ALGEBRAIC SYNTHESIS METHODS FOR LINEAR MULTIVARIABLE SYSTEMS: DECENTRALIZED STABILIZATION

THE CITY UNIVERSITY

LONDON EC1V 0HB

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BY

DAVID R. WILSON B.Sc (Hons)

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CONTENTS

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DECLARATION

ABSTRACT

A unifying approach for the study of solvability of algebraic synthesis problems defined on linear time invariant multivariable systems is given. The decentralized stabilization problem is formulated over the ring of proper and stable rational function and its solution reduces to the study of (sets of) matrix equations of the type AX = B. It is shown that many control problems can be described algebraically using matrices defined over special rings. The rings of importance are the Euclidean domains $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ and $\mathbf{R}_{\rho}(s)$ and these are used to investigate the structural and invariant aspects of system stability equations. The solvability of AX = B also provides conditions for the solvability of the generalised Diophantine equation.

The Diagonal Stabilization Problem (DSP) is defined over the ring of proper rational functions which have no poles inside a prescribed region of the finite complex plane. Solvability is intimately related to systems which exhibit the property of cyclicity. Necessary and sufficient conditions are established for the existence of solutions to the DSP. A complete parameterization of stabilizing controllers for 2x2 case is given. Conditions of nonsolvability and hence nonstabilizability yield an explicit expression for the fixed modes of the system.

The algebraic tools are given to investigate special type solutions such as realisable, stable and performance related controller designs as well as the more general case of multi-channel systems.

NOTATION AND ABBREVIATIONS

Throughout this thesis the following notation and abbreviations will be used:

$\mathbf{R}$:	Field of real numbers
C:	Field of complex numbers
$\mathbf{Z}$:	Ring of integers
$\mathbf{R}(s)$:	Field of rational functions
$\mathbf{R}[s]$:	Ring of polynomials
$\mathbf{R}_{pr}(s)$:	Ring of proper rational functions
$\mathbf{R}_{\rho}(s)$:	Ring of proper and Ω-stable rational functions
$\mathbf{F}$:	Field
$\mathbf{K}$:	Principal Ideal Domain P.I.D.
ϕ:	Empty set
$+$:	Addition
$.$:	Multiplication
$(\mathbf{K},+)$:	Additive group of $\mathbf{K}$
$(\mathbf{K},.)$:	Multiplicative group of $\mathbf{K}$
deg.a(s):	Degree of the polynomial a(s)
Σa_i:	Sum of the elements a_i
$\lvert a\rvert$:	Modules of the element a
a^{-1}:	Inverse of the element a
$a \Rightarrow b$:	a implies b
$a \Leftrightarrow b$:	a implies b and b implies a
$<, \leq$:	less than, less than or equal to
$>, \geq$:	greater than, greater than or equal to

$a=b$:	a equals b
$a^1 \neq b^1$:	a^1 does not equal b^1
$a \in A$:	a is an element of the set A
$a^1 \notin A^1$:	a is not an element of the set A
$a \forall A$:	for all elements a contained in the set A
$B \subset A$:	B is contained within A
$B \cap A$:	Intersection of A and B
$B \cup A$:	Union of A and B
Δ:	Defined by
■:	End of a proof
$A \sim B$:	The matrix A is equivalent to the matrix B
$\sigma(A)$:	The spectrum of the matrix A (equiv. the set of eigenvalues of A)
$A:=B$:	The set A is by definition the set B
$A \equiv B$:	A is equivalent to B
$x \mid y$:	x divides y, or y is a multiple of x
$i \in \underline{r}$:	$i=1,2,...,r$
diag.:	diagonal
min:	minimum
w.r.t.:	with respect to
l.r.d.:	left right divisor
g.l.r.d.:	greatest left right divisor
e.l.d.:	extended left divisor
e.r.d.:	extended right divisor
g.e.l.d.:	greatest extended left devisor
g.e.r.d.:	greatest extended right divisor
l.u.:	left unimodular

r.u.: right unimodular

g.c.e.l.d.: greatest common extended left divisor

g.c.e.r.d.: greatest common extended right divisor

c.p.: column projector

r.p.: row projector

r.a.: right annihilator

l.a.: left annihilator

l.i.: left inverse

r.i.: right inverse

l.u.: left unimodular

r.u.: right unimodular

PID: Principal Ideal Domain

MFD: Matrix Fraction Description

DSP: Diagonal Stabilization Problem

CDE: Centralized Diophantine Equation

DDE: Decentralized Diophantine Equation

GDE: Generalised Diophantine Equation

PDSP: Proper Diagonal Stabilization Problem

SDSP: Strong Diagonal Stabilization Problem

SEMMP: Stable Exact Model Matching Problem

EMMP: Exact Model Matching Problem

CHAPTER 1

INTRODUCTION

CHAPTER 1

INTRODUCTION

This dissertation is concerned with Linear Algebraic Synthesis Methods for Multivariable Systems and additional algebraic tools are developed on matrix divisors, projectors and solvability of matrix equations. The main problem studied is the Diagonal Stabilization Problem (DSP) and techniques are developed for solving the DSP as well as investigating the structural properties of solutions to the problem.

Recent work in this area is based on what is termed the Fractional Representation Approach to Linear Systems Theory [Des 1, Sae 1, Ant 1, Vid 1, Vis 1, Fra 3]. The impetus to study matrices having elements from special rings comes from the need to describe algebraically the familiar problems of stability, realizability and performance of linear systems. From a Control Theory viewpoint the rings of importance are $\mathbf{R}[s]$-polynomials, $\mathbf{R}_{pr}(s)$-proper rational functions which also have no poles inside a prescribed region of Ω of the finite complex place, denoted $\mathbf{R}_{\rho}(s)$. The detailed structure of the set $\mathbf{R}_{\rho}(s)$ have been thoroughly investigated [Var 3, Var 4, Var 8] and the structural and invariant aspects of $\mathbf{R}_{\rho}(s)$-modules and minimal bases have been defined. These notions generalise the structural tools and Algebraic Theory of

Rosenbrock [Ros 1], Wolovich [Wol 2] and Forney [For 1] to the case of the ring $\mathbf{R}_\rho(s)$.

Algebraically, many control systems problems are reduced to the solution of (sets of) matrix equations and a great deal of effort has been and continues to be exerted in this area.

The present approach seeks to provide a unifying approach for their analysis as well as establish deeper results concerning the structure of rational matrices and solvability of matrix equations and thus control problems. The techniques developed provide the means to tackle the main problem, the Diagonal Stabilization Problem (DSP), represented algebraically as (sets of) matrix Diophantine equations over the ring $\mathbf{R}_\rho(s)$.

Decentralized control results from the need to control individual parts of a system directly without interaction between them and thereby reducing the amount of centralized data acquisition and information transfers needed to effectively control large industrial complex plants. The essential features that distinguish between centralized and decentralized systems is the decoupling of subsystems and plants from each other thus eliminating interaction but retaining the ability to effect direct control.

A special case of Decentralized Stabilization is the problem of stabilization by a diagonally structured controller such that upon interconnection of unity feedback loop the system becomes internally stabilized. Conditions for solvability and the characterisation of solutions are given. It is demonstrated that systems which exhibit the property of cyclicity satisfy criteria for stabilization by dynamic controllers under unity feedback. The notion of cyclicity is obtained by reducing the system open loop transfer function matrix to its Hermite forms over the ring of proper and Ω-stable rational functions $\mathbf{R}_p(s)$.

For the simple two channel case where stabilization by two single input-single output (SISO) controllers can be achieved then a complete parameterisation is generated by a matrix 'T' defined on the plant. The 'T' matrix governs the interaction between each of the two stabilizing channels of the two-input-two output plant. Pairs of controllers which stabilize the closed loop system are defined from knowledge of the system T-matrix. These controller pairs which satisfy certain conditions are called mode-T-mutually stabilizing controllers and provide the main results. These results have been extended to three channel systems and the application to general multi channel system is discussed.

In chapter 2 some basic definitions and background results from Linear Algebra are given. Rings, Modules and the notion of Principal Ideal Domains (PID) are introduced. The rings $\mathbf{R}[s]$-polynomials, $\mathbf{R}_{pr}$(proper rational functions) and $\mathbf{R}_{\rho}(s)$ (proper and stable rational functions) are all principal ideal domains (PIDs). The Algebraic Theory of Linear time-invariant systems is based on the study of matrices over PIDs. One important aspect of that is the notion of the module. A module is the generalisation of the vector space where elements are defined from a commutative ring rather than from a field (as in the case of the vector space). A module provides a general setting for the purely algebraic aspects of linear control system problems. The above concepts and notions are used to develop synthesis techniques for solution of a number of important control problems.

In chapter 3 a summary of algebraic synthesis problem is given. These problems when defined over the ring $\mathbf{R}_{\rho}(s)$ lead to an elegant representation from which a number of synthesis techniques have developed. The generalisation of the Youla Bongiorno and Jabr (YBJ) stabilization theory [You 1, You 2, Kuc 1] in which a complete parameterisation of the set of stabilizing controllers over $\mathbf{R}_{\rho}(s)$ is defined established the fractional representation approach as a powerful algebraic synthesis technique [Des 1, Sae 1, Ant 1, Vid 1, Vis 1, Fra 3, Bra 1, Cal 2, Kuc 1]. The central issue of the stabilization problem is the solution of a (set of) matrix Diophantine equation formulated as a Stable Exact Model Matching Problem (SEMMP) over the ring

$\mathbf{R}_\rho(s)$. This provides the motivation to study further the structural aspects of matrices over Principal Ideal Domains.

The main aim of chapter 4 is to investigate further the structural and invariant aspects of matrices which are solutions to equations of the type AX=B and AX + BY = C where the given matrices A,B,C are in general rational and the solution matrices X,Y are to be determined from a PID $\mathbf{K}$ such that $\mathbf{R}(s)$ may be expressed as the field of fractions of $\mathbf{K}$. These equations are central to the solution of the more generalised Diophantine equation $A_1X_1 + A_2X_2 + \ldots + A_pX_p = B$ where B is generally a non square matrix. Thus solvability of AX=B also provides conditions for solvability of the more general set of equations. The results are given for a general rational matrix $A \in \mathbf{R}^{pxm}(s)$ using Smith-McMillan, Hermite-McMillan forms defined over a PID $\mathbf{K}$. Although the results are valid for PIDs in general we are concerned with the Euclidean rings $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ and $\mathbf{R}_\rho(s)$. Thus instead of PIDs we may say that $\mathbf{K}$ is a Euclidean ring (the difference is that in Euclidean rings, the unimodular matrices are expressed as products of elementary transformations).

In chapter 5 some general results are established as well as new solvability criteria presented for solution of matrix equations over PIDs. The algebraic tools developed in the previous chapters are used to develop a direct approach for solvability of equations which is algorithmic in nature.

In chapter 6 the decentralized stabilization problem is formulated over the PID ring $\mathbf{R}_p(s)$. A special case of decentralized control is the diagonal stabilization problem (DSP). The concept of fixed modes is introduced and it is shown that diagonal stabilization is possible if and only if the system is free from unstable hidden modes thus highlighting the important role they play and the need to characterise them.

Necessary and sufficient conditions are established for the solution of the DSP using dynamic compensation. The notion of cyclicity is introduced and the existence as well as the characterisation of solutions of DSP is shown to be related to the property of system cyclicity.

In chapter 7 the case of systems which exhibit strong cyclicity is examined. In general for an mxm system, strong cyclicity is a necessary condition for stabilization by a decentralized controller. In the restricted case m=2, two input-two output, strong cyclicity is demonstrated to be both necessary and sufficient for solution of DSP. This result demonstrates that strong cyclicity is equivalent to Ω-stabilizability by diagonal dynamic compensation. A complete parameterisation of diagonal stabilizing controllers is possible for the case m=2 using mode-T-mutually stabilizing pairs. The properness of solutions, thus realizability of controllers, is defined also and the existence of constant solutions, hence minimal design, is discussed. Conditions for non-solvability and hence non-stabilizability yield an explicit expression for the system fixed modes. Finally, a discussion on the integrity of system operation is given. This is an important quality of the system since the ability of a system to remain stable and controllable on failure of a control channel is an essential part of the design.

The results obtained for the simple case m=2 are extended to the more general case m=3. These results provide the means to generalise either to diagonal control of a general square system or to decentralized control of a two channel system.

CHAPTER 2

MATHEMATICAL BACKGROUND

2.1 Introduction

A great deal of what Systems Engineers do is based on the concepts of modern algebra and recent years have witnessed a growing awareness of the presence of algebra in systems theory. This recognition has led to further understanding of problems already solved and to unforeseen solutions of problems unsolved by other less formal methods.

The objective of this chapter is to introduce the concepts of Rings, Modules and Principal Ideal Domains (PIDs). Recent results in the area of linear multivariable control obtained using the so called factorization approach have highlighted the important role of PIDs [Des 1, Ham 2, Kuc 1, Var 7, Var 8, Sae 2, Vid 1, Var 3]. In particular, the PID $\mathbf{R}_\rho(s)$ (proper and stable rational functions which have no poles inside a prescribed region of the finite complex plane and at the point s: equal infinity) gives rise to elegant methods for to the resolution of several important control algebraic synthesis problems [Cal 1, Des 2, Fra 3, Kuc 1, Vid 1, Var 7, Var 8].

The Hermite and Smith-McMillan forms of a rational matrix over a general PID are defined and the notion of Matrix Fraction Description (MFD) introduced. The detailed structure of the set of proper and stable rational functions $\mathbf{R}_\rho(s)$ and the properties of matrices over $\mathbf{R}_\rho(s)$ are given. Finally, a number of important definitions and results are given on the algebraic structure of $\mathbf{R}_\rho(s)$-

vectors. These are known to be that of Noetherian $\mathbf{R}_\rho(s)$-modules [God 1] and have been classified according to properties of their McMillan degree. [Var 8, Var 4].

2.2 An Introduction to $R_\rho(s)$-MFDs and Matrices

The detailed structure of the set of proper and stable rational functions $\mathbf{R}_\rho(s)$ have been studied by Vardulakis and Karcanias [Var 3, Var 7] and a number of definitions and important properties are given in this section.

2.2.1 Proper and Stable Rational Functions

Let $\mathbf{R}$ be the field of reals, $\mathbf{R}[s]$ the ring of polynomials with coefficients in $\mathbf{R}$ and $\mathbf{R}(s)$ the field of rational functions. Then every rational function $t(s)=n(s)/d(s)$, $n(s)$, $d(s)\in\mathbf{R}[s]$, $d(s)\neq 0$ can be written as

$$t(s) = \left(\frac{1}{s}\right)^{q_\infty} \bar{n}(s)/\bar{d}(s) \tag{2.1}$$

where $q_\infty := \delta_\infty(t(s))$ a degree function and degree $\bar{n}(s)$ = degree $\bar{d}(s)$. If $q_\infty > 0$, we say that $t(s)$ has a zero at $s = \infty$ of order q_∞ conversely if $q_\infty < 0$, we say that $t(s)$ has a pole at $s = \infty$ of order $|q_\infty|$.

If $t(s) \in \mathbf{R}(s)$ has $q_\infty \geq 0$ then $t(s)$ is called a proper rational function and if the inequality is strict then $t(s)$ is called strictly proper. It can easily be verified [eg. see Var 7] that the set of proper rational functions which we denote by $\mathbf{R}_{pr}(s)$ is a Euclidean ring with degree function given by $\delta_\infty(t(s))$. Thus $\mathbf{R}_{pr}(s)$ is a principal ideal ring and both $\mathbf{R}[s]$ and $\mathbf{R}_{pr}(s)$ are subrings of $\mathbf{R}(s)$. The elements of $\mathbf{R}[s]$ can be regarded as rational functions with no poles in C (the finite complex plane) while the elements of $\mathbf{R}_{pr}(s)$ can be regarded as rational functions with no poles at $s = \infty$.

The units in $\mathbf{R}_{pr}(s)$ are proper rational functions $u(s)$ for which $\delta_\infty(u(s)) = 0$ (i.e. having no zeros at $s = \infty$) and are called biproper rational functions [Var 7].

<u>Example (2.1)</u> $\mathbf{R}[s]$: The units of $\mathbf{R}[s]$ are constants i.e. the polynomials $t(s) = c$, $c \in \mathbf{R} - \{0\}$. $\mathbf{R}_{pr}(s)$: the units of $\mathbf{R}_{pr}(s)$ are rational functions $t(s) = n(s)/d(s)$ where $\deg.n(s) = \deg.d(s)$.

(i) $\frac{s+1}{s(s+2)}$,	(ii) $\frac{s-1}{s+2}$,	(iii) 4
↓	↓	↓
Not a unit	Unit of $\mathbf{R}_{pr}(s)$	Unit of $\mathbf{R}_{pr}(s)$ Unit of $\mathbf{R}[s]$

■

The above can be generalised by defining a region Ω of the finite complex plane C symmetrically located with respect to the real axis $\mathbf{R}$ and which excludes at least one point $\propto$ on the real axis with Ω^c the complement of Ω with respect to C (i.e. $C = \Omega \cup \Omega^c$). Let $t(s) \in \mathbf{R}(s)$ and factorize it as

$$t(s) = t_\Omega(s)\ \hat{t}(s) = \frac{n_\Omega(s)}{d_\Omega(s)} \cdot \frac{\hat{n}(s)}{\hat{d}(s)} \tag{2.2}$$

where $n_\Omega(s)$, $d_\Omega(s)$ are coprime polynomials with <u>all their</u>

zeros in Ω and $\hat{n}(s)$, $\hat{d}(s)$ are coprime polynomials with all their zeros outside Ω. We define the map [Var 7] δ_Ω: $\mathbf{R}(s) \to \mathbf{Z} \cup \{\infty\}$ via:

$$\delta_\Omega(t(s)) = \begin{cases} \deg.\hat{d}(s) - \deg.\hat{n}(s) \in \mathbf{Z}, & t(s) \not\equiv 0 \\ \infty, & t(s) \equiv 0 \end{cases} \tag{2.3}$$

or equivalently,

$$\delta_\Omega(t(s)) = q_\infty + \deg.n_\Omega(s) - \deg.d_\Omega(s) \tag{2.4}$$

The subset of $\mathbf{R}(s)$ consisting of all rational functions which are proper (ie no poles at $s = \infty$) and have also no poles in the region Ω (i.e. Ω-stable) are called proper and Ω-stable rational functions denoted $\mathbf{R}_\rho(s)$ i.e. let

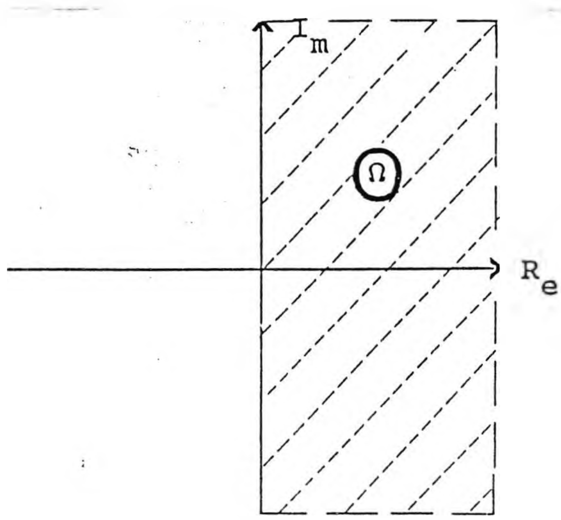
$$\mathbf{R}_\rho(s) = \{t(s) \in \mathbf{R}(s): t(s) \text{ has no poles in } \rho: = \Omega \cup (\infty)\}$$

The set $\mathbf{R}_\rho(s)$ endowed with the operations of addition and multiplication forms a commutative ring with unity element (the real number 1) and no zero divisors and thus it is an integral domain (see section 2.2).

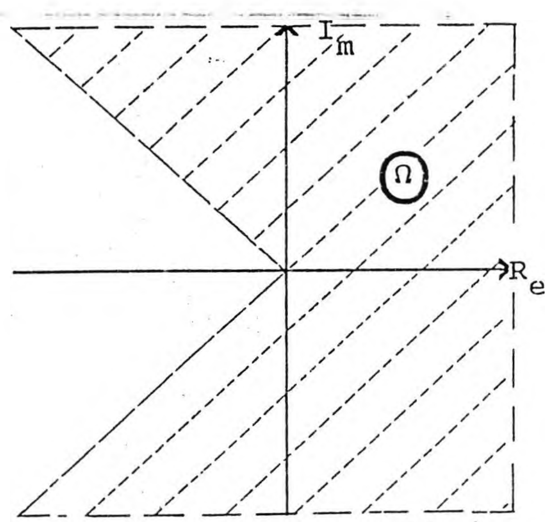
If $t(s) \in \mathbf{R}_\rho(s)$ then $t(s) = n_\Omega(s)\ \hat{n}(s)/\hat{d}(s)$ and since $\deg.(n_\Omega(s).\hat{n}(s)) \le \deg.\hat{d}(s)$ it follows that $\delta_\Omega(t(s)): = \deg.\hat{d}(s) - \deg.\hat{n}(s) \ge 0$; thus $\delta_\Omega(.)$ for the non zero elements of $\mathbf{R}_\rho(s)$ may serve as a degree function. The degree function denotes the number of zeros of the function in $\rho: = \Omega \cup \{\infty\}$.

The algebraic structure of the set $\mathbf{R}_\rho(s)$ was initially examined by Morse [Mor 1] and subsequently Hung and Anderson [Hun 1] established that with $\delta_{\Omega}(.)$ as degree function the set $\mathbf{R}_{\rho}(s)$ is a Euclidean ring and therefore a principal ideal domain (PID). In the following the function δ_{Ω} when restricted to the subdomain $\mathbf{R}_{\rho}(s) \subset \mathbf{R}(s)$ will be denoted by δ_{ρ} where $\delta_{\rho}: = \mathbf{R}_{\rho}(s) \rightarrow \mathbf{Z} \cup \{\infty\}$. The units of $\mathbf{R}_{\rho}(s)$ are biproper rational functions which have no poles and no zeros in $\delta: = \Omega \cup \{\infty\}$; equivalently $t(s) \in \mathbf{R}_{\rho}(s)$ is a unit if and only if $\delta_{p}(t(s)) = 0$.

<u>Example (2.3)</u>: $\mathbf{R}_{\rho}(s)$: the units of $\mathbf{R}_{\rho}(s)$ are biproper rational functions $t(s)= n(s)/d(s)$ where $n(s)$, $d(s)$ have no zeros in Ω i.e. <u>biproper rational functions which are Ω-stable and Ω-minimum phase</u>.



A. Ω is the undesired part of the complex plane. In the above case Ω is the right-half of the complex plane.



B. Ω is the undesired part of the complex plane. In the above case Ω is the right half of the complex plane and a selected part of the left half plane chosen such that a desired maximum damping factor is not exceeded.

$t_1(s) = \frac{s-1}{s+2}$: a unit of $\mathbf{R}_{pr}(s)$, not a unit of $\mathbf{R}_\rho(s)$.

$t_2(s) = \frac{s+1}{s+2}$: a unit of $\mathbf{R}_{pr}(s)$, a unit of $\mathbf{R}_\rho(s)$.

Equivalently,

$$t_1(s) = \frac{s-1}{s+2} = \frac{n_\Omega(s)}{d_\Omega(s)} \frac{\hat{n}(s)}{\hat{d}(s)} = \frac{s-1}{1} \frac{1}{s+2} \quad \text{with}$$

$\delta_\rho(t_1(s)) = \deg \hat{d} - \deg.\hat{n} = 1 - 0 = 1, \not\equiv 0$, not a unit of $\mathbf{R}_\rho(s)$

$$t_2(s) = \frac{s+1}{s+2} = \frac{n_\Omega(s)}{d_\Omega(s)} \frac{\hat{n}(s)}{\hat{d}(s)} = \frac{1}{1} \frac{s+1}{s+2} \quad \text{with}$$

$\delta_\rho(t_2(s)) = \deg \hat{d} - \deg.\hat{n} = 1 - 1 = 0$, a unit of $\mathbf{R}_\rho(s)$.

■

<u>Remark (2.1)</u>: If Ω coincides with the closed right half complex plane $C_+ := [s \in C, \mathrm{Re}(s) \geq 0]$ then $\rho \equiv C_+ \cup \{\infty\} =: \overline{C}_+$ and $\mathbf{R}_{\overline{C}+}(s)$ is the Euclidean ring of "<u>proper and stable</u>" rational functions. The units in $\mathbf{R}_{\overline{C}+}$ are biproper stable and minimum phase rational functions.

■

From (2.4) it follows that if $t(s) \in \mathbf{R}_\rho(s)$ then $q := \delta(t(s)) = q_\infty + \deg.n_\Omega(s)$ and $q_\infty \geq 0$ gives the order of the zero at $s = \infty$ of $t(s) \in \mathbf{R}_\rho(s)$ and $t(s)$ gives the number of finite zeros of $t(s)$ inside Ω. Thus a convenient factorization of rational functions $t(s) \in \mathbf{R}(s)$ is written as

$$t(s) = \frac{n_\Omega(s)}{d_\Omega(s)} \quad \frac{1}{(s+\alpha)^q} \quad \frac{\hat{n}(s)\ (s+\alpha)^q}{\hat{d}(s)} \tag{2.5}$$

where $-\alpha \in \mathbf{R}$ is outside Ω and otherwise arbitrary, $q := \delta_\Omega\ (t(s)) = \deg.\hat{d}(s) - \deg.\hat{n}(s)$ and $\hat{n}(s)(s+\alpha)^q/\hat{d}(s)$ is a unit in $\mathbf{R}_\rho(s)$. The term $(n_\Omega(s)/d_\Omega(s))(1/s+\alpha)^q)$ gives the pole zero structure of $t(s)$ in $\rho: = \Omega \cup \{\infty\}$. Thus the zeros of $n_\Omega(s)$ give the finite zeros of $t(s)$ in ρ and the zeros of $d\ (s)$ give the finite poles of $t(s)$ in ρ. Furthermore, if $q_\infty: = q + \deg.d_\Omega(s) - \deg.n_\Omega(s) > 0$ then $t(s)$ has a zero at $s = \infty$ of order q_∞ while if $q_\infty < 0$ then $t(s)$ has a pole at $s = \infty$ of order $|q_\infty|$ [Var 7]. From the above we see that every $t(s) \in \mathbf{R}_\rho(s)$ can be written as

$$t(s) = \frac{n_\Omega(s)}{(s+\alpha)^q} \cdot u(s) \tag{2.6}$$

where $n_\Omega(s)$ has no zeros outside Ω, $-\alpha \in \mathbf{R}$ is outside Ω, $q: = \delta_\rho(t(s))$ and $u(s)$ is a unit in $\mathbf{R}_\rho(s)$. Ω is the right half part of the finite complex plane symmetrically located with respect to the real axis.

<u>Example (2.3)</u>: Let $t(s) = \dfrac{(s+1)(s+2)}{s(s-3)(s+4)} \in \mathbf{R}(s)$ and factorize

it as $\dfrac{n_\Omega(s)}{d_\Omega(s)} \cdot \dfrac{1}{(s+\alpha)^q} \cdot \dfrac{\hat{n}(s)\ (s+\alpha)^q}{\hat{d}(s)}$

$$= \frac{s-1}{s(s-3)} \quad \frac{1}{(s+\alpha)^0} \cdot \frac{(s+2)(s+\alpha)^0}{(s+4)}$$

where $q = \delta_\rho(t(s)) = \deg \hat{d} = \deg \hat{n} = 0$, gives

$$t(s) = \frac{s-1}{s(s-3)} \; \frac{s+2}{s+4} = \frac{s-1}{s(s-3)} \, u(s)$$

where $\frac{s-1}{s(s-3)}$ gives the pole-zero structure of $t(s)$ in $\rho = \Omega \cup \{\infty\}$. With $q_\infty = q + \deg d_\Omega - \deg n_\Omega = 1$, $t(s)$ has a zero at $s = \infty$ of order 1.

■

The existence of euclidean division in the ring $\mathbf{R}_\rho(s)$ has been established [Hun 1]. The strong links between $\mathbf{R}_\rho(s)$ and $\mathbf{R}[W]$ where $W = 1/(s+\alpha)$ allow the reduction of a euclidean division in $\mathbf{R}_\rho(s)$ to a standard division of polynomials in $\mathbf{R}[W]$. This approach is more suitable for computational purposes. With $\delta_\rho(.)$ as degree function the following result [Var 7] provides a proof that $\mathbf{R}_p(s)$ is a Euclidean ring and hence a PID.

<u>Theorem (2.1)</u> [Var 7]: Let $t_1(s)$, $t_2(s) \in \mathbf{R}_\rho(s)$, $t_2(s) \not\equiv 0$ and let $w = 1/(s+\alpha)$, $-\alpha \in \mathbf{R}$, $-\alpha \notin \Omega$. If $t_i(s) = t_{i\alpha}(w)\, u_{i\alpha}(s)$, $i=1,2$ are (mod α) factorizations of $t_1(s)$, $t_2(s)$ where $t_{1\alpha}(w)$, $t_{2\alpha}(w) \in \mathbf{R}[w]$, $u_{1\alpha}(s)$, $u_{2\alpha}(s)$ are units in $\mathbf{R}_\rho(s)$ and $\delta_\rho(t_i(s)) = \deg . t_{1\alpha}(w)$ then:

(1) There exist polynomials $q_\alpha(w)$, $r_\alpha(w) \in \mathbf{R}[w]$ such that

$$t_{1\alpha}(w) = t_{2\alpha}(w) q_\alpha(w) + r_\alpha(w) \tag{2.7}$$

and either $r_\alpha(w) = 0$ or else $\deg r_\alpha < \deg . t_{2\alpha}(w)$

(2) The rational functions $q_\alpha(s)$, $r_\alpha(s) \in \mathbf{R}_p(s)$ defined by

$$q_\alpha(s) := u_{1\alpha}(s)\; u_{2\alpha}(s)^{-1}\; \tilde{q}_\alpha\left(\frac{1}{s+\alpha}\right) \tag{2.8}$$

$$r_\alpha(s) := u_{1\alpha}(s)\; \tilde{r}_\alpha\left(\frac{1}{s+\alpha}\right) \tag{2.9}$$

satisfy the euclidean division conditions for $t_1(s)$, $t_2(s)$ i.e.

$$t_1(s) = t_2(s) q_\alpha(s) + r_\alpha(s) \tag{2.10}$$

and either $r_\alpha(s) = 0$ or else $\delta_p(r_\alpha(s)) < \delta_p(t_2(s))$

<u>Proof</u> see [Var 7]

∎

The above result defines the mod α euclidean division of two elements of $\mathbf{R}_\rho(s)$ and indeed provides a proof that $\mathbf{R}_\rho(s)$ is a euclidean ring. For a given pair $t_{1\alpha}(w)$, $t_{2\alpha}(w) \in \mathbf{R}[w]$ the pair $q(w)$, $r(w) \in \mathbf{R}[w]$ is uniquely defined; thus q_α, $r_\alpha(s)$ are also uniquely defined for a mod α euclidean division. However different choices of α yield different pairs $q_\alpha(s)$, $r_\alpha(s)$ and thus the euclidean division in $\mathbf{R}_\rho(s)$ does not possess the uniqueness property for the quotient and remainder. The family of $(q_\alpha(s)$, $r_\alpha(s))$ obtained for various values α is not characterized by a uniquely defined "degree" remainder. The non uniqueness of $\delta_\rho(r_2(s))$ has motivated the study of the pair $(q(s), r(s))$ with $\delta_\rho(r)$ minimum [Vis 1, Var 7].

2.2.2 Proper and Stable Rational Matrices

The set of pxm matrices with elements in $\mathbf{R}(s)$ is denoted by $\mathbf{R}^{pxm}(s)$ and by $\mathbf{R}^{pxm}[s]$, $\mathbf{R}_{pr}{}^{pxm}(s)$ the subsets of $\mathbf{R}^{pxm}(s)$ consisting of pxm matrices with elements respectively in $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$. A matrix $T \in \mathbf{R}_{pr}{}^{pxp}(s)$ is called $\mathbf{R}_{pr}(s)$-unimodular (or biproper) if there exists a matrix $\hat{T} \in \mathbf{R}_{pr}{}^{pxp}(s)$ such that $T\hat{T} = I_p$

Example (2.4) Let T be the 2x2 proper rational matrix

$$T = \begin{bmatrix} 1 & \frac{s+1}{s-2} \\ \frac{s-2}{s+1} & 2 \end{bmatrix} \in \mathbf{R}_{pr}^{2x2}(s)$$

then

$$\begin{bmatrix} 1 & \frac{s+1}{s-2} \\ \frac{s-2}{s+1} & 2 \end{bmatrix} \begin{bmatrix} 2 & -(\frac{s+1}{s-2}) \\ -(\frac{s-2}{s+1}) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

where $$\hat{T} = \begin{bmatrix} 2 & -(\frac{s+1}{s-2}) \\ -(\frac{s-2}{s+1}) & 1 \end{bmatrix} \in \mathbf{R}_{pr}^{2x2}(s)$$

$T\hat{T} = I_2$, T is $\mathbf{R}_{pr}(s)$-unimodular

■

Denote now by $\mathbf{R}_\rho^{pxm}(s)$ the set of all pxm matrices with elements in $\mathbf{R}_\rho(s)$. These matrices are called proper and "Ω-stable" rational matrices (in the sense that they have no poles at infinity (proper) and also no poles inside Ω) [Var 7]. If $\Omega \equiv C_+$, then $\mathbf{R}_\rho^{pxm}(s)$ represents the set of proper and "stable" rational matrices. A matrix $T \in \mathbf{R}_\rho^{pxp}(s)$ is called $\mathbf{R}_\rho(s)$-unimodular (or biproper) if there exists a matrix $\hat{T} \in \mathbf{R}_\rho^{pxp}(s)$ such that $T\hat{T} = I_p$. A direct implication of a matrix $T \in \mathbf{R}_\rho^{pxp}$ and unimodular is it has also no zeros at $s = \infty$ and no finite zeros in Ω (i.e. $T \in \mathbf{R}_\rho^{pxp}(s)$, unimodular has no poles or zeros in $\rho: = \Omega \cup \{\infty\}$. A system theoretic interpretation of an $\mathbf{R}_\rho(s)$-unimodular matrix has been given by Vardulakis and Karcanias [Var 7]. In the particular case when $\Omega \equiv C_+$, then an $\mathbf{R}_\rho(s)$ unimodular matrix $T \in \mathbf{R}_\rho^{pxp}(s)$ represents a square, biproper, stable and minimum phase transfer function matrix. Elementary row and column operations on a $T \in \mathbf{R}^{pxm}(s)$ are defined in the usual way and can be accomplished by multiplying the given T matrix on the left (right) by "elementary" $\mathbf{R}_\rho(s)$-unimodular matrices obtained by performing elementary row (column) operations on the identity matrix $I_{p(m)}$. It can also be shown that every $\mathbf{R}_\rho(s)$-unimodular matrix can be represented as a product of a finite number of elementary $\mathbf{R}_\rho(s)$-unimodular matrices [Gan 1].

Definition (2.1) [Var 7]: Let $T_1 \in \mathbf{R}^{pxm}(s)$, $T_2 \in \mathbf{R}^{pxm}(s)$. Then T_1 and T_2 are called "equivalent in ρ" if there exist

$\mathbf{R}_\rho(s)$-unimodular matrices $T_l \in \mathbf{R}_\rho^{pxp}(s)$, $T_r \in \mathbf{R}_{\rho p}^{mxm}(s)$ such that:

$$T_\ell T_1 T_r = T_2 \tag{2.11}$$

If $T_\ell \equiv I_p \in \mathbf{R}_\rho^{pxp}(s)$ ($T_r \equiv I_m \in \mathbf{R}_\rho^{mxm}(s)$), then T_1, T_2 are called "column (row) equivalent in ρ".

■

Equation (2.11) defines an equivalence relation on $\mathbf{R}^{pxm}(s)$ which is denoted $\mathbf{E}^\rho$ and if T_1, T_2 are equivalent in ρ we write $(T_1,T_2) \in \mathbf{E}^\rho$. The $\mathbf{E}^\rho$ equivalence class of a fixed $T \in \mathbf{R}^{pxm}(s)$ is denoted by $[T]_{E_\rho}$. Let $T \in \mathbf{R}^{pxm}(s)$ with rank $T=r$ and consider the quotient of $\mathbf{R}^{pxm}(s)$ by $\mathbf{E}^\rho$, i.e. the set (denoted by) $\mathbf{R}^{pxm}(s)/\mathbf{E}^\rho$ of $\mathbf{E}^\rho$-equivalence classes $[T]_{E_\rho}$ where T runs through the elements of $\mathbf{R}(s)$. These equivalence classes are characterised by determining complete sets of invariants and canonical forms. The Smith-McMillan form of a rational matrix inside $\rho := \Omega \cup \{\infty\}$ (denoted by $\mathbf{S}^\rho$) has been given by Karcanias and Vardulakis [Var 2] and it is shown to be a canonical form for $\mathbf{E}^\rho$ on $\mathbf{R}(s)$ with a complete set of invariants for $\mathbf{E}^\rho$ on $\mathbf{R}(s)$ defined by the diagonal elements of $\mathbf{S}^\rho$. The Smith-McMillan form of a rational matrix T over a general principal ideal domain $\mathbf{K}$ is given in section (2.4). If $\mathbf{K}$ is now specialized to the case $\mathbf{K} \equiv \mathbf{R}_\rho(s)$ then $\mathbf{S}_T^\rho$ is defined by (2.11)

$$\mathbf{S}_T^{\ \rho} = [\operatorname{diag}\{\epsilon_1\psi_1^{-1},\ \epsilon_2\psi_2^{-1},\ \ldots,\ \epsilon_r\psi_r^{-1}\}\ 0_{p-r,m-r}] \tag{2.12}$$

where $\epsilon_i, \psi_i \in \mathbf{R}_\rho(s)$ form a complete set of invariants for $\mathbf{E}^\rho$ on $\mathbf{R}(s)$.

2.2.3 Coprimeness in ρ of Proper and Ω-Stable Rational Matrices

The notions of right or left coprimeness of a rational matrix in the set $\rho := \Omega \cup \{\infty\}$ follows from the definition of zeros of a rational matrix $T \in \mathbf{R}^{pxm}(s)$ inside ρ via its Smith-McMillan form $\mathbf{S}_T^{\rho}$ [Var 7].

Definition (2.2): Given two rational matrices $A \in \mathbf{R}^{\ell xm}(s)$, $B \in \mathbf{R}^{txm}(s)$ with $p := \ell + t \geq m$ and rank $\mathbf{R}(s)$ $[A\ B]^t = m$, then we may say that (the rows of) A and B are right coprime in $\rho = \Omega \cup \{\infty\}$ if $T := [A\ B]^t \in \mathbf{R}^{(\ell+t)xm}(s)$ has no zeros in ρ.

■

If we restrict ourselves to matrices that are proper and Ω-stable with $T := [A\ B]^t \in \mathbf{R}_{\rho}^{pxm}(s)$ and A, B right coprime in ρ then it has been shown [Var 7] there exists on $\mathbf{R}_{\rho}(s)$-unimodular matrix $T_1 \in \mathbf{R}_{\rho}^{pxp}(s)$ such that

$$T_1 T = [I_m \ O_{p-m,m}]^t = \mathbf{S}_T^{\rho} \tag{2.13}$$

and $T \in \mathbf{R}_{\rho}^{pxm}(s)$ may have zeros only in Ω^c.

Definition (2.3): A proper and Ω-stable rational matrix $T \in \mathbf{R}_{\rho}^{pxm}(s)$ $(p \geq m)$ satisfying condition (2.13) is defined as a $\mathbf{R}_{\rho}(s)$-left unimodular rational matrix. $\mathbf{R}_p(s)$-right unimodular rational matrices are defined in an analogous manner.

■

The notions of right (common) divisors in ρ and of greatest (common) right divisors in ρ of (the rows of two or more) rational matrices having the same number of columns have been defined by Vardulakis and Karcanias [Var 8] using the Smith-McMillan form of a rational matrix inside the region ρ.

Proposition (2.1) [Var 8]: Any rational matrix $T \in \mathbf{R}^{pxm}(s)$ with $p \geq m$ and $\text{rank}_{\mathbf{R}(s)} T = m$ can be factorised (in a non unique way as

$$T = T_1 T_{gr} \tag{2.14}$$

where $T_1 \in \mathbf{R}_\rho^{pxm}(s)$ is $\mathbf{R}_\rho(s)$-left unimodular and $T_{gr} \in \mathbf{R}^{mxm}(s)$ has pole-zero structure in $\rho = \Omega \cup \{\infty\}$ the same with that of T.

■

Definition (2.4) [Var 8]: Let the proper and Ω-stable rational matrices $T \in \mathbf{R}_\rho^{pxm}(s)$, $T_1 \in \mathbf{R}_\rho^{pxm}(s)$, $T_r \in \mathbf{R}_\rho^{mxm}(s)$ be related via

$$T = T_1 T_r \tag{2.15}$$

then T_r is a right divisor in ρ of T.

Definition (2.5) [Var 8]: Let $T \in \mathbf{R}_\rho^{pxm}(s)$ with $p \geq m$ and $\text{rank}_{\mathbf{R}(s)}\{T\} = m$. Then any rational matrix $T_{gr} \in \mathbf{R}_\rho(s)$ that satisfies (2.14) for some $\mathbf{R}_\rho(s)$-left unimodular rational matrix $T_1 \in \mathbf{R}_\rho^{pxm}(s)$ is called a greatest (common) right divisor in ρ of (the rows of) T. ■

Notice that if T_{gr} is a greatest (common) right divisor of T, then it follows from Proposition (2.1) that T_{gr} contains all the zeros of T in ρ (i.e. the finite ones in Ω and the infinite ones, if any). If T_{gr} happens to be $\mathbf{R}_\rho(s)$-unimodular then T has also no zeros in ρ and its rows are said to be right coprime in ρ. In such a case T might have finite zeros outside Ω.

The Matrix Fraction Description (MFD) of a rational matrix over a general Principal Ideal Domain (PID) $\mathbf{K}$ is given by Definition (2.2). If $\mathbf{K}$ is specialized to the case $\mathbf{K} \equiv \mathbf{R}_\rho(s)$ and let $T \in \mathbf{R}^{pxm}(s)$ then T can always be represented in a non unique way as

$$T = B_2 A_2^{-1} = A_1^{-1} B_1 \tag{2.15}$$

where $B_2 \in \mathbf{R}_\rho^{pxm}(s)$, $A_2 \in \mathbf{R}_\rho^{mxm}(s)$ are right coprime in ρ and $B_1 \in \mathbf{R}_\rho^{pxm}(s)$, $A_1 \in \mathbf{R}_\rho^{pxp}(s)$ are left coprime in ρ.

A systematic procedure to determine the family of coprime fractional representations - based on the Smith-McMillan form of T in ρ - is given by Vardulakis and Karcanias [Var 7]. The pairs (B_2,A_2), $(B_1,A_1,)$ satisfying (2.15) are respectively right, left coprime in ρ $\mathbf{R}_\rho(s)$-Matrix Fraction Descriptions ($\mathbf{R}_\rho(s)$-MFD). Notice that a rational matrix T is proper and Ω-stable i.e. $T \in \mathbf{R}_\rho^{pxm}(s)$ if and only if the denominator matrix $A_1,(A_2)$ is $\mathbf{R}_\rho(s)$-unimodular i.e no poles or zeros in $\rho = \Omega \cup \{\infty\}$.

The degree function δ_ρ introduced in section 2.5.1. for proper and Ω-stable rational functions has been generalized for the matrix case [Var 8] as follows. Let T $\in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R}(s)}T = r \leq \min(p,m)$. Define the map δ_ρ: $\mathbf{R}_\rho^{pxm}(s) \rightarrow \mathbf{Z} \cup \{\infty\}$ via:

$$\delta_\rho(T) = \begin{cases} \min \left\{ \begin{array}{l} \delta_\rho(.) \text{ among the } \delta_\rho(.)\text{'s of} \\ \text{all r-th order non-zero} \\ \text{minors of T.} \end{array} \right. & \text{if } r > 0 \\ + \infty & \text{if } r = 0 \end{cases}$$

Then,

(1) If $p \geq m$ then $\delta_\rho(T) \geq 0$. Moreover if $p = m = \text{rank}_{\mathbf{R}(s)}T$ then $\delta_\rho(T) = 0$ if and only if T is $\mathbf{R}_\rho(s)$-unimodular.

(2) Let $T_1 \in \mathbf{R}_\rho^{pxm}(s)$, $p \geq m$, $\text{rank}_{\mathbf{R}(s)}\ T_1 = m$ and $T_2 \in \mathbf{R}_\rho^{mxm}(s)$, rank $\mathbf{R}(s)\ T_2 = m$ and let $T = T_1T_2$, then $\delta_\rho(T) = \delta(T_1) + \delta_\rho(T_2)$.

2.3 Modules over the Ring of Proper and Stable Rational Functions

The algebraic structure of the set of all proper and stable rational vectors $\mathbf{R}_\rho^{px1}(s)$ which are contained within a given rational vector space ϑ is known to be that of a Noetherian $\mathbf{R}_\rho(s)$-module M*. The structure of the $\mathbf{R}_\rho(s)$-bases of the maximal $\mathbf{R}_\rho(s)$-module M* have been classified according to properties of their McMillan degree by Vardulakis and Karcanias [Var 8]. It is useful to conclude this section with a number of definitions and results from the theory of proper bases of a rational vector space.

Definition (2.6): Let $t \in \mathbf{R}_\rho^{px1}(s)$ be the set of all proper and Ω-stable rational vectors which are contained in the rational vector space ϑ spanned by the columns $\underline{t}_j \in \mathbf{R}_\rho^{px1}(s)$, $j \in \underline{m}$ of a general rational matrix $T \in \mathbf{R}^{pxm}(s)$. If T is a basis for ϑ, $p \geq m$, rank $T = m$, and is expressed as a (right coprime in ρ) $\mathbf{R}_\rho(s)$ - MFD, $T = BA^{-1}$, where $A \in \mathbf{R}_\rho^{mxm}(s)$, $B \in \mathbf{R}_\rho^{pxm}(s)$ then clearly B is a proper and Ω-stable basis of ϑ.

Let now $T_1 \in \mathbf{R}_\rho^{pxm}(s)$ be a basis for ϑ and consider the set of all linear combinations of the columns of $\underline{t}_j \in \mathbf{R}_\rho^{px1}(s)$, $j \in \underline{m}$ of T_1 with coefficients in the ring $\mathbf{R}_\rho(s)$. This set is a free and finitely generated $\mathbf{R}_\rho(s)$-module M_1

■

Notice from the above definition that any other basis for M_1 can be obtained from T_1 by post-multiplying T_1 by an $\mathbf{R}_\rho(s)$-unimodular matrix i.e. $U \in \mathbf{R}_\rho^{mxm}(s)$ is an $\mathbf{R}_\rho(s)$-unimodular and define

$$\hat{T}_1 = T_1 U \tag{2.16}$$

then $T \in \mathbf{R}_\rho^{pxm}(s)$ is also a basis for M_1. In such a case it can be shown [e.g. see Var 8] that the degree function $\delta_\rho(T_1)$ and all bases of M_1 have the same $\delta_\rho(.)$ and thus $\delta_\rho(.)$ of any basis T_1 is an invariant of the $\mathbf{R}_\rho(s)$-module M_1 generated by its columns. This is denoted $\delta_\rho(M_1)$ and is called the stathm in ρ of M_1.

Assume now that $T_1 \in \mathbf{R}_\rho^{pxm}(s)$ is <u>not</u> $\mathbf{R}_\rho(s)$-left unimodular and let $T_{1g} \in \mathbf{R}_\rho^{mxm}(s)$ be a non-$\mathbf{R}_\rho(s)$-unimodular right divisor in ρ of T_1 (not necessarily a g.(c.)r.d. in ρ of T_1), i.e. assume that

$$T_1 = T_2 T_{1g} \tag{2.17}$$

for some (not necessarily $\mathbf{R}_\rho(s)$-left unimodular) $T_2 \in \mathbf{R}_\rho^{pxm}(s)$ with $\delta_p(T_{1g}) \geq \delta_\rho(T_2)$ so that T_{1g} contains some of the zeros of T_1 in ρ. Now consider the $\mathbf{R}_\rho(s)$-module M_2 which is generated by (the columns of) T_2, then M_1 is a proper submodule M_2 i.e.

$$M_1 \subset M_2 \tag{2.18}$$

In general if T_{1g}, T_{2g} ,..., $T_{ig} \in \mathbf{R}_\rho^{pxm}(s)$ are (non-$\mathbf{R}_\rho(s)$-unimodular) right divisors in ρ of T_1 such that

$$0 < \delta_\rho(T_{1g}) < \delta_\rho(T_{2g}) < \ldots < \delta(T_{ig}) \tag{2.19}$$

and

$$T_1 = T_{i+1}\ T_{ig} \tag{2.20}$$

for some not necessarily $\mathbf{R}_\rho(s)$-left unimodular

rational matrix $T_{i+1} \in \mathbf{R}_\rho^{pxm}(s)$, then the $\mathbf{R}_\rho(s)$-modules M_{i+1}, i=1,2,...generated by (the columns of) T_{i+1}, i=1,2,... form an ascending sequence of sub modules (see section 2.3 or [Var 8]).

$$M_1 \subset M_2 \subset \ldots \subset M_{i+1}, \quad i=1,2,\ldots \tag{2.21}$$

and

$$\delta_\rho(M_1) = \delta_\rho(M_2) > \ldots > \delta_\rho(M_{i+1}), \quad i=1,2,\ldots \tag{2.22}$$

If now for some i=1,2,... $T_{1g} =: T_{gr}$ is a greatest right divisor of T_1 so that

$$T_1 = \hat{T}\, T_{gr}$$

for some $\mathbf{R}_\rho(s)$-left unimodular matrix $T \in \mathbf{R}_\rho^{pxm}(s)$, then the maximal $\mathbf{R}_\rho(s)$-module generated by (the columns of) $\hat{T}$, and which is denoted M*, satisfies an ascending chain condition on submodules [Bir 1] i.e.

$$M_1 \subset M_2 \subset \ldots \subset M_{i+1} \equiv M^* \tag{2.23}$$

for some i=1,2... and coincides with the set of all proper and Ω-stable rational vectors which are contained in the rational vector space ϑ.

<u>Remark (2.2)</u>: The extraction of right (left) divisors from an $\mathbf{R}_\rho(s)$-basis matrix T_1 sets up an ascending chain of $\mathbf{R}_\rho(s)$-modules which is terminated when the greatest right (left) divisor of T_1 has been extracted. This property is known as the <u>module inclusion property</u>.

Example (2.5): Let $\Omega \equiv C_+$ and let the rational matrix T where

$$T = \begin{bmatrix} \dfrac{(s+3)(s-2)}{(s+1)^2(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \\ \dfrac{s-1}{(s+1)^2} & 0 \\ \dfrac{s^3+s^2-3s+1}{(s+2)^2(s+1)} & \dfrac{s-1}{(s+2)^2} \end{bmatrix} \in \mathbf{R}^{3x2}(s)$$

be a basis for the rational vector space $\vartheta_{3,2}$ with $\text{rank}_{\mathbf{R}(s)}\{T\} = 2$. The rational matrix T may be expressed as a right coprime matrix fraction in ρ, $\mathbf{R}_\rho(s)$ - MFD, $T = BA^{-1}$. Then,

$$T = \underbrace{\begin{bmatrix} \dfrac{s-2}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)^2} \\ 0 & \dfrac{s-1}{(s+1)^2} \\ \dfrac{s-1}{(s+2)^2} & \dfrac{s-1}{(s+1)(s+2)} \end{bmatrix}}_{\triangleq B} \underbrace{\begin{bmatrix} \dfrac{1}{s+1} & 1 \\ 1 & 0 \end{bmatrix}}_{\triangleq A_1^{-1}}$$

where $B \in \mathbf{R}_\rho^{3x2}(s)$, $\text{rank}_{\mathbf{R}(s)}\{B\} = 2$; $A \in \mathbf{R}_\rho^{2x2}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = 2$.

Clearly B is a proper and Ω-stable basis of $\vartheta_{3,2}$. The columns of the basis matrix B generates an $\mathbf{R}_\rho(s)$-module M and any other basis $\hat{B}$ can be obtained using the $\mathbf{R}_\rho(s)$-unimodular transformation $U \in \mathbf{R}_\rho^{2x2}(s)$ where $\hat{B} = BU$. Consider the simple $\mathbf{R}_\rho(s)$-unimodular transformation

$$U = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+1}{s+2} \end{bmatrix} \in \mathbf{R}_{\rho}^{2x2}(s)$$

Then $\hat{B} = BU = \begin{bmatrix} \frac{s-2}{(s+1)^2} & \frac{s-2}{(s+1)(s+2)} \\ 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{s-1}{(s+2)(s+1)} & \frac{s-1}{(s+2)^2} \end{bmatrix} \in \mathbf{R}_{\rho}^{3x2}(s)$

The $\mathbf{R}_{\rho}(s)$ modules generated by the columns of $B = [b_1, b_2]$, $\hat{B} = [\hat{b}_1, \hat{b}_2]$ are given by M, $\hat{M}$ respectively as:

$$M = \mathrm{span}_{\mathbf{R}_{\rho}(s)}\{\underline{b}_1, \underline{b}_2\} = \mathrm{span}_{\mathbf{R}_{\rho}(s)} \left[\begin{bmatrix} \frac{s-2}{(s+1)(s+2)} \\ 0 \\ \frac{s-2}{(s+2)^2} \end{bmatrix}, \begin{bmatrix} \frac{s-1}{(s+1)^2} \\ \frac{(s-1)}{(s+2)^2} \\ \frac{(s-1)^2}{(s+1)(s+2)} \end{bmatrix} \right]$$

$$\hat{M} = \mathrm{span}_{\mathbf{R}_{\rho}(s)}\{\underline{\hat{b}}_1, \underline{\hat{b}}_2\} = \mathrm{span}_{\mathbf{R}_{\rho}(s)} \left[\begin{bmatrix} \frac{s-2}{(s+1)^2} \\ 0 \\ \frac{s-1}{(s+2)(s+1)} \end{bmatrix}, \begin{bmatrix} \frac{s-2}{(s+1)(s+2)} \\ \frac{(s-1)}{(s+1)(s+2)} \\ \frac{s-1}{(s+2)^2} \end{bmatrix} \right]$$

The function $\delta_\rho(B) = 4 = \delta_\rho(\hat{B})$ hence the stathm (see p.24) in ρ of $M = 4 = \hat{M}$. By extracting a right divisor in $\mathbf{P}$ from B we write

$$B = \begin{bmatrix} \frac{s-2}{s+1} & 0 \\ 0 & \frac{1}{s+1} \\ \frac{s-1}{s+2} & \frac{s-2}{s+2} \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s+1} \\ 0 & \frac{s-1}{s+1} \end{bmatrix} = B_1 B_{1g}$$

where the function $\delta_\rho(B_1) = 2$ so that $\delta_\rho(B) > \delta_\rho(B_1)$ and B_{1g} contains some of the zeros of B in ρ.

The $\mathbf{R}_\rho(s)$-module M_1 generated by the columns of B_1 is given by

$$M_1 = \text{span}_{\mathbf{R}_\rho(s)} \left[\begin{bmatrix} \frac{s-1}{s+2} \\ 0 \\ \frac{s-1}{s+2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{s+1} \\ \frac{s-2}{s+1} \end{bmatrix} \right]$$

In fact, B_{1g} is a greatest right divisor in ρ of B since rank (T) $= 2 = \delta_\rho(B_1)$ and

$$M \subset M_1 \equiv M^*$$

■

The structure of the various ($\mathbf{R}_\rho(s)$-left unimodular) bases of the "maximal" $\mathbf{R}_\rho(s)$-module M* are given next. It has been shown [Var 8] that these bases can be classified according to properties of their McMillan-degree and are the counterpart to the concept of minimal polynomial basis of a rational vector space, introduced by Forney [For 1], for the case of the $\mathbf{R}_p(s)$-module M*. The notion of column complexity and simple basis of a proper rational matrix are introduced first.

Definition (2.5): [Var 3]: Let $T \in \mathbf{R}_{pr}^{pxm}(s)$, $p \geq m$, $\text{rank}_{\mathbf{R}(s)}T = m$, with column vectors t_j and denote by $\delta_M(t_j)$ the McMillan degree of t_j. Then the column McMillan-degree complexity of T, denoted by $C_M^c(T)$ is the sum of the McMillan degrees of its columns, i.e.

$$C_M^{C}(T) := \sum_{j=1}^{m} \delta_M(t_j) \tag{2.24}$$

■

The conditions under which the column McMillan degree complexity of a column reduced at $s = \infty$ [Var 1, Var 3] proper rational matrix T coincides with its McMillan degree $\delta_M(T)$ have been established [Var 3] and defines the notion of a simple basis given below.

Definition (2.6) [Var 3]: A column reduced at $s = \infty$ proper rational matrix $T \in \mathbf{R}^{pxm}(s)$ with $p \geq m$ and $\text{rank}_{\mathbf{R}(s)}T = m$, which satisfies

$$\sum_{j=1}^{m} \delta_M(t_j) = \delta_M(T) \tag{2.25}$$

is defined as a simple basis of the $\mathbf{R}_{pr}(s)$-module M generated by (t_j)

<u>Proof</u> $T \in \mathbf{R}_{pr}{}^{pxm}$ and $\mathrm{rank}_{\mathbf{R}} E = m$ imply respectively that T has no poles and zeros at $s = \infty$.

Hence $\delta_M(T)$ is the number of finite poles of T. The result then follows as a particular case of the more general Theorem in [Var 4].

∎

Two important results from the theory of proper bases of rational vector spaces appear as Lemmas (1) and (2) in [Var 4] and are re-stated below. Lemma (2.1) is a particular case of a more general Theorem [e.g.see Var 2] concerning the relationship between (i) the total number of poles and zeros (finite and infinite) of a general rational matrix $T \in \mathbf{R}^{pxm}(s)$, and (ii) the sum of the invariant dynamical indices [For 1] of the left and right null spaces of T.

<u>Lemma (2.1)</u> [Var 4]: Let $T \in \mathbf{R}_{pr}{}^{pxm}(s)$, $p \geq m$, $\mathrm{rank}_{\mathbf{R}(s)} T = m$ and $\lim_{s\to\infty} (T) = E \in \mathbf{R}^{pxm}$ with $\mathrm{rank}_{\mathbf{R}}(E) = m$. Let $v_j \geq 0$, $j \in \mathbf{m}$ the (Forney) invariant dynamical indices of the rational vector space ϑ spanned by the columns of T, $\mathrm{ord}_F(\vartheta) := \sum_{j=1}^{m} v_j$ the (Forney) invariant dynamical order of T and $z_f(T)$ the number of finite zeros of T. Then

$$\delta_M(T) = \mathrm{ord}_F(\vartheta) + Z_f(T) \qquad (2.26)$$

Lemma (2.2) [Var 4]: Let $T_i \in \mathbf{R}_{pr}{}^{pxm}(s)$, i=1,2 with $\lim_{s\to\infty} T =: E_i \in \mathbf{R}^{pxm}$ and $\mathrm{rank}_{\mathbf{R}} E_i = m$. If there exists a $Q \in \mathbf{R}^{mxm}(s)$, $\mathrm{rank}_{\mathbf{R}(s)} Q = m$, such that $T_1 = T_2 Q$, then $Q \in \mathbf{R}_{pr}{}^{mxm}(s)$ and $\lim_{s\to\infty} Q =: Q_0 \in \mathbf{R}^{mxm}$ with $\mathrm{rank}_R\ Q_0 = m$.

Proof see [Var 4].

■

With the above definitions and results we now focus attention on the various $\mathbf{R}_\rho(s)$-left unimodular bases of the maximal $\mathbf{R}_\rho(s)$-module M*. We note first that unlike the proper submodules M_i of M* all bases of M* are column reduced at s = ∞ since (by the definition of M*) they are all $\mathbf{R}_\rho(s)$-left unimodular. Secondly, and due to the above fact, if $T \in \mathbf{R}_\rho{}^{pxm}(s)$ is a basis of M* we can always determine an $\mathbf{R}_\rho(s)$-unimodular matrix $U_r \in \mathbf{R}_\rho{}^{mxm}(s)$ such that

$$\hat{T}: = TU_r \in \mathbf{R}_\rho{}^{pxm}(s) \qquad (2.27)$$

is a simple basis of M* for which $\delta_M(\hat{T}) = \delta_M(T)$. Furthermore, it has been established [Var 4] that given a basis T of M* we can always determine an $\mathbf{R}_\rho(s)$-unimodular matrix $U \in \mathbf{R}_\rho{}^{mxm}(s)$ such that $\hat{T} = TU_r^{-1}$ is a simple basis of M* which has desired poles (in Ω^c) and whose McMillan degree is minimum among the McMillan degrees of all proper or proper and Ω-stable bases of the rational vector space ϑ spanned by the columns of T i.e. $\delta_M(\hat{T}) \le \delta_M(T)$ for all

$T \in \mathbf{R}_{pr}^{pxm}(s)$ basis of ϑ.

Theorem (2.2) [Var 4]: Let $T \in \mathbf{R}_\rho^{pxm}(s)$, $p \geq m$, $\mathrm{rank}_{\mathbf{R}(s)} T = m$ be $\mathbf{R}_\rho(s)$-left unimodular and let M^* be the $\mathbf{R}_\rho(s)$-module generated by its columns t_j. Then T can always be factorized (in a non unique way) as:

$$T = \hat{T}U_r$$

where $\hat{T} = [\underline{t}_1, \ldots, \underline{t}_m] \in \mathbf{R}_\rho^{pxm}(s)$ is an $\mathbf{R}_\rho(s)$-left unimodular and simple basis of M^* which has no finite zeros and $U \in \mathbf{R}_\rho^{mxm}(s)$ is $\mathbf{R}_\rho(s)$-unimodular and the set of its finite zeros contains as a subject the set of finite zeros (which, if any, are in Ω^c) of T. Furthermore, if $v_j \geq 0$, $j \in \underline{m}$ are the (Forney) invariant dynamical indices of the rational vector space ϑ spanned by the columns of T (and also of $\hat{T}$), $\mathrm{ord}_F(\vartheta) := \sum_{i=1}^{m} v_j$ is the Forney invariant dynamical order of ϑ then:

(i) $\delta_M(\hat{t}_j) = v_j, \quad j \in \underline{m}$

(ii) $\delta_M(\hat{T}) = \sum_{j=1}^{m} \delta_M(\hat{t}_j) = \sum_{j=1}^{m} v_j = \mathrm{ord}_F(\vartheta)$

and $\delta_M(\hat{T})$ is minimum among the McMillan degrees of all other proper bases of ϑ.

Proof see [Var 4]

■

Definition (2.7): An $\mathbf{R}_\rho(s)$-left unimodular and simple basis $\hat{T} \in \mathbf{R}_\rho^{pxm}(s)$ of M^* which has no finite zeros (and thus satisfies (i) and (ii) of Theorem (2.2)) is defined as a simple, minimal McMillan degree, proper and Ω-stable basis (SMMD-$\mathbf{R}_\rho(s)$ basis) of the rational vector space ϑ spanned by its columns.

■

Example (2.6): Let $\Omega \equiv C_+$ and consider the rational matrix

$$T = \begin{bmatrix} \frac{s-2}{(s+1)(s+2)} & \frac{s-2}{(s+1)^2} \\ 0 & \frac{s-1}{(s+1)^2} \\ \frac{s-1}{(s+2)^2} & \frac{(s-1)^2}{(s+1)(s+2)} \end{bmatrix} \in \mathbf{R}_\rho^{3x2}(s)$$

which has a finite zero at $s = 1$ and two zeros at $s = \infty$ each one is of order 1. Extracting a greatest right divisor in ρ from T we can write:

$$T = \begin{bmatrix} \frac{s-2}{s+1} & 0 \\ 0 & \frac{1}{s+1} \\ \frac{s-1}{s+2} & \frac{s-2}{s+2} \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s+1} \\ 0 & \frac{s-1}{s+1} \end{bmatrix} = T_1 T_{gr}$$

Now $T_1 \in \mathbf{R}_\rho^{3x2}(s)$ has no zeros in $C \cup \{\infty\}$ and therefore it is a proper and stable minimal McMillan degree basis of the rational vector space spanned by its columns with $\delta_M(T_1) = 3$. Notice that its column McMillan degree complexity is: $C_M^c(T_1) = \delta_M(t_{11}) + \delta_M(t_{12}) = 2 + 2 = 4 > \delta_M(T_1)$, i.e. T_1 is not simple. Multiplying T_1 on the right by the $\mathbf{R}_\rho(s)$-unimodular matrix.

$$U_r = \begin{bmatrix} 1 & 0 \\ -3/4 & 1 \end{bmatrix} \in \mathbf{R}_\rho^{2x2}(s) \quad \text{we obtain}$$

$$T_1 U_r = \begin{bmatrix} \frac{s-2}{s+1} & 0 \\ \frac{-3/4}{s+1} & \frac{1}{s+1} \\ 1/4 & \frac{s-2}{s+2} \end{bmatrix} =: \hat{T}$$

which is also a proper and stable minimal McMillan degree basis which is simple since $\delta_M(\hat{T}) = 3 = \delta_M(\hat{t}_1) + \delta_M(\hat{t}_2) = 1+2$.

■

The importance of the above concepts and results in the resolution of linear multivariable control algebraic synthesis problems has been highlighted by Vardulakis and Karcanias [Var 8]. In particular conditions for the solvability of the stable exact model matching problem have been derived and the difficulty in constructing solutions to the stable minimal design problem using SMMD-$\mathbf{R}_\rho(s)$ bases is given. These problems involve questions of properness, stability and/or minimality of solutions of rational matrix equations. Algebraic synthesis problems of this nature are formulated in Chapter 3.

CHAPTER 3

SUMMARY OF ALGEBRAIC SYNTHESIS PROBLEMS

3.1 Introduction

The stabilization theory of Youla, Bongiorno and Jabr (YBJ), [You 1, You 2] for the case of continuous systems and Kucera [Kuc 1] for the discrete systems case in which a parameterisation of the set of all stabilizing controllers for a general multi-variable system is formulated, has provided much interest in algebraic synthesis methods [Des 1, Sae 1, Ant 1, Vid 1, Vis 1, Fra 3, Bra 1, Cal 2]. The generalisation of the YBJ theory to a ring theoretic setting [Des 1] established the fractional approach as a powerful algebraic synthesis technique. The elegance of the approach results from the use of stable factors as opposed to polynomial factors since products and sums of such factors are themselves stable. As these algebraic operations correspond to cascade and parallel connections of a system's configuration then from a synthesis design view point the fractional approach leads to a highly desirable representation of linear multi variable control problems.

In this chapter the centralized and decentralized stabilization problems are formulated over the ring of proper and Ω-stable rational functions which have no poles inside a prescribed region Ω of the finite complex plane. The main difference between these two problems lies in the structure of the stabilizing controller. In the centralized case there is no restriction on the input-output connections between controllers whereas in the decentralized case a well defined input-output relationship must be maintained.

The central feature of the stabilization problems is the solution of a (set of) matrix Diophantine equation [Kuc 1] of the form $AX + BY = M$ where A,B are known matrices defined from the plant and X,Y are the matrices to be determined and form the resulting stabilizing controller.

In the centralized stabilization problem the characterization of solutions is given by the YBJ parameterisation. The parameterisation is chosen in such a way that the various system feedback gains are linear in the resulting design parameter thereby enabling selection of design parameter which also achieves a design constraint and/or optimizes same measure of system performance.

In the decentralized case the restriction on input-output connections between controllers gives rise to a highly structured form of stabilizing controller. The problem is reduced to the solution of a (set of) matrix Diophantine equations formulated as a Stable Exact Model Matching Problem (SEMMP) and provides the motivation to investigate the structure of matrices over a Principal Ideal Domain (PID).

Before presenting the stabilization problems a number of general aspects of a general feedback configuration are introduced. The definition of a well-posed system [Vid 1] is given. The importance of uncontrollable and unobservable hidden modes of a system is introduced through the notions of stabilizability and detectability respectively [Kai 1]. These standard state space notions when extended with respect to a symmetric region Ω of the finite complex plane lead to a convenient external system description which also guarantees the system to be internally stable i.e. the set of proper and Ω-stable closed loop transfer function matrices. It is interesting to note in the following that the plant P and Controller C, in the continuous case, are defined over $\mathbf{R}(s)$, whereas in the discrete case over $\mathbf{R}(d)$ ($d = z^{-1}$, delay operator). In the continuous case we define the set of proper and stable rational function $\mathbf{R}_p(s)$, and in the discrete case the ring of causal and stable rational in d-functions $\mathbf{R}_p(d)$. Both $\mathbf{R}_p(s)$, $\mathbf{R}_p(d)$ are Euclidean rings and $\mathbf{R}(s)$, $\mathbf{R}(d)$ may be considered field of fraction for $\mathbf{R}_p(s)$, $\mathbf{R}_p(d)$ respectively. The analysis that follows is valid for both continuous and discrete systems and no distinction need be made although for continuity the $\mathbf{R}_p(s)$ will be used.

3.2 General Aspects of the General Feedback Configuration

Consider the following general control system configuration:

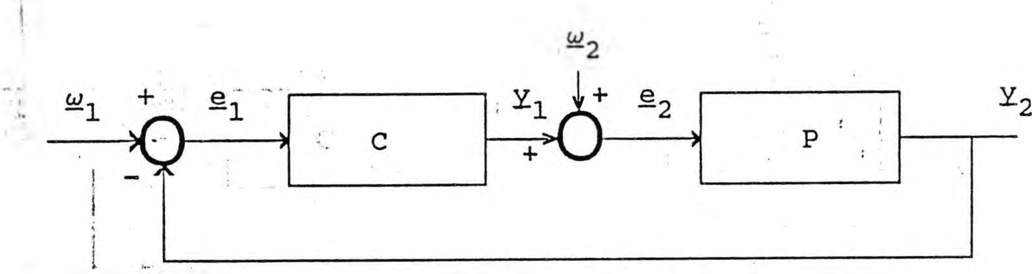


Figure (3.1): General Control System Configuration

where P represents the m x ℓ transfer matrix of the plant and C the ℓ x m transfer matrix of the controller. The vectors $\underline{\omega}_1$, $\underline{\omega}_2$ denote the externally applied inputs, $\underline{e}_1$, $\underline{e}_2$ denote the inputs to the controller, plant and $\underline{y}_1$, $\underline{y}_2$ the vector outputs of the controller, plant respectively. The transfer matrices P,C are both assumed to be rational and the set $\mathbf{R}_\rho^{p \times m}(s)$ will denote the set of p x m matrices with elements from $\mathbf{R}_\rho(s)$ (the ring of proper rational functions with no poles inside a prescribed region of the finite complex plane).

In the literature [Kuc 1], the configuration of Figure (3.1) also appears in the following equivalent form

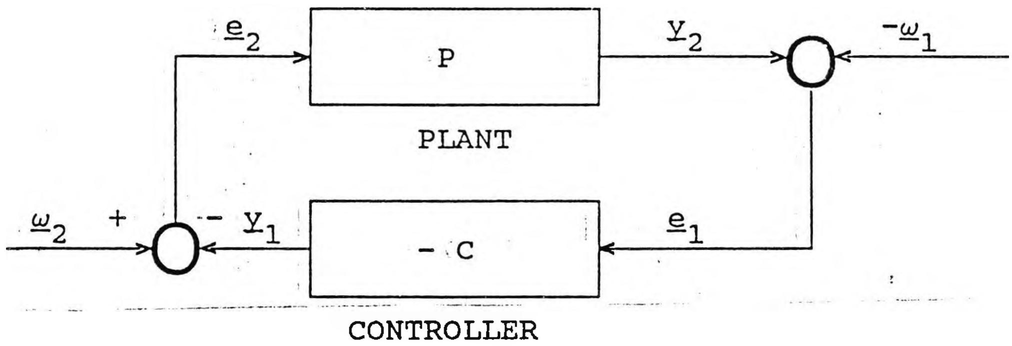


Figure (3.2): Alternative Control System Configuration

The configuration of Figure (3.1) will be considered here. Such a configuration is quite versatile and may accommodate several control problems. For instance, in a problem of tracking, $\underline{\omega}_1$ would be a reference signal to be tracked by the plant output $\underline{y}_2$. In a problem of disturbance rejection, or desensitization to noise, $\underline{\omega}_1$ would be the disturbance/noise vector. Depending on whether $\underline{\omega}_1$ or $\underline{\omega}_2$ is the externally applied control signal (as opposed to noise etc). The configuration can represent either feedback or cascade control.

The system under study is described by

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix} - \begin{bmatrix} 0 & P \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}, \quad \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \tag{3.1}$$

The system equations can be written as

$$\underline{e} = \underline{\omega} - FG\underline{e} \; , \quad \underline{y} = G\underline{e} \tag{3.2}$$

where

$$\underline{e} = \begin{bmatrix} \underline{e}_1 \\ \\ \underline{e}_2 \end{bmatrix}, \; \underline{\omega} = \begin{bmatrix} \underline{\omega}_1 \\ \\ \underline{\omega}_2 \end{bmatrix}, \; \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \\ \underline{y}_2 \end{bmatrix}, \; F = \begin{bmatrix} 0 & I \\ \\ -I & 0 \end{bmatrix}, \; G = \begin{bmatrix} C & 0 \\ \\ 0 & P \end{bmatrix} \tag{3.3}$$

It is easy to verify using the Shur-formula for determinants that

$$|I + FG| = |I + PC| = |I + CP| = t \tag{3.4}$$

The system of figure (3.1) is said to be <u>well-formed</u> [Cal 1] if t is a non zero rational function. This condition is necessary and sufficient to ensure that (3.1) has a unique rational solution for $\underline{e}_1$, $\underline{e}_2$, $\underline{y}_1$, $\underline{y}_2$ corresponding to vectors $\underline{\omega}_1$, $\underline{\omega}_2$ of appropriate dimensions.

If the system is well formed then

$$\underline{e} = (I + FG)^{-1} \, \underline{\omega} = H(P,C) \, \underline{\omega} \tag{3.5}$$

$$\underline{y} = G \, (I + FG)^{-1}\underline{\omega} = W \, (P,C)\underline{\omega} \tag{3.6}$$

where

$$\begin{bmatrix} \underline{\omega}_1 \\ \\ \underline{\omega}_2 \end{bmatrix} \xrightarrow{H(P,C)} \begin{bmatrix} \underline{e}_1 \\ \\ \underline{e}_2 \end{bmatrix} , \quad \text{denoted } H_{\underline{e}|\underline{\omega}}$$

$$\begin{bmatrix} \underline{\omega}_1 \\ \\ \underline{\omega}_2 \end{bmatrix} \xrightarrow{W(P,C)} \begin{bmatrix} \underline{y}_1 \\ \\ \underline{y}_2 \end{bmatrix} , \quad \text{denoted } H_{\underline{y}|\underline{\omega}}$$

A well formed system allows the existence of various closed loop functions. In the design of feedback systems the "properness" of these transfer functions is essential if no signal is to be unduly amplified or otherwise if the smoothness of signals throughout the system is to be preserved. Systems which exhibit this property are said to be "well-posed", a more formal definition is given below.

Definition (3.1): Let every sub-system of a composite system be described by a rational transfer function. Then the composite system is said to be well posed if the transfer function of every subsystem is proper and the closed loop transfer function from any point chosen as an input terminal to every other point along the directed path is well formed and proper.

■

The well posedness property is characterised by the following result.

<u>Theorem (3.1)</u> [Vid 1]: Consider the feedback system of figure (3.1) where P,C are proper rational matrices. The closed loop transfer function $H_{\underline{e}|\underline{u}}$ is proper if and only if

$$|I + C(\infty)\ P(\infty)| = |I + P(\infty)\ C(\infty)| \tag{3.7}$$

■

This result implies that if both P,C are proper then condition (3.7) is necessary and sufficient for $(I + PC)^{-1}$, $(I + CP)^{-1}$ to be proper and it follows that all transfer functions associated with the feedback configuration of figure (3.1) will be proper.

For systems that are well posed it is possible to obtain several equivalent expressions for H (P,C) and W(P,C). Thus for H(P,C) it is readily verified that

$$H(P,C) = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix} = \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \tag{3.8}$$

and the following identities hold true [Vid 1]

$$C(I + PC)^{-1} = (I + CP)^{-1}C, \quad P(I = CP)^{-1} = (I + PC)^{-1}P \tag{3.9}$$

$$(I + PC)^{-1} = I-P(I + CP)^{-1}C, \quad (I + CP)^{-1} = I-C(I + PC)^{-1}P \tag{3.10}$$

Using the above identities we can obtain the following equivalent expressions for H(P,C).

$$H(P,C) = \begin{bmatrix} I-P(I + CP)^{-1}C & -P(I + CP)^{-1} \\ (I + CP)^{-1}C & (I + CP)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} (I+PC)^{-1} & -(I + PC)^{-1}P \\ C(I + PC)^{-1} & I-C(I + PC)^{-1}P \end{bmatrix} \tag{3.11}$$

where the first involves only $(I + CP)^{-1}$ and the second $(I + PC)^{-1}$. For the W(P,C) transfer function we have similar expressions, that is

$$W(P,C) = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix} \tag{3.12}$$

or alternatively

$$W(P,C) = \begin{bmatrix} C-CP(I + CP)^{-1}C & -CP(I + CP)^{-1} \\ P(I + CP)^{-1}C & P(I + CP)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} C(I+PC)^{-1} & -C(I + PC)^{-1}P \\ PC(I + PC)^{-1} & P-PC(I + PC)^{-1}P \end{bmatrix} \qquad (3.13)$$

An interesting relationship between H(P,C), W(P,C) denoted in short by H,W is defined below

$$W = F\{H - I\} \qquad (3.14)$$

In fact, (3.14) readily follows from the following arguments

$$H = (I + FG)^{-1} = \{I + FG - FG\}(I + FG)^{-1}$$
$$= I - FG(I + FG)^{-1} = I - FW \qquad (3.15)$$

From (3.14) we have the following remark.

<u>Remark (3.1)</u>: The transfer function $W \in \mathbf{R}_p^{(m+p)(m+p)}(s)$ if and only if $H \in \mathbf{R}_p^{(m+p)(m+p)}(s)$

■

Thus in the investigation of stability (external) of the feedback configuration of figure (3.1), the transfer function H(P,C) may be used.

3.2.1 Characteristic Pole Function

The transfer function matrix of the plant and controller may be written as coprime matrix fraction descriptions (MFD's) over the appropriate ring of interest. Since P,C are generally non square we distinguish between left and right MFD's i.e.

$$P = A_1^{-1}B_1 = B_2A_2^{-1} \tag{3.16}$$

$$C = D_1^{-1}N_1 = N_2D_2^{-1} \tag{3.17}$$

By inserting (3.16) and (3.17) into the last of (3.8) we have

$$H(P,C) = \begin{bmatrix} A_1 & B_1 \\ -N_1 & D_1 \end{bmatrix}^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix} \tag{3.18a}$$

$$= \begin{bmatrix} D_2 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} D_2 & B_2 \\ -N_2 & A_2 \end{bmatrix}^{-1} \tag{3.18b}$$

Proposition (3.1) [Kai 1]: If (A_1,B_1), (A_2,B_2), (D_1,N_1), (D_2,N_2) are coprime pairs then (3.18a) defines a left coprime MFD and (3.18b) a right coprime MFD of H(P,C).

■

Assume now that both plant and controller transfer functions are represented by coprime MFD's then the expressions for H(P,C), i.e the transfer functions of the feedback configuration are coprime and the characteristic pole function of H(P,C) is given by the determinants of the denominator matrices

$$f \sim \det \begin{bmatrix} A_1 & B_1 \\ -N_1 & D_1 \end{bmatrix} \sim \det \begin{bmatrix} D_2 & B_2 \\ -N_2 & A_2 \end{bmatrix} \tag{3.19}$$

where "~" denotes equality modulo a non zero real constant.

<u>Proposition (3.2)</u>: The characteristic pole function of H(P,C) is given by

$$f \sim \det F_1 \sim \det F_2 \tag{3.20}$$

where

$$F_1 = A_1D_2 + B_1N_2 \tag{3.21a}$$

$$F_2 = D_1A_2 + N_1B_2 \tag{3.21b}$$

Proof

Using the Shur formula for expansion of determinants:

$$\det \begin{bmatrix} A_1 & B_1 \\ -N_1 & D_1 \end{bmatrix} = \det D_1 \det(A_1 + B_1 D_1^{-1} N_1)$$

$$= \det D_1 \det(A_1 + B_1 N_2 D_2^{-1})$$

$$= \det D_1 \det D_2^{-1} \det(A_1 D_2 + B_1 N_2)$$

$$\sim \det F_1$$

Similarly applying the Shur formula to

$$\det \begin{bmatrix} D_2 & B_2 \\ -N_2 & A_2 \end{bmatrix} = \det D_2 \det(A_2 + N_2 D_2^{-1} B_2)$$

$$= \det D_2 \det(A_2 + D_1 N_1^{-1} B_2)$$

$$= \det D_2 \det D_1^{-1} \det(D_1 A_2 + N\ B_2)$$

$$\sim \det F_2$$

The result follows by (3.19).

■

Note the importance of the assumption that both systems within the feedback loop be free of hidden modes [Kai 1]. If this assumption were violated, relation (3.19) would not be valid and the last step in the above proof could not be made. For systems with hidden modes

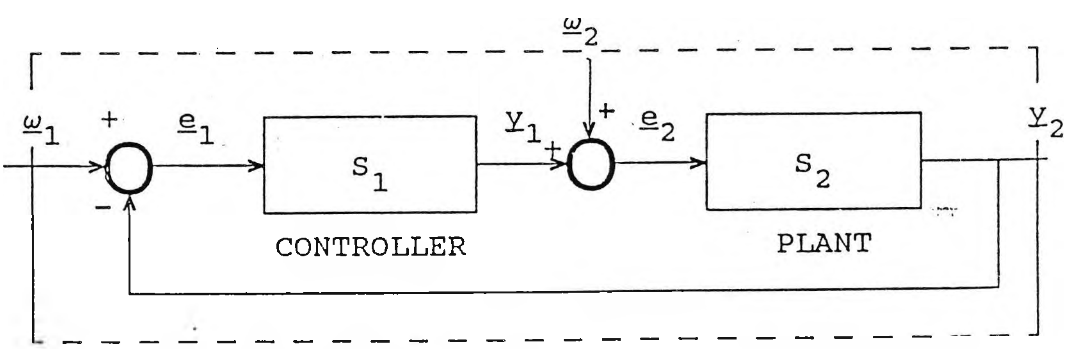
$$f \sim f_0 \; f_p \; f_c \tag{3.22}$$

where f_0 is defined by Proposition (3.2) and f_p, f_c are the hidden mode pole function of the plant, controller respectively.

Hidden modes play a key role in characterizing the internal stability of a feedback system in terms of the external description

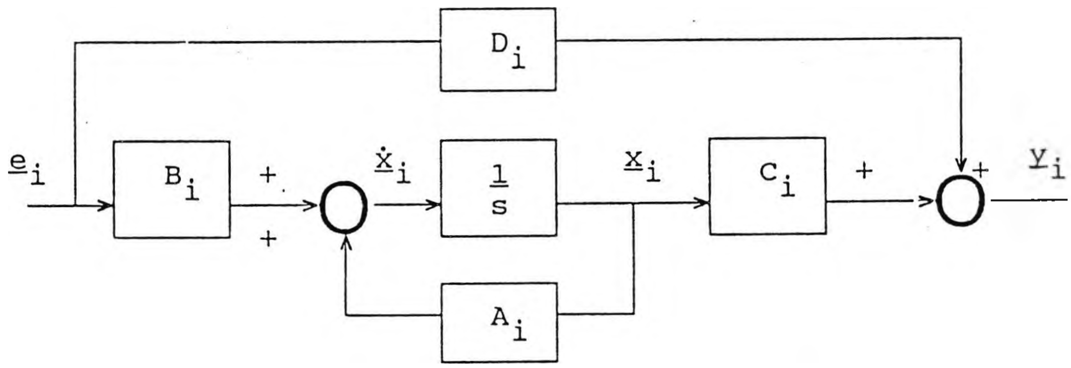
3.2.2 Internal Stability

Internal stability of the feedback system of figure (3.1) is related to its state space description.



S_f: = Feedback system

S_i, i=1,2 are state space representations of the controller, plant respectively:



S_f describes the state space representation of the feedback system:

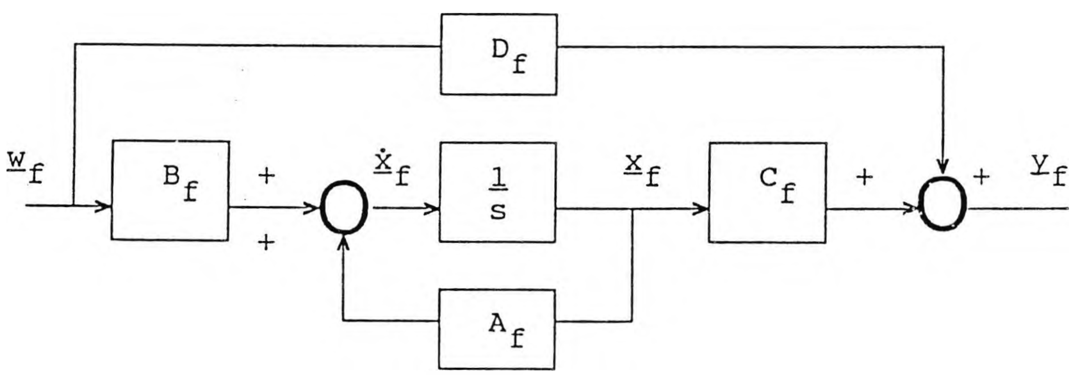


Figure (3.3): State Space representation of general feedback configuration

The plant and controller are characterised by the following sets of (not necessarily minimal) state space equations.

$$S_1: \underline{x}_1 = A_1\underline{x}_1 + B_1\underline{e}_1 \;,\; \underline{y}_1 = C_1\underline{x}_1 + D_1\underline{e}_1 \qquad (2.23a)$$

$$S_2: \underline{x}_2 = A_2\underline{x}_2 + B_2\underline{e}_2 \;,\; \underline{y}_2 = C_2\underline{x}_2 + D_2\underline{e}_2 \qquad (3.23b)$$

The feedback system is assumed to be well posed, so that $|I + D_1D_2| = |I + D_2D_1| \neq 0$, with the following constant matrices

$$\Delta_1 = (I + D_1D_2)^{-1} \ , \ \Delta_2 = (I + D_2D_1)^{-1} \tag{3.24}$$

The transfer function corresponding to the state space description with input vector $[\underline{\omega}_1{}^t, \ \underline{\omega}_2{}^t]^t$ and output vector of signals $[\underline{e}_1{}^t, \ \underline{e}_2{}^t]^t$ is clearly $H_{e|\underline{\omega}}$ Note that

$$\underline{e}_1 = \underline{\omega}_1 - \underline{y}_1 \ , \ \underline{e}_2 = \underline{\omega}_2 + \underline{y}_1 \tag{3.25a}$$

and thus

$$\underline{e}_1 = \underline{\omega}_1 - C_2\underline{x}_2 - D_2(\underline{\omega}_2 + C_1\underline{x}_1 + D_1\underline{e}_1)$$

or

$$\underline{e}_1 = -\Delta_2D_2C_1\underline{x}_1 - \Delta_2C_2\underline{x}_2 + \Delta_2\underline{\omega}_1 - \Delta_2D_2\underline{\omega}_2 \tag{3.25b}$$

$$\underline{e}_2 = \underline{\omega}_2 + C_1\underline{x}_1 + D_1\underline{e}_1 = \underline{\omega}_2$$

or

$$(I + D_1D_2)\underline{e}_2 = C_1\underline{x}_1 - D_1C_2\underline{x}_2 + D_1\underline{\omega}_1 + \underline{\omega}_2$$

or

$$\underline{e}_2 = \Delta_1C_1\underline{x}_1 - \Delta_1D_1C_2\underline{x}_2 + \Delta_1D_1\underline{\omega}_1 + \Delta_1\underline{\omega}_2 \tag{3.25c}$$

Substituting into the first of (3.23a), (3.23b) we obtain the state-space equations.

$$\underbrace{\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix}}_{\triangleq \, \dot{\underline{x}}_f} = \underbrace{\begin{bmatrix} A_1 - B_1\Delta_2 D_2 C_1 & -B_1\Delta_2 C_2 \\ B_2\Delta_1 C_1 & A_2 - B_2\Delta_1 D_1 C_2 \end{bmatrix}}_{\triangleq \, A_f} \underbrace{\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}}_{\triangleq \, \underline{x}_f} +$$

$$+ \underbrace{\begin{bmatrix} B_1\Delta_2 & -B_1\Delta_2 D_2 \\ B_2\Delta_1 D_1 & B_2\Delta_1 \end{bmatrix}}_{\triangleq \, B_f} \underbrace{\begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}}_{\triangleq \, \underline{u}_f} \tag{3.26a}$$

$$\underbrace{\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}}_{\triangleq \, \underline{y}_f} = \underbrace{\begin{bmatrix} -\Delta_{\bar{1}} D_2 C_1 & -\Delta_2 C_2 \\ \Delta_1 C_1 & -\Delta_1 D_1 C_2 \end{bmatrix}}_{\triangleq \, C_f} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} \Delta_2 & -\Delta_2 D_2 \\ \Delta_1 D_1 & \Delta_1 \end{bmatrix}}_{\triangleq \, D_f} \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix} \tag{3.26b}$$

The matrices (A_f, B_f, C_f, D_f) characterize the feedback system completely. The notion of internal and external stability are defined next.

<u>Definition (3.2)</u>: The feedback configuration of figure (3.3) will be called <u>internally stable</u> if the system

$$\dot{\underline{x}}_f = A_f \underline{x}_f \tag{3.27}$$

is asymptotically stable. It will be called <u>bounded-input, bounded-output stable</u> (BIBO) if the transfer function $H_{e|\omega}$ is BIBO stable.

■

Note that (3.27) is asymptotically stable if and only if the set of eigenvalues of A_f is contained within the closed left half complex plane $\sigma(A_f) \subset C\_$, and $H_{e|\omega}$ is BIBO stable if and only if all its poles are in $C\_$.

To examine the conditions under which the stability of $H_e|_\omega$ guarantees internal stability we introduce the following standard state space notions.

<u>Definition (3.3)</u>: [Won 1] (i) The pair (A,B) is <u>stabilizable</u> if the unstable subspace of $\underline{x} = A\underline{x} + B\underline{u}$ is contained in its controllable subspace. (ii) The pair (A,C) is <u>detectable</u> if the unreconstructable subspace of $\underline{x} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x}$ is contained in its observable subspace.

■

In the characterization of the properties of stabilizability and detectability the following results may be readily established.

<u>Lemma (3.1)</u>: [Won 1] (i) Any asymptotically stable system is stabilizable and detectable. (ii) Any completely controllable system is stabilizable. (iii) Any completely observable system is detectable.

■

<u>Lemma (3.2)</u>: [Won 1] (i) A system is stabilizable if and only if its uncontrollable eigenvalues are stable. For detectability we have the dual result. (ii) A system is detectable if its unobservable modes are stable.

■

<u>Definition (3.4)</u>: A quadruple (A,B,C,D) will be called <u>stabilizable and detectable</u> if (A,B) is stabilizable and (A,C) is detectable.

■

<u>Lemma (3.3)</u>: [Won 1] Suppose the quadruple (A,B,C,D) is stabilizable and detectable. Under these conditions the system

$$\underline{\dot{x}} = A\,\underline{x} \tag{3.28}$$

is asymptotically stable if and only if the transfer function

$$G(s) = C(sI - A)^{-1} B + D \tag{3.29}$$

is BIBO stable.

■

Remark (3.2): Under the stabilizability and detectability assumptions on a linear system, the notions of internal and external stability become equivalent.

■

The following result indicates the invariance of stabilizability and detectability under state feedback and output injection.

Lemma (3.4): [Won 1] Let (A,B,C) be a triple and let L,K,R,Q be arbitrary state feedback, output injection, input, output co-ordinate transformations respectively defined on (A,B,C). Then

(i) (A,B) is stabilizable, if and only if, (A + BL, BR) is stabilizable.

(ii) (A,C) is detectable, if and only if, (A + KC, QC) is detectable.

■

3.2.3 Guaranteed Internal Stability via Closed Loop External System Description

With the above standard state space notions we can establish the main result of this section in the study of the feedback system of figure (3.3).

Proposition (3.3): Consider the well-posed feedback system S_f of figure (3.3) with controller S_1 and plant S_2

represented by the quadruples (A_i, B_i, C_i, D_i) i = 1,2. Then

(i) S_f is controllable, observable if and only if both S_1 and S_2 are controllable, observable .

(ii) S_f is stabilizable, detectable if and only if both S_1 and S_2 are stabilizable, detectable.

Proof

The system S_f is represented by the quadruple (A_f, B_f, C_f, D_f) where

$$A_f = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} -\Delta D_2 C_1 & -\Delta_2 C_2 \\ \Delta_1 C_1 & -\Delta_1 D_1 C_2 \end{bmatrix}$$

$$\triangleq L \tag{3.30a}$$

$$= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} -B_1 \Delta_2 D_2 & -B_1 \Delta_2 \\ B_2 \Delta_1 & -B_2 \Delta_1 D_1 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

$$\triangleq K \tag{3.30b}$$

$$B_f = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} + \begin{bmatrix} \Delta_2 & -\Delta_2 D_2 \\ \Delta_1 D_1 & \Delta_1 \end{bmatrix} \tag{3.30c}$$

$$\triangleq R$$

$$C_f = \begin{bmatrix} -\Delta_2 D_2 & -\Delta_2 \\ \Delta_1 & -\Delta_1 D_1 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{3.30d}$$

Note that

$$|R| = |\Delta_2|.|\Delta_1 + \Delta_1 D_1 \Delta_2^{-1}\Delta_2 D_2| = |\Delta_2|.|\Delta_1 + \Delta_1 D_1 D_2|$$

$$= |\Delta_2|.|\Delta_1|.|I + D_1 D_2| = |\Delta_2|.|\Delta_1|.|\Delta_1|^{-1} =$$

$$= |\Delta_2| \neq 0$$

$$|Q| = |\Delta_2|.|\Delta_1 + \Delta_1 D_1 \Delta_2^{-1}\Delta_2 D_2| = |\Delta_2|.|\Delta_1|. \; |\Delta_1|^{-1}$$

$$= |\Delta_2| \neq 0$$

Thus R,Q are non singular and represent input, output co-ordinate transformations. Furthermore, L in (3.30a) represents state feedback and K in (3.30b) represents output injection.

(i) By the invariance of controllability properties under state feedback, input co-ordinate transformations and observability properties under output injection and output co-ordinate transformations we have:

(a) (A_f, B_f) is controllable if and only if (A_{12}, B_{12}) is controllable, where

$$A_{12} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad B_{12} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

(b) (A_f, B_f) is observable if and only if (A_{12}, C_{12}) is observable, where

$$C_{12} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

Similarly by the invariance of stabilizability, detectability (Lemma (3.4)) under state feedback, input co-ordinate transformations and output injection, output co-ordinate transformations respectively we have:

(c) (A_f, B_f) is stabilizable, if and only if (A_{12}, B_{12}) is stabilizable.

(d) (A_f, C_f) is detectable if and only if (A_{12}, C_{12}) is detectable.

By the block structure of A_{12}, B_{12}, C_{12} it follows that

(e) (A_{12}, B_{12}) is controllable (stabilizable), if and only if (A_1, B_1) and (A_2, B_2) are controllable (stabilizable).

(f) (A_{12}, C_{12}) is observable (detectable) if and only if (A_1, C_1) and (A_2, C_2) are observable (detectable).

From (a), (b), (c), (d), (e), (f) the result follows.

■

We may now state the main result of this section.

Theorem (3.2): Consider the well posed feedback system S_f with the controller S_1 and plant S_2 both stabilizable and detectable. Under these assumptions, S_f is internally stable if and only if the transfer function $H_e|_\omega$ is BIBO stable.

Proof

The result follows from Proposition (3.3) and Lemma (3.3).

■

Thus under the assumption of well posedness and stabilizability and detectability of plant and controller the closed loop transfer function $H_e|_\omega$ defines both internal and external stability. Clearly, since controllability implies stabilizability and observability implies detectability, if S_f is well posed and both plant and controller are free from hidden modes (controllable and observable) then $H_e|_\omega$ defines both internal and external stability.

Remark (3.3): The standard notions of detectability, stabilizability are connected with the notion of stability with $\Omega \equiv C\_$ closed. In many control synthesis applications Ω is a general symmetric region and the definitions and properties of stabilizability and detectability may be extended with respect to this region Ω and shall be referred to as Ω-stabilizability, Ω-detectability respectively. Theorem (3.2) should then be interpreted as: S_f Ω-stable is equivalent to $H_e|_\omega$ proper and Ω-stable.

■

By Theorem (3.2) and Proposition (3.3) it follows that

Corollary (3.1): Consider the well posed feedback system S_f of figure (3.3) with controller and plant systems S_1, S_2 minimal and the transfer functions P, represented by the $\mathbf{R}[s]$-coprime MFD's: $P_1 = A_1^{-1}B_1 = B_2A_2^{-1}$, $C = D_1^{-1}N_1 = N_2D_2^{-1}$ Then

$$|sI - A_f| \sim |A_1D_2 + B_1N_2| \sim |D_1A_2 + N_1B_2| \qquad (3.31)$$

■

The above formula provides the means for computing $|sI - A_f|$ and it is needed in the study of stability.

3.3 Control Systems Synthesis Problems

3.3.1 Centralized Stabilization

For a linear system the problem of Centralized Stabilization [You 2, Des 1, Vid 1] can be formulated in the frequency domain as follows.

Let the linear time invariant system in the feedback Configuration of figure (3.1) be well posed with Ω-stabilizable, Ω-detectable plant P described by the left, right $\mathbf{R}\rho(s)$-coprime MFD's

$$P = A_1^{-1}B_1 = B_2A_2^{-1} \qquad (3.32)$$

where

$$A_1 \in \mathbf{R}_\rho^{pxp}(s),\ B_1 \in \mathbf{R}_\rho^{pxm}(s);\ B_2 \in \mathbf{R}_\rho^{pxm}(s),\ A_2 \in \mathbf{R}_\rho^{mxm}(s)$$

The centralized stabilization problem is to determine conditions under which an Ω-stabilizable, Ω-detectable controller C defined by the left, right $\mathbf{R}\rho(s)$-coprime MFD's

$$C = D_1^{-1}N_1 = N_2D_2^{-1} \tag{3.33}$$

may be defined such that the closed loop system is stable i.e. the closed loop transfer function

$$H_{\underline{e}|\underline{\omega}} : \underline{\omega} := \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix} \longleftarrow \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} =: \underline{e}$$

is an element of $\mathbf{R}_\rho^{(m+p)x(m+p)}(s)$

The conditions under which such controllers exist are well known and a complete parameterization of all stabilizing controllers has been established [You 2, Des 1]. Indeed from (3.18) and Proposition (3.2) it is clear that for (A_1,B_1), (A_2,B_2) $\mathbf{R}\rho(s)$-coprime pairs, the centralized stabilization problem is reduced to the solution of the matrix Diophantine equation [Kuc 1],

$$A_1\ D_2 + B_1\ N_2 = U \in \mathbf{R}\rho^{pxp}(s) \tag{3.34a}$$

or

$$D_1\ A_2 + N_1\ B_2 = \overline{U} \in \mathbf{R}\rho^{mxm}(s) \tag{3.34b}$$

where U, $\bar{U}$ are arbitrary $\mathbf{R}_p(s)$-unimodular matrices, which characterises the centralized stabilization problem and is referred to as the Centralized Diophantine Equation (CDE). Notice that the above satisfy the Bezout identity [Kai 1]:

$$\begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -N_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{3.35}$$

where (D_2, N_2) is a solution to (3.34a) and (D_1, N_1) a solution to (3.34b).

Multiplying (3.35) on the left, right by the $\mathbf{R}_p(s)$ - unimodular matrices respectively

$$\begin{bmatrix} I_m & T \\ 0 & I_p \end{bmatrix}, \begin{bmatrix} I_m & -T \\ 0 & I_p \end{bmatrix}$$

where $T \in \mathbf{R}_p^{mxp}(s)$ is chosen such that

$|D_1 - TB_1| \neq 0$, $|D_2 - B_2T| \neq 0$ we obtain

$$\begin{bmatrix} (D_1-TB_1) & (N_1+TA_1) \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -(N_2+A_2T) \\ B_2 & (D_2-B_2T) \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{3.36}$$

The family of all stabilizing controllers $C \in \mathbf{R}^{m\times p}(s)$ is given by

$$C = (D_1 - TB_1)^{-1} (N_1 + TA_1) = N_2 + A_2T) (D_2 - B_2T)^{-1} \quad (3.37)$$

The arbitrary parameter $T \in \mathbf{R}_{\rho}^{m\times p}$ generates the family of stabilizing controllers for a given plant. The parameterization is defined in such a way that the various feedback system gains are linear in the design parameter T, thus the design parameter may be selected to meet a prescribed design constraint: tracking and disturbance rejection [Fra 1, Cal 2, Sae 1]; robust design [Vid 1, Vis 1]; pole placement [Bra 1, Vid 1]. Finally, if any remaining design latitude exists after the design constraints have been met it may be used to optimize some measure of system performance: sensitivity [Sae 1, Vid 1]; energy consumption etc.

The key to the parameterization is the solution of the Centralized Diophantine Equation. The solvability of the CDE is reduced to the solution of the following matrix equation over $\mathbf{R}_{\rho}(s)$.

$$\underbrace{[A_1 \quad B_1]}_{} \underbrace{\begin{bmatrix} D_2 \\ N_2 \end{bmatrix}}_{\triangleq X} = U \,, \quad [D_1 \quad N_1] \underbrace{\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}}_{\triangleq P^t} = \bar{U} \quad (3.38)$$

or the more general form

$$PX = U \,, \quad X^tP^t = \bar{U} \quad (3.39)$$

where $P(P^t)$, is a matrix defined on the plant, $U(\bar{U})$ an arbitrary biproper matrix and $X(X^t)$ the unknown matrix which defines the centralized controllers.

3.3.2 Decentralized Stabilization Problem

For a linear system the problem of decentralized stabilization has been examined in [Dav 1, Cor 1, Cor 2, Won 1] and can be formulated within the $\mathbf{R}_\rho(s)$-stabilization form over $\mathbf{K}$ as follows:

Let the linear time invariant system in the feedback configuration of figure (3.4) be well posed with Ω-stabilizable, Ω-detectable plant $P \in \mathbf{R}_{pr}^{pxm}(s)$ and controller $C \in \mathbf{R}^{mxp}(s)$. The decentralized stabilization problem is to determine conditions under which a decentralized stabilizing controller C may be defined such that the closed loop system is stable.

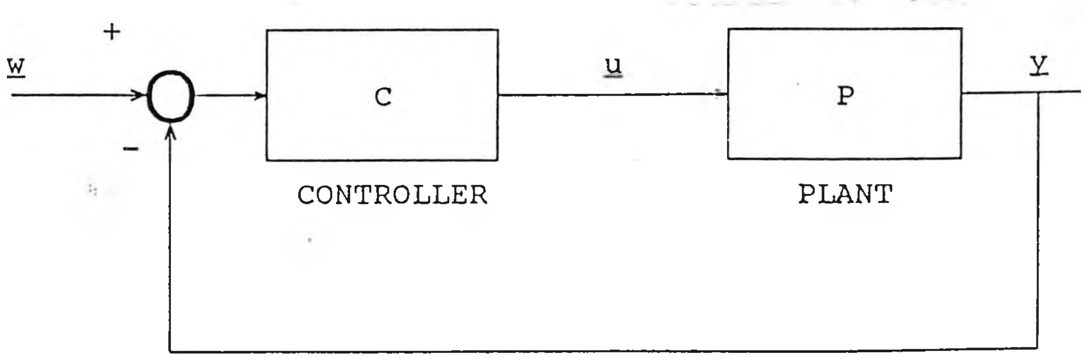


Figure (3.4): Basic feedback control configuration

Let the input-output description of the plant P by given by

$$\begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_i \\ \vdots \\ \underline{y}_r \end{bmatrix} = \begin{bmatrix} P_{11} & \cdots & P_{1i} & \cdots & P_{1r} \\ \vdots & \ddots & \vdots & & \vdots \\ P_{i1} & & P_{ii} & & P_{ir} \\ \vdots & & \vdots & \ddots & \vdots \\ P_{r1} & \cdots & P_{ri} & \cdots & P_{rr} \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_i \\ \vdots \\ \underline{u}_r \end{bmatrix} \tag{3.40}$$

where $P_{ii} \in \mathbf{R}_{pr}^{P_i x m_i}(s)$ represents the decentralized plants and $P_{ij} \in \mathbf{R}_{pr}^{P_i x \ell_i}(s)$, $i \neq j$, express the interconnections in the decentralized system.

The decentralized stabilization problem is to determine controllers of the type

$$C = \begin{bmatrix} C_1 & & & & 0 \\ & \ddots & & & \\ & & C_i & & \\ & & & \ddots & \\ 0 & & & & C_r \end{bmatrix} \tag{3.41}$$

such that the closed loop system under the feedback law

$$\underline{u}_i = C_i\{\underline{y}_i - \underline{\omega}_i\} \tag{3.42}$$

is stable.

Let the left, right $\mathbf{R}_\rho(s)$-coprime MFD's of the plant and local controllers be respectively

$$P = A_1^{-1}B_1 = B_2A_2^{-1} \tag{3.43}$$

$$C_i = D_{1i}^{-1}N_{1i} = N_{2i}D_{2i}^{-1} \;, \; i \in \underline{r} \tag{3.44}$$

where

$$A_1 \in \mathbf{R}_\rho^{p\times p}(s),\ B_1 \in \mathbf{R}_\rho^{p\times m}(s);\ B_2 \in \mathbf{R}_\rho^{p\times m}(s),\ A_2 \in \mathbf{R}_\rho^{m\times m}(s)$$

$$D_{1i} \in \mathbf{R}_\rho^{mi\times mi}(s),\ N_{1i} \in \mathbf{R}^{mi\times pi}(s);\ N_{2i} \in \mathbf{R}_\rho^{mi\times pi}(s),\ D_{2i} \in \mathbf{R}_\rho^{pi\times pi}(s)$$

Then,

$$C = \text{diag.}\{D_{1i}\}^{-1} \begin{bmatrix} N_{11} & & & & 0 \\ & \ddots & & & \\ & & N_{1i} & & \\ & & & \ddots & \\ 0 & & & & N_{1r} \end{bmatrix} = D_1^{-1}N_1$$

$$= \begin{bmatrix} N_{21} & & & & 0 \\ & \ddots & & & \\ & & N_{2i} & & \\ & & & \ddots & \\ 0 & & & & N_{2r} \end{bmatrix} \text{diag.}\{D_{2i}\}^{-1} = N_2 D_2^{-1} \in \mathbf{R}^{m\times p}(s)$$

(3.45)

defines the left, right $\mathbf{R}_\rho(s)$ - coprime MFD of the decentralized controller.

From (3.34) the centralized stabilization problem is reduced to the solution of the matrix Diophantine equation over $\mathbf{R}_\rho(s)$.

$$A_1 D_2 + B_1 N_2 = U \tag{3.46a}$$

or

$$D_1 A_2 + N_1 B_2 = \overline{U} \tag{3.46b}$$

With the further assumption that the $\mathbf{R}_\rho(s)$-coprime MFD of the controller C is of the form (3.45) the decentralized stabilization problem is reduced to the solution of the matrix Diophantine equation

$$A_1 \operatorname{diag}\{D_{2i}\} + B_1 \text{ block diag } \{N_{2i}\} = U \tag{3.47a}$$

or

$$\operatorname{diag}\{D_{1i}\} A_2 + \text{block diag } \{N_{1i}\} B_2 = U \tag{3.47b}$$

over $\mathbf{R}_\rho(s)$ where U, $\overline{U}$ arbitrary $\mathbf{R}_\rho(s)$-unimodular, characterises the decentralized stabilization problem and is referred to as the <u>Decentralized Diophantine Equation</u> (DDE).

Partitioning A_2, B_2 according to the block structure of diag $\{D_i\}$, block diag $\{N_i\}$ we have

$$\begin{bmatrix} D_{11} & & & & 0 \\ & \ddots & & & \\ & & D_{1i} & & \\ & & & \ddots & \\ 0 & & & & D_{1r} \end{bmatrix} \begin{bmatrix} A_{21} \\ \\ A_{2i} \\ \\ A_{2r} \end{bmatrix} + \begin{bmatrix} N_{11} & & & & 0 \\ & \ddots & & & \\ & & N_{1i} & & \\ & & & \ddots & \\ 0 & & & & N_{1r} \end{bmatrix} \begin{bmatrix} B_{21} \\ \\ B_{2i} \\ \\ B_{2r} \end{bmatrix} = \begin{bmatrix} U_1 \\ \\ U_i \\ \\ U_r \end{bmatrix} \tag{3.48}$$

from which

$$D_{1i}A_{2i} + N_{1i}B_{2i} = \overline{U}_i \tag{3.49}$$

where

$D_{1i} \in \mathbf{R}_\rho^{\,mixmi}(s)$, $N_{1i}^{\,mixpi} \in \mathbf{R}_\rho(s)$; $A_{2i} \in \mathbf{R}_\rho^{\,mixm}(s)$, $B_{2i} \in \mathbf{R}_\rho^{\,pixm}(s)$, and $\overline{U}_i \in \mathbf{R}_\rho^{\,mixm}(s)$ is an arbitrary (part of an arbitrary)

$\mathbf{R}_\rho(s)$-left unimodular matrix.

In a similar manner partitioning A_1, B_1 according to the block structure of diag $\{D_{2i}\}$, block diag $\{N_{2i}\}$ reveals

$$A_{1i}D_{2i} + B_{1i}N_{2i} = U_i \tag{3.50}$$

where

$A_{1i} \in \mathbf{R}_\rho^{pxpi}(s)$, $B_{1i} \in \mathbf{R}_\rho^{pxmi}(s)$, $D_{2i} \in \mathbf{R}_\rho^{pxpi}(s)$, $N_{2i} \in \mathbf{R}_\rho^{mixpi}(s)$ and $U_i \in \mathbf{R}^{pxpi}$ is an arbitrary (part of an arbitrary) $\mathbf{R}_\rho(s)$-right unimodular matrix.

Equations of the type (3.49), 3.50) will be referred to as <u>Generalised Diophantine Equations</u> (GDE) and systems of matrix equations over $\mathbf{R}_\rho(s)$.

$$\underbrace{[D_{1i} \quad N_{1i}]}_{\triangleq X^t} \underbrace{\begin{bmatrix} A_{2i} \\ B_{2i} \end{bmatrix}}_{\triangleq P^t} = U_i \quad , \quad i \in \underline{r} \text{ with the} \tag{3.51}$$

additional constraint that U_i arbitrary but $[U_i]$ $\mathbf{R}_\rho(s)$-unimodular.

or of the more general form

$$X^tP^t = U \tag{3.52}$$

where $P^t \in \mathbf{R}_\rho^{kxr}(s)$ is a matrix defined on the plant, $\overline{U} \in \mathbf{R}_\rho^{pxr}(s)$ an arbitrary biproper matrix and $X^t \in \mathbf{R}_\rho^{pxk}$ the unknown matrix which defines the decentralized controllers. Note that the term "generalised" refers to the non square nature of the $\overline{U}_i$'s in (3.51), (3.52).

Alternatively for an $\mathbf{R}_\rho(s)$-left MFD of the plant,

$$\underbrace{[A_{1i} \quad B_{1i}]}_{\triangleq P} \underbrace{\begin{bmatrix} D_{2i} \\ N_{2i} \end{bmatrix}}_{\triangleq X} = U_i \ , \quad i \in r \text{ with the additional constraint that } U_i \text{ arbitrary but } [\ldots U_i \ldots] \ \mathbf{R}_\rho(s)\text{-unimodular.} \tag{3.51}$$

reduces to the more general form

$$PX = U \tag{3.54}$$

where $P \in \mathbf{R}_\rho^{pxk}(s)$ is a matrix defined on the plant, $U \in \mathbf{R}_\rho^{pxr}(s)$ an arbitrary biproper matrix and $X \in \mathbf{R}_\rho^{kxr}(s)$ the unknown matrix which defines the decentralized controllers.

3.3.3 <u>Closed Loop Stabilization</u>

Let the linear time invariant and closed loop stabilizable (ie. free of unstable hidden modes) feedback system of figure (3.1) with plant transfer function $P \in \mathbf{R}_{pr}^{pxm}(s)$ be described in terms of $\mathbf{R}_\rho(s)$-left, right coprime MFD as

$$P = A_1^{-1}B_1 = B_2A_2^{-1} \tag{3.55}$$

where

$$A_1 \in \mathbf{R}_\rho^{pxp}(s), \ B_1 \in \mathbf{R}_\rho^{pxm}(s), \ B_2 \in \mathbf{R}_\rho^{pxm}(s), \ A_2 \in \mathbf{R}_\rho^{mxm}(s).$$

It is known [You 1, You 2, Sae 1, Des 1, Vid 2] that there exists a stabilizing compensator $C \in \mathbf{R}^{mxp}(s)$ which is free of unstable hidden modes such that the closed loop system is internally asymptotically stable i.e. all its hidden modes including the uncontrollable and unobservable lie in the closed left hand complex plane C_- and the closed loop transfer function matrix

$$H_{\underline{e}|\underline{\omega}} : \underline{e} := \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} =: \underline{y} \tag{3.56}$$

is an element of $\mathbf{R}_\rho^{(m+p)x(m+p)}(s)$ where $\rho \equiv C_+$. It is known that if $C_a \in \mathbf{R}^{mxp}(s)$ is any arbitrary compensator for the plant P then C_a has a non-unique left, right $\mathbf{R}_\rho(s)$-coprime MFD.

$$C_a = D_1^{-1}N_1 = N_2D_2^{-1} \tag{3.57}$$

where

$$D_1 \in \mathbf{R}_\rho^{mxm}(s),\ N_1 \in \mathbf{R}_\rho^{mxp}(s),\ N_2 \in \mathbf{R}_\rho^{mxp}(s),\ D_2 \in \mathbf{R}_\rho^{pxp}(s)$$

satisfying the Bezout identities:

$$\begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -N_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{3.58}$$

$$\begin{bmatrix} A_2 & -N_2 \\ B_2 & D_2 \end{bmatrix} \begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{3.59}$$

Multiplying the equation (3.58) on the left and right by the $\mathbf{R}_\rho(s)$-unimodular matrices.

$$\begin{bmatrix} I_m & T \\ 0 & I_p \end{bmatrix}, \begin{bmatrix} I_m & -T \\ 0 & I_p \end{bmatrix} \tag{3.60}$$

respectively, where $T \in \mathbf{R}_p^{mxp}(s)$ and such that $|D_1 - TB_1| \neq 0$, $|D_2 - B_2T| \neq 0$ we obtain the Bezout identity:

$$\begin{bmatrix} (D_1-TB_1) & (N_1+TA_1) \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -(N_2+A_2T) \\ B_2 & (D_2-B_2T) \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{3.61}$$

The set of all stabilizing compensators $C \in \mathbf{R}^{mxp}(s)$ for the plant P is given by

$$C = (D_1 - TB_1)^{-1} (N_1 + TA_1) = (N_2 + A_2T) (D_2 - B_2T)^{-1} \tag{3.62}$$

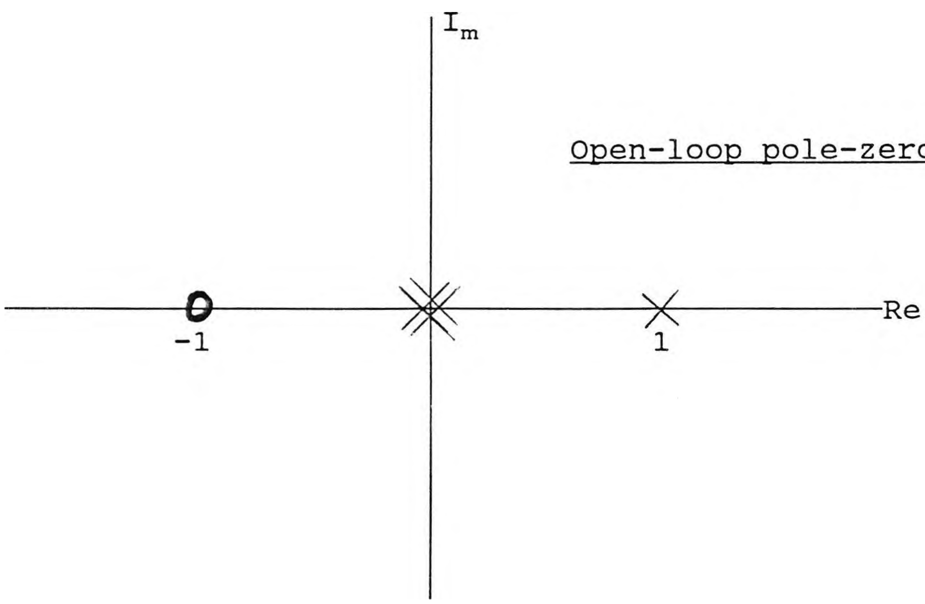
The arbitrary parameter $T \in \mathbf{R}_p^{mxp}(s)$ generates the family of stabilizing compensators but must be chosen such that $|D_1 - TB_1| \neq 0$, $|D_2 - B_2T| \neq 0$ so that the selected controller is realizable.

The parameterization of the set of stabilizing compensators for a given plant is the first step in the design process. Step two requires the selection of an appropriate parameter matrix T such that a prescribed design constraint is achieved e.g.: tracking and disturbance rejection [Fra 1, Cal 2, Sae 1); robust design [Vis 1, Vid 1]; pole placement design [Bra 1 Vid 1]. The parameterization is defined in such a way that the various feedback system gains are linear in the resultant design parameter T, Thus the design parameter may finally be selected to meet the design constraint and/or optimise some measure of system performance such as sensitivity, energy consumption or the like.

Example (3.1)

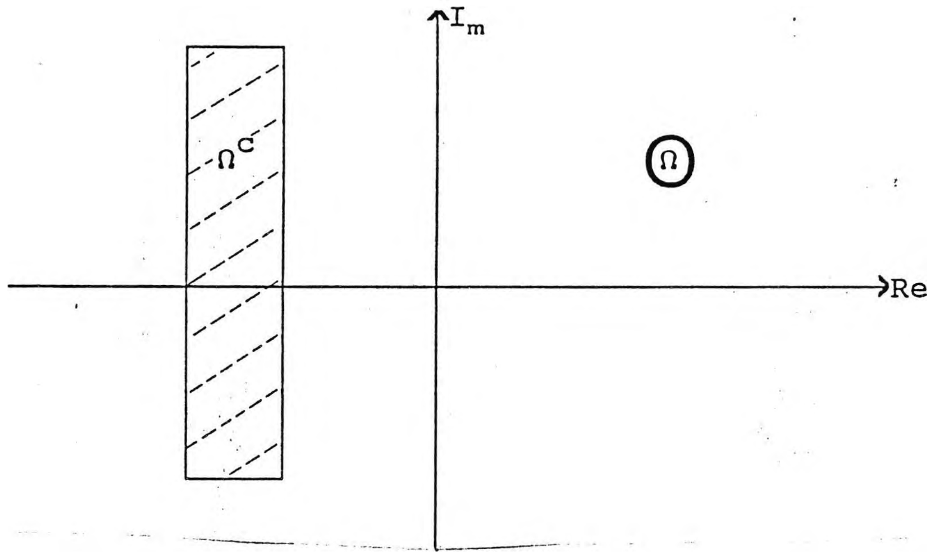
Let $T \in \mathbf{R}^{2x2}(s) = N_R D_R^{-1}$

$$T = \begin{bmatrix} \frac{s+1}{s^2} & 0 \\ \frac{1}{s(s-1)} & \frac{1}{1-s} \end{bmatrix} = \underbrace{\begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}}_{N_R} \underbrace{\begin{bmatrix} s^2 & 0 \\ 1 & 1-s \end{bmatrix}^{-1}}_{D_R^{-1}}$$



Open-loop pole-zero pattern

Determine the family of stabilizing compensators that place all the poles of the closed loop system in the desired area of the complex plane i.e.in Ω^c



$\rho = \Omega \cup \{\infty\}$

Let the desired set of poles inside Ω^c be $\{-1 \pm j, -1\}$ and others due to the compensator but inside Ω^c

Let

$$D_{R_d} = \begin{bmatrix} s^2 + 2s + 2 & 0 \\ -(s+1) & -(s+1) \end{bmatrix}$$

having zeros: $-1 \pm j, -1$

$$T = N_R D_R^{-1} = (N_R D_{R_d}^{-1}) (D_R D_{R_d}^{-1})^{-1}$$

$$\triangleq B_2 \qquad \triangleq A_2^{-1}$$

$T = B_2 A_2^{-1}$ is the $\mathbf{R}_\rho(s)$-coprime MFD of T where $B_2 \in \mathbf{R}_\rho^{pxm}(s)$, $A_2 \in \mathbf{R}_\rho^{mxm}(s)$ both proper and Ω-stable and coprime in the region $\rho := \Omega \cup \{\infty\}$. Thus by elementary row operations over $\mathbf{R}_\rho(s)$ reduce (B_2, A_2) to its Smith form over $\mathbf{R}_\rho(s)$ i.e. there exists a non unique $\mathbf{R}_\rho(s)$-unimodular such that

$$\begin{bmatrix} U_L \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

where U_L can be partitioned according to the block structure of equation (3.58) as

$$\left[\begin{array}{c|c} D_1 & N_1 \\ \hline -B_1 & A_1 \end{array}\right] \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

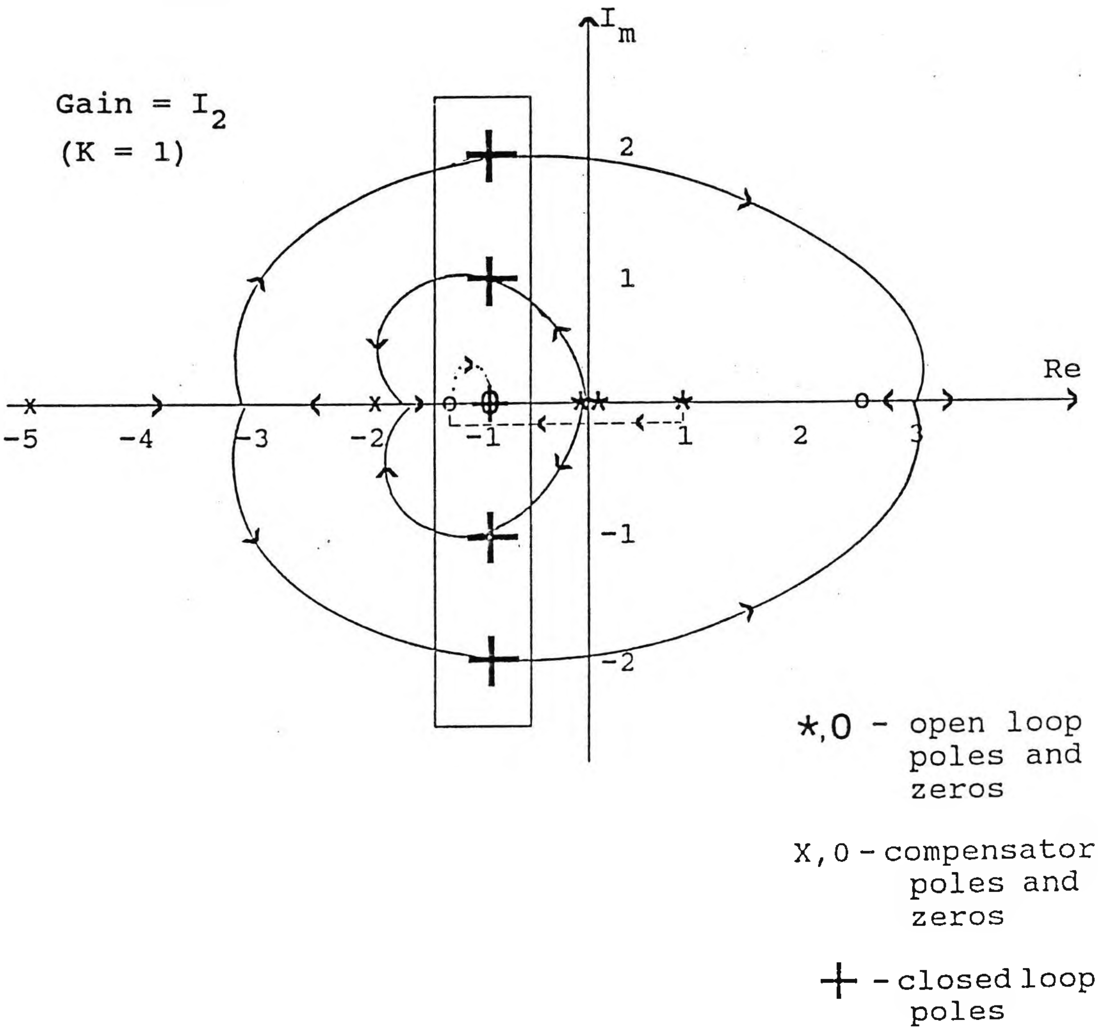
where $C_a = P_1^{-1} N_1$ gives rise to a (particular solution) stabilizing compensator for the closed loop system. Thus

$$\left[\begin{array}{cc|cc} \frac{(s+2)(s+5)}{(s^2+2s+5)} & 0 & \frac{-3s^2+4s+1}{s^2+2s+5} & 0 \\ \frac{s+5}{s^2+2s+5} & 1 & \frac{-2s^2-5s}{s^2+2s+5} & -2 \\ \hline \frac{1}{s+2} & \frac{1}{s+2} & \frac{-s}{s+2} & \frac{s-1}{s+2} \\ \frac{-1}{s^2+2s+2} & \frac{5}{s^2+2s+2} & 0 & \frac{s^2-}{s^2+2s+2} \end{array}\right] \left[\begin{array}{cc} \frac{s^2}{s^2+2s+2} & 0 \\ \frac{s}{s^2+2s+2} & \frac{s-1}{s+1} \\ \hline \frac{s+1}{s^2+2s+2} & 0 \\ 0 & \frac{-1}{s+1} \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow C_a = D_1^{-1}N_1 = \begin{bmatrix} \frac{(s+2)(s+5)}{(s^2+2s+5)} & 0 \\ \frac{(s+5)}{(s^2+2s+5)} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{-3s^2+4s+10}{s^2+2s+5} & 0 \\ \frac{-2s^2-5}{s^2+2s+5} & -2 \end{bmatrix}$$

$$C_a = \begin{bmatrix} \frac{-3s^2+4s+10}{(s+2)(s+5)} & 0 \\ \frac{-2(s+1)}{(s+2)} & -2 \end{bmatrix}$$ stabilizes the closed loop

system and places the closed loop system poles in Ω^c.



3.3.4 Model matching

Consider the closed loop transfer function matrix from $\underline{\omega}_1$ to $\underline{y}_2$ in Figure (3.1) denoted $H_{1,2}$.

$$H_{1,2} = H_{1,2}(P,C) = (I + PC)^{-1}PC \tag{3.63}$$

with C the selected stabilizing compensator such that $H_{1,2} \in \mathbf{R}_\rho^{pxp}(s)$. Then we have

Proposition (3.4): [Var 1] Let $P \in \mathbf{R}_{pr}^{pxm}(s)$ with $P = A_1^{-1}B_1 = B_2A_2^{-1}$ and C the set of stabilizing compensators with a selected compensator $C_a = D_1^{-1}N_1 = N_2D_2^{-1}$ such that (3.38) holds true. Then $H_{1,2}$ satisfies the relations.

$$H_{1,2} = B_2(N_1 + WA_1) \tag{3.64}$$

$$I_p - H_{1,2} = (D_2 - B_2W)A_1 \tag{3.65}$$

■

From the above proposition it follows that the matrices $X = N_1 + WA_1 \in \mathbf{R}_\rho^{mxp}(s)$ and $Y = D_2 - B_2W \in \mathbf{R}_\rho^{pxp}(s)$ represent a pair of solutions to the matrix equations

$$H_{1,2} = B_2X \tag{3.66}$$

$$I_p - H_{1,2} = YA_1 \tag{3.67}$$

If the matrices $B_2 \in \mathbf{R}_\rho^{pxm}(s)$, $A_1 \in \mathbf{R}_\rho^{pxp}(s)$ and $H_{1,2} \in \mathbf{R}_\rho^{pxp}(s)$ are all known then the problem of determining conditions under which the matrix equations (3.66), (3.67) have proper solution $X \in \mathbf{R}_{pr}^{mxp}(s)$ $Y \in \mathbf{R}_{pr}^{pxp}(s)$ is known as the

Exact Model Matching Problem (EMMP) or Model Following Problem (MFP) and has been the subject of numerous investigations. [Wol 2, Wol 3, For 1, Var 2].

With the further restriction that solutions of the above type problem are required to be proper and stable (ie. have no poles at $s = \infty$ and inside the region $C_+ = \{s \in C/\text{Re } s \geq 0\}$) the problem is termed the Stable Exact Model Matching Problem (SEMMP) and has been examined using various approaches [Wol 4, Sco 1, And 1, Kuc 2, Per 1, Emr, Kar 1].

Proposition (3.5): [Var 1] Let $B_2 \in \mathbf{R}_\rho^{p\times m}(s)$ and $H_{1,2} \in \mathbf{R}_\rho^{p\times p}(s)$ with $\text{rank}_{\mathbf{R}(s)} B_2 = \text{rank}_{\mathbf{R}(s)} H_{1,2} = p$. Let $T_R \in \mathbf{R}_\rho^{m\times m}(s)$ an $\mathbf{R}_\rho(s)$-unimodular matrix representing elementary column operations over $\mathbf{R}_\rho(s)$ reducing B_2 to $[B_{2\rho}, O_{p,m-p}]$ i.e. let

$$B_2 T_R = [B_{2\rho}, O_{p,m-p}] \tag{3.68}$$

Then,

i) $B_{2\rho} \in \mathbf{R}_\rho^{p\times p}(s)$ "is a structure matrix in ρ" of B_2 [ie.a g.c.l.d in ρ (of the columns of) of a "right fractional numerator" $B_2 \in \mathbf{R}_\rho^{p\times m}(s)$ of the plant P].

(ii) Eqn. (3.66) has a solution $X \in \mathbf{R}_\rho^{p\times m}(s)$ iff

$$H_1 := B_{2\rho}^{-1} H_{1,2} \in \mathbf{R}_\rho^{p\times p}(s) \tag{3.69}$$

iii) If condition (3.69) is satisfied then a general solution $X \in \mathbf{R}_\rho^{m\times p}(s)$ of (3.24) is given by

$$X = T_R[H_1 : Z]^t \in \mathbf{R}_\rho^{m \times p}(s) \tag{3.70}$$

where $Z \in \mathbf{R}_\rho^{(m-p) \times p}(s)$ and otherwise arbitrary.

■

Remark (3.4): If $p = m = \text{rank}_{\mathbf{R}(s)} P$ then $B_{2\rho} \equiv B_2$ and $T_R = I_p$. In such a case from (3.69) and (3.70) we observe that (3.66) has a solution $X \in \mathbf{R}_\rho^{p \times p}(s)$ iff

$$H_1 = X : B_2^{-1} H_{1,2} \in \mathbf{R}_\rho^{p \times p}(s) \tag{3.71}$$

which for fixed $H_{1,2}$ is also the "unique" solution of (3.66).

From the above it also follows that (3.67) has a solution $Y \in \mathbf{R}_\rho^{p \times p}(s)$ iff

$$H_2 = Y := (I_p - H_{1,2}) A_1^{-1} \in \mathbf{R}_\rho^{p \times p}(s) \tag{3.72}$$

which is also a "unique" solution (3.66).

The above results characterise the family of all "model" closed loop transfer function matrices $H_{1,2} \in \mathbf{R}_\rho^{p \times p}(s)$ which are obtainable from (3.67), (3.66) by some $X \in \mathbf{R}_\rho^{m \times p}(s)$, $Y \in \mathbf{R}_\rho^{p \times p}(s)$ or equivalently by some stabilizing compensators C (see Proposition (3.4)). Thus from (3.69) and (3.72) the family H* of all such $H_{1,2}$ is given by

$$H^* = \{H_{1,2} \in \mathbf{R}_\rho^{p \times p}(s) \mid H_{1,2} = B_{2\rho} H_1 - H_{1,2} = H_2 A_1$$
$$\text{where } H_i \in \mathbf{R}_\rho^{p \times p}(s) \text{ and } \text{rank}_{\mathbf{R}(s)} H_i = p,\ i=1,2\} \tag{3.73}$$

CHAPTER 4

STRUCTURE OF MATRICES OVER A PRINCIPAL IDEAL DOMAIN

4.1 Introduction

The main aim of this section is to investigate further the structural properties of matrices which provide solutions to matrix equations of the type

$$AX = B \tag{4.1a}$$

$$YA' = B' \tag{4.1b}$$

$$AXB = C \tag{4.1c}$$

where the given matrices A, B, C, A', B' are in general rational and the solution matrices X,Y are determined over a Euclidean ring $\mathbf{K}$ such that $\mathbf{R}(s)$ may be expressed as the field of fractions of $\mathbf{K}$.

Notice that equations (4.1a), (4.1b) are central to solution of the more generalised Diophantine equations

$$A_1X_1 + A_2X_2 +,\ldots, + A_pX_p = B \tag{4.2a}$$

$$Y_1A'_1 + Y_2A'_2 +,\ldots, + Y_pA_p'^1 = B' \tag{4.2b}$$

where B, B' are general non square matrices. Thus, solvability of (4.1) also provides conditions for solvability of the more general set of equations (4.2).

We have discussed in the previous chapter the importance of the Euclidean rings $\mathbf{R}[s]$-polynomials, $\mathbf{R}_{pr}(s)$-proper rational functions and $\mathbf{R}_{\rho}(s)$-proper and stable rational functions. The results presented here for a general rational matrix $A \in \mathbf{R}^{p \times m}(s)$ using Smith-McMillan, Hermite-McMillan forms may also be specialised to the case of matrices defined over a PID $\mathbf{K}$ using Smith and Hermite forms. Note that in the following $\mathbf{K}$ will denote one of the Euclidean rings $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ or $\mathbf{R}_{\rho}(s)$.

We begin by introducing the notions of square and non square divisors of a matrix. It is then possible to use these notions to define coprimeness conditions, matrix projectors and annihilators as well as generalised left and right inverses of a rational matrix.

Although the results are valid for PIDs in general, we are concerned here with the Euclidean rings thus instead of PIDs we may say that $\mathbf{K}$ is a Euclidean ring (the difference is that in Euclidean rings, the unimodular matrices are expressed as products of elementary transformations).

4.2 Matrix Divisors

Let $A \in \mathbf{R}^{p \times m}(s)$, $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = \mathbf{p} \leq \min\ (p,m)$. We may associate the following rational vector spaces with the matrix A:

$\mathbf{X}_A{}^r \triangleq$ row-span$_{\mathbf{R}(s)}\{A\}$ = row space of A

$\mathbf{X}_A{}^c \triangleq$ column-span$_{\mathbf{R}(s)}\{A\}$ = column-space of A

$\mathbf{N}_A{}^{\ell} \triangleq \mathbf{N}_{\ell}\{A\}$ = left-null space of A

$\mathbf{N}_A{}^r \triangleq \mathbf{N}_r\{A\}$ = right-null space of A and the following **K**-modules of the matrix A:

$\mathbf{M}_A{}^r \triangleq$ row-span$_{\mathbf{K}}\{A\}$ = **K**-row module of A

$\mathbf{M}_A{}^c \triangleq$ column-span$_{\mathbf{K}}\{A\}$ = **K**-column module of A.

4.2.1 Left, Right Square Divisors of a Rational Matrix

Definition (4.1) Let $A \in \mathbf{R}^{p\mathrm{x}m}(s)$, $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = r \le \min(p,m)$ and **K** be a P.I.D. A matrix $T = T_{N1}T_{D1}{}^{-1} = T_{D2}{}^{-1}T_{N2} \in \mathbf{R}^{r\mathrm{x}r}(s)$, T_{Ni}, $T_{Di} \in \mathbf{K}^{r\mathrm{x}r}$, i=1,2 will be called an **R**(s)-left, right divisor (l.r.d.) of A over **K** if there exist matrices $P \in \mathbf{K}^{p\mathrm{x}r}$, $Q \in \mathbf{K}^{r\mathrm{x}m}$, $\mathrm{rank}_{\mathbf{R}(s)}\{P\} = r$, $\mathrm{rank}_{\mathbf{R}(s)}\{R\} = r$ such that

$$A = PTQ \tag{4.3}$$

T will be called an **R**(s)-greatest left, right divisor (g.l.r.d) of A over **K** if it is a l.r.d. of A and P,Q are irreducible over **K** (i.e. no zeros). Note: since P,Q are over **K**, we talk about Smith forms and the matrices P,Q have the Smith forms over **K** of the type $[I_r,\ 0]^t$, $[I_r,\ 0]$ respectively. (4.4)

If we restrict ourselves to matrices from a PID, $A \in \mathbf{K}^{p\mathrm{x}m}$. Then, $T \in \mathbf{K}^{r\mathrm{x}r}$ (i.e.T_{Di} are **K**-unimodular matrices, i=1,2) defines a **K**-g.l.r.d of A. In this case a matrix for which a **K**-g.l.r.d is **K**-unimodular will be called prime.

■

The existence of a g.l.r.d is established in the following result.

Proposition (4.1): Let $A \in \mathbf{R}^{pxm}(s)$, $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = r \leq \min(p,m)$ and $\mathbf{K}$ be a PID.

There always exist matrices $P \in \mathbf{K}^{pxr}$, $Q \in \mathbf{K}^{rxm}$, $\mathrm{rank}_{\mathbf{R}(s)}\{P\} = r$, $\mathrm{rank}_{\mathbf{R}(s)}\{Q\} = r$, $T = T_{N1}T_{D1}^{-1} = T_{D2}^{-1}T_{N2} \in \mathbf{R}^{rxr}(s)$, T_{Ni}, $T_{Di} \in \mathbf{K}^{rxr}$, $i=1,2$, $\det(T) = t(s) \in \mathbf{R}(s) - \{0\}$ and the matrices P,Q have the Smith forms over $\mathbf{K}$ of the type $[I_r, O]_t$, $[I_r, O]$ respectively such that:

$$A = PTQ \tag{4.5}$$

Proof

Let S_A be the Smith-McMillan form of A over $\mathbf{K}$. Then there exist $\mathbf{K}$-unimodular matrices such that

$$A = U_L\, S_A U_R \tag{4.6}$$

or $$\hat{U}_L A \hat{U}_R = S_A \tag{4.7}$$

where

$$U_L,\ \hat{U}_L \in \mathbf{K}^{pxp},\ U_R\hat{U}_R \in \mathbf{K}^{mxm}$$

$$U_L\hat{U}_L = \hat{U}_L U = I_p$$

$$U_R\hat{U}_R = \hat{U}_R U_R = I_m$$

S_A is the Smith-McMillan form of A written as

$$S_A = \begin{bmatrix} S^*_r & O_{r,m-r} \\ O_{p-r,r} & O_{p-r,m-r} \end{bmatrix} \tag{4.8}$$

where

$S_r^* = \text{diag.}\{\epsilon_1\, \psi^{-1}, \ldots, \epsilon_r\, \psi_r^{-1}\} \in \mathbf{R}^{rxr}(s)$ is the non-zero part of the Smith-Mcmillan form of A over $\mathbf{K}$, ϵ_i, $\psi_i \in \mathbf{K}$, $\epsilon_i\, \psi_i^{-1} \in \mathbf{R}(s)$.

By partitioning U_R, U_L according to the partitioning of S_A we have

$$A = \begin{array}{c} p\updownarrow \end{array} \underbrace{\begin{bmatrix} U_1 & | & U_2 \end{bmatrix}}_{\underset{r}{\longleftrightarrow}\quad \underset{p-r}{\longleftrightarrow}} \begin{bmatrix} S_r^* & O \\ O & O \end{bmatrix} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\underset{m}{\longleftrightarrow}} \begin{array}{l} \uparrow r \\ \downarrow m-r \end{array} \tag{4.9}$$

from which

$$A = U_1 S_r^* \, V_1 \tag{4.10}$$

where

$U_1 \in \mathbf{K}^{pxr}$, $V_1 \in \mathbf{K}^{rxm}$ and since U_1, V_1 are parts of $\mathbf{K}$-unimodular matrices with $\text{rank}_{\mathbf{R}(s)}\{U_1\} = r$, $\text{rank}_{\mathbf{R}(s)}\{V_1\} = r$ then, the Smith forms of U_1, V_1 over $\mathbf{K}$ are $[I_r, O]^t$, $[I_r, O]$ respectively.

Clearly the non-zero part of the Smith-McMillan form $S_r^* = E\psi_1^{-1} = \psi_2^{-1}E \in \mathbf{R}^{rxr}(s)$ is a g.l.r.d of A. Furthermore if $A \in \mathbf{K}^{pxm}$ then $T = S_r^* \in \mathbf{K}^{rxr}$, ψ_i^{-1}, i=1,2 are $\mathbf{K}$-unimodular matrices, $\psi_i \in \mathbf{K}$ are units $\forall i \in r$. In such a case A and T have no poles in **K** and the same zero structure in **K**.

■

The above results establish the existence of g.l.r.d of any rational matrix $A \in \mathbf{R}^{pxm}(s)$. We note the following.

Remark (4.1): Let $A \in \mathbf{K}^{pxm}$ $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = r \le \min(p,m)$ and **K** be a PID, subring of **R**(s).

i) If $p \ge m$, then the notion of g.l.r.d (**K**-g.l.r.d) coincides with that of the standard notion of a right divisor (**K**-right divisor) of A, in this case R becomes an mxm **K**-unimodular matrix.

ii) If $p \le m$, then the notion of g.l.r.d. (**K** = g.l.r.d) coincides with that of the standard notion of a left divisor (**K**-left divisor) of A, in this case P becomes a pxp **K**-unimodular matrix.

■

For a general rational matrix the row module $\mathbf{M}_A{}^r$ (as defined over **K**) or column nodule $\mathbf{M}_A{}^c$ are well defined. The bases for such modules are general rational matrices and the canonical forms are defined via the Hermite-McMillan form i.e.

Let $A \in \mathbf{R}^{pxm}(s)$, $\mathrm{rank}_{\mathbf{R}(s)} \le \min(p,m)$ and let 'd' be the least common multiple of the elements of A over **K**. Then,

$$A = \frac{1}{d}\bar{A}, \qquad \bar{A} \in \mathbf{K}^{pxm}$$

Now the Column Hermite-Mcmillan form is given by $U_L\bar{A} = \bar{A}_H$, $\bar{A}_H$ is **K**-Hermite form with

$$U_L A = \frac{1}{d}\bar{A} = \frac{1}{d}\bar{A}_H \triangleq A^*_{H-M}$$

where A^*_{H-M} is the Hermite-McMillan form and it is a rational matrix. In these forms we can carry out all cancellations between factors of d and corresponding elements of $\bar{A}_H$.

Note that with $A \in \mathbf{R}^{pxm}(s)$ we have two different **K**-row modules associated with $\mathbf{M}_A{}^r$ = row $\text{span}_K\{A\}$ as defined in the text. $\bar{\mathbf{M}}_A{}^r$ = row $\text{span}_K\,\{\bar{A}\}$ where $\bar{A}$ is the numerator of any left MFDs over **K**. i.e. $\mathbf{M}_A{}^r$ is the module associated with all numerators (left) and it is also a **K**-module. Note that the vectors in $\mathbf{M}_A{}^r$ are general rational vectors and vectors in $\bar{\mathbf{M}}_A{}^r$ are from **K**. A similar set of results are defined for $\mathbf{M}_A{}^c$ = column $\text{span}_K\{A\}$ and associated $\bar{\mathbf{M}}_A{}^c$ modules.

Example (4.1): Let $\mathbf{K} \in \mathbf{R}[s]$

$$\underline{a}(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{s(s+1)}{(s+2)} \\ (s+1) \end{bmatrix} \quad \text{a rational vector.}$$

i) The $\mathbf{M}_A{}^r$ column-module is defined by $\{c(s), \underline{a}(s)\}$, $c(s) \in \mathbf{R}[s]$ arbitrary, and it is a general rational vector.

ii) The $\bar{\mathbf{M}}_A{}^r$ column module is defined by

$$\underline{a}(s) = \frac{1}{s(s+2)} \begin{bmatrix} s+2 \\ s^2(s+1) \\ s(s+1)(s+2) \end{bmatrix} = \frac{1}{d}\, \bar{\underline{a}}\ (s)$$

$\bar{\mathbf{M}}_A{}^r = \{c(s)\ \underline{a}(s)\}$, $c(s) \in \mathbf{R}[s]$ arbitrary.

Remark (4.2) Let $A \in \mathbf{R}^{pxm}(s)$, $\mathrm{rank}_{\mathbf{R(s)}}\{A\} = r \le \min\ (p,m)$ and $\mathbf{K}$ be a PID. If T is a g.l.r.d of A over $\mathbf{K}$ i.e.

$$A = PTQ = PT_{N1}T_{D1}{}^{-1}Q = PT_{D2}{}^{-1}T_{N2}Q$$

Then,

i) the rows of $T_{N2}Q$ define a basis for the module space $\bar{\mathbf{M}}_A{}^r$ and

the rows of $T_{D2}{}^{-1}T_{N2}Q$ define a basis for the module space $\mathbf{M}_A{}^r$

ii) the columns of PT_{N1} define a basis for the module space $\bar{\mathbf{M}}_A{}^C$ and

the columns of $PT_{N1}T_{D1}{}^{-1}$ define a basis for the module space $\mathbf{M}_A{}^c$.

By the above remark the characterization of all greatest left right divisors of a rational matrix A may be defined.

Proposition (4.2) Let $A \in \mathbf{R}^{pxm}(s)$ $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = r \leq \min(p,m)$ and $\mathbf{K}$ be a PID. If $T \in \mathbf{R}^{rxr}(s)$, $\hat{T} \in \mathbf{R}^{rxr}(s)$ are two g.l.r.d of A then T and $\hat{T}$ are equivalent over $\mathbf{K}$ denoted $T\ \mathbf{E}\ \hat{T}$.

Proof

Let T, $\hat{T}$ be two g.l.r.d of A over $\mathbf{K}$. Then,

$$A = PTQ \quad \text{and} \quad A = \hat{P}\,\hat{T}\,\hat{Q} \tag{4.11}$$

Since both P and $\hat{P}$ define basis matrices for M_A*^c and both Q and $\hat{Q}$ define basis matrices for M_A*^r, then

$$\hat{P} = PU_R \quad \text{and} \quad \hat{Q} = U_tQ \tag{4.12}$$

where U_R, $U_L \in \mathbf{K}^{rxr}$ are $\mathbf{K}$-unimodular matrices.

Thus by (4.11) and (4.12) we have

$$A = PTQ = \hat{P}\,\hat{T}\,\hat{Q} = PU_R\,\hat{T}\,U_LQ$$

and thus

$$P\{T - U_R\,\hat{T}\,U_L\}Q = 0 \tag{4.13}$$

Since P has no right-null space, $\mathbf{N}_r\{P\} = 0$ it follows that

$$\{T - U_R\,\hat{T}\,U_L\}Q = 0$$

Given that Q has no left-null space, $\mathbf{N}_L\{Q\} = 0$ it follows that

$$T - U_R\,\hat{T}\,U_L = 0$$

or

$$T = U_R \hat{T} U_L \Leftrightarrow T \mathbf{E} \hat{T}$$

■

Remark (4.3) If T is a g.l.r.d of A, then any other g.l.r.d of A may be obtained by

$$\hat{T} = U_R \, T \, U_L \tag{4.14}$$

where U_L, U_R are arbitrary unimodular matrices of dimensions rxr and all g.l.r.d. of A may be obtained via

$$T = U_R \, S^* \, U_L \tag{4.15}$$

where S* is the non zero part of the Smith McMillan form of A.

If we restrict ourselves to matrices from a PID i.e. $A \in \mathbf{K}^{pxm}$ then the notion of the greatest left right divisor provides the means for the canonical decomposition of A. This is established in the following result.

Proposition (4.3): Let $A \in \mathbf{K}^{pxm}$, $\mathrm{rank}_{\mathbf{R(s)}}\{A\} = r \leq \min\,(p,m)$ and $\mathbf{K}$ be a PID. The matrix A may be uniquely factorized as

$$A = P_H T_H Q_H \tag{4.16}$$

where P_H, Q_H are column, row Hermite (echelon type) minimal basis matrices for $M_A{*}^c$, $M_A{*}^r$ respectively and T_H is a $\mathbf{K}$-g.l.r.d of A.

Proof

If T is a **K**-g.l.r.d of A, then $A = PTQ$. We can always find **K**-unimodular matrices U_R, $U_L \in \mathbf{K}^{r\times r}$ such that

$$PU_R = P_H, \quad U_LQ = Q_H \tag{4.17}$$

where P_H, Q_H are the column, row Hermite form of P,Q respectively. It is known that P_H, Q_H are uniquely defined [Mar 1]. Thus from $A = PTQ$ we have

$$A = PU_RU_R^{-1}TU_L^{-1}U_LQ = P_H\{U_R^{-1}TU_L^{-1}\}Q_H = P_HT_HQ_H \tag{4.18}$$

The matrices P_H, Q_H uniquely characterize the strict equivalence classes $\mathbf{E}_R(P)$, $\mathbf{E}_L(Q)$. The uniqueness of T_H is established as follows. Let $\hat{T}_H$ be another **K**-g.l.r.d for which

$$A = P_H\hat{T}_HQ_H \tag{4.19}$$

Then from (4.14), (4.17) we have

$$A = P_H\hat{T}_HQ_H = P_HT_HQ_H \rightarrow P_H(\hat{T}_H - T_H)Q_H = 0$$

since $\mathbf{N}_r\ (P_H) = \{0\}$ and $\mathbf{N}_\ell\{Q_H\} = 0$, it follows that $T_H = \hat{T}_H$.

■

The notion of **K**-g.l.r.d defined above will be used to characterize the non square matrix divisors of a given matrix defined by Pernebo [Per 1].

4.2.2 Non Square Divisors of a Rational Matrix

Definition (4.2) Let $A \in \mathbf{R}^{p\times m}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = r \leq \min(p,m)$ and $\mathbf{K}$ be a PID. If A can be factorized as

$$A = LA_0 \tag{4.20}$$

where $L \in \mathbf{R}^{p\times r}(s)$, $\text{rank}_{\mathbf{R}(s)}\{L\} = r$ and $A_0 \in \mathbf{K}^{r\times m}$, then,

L is defined as an extended left divisor (e.l.d) of A over $\mathbf{K}$. L will be called a greatest extended left divisor (g.e.l.d) of A over $\mathbf{K}$ if L is an e.l.d. of A and every other e.l.d of A is also an e.l.d of L.

If we restrict ourselves to matrices from a PID, $A \in \mathbf{K}^{p\times m}$ then L is defined to be a $\mathbf{K}$-(greatest) extended left divisor ($\mathbf{K}$-(g).e.l.d) of A.

■

The notion of extended right divisor (e.r.d), $\mathbf{K}$-extended right divisor ($\mathbf{K}$-e.r.d) as well as the notion of greatest extended right divisor (g.e.r.d), $\mathbf{K}$-greatest extended right divisor ($\mathbf{K}$-g.e.r.d) of A may be introduced in a similar manner. The characterization of all e.l.d of a rational matrix A is considered next.

Theorem (4.1) Let $A \in \mathbf{R}^{p\times m}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = r \leq \min(p,m)$ and $\mathbf{K}$ be a PID. The matrix A has always a g.e.l.d. L which has the following properties

i) $L \in \mathbf{R}^{p\times r}(s)$ and may be expressed as

$$L = AX \tag{4.21}$$

for some appropriate $X \in \mathbf{K}^{m\times r}$

ii) If L and $\hat{L}$ are two g.e.l.d of A then $L \; \mathbf{E}_r \; \hat{L}$ i.e. a g.e.l.d is unique up to multiplication on the right by a $\mathbf{K}$-unimodular matrix.

<u>Proof</u>

i) By (4.6), (4.8) and (4.9) we have

$$A = [U_1 S*_r, \; O_{p-r}] U_R \tag{4.22}$$

and thus

$$A = U_1 \; S*_r \; V_1 = LV_1 \tag{4.23}$$

Clearly $L = U_1 S*_r \in \mathbf{R}^{p\times r}(s)$ and $\mathrm{rank}_{\mathbf{R}(s)}\{L\} = r$; thus, L is an e.l.d of A. By (4.22) we have

$$[L, \; O_{p-r}] = AU_R^{-1} = A\hat{U}_R \tag{4.24}$$

By partitioning the unimodular matrix $\hat{U}_R \in \mathbf{K}^{m\times m}$ according to the partitioning of $[L, \; O_{p-r}]$ then,

$$[L \mid O_{p-r}] \;\; = \;\; A \; [\underbrace{\hat{U}_1}_{r} \mid \underbrace{\hat{U}_2}_{m-r}] \;\; m \tag{4.25}$$

from which $L = A\hat{U}_1$ (4.26)

and thus L is expressed as in (4.22).

In order to show that L is a g.e.l.d of A, let us assume that L is an arbitrary e.l.d of A. Then,

$$A = L\ A_0 \tag{4.27}$$

and by inserting (4.27) into (4.26) we have

$$L = \hat{L}\ A_0\ \hat{U}_1$$

This clearly shows that $\hat{L}$ is also an e.l.d of L and thus L is a g.e.l.d of A.

ii) Let L_1 and L_2 be two g.e.l.d of A. Since L_1 is a g.e.l.d of A and L_2 is an e.l.d of A, then

$$L_1 = L_2\ U \tag{4.28}$$

for some matrix U over **K**. Analogously,

$$L_2 = L_1\ V \tag{4.29}$$

for some appropriate matrix V over **K**. By equations (4.28) and (4.29) we have

$$L_1 = L_1\ VU \tag{4.30}$$

$$L_2 = L_2\ UV \tag{4.31}$$

and since L_1, L_2 have linearly independent columns it follows that

$$VU = I \text{ and } UV = I \tag{4.32}$$

Thus the matrices V, U are both square and **K**-

unimodular. It has been shown that the g.e.l.d of A is unique up to multiplication on the right by a **K**-unimodular matrix. Given that there is one g.e.l.d L $\in \mathbf{R}^{pxr}(s)$ it follows that it is true for all g.e.l.d of A.

∎

In the case where $A \in \mathbf{K}^{pxm}$ the above result characterizes the **K**-e.l.d of A.

A similar statement can be made for the e.r.d (**K**-e.r.d) of $A \in \mathbf{R}^{pxm}(s)$ $(\in \mathbf{K}^{pxm})$.

From the proof of the above result the link between the g.e.l.d, g.e.r.d of A and the g.l.r.d of A and the corresponding decomposition of A is established. Thus, we may state

<u>Corollary (4.1)</u> Let $A \in \mathbf{R}^{pxm}(s)$, rank $\{A\} = r \leq \min(p,m)$, $T \in \mathbf{R}^{rxr}(s)$ be a g.l.r.d of A and let $A = PTQ$ where $P \in \mathbf{K}^{pxr}$, $Q \in \mathbf{K}^{rxm}$ and P, Q have the Smith forms $[I_r,\ 0]^t$, $[I_r,\ 0]$ respectively. Then,

i) A g.e.l.d of A, L and a g.e.r.d. of A, R are defined by

$$L = PT \in \mathbf{R}^{pxr}(s) \tag{4.34}$$

$$R = TQ \in \mathbf{R}^{rxm}(s) \tag{4.35}$$

and A may be factorized as

$$A = LQ = PR \tag{4.36}$$

ii) A g.e.l.d L and a g.e.r.d R of A may be expressed as

$$L = A\hat{Q} \tag{4.37}$$

$$R = \hat{P}A \tag{4.38}$$

where $\hat{Q} \in \mathbf{K}^{m \times r}$, $\hat{P} \in \mathbf{K}^{r \times p}$ are appropriate matrices with Smith forms $[I_r, 0]^t$, $[I_r, 0]$ respectively.

■

<u>Remark (4.4)</u> In the case where $A \in \mathbf{K}^{p \times m}$ with $\mathbf{K}$ a PID then a $\mathbf{K}$-g.e.l.d L and a $\mathbf{K}$-g.e.r.d R of A are defined by

$$L = PT = A\hat{Q} \tag{4.39}$$

$$R = TQ = \hat{P}A \tag{4.40}$$

From the above result the following interpretation of the $\mathbf{K}$-e.l.d ($\mathbf{K}$-e.r.d) and $\mathbf{K}$-g.e.l.d ($\mathbf{K}$-g.e.r.d) may be given.

<u>Remark (4.5)</u> Let $\mathbf{K}$ be a PID and L, L_g be a $\mathbf{K}$-e.l.d, $\mathbf{K}$-g.e.l.d of A and let R, R_g be a $\mathbf{K}$-e.r.d, $\mathbf{K}$-g.e.r.d of A respectively. Then,

$$A = L_g A_0 = L\bar{A}_0 \quad \text{and} \quad L_g = LM_0 \tag{4.41}$$

$$A = A'_0 R_g = A'_0 R \quad \text{and} \quad R_g = N_0 R \tag{4.42}$$

Given that L_g, L (R_g, R) have linearly independent columns (rows), then

$$M_A{}^c = M_{Lg}{}^c \subseteq M_L{}^c \qquad (4.43)$$

$$M_A{}^r = M_{Rg}{}^r \subseteq M_R{}^r \qquad (4.44)$$

that is the **K**-column module of A coincides with the **K**-column module of L_g and the **K**-row module of A coincides with the **K**-row module of R_g. Furthermore, the **K**-column module of any **K**-e.l.d L contains $M_A{}^c$ and the **K**-row module of any **K**-e.r.d R contains $M_A{}^r$.

■

From this remark, it is clear that the extraction of **K**-e.l.d of A is equivalent to the creation of an ascending chain of modules containing $M_A{}^c$; the minimal element in this chain $M_A{}^c$ itself. The extraction of the **K**-g.e.l.d is a procedure equivalent to the definition of a basis for $M_A{}^c$. The interpretation for the case of the **K**-e.r.d is similar.

<u>Remark (4.6)</u> All **K**-g.e.l.d of A have r columns where $r = \mathrm{rank}_{\mathbf{R}(s)}\{A\}$; however, a **K**-e.l.d of A in general has j columns where $p \le j \le r$.

Similarly, all **K**-g.e.r.d of A have $\bar{r}$ rows, but a **K**-e.r.d of A has in general i rows where $m \le i \le \bar{r}$.

<u>Definition (4.3)</u> Let **K** be a PID. A matrix $A \in \mathbf{K}^{p\times m}$ will be called <u>K-left unimodular</u> (**K**-l.u.) if the **K**-g.e.l.d. of A is **K**-unimodular. Similarly A will be called <u>K-right unimodular</u> (**K**-r.u.) if the **K**-g.e.r.d of A is **K**-unimodular.

The characterization of **K**-l.u, **K**-r.u matrices is given by the following result:

<u>Proposition (4.4)</u> Let $A \in \mathbf{K}^{p\times m}$, $\mathrm{rank}_{\mathbf{R}(s)}\{A\} = r \le \min(p,m)$ and **K** be a PID. The statements in sections (a), (b) are equivalent.

(a) i) A is **K**-left unimodular (**K**-l.u)

ii) The Smith form of A over **K** is $[I_r, 0]$

iii) p = r and A has no zeros in **K**

iv) p = r and the **K**-g.l.r.d of A is **K**-unimodular.

v) $\mathbf{M}_A{}^c = \mathbf{M}_A{*}^c$ and rank $\mathbf{M}_A{}^c$ is p, i.e. $\mathbf{M}_A{}^c = \mathbf{K}^p$

vi) The **K**-g.e.r.d of A is **K**-l.u

vii) These exists on **K**-r.u matrix $\mathbf{X} \in \mathbf{K}^{m\times p}$ such that

$$\mathbf{AX} = I_p \tag{4.45}$$

viii) There exists a **K**-l.u matrix $M \in \mathbf{K}^{(m-p)\times m}$ such that

$$U_L = \left[\begin{array}{c} A \\ \hline M \end{array}\right] \in \mathbf{K}^{r\times r}$$

is **K**-unimodular.

ix) There exists a $U_R \in \mathbf{K}^{r\times r}$, **K**-unimodular such that

$$AU_R = [I_p, 0] \tag{4.46}$$

(b) i) A is **K**-right unimodular (**K**-r.u)

ii) The Smith form of A over **K** is $[I_r, 0]^t$

iii) $r = m$ and A has no zeros in $\mathbf{K}$

iv) $r = m$ and the $\mathbf{K}$-g.l.r.d of A is $\mathbf{K}$-unimodular

v) $\mathbf{M}_A{}^r = \mathbf{M}_A{}^{*r}$ and the rank $\mathbf{M}_A{}^r$ is m, i.e. $\mathbf{M}_A{}^r \in \mathbf{K}^m$

vi) The $\mathbf{K}$-g.e.l.d of A is $\mathbf{K}$-r.u.

vii) There exists a $\mathbf{K}$-l.u matrix $Y \in \mathbf{K}^{rxp}$ such that

$$YA = I_r \tag{4.47}$$

viii) There exists a $\mathbf{K}$-r.u. matrix $N \in \mathbf{K}^{px(p-r)}$ such that

$$U_R = [A,\ N] \in \mathbf{K}^{pxp}$$

is $\mathbf{K}$-unimodular.

ix) There exists a $U_L \in \mathbf{K}^{pxp}$, $\mathbf{K}$-unimodular such that

$$U_L A = [I_r,\ O]^t \tag{4.48}$$

<u>Proof</u>

We shall prove (a) and the proof of (b) is similar.

ii) By (4.36) $A = L_g Q$ where $L_g \in \mathbf{K}^{pxp}$, $\mathbf{K}$-unimodular and $Q \in \mathbf{K}^{pxr}$ with Smith form $[I_r,\ O]$ Q.E.D.

iii) Follows directly from (ii).

iv) $A = PTQ = L_g Q$. Given that PT is $\mathbf{K}$-unimodular and $p = r$, then both P,T are square and $\mathbf{K}$-unimodular.

v) PT defines a basis for $\mathbf{M}_A{}^c$ and P defines a basis for $\mathbf{M}_A{}^{*c}$. Given that $P,T \in \mathbf{K}^{pxp}$ and $\mathbf{K}$-unimodular, then $\mathbf{M}_A{}^c = \mathbf{M}_A{}^{*c}$ and rank of $\mathbf{M}_A{}^c$ is p.

vi) $A = PTQ = PR_g$ where $R_g = TQ$. Given that T is **K**-unimodular and Q has Smith form $[I_r, O]$ then by (ii) R_g is **K**-unimodular.

ix) By (ii) there exists $U_L \in \mathbf{K}^{pxp}$, $U_R \in \mathbf{K}^{mxm}$, **K**-unimodular matrices such that

$$A = U_L\ [I_p,\ O]\ U_R = U_L\ [I_p,\ O] \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$$

$$= [I_p \quad O] \begin{bmatrix} U_L & O \\ O & I_{m-r} \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$$

$$= [I_p \quad O]\ \hat{U}_R$$

Clearly, $\hat{U}_R$ is an mxm **K**-unimodular matrix and thus $AU_R^{-1} = AU_R = [I_p,\ O]$.

vii) By (ix) and by partitioning of U_R as $U_R = [X,\ \hat{X}]$, then

$$A\ [X,\ \hat{X}] = [I_p,\ O],\ \mathbf{X} \in \mathbf{K}^{rxp}$$

and thus $A\mathbf{X} = I_p$

viii) From (ix) there exists $U_R \in \mathbf{K}^{rxr}$, **K**-unimodular such that

$$AU_R = [I_p,\ O] \text{ or } A = [I_p,\ O]\ \hat{U}_R,\ \hat{U}_R = U_R^{-1}.$$

Partition $\hat{U}_R$ as

$$\hat{U}_R = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} \in \mathbf{K}^{rxr} , \quad \mathbf{K}\text{-unimodular}$$

then

$$A = [I_p \quad O] \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = \hat{U}_1$$

take $M = \hat{U}_2$ and the result follows.

■

With the above results we now proceed to define the notions of divisors of two or more matrices.

4.2.3 <u>Non Square Divisors of Sets of Matrices</u>

<u>Definition (4.4)</u> Let $\mathbf{K}$ be a PID

i) Let $A_i \in \mathbf{R}^{pxmi}(s)$, $i \in \underline{n}$ and $L_g \in \mathbf{R}^{pxk}(s)$ for some k. L_g is a <u>greatest common extended left divisor</u> (g.c.e.l.d) of all A_i, $i \in \underline{n}$ if it is an e.d. of all A_i and if every other e.l.d of all A_i is also an e.l.d of L_g. If we restrict ourselves to matrices from a PID, $A_i \in \mathbf{k}_{pxmi}$, $i \in \underline{n}$, then L_g defines a $\mathbf{K}$-g.c.e.l.d of all A_i.

ii) Let $\bar{A}_i \in \mathbf{R}^{pixm}(s)$, $i \in \bar{\underline{n}}$ and $R_g \in \mathbf{R}^{\bar{k}xm}(s)$ for some $\bar{k}$. R_g is a <u>greatest common extended right divisor</u> (g.c.e.r.d.) of all $\bar{A}_i$, $i \in \bar{\underline{n}}$ if it is an e.r.d of all $\bar{A}_i$ and if every other e.r.d of all $\bar{A}_i$ is also an e.r.d of R_g. If we restrict ourselves to matrices from a PID, $A_i \in \mathbf{K}^{pixm}$ $i \in \bar{\underline{n}}$, then R_g defines a $\mathbf{K}$-g.c.e.r.d of all A_i.

The following result establishes the link between the g.c.e.l.d (g.c.e.r.d) and the notion of (g.e.l.d) (g.e.r.d) of a matrix.

<u>Lemma (4.1)</u> Let $A_i \in \mathbf{R}^{pxmi}$, $i \in \underline{n}$, $\bar{A}_j \in \mathbf{R}^{pixm}$, $j \in \bar{\underline{n}}$ and $\mathbf{K}$ a PID. The following statements hold true.

i) $L_g \in \mathbf{R}^{pxk}(s)$ is a g.c.e.l.d of A_i, $i \in \underline{n}$ if and only if it is a g.e.l.d. of the matrix

$$[A_1, A_2, \ldots, A_n].$$

ii) $R_g \in \mathbf{R}^{kxm}(s)$ is a g.c.e.r.d of $\bar{A}_j$, $j \in \bar{\underline{n}}$ if and only if it is a g.e.r.d of the matrix

$$[A_1, A_2, \ldots, A_j]$$

<u>Proof</u>

i) If L_g is a g.c.e.l.d of A_i, $i \in \underline{n}$, then

$$A_i = L_g A_{io},\ i \in \underline{n},\ A_{io} \in \mathbf{K}$$

and thus

$$[A_1, A_2, \ldots, A_n] = [L_g A_{10}, L_g A_{20}, \ldots, L_g A_{no}]$$

$$= L_g [A_{10} A_{20}, \ldots, A_{no}] \tag{4.49}$$

Thus, L_g is an e.l.d of the composite matrix; since, any other divisor of A_i, $i \in \underline{n}$ also satisfies an equation of the (4.49) type and it is an e.l.d of L_g, it follows that L_g is a g.e.l.d of $[A_1,\ldots,A_n]$. The sufficiency follows by a reversion of the arguments.

ii) The proof of part (ii) follows along similar lines.

■

<u>Corollary (4.2)</u> If A_i, $\bar{A}_j$ are matrices from a PID $\mathbf{K}$ respectively $A_i \in \mathbf{K}^{pxmi}$, $i \in \underline{n}$; $\bar{A}_j \in \mathbf{K}^{pixm}$, $j \in \underline{\bar{n}}$, then in the above statements:

i) holds true for a $\mathbf{K}$-g.c.e.l.d, $L_g \in \mathbf{K}^{pxk}$ of A_i and
ii) holds true for a $\mathbf{K}$-g.c.e.r.d, $\mathbf{R}_g \in \mathbf{K}^{kxm}$ of $\bar{A}_j$

■

The results presented for the g.e.l.d (g.e.r.d) of a matrix A over $\mathbf{K}$ can be used for the derivation of similar results for the g.c.e.l.d (g.c.e.r.d) of a set of matrices.

<u>Theorem (4.2)</u> Let $A_i \in \mathbf{R}^{p\times mi}(s)$, $i \in \underline{n}$, $\bar{A}_j \in \mathbf{R}_{pj\times m}(s)$, $j \in \bar{\underline{n}}$ and $\mathbf{K}$ can be a PID.

Denote by M and N the matrices

$$M = [A_1, A_2, \ldots, A_n] \in \mathbf{R}^{p\times\Sigma mi} \tag{4.50}$$

$$N = [\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{\bar{n}}]^t \in \mathbf{R}^{\Sigma pi\times m} \tag{4.51}$$

and let $\mathrm{rank}_{R(s)}\{M\} = r$, $\mathrm{rank}_{R(s)}\{N\} = \bar{r}$.

i) There exists a g.c.e.l.d $L_g \in \mathbf{R}^{p\times r}(s)$ of A_i, $i \in \underline{n}$ and this may be expressed as

$$L_g = [A_1\mathbf{X}_1 + A_2\mathbf{X}_2 + \ldots + A_n\mathbf{X}_n] \tag{4.52}$$

for some $\mathbf{X}_i \in \mathbf{K}^{mi\times r}$ matrices. Furthermore, if $\hat{L}_g$ is another g.c.e.l.d of A_i, $i \in \underline{n}$, then

$$L_g \; \mathbf{E}_r \; \hat{L}_g$$

ii) There exists a g.c.e.r.d $R_g \in \mathbf{R}^{\bar{r}\times m}(s)$ of $\bar{A}_j$, $j \in \bar{\underline{n}}$ and this may be expressed as

$$R_g = [Y_1\bar{A}_1 + Y_2\bar{A}_2 + \ldots + Y_{\bar{n}}\bar{A}_{\bar{n}}] \tag{4.53}$$

for some $Y_j \in \mathbf{K}^{\bar{r}\times Pj}$ matrices. Furthermore, if $\hat{R}_g$ is another g.c.e.r.d of $\bar{A}_j$, $j \in \bar{n}$, then $R_g \; \mathbf{E} \; \hat{R}_g$.

<u>Proof</u>

If the g.c.e.l.d of A_i, $i \in \underline{n}$ exists, then by Lemma (4.1) it is given by the g.e.l.d of M. By Theorem (4.1) the existence of such a divisor is established and we also have

$$L_g = [A_1, A_2, \ldots, A_n]\, T, \quad T \in \mathbf{K}$$

By partitioning T according to the partitioning of M expression (4.56) is derived.

The proof of part (ii) follows along similar lines.

■

The module interpretation of the **K**-g.e.l.d (**K**-g.e.r.d) of a matrix allows the module interpretation of the **K**-g.c.e.l.d (**K**-g.c.e.r.d) of a set of matrices.

<u>Corollary (4.3)</u> Let $A_i \in \mathbf{K}^{p\times mi}$, $i \in \underline{n}$, $L_g \in \mathbf{K}^{r\times r}$ be a **K**-g.c.e.l.d of A_i and let us denote by $\mathbf{M}_i^c$, $i \in \underline{n}$, $\mathbf{M}_L^c$ the **K**-column modules generated by the columns of A_i, $i \in \underline{n}$ and L_g respectively. Then the $\mathbf{M}_L^c$ is the smallest submodule that contains every $\mathbf{M}_i^c$ otherwise

$$\mathbf{M}_L^C = \sum_{i=1}^{m} \mathbf{M}_i^C$$

<u>Proof</u>

Since L_g is a **K**-g.c.e.l.d of A_i $i \in \underline{n}$, then for all $i \in \underline{n}$

$$A_i = L_g\, A_{io}, \quad i \in \underline{n}$$

and thus $\mathbf{M}_i^c \subset \mathbf{M}_L$ $\forall\, i \in \underline{n}$. Similarly if $\hat{L}$ is a **K**-g.c.e.l.d and if $\mathbf{M}_L^c$ is the corresponding **K**-module then

$$A_i = \hat{L}A_{io}, \quad i \in \underline{n} \quad \text{and} \quad L_g = \hat{L}T$$

Thus $\mathbf{M}_i^c \subset \mathbf{M}_{\hat{L}}^c$ and $\mathbf{M}_L^c \subset \mathbf{M}_{\hat{L}}^c$

In other words, $\mathbf{M}_L^c$ contains every $\mathbf{M}_i^c$ and submodule $\mathbf{M}_{\hat{L}}^c$ containing every $\mathbf{M}_i^c$ also contains $\mathbf{M}_L^c$. Thus $\mathbf{M}_L^c$ is the smallest submodule containing every $\mathbf{M}_i^c$.

On the other hand, let

$$\sum_{i=1}^{m} \mathbf{M}_i^c$$

be the sum of the submodules $\mathbf{M}_1^c, \ldots, \mathbf{M}_n^c$. This is obviously a submodule $\mathbf{M}^c$ containing each $\mathbf{M}_i^c$; moreover, any submodule $\overline{\mathbf{M}}^c$ of $\mathbf{M}^c$ containing each $\mathbf{M}_i^c$ must contain

$$\sum_{i=1}^{m} \mathbf{M}_i^c$$

Hence

$$\mathbf{M}_L^c = \sum_{i=1}^{m} \mathbf{M}_i^c$$

An alternative proof may be established by inspection of (4.52).

■

A similar interpretation may be given for the **K**-g.c.e.r.d, R_g of a set of matrices $\overline{A}_j$, $j \in \overline{\underline{n}}$. In this case, the module generated by the rows of R_g is the minimal module that contains all the modules generated by the $\bar{A}_j$, $\bar{A}_j \in \mathbf{K}^{pixm}$, $j \in n$ or otherwise is the sum of these modules.

The module interpretation of L_g as a basis for the "minimal cover module" of all modules generated by the columns of A_i will be used in the solution of matrix equations and generalised Diophantine equations.

With notion of g.c.e.l.d, g.c.e.r.d defined we proceed to define the notion of coprimeness of a set of matrices.

Definition (4.5) Let **K** be a PID.

i) Given a set of rational matrices $A_i \in \mathbf{R}^{pxmi}(s)$, $i \in \underline{n}$, $[A_1, \ldots, A_n] = M \in \mathbf{R}^{px\Sigma mi}(s)$, $\text{rank}_{\mathbf{R}(s)}\{M\} = r$ then we say that (the columns of) A_i, $i \in \underline{n}$ are **K**-left coprime (**K**-l.c.) if M has no zeros in **K**.

ii) Given a set of rational matrices $\bar{A}_j \in \mathbf{R}^{Pjxm}(s)$, $j \in \underline{n}$, $[\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{\bar{n}}]^t = N \in \mathbf{R}^{\Sigma Pjxm}(s)$, $\text{rank}_{\mathbf{R}(s)}\{N\} = \bar{r}$ then we say that (the rows of) $\bar{A}_j$, $j \in \underline{n}$ are **K**-right coprime (**K**-r.c.) if N has no zeros in **K**.

If we restrict this to rational matrices from a PID, then we have the following.

Proposition (4.5) Let **K** be a PID and let $A_i \in \mathbf{K}^{Pxmi}$, $i \in \underline{n}$, $\bar{A}_j \in \mathbf{K}^{Pjxm}$, $j \in \underline{n}$ be two sets of matrices. Then the following statements are equivalent.

(a)

i) The set A_i, $i \in \underline{n}$ is **K**-left coprime

ii) The matrix $M = [A_1, A_2, \ldots A_n]$ is **K**-left unimodular

iii) Let $\mathbf{M}_i^c$ denote the **K**-column modules generated by A_i, $i \in \underline{n}$. Then

$$\mathbf{M}^C = \sum_{i=1}^{n} \mathbf{M}_i^C = \mathbf{K}^{Px1}$$

(b)

i) The set $\bar{A}_j$, $j \in \underline{n}$ is **K**-right coprime

ii) The matrix $N = [\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{\bar{n}}]^t$ is **K**-right unimodular.

iii) Let $\mathbf{M}_i^c$ denote the **K**-row modules generated by $\bar{A}_j$, $j \in \underline{n}$. Then

$$\mathbf{M}^r = \sum_{j=1}^{\bar{n}} \mathbf{M}_j = \mathbf{K}^{1xm} \qquad (4.54)$$

<u>Proof</u>

(a) The **K**-g.e.l.d of A_i, $i \in \underline{n}$ is defined as the **K**-g.e.l.d of the matrix $M = [A_1, A_2, \ldots, A_p.]$. If L_g is unimodular then $\mathbf{M}_i^c$ is the **K**-module $\mathbf{K}^{px1}$, which is the maximal module of $\mathbf{R}^{px1}(s)$ rational vector space. The proof for (b) is similar.

■

The properties of **K**-left unimodular matrices may then be used to characterize coprimeness properties of a set of matrices if the above Proposition is used with Proposition (4.4).

4.3 Projectors, Annihilators and Left, Right Inverses of a Rational Matrix over a PID

For the analysis of matrix equations over principal ideal domains some further algebraic tools are needed. The notion of column, row projectors and left, right inverses of a rational matrix are introduced. These projectors and annihilators are shown to be generalizations of left, right inverses and are characterised using the properties of unimodular matrices defined over the appropriated ring.

4.3.1 Generalised column-row Projectors of a Rational Matrix

Definition (4.6) Let $A \in \mathbf{R}^{p\times m}(s)$, rank $\{A\} = r \leq \min(p.m)$, $\mathbf{K}$ be a PID and let $P_\ell \in \mathbf{K}^{r\times p}$, $\text{rank}_{\mathbf{R}(s)}\{P\ell\} = r$, $Q_r \in \mathbf{K}^{m\times r}$, rank $\{Q_r\} = r$.

i) P_ℓ is called a K-column projector (**K**-c.p.) of A over **K** if

$$P_\ell A = R_g \tag{4.55}$$

where $R_g \in \mathbf{R}^{r\times m}(s)$, $R_g = N_{gr}\, D_{gr}^{-1}$,

$N_{gr} \in \mathbf{K}^{rxm}$, $D_{gr} \in \mathbf{R}^{mxm}$ is a g.e.r.d of A over **K**. If A $\in$ $\mathbf{K}^{pxm}$ then $R_g \in \mathbf{K}^{rxm}$ and D_{gr} is a **K**-unimodular matrix.

ii) Q_r is called a <u>K-row projector</u> (**K**-r.p.) of A over **K** if

$$AQ_r = L_g \tag{4.56}$$

where $L_g \in \mathbf{R}^{pxr}(s)$, $L_g = D_{g\ell}^{-1} N_{g\ell}$, $D_{g\ell} \in \mathbf{K}^{pxp}$, $N_{g\ell} \in \mathbf{K}^{pxr}$, is a **K**-g.e.l.d of A over **K**. If A $\in \mathbf{K}^{pxm}$ then $L_g \in \mathbf{K}^{pxr}$ and $D_{g\ell}$ is a **K**-unimodular matrix.

<u>Remark (4.7)</u> By definition, P_ℓ produces a **K**-g.e.r.d of A and thus projects the column vectors of A onto the maximal **K**-column module $\mathbf{M}_A{*}^c$ of A. Alternatively, $P_\ell A$ produces a basis for $\mathbf{M}^r{}_A$. Similarly, Q_r produces a **K**-g.e.l.d of A and thus projects the row vectors of A onto the maximal **K**-row module $\mathbf{M}_A{*}^r$ of A. Alternatively AQ_r produces a basis for $\mathbf{M}_A{}^c$.

<u>Proposition (4.6)</u> Let **K** be a PID. Then, every matrix A $\in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = r \le \min(p,m)$ has a **K**-c.p., $P_\ell \in \mathbf{K}^{rxp}$ and a **K**-r.p., $Q_r \in \mathbf{K}^{mxr}$.

<u>Proof</u>

By the Smith-McMillan decomposition of A over **K** we have

$$A = U\ S^*\ V = [U_1 \quad U_2] \begin{bmatrix} S^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= U_1 \; S^* \; V_1 \;\; = \;\; U_1 \; R_g = L_g \; V_1 \tag{4.57}$$

Let $\hat{U} = U^{-1}$, $\hat{V} = V^{-1}$, then

$$\hat{U} \; U \;\; = \begin{bmatrix} \hat{U}_1 \\ \\ \hat{U}_2 \end{bmatrix} [U_1 \quad U_2] \; = \begin{bmatrix} I_r & & O \\ & & \\ O & & I_{p-r} \end{bmatrix}$$

$$\hat{U}_1 \; U_1 \; = \; I_r \tag{4.58}$$

$$V \; \hat{V} \;\; = \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix} [\hat{V}_1 \quad \hat{V}_2] \; = \begin{bmatrix} I_r & & O \\ & & \\ O & & I_{m-r} \end{bmatrix}$$

$$V_1 \; \hat{V}_1 \; = \; I_r \tag{4.59}$$

By defining $P_\ell = \hat{U}_1$ and $Q_r = \hat{V}_1$ and using (4.57), (4.58), (4,59) we have:

$\hat{U}_1 A = \hat{U}_1 \; U_1 \; R_g = R_g$

$A\hat{V}_1 = L_g \; V_1 \; \hat{V}_1 = L_g$

QED.

■

<u>Proposition (4.7)</u> If P_ℓ, Q_r are **K**-c.p, **K**-r.p of A respectively, then P_ℓ is **K**-left unimodular and Q_r is a **K**-right unimodular matrix.

Proof

Assume that P_ℓ is not a **K**-left unimodular matrix. Then we may factorize it as $P_\ell = Z\hat{P}_\ell$, where $\hat{P}_\ell$ is a **K**-left unimodular matrix and $Z \in \mathbf{K}^{rxr}$ is a nontrivial (non unimodular) greatest left divisor of A over **K**. Given that A may be expressed as $A = PR_g$ then $P_\ell A = Z\hat{P}_\ell PL$ where $\hat{P}_\ell P = W \in \mathbf{K}^{rxr}$ and thus $ZWR_g = P_\ell A$. No matter what the matrix W is, ZW cannot be a **K**-unimodular matrix since Z is not. Thus ZWL is not a **K**-g.e.r.d of A, since all **K**-g.e.r.d of A are left equivalent then the result follows by contradiction.

■

An alternative characterization of column, row projectors of a matrix A is given by the following result.

Proposition (4.8) Let **K** be a PID and $A \in \mathbf{R}^{pxm}(s)$, $A = PTQ$ be a g.l.r.d decomposition of A over **K** where $T \in \mathbf{R}^{rxr}(s)$, $T = T_{N1}\, T_{D1}^{-1} = T_{D2}^{-1}\, T_{N2}$, is a **K**-g.l.r.d of A, $P \in \mathbf{K}^{pxr}$ is a basis matrix for $\mathbf{M}_A{*}^c$ and $Q \in \mathbf{K}^{rxm}$ is a basis matrix for $\mathbf{M}_A{*}^r$. Then,

i) $P_\ell \in \mathbf{K}^{rxp}$ is a **K**-c.p. of A if and only if

$$P_\ell P = W_\ell \in \mathbf{K}^{rxr}, \text{ and } \mathbf{K}\text{-unimodular} \tag{4.60}$$

ii) $Q_r \in \mathbf{K}^{rxm}$ is a **K**-r.p of A if and only if

$$QQ_r\ W_r \in \mathbf{K}^{rxr}, \text{ and } \mathbf{K}\text{-unimodular} \tag{4.61}$$

Proof

The sufficiency is obvious. To prove the necessity assume that $P_\ell P = W_\ell$ but not unimodular then

$$P_\ell A = P_\ell \; PTQ = P_\ell PR_g = W_r R_g$$

since all g.e.r.d are correct by left unimodular transformation then W_r must be **R**-unimodular hence this result follows by contradiction.

■

Corollary (4.4) Let P_ℓ, Q_r be a pair of **K**-c.p, **K**-r.p of A respectively. Then $P_\ell A Q_r$ is a g.l.r.d of A over **K**.

4.3.2 Prime left, Right Annihilators of a Rational Matrix

Definition (4.7) Let $A \in \mathbf{R}^{pxm}(s)$, $\mathrm{rank}_{\mathbf{R}(s)} \{A\} = r \leq \min(p,m)$ and **R** be a PID.

i) Let $p > r$ and $N_\ell \in \mathbf{K}^{(p-r)xp}$

N_ℓ will be called a K-prime left annihilator (**K**-p.ℓ.a) of A over **K** if n_ℓ is a **K**-unimodular matrix and

$$N_\ell A = O_{p-r,m} \tag{4.62}$$

ii) Let $m > r$ and $N_r \in \mathbf{K}^{mx(m-r)}$

N_r will be called a K-prime right annihilator (**K**-p.r.a) of A over **K** if N_r is a **K**-unimodular matrix and

$$AN_r = O_{p,m-r} \tag{4.63}$$

<u>Proposition (4.9)</u> Let $A \in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R(s)}}\{A\} = r \leq \min(p.m)$ and $\mathbf{K}$ be a PID. Then

i) if $p > r$, A has always a $\mathbf{K}$-p.ℓ.a N_ℓ furthermore if N_ℓ, $\hat{N}_\ell$ are two $\mathbf{K}$-p.ℓ.a of A then

$$N_\ell \; \mathbf{E}_\ell \; \hat{N}_\ell$$

ii) if $m > r$, A has always a $\mathbf{K}$-p.r.a N_r furthermore if N_r, N_r are two $\mathbf{K}$-p.r.a of A then

$$N_r \; \mathbf{E} \; \hat{N}_r$$

<u>Proof</u>

By the Smith-McMillan reduction of A over $\mathbf{K}$ we have

$$\hat{U} \, A \, \hat{V} = \begin{bmatrix} S^* & 0 \\ 0 & 0 \end{bmatrix} = S_A$$

By partitioning $\hat{U}$, $\hat{V}$ the pxp, mxm $\mathbf{k}$-unimodular matrices respectively according to the partitioning of S_A we have

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} A \, [V_1 \quad V_2] = \begin{bmatrix} S^* & 0 \\ 0 & 0 \end{bmatrix} = S_A \tag{4.64}$$

Clearly $\hat{U}_2$, $\hat{V}_2$ are **K**-unimodular matrices and since $\hat{U}_2 \in \mathbf{K}^{(p-r)xp}$ is left unimodular, $V_2 \in \mathbf{K}^{mx(m-r)}$ is right unimodular with $\hat{U}_2 A = O_{p-r,m}$, $A\hat{V}_2 = O_{p,m-r}$ then the rows of $\hat{U}_2$ belong to $\mathbf{N}_A^{\ell}$ and the columns of $\hat{U}_2$ belong to $\mathbf{N}_A^{r}$. Given that $\mathbf{N}_A^{r}$ has dimension m-r, $\hat{U}_2$ is a **K**-left unimodular basis matrix for $\mathbf{N}_A^{r}$ and similarly $\hat{V}_2$ is a **K**-right unimodular basis matrix for $\mathbf{N}_A^{r}$. In fact, $\hat{U}_2$ is a basis matrix for the maximal **K**-module in $\mathbf{N}_A^{\ell}$ and $\hat{V}_2$ is a basis matrix for the maximal **K**-module in $\mathbf{N}_A^{r}$. By definition any other N_ℓ is a **K**-left unimodular basis matrix for $\mathbf{N}_A^{\ell}$ and thus N_ℓ and $\hat{U}_2$ are left equivalent.

■

<u>Corollary (4.5)</u> Let $A \in \mathbf{R}^{pxm}(s)$, $\text{rank}_{R(s)}\{A\} = r \le \min\ (p,m)$ and **K** be a PID.

i) If p>r then there exists a pair (P_ℓ, N_ℓ) where $P_\ell \in \mathbf{K}^{rxp}$ is a **K**-c.p. and $N_\ell \in \mathbf{K}^{(p-r)xp}$ is a **K**-p.ℓ.a of A over **K** such that

$$\Psi_\ell = \begin{bmatrix} P_\ell \\ \text{-----} \\ N_\ell \end{bmatrix} \in \mathbf{K}^{pxp} \qquad (4.65)$$

is **K**-unimodular.

ii) If m > r then there exists a pair (Q_r, N_r) where $Q_r \in \mathbf{K}^{mxr}$ is a **K**-r.p and $N_r \in \mathbf{K}^{mx(m-r)}$ is a **K**-p.r.a of A over **K** such that

$$\Psi_r = [Q_r \quad N_r] \in \mathbf{K}^{mxm} \tag{4.66}$$

is **K**-unimodular

Proof

By (4.64) we have

$$\begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} A\ S_A\ \hat{V}^{-1} = S\ V_A = \begin{bmatrix} S* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{4.67}$$

and

$$A[\hat{V}_1 \quad \hat{V}_2] = \hat{U}^{-1}\ S_A = U\ S_A = [U_1 \quad U_2] \begin{bmatrix} S* & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly,

$$\hat{U}_1\ A = S*\ V_1, \quad \hat{U}_2\ A = 0 \tag{4.68}$$

$$A\hat{V}_1 = U_1\ S*,\ A\ \hat{V}_2 = 0 \tag{4.69}$$

and thus: $\hat{U}_1$ is a **K**-c.p, $\hat{U}_2$ is a **K**-p.ℓ.a; $\hat{V}_1$ is a **K**-r.p., and $\hat{V}_2$ is a **K**-p.r.a of A over **K**.

By contradiction $\Psi_\ell = \hat{U}$ is a **K**-unimodular and $\Psi_r = \hat{V}$ is also a **K**-unimodular matrix

■

Proposition (4.10) Let $A \in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\}$ r ≤ min (p,m) and **K** be a PID with P_ℓ a **K**-c.p, Q_r a **K**-r.p, N_ℓ be a **K**-p.ℓ.a and N_r be a **K**-p.r.a. of A over **K**.

i) The general family of $\mathbf{K}$-c.p of A is given by

(a) $P_\ell' = UP_\ell + YN_\ell$ ... if $p > r$ (4.70)

where $U \in \mathbf{K}^{r\times r}$ arbitrary unimodular and $Y \in \mathbf{K}^{r\times(p-r)}$ an arbitrary matrix

(b) $P_\ell' = UP_\ell$ ... if $p = r$ (4.71)

ii) The general family of $\mathbf{K}$-r.p of A is given by

(a) $Q_r' = Q_r V + N_r X$ ... if $m > r$ (4.72)

where $V \in \mathbf{K}^{r\times r}$ arbitrary unimodular and $X \in \mathbf{K}^{(m-r)\times r}$ an arbitrary matrix

(b) $Q_r' = Q_r V$ ... if $m = r$ (4.73)

Proof

(a) Let P_ℓ , P_ℓ' be two $\mathbf{K}$-c.p of A. Then

$$P_\ell A = L_r \ , \ P_\ell' A = L_r'$$

Given that L_r **E** L_r', then $L_r = \hat{U} L_r'$, $\hat{U} \in \mathbf{K}^{r\times r}$ unimodular and thus

$$L_r' = P_\ell' A = UP_\ell A$$

From the above condition we have that

$$(P_\ell' - UP_\ell)A = 0$$

and thus if $p > r$ then,

$$P_\ell' - UP_\ell = YN_\ell, \quad Y \in \mathbf{K}^{r\times(p-r)}$$ arbitrary and if $p = r$ then,

$$P_\ell' - UP_\ell = 0$$

(b) The proof of part (b) follows along similar lines.

■

The following result may also be stated.

<u>Corollary (4.6)</u> If (P_ℓ', N_ℓ') is any pair of **K**-c.p, p.ℓ.a and (Q_r', N_r') is any pair of **K**-r.p., p.r.a of A then

i) The matrix

$$\Psi_\ell' = \begin{bmatrix} P_\ell' \\ \hline N_\ell \end{bmatrix}$$

is **K**-unimodular.

ii) The matrix

$$\Psi_r' = [Q_r' \quad N_r'] \in \mathbf{K}^{m\times m} \tag{4.75}$$

is **K**-unimodular.

<u>Proof</u>

By corollary (4.5) there exists a pair (P_ℓ, N_ℓ) such that the matrix Ψ_ℓ is **K**-unimodular. A general pair may be expressed as

$$P_\ell' = UP_\ell + YN_\ell \ , \quad N_\ell' = U'N_\ell$$

where U, U' are **K**-unimodular and $Y \in \mathbf{K}^{r\times(p-r)}$ arbitrary. Then,

$$\Psi_\ell' = \begin{bmatrix} P_\ell' \\ \hline N_\ell' \end{bmatrix} = \begin{bmatrix} UP_\ell + YN_\ell \\ \hline UN_\ell \end{bmatrix} = \begin{bmatrix} U & Y \\ O & U' \end{bmatrix} \begin{bmatrix} P_\ell \\ N_\ell \end{bmatrix}$$

$= \begin{bmatrix} U & Y \\ O & U' \end{bmatrix} \Psi_\ell$ given that Ψ_ℓ is by construction

K-unimodular and $\begin{bmatrix} U & Y \\ 0 & U' \end{bmatrix}$ is obviously unimodular

then the rest follows.

∎

The notions of **K**-c.p, **K**-r.p. are generalizations of left, right inverses of a matrix A over **K**. Such inverses are considered next.

4.3.3 Left, Right Inverses of a Rational Matrix

Definition (4.8) Let $A \in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = r \leq \min(p,m)$, $\mathbf{K}$ be a PID and let $A_\ell \in \mathbf{K}^{pxm}$, $A_r \in \mathbf{K}^{mxp}$. Then

i) A_ℓ^t will be called a K-left inverse ($\mathbf{K}$-ℓ.i) of A over $\mathbf{K}$ if

$$A_\ell^t A = I_m \tag{4.76}$$

ii) A_r will be called a K-right inverse ($\mathbf{K}$-r.i) of A over $\mathbf{K}$ if

$$A A_r^t = I_p \tag{4.77}$$

The conditions under which a $\mathbf{K}$-ℓ.i, $\mathbf{K}$-r.i exist are examined next. We first state the following result.

Lemma (4.2) [Pri 1]: Let $A \in \mathbf{R}^{pxm}(s)$, then

i) A left inverse $A_\ell^g \in \mathbf{R}^{mxp}(s)$ $(A_\ell^g A = I_m)$ exists if and only if $\text{rank}_{\mathbf{R}(s)}\{A\} = m$.

ii) A right inverse $A_r^g \in \mathbf{R}^{pxm}(s)$ $(AA_r^g = I_p)$ exists if and only if $\text{rank}_{\mathbf{R}(s)}\{A\} = p$.

■

Remark (4.8): Any $\mathbf{K}$-ℓ.i. A_ℓ^t or any $\mathbf{K}$-r.i. A_r^t of A is by definition also on inverse over the field $\mathbf{R}(s)$. Thus, by Lemma (4.2) is follows that a necessary but not sufficient condition for the existence of A_ℓ^t is that the $\text{rank}_{\mathbf{R}(s)}\{A\} = m$ and for the existence of A_r^t is that the $\text{rank}_{\mathbf{R}(s)}\{A\} = p$.

Theorem (4.3): Let $A \in \mathbf{R}^{pxm}(s)$, $\text{rank}_{\mathbf{R}(s)}\{A\} = r \le \min(p,m)$ and $\mathbf{K}$ be a PID. Then,

i) Necessary and sufficient condition for the existence of a $\mathbf{K}$-left inverse A_ℓ^t of A over $\mathbf{K}$ is that A is $\mathbf{K}$-right unimodular.

ii) Necessary and sufficient condition for the existence of a $\mathbf{K}$-right inverse A_r^t of A over $\mathbf{K}$ is that A is $\mathbf{K}$-left unimodular.

Proof

By Remark (4.8) necessary and sufficient condition for the existence of A_ℓ^t is that $\text{rank}_{\mathbf{R}(s)}\{A\} = m$. Thus $p \ge m$ and the g.l.r.d. decomposition of A over $\mathbf{K}$ is of the type

$$A = PTQ$$

where $P \in \mathbf{K}^{pxm}$, $T \in \mathbf{R}^{mxm}(s)$, $Q \in \mathbf{K}^{mxm}$ and Q is $\mathbf{K}$-unimodular. Since A_ℓ^t exists, then

$$A_\ell^t\ A = A_\ell^t\ PTQ = I_m = UTQ$$

where $U = A_\ell^t\ P \in \mathbf{K}^{mxm}$. Clearly, UTQ must be $\mathbf{K}$-unimodular and then both U and T must be unimodular. However, $TQ = L_r$ and thus A is $\mathbf{K}$-right unimodular. To prove the sufficiency assume A is $\mathbf{K}$-right unimodular then,

$$A = PL_r$$

where $L_r \in \mathbf{K}^{m\times m}$ and $\mathbf{K}$-unimodular. Choose a $\mathbf{K}$-c.p. of A, say P_ℓ. Then,

$$P_\ell A = P_\ell P L_r = U L_r$$

where U, $L_r \in \mathbf{K}^{m\times m}$ and $\mathbf{K}$-unimodular. By choosing

$$A_\ell^t = (P_\ell \ A)^{-1} \ P_\ell$$

the result is established. The proof of the case for right inverses is similar.

■

The intimate link of $\mathbf{K}$-ℓ.i to $\mathbf{K}$-c.p. and of the $\mathbf{K}$-r.i. to the $\mathbf{K}$-c.p., $\mathbf{K}$-r.p. clearly suggest that the results stated for the $\mathbf{K}$-c.p., $\mathbf{K}$-r.p. carry over for the inverses defined on $\mathbf{K}$-right unimodular, $\mathbf{K}$-left unimodular matrices. The family of inverses is characterised by the following result.

Corollary (4.7): Let $A \in \mathbf{R}^{p\times m}(s)$, $\text{rank}_{\mathbf{R(s)}}\{A\} = r \leq \min(p,m)$ and $\mathbf{K}$ be a PID. Then,

i) If A is a $\mathbf{K}$-right unimodular then the $\mathbf{K}$-ℓ.i exist or else the family is defined by

$$A_\ell^t = (P_\ell A)^{-1} \ P_\ell + Y N_\ell \qquad (6.49)$$

where $P_\ell \in \mathbf{K}^{m\times p}$ is a $\mathbf{K}$-c.p., $N_\ell \in \mathbf{K}^{(p-m)\times m}$ is a $\mathbf{K}$-p.l.a and $Y \in \mathbf{K}^{m\times(m-p)}$ is an arbitrary matrix.

ii) If A is a $\mathbf{K}$-left unimodular, then the $\mathbf{K}$-r.i exists and the family is given by

$$A_r^t = Q_r\ (AQ_r)^{-1} + N_rX \tag{4.80}$$

where $Q_r \in \mathbf{K}^{m\times p}$ is a **K**-r.p., $N_r \in$ **K**-p.r.a and $X \in \mathbf{K}^{(m-p)\times p}$ is an arbitrary matrix.

Proof

$(P_\ell A)^{-1}$ is clearly a **K**-l.i. If A_ℓ^t is another inverse then $A_\ell^t\ A = \{(P_\ell A)^{-1}\ P_\ell\}\ A = I_m$ and thus $\{A_\ell^t - (P_\ell A)^{-1}\ P_\ell\}A = 0$ from which

$$A_\ell^t = (P_\ell\ A)^{-1}\ P_\ell + YN_\ell$$

The case for right inverses is similar.

■

Remark (4.9): A matrix $A \in \mathbf{R}^{p\times m}(s)$ possesses both inverses (left and right) if and only if $p = m$ and it is **K**-unimodular. In this case the inverses are uniquely defined and $A_\ell^t = A_r^t = A^{-1}$. In the case where $p \neq m$ then the matrix A has only one of the two types of inverse defined above. Since a **K**-ℓ.r.i is a special case of a **K**-c.p. and a **K**-r.i. is a special case of a **K**-r.p. we may state.

Remark (4.10):

i) Let $A \in \mathbf{R}^{p\times m}(s)$ be a **K**-right unimodular A_ℓ^t be a **K**-ℓ.i and N_ℓ a **K**-p.ℓ.a of A over **K**. Then,

$$\Phi_\ell = \begin{bmatrix} A_\ell{}^t \\ \hline N_\ell \end{bmatrix} \in \mathbf{K}^{p\times p} \quad \text{and} \quad \mathbf{K}\text{-unimodular}$$

ii) Let $A \in \mathbf{R}^{p\times m}(s)$ be a **K**-left unimodular, $A_r{}^t$ be a **K**-r.i and N_r a **K**-p.r.a of A over **K**. Then,

$$\Phi_r = [A_r{}^t \quad N_r] \in \mathbf{K}^{m\times m} \quad \text{and } \mathbf{K}\text{-unimodular.}$$

■

4.4 Summary of Matrix Structure over a PID

For control systems applications the rings of importance and the principal ideal domains $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ and $\mathbf{R}_\rho(s)$, polynomials, proper rational functions and proper and stable rational functions respectively. A summary of the structure and properties of these matrices is given below and the analogies between the rings are highlighted.

Table 4.1 Comparison between Properties of Euclidean Rings $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ and $\mathbf{R}_{\rho}(s)$

Ring: $\mathbf{R}[s]$-Polynomials	Ring: $\mathbf{R}_{pr}(s)$-Proper Rational Functions	Ring: $\mathbf{R}_{\rho}(s)$-Proper and Ω-Stable Rational Functions
$a(s) \in \mathbf{R}[s]$	$t(s) \in \mathbf{R}_{pr}(s)$	$t(s) \in \mathbf{R}_{\rho}(s)$
$\deg a(s) \begin{cases} \geq 0 \ , & a(s) \quad 0 \\ -\infty \ , & a(s) = 0 \end{cases}$	$\delta_{\infty}(t(s)) \begin{cases} \geq 0 \ , & t(s) \quad 0 \\ +\infty \ , & t(s) = 0 \end{cases}$	$\delta_{\rho}(t(s)) \begin{cases} \geq 0 \ , & t(s) \quad 0 \\ +\infty \ , & t(s) = 0 \end{cases}$
Polynomials may have:	Proper rational functions may have:	Proper and Ω-stable rational functions may have:
f.z. if deg $(\cdot) > 0$	f.z.	f.z.
no iz.	i.z. if $\delta_{\infty}(\cdot) > 0$	i.z. if $\delta_{\infty}(\cdot) > 0$
no f.p.	f.p.	f.p. will have outside Ω (property)
i.p. if deg $(\cdot) > 0$	no i.p. (property)	no i.p. property
# f.z. = # i.p.	# i.z. = # f.p. - #f.z.	# i.z. = #f.p. - #f.z.
Units in $\mathbf{R}[s]$ are constants in $\mathbf{R}$	Units in $\mathbf{R}_{pr}(s)$ are biproper rational functions.	Units in $\mathbf{R}_{\rho}(s)$ are biproper rational functions with no f.p. or i.z. in Ω.
$\deg a(s) = 0$	$\delta_{\infty}(t(s)) = 0$	$\delta_{\rho}(t(s)) = 0$

Cont ...

<u>Units</u> in $\mathbf{R}[s]$ have:	<u>Units</u> in $\mathbf{R}_{pr}(s)$ have:	<u>Units</u> in $\mathbf{R}_{\mathcal{P}}(s)$ have:
no f.z.	may have f.z.	may have f.z. outside Ω
no i.z.	have no i.z.	have no i.z.
no f.p.	may have f.p.	may have f.p. outside Ω
no i.p.	have no i.p.	have no i.p.
	# f.z. = # f.p.	# f.z. = # f.p.
<u>Matrices</u> T(s) $\mathbf{R}^{pxm}[s]$	<u>Matrices</u> T(s) $\mathbf{R}_{pr}{}^{pxm}(s)$	<u>Matrices</u> T(s) $\mathbf{R}_{\mathcal{P}}{}^{pxm}(s)$
degree T(s) = max. (degree among max. order non zero minors)	$\delta_\infty(T)$ = min. ($\delta_\infty(\cdot)$ among the $\delta_\infty(\cdot)$ of all max order non zero minors)	$\delta_{\mathcal{P}}(T)$ = min($\delta_{\mathcal{P}}(\cdot)$ among the $\delta_{\mathcal{P}}(\cdot)$ of all max. order non zero minors)
deg. $T(s) \geq 0$	$\delta_\infty(T(s)) \geq 0$	$\delta_{\mathcal{P}}(T(s)) \geq 0$
<u>Polynomial matrices</u>:	<u>Proper rational matrices</u>:	<u>Proper and Ω-stable rational matrices</u>:
may have f.z.	may have f.z.	may have f.z.
may have i.z.	may have i.z.	may have i.z.
have no f.p.	may have f.p.	may have f.p. outside Ω
may have i.p.	have no i.p.	have no i.p.

Cont ...

$\mathbf{R}[s]$-unimodular matrices (p = m)	$\mathbf{R}_{pr}(s)$-unimodular matrices (p = m biproper)	$\mathbf{R}_{\wp}(s)$-unimodular matrices (p = m biproper)
degree T(s) = 0	$\delta_{\infty}(T(s)) = 0$	$\delta_{\wp}(T(s)) = 0$
have no f.z.	may have f.z.	may have f.z. outside Ω
may have i.z.	have no i.z.	have no i.z.
have no f.p.	may have f.p.	may have f.p. outside Ω
may have i.p.	have no i.p.	have no i.p.
# i.z. = # i.p.	# f.z. = # f.p.	# f.z. = # f.p.
<u>Modules over the ring $\mathbf{R}[s]$</u>:	<u>Modules over the ring $\mathbf{R}_{pr}(s)$</u>:	<u>Modules over the ring $\mathbf{R}_{\wp}(s)$</u>:
$T_i(s)$ $\mathbf{R}^{pxm}[s]$, $p \geq m$, $\mathrm{rank}_{\mathbf{R}(s)}$	$T_i(s)$ $\mathbf{R}_{pr}{}^{pxm}(s)$, $p \geq m$, $\mathrm{rank}_{\mathbf{R}(s)}$	$T_i(s)$ $\mathbf{R}_{\wp}{}^{pxm}(s)$, $p \geq m$, $\mathrm{rank}_{\mathbf{R}(s)}$
$T_i(s)$ = m is a polynomial basis of a rational vector space $\mathbf{T}(s)$	$T_i(s)$ = m is a proper rational basis of a rational vector space $\mathbf{T}(s)$	$T_i(s)$ = m is a proper and Ω-stable rational basis of a rational vector space $\mathbf{T}(s)$
If $T_{Ri}(s)$ $\mathbf{R}^{mxm}[s]$ (non unimodular) right divisor of $T_i(s)$ i.e	If $T_{Ri}(s)$ $\mathbf{R}_{pr}{}^{mxm}(s)$ (non biproper) right divisor of $T_i(s)$ i.e.	If $T_{ri}(s)$ $\mathbf{R}_{\wp}{}^{mxm}(s)$ (non biproper) right divisor of $T_i(s)$ i.e.
$T_i(s) = T_{i+1}(s).T_{Ri}(s)$	$T_i(s) = T_{i+1}(s).T_{Ri}(s)$	$T_i(s) = T_{i+1}(s).T_{Ri}(s)$
and $\mathbf{M}$ the $\mathbf{R}[s]$-module generated by $T_i(s)$ then	and $\mathbf{M}$ the $\mathbf{R}_{pr}(s)$-module generated by $T_i(s)$ then	and $\mathbf{M}$ the $\mathbf{R}_{\wp}(s)$-module generated by $T_i(s)$ then
M_i M_{i+1} if	M_i M_{i+1} if	M_i M_{i+1} if
deg T_i > deg T_2 > ... > deg T*	$\delta_{\infty}(T_1) > \delta_{\infty}(T_2) > \ldots > \delta_{\infty}(T^*)$	$\delta_{\wp}(T_1) > \delta_{\wp}(T_2) > \ldots > \delta_{\wp}(T^*)$
and T* is $\mathbf{R}[s]$-left unimodular then:	and T* is $\mathbf{R}_{pr}(s)$-left unimodular then:	and T* is $\mathbf{R}_{\wp}(s)$-left unimodular then:

Cont ...

$M_1 \quad M_2 \quad \ldots \quad M^*$ M^* is the maximal **R**[s]-module generated by the columns of the **R**[s]-left unimodular matrix T^* and T_{GR} is the greatest right divisor of T_1 i.e. $T_1 = T^*T_{GR}$	$M_1 \quad M_2 \quad \ldots \quad M^*$ M^* is the maximal $\mathbf{R}_{pr}(s)$-module generated by the columns of the $\mathbf{R}_{pr}(s)$-left unimodular matrix T^* and T_{GR} is the greatest right divisor at $s = \infty$ of T_1 i.e. $T_1 = T^*T_{GR}$	$M_1 \quad M_2 \quad \ldots \quad M^*$ M^* is the maximal $\mathbf{R}_{\Gamma}(s)$-module generated by the columns of the $\mathbf{R}_{\Gamma}(s)$-left unimodular matrix T^* and T_{GR} is the greatest right divisor in of T_1 i.e. $T_1 = T^*T_{GR}$

Summary of divisors projectors annihilators and inverses of a matrix over a principal Ideal domain $\mathbb{K}$

$\mathbb{K}$-extended left divisor ($\mathbb{K}$-e.ℓ.d.)	$T \in \mathbb{K}^{pxm}$, $\text{rank}_{\mathbb{R}(s)}(T) = \rho \leq \min(p,m)$ $L_1 \in \mathbb{K}^{pxq}$ is a $\mathbb{K}$-e.ℓ.d of T if $T = L_1 T_0$
$\mathbb{K}$-greatest extended left divisor ($\mathbb{K}$-g.e.ℓ.d)	$T \in \mathbb{K}^{pxm}$, $\text{rank}_{\mathbb{R}(s)}(T) = \rho \leq \min(p,m)$ $L_1 \in \mathbb{K}^{pxq}$ is a $\mathbb{K}$-g.e.ℓ.d of T if $T = L_1 T_0$ } L_1 is a $\mathbb{K}$-e.ℓ.d. of T and $L_1 = L_2 K$ } any other $\mathbb{K}$-e.ℓ.d of T, L_2 is a $\mathbb{K}$-e.ℓ.d. of L_1
$\mathbb{K}$-greatest common extended left divisor ($\mathbb{K}$-g.c.e.ℓ.d)	$T \in \mathbb{K}^{pxmi}$, $i \in \underline{\mu}$ be a set of matrices $L_{\ell 1} \in \mathbb{K}^{pxq}$ is a $\mathbb{K}$-g.c.e.ℓ.d of T if $T = L_{\ell 1} T_{io}$ } $L_{\ell 1}$ is a $\mathbb{K}$-e.ℓ.d. of all T_i and $L_{\ell 1} = L_{\ell 2} K$ } any other $\mathbb{K}$-e.ℓ.d of T_i, $L_{\ell 2}$ is a $\mathbb{K}$-e.ℓ.d. of $L_{\ell 1}$ The set is $\mathbb{K}$-left coprime if the $\mathbb{K}$-g.c.e.l.d of the set is $\mathbb{K}$-unimodular.
$\mathbb{K}$-row projector ($\mathbb{K}$-r.p.)	$T \in \mathbb{K}^{pxm}$, $\text{rank}_{\mathbb{R}(s)}(T) = \rho \leq \min(p,m)$ $R_r \in \mathbb{K}^{mx\rho}$, $\text{rank}_{\mathbb{R}(s)}(R_r) = \rho$ R_r is a $\mathbb{K}$-r.p. of T if $TR_r = L_1$ L_1 is a $\mathbb{K}$-g.e.ℓ.d. of T.

K-<u>prime left annihilator</u> (K-p.ℓ.a.)	$T \in K^{pxm}$, $\text{rank}_{\mathbf{R}(s)}(T) = m$ $N_\ell \in K^{(p-m)xp}$ N_ℓ is a K-p.l.a. of T if $N_\ell T = 0_{(p-m),m}$ and N_ℓ is K-left unimodular
K-<u>left inverse</u> (K-ℓ.i.)	$T \in K^{pxm}$, $\text{rank}_{\mathbf{R}(s)}(T) = \rho \le \min(p,m)$ $T_\ell{}^t \in K^{mxp}$ $T_\ell{}^t$ is a K-l.i. of T if $T_\ell{}^t T = I_m$ T_ℓ is K-left unimodular

CHAPTER 5

SOLUTION OF MATRIX EQUATIONS OVER A PRINCIPAL IDEAL DOMAIN

5.1 Introduction

Many control systems problems are reduced to the study of matrix equations of the type

$$AX = B \tag{5.1}$$

or

$$YA' = B' \tag{5.2}$$

where the given matrices A, B, A', B' are in general rational matrices and the solution matrices X,Y are to be determined from a given Euclidean ring $\mathbf{K}$.

Notice that the equation YA' = B' is the dual of AX = B and hence results stated for one equation may readily be derived for the other by transposition. Results are established here for the case AX = B.

In recent years it has become clear that from a control synthesis view point the rings of importance for control applications are the Euclidean rings. $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$ and $\mathbf{R}_{\rho}(s)$. Continuing work by many researchers has highlighted the special aspects of the set $\mathbf{R}_{\rho}(s)$ of proper and stable rational functions [(section 2.3), Vid 1, Var 8, Var 3]. In recent algebraic synthesis methods the importance of the set $\mathbf{R}_{\rho}(s)$ has established its use as a powerful synthesis tool for the exposition and solution of many control problems. It is useful then, in the examples, to highlight the solution of matrix equations in this section to specialise the ring $\mathbf{K}$ to be the set $\mathbf{R}_{\rho}(s)$ and so make use of the P.I.D. properties of $\mathbf{R}_p(s)$ although, the analysis equally applies to any other Principal Ideal Domain $\mathbf{K}$.

Equations of the type (5.1 and (5.2) have been discussed in the literature [Rot 1, Kuc 1, Per 2]. The present attempt is to provide a unifying approach for their analysis as well as establish deeper results concerning solvability. The machinery developed in the previous section will be used and in the following it will be assumed that $A \in \mathbf{R}^{pxm}(s)$, $B \in \mathbf{R}^{pxr}(s)$, are given rational matrices and $X \in \mathbf{K}^{mxr}$ where $\mathbf{K}$ is a Euclidean ring, such that $\mathbf{R}(s)$ is the field of fractions of $\mathbf{K}$. Furthermore we shall denote by

M_A^c, M_B^c, the following $\mathbf{K}$-modules

$$M_A^c \triangleq \text{col.span}_{\mathbf{K}}\{A\}, \qquad M_B^C \triangleq \text{col.span}_{\mathbf{K}}\{B\} \tag{5.3}$$

and let

$$\mu_A = \text{rank}_{\mathbf{R}(s)}\{A\} \leq (p,m)$$

$$\mu_B = \text{rank}_{\mathbf{R}(s)}\{B\} \leq (p,r)$$

Recall from section 2 that since $\mathbf{K}$ is a P.I.D. the $\mathbf{K}$-modules M_A^c, M_B^c are finitely generated and free modules with ranks respectively μ_A, μ_B, [Mar, 1].

The analysis begins with a statement which is a generalisation of (5.1 (5.2) over a field.

5.2 General Results on Solvability Conditions of Matrix Equations

Theorem (5.1): Let $\mathbf{K}$ be a PID. Then, the equation $AX = B$ has a solution over $\mathbf{K}$ if and only if

$$M_B{}^c \subseteq M_A{}^c \tag{5.4}$$

Proof

Assume that $X \in \mathbf{K}^{mxr}$ exists

If $A = [\underline{a}_1, \ldots, \underline{a}_n]$

$B = [\underline{b}_1, \ldots, \underline{b}_r]$

then $AX = B$ implies that

$$\underline{b}_i = \sum_{j=1}^{m} x_{ji}\, \underline{a}_j \qquad \text{for } \forall\ i \in \underline{r} \tag{5.5}$$

thus since every column $\underline{b}_i$ of B may be expressed as a linear combination of the columns of A the necessity is established i.e $M_B{}^c \subseteq M_A{}^c$.

To prove the sufficiency assume that $M_B{}^c \subseteq M_A{}^c$ then every column $\underline{b}_i$ of B may be written in the form of (5.5) for same $X_{ji} \in \mathbf{K}$ and thus the result follows.

Note that if $\mu_A < \min(p,m)$ then (5.5) is not unique since

$\underline{b}_i$ are not linearly independent.

■

Now if M is a free **K**-module with a basis $\{\underline{s}_1, \ldots, \underline{s}_m]$ and W is a submodule of M then W is also free with rank r ($r \leq m$) [Mar 1] with this property we may then state.

<u>Remark (5.1)</u>: A necessary condition for equation (5.1) to have a solution is that $\mu_B \leq \mu_A$

■

This module inclusion property (5.4) forms the basis for our analysis. In the following, conditions for the characterisation of these properties will be derived.

In the previous chapter we defined the notion of non square matrix divisors. The following result, due to Pernebo [Per 1], defines solutions of matrix equations using the non square divisors of a matrix over a PID **K**.

<u>Theorem (5.2)</u> [Per 1]: Let **K** be a P.I.D. Then,

the equation AX = B has a solution X over **K** if and only if the g.e.l.d. of A is also an e.l.d. of B.

Proof

Let L_ℓ be a g.e.l.d. of A. Then

$$A = L_\ell \; A_0$$

If a solution exists then

$$B = AX = L_\ell \; A_0 X \text{ and thus}$$

L_ℓ is an e.l.d. of B, this proves the necessity. To prove the sufficiency assume that L_ℓ is the g.e.l.d. of A and that L_ℓ is an e.l.d of B. By Theorem (4.1.) L_ℓ may be expressed as $L_\ell = AK$ for some appropriate K. Furthermore $B = L_\ell B_0$ and thus $B = AKB_0$. By taking $X = KB_0$ the solution is defined.

■

Notice that the above result is almost identical in nature to the previous result in fact from the analysis of the previous section it is clear that the columns of a g.e.l.d define a basis for the column module of A, $M_A{}^c$. The condition that the g.e.l.d. of A is also an e.l.d. of B implies that the $B = L_\ell B_0$ and thus the column module of B, $M_B{}^c$ is covered by $M_{L\ell}{}^c = M_A{}^c$. Thus Theorem (5.2) is an alternative formulation of the Central Theorem (5.1).

5.2.1 Characterisation of Families of Solutions

If a solution of (5.1) (5.2) exists then it is important to determine if the family of solutions can be generated.

The following corollary provides a characterization of the family whenever a solution is found to exist.

Corollary (5.1): Let $\mathbf{K}$ be a P.I.D. Then,

If equation $AX = B$ has a particular solution X_0 over $\mathbf{K}$ then the family of solutions is characterized by the following properties.

(a) If $\mathbf{N}_A{}^r = \{0\}$ then X_0 is uniquely defined

(b) If $\mathbf{N}_A{}^r \neq \{0\}$ and $N_r \in \kappa^{\,m x(m-p)}$ is a p.r.a of A, then the whole family is defined by

$$X = X_0 + N_r K, \quad K \in \mathbf{K}^{(m-p)xr} \text{ arbitrary} \tag{5.6}$$

Proof

Let X and X_0 be the solutions of $AX = B$. Then $B = AX = AX_0$ and thus

$$A(X - X_0) = 0 \tag{5.7}$$

(i) If $\mathbf{N}_A{}^r = \{0\}$ then $X - X_0 = 0$

ie. $X = X_0$ and the solution is uniquely defined.

(ii) If $\mathbf{N}_A{}^r \neq \{0\}$ choose a p.r.a. of A as a basis of $\mathbf{N}_A{}^r$ and thus $(X - X_0) = N_r K$

$$X = X_0 + N_r K, \qquad K \in \mathbf{K}^{mx(m-p)} \text{ arbitrary.}$$

∎

We now give a further result on the characterization of the family of solutions. This result follows directly from corollary (5.2) and the proof of Theorem (5.2).

Corollary (5.2): Let $\mathbf{K}$ be a P.I.D.

Let R_r be a r.p, of A over $\mathbf{K}$ and assume that B may be factorized as $B = L_\ell B_0$

The equation $AX = B$ has a solution of the type

$$X_0 = R_r B_0 \tag{5.8}$$

This solution is uniquely defined if $\mathbf{N}_A{}^r = \{0\}$

If $\mathbf{N}_A{}^r \neq \{0\}$ and N_r is a p.r.a of A then the whole family of solutions is given by

$$X = R_r B_0 = + N_r K \tag{5.9}$$

where $K \in \mathbf{K}^{(m-p)xr}$ is arbitrary.

Proof

By definition $L_\ell = AR_r$ and since $B = L_\ell B_0$ a solution exists. By multiplying L_ℓ on the right by B_0 we have

$$B = L_\ell B_0 = AR_r B_0 = AX$$

and thus $X_0 = R_r B_0$ is a particular solution. The remainder of the proof is similar to that of Corollary (5.2).

∎

The results so far establish the conditions under which a solution exists and whenever a solution exists then define the family of solutions. Notice however that these are not readily verifiable and simpler conditions are sought.

5.2.2 A Practical Approach to Solvability Conditions

In the search for simple criteria to determine solubility of the above defined matrix equations the source of difficulty is determining the conditions under which the matrix B may be factorized as $B = L_\ell B_0$. A useful result is given next.

Theorem (5.3): Let $\mathbf{K}$ be a P.I.D. Then,

The equation $AX = B$ has a solution X over $\mathbf{K}$ if and only if matrices

$$[A,B] \text{ and } [A,0_{p,r}]$$

are right equivalent.

Proof

If a solution X exists then,

$$[A,B] = [A,AX]$$

$$= [A,\ 0_{p,r}] \begin{bmatrix} I_m & X \\ 0 & I_r \end{bmatrix}$$

$$= [A,\ 0_{p1r}]\ U_r$$

Clearly $U_r \in \mathbf{K}^{(mxr)x(mxr)}$ is unimodular and thus

$$[A,B] \; \mathbf{E}_r \; [A, \; O_{p,r}]$$

establishes the necessity.

To prove the sufficiency assume the above equivalence, then there exists a unimodular matrix U_r over $\mathbf{K}$ such that

$$[A,B] = [A, \; O_{p,r}] \; U_r$$

$$= [A, \; O_{p,r}] \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_3 & \mathbf{U}_4 \end{bmatrix} \qquad (5.10a)$$

where U_r is partitioned according to the partition of $[A, \; O_{p,r}]$. By equation (5.10a) we have

$$A = A\mathbf{U}_1 \qquad (5.10b)$$

$$B = A\mathbf{U}_2 \qquad (5.10c)$$

Note that $\mathbf{U}_2 \in \mathbf{K}^{mxr}$ and thus by setting $X = \mathbf{U}_2$ a solution exists.

■

<u>Corollary (5.3)</u>: Let $\mathbf{K}$ be a P.I.D. Then,

Let $[A,B] \; \mathbf{E}_r \; [A, \; O_{p,r}]$ and let

$U_r \in \mathbf{K}^{(m-r)x(m-r)}$ be an unimodular matrix over $\mathbf{K}$ for which

$$[A,B] = [A, O_{p,r}]\, U_r = [A, O_{p,r}] \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \tag{5.11}$$

Then equation AX = B has a solution and a particular solution is defined by $X_0 = \mathbf{U}_2$.

■

An alternative way of expressing the result of Theorem (5.3) is then as follows.

Corollary (5.4): Let **K** be a P.I.D. Then,

Let A_H^c be the column Hermite McMillan form of A. The equation AX = B has a solution over **K** if and only if the column Hermite McMillan of [A,B] is

$$[A_H^c \;\; O_{p,r}] \tag{5.12}$$

Proof

Necessary and sufficient conditions for $[A,B]\ \mathbf{E}_r\ [A, O_{p,r}]$ is that they have the same column Hermite form. Thus if $[A,B]\ \mathbf{E}_r\ [A, O_{p,r}]$ either of the two matrices may be used to determine the column Hermite form. Thus we start from $[A, O_{p,r}]$ and let $\mathbf{U} \in \mathbf{K}^{m\times m}$ be the unimodular matrix over **K** for which $A\mathbf{U} = A_H^c$. Then

$$[A,\ O_{p,r}]\begin{bmatrix} U & O \\ O & I \end{bmatrix}[A_H{}^C,\ O_{p,r}]$$

which is the column Hermite form of $[A,\ O_{p,r}]$ and thus of $[A,B]$. The necessity is obvious.

■

The above result may be used for the derivation of a more practical check for solvability of AX = B (YA' = B'). Attention is now focused on a more direct approach to solvability involving the machinery developed in section 4.

5.3 A Direct Approach to Solvability of Matrix Equations over a PID Ring K

Definition (5.1):

Let $A \in \mathbf{R}^{pxm}(s)$ and the $\text{rank}_{\mathbf{R}(s)}\{A\} = \mu_A$, $\mu_A \leq \min(p,m)$ and $\mathbf{K}$ be a P.I.D.

A will be called left regular if $p = \mu_A$ and right-regular if $m = \mu_A$ otherwise if $p > \mu_A$ then it will be called left irregular and if $m > \mu_A$ it will be called right irregular. If a matrix is both left and right regular it will be called regular.

Remark (5.2)

If A is left regular matrix, then $\mathbf{N}_A^{\ell} = \{0\}$ and any $\mathbf{K}$ c.p. is a $\mathbf{K}$-unimodular matrix. If A is left irregular matrix then $N_A^{\ell} \neq \{0\}$ and a $\mathbf{K}$-p.ℓ.a, $\mathbf{N}_A^{r} = \{0\}$ and a $\mathbf{K}$-p.ℓ.a exists. Similarly if A is right regular matrix then $\mathbf{N}_A^{r} = \{0\}$ and any $\mathbf{K}$-r.p is $\mathbf{K}$-unimodular matrix. If A is right irregular matrix then $\mathbf{N}_A^{r} \neq \{0\}$ and a $\mathbf{K}$-p.r.a exists.

Remark (5.3)

If L_{ℓ} is a $\mathbf{K}$-g.e.l.d. of A, then L_{ℓ} is right regular. Similarly if L_r is a $\mathbf{K}$-g.e.r.d. of A then L_r is left regular. If A is left regular then A is also a $\mathbf{K}$-g.e.r.d. of itself. Similarly if A is right regular then, A is also a $\mathbf{K}$ g.e.l.d. of itself.

Proposition (5.1)

Let $A \in \mathbf{R}^{pxm}(s)$ be a left irregular matrix $P_A^{\ell} \in \mathbf{K}^{\mu A x P}$ be a $\mathbf{K}$-c.p and $N_A^{\ell} \in \mathbf{K}^{(\mu-\mu A)xP}$ be a $\mathbf{K}$-p.ℓ.a of A. Then equation AX = B has a solution if and only if

$$N_A^{\ell}B = 0 \qquad (5.13)$$

and the following equation has a solution

$$P_A^{\ell}AX = P_A^{\ell}B \qquad (5.14)$$

Proof

The matrix

$$\Psi_{\ell} = \begin{bmatrix} P_A^{\ell} \\ \hline N_A^{\ell} \end{bmatrix} \in \mathbf{K}^{p \times p}$$

is $\mathbf{K}$-unimodular and thus

$$AX = B \longleftrightarrow \Psi_{\ell} B \longleftrightarrow \begin{bmatrix} P_A^{\ell} \\ \hline N_A^{\ell} \end{bmatrix} AX = \begin{bmatrix} P_A^{\ell} \\ \hline N_A^{\ell} \end{bmatrix} B$$

we have

$$P_A^{\ell} AX = P_A^{\ell} B \quad \text{and} \quad N_A^{\ell} AX = 0 = N_A^{\ell} B$$

Given that the steps are reversible then the proof of the proposition is established.

■

Corollary (5.5)

If A is left-irregular and P_A^{ℓ}, $P_A^{-\ell}$ are two $\mathbf{K}$-c.p. and if equation (5.13) is satisfied then equation (5.14) which corresponds to the two different P_A^{ℓ}, $P_A^{-\ell}$ $\mathbf{K}$-c.p., are equivalent.

Proof

By proposition (4.10) the general family of **K**-c.p is given by

$$P_A^{-\ell} = QP_A^{\ell} + YN_A^{\ell}$$

where U is **K**-unimodular and Y arbitrary.

Thus,

$$\overline{P}_A^{\ell}B = (\mathbf{U}P_A^{\ell} + YN_A^{\ell})AX = \mathbf{U}P_A^{\ell}AX + \underbrace{YN_A^{\ell}AX}_{=\,0}$$

$$= \mathbf{U}P_A^{\ell}AX = UP_A^{\ell}B + \underbrace{YN_A^{\ell}B}_{=\,0} = \mathbf{U}P_A^{\ell}\ B$$

Thus

$$P_A^{\ell}B = UP_A^{\ell}B = \mathbf{U}P_A^{\ell}AX$$

But the above equation is obtained by multiplying

$$P_A^{\ell}B = P_A^{\ell}AX$$

on the left by the **K**-unimodular **U**

$\blacksquare$

Note that $P_A^{\ell}A = L_A^{r}$ where L_A^{r} is a **K**-g.e.r.d of A. In the case where A is left-irregular. $P_A^{\ell} = \mathbf{U}$ where **U** is any **K**-unimodular in fact we may also assume that $P_A^{\ell} = I_{\mu A}$ in this case.

The analysis so far may be summarized by the following remarks.

Remarks (5.4)

The solvability of AX = B is reduced to the solvability of

$$L_A^r X = P_A^\ell B \tag{5.15}$$

where L_A^r is a **K**-g.e.r.d of A, that corresponds to the **K**-c.p. P_A^ℓ of A. Especially, if

(a) A is left irregular, P_A^ℓ is a **K**-c.p. and N_A^ℓ is a **K**-p.ℓ.a, then $L_A^r = P_A^\ell A$ and equation (5.15) together with

$$N_A^\ell B = 0 \tag{5.16}$$

are equivalent to the original set of equations.

(b) A is left regular, P_A^ℓ may be chosen as I_p and thus $L_A^r = A$ and $P_A^\ell B = B$.

A similar remark may be stated for the solvability of YA' = B'.

Remark (5.5)

The solvability of the equation YA' = B' is reduced to the solvability of

$$YL_{A'}^\ell = B'R_{A'}^r \tag{5.17}$$

where $L_{A'}^\ell$ is a **K**-g.e.l.d of A', that corresponds to the **K**-r.p. $R_{A'}^r$ of A'.

Especially,

(d) If A' is right irregular, $R_{A'}^{r}$ is a **K**-r.p. and $N_{A'}^{r}$ is a **K**-p.r.a then, $L_{A'}^{\ell} = A'R_{A'}^{r}$ and equation (5.15) together with

$$B'N_{A'}^{r} = 0 \tag{5.18}$$

are equivalent to the original set of matrices.

■

The solvability of the original set of equations AX = B (YA' = B') is thus reduced to the study of the solvability of the equations

$$L_A^{r}X = P_A^{\ell}B \tag{5.19}$$

where

$$L_A^{r} \in \mathbf{K}^{\mu A x m}, \ \mu_A \leq m, \ P_A^{\ell} \in \mathbf{K}^{\mu A x P}$$

and with

$$\mathrm{rank}_{\mathbf{R}(s)}\{L_A^{r}\} = \mu_A$$

Note that L_A^{r} may be factorised as

$$L_A^{r} = TR \tag{5.20}$$

where $T \in \mathbf{K}^{\mu A x \mu A}$ is a **K**-g.l.r.d of A and R is a **K**-left unimodular.

Proposition (5.2):

Let $L_A^r \in \mathbf{R}^{\mu A x m}(s)$, $\mu_A \leq m$ be a $\mathbf{K}$-g.e.r.d. of A, $R_A^r \in \mathbf{K}^{m x \mu A}$ be a $\mathbf{K}$-r.p. of A and let $L_A^r\ R_A^r = T_A$ be the corresponding $\mathbf{K}$-g.l.r.d of A.

Equation $L_A^r\ X = P_A^{\ell} B$ has a solution if and only if the following equation has a solution

$$T_A \overline{X} = P_A^{\ell} B \tag{5.21}$$

where $\overline{X}$ is related to X in the following way:

(a) If A is right regular, $\mu_A = m$ then R_A^r may be chosen to be $R_A^r = I_m$ and thus $L_A^r\ R_A^r = L_A^r = T_A$ and $X = \overline{X} \in \mathbf{K}^{\mu A x m}$.

(b) If A is right irregular, $\mu_A < m$ and $N_A^r \in \mathbf{K}^{m x (m-\mu A)}$ is a $\mathbf{K}$-p.r.a of A, then

$$X = R_A^r \overline{X} + N_A^r\ X' \tag{5.22}$$

where $X' \in \kappa^{(m-\mu A) x r}$ is arbitrary.

Proof

(a) If A is right regular then $\mu_A = m$ and R_A^r may be chosen to be I_m and thus $L_A^r\ R_A^r = L_A^r = T_A$ and $X = \overline{X}$

(b) If A is right irregular then a pair $R_{A'}^r = N_A^r$ of a $\mathbf{K}$-r.p. and a $\mathbf{K}$-p.r.a. may be defined and the matrix

$\Psi_r = [R_A^r, N_A^r] \in \mathbf{K}^{m x m}$ and $\mathbf{K}$-unimodular

Define the transformation

$$X = [R_A^{\ r} \quad N_A^{\ r}] \begin{bmatrix} \bar{X} \\ X' \end{bmatrix}$$

and substitute in $L_A^{\ r} X = P_A^{\ \ell} B$. Then

$$P_A^{\ \ell} B = L_A^{\ r} [R_A^{\ r} \quad N_A^{\ r}] \begin{bmatrix} \bar{X} \\ X' \end{bmatrix} = L_A^{\ r} R_A^{\ r} \bar{X} + L_A^{\ r} N_A^{\ r} X'$$

Given that $L_A^{\ r} R_A^{\ r} = T_A$ and $L_A^{\ r} N_A^{\ r} = 0$ we have

$$P_A^{\ \ell} B = T_A \bar{X}$$

The transformations are reversible and the result is established.

■

From the above analysis it is clear that the solvability of the original equations is reduced to the solvability of the equation

$$T_A \bar{X} = P_A^{\ \ell} B \tag{5.23}$$

The general results derived so far may be applied to the above equations. We summarise the analysis by stating the following results.

Theorem (5.4):

Let $A \in \mathbf{R}^{pxm}(s)$, $\mu_A = \text{rank}_{\mathbf{R}(s)}\{A\} \leq \min(p,m)$ and $\mathbf{K}$ be a P.I.D.

$P_A^{\ell} \in \mathbf{K}^{\mu A x P}$ be a $\mathbf{K}$-c.p. and R_A^r be a $\mathbf{K}$-r.p.,

$T_A = P_A^{\ell} A R_A^r$ be a κ-g.l.r.d. of A.

(i) If A is left-right regular ($p = m = \mu_A$) then the equation $AX = B$ has a solution if and only if A is a left divisor of B. If a solution exists then it is uniquely defined.

(ii) If A is left regular ($p = \mu_A$), but right irregular ($m > \mu_A$), then equation $AX = B$ has a solution if and only if T_A is a left divisor of B. If a solution exists then a family of solutions exists and it is defined by

$$X = R_A^r \overline{X} + N_A^r X'$$

where $\overline{X}$ is a particular solution of

$$T_A \overline{X} = B$$

$N_A^r \in \mathbf{K}^{mx(m-\mu A)}$ is a $\mathbf{K}$-p.r.a of A and

$X' \in \kappa^{(m-\mu A)xr}$ is arbitrary.

(iii) If A is left-irregular ($p > \mu_A$), then equation $AX = B$ has a solution if and only if

$$N_A^{\ell} B = 0 \tag{5.26}$$

where $N_A^{\ell} \in \mathbf{K}^{(P-\mu A)xP}$ is a $\mathbf{K}$-p.l.a. of A and T_A is a left divisor of $P_A^{\ell} B$. If a solution exists then

(a) If A is right regular ($m = \mu_A$) then the solution is unique and is defined by solving the equation

$$T_A X = P_A^{\ell} B \tag{5.27}$$

(b) If A is right irregular ($m > \mu_A$) then a family of solutions exists and this family is defined by

$$X = R_A^{r} \bar{X} + N_A^{r} X' \tag{5.28}$$

where $\bar{X}$ is a particular solution of

$$T_A \bar{X} = P_A^{\ell} B \tag{5.29}$$

$N_A^{r} \in K^{mx(m-\mu A)}$ is a $\mathbf{K}$-p.r.a. of A

$X' \in K^{(m-\mu A)xr}$ is arbitrary.

■

<u>Proof</u>

The result immediately follows by propositions (5.1), (5.2), corollary (5.5) and remarks (5.4), (5.5). Given that in all cases the central equation is $T_A \bar{X} = P_A^{\ell} B$, then by Theorem (5.2) the result follows. Note that since

T_A is left-right regular then the notion of $\mathbb{K}$-g.e.l.r.d. coincides with the matrix itself.

$\blacksquare$

A similar result may be stated for the equation $YA' = B'$.

CHAPTER 6

DECENTRALIZED DIAGONAL STABILIZATION: A FRACTIONAL REPRESENTATION APPROACH

6.1 Introduction

A special case of decentralized stabilization is the problem of diagonal stabilization of a linear invariant system P. In this special case the problem is to determine a stabilizing compensator C = diag{C_i} such that upon interconnection of the feedback loop $\underline{u} = -C\underline{y}$, where $\underline{u}$ is the input and y the output of P. The closed loop system becomes internally stable ie. P is internally stabilized by C. Alternatively the internal stability requirement may be expressed in terms of transfer matrices (section 3.2) thus: the closed loop system is internally stable if (I + PC) is non singular and the poles of the transfer matrix (Eqn 3.7) H(P,C) where

$$H(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP) \end{bmatrix}$$

are stable [Des 3].

The diagonal stabilization of P is possible if and only if P is free of unstable fixed modes under diagonal output feedback [Wan 1]. This fundamental result of decentralized stabilization highlights the important role and the need for characterisation of fixed modes. Various researchers have provided such characterisations. [Wan 1, Cor 2, And 1, And 2, Vis 1, Dav 3].

The objective of this section is to give new necessary and sufficient conditions for the solution of the decentralized stabilization problem using diagonal dynamic compensators. The notion of cyclicity is introduced using the unique Hermite form of a matrix defined over the ring of proper and stable rational functions. The existence and characterisation of solutions is intimately related to systems that exhibited the property of cyclicity.

It is useful to begin this section with a brief outline of the essential difference between centralized and decentralized structure and to review the decentralized stabilization problem to date.

6.2 <u>Decentralized Control Structure and Stabilization</u>

6.2.1 <u>Decentralized Control</u>

Control system design techniques such as linear quadratic LQ (optimal) and pole-placement design use state feedback to improve system behaviour. It is often impossible to instrument a system to the extent required for full state feedback and to overcome this difficulty techniques such as linear-quadratic Gaussian (LQG), observer based control and time-domain compensator design have been evolved. However, a key feature of all these techniques is that a design results in which every sensor output affects every actuator input. We term this situation <u>Centralized Control</u> and the structure of such a scheme is shown in

Fig.6.1 where the interaction between sensors and actuators results in a coupling between the controllers.

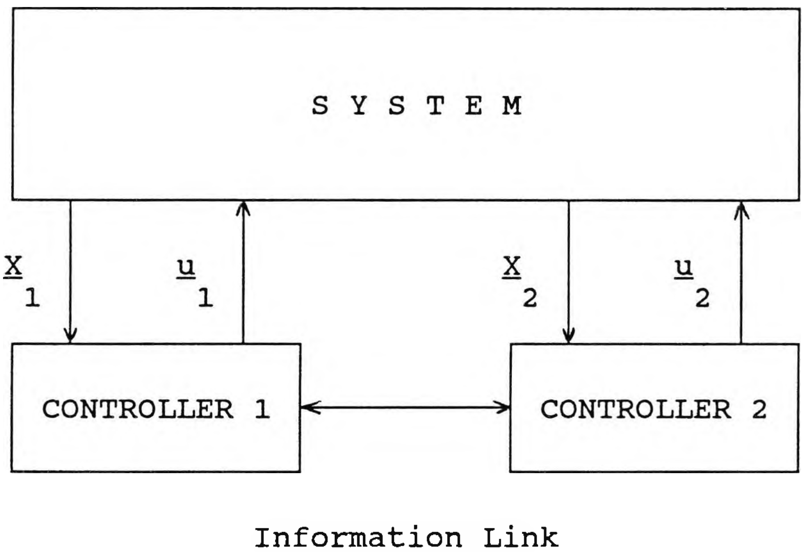


Fig. 6.1 <u>Centralized Control System Structure</u>

The disadvantage of centralized control arises in many systems but particularly in large scale systems through the impossibility to incorporate so many feedback loops in the design. To overcome this difficulty <u>Decentralized Control</u> theory has developed [San 1]. The basic feature of decentralized control is the restriction on information transmission between certain groups of sensors and actuators as shown in Fig. 6.2 below. Notice that only state vector $\underline{X}_1$ is used to form control law $\underline{u}_1$ and similarly only state vector $\underline{X}_2$ forms control law $\underline{u}_2$. Thus decentralized control refers to the implemented <u>control structure</u> and not the <u>control law</u> which may be designed in a completely centralized way.

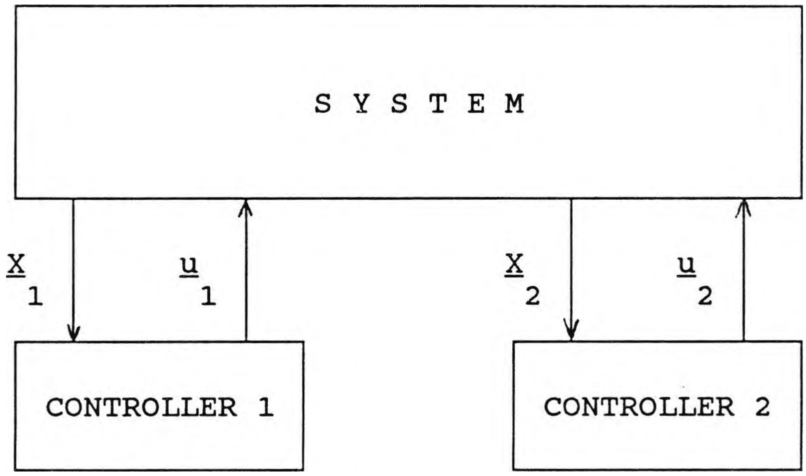


Fig. 6.2 Decentralized Control Structure

6.2.2 Decentralized Stabilization and Pole Placement

It is known that the poles of a controllable linear system can be arbitrarily assigned (subject to complex pole-pairing constraints) by state feedback [Kai 1]. This fundamental result has been extended to show that the closed loop poles of a completely controllable and observable system can be freely assigned using a dynamic compensator of defined order [Bra 1]. These results have been used as a basis of practical synthesis procedures.

Although several authors [McF 1, Aok 1, Aok 2, Cor 3] had investigated the generalization of the pole placement question under the restriction of decentralized feedback control the most authoritative results are those of Wang and Davison [Wan 1], and Corfmat and Morse [Cor 1, Cor 2] and form the basis of synthesis methods for the design of stabilizing controllers. The procedure of Wang and Davison [Wan 1] in state space setting consists of sequentially moving unstable modes to the defined stability region using feedback around each channel. This procedure can be tedious and does not yield an explicit expression for the stabilizing compensator. Corfmat and Morse [Cor 1] define a method for "strongly connected" systems to make the system stabilizable and detectable through the remaining channel. An alternative version of this method is given by Vidyasagar and Viswanadham [Vis 1]. A recent synthesis method [Ozg 1] provides a procedure for the construction of stabilizing compensator using simple polynomial algebra and yields an explicit expression for the compensator transfer functions. It is applicable to not necessarily strongly connected systems and to systems of arbitrary causality degree.

6.2.3 <u>Decentralized Stabilization Problem</u>

In Chapter 3 (section 3.2) the decentralized stabilization problem was formulated and the problem reduced to the solution of the matrix diophantine equation.

$$A_1 \text{ diag } \{D_i\} + B_1 \text{ block diag } \{N_i\} = U \tag{6.1a}$$

over the ring of proper and stable rational functions $\mathbf{R}_\rho(s)$ and is referred to as the <u>Decentralized Diophantine Equation</u> (DDE).

Where (A_1, B_1) is an $\mathbf{R}_\rho(s)$-left coprime MFD of the plant P.

$$P = A_1^{-1} B_1;\ A_1 \in \mathbf{R}_\rho^{p\times p}(s);\ B_1 \in \mathbf{R}_\rho^{p\times m}(s),\ P \in \mathbf{R}^{p\times m}(s)$$

Partitioning (A_1, B_1) according to the block structure of diag $\{D_i\}$, block diag $\{N_i\}$ we have

$$\begin{bmatrix} A_{1_1} \\ \vdots \\ A_{1_i} \\ \vdots \\ A_{1_r} \end{bmatrix} \begin{bmatrix} D_1 & & & & 0 \\ & \ddots & & & \\ & & D_i & & \\ & & & \ddots & \\ 0 & & & & D_r \end{bmatrix} + \begin{bmatrix} N_1 & & & & 0 \\ & \ddots & & & \\ & & N_i & & \\ & & & \ddots & \\ 0 & & & & N_r \end{bmatrix} = \begin{bmatrix} U_1 \\ \vdots \\ U_i \\ \vdots \\ U_r \end{bmatrix} = U$$

(6.1b)

from which

$$A_{1i}\ D_i + B_{1i}\ N_i = U_i$$

with

$D_i \in \mathbf{R}_\rho(s)$, $N_i \in \mathbf{R}_\rho(s)$, $A_{1i} \in \mathbf{R}_\rho(s)$, $B_{1i} \in \mathbf{R}_\rho(s)$ and $U_i \in \mathbf{R}_\rho(s)$ is part of an arbitrary $\mathbf{R}_\rho(s)$-left unimodular matrix.

The decentralized stabilization problem is then defined to be the problem of determining precompensators C of the type

$$C = \begin{bmatrix} C_1 & & & & 0 \\ & \ddots & & & \\ & & C_i & & \\ & & & \ddots & \\ 0 & & & & C_r \end{bmatrix} = N_2\ D_2^{-1} \qquad (6.1c)$$

(where $C_i = N_i\ D_i^{-1}$ is an $\mathbf{R}_\rho(s)$ MFD of the diagonal controller) such that the closed loop system (Fig 3.1) under the feedback

$$\underline{u}_i = C_i\{\underline{y}_i - \underline{W}_i\} \tag{6.1d}$$

is closed loop stable.

6.2.4 Stabilization by Permutation of Diagonal Controllers

Stabilization by direct connection of corresponding inputs to outputs may not always be possible and, or produce the optimum performance of the control system. This problem is characterised by system design indicators [Mac 2] which may be used during the design phase to evaluate the merits of a particular control scheme. Consider the control scheme shown in figure 6.1 below

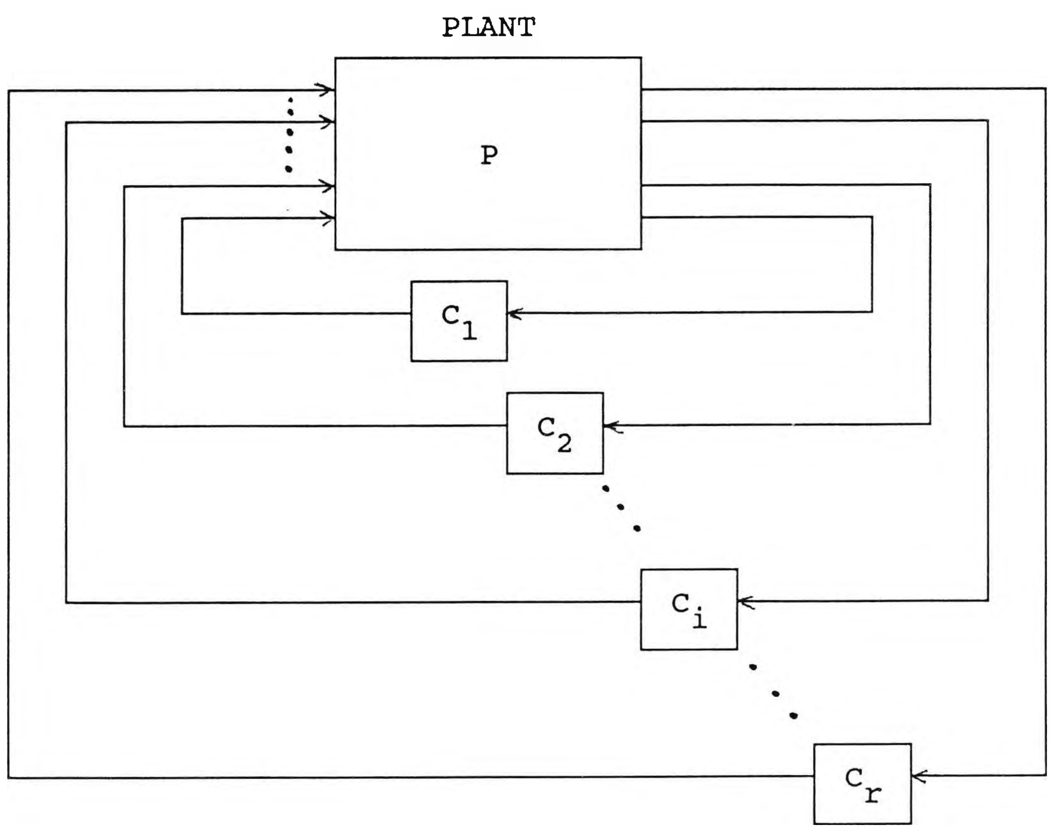


Fig. 6.3 Feedback Control Scheme: Diagonal Controller

The closed loop transfer function matrix H(P,C) is given by Eqn (3.20) as:

$$I + PC_{diag} \tag{6.2}$$

where

$$P = D_p^{-1} N_p, \; N_p \in \mathbf{R}_\rho^{mxp}(s) ; \; D_p \; \mathbf{R}_\rho^{mxm}(s)$$

$$C_{diag} = N_c D_c^{-1}, \; N_c \in \mathbf{R}_\rho^{pxm}(s) ; \; D_c \in \mathbf{R}_\rho^{mxm}(s)$$

$$C_{diag} = \begin{bmatrix} c_1 & & & & o \\ & \ddots & & & \\ & & c_i & & \\ & & & \ddots & \\ o & & & & c_r \end{bmatrix}$$

The stabilization equation reduces to Eqn. (6.1a)

$$D_p D_c + N_p N_c = U, \; U \in \mathbf{R}_\rho^{mxm}(s), \text{ unimodular}$$

and the diagonal controllers are corrected between respective input-output channels.

Consider now the control shown in figure 6.4 below

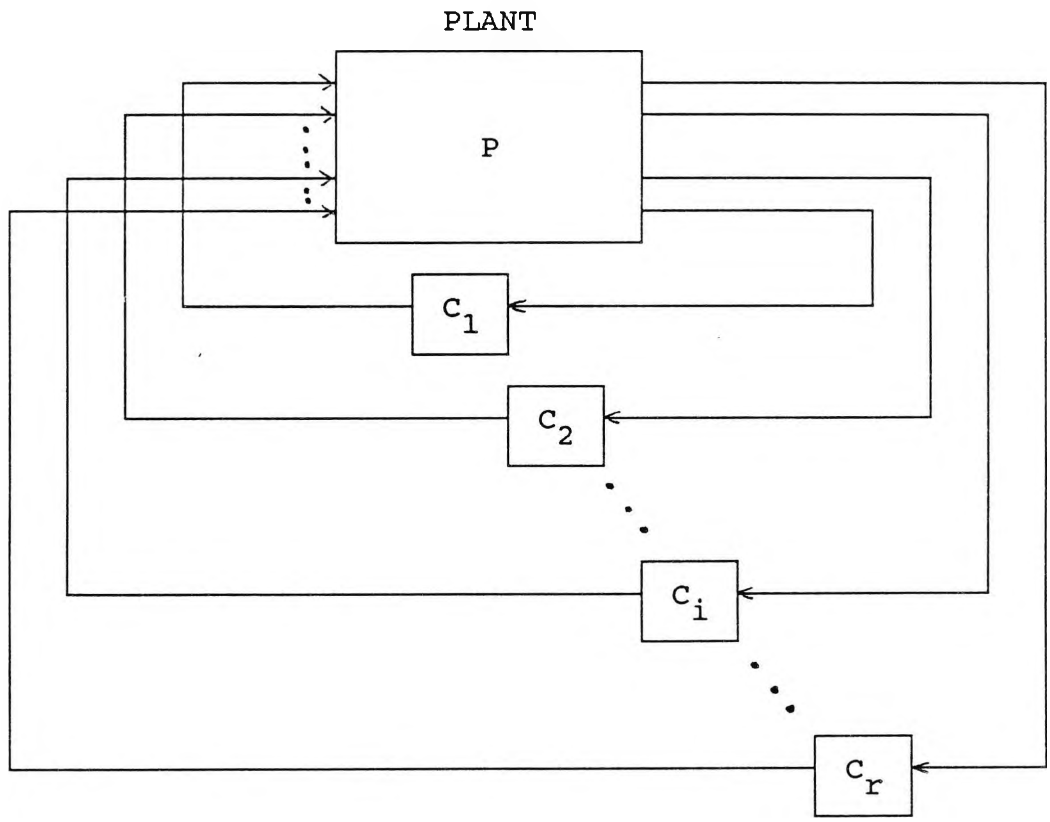


Fig. 6.4 <u>Feedback Control Scheme: Permutation of Diagonal Controllers</u>

The closed loop transfer function matrix is given by Eqn (3.20) as $I + P\bar{C}_{diag}$, where

$$\bar{C}_{diag} = \begin{bmatrix} 0 & & & & c_r \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & c_i & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ c_1 & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & & 1 \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & 0 \end{bmatrix} \begin{bmatrix} c_1 & & & & \\ & \ddots & & & \\ & & c_i & & \\ & & & \ddots & \\ & & & & c_r \end{bmatrix}$$

$$= I + D_p^{-1} N_p \bar{I} N_c D_c^{-1} \tag{6.3}$$

$$= D_p D_c + N_p \bar{I} D_c = U \tag{6.4}$$

where $U \in \mathbf{R}_{\rho}^{m\times m}(s)$, unimodular

$\overline{I} \in \mathbf{R}_{\rho}^{m\times m}(s)$, unimodular matrix describing the interconnections between inputs and outputs and is known as the <u>diagonal controller permutation matrix</u>.

6.3 <u>Stabilization of mxm plants with diagonal controller</u>

Consider a plant transfer function $P = A_1^{-1} B_1 \in \mathbf{R}^{m\times m}(s)$, where A_1, B_1 is an $\mathbf{R}_{\rho}(s)$-coprime pair and let $C = \mathrm{diag}\{c_1,\ldots,c_m\} = N_2 D_2^{-1}$ be an $\mathbf{R}_{\rho}(s)$ coprime MFD of the diagonal controller, where $c_i = n_i d_i^{-1}$, $i \in m$, is an $\mathbf{R}_{\rho}(s)$-coprime MFD of c_i. Then $N_2 = \mathrm{diag}\{n_1,\ldots,n_m\}$, $D_2 = \mathrm{diag}\{d_1,\ldots,d_m\}$. For stabilization of the plant by the diagonal controller C we must have (eqn. 6.1):

$$A_1 D_2 + B_1 N_2 = U, \tag{6.5}$$

where U is an $\mathbf{R}_{\rho}$-unimodular matrix. Let us partition A_1, B_1 in terms of columns, then (6.5) yields

$$[\underline{a}_1,\underline{a}_2,\ldots\underline{a}_m]\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{bmatrix} + [\underline{b}_1,\underline{b}_2,\ldots,\underline{b}_m]\begin{bmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_m \end{bmatrix}$$

$$= [\underline{u}_1,\ \underline{u}_2,\ldots,\ \underline{u}_m]$$

or equivalently,

$$\underbrace{[\underline{a}_i, \underline{b}_i]}_{\triangle\, P_i} \underbrace{\begin{bmatrix} d_i \\ n_i \end{bmatrix}}_{\triangle\, \underline{q}_i} = \underline{u}_i \,, \qquad i=1,2,\ldots,m \tag{6.6}$$

where $P_i \in \mathbf{R}_\rho^{mx2}(s)$ are matrices defined by the plant and the $\underline{q}_i \in \mathbf{R}_\rho^{1x2}(s)$ characterise the SISO controllers.

Remark (6.1): The solvability of (6.5) is independent of the particular $\mathbf{R}_\rho(s)$-coprime MFD of the plant, in fact, if (6.5) is solvable for the given (A_1, B_1) pair, then any other pair is defined by $(A_1, B_1) = (U_L A_1, U_L B_1)$, where $U_L \in \mathbf{R}_\rho^{mxm}(s)$, unimodular and thus the problem is solvable for the new pair (A, B) also.

■

Thus the set of matrices $\{P_i, i \in \underline{m}\}$ is characteristic of the plant, since for any other coprime MFD of the plant the corresponding set is $\{U_L P_i, i \in \underline{m}, U_L \in \mathbf{R}_\rho^{mxm}(s)$, unimodular$\}$. This leads to the following obvious result.

Proposition (6.1): The sets $\{P_i, i \in \underline{m}\}$, $\{p_i', i \in \underline{m}\}$ of matrices associated with two different $\mathbf{R}_\rho(s)$-coprime MFD's of the plant P are $\mathbf{R}_\rho(s)$-left equivalent. That is, there exists $U_L \in \mathbf{R}_\rho^{mxm}(s)$, unimodular such that

$$P_i' = U_L P_i, \; i \in \underline{m} \tag{6.7}$$

■

In the following we shall use the above property for the development of solvability criteria, which are independent from the particular MFD. A set $\rho \triangle \{P_i,\ i \in \underline{m}\}$ will be referred to as a representative decentralized matrix (RDM) set of the plant.

6.3.1 Cyclic and Non-Cyclic sets of Matrices

Before we proceed with the above solvability conditions we define the notion of cyclicity.

Definition (6.1): Let $T \in \mathbf{R}_\rho^{m\times k}(s)$, $m \geq k$, $\text{rank}_{\mathbf{R}(s)}\{T\} = k$ and let $S(T)$ be the Smith form of T over $\mathbf{R}_\rho(s)$. Then,

(i) T will be called cyclic if,

$$S(T) = \begin{bmatrix} \begin{matrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ & & & & f(s) \end{matrix} \\ \hline 0 \end{bmatrix} = \begin{bmatrix} S^*(T) \\ \hline 0 \end{bmatrix} \tag{6.8}$$

i.e the first k-1 invariant functions are trivial and $f(s) \in \mathbf{R}_\rho(s)-\{0\}$; otherwise, ie. more than one invariant function is non trivial, it will be called noncyclic.

(ii) T will be called complete, if it is cyclic and $f(s) = 1$. In this case $S^*(T) = I_k$ and T has no zeros in $\mathbf{R}_\rho(s)$.

■

Note that in the present study it is assumed that the plant is non degenerate ($|p| \not\equiv 0$) and thus in any RDM set $\rho = \{p_i,\ i \in \underline{m}\}$ the matrices P_i have full rank.

Definition (6.2): An RDM set $\rho = \{P_i,\ i \in \underline{m}\}$ of the plant P will be called cyclic if for all $i \in \underline{m}$ the matrices P_i are cyclic; if at least one of the P_i's is non cyclic, then ρ will be called non-cyclic. The set ρ will be called complete, if for all $i \in \underline{m}$, the matrices P_i are complete.

■

Let us denote by $\mathbf{F}(P_i) \triangleq \{) f_{1i}(s),\ f_{1i}(s)/f_{2i}(s)\}$ the invariant functions of P_i and by $\mathbf{F}(\rho) \triangleq \{\mathbf{F}(P_1); \ldots; \mathbf{F}(P_m)\}$ the ordered set of invariant functions of ρ; furthermore, let $Q = [P_i; \ldots; P_m]$ and

$$R(\rho) \triangleq \mathbf{R}_\rho(s)\text{-row module } \{Q\} \tag{6.9}$$

By proposition (6.1) it is readily shown that:

Proposition (6.2) Let ρ and ρ' be two RDM sets associated with the plant P. Then,

(i) $\mathbf{F}(\rho) = \mathbf{F}(\rho')$

(ii) $\mathbf{R}(\rho) = \mathbf{R}(\rho')$

■

Because of the above property the set $\mathbf{F}(\rho)$ and the module $\mathbf{R}(\rho)$ are invariants of the plant p and do not characterise the individual ρ set only. Because of the above property, if for a set ρ the cyclic, noncyclic, or complete property holds true, then the corresponding property characterises the plant and not just the individual set ρ. Thus, the system will be called <u>cyclic</u>, <u>non cyclic</u>, or <u>complete</u> if for some RDM set ρ the corresponding property holds true. The set $\mathbf{F}(\rho)$ and the module $\mathbf{R}(\rho)$ will be denoted by $\mathbf{F}_\rho$, $\mathbf{R}_\rho$ respectively. Clearly, the system is cyclic if $f_{1i}(s) = 1$ for all $i \in \underline{m}$ and complete if $(f_{1i}(s) = 1,\ f_{1i}(s) = 1)$ for all $i \in \underline{m}$. The importance of the above notions in the study of decentralized SISO stabilization is examined next.

6.3.2 <u>Cyclic Plants and Diagonal Stabilization</u>

<u>Proposition (6.3)</u> Let P be a non cyclic plant. There exists no diagonal compensator $C = \text{diag}\{C_i,\ i \in \underline{m}\}$ that stabilizes the closed loop system.

<u>Proof</u>

Let p_j be a non cyclic matrix in an RDM set ρ of P and assume that there exists a diagonal controller C that stabilizes the plant. Then, by (6.6),

$$P_j \begin{bmatrix} d_j \\ n_j \end{bmatrix} = \underline{u}_j \qquad (6.10a)$$

where $\underline{u}_j$ must be a coprime $\mathbf{R}_\rho(s)$ vector (as a column of an $\mathbf{R}_\rho(s)$-unimodular matrix). Let U_L^{-1}, U_R^{-1} be a pair of $\mathbf{R}_\rho(s)$-unimodular matrices that reduce P_j to its Smith form over $\mathbf{R}_\rho(s)$. Then we have $P_j = U_L\ S'(P_j)U_R$ and by (6.10a)

$$U_L = \begin{bmatrix} f_{1j} & 0 \\ 0 & f_{2j} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} U_R\ \underline{q}_j = \underline{u}_j \tag{6.10b}$$

By partitioning U_L according to the partitioning of $S(P_j)$ we have

$$[\underline{v}_1,\ \underline{v}_2,\ U_L'] \begin{bmatrix} f_{1j} & 0 \\ 0 & f_{2j} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} \tilde{\underline{q}} = \underline{u}_j,\quad \tilde{\underline{q}}_j = U_R\ \underline{q}_j = \begin{bmatrix} \tilde{d}_j \\ \tilde{n}_j \end{bmatrix} \tag{6.10c}$$

and thus,

$$\underline{v}_1 f_{1j}\ \tilde{d}_j + \underline{v}_2\ f_{2j}\ \tilde{n}_j = \underline{u}_j \tag{6.10d}$$

Clearly, since f_{1j}/f_{2j}, for all choices of $(\tilde{d}_j,\ \tilde{n}_j)$ and thus $(d_j,\ n_j)$, f_{ij} must divide $\underline{u}_j$ and thus $\underline{u}_j$ is not coprime (since $f_{ij} \not\equiv 1$). This contradicts our assumption.

■

The above result clearly implies the following:

<u>Corollary (6.1)</u>: A necessary condition for diagonal closed loop stabilization of a plant P, is that P be

cyclic.

∎

<u>Definition (6.3)</u> Let $f_G \triangleq \{f_{11}(s), \ldots, f_{1m}(s)\}$ and

$$G_p(s) \triangleq \prod_{i=1}^{m} f_{1i}(s);$$ $G_p(s)$ will be called the first invariant function of P. The properties of $G_p(s)$ are summarized below.

<u>Proposition (6.4)</u>: Let P be an mxm plant and G_p be its first invariant function. Then,

(i) $G_p(s)$ is an invariant of the plant

(ii) The zeros in $\rho := \Omega \cup \{\infty\}$ of $G_p(s)$ are fixed closed loop poles of any closed loop system obtained by diagonal pre compensation and unity feedback.

<u>Proof</u>

Part (i) follows from proposition (6.2). From the proof of proposition (6.3), it is clear that for every $j \in \underline{m}$ the vectors $\underline{u}_j$ must be written as $\underline{u}_j = f_{1j}(s)\, \underline{u}'_j$ for a solution of (6.10c) to exist. Then

$$\underbrace{[\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m]}_{\triangleq\, u} = \begin{bmatrix} f_{11} & & & \\ & f_{12}(s) & & \\ & & \ddots & \\ & & & f_{1m}(s) \end{bmatrix} \underbrace{[\underline{u}'_1\ \underline{u}'_2, \ldots, \underline{u}'_m]}_{\triangleq\, u'} \qquad (6.11)$$

and for all choices $C = \text{diag}\ \{C_i,\ i \in \underline{m}\}$ and u', $\mathbf{R}_\rho(s)$-unimodular, diag $\{f_{11}(s), \ldots, f_{1m}(s)\}$ will be a factor of in the denominator (U) of the closed-loop plant and thus the zeros of $G_p(s)$ define fixed-unstable closed-loop poles.

■

If $G_p^f(s)$ denotes the <u>fixed pole function</u> of the closed-loop system obtained under any diagonal precompensation and unity output feedback, then we have:

<u>Remark (6.2)</u>: The first invariant function $G_p(s)$ divides the fixed pole function $G_p^f(s)$.

■

We now give some criteria for cyclicity of a plant. From the definition of the Smith form [(Chapter 2)] we have the following remark:

<u>Remark (6.3)</u>: Let $d_i(s)$ be a greatest common divisor of the elements of p_i (1x1 minors). Then, $f_{1i}(s)$ is an associate of $d_j(s)$ and

$$G_p(s) = u \prod_{i=1}^{m} d_i(s), \text{ where } u \text{ is an } \mathbf{R}_\rho(s) \text{ unit.}$$

■

From the above remark we have:

<u>Proposition (6.5)</u>: The system defined by the transfer function G is cyclic, if and only if for every fixed i,

$i = 1,2,\ldots,m$, the elements of P_i are coprime.

■

<u>Definition (6.4)</u>: Consider now the case of cyclic plants. The system represented by G will be called <u>diagonally stabilizable</u> (D-stabilizable) if equation (6.5) holds true for some U, $\mathbf{R}_\rho(s)$-unimodular matrix and with $N_2 = \text{diag}\{n_i, i \in \underline{m}\}$, $D_2 = \text{diag}\{d_i, i \in \underline{m}\}$, (n_i, d_i) coprime. From equations (6.5), (6.6) it is clear that our investigation is reduced to the following problem.

<u>Problem (6.1)</u>: Given a set of full rank cyclic matrices P_i, $P_i \in \mathbf{R}_\rho^{m\times 2}(s)$, $i \in \underline{m}$, we may define the following problems:

(i) Determine solvability of the following equations over $\mathbf{R}_\rho(s)$:

$$P_i \, \underline{q}_i = \underline{u}_i, \ \underline{q}_i = \begin{bmatrix} d_i \\ n_i \end{bmatrix} \in \mathbf{R}_\rho^{2\times 1}(s), \quad d_i \neq 0, \ i \in \underline{m} \tag{6.12}$$

$\underline{u}_i \in \mathbf{R}_\rho^{m\times 1}(s)$ arbitrary, constrained however by the <u>coupling condition</u> that $U = [\underline{u}_i, \ldots, \underline{u}_m]$ is $\mathbf{R}_\rho(s)$-unimodular. This problem will be referred to as <u>D-stabilization problem</u> (DSP).

(ii) If we further constrain the problem in (i) by the extra condition that for all $i \in \underline{m}$, the d_i solutions have no zeros at infinity (i.e they are also units of

$\mathbf{R}_{pr}(s)$, then the problem will be referred to as <u>proper D-stabilization problem</u> (PDSP).

(iii) By constraining further the problem in (ii) by requiring that for all $i \in \underline{m}$, the d_i solutions are units of $\mathbf{R}_p(s)$, then we define the <u>strong-D-stabilization problem</u> (SDSP).

■

6.3.3 <u>Solvability of the Diagonal Stabilization Problem</u>

We consider first the general case of DSP. We shall examine some general solvability conditions. Notice that equation (6.12) may be written in an equivalent manner as

$$\underbrace{[P_1, P_2, \dots, P_m]}_{\triangleq\, P_m} \underbrace{\begin{bmatrix} \underline{q}_1 & & & 0 \\ & \underline{q}_2 & & \\ & & \ddots & \\ 0 & & & \underline{q}_m \end{bmatrix}}_{\triangleq X_m} = U \tag{6.13}$$

Let us define $\underline{q}_i = [d_i \; n_i]^t = [X_{i1}, X_{i2}]^t$. Since U is an $\mathbf{R}_p(s)$-unimodular, by using the Binet-Cauchy Theorem [Mar 2] we have:

$$|U| = u = \mathbf{C}_m(P) \; \mathbf{C}_m \; (X_m) \tag{6.14}$$

where $u \in \mathbf{R}_p(s)$ is a unit, and $\mathbf{C}_m$ $(\cdot)$ defines the m-th compound matrix of $(\cdot)$.

The above equation is multilinear as far as the parameters X_{ij}, $i \in \underline{m}$, $j \in \underline{2}$ in $\mathbf{C}_m\ (X_m)$; their exact location is to be investigated.

Consider just the simple case m = 2. Then $\mathbf{X}_2$ may be written as

$$\mathbf{X}_2 = \begin{array}{c} \begin{bmatrix} X_{11} & 0 \\ X_{12} & 0 \\ 0 & X_{21} \\ 0 & X_{22} \end{bmatrix} \end{array} \begin{array}{l} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \end{array} = [\underline{X}_1\ ,\ \underline{X}_2] \qquad (6.15a)$$

The second compound matrix is defined by

$$\mathbf{C}_2(\mathbf{X}_2) = \underline{X}_1 \wedge \underline{X}_2 = \begin{bmatrix} 0 \\ X_{11}X_{21} \\ X_{11}X_{22} \\ X_{12}X_{21} \\ X_{12}X_{22} \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow (1,2) \\ \leftarrow (1,3) \\ \leftarrow (1,4) \\ \leftarrow (2,3) \\ \leftarrow (2,4) \\ \leftarrow (3,4) \end{array} = \begin{bmatrix} 0 \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ 0 \end{bmatrix}$$

(6.15b)

If $\mathbf{P}_2$ is defined by

$$\mathbf{P}_2 = [\underline{P}_{11}, \underline{P}_{12}; \underline{P}_{21} \underline{P}_{22}] \tag{6.16a}$$

Then

$$\mathbf{C}_2(\mathbf{P}_2) = [a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}] \tag{6.16b}$$

where

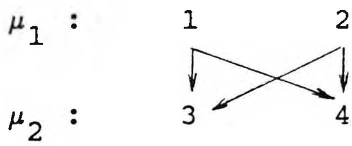
$$a_{12} = |\underline{P}_{11}, \underline{P}_{12}|, \quad a_{13} = |\underline{P}_{11}, \underline{P}_{21}|, \quad a_{14} = |\underline{P}_{11}, \underline{P}_{22}|$$

$$a_{23} = |\underline{P}_{12}, \underline{P}_{21}|, \quad a_{24} = |\underline{P}_{12}, \underline{P}_{22}|, \quad a_{34} = |\underline{P}_{21}, \underline{P}_{22}|$$

Then equations (6.14) becomes

$$a_{13}\ \lambda_{13} + a_{14}\ \lambda_{14} + a_{23}\ \lambda_{23} + a_{24}\ \lambda_{24} = u \tag{6.17}$$

Because of the structural form of $\mathbf{X}_2$ a number of zeros appear in $\mathbf{C}_2(\mathbf{X}_2)$. The elements of $\mathbf{C}_2(\mathbf{X}_2)$ are indexed by the sequences $Q_{2,4} = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$. Note that if the integers $\{1,2,3,4\}$ are grouped as $\{\mu_1 = (1,2),\ \mu_2 = (3,4)\}$, then the element a_ω in $\mathbf{C}_2(\mathbf{X}_2)$, $\omega \in Q_{2,4}$ is zero if and only if more than one index in $\omega = (i_1, i_2)$ is taken from the same μ_i. The location of non zero elements is defined by these sequences $\omega \in Q_{2,4}$ for which only one index is defined from each of the μ_1, μ_2. A table which allows the computation of non zero elements in $\mathbf{C}_2(\mathbf{X}_2)$ is given below:



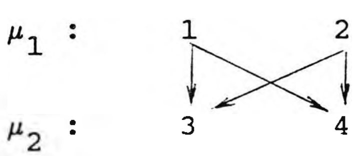
$\{(1,3),\ (1,4),\ (2,3),\ (2,4)\} = \Omega_2$

the Ω_2 set will be referred to as the <u>essential subset</u> of $Q_{2,4}$.

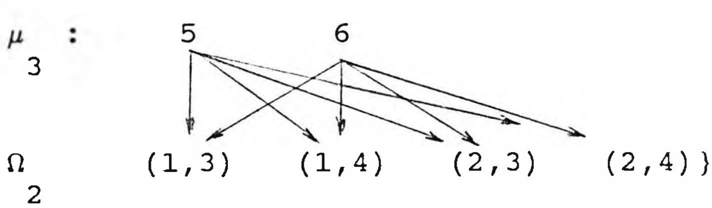
Consider now the case m = 3. The matrix X_3 is defined by

$$
\mathbf{X}_3 = \begin{array}{c} \left[\begin{array}{ccc} X_{11} & & \\ X_{12} & & \\ \hline & X_{21} & \\ & X_{22} & \\ \hline & & X_{31} \\ & & X_{32} \end{array}\right] \end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

To complete the essential subset Ω_3 of sequences in $Q_{3,6}$ we form the composite table shown below



$\Omega_2 = \{(1,3),\ (1,4),\ (2,3),\ (2,4)\}$



Ω_3 $\{(1,3,5)(1,3,6)(1,4,5)(1,4,6)(2,3,5)(2,3,6)(2,4,5)(2,4,6)\}$

From the above two cases a general pattern indicating the presence of fixed zeros in $\mathbf{C}_m$ $(\mathbf{X}_m)$ has emerged. To generalise these observations and prove results we introduce some useful notation.

Definition (6.5): Let $Q_{m,2m}$ denote the set of strictly increasing and lexicographically ordered sequences of m integers taken from 1,2,...,2m. For the set of integers {1,2,...2m} a pair partitioning is defined as the set of ordered pairs $\Phi = \{\mu_1 = \{1,2\};\ \mu_2 = \{3,4\};\ldots;\ \mu_m = \{2m-1, 2m\}\}$. A sequence $\omega = \{i_1, i_2,\ldots i_m\} \in Q_{m,2m}$ will be called Φ-prime, if there is no pair of indices $(i_j, i_k) \in \omega$ which is taken from the same $\mu_\alpha \in \Phi$. The set of all Φ-prime sequences of $Q_{m,2m}$, will be denoted by $\Omega_{m,2}$ and shall be referred to as the (m,2)-prime set of $Q_{m,2m}$.

■

With the above definition, we may state

Proposition (6.6): Let $\mathbf{X}_m \in \mathbf{R}_p^{2m\times m}(s)$, $\mathbf{C}_m$ $(\mathbf{X}_m) = [\ldots,\lambda_\omega,\ldots]$, $\omega \in Q_{m,2m}$, $\Omega_{m,2}{}^c$ the (m,2)-prime set of $Q_{m,2m}$ and $\Omega_{m,2}{}^c$ the complement of $\Omega_{m,2}$ in $Q_{m,2m}$. Then,

(i) A co-ordinate $\lambda_\omega = 0$ for generic values of the non zero elements in $\mathbf{X}_m$ if and only if $\omega \in \Omega_{m,2}{}^c$.

(ii) The non zero coordinates λ_ω that correspond to generic values of the elements in $\mathbf{X}_m$ are those corresponding to $\omega \in \Omega_{m,2}$.

Proof

Note that λ_ω is an mxm minor of $\mathbf{X}_m$. Because of the block diagonal structure of $\mathbf{X}_m$, with these blocks 2x1 vectors, if $\omega = (i, i_2, \ldots, i_m) \in Q_{m,2m}$ has at least two indices taken from the same $\mu_\alpha \in \phi$, then there exists a zero column in the corresponding submatrix and thus $\lambda_\omega = 0$ independent of the values of elements in $\mathbf{X}_m$. By noting that every row in $\mathbf{X}_m$ has only one non zero element and that every mxm submatrix is always lower triangular the necessity follows.

Part (ii) follows along similar lines.

The set of coordinates in $\mathbf{C}_m\ (\mathbf{P}_m) = [\ldots, a_\omega, \ldots]$ and the coordinates in $\mathbf{C}_m\ (\mathbf{X}_m) = [\ldots, \lambda_\omega, \ldots]^t$ which correspond to $\omega \in \Omega_{m,2}$ will be called (m,2)-prime coordinates and will be denoted by $\rho_m \ \Delta \ \{a_\sigma\colon \ \sigma \in \Omega_{m,2}\}$, $\mathbf{X}_m \ \Delta \ \{\lambda_\sigma\colon \ \sigma \in \Omega_{m,2}\}$. To proceed with the analysis of the original problem it is essential to be able to compute the set $\Omega_{m,2}$ for any $m \geq 2$. This may be done by the following algorithm.

The computation of $\Omega_{m,2}$ set may be systematically carried out as follows:

Step (1): Set m = 2. Then, the set $\Omega_{2,2}$ is clearly

$$\Omega_{2,2} = \{(1,3),\ (1,4),\ (2,3),\ (2,4)\}$$

Step (2): For every sequence $\omega_2 = (i_1, i_2) \in \Omega_{2,2}$ generate the two sequences of $\Omega_{3,2}$ as $(i_1,i_2,3)$, $(i_1,i_2,4)$. This process generates all sequences in $\Omega_{3,2}$.

The general step is then described as:

<u>Step m</u>: For every $\omega_{m-1} = (i_1,i_2,...,i_{m-1}) \in \Omega_{m-1,2}$ generate two sequences of $\Omega_{m,2}$ as: $(i_1,i_2,...,i_{m-1},2m-1)$, $(i_1,i_2,...,i_{m-1},2m)$. This process generates all sequences in $\Omega_{m,2}$.

■

The above algorithm clearly generates sequences in $\Omega_{m,2}$. An important observation that follows from the above procedure is.

<u>Remark (6.4)</u>: The cardinality of $\Omega_{m,2}$ is 2^m for all $m \geq 1$.

■

We now return to our original problem, which was the solution of equation (6.14) for some set $\{x_{ij}, \in \mathbf{R}_\rho(s), i \in \mathbf{m}, j \in \mathbf{2}\}$. For convenience of notation we set

$$y_{2i} = x_{ij}, \text{ when } j = 2$$

$$y_{2i-1} = x_{ij}, \text{ when } j=1 \tag{6.18}$$

With this notation, for every $\sigma = (j_i,j_2,...,j_m) \in \Omega_{m,2}$ we have that

$$\lambda_\sigma = y_{j1} \quad y_{j2} \cdots y_{jm}, \ \sigma \in \Omega_{m,2} \tag{6.19}$$

Because of the fixed zeros in the $\Omega_{m,2m}{}^c$ locations in $\mathbf{C}_m(\mathbf{X}_m)$, eqn (6.13) may be reduced to the form

$$\sum_{\sigma\in\Omega_{m,2}} a_\sigma \lambda_\sigma = u \ , \qquad u \in \mathbf{R}_\rho(s) \text{ unit} \tag{6.20}$$

The above equation is a Diophantine equation over $\mathbf{R}_\rho(s)$ with parameters $\rho_m = \{a_\sigma \in \mathbf{R}_\rho(s), \ \sigma \in \Omega_{m,2}\}$ and unknowns $\mathbf{X}_m = \{\lambda_\sigma \in \mathbf{R}_\rho(s), \ \sigma \in \Omega_{m,2}\}$. The set of (m,2)-prime coordinates is crucial for the solvability of eqn. (6.16) and its properties are examined next.

<u>Proposition (6.7)</u>: The ρ_m (m,2)-prime coordinates set is an invariant (modulo $\mathbf{R}_\rho(s)$ units) of the transfer function P of the plant.

<u>Proof</u>

For the plant P the matrix

$$T_\ell = [A_1, \ B_1] \tag{6.21a}$$

is not uniquely defined, but if (A'_1, B'_1) is another left $\mathbf{R}_\rho(s)$-coprime MFD pair, then

$$T'_\ell = [A'_1, \ B'_1] = UT_\ell \tag{6.21b}$$

where U is an $\mathbf{R}_\rho(s)$-unimodular matrix. We first note that the matrices

$$\mathbf{P}_m = [P_1, P_2, ..., P_m] = [A_1, B_1] S_p$$

(6.22a)

$$\mathbf{P}_m' = [P'_1, P'_2, ..., P'_m] = [A'_1, B'_1] S_p$$

where S_p is a permutation matrix. From (6.18a) it follows that

$$\mathbf{P'}_m = U\mathbf{P}_m \quad (6.22b)$$

and thus $\mathbf{C}_m(\mathbf{P'}_m) = |U|\mathbf{C}_m(\mathbf{P}_m) = U\mathbf{C}_m(\mathbf{P}_m)$, where u is an $\mathbf{R}_\rho(s)$ unit. The above relationship implies that if $\{a_\omega, a_{\omega'}\}$ are the coordinates in $\mathbf{C}_m(\mathbf{P}_m)$, $\mathbf{C}_m(\mathbf{P'}_m)$ then

$$a'_\omega = u.a_\omega \text{ for all } \omega \in Q_{m,2m} \quad (6.22c)$$

and thus the corresponding elements of ρ_m, ρ'_m are associates.

■

The set ρ_m, thus characterises the plant and not the particular description and will be referred to as a <u>general set</u> of DSP. From the above analysis we have the following result.

<u>Theorem (6.1)</u>: Let P be a plant and ρ_m be a generator set of DSP. Then,

(i) A necessary and sufficient condition for solvability of eqn (6.20) is that the set ρ_m is $\mathbf{R}_\rho(s)$-coprime.

(ii) A necessary condition for solvability of DSP is that the set ρ_m is $\mathbf{R}_\rho(s)$-coprime.

Proof

The proof of part (i) is a well known result and is omitted. Clearly, coprimeness of ρ_m is a necessary condition for solvability of DSP; it is not sufficient, since for every set of solutions $\{\lambda_\sigma, \sigma \in \Omega_{m,2}\}$ conditions (6.19) must also be satisfied plus the condition that the denominators must be non zero.

■

The non coprimeness of ρ_m implies that there is no solution of DSP. If the plant is non cyclic, then clearly ρ_m is not coprime. We denote by $f_G(s)$ the greatest common divisor of ρ_m; $f_G(s)$ will be referred to as the prime invariant function of P. The previous analysis clearly also applies to non cyclic plants. The following result readily follows from the previous analysis.

Corollary (6.2): Let G be a plant, and let $G_p(s)$, $f_G(s)$ be the first, prime invariant functions respectively. Then,

(i) $G_p(s)$ divides $f_G(s)$

(ii) The zeros of $f_G(s)$ are fixed modes of any closed-loop system obtained by diagonal dynamic precompensation and unity output feedback.

■

The question that naturally arises is whether the $f_G(s)$ invariant function defines all closed-loop schemes. The relevant question may be phrased as:

Question (6.1): Are there fixed modes which are not zeros of $f_G(s)$?

The essence of the question posed above is that although coprimeness of ρ_m implies that there exists $\{\lambda_\sigma\}$ such that (6.20) has a solution for $u \in \mathbf{R}_\rho(s)$ unit, this does not necessarily imply that we restrict ourselves to those $\{\lambda_\sigma\}$ obtained via (6.19) (a subset of the solutions of (6.20) we have not additional fixed modes. This question will be examined later on.

A system for which $f_G(s)$ is an $\mathbf{R}_\rho(s)$ unit will be called strongly cyclic. Note that strong cyclicity implies cyclicity, but not the other way round. An interesting observation about the properness of the closed loop system that follows from the above analysis is

Remark (6.5): If $f_G(s)$ is not an $\mathbf{R}_\rho(s)$ unit, ie. $\delta_\infty(f_G) > 0$ (has no zeros at $s = \infty$), then all closed loop systems obtained by diagonal dynamic precompensation and unity feedback have fixed poles at $s = \infty$ with their total numbers defined by $\delta_\infty(f_G)$. In this case the closed loop system exhibits impulsive behaviour for all compensator schemes of the above type and it is not stable.

■

Strong cyclicity is a necessary condition for stabilization by a diagonal compensator. The case of strongly cyclic systems is presented in Chapter 7.

CHAPTER 7

DIAGONAL STABILIZATION OF TWO INPUT – TWO OUTPUT PLANTS

7.1 Introduction

The case of strongly cyclic systems is examined next, since for stabilization strong cyclicity is a necessary condition for stability. We shall examine the following problems:

i) Solvability of the diagonal stabilization problem;

ii) Parameterization of the family of solutions;

iii) Investigation of the presence of fixed modes

To illustrate the approach we examine the simple case $m = 2$. The results for the simple case provide the means to generalize either to diagonal control of a general square system or to decentralized control of a two channel system [Kar 1, Kar 2].

Conditions for non solvability and hence non stabilizability yield an explicit expression for the fixed modes [Kar 1] of the system.

7.2 Solvability Conditions

For the case of $m = 2$ conditions (6.15) and (6.16) are reduced to

$$a_{13}\ \lambda_{13} + a_{14}\ \lambda_{14} + a_{23}\ \lambda_{23} + a_{24}\ \lambda_{24} = 1 \qquad (7.1a)$$

where

$$a_{13} = d_1 d_2, \quad \lambda_{14} = d_1 n_2, \quad \lambda_{23} = n_1 d_2, \quad \lambda_{24} = n_1 n_2 \tag{7.1b}$$

by substituting (7.1b) into (7.1a) we have

$$a_{13} d_1 d_2 + a_{14} d_1 n_2 + a_{23} n_1 d_2 + a_{24} n_1 n_2 = 1 \tag{7.2a}$$

Equation (7.2a) may be written as

$$(a_{13} d_2 + a_{14} n_2) d_1 + (a_{23} d_2 + a_{24} n_2) n_1 = 1 \tag{7.2b}$$

or

$$(a_{13} d_1 + a_{23} n_1) d_2 + (a_{14} d_1 + a_{24} n_1) n_2 = 1 \tag{7.2c}$$

The above interpretations of conditions (7.2a) leads to the following result:

<u>Proposition (7.1)</u>: Necessary and sufficient conditions for solvability of equation (7.2a) over $\mathbf{R}_\rho(s)$ are:

i) The following equation has a solution over $\mathbf{R}_\rho(s)$

$$\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} d_n \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \tag{7.3a}$$

with some r_α, $r_\beta \in \mathbf{R}_\rho(s)$ and coprime

or equivalently

ii) The following equation has a solution over $\mathbf{R}_{\rho}(s)$.

$$\begin{bmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} \begin{bmatrix} d_1 \\ n_1 \end{bmatrix} = \begin{bmatrix} r_{\gamma} \\ r_{\delta} \end{bmatrix} \tag{7.3b}$$

with some r_{γ}, $r_{\delta} \in \mathbf{R}_{\rho}(s)$ and coprime.

<u>Proof</u>

i) Assume that a solution (n_1, d_1), (n_2, d_2) exists. From eqn (7.2a) we may set

$$a_{13}d_2 + a_{14}n_2 = r_{\alpha}, \quad a_{23}d_2 + a_{24}n_2 = r_{\beta} \tag{7.4a}$$

where clearly r_{α}, $r_{\beta} \in \mathbf{R}_{\rho}(s)$. By substituting (7.4a) into (7.2a) we have

$$r_{\alpha}d_1 + r_{\beta}n_1 = 1 \tag{7.4b}$$

Since n_1, d_1 exist that satisfy (7.4b), it follows that r_{α}, r_{β} must be coprime, otherwise there exists no d_1, n_1 for which (7.4b) is satisfied (classical result for solvability of linear, scaler Diophantine equations [Kai 1]). This proves the necessity. The sufficiency is proved by a mere reversion of the steps.

Following identical arguments part(ii) may also be established. Note that since part (i) and part (ii) provide alternative necessary and sufficient conditions for solvability of (7.2a) and their equivalence is obvious.

■

Conditions (7.3a), or (7.3b) provide simple criteria for solvability of the eqn (7.2), a restricted Diophantine equation, which are of a linear nature. They clearly show that the problem for the m = 2 case is essentially linear. Conditions (7.3a) will now be used to provide a deeper characterization of the solvability of DSP for the m = 2 case.

7.2.1 Necessary and sufficient conditions for solvability of DSP

Let us define the matrices:

$$T = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}, \quad T^t = \begin{bmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} \tag{7.5a}$$

Then, (7.3a), (7.3b) may be written as

$$T\,\underline{q}_2 = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = \underline{r}_2 \tag{7.5b}$$

$$T^t\underline{q}_1 = \begin{bmatrix} r_\gamma \\ r_\delta \end{bmatrix} = \underline{r}_1 \tag{7.5c}$$

By proposition (6.8) we have:

Remark (7.1): The matrix T defined by (7.5a) is an invariant modulo $u \in \mathbf{R}_\rho(s)$ of the system.

■

The strong cyclicity assumption implies cyclicity of the T matrix. Thus, if U_L, U_R are 2x2 $\mathbf{R}_\rho(s)$-unimodular matrices that reduce T to its Smith form, we have

$$T = U_L\ S_T\ U_R \tag{7.6a}$$

and then equation (7.5b) becomes

$$U_L\ S_T\ U_R\ \underline{q}_2 = \underline{r}_2$$

or

$$U_L\ S_T\ \tilde{\underline{q}}_2 = r_2,\ \tilde{\underline{q}}_2 = U_R\ \underline{q}_2,\ U_L = [\underline{u}_1,\ \underline{u}_2] \tag{7.6b}$$

Under the assumption of strong cyclicity, S_T may take either of the following two forms.

$$S_T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.6c}$$

$$S_T = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \tag{7.6d}$$

The matrix T characterises the solvability of the problem and will be referred to as the <u>fundamental matrix</u> of DSP. T will be called <u>degenerate</u>, <u>non degenerate</u> if S_T is of the form (7.6c), (7.6d) respectively and will be called complete if it is non degenerate and $\phi = 1$. Clearly, T is complete if it is $\mathbf{R}_\rho(s)$-unimodular. The solvability of (7.5b) is examined next.

Theorem (7.1): Necessary and sufficient condition for solvability of DSP for the case m = 2 is that the plant is strongly cyclic.

Proof

Necessity is proved by contradiction. Assume that $f_G(s)$ is a g.c.d. of $(a_{13}, a_{14}, a_{23}, a_{24})$ and that it is not a unit of $\mathbf{R}_\rho(s)$. By proposition (7.1) for solvability of DSP it is necessary that equation (7.5b) has a solution for some (r_α, r_β) coprime pair. Thus, (7.5b) yields

$$f_G \begin{bmatrix} \bar{a}_{13} & \bar{a}_{14} \\ \bar{a}_{23} & \bar{a}_{24} \end{bmatrix} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix}$$

From the above it is clear that f_G must be a divisor of r_α, r_β and thus (r_α, r_β) cannot be coprime for any choice of (d_2, n_2) i.e. contradiction.

To prove sufficiency, we assume strong cyclicity and distinguish the following cases.

Case (I): T is complete; Then T is $\mathbf{R}_\rho(s)$ unimodular and for all (r_α, r_β) arbitrary $\mathbf{R}_\rho(s)$ coprime pairs

$$\begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = T^{-1} \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \tag{7.7}$$

By proposition (8.1) a solution always exists.

<u>Case (II): T is degenerate</u>; Then T may be factorized as

$$T = \underline{u}_1 \, \underline{v}_1^{\,t} \tag{7.8a}$$

where $\underline{u}_1$ is a minimal basis vector for the column $\mathbf{R}_\rho(s)$-module and v^t a minimal basis vector for the row $\mathbf{R}_\rho(s)$-module of T. Then from (7.5b) we have

$$\underline{u}_1 \, \underline{v}_1^{\,t} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \tag{7.8b}$$

It follows that a solution with $(r_\propto, r_B)$ $\mathbf{R}_\rho(s)$ coprime always exists as long as

$$\begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = \underline{u}_1 \lambda, \quad \lambda \in \mathbf{R}_\rho(s) \text{ unit} \tag{7.8c}$$

By (7.8b) and (7..8c) we have

$$\underline{u}_1 \{ \underline{v}_1^{\,t} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} - \lambda \} = 0 \rightarrow \underline{v}_1^{\,t} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \lambda \tag{7.8d}$$

which has always a solution, since $v_1^{\,t}$ is an $\mathbf{R}_p(s)$ coprime vector.

Case (III): T is non degenerate but not complete;

let U_L be an $\mathbf{R}_\rho(s)$ unimodular that reduces T to its row Hermite form i.e.

$$U_L T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = U_L \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = \begin{bmatrix} \tilde{r}_\alpha \\ \tilde{r}_\beta \end{bmatrix} \tag{7.9a}$$

where

$$U_L T = \begin{bmatrix} \alpha & 1 \\ \phi & 0 \end{bmatrix} \tag{7.9b}$$

Then, by (7.9a) and (7.9b) we have

$$\begin{bmatrix} \alpha & 1 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} \tilde{r}_\alpha \\ \tilde{r}_\beta \end{bmatrix} \tag{7.9c}$$

and thus

$$\alpha d_2 + n_2 = \tilde{r}_\alpha, \quad \phi d_2 = \tilde{r}_\beta \tag{7.9d}$$

from which, if we chose $d_2 = t_1 \in \mathbf{R}_\rho(s)$ arbitrary and $\tilde{r}_\beta = \phi t_1$, $\tilde{r}_\alpha = t_2 \in \mathbf{R}_\rho(s)$ arbitrary,

$$n_2 = t_2 - \alpha t_1 \tag{7.9e}$$

then,

$$\begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = U_L^{-1} \begin{bmatrix} t_2 \\ \phi t_1 \end{bmatrix}, \text{ with } (t_2, \phi t_1) \text{ coprime}$$

This proves the result.

■

Remark (7.2): The zeros of the prime invariant function $f_G(s)$ define the set of fixed modes of all systems obtained by diagonal dynamic stabilization and unity negative feedback.

■

The above result demonstrates that strong cyclicity is equivalent to Ω-stabilizability by diagonal dynamic compensation. The problem examined next is the nature of the solutions. The specific questions we are addressing are:

i) Parameterization of the family of solutions
ii) Existence of proper solutions (realizability)
iii) Reliable solutions
iv) Minimal design, existence of constant solutions
v) Strong stabilizability, existence of stable solutions

7.3 Parameterization of the family of solutions

The problem under investigation may be formulated as: Find $(r_\alpha, r_\beta) \in \mathbf{R}_\rho(s)$, such that the following equations are solvable over $\mathbf{R}_\rho(s)$.

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix}, \quad [d_1, n_1] \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = 1 \tag{7.10}$$

The above equations may be equivalently expressed as:

$$[d_1, n_1] \, T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \tag{7.11}$$

where the (r_α, r_β) parameters have been eliminated. We may distinguish the following cases:

<u>Case (I): T is degenerate</u>;

By (7.8a), eqn. (7.11) may be written as

$$[d_1, n_1] \, [\underline{u}_1, \underline{v}_1^t] \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \tag{7.12}$$

where $\underline{u}_1$, $\underline{v}_1^t$ are $\mathbf{R}_\rho(s)$-coprime vectors, uniquely defined modulo units of $\mathbf{R}_\rho(s)$.

<u>Theorem (7.2)</u>: For strong cyclic systems with T degenerate the family of diagonal stabilizing compensators is given

by the families of solutions of the following scalar Diophantine equations:

$$[d_1, n_1] \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 1 \quad [\mathbf{u}_{11}\ \mathbf{u}_{12}] \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \tag{7.13}$$

<u>Proof</u>

Let (d_1, n_1), (d_2, n_2) be a solution of DSP. Then,

$$[d_1, n_1] \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = r \quad [\mathbf{u}_{11}\ \mathbf{u}_{12}] \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = r'$$

$r, r' \in \mathbf{R}_\rho(s)$

By (7.12) we have that

$$r\underline{v}_1{}^t] \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1$$

and thus r must be a divisor of 1. i.e. an $\mathbf{R}_\rho(s)$ unit. Similarly r' must be a divisor of 1 and thus r' must also be an $\mathbf{R}_\rho(s)$ unit. This proves the necessity, the proof of sufficiency is obvious.

■

The above result indicates that, if T is degenerate, diagonal stabilization is reduced to stabilization of two independent SISO plants defined by the vectors $\underline{u}_1$, $\underline{v}_1^t$; that is, if we write

$$\underline{u}_1 = [u_{11},\ u_{12}],\quad \underline{v}_1^t = [v_{11},\ v_{12}] \tag{7.14}$$

then the SISO controllers (d_1, n_1), (d_2, n_2) that stabilize the overall plant are solutions of the SISO stabilization problems defined by

$$\begin{aligned} u_{11}\, d_1 + u_{12}\, n_1 &= 1 \\ v_{11}\, d_2 + v_{12}\, n_2 &= 1 \end{aligned} \tag{7.15}$$

Note that $\underline{u}_1$, $\underline{v}_1^t$ are basis vectors for the maximal $\mathbf{R}_\rho(s)$ column, row modules of T and thus uniquely defined modulo units of $\mathbf{R}_\rho(s)$.

The generic case of non degenerate T is considered next.

<u>Case II: T is nondegenerate</u>

From eqn (7.11)

$$[d_1,\ n_1]\ T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \tag{7.16}$$

an interesting characterization of diagonal stabilizability is provided by the following result.

<u>Proposition (7.2)</u>: Consider a strongly cyclic plant with $m = 2$ and let

$$T = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \tag{7.17}$$

The pair of $\mathbf{R}_\rho(s)$-coprime descriptions (n_1,d_1), (n_2,d_2) that define the diagonal controller, stabilize the plant iff

i) (d_1,n_1) stabilizes the plant defined by the vector

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} a_{13}d_2 + a_{14}n_2 \\ a_{23}d_2 + a_{24}n_2 \end{bmatrix} \tag{7.18a}$$

or equivalently

ii) (d_2,n_2) stabilizes the plant defined by the vector

$$[d_1,n_1]T = [a_{13}d_1 + a_{23}n_1,\ a_{14}d_1 + a_{24}n_1] \tag{7.18b}$$

■

Proposition (7.2) is a restatement of condition (7.16). It describes the important property that stabilization of the plant by a pair (d_1,n_1), (d_2,n_2) is equivalent to stabilization of the SISO plant.

$$(a_{13}d_1 + a_{23}n_1,\ a_{14}d_1 + a_{24}n_1) = (\tilde{d}_1,\tilde{n}_1) \tag{7.19a}$$

by (d_2,n_2), or equivalently to stabilization of the SISO plant

$$(a_{13}d_2 + a_{14}n_2,\ a_{23}d_2 + a_{24}n_2) = (\tilde{d}_2, \tilde{n}_2) \tag{7.19b}$$

by (d_1,n_1). A pair of SISO compensators (d_1,n_1), (d_2,n_2) that satisfies (7.16) will be referred to as a <u>mode T mutually stabilizing pair</u>. Thus, diagonal stabilization is equivalent to the existance of mode T mutually stabilizing pairs. The characterisation of all such pairs is intimately related to the parameterization of stabilizing compensators and is considered next.

7.3.1 <u>Characterisation of mode T mutually stabilizing pairs</u>

Let us assume that the Smith form of T over $\mathbf{R}_\rho(s)$ is

$$S_T = \begin{bmatrix} 1 & 0 \\ 0 & \phi(s) \end{bmatrix} \tag{7.20}$$

and let $\Lambda_T = \{\lambda_i,\ i \in \underline{\mu}\}$ be the distinct values of the zeros of $\phi(s)$ (roots of $\mathbf{R}_\rho(s)$ elementary divisors of $\phi(s)$). Λ_T will be referred to as the <u>$\mathbf{R}_\rho(s)$-root range of T</u>.

<u>Definition (7.1)</u>: Let $T \in \mathbf{R}_\rho^{2\times 2}(s)$ be a nondegenerate cyclic matrix and let (d,n) be an $\mathbf{R}_\rho(s)$-coprime pair. Then, (d,n) will be called <u>mode T (T^t), $\mathbf{R}_\rho(s)$-coprime</u>, if the pair $(\tilde{d},\tilde{n})$, $(\hat{d},\hat{n})$ defined by the vectors respectively

$$[\tilde{d},\tilde{n}] = [d,n]\,T\ , \qquad \begin{bmatrix} \hat{d} \\ \hat{n} \end{bmatrix} = T \begin{bmatrix} d \\ n \end{bmatrix} \tag{7.21}$$

are $\mathbf{R}_{\rho}(s)$-coprime

The set of mode $T(T^t)$ coprime vectors is characterised by the following result.

<u>Proposition (7.3)</u>: Let $T \in \mathbf{R}_{\rho}^{2x2}(s)$ be a nondegenerate cyclic matrix and let Λ_T be its root range. An $\mathbf{R}_{\rho}(s)$-coprime pair (d,n) is:

i) mode T coprime, iff for $\forall\ \lambda \in \Lambda_T$

$$[d(\lambda),\ n(\lambda)]\ T(\lambda) \neq \underline{O}^t \tag{7.22a}$$

ii) mode T^t coprime, iff for $\forall\ \lambda \in \Lambda_T$

$$T(\lambda) \begin{bmatrix} d(\lambda) \\ \\ d(\lambda) \end{bmatrix} \neq \underline{O} \tag{7.22b}$$

<u>Proof</u>:

i) Assume that $(\bar{d},\bar{n})$ is not $\mathbf{R}_{\rho}(s)$coprime. Then there exists $\mu \in \mathbf{C}$, zero of the greates common divisor such that $[\bar{d}(\mu),\ \bar{n}(\mu)] = \underline{O}^{+}$. Then from the definition

$$[d(\mu),\ n(\mu)]\ T\ (\mu) = \underline{O} \tag{7.23}$$

and since (d,n) is $\mathbf{R}_{\rho}(s)$ coprime, $[d(\mu),\ n(\mu)] \neq \underline{O}^t$ and thus for (7.23) to be true we must have $|T(\mu)| = 0$, ie. $\mu \in \Lambda_T$. Note that for $\forall\ \mu \notin \Delta_T$, $|T(\mu)| \neq 0$ and since $[d(\mu),\ n(\mu)] \neq 0$, it follows that $[\bar{d}(\mu),\ \bar{n}(\mu)]$

$\neq \underline{0}$, ie. $(\tilde{d},\tilde{n})$ has no zeros outside the set Λ_T. This proves part (i). Part (ii) follows along similar lines.

■

<u>Remark (7.3)</u>: If T is complete, ie $\mathbf{R}_\rho(s)$ unimodular, then every coprime pair (d,n) is mode T and T^t coprime.

■

7.3.2 <u>Parameterization of stabilizing diagonal controllers</u>

With the preliminary results developed above we may state the following main result regarding the parameterization of stabilizing diagonal controllers.

<u>Theorem (7.3)</u>: Let G be a 2x2 strongly cyclic system with T nondegenerate. Then,

(a) Let (c_1, c_2) be a pair of SISO controllers, described by the $\mathbf{R}_\rho(s)$-coprime pairs (d_1,n_1), (d_2,n_2). The following statements are equivalent.

i) (c_1, c_2) stabilizes the plant

ii) (d_1,n_1), (d_2,n_2) is a mode T mutually stabilizing pair

iii) (d_1,n_1) is mode T coprime and (d_2,n_2) stabilizes $(\tilde{d}_1,\tilde{n}_1)$. Equivalently, (d_2,n_2) is mode T coprime and (d_1,n_1) stabilizes $(\hat{d}_2,\hat{n}_2)$.

(b) The family of all (c_1, c_2) stabilizing controllers is defined by:

i) For any (d_1, n_1) mode T coprime pair a subfamily of $\{(d_2, n_2)\}$ controllers that together with (d_1, n_1) fixed, stabilizes the plant is given by the solution of

$$\tilde{d}_1 d_2 + \tilde{n}_1 n_2 = 1 \tag{7.24a}$$

where

$$[\tilde{d}_1, \tilde{n}_1] = [d_1, n_1]\ T$$

ii) For any (d_2, n_2) mode T^t coprime pair a subfamily of $\{(d_1, n_1)\}$ controllers that together with (d_2, n_2) fixed, stabilizes the plant, is given by the solution of

$$\hat{d}_2 d_1 + \hat{n}_2 n_1 = 1 \tag{7.24b}$$

where

$$[\hat{d}_2, \hat{n}_2] = [d_2, n_2]\ T^t$$

<u>Proof</u>:

(a) The equivalence of (i) and (ii) follows from Proposition (7.2). If (d_1, n_1), (d_2, n_2) is a mode T mutually stabilizing pair, then eqn (7.11) has a solution ie.

$$[d_1, n_1]\ T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1$$

Clearly, if (d_1,n_1) is not mode T coprime, then $[d_1,n_1]$ $T = [\tilde{d}_1,\tilde{n}_1]$ is not comprime and the equation $\tilde{d}_1 d_2 + \tilde{n}_1 n_2 = 1$ has no solution. Thus mutual stabilization implies mode T coprimeness of (d_1,n_1). By reversing the arguments, mode T coprimeness of (d_1,n_1), implies that there exists (d_2,n_2) that stabilizes $(\hat{d}_1,\hat{n}_1)$. Furthermore, if (d_1,n_1) is not mode T coprime, then there exists no (d_2,n_2) solution of $\tilde{d}_1 d_2 + \tilde{n}_1 n_2 = 1$ and this completes the proof. The statement for (d_2,n_2) being mode T coprime follows along similar lines.

(b)

i) For any fixed (d_1, n_1) mode T coprime, the solution of (7.24a) clearly defines $\{(d_1,n_1), (d_2,n_2)\}$ stabilizing pairs. What has to be proved is that all stabilizing pairs are generated by this process. Thus assume that there exists a pair (d_2,n_2) which together with some (d_1,n_1) stabilizes the plant. Then clearly

$$[d_1, n_1]\ T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1$$

and (d_2,n_2) has to be T^t coprime, otherwise the above equation cannot be solved. If we assume that (d_1,n_1) is not mode T coprime, then clearly the above equation also is not soluble. Thus, a stabilizing pair $\{(d_1,n_1), (d_2,n_2)\}$ must always have the property that (n_1,d_1) must be mode T coprime and (n_2,d_2) must be mode T^t coprime. Therefore the above process generates the complete family. The proof of part (ii) is identical.

■

Corollary (7.1): Let G be a 2x2 strongly cyclic plant with a nondegenerate T and let Λ_T be the root range of T.

(a) If T is complete, ie. $\Lambda_T = \emptyset$, then

i) For any (d_1,n_1) $\mathbf{R}_\rho(s)$-coprime pair defining a controller for channel (1), the family of $\{(d_2,n_2)\}$ controllers of channel (2) which together with (d_1, n_1) stabilize the plant is given by

$$\begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = T^{-1} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + t \begin{bmatrix} -n_1 \\ d_1 \end{bmatrix} \quad , \ t \in \mathbf{R}_\rho(s) \text{ arbitrary} \tag{7.25a}$$

where (a_1,b_1) is a SISO plant that stabilizes (d_1,n_1)

ii) For any (d_2,n_2) $\mathbf{R}_\rho(s)$ coprime pair defining a controller for channel (2), the family of $\{(d_1,n_1)\}$ controllers for channel (1) which together with (d_2, n_2) stabilize the plant is given by

$$[d_1,n_1] = T^{-1} \{[a_2,b_2] + t'[-n_2,d_2]\},\ t \in \mathbf{R}_\rho(s) \text{ arbitrary} \tag{7.25b}$$

where (a_2,b_2) is a SISO plant that stabilizes (d_2,n_2)

(b) If T is not complete, ie. $\Lambda_T \neq \varnothing$, then,

i) For any controller (d_1,n_1) for channel (1), such that

$$[d_1(\lambda),\ n_1(\lambda)]\ T(\lambda) \neq \underline{0},\ \forall\ \lambda \in \Lambda_T \tag{7.26a}$$

there exists a controller (d_2,n_2) for channel (2), which together with (d_1,n_1) stabilizes the plant.

ii) For any controller (d_2,n_2) for channel (2), such that

$$T(\lambda) \begin{bmatrix} d_2(\lambda) \\ n_2(\lambda) \end{bmatrix} \neq \underline{0}\ , \quad \forall\ \lambda \in \Lambda_T \tag{7.26b}$$

there exists a controller (d_1,n_1) for channel (1), which together with (d_2,n_2) stabilizes the plant.

$\blacksquare$

The above results provide the answer to the parameterization problem and indeed this provides the tools for the investigation of special type of solutions, such as proper, stable, reliable and minimal design.

7.4 Proper solutions of diagonal stabilization problem

7.4.1 Properness of solutions of scalar Diophantine equations

Let (b,a) be an $\mathbf{R}_\rho(s)$-coprime pair. The pair (b,a) will be called proper, nonproper, strictly proper, if the transfer function $P = ba^{-1}$ is respectively proper, nonproper, strictly proper. For the general given pair (coprime) we define the scalar Diophantine equation.

$$bn + ad = 1 \tag{7.27}$$

where the solution (n,d) over $\mathbf{R}_\rho(s)$ always exists because of the $\mathbf{R}_\rho(s)$-coprimeness of (b,a). The solution pairs (n,d) are always $\mathbf{R}_\rho(s)$-coprime and if (n_0, d_0) is a particular solution then the general solution is expressed by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ -b \end{bmatrix}, \quad t \in \mathbf{R}_\rho(s) \text{ arbitrary} \tag{7.28}$$

In the study of diagonal stabilization, Diophantine equations of the type (7.27) always emerge, where (b,a) is not necessarily proper; however, since (n,d) represent compensators the question of properness (realizability) is always an important aspect to be examined. A pair {(b,a), (n,d)} that satisfies (7.27) will be referred to as mutually stabilizing pair; in particular (n,d) (or (b,a)) will be called a dual of (b,a) (or (n,d)). The existance of proper dual pairs for a given (b,a) is examined next.

The following result establishes a useful general property of a mutually stabilizing pair.

Lemma (7.1): Let (b,a), (n,d) be $\mathbf{R}_\rho(s)$-coprime and mutually stabilizing pairs. Then,

$$\min\{\delta_\infty(b) + \delta_\infty(n),\ \delta_\infty(a) + \delta_\infty(d)\} = 0 \qquad (7.29)$$

Proof:

Since bn + ad = 1, by taking valuations we have:

$$\delta_\infty(bn + ad) = 0$$

By the properties of $\delta_\infty(\cdot)$ valuation it follows that

$$0 = \delta_\infty(bn + ad) \geq \min\{\delta_\infty(bn),\ \delta_\infty(ad)\} =$$
$$= \min\{\delta_\infty(b) + \delta_\infty(n),\ \delta_\infty(a) + \delta_\infty(d)\} \geq 0$$

since (b,a), (n,d) are from $\mathbf{R}_\rho(s)$ and thus have non negative valuations. The last condition clearly implies (7.29).

■

Using the above lemma we have:

<u>Proposition (7.4)</u>: If (b,a) is a strictly proper pair, then all dual (n,d) are proper.

<u>Proof</u>:

Since (b,a) is coprime and strictly proper, it follows that $\delta_\infty(b) = \epsilon > 0$ and $\delta_\infty(a) = 0$. By condition (7.29) (necessary condition which all duals (n,d) must satisfy) we have

$$\min\{\epsilon + \delta_\infty(n),\ \delta_\infty(d)\} = 0$$

and this clearly implies (since $\epsilon > 0$ and $\delta_\infty(n) \geq 0$) that $\delta_\infty(d) = 0$, ie. all duals have d biproper, ie. they are proper.

■

The above result provides an alternative proof of a well known result that the family of stabilizing compensators of strictly proper plants has all its elements proper. For the case of nonproper plants we have:

<u>Proposition (7.5)</u>: Let (b,a) be an $\mathbf{R}_p(s)$-coprime nonproper pair. Then,

i) For all (n,d) dual pairs, $\delta_\infty(n) = 0$

ii) If a proper dual exists, it has to be biproper

iii) There always exists a family of biproper (n,d) duals.

<u>Proof</u>

i) Since (b,a) is coprime and nonproper, it follows that $\delta_\infty(b) = 0$ and $\delta_\infty(a) = \epsilon' > 0$. Thus, by condition (7.29) we have

$$\min\{\delta_\infty(n),\ \epsilon' + \delta_\infty(d)\} = 0$$

Clearly, since $\epsilon' > 0$ and $\delta_\infty(d) \geq 0$, follows that $\delta_\infty(n) = 0$.

ii) Since for all dual $\delta_\infty(n) = 0$, then if a proper dual exists we must have $0 \leq \delta_\infty(c) = \delta_\infty(nd^{-1}) = \delta_\infty(n) - \delta_\infty(d) = 0 - \delta_\infty(d) = -\delta_\infty(d)$ and thus $\delta_\infty(d) = 0$. Thus, if a proper dual exists it must be biproper.

iii) Consider the family of duals as defined by (7.28). At $s = \infty$ we have

$$\begin{bmatrix} n^\infty \\ d^\infty \end{bmatrix} = \begin{bmatrix} n_0^{\ \infty} \\ d_0^{\ \infty} \end{bmatrix} + t \begin{bmatrix} a^\infty \\ -b^\infty \end{bmatrix}$$

Since (b,a) is nonproper if follows that $b^{\infty} = \beta \neq 0$ and $a^{\infty} = 0$. Furthermore, by part (ii) $n_0^{\infty} = \alpha \neq 0$ and thus the above may be written as:

$$\begin{bmatrix} n^{\infty} \\ d^{\infty} \end{bmatrix} = \begin{bmatrix} \alpha \\ d_0^{\infty} - t^{\infty}\beta \end{bmatrix} \tag{7.30}$$

We may distinguish the following cases:

(a) Particular solution is nonproper

(b) Particular solution is biproper

(a) If particular solution is nonproper, then $\delta_{\infty}(d_o) > 0$ and thus $d_0^{\infty} = 0$. By (7.29) we have

$$\begin{bmatrix} n^{\infty} \\ d^{\infty} \end{bmatrix} = \begin{bmatrix} \alpha \\ -t^{\infty}\beta \end{bmatrix} \tag{7.30a}$$

and thus for any biproper $t \in \mathbf{R}_{\rho}(s)$, ie. $\delta_{\infty}(t) = 0$, $d^{\infty} \neq 0$ and thus the corresponding d has $\delta_{\infty}(s) = 0$ ie. there exist biproper duals for all biproper parameters $t \in \mathbf{R}_{\rho}(s)$.

(b) If particular solution is biproper, then $\delta_{\infty}(d_0) = 0$ and $d_0^{\infty} = \gamma \neq 0$. By (7.30) we have:

$$\begin{bmatrix} n^{\infty} \\ d^{\infty} \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma - t^{\infty}\beta \end{bmatrix} \tag{7.30b}$$

Clearly for all $t \in \mathbf{R}_\rho(s)$ parameters such that

$$[\gamma \quad -\beta] \begin{bmatrix} 1 \\ t^\infty \end{bmatrix} \neq 0 \tag{7.30c}$$

$d^\infty \neq 0$ and thus $\delta_\infty(d) = 0$, ie. solution is biproper.

■

An important remark that follows immediately from the above proof is stated next.

Remark (7.4): If (b,a) is an $\mathbf{R}_\rho(s)$-coprime, nonproper pair, then there exists no strictly proper dual.

■

The biproper duals of a nonproper (b, a) pair may be parameterised by the following result.

Corollary (7.2): Let (b,a) be an $\mathbf{R}_\rho(s)$-coprime, nonproper pair.

(a) There always exists a biproper dual (n_0, d_0)

(b) Let $n_0^\infty = \alpha \neq 0$, $d_0^\infty = \beta \neq 0$ and $b^\infty = \gamma \neq 0$. The family of biproper duals is defined by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ -b \end{bmatrix}, \quad t \in \mathbf{R}_\rho(s) \tag{7.31a}$$

where t^{∞} is constrained by the condition

$$\beta - t^{\infty}\nu \neq 0 \qquad (7.31b)$$

■

The proof clearly follows from the proof of Proposition (7.5) and is omitted. Note that those $t \in \mathbf{R}_{\rho}(s)$ for which nonproper solutions are obtained are characterised by the $t^{\infty} = \gamma/B$ and thus belong to a hyperplane; that is generically, the duals of a nonproper (b, a) are biproper.

<u>Remark (7.5)</u>: The duals of a coprime, nonproper (b,a) are generically biproper.

■

The case of biproper pairs (b,a) is considered next.

<u>Proposition (7.6)</u>: Let (b,a) be an $\mathbf{R}_{\rho}(s)$-coprime biproper pair.

(a) There always exists a family of biproper duals and a family of strictly proper duals.

(b) Let (n_0, d_0) be a biproper dual

i) The family of biproper duals is defined by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ -b \end{bmatrix}, \quad t \in \mathbf{R}_{\rho}(s) \qquad (7.32a)$$

where t^{∞} is constrained by the condition

$$d_0^{\infty} - t^{\infty} b^{\infty} \neq 0 , \quad n_0^{\infty} + t^{\infty} a^{\infty} \neq 0 \qquad (7.32b)$$

ii) A family of strictly proper duals is defined by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \end{bmatrix} , \quad t \in \mathbf{R}_{\rho}(s) \qquad (7.33a)$$

where t^{∞} is constrained by the conditions

$$d_0^{\infty} - t^{\infty}b^{\infty} \neq 0, \; n_0^{\infty} - t^{\infty}a^{\infty} = 0 \qquad (7.33b)$$

<u>Proof</u>:

(a) The general family of duals is given by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ -b \end{bmatrix} , \quad t \in \mathbf{R}_{\rho}(s)$$

where from the comprimeness of every dual it follows that $(n_0, \; d_0)$ may have one of the following properties:

i) $\delta_{\infty} (n_0) = 0, \quad \delta_{\infty}(d_0) > 0$: nonproper dual

ii) $\delta_{\infty} (n_0) > 0, \quad \delta_{\infty}(d_0) = 0$: strictly proper dual

iii) $\delta_\infty(n_0) = 0, \quad \delta_\infty(d_0) = 0$: biproper dual

i) If (n_0,d_0) is nonproper, then $n^\infty \neq 0$ and $d^\infty = 0$ and thus

$$\begin{bmatrix} n^\infty \\ \\ d^\infty \end{bmatrix} = \begin{bmatrix} n_0^{\ \infty} \\ \\ d_0 \end{bmatrix} + t^\infty \begin{bmatrix} a^\infty \\ \\ -b^\infty \end{bmatrix}$$

by selecting t such that $t^\infty \neq -n_0^{\ \infty}/a^\infty$, 0 then a biproper solution is defined.

ii) If (n_0,d_0) is strictly proper, then $d_0^{\ \infty} \neq 0$ and $n_0^{\ \infty} = 0$ and thus

$$\begin{bmatrix} n^\infty \\ \\ d^\infty \end{bmatrix} = \begin{bmatrix} 0 \\ \\ d_0^{\ \infty} \end{bmatrix} + t^\infty \begin{bmatrix} a^\infty \\ \\ -b^\infty \end{bmatrix}$$

by selecting t such that $t^\infty \neq -d_0^{\ \infty}/b^\infty$, 0, then a biproper solution is defined.

iii) If (n_0,d_0) is biproper, then at least one biproper solution exists

(b) The analysis of the above cases demonstrates that there always exists a biproper dual say $(n_0,\ d_0)$. Using this, the whole family of duals is given by

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \begin{bmatrix} a \\ -b \end{bmatrix}, \quad t \in \mathbf{R}_\rho(s)$$

At $s = \infty$, the above yields

$$\begin{bmatrix} n^\infty \\ d^\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty \\ d_0^\infty \end{bmatrix} + t^\infty \begin{bmatrix} a^\infty \\ -b^\infty \end{bmatrix}$$

where n_0^∞, d_0^∞, a^∞, $b^\infty \neq 0$. By restricting the parameters t such that $d_0^\infty - t^\infty b^\infty \neq 0$ and $n_0^\infty + t^\infty a^\infty \neq 0$, n^∞, d^∞ become $\neq 0$ and (n,d) is biproper. This proves part (i). Part (ii) follows along similar lines.

■

Starting from a biproper dual (n_o, d_o) the conditions for the existance of nonproper duals are:

$$d_0^\infty - t^\infty b^\infty = 0, \qquad n_0^\infty + t^\infty a^\infty \neq 0 \tag{7.34}$$

and for strictly proper duals are:

$$d_0^\infty - t^\infty b^\infty \neq 0, \qquad n_0^\infty + t^\infty a^\infty = 0 \tag{7.35}$$

given that t^∞ is constrained by equations we have:

Remark (7.6): The duals of a coprime, biproper (b, a) are generically biproper. The existence of nonproper and strictly proper duals is nongeneric.

■

The above results are used next for the study of proper diagonal stabilizing compensators.

7.4.2 Properness of solutions to DSP, m = 2 case

To generate realizable controllers we must be able to determine the properness of solutions to -

$$[d_1, n_1] \, T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \tag{7.36}$$

$(d_1 \; n_1)$, $(d_2 \; n_2)$ represent the SISO controllers hence the question of properness (realizability) is always important.

In the generic case, T nondegenerate, and $(d_1 \; n_1)$, $(d_2 \; n_2)$ the $\mathbf{R}_p(s)$-coprime descriptions of the stabilizing controllers then,

(1) $(d_1 \; n_1)$ stabilizes the plant defined by the vector

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} \hat{d}_2 \\ \hat{n}_2 \end{bmatrix} \longrightarrow [d_1 \; n_1] \begin{bmatrix} \hat{d}_2 \\ \hat{n}_2 \end{bmatrix} = 1 \tag{7.37}$$

(2) $(d_2\ n_2)$ stabilizes the plant defined by the vector

$$[d_1\ n_1]\ T = [\tilde{d}_1\ \tilde{n}_1] \longrightarrow [\tilde{d}_1\ \tilde{n}_1] \begin{bmatrix} d_2 \\ \\ n_2 \end{bmatrix} = 1 \tag{7.38}$$

Stabilization of a plant by a pair of SISO controllers $(d_1\ n_1)$, $(d_2\ n_2)$ is equivalent to stabilization of a SISO plant: $[\hat{d}_2\ \hat{n}_2]^t$ by $(d_1\ n_1)$; $[\tilde{d}_1\ \tilde{n}_1]$ by $(d_2\ n_2)$ ie. mode T $\mathbf{R}_\rho(s)$-mutually stabilizing pairs.

The selection of $(d_1\ n_1)$ such that $[\tilde{d}_1\ \tilde{n}_1] = [d_1\ n_1]\ T$ are $\mathbf{R}_\rho(s)$-coprime, channel (1) fixed, generates a subfamily of controllers $(d_2\ n_2)$ that together with the fixed channel (1) stabilizes the closed loop system. The subfamily of controllers for channel (2) with fixed channel (1) is generated via the solution of the scalar diophantine equation -

$$\tilde{d}_1\ d_2 + \tilde{n}_1\ n_2 = 1 \tag{7.39}$$

with $(d_1\ n_1)$, $(d_2\ n_2)$ mode T mutually stabilizing pairs.

The selection of a realizable controller for the fixed channel is ensured if $k_f = n_f\ d_f^{-1}$, $\delta_\infty(d_f) \leq \delta_\infty(n_f)$ ie. k_f is proper (strictly proper).

Consider channel (1) fixed: ie. select a realizable controller $k_1 = n_1 d_1^{-1}$ such that channel (1) $c_1 = \tilde{n}_1\ \tilde{d}_1^{-1}$ is

$\mathbf{R}_\rho(s)$-coprime. Then the pair $(\tilde{d}_1\ \tilde{n}_1)$ will be called proper, strictly proper or nonproper if its respective transfer function $c_1 = \tilde{n}_1\tilde{d}_1$ is so defined.

There are three cases which may be distinguished

(1) <u>$(\tilde{d}_1,\ \tilde{n}_1)$ Strictly Proper</u>

If the $\mathbf{R}_\rho(s)$ coprime plant generated by selecting a realizable fixed controller $(d_1\ n_1)$ is strictly proper, then from Proposition (7.4) all solutions of $\tilde{d}_1 d_2 + \tilde{n}_1 n_2 = 1$ are proper ie.

<u>Channel (1) fixed</u>:

$$\begin{array}{ccccc} (d_1\quad n_1) & \longrightarrow & (\tilde{d}_1\quad \tilde{n}_1) & \longrightarrow & (d_2\quad n_2) \end{array}$$

realizable Strictly proper proper i.e. realizable

(2) <u>$(\tilde{d}_1\ \tilde{n}_1)$ nonproper</u>

If the $\mathbf{R}_\rho(s)$-coprime plant $(\tilde{d}_1\ \tilde{n}_1)$ is non proper.i.e $(d_1\ n_1)$ selected to be realizable generates $(\tilde{d}_1\ \tilde{n}_1)$ non proper then by Proposition (7.5) there exists no strictly proper solution to $\tilde{d}_1\ d_2 + \tilde{n}_1\ n_2 = 1$. If a solution exists then generically it will be biproper. The family of biproper solutions (generic case) is defined by (7.31a)

$$\begin{bmatrix} n_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} n_{2_0} \\ d_{2_0} \end{bmatrix} + t \begin{bmatrix} \tilde{d}_1 \\ -\tilde{n}_1 \end{bmatrix}, \quad t \in \mathbf{R}_\rho(s) \text{ arbitrary}$$

but where t^∞ constrained by the condition $\beta - t^\infty\gamma \neq 0$, $d_2^\infty 0 = \beta \neq 0$, $\tilde{n}_1^\infty = \gamma \neq 0$.

Channel (1) fixed

$$(d_1 \quad n_1) \longrightarrow (\tilde{d}_1 \quad \tilde{n}_1) \longrightarrow (d_2 \quad n_2)$$

realizable non proper biproper i.e. realizable (generic case)

(3) $(\tilde{d}_1\ \tilde{n}_1)$ biproper

If the $\mathbf{R}_\rho(s)$ coprime plant generated by selecting a realizable controller $(d_1\ n_1)$ generates the biproper plant $(\tilde{d}_1\ \tilde{n}_1)$ then by Proposition (7.6) the resultant controller $(d_2\ n_2)$ will generically be biproper. The family of all such controller is given by (7.32a)

$$\begin{bmatrix} n_2 \\ \\ d_2 \end{bmatrix} = \begin{bmatrix} n_{2_0} \\ \\ d_{2_0} \end{bmatrix} + t \begin{bmatrix} \tilde{d}_1 \\ \\ -\tilde{n}_1 \end{bmatrix}, \quad t \in \mathbf{R}_\rho(s)$$

where t is constrained by the condition

$$d^\infty - t^\infty \neq 0, \qquad n_{20}^\infty + t^\infty \tilde{d}_1^\infty \neq 0$$

The existence of non proper and strictly proper solutions is non generic.

Channel fixed

$$(d_1 \quad n_1) \longrightarrow (\tilde{d}_1 \quad \tilde{n}_1) \longrightarrow (d_2 \quad n_2)$$

realizable non proper biproper i.e. realizable (generic case)

7.5 Reliable Solutions of the Diagonal Stabilization Problem

Reliable stabilization is the ability of the system to maintain closed loop stability with the loss of one or more of its channels.

For the case of a two channel system either or both channels may fail. Failure of a channel is equivalent to the loss of a SISO controller $c_i = n_i\, d_i^{-1} \rightarrow n_i = 0,\ d_i \neq 0$.

There are three cases to distinguish:

(1) failure of channel (1)

(2) failure of channel (2)

(3) failure of channels (1) & (2).

In each case the system is said to be reliably stabilizable if (a) the system is closed loop stable with a pair of controllers $(d_1 \quad n_1)$, $(d_2 \quad n_2)$ (b) the system remains stable with failures (1), (2) or (3) above.

We have seen from the parameterization of the family of solutions to the DSP that condition (a) is satisfied by selecting controllers to be mode-T-$\mathbf{R}_\rho(s)$-coprime mutually stabilizing. The question that remains to be answered is under what constraints do such selected controllers satisfy condition (b) for reliable stabilization.

Let $(d_1 \ \ n_1)$, $(d_2 \ \ n_2)$ be mode-T-$\mathbf{R}_\rho(s)$-coprime mutually stabilizing pair such that

$$[d_1 \quad n_1] \ T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1$$

Then,

i) failure of channel (1), $n_1 = 0$

$$[d_1, \ 0] \ T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = u$$

$$[d_1, \ 0] \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = u$$

$$[d_1 a_{13} \quad d_1 a_{14}] \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = u$$

$$d_1 \, [a_{13} \quad a_{14}] \begin{bmatrix} d_2 \\ \\ n_2 \end{bmatrix} = u \tag{7.40}$$

a_{13}, a_{14} $\mathbf{R}_\rho(s)$-coprime $\rightarrow$ $(d_2 \quad n_2)$ exist and for stabilization d_1 must be a divisor of $u \in \mathbf{R}_\rho(s)$ $\rightarrow$ d_1 must be a unit.

System remains closed loop stable with loss of channel (1) if d_1 = unit $\in \mathbf{R}_\rho(s)$.

ii) failure of channel (2). $n_2 = 0$

$$[d_1 \quad n_1] \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} d_2 \\ 0 \end{bmatrix} = u$$

$$[d_1 \quad n_1] \begin{bmatrix} a_{13} \\ \\ a_{23} \end{bmatrix} = d_2 = u \tag{7.41}$$

System remains closed loop stable with loss of channel (2) if d_2 = unit $\in \mathbf{R}_\rho(s)$.

iii) failure of channels (1) & (2)

$$[d_1 \quad 0]\ T \begin{bmatrix} d_2 \\ 0 \end{bmatrix} = u \tag{7.42}$$

$d_1 a_{13} d_2 = u \longrightarrow d_1,\ d_2,\ a_{13},\ u \in \mathbf{R}_\rho(s)$ units.

System remains closed loop stable with loss of channel (2) & (1) if d_1, d_2 = units $\in \mathbf{R}_\rho(s)$.

A necessary condition for reliable stabilization is that d_1, d_2 units $\in \mathbf{R}_\rho(s)$.

7.6 Diagonal Stabilization of Higher Order Systems

For the case m = 3 the scalar Diophantine equation becomes:

$$\begin{aligned} & a_{135}d_1d_2d_3 + a_{136}d_1d_2n_3 + a_{145}d_1n_2d_3 + a_{146}d_1n_2n_3 + \\ & + a_{235}n_1d_2d_3 + a_{236}n_1d_2n_3 + a_{245}n_1n_2d_3 + a_{246}n_1n_2n_3 = 1 \end{aligned} \tag{7.43}$$

$$\begin{aligned} & [a_{135}d_2d_3 + a_{136}d_2n_3 + a_{145}n_2d_3 + a_{146}n_2n_3]\ d_1 + \\ & + [a_{235}d_2d_3 + a_{236}d_2n_3 + a_{245}n_2d_3 + a_{246}n_2n_3]\ n_1 = 1 \end{aligned}$$

$$\begin{aligned} & [(a_{135}d_3 + a_{136}n_3)d_2 + (a_{145}d_3 + a_{146}n_3)n_2]\ d_1 + \\ & + [(a_{235}d_3 + a_{236}n_3)d_2 + (a_{245}d_3 + a_{246}n_3)n_2]n_1 = 1 \end{aligned} \tag{7.44}$$

SET: $$\left[\begin{array}{l} a_{135}d_3 + a_{136}n_3 = r_1^{\,0} \\ a_{145}d_3 + a_{146}n_3 = r_2^{\,0} \\ a_{235}d_3 + a_{236}n_3 = r_3^{\,0} \\ a_{245}d_3 + a_{246}n_3 = r_4^{\,0} \end{array}\right. \tag{7.45a}$$

Then

$$[r_1^{\,0}d_2 + r_2^{\,0}n_2]\; d_1 + [r_3^{\,0}d_2 + r_4^{\,0}n_2]\; n_1 = 1 \tag{7.45b}$$

SET: $$\left[\begin{array}{l} r_1^{\,0}\; d_2 + r_2^{\,0}\; n_2 = r_1^{\,1} \\ r_3^{\,0}\; d_2 + r_4^{\,0}\; n_2 = r_2^{\,1} \end{array}\right. \tag{7.46a}$$

Then

$$r_1'd_1 + r_2'n_1 = 1 \tag{7.46b}$$

(r_1', r_2') must be $\mathbf{R}_\rho(s)$-coprime.

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \tag{7.46c}$$

<u>Case (I)</u>:

$S_T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftrightarrow$ T is an $\mathbf{R}_\rho(s)$ unimodular

$\longrightarrow \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = T^{-1} \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix}_{(2)}$ with (r_α, r_β) arbitrary coprime elements of $R_\rho(s)$

(7.47)

Case (II):

$$S_T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we may decompose T as

$T = \underline{u}_1, \underline{v}_1^t$ where $\underline{u}_1, \underline{v}_1$ are minimal bases vectors for the column, row modules ($\mathbf{R}_\rho(s)$) of T.

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \Leftrightarrow \underline{u}_1 \underline{u}_1^t \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \tag{7.48}$$

(7.48) has a solution if and only if $\begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = \underline{u}_1 \lambda,$

$$\lambda \in \mathbf{R}_\rho(s) \text{ unit, arbitrary} \tag{7.49}$$

Under this assumption

$$\underline{u}_1 \underline{v}_1^t \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \underline{u}_1 \lambda \longrightarrow \underline{u}_1 \{ \underline{v}_1^t \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} - \lambda \} = 0$$

$$\longrightarrow \underline{v}_1^t \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \lambda$$ and solution always exists

Case (III):

$$S_T \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix}$$

$$T \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} \Leftrightarrow \underbrace{U_L\, T}_{\triangleq\, T_H} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = U_L \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix}$$

$$T_H = \begin{bmatrix} 1 & \alpha \\ 0 & \phi \end{bmatrix} \text{———} \begin{bmatrix} 1 & \alpha \\ 0 & \phi \end{bmatrix} \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} \tilde{r}_\alpha \\ \tilde{r}_\beta \end{bmatrix} \tag{7.50}$$

$$\{d_2 + \alpha n_2 = \tilde{r}_\alpha, \quad \phi n_2 = \tilde{r}_\beta\}$$

$$\Leftrightarrow \quad \{d_2 = \tilde{r}_\alpha - \alpha n_2, \quad \phi n_2 = \tilde{r}_\beta\}$$

i) Choose: $\underline{\tilde{r}_\beta = \phi\, t_1}$ where t_1 arbitrary element of $\mathbf{R}_\rho(s)$

Then $\phi n_2 = \phi\, t_1$ gives $n_2 = t_1$

thus: $d_2 = \tilde{r}_\alpha - \alpha n_2 = \tilde{r}_\alpha - \alpha t_1$

ii) Choose: $\tilde{r}_\alpha = t_2$ where t_2 arbitrary element of $\mathbf{R}_\rho(s)$ such that

$(\phi\, t_1, t_2)$ $\mathbf{R}_\rho(s)$ comprime

$$\begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = U_L^{-1} \begin{bmatrix} t_2 \\ \phi t_1 \end{bmatrix} \qquad (t_2, \quad \phi t_1) \quad \mathbf{R}_\rho(s) \text{ comprime}$$

(7.51)

$$r_\alpha d_1 + r_\beta n_1 = 1 \longrightarrow [r_\alpha \quad r_\beta] \begin{bmatrix} d_1 \\ n_1 \end{bmatrix} = 1$$

$$[d_1 \quad n_1] \begin{bmatrix} r_\alpha \\ r_\beta \end{bmatrix} = 1 \qquad (7.52)$$

CHAPTER 8

CONCLUSIONS

CONCLUSIONS

The main aim of this thesis was to provide a unifying approach for the study of solvability of algebraic problems defined as linear time invariant multivariable systems. The formulation of the generalised Diophantine equation, over the ring $\mathbf{R}_{\rho}(s)$ of proper and stable rational functions, as the unifying stabilization problem for both centralized and decentralized control provides the means for the reduction of the Decentralized Stabilization Problem to the study of (sets of) matrix equations of the type AX = B.

It is shown that many control problems can be reduced to the study of matrices defined over special rings which describe in an algebraic context the problems of system stabilizability, realizability and performance. The rings of importance are the Euclidean domains $\mathbf{R}[s]$-polynomials, $R_{pr}(s)$-proper rational functions and $\mathbf{R}_{\rho}(s)$-proper and Ω-stable rational functions and these are used to investigate the structural and invariant aspects of system stability equations. The solvability of AX = B also provides conditions for the solvability of the generalised Diophantine equation $A_1X_1 + A_2X_2 + \ldots + A_mX_p = B$ where B is in general non square. The solvability of AX = B also provides conditions for solvability of the more general set of equations.

The diagonal stabilization problem (DSP) has been defined over the ring $\mathbf{R}_{\rho}(s)$ and conditions for solvability have been given. For the case of two-channel systems stabilization by a diagonal dynamic controller is possible if and only if the system exhibits the property of cyclicity. In that case the system is free from unstable hidden modes. A complete parameterisation of diagonally stabilizing controllers is given for the case of two-channel systems.

This work provides the tools to investigate special type solutions such as realizable, stable and performance related controller designs as well as the more general case of n-channel systems. the general case of solving the Diophantine equation simultaneously with the multilinear system requires further investigation.

Although a clear algorithmic procedure has been developed for the case of 2 or 3-channel systems the general n-channel case becomes less manageable using the same approach. The main problem is that as the number of channels increases so too does the number and the order of the diophantine equations to be solved simultaneously. The developed theory also applies to discrete time systems.

This work has concentrated on the analysis of how problems are solved and the parameterisation of solutions. The freedom of choice to select a parameter which determines the final structure of the decentralized control scheme needs further investigation. Future research work should also include investigation into the general case of diagonal stabilization and how this work can be extended to take account of simple structured control schemes. The general case of block diagonal stabilization and how we solve the more general Diophantine equations needs further work. The links between the present approach and systems which have no fixed (unstable) modes need to be better understood.

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APPENDICES

APPENDIX A

Rings, Fields and Integral Domains [God 1, Mar 1]

A ring is an abstract notion which provides a uniform and convenient terminology for the representation of binary operations on a set.

Definition (A.1) [God 1]: A ring is a non empty set **K** together with two operations + (addition) and $\cdot$ (multiplication) such that the following conditions are satisfied.

(R1) (K,+) is a commutative group. This means that

i) $a+(b+c) = (a+b)+c, \quad \forall\ a,b,c \in \mathbf{K}$

ii) $a+b = b+a, \quad \forall\ a,b \in \mathbf{K}$

iii) there exists an element $0 \in \mathbf{K}$ such that

$a+0 = 0+a, \quad \forall\ a \in \mathbf{K}$

iv) for every element $a \in \mathbf{K}$ there exists a corresponding element $-a \in \mathbf{K}$ such that $a+(-a)=0$

(R2) $(\mathbf{K}, \cdot)$ is a semi group. This means that

$a\cdot(b\cdot c) = (a\cdot b)\cdot c, \quad \forall\ a,b,c \in \mathbf{K}$

(R3) Multiplication is distributative over addition i.e.

i) $a\cdot(b+c) = a\cdot b + a\cdot c$

ii) $(a+b)\cdot c = a\cdot c + b\cdot c$, for all $a,b,c \in \mathbf{K}$

As is customary, $a\cdot b$ is denoted by ab and $a + (-b)$ denoted $a + b$.

■

<u>Example (A.1)</u> The set $\mathbf{K} = \{A: = A \in \mathbf{R}^{nxn}$, matrices, $+$ addition of matrices, $\cdot$ multiplication of matrices$\}$. $\mathbf{K}$ is a ring.

A ring $\mathbf{K}$ is said to be <u>commutative</u> if $ab = ba$, for all $a,b \in \mathbf{K}$ and is said to have an identity if there exists an element $1 \in \mathbf{K}$ such that $1\cdot a = a\cdot 1$, for all $a \in \mathbf{K}$.

<u>Example (A.2)</u> The set of real nxn matrices in the above example is not, in general, a commutative ring, i.e.

$$A\ \bar{A} \neq \bar{A}\ A \text{ where } A,\ \bar{A} \in \mathbf{R}^{nxn}$$

The set of polynomials $\mathbf{R}[s]$ in s-variable with real coefficients is a commutative ring e.g. $t_1(s)=(s+1)$, $t_2(s) = (s^2 + 2s + 1)$, $t_1(s)$, $t_2(s) \in \mathbf{R}[s]$ and clearly $t_1(s)\ t_2(s) = t_2(s)\ t_1(s) \in \mathbf{R}[s]$.

■

<u>Definition (A.2)</u> [God 1] : Let $\mathbf{K}$ be a ring. A subring of $\mathbf{K}$ is a subset of $\mathbf{K}$ which is also a ring with respect to binary operations on $\mathbf{K}$ i.e. $\mathbf{S}$ is a subring of $\mathbf{K}$ if and only if

(SR1) $\mathbf{S}$ is closed with respect to binary operations on $\mathbf{K}$.

(SR2) For all $a \in \mathbf{S}$, we have $(-a) \in \mathbf{S}$.

■

Example (A.3) Let **K** be the ring of proper rational functions $\mathbf{R}_{pr}(s)$ (rational functions which have no poles at $s = \infty$) and **S** be the ring of proper rational functions which are also stable denoted $\mathbf{R}_{\rho}(s)$. Then $\mathbf{R}_{\rho}(s)$ is a subring of $\mathbf{R}_{pr}(s)$.

■

An element u of a ring **K** which has a multiplicative inverse $u^{-1} \in \mathbf{K}$ is called a unit of **K**. An element $b \in \mathbf{K}$ is called an associate of $a \in \mathbf{K}$ if $b = ua$ where $u \in \mathbf{K}$ is a unit.

An element $d \in \mathbf{K}$ is a divisor of $e \in \mathbf{K}$ provided there exists an element f such that $e = df$. Every non-zero element of $e \in \mathbf{K}$ has as divisors its associates in **K** and the units of **K**. These divisors are called trivial all others are non trivial. A non zero, non unit element $e \in \mathbf{K}$ having only trivial divisors is called prime or irreducible element of **K**.

Example (A.4) The units of $\mathbf{R}[s]$ are the constants i.e the polynomials $t(s) = c$, $c \in \mathbf{R}-\{0\}$. If $q(s) \in \mathbf{R}[s]$ then clearly the associates $r(s)$ of $q(s)$ are given by

$$r(s) = t(s)\ q(s)$$

If $q(s) = (s^2 + 3s + 2)$ then $(s+1)$, $(s+2)$ are non trivial divisors of $q(s)$ and $(s+1)$, $(s+2)$ are primes of $\mathbf{R}[s]$.

■

Definition (A.3) [God 1]: Let **K** be a ring with neutral element "0". An element $a \neq 0$ of **K** is called a divisor of zero if there exists an element $b \neq 0$ of **K** such that ab <u>or</u> ba = 0.

A ring **K** which has no divisors of zero is called an <u>integral domain</u>. Thus, if $a,b \in \mathbf{K}$ with $ab = 0$ implies either $a = 0$ or $b = 0$. Furthermore, for every integral domain where $ac = bc$ and $c \in \mathbf{K}$, but $\neq 0$, then $a = b$ (cancellation law).

■

<u>Example (A.5)</u> Let $\mathbf{K} = \{A: A \in \mathbf{R}^{2x2}$, ring of 2x2 real matrices$\}$ with $A, \bar{A} \in \mathbf{R}^{2x2}$ given by

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \neq 0 , \quad \bar{A} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \neq 0$$

$$A\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 , \quad A,\bar{A} \text{ are divisors of zero}$$

in $\mathbf{K} \equiv \mathbf{R}^{2x2}$, **K** is <u>not</u> an integral domain.

The set of polynomials is an integral domain. If $t_1(s)$, $t_2(s) \in \mathbf{R}[s]$ and $t_1(s)\, t_2(s) = 0$ implies that

$t_1(s) = 0$ or $t_2(s) = 0$. Let

$$t_1(s) = as^2 + bs + c, \qquad t_2(s) = ds + e$$

$$t_1(s)\, t_2(s) = ads^3 + (bd + ae)s^2 + (cd + be)s + ce \equiv 0$$

$$\Rightarrow \quad \left[\begin{array}{l} ad = 0 \;\rightarrow\; a = 0, \text{ or } b = 0 \\ bd = ae \\ cd = be \\ ce = 0 \;\rightarrow\; c = 0, \text{ or } e = 0 \end{array}\right.$$

$$\Rightarrow \quad \left[\begin{array}{c} a = b = c = 0 \\ \underline{\text{or}} \\ d = e = 0 \end{array}\right] \quad \begin{array}{l} \Leftrightarrow \; t_1(s) = 0 \\ \\ \Leftrightarrow \; t_2(s) = 0 \end{array}$$

■

In the case where every non zero element of a commutative ring is a unit we have

<u>Definition (A.4)</u> [God 1]: A <u>field</u> **F** is a commutative ring with an identity (the element 1) which satisfies the following conditions.

(F1) **F** contains at least two elements

(F2) Every non zero element of **F** is a unit.

■

<u>Example (A.6)</u> The rational numbers **Q**, real numbers **R**, complex numbers **C** and the rational functions **R**(s) are all

well known examples of fields.

The Field of Fractions of an Integral Domain

If every subring of a field is a commutative integral domain then we can construct a field of fractions.

Definition (A.5) [God 1]: Let **F** be a field, **K** be a subring of **F** and suppose that every element $x \in$ **F** can be represented in the form $y/z = yz^{-1}$ where $y, z \in$ **K**, $z \neq 0$. Then **F** is called the field of fractions of **K**, denoted **F/K**.

■

Example (A.7) Let **F** be the field of rational functions $\mathbf{R}(s)$ and **K** be the ring of polynomials $\mathbf{R}[s]$. Any element $t(s) = \frac{n(s)}{d(s)} \in \mathbf{R}(s)$ where $n(s), d(s) \in \mathbf{R}[s]$, thus $\mathbf{R}(s)$ is the field of fractions of $\mathbf{R}[s]$. This is known as the $\mathbf{R}[s]$-fractional representation of a rational function.
Similarly, if $\mathbf{K} \equiv \mathbf{R}_{pr}(s)$, $t(s) = \frac{n(s)}{d(s)} \in \mathbf{R}(s)$, $n(s), d(s) \in \mathbf{R}_{pr}(s)$, then $\mathbf{R}_{pr}(s)$-fractional representation of a rational function.
If $\mathbf{K} \equiv \mathbf{R}_{\rho}(s)$-, $t(s) = \frac{n(s)}{d(s)} \in \mathbf{R}(s)$, $n(s), d(s) \in \mathbf{R}_{p}(s)$, then $\mathbf{R}_{\rho}(s)$-fractional representation of a rational function.

The field **F/K** is the quotient field of **F** by the integral domain **K**. Generally, if **K** is any ring and **D** is any integral domain then **K/D** is termed the quotient ring of **K** by **D**.

The embedding of the ring **K** in its field of fractions forms the basis of what is known as the "fractional representation approach" to control systems synthesis [Vid 1]. This approach is discussed later on but in order to do so the notion of a discrete valuation on a field needs to be defined.

<u>Definition (A.6)</u> [Mar 1]: Let **F** be a field. A discrete valuation of **F** is a function γ defined on **F** whose values are integers or the symbol $+\infty$ such that

(DV1) $\gamma(0) = +\infty$; $\gamma(x) \in \mathbf{Z}$ if $x \neq 0$

(DV2) $\gamma(xy) = \gamma(x) + \gamma(y)$, $\forall\ x,y \in \mathbf{F}$

(DV3) $\gamma(x+y) \geq \min\,[\gamma(x); \gamma(y)]$, $\forall\ x,y \in \mathbf{F}$

<u>Example (A.8)</u> Let $t(s) = \frac{n(s)}{d(s)} \in \mathbf{R}(s)$, $n(s)$, $d(s) \in \mathbf{R}_{pr}(s)$ i.e. $\mathbf{K} \equiv \mathbf{R}_{pr}(s)$, the ring of proper rational functions (no poles at $s = \infty$) and $(n(s), d(s))$ defines an $\mathbf{R}_{pr}(s)$-fractional representation of $t(s)$. Define the map

$$\delta_\infty : \mathbf{R}(s) \rightarrow \mathbf{Z} \cup \{+\infty\} \text{ via}$$

$$\delta_\infty(t(s)) \begin{cases} \deg.\, d(s) - \deg.\, n(s), & t(s) \neq 0 \\ +\infty, & t(s) = 0 \end{cases}$$

Now $t(s)$ can be factorized as

$$t(s) = \left(\frac{1}{s}\right)^{q_\infty} \frac{n(s)}{d(s)} \quad \text{where } q_\infty = \delta_\infty(t(s))$$

and deg. n(s) = deg.d(s). Then,

i) if $q_\infty > 0$ we say that t(s) has a zero at $s = \infty$ of order q_∞

ii) if $q_\infty < 0$ we say that t(s) has a pole at $s = \infty$ of order $|q_\infty|$

iii) if $q_\infty = 0$ we say that t(s) is biproper or a unit of $\mathbf{R}_{pr}(s)$.

The discrete valuation $\delta_\infty(\cdot)$ may serve as a degree function for the set $t(s) \infty \mathbf{R}_{pr}(s)$ of proper rational functions where $\delta_\infty(t(s)) \geq 0$.

■

Principal Ideal Domains (PID's)

A subset **I** in a ring **K** is said to be a <u>left ideal</u> if **I** is a subgroup of the additive group of **K** and elements a ∈ **I**, x ∈ **K** imply that xa ∈ **I**. Similarly, **I** is a <u>right ideal</u> if **I** is a subgroup of the additive group of **K** and elements a ∈ **K** imply that ax ∈ **I**, in other words if the <u>right multiples</u> of every x ∈ **I** belong to **I**. It comes to the same thing to say that **I** is a non empty subset of **K** which has the following property: For a right ideal, ux+vy ∈ **I** for all u, v ∈ **K** and all x, y ∈ **I**. Clearly if **K** is a commutative ring the notions of left and right ideal coincide and we speak simply of an <u>ideal</u>.

Example (A.9) Let **K** be a commutative ring. For each x $\in$ **K** let x**K** (or **K**x) denote the set of all multiples of x in **K**. Thus, **I** = x**K** = {ux: u $\in$ **K**, u:units} is an ideal of **K**.

■

Ideals of the type given in example (2.7) are called principal ideals of **K**. An ideal **I** in a commutative ring **K** is a prime ideal if a $\in$ **K**, b $\in$ **K**, ab $\in$ **I** implies that either a $\in$ **I** or b $\in$ **I**.

Example (A.10) Consider the ring of integers **Z** and let a,b,n be any integers. Then, the set of multiples of n denoted $\bar{n}$ is the principal ideal generated by n. It is a prime ideal if and only if n divides ab implies that n divides a or n divides b, which is true if and only if n is a prime number.

■

Remark (A.1): In the remainder of this section and therefore the ring **K** is assumed to be a commutative integral domain with identity element.

Definition (A.7) [Mar 1]: Let **K** be a ring and **I** be an ideal of **K**. If **I** = x**K** for some x $\in$ **K**, then **I** is called a principal ideal and x is called the generator of **I**. A ring **K** is said to be a principal ideal ring if every ideal in **K** is principal and if **K** is also an integral domain then **K** is a <u>principal ideal domain</u> (PID) denoted **K**.

■

Recall that a principal ideal **I** consists of all multiples of some element $a \in \mathbf{I}$, i.e. $\mathbf{I} = \{xa: x \in \mathbf{K}\}$. Thus in a PID every ideal is generated by a single element.

<u>Example (A.11)</u> The rings $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$, $\mathbf{R}_{\rho}(s)$ respectively: polynomials, proper rational functions, proper and stable rational functions, are all PID's.

■

<u>The Algebraic Theory of Linear Time Invariant Systems is Based on the Study of Matrices over PID's</u>

If **K** is a P.I.D and x,y be any two elements of **K** which are not both zero. Then a non zero element $d \in \mathbf{K}$ is a <u>greatest common divisor</u> (g.c.d) of x,y if d divides x and y, denoted $d|x, d|y$. A g.c.d is not unique, since another g.c.d is given by du whenever u is a unit of **K**. Thus, every g.c.d d_1 of x and y is of the form $d_1 = du$ for some unit u and once we have found one g.c.d we can quickly find them all. The existence of a g.c.d is given by the following well known result.

<u>Theorem (A.1)</u> [God 1]: Let **K** be a PID. Then every pair of elements $x, y \in \mathbf{K}$, not both of which are zero, has a greatest common divisor (g.c.d.) d which can be expressed in the form

$$d = px + qy \tag{A.1}$$

for appropriate elements $p,q \in \mathbf{K}$.

Two elements $x,y \in \mathbf{K}$ are called <u>relatively prime</u> or simply <u>coprime</u> if every g.c.d. of x,y is a unit. This is equivalent to saying that x and y are coprime if and only if 1 is a g.c.d. of x and y. In view of expression (2.1) $x,y \in \mathbf{K}$ are coprime if and only if there exists $p,q \in \mathbf{K}$ such that $px + qy = u$, $u \in \mathbf{K}$ a unit.

■

A parallel concept to the greatest common divisor of a set of elements is that of the least common multiple. If $(x_1,...,x_n)$ is a set of elements from a PID $\mathbf{K}$, none of which is zero. We say that y is a <u>least common multiple</u> (lcm) of this set if, $x_i|y$ and $x_i|z$ implies $y|z$, for all $i = 1,2,...,n$.

<u>Unique Factorization and Euclidean Domains</u>

The coprimeness between elements of ring is characterized by a Euclidean division process. Rings in which a euclidean algorithm can be defined are given below. First, the conditions under which an element from a ring can be uniquely factorized are given.

<u>Definition (A.8)</u> [God 1]: Let $\mathbf{K}$ be a commutative integral

domain. An element $p \in \mathbf{K}$ is <u>irreducible</u> (prime) if it is not a unit and it has no divisors in $\mathbf{K}$ other than trivial ones (i.e. units of $\mathbf{K}$ and products of p by units). The ring $\mathbf{K}$ is said to be a <u>unique factorization domain</u> if it possesses the following two properties

(UFD1) Every element of $\mathbf{K}$ which is neither zero nor a unit is the product of a finite number of irreducible elements.

(UFD2) If $p_1, \ldots, p_r = q_1, \ldots, q_5$ where the p_i and q_j are irreducible elements of $\mathbf{K}$. Then, r=s and the orders of the q_j can be changed so that $\mathbf{K}q_j$ for $1 \leq i \leq r$ i.e. $q_i = u_i p_i$ with u_i a unit of $\mathbf{K}$.

■

The above conditions imply that if an irreducible element $p \in \mathbf{K}$ divides a product $xy \in \mathbf{K}$ then it divides x or y or both and for every irreducible $p \in \mathbf{K}$ the ideal generated is prime. <u>Every principal ideal domain is a unique factorization domain although the converse is not true</u>.

<u>Definition (A.9)</u> [Mar 1]: A commutative integral domain is said to be a <u>Euclidean ring or Euclidean domain</u> if there exists a degree function δ with non zero integer value which satisfies the following conditions

(ED1) For every $x, y \in \mathbf{K}$ with $y \neq 0$ there exists a $q \in \mathbf{K}$ such that either

$r := x - qy = 0$, or else $\delta(r) < \delta(y)$.

(ED2) If $x|y$ then $\delta(x) \le \delta(y)$.

■

One can think of q as a quotient and r as a remainder after dividing x by y. Condition (ED1) states that we can always obtain a remainder that is either zero or else has a smaller degree than the divisor y. We speak of a quotient and a remainder because q and r are not necessarily unique.

Condition (ED2) implies that $\delta(1) \le \delta(x)$, $\forall x \ne 0$, since 1 divides every non zero element. Hence it can be assumed without loss of generality that $\delta(1) = 0$. The same condition implies that if x and y are associates then they have the same degree since in that case $x|y$ and $y|x$. In particular $\delta(u)=0$ whenever u is a unit.

If **K** is a Euclidean domain with degree function $\delta(\cdot)$ where

$$\delta(x+y) \le \max\{\delta(x), \delta(y)\} \tag{A.2}$$

$$\delta(x,y) = \delta(x) + \delta(y) \tag{A.3}$$

then for every $x,y \in \mathbf{K}$ with $y \ne 0$ there exists a unique $q \in \mathbf{K}$ such that $\delta(x-yq) < \delta(y)$ where the degree of zero is taken as $(-\infty)$. A Euclidean domain **K** with degree function $\delta(\cdot)$ is called a proper Euclidean domain if **K** is not a field and the degree function $\delta(\cdot)$ satisfies condition (2.3). Note that in a proper Euclidean domain the division process may still produce non unique quotient, remainders because (2.2) is not assumed to hold. This is the case, for example, in the ring of proper and stable rational functions which are studied in chapter 2.

Example (A.13): Let **K** be a ring. Then a polynomial over **K** is an infinite sequence $\{a_0, a_1 ...\}$ such that only finitely many terms are non zero. The sum and product of two polynomials $a = \{a_i\}$ and $b = \{b_i\}$ are defined by

$$(a+b)_i = a_i + b_i$$

$$(ab)_i = \sum_{j=0}^{i} a_{i-j}\, b_j = \sum_{j=0}^{i} a_j b_{i-j}$$

For notational convenience a polynomial $a = \{a_i\}$ can be represented by $a_0 + a_1 + a_1 s + a_2 s^2 + \ldots$ where s is called the indeterminate. The highest value of the index i such that $a_i \neq 0$ is called the degree of a polynomial $a = \{a_0, a_1, a_2, ...\}$. Thus, if a is polynomial of degree m we write $a(s) = a_0 + a_1 s + \ldots + a_m s^m = \sum_{i=0}^{m} a_i s^i$

The set of polynomials over **K** is denoted **R**[s] and is a commutative ring with identity. Moreover, if **K** is a domain, so is **R**[s]. If **R** is a field then **R**[s] is a Euclidean domain if the degree of a polynomial in **R**[s] is defined as in (2.2), (2.3). The field of fractions associated with **R**[s] is denoted by **R**(s) and is called the set of rational functions over **R**. Note that every element of **R**(s) is a ratio of two polynomials.

■

APPENDIX B

Modules over a Ring

The notion of a module over a ring provides a general setting for the purely algebraic aspects of linear problems which arise in control systems theory: linear algebra, differential equations, algebraic geometry, analytic functions, algebraic topology etc.

The definition of a module over a commutative ring is given first and the notion of finitely generated and free modules is introduced. Finally, the structure of Noetherian modules is given and these modules when defined over the appropriate ring provide an important setting for control algebraic synthesis problems.

Definition (B1) [God 1]: Let **K** be a commutative ring. A **K**-module is defined to be an object consisting of a set **M** together with the binary operations + (addition) and $\cdot$ (multiplication) which satisfy the following two conditions.

(M1) (**M**, +) is a commutative group. This means that the law of composition $(x,y) \to (x+y)$ is commutative.

(M2) $\forall$ $x,y \in$ **M** and $\forall$ λ, $\mu \in$ **K** we have:

$$\lambda(\mu x) = (\lambda\mu)x; \quad 1x = x$$

$$(\lambda + \mu)x = (\lambda x + \mu x); \quad \lambda(x + y) = \lambda x + \lambda y$$

■

In the theory of modules, the ring **K** is fixed and is generally called the ground ring. The elements of the ring are called scalars and the elements of the **K**-module are called vectors.

If the ring **K** is a field we speak of a vector space over **K**. In particular a vector space over the field **R** of real number is called a real vector space. A vector space over the field of complex numbers **C** is called a complex vector space. Vector spaces over $\mathbf{R}(s)$ rational vector spaces and modules over the rings $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$, $\mathbf{R}_{\rho}(s)$ polynomials, proper rational and proper, stable rational functions respectively play on important role in algebraic synthesis methods [Var 2, Var 3, Var 7, Var 8, Per 1].

Example (B1): Let **K** be the ring $\mathbf{R}[s]$ with $t(s) \in \mathbf{R}[s]$ then an $\mathbf{R}[s]$ module is given by

$$M = t(s) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If now the ring **K** is the field of rational functions $\mathbf{R}(s)$, then an $\mathbf{R}(s)$ vector space is given by

$$V = r(s) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad r(s) \in \mathbf{R}(s)$$

■

Definition (B2) [God 1]: Let **K** be a commutative ring and **M** be a module over **K**. A submodule of **M** is defined to be a subset **S** of **M** that satisfies the following two conditions:

(SM1) **S** is a subgroup of the addition group **M**

(SM2) If $\underline{x} \in \mathbf{S}$ and $\lambda \in \mathbf{K}$ then $\lambda\underline{x} \in \mathbf{S}$.

■

Remark (B1) If **K** is a field, then in the above definition **M** is a vector space over **K** and **S** is a subspace of **M**.

Example (B2) Let $\mathbf{K} \equiv \mathbf{R}[s]$ and let M_1 and M_2 be two modules over $\mathbf{R}[s]$ with

$$M_1 = \begin{bmatrix} 0 \\ 1 \\ (s+3) \end{bmatrix}, \quad M_2 = \begin{bmatrix} s+3 \\ 0 \\ (s+3)^2 \end{bmatrix}$$

Clearly, $M_2 = (s+3)M_1$ thus $M_2 \subset M_1$ i.e. M_2 is an $\mathbf{R}[s]$-submodule of M_1.

■

Finitely Generated Modules

Definition (B3) Let **M** be a **K**-module and **S** a submodule of **M**. Then **S** is said to be finitely generated if a finite set of m elements $\{\underline{a}_1,...,\underline{a}_m\} \in \mathbf{M}$ generate the submodule **S**. The vectors $[\underline{a}_1,...,\underline{a}_m]$ are said to be a system of generators of **S**.

■

The above applies in particular to the module M itself; thus a module **M** is finitely generated if it contains a finite set of m vectors $\{\underline{a}_1,...,\underline{a}_m\}$ such that every vector $\underline{x} \in \mathbf{M}$ is a linear combination of $\{\underline{a}_1,...,\underline{a}_m\}$ i.e. there exists scalars $(\lambda_1,...\lambda_m) \in \mathbf{K}$ such that

$$\underline{x} = \lambda_1\underline{a}_1 + \lambda_2\underline{a}_2 + ... + \lambda_m\underline{a}_m \qquad \text{(B1)}$$

$$= \sum_{i=1}^{m} \lambda_i \underline{a}_i \qquad \text{(B2)}$$

The collection of all linear combinations $\underline{x}$ is called the module spanned by $\{\underline{a}_1,...,\underline{a}_m\}$ denoted $\mathbf{M} = \text{span}_K \{\underline{a}_1,...,\underline{a}_m\}$. If a_i is a column, row vector then the module generated is called a column, row-module of **M** denoted $\mathbf{M}^c$, $\mathbf{M}^r$ respectively.

If the ring **K** is a field a finitely generated module coincides with the standard notion of a finite dimensional vector space ϑ and the collection of linear combinations $\underline{x}$ of the given vectors $\{\underline{a}_1,...,\underline{a}_m\}$ is called the range space of ϑ . In particular if $\underline{a}_i$ is a column, row vector, then the range space generated is called the column, row range space of ϑ denoted $\mathbf{X}^c$, $\mathbf{X}^r$ respectively. Furthermore, the collection of all linear combinations x such that $\sum_{i=1}^{m} \lambda_i \underline{a}_i = 0$ is called the <u>null space</u> of the vector space ϑ spanned by the columns, rows of $\underline{a}_i$, $i \in m$ denoted $\mathbf{N}_\ell$ (left-null space), $\mathbf{N}_r$ (right-null space) respectively. The corresponding notion for a module is that of the <u>torsion module</u> [God 1]. Modules which are <u>torsion free</u> are considered next.

<u>Free Modules Bases</u>

<u>Definition (B4)</u> Let **M** be a **K**-module. Then **M** is said to be a <u>finitely generated free module</u> if there exists a set of vectors $\{b_1, \ldots b_m\}$ in **M** which are linearly independent and generate **M** i.e.

$$\lambda_1 \underline{b}_1 + \ldots + \lambda_m \underline{b}_m = \sum_{i=1}^{m} \lambda_i \underline{b}_i = 0 \text{ implies}$$

$$\lambda_1 = 0 \ldots = \lambda_m = 0 \qquad \text{(B3)}$$

The vectors $[b_1, \ldots, b_m]$ are said to form a basis of **M** so that a basis is a <u>finite set of linearly independent generators</u>.

■

All the bases of a **K**-module **M** have m linearly independent generators. This number is called the <u>rank</u> of **M** over **K**. In the case where **K**-is a field then M is called the <u>dimension</u> of the vector space and corresponds to the notion of rank "if and only if" the dimension of the null space of the vector space is zero.

<u>Example (B3)</u>. Let **K** be the ring **R**[s] and

$$M_1 = \begin{bmatrix} 0 & 0 \\ s+2 & 0 \\ 0 & s+3 \end{bmatrix}$$ be an **R**[s]-module. Then

$$M^c = \text{span}_{\mathbf{R}[s]} \begin{bmatrix} 0 \\ s+2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ s+2 \end{bmatrix}$$ are

clearly linearly independent, rank of M = 2, and form a basis for the **R**[s]-module M.

<u>Finitely Generated Modules over a Noetherian Ring</u>

Noetherian rings were invented around 1920 by Emmy Noether and have been one of the principal starting points for modern algebra.

Definition (B5) [God 1]: Let **K** be a commutative ring. Then **K** is said to be Noetherian if every ideal in **K** is finitely generated. The **K**-module generated over a Noetherian ring is called a Noetherian **K**-module.

■

All principal ideal domains are Noetherian and thus all modules over PID's are Noetherian **K**-modules. The most important rings in algebraic control theory are the PID's $\mathbf{R}[s]$, $\mathbf{R}_{pr}(s)$, $\mathbf{R}_{\rho}(s)$, polynomials, proper rational and proper, stable rational respectively.

Before giving some characteristic properties of Noetherian rings we first introduce the following terminology.

Let (A_n) be a set of subsets of a set X. Then X is said to be an increasing sequence if

$$A_n \subset A_{n+1} \quad , \qquad \forall\ n \tag{B4}$$

and a stationary sequence if there exists an integer p such that

$$A_n = A_{n+1}, \ \forall\ n > p \tag{B5}$$

so that $A_p = A_{p+1} = A_{p+2} = \ldots.$. On the other hand, let F be a set of subsets of X. An $A \in F$ is said to be a maximal element of F if the relations

$$A \subset B \text{ and } B \in F$$

imply $\quad A = B$

In other words if F contains no set strictly larger than A (this does not mean that every $B \in F$ is necessarily contained in A).

Properties:

i) $\mathbf{K}$ is Noetherian i.e. every ideal of $\mathbf{K}$ is finitely generated.

ii) Every increasing sequence of ideals of $\mathbf{K}$ is stationary.

iii) Every non empty set of ideals has at least one maximal element.

APPENDIX C

Normal Forms and Coprime Matrix Fraction Descriptions over a Principal Ideal Domain

Many of the results in the section were first obtained (e.g.the Smith form (1861)) for matrices with integer entries; it was only realised much later that they also hold for entries drawn from any PID. This generality is highly useful in systems problems. Throughout, $\mathbf{K}$ denotes a PID with $\mathbf{K}^{pxm}$ the $\mathbf{K}$-module of pxm matrices with elements from $\mathbf{K}$.

Hermite and Smith-McMillan Forms of a Rational Matrix over a PID

Definition (C1) Let T and $\hat{T}$ be two matrices in $\mathbf{K}^{pxm}$.
We say that $\hat{T}$ is left equivalent to T, written $T\ \mathbf{E}_\ell\ \hat{T}$ if there exists a matrix $U_\ell \in \mathbf{K}^{pxp}$ which satisfies the following two conditions:

(U1) $|U|$ is a unit in the ring $\mathbf{K}$

(U2) $T = U_\ell \hat{T}$

Similarly we say that T and $\hat{T}$ are right equivalent, written $T\ \mathbf{E}\ \hat{T}$ if there exists a matrix $U_r \in \mathbf{K}^{mxm}$ which satisfies (U1) and

(U3) $T = \hat{T}U_r$.

Furthermore, T and $\hat{T}$ are called <u>equivalent</u> written, $T \; \mathbf{E} \; \hat{T}$ if there exists $U_\ell \in \mathbf{K}^{pxp}$, $U_r \in \mathbf{K}^{mxm}$ which satisfy (U1) and

(U4) $T = U_\ell \hat{T} U_r$

■

<u>Unimodular matrices</u> are elementary matrices that correspond to elementary row (left), column (right) operations performed on the identity matrix I. There are three types of row (column) operation that can be performed on a matrix in $\mathbf{K}^{pxm}$.

I Interchange row (column) i and row (column) j.

II To row column i add (in $\mathbf{K}$) r times row column j, $i \neq j$, $r \in \mathbf{K}$.

III Multiply row column i by a unit $u \in \mathbf{K}$.

If any of these operations is performed on the identity matrix the resulting matrix is an elementary matrix of the corresponding type.

By elementary operations we can convert matrices to several "standard forms". We next describe the Hermite normal form of a rational matrix over a PID $\mathbf{K}$ which is obtained by applying row or column operations only. When both row and column operations are performed on a matrix the Smith, Smith-MacMillan is then defined.

<u>Theorem (C1)</u> [Mar 1] (Hermite Normal Form): Let **K** be a PID. Any rational matrix $T \in {}^{pxm}(s)$, rank $\{T\} = r$ with $r \leq \min(p,m)$ can be reduced by operations in **K** (premultiplication by a **K**-unimodular matrix) to a (lower or upper) quasi-triangular form-the <u>column Hermite form</u> - in which:

i) the last p-r rows are identically zero;

ii) in column j, $1 \leq j \leq r$ the diagonal element is prime and of higher degree than any non zero element above it;

iii) in column j, $1 \leq j \leq r$ if the diagonal element is a unit in **K** then all elements above it are zero; and

iv) no particular statement can be made about the elements in the last m-r columns and the first r rows.

<u>Note</u>: The above form is uniquely defined modulo units, i.e. its elements are uniquely defined.

<u>Remark (C1)</u> By interchanging the roles of rows and columns a similar <u>row Hermite form</u> can be obtained. If the matrix T is defined as the field **R**(s) then reduction to (lower, upper) quasi-triangular form defines the (column, row) <u>Hermite MacMillan form of T</u> over **R**(s).

<u>Example (C1)</u> The following steps are self-explanatory. (Consider a matrix from $\mathbf{R}_{pr}(s)$)

$$T = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{s^2} \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{s^2} \\ \frac{1}{(s+1)^2} & 0 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ 0 & \frac{-1}{s^2} \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{s^2} \\ 0 & 0 \end{bmatrix} = H$$

The corresponding unimodular matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{s^2}{(s+1)^2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{-1}{(s+1)^2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{s^2}{(s+1)^2} & (s+1)^2 \end{bmatrix} \in \mathbf{R}_{pr}^{3x3}(s)\text{-unimodular}$$

$$\triangleq U_\ell$$

i.e. $U_\ell T = H$ where H is the column Hermite form of $T \in \mathbf{R}^{3x2}(s)$ over $\mathbf{R}_{pr}(s)$.

<u>Theorem (C2)</u> (Smith-McMillan Form) [Mar 1]:

Let **K** be a PID and **K** be such that **R**(s) is the field of fractions of **K**. For any rational matrix $T \in \mathbf{R}^{pxm}(s)$, $\mathrm{rank}_{\mathbf{R}(s)}\{T\} = r$, $r \leq \min(p,m)$ there exists **K**-unimodular matrices $U_\ell \in \mathbf{K}^{pxp}$, $U_r \in \mathbf{K}^{mxm}$ corresponding to elementary row, column operations respectively such that

$$U_\ell \; T \; U_r = S_T \tag{C1}$$

where S_T is a diagonal matrix having the form:

$$S_T = [\mathrm{diag}\{\epsilon_1\psi_1^{-1}, \; \epsilon_2\psi_2^{-1}, \ldots, \epsilon_r\psi_r^{-1}\} \;\; O_{p-r,m-r}] \tag{C2}$$

where $\epsilon_i \in \mathbf{K}$, $\psi_i \in \mathbf{K}$ are coprime in **K**, $i \in r$ and obey the division property

$$\begin{aligned} &\text{i)} \quad \psi_{i+1} \mid \psi_i \qquad i \in \underline{r-1}, \qquad \psi_1 = d \\ &\text{ii)} \quad \epsilon_i \mid \epsilon_{i+1} \qquad i \in \underline{r-1} \end{aligned} \tag{C3}$$

"|" means divide and "d" the least common multiple of the denominators of the entries of T.

The matrix S_T can also be written as

$$S_T = E \; \Psi_r^{-1} = \Psi_\ell^{-1} \; E \tag{C4a}$$

where

$$E = [\mathrm{diag}\{\epsilon_1,\ \epsilon_2, \dots ,\epsilon_r\},\ O_{p-r,m-r}] \tag{C4b}$$

$$\Psi_r = \mathrm{diag}^{\cdot}\{\psi_1,\ \psi_2, \dots ,\psi_r,\ I_{m-r}\} \tag{C4c}$$

$$\Psi_\ell = \mathrm{diag}^{\cdot}\{\psi_1,\ \psi_2, \dots ,\psi_r,\ I_{p-r}\} \tag{C4d}$$

In the case where the ψ_i's are all units of $\mathbf{K}$ i.e. $|\Psi_r| = I_m$, $|\Psi_{\ell'}| = I_p$ then S_T is called the Smith Form of T over $\mathbf{K}$. Otherwise some of the ψ_i's will be different from units and S_T is called the Smith-McMillan form of T over $\mathbf{K}$.

Example (C2) Let $T \in \mathbf{R}^{2x2}(s)$ be the rational matrix.

$$T = \begin{bmatrix} \dfrac{s}{(s+1)^2\ (s+2)^2} & \dfrac{s}{(s+2)^2} \\ \\ \dfrac{-s}{(s+2)^2} & \dfrac{-s}{(s+2)^2} \end{bmatrix}$$

Then we can check that

$$\begin{bmatrix} 1 & 0 \\ \\ -(s+1)^2 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{s}{(s+1)^2(s+2)^2} & 0 \\ \\ 0 & \dfrac{s^2}{s+2} \end{bmatrix} \begin{bmatrix} 1 & (s+1)^2 \\ \\ 0 & 1 \end{bmatrix}$$

or $T = U_\ell^{-1}\ S_T\ U_r^{-1}$, $d = (s+1)^2\ (s+2)^2$ where U_ℓ, U_r are the $\mathbf{R}[s]$-unimodular matrices which reduce T to its Smith-

McMillan form over **R**[s].

$$S_T = \text{diag.}\ \{\frac{s}{(s+1)^2\ (s+2)^2}\ ,\ \frac{s^2}{s+2}\}$$

where: $\epsilon_1 = s,\ \ \psi_1 = (s+1)^2\ (s+2)^2;$

$\epsilon_2 = s^2,\ \psi_2 = s+2$ are coprime in **R**[s].

■

<u>Definition (C1)</u> [Kai 1]: The sum of the deg.ψ_i, $i \in r$ is called the <u>McMillan degree of T</u>.

■

<u>Matrix Fraction Description of a Rational Matrix over a Principal Ideal Domain</u>

The generalization of the representation of a rational function t(s) as a quotient of relatively prime (coprime) elements over a PID to the matrix case is given below.

Recall that if **F** is the field of fractions associated with a PID **K** and t(s) = a/b is a function in **F** then it is always possible to express a/b as an equivalent fraction f/g where f,g are coprime in **K**. Thus, their greatest common divisor is a unit in **K**. This notion is extended to the matrix. We begin by defining matrix divisors and multiple common divisors for the set $\mathbf{K}^{p\times m}$. Since matrix multiplication is in general non commutative it is necessary to distinguish between left and right multiples.

Definition (C2): Let $\mathbf{K}$ be a PID. Any rational matrix $T \in \mathbf{K}^{pxm}(s)$ with $p \geq m$, rank $\{T\} = m$ can be factorized in a non unique way as

$$T = T_1 \, T_r \tag{C5}$$

where $T_r \in \mathbf{K}^{mxm}$ is a right divisor of T and T is a left multiple of T_r. If T_r is a greatest right divisor ($\mathbf{K}$-g.r.d.) of T this is denoted by T^{gr}. If $\hat{T} \in \mathbf{K}^{pxm}(s)$, rank$\{\hat{T}\}$ = m, p ≥ m is any other matrix factorized as in (2.13). Then a square matrix $T_{gr} \in \mathbf{K}^{mxm}$ is called a greatest common right divisor ($\mathbf{K}$ = g.c.r.d) of T and $\hat{T}$ if:

(GCRD1) T_{gr} is a right divisor of both T and $\hat{T}$

(GCRD2) T_{gr} is a left multiple of every common right divisor of T and $\hat{T}$.

Two (or more) matrices T and T are called right coprime over $\mathbf{K}$ if every $\mathbf{K}$-g.c.r.d of T and $\hat{T}$ is $\mathbf{K}$-unimodular.

The definition for left divisor, greatest common left divisor and left coprimeness follows in an analogous way.

■

Proposition (C1) [Kar 3]: Let $T \in \mathbf{K}^{pxm}(s)$, rank $\{T\} = r$, $r \leq \min(p,m)$ and $\mathbf{K}$ be a PID. Then T can always be expressed (in a non unique way) as:

$$T = B_2 A_2^{-1} = A_1^{-1} B_1 \tag{C6}$$

where

$B_2 \in \mathbf{K}^{pxm}$, $A_2 \in \mathbf{K}^{mxm}$; $A_1 \in \mathbf{K}^{pxp}$, $B_1 \in \mathbf{K}^{pxm}$

and (B_2, A_2) right coprime over $\mathbf{K}$

$(B_1\ A_1)$ left coprime over $\mathbf{K}$.

<u>Proof</u> Let $U_\ell \in \mathbf{K}^{pxp}$, $U_r \in \mathbf{K}^{mxm}$ be $\mathbf{K}$- $\mathbf{K}$-unimodular matrices that reduce $T \in \mathbf{R}^{pxm}(s)$ to its Smith-McMillan form over $\mathbf{K}$. Then from (C3) we have:

$$U_\ell\ T\ U_r = S_T = E\ \Psi_r^{-1} = \Psi_\ell^{-1}\ E$$

Then let

$$B_2 = U_\ell E,\ A_2 = U_r\ \Psi_r$$

$$B_1 = E\ U_r,\ A_1 = \Psi_\ell\ U_\ell$$

from which

$$\begin{bmatrix} B_2 \\ A_2 \end{bmatrix} = \begin{bmatrix} U_\ell^{-1} & 0 \\ 0 & U_r^{-1} \end{bmatrix} \begin{bmatrix} E \\ \Psi_r \end{bmatrix}$$

and

$$[B_1\quad A_1] = [E\quad \Psi_\ell] \begin{bmatrix} U_r^{-1} & 0 \\ 0 & U_\ell \end{bmatrix}$$

and thus it follows that (B_2, A_2) are right and (B_1, A_1) are left coprime in $\mathbf{K}$.

■

A pair (B_2, A_2) satisfying the above proposition is called <u>right coprime **K**-matrix fraction description</u> ($\mathbf{K}_r$-MFD) of a rational matrix T. Similarly, a pair (B_1, A_1) is referred to as a <u>left coprime **K**-matrix fraction description</u> ($\mathbf{K}_\ell$-MFD).

Proposition (2.1) describes the well known fact that every rational matrix T can be expressed as a ratio of coprime over **K** matrix fractions. This representation of rational matrices was first introduced by Vidyasagar [Vid 1] for the case $\mathbf{K} \equiv \mathbf{R}_\rho(s)$, the set of proper rational functions which are also stable (i.e. have no poles at $s = \infty$ or in a prescribed region of the finite complex plane). Matrix fraction descriptions of this type forms the basis of what is termed the fractional approach to analysis and synthesis of linear multivariable control algebraic problems.