TO

my wife, HELENA

AND

my mother, ÖZEL

THE KORTEWEG-DEVRIES EQUATION AND ITS HOMOLOGUES III ANALYTICAL STRUCTURE

BY

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A THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

THE DEPARTMENT OF MATHEMATICS
THE CITY UNIVERSITY
LONDON, ENGLAND

JANUARY 1990

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ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my Supervisors, Mr. Alan C. Bryan, whose constant technical help and moral support has made the work possible, and Dr. Allan E. G. Stuart, whose advice and discussions on many technical difficulties have been invaluable.

I also record my thanks to the Library staff and in particular John Adams for his help in obtaining research material.

A special thanks to Ceinwen Sanderson for her professional typing of the manuscript.

Finally, but not less significantly, I wish to thank my wife for her encouragement, understanding and patience during this work.

DECLARATION

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ABSTRACT

This thesis is the third in a series of studies on the Korteweg-de Vries equation (KdV) and its homologues, the objective being to understand its distinguished position when embedded in a class of similar equations.

Now the KdV is a partial differential equation which is well-known to have some remarkable mathematical properties. Furthermore, it also appears as a useful model in a great many physical situations. Thus, although it was originally obtained as an approximation in fluid dynamics, it was reinterpreted as a canonical field theory for weakly dispersive and weakly nonlinear systems. This reinterpretation led to the hypothesis that the properties of the KdV could be understood in terms of a balance between the competing effects of dispersion and nonlinearity. Alternatives to the KdV were proposed on the basis that their dispersive properties were physically and mathematically preferable to those of the KdV.

The first study, which was undertaken by Abbas, was to test the hypothesis described above that dispersion is a useful criterion for constructing nonlinear equations. By introducing a general class of equations which includes the KdV and all its proposed alternatives as special cases, he investigated in detail the predictions based on the dispersion relation and compared them with the actual properties of the equation, particularly in regard to the existence of solitary waves. He found little correlation and some contradictions and concluded that the idea of a balance between nonlinearity and dispersion is not a useful way of understanding these equations. This meant that other criteria must be developed to obtain this understanding.

The criteria we are looking for would have to account for the existence of families of solitary waves in the general class and, in the case of the KdV, for solitons. However, before doing this it was important to establish the mathematical validity of the equations, i.e. well-posedness and existence of conservation laws. This was carried out by El-Sherbiny in the second study of the series. By partitioning the set of equations into equivalence classes, he proved existence for most of the equations and well-posedness for some. He also showed that, with the exception of the KdV, all the equations have at least two and at most three conservation laws.

At the time that this third study was started interest was focussed on the integrability of nonlinear evolution equations and, through the Painlevé conjectures, this was reformulated in terms of the analytic structure of the solutions of these equations. It seemed natural, therefore, to look at the Abbas' class of equations from this point of view as the major objective. In addition, we critically examine the structure of the solitary waves themselves in order to answer the question of when a solitary wave is a soliton.

The first part of this thesis contains the introduction and relevant reviews of the inverse scattering method, integrability and the work of Abbas and El-Sherbiny.

The second part of the thesis contains our main contributions and we begin by obtaining all the similarity reductions to ordinary differential equations for the general class of partial differential equations using one-parameter Lie groups. We derive the singularity structure of the general solutions of the similarity equations and use this analysis to initiate a classification of third order nonlinear ordinary differential equations. Next we obtain the singularity structure of classes of general solutions of the partial differential equations directly in terms of Laurent-type expansions. These results are compared with those obtained via symmetry groups and equations which are or are not Painlevé-type identified. We also look for special cases of the general solution which may be restricted solitons. We do not find any, and, in the case of the regularised long wave equation, we prove that it does not have any. Finally, we develop a classification of the solitary waves of the general class and use this to develop necessary criteria for a solitary wave to be a soliton.

The thesis ends with a resumé and suggests avenues for continuing this investigation.

CHAPTER ONE

INTRODUCTION

The concept of a solitary wave was first introduced in 1834 by Scott-Russell and his famous horseback observation. He was observing the motion of a boat drawn rapidly along a narrow channel by a pair of horses. When the boat suddenly stopped, the mass of water which the boat had set in motion '... rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speeds ...'.

In 1895 Korteweg and deVries [19] constructed a nonlinear partial differential equation (PDE) for studying solitary waves on shallow water. This equation which was derived as an approximation from the equations of hydrodynamics, includes both nonlinear and dispersive effects, but ignores dissipation. It has the well-known form

$$u_t + u_x + uu_x + u_{xxx} = 0 \qquad (1.1)$$

A solitary wave is a localized travelling wave and consequently can be obtained by assuming

$$u(x, t) = f(\eta) \qquad (1.2)$$

where $\eta = x - (1+c)t$ represents the position in a coordinate system moving at a velocity $(1+c)$ for which the wave appears stationary. The resulting ordinary differential equation (ODE) can then be solved to give the shape of the solitary wave, if such a solution exists. In the case of the Korteweg-deVries equation (KdV) this ODE can be integrated exactly to give the solitary waves

$$u = 3c \operatorname{sech}^2 \left[\frac{\sqrt{c}}{2} (x - (1+c)t) \right], \; c \in \mathbb{R}^+ \qquad (1.3)$$

Since c is not negative, an important property of the solitary waves of the KdV is that they are unidirectional with speeds ranging from 1 to ∞.

Since the work of Korteweg and deVries the existence of solitary waves has been shown for several PDEs and some of these will be given in the next chapter.

A different kind of solitary wave occurs for the sine-Gordon equation (SGE)

$$\phi_{xx} - \phi_{tt} = \sin\phi \qquad (1.4)$$

where $\phi \in [0, 2\pi]$. For this equation, the solitary waves correspond to a rotation in ϕ by 2π and have the forms

$$\phi = 4\tan^{-1}\left[\exp \pm \left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right], \quad c \in \mathbb{R} \tag{1.5}$$

These solutions exhibit some of the properties of classical particles and hence have potential applications in elementary particle theory [25]. In fact it was their interest in how elementary particles would scatter upon collision which led Perring and Skyrme [24], in 1962, to initiate computer experiments on the solitary waves of the SGE. These experiments indicated that the solitary waves emerge from the collision having the same shapes and velocities with which they enter. This led Perring and Skyrme to find analytical expressions describing collision events which, interestingly, had been derived a decade earlier by Seeger, Donth and Kochendörfer [26].

Soon after the work of Perring and Skyrme had appeared, Zabusky and Kruskal [31] published results of a completely independent study of the application of the KdV equation to the investigation of plasma waves. Once again, computer experiments indicated that the solitary waves emerge from a collision having the same shapes and velocities with which they enter. Zabusky and Kruskal coined the term SOLITON to describe the solitary wave which exhibits this phenomenon. The question then arose as to whether there exist solitons which, asymptotically, are a superposition of an arbitrary number of solitary waves. The earlier evidence that the work of Perring and Skyrme provided seems to have gone unnoticed at this stage.

An analytic procedure for solving the initial value problem of the KdV equation for data which vanishes rapidly as $|x| \longrightarrow \infty$ was discovered by Gardner, Green, Kruskal and Muira (GGKM) in 1967 [12]. The initial value problem they considered was

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.6a}$$

$$u(x, 0) = u_o(x) \tag{1.6b}$$

for $x \in (-\infty, \infty)$, $t > 0$. The existence of classical solutions for this class of data was proved by Bona and Smith in 1976 [6]. The key step in developing the method came from an observation of Muira [22] that if v is a solution of the modified KdV equation (MKdV)

$$v_t - 6v^2v_x + v_{xxx} = 0 \tag{1.7}$$

then

$$u = v^2 + v_x \qquad (1.8)$$

is a solution of the KdV equation (1.6). Now (1.8) is a Ricatti equation and can be linearized by the substitution $v = \phi_x/\phi$. Using this together with a symmetry transformation of the KdV leads to the eigenvalue ODE

$$\phi_{xx} - (u-\lambda)\phi = 0 \qquad (1.9)$$

where λ is a constant, i.e. independent of time for each t, equation (1.9) is a time-independent Schrödinger equation with potential $u(x, t)$, energy level λ and wave formation $\phi(x, t)$.

The initial value problem (1.6a, b) was thus reduced to an inverse scattering problem i.e., computing $u(x, t)$ from a knowledge of the scattering data comprising the discrete eigenvalues, normalised coefficients of the corresponding eigenfunctions and the reflection coefficients. The problem is then solved in terms of a linear integral equation.

Solving this linear integral equation was then equivalent to solving the initial-value problem. This was a significant addition to methods for solving nonlinear PDEs and became known as the inverse scattering method. This method will be reviewed in much greater detail in the next chapter.

Although GGKM developed the inverse scattering method (ISM) for the KdV in 1967, exact solutions for solitons were not obtained from it or any other method until 1971. In the meantime, however, Lax [21] followed up the computer experiments of Zabusky and Kruskal and proved analytically the existence of two-soliton solutions of the KdV, thus confirming numerical predictions. Lax also generalized the GGKM method to a general class of nonlinear PDEs of the form

$$u_t = K[u] \tag{1.10}$$

where K is a nonlinear operator. Using this approach Zakharov and Shabat [33] solved the nonlinear Schrödinger equation (NLS)

$$u_t = i(u_{xx} + 2u^2u^*) \tag{1.11}$$

where u^* is the complex conjugate of u, and thus showed that the inverse scattering method was not limited to the KdV hierarchy. Furthermore, they obtained exact N-soliton solutions.

The exact N-soliton solutions of the KdV equation using ISM were first obtained by Wadati and Toda [29] in 1972. Following Zakharov and Shabat, Wadati extended the ISM to the MKdV and found the N-soliton solutions [28]. The extension to the SGE was made by Ablowitz, Kaup, Newell and Segur (AKNS) [2] who also provided a procedure for solving a number of other nonlinear PDEs.

The inverse scattering method, which by now had become known as the inverse scattering transform (IST), is not the only way of obtaining N-soliton solutions. Hirota found exact N-soliton solutions to the KdV, MKdV and SGE [15-17] via a completely different approach in which he transformed the PDE into homogenous bilinear forms of degree two. In the case of the KdV, the Hirota solutions preceeded those of Wadati and Toda. More recently Bryan [8] put forward a much simpler algorithm for the SGE and then extended it to the other equations [9, 10].

We note that the existence of solitons in an IST solution corresponds to the presence of discrete eigenvalues in the scattering data. Therefore, an equation which reduces to linear integral equations via IST may not have soliton solutions (we give examples of this in the following chapter). Equations which have been shown to possess solitons and which can be solved by IST also have the property that they possess an infinite number of local conservation laws.

These developments have led workers in the field to consider the concept of integrability for PDEs. For an n-dimensional Hamiltonian system (ODEs) the definition of integrability is provided by Liouville's theorem i.e. there must exist n-constants of the motion (first integrals) which are in involution. For PDEs, which are infinite dimensional, there is no theorem which corresponds to Liouville's. Thus, the existence of an infinite number of conservation laws, the existence of an IST and the existence of soliton solutions are distinct properties. However, in every case equations with soliton solutions are found to possess an infinite number of conservation laws and thus it is generally supposed that these properties are complementary in the sense that either one implies the other, but this has yet to be rigorously formulated and proved.

On the other hand by making certain assumptions it may be possible to represent an equation which has both solitons and an infinite number of conservation laws as an infinite dimensional Hamiltonian system and hence imply that IST is equivalent to complete integrability. This was done by Zakharov and Faddeev [32], but the assumptions are specific to that equation.

A general scheme for obtaining solutions to nonlinear differential equations without first finding an associated scattering problem was first presented by

Zakharov and Shabat [34]. This scheme was later used by Lambert [20] to obtain exact solutions for the KdV equation which are not recoverable by the IST method. Using this approach Ablowitz, Ramani and Segur [3] developed a scheme for solving nonlinear PDEs via linear integral equations and this work will be reviewed in Chapter Three.

In 1977, Ablowitz and Segur [4] demonstrated a close connection between nonlinear PDEs which are soluble by IST and Painlevé transcendents. The Painlevé transcendents were first discovered by Painlevé and his colleagues around the turn of the century [14], [18]. They showed that of all possible equations of the form

$$\frac{d^2w}{dz^2} = F\left(z,\ w,\ \frac{dw}{dz}\right), \tag{1.12}$$

where F is rational in z and $\frac{dw}{dz}$, and analytic in z, only 50 canonical equations possess the property of having no movable singularities except poles in their general solutions. Furthermore, they showed that 44 of these were soluble in terms of elementary and elliptic functions. The remaining six defined new transcendental functions which are the Painlevé transcendents. The first and the simplest of these is

$$\frac{d^2w}{dz^2} = 6w^2 + z \qquad P_I$$

Thus an ODE is said to have the Painlevé property or be of Painlevé type if all movable singularities of the general solution are poles [18]. In [3] it was conjectured that all the possible reductions (e.g. similarity transformations) of a completely integrable PDE will have the Painlevé property.

More recently Weiss, Tabor and Carnevale [30] have formulated a 'Painlevé test' that can be applied directly to a PDE without any need for reductions. These tests for integrability will be reviewed in Chapter Four and, as we shall see, although they are by no means unambiguous, they do seem to provide a valuable first test. The work of Gibbon, Radmore, Tabor and Wood [13] and Tabor and Gibbon [27] also shows an interesting connection between the analysis of Weiss et al. and Hirota's method.

Now it is accepted convention to say that a nonlinear PDE is integrable if it can be reduced to a linear system. As we have described above the IST is a method which linearises certain equations and so such equations can then be classified as integrable. Furthermore, it is found that every equation which has soliton solutions can be linearized by this method which implies that a necessary condition for the existence of soliton solutions is that the equation is integrable. However, this definition of integrability is not constructive in the sense that there is no systematic procedure for finding the Lax operators for a given equation. Nor is there a general theorem which can be used to decide whether an equation has soliton solutions or not. Thus, the problem of identifying integrable systems can be more simply stated as follows: when is a solitary wave a soliton? In this formulation of integrability, the criteria for the identification of soliton solutions should be a priori theorems based on the structure of the equation and its family of solitary waves.

One way of implementing this idea is to analyse a general class of equations which form a neighbourhood of a soliton equation. The parameter of the equations in this neighbourhood, e.g., the shape of the solitary waves, may then provide indicators of necessary and sufficient conditions for the existence of soliton solutions.

This approach was first used by Abbas [1] when he considered a class of third order PDEs with quadratic nonlinearities defined by

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0, \tag{1.13}$$

where $a_i \in \mathbb{R}$. This class forms a neighbourhood of the KdV and it also contains another well-known equation, namely the regularized long wave equation (RLW).

$$u_t + u_x + uu_x - u_{xxt} = 0 . \tag{1.14}$$

The basis on which this class was constructed was that suggested by Broer [7] in 1964 and subsequently (1972) used by Benjamin et al. [5] in their construction of the RLW. Note that this equation was derived as an approximation to the equations of hydrodynamics by Peregrine [23]. Broer's suggestion was that the essential properties of equations such as the KdV were a result of the interaction between dispersion and nonlinearity so that the equation can be analysed in terms of its linear and nonlinear parts. As a consequence, the introduction of these terms could be based on independent physical considerations. This point was clearly emphasised in the paper of Benjamin et al. who argued on the grounds of physically allowable dispersion relations what the RLW was a better model than the KdV. Furthermore, at this time it was generally believed that the existence of stable solitary waves was due to a balance between nonlinearity and dispersion (see Scott et al. [25]).

Abbas' work was concerned with testing this hypothesis on the general class of equations (1.13). Firstly, he found that the predicted properties of the solitary wave of the KdV i.e. on the basis of the dispersion relation, contradicted the actual properties. Secondly, he obtained periodic and solitary wave solutions of the equations in (1.13) and classified them in terms of their dispersion relations. He showed that linearly stable solitary wave solutions with the KdV profile ($sech^2$) exist for a variety of dispersion relations including a formally nondispersive subclass. This indicated that the existence of stable solitary waves could not be interpreted as a balance between nonlinearity and dispersion. The general conclusion that Abbas drew from these results was that the dispersion relation was not a useful predictor of the properties of a nonlinear equation.

Before developing other criteria for understanding the existence of solitary waves and solitons in the class of equation (1.13), it was important to establish the well-posedness of the class for reasonable data. This was done by El-Sherbiny [11] who showed that equations (1-13) formed four equivalence classes. Using the method of characteristics he proved well-posedness for a subset of equations in these equivalance classes and for the remaining equations, where the method of characteristics fails, he proved well-posedness for some cases and established necessary existence theorems for most of the others. Note that the well-posedness of the KdV and RLW were proved earlier by Bona and Smith [6] and Benjamin et al. [5]. Having established a well-defined neighbourhood of the KdV, El-Sherbiny then looked at the existence of conservation laws. Apart from the KdV with its infinite number of conservation laws, he showed that the rest of the class have at least two conservation laws, but not more than three. This established the uniqueness of the KdV in the class (1.13) on the basis of the number of conservation laws.

The next step was to look at other properties of the class. Now at that time considerable interest had developed in the analytic structure of general solutions of nonlinear evolution equations as a result of the Painlevé conjectures [4, 30] which related this property to the integrability of the equations. Therefore it seemed appropriate to investigate this aspect of the class of equations (1.13).

In this thesis we examine the analytic structure of the general solutions of the equations in (1.13) in order to establish further evidence of the uniqueness of the

KdV in this class. Following this we return to our original question of determining when a solitary wave is a soliton and look at the solitary waves in the class to establish criteria for identifying the KdV. Our original contributions are as follows:

(1) Using symmetry reductions we obtain all the ODEs of similarity solutions for the equations in (1.13).

(2) We determine the analytic structure of general solutions of the equations obtained in (1) and hence initiate a classification of third order nonlinear ODEs.

(3) Using Laurent-type expansions we determine the analytic structure of the general solutions of the PDEs in (1.13).

(4) We compare the results obtained in (2) and (3) and use them to identify equations of Painevé type.

(5) We analyse the detailed structure of the solitary waves of the equations in (1.13) and establish criteria for a solitary wave to be a soliton.

The layout of this thesis is as follows: Together with the introduction, the thesis consists of ten chapters and four appendices, the references for each chapter being presented separately at the end of the thesis. The thesis falls naturally into two parts. Following the introduction, Chapters 2, 3, 4 and 5 contain essentially known results. A part of Chapter 3 contains an original contribution and the presentation, discussions and criticisms in all chapters are our own.

Our own contributions are presented in Chapters 6, 7, 8 and 9 and in Chapter 10 we give our concluding remarks. A summary of chapters 2 to 9 is as follows:

Chapter 2: We review the inverse scattering method, solitons and conservation laws with particular reference to the KdV. We also discuss a connection between conservation laws and the inverse scattering method.

Chapter 3: We define integrability for both ODEs and PDEs and implement this for a nontrivial Hamiltonian system of ODEs. In the case of PDEs we show how they can be integrated via linear integral equations obtaining a wider class of solutions than via IST. Finally, we discuss the Painlevé conjectures for PDEs.

Chapter 4: An outline of the Lie-group method for finding similarity transformations of PDEs is given and applied to the KdV. A review of the singularity structure of the solutions of ODEs is then presented together with techniques for analysing their structure. Finally, a direct method of analysing the general solution of an integrable PDE is introduced.

Chapter 5: We review the work of Abbas and El-Sherbiny on the equations in (1.13).

Chapter 6: We obtain the one-parameter Lie groups of local symmetries of the equations in (1.13). We refine the classification of the equations according to the infinitesimal generators of these groups and use them to obtain the corresponding similarity reductions to third order ODEs.

Chapter 7: We determine the analytic structure of the solutions of the ODEs obtained in Chapter 6. Those that can be integrated to become second order are classified using the Painlevé list. The remainder, which are irreducible third order, we investigate using local power series expansions and classify as Painlevé type or not. We discuss our findings and suggest necessary conditions for the existence of non-analytic general solutions and the non-existence of logarithmic branch points. Finally, we initiate a classification for third order nonlinear ODEs.

Chapter 8: We determine directly the analytic structure of general solutions of the PDEs in each of the four equivalence classes by using local Laurent-type expansions. The results are used to refine the classification of Chapter 7 in terms of the ODEs and to establish the theorem that the KdV is the only equation in (1.13) of Painlevé type. Next we look for special meromorphic solutions with a view to identifying equations which may have restricted N-soliton solutions. This led to a rigorous proof that the RLW has no soliton solutions.

Chapter 9: We analyse the detailed structure of the solitary waves of the equations in (1.13) in terms of their width and amplitude parameters and reduce them to generic forms by using certain reasonable selection rules. Using these generic forms we establish necessary criteria for a solitary wave to be a soliton.

CHAPTER TWO

INVERSE SCATTERING TRANSFORM, SOLITONS AND CONSERVATION LAWS

In this chapter we describe the development of the IST method for solving the KdV equation. The method consists of transforming the equation to a linear integral equation via a scattering problem defined by the time-independent Schrödinger equation. We discuss the method as orginally developed by Gardner, Greene, Kruskal and Miura (GGKM) and then consider its generalization by other authors. As an illustration, the method is used to obtain the N-soliton solutions of the KdV equation. Finally, we review results on conservation laws and show a relationship between the IST method and the infinite number of local conservation laws possessed by the equation.

2.1 The Inverse Scattering Method of GGKM [7]

Consider the solution $u(x, t)$ of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in (-\infty, \infty), \quad t > 0, \tag{2.1.1}$$

$$u(x, 0) = g(x) \tag{2.1.2}$$

and let $u(x, t)$ be the potential in the Schrödinger equation

$$v_{xx} - [u(x, t) - \lambda]v = 0, \qquad (2.1.3)$$

where t is considered to be a parameter.

The method consists of using the properties of the given function $u(x, 0)$ to determine the initial eigenvalues and eigenfunctions of (2.1.3) and then using the resulting scattering data to compute $u(x, t)$ from the Gelfand-Levitan-Marchenko equations.

For initial values $u(x, 0) = g(x)$, to obtain the values of λ for which there exist solutions $v(x)$ that are bounded as $|x| \longrightarrow \infty$, we require that the growth condition

$$\int_{-\infty}^{\infty} |g(x)|(1 + |x|)dx < \infty \qquad (2.1.4)$$

is to be satisfied [6].

The set of all eigenvalues λ is called the spectrum corresponding to a given potential $g(x)$. Computing the spectrum for $t = 0$ gives, in general, a finite number of discrete eigenvalues $\lambda = - K_n^2(0)$ and a continuous part $\lambda = k^2(0)$.

The behaviour of the eigenfunctions corresponding to the discrete eigenvalues $\lambda(t) = - K_n^2(t)$ may be assumed to be given by

$$v_n(K_n, x; t) = \begin{cases} c_n(t)e^{-K_n(t)x} & \text{for } x \to \infty \\ \tilde{c}_n(t)e^{K_n(t)x} & \text{for } x \to -\infty \end{cases} \tag{2.1.5}$$

where $c(t)$ are the normalized coefficients and (2.1.5) is known to be true for $t = 0$.

For the continuous spectrum the asymptotic behaviour of the eigenvalues which correspond to $\lambda(t) = k^2(t)$ may be assumed to be given by

$$v_n(k, x; t) = \begin{cases} e^{-ikx} + b(k, t)\, e^{ikx} & \text{for } x \to \infty \\ a(k, t)e^{-ikx} & \text{for } x \to -\infty \end{cases} \tag{2.1.6}$$

where $a(k, t)$ are the transmission coefficients and $b(k, t)$ the reflection coefficients (a and b occur when a wave sent in from $-\infty$ interacts with a potential and some is reflected and the remainder transmitted). Equation (2.1.6) is known to be true when $t = 0$ and $a(k, 0)$, $b(k, 0)$ can be computed from $g(x)$. Further, a and b are related by the conservation law

$$|a|^2 + |b|^2 = 1.$$

The spectrum of the Schrödinger equation, together with the coefficients $c_n(t)$, $a(k, t)$ and $b(k, t)$ are called the scattering data of the potential $u(x, t)$. To obtain the scattering coefficients for any $t > 0$ we impose on the eigenfunctions a specific evolution which keeps the spectrum time-invariant. These results are given by the following theorem.

Theorem 2.1 [7], [11]

If $u(x, t)$ evolves according to the KdV equation (2.1.1) with initial data satisfying (2.1.4), then $u(x, 0) \to 0$ as $|x| \to \infty$ and the following relations are satisfied

(i) $\lambda(t) = \lambda(0)$

(ii) $b(k, t) = b(k, 0)\exp(8ik^3t)$

(iii) $c_m(t) = c_m(0)\exp(4K_m^3t)$

(iv) $a(k, t) = a(k, 0)$

where $c_m(0)$, $b(k, 0)$ and $a(k, 0)$ are determined from the initial data $g(x)$ □

The potential of the Schrödinger equation can now be recovered from the scattering data for any $t > 0$ by solving the inverse scattering problem. For this we use the Gelfand-Levitan-Marchenko integral equation

$$K(x,y;t) + B(x+y;t) + \int_x^{\infty} B(z+y;t)K(x,z;t)dz = 0 \tag{2.1.7}$$

where

$$B(\xi,t) = \sum_{n=1}^{N} c_n^2(t)e^{-K_n(\xi)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)e^{ik\xi}d\xi \tag{2.1.8}$$

The solution of the initial value problem for the KdV equation is now obtained from the formula

$$u(x, t) = -2\frac{d}{dx} K(x, x; t) \tag{2.1.9}$$

where $K(x, y; t)$ is the solution of the integral equation (2.1.7).

Thus the original problem of the nonlinear PDE (2.1.1-2) is transformed and reduced, in this way, to the problem of solving a one-dimensional linear integral equation. This linearization problem has been studied by many people including [11], [16].

2.2 Lax's Development of Inverse Scattering.

Peter Lax [10] generalized this method to any evolution equation of the type

$$u_t = K(u) \tag{2.2.1}$$

where K is a nonlinear operator. Lax's idea was that if a PDE could be cast into the form

$$iL_t = [B, L] = BL - LB \quad (2.2.2)$$

where L and B are linear, differentiable operators on some Hilbert space of functions then the eigenvalues λ of L would be independent of time and its eigenfunctions v would evolve according as

$$iv_t = Bv. \quad (2.2.3)$$

For the KdV equation the operators in the Lax representation are

$$L = -\frac{\partial^2}{\partial x^2} + u$$

$$B = -4i\frac{\partial^3}{\partial x^3} + 3i\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right)$$

Then $iv_t = Bv$ gives the time dependence of the spectral data as in Theorem 2.1.

Details of the work of Lax can be found in [6], [10] and [13].

Lax further stated that other choices of the operator L should lead to other classes of equations. This led the way to the important breakthrough in the development of the method of inverse scattering by Zakharov and Shabat [18] in 1972. Using two component eigenfunctions these authors showed that one can find a pair of operators L and B satisfying the conditions of the Lax method such that the potentials $u(x, t)$ are solutions of the nonlinear Schrödinger equation (NSE)

$$u_t = i(u_{xx} + 2u^2u^*), \qquad (2.2.4)$$

where u^* denotes the complex conjugate of u. Solutions of this equation by the method of inverse scattering was the first demonstration that the method is not limited to the KdV family. Motivated by the work of Zakharov and Shabat, Wadati [14] solved the modified KdV equation (1.7). Later, in 1973, Ablowitz, Kaup, Newell and Segur (AKNS) [1] solved the SGE and went on to generalize the method to include all the above mentioned equations.

2.3 The Generalization of AKNS. [2]

AKNS found that many equations can be solved by the two component scattering problem of Zakharov and Shabat

i.e. $$Lv = \lambda v \qquad (2.3.1)$$

where

$$L \equiv \begin{bmatrix} i\frac{\partial}{\partial x} & -iq \\ ir & -i\frac{\partial}{\partial x} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad (2.3.2)$$

and the coefficients $q(x, t)$, $r(x, t)$ are arbitrary.

Choosing the time dependence of the eigenfunctions (v_1, v_2) to be

$$iv_t = Bv \tag{2.3.3}$$

where

$$B = \begin{bmatrix} A(x, t; \lambda) & D(x, t; \lambda) \\ C(x, t; \lambda) & -A(x, t; \lambda) \end{bmatrix}, \tag{2.3.4}$$

the eigenvalues λ are time invariant when

$$\frac{\partial A}{\partial x} = qC - rD \tag{2.3.5a}$$

$$\frac{\partial D}{\partial x} + 2i\lambda D = i\frac{\partial q}{\partial t} - 2Aq \tag{2.3.5b}$$

$$\frac{\partial C}{\partial x} - 2i\lambda C = i\frac{\partial r}{\partial t} + 2Ar \tag{2.3.5c}$$

Equations (2.3.5) are obtained by cross differentiating (2.3.1) and (2.3.3). By various choices of the elements of B the conditions (2.3.5) generate a large class of evolution equations which are solvable by inverse scattering.

For example, take

$$A = 4\lambda^3 + 2qr\lambda + irq_x - iqr_x .$$

Conditions (2.3.5) yield

$$q_t - 6rqq_x + q_{xxx} = 0, \qquad (2.3.6a)$$

$$r_t - 6rqr_x + r_{xxx} = 0 . \qquad (2.3.6b)$$

When $r = -1$, (2.3.6) reduces to the KdV equation and the system of equations (2.3.1) reduces to the Schrödinger equation

not squared.

$$v_{2xx} + [q(x, t) + \lambda]v_2 = 0 . \qquad (2.3.7)$$

When $r = \pm q$, we obtain the modified KdV equation

$$q_t \mp 6q^2q_x + q_{xxx} = 0 \qquad (2.3.8)$$

Other well known PDEs such as the SGE and the NSE can be obtained by different choices of A [3].

In addition the following equations have been solved by IST.

(1) Higher order KdV equation [12]

$$u_t = \frac{1}{4}(u_{xxxx} + 5u_x^2 + 10uu_x + 10u^3)_x \qquad (2.3.9)$$

(2) The Boussinesq Equation [19]

$$(u_{xx} + 6u^2 + u)_{xx} - u_{tt} = 0 \qquad (2.3.10)$$

(3) The Derivative NSE [4]

$$iu_t = u_{xx} - 4iu_x^* + 8|u|^4u \qquad (2.3.11)$$

(4) The Kadomtsev-Petiashili (KP) or 2D KdV Equation [5]

$$(u_{xxx} + 12uu_x + u_t)_x \pm u_{yy} = 0 \qquad (2.3.12)$$

2.4 The N-Soliton solutions of the KdV Equation

The solitary wave of any nonlinear evolution equation is called a soliton if there exist solutions for this equation which approach a linear superposition of its solitary waves as $|t| \longrightarrow \infty$. The interaction between solitary waves for the KdV equation (2.1.1) was first observed numerically by Zabusky and Kruskal [17]. They showed that if two solitary waves are placed on the real line with the taller to the left of the shorter at $t = 0$ and are travelling to the right, then, after some time, they interact and the taller overtakes the shorter and they both regain their original shapes and velocities as $t \longrightarrow \infty$. The only change is that a phase shift occurs. Lax [10] discussed the same phenomena analytically and confirmed Zabusky and Kruskal's observations.

The exact solution for the case of multiple collisions of N-solitons with different amplitudes for the KdV equation was first found by Hirota [8]. However, Hirota's method, which we do not cover here, was heuristic. Wadati and Toda [16] were the first to give the exact N-soliton solution of the KdV equation through the procedure suggested by GGKM.

N-soliton solutions for the modified KdV and other well known evolution equations may be found in many references including [2], [14] and [18].

To obtain N-soliton solutions, it is required that the spectrum of the initial profile is discrete and this is equivalent to the reflection coefficient in (2.1.8) being zero i.e. $b(k, t) = 0$. This reduces the Gelfand-Levitan-Marchenko equation (2.1.7) to

$$K(x, y) + \sum_{n=1}^{N} c_n^2 \exp\left\{- \kappa_n (x + y)\right\}$$

$$+ \sum_{n=1}^{N} \int_x^{\infty} c_n^2 \exp\left\{- \kappa_n (z + y)\right\} K(x, z)dz = 0 \tag{2.4.1}$$

where $c_n = c_n(t) = c_n(0)\exp(4\kappa_n^3 t)$.

It follows from (2.4.1) that $K(x, y)$ is necessarily of the form

$$K(x, y) = - \sum_{n=1}^{N} c_n f_n(x)\exp(-\kappa_n y) \tag{2.4.2}$$

[Note: c_n have been introduced so that the f_n turn out to be normalized eigenfunctions of the Schrödinger equation.]

Substituting (2.4.2) into (2.4.1) enables us to factor out the y-dependence and equating the coefficients of $\exp(-K_n y)$ to zero, we obtain the following N linear equations in f:

$$f_n(x) + \sum_{\ell=1}^{N} c_n c_\ell f_\ell(x) \exp \frac{\left[-(K_n+K_\ell)x\right]}{K_n+K_\ell} = c_n \exp(-K_n x)$$

$$n = 1, 2, \ldots, N \qquad (2.4.3)$$

These can be rewritten in the form

$$(I + C)f = E \qquad (2.4.4)$$

where I is the unit matrix of order N,

$$C = [C_{\ell n}], \; c_{\ell n} = c_\ell c_n \left\{ \exp \frac{\left[-(K_n+K_\ell)x\right]}{K_n+K_\ell} \right\}$$

$$(2.4.5)$$

is an $N \times N$ matrix and

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{bmatrix} \qquad \text{and} \qquad E = \begin{bmatrix} c_1 \exp(-K_1 x) \\ c_2 \exp(-K_2 x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_N \exp(-K_N x) \end{bmatrix} \tag{2.4.6}$$

are column vectors.

A sufficient condition under which (2.4.4) has a unique solution is that C is positive definite. This follows since the quadratic form corresponding to C is

$$\sum_{n=1}^{N} \sum_{\ell=1}^{N} c_\ell c_n \exp \frac{\left[-(K_n+K_\ell)x\right]}{K_n+K_\ell} x_n x_\ell$$

$$= \int_x^\infty dz \left[\sum_{n=1}^{N} c_n \exp(-K_n z) x_n \right]^2$$

which is positive. Now,

$$\det C = \left[\prod_{n=1}^{N} c_n^2 \right] \exp\left[-2 \left(\sum_{n=1}^{N} K_n \right) x \right]$$

$$\times \det\left[(K_n + K_\ell)^{-1} \right] > 0 \qquad (2.4.7)$$

so that $\det\left[(K_n + K_\ell)^{-1} \right] > 0$.

From (2.4.7) we can write C as $\det C = \alpha \exp(-\beta x)$, where α and β are positive. Then expanding along the nth column we have

$$\Delta = \det(I + C) + \sum_{n=1}^{N} \left[\delta_{n\ell} + c_n c_\ell \exp \frac{\left[-(K_n + K_\ell)x\right]}{K_n K_\ell} \right] Q_{n\ell} \qquad (2.4.8)$$

where $Q_{n\ell}$ is the co-factor of the coefficient matrix $I + C$.

Using Cramer's rule to solve (2.4.4) gives

$$f_n = \frac{1}{\Delta} \sum_{n=1}^{N} c_\ell \exp(-K_n x) Q_{n\ell} \qquad (2.4.9)$$

Replacing y by x in the expression of $K(x, y)$ in (2.4.2) and using (2.4.9), we have

$$K(x, x) = - \sum_{n=1}^{N} c_n f_n \exp(-\kappa_n x)$$

$$+ \frac{1}{\Delta} \sum_{n=1}^{N} \sum_{\ell=1}^{N} c_n c_\ell \exp\left[-(\kappa_\ell + \kappa_n\right] Q_{n\ell}$$

$$= \frac{1}{\Delta} \frac{d}{dx} \Delta = \frac{d}{dx} \ln\Delta \qquad (2.4.10)$$

Substituting (2.4.10) into (2.1.9)

$$u(x, t) = -2 \frac{d}{dx} K(x, x) = -2 \frac{d^2}{dx^2} \ln\Delta$$

$$= -2 \frac{d^2}{dx^2} \ln[\det(I + C)] \qquad (2.4.11)$$

which is a solution of the KdV equation corresponding to a reflectionless potential.

If we now consider a single soliton solution, i.e. $N = 1$, then

$$B(x, t) = c^2(t) e^{-\kappa(x+y)} \qquad (2.4.12a)$$

$$c(t) = c(0) e^{4\kappa^3 t} \qquad (2.4.12b)$$

and we put $K(x, y) = f(x) e^{-\kappa y}$ (2.4.13)

Substituting (2.4.12) and (2.4.13) into (2.1.7) gives

$$f(x)e^{-Ky} + c^2e^{-K(x+y)} + \int_x^{\infty} c^2e^{-K(x+z)}f(x)e^{-Kz}dz = 0. \tag{2.4.14}$$

Thus, $f(x) = \dfrac{-c^2(t)e^{Kx}}{1+\dfrac{c^2(t)e^{2Kx}}{2K}}$

It follows that

$$u(x,\ t) = 2K^2\text{sech}^2[K(4K^2t-x) + \delta] \tag{2.4.15}$$

where $\delta = \frac{1}{2}\ln\left(\frac{c^2(0)}{2K}\right)$.

For a two soliton solution, i.e. N = 2, the procedure is the same where we now put

$$K(x,\ y) = f_1(x)e^{-K_1y} + f_2(x)e^{-K_2y} \tag{2.4.16}$$

and proceed as above. Then it is easy to show that as $t \to \infty$ the two solutions approach a linear superposition and the solution asymptotically becomes

$$u(x,\ t) = 2\left\{\left[K_1^2\mathrm{sech}^2[K_1(x-4K_1^2t) - \delta_1]\right.\right.$$

$$\left.\left. + K_2^2\mathrm{sech}^2[K_2(x-4K_2^2t - \delta_2]\right]\right\} \qquad (2.4.17)$$

where $\delta_i = \frac{1}{2}\ln\frac{c_i^2(0)}{2K_i} \quad i = 1,\ 2\ .$

Thus to summarize the work of this chapter so far, solitons are obtained from the IST method as follows:

(1) Set up an appropriate linear scattering eigenvalue problem in the space variable where the solution of the nonlinear evolution equation plays the role of the potential.

(2) Choose the time dependence of the eigenfunctions in such a way that the eigenvalues remain time invariant as the potential evolves according to the evolution equation.

(3) Solve the direct scattering problem at the initial time and determine the time dependence of the scattering data.

(4) Considering discrete eigenvalues only, corresponding to the bound states, and knowing the time dependence of the other scattering data,

reconstruct the potential i.e. the inverse scattering problem.

Clearly if the scattering problem does not have discrete eigenvalues then solitons do not exist.

For example, if we take $r = +q$ and q real in (2.3.1), we obtain the modified KdV equation

$$q_t - 6q^2q_x + q_{xxx} = 0 .$$

However, the eigenvalue problem posed by (2.3.1) becomes self-adjoint and hence all eigenvalues are real. In this case no solitons arise and the final state can be shown to decay algebraically in time [2].

Note: The choice $r = -q$ corresponds to what is usually known as the modified KdV equation which does have solitons

$$\text{i.e.} \quad q_t + 6q^2q_x + q_{xxx} = 0 .$$

A second example is given with the choice

$$A = -\frac{1}{4\lambda}\cosh u \qquad r = -q = \frac{1}{2}u_x \tag{2.4.18}$$

yielding the sinh-Gordon equation

$$u_{xy} = \sinh u \tag{2.4.19}$$

Again the eigenvalue problem posed by (2.3.1) is self-adjoint and hence all eigenvalues are real. Thus no solition solutions exist.

2.5 Conservation Laws

Consider the evolution equation

$$u_t = K(u), \quad x \in (-\infty, \infty), \quad t > 0 . \tag{2.5.1}$$

When a functional $I[u]$ of a solution $u(x, t)$ of (2.5.1) satisfies

$$\frac{d}{dt} I[u] = 0 \tag{2.5.2}$$

then the functional I is said to be a constant of motion or an integral of (2.5.1). Usually constants of motion are derived from local conservation laws. These have the form

$$\frac{\partial}{\partial t} D[u(x, t)] + \frac{\partial}{\partial x} F[u(x, t)] = 0 \tag{2.5.3}$$

where D and F are called the conserved density and conserved flux respectively and $I[u]$ is defined by

$$I[u] = \int_{-\infty}^{\infty} D[u(x, t)]dx \tag{2.5.4}$$

The existence of $I[u]$ depends on the convergence of this integral and F tending to zero as $|x| \to \infty$.

It has been proved that the KdV equation [11], the modified KdV equation [12] and the SGE [9] each have an infinite number of independent local conservation laws

$$\frac{\partial D_i}{\partial t} + \frac{\partial F_i}{\partial x} = 0 \qquad (2.5.5)$$

For the multisoliton solutions (N = 1, 2, ...) these lead to an infinite number of independent constants of the motion. Although it has not been proved, it appears that an infinite number of independent conservation laws is necessary for the existence of an IST. We will come back to this point later.

As examples of conservation laws and constants of the motion for solutions, we give the first three constants of motion and flux for the KdV, modified KdV and the SGE:

(1) The KdV Equation

$$I_1 = \int_{-\infty}^{\infty} u \, dx, \qquad F_1 = 3u^2 + u_{xx} \qquad (2.5.6a)$$

$$I_2 = \int_{-\infty}^{\infty} \frac{u^2}{2} \, dx, \qquad F_2 = 2u^3 + uu_{xx} - \frac{u_x^2}{2}$$

(2.5.6b)

$$I_3 = \int_{-\infty}^{\infty} \left[\frac{u^3}{3} - \frac{u_x^2}{6}\right] dx, \quad F_3 = \frac{3}{2} u^4 + u^2 u_{xx} - 2uu_x^2$$

$$- \frac{1}{3} u_x u_{xxx} + \frac{1}{6} (u_{xx})^2$$

(2.5.6c)

Note: F_n includes $\partial_x^n u$ and hence an infinite number of constants of the motion will only exist for infinitely differentiable solutions whose derivatives all go to zero as $|x| \to \infty$.

(2) <u>The Modified KdV Equation</u>

$$I_1 = \int_{-\infty}^{\infty} u \, dx, \quad F_1 = 2u^3 + u_{xx} \qquad (2.5.7a)$$

$$I_2 = \int_{-\infty}^{\infty} \frac{u^2}{2} dx, \quad F_2 = \frac{3}{2} u^4 + uu_{xx} - \frac{1}{2} u_x^2$$

(2.5.7b)

$$I_3 = \int_{-\infty}^{\infty} \left[\frac{u^4}{4} - \frac{u_x^2}{4}\right] dx, \quad F_3 = u^6 + u^3 u_{xx}$$

$$- 3u^2 u_x^2 - \frac{u_x}{2} u_{xxx} + \frac{u_{xx}^2}{4}$$

(2.5.7c)

(3) The sine-Gordon Equation

$$I_1 = \int_{-\infty}^{\infty} \frac{u_x^2}{2}\, dx, \qquad F_1 = \cos u \qquad (2.5.8a)$$

$$I_2 = \int_{-\infty}^{\infty} \left[\frac{u_x^2}{4} - u_{xx}^2\right] dx, \qquad F_2 = u_x^2 \cos u \qquad (2.5.8b)$$

$$I_3 = \int_{-\infty}^{\infty} \left[\frac{1}{6}\, u_x^6 - \frac{2}{3}\, u_x^2 u_{xx}^2 + \frac{8}{9}\, u_x^3 u_{xxx} + \frac{4}{3}\, u_{xxx}^2\right] dx,$$

$$F_3 = \cos u \left[\frac{1}{9}\, u_x^4 - \frac{4}{3}\, u_{xx}^2\right] \qquad (2.5.8c)$$

2.5.1 Relationship between the IST method and Conservation Laws.

In 1974, Konno, Sanuki and Ichikawa [9] used the scattering problem (2.3.1) devised by AKNS and proved the following theorem.

Theorem 2.2 [9]

The conservation laws of the KdV equation (2.1.1) can be obtained from the IST method □

Later Wadati, Sanuki and Konno [15] extended this work to a general class of equations. Here, we give an outline of their method.

Introducing the variables

$$\Gamma_1 = \frac{v_2}{v_1} \quad \text{and} \quad \Gamma_2 = \frac{v_1}{v_2} \tag{2.5.1.1}$$

Equations (2.3.1) and (2.3.3) can then be written as

$$\frac{\partial \Gamma_1}{\partial x} = -\,2\eta\Gamma_1 + r - q\Gamma_1^2 \tag{2.5.1.2a}$$

$$\frac{\partial \Gamma_1}{\partial t} = C - 2A\Gamma_1 - B\Gamma_1^2 \tag{2.5.1.2b}$$

and

$$\frac{\partial \Gamma_2}{\partial x} = 2\eta\Gamma_2 + q - r\Gamma_2^2 \tag{2.5.1.3a}$$

$$\frac{\partial \Gamma_2}{\partial t} = B + 2A\Gamma_2 - C\Gamma_2^2 \tag{2.5.1.3b}$$

where we have put $\eta = -i\lambda$

Clearly, the systems (2.5.1.2) and (2.5.1.3) are not independent. For the derivation of conservation laws from the inverse method, the 'time' parts may be replaced by

$$\frac{\partial}{\partial t}(q\Gamma_1) = \frac{\partial}{\partial x}(A + B\Gamma_1) \tag{2.5.1.4a}$$

$$\frac{\partial}{\partial t}(r\Gamma_2) = \frac{\partial}{\partial x}(-A + C\Gamma_2) \tag{2.5.1.5a}$$

which can be verified from (2.3.1) and (2.3.3).

Corresponding to these expressions, the 'space' parts are written as

$$2\eta(q\Gamma_1) = rq - (q\Gamma_1)^2 - q\Big[(q\Gamma_1)/q\Big]_x \qquad (2.5.1.4b)$$

$$2\eta(r\Gamma_2) = -rq + (r\Gamma_2)^2 + r\Big[(r\Gamma_2)/r\Big]_x \qquad (2.5.1.5b)$$

We can now derive conservation laws from the modified Riccati form of inverse method equation (2.5.1.4) and (2.5.1.5). The same argument is possible for $r\Gamma_2$.

Expand (2.5.1.4b) in power series of $\frac{1}{\eta}$

$$q\Gamma_1 = \Big[rq - q^2\Gamma_1^2 - q\{(q\Gamma_1)q\}_x\Big]/2\eta \qquad (2.5.1.6)$$

$$q\Gamma_1 = \sum_{n=1}^{\infty} f_n\ \eta^{-n} \qquad (2.5.1.7)$$

Equating powers of $(1/\eta)$ we obtain

$$f_{n+1} = \frac{1}{2}\left[(rq)\delta_{n,o} - \sum_{k=1}^{n-1} f_k f_{n-k} - q(f_n/q)_x\right] \qquad (2.5.1.8)$$

for which

$$f_1 = \frac{1}{2}\,rq$$

$$f_2 = -\left(\frac{1}{2}\right)^2 qr_x$$

$$f_3 = -\left(\frac{1}{2}\right)^3 \left[r^2q^2 + q_x r_x - (qr_x)x\right] \qquad (2.5.1.9)$$

. . . .

We note that (2.5.1.4a) is the form of conservation law

$$\frac{\partial}{\partial t}(q\Gamma_1) = \frac{\partial}{\partial x}(A + Bq\Gamma_1/q) \qquad (2.5.1.10)$$

Therefore for a given nonlinear equation i.e. for a given A and B, we can write down conservation laws by using a recurrence formula (2.5.1.8) and equating the terms of the same powers of $(1/\eta)$ in (2.5.1.5b).

2.6 Conclusion

In this chapter we have presented a review of the inverse scattering transform methods. We have shown how N-soliton solutions are obtained, via IST, for the KdV equation and we have looked at conservation laws and their relationship to the IST method. Although it has not been proved, it is believed that an infinite number of conservation laws is a necessary condition for the applicability of the IST method. As we shall see, the RLW equation has been shown to possess only three independent conservation laws and so far no inverse scattering method has been found for it. The conjecture above implies such a method of solution does not exist for this equation. This is true for many other equations, including many equations of the general class which we shall study later.

Given the difficulty of obtaining an associated scattering problem for an arbitrary PDE, the question arises as to whether there is any way of testing the PDE to see whether it is of IST class or not. Tests for integrability have been proposed, but, before reviewing them, we look at integrability and an 'extension' of the IST method in the following chapter.

CHAPTER THREE

INTEGRABILITY AND TESTS FOR INTEGRABILITY

In this chapter we discuss the concept of integrability for nonlinear PDEs and review the work which led to the "Painlevé Conjecture" of Ablowitz, Ramani and Segur (ARS) [1-4]. We start with the definition of integrability for ODEs and PDEs. In the case of ODEs, for Hamiltonian systems, we have Liouville's theorem as a test for integrability, whereas in the case of nonlinear PDEs we have IST. Furthermore, IST can be used to define some sort of Hamiltonian structure for nonlinear PDEs and in this sense ties up with the Liouville theorem. On the other hand it may be possible to map the linear integral equations of Gelfand-Levitan to a nonlinear PDE directly without the need for a scattering problem. In this case we will obtain a wider class of solutions than IST because there are no restrictions imposed on the solutions by the scattering problem. It turns out that this can be done for a variety of PDEs and we demonstrate it for the KdV. However, we doubt that this method could be applied if one did not have prior knowledge of an associated scattering problem.

The main question, therefore, is how one can know, a priori, that a system of PDEs can be reduced to a linear integral equation. Attempts have been made in this direction by analyzing the analytic structure of the solution manifold. In particular, this has led to two conjectures known as the Painlevé conjectures and we shall discuss these also.

3.1 Integrability

We introduce the concept of integrability for differential equations in a constructive manner by means of the following definition.

Definition 3.1

Let $u_t = K(u)$ be a system of ordinary or partial differential equations. The system is said to be integrable if it can be reduced to a linear system which can be represented as an integral equation.

For example, if a nonlinear system of ODEs is reduced to the linear system

$$\frac{dx}{dt} = A(t)x(t) \qquad (3.1.1)$$

then clearly this can be written in the equivalent integral equation form

$$x(t) = x(0) + \int_0^t A(s)x(s)dt \qquad (3.1.2)$$

where we maintain the requirement for differentiable solutions.

Now we have seen, in the previous chapter, that the KdV and other systems can be reduced to the linear integral equation

$$K(x, y) = F(x + y) + \int_x^{\infty} K(x, z)N(x; z, y)dz \qquad y \geq z$$

(3.1.3)

where N and F are given functions. This is a two-dimensional generalization of (3.1.2).

Thus our definition is a constructive one for both ODEs and PDEs.

The question then arises as to how we can implement this definition. In the case of Hamiltonian systems of ODEs this is a theorem, namely Liouville's theorem, which gives conditions which identify integrable systems. Such systems are described as completely integrable, whereas a PDE reducible to (3.1.3) is called integrable as (3.1.3) may only apply to part of the solution space. Before stating Liouville's theorem, we define some of the terms used in the theorem.

A Hamiltonian system is a system of first order PDE's of the form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, \ldots, n$$

(3.1.4)

where H is a differentiable function of the 2n variables x_i and y_i.

Consider the functions $F(x_i, y_i)$, $G(x_i, y_i)$. The Poisson bracket of these two functions is defined as

$$[F(x_i, y_i), G(x_i, y_i)] = \sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial y_i} - \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial x_i}\right]$$

(3.1.5)

If $[F, G] = 0$ then F and G are said to be in involution.

Now consider $G = H$. Then

$$[F, H] = \sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial F}{\partial y_i}\frac{\partial H}{\partial x_i}\right]$$

$$= \sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial F}{\partial y_i}\frac{dy_i}{dt}\right] = \frac{dF}{dt}(x_i, y_i) \; .$$

(3.1.6)

Thus $[F, H] = 0$ implies that F is a constant of the motion.

Two functions F and G are independent over the 2n variables if gradF is nowhere parallel to gradG.

A canonical transformation is a transformation of the variables $(x_i, y_i) \to (\xi_i, \eta_i)$ with $H \to K$ such that

$$\frac{d\xi_i}{dt} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial K}{\partial \xi_i}. \qquad (3.1.7)$$

We are now ready to state Liouville's theorem.

<u>Theorem 3.1</u> (Liouville's Theorem)

Suppose that the Hamiltonian system (3.1.4) has n independent constants of the motion F_i, $i = 1, 2, , n$ which are in involution. Then there exists a canonical transformation which determines a new set of variables ξ_i, conjugate to $\eta_i = F_i$ such that the Hamiltonian K is only a linear function of the η_i. Thus the equations of motion in the new variables take the simple form

$$\frac{d\eta_i}{dt} = 0, \quad \frac{d\xi_i}{dt} = \text{constant} \qquad \square.$$

Note: If η is a constant of motion so is any function of η_i. Thus the choice of constants for the new variables is modulo this property.

We illustrate the above theorem with a non-trivial example.

<u>Example</u>

Consider the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{(x_1-x_2)^2} \qquad (3.1.8)$$

The equations of motion are

$$\frac{dx_1}{dt} = y_1, \quad \frac{dx_2}{dt} = y_2 \qquad (3.1.9a)$$

$$\frac{dy_1}{dt} = \frac{2}{(x_1-x_2)^3}, \quad \frac{dy_2}{dt} = -\frac{2}{(x_1-x_2)^3} \qquad (3.1.9b)$$

and two constants of motion are given by

$$y_1 + y_2 = \text{const.}$$

$$H = \text{const.}$$

Taking the new η variables as

$$\eta_1 = \frac{1}{2}(y_1 + y_2) = \alpha_1(\text{const.}) \qquad (3.1.10)$$

$$\eta_2 = \frac{1}{4}(y_1-y_2)^2 + \frac{1}{(x_1-x_2)^2} = \alpha_2(\text{const}) \qquad (3.1.11)$$

their conjugate ξ can now be obtained by solving the fundamental Poisson brackets

$$[\xi_i, \xi_j] = 0, \quad [\eta_i, \eta_j] = 0.$$

$$[\xi_i, \eta_i] = \pm\, \delta_{ij}.$$

This gives

$$\xi_1 = \frac{1}{2}(x_1+x_2) \qquad (3.1.12)$$

and

$$\xi_2 = \frac{(y_1-y_2)(x_1-x_2)}{(y_1-y_2)^2 + \dfrac{4}{(x_1-x_2)^2}} \qquad (3.1.13)$$

t_1, t_2 constants.

Then for the new variables:

$$K = \eta_1^2 + \eta_2$$

$$\left.\begin{aligned} \frac{d\xi_1}{dt} &= \frac{\partial K}{\partial \eta_1} = 2\eta_1 = \text{const.} \Rightarrow \xi_1 = \alpha_1(t-t_1) \\ \frac{d\xi_2}{dt} &= \frac{\partial K}{\partial \eta_2} = 1 = \text{const.} \Rightarrow \xi_2 = t - t_2 \end{aligned}\right\} \qquad (3.1.14a)$$

$$\frac{d\eta_1}{dt} = -\frac{\partial K}{\partial \xi_1} = 0, \quad \frac{d\eta_2}{dt} = -\frac{\partial K}{\partial \xi_2} = 0. \qquad (3.1.14b)$$

Solving for x_i, y_i, $i = 1, 2$ from (3.1.8) - (3.1.11)

$$x_1 = \xi_1 + \sqrt{\frac{1+4\eta_2^2\xi_2^2}{4\eta_2}} = \alpha_1(t-t_1) + \sqrt{\frac{1+4\alpha_2^2(t-t_2)^2}{4\alpha_2}}$$

$$x_2 = \xi_1 - \sqrt{\frac{1+4\eta_2^2\xi_2^2}{4\eta_2}} = \alpha_1(t-t_1) - \sqrt{\frac{1+4\alpha_2^2(t-t_2)^2}{4\alpha_2}}$$

$$y_1 = \eta_1 + \eta_2\xi_2 \sqrt{\frac{4\eta_2}{1+4\eta_2^2\xi_2^2}}$$

$$= \alpha_1 + \alpha_2(t-t_2)\sqrt{\frac{4\alpha_2}{1+4\alpha_2^2(t-t_2)^2}}$$

$$y_2 = \eta_1 - \eta_2\xi_2 \sqrt{\frac{4\eta_2}{1+4\eta_2^2\xi_2^2}}$$

$$= \alpha_1 - \alpha_2(t-t_2)\sqrt{\frac{4\alpha_2}{1+4\alpha_2^2(t-t_2)}} \qquad \square.$$

Note: Existence of these constants of the motion enabled us to transform the original nonlinear equations of motion (3.1.9) to simple linear equations (3.1.14).

As we have seen in the previous chapter, for certain nonlinear PDEs there exists a method, namely IST, which transforms them to a linear integral equation. Furthermore, all these equations have an infinite number of local conservation laws which, for those cases possessing soliton solutions, leads to a corresponding infinite set of constants of the motion. These results suggest that Liouville's theorem may be extended to these PDEs. In fact Zakharov and Faddeev [13] have shown how the KdV can be fitted into a Hamiltonian structure. However, this is done in the space of scattering data of the associated linear problem rather than in the phase space of the KdV field.

3.2 Direct maps between nonlinear PDEs and linear integral equations

The use of IST in the case of PDEs leads us to a restriction on the class of solutions available. This class could possibly be widened if the PDE can be mapped directly onto the linear integral equation i.e. without the need for defining associated problems. One such method was introduced by Zakharov and Shabat [14] who showed, via Volterra operators, that solutions and explicit solutions in addition to those given by IST could be obtained. Lambert [8] used this method to obtain solutions not recoverable by IST. Based on Zakharov and Shabat's work, ARS[2] developed a more direct method of mapping nonlinear PDEs to linear integral equations. As part of our original contribution we show how the method can be used to obtain the Lambert solutions.

The ARS method leads to extra solutions because it requires only that the solutions decay rapidly enough as $x \to \pm\infty$ so that the integral operators are defined. In particular, it does not require any analytic properties of an associated scattering problem.

The method is described in the following steps:

(i) We start with the Gelfand-Levitan-Marchencko equation (3.1.3) and define N explicitly in terms of F. For instance

$$N(x; z, y) = F(z, y) \quad \text{(KdV)}$$

$$N(x; z, y) = \pm \int_x^\infty F(z, s)F(s, y)ds \quad \text{(mKdV, sGE)}.$$

(ii) In the method of IST F is controlled from the scattering data. However in this case we require only that F satisfies two linear PDEs.

$$\text{i.e.} \quad L_i F = 0 \qquad i = 1, 2 \tag{3.2.1}$$

(iii) Now define an operator A_x by

$$A_x f(y) = \begin{cases} \int_x^\infty f(z)N(x; z,y)dz & y \geq z \\ 0 & y < z \end{cases} \tag{3.2.2}$$

Then the linear integral equation (3.1.3) becomes

$$(I - A_x)K = F \tag{3.2.3}$$

(iv) Applying L_i $i = 1, 2$ to this equation yields

$$L_i(I - A_x)K = 0 \qquad (3.2.4)$$

and this is rewritten as

$$(I - A_x)(L_iK) = [L_i, A_x]K \qquad (3.2.5)$$

where $[L_i, A_x] = L_iA_x - A_xL_i$ is the commutator of L_i and A_x. Now (3.1.3) and (3.2.1) are chosen such that

$$[L_i, A_x]K = (I - A_x)M_i(K) \quad i = 1, 2 \qquad (3.2.6)$$

where $M_i(K)$ is, in general, a nonlinear function of K.

(v) We assume that for N there is a function space on which $(I - A_x)$ is invertible and $(I - A_x)^{-1}$ is continuous. (The proof that $(I - A_x)$ is invertible in some L^2 space to which N belongs is given in [2].) Moreover, it is assumed that the operators obtained by differentiating (3.2.4) with respect to x or y are also defined on this function space.

Then from (iv)

$$(I - A_x)[L_iK - M_i(K)] = 0 \qquad i = 1, 2.$$

But $(I - A_x)$ is invertible so K must satisfy the nonlinear differential equations

$$L_i K - M_i(K) = 0, \quad i = 1, 2 \tag{3.2.7}$$

Therefore, every solution of the linear integral equation (3.1.3) is also a solution of the nonlinear differential equations (3.2.7).

So far the method is general in that N and L_i are arbitrary. To recover a specific equation we have to make a definite choice of these operators. We now demonstrate the above on the KdV and construct a solution which cannot be obtained directly by IST. The proof of the theorem that follows is in Appendix A.

<u>Theorem 3.2</u>

Choose $N(x; z, y) = F(z, y)$ and let $F(x, y; t)$ satisfy the PDEs

$$L_1 F = F_{xx} - F_{yy} = 0 \tag{3.2.8a}$$

$$L_2 F = F_t + F_{xxx} + 3F_{xxy} + 3F_{xyy} + F_{yyy} = 0 \tag{3.2.8b}$$

and for each t vanish rapidly as $x \to \pm\,\infty$. Then the solution $K(x, y)$ of the linear integral equation (3.1.3) is also a solution of the nonlinear PDEs

(3.2.7) which for $x = y$ gives

$$K_t + K_{xxx} + 6(K_x)^2 = 0$$

so that $q = 2 \frac{d}{dx} K(x, x)$ is a solution of the KdV

$$q_t + 6qq_x + q_{xxx} = 0 \qquad \square.$$

3.2.1 Explicit solutions of the KdV equation

The linear system (3.2.8) is homogenous with constant coefficients and we expect solutions under the form of a Fourier series or integral.

Introducing characteristic coordinates $u = x + y$, $\nu = x - y$ in (3.2.8) we have

$$F_{uv} = 0 \tag{3.2.1.1a}$$

$$F_t + 8F_{uuu} = 0 \tag{3.2.1.1b}$$

We look for solutions which are independent of v i.e. $F(u, v; t) = \psi(u, t)$. Such solutions automatically satisfy (3.2.1.1a) for any ψ and hence we are looking for solutions of (3.2.1.1b)

i.e. $\psi_t + 8\psi_{uuu} = 0$ (3.2.1.2)

Separating the variables in this equation, we can obtain solutions of the form

$$\psi_k(u, t) = \exp(-8k^3 t)$$

$$\left[A_k \exp(ku) + \exp\left(-\frac{k}{2}u\right)\left(c_k \cos\frac{\sqrt{3}}{2} ku + s_k \sin\frac{\sqrt{3}}{2} ku\right)\right]$$

(3.2.1.3)

where k is real or imaginary.

This allows, for a particular subset, solutions of the IST type:

$$\psi(u,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} R(k)\exp(-8ik^3 t - iku)dk$$

$$\sum_{n=1}^{N} c_n \exp[\chi_n u - 8\chi_n^3 t] \qquad \chi_n, c_n > 0.$$

(3.2.1.4)

The solutions given by (3.2.1.3) are not new. Lambert [8] obtained them via Volterra operators where the only slight difference from our solution was due to the PDE (3.2.1.2) in his case being

$$\psi_t - 2\psi_{uuu} = 0.$$

Following Lambert it can be shown that, if we look for a simple explicit solution of the KdV equation for negative k by considering the contribution of a single cosine term in (3.2.1.3)

i.e. $F_{cos}(x+y,\ t) =$

$$c \exp\left[8k^3 t + \frac{k}{2}(x+y)\right]\cos\left[\frac{\sqrt{3}}{2}\, k(x+y)\right],\quad k > 0 \tag{3.2.1.5}$$

then inserting (3.2.1.5) into the Gelfand-Levitan equation (3.1.3) leads, for $x = y$, to the solution

$$K_{cos}(x,\ x;\ t) = -\frac{d}{dx}\log \Delta_{cos}(x,\ t)$$

where
$$\Delta_{cos}(x,\ t) = 1 + \frac{c}{8k}\exp(8k^3t+kx)\cos(\sqrt{3}x + {}^{\pi}/3) + \frac{3c^2}{(16k)^2}\exp(16k^3t+2kx) \tag{3.2.1.6}$$

This gives the following solution of the KdV equation

$$q_{cos}(x,\ t) = -2\frac{d^2}{dx^2}\log\Delta_{cos}(x,t) \tag{3.2.1.7}$$

which can not be obtained by the IST method.

These solutions were studied, in some detail, by Lambert and it was shown that they possess one pole of order two whose location varies with time.

The major problem with this direct approach seems to be choosing the function N and the linear operators L_1 and L_2. For the KdV above, the MKdV, SGE and nonlinear Schrödinger equations considered by ARS[2], these functions are chosen with the benefit of hindsight from IST. It might be that for a given equation where the scattering problem is not known, choosing N, L_1, and L_2, is more difficult than finding the scattering problem. Thus although the direct method can be used to obtain solutions which cannot be obtained by the IST method, it has not been shown to work for equations where IST is not already known to apply.

3.3 Test for Integrability - the Painlevé conjectures

We have seen that for ODEs we have Liouville's theorem and for PDEs the IST for certain classes of equations. The difficulty for Liouville's approach is showing that there are n constants of the motion because one has to proceed by first finding them. The difficulty of IST is finding the associated linear problem and, as we have discussed, the direct method relies on the IST. Thus for a given ODE or PDE the above methods are difficult to implement. This led people to look for other ways of identifying integrable systems which do not depend either on constants of the motion or scattering problems.

Recent work suggests that there is an intimate connection between the analytic structure of a system and its integrability. Ablowitz et. al [1-2] have pointed this out in the context of PDEs. They conjectured that every nonlinear ODE obtained by an exact reduction of a nonlinear PDE, soluble by IST, will have a solution structure which has at most movable poles.

The origin of this conjecture goes back to Sofya Kovalevsky's work in 1889 [7]. Kovalevsky worked on the Euler-Poisson equations and she was able to identify the known integrable cases together with one new case by looking for those system parameter values for which the only movable singularities exhibited by the solution in the complex time plane were ordinary poles. While this work has been neglected for more than fifty years, the discovery sixteen years ago that the KdV equation could be integrated via spectral methods has generated an enormous amount of study in the area of completely integrable Hamiltonian systems.

In the last few years, Kovalevsky's approach has been used to predict integrable cases of a variety of systems such as Lorenz equations [9] and the Henon-Heiles Hamiltonian [5]. More recent work has been undertaken by Van Moerbeke [10] and Yoshida [12]. These works have led to the conjecture that if the solution structure of a given system of ODEs is at most movable poles, then integrability is implied. Ward [11] points out that the reverse does not hold since a single ODE of the form $x' = f(x)$ is integrable by quadratures but does not, in general, have only movable pole structure. This requirement on the solution structure has become known, in current literature, as the "Painlevé Property" - after Painlevé's extensive study of classes of second order ODEs [6]. Although on reflection it might have been called the "Kovalevsky Property".

3.4 Conclusion

In this chapter we have concentrated on integrability and have demonstrated how the available methods - Liouville's theorem for ODEs and IST/direct mapping of integral equations for PDEs - can be used to solve nonlinear differential equations. We highlighted the difficulties with applying these methods to a given equation and went on to discuss work on the connection between Hamiltonian systems, their analytic structure and integrability. We have noted that a conjecture implying that the "Painlevé property" is sufficient but not necessary for integrability of ODEs has emerged.

A previous conjecture made by Ablowitz et al [1-4] for PDEs states that if a given PDE is IST soluble then all its ODE reductions will possess the "Painlevé

Property". That is the "Painlevé Property"is at least a necessary condition for integrability of non linear PDEs.

In the next chapter we review the development of this conjecture and look at the methods which have been proposed to test the analytic structure of a given PDE.

CHAPTER FOUR

THE ANALYTIC STRUCTURE OF THE SOLUTIONS OF INTEGRABLE PDES

In the previous chapter we indicated how the work of Kovalevsky, Moerbeke, Yoshida and others showed how the analytic structure of solutions of Hamiltonian systems of ODEs can be used to determine whether they are completely integrable. In this chapter we shall review attempts which have been made to extend these tests to decide on the integrability of nonlinear PDEs. There are two avenues (i) by looking at the meromorphic structure of special solutions of the PDE known as similarity solutions and (ii) by looking at the general solution of the PDE.

4.1 Similarity Transformations

The major application of similarity transformations is the reduction of certain classes of nonlinear PDEs to ODEs. We give the following definition.

Definition 4.1

The term "similarity transformation" of a PDE is a transformation of dependent and independent variables occurring in the equation such that the number of independent variables appearing in the transformed equation is at least one less than in the original equation. The transformed variables are referred to as similarity variables.

Before going into similarity transformations further, we give an example, due to Ablowitz and Segur [2] which demonstrates a connection between the nonlinear modified KdV and the second Painlevé transcendent.

The modified KdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{4.1}$$

has a similarity solution

$$u(x, t) = f(\eta) \quad \text{where} \quad \eta = x(3t)^{-\frac{1}{3}} \tag{4.2}$$

The transformation takes $(x, t, u(x,t))$ to $(\eta, t, f(\eta))$. The variables $f(\eta)$ and η are the similarity variables and the modified KdV (4.1) is reduced to

$$\frac{d^2 f}{d\eta^2} = 2f^3 + \eta f \tag{4.3}$$

which is the equation of the second Painlevé transcendent.

By seeking a purely self-similar solution to the Gelfand-Levitan integral equation (3.1.3), Ablowitz and Segur proved that the connection between (4.1) and (4.3) reduces (4.3) to the set of linear integral equations

$$K_1(x, y) - rAi\left(\frac{x+y}{2}\right) + \frac{r}{2}\int_x^\infty K_2(x, s)A\left(\frac{s+y}{2}\right)ds = 0$$

$$K_2(x, y) + \frac{r}{2}\int_x^\infty K_1(x, \eta)Ai\left(\frac{x+y}{2}\right)d\eta = 0 \qquad (4.4)$$

for $y > x$, r real.

where $f(x) = K_1(x, x)$ and the Airy function Ai is given by $rAi(z) \sim \frac{r}{2\sqrt{\pi}} z^{-1/4} \exp\left[-\frac{2}{3} z^{\frac{3}{2}}\right]$, $z \to \infty$.

The important point is that the Painlevé transcendent defined by (4.3) is known to be free from movable algebraic logarithmic and essential singularities. Later we shall list a number of integrable PDEs which have been reduced to Painlevé transcendents via similarity reductions.

It should be made clear that there is no unique way of obtaining similarity transformations. The two most well known methods are (i) using infinitesimal symmetries of the PDE and (ii) using separation of the variables, although both have been shown to agree except in certain degenerate cases [3]. Here we concentrate on the first approach.

4.1.1 Infinitesimal Transformations

In this subsection we give a brief outline of the theory of Lie's one-parameter(ε) group of transformations for invariance of a PDE with two independent variables [4], [5], [16], [18].

Consider a PDE with one dependent variable u and two independent variables x and t.

$$L[u] = L(x, t, u, u_x, u_t\ u_{xx}, u_{xt}, u_{tt}, \ldots) \tag{4.1.1.1}$$

We now make the following definition.

Definition 4.1.1

If $x' = f(x, t, u; \varepsilon)$, $t' = g(x, t, u; \varepsilon)$ and $v = h(x, t, u; \varepsilon)$ are a sufficiently differentiable set of infinitesimal transformations which leave the PDE invariant, then this group of transformations parameterized by ε is a Lie group and is the local symmetry group of the equation.

We seek f, g, h such that operator (4.1.1.1) has the same form when the variables x, t, u are replaced by x', t', v respectively.

i.e.

$$L(x', t', v, v'_{x'}, v'_{t'}, v_{x'x'}, v_{x't'}, v_{t't'}, \ldots) = 0 \qquad (4.1.1.2)$$

We look for infinitesimal transformations, i.e. $\varepsilon \ll 1$, and assume that the functions f, g, h have the following expansions about $\varepsilon = 0$:

$$x' = x + \varepsilon X(x, t, u) + O(\varepsilon^2)$$

$$t' = t + \varepsilon T(x, t, u) + O(\varepsilon^2)$$

$$v = u + \varepsilon U(x, t, u) + O(\varepsilon^2) \qquad (4.1.1.3)$$

The functions X, T and U are the infinitesimal generators of the transformations for the variables x, t and u respectively and are determined by the condition that the equation is invariant. Thus, we substitute in (4.1.1.2) for x', t', v and the derivatives of v in terms of x, t and u and its derivatives and use the fact that u satisfies (4.1.1.1) to obtain explicit expressions for X, T and U.

We show below how the derivatives are computed. For example consider $\frac{\partial v}{\partial x'}$.

Writing (4.1.1.3) in differential form and using the fact that $du = u_x dx + u_t dt$ we obtain, to first order in ε,

$$dx' = (1 + \varepsilon X_x + \varepsilon X_u u_x)dx + \varepsilon(X_t + X_u u_t)dt$$

$$dt' = \varepsilon(T_x + T_u u_x)dx + (1 + \varepsilon T_t + \varepsilon T_u u_t)dt$$

$$dv = (u_x + \varepsilon U_x + \varepsilon U_u u_x)dx + (u_t + \varepsilon U_t + \varepsilon U_u u_t)dt.$$

Since we are working to first order in ε the first two equations can be inverted by means of the transformations $dx' \to dx$, $dt' \to dt$, and $\varepsilon \to -\varepsilon$ giving

$$dx = (1 - \varepsilon X_x - \varepsilon X_u u_x)dx' - \varepsilon(X_t + X_u u_t)dt'$$

$$dt = -\varepsilon(T_x + T_u u_x)dx' + (1 - \varepsilon T_t - \varepsilon T_u u_t)dt' .$$

The derivatives $\frac{\partial x}{\partial x'}$ etc. are given by the coefficients of the derivatives on the right hand side. They can now be used to obtain

$$\frac{\partial v}{\partial x'} = \left[u_x + \varepsilon U_x + \varepsilon U_u u_x\right]\frac{\partial x}{\partial x'} + \left[u_t + \varepsilon U_t + \varepsilon U_u u_t\right]\frac{\partial t}{\partial x'} .$$

$$\to v_{x'} = u_x + \varepsilon[U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t] \tag{4.1.1.4}$$

In a similar manner the second derivative is

$$v_{x'x'} = u_{xx}+\epsilon[U_{xx}+(2U_{xu}-X_{xx})u_x-T_{xx}u_t+(U_{uu}-2X_{xu})u_x^2$$

$$- T_{xu}u_xu_t-X_{uu}u_x^3-T_{uu}u_x^2u_t+(U_u-2X_x)u_{xx}-2T_xu_{xt}$$

$$- 3X_uu_{xx}u_x-T_uu_{xx}u_t-2T_uu_{xt}u_x] + 0(\epsilon^2) \qquad (4.1.1.5)$$

Expressions for the time derivatives $v_{t'}$, $v_{t't'}$ etc. can be obtained by making the transformations $x' \leftrightarrow t'$. $X \leftrightarrow T$, $x \leftrightarrow t$ in the above equations for $v_{x'}$, $v_{x'x'}$ and so on.

We are now in a position to define what is meant by similarity solutions.

<u>Definition 4.1.2</u> (Similarity Solution)
Let X, T, U be the generators of an infinitesimal symmetry transformation of the PDE $L(u(x, t)) = 0$ which maps it to $L(v(x', t')) = 0$.

Then a solution $u = \theta(x, t)$ such that $v = \theta(x', t')$ is called a similarity solution.

Implementing the above definition in (4.1.1.3) leads to the functional equation

$$\theta(x + \epsilon X + 0(\epsilon^2), t + \epsilon T + 0(\epsilon^2))$$

$$= \theta(x, t) + \epsilon U(x, t, u) + 0(\epsilon^2) \qquad (4.1.1.6)$$

and expanding the left-hand side of (4.1.1.6) shows that θ also has to satisfy the first order quasilinear PDE

$$X(x, t, \theta) \frac{\partial \theta}{\partial x} + T(x, t, \theta) \frac{\partial \theta}{\partial t} = U(x, t, \theta) \tag{4.1.1.7}$$

which is the equation of an invariant surface for θ.

We now have the following theorem.

<u>Theorem 4.1.1</u> [3]

The general solution of (4.1.1.7) is

$$F(p, q) = 0 \tag{4.1.1.8}$$

where F is an arbitrary, sufficiently differentiable function and $p(x, t, \theta) = a$, $q(x, t, \theta) = b$ form independent solutions of the Lagrange system

$$\frac{dx}{X(x,t,\theta)} = \frac{dt}{T(x,t,\theta)} = \frac{du}{U(x,t,\theta)} \tag{4.1.1.9}$$

□

The method described above has been applied to obtain similarity solutions to many nonlinear PDEs as we shall discuss in section 4.5. Here we give the results for the KdV equation.

<u>Example 4.1.1</u>

For the KdV equation

$$u_t + uu_x + u_{xxx} = 0 \tag{4.1.1.10}$$

Shen and Ames [24] derived that

$$X(x,\ t,\ u) = \alpha x + \beta t + \gamma$$

$$T(x,\ t,\ u) = 3\alpha t + \delta$$

$$U(x,\ t,\ u) = -2\alpha u + \beta \qquad (4.1.1.11)$$

where α, β, γ and δ are arbitrary constants.

When $\alpha = \gamma = o$, $\delta = 1$. We have $X = \beta t$, $T = 1$, $U = \beta$ and (4.1.1.9) leads to

$$p = x - \frac{1}{2}\beta t^2 = \text{constant} = \eta$$

$$q = \theta - \beta t = \text{constant} = f(\eta)$$

where f is an arbitrary function. The similarity variables are η and $f(\eta)$ and substituting into (4.1.1.10) reduces it, after one integration, to

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2} f^2 + \beta\eta = \text{const.} \qquad (4.1.1.12)$$

Thus f is the first Painlevé transcendent.

Similarly when $\alpha = \beta = 0$ we obtain the ODE

$$\frac{d^3f}{d\eta^3} + f\frac{df}{d\eta} - c\frac{df}{d\eta} = 0 \qquad (4.1.1.8)$$

where $c = \gamma/\delta$.

This ODE gives travelling wave solutions which can be directly obtained by elliptic functions.

If we use the full four parameter group (4.1.1.11) the KdV reduces to the second Painlevé transcendent

$$\frac{d^2f}{d\eta^2} = 2f^3 + \eta f \qquad (4.1.1.14)$$

We note that to obtain all the one-parameter reductions it is necessary to consider all possible values of the arbitrary constants α, β, γ and δ which arise in the infinitesimals [11].

Lakshmanan and Kaliappan [18], Shen and Ames [24], Fokas and Ablowitz [14] and Clarkson and McLeod [11] among many other authors have applied the method of one-parameter Lie groups of infinitesimal transformations to several nonlinear PDEs.

Before concluding this section, we list a few important points.

(i) Lie's theory of one-parameter infinitesimal transformations is not a unique way of obtaining similarity variables [3], [4].

(ii) The similarity solutions obtained via one-parameter groups or separation of variables or some other method represent a special class of solutions for the original equation.

(iii) If we cannot obtain a similarity variable for a given equation it cannot be concluded that none exists. We can only conclude that there is no similarity variable under that class of transformations [3].

Having used similarity transformations to reduce a given nonlinear PDE to a set of ODEs, we need to be familiar with the theory of ODEs in order to obtain information about the PDE.

4.2 Brief review of linear ODEs (Ince [17] Ch. XV).

Consider the nth order ODE

$$\frac{d^n w}{dz^n} + P_1(z)\frac{d^{n-1} w}{dz^{n-1}} + \ldots + P_{n-1}(z)\frac{dw}{dz} + P_n(z)w = 0 \tag{4.2.1}$$

If z_o is any point in the neighbourhood of which the n coefficients are analytic, the equation possesses a fundamental set of n solutions regular at z_o. Any singularities of the solution are the singularities of the equation. If the locations of such singularities are independent of the n constants of integration, then these singularities are called fixed.

A general property of linear ODEs in the complex plane is that their solutions have only fixed singularities. However, this is not necessarily the case for nonlinear ODEs to which we now turn.

4.3 Nonlinear ODEs and Movable Singularities

Consider the following examples:

Example 4.3.1

$$\frac{d^2w}{dz^2} = \frac{2w}{w^2-1}\left(\frac{dw}{dz}\right)^2 \qquad (4.3.1)$$

rewriting the equation

$$\frac{d^2w}{dz^2} = \left(\frac{1}{w+1} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2$$

Integrating once

$$\left(\frac{dw}{dz}\right)^2 = 2k^2(w^2 - 1)^2$$

$$Az = \int_0^w \frac{1}{(q^2-1)}\,dq + B$$

then $w(z) = \tanh(Az + B)$ □

Here A and B are the constants of integration: they also define the locations of the singularities i.e. $Az + B = \frac{(2n+1)i\pi}{2}$. These singularities are called MOVABLE because their location depends on the constants of integration. Thus the singularities may be placed anywhere in the complex plane.

Example 4.3.2

$$\frac{dw}{dz} + w(\log w)^2 = 0 \qquad (4.3.2)$$

is satisfied by $w = \exp\left(\frac{1}{z-A}\right)$

where A is an arbitrary constant. If $z = A$ the solution has neither a finite nor an infinite limit. That is, at $z = A$ there is a movable essential singularity.

Example 4.3.3 (Ince P. 317)

For the equation

$$\frac{d^2w}{dz^2} = \left(\frac{dw}{dz}\right)^2 \frac{2w-1}{w^2+1} \qquad (4.3.3)$$

the general solution is

$$w(z) = \tan\{\ln(Az + B)\}$$

where A and B are arbitrary constants. At

$z = -A/B$, w tends to no limit (finite or infinite). In fact, here we have both movable branch points and essential singularities.

We now make the following definitions.

Definition 4.3.1

Any singularity of a solution of an ODE that is not a pole is called a critical point.

Definition 4.3.2

An ODE is of Painlevé type (or P-type) if the only movable singularities of its solutions in the finite complex plane are poles.

Such ODEs are said to possess the Painlevé Property.

4.3.1 Painlevé type ODEs

In 1884 Fuchs showed that out of all first order equations of the form

$$\frac{dw}{dz} = F(w, z), \qquad (4.3.1.1)$$

where F is rational in w and analytic in z, the only equations without movable critical point are generalized Riccati equations:

$$\frac{dw}{dz} = P_0(z) + P_1(z)w + P_2(z)w^2 \qquad (4.3.1.2)$$

The class of second order nonlinear ODEs

$$\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right) \qquad (4.3.1.3)$$

where F is rational in $\frac{dw}{dz}$ and w and analytic in z, was examined by Painlevé [17], who obtained the subset of equations free from movable critical points. He introduced a parameter α into the equation such that for $\alpha = 0$ the resulting equation could be seen, by inspection, to be with or without movable critical points. Using this method, Painlevé obtained two necessary conditions

(i) F must be a polynomial in $\frac{dw}{dz}$ of degree ≤ 2 of the form

$$F(w',w,z) = L(w,z)(w')^2+M(w,z)(w')+N(w,z)$$

(Ince Ch. XIV) (4.3.1.4)

where L, M, N are rational in w, analytic in z. Furthermore, L is required to be either identically zero or one of five specific forms which are listed in Ince.

(ii) If $L \equiv 0$ then M must be linear in w and N a polynomial in w of degree ≤ 3.

If $L \not\equiv 0$, then M and N must have the form

$$M(w, z) = \frac{m(w,z)}{D(w,z)}, \quad N(w, z) = \frac{n(w,z)}{D(w,z)}$$

where $D(w, z)$ is the least denominator of the partial fractions in $L(w, z)$ and is a polynomial of w of degree d, $2 \leq d \leq 4$, while m and n are polynomials in w of degree $\leq d + 1$ and $\leq d + 3$ respectively.

Implementation of the necessary conditions (i) and (ii) leads to 50 canonical equations with the property of having no movable critical points. Of these equations, it was shown that 44 were soluble in terms of elementary functions or elliptic functions. The remaining equations define six new transcendental functions - the Painlevé transcendents $P_I - P_{VI}$.

Since the 50 ODEs are of canonical type, it does not necessarily follow that a given second order ODE satisfies, directly, the necessary conditions. It may be possible to make the ODE one of the 50 by a transformation of variables. To demonstrate this we present the following example.

Example 4.3.4

Consider the equation

$$z^m(w' + zw'') = \sin w, \quad \text{where} \quad m \in z^+ \cup \{0\}.$$
(4.3.1.5)

If $m = 1$ then $\ln(\alpha z) = \pm \int \frac{dw}{(\beta - 2\cos w)^{1/2}}$ [22]

and the solution is given in terms of elliptic functions.

If $m \neq 1$ then $\phi = e^{iw}$ transforms the ODE to

$$\frac{d^2\phi}{dz^2} = \frac{1}{\phi}\left(\frac{d\phi}{dz}\right)^2 - \frac{1}{z}\frac{d\phi}{dz} + \frac{1}{2iz^{m+1}}(\phi^2 - 1) \qquad (4.3.1.6)$$

If $m = 0$ then (4.3.1.6) becomes P_{III} with $\alpha = 1$, $\beta, \phi, \delta = 0$

If $m > 1$ we change the independent variable to

$\psi = \frac{z^{1-m}}{(1-m)^2}$ and obtain

$$\frac{d^2\phi}{d\psi^2} = \frac{1}{\phi}\left(\frac{d\phi}{d\psi}\right)^2 - \frac{1}{\psi}\frac{d\phi}{d\psi} + \frac{1}{2i\psi}(\phi^2 - 1) \qquad P_{III}$$

It is important to note that extensive results for nonlinear ODEs are only available for first and second order equations. The main difficulty with determining whether equations of third and higher orders are of P-type is showing that the equation is free from movable essential singularities. Painlevé

[17] showed by a separate analysis that the Painlevé transcendents are free from such singularities. The analysis, however, becomes very complicated when considering third and higher orders and this difficulty has prevented workers in the field from giving a classification of even third order nonlinear ODEs.

However, in the following section we review an algorithm which is generally easier to apply than Painlevé's α-method and determine whether a given nonlinear ODE admits movable branch points. This algorithm can be applied to equations of all orders.

4.4 Singular Point Analysis

Given a nonlinear ODE, how do we determine whether it is of Painlevé type? Since extensive results are available for first and second order equations, it is the nature of the singularities of the third and higher order equations that are of particular interest.

The algorithm we outline below, given by Ablowitz, Ramani and Segur [1] provides necessary conditions for an ODE to be of P-type. The algorithm is based on two assumptions.

(i) The nth order ODE has the form

$$\frac{d^n f}{d\eta^n} = F(\eta,\ f',\ f'',\ \ldots,\ f^{n-1}) \qquad (4.4.1)$$

where F is analytic in η and rational in its other arguments.

(ii) The dominant behaviour of the function in a sufficiently small neighbourhood of a movable singularity, if it exists, is algebraic

$$\text{i.e.} \quad f(\eta) \sim \alpha(\eta - \eta_o)^{\rho} \qquad (4.4.2)$$

$Re(\rho) < 0$ and η_o is arbitrary.

We demonstrate how this algorithm is applied by means of an example.

Consider the ODE

$$\frac{d^2 f}{d\eta^2} = \left(\frac{df}{d\eta}\right)^2 \left(\frac{2f-1}{f^2+1}\right) \qquad (4.4.3a)$$

Rearranging (4.4.3a) in a suitable form gives

$$f^2\frac{d^2 f}{d\eta^2} + \frac{d^2 f}{d\eta^2} - 2f\left(\frac{df}{d\eta}\right)^2 + \left(\frac{df}{d\eta}\right)^2 = 0 \qquad (4.4.3b)$$

Step 1: Leading term

It is clear that all solutions have the form $f(\eta - \eta_o)$

where η_o is arbitrary. Find the dominant behaviour of the solution in the neighbourhood of a movable singularity of $\eta = \eta_o$. If η_o is a singular point, then this means that it is movable and we assume that in a neighbourhood of this point

$$f(\eta) = \sum_{j=0}^{\infty} b_j(\eta - \eta_o)^{\rho+j}$$

where α and ρ are constants to be determined. Substituting into (4.4.3b) it is seen that the dominant terms are the first and the third terms and these give

$$\rho = -1 \quad \text{and} \quad b_o = \alpha \quad \text{is arbitrary.}$$

Hence, in a neighbourhood of η_o,

$$f(\eta) = \alpha(\eta-\eta_o)^{-1} + \sum_{j=0}^{\infty} a_j(\eta-\eta_o)^j \qquad (4.4.4)$$

where $a_o = b_1$ etc.

If ρ had been fractional, then the singularity would have been an algebraic branch point.

<u>Step 2: Arbitrary constants</u>

Now the general solution of (4.4.3a) has two arbitrary constants and these have already been determined in Step 1 as η_o and α. Consequently, in this problem the remaining coefficients a_j, $j = 1, 2, \ldots$ are determined in terms of α by recurrence relations. However, if in Step 1 not all arbitrary constants were determined, then we substitute

$$g(\eta) = \alpha t^{-1} + \beta t^{r-1} + \gamma t^{s-1}, \qquad t = \eta - \eta_0$$

where β and γ are the required arbitrary constants, and three terms are included because the dominant nonlinearities are cubic. Balancing the equation for powers of t gives values for r and s which determines the positions in the series of all arbitrary constants not determined in Step 1.

<u>Step 3: Coefficients</u>

We substitute the whole series (4.4.4) with the arbitrary coefficients in their correct positions and check for consistency. If we cannot obtain consistency then the solution has movable branch points.

In this example the equation (4.4.3a) is free from algebraic and logarithmic (movable) singularities. However, the algorithm tells us nothing about the existence of essential movable singularities. In fact we know that the general solution of (4.4.3a) is not free from movable singularities and is given by

$$f(\eta) = \tan(\ln(A\eta + B)) \qquad (4.4.7)$$

The difficulty of detecting essential singularities means that this algorithm is powerful for eliminating candidates for Painlevé rather than discovering them.

4.5 Similarity solutions of integrable PDEs

Example 4.1.1 shows that the similarity reductions of the KdV equation, with one-parameter infinitesimals are all of Painlevé type. The work of Ablowitz and Segur on the modified KdV equation had encouraged many workers to look for similarity solutions of other well known IST integrable equations and it turns out that all the integrable PDEs that we know have similarity solutions which are of Painlevé type. Furthermore, they all reduce to one of the Painlevé transcendents. Below we give a table of IST equations together with their similarity variables and ODE reductions.

Equation	Invariant variable (η)	Invariant form of sol^{s} (u)	Reduced Form
KdV	$x(3t)^{-\frac{1}{3}}$	$f(\eta)t^{-\frac{1}{3}}$	P_{II}
mKdV	$x(3t)^{-\frac{1}{3}}$	$f(\eta)t^{-\frac{1}{3}}$	P_{II}
SG	xt	$f(\eta)$	P_{III}
Boussinesq	$x - t/_{2}$	$f(\eta)$	P_{I}
NLS	$\left[x+\frac{t^2}{2}\right] - bt + (-1)^{\frac{2}{3}}\left(\frac{b^2}{2}\right)$	$\exp\left(-\frac{i}{2}\left[xt+\frac{t^3}{3}-\frac{bt^2}{2}\right)\right. \times p(\eta)e^{i\int q(\eta)d\eta}$	P_{II}
Derivative NLS	$x(2t)^{-\frac{1}{2}}$	$\dfrac{p(\eta)^{i\int q(\eta)d\eta}}{(2t)^{\frac{1}{4}}}$	P_{IV}

For more examples and for equations with higher special dimensions see reference [18].

On the other hand, several PDEs which are thought not to be integrable - either because they only have a finite number of conservation laws, or that numerical evidence shows that their solitary waves are not solitons - have been reduced to ODEs which are not free from movable critical points.

For example, consider the BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (4.5.1)$$

The following evidence suggests that the equation is not integrable.

(i) It has a family of solitary wave solutions

$$u(x, t) = 3c \operatorname{sech}^2 \left[\frac{1}{2} \sqrt{\frac{c}{1+c}} \; (x - (1+c)t) \right] . \qquad (4.5.2)$$

where c is a constant. However, numerical studies by Bona, Pritchard and Scott [7] have shown that the interaction of two solitary waves is inelastic and therefore not soliton-like.

(ii) The equation has only three local conservation laws of polynomial type (Olver [21], Duzkin and Tsujishita [13]).

$$u_t + \left(u + \frac{1}{2} u^2 - u_{xt}\right)_x = 0$$

$$\left(\frac{1}{2}u^2 + \frac{1}{2}u_x^2\right)_t + \left(\frac{1}{2}u^2 + \frac{1}{3}u^3 - uu_{xt}\right)_x = 0$$

$$\left(\frac{1}{3}u^3 - u_x^2\right)_t + \left(\frac{1}{3}u^3 + \frac{1}{4}u^4 + u_{xt}^2 - u_t^2 - u^2u_{xt}\right)_x = 0 .$$

On the other hand, the KdV, modified KdV, SGE all have an infinite number of conservation laws.

(iii) The equation has general travelling wave solutions

$$u(x, t) = f(\eta),$$

where $\eta = x - (1 + c)t$ and $f(\eta)$ satisfies

$$(1 + c) \frac{d^3 f}{d\eta^3} = c \frac{df}{d\eta} - f \frac{df}{d\eta} \qquad (4.5.3)$$

Integrating twice gives

$$\frac{(1+c)}{2} \left(\frac{df}{d\eta}\right)^2 = \frac{c}{2} f^2 - \frac{f^3}{6} + Af + B, \qquad (4.5.4)$$

where A and B are constants, and since (4.5.4) can be solved by elliptic functions, equation (4.5.3) is of P-type.

In addition to the self-similar travelling wave solution, the BBM also has the similarity solution

$$u(x, t) = t^{-1} y(x) - 1$$

where $y(x)$ satisfies

$$\frac{d^2 y}{dx^2} = y - y \frac{dy}{dx} . \qquad (4.5.5)$$

In the vicinity of a pole, $y(x)$ is given by

$$y(x) = \frac{2}{x-x_o} + \left\{\frac{2}{3}\ln(x-x_o) + A\right\}(x-x_o) + O((x-x_o))$$

where A is a constant. Therefore $y(x)$ has a movable logarithmic branch point and hence (4.5.5) is not P-type.

Other partial differential equations which are believed not to be integrable also have similarity reductions to equations which are not of P-type. These include:

Generalized KdV $\quad u_t + u^n u_x + u_{xxx} = 0, \quad n > 2$

(4.5.6)

KdV Burger's Equation $\quad u_t - \mu u u_x + \nu u_{xxx} = \gamma u_{xx}$

(4.5.7)

Fisher's Equation $\quad u_{xx} - u_t + u - u^n = 0 \quad$ (4.5.8)

Phi-Four(ϕ^4) Equation $\quad u_{tt} - u_{xx} + u - u^3 = 0$

(4.5.9)

See reference [18].

Ablowitz, Ramani and Segur's work on the connection between IST soluble nonlinear PDEs and Painlevé transcendents led them to make the following conjecture.

The Painlevé Conjecture [1]

Suppose a nonlinear PDE is soluble by IST. Then all the solutions of every nonlinear ODE obtained by exact reductions (perhaps after a transformation of variables) are free from movable algebraic, logarithmic and essential singularities.

If true, this conjecture provides a powerful necessary condition for testing whether a given PDE is soluble by IST since it means that if there is a reduction which is not P-type then the equation is not soluble by IST. It does not claim to be a sufficient condition for integrability. A similar conjecture has been given by McLeod and Olver [20]. Although neither of these conjectures has been fully proved, both ARS [1] and McLeod and Olver have given "partial proofs". Despite the lack of complete proofs we have already presented some of the considerable evidence in support of the conjecture.

Recent work by Clarkson and McLeod [11] indicates that the above conjecture may remain only a necessary condition. They proved that the only similarity solutions to the following equations:

$$u_t = u_x + u^2 u_x + u_{xxt} \quad \text{(Modified BBM)} \quad (4.5.10)$$

and

$$u_{tt} = u_{xx} - \frac{1}{2}(u^2)_{xt} + u_{xxtt} \quad \text{(Symmetric RLW)} \quad (4.5.11)$$

that can be obtained by the Lie group method are travelling wave solutions $u(x, t) = f(x - ct)$, where c is a constant and f is of P-type.

On the other hand, numerical studies by Makhankov [19], Seyler and Fenstermaker [23] and Bogolubsky [6] show that the interaction of the solitary waves of (4.5.10-11) are inelastic and therefore, they are probably not integrable.

Clarkson and McLeod assume that all the possible reductions of a given nonlinear PDE can be obtained via one-parameter groups whereas Ames [1] has pointed out that this may not be the case. Furthermore, Ward [25] has remarked that since most nonlinear PDEs do not have local symmetries it follows that they cannot have similarity solutions obtained by the Lie group method. In these cases the question arises as to how we decide on integrability by IST.

4.6 Analytic Structure of the General Solution of Integrable PDEs.

Developing the work of Ablowitz, Ramani and Segur, Weiss [26] and Weiss, Tabor and Carnavale [27] conjectured that every solution of an integrable nonlinear PDE has a meromorphic structure. This they called the "Painlevé Property" and claimed that it was a sufficient, rather than a necessary, condition for integrability. We now give the definition of the "Painlevé Property".

Definition 4.6.1

Consider the PDE in the form

$$u_t = K(u), \qquad (4.6.1)$$

where K is a nonlinear operator and $u: \mathbb{R}^{n+1} \to \mathbb{R}$ is a solution determined by some initial data.

Consider now the complex extension $(x_1, \ldots, x_n, t) \to (z_1, \ldots, z_n, \tau)$ so that now $u: \mathbb{C}^{n+1} \to \mathbb{C}$ and assume that u can be expanded in the form

$$u(z_1, \ldots, z_n, \tau) = \phi^{-\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \quad (\alpha \geq 0)$$

(4.6.2)

where $\phi = \phi(z_1, \ldots z_n, \tau)$ and $u_j = u_j(z_1, \ldots z_n, \tau)$ are analytic functions of $z_1, \ldots, z_n, \tau$.

If $\alpha > 0$ then the singularities of u are determined by the zeros of ϕ. Thus we call

$$\phi(z_1, \ldots, z_n, \tau) = 0 \qquad (4.6.3)$$

the singularity manifold of u.

The equation is said to possess the Painlevé Property if the equation is at most meromorphic on $\phi = 0$. That is, α is a non-negative integer.

The expansion (4.6.2) is motivated by Cauchy-Kovalevsky theorem [12], [15] which requires that the manifold (4.6.3) is non-characteristic [15]. It is also required that the number of arbitrary functions for a general solution is equal to the order of the equation and the arbitrary functions involve one less independent variable than the number occurring in the equation [15].

The procedure for checking the property is analagous to that for finding series solutions of ODEs [1]. Substituting (4.6.2) into the PDE determines α and defines recursion relations for u_j, $j = 0, 1, 2 \ldots$. If α is a non-negative integer and the expansion (4.6.2) gives a general solution of the PDE then the PDE has the Painlevé Property.

Weiss et al, Chudnovsky and Chudnovsky [8] and others, have demonstrated that several PDEs known to be integrable have the Painlevé Property. These include the PDEs listed in Table 4.5.1 as well as equations with higher special dimensions such as the Kadamtsev-Petviashvile equation.

Furthermore, Clarkson and McLeod [11] have shown that equations (4.5.10-11), which were only reducible via one-parameter groups to travelling waves, do not possess the Painlevé Property. Since it is believed that these equations are not integrable, this suggests that the Painlevé Property may provide a "better test" than reduction to ODEs.

However, Ward [25] pointed out that the analysis involved in determining the Painlevé Property is based on the assumption that the solution does not have essential singularities. Thus, it may be only detecting a subset of solutions of the PDE and not the general solution. This was later demonstrated by Clarkson [9], [10] when he showed that the equation

$$u_t^2 = 2uu_x^2 - (1+u^2)u_{xx} \qquad (4.6.4)$$

possesses the Painlevé Property but is known also to possess movable essential singularities. Thus, if we look for a travelling wave solution to (4.6.4) in the form

$$u(x, t) = f(x - ct)$$

we find that $f(\eta)$, where $\eta = x - ct$, satisfies

$$c^2\left(\frac{df}{d\eta}\right)^2 = 2f\left(\frac{df}{d\eta}\right)^2 - (1 + f^2)\,\frac{d^2f}{d\eta^2} . \qquad (4.6.15)$$

This equation has the general solution (see Example 4.3.3)

$$f(\eta) = \tan\left\{\frac{1}{c^2}\ln(A\eta + b)\right\},$$

which shows that (4.6.4) has a class of solutions with movable essential singularities.

This example demonstrates that the method proposed for detecting the Painlevé Property is inadequate in that it does not necessarily prove that the equation has the Painlevé Property. Thus it can only be used in the negative sense to eliminate equations if they turn out to have branch points, as in the case of ODEs. Thus, Clarkson and McLeod argue that as far as testing the analytic structure of a given PDE is concerned we cannot say which approach is better, i.e. testing the PDE directly or testing its similarity solutions when they exist. However, we would state the position as follows.

(i) If a given PDE has a similarity reduction to a first or second order ODE belonging to the Painlevé classification, then we immediately have the ODEs complete analytic structure including the existence of essential singularities which the direct method misses. In this instance the reduction method is better.

(ii) If we do not know that a PDE has similarity reductions to ODEs of first or second order, i.e. we are unable to obtain any by the Lie method, then the direct method is the best we can do.

4.7 Conclusion

Although the Painlevé conjectures of ARS and McLeod and Olver have not been proven, some important points about the proposed tests have emerged. We summarize them below.

(1) All the known IST soluble equations reduce to one or more Painlevé transcendents [18].

(2) The test for P-type ODEs via similarity reductions and the direct test as proposed by Weiss et al. are not identical.

(3) Strong evidence exists that neither the Painlevé test via similarity reductions nor the direct test are sufficient tests for integrability although both may be necessary.

(4) As we have just commented above, the direct test of Weiss et al. is stronger unless there is a similarity reduction to a first or second order ODE.

To complete our review and to set our work in context, we consider the work of Abbas and El-Sherbiny on a general class of PDEs with quadratic nonlinearities which we shall study further in later chapters.

CHAPTER FIVE

THE GENERAL CLASS OF EQUATIONS

In previous chapters we have noted the following important points:

(i) There is no systematic way of obtaining the linear eigenvalue problem necessary for IST from the nonlinear PDE. It must be guessed.

(ii) Although there are systematic ways of obtaining conservation laws, the methods are tedious and may not always work on arbitrary equations.

(iii) The Painlevé Conjecture, if true, is at most a necessary condition for IST and its strength is not in indentifying, but rather in eliminating equations which do not possess solitons. Thus, even if the solution structure is at most poles, we are again faced with finding an associated eigenvalue problem.

(iv) If a nonlinear PDE is soluble by IST and has solitary wave solutions, then the solitary waves are solitons.

Clearly, therefore, we are interested in knowing when a solitary travelling wave solution of a nonlinear PDE is also a soliton. That is, what properties does this wave and the equation have that guarantee the existence of soliton solutions. This study was initiated by Abbas [1] and developed by El-Sherbiny [3].

Broer's hypothesis [2] that the KdV could be intepreted as a field equation for a general field when properties could be thought of as arising from nonlinear and dispersive effects, had led to the belief that both dispersion and nonlinearity were necessary for evolution equations to have stable solitary waves. Abbas set about to test this belief by analysing the effects of dispersion and nonlinearity separately for selected initial profiles which included the sech^2 solitary wave. Then, by comparing the predictions of this analysis with the properties of the solitary wave solutions of the complete KdV equation, he found several contradictions. Abbas then considered a general class of third order equations with quadratic nonlinearities which included the KdV equation. He showed that this class has solitary wave solutions for a variety of dispersion relations, including a subclass of formally nondispersive equations, clearly contradicting the belief that both nonlinearity and dispersion are necessary for solitary waves. Finally, Abbas began the classification of the general class in terms of solitary wave solutions.

since dispersion is not a useful criterion in understanding the properties of the KdV equation, the question arises as to whether it is possible to develop other criteria for such understanding. El-Sherbiny took up this problem by asking whether the properties of the KdV are unique in this general class and whether this can be spotted from the equation and its elementary properties. He started by considering the question of whether the embedding of the KdV in the general class is reasonable. To do this he studied well-posedness, the existence of solitary waves and the existence of conservation laws. To establish well-posedness El-Sherbiny partitioned the general class into four equivalence classes and managed to show well-posedness for three of them and existence for a fourth. He also found that while all the equations have at least two conservation laws, unless an equation is in the KdV class it has at most three conservation laws.

In the following chapters we shall extend the work of Abbas and El-Sherbiny. This chapter gives a review of their results.

5.1 Broer's hypotheses and the KdV equation

Broer [2] suggested that the KdV equation could be approached from the point of view of field theory where the properties of the field are obtained from nonlinear and dispersive effects corresponding to the terms uu_x and u_{xxx} respectively. Since these terms appear additively in the equation, their interaction could be observed only in the space of solutions. Hence the general scheme proposed by Broer was to write the field equation as a structural perturbation

$$u_t + u_x + N(u) + D(u) = 0 \qquad (5.1.1)$$

of the basic unidirectional linear nondispersive equation

$$u_t + u_x = 0 \qquad (5.1.2)$$

In this perturbation N and D are the nonlinear and dispersive components respectively. This interpretation of the KdV equation led to the belief that its properties could be understood in terms of a balance between the nonlinear and dispersive effects, see Scott et al. [6].

Abbas [1] set out to test this hypothesis by analysing, separately, the effects of dispersion and nonlinearity on selected initial profiles. He considered the equations

$$u_t + u_x + u_{xxx} = 0, \qquad (5.1.3)$$

$$u_t + u_x + Auu_x = 0, \qquad (5.1.4)$$

with the initial profiles

$$g(x) = \operatorname{sech}^2 bx; \; g(x) = \exp(-a^2x^2); \; g(x) = (1+\lambda^2x^2)^{-1}$$

His analysis indicated that the sech^2 profile was the least dispersive of these three, while the nonlinear analysis did not distinguish between the profiles. Abbas proceeded to compare the properties of the solitary waves of the KdV with the predictions of its component parts. To vary the amounts of dispersion and nonlinearity, independently, the KdV was considered as

$$u_t + u_x + Auu_x + Bu_{xxx} = 0 \qquad (5.1.5)$$

so that the solitary waves are

$$u_s(x, t) = \frac{3c}{A} \operatorname{sech}^2 \left\{ \frac{1}{2} \sqrt{\frac{c}{B}} \, (x - (1+c)t) \right\}$$

$$(c > 0)$$

$$(5.1.6)$$

This comparison showed some contradictions of the Broer hypotheses. The solution indicates that solitary waves exist for all A, B, c > 0, subject to the constraints of unidirectionality, whereas the assumptions of a perturbation would imply that solitary waves will only exist if A, B and c are small. Furthermore, we see that for fixed c the amplitude and width are proportional to 1/A and $\sqrt{B}$ respectively, meaning that small amplitude, long wavelength regime is reached only for large values of A and B. Since (5.1.2) is a zero-order approximation for propagation of waves of small amplitude and long wavelength, this property is contrary to the assumptions. This demonstrates that the Broer hypothesis is not a useful way of understanding the KdV.

5.2 The General Class of Equations

To further test the general belief that the existence of solitary waves is due to a balance between nonlinearity and dispersion and that the KdV does not simply define a special case, Abbas considered a general class of third order equations with quadratic nonlinearity

i.e. $$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0 \qquad (5.2.1)$$

where $a_i \in \mathbb{R}$ ($i = 1, 2, \ldots, 6$). This contains the KdV and some proposed alternatives such as the regularized long wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (5.2.2)$$

and the Joseph and Egri equation (J.E.) [5]

$$u_t + u_x - uu_t + u_{xtt} = 0 \tag{5.2.3}$$

Abbas showed that solitary waves with $sech^2$ profiles exist for a wide variety of dispersion relations. However, he also showed the existence of a formally nondispersive subclass of the general class which has stable solitary waves

i.e. $$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt} = 0 \tag{5.2.4}$$

where $a_3 = a_4 < 0$.

The solitary waves of (5.2.4) are given by

$$u_s(x, t) = \frac{3c}{a_1-a_2(1+c)} \operatorname{sech}^2 \frac{1}{2\sqrt{|a_3|}} (x-(1+c)t) \tag{5.2.5}$$

and thus have a fixed width, i.e. independent of the speed c.

The existence of this formally nondispersive subclass clearly contradicts the belief that the formation and properties of solitary waves can be understood in terms of a balance between nonlinearity and dispersion.

Abbas concluded that dispersion is not necessary for the existence of solitary waves [1].

The inability of the dispersion terms to provide a criterion for understanding the properties of the KdV, leads to the question as to whether it is possible to develop other criteria for such understanding. El-Sherbiny took up this question and we will consider his work in section 5.5.

We now take a closer look at the solitary wave solutions of (5.2.1).

5.3 Solitary waves of the general class

Solitary waves are special instances of travelling waves, i.e. self-similar solutions, which are obtained by transforming the evolution equation (5.2.1) to the frame of reference in which the waves appear stationary (rest frame). Using the transformation

$$x \longrightarrow x - (1+c)t, \quad t \longrightarrow t \quad \text{and} \quad u(x, t) \longrightarrow v(x) \tag{5.3.1a}$$

(5.2.1) reduces to the O.D.E.

$$\beta v''' + \alpha v v' - v' = o \tag{5.3.1b}$$

where the primes denote x-derivatives and

$$\alpha c = a_1 - a_2(1+c), \quad \beta c = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3$$

Integrating equation (5.3.1) gives

$$\beta \frac{d^2v}{dx^2} + \frac{\alpha}{2}v^2 - v + A = 0, \qquad (5.3.2)$$

where A is a real constant of integration.

Multiplying (5.3.2) by v' and integrating again gives

$$\frac{\beta}{2}\left(\frac{dv}{dx}\right)^2 + \frac{\alpha v^3}{6} - \frac{v^2}{2} + Av + B = 0, \qquad (5.3.3)$$

where B is the second integration constant.
Equation (5.3.3) can be rewritten in the form:

$$\frac{3\beta}{\alpha}\left(\frac{dv}{dx}\right)^2 = - v^3 + \frac{3}{\alpha}v^2 - \frac{6Av}{\alpha} - \frac{6}{\alpha}B \qquad (5.3.4)$$

If we now make the substitutions

$$x \to \xi = \sqrt{\frac{1}{12\beta}}\, x, \qquad v \longrightarrow w = \frac{1}{\alpha} - v,$$

equation (5.3.4) reduces to the form

$$\left(\frac{dw}{d\xi}\right)^2 = 4w^3 - g_2 w - g_3 \qquad (5.3.5)$$

where $g_2 = \left(\frac{12}{\alpha^2}\right)(1-2A\alpha)$, $g_3 = \left(-\frac{4}{\alpha^3}\right)(2+6\alpha A+6\alpha^2 B)$.

We note also that α and β have the same sign.

The general solution of (5.3.5) is the Weiestrassian elliptic function $q(\xi)$ which can be written in terms of one of its Jacobian counterparts as follows [4]

$$q(\xi) = r_1 + (r_3 - r_2)cn^2(\lambda\xi;\ k) \qquad (5.3.6)$$

where cn is the Jacobian elliptic cosine amplitude

with modules k, $\lambda^2 = r_1 - r_3$, $k^2 = \dfrac{r_2 - r_3}{r_1 - r_3}$

and r_1, r_2 and r_3 are the roots of the equation

$$4r^3 - g_2 r - g_3 = 0 \ . \qquad (5.3.7)$$

Then it can be shown that for $k^2 = 1$ the solution (5.3.6) reduces to solitary wave form [1], [3]

$$u_s(x,t) = \frac{3}{\alpha}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{1}{\beta}}\,(x - (1+c)t)\right] \qquad (5.3.8)$$

Some of the consequences of the above analysis, due to Abbas, may be given in the following theorem:

<u>Theorem 5.1</u>

(i) The equation (5.2.1), with possible constraints on the parameters to keep the solutions real, has periodic waves which are Weierstrassian elliptic functions.

(ii) The necessary condition for the existence of real solitary waves is that $\beta > 0$ and these waves necessarily satisfy the boundary conditions

$$u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty .$$

(iii) The solitary waves all have the sech^2 profile □

Abbas also began a classification of the equations (5.2.1) in terms of their solitary waves by considering the properties of the width parameter β

i.e. $\beta c = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3$

(5.3.9)

and assuming that the amplitudes were positive with $c > 0$.

He classified the existence of solitary waves as follows:

(i) $a_6 = 0 \quad a_5 \neq 0, \quad \beta c = a_3 - a_4(1+c) + a_5(1+c)^2$ (5.3.10)

(ii) $a_6 = 0 = a_5 \quad , \quad \beta c = a_3 - a_4(1+c)$ (5.3.11)

Case (i) further gives rise to two subcases i.e. $a_5 < 0$ and $a_5 > 0$. The results of this classification are as follows:

(i) $a_5 \neq 0$. $\beta c = f(\gamma) = a_3 - a_4\gamma + a_5\gamma^2$ $(\gamma = (1+c) > 1)$

(a) $a_5 < 0$. A family of solitary waves exists if the maximum of $f(\gamma)$ is positive. There are two possible graphs as indicated in Figures 1(a) and 1(b).

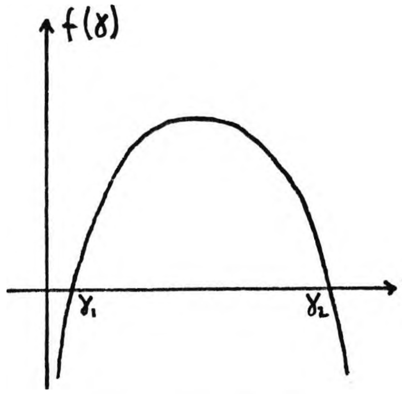


Fig. 1(a)

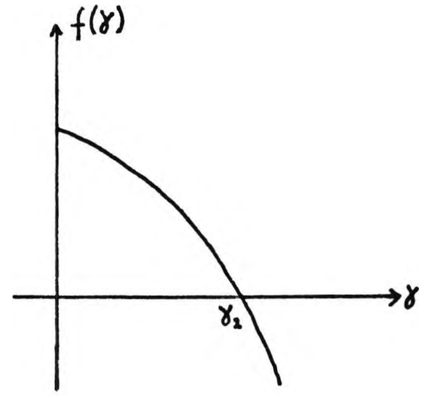


Fig. 1(b)

In the first case, Fig. 1(a), the speeds lie in the open interval (γ_1, γ_2), and in the second case, Fig. 1(b), they lie in $(1, \gamma_2)$.

(b) $a_5 > 0$. A family of solitary waves always exists, but there may be a gap in the allowed range of speeds. Consider the graphs of $f(\gamma)$.

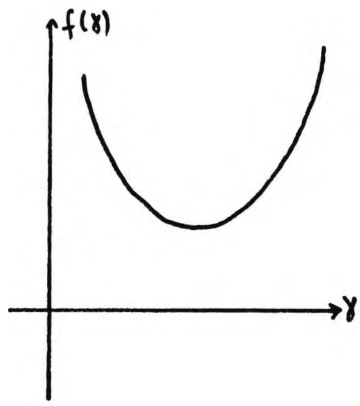


Fig. 1(c)

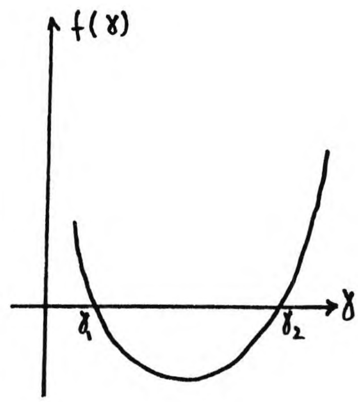


Fig. 1(d)

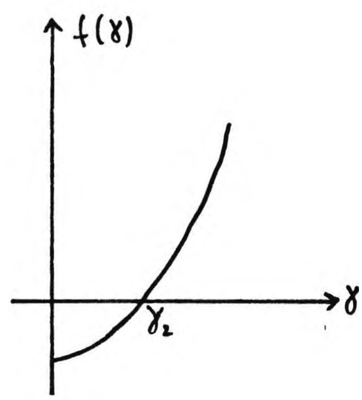


Fig. 1(e)

If the minimum is positive as in Fig. 1(c) then solitary waves exist for all speeds.

If the minimum is negative as in Figures 1(d) and 1(e) then excluded speeds are those in the closed intervals $[\gamma_1, \gamma_2]$ (including the case $\gamma_1 = \gamma_2$) and $[1, \gamma_2]$.

The results for $c < 0$ follow from the above by reversing the direction of the γ-axis and interchanging the interpretation of the figures.

(ii) <u>$a_5 = 0$:</u> Existence and speeds of the solitary wave is summarized in the Figure 2 below:

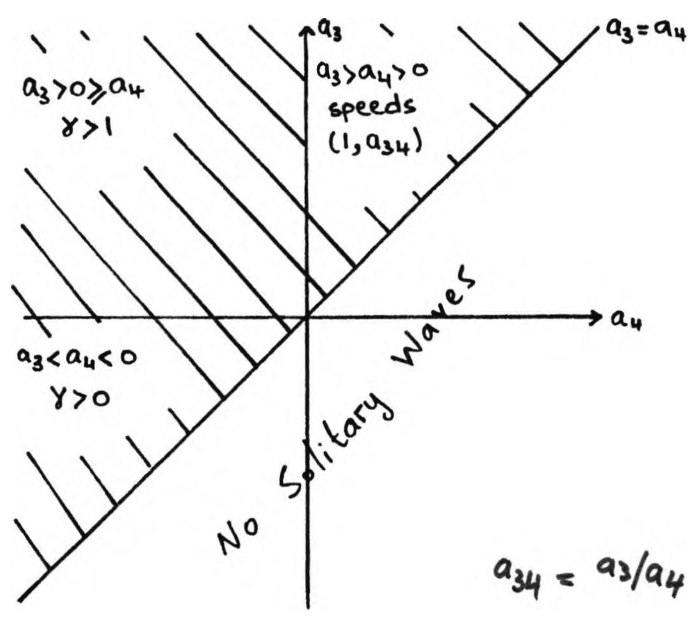


Fig. 2

This classification was completed by El-Sherbiny [3] when he considered the case $a_6 \neq 0$. The types of roots of (5.3.9) are obtained according to the properties of the discriminant Δ of the cubic (5.3.9) i.e.,

$$\Delta = \frac{1}{108a_6}\left(4a_3a_5^3-18a_3a_4a_5a_6+27a_3^2a_6^2+4a_4^3a_6-a_4^2a_5^2\right) \tag{5.3.11}$$

By considering the cases $\Delta = 0$, $\Delta > 0$ and $\Delta < 0$ and the graphs of (5.3.9) in the same manner as Abbas, El-Sherbiny completed the classification of the general class (5.2.1) [3].

The above classification shows that for quadratic nonlinearities and third order dispersive terms, solitary waves, where they exist, have the sech^2 form. Furthermore, there is a variety of equations which have the same nonlinear but different dispersion terms. A major result of Abbas's work is that the linear part of the equation is an unreliable indicator of the properties of the full nonlinear equation.

Other extensions of Abbas's results on the general class necessary to understand the special properties of the KdV were completed by El-Sherbiny. His starting point was to show that the class of equations (5.2.1) could in fact be considered as homologues of the KdV.

Thus, he studied the well-posedness of the class.

5.4 Well-Posedness of the general class

Consider the initial value problem which corresponds to the general class (5.2.1) where u, u_t and u_{tt} are given on an arbitrary space-time curve $x = x(s)$, $t = t(s)$, where s is the parameter. This initial value problem is said to be well-posed if it has a unique solution which depends continuously on the initial data.

For the initial value problem for the general equation

$$u_t+u_x+a_1uu_x+a_2uu_t+ a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+ a_6u_{ttt} = 0 \qquad (5.4.1a)$$

$$u(x, 0) = f(x), \; u_t(x, 0) = g(x), \; u_{tt}(x, 0) = h(x) \qquad (5.4.1b)$$

El-Sherbiny proved the following theorem.

Theorem 5.2 [3]

The initial value problem (5.4.1) for the general class of equations with non-characteristic data can be reduced to the non-characteristic initial value problem for a first order system of semi-linear PDEs given by

$$U_t + AU_x + C = 0 \qquad (5.4.2)$$

$$U(x,\ 0) = H(x)$$

where the U, C and A are

$$U^T = \left[u \ u_t u_x u_{tt} u_{xt} u_{xx} u_{ttt} u_{xtt} u_{xxt} u_{xxx}\right] \tag{5.4.3a}$$

$$C^T = \left[-u_t - u_{tt} - u_{xt} - u_{ttt} \ \ 0 \ 0 \ m \ 0 \ 0 \ 0\right]$$

with $m = \dfrac{1}{a_\sigma}\left[(a_1 u_x + a_2 u_t)u_t - (1+a_2 u)u_{tt} + 4a_1 u u_{xt}\right]$,

$a_\sigma \neq 0.$

and

$$A = \left[\begin{array}{cccc|cc|cccc}
 & & & & 0 & 0 & & & & \\
 & 0(4\times 4) & & & 0 & 0 & & 0(4\times 4) & & \\
 & & & & 0 & 0 & & & & \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 0 & 0 & a_5/a_\sigma & a_4/a_\sigma & a_3/a_\sigma & 0 \\
 & & & & 0 & 0 & -1 & 0 & 0 & 0 \\
 & 0(4\times 4) & & & 0 & 0 & 0 & -1 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right] \ \square \tag{5.4.3c}$$

When $a_\sigma = 0$, A becomes singular and so El-Sherbiny studied well-posedness for two separate classes i.e. the nonsingular class $(a_\sigma \neq 0)$ and the singular class $(a_\sigma = 0)$.

For the nonsingular class, well-posedness was obtained via the traditional method of characteristics. That is, by finding the characteristics of the system in (5.4.2) and showing that it is equivalent to a system of ODEs in which differentiation is along a characteristic direction. These equations can be integrated to give the solution of the system, provided the data is not specified on a characteristic.

Now the characteristics of the system (5.4.2), corresponding to the general class, are given by the roots of the cubic

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0. \qquad (5.4.4)$$

where $\lambda = \frac{dx}{dt}$.

The method of characteristics fails when either A is singular or the data are characteristic. In these cases El-Sherbiny was able to reduce the general class to four equivalence classes as defined by the following theorem.

<u>Theorem 5.3</u>

Consider the initial value problem corresponding to the general class of equations

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0 \qquad (5.4.5)$$

where the initial data u, u_t and u_{tt} are given on a characteristic line $x = mt$ $m \neq 0$. Then this problem reduces to the four equivalence classes KdV, RLW, W_{54} and W_{53} defined as follows:

$$\text{KdV:} \quad v_t + v_x + c_1 v v_x + c_2 v v_t + c_3 v_{xxx} = 0$$

$$\text{RLW:} \quad v_t + v_x + d_1 v v_x + d_2 v v_t + d_4 v_{xxt} = 0$$

$$W_{54}: \quad v_t + v_x + \gamma_1 v v_x + \gamma_2 v v_t + \gamma_4 v_{xxt} + \gamma_5 v_{xtt} = 0$$

$$W_{53}: \quad v_t + v_x + \delta_1 v v_x + \delta_2 v v_t + \delta_3 v_{xxx} + \delta_4 v_{xtt} = 0 \qquad (5.4.6)$$

and the corresponding characteristic data u, u_t and u_{tt} reduce to v, v_t and v_{tt} on $t = 0$ □

To investigate well-posedness of the singular class, El-Sherbiny considered well-posedness of each of the four classes in (5.4.6). Using known theory on the well-posedness of the KdV and RLW equations, he was able to show well-posedness for a variety of cases including the RLW class with a uu_t term. However, he had limited success with the classes W_{54} and W_{53}, where he gave restricted existence proofs, and was unable to obtain results for the KdV class when the uu_t term was included.

5.5 Conservation Laws of the general class

The third part of El-Sherbiny's contribution was to examine the conservation law property of the general class (5.2.1). The first two conservation laws of this class were obtained via elementary operations and it was shown that if (a_1/a_2) satisfy the cubic equation

$$a_3 - a_4 \left(\frac{a_1}{a_2}\right) + a_5 \left(\frac{a_1}{a_2}\right)^2 - a_6 \left(\frac{a_1}{a_2}\right)^3 = 0 \tag{5.5.1}$$

then the corresponding subset of equations would have a third conservation law. By applying (5.5.1), the problem was reduced to the four equivalence classes of Theorem 5.3, but with uu_t terms removed. These classes were then studied separately.

According to the nature of the roots of the cubic (5.5.1) El-Sherbiny obtained the following results:

(i) If (a_1/a_2) is a triple root then we have the KdV, which has an infinite number of conservation laws.

(ii) If (a_1/a_2) is a double root, then we have the RLW which has only three conservation laws.

(iii) If (a_1/a_2) is one of three real roots we obtain the class W_{54}.

(iv) If (a_1/a_2) is the only real root we obtain the class W_{53}.

In cases (iii) and (iv) the existence and number of conservation laws is given in the following theorem:

Theorem 5.4

Each of the equations

$$u_t + u_x + b_1 u u_x + b_4 u_{xxt} + b_5 u_{xtt} = 0 \qquad (5.5.2)$$

and

$$u_t + u_x + b_1 u u_x + b_3 u_{xxx} + b_5 u_{xtt} = 0 \qquad (5.5.3)$$

has only three conservation laws.

5.6 Conclusion

In this chapter we reviewed the work of Abbas and El-Sherbiny on the general class of equations (5.2.1). This class was considered as forming a neighbourhood of the KdV. Abbas showed that the idea of a balance between the nonlinear and dispersion terms does not provide an understanding of why solitary waves occur. This contradiction of Broer's hypothesis meant that other criteria for such understanding was necessary.

El-Sherbiny considered the well-posedness and the number of conservation laws of the general class.

Both his and Abbas's work show that the KdV has a number of exceptional properties which are not shared by any of its alternatives. The KdV has an infinite number of conservation laws, can be solved by IST and possesses solitons. Abbas and El-Sherbiny's work suggests that none of the alternatives possesses any of these properties.

In the following chapters we continue the study of the general class by investigating the analytic structure of solutions, both special and general, of (5.2.1). In this context we pursue the question of why the KdV is a unique equation in the class. In particular we review and apply criteria based on the analytic structure of solutions i.e. the Painlevé conjectures.

CHAPTER SIX

THE SIMILARITY REDUCTIONS OF THE GENERAL CLASS OF EQUATIONS

In this chapter we begin our study of the general class of equations first defined by Abbas [1]. That is

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0$$

where $a_i (i = 1, 2, ..., 6)$ are real numbers and $u(x, t)$ is a real scalar field for all $(x, t) \in \mathbb{R}^2$. Our work, which has been motivated by similarity solutions and the Painlevé conjecture will be to investigate the analytic structure of this class. The work of Abbas [1] and El-Sherbiny [4] shows that this is a reasonable class to consider since it can be viewed as forming a neighbourhood of the KdV equation. We have seen that the KdV appears to be unique in admitting IST solutions, solitons and possessing an infinite number of local conservation laws. It is also known that the similarity reductions of the KdV are Painlevé type. In this chapter we obtain the similarity reductions of the general class of equations, the solution structure of which will be considered in the following chapter. The technique we have reviewed in chapter four of Lie's one-parameter (ε) group of transformations [2] will be used. Our approach will be to split the class into three subclasses in accordance with Cauchy's problem and to obtain the symmetry groups for each subclass separately. Then, using these, we proceed to obtain their similarity reductions.

6.1 Subclasses of the general class

The general class of equations

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0 \tag{6.1.1}$$

splits with respect to Cauchy's problem into three distinctive subclasses.

(i) The class $W_6(a_6 \neq 0)$ for which (6.1.1) is third order in t and three bits of data u, u_t and u_{tt} are given at $t = 0$.

(ii) The class $W_5(a_6 = 0,\ a_5 \neq 0)$ for which (6.1.1) is second order in t and two bits of data u and u_t are given at $t = 0$.

(iii) The class $W_{43}(a_6 = 0 = a_5,\ a_4 \neq 0)$ for which (6.1.1) is first order in t and only u is given at $t = 0$.

6.2 The local symmetry group of the class W_σ

The equation of this class is (6.1.1) and the one-parameter (ϵ) group of infinitesimal transformations in (x, t, u) are given by

$$x' = x + \epsilon X(x, t, u) + O(\epsilon^2) \qquad (6.2.1a)$$

$$t' = t + \epsilon T(x, t, u) + O(\epsilon^2) \qquad (6.2.1b)$$

$$v = u + \epsilon U(x, t, u) + O(\epsilon^2) \qquad (6.2.1c)$$

Then following the work in chapter four

$$v_{x'} = u_x + \epsilon U^x + O(\epsilon^2) \qquad (6.2.2a)$$

$$v_{t'} = u_t + \epsilon U^t + O(\epsilon^2) \qquad (6.2.2b)$$

$$v_{x'x'x'} = u_{xxx} + \epsilon U^{xxx} + O(\epsilon^2) \qquad (6.2.2c)$$

$$v_{x'x't'} = u_{xxt} + \epsilon U^{xxt} + O(\epsilon^2) \qquad (6.2.2d)$$

$$v_{x't't'} = u_{xtt} + \epsilon U^{xtt} + O(\epsilon^2) \qquad (6.2.2e)$$

$$v_{t't't'} = u_{ttt} + \epsilon U^{ttt} + O(\epsilon^2) \qquad (6.2.2f)$$

where the functions U^x, U^t, U^{xxx}, U^{xxt}, U^{xtt} and U^{ttt} are determined from equations (6.2.1a-c) as

$$U^x = U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t \tag{6.2.3}$$

$$U^t = U_t + (U_u - T_t)u_t - X_t u_x - T_u u_t^2 - X_u u_x u_t \tag{6.2.4}$$

with U^{xxx}, U^{xxt}, U^{xtt} and U^{ttt} given in Appendix B.

Equation (6.1.1) is invariant under the transformations (6.2.1-2) if

$$v_{t'}+v_{x'}+a_1 vv_{x'}+a_2 vv_{t'}+a_3 v_{x'x'x'}+a_4 v_{x'x't'}+a_5 v_{x't't'} +a_6 v_{t't't'} = 0 \tag{6.2.5}$$

Substituting (6.2.1-2) into (6.1.1) gives, to first order in ϵ,

$$(1+a_1 u)U^x+(1+a_2 u)U^t+a_1 u_x U+a_2 u_t U+a_3 U^{xxx}+a_4 U^{xxt} +a_5 U^{xtt}+a_6 U^{ttt} = 0 \tag{6.2.6}$$

which is the invariance equation. The infinitesimals $X(x, t, u)$, $T(x, t, u)$ and $U(x, t, u)$ are determined by equating the coefficients of the derivatives of u and terms free from derivatives of u to zero.

Equating to zero the coefficients of $u_t u_{ttt}$, $u_x u_{xxx}$ and $u_t u_{tt}$ gives

$$T_u = X_u = 0, \quad U_{uu} = 0 \tag{6.2.7}$$

Now writing out (6.2.5) in full we have

$$(u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt}) U_u$$

$$+ (1+a_1 u)\left[U_x - X_x u_x - T_x u_t\right] + a_1 u_x U + a_2 u_t U + (a_1 + a_2 u)$$

$$\left[U_t - T_t u_t - X_t u_x\right] + a_3 \left[U_{xxx} + (3U_{xxu} - X_{xxx}) u_x - T_{xxx} u_t\right.$$

$$\left. + (3U_{xu} - 3X_{xx}) u_{xx} - 3X_x u_{xxx} - 3T_{xx} u_{xt} - 3T_x u_{xxt}\right]$$

$$+ a_4 \left[U_{xxt} + (2U_{xtu} - X_{xxt}) u_x + (U_{xxu} - T_{xxt}) u_t + (U_{ut} - 2X_{xt}) u_{xx}\right.$$

$$\left. + (2U_{xu} - X_{xx} - 2T_{xt}) u_{xt} - T_{xx} u_{tt} - X_t u_{xxx} - (2X_x + T_t) u_{xxt} - 2T_x u_{xtt}\right]$$

$$+ a_5 \left[U_{xtt} + (2U_{xtu} - T_{xtt}) u_t + (U_{ttu} - X_{xtt}) u_x + (U_{ux} - 2T_{xt}) u_t\right.$$

$$\left. + (2U_{tu} - T_{tt} - 2X_{xt}) u_{xt} - X_{tt} u_{xx} - T_x u_{ttt} - (2T_t + X_x) u_{xtt} - 2X_t u_{xxt}\right]$$

$$+ a_6 [U_{ttt} + (3U_{ttu} - T_{ttt}) u_t - X_{ttt} u_x + (3U_{tu} - 3T_{tt}) u_{tt}$$

$$\left. - 3T_t u_{ttt} - 3X_{tt} u_{xt} - 3X_t u_{xtt}\right] = 0 \tag{6.2.8}$$

Using the PDE (6.1.1) to eliminate u_{ttt}

i.e.,

$$u_{ttt} = -\frac{1}{a_6}(u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt})$$

in (6.2.8) and equating the coefficients of third order to zero gives

$$u_{xxx}: \frac{a_5}{a_6}T_x - \frac{a_4}{a_3}X_t - 3X_x + 3T_t = 0 \qquad (6.2.9)$$

$$u_{xxt}: \left[\frac{a_5}{a_6} - 3\frac{a_3}{a_4}\right]T_x - 2\frac{a_5}{a_4}X_t + 2T_t - 2X_x = 0 \qquad (6.2.10)$$

$$u_{xtt}: \left[\frac{a_5}{a_6} - 2\frac{a_4}{a_5}\right]T_x - 3\frac{a_6}{a_5}X_t + T_t - X_x = 0 \qquad (6.2.11)$$

From (6.2.11) and (6.2.9) we obtain

$$\left[2\frac{a_5}{a_6} - 6\frac{a_4}{a_5}\right]T_x + \left[\frac{a_4}{a_3} - 9\frac{a_6}{a_5}\right]X_t = 0 \qquad (6.2.12)$$

and (6.2.10) and (6.2.9) give

$$\left[\frac{a_5}{a_6} - 9\frac{a_3}{a_4}\right]T_x + \left[2\frac{a_4}{a_3} - 6\frac{a_5}{a_4}\right]X_t = 0 \qquad (6.2.13)$$

Then if $\Delta = 4a_3a_5^3-18a_3a_4a_5a_6+27a_3^2a_6^2+4a_4^3a_6-a_4^2a_5^2 \neq 0$

(6.2.14)

equations (6.2.12) and (6.2.13) lead to $X_t = T_x = 0$ i.e. $X = X(x)$, $T = T(t)$. Now from (6.2.9-11) $X_x = T_t$, and it follows that $X_{xx} = T_{tt} = X_{xt} = T_{tx} = 0$. Hence, $X = \alpha x + \beta$, $T = \alpha t + \gamma$ where α, β and γ are arbitrary constants.

Returning to equation (6.2.8) and equating the coefficients of u_{xx}, u_{tt} and u_{xt} to zero we obtain

$$3a_3 U_{ux} + a_4 U_{ut} = 0 \qquad (6.2.15)$$

$$a_5 U_{ux} + 3a_6 U_{ut} = 0 \qquad (6.2.16)$$

$$a_4 U_{ux} + a_5 U_{ut} = 0 \qquad (6.2.17)$$

Now there are four possibilities all of which lead to

$$U_{ux} = U_{ut} = 0 \qquad (6.2.18)$$

These are:

(i) $a_4 a_5 \neq 9a_3 a_6$, $a_5^2 \neq 3a_4 a_6$, $a_4^2 \neq 3a_3 a_5$

(ii) $a_4 a_5 = 9a_3 a_6$, $a_5^2 \neq 3a_4 a_6$, $a_4^2 \neq 3a_3 a_5$

(iii) $a_5^2 = 3a_4 a_6$, $a_4 a_5 \neq 9a_3 a_6$, $a_4^2 \neq 3a_3 a_5$

(iv) $a_4^2 = 3a_3 a_5$, $a_4 a_5 \neq 9a_3 a_6$, $a_5^2 \neq 3a_4 a_6$.

All other possibilities do not satisfy the condition $\Delta = 0$.

To determine U from the invariance equation we equate the coefficients of u_x and u_t to zero to obtain

$$u_x:\ 2(1+a_1u)X_x + a_1U = 0 \qquad (6.2.19a)$$

$$u_t:\ 2(1+a_2u)X_x + a_2U = 0 \qquad (6.2.19b)$$

We consider the following two cases: (i) $a_1 \neq a_2$, (ii) $a_1 = a_2 \neq 0$.

(i) When either a_1 or a_2 is zero in (6.2.19) we are led to $X_x = 0$ which gives the generator $U = 0$.

When neither a_1 nor a_2 is zero then we have

$$U = -\frac{2\alpha}{a_1}(1 + a_1u) = -\frac{2\alpha}{a_2}(1 + a_2u)$$

and we are forced to $\alpha = 0$. Hence we obtain the same generator $U = 0$.

(ii) When $a_1 = a_2 \neq 0$ then clearly $U = -\dfrac{2\alpha}{a_1}(1+a_1u)$.

We now state our result in the following theorem:

<u>Theorem 6.1</u>

If $\Delta \neq 0$ then the generators of the local symmetry group of the equation $W_\sigma (a_\sigma \neq 0)$ are as follows:

(a) $a_1 \neq a_2$: $X = \beta, \quad T = \gamma, \quad U = 0.$

(b) $a_1 = a_2 \neq 0$: $X = \alpha x + \beta, \quad T = \alpha t + \gamma,$

$$U = -\frac{2\alpha}{a_1}(1 + a_1 u)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ □

When $a_1 \neq a_2$, the general class W_σ can always be reduced to a class in which one of the nonlinear terms is eliminated. We give the theorem below, the proof of which is in Appendix C.

<u>Theorem 6.2</u>

When $a_1 \neq a_2$, the general class of equations W_σ reduces to the general class of equations

$$v_\xi + v_\eta + b_1 v v_\eta + b_2 v v_\xi + b_3 v_{\eta\eta\eta} + b_4 v_{\eta\eta\xi} + b_5 v_{\eta\xi\xi} + b_6 v_{\xi\xi\xi} = 0 \tag{6.2.20}$$

where $b_i (i = 1, 2, \ldots, 6)$ are constants with either b_1 or b_2 zero. □

<u>Corollary</u> If $a_1 = a_2$ then $b_1 = b_2$ and neither can be eliminated. □

Note that in the above we have assumed that a_3, a_4 and a_5 are nonzero. It is easy, however, to show that the generators of the local symmetry group of W_σ are those given in theorem 6.1 for all a_3, a_4, a_5. This can be demonstrated by putting in required values of a_3, a_4, a_5 in (6.2.8) and proceeding as above. Clearly, the restrictions on the dispersive coefficients will change for different choices.

To complete this section on the class W_σ it is necessary to consider the case $\Delta = 0$ where Δ was defined in (6.2.14).

The following theorem, due to El-Sherbiny, simplifies our analysis.

<u>Theorem 6.3</u>

If $\Delta = 0$, then the class W_σ can be reduced to the following subclasses via an equivalence relation:

$$v_t + v_x + c_1 v v_x + c_2 v v_t + c_3 v_{xxx} = 0 \qquad \text{KdV class}$$

$$v_t + v_x + d_1 v v_x + d_2 v v_t + d_4 v_{xxt} = 0 \qquad \text{RLW class} \quad \square$$

Note: The equivalence relation is as follows: two equations are said to be equivalent if they are connected by a non-singular linear transformation.

The local symmetry group of the KdV and RLW classes will be found later. This aside, we have now obtained the local symmetry groups of the class W_6.

6.3 The local symmetry group of the class W_5

The equation of this class is

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt} = 0 \quad (6.3.1)$$

and the following theorems (El-Sherbiny [4]) enable us to obtain the generators of the local symmetry groups of W_5 from those of W_6.

Theorem 6.4

If $\Delta \neq 0$ then the class W_6 is equivalent to the subclass of W_5 defined by the condition $a_4^2 \neq 4a_3a_5$ (i.e., a_6 can always be eliminated by a nonsingular linear transformation). □

From this theorem it follows that the generators of this subclass of W_5 are also given by those stated in theorem 6.1 as summarised in the following theorem.

Theorem 6.5

The local symmetry groups of the class of equations (6.3.1) where $a_4^2 \neq 4a_3a_5$ are as follows:

(i) If $a_1 \neq a_2$, then the equation is equivalent to one in which either a_1 or $a_2 = 0$ and the generators are

$$X = \beta, \quad T = \gamma, \quad U = 0.$$

(ii) If $a_1 = a_2$, then neither of the nonlinear terms can be eliminated by an equivalence transformation as defined before and the generators are

$$X = \alpha x + \beta, \quad T = \alpha t + \gamma, \quad U = -\frac{2\alpha}{a_1}(1 + a_1 u)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ □

<u>Theorem 6.6</u>

If $a_4^2 = 4a_3a_5$ then W_5 reduces to the same classes that W_6 reduces to when $\Delta = 0$, i.e., those given in theorem 6.3 above □

<u>6.4 The local symmetry group of the class W_{43}</u>

The equation of this class is

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} = 0$$

(6.4.1)

Following the work on W_6, the invariance equation under one-parameter (ϵ) infinitesimal transformations is

$$(1+a_1 u)U^x + (1+a_2 u)U^t + a_1 u_x U + a_2 u_t U + a_3 U^{xxx} + a_4 U^{xxt} = 0$$

(6.4.2)

together with the conditions

$$X_u = T_u = U_{uu} = 0 \qquad (6.4.3a)$$

Furthermore, equating the coefficient of the derivative of u_{xtt} to zero gives

$$T_x = 0 \qquad (6.4.3b)$$

Using (6.4.3) and

$$u_{xxt} = -\frac{1}{a_4}\left[(1+a_1u)u_x+(1+a_2u)u_t+a_3u_{xxx}\right]$$

gives the invariance equation in the form

$$(1+a_1u)\left[U_x+(X_x+T_t)u_x\right]+(1+a_2u)\left[U_t+2X_xu_t-X_tu_x\right]+a_1u_xU$$

$$+a_2u_tU+a_3\left[U_{xxx}+(3U_{xxu}-X_{xxx})u_x+(3U_{xu}-3X_{xx})u_{xx}-3X_xu_{xxx}\right]$$

$$+a_4\left[U_{xxt}+(2U_{xtu}-X_{xxt})u_x+U_{xxu}u_t+(U_{tu}-2X_{xt})u_{xx}\right.$$

$$\left.+(2U_{xu}-X_{xx})u_{xt}-X_tu_{xxx}\right]+a_3(2X_{xt}+T_t)u_{xxx} = 0.$$

(6.4.4)

Furthermore, from the terms u_{xxx}, u_{xx} and u_{xt} we have

$$-a_3X_x - a_4X_t + a_3T_t = 0 \qquad (6.4.5)$$

$$a_3(3U_{xu}-3X_{xx}) + a_4(U_{tu}-2X_{xt}) = 0 \qquad (6.4.6)$$

$$a_4(2U_{xu}-X_{xx}) = 0 \qquad (6.4.7)$$

From (6.4.7) we obtain $U_{xu} = \frac{1}{2} X_{xx}$ and substituting in (6.4.6) gives $-\frac{3}{2} a_3X_{xx} + a_4U_{tu} - 2a_4X_{xt} = 0$.
Differentiating (6.4.5) with respect to x gives $a_3X_{xx} + a_4X_{xt} = 0$ and hence $\frac{3}{2} a_4X_{xt} + a_4U_{tu} - 2a_4X_{xt} = 0$ and $U_{tu} = \frac{1}{2} X_{xt}$.

Then $U_{xu} = \frac{1}{2} X_{xx}$, $U_{tu} = \frac{1}{2} X_{xt}$, $a_3X_{xx} + a_4X_{xt} = 0$

(6.4.8)

Using (6.4.8) reduces the invariance equation (6.4.4) to:

$$(1+a_1u)\left[U_x+(X_t+T_t)u_x\right]+(1+a_2u)\left[U_t+2X_xu_t-X_tu_x\right]+a_1u_xU$$

$$+a_2u_tU+a_3\left[U_{xxx}+\frac{1}{2}X_{xxx}u_x\right]+a_4\left[U_{xxt}+\frac{1}{2}X_{xxx}u_t\right] = 0$$

(6.4.9)

Equating coefficients of u_x and u_t to zero:

$$u_x: \ (1+a_1u)(X_x+T_t) - (1+a_2u)X_t + a_1U + \frac{1}{2}a_3X_{xxx} = 0$$

(6.4.10)

$$u_t: \ 2(1+a_2u)X_x + a_2U + \frac{1}{2}a_4X_{xxx} + 0 \qquad (6.4.11)$$

Differentiating (6.4.10-11) with respect to u and using $X_u = T_u = 0$ gives

$$a_1(X_x + T_t) - a_2X_t + a_1U_u = 0 \qquad (6.4.12)$$

and $2a_2X_x + a_2U_u = 0$ (6.4.13)

From (6.4.13) $U_{ux} = -2X_{xx}$. Thus, from (6.4.8) we deduce that

$$U_{ut} = U_{ux} = X_{xt} = X_{xx} = 0 \qquad (6.4.14)$$

Now using (6.4.13) and (6.4.14) in (6.4.11) to eliminate the X term we obtain

$$\frac{\partial}{\partial u}\left[\frac{a_2U}{2(1+a_2u)}\right] = 0.$$

Integrating with respect to u gives $a_2U = 2(1+a_2u)f(x, t)$ i.e. $U_u = 2f(x, t)$. Since $U_{xu} = 0 = U_{tu}$ then U_u = constant. Let $f(x, t) = -\alpha$ then $X_x = \alpha$ and

$$a_2 U = -2\alpha(1 + a_2 u) \qquad (6.4.15)$$

Equation (6.4.12) becomes $a_1 T_t - a_2 X_t = a_1\alpha$

$$(6.4.16)$$

Equation (6.4.5) becomes $a_3 T_t - a_4 X_t = a_3\alpha$

$$(6.4.17)$$

Now if $\frac{a_1}{a_2} \neq \frac{a_3}{a_4}$ $(a_2 \neq 0)$ then $X_t = 0$ and $T_t = \alpha$ and (6.4.10) gives

$$a_1 U = -2\alpha(1 + a_2 u) \qquad (6.4.18)$$

There are two cases to consider: (i) $a_1 \neq a_2$, (ii) $a_1 = a_2$.

(i) When $a_1 = 0$, $a_2 \neq 0$ and $a_1 \neq a_2 \neq 0$, from (6.4.15) and (6.4.18) we see that $\alpha = 0$ and so the generators are $X = \beta$, $T = \gamma$, $U = 0$.

(ii) When $a_1 = a_2$ i.e. $a_3 \neq a_4$ we have the generators $X = \alpha x + \beta$, $T = \alpha t + \gamma$, $U = -\frac{2\alpha}{a_1}(1 + a_1 u)$.

To consider the case $a_2 = 0$, $a_1 \neq 0$ we refer back to equations (6.4.10-13). From (6.4.11) it is easy to

show that $X_x = 0$. Then from (6.4.12) $U_u = -T_t$.
But $U_{ux} = U_{ut} = T_{xt} = T_{tt} = 0$ so $T_t = \alpha$.
Substituting this value into (6.4.5) we obtain

$$X_t = \frac{a_3}{a_4}\alpha.$$

We now state our result in the following theorem.

<u>Theorem 6.7</u>

If $\frac{a_1}{a_2} \neq \frac{a_3}{a_4}$ $(a_2 \neq 0)$ then the generators of the local symmetry group of the equation W_{43} are as follows:

(a) $a_1 \neq a_2 \neq 0 \neq 0$ or $a_1 = 0$, $a_2 \neq 0$: $X = \beta$, $T = \gamma$, $U = 0$.

(b) $a_1 = a_2 \neq 0$: $X = \alpha x + \beta$, $T = \alpha t + \gamma$,

$$U = -\frac{2\alpha}{a_1}(1 + a_1 u)$$

If $a_2 = 0$, $a_1 \neq 0$: $X = \frac{a_3}{a_4}\alpha t + \beta$, $T = \alpha t + \gamma$,

$$U = -\frac{\alpha}{a_1}\left[a_1 u - \left(\frac{a_3}{a_4} - 1\right)\right] \square$$

when $\frac{a_1}{a_2} = \frac{a_3}{a_4} = \frac{1}{k}$ and neither a_1 nor a_2 is zero then we have the following lemma:

Lemma 6.1

The class of W_{43} equations

$$u_t + u_x + a_1uu_x + ka_1uu_t + a_3u_{xxx} + ka_3u_{xxt} = 0,$$
$$(k \neq 0, 1)$$

are equivalent to a class of RLW equations

$$(k-1)a_3u_X + u_T + ka_1uu_T + ka_3u_{XXT} = 0 \quad \square.$$

Proof. Make the non-singular linear transformation

$X = x - \frac{1}{k}t$, $T = t/ka_3$ to obtain result.

To obtain all the local symmetry groups for the general class, it remains to study the RLW and the KdV classes.

6.5 The local symmetry group of the RLW class

The equation of this class is

$$u_t + u_x + a_1uu_x + a_2uu_t + a_4u_{xxt} = 0 \quad (6.5.1)$$

The invariance equation under one-parameter (ϵ) infinitesimal transformations is

$$(1+a_1u)U^x + (1+a_2u)U^t + a_1u_xU + a_2u_tU + a_4U^{xxt} = 0$$
$$(6.5.2)$$

As in the previous derivations we obtain

$$T_u = 0 = T_x, \quad X_u = 0 = X_t, \quad U_{uu} = 0 = U_{ut} \tag{6.5.3}$$

$$2U_{xu} - X_{xx} = 0 \tag{6.5.4}$$

$$(1+a_1u)(X_x+T_t) + a_1U = 0 \tag{6.5.5}$$

$$2X_x(1+a_2u) + a_2U + a_4U_{xxu} = 0 \tag{6.5.6}$$

We now look at the following cases: (i) $a_1 = 0$, $a_2 \neq 0$, (ii) $a_1 \neq 0$, $a_2 = 0$, (iii) $a_1 \neq 0$, $a_2 \neq 0$.

(i) When $a_1 = 0$, $a_2 \neq 0$ we obtain, from (6.5.4) that $X_x + T_t = 0$, i.e. $X_{xx} = T_{tt} = 0$ and from (6.5.4) $U_{xu} = 0$. Since $X = X(x)$, $T = T(t)$ it follows that $T = \alpha t + \gamma$ and $X = -\alpha x + \beta$. Then from (6.5.6) $U = -\frac{2\alpha}{a_2}(1+a_2u)$.

(ii) When $a_1 \neq 0$, $a_2 = 0$ we obtain from (6.5.4) and (6.5.6) that $2U_{xu} = X_{xx}$ and $2X_x = -a_4U_{xxu}$ respectively. Then $X_x = 0$ and $X = \beta$. Also differentiating (6.5.5) with respect to u first and then with respect to t we see that $T_{tt} = 0$ i.e., $T = \alpha t + \gamma$. Then from (6.5.5) $U = -\frac{\alpha}{a_1}(1+a_1u)$.

(iii) If $a_1 \neq 0$, $a_2 \neq 0$ then (6.5.5) gives

$$X_x + T_t = - \frac{a_1 U}{1+a_1 u} \qquad (6.5.7)$$

which implies that

$$a_1 U = (1+a_1 u) f(x, t) \qquad (6.5.8)$$

Thus from (6.5.3) $U_{ut} = f_t(x, t) = 0$ so that f is a function of x only. Returning to (6.5.8), $X_x + T_t = - f(x)$ and since $X = X(x)$ and $T = T(t)$ we must have $T = \alpha t + \gamma$.

Now from (6.5.4) we have $2U_{xu} = -f'(x)$ and substituting into (6.5.8) leads to $f'(x) = 0$ and $U_{xxu} = 0$. Letting $f(x) = \delta$ we have $X_x = -(\delta+\alpha)$ and $a_1 U = (1+a_1 u)\delta$ implying that $U_u = \delta$. Thus from (6.5.7) we have that $2X_x = - \frac{a_2 U}{1+a_2 U} = -2(\delta+\alpha)$ and also by differentiating with respect to u, that $X_x = - \frac{1}{2} U_u = - \frac{\delta}{2}$. This gives $\delta = -2\alpha$ and consequently, $X = \alpha x + \beta$, $U = - \frac{2\alpha}{a_1} (1+a_1 u) = - \frac{2\alpha}{a_2} (1+a_2 u)$. We note that these generators are obtained when $a_1 = a_2$ since if $a_1 \neq a_2$ then $\alpha = 0$ and $X = \beta$, $T = \gamma$ and $U = 0$.

We summarize our result by the following theorem:

Theorem 6.8

The generators of the local symmetry group of the RLW class of equations are as follows:

(a) $a_1 = 0,\ a_2 \neq 0$: $X = -\alpha x+\beta,\ T = \alpha t+\gamma,$

$$U = -\frac{2\alpha}{a_2}(1+a_2 u)$$

(b) $a_1 \neq 0,\ a_2 = 0$: $X = \beta,\ T = \alpha t+\gamma,\ U = -\frac{\alpha}{a_1}(1+a_1 u)$

(c) $a_1 \neq a_2 \neq 0$: $X = \beta,\ T = \gamma,\ U = 0$

(d) $a_1 = a_2 \neq 0$: $X = \alpha x+\beta,\ T = \alpha t+\gamma,$

$$U = -\frac{2\alpha}{a_1}(1+a_1 u) \quad \square$$

6.6 The local symmetry group of the KdV class

The equation of this class is

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} = 0 \qquad (6.6.1)$$

The invariance equation under one-parameter (ϵ) infinitesimal transformations is

$$(1+a_1 u)U^x + (1+a_2 u)U^t + a_1 u_x U + a_2 u_t U + a_3 U^{xxx} = 0 \qquad (6.6.2)$$

leading to the results:

$$T_u = T_x = X_u = U_{uu} = 0 \qquad (6.6.3)$$

$$U_{xu} = X_{xx} \tag{6.6.4}$$

$$2(1+a_1u)X_x - (1+a_2u)X_t + a_1U + 2a_3X_{xxx} = 0 \tag{6.6.5}$$

$$(1+a_2u)(3X_x-T_t) + a_2U = 0 \tag{6.6.6}$$

$$(1+a_1u)U_x + (1+a_2u)U_t + a_3U_{xxx} = 0 \tag{6.6.7}$$

From (6.6.3) we have that $X = X(x, t)$ and $T = T(t)$.

For the KdV class of equations we consider the following two cases: (i) a_1 arbitrary, $a_2 \neq 0$. (ii) $a_1 \neq 0$, $a_2 = 0$.

(i) This case includes (a) $a_1 = 0$, $a_2 \neq 0$, (b) $a_1 = a_2$, (c) $a_1 \neq a_2 \neq 0$. Differentiating (6.6.5) with respect to u first and then with respect to x we obtain

$$2a_1X_{xx} - a_2X_{xt} + a_1U_{ux} = 0 \tag{6.6.8}$$

Differentiating (6.6.6) with respect to u first and then with respect to x we obtain

$$U_{ux} = -3X_{xx} \tag{6.6.9}$$

Then from (6.6.4) and (6.6.9) $U_{ux} = X_{xx} = 0$ and (6.6.8) gives $X_{xt} = 0$.

Differentiating (6.6.6) with respect to u gives $U_u = -(3X_x - T_t)$ and differentiating (6.6.6) with respect to x gives $U_x = 0$. From (6.6.7) $U_t = 0$.

Differentiating (6.6.6) with respect to t gives $T''(t) = 0$. Therefore T is a linear function of t and X is a linear function of (x, t).

Thus $3X_x - T_t$ = constant = $-\alpha$ say, and substituting in (6.6.6) gives $U = \frac{\alpha}{a_2}(1+a_2 u)$. Letting $T = \beta t + \gamma$ gives $X = \frac{1}{3}(\beta-\alpha)x + \delta t + \epsilon$ and substituting this into (6.6.5) gives $\delta = \frac{1}{3}\frac{a_1}{a_2}(\alpha+2\beta)$. Furthermore, The invariance equation (6.6.2) requires that we have the consistency condition $(a_1-a_2)(\alpha-\beta) = 0$. We summarize our first result by the following theorem:

<u>Theorem 6.9</u>

If $(a_1-a_2)(\alpha-\beta) = 0$, $a_2 \neq 0$ then the generators of the local symmetry group of the KdV class of equations are as follows:

$$X(x, t) = \frac{1}{3}(\beta - \alpha)x + \frac{1}{3}\frac{a_1}{a_2}(\alpha + 2\beta)t + \epsilon$$

$$T(t) = \beta t + \gamma$$

$$U(x, t, u) = \frac{1}{a_2}(1 + a_2 u)\alpha$$

where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$ □

Clearly, if $a_1 \neq a_2$ then the generators of the local symmetry group become

$$X(t) = \frac{a_1}{a_2}\alpha t + \varepsilon, \quad T(t) = \alpha t + \gamma,$$

$$U(x, t, u) = \frac{1}{a_2}(1+a_2 u)\alpha$$

Note that this includes the case with the single nonlinear term $a_2 u u_t$.

If $a_1 = a_2$ then we obtain the generators in theorem 6.9 with $\frac{a_1}{a_2} = 1$.

(ii) For $a_1 \neq 0$, $a_2 = 0$ we have the results:

$$U_{xu} = X_{xx} \qquad (6.6.10)$$

$$2(1 + a_1 u)X_x - X_t + a_1 U + 2a_3 X_{xxx} = 0 \qquad (6.6.11)$$

$$3X_x - T_t = 0 \qquad (6.6.12)$$

$$(1 + a_1 u)U_x + U_t + a_3 U_{xxx} = 0 \qquad (6.6.13)$$

together with (6.6.3). Then from (6.6.12) $X_{xx} = 0$

and so $U_{xu} = 0$. From (6.6.11) we obtain $U_u = -2X_x$ and differentiating (6.6.11) with respect to x gives $a_1 U_x = X_{xt}$.

From the same equation we can show that $U_{tu} = -2X_{xt}$. But (6.6.13) gives $(1 + a_1 u)U_x + U_t = 0$ which upon differentiating with respect to u gives $a_1 U_x + U_{ut} = -X_{xt} = 0$. From (6.6.12) $T''(t) = 0$.

Therefore, T is a linear function of t and X is a linear function of (x, t). If $T = \beta t + \gamma$ then from (6.6.12) $X_x = \frac{1}{3}\beta$ and $X = \frac{1}{3}\beta x + \delta t + \varepsilon$ so that

from (6.6.11) $U = -\frac{2}{3}\left[\frac{1+a_1 u}{a_1}\right]\beta + \frac{\delta}{a_1}$.

Thus we have the following theorem:

Theorem 6.10

The generators of the local symmetry group of the KdV class of equations

$$u_t + u_x + a_1 u u_x + a_3 u_{xxx} = 0$$

are as follows:

$$X(x, t) = \frac{1}{3}\beta x + \delta t + \varepsilon$$

$$T(t) = \beta t + \gamma$$

$$U(x, t, u) = -\frac{2}{3}\left[\frac{1+a_1 u}{a_1}\right]\beta + \frac{\delta}{a_1}$$

where $\beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ □

The infinitesimals obtained for the classes of equations we have considered can now be used to obtain similarity solutions. Before doing this we summarize our results in the following section.

6.7 Generators of Infinitesimal Transformations for the general class: a complete classification

We present our results in Table 6.1. For simplicity we have introduced the following notation:

$$\Delta_6 = 4a_3a_5^3 - 18a_3a_4a_5a_6 + 27a_3^2a_6^2 + 4a_4^3a_6 - a_4^2a_5^2$$

$$\Delta_5 = a_4^2 - 4a_3a_5, \qquad \Delta_4 = \frac{a_1}{a_2} - \frac{a_3}{a_4}.$$

We have also the following equivalence classes:

(1) (a) $W_6(a_6 \neq 0)$, $\Delta_6 \neq 0 \longrightarrow$ Class A

(b) $W_6(a_6 \neq 0)$, $\Delta_6 = 0$ $\nearrow$ Class C (RLW) $\searrow$ Class D (KdV)

(2) (a) $W_5(a_6 = 0,\ a_5 \neq 0),\quad \Delta_5 \neq 0 \longrightarrow$ Class A

(b) $W_5(a_6 = 0,\ a_5 \neq 0),\quad \Delta_5 = 0$ $\nearrow$ Class C, $\searrow$ Class D

(3) (a) $W_{43}(a_6 = 0 = a_5,\ a_4 \neq 0),\quad \Delta_4 \neq 0 \longrightarrow$ Class B

(b) $W_{43}(a_6 = 0 = a_5,\ a_4 \neq 0),\quad \Delta_4 = 0 \longrightarrow$ Class C.

(4) $RLW(a_6 = a_5 = a_3 = 0,\ \ a_4 \neq 0) \longrightarrow$ Class C.

(5) $KdV(a_6 = a_5 = a_4 = 0,\ \ a_3 \neq 0) \longrightarrow$ Class D.

We now present our table.

Table 6.1

Class of Equations		Conditions on a_1, a_2	Generators
A	A1	$a_1 \neq a_2$	$X = \beta,\ T = \gamma,\ U = 0$
	A_2	$a_1 = a_2$	$X = \alpha x + \beta,\ T = \alpha t + \gamma$ $U = -\frac{2\alpha}{a_1}(1+a_1 u)$
B	B1	$a_1 \neq a_2 \neq 0$	as in A1
	B2	$a_1 = 0,\ a_2 \neq 0$	as in A1
	B3	$a_2 = 0,\ a_1 \neq 0$	$X = \frac{a_3}{a_4}\alpha t + \beta,\ T = \alpha t+\gamma$ $U = -\frac{\alpha}{a_1}\left[a_1 u - \left(\frac{a_3}{a_4} - 1\right)\right]$
	B4	$a_1 = a_2$	as in A2
C	C1	$a_1 \neq a_2 \neq 0$	as in A1
	C2	$a_1 = 0,\ a_2 \neq 0$	$X = -\alpha x + \beta,\ T = \alpha t + \gamma$ $U = -2\alpha(1 + a_2 u)/a_2$
	C_3	$a_2 = 0,\ a_1 \neq 0$	$X = \beta,\ T = \alpha t + \gamma,$ $U = -\alpha(1+a_1 u)/a_1$
	C_4	$a_1 = a_2$	as in A_2
D*	D1	$a_1 \neq a_2$	$X = \frac{a_1}{a_2}\alpha t + \epsilon,\ T = \alpha t+\gamma$ $U = \alpha(1+a_2 u)/a_2$
	D2	$a_2 = 0,\ a_1 \neq 0$	$X = \frac{1}{3}\beta x + \delta t + \epsilon,$ $T = \beta t + \gamma$ $U = -\frac{2}{3a_1}(1+a_1 u)\beta + \frac{\delta}{a_1}$
	D3	$a_1 = a_2$	$X = \frac{1}{3}(\beta-\alpha)x +$ $\frac{1}{3}(\alpha+2\beta)t + \epsilon,\ T = \beta t+\gamma$ $U = \alpha(1+a_2 u)/a_2$

* Note that $D_1 = A_1$ for $\alpha = 0$ and $D_3 = A_2$ for $\alpha = -2\beta$.

6.8 The similarity reductions of the general class

In this section we obtain the similarity reductions of the classes listed in Table 6.1. This will be done in a systematic manner starting with the first set of equations in the Table and finishing with the KdV classes.

Similarity reductions are obtained by solving the characteristic equations

$$\frac{dx}{X(x,t,u)} = \frac{dt}{T(x,t,u)} = \frac{du}{U(x,t,u)}$$

Then for the first set of equations in Table 6.1 we integrate the ODEs

$$\frac{dx}{\alpha x+\beta} = \frac{dt}{\alpha t+\gamma} = \frac{-\, a_1 du}{2\alpha(1+a_1 u)} \tag{6.8.1}$$

for the cases $\alpha = 0$ and $a_1 = a_2$.

Case A1 $\alpha = 0$. In this case solving the characteristic equations is easy and we obtain the similarity variables

$$u(x,\ t) = f(\eta) \tag{6.8.2a}$$

where $\eta = x - (1+c)t$ and $(1+c) = \beta/\gamma$ (6.8.2b) and $f(\eta)$ satisfies the ODE

$$- cf' + \left[a_1 - a_2(1+c)\right]ff' +$$

$$\left[a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3\right]f''' = 0 \qquad (6.8.3)$$

where $' \equiv \dfrac{d}{d\eta}$.

Note that this reduction is also obtained for the classes B1, B2, C1 and D1 ($\alpha = 0$) with the appropriate $a_j (j = 1, \ldots, 4)$.

<u>Case A2</u> $a_1 = a_2$. In this case we show how theorem 4.1.1 is applied to solve (6.8.1).

Solving $\dfrac{dx}{\alpha x+\beta} = \dfrac{dt}{\alpha t+\gamma}$

gives $\ln(\alpha x+\beta) = \ln(\alpha t+\gamma)$

which leads to $\eta = (\alpha x+\beta)(\alpha t+\gamma)^{-1}$.

Solving $\dfrac{dt}{\alpha t+\gamma} = - \dfrac{a_1 du}{2\alpha(1+a_1 u)}$

gives $2\ln(\alpha t+\gamma) = - \ln(1+a_1 u)k$

which leads to $k = (1+a_1 u)(\alpha t+\gamma)^2$.

From theorem 4.1.1 the general solution is given by

$$F(k, \eta) = F\left[(1+a_1u)(\alpha t+\gamma)^2, \ (\alpha x+\beta)(\alpha t+\gamma)^{-1}\right]$$

which may be written in the form

$$(1+a_1u)(\alpha t+\gamma)^2 = f\left[(\alpha x+\beta)(\alpha t+\gamma)^{-1}\right]$$

so that $u = \frac{1}{a_1}\left[(\alpha t+\gamma)^{-2}f(\eta) - 1\right]$.

Hence in this case the similarity variables are

$$u(x, t) = \frac{1}{a_1}\left[(\alpha t+\gamma)^{-2}f(\eta) - 1\right] \qquad (6.8.4a)$$

where $\eta = (\alpha x+\beta)(\alpha t+\gamma)^{-1}$ (6.8.4b)

and $f(\eta)$ satisfies the ODE:

$$\alpha^2(a_3-a_4\eta+a_5\eta^2-a_6\eta^3)f''' + 4\alpha^2(-a_4+2a_5\eta-3a_6\eta^2)f''$$

$$+6\alpha^2(2a_5-6a_6\eta)f' + (1-\eta)ff' - 2f^2 - 24\alpha^2a_6f = 0$$

(6.8.5)

From Table 6.1 we see that this reduction is also obtained for the classes B4, C4 and D3 ($\alpha = -2\beta$) with the appropriate $a_j (j = 1, \ldots, 4)$.

<u>Case B3</u> Applying the above analysis, we obtain the similarity variables:

$$u(x,\ t) = \frac{1}{a_1}\left[(\alpha t+\beta)^{-1}f(\eta) - k\right],\ k = 1 - \frac{a_3}{a_4}$$

(6.8.6a)

where $\eta = \alpha\left[x - \frac{a_3}{a_4}t\right] - \left[\gamma - \frac{a_3}{a_4}\beta\right]\ln(\alpha t+\beta)$

(6.8.6b)

and $f(\eta)$ satisfies the ODE

$$\alpha^2 a_4\left[\gamma - \frac{a_3}{a_4}\beta\right]f''' + \alpha^2 a_4 f''-ff' + \left[\gamma - \frac{a_3}{a_4}\beta\right]f' + f = 0$$

(6.8.7)

<u>Case C2</u> In this case we obtain the similarity variables

$$u(x,\ t) = \frac{1}{a_2}\left[(\alpha t+\gamma)^2 f(\eta) - 1\right] \qquad (6.8.10a)$$

where $\eta = (\alpha t+\gamma)(\alpha x+\beta)$

and $f(\eta)$ satisfies the ODE

$$a_4\alpha^2\eta f''' + 4a_4\alpha^2 f'' + \eta ff' + 2f^2 + f' = 0$$

(6.8.10b)

<u>Case C3</u> For this case we consider (i) $\gamma = 0$, (ii) $\gamma \neq 0$.

(i) For $\gamma = 0$ we obtain the similarity variables

$$u(x, t) = \frac{1}{a_1}\left[(\alpha t+\beta)^{-1}f(x) - 1\right] \qquad (6.8.8a)$$

where $\eta = x$ (6.8.8b)

and $f(x)$ satisfies the ODE

$$a_4\alpha f'' - ff' + f = 0 \qquad (6.8.9)$$

(ii) For $\gamma \neq 0$ we obtain the result of case B3 where $a_3 = 0$.

<u>Case D1</u> In this case we obtain the similarity variables

$$u(x, t) = \frac{1}{a_2}\left[(\alpha t + \gamma)f(\eta) - 1\right] \qquad (6.8.11a)$$

where

$$\eta = \left[x - \frac{a_1}{a_2}t\right] + \left[\frac{(a_1/a_2)\gamma-\lambda}{\alpha}\right]\ln(\alpha t+\gamma)$$

(6.8.11b)

and $f(\eta)$ satisfies the ODE:

$$a_3 f''' + \left[1 - \frac{a_1}{a_2}\right]f' - \left[\lambda - \frac{a_1}{a_2}\gamma\right]ff' + \alpha f^2 = 0$$

(6.8.12)

Clearly we can choose $a_1 = 0$ to obtain a reduction for the KdV class with the nonlinear term $a_2 uu_t$.

<u>Case D2</u> For this case we consider (i) $\beta = \varepsilon = 0$, $\gamma = 1$, (ii) β, δ, γ, ε all arbitrary.

(i) When $\beta = \varepsilon = 0$, $\gamma = 1$ the similarity variables are

$$U(x,\ t) = \frac{1}{a_1}\,[f(\eta) + \delta t] \tag{6.8.13a}$$

where

$$\eta = x - \frac{\delta}{2}\,t^2 \tag{6.8.13b}$$

and $f(\eta)$ satisfies the ODE

$$a_3 f''' + ff' + f' + \delta = 0 \tag{6.8.14}$$

We shall show in the following chapter that this ODE in fact reduces to the first Painlevé transcendent.

(ii) When β, δ, γ, and ε are arbitrary, the similarity variables are

$$u(x,\ t) = \frac{1}{a_1}\left[\left[\frac{3}{2}\,\frac{\delta}{\beta} - 1\right] - \frac{9}{4}\,\beta^{-5/3}(\beta t+\gamma)^{-2/3} f(\eta)\right] \tag{6.8.15a}$$

where

$$\eta = \frac{1}{2}\beta^{-5/3}\left[2\beta^2 x - 3\delta(\beta t+\gamma) + 6(\beta\epsilon-\gamma\delta)\right](\beta t+\gamma)^{-\frac{1}{3}}$$

(6.8.15b)

and $f(\eta)$ can be shown to satisfy the ODE

$$3a_3 f''' + 3ff' - \eta f' - 2f = 0 \qquad (6.8.16)$$

We shall see in the following chapter that this ODE reduces to the second Painlevé transcendent.

Case D3 In this case we obtain six different reductions. However three of them are special cases of equations already obtained and we take care of these first.

(i) $\alpha = \beta = 0$. D3 $\longrightarrow$ A1 with $a_4 = a_5 = a_6 = 0$.

(ii) $\alpha = -2\beta$. D3 $\longrightarrow$ A2 with $a_4 = a_5 = a_6 = 0$.

(iii) $\alpha = \beta \neq 0$. D3 $\longrightarrow$ D1.

(iv) $\alpha \neq \beta \neq 0$. In this case the similarity variables are

$$u(x, t) = \frac{1}{a_1}\left[(\beta t + \gamma)^{\alpha/\beta} f(\eta) - 1\right] \qquad (6.8.17a)$$

where

$$\eta = (\beta t + \gamma)^{(\alpha-\beta)/3\beta}\left[\beta x - (\beta t+\gamma) - \frac{1}{\beta-\alpha}(\gamma(\alpha+2\beta)+3\beta\epsilon)\right]$$

(6.8.17b)

and $f(\eta)$ satisfies the ODE

$$a_3\beta^3 f''' + \frac{1}{3}(\alpha-\beta)\eta f f' + \alpha f^2 = 0 \qquad (6.8.18)$$

(v) $\alpha = 0,\ \beta \neq 0$. In this case the similarity variables are

$$u(x, t) = f(\eta) \qquad (6.8.19a)$$

where

$$\eta = \beta^{-2/3}(\beta t+\gamma)^{-\frac{1}{3}}\left[(\beta t+\gamma) - (\beta x+\delta-2\gamma)\right]$$

(6.8.19b)

and $f(\eta)$ satisfies the ODE

$$3a_3 f''' + \eta f f' + \eta f' = 0 \qquad (6.8.20)$$

(vi) $\alpha \neq 0,\ \beta = 0$. In this case the similarity variables are

$$u(x, t) = \frac{1}{a_1}\left\{e^{\frac{\alpha}{\gamma}t} f(\gamma) - 1\right\} \qquad (6.8.21a)$$

where

$$\eta = (\alpha x - \alpha t + 3\gamma - \delta)e^{\frac{\alpha}{3\gamma}t} \qquad (6.8.21b)$$

and $f(\eta)$ satisfies the ODE

$$3a_3\alpha\gamma f''' + \eta f f' + f^2 = 0. \qquad (6.8.22)$$

We have now completed the similarity reductions of our general class via one-parameter (ϵ) Lie transformations. For easy access to our results we summarize the ODE reductions in the following table.

Class of equations	Similarity ODEs
A1,B1,B2,C1,D1($\alpha=0$)	$Kf''' - f' + \lambda ff' = 0$ $\lambda c = a_1 - a_2(1+c)$, $Kc = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3$
A2,B4,C4,D3($\alpha=-2\beta$)	$\mu f''' + 4\mu' f'' + 6\mu'' f' + 4\mu''' f$ $+ (1-\eta)ff' - 2f^2 = 0$ $\mu(\eta) = \alpha^2(a_3 - a_4\eta + a_5\eta^2 - a_6\eta\ )$
B3	$\alpha^2 a_4 kf''' + \alpha^2 a_4 f'' + kf' + f - ff' = 0$, $k = \gamma - \dfrac{a_3}{a_4}\beta$
C3(a)	$\alpha a_4 f'' + f - ff' = 0$
C3(b)	B3 with $a_3 = 0$.
C2	$\alpha^2 a_4 \eta f''' + 4\alpha^2 a_4 f'' + f' + \eta ff'$ $+ 2f^2 = 0$
D1	$a_3 f''' + \left(1 - \dfrac{a_1}{a_2}\right) f'$ $- \left(\lambda - \dfrac{a_1}{a_2}\gamma\right) ff' + \alpha f^2 = 0$
D2(a)	$a_3 f''' + f' + \gamma + ff' = 0$
D2(b)	$3a_3 f''' - 2f - \eta f' + 3ff' = 0$
D3(a)	$a_3 \beta^3 f''' + \frac{1}{3}(\alpha-\beta)\eta ff' + \alpha f^2 = 0$ $\alpha \neq \beta$
D3(b)	$3a_3 f''' + \eta f' + \eta ff' = 0$

Table 6.2

6.9 Conclusion

In this chapter we have applied one-parameter (ϵ) Lie group of transformations to obtain a complete classification of the local symmetry groups of the general glass of equations (6.1.1). This was done by splitting the general class into three distinct subclasses W_6, W_5 and W_{43} according to Cauchy's problem and considering them separately. We first obtained the infinitesimals X, T and U and listed them in Table 6.1. Then using these we obtained the similarity reductions.

We noted that for any a_1, a_2, a similarity reduction leads to the class of ODEs (6.8.3). In particular when $a_1 \neq a_2 \neq 0$, (6.8.3) is the only reduction possible, i.e. the self-similar or travelling wave solutions. Symmetries giving rise to additional reductions appear, firstly when $a_1 = a_2$ and secondly when $a_6 = a_5 = 0$.

To obtain information about the analytic structure of the general class (6.1.1) we study the analytic structure of its ODE reduction i.e. the ODEs in Table 6.2. This is the work of the following chapter.

CHAPTER SEVEN

THE ANALYTIC STRUCTURE OF THE SIMILARITY SOLUTIONS OF THE GENERAL CLASS OF PDEs

In this chapter we begin our study of the analytic structure of the general class of equations

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0 \tag{7.1}$$

by investigating the solution structure of the nonlinear ODEs obtained in the previous chapter i.e. the ODEs listed in Table 6.2. We first look at equations which are or can be reduced to second order. This is done because the solution structure of second order ODEs is easier to examine because of the existence of the Painlevé classification. The remaining equations are studied via local analysis based on the Singular Point Analysis of Ablowitz, Romani and Segur [2]. This will involve some complex variable theory and a resumé of some complex variable definitions will be our starting point.

7.1 Resumé of complex variable definitions

Definition 7.1

If G is an open set in $\mathbb{C}$, a function $f: G \rightarrow \mathbb{C}$ is analytic if f is continuously differentiable on G [3].

In fact every analytic function is infinitely differentiable and furthermore has a power series expansion about each point of its domain.

Thus we make the following definition.

Definition 7.2

A function $f: G \longrightarrow \mathbb{C}$ is said to be analytic on G if for each $z_o \in G$ it has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_o)^n$$

which converges for all $z \in G$.

Definition 7.3

Let f have an isolated singularity at $z = z_o$.
Then z_o is a pole of f if $\lim_{z \to z_o} |f(z)| = \infty$.

Definition 7.4

If f has a pole at $z = z_o$ and m is the smallest positive integer such that $f(z)(z-z_o)^m$ has a removable singularity at $z = z_o$ then f is said to have a pole of order m at $z = z_o$.

Definition 7.5

If G is open and f is a function defined and analytic in G except for poles, then f is a meromorphic function on G. For example

$f(z) = \frac{1}{z}$ is analytic in $\mathbb{C} - \{0\}$ and hence meromorphic in $\mathbb{C}$

$f(z) = \frac{1}{\sin z}$ is analytic in $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$ and hence meromorphic in $\mathbb{C}$.

We represent a meromorphic function in the usual manner by a Laurent series as defined in the following theorem.

Theorem 7.1

Let $f: G \longrightarrow \mathbb{C}$ be meromorphic and have a pole of order m at $z_o \in G$. Then

$$f(z) = \sum_{r}^{-1} b_r (z - z_o)^r + g(z) \quad (r < \infty)$$

where $g(z)$ is analytic □

The first term in the expression on the right hand side is called the principal part. The representation in this theorem is local and has a disc of convergence radius from z_o to the closest pole.

Non-Meromorphic functions

Non-Meromorphic functions fall into two classes.

(a) Those with branch points e.g. $z^{3/4}$, $z^2 \ln z$ which may or may not be singular at $z = 0$.

(b) Functions as defined by Definition 7.4 but with $m = \infty$ e.g. $e^{1/z}$. These are called essentially singular functions.

We now state a theorem due to Picard.

Theorem 7.2

Suppose f is analytic except for an essential singularity at $z = z_o$. Then in each neighbourhood of z_o, f takes every value in $\mathbb{C}$, with one possible exception, an infinite number of times.

Our analysis of the nonlinear ODEs obtained in Chapter Six will concentrate on locating their movable singularities and our approach is outlined below.

7.2 The analytic structure of the solutions of the ODEs.

In order to locate the existence of analytic and meromorphic solutions, we assume that the solution can be represented in a series with argument $t = \eta - \eta_o$

i.e., $$f(t) = t^p \sum_{r=0}^{\infty} b_r t^r, \qquad p \in \mathbb{R} \qquad (7.2.1)$$

where the b_r are constant coefficients with $b_o \neq 0$. (We have changed from using z to η for consistency with Chapter Six).

If the solution has a pole or a branch point at η_o and if η_o is fixed by the coefficients of the equation then we call this a fixed singularity. If η_o can be chosen arbitrarily then we call this a movable singularity. If a singularity is movable and is not a pole we call it a movable critical point. In what follows equations which do not possess movable critical points we shall refer to as P-type.

Since the ODEs are, in general, third order a general solution would have three arbitrary constants. We look for solutions in which η_o is one of these arbitrary constants and also when it is not. Thus, if f is the general solution, then the set of coefficients $\{b_r\}$ contains two or three arbitrary elements depending if η_o is arbitrary or not. Four cases may occur which are as follows:

(i) If p is a positive integer then $f(\eta)$ is analytic almost everywhere.

(ii) If p is positive, but non-integral, then $f(\eta)$ has a singular branch point.

(iii) If p is a negative integer then $f(\eta)$ is meromorphic almost everywhere.

(iv) If p is negative, but non-integral, then $f(\eta)$ has a singular branch point.

(Note: the term almost everywhere is used to exclude fixed singularities.)

If the set $\{b_r\}$ does not contain two (or three) arbitrary elements then $f(\eta)$ is not the general solution. In this case if the general solution has a movable logarithmic singularity or a movable algebraic branch point then it can be recovered by supplementing the original series (7.2.1) with a series of the form

$$\sum_{r=1}^{\infty} g_r(t)(\ln t)^r \quad \text{or} \quad \sum_{r=1}^{\infty} h_r(t)(t^{\alpha})^r, \quad \text{where} \quad g_r(t)$$

and $h_r(t)$ are analytic functions.

However if $f(\eta)$ has an essential singularity then a series expansion cannot be used to detect it. Thus, this analysis is incomplete insofar as it does not detect the existence of solutions with movable essential singularities.

Now we know from the work of Painlevé (reviewed in Chapter Four) that the general solution of a second order ODE will be meromorphic if it can be transformed to one of 50 canonical equations which are either soluble by elliptic functions or define one of the six Painlevé transcendents. Thus, before applying the above method, we consider the third order ODEs, which can be directly integrated together with the second order ODE given in Table 6.2.

In chapter five we considered the work of Abbas [1] and El-Sherbiny [4] on the third order class

$$Kf''' - f' + \lambda ff' = 0. \tag{7.2.2}$$

In particular we noted, in theorem 5.1, that this equation has solutions which are Weierstrassion elliptic functions. Furthermore, when $K > 0$ we obtain solitary wave solutions which all have the sech^2 profile. Therefore the solution structure of (7.2.2) is clearly meromorphic with poles of order two. We come back to this equation in the next chapter where we extend the work of Abbas and El-Sherbiny on the solitary wave solutions.

There is one second order ODE in Table 6.2.:

$$\underline{C3: \quad \alpha a_4 f'' + f - ff' = 0} \tag{7.2.3}$$

If we check this equation against the 50 canonical equations listed in Ince [5] we see that none of the 50 represents this equation. This leads us to the conclusion that (7.2.3) has solutions which possess movable algebraic, logarithmic or essential singularities. We come back to this equation later.

Also from Table 6.2 we are able to transform D2(a) and D2(b) into second order ODEs:

D2(a): $a_3 f''' + f' + \gamma + ff' = 0$ (7.2.4)

Make the transformation $f = -1 - \frac{w}{3}$ and integrate once. This reduces the third order ODE to

$$a_3 \frac{d^2 w}{d\eta^2} = \frac{w^2}{6} + 3\gamma\eta$$

which is the first Painlevé transcendent i.e. its solution is meromorphic.

D2(b): $3a_3 f''' - \eta f' - 2f + 3ff' = 0$ (7.2.5)

Note that $f = \eta$ is a solution.

For convenience we let $a_3 = 1$. Making the substitution

$$f = w' - \frac{1}{6} w^2$$

gives $\left[w'' - \frac{\eta}{3} w - \frac{1}{18} w^3\right]'' - \frac{1}{3} w \left[w'' - \frac{\eta}{3} w - \frac{1}{18} w^3\right]' = 0$

Integrating once

$$\left[w'' - \frac{\eta}{3} w - \frac{1}{18} w^3\right]' = k \exp\left[\frac{1}{3} \int_0^w F(y)dy\right]$$

where k is an arbitrary constant.

Looking for solutions which are bounded as $\eta \longrightarrow \infty$ we have $k = 0$. Another integration then gives

$$w'' = \eta w + \frac{1}{18} w^3 .$$

This second order ODE is the second Painlevé transcendent. Thus, once again the solution is meromorphic.

Thus the equations A1, B1, B2, C1, D1 ($\alpha = 0$), D2(a) and D2(b) are all free from critical points.

We now turn to the remaining equations in Table 6.2 which are all irreducible third order. They can all be written in the form

$$L(f) + N(f) = 0 \qquad (7.2.6)$$

where $L(f) = \gamma_3(\eta)f''' + \gamma_2(\eta)f'' + \gamma_1(\eta)f' + \gamma_0(\eta)f$

$N(f) = \rho(\eta)ff' + \sigma f^2,$

with $\gamma_j(\eta)$ a polynomial of degree at most j, with non-zero constant term, $\rho(\eta)$ a polynomial of degree ≤ 1 and σ constant.

7.2.1 Analytic Solutions

Theorem 7.3

The class of third order ODEs

$$\alpha^2(a_3-a_4\eta+a_5\eta^2-a_6\eta^3)f'''+4\alpha^2(-a_4+ 2a_5\eta-3a_6\eta^2)f''$$
$$+6\alpha^2(2a_5-6a_6\eta)f'-24\alpha^2a_6f+(\rho_1+\rho_2\eta)ff'+\sigma f^2 = 0 \quad (7.2.1.1)$$

has a formally analytic general solution □

Proof

Assume that $$f(\eta) = \sum_{j=0}^{\infty} b_j(\eta-\eta_0)^j \quad (7.2.1.2)$$

If we make the substitution $t = \eta - \eta_0$ in the ODE we see that the form of the equation remains unchanged. Hence, without loss of generality, we may take $\eta_0 = 0$ and $\alpha = 1$.

Then substituting the expansion into the ODE and equationg powers of η we obtain the following recurrence relations

$$\eta^{0}: \quad 6a_3b_3 - 8a_4b_2 + 12a_5b_1 - 24a_6b_0 + \rho_1b_0b_1 + \sigma b_0^2 = 0 \qquad (7.2.1.3a)$$

$$\eta^{1}: \quad 24a_3b_4 - 30a_4b_3 + 40a_5b_2 - 60a_6b_1 + \rho_1(2b_0b_2+b_1^2) + \rho_2b_0b_1 + 2\delta b_0b_1 = 0 \qquad (7.2.1.3b)$$

$$\eta^{n}: \quad a_3(n+3)(n+2)(n+1)b_{n+3} - a_4(n+1)(n+2)(n+4)b_{n+2}$$
$$+ a_5(n+1)(n+3)(n+4)b_{n+1} - a_6(n+2)(n+3)(n+4)b_n$$
$$+ \rho_1 \sum_{j=1}^{n} jb_jb_{n+1-j} + \sigma b_0b_n + \sum_{j=1}^{n} (j\rho_2+\sigma)b_jb_{n-j} = 0 \qquad (7.2.1.3c)$$

The general solution of this equation will have three arbitrary constants, none of which is η_0. We are, therefore, free to choose three arbitrary constants. If we choose b_0, b_1 and b_2 then b_3 is determined by (7.2.1.3a), b_4 is determined by (7.2.1.3b) and so on. Therefore the expansion (7.2.1.2) is a formal solution of the ODE □

We mention that we have not been able to obtain any special classes of analytic solutions e.g. polynomials and while we have not proved it we do not believe that they exist.

7.2.2 Meromorphic solutions

Theorem 7.4

If any of the ODEs (7.2.6) has a meromorphic solution then it has the form

$g(\eta)\eta^p$ where $g(\eta)$ is analytic and $p = -2$ or -3 □

Proof

Assume that $f(\eta-\eta_o) = \sum_{r=0}^{\infty} b_r(\eta-\eta_o)^{p+r} \quad p \in \mathbb{R}$

and that $t = \eta - \eta_o$. Substituting in the differential equation (7.2.6) leads to the following observations.

There exists one lowest power term in $L(f)$ which is t^{p-3}. If p is positive this term cannot be balanced by any term in $N(f)$ so that p must be negative. As a consequence the term of lowest power in $N(f)$ is either t^{2p} or t^{2p-1}.
If $\rho(\eta) \neq 0$ then, in general, $2p - 1 = p - 3 \Rightarrow p = -2$. If $\rho(\eta) = 0$ or η_o is chosen such that $\rho(\eta) \propto t$ then $2p = p - 3 \Rightarrow p = -3$. Therefore if the ODE has a meromorphic solution it is of the form

$$f(\eta) = \frac{1}{(\eta-\eta_o)^p} \sum_{r=0}^{\infty} b_r(\eta-\eta_o)^r, \quad (p = 2 \text{ or } 3)$$

(7.2.2.1) □

Corollary 1 $b_o = -12\gamma_3(\eta_o)/\rho(\eta_o)$ when $p = -2$, and $b_o = 60\gamma_3(\eta_o)/\sigma$ when $p = -3$ □

Proof

This follows by equating the sum of the coefficients of the lowest powers to zero □

Corollary 2 If $\rho(\eta) \neq 0$, with a non-zero constant term and σ constant then $p = -2$ □

Proof If it were otherwise i.e. $p = -3$ and $\rho(\eta) \neq 0$ then the coefficient of the lowest power is $\rho(\eta)pb_o^2$ which upon equating to zero gives $b_o = 0$ and thus a solution of the form (7.2.2.1) does not exist □

Theorem 7.5

If any of the ODEs (7.2.6) has a meromorphic solution with $p = -3$ then this solution is not the general solution □

Proof

Suppose that the arbitrary constants of the general solution of the ODE are η_o, b_{r_1} and b_{r_2} where $0 < r_1 < r_2$. Then terms in $t^{r_i - \sigma}$ only occur in f''' and f^2, provided $r_2 < r_1 + 3$ and must balance. Furthermore, since the term f^2 is quadratic it is sufficient to substitute the two term expression $b_o t^{-3} + b_r t^{r-3}$ to effect this balance. The coefficient of t^{r-6} obtained in this way from these two sources is

$$120\gamma_3(\eta_o)b_r+\gamma_3(\eta_o)(r-3)(r-4)(r-5)b_r$$

$$= b_r\gamma_3(\eta_o)(r+1)(r^2-13r+60) = 0 \ .$$

The roots of this equation are not nonnegative distinct integers. This implies that none of the coefficients b_r are arbitrary □

Theorem 7.6

If any of the ODEs (7.2.6) has a general meromorphic solution with $p = -2$, then the arbitrary constants are η_o, b_4 and b_6 □

Proof

The proof of this theorem is identical to the previous proof where here we are balancing the coefficients of t^{r-5}. The coefficient in this case is given by

$$\gamma_3(\eta_o)b_r(r-2)(r-3)(r-4) + \rho(\eta_o)\{b_o b_r(r-2)-2b_o b_r\}$$

$$= \gamma_3(\eta_o)b_r(r+1)(r-4)(r-6) = 0.$$

Since b_r is arbitrary and $\gamma_3 \neq 0$ it follows that $r = 4$ or $r = 6$.

Thus b_4 and b_6 can be specified arbitrarily □

The remaining coefficients of the series (7.2.2.1) may be obtained by substituting the series into the ODE and equating powers of $(\eta-\eta_0)$.

In our work we are concerned with finding the general non analytic solutions of the equations represented by (7.2.6). In those cases where we are unable to obtain the general solution, i.e. $\rho(\eta) = 0$, we will give a particular solution. Thus, when we refer to (7.2.2.1) we assume $p = -2$ unless we state otherwise.

We are now ready to look for general solutions of the ODEs represented by (7.2.6).

<u>A2, B4, C4, D3(α = -2β)</u>:

$$\mu f''' + 4\mu' f'' + 6\mu'' f' + 4\mu''' f + (1-\eta)ff' - 2f^2 = 0 \tag{7.2.2.2}$$

For this equation

$$b_0 = \frac{-12\alpha^2(a_3-a_4\eta_0+a_5\eta_0^2-a_6\eta_0^3)}{1-\eta_0} \tag{7.2.2.3}$$

<u>Check for consistency on the arbitrariness of b_4 and b_6.</u>

We substitute the expansion with $t = \eta - \eta_0$, i.e.

$$f(t) = \frac{b_0}{t^2} + \frac{b_1}{t} + b_2 + b_3 t + b_4 t^2 + b_5 t^3 + b_6 t^4 + 0(t^4)$$

(7.2.2.4)

into (7.2.2.2) to verify that the coefficients b_1, b_2, b_3 and b_5 are fixed and that b_4 and b_6 are arbitrary. The required derivatives are:

$$f' = -\frac{2b_0}{t^3} - \frac{b_1}{t^2} \ldots + b_3 + 2b_4 t + 3b_5 t^2 + 4b_6 t^3 + \ldots$$

$$f'' = \frac{6b_0}{t^4} + \frac{2b_1}{t^3} \ldots + 2b_4 + 6b_5 t + 12b_6 t^2 + \ldots$$

$$f''' = -\frac{24b_0}{t^5} - \frac{6b_1}{t^4} \quad + 6b_5 + 24b_6 t + \ldots$$

Collecting powers of t and equating coefficients to zero leads to the following results:

$$t^{-5}: \ -24b_0\alpha^2\left[a_3 - a_4\eta_0 + a_5\eta_0^2 - a_6\eta_0^3\right] - 2b_0^2(1-\eta_0) = 0,$$

which gives us $b_0 \neq 0$ as obtained at (7.2.2.3).

$$t^{-4}: \ -6\alpha^2 b_1\left[a_3 - a_4\eta_0 + a_5\eta_0^2 - a_6\eta_0^3\right] - 3b_0 b_1(1-\eta_0) = 0$$

which gives $b_1 = 0$.

Similarly from the coefficients of t^{-3} and t^{-2} we obtain that $b_2 = b_3 = 0$.

$$t^{-1}: \; b_0(1-\eta_0)(2b_4-2b_4) = 0 \; .$$

Thus b_4 can be chosen arbitrarily.

$$t^0: \; 6\alpha^2 b_5\left[a_3-a_4\eta_0+a_5\eta_0^2-a_6\eta_0^3\right] - 8\alpha^2 b_4\left[a_4-2a_5\eta_0+3a_6\eta_0^2\right]$$

$$+ b_0 b_5(1-\eta_0) + 4b_0 b_4 = 0$$

giving $b_5 = \dfrac{8b_4}{1-\eta_0} - \dfrac{4b_4\left[a_4-2a_5\eta_0+3a_6\eta_0^2\right]}{3\left[a_3-a_4\eta_0+a_5\eta_0^2-a_6\eta_0^3\right]}$

$$t: \; 0.b_6 + A(\eta_0)b_4 = 0 \qquad (7.2.2.5)$$

$$A(\eta_0) = \frac{40\alpha^2 B(\eta_0)}{\left[1-\eta_0\right]^2\left[a_3-a_4\eta_0+a_5\eta_0^2-a_6\eta_0^3\right]}$$

where $B = P\eta_0^4 + Q\eta_0^3 + R\eta_0^2 + S\eta_0 + T$

with $P = a_5^2 + 12a_6^2 + a_4a_6 - 8a_5a_6$

$$Q = 6a_5^2 - 5a_4a_5 - 3a_3a_6 + 14a_4a_6 - 16a_5a_6$$

$$R = 5a_4^2 + 5a_5^2 + 9a_3a_5 - 14a_4a_5 - 18a_3a_6 + 9a_4a_6$$

$$S = 6a_4^2 - 16a_3a_4 + 14a_3a_5 - 5a_4a_5 - 3a_3a_6$$

$$T = 12a_3^2 + a_4^2 + a_4^2 - 8a_3a_4 + a_3a_5 \; .$$

Thus for b_4 to be arbitrary we require that $A \equiv 0$, which implies that the coefficients of the quartic must all be zero. However, this cannot be achieved as the following lemma shows:

Lemma 7.1 The equations $P = Q = R = S = T = 0$ have no solution in the space of coefficients represented by class A2.

Proof

We recall that in the class A2, the coefficient $a_6 \neq 0$ and $\Delta \neq 0$. Writing $\bar{a}_i = a_i/a_6$, $i = 3, 4, 5$ solving $P = 0$ for $\bar{a}_4$ gives

$$\bar{a}_4 = -\bar{a}_5^2 + 8\bar{a}_5 - 12$$

and solving $Q = 0$ for $\bar{a}_3$ and substituting for $\bar{a}_4$ gives

$$3\bar{a}_3 = 5a_5^3 - 48\bar{a}_5^2 + 156\bar{a}_5 - 168 \ .$$

Using these values, the expression for R gives

$$R = 20\bar{a}_5^4 - 240\bar{a}_5^3 + 1080\bar{a}_5^2 - 2160\bar{a}_5 + 1620$$

$$= 20\left[\bar{a}_5 - 3\right]^4 = 0 \ .$$

Thus, $\bar{a}_5 = 3$, which implies that $\bar{a}_4 = 3$ and $\bar{a}_3 = 1$. Thus every solution of the set of equations has the form $k(1, 3, 3, 1)$.

However, this implies that $\Delta = 0$ (see 6.2.14) and so does not belong to the space of coefficients of class A2 □

Corollary Equation (7.2.2.5) is only satisfied if $b_4 = 0$ □

Hence b_4 cannot be chosen arbitrarily and as a consequence the general solution of (7.2.2.2) is not represented by the series (7.2.2.1). Thus we have the following theorem:

Theorem 7.7

If the equation (7.2.2.5) has a meromorphic solution then this solution can contain at most two arbitrary constants, i.e. η_0 and b_6, with b_4 fixed at zero, and hence is not a general solution □

To reinstate the arbitrariness of b_4 and hence obtain a general solution it is necessary to supplement the series (7.2.2.1) with algebraic or logarithmic terms at b_6.

If we try to add the algebraic term $c_6 t^{p/q}$, $p \neq q$, $q \neq \pm 1$, $p, q \in \mathbb{Z}$ to b_6, we can easily show that the coefficient of t remains unchanged i.e. equation (7.2.2.5) is unchanged. Therefore the arbitrariness of b_4 cannot be recovered. Thus we have the following theorem.

Theorem 7.8

The general solution of (7.2.2.5) cannot have movable algebraic singularities □

Hence we must supplement (7.2.2.1) with logarithmic terms i.e., consider the truncated solution

$$f(t) = t^{-2} \sum_{r=0}^{\infty} b_r t^r + t^{-2} \ln t \sum_{r=6}^{\infty} c_r t^r + \ldots^{*} \tag{7.2.2.6}$$

Substituting (7.2.2.6) into the ODE the terms in $t^{-3}, \ldots, t^0$ are unchanged and so $b_0, b_1, \ldots b_5$ are as above. However, the t term is now as follows:

$$t^1: \quad 0.b_6 + A.b_4 + 14\alpha^2 c_6 \left[a_3 - a_4\eta_0 + a_5\eta_0^2 - a_6\eta_0^3\right] = 0 \tag{7.2.2.7}$$

This equation is satisfied with b_4 chosen arbitrarily and

* see phi series by Hiller, Tabor and Weiss

$$c_6 = -\frac{Ab_4}{14\mu(\eta_0)} \tag{7.2.2.8}$$

where $\mu(\eta) = \alpha^2\left[a_3 - a_4\eta_0 + a_5\eta_0^2 - a_6\eta_0^3\right]$. This restores the arbitrariness of b_4 at this point and the first few of the solutions are given by

$$f(\eta) = \frac{b_0}{(\eta-\eta_0)^2} + b_4(\eta-\eta_0)^2 + b_5(\eta-\eta_0)^3$$

$$+ \left\{b_6 + c_6\ln(\eta-\eta_0)\right\}(\eta-\eta_0)^4 + 0\left[(\eta-\eta_0)^5\right] \text{ as } \eta \to \eta_0 \tag{7.2.2.9}$$

Thus $f(\eta)$ may have movable logarithmic branch points. However, we have to check for consistency of the arbitrariness of b_6 by going further down the series. Because of the quadratic nonlinearity we have to include all powers of $(\ln t)$ in the expansion in order to achieve balance. If we substitute (7.2.2.6) into the equations we see that the lowest power of t at which the $[\ln(\eta-\eta_0)]^2$ term appears is ten and is produced by ff'. Higher powers of $\ln(\eta-\eta_0)$ enter successively at $t^{6r-2}(r \geq 3)$. Thus the general solution will have the following form.

$$f(\eta) = \frac{b_0}{(\eta-\eta_0)^2} + \sum_{r=4}^{\infty} b_r(\eta-\eta_0)^{r-4}$$

$$+ \sum_{j=1}^{\infty} \left\{ \ln(\eta-\eta_o)^j \sum_{r=6_j}^{\infty} K_{rj}(\eta-\eta_o)^{r-2} \right\} .$$

(7.2.2.10)

Note that in terms of previous notation $K_{r1} = c_r$, $r \geq 6$ and below we shall use $K_{r2} = d_r$, $r \geq 12$.

So far we have determined the coefficients $b_r (r = 1, \ldots, 6)$ and c_6. To obtain a solution whose terms balance we need to go as far as b_{12}, c_{12} and d_{12}. This may also provide relationships between the coefficients which may lead to other results. In order to obtain these higher coefficients we make two simplifications which do not affect the generality of our work, i.e. we choose $\alpha = 1$ and $\eta_o = 0$. Then (7.2.2.2) becomes

$$a_3 f''' - 4a_4 f'' + 12a_5 f' - 24f + (1-\eta)ff' - 2f^2 = 0$$

(7.2.2.11)

and in (7.2.2.10) $b_o = -12$, $b_1 = b_2 = b_3 = 0$, b_4 arbitrary, $b^5 = 4b_4\left[2 - \frac{a_4}{3a_3}\right]$, b_6 arbitrary,

$$c_6 = \frac{-20}{7a_3^2} (12a_3^2 + a_4^2 - 8a_3a_4 + a_3a_5).$$

Determination of the coefficient d_{12} This coefficient is determined by substituting (7.2.2.10) into (7.2.2.11) and equating the coefficient of $t^7(\ln t)^2$, to zero. Contributions come from the f''' and the ff' terms and the result is

$$d_{12} = -c_6^2/150a_3.$$

<u>Determination of the coefficients c_{j+1}, $6 \le j \le 11$</u>

These are determined by equating the coefficients of $t^{j-4}\ln t$ to zeros beginning with $t^2 \ln t$. The results are as follows:

$$c_7 = \frac{3c_6}{a_3}(a_4 - a_3) \quad \text{or} \quad c_7 = k_7 c_6, \quad k_7 = k_7(a_3, a_4)$$

$$c_8 = \frac{c_6}{6a_3}\left\{(a_4-a_3)(35a_4-21a_3) - 10a_5\right\} \quad \text{or} \quad c_8 = k_8 c_6,$$

$$k_8 = k_8(a_3, a_4, a_5)$$

$$c_9 = k_9 c_6, \quad k_9 = k_9(a_3, a_4, a_5)$$

$$c_{10} = k_{10} c_6, \; k_{10} = k_{10}(a_3, a_4, a_5, b_4)$$

$$c_{11} = k_{11} c_6, \; k_{11} = k_{11}(a_3, a_4, a_5, b_4)$$

$$c_{12} = k_{12} c_6, \; k_{12} = k_{12}(a_3, a_4, a_5, b_4, b_6) \; .$$

<u>Determination of the coefficients b_{j+1}, $j \ge 6$</u>

These are determined by equating the coefficients of t^{j-4} to zero and it is straightforward to show that they are consistently functions of b_4 and b_6.

The above results show that we have consistently extended the series to t^{12} and it is routine, but tedious to continue the evaluation of the series. As expected we find that the $t^{16}(\ln t)^3$ comes in at b_{18}, $t^{22}(\ln t)^4$ at b_{24} and so on.

We state out main result on the ODE (7.2.2.2) in the following theorem:

<u>Theorem 7.9</u>

The third order nonlinear ODE A2:

$$\mu(\eta)f'''+4\mu'(\eta)f''+6\mu''(\eta)f'+4\mu'''(\eta)f+(1-\eta)ff'-2f^2 = 0$$

where $\mu(\eta) = \alpha^2(a_3-a_4\eta+a_5\eta^2-a_6\eta^3)$ has a formal general solution

$$f(\eta) = \frac{b_0}{t^2} + \sum_{r=4}^{\infty} b_r t^{r-4} + \sum_{j=1}^{\infty} g_j(t)(\ln t)^j \tag{7.2.2.12}$$

where $g_j(t) = \sum_{j=6_j}^{\infty} K_{jr}t^{r-2}$, $t = \eta-\eta_0$ and η_0, b_4 and b_6 are arbitrary □

Subclasses of solutions

Since b_4 and b_6 are arbitrary we can look for subclasses of solutions for specific choices of these constants. Our results are given below:

(i) If $b_4 = 0$ and b_6 is arbitrary then $K_{rj} = 0$ for all r, j and hence

$$f(\eta) = \frac{b_0}{t^2} + \sum_{r=6}^{\infty} b_t t^{r-2} .$$

(ii) If $b_4 = 0 = b_6$ then $b_r = 0$, $K_{rj} = 0$ for all r, j and hence

$$f(\eta) = \frac{b_0}{t^2} .$$

(iii) If $b_6 = 0$ and b_4 is arbitrary then the solution given in theorem 7.9 is fundamentally unchanged.

We have carried out the above analysis on the remaining equations in Table 6.2 and the results are summarized below:

c3(a): $\alpha a_4 f'' - f + ff' = 0$ (7.2.2.13)

We have already noted that this equation is not of P-type. Furthermore, theorems 7.4 and 7.6 do not apply to this equation which, by a similar analysis, can be shown to possess a pole of order one in the leading term. Also one arbitrary constant is η_o and the other can be shown to be b_z. However we cannot satisfy the requirement that b_z is arbitrary and we have to supplement the solution with logarithmic terms as before. Thus $f(\eta)$ has a movable critical point. The general solution is formalised in the theorem below:

<u>Theorem 7.10</u>

The equation (7.2.2.13) has a formal general solution

$$f(\eta) = \frac{b_o}{t} + \sum_{r=1}^{\infty} b_{2r} t^{2r-1} + \sum_{j=1}^{\infty} g_j(t)(\ln t)^j \tag{7.2.2.14}$$

where $g_j(t) = \sum_{r=2j}^{\infty} K_{jr} t^{2r-1}$, $t = \eta - \eta_o$ and

η_o and b_z are arbitrary □

<u>Corollary</u> Equation C3(a) does not have any single-valued meromorphic solutions □

We now look at the third-order equations in Table 6.2 which are special cases of (7.2.6) but are not covered by theorem 7.9.

$\underline{C3(b): \quad a_3\gamma\alpha^2 f''' + a_4\alpha^2 f'' + \gamma f' + f - ff' = 0}$

(γ is constant) (7.2.2.15)

The series solution has $b_0 = 12a_4\alpha^2\gamma$ with arbitrary constants at b_4 and b_6. Checking for consistency on the arbitrariness of b_4 and b_6 we find that

$$b_1 = -\frac{36}{15}a_4\alpha^2, \quad b_2 = -\frac{a_4\alpha^2}{25\gamma} + \gamma, \quad b_3 = -\frac{a_4\alpha^2}{125\gamma^2} - 1,$$

$$0.b_4 = b_1 .$$

Hence b_4 cannot be chosen arbitrarily and the general solution of (7.2.2.15) is not represented by (7.2.2.1). To reinstate the arbitrariness of b_4 we supplement the series (7.2.2.1) with logarithmic terms. Then the first few terms of the solution have the form:

$$f(\eta) = \frac{b_0}{t^2} + \frac{b_1}{t} + b_2 + b_3 t + (b_4 + c_4 \ln t)t^2 + 0(t^2)$$

(7.2.2.16)

Thus $f(\eta)$ has a movable critical point.

Consistency can be shown for the higher powers by the method used previously and we formalize the result in the following theorem:

Theorem 7.11

The equation (7.2.2.15) has a formal general solution

$$f(\eta) = \frac{b_0}{t^2} + \frac{b_1}{t} + \sum_{r=2}^{\infty} b_r t^{r-2} + \sum_{j=1}^{\infty} g_j(t)(\ln t)^j$$

(7.2.2.17)

where $g_j(t) = \sum_{j=4r}^{\infty} K_{rj} t^{r-2}$, $t = \eta - \eta_0$, η_0, b_4

and b_6 arbitrary □

Corollary Equation C3(b) does not have any single-valued meromorphic solutions □

C2: $a_4\alpha^2\eta f''' + 4a_4\alpha^2 f'' + f' + \eta f f' + 2f^2 = 0$

(7.2.2.18)

Let $a_4 = -1$, $\alpha = \eta_0 = 1$. Then the equation becomes

$$\eta f''' + 4f'' - f' - \eta f f' - 2f^2 = 0 .$$

The solution has a pole of order two and the arbitrary constants are once again b_4 and b_6. Checking for consistency on the arbitrariness of b_4 and b_6 gives

$b_0 = 12$, $b_1 = 0$, $b_2 = -1$, $b_3 = -2$, $0.b_4 = -36b_3$.

Hence the general solution is not represented by (7.2.2.1) and we have to supplement the solution with logarithmic terms. The general solution can now be obtained consistently and the result is stated in the theorem below:

Theorem 7.12

The equation (7.2.2.18) has a formal general solution

$$f(\eta) = \frac{b_o}{t^2} + \sum_{r=2}^{\infty} b_r t^{r-2} + \sum_{j=1}^{\infty} g_j(t)(\ln t)^j \tag{7.2.2.19}$$

where $g_j(t) = \sum_{j=4r}^{\infty} K_{rj} t^{r-2}$, $t = \eta - \eta_o$, η_o, b_4 and b_6 arbitrary □

Corollary Equation C2 does not have any single-valued meromorphic solutions.

$$\text{D1:} \quad a_3 f''' + \left(1 - \frac{a_1}{a_2}\right) f' - \left(\lambda - \frac{a_1}{a_2}\gamma\right) f f' + \alpha f^2 = 0 .$$

$$\alpha \neq 0, \quad a_2 \neq 0 \qquad (7.2.2.20)$$

Without any loss of generality we may take $a_3 = \alpha = 1$.

This equation has the interesting feature that the coefficients of the nonlinear ff' term may be chosen to be zero. Thus we consider the two cases (i) $\lambda = \gamma a_1/a_2$ and $\lambda \neq \gamma a_1/a_2$. Furthermore, in both cases (i) and (ii) we have $a_1 = a_2$ and $a_1 \neq a_2$ and these will have to be studied separately.

<u>(i) $\lambda = \gamma a_1/a_2$</u>

(a) $a_1 \neq a_2$: In this case equation (7.2.2.20) reduces to the nonlinear ODE

$$f''' + kf' + f^2 = 0, \quad k = 1 - \frac{a_1}{a_2} \qquad (7.2.2.21)$$

We know from theorem 7.5 that the method we have been using is unable to obtain the general solution of this equation. However a formal particular solution is

$$f(\eta) = \frac{b_o}{t^3} + \frac{b_3}{t} + \sum_{r=2}^{\infty} b_r t^{2r-3} \qquad (7.2.2.22)$$

where $b_o = 60$ and none of the b_{2r} are arbitrary.

(b) $a_1 = a_2$: In this case the equation (7.2.2.20) reduces to the nonlinear ODE

$$f''' + f^2 = 0 \qquad (7.2.2.23)$$

Again we are unable to obtain a general solution for this equation but note that a particular solution is

$$f(\eta) = \frac{60}{t^3} .$$

This is meromorphic. However, we are also able to show that there is a solution which is essentially singular with two arbitrary constants and is given by:

$$f(\eta) = \ldots . + \frac{\alpha_1}{t^5} + \frac{\alpha_0}{t^4} + \frac{60}{t^3}$$

where α_0 is arbitrary and α_j, $j \geq 1$ are given in terms of α_0.

(ii) $\lambda \neq \gamma a_1/a_2$

(a) $a_1 \neq a_2$: In this case the equation (7.2.2.20) becomes

$$f''' + kf' - mff' + f^2 = 0, \quad k = 1 - \frac{a_1}{a_2} ,$$

$$m = \lambda - \frac{a_1}{a_2} \gamma \qquad (7.2.2.24)$$

The solution has a pole of order two and arbitrary constants are b_4 and b_6. Checking for consistency on the arbitrariness of b_4 and b_6 gives:

$$b_0 = \frac{12}{m} , \; b_1 = - \frac{24}{5m^2} , \; b_2 = \frac{k}{m} + \frac{96}{25m^2} ,$$

$$b_3 = - \frac{2k}{m^2} - \frac{1008}{125m^4} \; 0.b_4 = - \frac{1}{m^3} \left[k + \frac{4}{m^2}\right] .$$

Thus if we assume that $k = -4/m^2$ then b_4 may be chosen arbitrarily. We make this assumption and proceed to obtain

$$b_5 = \frac{4}{5m}\left[6b_4 + \frac{346}{1875m^5}\right] \quad \text{and}$$

$$o.b_6 = \frac{2}{25m^2}\left[840b_4 + \frac{3359}{125m^5}\right].$$

Thus both b_4 and b_6 cannot be chosen arbitrarily and as a consequence the general solution of (7.2.2.24) is not represented by the series (7.2.2.1). The general solution is given in the following theorems.

Theorem 7.13A

The equation (7.2.2.24) with $k = -4m^{-2}$ has a formal general solution

$$f(\eta) = \frac{b_0}{t^2} + \frac{b_1}{t} + \sum_{r=2}^{\infty} b_r t^{r-2} + \sum_{j=1}^{\infty} g_j(t)(\ln t)^j, \tag{7.2.2.25}$$

where $g_j(t) = \sum_{r=6j}^{\infty} K_{rj} t^{r-2}$, η_0, b_4 and b_6 are arbitrary □

Corollary If $b_4 = \frac{-3359}{10500} m^{-5}$ and b_6 is arbitrary then the equation has the single-valued meromorphic solution

$$f(\eta) = \frac{b_o}{t^2} + \frac{b_1}{t} + \sum_{r=2}^{\infty} b_r t^{r-2} \quad \Box$$

Theorem 7.13B

The equation (7.2.2.24) with $k \neq -4m^{-2}$ has a formal general solution (7.2.2.17) □

(b) $a_1 = a_2$: In this case $0.b_4 = -4/m^3$ and we are forced to supplement the solution with logarithmic terms to reinstate the arbitrariness of b_4. The result is given in the following theorem.

Theorem 7.14

The equation (7.2.2.24) with $a_1 = a_2$ has a formal general solution (7.2.2.17) □

D3(a): $a_3\beta^3 f''' + \frac{1}{3}(\alpha-\beta)\eta f f' + \alpha f^2 = 0,$

$$\alpha \neq \beta \neq 0, \ \alpha \neq -2\beta \qquad (7.2.2.26)$$

Without loss of generality we may take $a_3 = 1$ and $\eta = 1$. The solution has a pole of order two and the arbitrary constants are b_4 and b_6. Checking for consistency on the arbitrariness of b_4 and b_6 we obtain:

$$b_o = \frac{-36}{\alpha-\beta}, \quad b_1 = \frac{-72(\alpha+2\beta)}{5(\alpha-\beta)^2}, \quad b_2 = \frac{-36(\alpha+2\beta)(13\alpha+11\beta)}{25(\alpha-\beta)^3}$$

$$b_3 = \frac{-72(\alpha+2\beta)}{75(\alpha-\beta)^4}(13\alpha^2 + 188\alpha\beta + 53\beta^2),$$

$$0.b_4 = \frac{1}{3}(5\alpha+\beta)[b_0 b_3 + b_1 b_2]$$

Now $b_0 b_3 + b_1 b_2 = \dfrac{2592(\alpha+2\beta)(2\alpha+\beta)(\alpha+\beta)}{(\alpha-\beta)^5}$ and thus it follows that if $\alpha \neq -\beta$ and $\alpha \neq -\beta/2$ and $\alpha \neq -\beta/5$ then the general solution is of the form (7.2.2.17).

If $\alpha = -\beta$ or $\alpha = -\beta/2$ or $\alpha = -\beta/5$ then b_4 can be chosen as arbitrary and we have

$$b_5 = kb_4 + l, \quad k \neq l, \quad k, l \in \mathbb{R} - \{0\}$$

$$b_6 = mb_4 + n, \quad m \neq n, \quad m, n \in \mathbb{R} - \{0\} .$$

But since b_4 is arbitrary, b_6 cannot be arbitrary and the general solution may be shown to be of the form (7.2.2.25).

We state our result in the following theorem:

<u>Theorem 7.15</u>

Consider the equation (7.2.2.26).

(i) When $\alpha \neq -\beta$, $\alpha \neq -\beta/2$, $\alpha \neq -\beta/5$ the general solution is (7.2.2.17).

(ii) When $\alpha = -\beta$ or $\alpha = -\beta/2$ or $\alpha = -\beta/5$ the general solution is (7.2.2.25) □

Corollary For the case of theorem 7.15(ii), if we choose $b_4 = -n/m$ and leave b_6 arbitrary then the equation has the single-valued meromorphic solution

$$f(\eta) = \frac{b_0}{t^2} + \frac{b_1}{t} + \sum_{r=2}^{\infty} b_r t^{r-2} \quad \square$$

D3(b): $3a_3 f''' + \eta f' + \eta f f' = 0$ (7.2.2.27)

Proceeding as before we obtain the following theorem:

Theorem 7.16

The equation (7.2.2.27) has a formal general solution (7.2.2.17) □

We have now completed our analysis on the classes of ODEs, represented by (7.2.6) determining the similarity solutions of the general class of PDEs. We now collect our results and summarize them in the following table. The table is made up by referring first to the subclass of PDEs and then to the ODE giving the similarity solution. A table giving the ODEs themselves has been given earlier (Table 6.2). In the final column the notation is as follows:

G ≡ general solution,   P ≡ particular solution

SVM(n) = single-valued meromorphic solutions with pole order n.

MVM(n) = multiple-valued meromorphic solutions with pole order n which is also a logarithmic branch point.

Note: By theorem 7.3 all the equations also have analytic solutions which are not special cases of the singular solutions given in the table.

Class of PDE Equations	ODE Reductions	Analytic Structure
W_σ	A1	G: SVM(2)
KdV ($a_2=0$)	D2(a)	G: SVM(2)
	D2(b)	G: SVM(2) P: $f=\eta-\eta_0$
KdV ($a_2\neq0$)	D1 $\lambda\neq\gamma a_1/a_2$	G: MVM(2) P: SVM(2)
	D1 $\lambda=\gamma a_1/a_2$	P: SVM(3)
KdV ($a_1=a_2$)	D3(a)	G: MVM(2) P: SVM(2)
	D3(b)	G: MVM(2)
$W_\sigma(a_1=a_2)$	A2	G: MVM(2) P: SVM(2)
RLW($a_1=0$)	C3(a)	G: MVM(2)
	C3(b)	G: MVM(2)
RLW($a_1=0$)	C2	G: MVM(2)

Table 7.1

These results lead to some important observations on the general class of PDEs (7.1). However, before considering this we highlight the main results on the class of third order nonlinear ODEs (7.2.6).

(1) All the ODEs represented by (7.2.6) have solutions which are analytic functions.

(2) Third order nonlinear ODEs of the form (7.2.6) where $\rho(\eta) \neq 0$ i.e., which have an ff' term, have a second general solution which is either single-valued meromorphic or multiple-valued meromorphic (via logarithmic branching). This conclusion applies to equation D2(a) which is not in the class (7.2.6).

(3) When the ff' term is absent from the ODE then we have not been able to obtain a second class of general solutions by the series solution.

(4) Only the ODEs A1, D2(a) and D2(b) have general solutions which are single-valued and meromorphic. All the others have general solutions which are multiple-valued meromorphic maps with logarithm branch points. We note that these three equations do not contain any of the terms f'', f^2 and $\eta ff'$. We believe that this is interesting because they may be individually responsible for the appearance of logarithmic branch points in C3(b), D1 and D3(b) respectively.

7.3 Concluding remarks on the similarity reductions

We note that every PDE represented by W_σ has subclasses of analytic and single-valued meromorphic solutions. The meromorphic solutions have second order poles although the KdV equation $(a_1 = a_2)$ also has poles of order three and essential singularities.

Also all the equations represented by W_6, except for the $KdV(a_2=0)$ possess a subclass of logarithmic branch points. We present this important result in the following theorem:

Theorem 7.17

Consider the general class of PDEs

$$u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}+a_6u_{ttt} = 0 \tag{7.3.1}$$

with $a_j \in \mathbb{R}$ $j = 1, \ldots, 6$.

If, in addition to travelling wave solutions, there exist similarity solutions obtained by the Lie group method, then these solutions will be free from movable critical points if and only if $a_2 = a_4 = a_5 = a_6 = 0$ □

Corollary The following PDEs are not of P-type.

W_6, W_5, W_{43}: $u_t+u_x+a_1u(u_x+u_t)+a_3u_{xxx}+a_4u_{xxt}+a_5u_{xtt}$

$$+ a_6u_{ttt} = 0, \; a_j \in \mathbb{R}.$$

RLW: $u_t+u_x+a_1uu_x+a_2uu_t+a_4u_{xxt} = 0$ $(a_1=a_2)$ or

$(a_1=0,\; a_2\neq 0;\; a_1\neq 0,\; a_2= 0)$
$a_j \in \mathbb{R}$.

KdV: $u_t+u_x+a_1uu_x+a_2uu_t+a_3u_{xxx} = 0$ $(a_2 \neq 0)$ $a_j \in \mathbb{R}$ □

Thus, only the KdV equation

$$u_t + u_x + a_1 uu_x + a_3 u_{xxx} = 0 \qquad (7.3.2)$$

has similarity reductions, in addition to travelling waves, which are of Painlevé type. In all other cases the critical points appear in the form of movable logarithmic branch points.

While we have obtained only analytic and meromorphic solutions for the KdV equation (7.3.2) we have not shown that these are the only solutions. The complete analytic stucture of a nonlinear PDE may not be represented by its ODE reductions alone. We have noted in Chapter Four that some PDEs may not have symmetries at all and, as we have shown in Chapter Six, many subclasses of the general class we are considering have a single ODE reduction A1. These include the W_6 and W_5 classes when $a_1 \neq a_2$, the W_{43} class when $a_1 = 0$ and the RLW class when $a_1 \neq a_2 \neq 0$. To obtain further information about these equations we need to study them by a different method.

In the following chapter we study the general solutions of the general class of PDEs by examining their analytic structure directly. This direct approach will add to our knowledge of the analytic structure and may identify further equations which are not of P-type.

CHAPTER EIGHT

ANALYTIC STRUCTURE OF SOLUTIONS OF THE GENERAL CLASS OF PDES.

In this chapter we continue our study of the analytic structure of the solutions of the general class of PDEs by extending the analysis from those of similarity solutions, i.e. based on reductions to ODEs, to general meromorphic solutions of the PDE itself. This set of solutions should be wider than the similarity solutions discussed before and, in principle, will enable us to decide whether the analysis of the similarity solutions is sufficient to decide whether the PDE is of P-type or not. As we have seen earlier this property is conjectured to be connected with the existence of soliton solutions.

8.1 Meromorphic solutions of a PDE

The method for finding meromorphic solutions was originally put forward by Weiss et al [4] and is referred to as the direct method. However, we found certain mathematical redundancies in their presentation arising from assumptions which they do not later use. These are as follows:

(i) we see no reason for introducing complex time and so we work with time as a real parameter;

(ii) since the equations we deal with have only one space variable then we see no reason for introducing coefficients which depend on z as well as t.

Our method for dealing with evolution equations in one space variable is based on the following assumptions:

(i) the equation and its solution can be analytically continued from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C} \times \mathbb{R}$, i.e., $(x, t) \longrightarrow (z, t)$;

(ii) for each t the solution is a function of a single complex variable z.

We assume that the solution $u(z, t)$ has the form

$$u(z, t) = \frac{F(z,t)}{(z+\psi(t))^{\alpha}} , \quad \alpha \in \mathbb{R}^{+} ,$$

(due to Kruskal [11])

where $z(t) = -\psi(t)$ is the orbit of an isolated singularity which is either a pole or an algebraic branch point. Then $F(z, t)$ is analytic in some neighbourhood of this orbit and the radius of this neighbourhood will be determined by the positions of any other singularities of $u(z, t)$. Therefore, for each t the solution has the expansion

$$u(z, t) = (z - z(t))^{-\alpha} \sum_{j=0}^{\infty} c_j(t)(z-z(t))^j \tag{8.1.1}$$

about the singularity $z(t)$. Since the method we are using involves analytic expansions about points (z, t), the Cauchy-Kowalewski existence theorem tells us that this is only possible if the orbit $z(t)$ is not a characteristic of the PDE. We also assume that $\psi(t)$ has the smoothness required by the PDE.

The value of α in (8.1.1) is determined by the dominant terms and the coefficients $c_j(t)$ are defined recursively where such a solution exists. If the solution does not exist, this will be shown by inconsistencies arising in the evaluation of the $c_j(t)$.

Note: (i) In cases where there is inconsistency the series can be made to work if we supplement it by terms including powers of $\ln(z - z(t))$ as in the previous chapter. We shall not demonstrate this in this chapter.

(ii) If we get a consistent solution with α a positive integer then we may be able to conclude that the PDE satisfies the necessary condition to be P-type. However, the method fails to detect movable essential singularities and, thus, the fact that it works is not sufficient to deduce it is of P-type.

We now proceed to analyse the general class of equations (7.1).

8.2 Meromorphic solutions of the general class

We begin with a theorem on the value of α:

Theorem 8.1

If the general class (7.1) has general meromorphic solutions then its movable poles can only be second order, i.e. $\alpha = 2$ □

Proof The result follows by substituting the expansion (8.1.1) into the PDE (7.1) and balancing dominant terms □

Corollary The general class (7.1) has no solutions with movable algebraic singularities.

The dominant term analysis also gives that

$$(a_1 + a_2\xi)c_0 = -12(a_3 + a_4\xi + a_5\xi^2 + a_6\xi^3) \tag{8.2.1}$$

where $\xi = \psi'$. The right hand side is non-zero since we have assumed that $-\psi(t)$ is not a characteristic of the equation. This is covered by the following theorem (Sherbiny [3]).

Theorem 8.2

The characteristics of the general class (7.1) are defined by the cubic

$$a_3 - a_4 \frac{dz}{dt} + a_5\left(\frac{dz}{dt}\right)^2 - a_6\left(\frac{dz}{dt}\right)^3 = 0 \quad \square$$

Note 1: Equation (8.2.1) also implies that $a_1 + a_2\xi \neq 0$ and defines $c_0 \neq 0$.

Note 2: In theorem 8.1 above we emphasised general solutions because, as we have seen in the previous chapter, there are particular solutions with movable third order poles. The explanation for this is as follows: if $a_1 + a_2\xi$ is not a root of the cubic on the right hand side then one of its possible values is zero. In this case c_0 will have to be infinite which indicates that the pole has higher order than two.

Returning to the series expansion (8.1.1) we see that it is a Laurent series about a pole of order two and in order to substitute into (7.1) we need the derivatives of u. These are given as follows: where

$\phi = z - z(t)$, $\xi = -z'(t)$ and $\sum = \sum_{0}^{\infty}$.

$$u = \sum c_j(t)\phi^{j-2} \qquad (8.2.2a)$$

$$u_x = \sum (j-2)c_j\phi^{j-3} \qquad (8.2.2b)$$

$$u_t = \sum \left[c_j' + (j-1)\xi c_{j+1}\right]\phi^{j-2} - 2\xi c_o\phi^{-3}$$

(8.2.2c)

$$u_{xxx} = \sum (j-2)(j-3)(j-4)c_j\phi^{j-5} \qquad (8.2.2d)$$

$$u_{xxt} = \sum \Big[(j-2)(j-3)\{(j-1)c_{j+1}\xi + c_j'\}\phi^{j-4}$$

$$- 24c_o\xi\phi^{-5}\Big]$$

(8.2.2e)

$$u_{xtt} = \sum (j-2)\Big[c_j'' + (j-1)\{2c_{j+1}'\xi + c_{j+1}\xi' +$$

$$jc_{j+2}\xi^2\}\phi^{j-3} - 3(2c_1\xi^2 - 4c_o'\xi - 2c_o\xi')\phi^{-4} - 24c_o\xi^2\phi^{-5}\Big]$$

(8.2.2f)

$$u_{ttt} = \sum [c_j''' + (j-1)\Big\{3c_{j+1}''\xi + 3c_{j+1}'\xi' + c_{j+1}\xi''$$

$$+ 3jc_{j+2}'\xi^2 + 3jc_{j+2}\xi\xi' + j(j+1)c_{j+3}\xi^2\Big\}\phi^{j-2}$$

$$+ \Big\{6c_1\xi\xi' - 6c_o'\xi' - 2c_o\xi'' + 6c_1'(\xi')^2$$

$$- 6c_o''\xi\Big]\phi^{-3} + (18c_o\xi\xi' + 18c_o'\xi^2 - 6c_1\xi^3)\phi^{-4}$$

$$- 24c_o\xi^3\phi^{-5} \qquad (8.2.2g)$$

Defining $v_j(t) = \sum_{k=0}^{j} c_{j-k}(t)c_k(t)$

then $u^2 = \sum v_j\phi^{j-4}$

and differentiating this expression with respect to x and t leads to

$$uu_x = \frac{1}{2}\sum (j-4)v_j\phi^{j-5} \qquad (8.2.2h)$$

$$uu_t = \frac{1}{2}\sum \Big[(j-3)v_{j+3}\xi + v_j'\Big]\phi^{j-4} - 2v_o\xi\phi^{-5} \qquad (8.2.2i)$$

Substituting (8.2.2.a-i) into (7.1) and equating powers of ϕ gives the following recursion relations:

$$\phi^{-5}: \quad 2(a_1+a_2\xi)v_o + 24(a_3+a_4\xi)c_o + 24a_5c_o\xi^2 + 24a_6c_o\xi^3 = 0 \qquad (8.2.3a)$$

$$\phi^{-4}: \frac{a_2}{2} V_0' - \frac{3}{2}(a_1+a_2\xi)V_1 - 6(a_3+a_4\xi)c_1 + 6a_4c_0'$$

$$- 3a_5[2c_1\xi^2-4c_0'\xi-2c_0\xi']$$

$$+ a_6[18c_0\xi\xi'+18c_0'\xi^2-6c_1\xi^3] = 0 \qquad (8.2.3b)$$

$$\phi^{-3}: - 2(1+\xi)c_0 - (a_1+a_2\xi)V_2 + \frac{a_2}{2}V_1' + 2a_4c_1'$$

$$- 2a_5(c_0''-2c_1'\xi-2c_1\xi')$$

$$+ a_6(6c_1\xi\xi'-6c_0'\xi'-2c_0\xi''+6c_1'\xi^2-6c_0''\xi) = 0$$

(8.2.3c)

and for any $j \geq 3$,

$$\phi^{j-3}: a_6c_{j-3}''' + c_{j-3}' + (j-4)\Big[c_{j-2}\xi+c_{j-2}+a_5c_{j-2}''$$

$$+3a_6c_{j-2}''\xi+3a_6c_{j-2}'\xi'+a_6c_{j-2}\xi''\Big] + (j-3)(j-4)$$

$$\Big[a_4c_{j-1}'+2a_5c_{j-1}'\xi+a_5c_{j-1}\xi'+3a_6c_{j-1}'\xi^2$$

$$+3a_6c_{j-1}\xi\xi'\Big] + \frac{a_2}{2} V_{j-1}' + (j-2)(j-3)(j-4)$$

$$\Big[a_3+a_4\xi+a_5\xi^2+a_6\xi^3\Big]c_j + \frac{(j-4)}{2}(a_1+a_2\xi)V_j = 0$$

(8.2.3d)

From (8.2.3d) the terms involving $c_j (j \geq 3)$ are given by

$$(j-6)(j-4)(j+1)(a_3+a_4\xi+a_5\xi^2+a_6\xi^3)c_j = F_j \tag{8.2.4}$$

where $F_j = F(c_{j-1}, c_{j-2}, \ldots, c_0, \xi, \xi', \ldots)$.

Consequently the left-hand side of (8.2.4) is zero when $j = 4$ or 6 so that $F_4 = F_6 = 0$.

From (8.2.3a, b, c) the first few recurrence relations are as follows:

$$c_0 = \frac{-12\left[a_3+a_4\xi+a_5\xi^2+a_6\xi^3\right]}{a_1+a_2\xi} \tag{8.2.5a}$$

$$c_1 = \frac{12c_0'\left[a_4+2a_5\xi+3a_6\xi^2\right]}{5c_0(a_1+a_2\xi)} + \frac{12\xi'\left[a_5+3a_6\xi\right]+2a_2c_0'}{5(a_1+a_2\xi)} \tag{8.2.5b}$$

$$c_2 = -\frac{c_1^2}{2c_0} + \frac{a_2(c_0c_1)'}{2(a_1+a_2\xi)c_0}$$

$$+ \frac{\left[a_4+2a_5\xi+3a_6\xi^2\right]c_1' + \left[a_5+3a_6\xi\right]c_1\xi'}{(a_1+a_2\xi)c_0}$$

$$- \frac{\left[3a_6c_0'\xi'+a_5c_0\xi''\right]}{(a_1+a_2\xi)c_0} - \frac{(1+\xi)}{a_1+a_2\xi} - \frac{\left[a_5+3a_6\xi\right]c_0''}{(a_1+a_2\xi)c_0}$$

(8.2.5c)

and from (8.2.3d we obtain:

$$c_3 = \frac{a_6c_0'''+c_0'-\left[c_1\xi+c_1+a_5c_1''+3a_6(c_1'\xi)'+a_6c_1\xi''\right]}{(a_1+a_2\xi)c_0}$$

$$\frac{+a_2\left[(c_0c_2)'+c_1c_1'\right]}{(a_1+a_2\xi)c_0}$$

$$- \frac{c_1c_2}{c_0} \qquad (8.2.5d)$$

$$F_4 = a_2(c_0c_3+c_1c_2)' + (a_6c_1'''+c_1') = 0 \qquad (8.2.5e)$$

$$c_5 = -2\Big\{a_6c_2'''+c_2'+c_3\xi+c_3+a_5c_3''+3a_6(c_3'\xi)'+a_6c_3\xi''$$

$$+ 2\left[a_4c_4'+2a_5c_4'\xi+a_5c_4\xi'+3a_6c_4'\xi^2+3a_6c_4\xi\xi'\right]$$

$$+ a_2\left[(c_0c_4)'+(c_1c_3)'+c_2c_2'\right]$$

$$+ (a_1+a_2\xi)(c_2c_3+c_4c_1)\Big\}/c_0(a_1+a_2\xi) \qquad (8.2.5f)$$

$$F_6 = 6\left[a_4c_5'+2a_5c_5'\xi+a_5c_5\xi'+3a_6c_5'\xi^2+3a_6c_5\xi\xi'\right]$$

$$+ a_2\left[(c_0c_5)' + (c_1c_4)' + (c_2c_3)'\right]$$

$$+ \left[a_6 c_3''' + c_3' + 2(c_4\xi + c_4 + a_5 c_4'' + 3a_6(c_4'\xi) + a_6 c_4 \xi'')\right]$$

$$+ (a_1 + a_2\xi)(2c_1c_5 + 2c_2c_4 + c_3^2) = 0 \qquad (8.2.5g)$$

Equations (8.2.5a-d) determine c_1, c_2, c_3 in terms of ξ and its derivatives and hence allows us to compute F_4. Relation (8.2.5f) gives c_5 in terms of c_4 etc. and since the previous equations do not define c_4 we assume that c_4 is arbitrary. We can now compute c_5 from (8.2.5f) and obtain F_6 in terms of c_4 and ξ and its derivatives from equation (8.2.5g). The next relation ($j = 7$) will determine c_7 in terms of c_6 etc. and since c_6 has not been defined above we shall assume that c_6 is an arbitrary function. The fact that c_4 and c_6 can be chosen arbitrarily is also a consequence of equation (8.2.4).

Working out F_4 and F_6 is extremely cumbersome in general and we have approached this problem in two ways. Firstly, we used the computer with the REDUCE package for symbolic manipulation - although with this package the algebra for equations involving more than one linear term is very long and tedious. Secondly, we simplified the problem by using the classification theorem presented by El-Sherbiny and then computed by hand for each equivalence class. The classification theorem is as follows:

Theorem 8.3

Given an equation from the general class (7.1) there exists a nonsingular real linear transformation (which may be the identity) which takes it into one of the following four equivalence classes:

KdV class $(a_1, a_2, a_3, 0, 0, 0)$

RLW class $(a_1, a_2, 0, a_4, 0, 0)$

W_{54} class $(a_1, a_2, 0, a_4, a_5, 0)$

W_{53} class $(a_1, a_2, a_3, 0, a_5, 0)$ □

The reductions in the above theorem are obtained by considering the cubic

$$a_3 + a_4\xi + a_5\xi^2 + a_6\xi^3 = 0.$$

El-Sherbiny showed that when this cubic has three equal roots the general class reduces to the KdV class; when it has two equal and one distinct root it reduces to the RLW class; when it has three distinct roots it reduces to the W_{54} class and when it has one real and two complex conjugate roots it reduces to the W_{53} class.

8.2.1 Results via the REDUCE package

For this analysis we wrote the program given in Appendix D. In the case when $a_1 + a_2\xi$ is a triple root of $-12(a_3+a_4\xi+a_5\xi^2+a_6\xi^3)$ i.e. the KdV equation, we showed that $F_4 \equiv 0$ and $F_6 \equiv 0$. When we assumed that $a_1 + a_2\xi$ is a double root, i.e. the RLW equation, we had both $F_4 \not\equiv 0$ and $F_6 \not\equiv 0$. However, we did not proceed further since the version of REDUCE we were using was unable to cope, easily, with equations with more than one linear third order term.

8.2.2 Results with equivalence classes

Before we embark on these calculations we would like to clarify the role played by the constraints $F_4 \equiv 0$, $F_6 \equiv 0$ regarding the arbitrariness of $z(t)$, $c_4(t)$ and $c_6(t)$. The constraint $F_4 = 0$ is a condition on $\xi = z(t)$ which does not contain c_4 and c_6. If this condition is identically satisfied then ξ is an arbitrary function, whilst, if it is not, then it is an ODE defining ξ in which case the pole orbits are fixed by the constraint. The condition $F_6 \equiv 0$ includes both ξ and c_4. If it is identically satisfied then c_4 is arbitrary and ξ is not restricted by this condition. If it is not identically satisfied then it gives an ODE for c_4 where the coefficients are functions of ξ. We note that there is no restriction on c_6 and therefore c_6

can be chosen freely in every case.

We now consider the condition $F_6 = 0$. Substituting (8.2.5f) into (8.2.5g) gives us that

$$F_6 \equiv A + Bc_4 + Cc_4' + Dc_4'' = 0, \quad t \in \mathbb{R}^+$$

(8.2.2.1)

where A, B, C, D are functions of c_0, c_1, c_2, c_3 and their derivatives. If c_4 is arbitrary it is clear that c_4', c_4'' are independent of c_4 and hence (8.2.2.1) is satisfied if and only if A = B = C = D = 0 for all t. This gives us four differential equations for ξ. In addition, equation (8.2.5e) gives a fifth, i.e. $F_4 \not\equiv 0$. We now look at each equivalence class.

(1) KdV class: $(a_1, a_2, a_3, 0, 0, 0)$, $(a_3 \neq 0)$

For this class $D = \dfrac{24a_2^2 a_3}{(a_1 + a_2\xi)^2}$. Thus, D = 0 if and only if $a_2 = 0$. Hence, the only equations in this class which may have general meromorphic solutions are the elements of the subclass $(a_1, 0, a_3, 0, 0, 0)$. To verify whether this is the case or not we go back to equation (8.2.6) and put $a_2 = a_4 = a_5 = a_6 = 0$. This gives the following results:

$$c_0 = -12a_3/a_1, \quad c_2 = -(1+\xi)/a_1 \quad c_1 = c_3 = c_5 = 0,$$

$$F_4 \equiv 0 \equiv F_6.$$

Thus for this class we have the following theorem:

Theorem 8.4

The only equations in the KdV class which have meromorphic solutions in which $z(t)$, $c_4(t)$ and $c_6(t)$ are arbitrary functions are those with coefficients $(a_1, 0, a_3, 0, 0, 0)$. Consequently, for each t, the corresponding ODE has three arbitrary constants. Hence this meromorphic solution is a general solution of the PDE □

Corollary. The PDEs $(a_1, 0, a_3, 0, 0, 0)$ are P-type modulo the existence of essential singularities □

Turning to the cases where $a_2 \neq 0$ we note that the conditions $F_4 \equiv 0$ and $F_6 \equiv 0$ lead to very complicated differential equations. For example, for the case $(0, -1, 1, 0, 0, 0)$

$$F_4 \equiv \frac{d}{dt}\left\{ \frac{144}{5}\frac{1}{\xi^2}\left[\frac{\xi'}{\xi^2}\left[1 - \frac{13\xi''}{\xi^3} + \frac{24(\xi')^2}{\xi^4}\right] + \frac{\xi^{iv}}{\xi^4}\right]\right\} \tag{8.2.2.2}$$

We do not feel that this case is sufficiently interesting to pursue any further.

(2) RLW class: $(a_1, a_2, 0, a_4, 0, 0)$. $(a_4 \neq 0)$

In this case $D = \dfrac{2a_4(a_1-a_2\xi)(a_1-5a_2\xi)}{\xi(a_1+a_2\xi)^2}$

Hence, $D = 0$ if and only if $\xi = a_1/a_2$ or $\xi = a_1/5a_2$. Thus, the RLW class has no general meromorphic solution.

However, we have the following theorem for special cases:

Theorem 8.5

If a_1 and a_2 are non-zero then the RLW class has meromorphic solutions in which c_4 and c_6 are arbitrary, but $\xi = a_1/a_2$ or $a_1/5a_2$, i.e. the poles of the solution move with one of two constant speeds □

Note: This seems to imply that these solutions would have at most two poles (modulo periodicity).

The interesting equations in this class are those with $a_2 = 0$, which correspond to the RLW itself. In this case the differential equation $F_4 \equiv 0$ can be solved exactly for ξ. Going back to the equation (8.2.6) and putting $a_2 = a_3 = a_5 = a_6$ gives the following:

$$c_0 = -12 \frac{a_4}{a_1} \xi, \quad c_1 = \frac{12}{5} \frac{a_4}{a_1} \frac{\xi'}{\xi},$$

$$c_2 = \frac{6a_4}{25a_1} \frac{(\xi')^2}{\xi^3} - \frac{(1+\xi)}{a_1},$$

$$c_3 = \frac{\xi'}{a_1 \xi} + \frac{6a_4}{125a_1} \frac{(\xi')^3}{\xi^5}, \quad F_4 \equiv c_1' .$$

Hence, $F_4 \equiv 0$ has the general solution

$$\xi = e^{\alpha t + \beta}, \qquad \alpha, \beta \in \mathbb{C} \qquad (8.2.2.3)$$

From this we conclude that the pole motions of meromorphic solutions of the RLW are at most of the form given by (8.2.2.3). Thus we have the following theorem:

Theorem 8.6 The general solution of the RLW is not meromorphic □

We look for particular meromorphic solutions with pole orbits given by (8.2.2.3). This means that we have to find values of $\alpha, \beta \in \mathbb{C}$ and functions $c_4(t)$ which identically satisfy $F_6 \equiv 0$. Referring to equation (8.2.2.1) and using (8.2.2.3) gives the following values of the coefficients:

$$A = \frac{1}{5}\frac{\alpha^2}{a_1}\left[1 - \frac{108}{625}a_4^2\frac{\alpha^4}{\xi^4}\right] , \quad B = 0,$$

$$C = \frac{2a_4\alpha}{\xi} , \quad D = \frac{2a_4}{\xi} ,$$

(note $\xi'/\xi = \alpha$).

For F_6 to be identically zero it is necessary for A to be identically zero and this is only possible by choosing $\alpha = 0$. Note that this choice also makes $C = 0$ so that

$$F_6 = 2a_4 e^{-\beta} c_4'' .$$

If we now choose $c_4(t)$ to be at most linear in t then $F_6 \equiv 0$ and we have the following theorem:

<u>Theorem 8.7</u>

The meromorphic solutions of the RLW class $(a_1, a_2, 0, a_4, 0, 0)$ are given by

$$u(z, t) = (z - z(t))^{-2} \sum_{j=0}^{\infty} c_j(t)(z-z(t))^j$$

when $z(t) = \delta t + \varepsilon$, $\delta, \varepsilon \in \mathbb{C}$, $c_4(t)$ is linear in t and $c_6(t)$ is arbitrary □

(3) W_{54} and W_{53} classes. ($a_5 \neq 0$)

In these cases it is easily shown that for any allowed choices of coefficients $D = 0$ for at most a few constant values of ξ. Hence these classes of equations have no general meromorphic solutions. They do, however, have special meromorphic solutions when ξ = constant and $c_4(t)$ is an arbitrary linear function i.e. the result obtained for the RLW class. We state our result for these classes in the following theorem:

Theorem 8.8

The meromorphic solutions of the W_5 class $(a_1, a_2, a_3, a_4, a_5, 0)$ are given by

$$u(z, t) = (z - z(t))^{-2} \sum_{j=0}^{\infty} c_j(t)(z - z(t))^j$$

where $z(t) = \delta t + \varepsilon$, $\delta, \varepsilon \in \mathbb{C}$, $c_4(t)$ is linear in t and $c_6(t)$ is arbitrary □

Corollary. For all the equivalence classes, $F_4 \equiv 0$, $F_6 \equiv 0$ if ξ = constant and c_4 is linear. This accounts for the existence of solitary waves in all classes □

Furthermore, we recall from Chapter Seven that the only similarity reductions for the classes W_{54} and W_{58} in the cases when $a_1 = 0$ or $a_2 = 0$ were travelling waves. Thus, we were unable to conclude whether these equations are P-type. Here we have shown that all meromporphic solutions must have ξ = constant and consequently the general solution cannot be meromorphic. We therefore conclude that these equations are not P-type.

The classification is now complete and the results lead us to the following uniqueness theorem:

Theorem 8.9

The only elements of the general class of equations

$$u_t+u_z+a_1uu_z+a_2uu_t+a_3u_{zzz}+a_4u_{zzt}+a_5u_{ztt}+a_6u_{ttt} = 0 \qquad (8.2.2.5)$$

for which the Laurent expansion

$$u(z, t) = \sum_{j=0}^{\infty} c_j(t)(z-z(t))^{j-2} \qquad (8.2.2.6)$$

where $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $z(t)$ is an arbitrary function, is a general solution are the equations defined by $(a_1, 0, a_3, 0, 0, 0)$ □

8.3 Conclusion

One of the conjectures about evolution equations that has generated considerable interest are the so-called Painlevé conjectures. We have already discussed these in Chapter Four but we repeat some of that discussion here in the setting of this chapter. A general definition that a PDE is of Painlevé or P-type is that all the movable singularities of general solutions should be poles and that these solutions should be single-valued. There are two Painlevé conjectures. The first assumes that all the relevant properties of the general solution are contained in the special similarity solutions and so the conjecture is made on reduced ODEs rather than the original PDE. The second conjecture states that the similarity solutions are not sufficient to cover all general solutions of the PDE and that it is necessary to check the PDE itself.

In this chapter we have verified this fact (i.e. the insufficiency of the similarity solutions) for subsets of the general class whose ODE reductions were of P-type. Specifically, we found that the only similarity reductions of the subclasses A1, B1 and C1 (see Table 6.1) were the solitary waves and consequently that they were of P-type. However, the analysis of the present chapter has shown that the general solution of the PDEs in these subclasses are multivalued in the sense that the pole positions are also logarithmic branch points. Thus it is clear that the equations considered either have similarity solutions which cannot be reached by the Lie-group approach or else that the first Painlevé conjecture is inadequate. If the latter is true then the similarity solutions do not capture all the properties of the general solutions. Our conclusion about meromorphic solutions is that it is only the KdV equations, i.e. $(a_1, 0, a_3, 0, 0, 0)$, which have movable poles, in its general solution, which are not also logarithmic branch points. Thus the KdV is the only equation in the general class which is P-type (modulo essential singularities) as a PDE. The implication of this, according to the Painlevé conjecture, is that this is the only equation which can be solved by the inverse scattering method and also the only one which could have soliton solutions. For the rest of the class we have also looked at special solutions which do not have the full freedom of the general solution, with a view to identifying equations which may have restricted N-soliton solutions, e.g. $N \leq K$ with fixed speed ratios. We do this because we have not come across any information in the literature which requires that an equation which has N-soliton solutions has to have them for every integer

N, although all known soliton equations have this property.

Now clearly the soliton solutions, if they exist, of any equation are special meromorphic solutions and consequently the pole motions of these solitons will be revealed through the function $\psi(t)$ which satisfies the differential equation $F_4 = 0$. Earlier work on the KdV pole motions [1] for 2-soliton solutions indicated that the poles exchanged their asymptotic speeds and that the pole orbits were continuous from $t = -\infty$ to $t = +\infty$. Consequently a necessary condition for an equation to have a 2-soliton solution is that the pole-orbits $\psi(t)$ should be non-linear functions of t.

The only equation which could be solved explicitly for $\psi(t)$, i.e. all possible pole motions, was the RLW equation i.e. $(a_1, 0, 0, a_4, 0, 0)$. For this equation we found that $\psi(t)$ was a linear function. This enables us to conclude that if the pole picture of the KdV is representative of soliton solutions for this class of equations then the RLW has no soliton solutions at all. We note that previous justifications of this result were based on numerical proofs but our proof is analytical and, we believe, rigorous. [2].

However, the methods of finding the singularity structure of general solutions do not tell us when a solitary wave is a soliton. Furthermore, there is no theorem which links Painlevé property with the existence of soliton solutions except, perhaps, the necessary conditions we just mentioned on the pole-orbits. Thus the question of when a solitary wave is a soliton is a more direct one and one that we feel is connected to the structure of the solitary wave itself. To explore this idea it is necessary to look more closely at the properties of the solitary waves to see if they provide sufficient information to deduce whether they can be combined into soliton solutions or not. This approach is also novel inasmuch as it takes the focus off the PDE and onto the construction of functions which asymptotically are linear combinations of the solitary waves. In the next chapter we make a preliminary attempt at establishing a classification of the solitary waves of the general class as a preparation for implementing this procedure.

CHAPTER NINE

COMPARATIVE ANALYSIS OF THE SOLITARY WAVES OF THE GENERAL CLASS OF PDEs

It is well-known that one of the significant properties of integrable equations is that, in general, they have soliton solutions. In previous chapters we have looked at ways of identifying such equations by using the differential equation itself either as a PDE or as a reduced ODE. On the other hand, the basic question that we asked at the beginning of this thesis, i.e., "when is a solitary wave a soliton?", can be looked at as a problem in the combination of functions as follows: Given past and future asymptotic states, i.e. $t = -\infty$ and $t = +\infty$, as linear combinations of solitary waves is it possible to construct a function which connects the two states in a smooth way? If this can be done then this function will be a multisoliton solution. Clearly this approach is a priori independent of the differential equation and appears to be difficult to implement. However, a start can be made by examining the properties of the functional form of the solitary waves to see if we can establish necessary criteria for the solitary wave to be a soliton. If this method is successful then it is more straightforward than the previous analytic structure analysis and is easier to apply in those cases when the functional form of the solitary wave is known. This is the case for the general class and it is this aspect that we concentrate on in this chapter. More specifically, since we know the KdV has soliton solutions, whereas the RLW has not (see chapter 8.) we analyse the properties of profiles of solitary waves of the general class and compare them with those of the KdV and RLW.

9.1 The general solitary wave

We recall from Chapter Five that the solitary waves of the general class (7.1) are given by

$$u(x,\ t) = \frac{3}{\alpha}\,\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{1}{\beta}}\ (x-(1+c)t)\right] \qquad (9.1.1a)$$

where $\alpha = \dfrac{a_1 - a_2(1+c)}{c}$ and

$$\beta = \frac{a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3}{c} \qquad (9.1.1b)$$

The amplitude is given by $A(c) = 3/\alpha$ and we define the width $w(c)$ to be the width at half-height, i.e., $w = 2\eta$ where $\text{sech}^2[\eta/2\sqrt{\beta}] = \dfrac{1}{2}$. Hence $w(c) = \delta\sqrt{\beta}$ where $\eta = x - (1+c)t$ and $\delta = 4\ln(1+\sqrt{2})$.

We feel that the width and the amplitude are significant parameters of the solitary wave and hence our first classification should be in terms of them. We note that the amplitude as a function of c is determined by the values of the coefficients of the nonlinear terms in the PDE i.e. a_1, a_2. Furthermore, the requirement of positive amplitudes means that the range of speeds is also restricted by these coefficients. On the other hand, the width is entirely a function of the linear coefficients a_3, a_4, a_5, a_6 and the requirement that the width is a real function of c may introduce further constraints on the allowed values of the speeds.

Before we classify the generic forms in terms of these parameters we detail the special cases of the KdV and RLW.

9.2 The KdV and RLW solitary waves

The data for the KdV: $(a_1, 0, a_3, 0, 0, 0)$ and the RLW: $(a_1, 0, 0, a_4, 0, 0)$, where $a_1, a_3, (-a_4) \in \mathbb{R}^+$ are as follows:

	Range of speeds	A(c)	W(c)	A(w)
KdV:	$(1, \infty)$	$\frac{3c}{a_1}$	$\delta\sqrt{\frac{a_3}{c}}$	$3\delta^2 \frac{a_3}{a_1} \frac{1}{w^2}$
RLW:	$(1, \infty)$	$\frac{3c}{a_1}$	$\delta\sqrt{\frac{\vert a_4\vert(1+c)}{c}}$	$\frac{3\delta^2\vert a_4\vert}{a_1\left(w^2-\delta^2\vert a_4\vert\right)}$

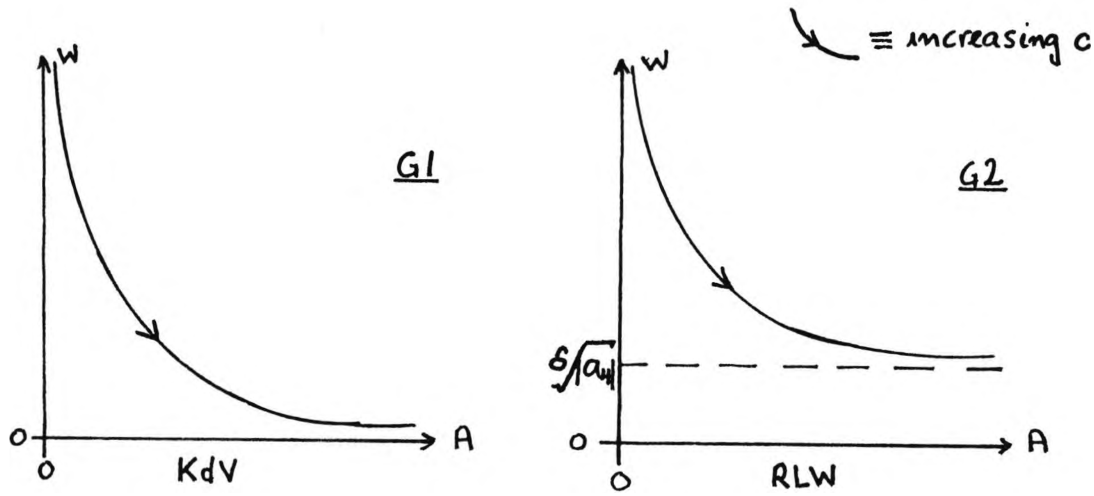


Fig 1: Graphs 1, 2

If the amplitude and width parameters are significant then the two graphs given above would lead us to make the following predictions for the RLW:

(i) The small-amplitude large-width solitary waves would act more like solitons than the large-amplitude, small-width solitary waves (see [1], [2] for verification of this prediction).

(ii) We would expect the range of amplitudes over which the solitary waves are soliton-like to increase as $|a_4|$ decreases.

Note: Soliton-like means that if two solitary waves are made to collide numerically, then the solitary waves would emerge with a tail that is extremely small, i.e., the interaction would be almost elastic.

9.3 Amplitude-speed relationship

Since the KdV and the RLW are both unidirectional we restrict our considerations to subclasses of the general class which have this property by setting $\bar{c} = 1 + c > 0$. The requirement that the amplitude

$$A(c) = \frac{3c}{\left(a_1 - a_2\bar{c}\right)}$$

be positive leads to the following classification of the speed ranges in terms of a_1, a_2 where $\bar{c} \in (\alpha, \beta)$ is indicated by (α, β):

$a_1 = 0$: $(1, \infty)$ — 0 — $(0, 1)$ → a_2

$a_2 = 0$: $(0, 1)$ — 0 — $(1, \infty)$ → a_1

$a_1 = a_2 \neq 0$: $\xrightarrow[0 \qquad\qquad a_1 = a_2]{(0,\ \infty)}$ $A = -\dfrac{1}{a_2} =$

constant.

For the case $a_1 \neq a_2 \neq 0$ the range of speeds are displayed in the (a_1, a_2) plane below:

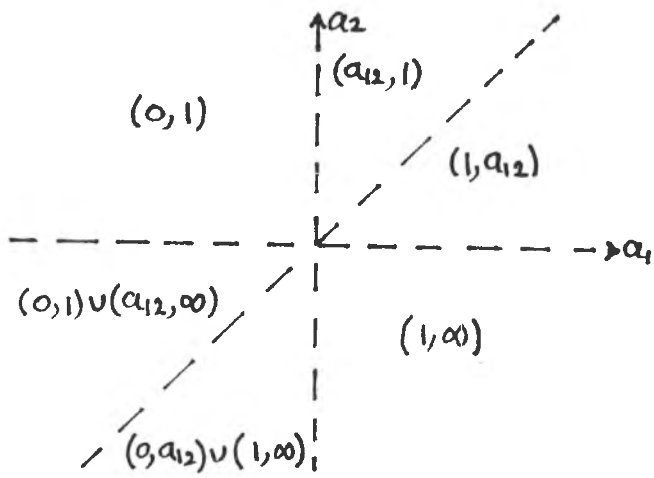


Fig. 2 $a_{12} = a_1/a_2$

From the figures above the generic form can be recognised as: unbounded above, bounded above and split range.

In proceeding with our classification we have to make a selection from these generic forms and we do this as follows:

(i) We require the amplitude to be a nonconstant function of c. This eliminates the case $a_1 = a_2$.

(ii) We require the equation to be in a neighbourhood of the KdV or RLW which eliminates the sectors $a_2 < a_1 \le 0$ and $0 \le a_1 < a_2$.

(iii) We require the range of speeds to be either unbounded above or have the upper bound a_1/a_2. The latter is chosen since it becomes large as $a_2 \to 0$. This eliminates the sector $a_1 < 0 < a_2$. The amplitude-speed curves for the allowed sectors are sketched below:

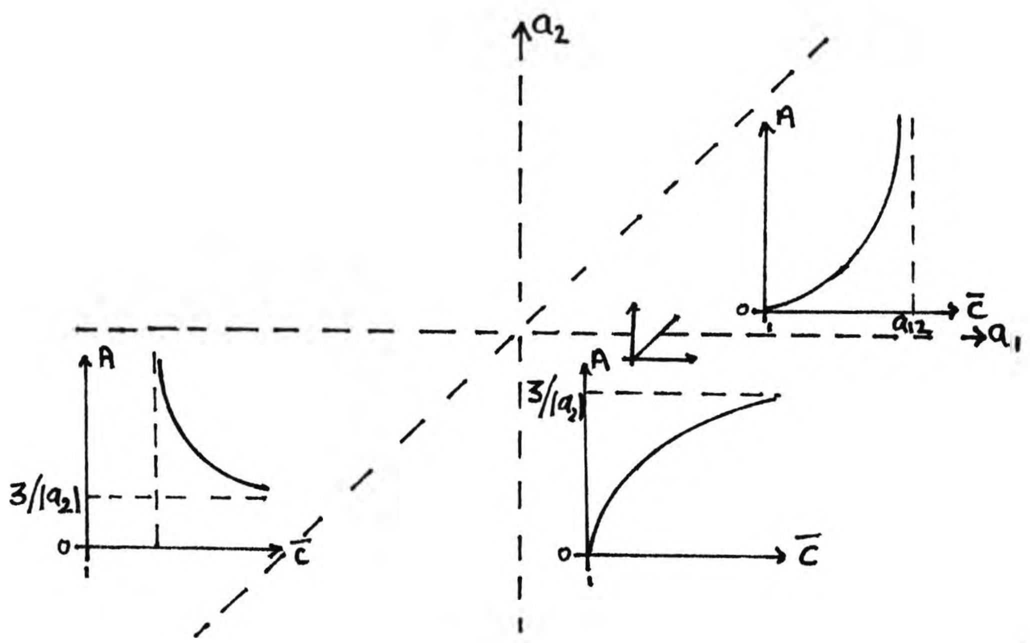


Fig. 3

9.4 Width-speed relationship

We showed earlier that the width-speed relationship is given by $w(c) = \delta\sqrt{\beta}$ where δ and β have been defined. We know from the previous chapter, Theorem 8.3, that the general class of equations (7.1) may be reduced to four equivalence classes. We now look at each class in turn and obtain the ranges of speeds for which the width is real.

(a) KdV class: $(a_1, a_2, a_3, 0, 0, 0)$

$$w(c) = \delta\sqrt{a_3/c}$$

$$\underline{(0,\ 1)} \quad \underset{0}{\cdot} \quad \underline{(1,\ \infty)} \rightarrow a_3 \ .$$

(b) RLW class: $(a_1, a_2, 0, a_4, 0, 0)$

$$w(c) = \delta\sqrt{\frac{-a_4(1+c)}{c}}$$

$$\underline{(1,\ \infty)} \quad \underset{0}{\cdot} \quad \underline{(0,\ 1)} \rightarrow a_4$$

(c) W_{54} class: $(a_1, a_2, 0, a_4, a_5, 0)$

$$w(c) = \delta\sqrt{\frac{-a_4(1+c)+a_5(1+c)^2}{c}} \rightarrow \frac{a_4 - a_5(1+c)}{c} < 0 \ .$$

The ranges of speeds are displayed in the

(a_4, a_5)-plane below:

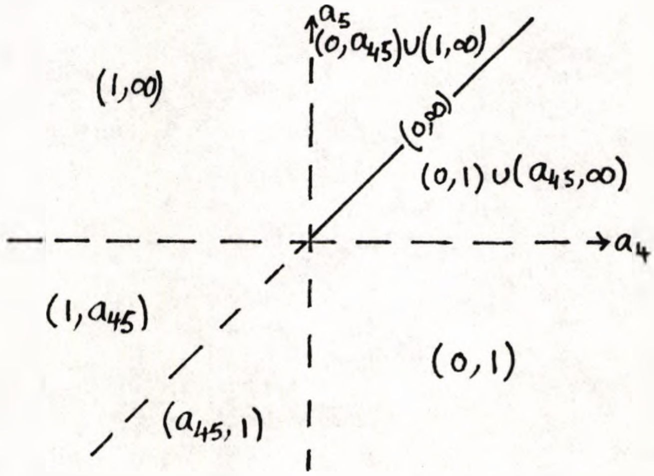


Fig. 4 $a_{45} = a_4/a_5$

The width-speed curves for those sectors which obey the restrictions (ii) and (iii) of 9.3 with (a_1, a_2) replaced by (a_4, a_5) are sketched below:

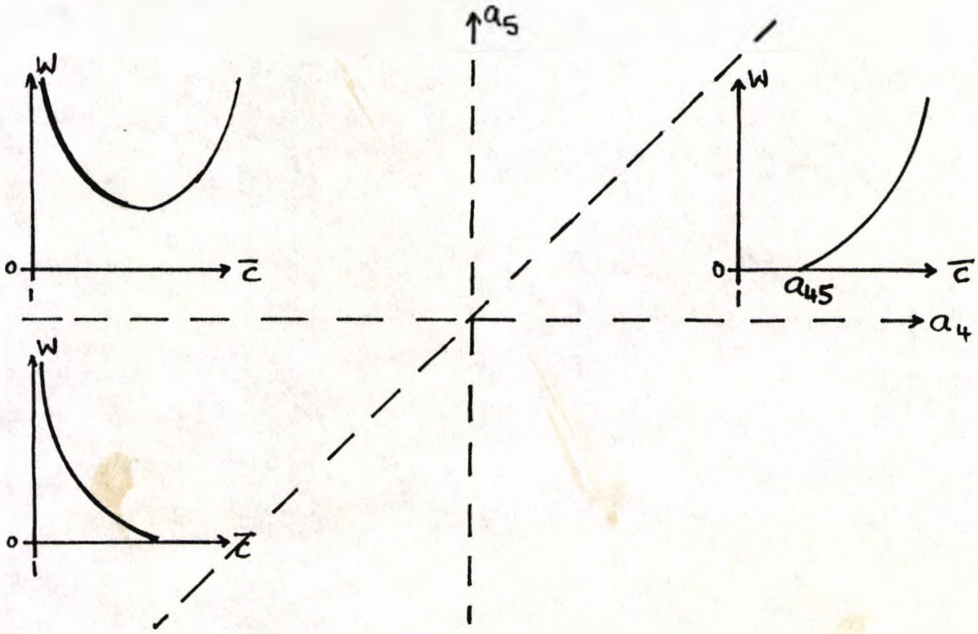


Fig 5

(d) W_{53} class: $(a_1, a_2, a_3, 0, a_5, 0)$

$$w(c) = \delta\sqrt{\frac{a_3+a_5(1+c)^2}{c}}$$

The ranges of speeds are displayed in the (a_3, a_5)-plane below:

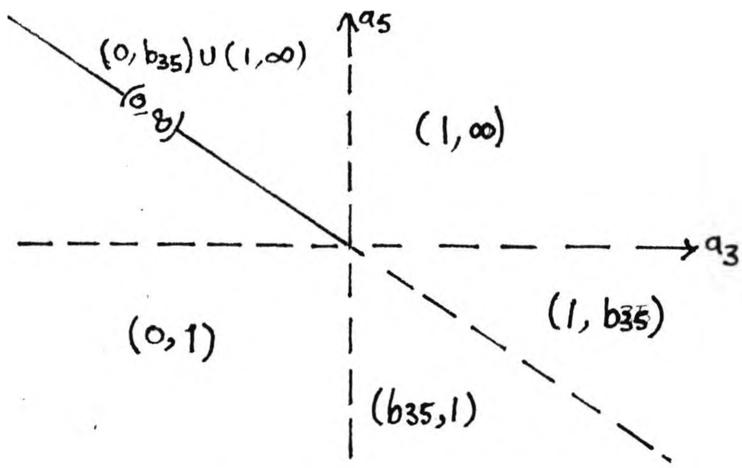


Fig 6. $b_{35} = \sqrt{|a_3/a_5|}$

The width-speed curves for those sectors which obey the restrictions (ii) and (iii) of 9.3 with (a_1, a_2) replaced by (a_3, a_5) are sketched below:

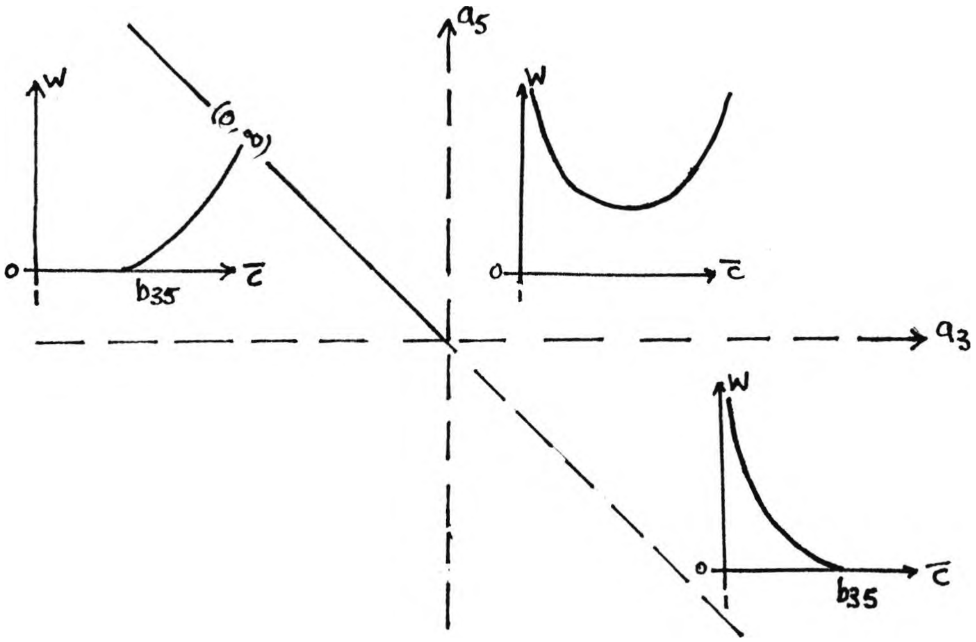


Fig. 7

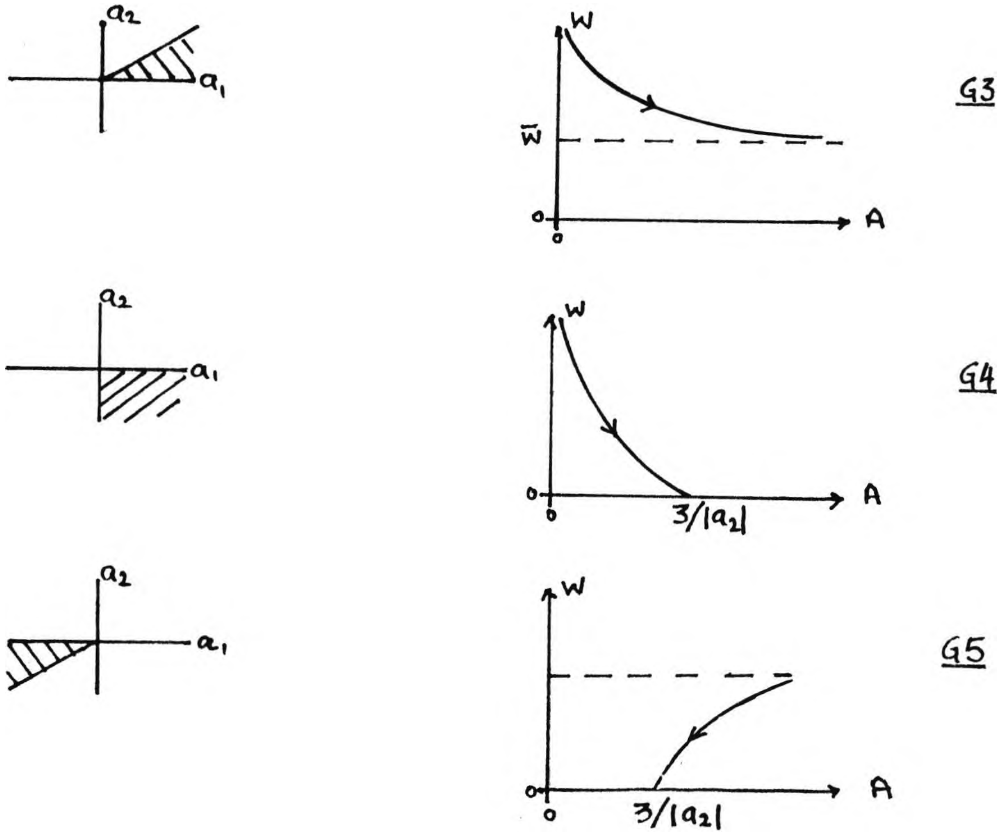
9.5 Amplitude-width relationship

We now combine the results of the last two sections to obtain generic forms in each class and plot their amplitude-width relationships.

(a) KdV class: $(a_1, a_2, a_3, 0, 0, 0)$, $a_3 > 0$

In this case $$w^2 = \frac{a_3\delta^2(3+a_2A)}{(a_1-a_2)A} \qquad (9.5.1)$$

and the amplitude-width curves for the various sectors are sketched below:

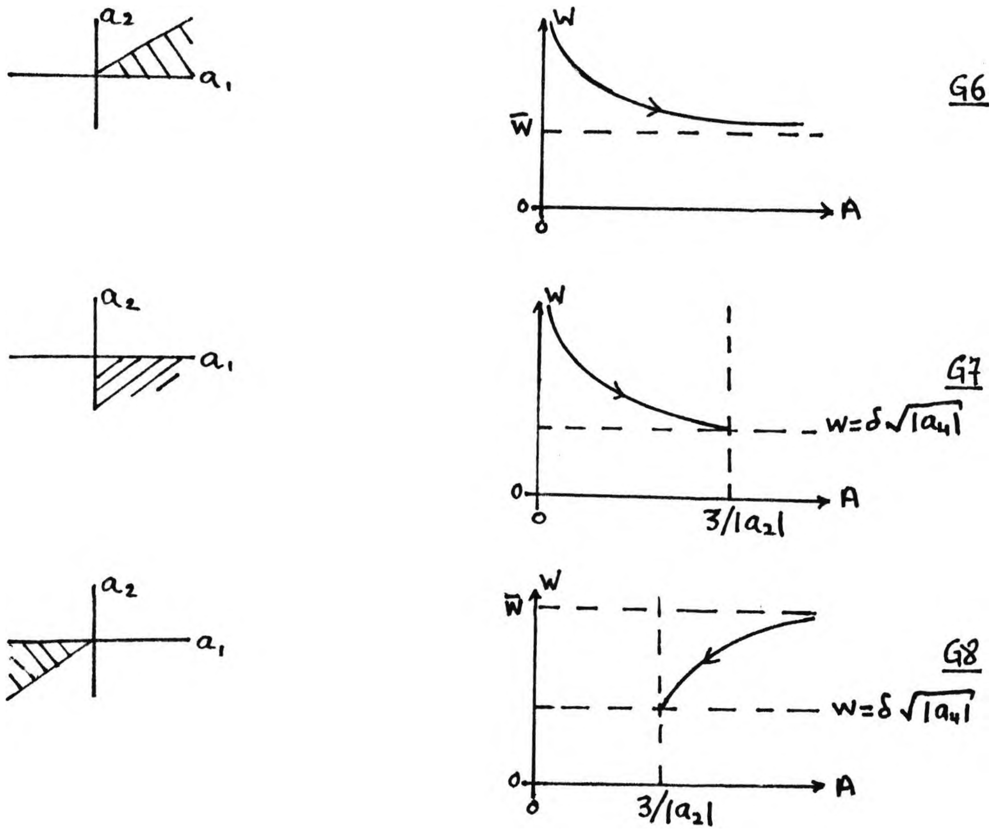


G ≡ Graph. $\bar{w} = \delta\sqrt{a_3/(a_{12}-1)}$

(b) RLW class: $(a_1, a_2, 0, a_4, 0, 0)$, $a_4 < 0$

Here we have
$$w^2 = \frac{|a_4|\delta^2(3+a_1A)}{(a_1-a_2)A} \qquad (9.5.2)$$

The amplitude-width curves for the various sectors are sketched below:



$$\bar{w} = \delta\sqrt{|a_4|/(a_{12}-1)}$$

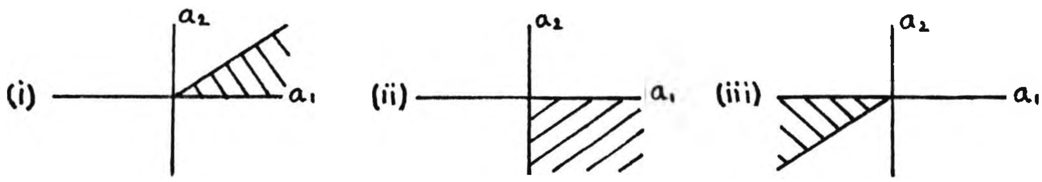
(c) W_{34} class: $(a_1,\ a_2,\ 0,\ a_4,\ a_5,\ 0)$

For this class of equations the range of speeds given in Fig. 2 are further restricted because of the additional requirement on the coefficients $(a_4,\ a_5)$. The range of speeds in this case are given by the intersection of the ranges in Fig. 2 and Fig. 4. We note that for each sector in Fig. 2 there are three corresponding sectors in Fig. 4.

The amplitude-width function is given by

$$w^2 = \delta^2 \left\{ \frac{(a_1 A+3)\left[(a_1 a_5 - a_2 a_4)A + 3(a_5 - a_4)\right]}{(a_1 - a_2)A(3 + a_2 A)} \right\} \qquad (9.5.3)$$

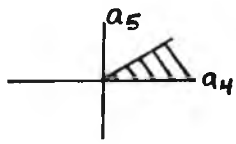
and the amplitude-width curves are sketched for the following sectors.

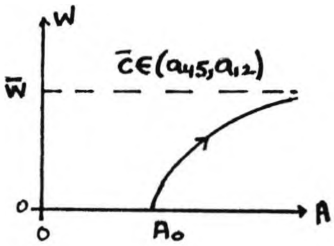


In the following graphs: $\bar{w} = \delta\sqrt{\dfrac{a_{12}(a_5 a_{12} - a_4)}{a_{12} - 1}}$,

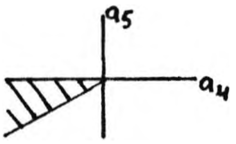
$$A_o = \frac{3(a_{45} - 1)}{a_2(a_{12} - a_{45})} \quad .$$

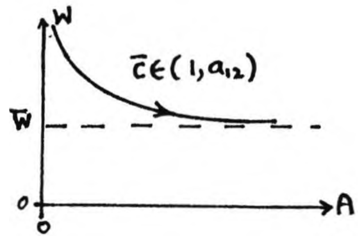
In (a_1 , a_2)-sector (i) we have the following graphs:



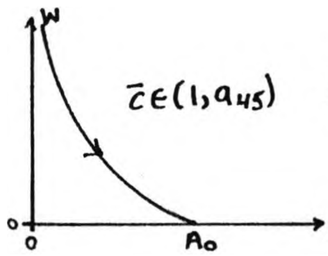


G9

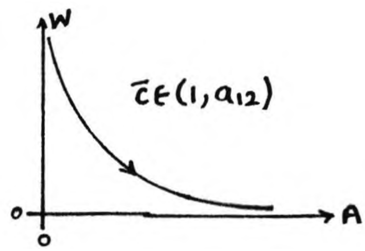




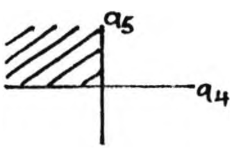
G10



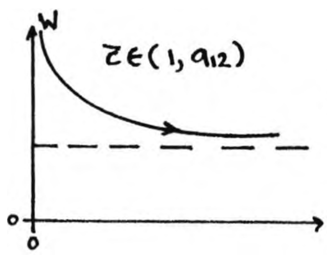
G11



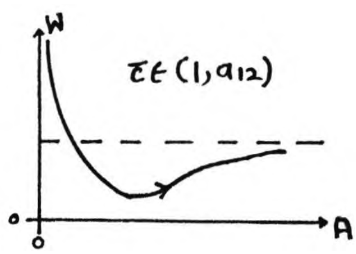
G12



G13 $1+\sqrt{1-a_{45}} > a_{12}$ G14 $1+\sqrt{1-a_{45}} < a_{12}$

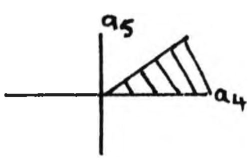


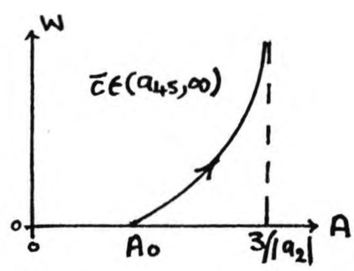
G13



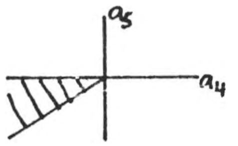
G14

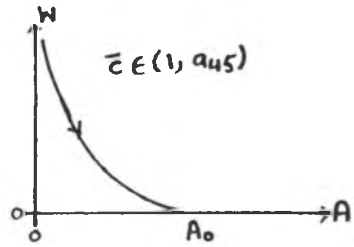
In (a_1 , a_2)-sector (ii) we have the following graphs:



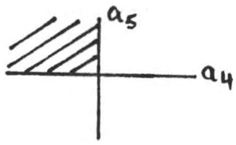


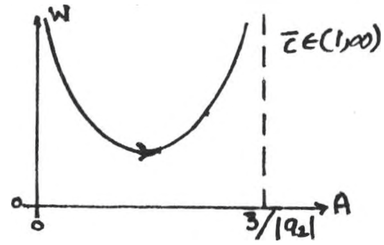
G15





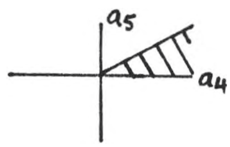
G16

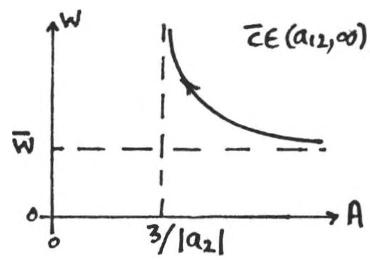




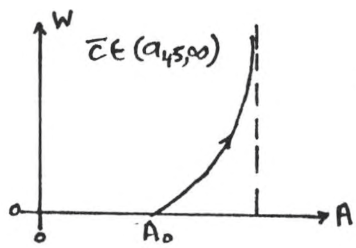
G17

In (a_1, a_2)-sector (iii) we have the following graphs:

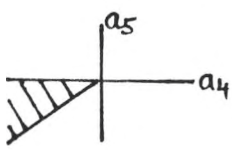


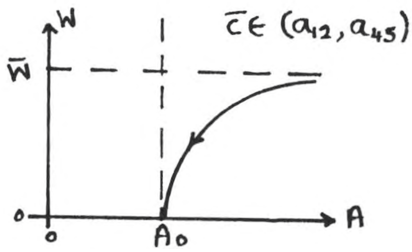


G18

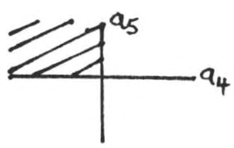


G19

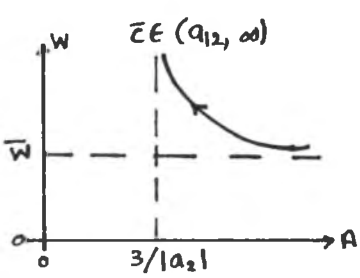




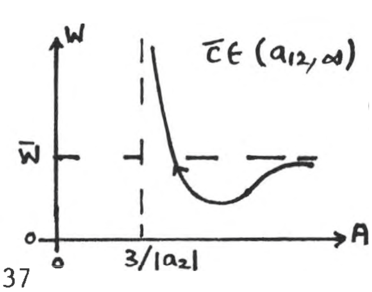
G20



G21 $1+\sqrt{1-a_{45}} < a_{12}$ G22 $1+\sqrt{1-a_{45}} > a_{12}$



G21



G22

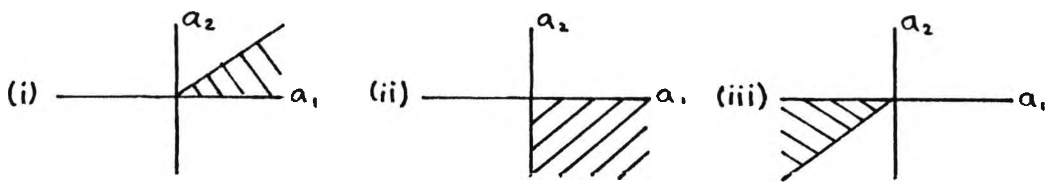
(d) W_{53} class: $(a_1, a_2, a_3, 0, a_5, 0)$

For this class the range of speeds are given by the intersection of the ranges in Fig. 2 and Fig. 6. Again for each sector in Fig. 2 there are three corresponding sectors in Fig. 6.

The amplitude-width function is given by

$$w^2 = \delta^2 \left\{ \frac{a_3(a_2A+3)^2+a_5(a_1A+3)^2}{(a_1-a_2)A(2+a_2A)} \right\} \qquad (9.5.4)$$

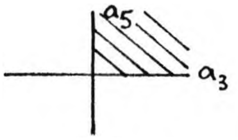
and the amplitude-width curves are sketched for the following sectors:



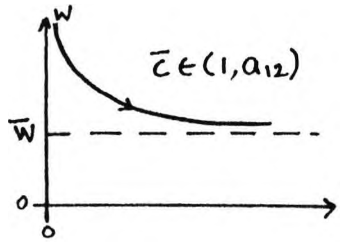
In the following graphs $\bar{w} = \delta \sqrt{\dfrac{a_3+a_5a_{12}^2}{a_{12}-1}}$,

$$A_0 = \frac{3(b_{35}-1)}{a_1-a_2b_{35}} .$$

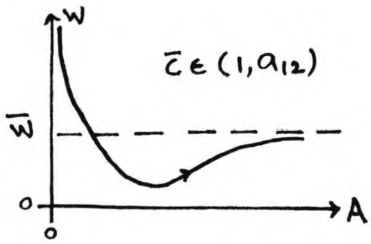
In (a_1, a_2)-sector (i) we have the following graphs:



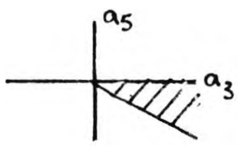
G23 $1+\sqrt{1+a_{35}} > a_{12}$ G24 $1+\sqrt{1+a_{35}} < a_{12}$

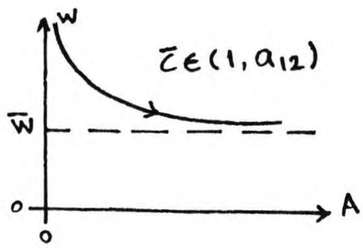


G23

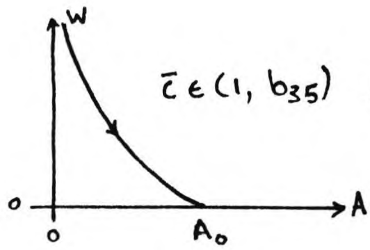


G24

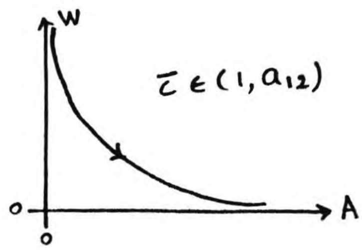




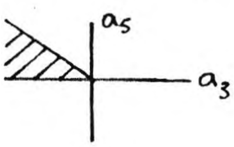
G25

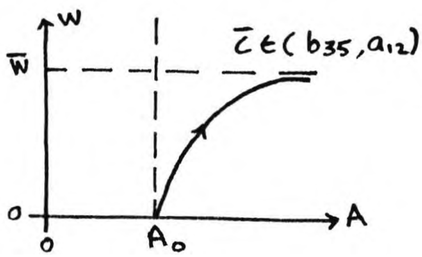


G26



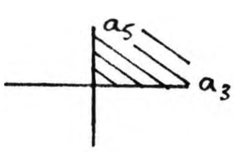
G27

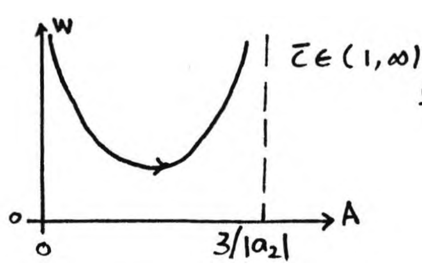




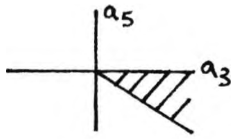
G28

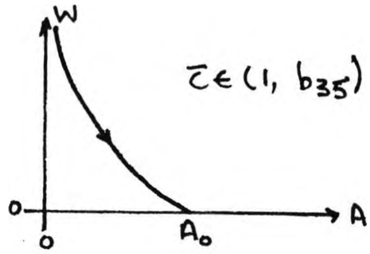
In (a_1, a_2)-sector (ii) we have the following graphs:



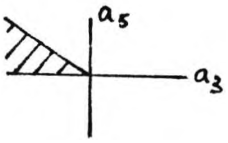


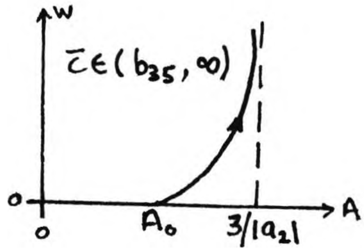
G29





G30

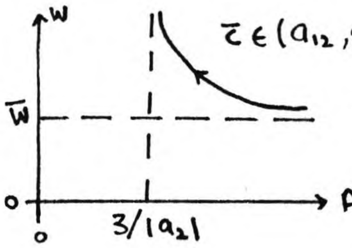




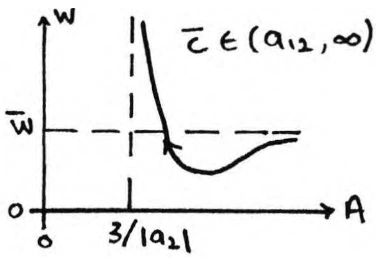
G31

In (a_1, a_2)-sector (iii) we have the following graphs:

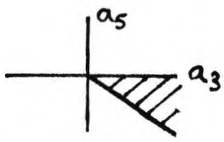
G32 $1+\sqrt{1+a_{35}} < a_{12}$ G33 $1+\sqrt{1+a_{35}} > a_{12}$

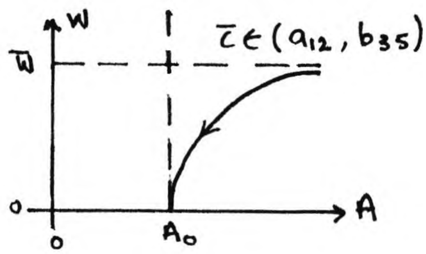


G32

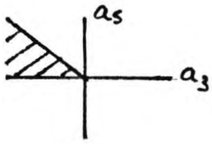


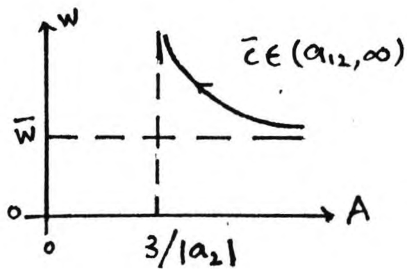
G33



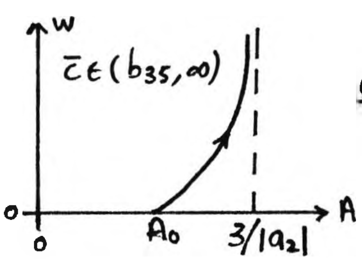


G34





G35



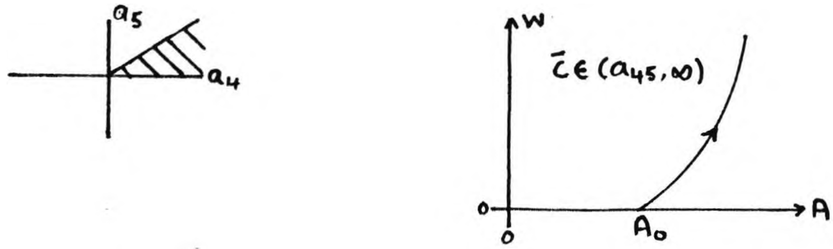
G36

Finally we give the amplitude-width curves for the W_{54} and W_{53} classes when $a_2 = 0$. Note that in these cases $a_1 > 0$.

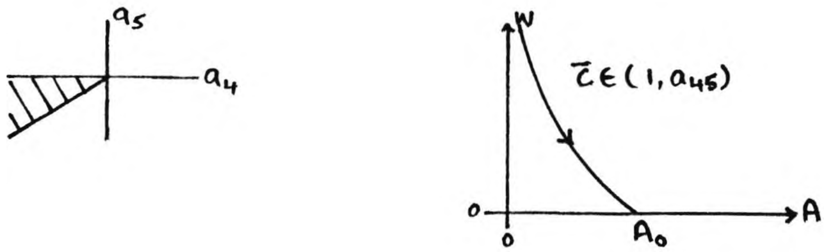
<u>W_{54} class: $(a_1, 0, 0, a_4, a_5, 0)$</u>

The amplitude-width function is obtained by putting $a_2 = 0$ in (9.5.3) and the amplitude-width curves are sketched below. Note that

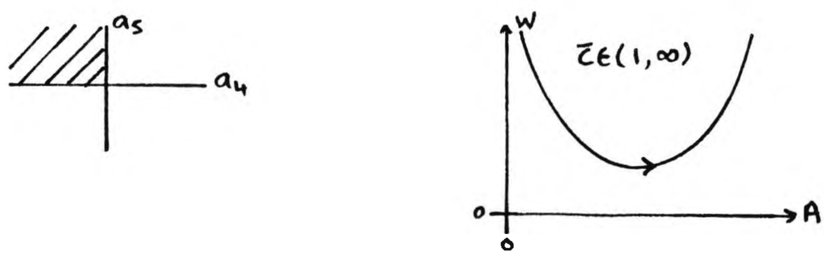
$$A_0 = \frac{3(a_{45}-1)}{a_1}$$



G37



G38



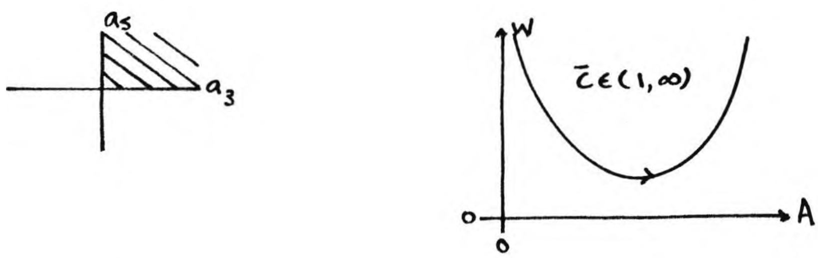
G39

W_{53} class: $(a_1, 0, a_3, 0, a_5)$

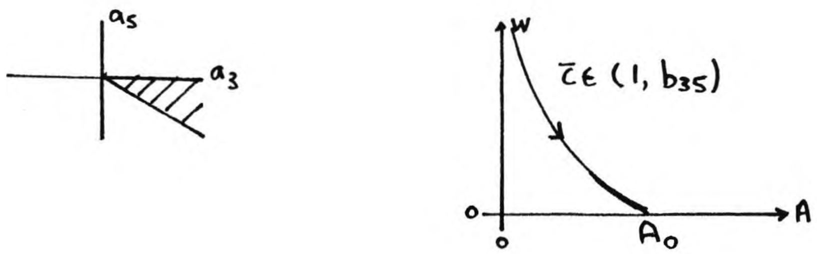
The amplitude-width function is obtained by putting $a_2 = 0$ in (9.5.4) and the amplitude-width curves are sketched below:

Note that

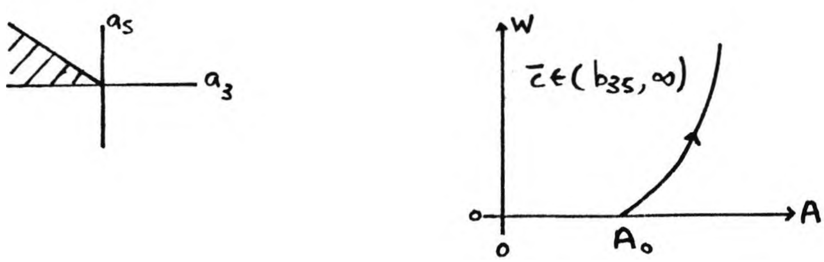
$$A_0 = \frac{3(b_{35}-1)}{a_1}$$



G40



G41



G42

9.6 Classification of solitary waves

Before we comment on the data given in the graphs above we first classify them into generic types. We concentrate on the width-amplitude graphs since, as we have mentioned before, we think that these are significant parameters. In drawing these graphs we laid down certain selection criteria which are specified on Page 227, 228. Using these criteria we end up with a total of 42 graphs which form 12 generic equivalence classes. Representatives of these equivalance classes are G1, G2, G4, G5, G7, G8, G14, G15, G17, G18, G22 and G37. We now further refine the class of admissible solutions by imposing the following two conditions. Firstly, we reject all double-valued graphs i.e. those equations which have solitary waves with two different amplitudes for the same width. This is done because we do not understand the significance of this property and we have left it to be investigated later. Secondly, we require that there must be a graph in the limits $a_2 \longrightarrow 0$ and $a_5 \longrightarrow 0$. Using these criteria eliminates 8 generic types: G14, G17, G22 and G5, G8, G15, G18, G37 respectively. The limits of the remaining graphs are either KdV or RLW-like. We note that we eventually end up with the KdV since the RLW graph tends to that of the KdV as $|a_4|$ gets smaller. We could not have imposed this condition earlier because we were dealing separately with the amplitude-speed and the width-speed graphs. The remaining 4 generic types represent 18 of the original 42 equations. We summarize this in the following table.

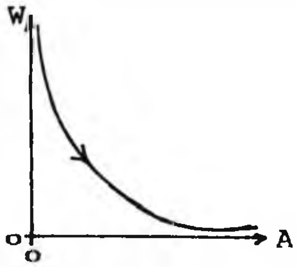
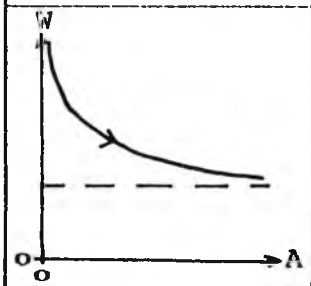
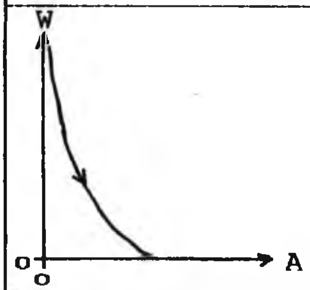
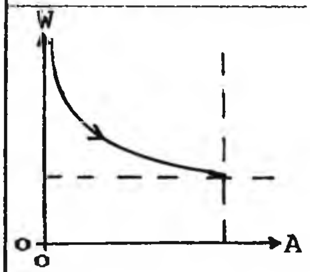
	KdV	RLW	W_{54}	W_{53}
	G1		G12	G27
	G3	G2, G6	G10, G13	G23, G25
	G4		G11, G16 G38	G26, G30 G41
		G7		

Table 9.1

We now introduce the quantity Aw^2 which seems to have some significance. The value for the equivalence classes of the general equation is as follows:

KdV: $$\frac{\delta^2 a_3}{(a_1 - a_2)}(3+|a_2|A)$$

RLW: $$\frac{\delta^2 |a_4|}{(a_1 - a_2)}(3+a_1 A)$$

W_{54}: $$\delta^2 \left\{ \frac{(a_1 A+3)\left[(a_1 a_5 - a_2 a_4)A + 3(a_5 - a_4)\right]}{(a_1 - a_2)(3+a_2 A)} \right\}$$

W_{53}: $$\delta^2 \left\{ \frac{a_3(a_2 A+3)^2 + a_5(a_1 A+3)^2}{(a_1 - a_2)(3+a_2 A)} \right\}$$

9.7 Summary of structural properties of admissible equations

The graphical analysis of the solitary waves presented in the previous sections and the criteria used for selecting admissible forms leads to the following observations of changes in the allowed graphs as functions of the parameters a_2 and a_5. These correspond to perturbations of the nonlinear and linear terms respectively of the KdV and RLW.

(1) Global restriction: $a_1 > 0$, $a_3 \geq 0$, $a_4 \leq 0$

(2) KDV and RLW classes ($a_5 = 0$):

$a_2 < 0$: puts an upper bound on the amplitude which leads to a singular limit.

$a_2 > 0$: speeds bounded above introduces (KdV) or changes (RLW) lower bound on width.

(3) W_{54} and W_{53} classes ($a_5 \neq 0$)

(a) Speeds bounded above for any value of a_2.

(b) $a_5 < 0$: $a_2 \leq 0$: amplitude bounded above with a singular limit

$a_2 > 0$: three types of graphs occur which, as a_2 increases, appear in the order singular, KdV, RLW.

(c) $a_5 > 0$: $a_2 \leq 0$: no allowed graphs

$a_2 > 0$: RLW type only, i.e. lower bound on width.

Implication: W_{54} and W_{53} are not distinguished by their graphs.

(d) Two special cases occur under (b) when $a_{12} = a_{45}$ and $a_{12} = b_{35}$. Both those conditions lead to the KdV graph G1. However, in these cases

$Aw^2 \neq$ constant, and so the quantity Aw^2 distinguishes them from the KdV itself.

(4) If we consider equations in which $a_3 + a_5 > 0$ or $|a_4| + a_5 > 0$ then, for small wavenumbers k (i.e. the approximation used in deriving the KdV and the RLW [1], the linear dispersion relation is given by

$$w(k) = k - \left[a_3 + |a_4| - a_5\right] k^3$$

which is in fact the dispersion relation of the KdV. Thus, the linear dispersion relation does not distinguish between these equations. On the other hand the function Aw^2 does, which reinforces the significance of this quantity in the classification.

We conclude with the following theorem:

<u>Theorem 9.1</u>

In the class of admissible PDEs the KDV is unique in having a family of solitary waves with the properties

(a) speed unbounded above,

(b) Aw^2 = constant for $0 < A < \infty$ □

CHAPTER TEN

CONCLUDING REMARKS

This thesis is the third in a series of studies on a comparative analysis of the Korteweg-deVries equation and its homologues. The general discussion is centred on a class of third-order semi-linear equations with quadratic nonlinearities of which the KdV and RLW are members. All the equations considered have stable solitary wave solutions. However, it is well-known that the KdV has a number of exceptional properties which do not appear to be shared by equations such as the RLW which have been proposed as alternative models. This raises the question as to whether the KdV is a unique equation in this class and whether we can develop criteria for understanding this uniqueness.

In the first study, Abbas looked at the class from the point of view prevailing at the time that the existence of solitary waves was due to a balance between nonlinearity and dispersion which, in special cases like the KdV, would also produce solitons. He found that, for a fixed nonlinearity, stable solitary waves exist for a wide variety of dispersion relations including formally nondispersive equations. Thus, he concluded that the idea of balance beween nonlinearity and dispersion is not a useful one for explaining the appearance of solitary waves and hence, by implication, for understanding the existence of solitons.

Since the physical hypothesis of dispersion is not a valid indicator of properties of the nonlinear equation, the discussion shifts to a consideration of the specific properties of this class of equations. This required a much more rigorous mathematical approach to these equations and, in particular, to the question of their validity as genuine evolution equations. The foundations for this were established in the second study of the series in which El-Sherbiny investigated the well-posedness of the general class of equations and also the existence of conservation laws. The result of this work showed that the equations split quite naturally into four equivalence classes and theorems relating to well-posedness were produced for each. Conservation laws were derived for these equations and it was established that, with the exception of the KdV which has an infinite number, all the equations have at least two and at most three conservation laws. Furthermore, El-Sherbiny also showed that the characteristic equation of the KdV has a triple root, i.e., although the PDE is of third order, it has only one characteristic instead of three. As a consequence, El-Sherbiny concluded that the existence of an infinite number of conservation laws is related to the occurrence of degenerate characteristics. Once again, these facts demonstrated uniqueness of the KdV in the class.

In this thesis we have continued the study of this class of equations by looking at the analytic structure of their solutions. Specifically, we have considered the similarity solutions of reduced ODEs as well as general solutions of the PDE itself. We have also extended the work on the solitary waves, initiated by Abbas and completed by El-Sherbiny, to a more general and, we feel, a more useful classification.

Using one-parameter Lie groups we obtained all the corresponding local symmetries of the equations and classified them in terms of the infinitesimal generators of their groups. These generators were then used to reduce the PDEs to the corresponding similarity ODEs. The total number of third order ODEs obtained was 10 of which 6 were irreducibly third order and the rest were or could be integrated to second order. We also proved that if $a_1 \neq a_2 \neq 0$ then the only reduction is to an ODE with self-similar solutions, i.e. travelling waves.

The analytic structures of the solutions of the second order equations were obtained from the existing Painlevé classification, while those of the third order were studied using singular point analysis. We first showed that all the ODEs have a class of analytic general solutions. In those cases in which the nonlinear from ff' is present there was a second class of general solutions which are meromorphic, but could be either single-valued or multiple-valued. Of these general solutions, only the travelling waves and the solutions of all reductions of the KdV equation itself are meromorphic as well as being single-valued. All the other equations have solutions which are multiple-valued meromorphic maps with logarithmic branch points. We also noted that solutions which were free from branch points come from equations in which ff' was the only nonlinear term and were in addition free from f'' term. These properties enabled us to initiate a classification of third order nonlinear ODEs.

The poles in the meromorphic solutions were found, in general, to be of order two. However, the general KdV equation with nonlinearities $a_1 u(u_x + u_t)$ also has solutions with a pole of order three and essential singularities. The KdV equation is thus unique in that class in that all its similarity ODEs have single-valued meromorphic solutions i.e., are Painlevé-type. However, for many other equations in the class there was only one reduced ODE, i.e. the self-similar ODE with travelling wave solutions. Hence, on the basis of this test, we must either conclude that the PDE is of Painlevé type or, what is more likely, that the one-parameter Lie groups do not give the symmetries of these equations, i.e. they may have other similarity reductions.

We next looked directly at the PDEs themselves and showed that the only class of equations which possessed meromorphic general solutions is the KdV. All the others had logarithmic branch point singularities in their expansions. We verified that the similarity ODEs obtained above are not sufficient to cover all general solutions of the corresponding PDE. This was done by showing that the subclass of PDEs which only had one reduction, i.e. to self-similar ODEs had general solutions which included logarithmic branch points. Consequently, they are not of Painlevé type and hence the only equation which is of this type both in terms of the ODEs as well as the PDEs is the KdV (modulo essential singularities).

Since soliton solutions where they exist are special single-valued meromorphic functions, we then tried to identify equations with restricted soliton solutions, i.e. as special cases of the general solution. We did not find any and, in particular in the case of the RLW, we were able to prove rigorously, using arguments based on pole motions, that it does not have any soliton solutions.

Since there are no theorems linking the analytic structure of general solutions to the existence of solitons we decided that another approach to this question was needed. Our idea is to analyse directly the functional form of the solitary wave to establish criteria for it to be a soliton. We conjectured that the width and the amplitude are significant parameters for solitary waves and classified them graphically using the KdV and RLW as standards, since the KdV has soliton solutions and we have established that the RLW has not. By imposing reasonable selection rules we were able to reduce the width-amplitude graphs to 4 generic forms and we introduced the quantity Aw^2 as being a significant functional of the solitary wave. The importance of this functional lies in the fact that firstly, it is a constant for the family of solitary waves of the KdV and, secondly, we were able to use it to distinguish between the KdV and another equation which could not be distinguished by the linear dispersion relation. This functional may explain the existence of soliton solutions in at least the KdV family of equations (i.e. same dispersion relation), although we have yet to check it on higher order nonlinear terms, e.g. the MKdV with u^2u_x. If it is a useful functional then it is more straightforward and easier to apply than the analytic structure analysis.

We come now to the question of future research. This thesis has further established the uniqueness of the KdV in the class considered. However, while we have shown that every equation in the class, apart from the KdV, is not of Painlevé type, we have not been able to prove that the KdV is. This is because of the difficulty of showing whether the general solution of a PDE has movable essential singularities or not. This restriction has prevented our analysis of the analytic structure from being complete and is therefore still an open question. Our method of proving that the RLW does not have soliton solutions does not easily extend to the other equations in the class. Nevertheless, we feel that it is a good method and has potential for development to more general situations. Finally, our preliminary analysis on the width-amplitude behaviour of the solitary waves produced some interesting results and warrants further development. In particular, we believe that the functional Aw^z is in effect a "nonlinear dispersion" relation, for the equation which is significant in explaining the existence of solitons. The next stage in understanding this functional would be to conduct a series of numerical experiments on the interaction of solitary waves in regions where Aw^z is very slowly varying.

We are now working on these developments as well as trying to attain the objectives set out in this thesis.

APPENDIX A

In this appendix we present a proof of Theorem 3.2. Before doing so, following ARS[2], we establish results which we shall need for the proof.

Shifting the origin, we rewrite (3.1.3) as

$$K(x,y) = F(x,y) + \int_0^\infty K(x,x+z)F(x+z,y)dz \qquad (A.1)$$

Now $\frac{\partial^n}{\partial x^n} \int_0^\infty K(x,x+z)F(x+z,y)dz =$

$$\int_0^\infty \left[\partial_1^n K(x,x+z)\right] F(x+z,y)dz + \Psi_n, \quad n = 1,\ 2,\ 3. \qquad (A.2a)$$

with $\Psi_1 = -K(x,x)F(x,y)$ (A.2b)

$$\Psi_2 = -\frac{\partial}{\partial x}\left[K(x,x)F(x,y)\right] - F(x,y)\left[\partial_1 K(x,x+z)\right]_{z=0} \qquad (A.2c)$$

$$\Psi_3 = -\frac{\partial^2}{\partial x^2}\left[K(x,x)F(x,y)\right] - \frac{\partial}{\partial x}\left[F(x,y)\partial_1 K(x,x+z)\right]_{z=0} - F(x,y)(\partial_1^2 K(x,x+z))_{z=0} \qquad (A.2d)$$

where ∂_1 and ∂_2 are used to denote derivatives with respect to the first and second argument, respectively.

Similarly, integrating by parts we have

$$\int_o^\infty K(x,x+z)\partial_1^n F(x+z,y)dz =$$

$$(-1)^n \int_o^\infty \left[\partial_2^n K(x, x+z)\right] F(x+z,y)dz + \Phi_n, \quad n = 1, 2, 3.$$

(A.3a)

with $\Phi_1 = - K(x, x)F(x, y)$ (A.3b)

$$\Phi_2 = - K(x,x)\frac{\partial}{\partial x} F(x,y) + \left[\partial_2 K(x,x+z)\right]_{z=o} F(x,y)$$

(A.3c)

$$\Phi_3 = - K(x,x)\frac{\partial^2}{\partial x^2} F(x,y) + \left[\partial_2 K(x,x+z)\right]_{z=o}\frac{\partial}{\partial x} F(x,y)$$

$$- \left[\partial_2^2 K(x,x+z)\right]_{z=o} F(x,y) \quad \text{(A.3d)}$$

Note that $\frac{d}{dx} K(x,x) = (\partial_1 K(x,x+z)+\partial_2 K(x,x+z))_{z=o}$

Then we obtain

$$\Psi_1 - \Phi_1 = 0 \quad \text{(A.4a)}$$

$$\Psi_2 - \Phi_2 = - 2F(x,y)\frac{d}{dx} K(x,x) \quad \text{(A.4b)}$$

$$\Psi_3 - \Phi_3 = -3\frac{\partial}{\partial x}F(x,y)\frac{d}{dx}K(x,x)$$

$$- 3F(x,y)\left[(\partial_1^2+\partial_1\partial_2)K(x,x+z)\right]_{z=0} \qquad (A.4c)$$

We now present the proof.

Lemma. $(I - A_x)$ is invertible. □

Proof. See Ref. 2.

Proof of Theorem.

We first construct L_1K by operating on (A.1) by L_1. Thus,

$$L_1K = L_1F + (\partial_x^2 - \partial_y^2)\int_0^\infty K(x,x+z)F(z+x,y)dz$$

$$= \int_0^\infty F(z+x,y)\partial_1^2 K(x,x+z)dz + \Psi_2 - \int_0^\infty K(x,x+z)F_{yy}dz$$

from (A.2a) and $L_1F = 0$.

Using $F_{x+z,x+z} = F_{yy}$, from $L_1F = 0$, (A.3a) and (A.4b) gives

$$L_1 K = (\partial_x^2 - \partial_y^2)K =$$

$$\int_0^\infty F(z+x,y)(\partial_1^2 - \partial_2^2)K(x,x+z)dz - 2F(x,y)\frac{d}{dx} K(x,x)$$

(A.5)

Using (3.2.2) the first term on the right hand side can be written as

$$A_x(\partial_x^2 - \partial_y^2)K(x,y)$$

and using (3.2.3) the second term on the right hand side becomes

$$2(I - A_x)K(x,y)\frac{d}{dx} K(x,x),$$

Hence from (A.5) we obtain

$$(I-A_x)\left\{(\partial_x^2-\partial_y^2)K(x,y) + 2\left[\frac{d}{dx}K(x,x)\right]K(x,y)\right\} = 0$$

(A.6)

and from the lemma on the invertibility of $(I - A_x)$ it follows that

$$L_1 K - M_1(K) \equiv (\partial_x^2 - \partial_y^2)K(x,y)$$

$$+ 2\left[\frac{d}{dx} K(x,x)\right]K(x,y) = 0 \qquad (A.7)$$

We now turn to determining the second of the differential equations (3.2.7).

Operating on (A.1) by L_2 gives

$$(\partial_t + (\partial_x+\partial_y)^2 K(x,y)$$

$$= \left[\partial_t + (\partial_x+\partial_y)^3\right] \int_0^\infty K(x,x+z)F(z+x,y)dz$$

(A.8)

Suppressing the arguments, the right hand side gives

$$\int_0^\infty K_t F dz + \int_0^\infty KF_t dz + (\partial_x + \partial_y)^3 \int_0^\infty KFdz = 0$$

(A.9)

$$\rightarrow \int_0^\infty K_t F dz - \int_0^\infty K(\partial_1+\partial_2)^3 F dz + (\partial_x+\partial_y)^3 \int_0^\infty KFdz = 0$$

(A.10a)

If we write $\int_0^\infty K_t F dz + I = 0$ where $I = I_1 + I_2 + I_3$

then I_1 is given by

$$I_1 = \partial_x^3 \int_0^\infty KFdz - \int_0^\infty K\partial_1^3 F dz \qquad \text{(A.10b)}$$

from (A.3) and (A.4)

$$I_1 = \int_0^\infty (\partial_1^3 K + \partial_2^3 K) F dz + \Psi_3 - \Phi_3 \tag{A.10c}$$

I_2 is given by

$$I_2 = 3\partial_x^2 \partial_y \int_0^\infty KF dz - 3\int_0^\infty K\partial_1^2 \partial_2 F dz$$

$$= 3\int_0^\infty (\partial_1^2 K - \partial_2^2 K) F_y dz + 3\partial_y(\Psi_2 - \Phi_2)$$

$$= -3q(x)\int_0^\infty KF_y dz + 3\partial_y(\Psi_2 - \Phi_2)$$

$$= -3q(x)\partial_y(K(x,y) - F(x,y)) + 3\partial_y(\Psi_2 - \Phi_2) \tag{A.11}$$

where we have put $\int_0^\infty KF = K - F$ and $q(x) = 2\frac{d}{dx}[K(x,x)]$.

Finally for I_3 we have

$$I_3 = 3\partial_x \partial_y^2 \int_0^\infty KF dz - 3\int_0^\infty K\partial_1 \partial_2^2 F dz$$

$$= 3\partial_x \int_0^\infty KF_{yy} dz + 3\int_0^\infty K\partial_1 F_{yy} dz$$

$$= 3\int_0^\infty (\partial_1 K) F_{yy} dz + 3\Psi_{1yy} + 3\int_0^\infty (\partial_2 K) F_{yy} dz - 3\Phi_{1yy}$$

$$= 3 \int_0^\infty (\partial_1 K) F_{yy} dz + 3 \int_0^\infty (\partial_2 K) F_{yy} dz$$

$$= 3 \int_0^\infty \left[(\partial_1 + \partial_2) K \right] F_{x+z,x+z} dz \ .$$

Integrating by parts twice

$$= \left[3(K_{x,x+z} + K_{x+z,x+z})F - 3(K_x + K_{x+z})F_x \right]_{z=0}$$

$$+ 3 \int_0^\infty \left[(\partial_2^2 \partial_1 + \partial_1^3) K \right] F dz \ .$$

Summing the results and using $K_{x+z,x+z} = K_{xx} + qK$

$$I = I_1 + I_2 + I_3 = \int_0^\infty (\partial_1^3 K + \partial_2^3 K) F dz + \Psi_3 - \Phi_3 - 3q\partial_y (K-F)$$

$$+ 3\partial_y (\Psi_2 - \Phi_2) + \left[3(K_{x,x+z} + K_{x+z,x+z})F - 3(K_x + K_{x+z})F_x \right]_{z=0}$$

$$+ 3q \int_0^\infty (\partial_2 K) F dz \ .$$

Then substituting into (A.8) we have

$$(\partial_t + (\partial_x + \partial_y)^3 + 3q\partial_y) K = \int_0^\infty (K_t + (\partial_1^3 + \partial_2^3) K + 3q\partial_2 K) F dz$$

$$+ Q$$

where $Q = (\Psi_3 - \Phi_3 + 3\partial_y(\Psi_2 - \Phi_2) + 3qF_y$

$$+ \left[3(K_{x,x+z} + K_{x+z,x+z})F - 3(K_x + K_{x+z})F_x\right]_{z=0} .$$

Substituting for $\Psi_3 - \Phi_3$, $\Psi_2 - \Phi_2$ and using (A.7) we find

$$Q = 3q(x)K(x,x)F(x,y) - 3q(x)F_x(x,y)$$

$$= -3q(x)K_x(x,y) + 3q(x)\int_0^\infty \left[\partial_1 K(x,x+z)\right]F(x+z,y)dz$$

Then $\left[\partial_t + (\partial_x+\partial_y)^3 + 3q(\partial_x+\partial_y)\right]K$

$$= \int_0^\infty (K_t + (\partial_1+\partial_2)^3 K + 3q\partial_2 K)Fdz + 3q\int_0^\infty (\partial_1 K)F$$

$$= \int_0^\infty \left[K_t + (\partial_1+\partial_2)^3 K + 3q(\partial_1+\partial_2)K\right]Fdz$$

Hence $(I - A_x)[\partial_t + (\partial_x+\partial_y)^3 + 3q(\partial_x+\partial_y)]K = 0$

(A.12)

On $y = x$ using $q(x) = 2\frac{d}{dx}K(x,x)$ after taking a derivative of (A.12) we have the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0 \qquad \text{(A.13)}$$

hence result □.

Appendix B

The calculation of the terms U^x, U^t, U^{xx}, U^{tt}, U^{xxx}, U^{ttt}, U^{xxt} and U^{xtt}.

We have already calculated the functions U^x, U^t and U^{xx}. Here the remaining functions are simply given with a note that the procedure is entirely mechanical.

$$U^{tt} = U_{tt} + (2U_{tu} - T_{tt})u_t - X_{tt}u_x + (U_{uu} - 2T_{tu})u_t^2$$

$$- 2X_{tu}u_t u_x - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + (U_u - 2T_t)u_{tt} - 2X_t u_{xt}$$

$$- 3T_u u_t u_{tt} - X_u u_{tt} u_x - 2X_u u_{xt} u_t \qquad (B.1)$$

$$U^{xxx} = U_{xxx} + (3U_{xxu} - X_{xxx})u_x - T_{xxx}u_t$$

$$+ (3U_{xuu} - 3X_{xxu})u_x^2 + (U_{uuu} - 3X_{xuu})u_x^3 - X_{uuu}u_x^4$$

$$+ (3U_{xu} - 3X_{xx})u_{xx} + (3U_{uu} - 9X_{ux})u_x u_{xx}$$

$$- 6X_{uu}u_x^2 u_{xx} - 3X_u u_x^2 - 4X_u u_x u_{xxx} + (U_u - 3X_x)u_{xxx}$$

$$- 3T_{xxu}u_x u_t - 3T_{xx}u_{xt} - 3T_{xuu}u_x^2 u_t - 3T_{xu}u_t u_{xx}$$

$$- T_{uuu}u_x^3 u_t - 3T_{uu}u_x u_t u_{xx} - 3T_u u_{xx} u_{xt} - 6T_{ux}u_x u_{xt}$$

$$- 3T_x u_{xxt} - 3T_{uu} u_x^2 u_{xt} - 3T_u u_x u_{xxt} - T_u u_t u_{xxx}.$$

(B.2)

$$U^{ttt} = U_{ttt} + (3U_{ttu} - T_{ttt})u_t - X_{ttt}u_x$$

$$+ (3U_{tuu} - 3T_{ttu})u_t^2 + (U_{uuu} - 3T_{tuu})u_t^3 - T_{uuu}u_t^4$$

$$+ (3U_{tu} - 3T_{tt})u_{tt} + (3U_{uu} - 9T_{ut})u_t u_{tt} - 6T_{uu}u_t^2 u_{tt}$$

$$- 3T_u u_{tt}^2 - 4T_u u_t u_{tt} + (U_u - 3T_t)u_{ttt} - 3X_{ttu}u_t u_x$$

$$- 3X_{tt}u_{tx} - 3X_{tuu}u_t^2 u - 3X_{tu}u_x u_{tt} - X_{uuu}u_t^3 u_x$$

$$- 3X_{uu}u_t u_x u_{tt} - 3X_u u_{tt} u_{tx} - 6X_{ut}u_t u_{xt} - 3X_t u_{xtt}$$

$$- 3X_{uu}u_t^2 u_{xt} - 3X_{uut}u_{xtt} - X_u u_x u_{ttt}$$

(B.3)

$$U^{xxt} = U_{xxt} + (2U_{xtu} - X_{xxt})u_x + (U_{xxu} - T_{xxt})u_t$$

$$+ (U_{uut} - 2X_{xtu})u_x + (2U_{xuu} - X_{xxu} - 2T_{xtu})u_x u_t$$

$$- T_{xxu}u_t^2 - X_{uut}u_x^3 + (U_{uuu} - 2X_{xuu} - T_{uut})u_x^2 u_t$$

$$- 2T_{xuu}u_x u_t^2 - X_{uuu}u_x^3 u_t - T_{uuu}u_x^2 u_t^2$$

$$+ (U_{ut} - 2X_{xt})u_{xx} + (2U_{xu} - X_{xx} - 2T_{xt})u_{xt}$$

$$- T_{xx}u_{tt} - 3X_{ut}u_{x}u_{xx} + (U_{uu} - 2X_{xu} - T_{ut})u_{t}u_{xx}$$

$$- 4T_{xu}u_{t}u_{xt} + (2U_{uu} - 4X_{xu} - 2T_{tu})u_{x}u_{xt}$$

$$- 2T_{xu}u_{x}u_{tt} - 3X_{uu}u_{x}u_{t}u_{xx} - T_{ux}u_{xx}u_{t}^{2} - 3X_{uu}u_{x}^{2}u_{xt}$$

$$- 4T_{uu}u_{x}u_{t}u_{xt} - T_{uu}u_{x}^{2}u_{tt} - 3X_{u}u_{xx}u_{xt} - 2T_{u}u_{xt}^{2}$$

$$- T_{u}u_{xx}u_{tt} - X_{t}u_{xxx} + (U_{u} - 2X_{x} - T_{t})u_{xxt}$$

$$- 2T_{x}u_{xtt} - X_{u}u_{t}u_{xxx} - 3X_{u}u_{x}u_{xxt} - 2T_{u}u_{t}u_{xxt}$$

$$- 2T_{u}u_{x}u_{xtt}.$$

(B.4)

$$U^{ttx} = U_{ttx} + (2U_{xtu} - T_{ttx})u_{t} + (U_{ttu} - X_{ttx})u_{x}$$

$$+ (U_{uux} - 2T_{xtu})u_{t}^{2} + (2U_{tuu} - T_{ttu} - 2X_{xtu})u_{x}u_{t}$$

$$- X_{ttu}u_{x}^{2} - T_{uux}u_{t}^{3} + (U_{uuu} - 2T_{tuu} - X_{uux})u_{t}^{2}u_{x}$$

$$- 2X_{xuu}u_{t}u_{x}^{2} - T_{uuu}u_{t}^{3}u_{x} - X_{uuu}u_{t}^{2}u_{x}^{2}$$

$$+ (U_{ux} - 2T_{xt})u_{tt} + (2U_{tu} - T_{tt} - 2X_{xt})u_{xt}$$

$$
\begin{aligned}
& - X_{tt}u_{xx} - 3T_{ux}u_t u_{tt} + (U_{uu} - 2T_{tu} - X_{ux})u_x u_{tt} \\
& - 4X_{tu}u_x u_{xt} + (2U_{uu} - 4T_{tu} - 2X_{xu})u_t u_{xt} \\
& - 2X_{tu}u_t u_{xx} - 3T_{uu}u_t u_x u_{tt} - X_{uu}u_{tt}u_x^2 - 3T_{uu}u_t^2 u_{xt} \\
& - 4X_{uu}u_t u_x u_{xt} - X_{uu}u_t^2 u_{xx} - 3T_u u_{tt}u_{xt} - 2X_u u_{xt}^2 \\
& - X_u u_{tt}u_{xx} - T_x u_{ttt} + (U_u - 2T_t - X_x)u_{ttx} - 2X_t u_{txx} \\
& - T_u u_x u_{ttt} - 3T_u u_t u_{ttx} - 2X_u u_x u_{ttx} - 2X_u u_t u_{txx}.
\end{aligned}
\tag{B.5}
$$

Appendix C

In this appendix we present the proof of Theorem 6.1 which says that when $a_1 \neq a_2$ the general class (6.1.1) may be reduced to another general class such that one of the nonlinear terms can be eliminated.

Proof

Consider the nonsingular linear transformation

$$x \longrightarrow \eta = \frac{x}{1-n} - \frac{nt}{1-n}, \quad t \longrightarrow \xi = \frac{x}{1-m} - \frac{mt}{1-m} \quad \text{(C1)}$$

and $u(x, t) \longrightarrow v(\eta, \xi)$ where $n \neq m \neq 1$.

Applying the transformation to (6.1.1) reduces it to (6.2.20) with

$$b_1 = \frac{a_1 - na_2}{1-n}, \quad b_2 = \frac{a_1 - ma_2}{1-m}, \quad b_3 = \frac{a_3 - a_4 n + a_5 n^2 - a_6 n^3}{(1-n)^3}$$

$$b_4 = \frac{3a_3 - (m+2n)a_4 + n(n+2m)a_5 - 3mn^2 a_6}{(1-n)^2(1-m)}$$

$$b_5 = \frac{3a_3 - (2m+n)a_4 + m(2n+m)a_5 - 3nm^2 a_6}{(1-n)(1-m)^2}$$

$$b_6 = \frac{a_3 - a_4 m + a_5 m^2 - a_6 m^3}{(1-m)^3}$$

Now since $a_1 \neq a_2 \neq 0$, we may choose $m = \dfrac{a_1}{a_2}$.

Then $b_2 = 0$.

Similarly if we choose $n = \frac{a_1}{a_2}$ we obtain $b_1 = 0$ □

```
% This program calculates the  functions  uj(t),  j=1,..6
% of the  Laurent  series  expansion (8.1.1)  using  the
% expressions (8.2.5a - g)              If the Painleve test
% is to  be  satisfied  then  it  is  required  that  the
% compatability conditions at j=4 and j=6 are identically
% satisfied  This program has been used to show that only
%  the  KdV  equation  satisfies  the  test  ie  when
% u=(a1+a2f)**2
%
depend(f,t)$
u:=-12*(a3+a4*f+a5*f**2+a6*f3)/(a1+a2*f)$
%
% set derivatives of f
%
let df(f,t)=fp$
let df(fp,t)=fpp$
let df(fpp,t)=fppp$
let df(fppp,t)=fpppp$
let df(fpppp,t)=fppppp$
%
% calculate first, second and third derivatives of u
%
up:= df(u,t)$
upp:= df(up,t)$
uppp:= df(upp,t)$
%
u1:=   (12*up*(a4+2*a5*f+3*a6*f**2))/(5*(u*(a1+a2*f))   +
(12*fp*(a5+3*a6*f))/(5*(a1+a2*f))+(2*a2*up)/(5*(a1+a2*f))$
%
u1p:= df(u1,t)$
u1pp:= df(u1p,t)$
u1ppp:= df(u1pp,t)$
%
% calculate u2 and its derivatives
%
p1:= ((a2*u*u1p) + (a2*up*u1))/(2*u*(a1+a2*f))$
q1:= ((a4+2*a5*f+3*a6*f**2)*u1p)/(u*(a1+a2*f))$
r1:= ((a5+3*a6*f)*u1*fp)/(u*(a1+a2*f))-(1+f)/(a1+a2*f))$
t1:= (((a5+3*a6*f)*upp) + 3*a6*up*fp+a6*u*fpp)/
                                               (u*(a1+a2*f))$
u2:= (-u1**2)/(2*u) + p1 + q1 + r1 - t1$
%
p2:= (a6*uppp)/(u*(a1+a2*f)) + up/(u*(a1+a2*f))$
q2:= (u1*f+u1+a5*u1pp+3*a6*u1pp*f+3*a6*u1p*fp+a6*u1*fpp)/
                                               (u*(a1+a2*f))$
r2:= a2*(u*u2p+up*u2+u1*u1p)/(u*(a1+a2*f))$
u3:= p2 - q2 + r2 -(u1*u2)/u$
u3p:= df(u3,t)$
%
% check whether the compatibility  condition  at  j=4  is
% identically satisfied ie whether comp1=0
%
K:= a2*(u*u3p+up*u3+u1*u2p+u1p*u2)$
L:= (a6*u1ppp+u1p)$
comp1:= K + L$
%
```

```
% No need to check for compatability at j=6 if comp1<>0
%
If comp1 NEQ 0 then quit;
%
depend(u4,t)$
depend(u6,t)$
u2pp:= df(u2p,t)$
u2ppp:= df(u2pp,t)$
u3pp:= df(u3p,t)$
u3ppp:= df(u3pp,t)$
let u4p=df(u4,t)$
let u4pp=df(u4p,t)$
%
u5:= -(a6*u2ppp+u2p+u3*f+u3+a5*u3pp+3*a6*u3pp*f +
      3*a6*u3p*fp+a6*u3*fpp+2*(a4*u4p+2*a5*u4p*f +
      a5*u4*fp+3*a6*u4p*f**2+3*a6*u4*f*fp) + a2*(u4*up
      +u4p*u+u3*u1p+u3p*u1+u2*u2p) + (a1+a2*f)*(u2*u3+
      u4*u1))/((a1+a2*f)*u+6*(a3+a4*f+a5*f**2+a6*f**3))$
u5p:= df(u5,t)$
%
% calculate the expression at j=6 For compatability it is
% required that 0*u6=0
%
p3:= (2*u*(a1+a2*f)+24*(a3+a4*f+a5*f**2+a6*f**3))$
q3:= (a6*u3ppp+u3p+2*(u4*f+u4+a5*u4pp+3*a6*u4pp*f+
      3*a6*u4p*fp+a6*u4*fpp))$
r3:= 6*(a4*u5p+2*a5*u5p*f+a5*u5*fp+3*a6*u5p*f**2+
      3*a6*u5*f*fp)$
t3:= a2*(u5*up+u5p*u+u4*u1p+u4p*u1+u3*u2p+u3p*u2)$
%
p3*u6:= r3 + t3 - q3 + (a1+a2*f)*(2*u1*u5+2*u2*u4+u3**2)$
quit;
```

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