t)$, as may some of their products and powers. In general the use of $O(t)$ for the error term in this section is meant to include terms such as $(\theta / t^2)^2$, $(\theta' / t)^2$, $(\theta\theta') / t^3$, etc.

3. To prove that $\int_0^\delta \frac{\| \theta'(t) \|}{t^2} dt$ and $\int_0^\delta \frac{\| \theta(t) \|}{t^3} dt$ are bounded by $\int_0^\delta \frac{w(t)}{t} dt < \infty$

Proof. (i)

Since $\theta'(0) = 0$, $\theta'(t) = \int_0^t \theta''(\tau)\, d\tau$, therefore

$$\| \theta'(t) \| = \left\| \int_0^t \theta''(\tau)\, d\tau \right\| \le \int_0^t \| \theta''(\tau) \|\, d\tau = \int_0^t w(\tau)\, d\tau$$

Thus

$$\int_0^\delta \frac{\| \theta'(t) \|}{t^2} dt \le \int_0^\delta \frac{1}{t^2} \int_0^t w(\tau)\, d\tau\, dt$$

$$= \int_0^\delta \left(\frac{1}{\tau} - \frac{1}{\delta} \right) w(\tau)\, d\tau$$

on changing the order of integration and integrating over t. Thus we have

$$\int_0^\delta \frac{\| \theta'(t) \|}{t^2} dt \le \int_0^\delta \left(\frac{1}{\tau} - \frac{1}{\delta} \right) w(\tau)\, d\tau \le \int_0^\delta \frac{w(\tau)}{\tau} d\tau ,$$

as we wished to show.

(ii)

Since $\theta(0) = \theta'(0) = 0$, $\theta(t) = \int_0^t (t - \tau)\, \theta''(\tau)\, d\tau$ and hence

$$\| \theta(t) \| = \left\| \int_0^t (t - \tau)\, \theta''(\tau)\, d\tau \right\| \le \int_0^t (t - \tau) \| \theta''(\tau) \|\, d\tau = \int_0^t (t - \tau)\, w(\tau)\, d\tau$$

Thus, on changing the order of integration and integrating over t, we obtain

$$\int_0^{\delta} \frac{\| \theta(t) \|}{t^3}\, dt \ \le \int_0^{\delta} \frac{1}{t^3} \int_0^{t} (t - \tau)\, w(\tau)\, d\tau\, dt$$

$$= \int_0^{t} \left\{ \frac{1}{2\tau} - \frac{2\delta - \tau}{2\delta^2} \right\} w(\tau)\, d\tau\, dt \ \le \int_0^{\delta} \frac{w(\tau)}{2\tau}\, d\tau \ < \infty$$

as was to be proved.

Appendix B

(a) <u>The conditional distribution of $\mathbf{X}(t)$, $\mathbf{X}'(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$</u> .

The joint distribution of $\mathbf{X}(0)$, $\mathbf{X}'(0)$, $\mathbf{X}(t)$, $\mathbf{X}'(t)$ is multivariate normal with zero means and covariance matrix

$$\Lambda = \begin{bmatrix} \begin{matrix} R(0) & R'(0) \\ -R'(0) & -R''(0) \end{matrix} & \begin{matrix} R(t)^T & -R'(t)^T \\ R'(t)^T & -R''(t)^T \end{matrix} \\ \\ \begin{matrix} R(t) & R'(t) \\ -R'(t) & -R''(t) \end{matrix} & \begin{matrix} R(0) & R'(0) \\ -R'(0) & -R''(0) \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_0 & B^T \\ B & \Sigma_0 \end{bmatrix} .$$

From the general argument of appendix A, the conditional distribution of $\mathbf{X}(t)$, $\mathbf{X}'(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$ is

$$N\left(B^T\Sigma_0^{-1}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{bmatrix} , \Sigma_0 - B^T\Sigma_0^{-1}B \right) . \qquad \text{(B.1)}$$

(b) <u>Inverse of Σ_0</u>

Write $\Sigma_0^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$, where σ_{ij} are 2 x 2 matrices. Since Σ_0 is symmetric, σ_{11} and σ_{22} are symmetric and $\sigma_{12} = \sigma_{21}^T$.

Since we must have

$$\begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} ,$$

we obtain, equating component sub-matrices,

$$\lambda_0\sigma_{11} + \lambda_1\sigma_{21} = I \qquad \text{(i)}$$

$$-\lambda_1\sigma_{11} + \lambda_2\sigma_{21} = 0 \qquad \text{(ii)}$$

$$\lambda_0\sigma_{12} + \lambda_1\sigma_{22} = 0 \qquad \text{(iii)}$$

$$-\lambda_1\sigma_{12} + \lambda_2\sigma_{22} = I \qquad \text{(iv)}$$

Eliminating σ_{21} between (i) and (ii), we obtain

$$(\lambda_0 + \lambda_1\lambda_2^{-1}\lambda_1)\, \sigma_{11} = I \quad . \tag{B.2}$$

Equations (ii) and (iii) give

$$\sigma_{21} = \lambda_2^{-1}\lambda_1\, \sigma_{11} \quad , \tag{B.3}$$

and

$$\sigma_{12} = -\lambda_0^{-1}\lambda_1\, \sigma_{22} \quad . \tag{B.4}$$

Also, since $\sigma_{21} = \sigma_{12}^T$, we get

$$\sigma_{11}\, \lambda_1\lambda_2^{-1} = \lambda_0^{-1}\lambda_1\, \sigma_{22} \quad , \tag{B.5}$$

or

$$\sigma_{22}\, \lambda_1\lambda_0^{-1} = \lambda_2^{-1}\lambda_1\, \sigma_{11} \quad . \tag{B.6}$$

(c) <u>Expansion of $B^T\Sigma_0^{-1}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\Sigma_0 - B^T\Sigma_0^{-1}B$, for small t</u>

We assume that, for small t , R(t) has the expansion

$$R(t) = \lambda_0 + \lambda_1 t - \frac{1}{2!}\lambda_2 t^2 + \frac{1}{3!}\lambda_3 t^3 + \frac{1}{4!}\lambda_4 t^4 + \phi(t) \quad , \tag{B.7}$$

where $\phi(0) = \phi'(0) = \phi''(0) = \phi'''(0) = 0$ and $\phi^{iv}(t) = o(1)$. Thus

$$B = \begin{bmatrix} R(t) & R'(t) \\ -R'(t) & -R''(t) \end{bmatrix} = \Sigma_0 + \Sigma_1 t + \frac{1}{2}\Sigma_2 t^2 + \Theta \quad , \tag{B.8}$$

where

$$\Sigma_1 = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & -\lambda_3 \end{bmatrix} \quad , \quad \Sigma_2 = \begin{bmatrix} -\lambda_2 & \lambda_3 \\ -\lambda_3 & -\lambda_4 \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} \frac{1}{3!}\lambda_3 t^3 + \frac{1}{4!}\lambda_4 t^4 + \phi , & \frac{1}{3!}\lambda_4 t^3 + \phi' \\ -\frac{1}{3!}\lambda_4 t^3 - \phi' , & -\phi'' \end{bmatrix} .$$

Bearing in mind Σ_0 and Σ_2 are symmetric matrices of order four, while Σ_1 is a skew-symmetric matrix of order four, we have

$$B^T\Sigma_0^{-1}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \left[I - \Sigma_1\Sigma_0^{-1} t + \frac{1}{2}\Sigma_2\Sigma_0^{-1} t^2 + \Theta^T\Sigma_0^{-1} \right]\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} ,$$

where I is the identity matrix of order four.

Equations (ii) and (iv) preceeding (B.2), give

$$\Sigma_1\Sigma_0^{-1} = \begin{bmatrix} 0 & -I \\ \lambda_2\sigma_{11} - \lambda_3\sigma_{21}\,, & \lambda_2\sigma_{12} - \lambda_3\sigma_{22} \end{bmatrix} \tag{B.9}$$

and, using (B.3) and (B.4) to express σ_{21} and σ_{12} in terms of σ_{11} and σ_{22},

the expansions of the conditional means, for small t, can be written

$$\begin{aligned} E[\mathbf{X}(t) \mid \mathbf{x}_1, \mathbf{y}_1] &= \mathbf{x}_1 + t\,\mathbf{y}_1 + \\ &+ \frac{1}{2}t^2\left[(-\lambda_2 + \lambda_3\lambda_2^{-1}\lambda_1)\,\sigma_{11}\mathbf{x}_1 + (\lambda_3 + \lambda_2\lambda_0^{-1}\lambda_1)\,\sigma_{22}\mathbf{y}_1\right] + \\ &+ \mathbf{x}_1\,O(t^3) + \mathbf{y}_1\,O(t^3) \end{aligned} \tag{B.10}$$

$$\begin{aligned} E[\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1] &= \mathbf{y}_1 + \mathbf{x}_1\,O(t^2) + \mathbf{y}_1\,O(t^2) + \\ &+ t\left[(-\lambda_2 + \lambda_3\lambda_2^{-1}\lambda_1)\,\sigma_{11}\mathbf{x}_1 + (\lambda_3 + \lambda_2\lambda_0^{-1}\lambda_1)\,\sigma_{22}\mathbf{y}_1\right]. \end{aligned} \tag{B.11}$$

<u>Expansion of $\Sigma_0 - B^T\Sigma_0^{-1}B$</u>

Expanding the second term in powers of t, we get

$$\begin{aligned} B^T\Sigma_0^{-1}B &= \Sigma_0 + (\Sigma_2 - \Sigma_1\Sigma_0^{-1}\Sigma_1)\,t^2 + \frac{1}{2}(\Sigma_2\Sigma_0^{-1}\Sigma_1 - \Sigma_1\Sigma_0^{-1}\Sigma_2)\,t^3 + \\ &+ \frac{1}{4}\Sigma_2\Sigma_0^{-1}\Sigma_2\,t^4 + \left\{I - \Sigma_1\Sigma_0^{-1}\,t + \frac{1}{2}\Sigma_2\Sigma_0^{-1}\,t^2\right\}\Theta + \\ &+ \Theta^T\left\{I + \Sigma_0^{-1}\Sigma_1\,t + \frac{1}{2}\Sigma_0^{-1}\Sigma_2\,t^2\right\} + \Theta^T\Sigma_0^{-1}\Theta\,. \end{aligned} \tag{B.12}$$

Since

$$\Sigma_0\Sigma_0^{-1} = \Sigma_0^{-1}\Sigma_0 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

it follows that $\Sigma_1\Sigma_0^{-1}$ and $\Sigma_0^{-1}\Sigma_1$ are of the forms

$$\Sigma_1\Sigma_0^{-1} = \begin{bmatrix} 0 & -I \\ * & * \end{bmatrix}, \qquad \Sigma_0^{-1}\Sigma_1 = \begin{bmatrix} 0 & * \\ I & * \end{bmatrix}$$

and hence

$$\Sigma_1\Sigma_0^{-1}\Sigma_1 = \begin{bmatrix} -\lambda_2 & \lambda_3 \\ -\lambda_3 & * \end{bmatrix},$$

where the asterisks denote undetermined sub-matrices. Thus

$$\Sigma_2 - \Sigma_1\Sigma_0^{-1}\Sigma_1 = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix},$$

and, as a little algebra shows,

$$d = \lambda_4 - (\lambda_2 \sigma_{11} \lambda_2 + \lambda_2 \sigma_{12} \lambda_3 - \lambda_3 \sigma_{21} \lambda_2 - \lambda_3 \sigma_{22} \lambda_3) . \qquad (B.13)$$

Similarly, we can show

$$\frac{1}{2} (\Sigma_2 \Sigma_0^{-1} \Sigma_1 - \Sigma_1 \Sigma_0^{-1} \Sigma_2) = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} ,$$

and, as t tends to infinity,

$$\Sigma_1 \Sigma_0^{-1} \Theta - \Theta^T \Sigma_0^{-1} \Sigma_1 = \begin{bmatrix} \frac{1}{3} \lambda_4 t^3 + \phi' + \phi'^T , & o(t^2) \\ o(t^2) , & o(t^2) \end{bmatrix} .$$

Further

$$\Theta + \Theta^T = \begin{bmatrix} \frac{1}{12} \lambda_4 t^4 + \phi + \phi^T , & \dot{\phi}' - \phi'^T \\ - \phi' + \phi'^T , & - \phi'' - \phi''^T \end{bmatrix} = \begin{bmatrix} O(t^4), & o(t^3) \\ o(t^3), & o(t^2) \end{bmatrix} ,$$

for small t .

Substituting the above results into (B.12) and retaining terms up to the fourth degree in t, we find

$$\Sigma_0 - B^T \Sigma_0^{-1} B = (\Sigma_1 \Sigma_0^{-1} \Theta - \Theta^T \Sigma_0^{-1} \Sigma_1) t - (\Sigma_2 - \Sigma_1 \Sigma_0^{-1} \Sigma_1) t^2 +$$
$$- \frac{1}{2} (\Sigma_2 \Sigma_0^{-1} \Sigma_1 - \Sigma_1 \Sigma_0^{-1} \Sigma_2) t^3 + \frac{1}{4} \Sigma_2 \Sigma_0^{-1} \Sigma_2 t^4 - (\Theta + \Theta^T) + o(t^4) ,$$

or $$\Sigma_0 - B^T \Sigma_0^{-1} B = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix} , \qquad (B.14)$$

where the 2 x 2 sub-matrices a, b, c are given by

$$a = \frac{1}{4} t^4 d + o(t^4) \qquad (B.15)$$

$$b = \frac{1}{2} t^3 d + o(t^3) = b^T \qquad (B.16)$$

$$c = t^2 d + o(t^2) \qquad (B.17)$$

and the sub-matrix d is defined in (B.13) .

(d) The conditional distribution of $\mathbf{X}'(t)$, given $\mathbf{X}(t) = \mathbf{x}_2$, $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$

From the results of sections (a) and (c) it follows that the distribution of $\mathbf{X}(t)$, given $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$, is $N(E[\mathbf{X}(t) \mid \mathbf{x}_1, \mathbf{y}_1], a)$. The general considerations of appendix A applied here, show that the distribution of $\mathbf{X}'(t)$, conditional on $\mathbf{X}(t) = \mathbf{x}_2$, $\mathbf{X}(0) = \mathbf{x}_1$, $\mathbf{X}'(0) = \mathbf{y}_1$, is $N(\mathbf{m}, c - b^T a^{-1} b)$,

where the mean $\mathbf{m}$ is given by

$$\mathbf{m} = E[\mathbf{X}'(t) \mid \mathbf{x}_1, \mathbf{y}_1] + b^T a^{-1}(\mathbf{x}_1 - E[\mathbf{X}(t) \mid \mathbf{x}_1, \mathbf{y}_1]) . \tag{B.18}$$

The results of the last section show that, as $t \to 0$,

$$b^T a^{-1} = O(t^{-1}) , \tag{B.19}$$

and

$$c - b^T a^{-1} b = o(t^2) . \tag{B.20}$$

(e) The conditional distribution of $\mathbf{X}'(0)$, given $\mathbf{X}(0) = \mathbf{x}_1$.

The joint distribution of $\mathbf{X}(0)$, $\mathbf{X}'(0)$ is $N(\underline{0}, \Sigma_o)$, where $\Sigma_o = \begin{bmatrix} \lambda_o & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{bmatrix}$

is the covariance matrix, as follows from section (a).

Again, from the general considerations of appendix A, the marginal distribution of $\mathbf{X}(0)$ is $N(\underline{0}, \lambda_o)$ and the conditional distribution of $\mathbf{X}'(0)$, given $\mathbf{X}(0) = \mathbf{x}_1$, is

$$N(-\lambda_1 \lambda_0^{-1} \mathbf{x}_1 , \lambda_2 + \lambda_1 \lambda_0^{-1} \lambda_1) . \tag{B.21}$$

REFERENCES

Ahern, D B , Lamont, G B & Peterson, J B. (1976) Microprocessor development for a digital flight control system voter / monitor. Proceedings of the IEE 1976 National Aerospace and Electronics Conference.

Aronowich, M & Adler, R J. (1985) Behaviour of χ^2 processes at extrema. *Adv. Appl. Prob.* vol **17**, pp 280-297.

Aronowich, M & Adler, R J. (1986) Extrema and level crossings of χ^2 processes. *Adv. Appl. Prob.* vol **18**, pp 901-920.

Belyaev, Yu K. & Nosko, V P. (1969) Characteristics of excursions above a high level for a Gaussian process and its envelope. *Theory Prob. Appl.* vol **14**, pp 296-309.

Belyaev, Yu K. (1966) On the number of intersections of a level by a Gaussian stochastic process, I . *Theory Prob. Appl.* vol **11**, pp 106-113.

Belyaev, Yu K. (1968) On the number of exits across the boundary of a region by a vector stochastic process. *Theory Prob. Appl.* vol **13**, pp 320-324.

Berman, S M. (1971) Excursions above high levels for stationary Gaussian processes. *Pacific J. Math.* vol **36**, pp 63-79.

Blake, I F & Lindsey, W C. (1973) Level-crossing problems for random processes. *IEEE Trans. on Information Theory*, vol **IT-19**, pp 295-315.

Bolotin, V V. (1971) Primenenie Metodov Teorii Verovatnostei i Teorii Nadejnosti v Raschetah Soornjenni, Izdatelstvo Literativo po Stroitelstvi, Moskov.

Breitung, K. (1988) Asymptotic crossing rates for stationary Gaussian vector processes. *Stochastic Process Appl.* vol **29**, pp 195-207.

Bulinskya, E V. (1961) On the mean number of crossings of a level by a stationary Gaussian process. *Theory Prob. Appl.* **6**, pp 474-477.

Cramer, H & Leadbetter, M R. (1965) The moments of the number of crossings of a level by a stationary normal process. *Ann. Math. Statist.* vol **36**, pp 1656-1663.

Cramer, H & Leadbetter, M R. (1967) Stationary and Related Stochastic Processes. John Wiley & Sons.

Cramer, H. (1940) On the theory of stationary random processes. *Ann. Math.* vol **41**, pp 215-230.

Cramer, H. (1966) On the intersections between the trajectories of a normal stationary stochastic process and a high level. *Arkiv. Mat.* vol **6**, pp 337-349.

Cuzick, J. (1975) Conditions for finite moments of the number of zero crossings for Gaussian processes. *Ann. Prob.* vol **3**, pp 849-858.

Geman, D. (1972) On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **43**, pp 977-982.

Gill, F R. (1977) Engineering of control systems and implications on control law design. AGARD Lecture Series No. 89.

Hasofer, A M. (1974) 'The upcrossing rate of a class of stochastic processes.' In: Studies in Probability and Statistics (papers in honour of E J G Pitman), edited by E J Williams. Jerusalem Academic Press, pp 153-160.

Ito, K. (1964) The expected number of zeros of continuous stationary Gaussian processes. *J. Math. Kyoto Univ.* **3-2**, pp 207-216.

Ivanov, V A. (1960) On the average number of crossings of a level by sample functions of a stochastic process. *Theory Prob. Appl.* **5**, pp 319-323.

Kac, M & Slepian, D. (1959) Large excursions of Gaussian processes. *Ann. Math. Statist.* **30**, pp 1215-1228.

Kac, M. (1943) On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49**, pp 314-320.

Leadbetter, M R & Cryer, J D. (1965) The variance of the number of zeros of a stationary normal process. *Bull. Amer. Math. Soc.* **71**, pp 561-563.

Lindgren, G. (1980a) Extreme values and crossings for the χ^2 - process and other functions of multidimensional Gaussian processes, with reliability applications. *Adv. Appl. Prob.* **12**, pp 746-774.

Lindgren, G. (1980b) Model processes in nonlinear prediction with application to detection and alarm. *Ann. Prob.* **8,** pp 775-792.

Marcus, M B. (1977) Level crossings of a stationary process with absolutely continuous sample paths. *Ann. Probab.* **5**, pp 52-71.

Miroshin, R N. (1973) A sufficient condition for finiteness of moments of the number of zeros of a differentiable stationary Gaussian process. *Theory Probab. Appl.* **18**, pp 454-463.

Rice, S O. (1945) Mathematical analysis of random noise. *Bell System Tech. J.* **24**, pp 46-156.

Rice,S O. (1958) Distribution of the duration of fades in radio transmission. *Bell System Tech. J.* **37**, pp 581-635.

Sharpe, K. (1978) Some properties of the crossings process generated by a stationary χ^2 - process. *Adv. Appl. Prob.* **10**, pp 373-391.

Veneziano, D, Grigoriu, M & Cornell, C A. (1977) Vector-process models for system reliability. *J. Engineering Mechanics Division ASCE* **103, EM3**, pp 441-460.

Volkonskii, V A & Rozanov, Yu A. (1959) Some limit theorems for random functions, I. *Theory Prob. Appl.* **4**, pp 178-197.

Volkonskii, V A & Rozanov, Yu A. (1961) Some limit theorems for random functions, II. *Theory Prob. Appl.* **6**, pp 186-198.

Ylvisaker, N D. (1965) The expected number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **36**, pp 1043-1046.

Ylvisaker, N D. (1966) On a theorem of Cramer and Leadbetter. *Ann, Math. Statist.* **37**, pp 682-685.