On optimal constrained investment strategies for long-term savers in stochastic environments and probability hedging

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Abstract

In this paper, we derive constrained optimal investment strategies for long-term savers who are interested in investing their funds in stocks, but are afraid of potentially losing, for example, their retirement income. We call this probability hedging as it is determined by the probability of landing up within bounds that are agreed from interaction with the investor. We show that our strategies can be derived under different utility functions and multifactor model assumptions. We prove that the probability measure varies with the utility function choice and that the logarithmic utility, in particular, results in an intuitive probability hedge under the physical measure. This makes it easier to communicate, without putting at risk the financial advice conducted by potentially misrepresenting the realism of the theoretical results. Our strategy is also shown to yield a better distribution of the terminal wealth than traditional hedging approaches.

Keywords: investment analysis, finance, utility theory

1. Introduction

When a financial advisor in the selection process of a financial plan for a consumer asks questions like "how much risk are you willing to take on your accumulated savings?", the long-term investor often finds it difficult to provide an adequate individual risk preference or their targeted saving goal. A natural answer would be: "I want to invest all my money in stocks to get a high return. But I am afraid of losing money. What should I do?". In fact, trying to quantitatively translate the risk profile of the investor and select the right investment proportions in risky and risk-free assets is not trivial at all.

As in Gerrard *et al.* (2019), we set our sights on a system whereby the fund manager (financial advisor/optimiser) offers the option to the long-term investor (pension saver/consumer) to self-select their individual financial risk preference via a simple application run on a smart device. The original question is modified as: "how much of your invested capital are you willing to lose?", prompting this way the consumer to identify a lower bound G_L. The upside potential is also restricted by placing a cap G_U to the possible investment outcomes. As we explain next, this aims to simplify the customer's choice of risk, consistently with Merton's (2014) vision, which we materialize in this paper via the use of upper and lower bounds. More specifically, G_L corresponds to the investor's worst-case scenario, the guarantee, and it is chosen directly by them. On the other hand, G_U is the best-case outcome; it is possible to set this to be achieved half of the time,

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i.e., to correspond to the median terminal reward, so that this becomes the most likely of all the whole-life-cycle investment outcomes for the investor. This choice is certainly not intended to be a restriction and it can be chosen differently by the fund manager (or even be periodically reviewed contingent on the fund's performance). The consumer can easily look into different G_L and that should be possible via an application on the financial advisor's or pension provider's website, or a slider on their smart device application. The consumers should be able to see that a lower guarantee is in tune with a unique higher upper bound and the remaining investment outcomes lie within these bounds. They should also be able to notice that the lower the lower bound they pick, the higher their most likely upper bound becomes.

In this paper, we are concerned with one aspect of the asset allocation decision, that is, the optimisation of constrained allocation strategies. We will present a framework that is able to accommodate different underlying utility functions which are increasing and concave, including, for example, the power, logarithmic and exponential[1], and multifactor model assumptions, resulting in a general structure. Our foremost theoretical result is a general expression which unravels the intimate link between the optimal constrained and unconstrained strategies by means of the probability of the terminal wealth to lie within the range $[G_L, G_U]$: we call this *probability hedging* and this is properly mathematically formulated later in the paper. There is a rich literature on the wider field, from which we will borrow some principles for our own development that incorporates constraints the way just described. Bajeux-Besnainou *et al.* (2003) acknowledge in their concluding remarks the importance of a potential study with constraints on investor behaviour such as a minimum terminal wealth when this is implied by an institutional constraint. Other authors have also considered constrained optimisation for the long-term saver including, for example, Ameur and Prigent (2018) with time-varying floors, Barucci *et al.* (2021) and Dong and Zheng (2019) with minimum guarantee constraints. In this paper, we impose double bounds under different stochastic environments and arrive at explicit – or nearly explicit in a multidimensional financial market – probability hedging expressions.

Without aiming to be exhaustive, we attempt next to hand-pick some important related works and briefly discuss them. The Merton (1969, 1971) papers are admittedly deemed classical in setting consumption and portfolio rules in continuous time under uncertainty. We have seen several other seminal contributions since then. With a focus on optimal consumption policy, Cuoco (1997) studies existence in the presence of nontraded stochastic income over finite horizon. The investment opportunities are represented by long-lived securities, including a bond and the remaining risky assets. The case of constrained dollar amounts invested in the traded assets is considered; short-sale and borrowing constraints and nontradeable assets are modelled as special cases. Cuoco (1997) shows that the conditions for existence remain similar to the unconstrained case studied originally by Cox and Huang (1989, 1991). Campbell *et al.* (2001) consider a discrete-time, time-homogeneous model where the decision variables are the consumption and the proportion invested in the risky asset. Equations for the optimal values are obtained for an infinitely lived investor with a time-varying equity premium, based on a first-order autoregressive model for the Sharpe ratio, including the constrained version with borrowing and short-sales restrictions, which can be solved numerically. In incomplete models (e.g., see Karatzas *et al.*, 1991; He and Pearson, 1991b,a;

[1] We have also investigated the case of the mean-variance utility, and more theoretical results can be made available upon request in relation to that.

Cvitanić and Karatzas, 1992; Kramkov and Schachermayer, 1999; Schachermayer, 2002), it can be shown by applying duality methods that, roughly speaking, the optimal portfolio and wealth process coincide with those in a fictitious completed market if the completion is done in the least favourable manner (see also Kallsen, 1998). In a multi-asset market, Lioui and Poncet (2001) find an optimal strategy where two non-equity assets are sufficiently used as hedging instruments and are associated with the interest rate risk and the combined spot interest rate risk and market price of risk. Bajeux-Besnainou *et al.* (2001) also optimise over the consumption and the terminal fund size. The market comprises a growth-optimal portfolio of equities, a bond and the money market account, and is complete with the risk-free rate assumed stochastic and the market price of risk for equities and the bond constant. Kamma and Pelsser (2022) optimise over the consumption and the retirement wealth, based on a different utility function for each of them, in a multidimensional financial market with usual trading constraints and non-traded assets. Several papers discuss the effect of stochastic interest rates on the optimal portfolio allocation. For example, Campbell and Viceira (2001) use numerical approximations to analyze the optimal allocation of an investor with an infinite horizon. Instead, Brennan and Xia (2000) provide analytical results for different horizons for a model with constant risk premium and volatility, while their two-factor model for interest rates captures independent long and short-term yield variations. Although the framework of Liu (2007) assumes a single-factor interest rate model, it does offer analytic allocation results under additional stochastic risk premium and volatility. Sørensen (1999) also adopts a single-factor term-structure model in optimal discrete-time portfolio selection.

It is worth pointing out that our research is not directly related to optimal consumption problems with stochastic expenses and labor income (e.g., see Cuoco, 1997, El Karoui and Jeanblanc-Picqué, 1998, Kamma and Pelsser, 2022); in fact, neither the contribution rate nor the pension rate are considered by the fund manager. We contribute in the following ways. First, we introduce probability hedging under constant risk premium. Using the martingale approach, we prove that a constrained strategy with underlying logarithmic utility makes the financial hedging a probability hedging based on unadjusted probability measure. This makes it a popular piece of advice that is intended for a wide spectrum of investors. Instead, a constrained strategy based on a general power or exponential utility function requires a translated probability measure which might be common in financial mathematics, however it loses its appeal when having to explain it to non-financially literate people who are the majority of the consumers. In those cases, any deliberately simplified financial advice by the fund manager will misrepresent the realism of the theoretical results (see also Bajeux-Besnainou *et al.*, 2003 for an important analogous conclusion).

Second, with our solutions, we are able to investigate the implications of the different utility functions. The upper and lower bounds chosen shape the risk portrayal of the financial strategy of the long-term saver. If the unconstrained strategy – used before the introduction of bounds – has a high-risk profile, the long-term saver can moderate this to the appropriate level of individual risk appetite via the bounds. This is the case with the logarithmic utility, which implies more risk than most long-term savers would like to have: under realistic market assumptions, this corresponds, as we show later in the paper, to an unconstrained portfolio comprising a close to 100% investment in the risky asset, consistently with the immediate and, perhaps, naive desire of a long-term saver as stated at the very beginning of the paper. By constraining strategies, we end up with minimal reduction in terms of certainty equivalence, and lower and upper bounds which, when compared with the initial investment, yield most-likely gains that are at least about equal to the low-chance worst-case losses; the gains increase with risk-aversion up to four times. The certainty equivalent and the bounds do not vary importantly with the choice of the utility function.

Third, following the aforementioned literature in terms of their choice of stochastic environments, we enlarge the scope of our application in a general affine model framework with stochastic risk premium and interest rates; the correlation with the wealth process does not need to be perfect, so the market is incomplete. We focus on the logarithmic utility case for the sake of illustration. As a by-product, we also develop our own simulation scheme for the derived optimally controlled constrained process, with further uses in financial engineering as this boils down to the celebrated Ornstein–Uhlenbeck driven stochastic volatility model studied in, e.g., Scott (1987), Stein and Stein (1991) and Schöbel and Zhu (1999). The major challenge in this simulation is the conditional sampling of the integrated squared process given its marginal terminal state. The proposed method works efficiently bypassing the most time-consuming element involving Fourier transform inversion in the, otherwise elegant, method of Li and Wu (2019) mainly when generating entire sample trajectories. Accounting for additional risk factors, such as a stochastic risk premium, results in a more flexible modelling of the optimal constrained strategy whose distributional characteristics vary with the level of randomness of the risk premium and are particularly magnified for risk-seeking savers.

Our preference in the logarithmic utility when designing the financial hedge of the long-term savers' risk is also corroborated by its relevance in optimal hedging theory as underlined in Merton (1973), Kraus and Litzenberger (1975), Breeden (1979) and Adler and Detemple (1988). The hyperbolic absolute risk aversion (HARA) class of utility functions, which includes the logarithmic utility, is considered also in Lioui and Poncet (2001) and Bajeux-Besnainou *et al.* (2003) in portfolio optimisation under stochastic interest rates, in Cuoco (1997) in a finite-horizon economy and El Karoui and Jeanblanc-Picqué (1998) in infinite-horizon consumption-portfolio problems, in Goll and Kallsen (2000) in maximizing the expected utility from consumption or terminal wealth in a general semimartingale market model, Marín-Solano and Navas (2010) in consumption and portfolio rules for decision-makers with time-inconsistent preferences, or in Chen and Vellekoop (2017) with allowed terminal debt.

Our new approach provides a rock-hard investment bottom while adding considerably to the median return: at least 20% for highly and sometimes moderate risk-takers and up to 10% for more risk-averse investors, depending on the utility function. How are these results possible in a tough financial environment without any free lunch? The answer to this is that we sell off extra returns in extraordinary good scenarios. We do not think that the long-term saver should gamble with security or median returns to be able to get some fantastic returns in rare scenarios. Our recommendation, therefore, to commercial and non-commercial pension providers is to adopt such a cautious approach of providing for the basic living of long-term savers. Our approach takes the gamble out of long-term saving while providing worst-case scenarios and most likely best scenarios. Our advice is that this should be applied to that part of any long-term saving that is meant to cover the necessary long-term expenses, whereas any additional savings could be then invested more freely in the stock market disregarding our more disciplined approach. Finally, we note that, whilst we present explicitly here the lump-sum case, the ideas for generalizing to annuities are delineated in Gerrard *et al.* (2018).

The rest of the paper is structured as follows. In Section 2, we present the basic complete market model framework and all the theoretical workings that lead to the formation of the probability hedging; Section 3 focuses on its empirical implementation and analysis of our results. In Section 4, we extend to a multi-stochastic environment with the aid of an auxiliary market resulting from a fictitious completion of assets. We present the formulation of the probability hedging in a general affine model setting and implement this. Section 5 concludes the paper. All our proofs of theoretical results are gathered in the online supplement, including a simulation scheme for use when carrying out our application.

2. Probability hedging: the case of constant market risk premium

We start by assuming a basic stock price model

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where the constant μ is the expected annual growth rate and σ the price volatility. $W = \{W_t : t \geq 0\}$ is the standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is an observable, unadjusted probability measure. The available information is represented by the filtration $\mathcal{F}_t = \sigma\{W_s,\ s \in [0,t]\} \vee \mathcal{N}(\mathbb{P})$, where $t \in [0,T]$, $T > 0$ and $\mathcal{N}(\mathbb{P})$ is the collection of all $\mathbb{P}$-null sets so that the filtration obeys the *usual conditions*. We will denote by $\{X_t : 0 \leq t \leq T\}$ the wealth process, based on an investment of π_t in the risky asset and the remainder in the risk-free asset, with dynamics given by

$$\begin{aligned} dX_t &= r\left(X_t - \pi_t\right) dt + \left(\mu dt + \sigma dW_t\right) \pi_t \\ &= r X_t \, dt + \sigma \pi_t \left(\theta \, dt + \, dW_t\right), \end{aligned} \tag{1}$$

where $X_0 = x_0$, r is the annual risk-free rate of return and $\theta := (\mu - r)/\sigma$ is the constant market price of risk. In this model, any update of the investor's allocation choice affects only the fund process, not the dynamics of the stock price they are trading, which makes it a popular model in a whole-life-cycle context.

Further, we introduce a generic probability measure $\mathbb{S}^{(\alpha)}$ equivalent to $\mathbb{P}$, which has Radon–Nikodým derivative

$$\left.\frac{d\mathbb{S}^{(\alpha)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} =: L_t^{(\alpha)} = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}. \tag{2}$$

By the Girsanov theorem,

$$W_t^{(\alpha)} := W_t + \alpha t$$

is a Brownian motion under $\mathbb{S}^{(\alpha)}$. The value of α will vary with the choice of the utility function as we will show later in Section 2.4. Note that $\mathbb{S}^{(0)} \equiv \mathbb{P}$ and $\mathbb{S}^{(\theta)} \equiv \mathbb{Q}$, where the latter is the risk neutral measure.

2.1. Optimising unconstrained and constrained wealth processes

The long-term saver with the aid of the fund manager seeks to maximize

$$\mathbb{E}^{\mathbb{P}}\left[U(X_T)\right] \text{ subject to } \mathbb{E}^{\mathbb{Q}}\left[X_T\right] = e^{rT} x_0, \tag{3}$$

where U denotes a utility function which is assumed to be increasing and concave. Problem (3) can be approached by the Lagrangian method (see Björk, 2009). More specifically, for each $\omega \in \Omega$, we choose $X_T(\omega)$ to be the value of x which maximizes

$$U(x) + \lambda_0 \left(e^{rT} x_0 - L_T^{(\theta)}(\omega) x \right), \tag{4}$$

where λ_0 is the Lagrange multiplier, also known as the shadow price, to be determined later, and $L_T^{(\theta)}$ is given by (2). The derivative of (4) is

$$U'(x) - \lambda_0 L_T^{(\theta)}(\omega),$$

which is a decreasing function of x.

If there are no constraints on the value X_T can take, the saver's optimal terminal wealth is

$$X_T = X_T^* = \Upsilon \left(\lambda_0 L_T^{(\theta)} \right), \tag{5}$$

where Υ is the inverse of U' and is a decreasing function. The unconstrained process $\{X_t^* : 0 \leq t \leq T\}$ reflects the investment options pursued by the fund manager and are driven by a belief in a certain utility which may agree or not with the actual utility of the consumer. This is given at each time t by

$$X_t^* = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[X_T^* \right], \tag{6}$$

where the expected value is taken conditional on $\mathcal{F}_t$. In addition, we define the constrained process $\tilde{X}$ which reflects the behaviour of X subject to the restriction $\mathbb{P}\left(G_L \leq X_T \leq G_U \right) = 1$. The optimal terminal value of the consumer's portfolio of assets then becomes

$$\begin{aligned} X_T = \tilde{X}_T &= \min\left(G_U, \max\left(G_L, X_T^* \right) \right) \\ &= \left(G_L - X_T^* \right)^+ - \left(X_T^* - G_U \right)^+ + X_T^*, \end{aligned} \tag{7}$$

where $z^+ := \max(z, 0)$. The trajectory of the constrained process $\left\{ \tilde{X}_t : 0 \leq t \leq T \right\}$ is given by

$$\tilde{X}_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[\tilde{X}_T \right], \tag{8}$$

which satisfies the budget constraint $\tilde{X}_0 = x_0$. The relationship between x_0 and x_0^* can be written in the form

$$x_0 = x_0^* + e^{-rT} \mathbb{E}_0^{\mathbb{Q}} \left[\left(G_L - X_T^* \right)^+ \right] - e^{-rT} \mathbb{E}_0^{\mathbb{Q}} \left[\left(X_T^* - G_U \right)^+ \right], \tag{9}$$

which implies that x_0^* is, in general, *not* equal to x_0. From (9), the difference $x_0 - x_0^*$ actually corresponds to the cost of buying a vanilla put option with strike price G_L less the proceeds from the sale of a vanilla call option with strike G_U.

2.2. Dynamical behaviour of optimal unconstrained and constrained processes

In this section, we study the behaviour of the unconstrained and constrained processes X^* and $\tilde{X}$, respectively. We do this in Proposition 2, but first we present a key result in Proposition 1. Having proved the results under general utility assumptions made by the fund manager, we then exemplify cases corresponding to specific utility functions.

Proposition 1. *Let $F^{(\alpha)}(x,t)$ be the conditional distribution function of X_T^* under the generic measure $\mathbb{S}^{(\alpha)}$ defined in (2) given $\mathcal{F}_t$. Then,*

$$F^{(\alpha)}(x,t) := \Phi\left(\mathcal{K}\left(x,t;W_t,\lambda\right) + \alpha\sqrt{T-t}\right), \tag{10}$$

where $\Phi\left(.\right) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{.} e^{-z^2/2}dz$,

$$\mathcal{K}\left(x,t;W_t,\lambda\right) := \frac{\ln\lambda - \ln U'(x) - \theta W_t}{\theta\sqrt{T-t}}$$

and

$$\lambda := \lambda_0 e^{-\frac{1}{2}\theta^2 T}.$$

Proof. See Section EC.2 of the e-companion of the paper. ■

Proposition 2. *i) The value at time t of a fund invested in the unconstrained process is given by*

$$X_t^* = e^{-r(T-t)}\int_{-\infty}^{\infty}\Upsilon\left(\lambda\mathcal{H}\left(x,t;W_t\right)\right)\phi(x)\,dx, \tag{11}$$

where $\phi(z) := \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ *and*

$$\mathcal{H}\left(x,t;W_t\right) := \exp\left(-\theta\sqrt{T-t}x - \theta W_t + \theta^2(T-t)\right). \tag{12}$$

ii) The optimal constrained process is given by

$$\tilde{X}_t = e^{-r(T-t)}\left(G_L + \int_{G_L}^{G_U}\left[1 - F^{(\theta)}(x,t)\right]dx\right), \tag{13}$$

where $F^{(\theta)}$ is given in (10).

Proof. See Section EC.2 of the e-companion of the paper. ■

Based on the beliefs of the fund manager for the driving utility, the dynamics of X^* varies according to (11). The utility functions considered in this paper are the power, exponential and logarithmic. Table 1 summarizes various quantities of interest in our study related to different utility functions. For the family of power utility functions, we introduce, for convenience, also the notation $\eta := \gamma/(1-\gamma)$ to distinguish the special case of the logarithmic utility for $\eta = 0$ from the general case of power utilities for $\eta > -1$. We shall see in Section 2.4 that there are circumstances in which the limiting behaviour as $\eta \to -1$ is that under the exponential utility.

Utility	$U(x)$	Parameter	$\Upsilon(a)$	$\Upsilon'(a)$
Power	$\gamma^{-1}x^{\gamma}, \quad x > 0$	$\gamma \in (-\infty, 1)\setminus\{0\}$	$a^{-(1+\eta)}$	$-(1+\eta)a^{-(2+\eta)}$
Logarithmic	$\ln x, \quad x > 0$	–	a^{-1}	$-a^{-2}$
Exponential	$-\xi^{-1}e^{-\xi x}, \quad x \in \mathbb{R}$	$\xi > 0$	$-\xi^{-1}\ln a$	$-\xi^{-1}a^{-1}$

Table 1: Fund manager's utility function choices. For the power utility, $\eta = \gamma/(1-\gamma)$.

If the optimiser uses an exponential utility, then based on the information in Table 1 we get from (11) that

$$X_t^* = \frac{e^{-r(T-t)}\left(-\ln\lambda + \theta W_t - \theta^2(T-t)\right)}{\xi} = x_0^* e^{rt} + \frac{e^{-r(T-t)}\left(\theta W_t + \theta^2 t\right)}{\xi}, \tag{14}$$

from which

$$dX_t^* = rX_t^*\,dt + \frac{\theta}{\xi}e^{-r(T-t)}\left(dW_t + \theta\,dt\right). \tag{15}$$

Similarly, for the power utility,

$$\begin{aligned} X_t^* &= \frac{e^{-r(T-t)}}{\lambda^{\eta+1}}\exp\left(\frac{\theta W_t - \theta^2(T-t)}{(\eta+1)^{-1}} + \frac{\theta^2(T-t)}{2(\eta+1)^{-2}}\right) \\ &= x_0^*\exp\left(rt + \frac{\theta^2(1-\eta^2)t}{2} + \theta(\eta+1)W_t\right) \end{aligned} \tag{16}$$

and

$$dX_t^* = rX_t^*\,dt + \theta(\eta+1)X_t^*\left(dW_t + \theta\,dt\right). \tag{17}$$

The special case of the logarithmic utility follows from (16)–(17) by setting $\eta = 0$.

2.3. Choosing the shadow price, the lower and the upper bound

Based on the distributional properties of the processes X^* and $\tilde{X}$ we have established in the previous section, we obtain next the optimal shadow price λ and the lower bound G_L for a given upper bound G_U.

As we have discussed in the introduction, we aim for G_U to be achieved half of the time, i.e., to be the most likely of all the investment outcomes; this is not intended to be a limitation and a different choice can be made by the fund manager. Therefore, we choose G_U to be the median such that

$$\mathbb{P}\left(\tilde{X}_T \geq G_U\right) = \mathbb{P}\left(X_T^* \geq G_U\right) = \frac{1}{2}. \tag{18}$$

From (5) and (18), $G_U = \Upsilon(\lambda)$ or, equivalently,

$$\lambda = U'(G_U).$$

In addition, from (14), for the exponential utility, $x_0^* e^{rT} = G_U - \theta^2 T/\xi$ and from (16), for the power utility, $x_0^* e^{rT} = G_U \exp\left(-\theta^2(1-\eta^2)T/2\right)$.

Given the budget constraint $\tilde{X}_0 = x_0$, we have from (13) that

$$x_0 e^{rT} = G_L + \int_{G_L}^{G_U}\left[1 - \Phi\left(\frac{\ln\lambda - \ln U'(x) + \theta^2 T}{\theta\sqrt{T}}\right)\right]dx \tag{19}$$

as long as $G_L \leq x_0 e^{rT} \leq G_U$ by definition of $\tilde{X}$.

Proposition 3. *Let G_U satisfy (18). Then, G_L exists such that the generic equation (19) is satisfied if*

$$G_U - x_0 e^{rT} \leq \int_l^{G_U}\Phi\left(\frac{\ln U'(G_U) - \ln U'(x) + \theta^2 T}{\theta\sqrt{T}}\right)dx, \tag{20}$$

where l is the minimal value that x can take[2].

Proof. The result follows from (19). ■

[2]For example, $l = -\infty$ for the exponential utility and $l = 0$ for the power utility, according to the support of the distribution of the resulting X_T^*.

So far, we have two equations relating the three unknowns λ, G_L and G_U. There is one remaining degree of freedom and we exercise it by choosing the best value from the point of view of maximizing the expected terminal utility of the consumer. We will assume, without this being restrictive, that our consumers assess wealth according to a power-law utility with parameter ρ. This is practically unknown but it is a convenient assumption due to the flexibility that it offers in explicitly accounting for different levels of risk-aversion, although our machinery can be rebuilt on a different utility basis and more results can be made available for alternative cases such as for an exponential utility. Therefore, we will choose the maximizing G_L of the expected utility

$$\frac{\mathbb{E}^{\mathbb{P}}\left[\tilde{X}_T^{\rho}\right]}{\rho}, \tag{21}$$

where $\rho < 1$.

The process $\tilde{X}$ depends on the optimiser's belief about the utility and that may differ from the consumer's. Depending on the optimiser's use of a certain utility function (see Table 1), the upper bound for G_U can be explicitly derived. For example, for the exponential utility, we have from (18) and (14) that

$$G_U = -\xi^{-1}\ln\lambda,$$

and (19) then needs to be satisfied:

$$G_U - x_0e^{rT} = \int_{G_L}^{G_U}\Phi\left(\frac{-\xi(G_U - x)}{\theta\sqrt{T}} + \theta\sqrt{T}\right)dx = \frac{\theta\sqrt{T}}{\xi}\int_{\theta\sqrt{T}-\frac{\xi\left(G_U - G_L\right)}{\theta\sqrt{T}}}^{\theta\sqrt{T}}\Phi\left(x\right)dx.$$

For a solution to exist, we have to have from (20) that

$$G_U - x_0e^{rT} \leq \frac{\theta\sqrt{T}}{\xi}\int_{-\infty}^{\theta\sqrt{T}}\Phi\left(x\right)dx = \frac{\theta^2T\Phi(\theta\sqrt{T}) + \theta\sqrt{T}\phi(\theta\sqrt{T})}{\xi}.$$

Therefore, if $x_0e^{rT} \leq G_U \leq x_0e^{rT} + \xi^{-1}\left(\theta^2T\Phi(\theta\sqrt{T}) + \theta\sqrt{T}\phi(\theta\sqrt{T})\right)$, then it is possible to find a valid value for $G_L > -\infty$. In addition, from (21)

$$\frac{\mathbb{E}^{\mathbb{P}}\left[\tilde{X}_T^{\rho}\right]}{\rho} = \frac{G_U^{\rho}}{\rho} - \frac{\theta\sqrt{T}}{\xi}\int_{-\frac{\xi\left(G_U - G_L\right)}{\theta\sqrt{T}}}^{0}\left(G_U + \frac{\theta\sqrt{T}}{\xi}x\right)^{-1+\rho}\Phi\left(x\right)dx,$$

where $G_L > 0$ ensures that the integral above does not become infinite.

Based on similar arguments, for the power utility, we require from (18) and (16) that

$$G_U = \lambda^{-(1+\eta)},$$

and

$$G_U - x_0e^{rT} = \int_{G_L}^{G_U}\Phi\left(-\frac{1+\eta}{\theta\sqrt{T}}\ln\frac{G_U}{x} + \theta\sqrt{T}\right)dx \tag{22}$$

must be satisfied[3]. Since G_L must be non-negative, a solution exists if

$$\begin{aligned} G_U - x_0 e^{rT} &\leq \int_0^{G_U} \Phi\left(-\frac{1+\eta}{\theta\sqrt{T}}\ln\frac{G_U}{x} + \theta\sqrt{T}\right) dx \\ &= G_U\left[\Phi\left(\theta\sqrt{T}\right) - \exp\left(-\frac{(1+2\eta)\theta^2 T}{2(1+\eta)^2}\right)\Phi\left(\frac{\eta\theta\sqrt{T}}{1+\eta}\right)\right]. \end{aligned}$$

A possible value for G_L is, therefore, implied when

$$x_0 e^{rT} \leq G_U \leq \frac{x_0 e^{rT}}{1 - \Phi(\theta\sqrt{T}) + \exp\left(-\frac{1+2\eta}{2(1+\eta)^2}\theta^2 T\right)\Phi\left(\frac{\eta\theta\sqrt{T}}{1+\eta}\right)}.$$

The best value of G_L is the one which maximizes (21). The special case of a fund manager believing in the logarithmic utility function follows by setting $\eta = 0$.

Note that if the upper bound condition for G_U specified in Proposition 3 is not satisfied, it will not be possible to find a strategy which has G_U as the median terminal value. However, we will still be able to choose G_U differently and, subsequently, $G_L < x_0 e^{rT}$, with λ given by the general expression (19) which satisfies the budget constraint.

2.4. Optimal investment strategies and probability hedging

In the last part of this section, we assemble all the theoretical results we have developed so far to arrive finally at a neat result which lays the cornerstone for the contribution of this paper. More specifically, both X^* and $\tilde{X}$, as we have seen in the previous sections, are admissible portfolio processes, so can be generated from (1) using an appropriate asset allocation strategy; these strategies are denoted by $\{\pi_t^* : 0 \leq t \leq T\}$ and $\{\tilde{\pi}_t : 0 \leq t \leq T\}$, respectively. We show here that the optimal control for the constrained process for universal underlying utility functions can be derived from the optimal control for the unconstrained process. The ratio of the two optimal strategies is given by the probability, with respect to a suitable measure and conditional on $\mathcal{F}_t$, of X_T^* lying in the interval $[G_L, G_U]$ obtained in Section 2.3. We use a newly coined financial hedging term to describe this phenomenon and that is probability hedging.

Theorem 4. *Let U be a utility function governing the decisions of the fund manager. Then, the associated optimal asset allocation strategy at time t is given by*

$$\tilde{\pi}_t := \frac{1}{\sigma} e^{-r(T-t)} \mathcal{I}(t; W_t, \lambda), \tag{23}$$

where

$$\mathcal{I}(t; W_t, \lambda) := \theta \int_{\mathcal{K}(G_L,t;W_t,\lambda)+\theta\sqrt{T-t}}^{\mathcal{K}(G_U,t;W_t,\lambda)+\theta\sqrt{T-t}} \frac{\phi(x)}{A\left(\Upsilon\left(\lambda\mathcal{H}\left(x,t;W_t\right)\right)\right)} dx, \tag{24}$$

[3] The integral on the right-hand side of equation (22) has an explicit solution given via

$$\begin{aligned} \int_{G_L}^{G_U} \Phi\left(\alpha \ln z + \beta\right) dz &= G_U \Phi\left(\alpha \ln G_U + \beta\right) - G_L \Phi\left(\alpha \ln G_L + \beta\right) \\ &\quad - \exp\left(\frac{-\beta}{\alpha} + \frac{1}{2\alpha^2}\right)\left[\Phi\left(\alpha \ln G_U + \beta - \frac{1}{\alpha}\right) - \Phi\left(\alpha \ln G_L + \beta - \frac{1}{\alpha}\right)\right]. \end{aligned}$$

$\mathcal{H}$ is given by (12), and

$$A(x) := -\frac{U''(x)}{U'(x)}$$

is the coefficient of absolute risk-aversion associated with U.

Proof. See Section EC.2 of the e-companion of the paper. ∎

Based on the general result presented in Theorem 4, we obtain next the exact probability hedge corresponding to different cases of utility functions.

Corollary 5. *Under general assumptions for the optimiser's choice of utility function and for constant market price of risk θ, the optimal constrained strategy is given by*

$$\tilde{\pi}_t = \pi_t^* \mathbb{S}_t^{(-\eta\theta)} \left(X_T^* \in [G_L, G_U] \right),$$

where the probability measure $\mathbb{S}_t^{(-\eta\theta)}$ satisfies (2) (see also Proposition 1). In addition, the unconstrained optimal strategy and measure are given by

$$\begin{cases} \pi_t^* = \frac{\theta e^{-r(T-t)}}{\xi\sigma}, & \eta = -1 \text{ (exponential)} \\ \pi_t^* = \frac{\theta(\eta+1)}{\sigma} X_t^* = \frac{\theta e^{\frac{\theta W_t - \theta^2(T-t)}{(\eta+1)^{-1}} + \frac{\theta^2(T-t)}{2(\eta+1)^{-2}} - r(T-t)}}{\sigma(\eta+1)^{-1}\lambda^{\eta+1}}, & \eta = \frac{\gamma}{1-\gamma} \text{ (power)} \\ \pi_t^* = \frac{\theta}{\sigma} X_t^* = \frac{\theta e^{\theta W_t - \left(r + \frac{1}{2}\theta^2\right)(T-t)}}{\lambda\sigma}, & \eta = 0 \text{ (logarithmic)} \end{cases} . \tag{25}$$

Proof. See Section EC.2 of the e-companion of the paper. ∎

What is remarkable from Corollary 5 is that, for an underlying logarithmic utility, the probability hedging for a long-term investor does not require departing from the physical probability measure, rendering the notion of probability hedging with the lower and upper bounds more intuitive. The exponential utility, instead, requires an adjustment to the risk neutral measure, whereas a new change of measure is required in the case of the power-law utility.

From (25), it is obvious that the family of power utility functions, including the logarithmic special case, yields constant relative amounts of wealth invested in a risky asset in the optimal unconstrained strategies. On the other hand, the exponential utility gives rise to a deterministic nominal amount invested in the risky asset, therefore, in this case, the only relevant relationship is that between $\tilde{\pi}$ and π^* and we know from Corollary 5 that $\tilde{\pi} \leq \pi^*$ as they are connected by means of a probability. Our last theoretical result of this section is devoted to the trickier power family case and shows that the risky investment proportion in the optimal constrained strategy is bounded from above by the unconstrained.

Corollary 6. *The proportion of wealth invested in a risky asset in the optimal constrained and unconstrained strategies, when the optimiser assumes a utility function from the power family, satisfies*

$$\frac{\tilde{\pi}_t}{\tilde{X}_t} < \frac{\pi_t^*}{X_t^*}.$$

Proof. See Section EC.2 of the e-companion of the paper. ∎

3. Implementing the probability hedging

As we concluded in Section 2.4, the distinguishing feature of the different hedged strategies based on different utilities is the relevant probability measure. But this is a principal aspect that drives the quality of communication with pension savers; having to explain, from a fund manager's perspective, and understand, from an investor's side, probability measure changes can obscure communication and, thus, our ultimate goal. The solution is given by the logarithmic utility, which leads to a probability hedge under the physical measure. This brings with it additional flexibility. Imagine an investment strategy involving a capital placement entirely in stocks with an optimistic vision of receiving a high return at retirement. This sounds like a gamble except that the investor is reluctant to lose a significant proportion of their invested capital, especially in such a case of long-term planning.

From equation (25) in Corollary 5, the unconstrained optimal strategies under power utility are given by $(\mu - r)/(\sigma^2(1-\gamma))$, from which it is obvious how the percentage investment in stocks varies with the risk appetite. We consider the cases of $\gamma = -0.25$, -1, -4 and -10, in addition to $\gamma = 0$ corresponding to the logarithmic utility. Based on the empirical analysis of Kyriakou et al. (2020, 2021b), we adopt a yearly mean excess return of 2.5% and standard deviation of 16% and obtain the optimal investment strategies reported in Table 2. We observe that under the logarithmic utility an implied 98% of capital invested in stocks almost matches the investment portrayal we are after. While the logarithmic utility assumption sounds to be workable, we need to explore the cost from transitioning from an unconstrained hypothetical power strategy to a strategy with restrained terminal reward based on different utility assumptions made by the fund manager. In other words, we need to study the effect of different mismatched hedges, even if intended in order to accommodate the limited financial training of the consumers.

Risk appetite γ	0	-0.25	-1	-4	-10
Risky asset investment	97.7%	78.1%	48.8%	19.5%	8.9%

Table 2: Optimal unconstrained strategies under the power utility: proportion of wealth invested in risky asset $\pi^*/X^* = (\mu - r)/(\sigma^2(1-\gamma))$, $\mu - r = 2.5\%$ and $\sigma = 16\%$.

3.1. Unconstrained versus constrained strategies and the mismatched hedge effect

To this end, we revisit the four long-term investors from Gerrard *et al.* (2019), Lisa, John, Susan and James, with power utility parameter values $\rho = -0.25$, -1, -4 and -10, respectively, corresponding to a pattern of increasing risk-aversion. Each of them invests a total of $x_0 = 10{,}000$ in stocks and a risk-free inflation bond with an investment horizon of 30 years. We consider first their optimal strategies in an unhedged power utility world. On the left panel of Table 3, we report for varying ρ the certainty equivalent (CE), that is, the certain amount of money they would trade off against their uncertain terminal reward X_T^* given by

$$CE = \mathbb{E}^{\mathbb{P}}\left[X_T^{*\rho}\right]^{\frac{1}{\rho}}.$$

In addition, we report the median of X_T^* and, as a lower bound is not properly defined for the unconstrained strategy, the Value-at-Risk (VaR) for a given probability (e.g., 5% VaR). By analogy, we then compute for different utility hedges for each of our consumers, based on the procedure delineated in Section 2.3, for the constrained terminal reward $\tilde{X}_T$ the

$$CE = \left(\rho \max_{G_L} \frac{\mathbb{E}^{\mathbb{P}}\left[\tilde{X}_T^{\rho}\right]}{\rho}\right)^{\frac{1}{\rho}}, \tag{26}$$

as well as the upper bound G_U, given by the median of $\tilde{X}_T$, and the lower bound G_L. The different utility hedges are reflected by the process $\tilde{X}$. The calculations are reported on the right panel of Table 3, whereas an example of the outcome from the maximization in (26) is shown in Figure EC.1 in the paper's e-companion.

The results are most enlightening. First, as expected, we observe a reduction in the CE compared to the unconstrained strategy. However, this is overall fairly small and largest for a risk-lover such as Lisa with $\rho = -0.25$: 6.6%, 5.5% and 7.0%, respectively, under the constrained power, logarithmic and exponential (parameter $\xi := \theta/(\sigma x_0)$) strategy. This drops with increasing risk-aversion to reach, respectively, 0.5%, 1.4% and 1.3% when $\rho = -10$ (James' case). The second important observation is that, for a given level of risk-aversion, the CE amongst the different utility hedges does not vary much. This suggests that the level of risk-aversion plays some role when we change from an unconstrained to a constrained strategy, while the actual kind of utility hedge a far lesser role.

Furthermore, for consistency between the unconstrained and constrained strategies, we compare the median of the terminal rewards and find perceptible increases, such as 20% (approx.) for the exponential and power hedges and up to 30% for the logarithmic hedge in the case of a risk-taker like Lisa. Reflecting Lisa's risk appetite, her G_L guarantee reduces, sometimes even below the 5% VaR of the unhedged position, whilst her most likely reward G_U increases, resulting in the largest $(G_U - G_L)$ spread. Both effects generally weaken with increasing risk-aversion, still the median remains fairly high and the range $[G_L, G_U]$ narrows mostly for the exponential and logarithmic hedges.

We conclude this section with some favourable news about the logarithmic utility. Contrary to the power utility for $\gamma = -0.25, -1, -4$ and -10 which implies unconstrained investments of wealth in the risky asset up to 78% as shown in Table 2, the logarithmic utility ($\gamma = 0$) amounts to an investment of almost 100%. By constraining this strategy, we achieve a minimal CE reduction for a relatively high lower guarantee and a high best-case outcome. Therefore, a risky investment like this can lead, by means of our proposed hedged strategy, to favourable investment opportunities even for less risky investors such as Susan ($\rho = -4$) and James ($\rho = -10$). Similar hedging performances are reported for the other utility functions.

We have seen that the loss in terms of CE can be up to 7% against the unconstrained power strategy for a risk-taker like Lisa, but this is considerably less than the loss from potential consumer-advisor miscommunication and consequent inaccurate assessment of the risk preference and the intended saving by retirement, as we see next.

3.2. Constrained strategies and the misspecified risk profile effect

Misspecification of the risk profile of the investor often manifests itself in the interview with the financial advisor and can cause a significant loss. This can be in the form of a mismatched fund manager's decision driven by a belief in a power-law utility with parameter γ and consumer's own assessment of the worth of an outcome according to a power-law utility with parameter ρ. The consumer has no choice but to accept the decisions of the fund manager. Therefore, the process $\tilde{X}$ in equation (26) will reflect the investment options pursued by the fund manager. We focus on this case on the left panel of Table 4 where we take account of the CE of each investor, if their risk selection were mistaken. The worst possible loss of 16.7% is induced when the risk-taker Lisa is wrongly assessed to be the risk-averse James (italicized entry). When the moderately risk-taker

Strategy	Risk appetite	CE	5% VaR	Median	Strategy	CE	G_L	G_U	$\%CE$ reduction	$G_U - G_L$	G_L vs. uncon. VaR	G_U vs. uncon. median
Unconstrained power	$\rho = -0.25$	13,398	4,656	14,191	Constrained power ($\gamma = \rho$)	12,509	1,600	16,788	6.6%	15,188	66%	18%
	$\rho = -1$	12,010	6,552	13,148		11,589	3,400	14,249	3.5%	10,849	48%	8%
	$\rho = -4$	10,763	8,632	11,405		10,639	5,436	11,679	1.2%	6,243	37%	2%
	$\rho = -10$	10,340	9,387	10,654		10,290	7,600	10,757	0.5%	3,157	19%	1%
	$\rho = -0.25$	13,398	4,656	14,191	Constrained logarithmic	12,666	1,554	18,411	5.5%	16,857	67%	30%
	$\rho = -1$	12,010	6,552	13,148		11,547	6,049	16,827	3.9%	10,778	8%	28%
	$\rho = -4$	10,763	8,632	11,405		10,491	9,081	12,717	2.5%	3,636	-5%	12%
	$\rho = -10$	10,340	9,387	10,654		10,198	9,690	11,108	1.4%	1,418	-3%	4%
	$\rho = -0.25$	13,398	4,656	14,191	Constrained exponential ($\xi = \theta/(\sigma x_0)$)	12,463	3,458	17,236	7.0%	13,778	26%	21%
	$\rho = -1$	12,010	6,552	13,148		11,560	6,125	16,028	3.7%	9,903	7%	22%
	$\rho = -4$	10,763	8,632	11,405		10,501	9,043	12,715	2.4%	3,671	-5%	11%
	$\rho = -10$	10,340	9,387	10,654		10,200	9,677	11,144	1.3%	1,467	-3%	5%

Table 3: Unconstrained power strategy versus probability hedges under power, logarithmic and exponential utility functions. We assume that our consumers assess wealth according to a power-law utility with parameter ρ. The left panel reports the certainty equivalent (CE), median and 5% Value-at-Risk (VaR) of X_T^* for varying power utility parameter ρ. The right panel focuses on different constrained strategies where the process $\tilde{X}$ depends on the optimiser's belief about the utility which may match the consumer's (power with parameter $\gamma = \rho$) or differ (logarithmic or exponential). We report the CE, the lower and upper bounds, G_L and G_U, obtained as explained in Section 2.3, their spread $(G_U - G_L)$, the $\%CE$ reductions for each hedged strategy with respect to the unconstrained strategy, the "G_L vs. uncon. VaR", i.e., the reductions (increases if negative sign) of G_L with respect to the 5% VaR, and the "G_U vs. uncon. median", i.e., the increases of G_U with respect to the median of the terminal reward. Other parameter values: $\mu - r = 2.5\%$, $r = 0$, $\sigma = 16\%$, $\theta = (\mu - r)/\sigma$, $x_0 = 10{,}000$, $T = 30$ years.

John is given James' plan, the loss reduces at most to 10.1%, whereas when the more risk-averse Susan's and James' plans are misspecified the worst percentage loss reduces further. However, in James' case this leads to a loss of 1% of his initial capital investment of 10,000. Contrary to an unconstrained strategy where misspecification can have a detrimental effect on the CE against the initial capital (e.g., see Gerrard *et al.*, 2019), the good news here is that a hedged strategy can bring this substantially down. Naturally, if the risk profile assessments are correctly matched, the CE restores to its maximum level (boldfaced entries).

Alternatively, risk profile misspecification can arise when the fund manager's decision is driven by a belief in a logarithmic or exponential utility, contrary to the consumer's power-law utility risk profile. This is studied on the right panel of Table 4. Here, in support of both the logarithmic and exponential hedges, the reduction in the CE from the correctly matched power hedges is very small (on average 1%). In fact, in Lisa's case we instead observe an increase by 1.3% for the logarithmic hedge. The upper (lower) bounds of the logarithmic and exponential hedges also increase above those of the power hedges especially for risky (risk-averse) consumers. The differences between the two hedges are very small.

3.3. A rock-hard bottom while increasing the median return

A major advantage of our method is the replacement of the VaR with a more solid downside, while the median return is significantly improved. These two important outputs are paid for by selling off the upside in a few cases where returns would be higher in an unhedged strategy than the inflated median provided by our proposed hedged strategy.

We focus the spotlight on our preferred constrained logarithmic strategy in Table 3. The most risk-averse long-term saver ($\rho = -10$) is most likely to end up with a long-term reward of $G_U = 11{,}108$, that is 1,108 in excess of the 10,000 initial investment, from risking just about $10{,}000 - G_L = 300$ at worst, which is actually 300 less than what is put at risk in the unconstrained strategy according to the VaR measure. The outcome of 1,108 is also almost four times the money at risk and we are in a situation of a much higher probability of a gain than of a loss (recall that G_U corresponds to the median terminal reward); any gambler would be excited by seeing these odds! We are, nevertheless, following the risk-and-return laws of arbitrage-free long-term finance. Our second, in the risk-aversion ranking, long-term saver ($\rho = -4$) is interested in a slightly higher gain given the beneficial circumstances. The quadruple of gains is beyond reach in this case, but the pot for this favourable game is bigger. More specifically, this long-term saver decides to risk 919 for a more likely gain of 2,717, which is about three times the amount at risk. The two most risk-seeking long-term savers arrive at most likely gains of, respectively, 8,411 and 6,827, with corresponding amounts at risk of 3,951 and 8,446. While all four of them are receiving a favourable game, it is clear that the more risk-averse savers benefit relatively more from their risky exposure: the most likely gains are roughly four, three, two, one times the worst-case loss for our four investors. The advantage of our new approach is that they can self-select the participation in the risky but favourable game provided to them. We also think that our methodology with a most likely high gain for an unlikely worst case, that is still better than a very uncertain bottomless approach, is what most investors would be looking for.

3.4. Analysis of simulated optimal investment strategies

In what follows, we present the detailed pathwise properties of our hedging methodology varying with utility function and risk preference. There are clear differences between these paths for varying

Risk appetite	Constrained power CE				Power-power	Constrained	Logarithmic-	Constrained	Exponential-
	$\gamma = -0.25$	$\gamma = -1$	$\gamma = -4$	$\gamma = -10$	$\%CE$	logarithmic CE	power $\%CE$	exponential CE	power $\%CE$
$\rho = -0.25$	**12,509**	11,916	10,894	*10,425*	-16.7%	12,666	1.3%	12,463	-0.4%
$\rho = -1$	11,497	**11,589**	10,851	*10,417*	-10.1%	11,547	-0.4%	11,560	-0.3%
$\rho = -4$	*10,334*	10,355	**10,639**	10,381	-2.9%	10,491	-1.4%	10,501	-1.3%
$\rho = -10$	10,152	10,124	*9,935*	**10,290**	-3.4%	10,198	-0.9%	10,200	-0.9%

Table 4: Constrained strategies and the misspecified risk profile effect. We assume that our consumers assess wealth according to a power-law utility with parameter ρ. The "constrained CE" entries correspond to the certainty equivalent (CE) of the constrained strategies where the process $\tilde{X}$ depends on the optimiser's belief about the utility which may match the consumer's (power with parameter $\gamma = \rho$) or differ (logarithmic or exponential). The boldfaced entries correspond to correctly matched power utility profiles; the italicized entries correspond to the worst (lowest CE) constrained power strategy. Power-power $\%CE$ changes are from the correctly specified (best-performing) power strategy ($\rho = \gamma$) to the worst-performing. Logarithmic-power $\%CE$ changes are from the correctly specified (best-performing) power strategy to the logarithmic strategy. Similarly, exponential-power $\%CE$ changes are from the correctly specified (best-performing) power strategy to the exponential (parameter $\xi = \theta/(\sigma x_0)$) strategy. Other parameter values: $\mu - r = 2.5\%$, $r = 0$, $\sigma = 16\%$, $\theta = (\mu - r)/\sigma$, $x_0 = 10{,}000$, $T = 30$ years.

risk preferences, whereas the important message that the utility function matters less remains, especially in later years.

More specifically, we study the asset allocation strategies we have derived in Section 2.4 for the logarithmic and exponential utilities. We simulate the dynamics of the ratio $\tilde{\pi}/\pi^*$ of the investment in the risky asset under the constrained and unconstrained strategies, i.e., of the probability that the unconstrained wealth at the end of the 30-year cycle will be within the lower and upper bounds under the appropriate measure (see Corollary 5). In addition, we inspect the distributional characteristics of the proportion of wealth invested in the risky asset.

We consider first the case of a driving logarithmic utility. The left side of Figure 1 exhibits the evolution of the distribution of the ratio $\tilde{\pi}/\pi^*$. For the risky investor ($\rho = -0.25$) this ratio averages around half every year, while its standard deviation increases year on year up to half and the interquartile range is one. As the probability of the unconstrained wealth to exceed the upper bound is, by definition of the median wealth, set equal to 50% and the mean probability to lie within the bounds is also 50%, it is therefore implied that the probability to drop below the lower bound is on average zero (as for the constrained wealth). This is sensible given the fairly wide lower-upper bound spread of a risk-taker (see also Table 3). The distribution has small negative skewness most of the time, whereas closer to the terminal time this becomes positive; it generally also exhibits light tails. This quite changes when additional risk factors are accounted for, such as a random market price of risk, as we will see later in Section 4.3.

With increasing levels of risk-aversion, the relative optimal investment $\tilde{\pi}/\pi^*$ averages at gradually decreasing levels of a $1/2$, $1/3$, $1/6$ and $1/16$ for $\rho = -0.25$, -1, -4 and -10, respectively. This is connected with the increasing chance of the unconstrained wealth to fall below the lower bound while the two bounds get close to each other (see Table 3) and the decreasing desire to allocate wealth to the risky asset. The dispersion increases over time for all consumers but remains larger for the riskier. The skewness is negative most of the time but in the later years it becomes positive, especially for the more risk-averse consumers who also exhibit some positive excess kurtosis.

When we switch to the exponential utility (see the left side of Figure 2), the differences, in general, with the logarithmic utility in the distributional properties of $\tilde{\pi}/\pi^*$ are more obvious for risky investors and especially up to, approximately, the twentieth year. More specifically, during the first 20 years, the exponential case exhibits a slightly smaller standard deviation and larger mean, as well as a significant negative skewness and a fat tail for risky and moderately risky investors. Over the last 10 years, the equality of the two distributions improves perceptibly and this is confirmed by the outcome of a two-sample Kolmogorov–Smirnov test with a p-value of at least 30%, depending on the level of risk-aversion, at the terminal time.

Finally, on the right sides of Figures 1 and 2, we focus on the percentage investment in the risky asset in the constrained strategies, $\tilde{\pi}/\tilde{X}$, for both utilities. For the logarithmic utility, $\tilde{\pi}/\tilde{X}$ lies, on average, within (approx.) 62% to 72%, 44% to 59%, 13% to 21%, and 5% to 9%, respectively, for $\rho = -0.25$, -1, -4 and -10. For the exponential utility, the relevant ranges are 42% to 56%, 33% to 41%, 14% to 16%, and 5% to 6.5%. The variability increases over time, but shifts downwards with increasing risk-aversion; this is slightly larger for the logarithmic utility. In addition, the right skewness of the percentage risky asset investment is more noticeable for risk-averse consumers and increases over time; the skewness is larger for the logarithmic utility. Similar patterns apply for the positive excess kurtosis, but again are more inconspicuous for the exponential utility.

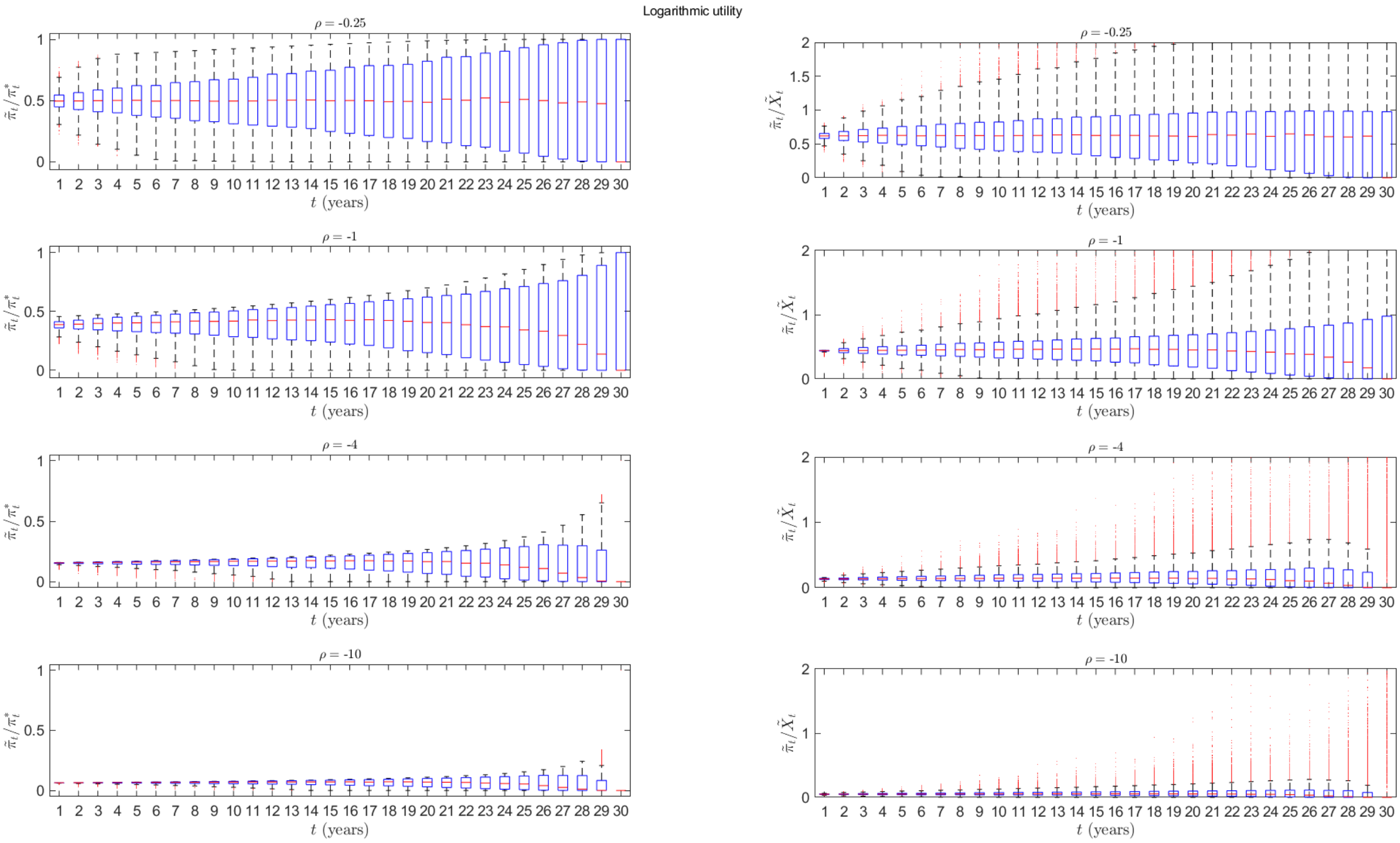

Figure 1: *Left-hand side.* Probability hedge distribution evolution, $\tilde{\pi}/\pi^*$ (see Corollary 5), under logarithmic utility for investors with different level of risk-aversion controlled by parameter ρ and resulting bounds G_L and G_U as in Table 3. *Right-hand side.* Corresponding evolutions of proportional-investment-in-stock distributions, $\tilde{\pi}/\tilde{X}$ (see equations 13 and 23). Other parameter values: $\mu - r = 2.5\%$, $r = 0$, $\sigma = 16\%$, $\theta = (\mu - r)/\sigma$, $x_0 = 10{,}000$, $T = 30$ years.

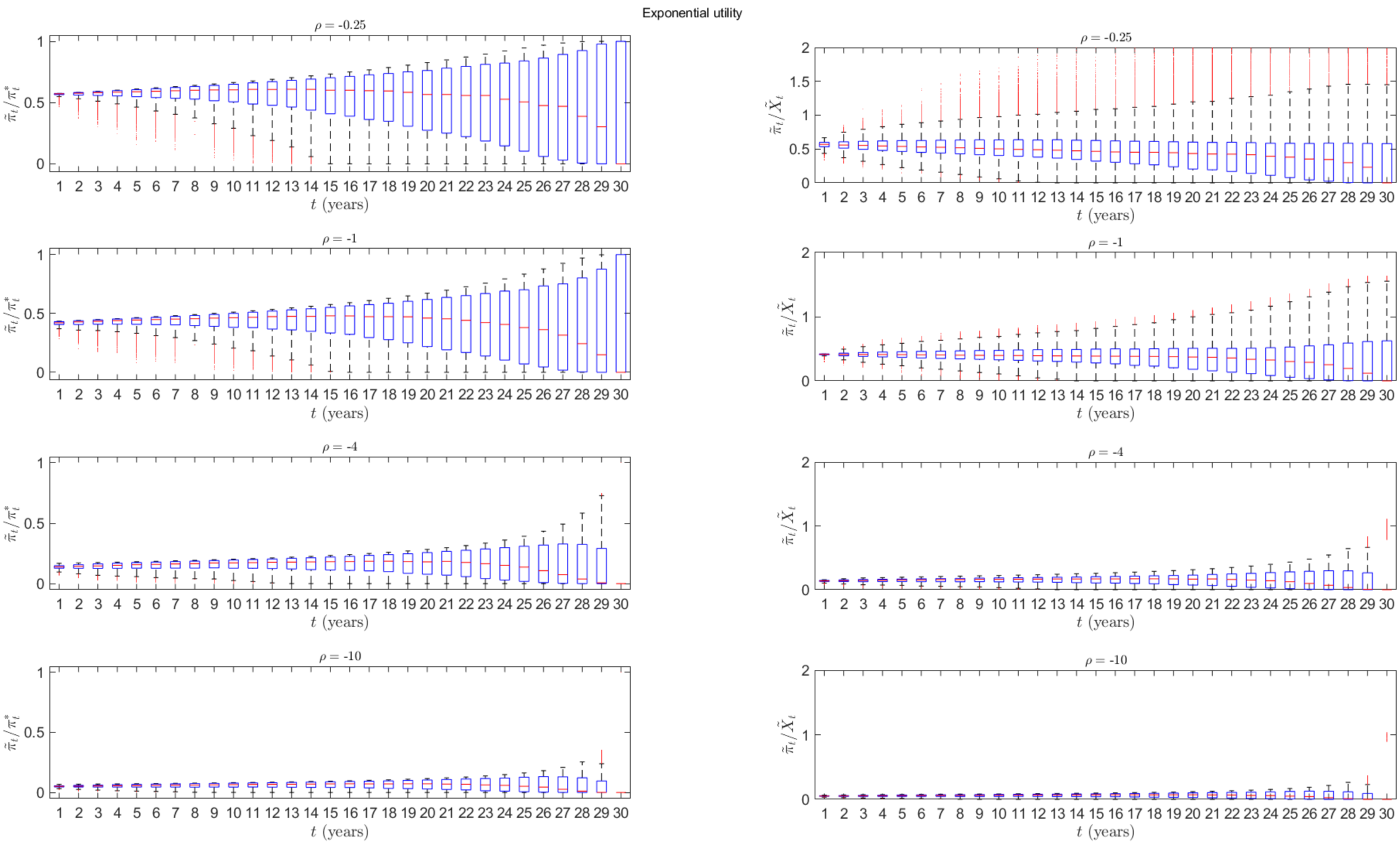

Figure 2: *Left-hand side.* Probability hedge distribution evolution, $\tilde{\pi}/\pi^*$ (see Corollary 5), under exponential utility (with parameter $\xi = \theta/(\sigma x_0)$) for investors with different level of risk-aversion controlled by parameter ρ and resulting bounds G_L and G_U as in Table 3. *Right-hand side.* Corresponding evolutions of proportional-investment-in-stock distributions, $\tilde{\pi}/\tilde{X}$ (see equations 13 and 23). Other parameter values: $\mu - r = 2.5\%$, $r = 0$, $\sigma = 16\%$, $\theta = (\mu - r)/\sigma$, $x_0 = 10{,}000$, $T = 30$ years.

4. Multi-stochastic environment

So far, we have presented and illuminated the notion of probability hedging under different strategies in a basic stochastic model. Introducing extra factors of randomness leads to incomplete market models. It would be interesting to extend our approach to situations where markets are not dynamically complete. To this end, we generalize our model of uncertainty developed in Section 2 to a model with multiple risk factors. The assumption of market completeness is strong and is not satisfied in many model specifications where the whole risk cannot be hedged. By introducing sufficient artificial securities though, we allow the investor to allocate and work on a complete market. In light of the results of the previous sections in favour of the logarithmic utility, in the interest of space, but also the interest of the relevant literature, as mentioned in the introduction, in the class of HARA utility functions, we derive expressions for our optimal strategies under the logarithmic utility. (More results can be made available upon request in relation to the power, exponential, but also the mean-variance utility as encouraged by a comment of a referee.)

Based on the martingale approach, we derive the optimal portfolio strategy which maximizes the expected utility of terminal wealth of the investor. We consider the usual probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : 0 \le t \le T\}, \mathbb{P})$ and an n-dimensional Brownian motion $\{W_t : 0 \le t \le T\}$ adapted to $\mathbb{F}$. Our market comprises a bond with instantaneous return r_t, a collection of real risky assets and, where necessary, a collection of artificial risky assets whose role is to complete the market. The price processes of the risky assets $\{S_{j,t} : 0 \le t \le T, 1 \le j \le n\}$ are given by

$$\frac{dS_{j,t}}{S_{j,t}} = r_t dt + \sum_k \sigma_{jk,t} \left(dW_{k,t} + \theta_{k,t} dt \right), \tag{27}$$

where the vector θ_t denotes the market price of risk.

Similarly to the single-factor model case in Section 2, if there are no constraints on the value X_T can take, we are looking into maximizing

$$\mathbb{E}^{\mathbb{P}} \left[U(X_T) \right] \text{ subject to } \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_s \, ds} X_T \right] = x_0,$$

i.e., we are interested in solving

$$\max_x \left\{ U(x) + \lambda_0 \left(x_0 - L_T(\omega) e^{-\int_0^T r_s(\omega) \, ds} x \right) \right\} \tag{28}$$

for each $\omega \in \Omega$, where L is the Radon–Nikodým derivative of the risk neutral measure $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfying

$$dL_t = -L_t \theta_t^\top dW_t. \tag{29}$$

The maximizing value of x in (28) is given by

$$U'(x) = \lambda_0 L_T(\omega) e^{-\int_0^T r_s(\omega) \, ds}, \text{ or } X_T^*(\omega) = \Upsilon \left(\lambda_0 L_T(\omega) e^{-\int_0^T r_s(\omega) \, ds} \right);$$

we postpone our discussion of the determination of the value of λ_0 until after equation (38). In the case of the logarithmic utility, $\Upsilon(a) = 1/a$ hence

$$X_T^* = \frac{e^{\int_0^T r_s \, ds}}{\lambda_0 L_T},$$

and

$$X_t^* = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} X_T^*\right] = \frac{e^{\int_0^t r_s\,ds}}{\lambda_0 L_t}$$

implying

$$\frac{dX_t^*}{X_t^*} = r_t\,dt + \theta_t^\top\,dW_t + \theta_t^\top\theta_t\,dt.$$

Let π_t be an $\mathcal{F}_t$-measurable vector process representing an allocation strategy, that is, the amount invested in each of the risky assets at time t. It follows from (27) that

$$\begin{aligned} dX_t &= \sum_j \pi_{j,t}\left(r_t dt + \sum_k \sigma_{jk,t}\left(dW_{k,t} + \theta_{k,t}dt\right)\right) + \left(X_t - \sum_j \pi_{j,t}\right) r_t dt \\ &= r_t X_t\,dt + \pi_t^\top \Sigma_t\left(dW_t + \theta_t\,dt\right). \end{aligned} \tag{30}$$

As X^* is an admissible portfolio process, it can be generated from (30) using the asset allocation strategy which satisfies

$$(\pi_t^*)^\top \Sigma_t = X_t^* \theta_t^\top, \text{ or } \pi_t^* = \left(\Sigma_t \Sigma_t^\top\right)^{-1} \Sigma_t \theta_t X_t^*. \tag{31}$$

This strategy implies that the asset allocations are not linearly independent and vary with time, consistently with the so-called bond-stock allocation puzzle (Canner *et al.*, 1997), which is theoretically supported by Bajeux-Besnainou *et al.* (2001) but also Bajeux-Besnainou *et al.* (2003) for any HARA investor. If some of the assets in the mix are artificial, we ignore those components of the vector π^*. Suppose, for example, that assets 1 to m are tradeable and $m+1$ to n are artificial; then, defining $\mathbf{I}_m$ to be the $n \times n$ matrix which is zero except for an $m \times m$ identity matrix in the top-left corner, the feasible strategy we implement is

$$\pi_t^{*\dagger} := \mathbf{I}_m \pi_t^*. \tag{32}$$

4.1. The constrained strategy

So now we come to the constrained case, and we have

$$\begin{aligned} \tilde{X}_t &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \max\left\{G_U, \min\left\{G_L, X_T^*\right\}\right\}\right] \\ &= \frac{1}{L_t}\mathbb{E}_t^{\mathbb{P}}\left[\max\left\{e^{-\int_t^T r_s\,ds} L_T G_U, \min\left\{e^{-\int_t^T r_s\,ds} L_T G_L, \frac{e^{\int_0^t r_s\,ds}}{\lambda_0}\right\}\right\}\right]. \end{aligned}$$

Our aim is to work out the dynamics of $\tilde{X}$ and, by comparing it with the equation

$$d\tilde{X}_t = r_t \tilde{X}_t\,dt + \tilde{\pi}_t^\top \Sigma_t(\theta_t\,dt + dW_t), \tag{33}$$

derive the optimal strategy $\tilde{\pi}$, at least in the complete market case.

Upon defining

$$\Psi_{t,T} = \int_t^T r_s\,ds + \int_t^T \theta_s^\top\,dW_s + \frac{1}{2}\int_t^T \theta_s^\top \theta_s\,ds,$$

we have that

$$e^{-\Psi_{t,T}} = \frac{X_t^*}{X_T^*} \tag{34}$$

and

$$\tilde{X}_t = G_U \mathbb{E}_t^{\mathbb{P}} \left[e^{-\Psi_{t,T}} \mathbf{1}_{\{\Psi_{t,T} > g_{U,t}\}} \right] + G_L \mathbb{E}_t^{\mathbb{P}} \left[e^{-\Psi_{t,T}} \mathbf{1}_{\{\Psi_{t,T} < g_{L,t}\}} \right] + \frac{e^{\Psi_{0,t}}}{\lambda_0} \mathbb{E}_t^{\mathbb{P}} \left[\mathbf{1}_{\{g_{L,t} < \Psi_{t,T} < g_{U,t}\}} \right], \quad (35)$$

where

$$g_{U,t} := \ln \lambda_0 + \ln G_U - \Psi_{0,t}, \quad g_{L,t} := \ln \lambda_0 + \ln G_L - \Psi_{0,t}. \quad (36)$$

Then, upon denoting by $f_{\Psi,t}(\cdot)$ and $F_{\Psi,t}(\cdot)$, respectively, the conditional density and distribution function of $\Psi_{t,T}$ given $\mathcal{F}_t$, we get

$$\begin{aligned} \tilde{X}_t &= G_U \int_{g_{U,t}}^{\infty} e^{-\psi} f_{\Psi,t}(\psi)\, d\psi + G_L \int_{-\infty}^{g_{L,t}} e^{-\psi} f_{\Psi,t}(\psi)\, d\psi + \frac{e^{\Psi_{0,t}} \mathbb{P}_t \left(g_{L,t} < \Psi_{t,T} < g_{U,t} \right)}{\lambda_0} \quad (37) \\ &= G_U \int_{g_{U,t}}^{\infty} e^{-\psi} F_{\Psi,t}(\psi)\, d\psi + G_L \int_{-\infty}^{g_{L,t}} e^{-\psi} F_{\Psi,t}(\psi)\, d\psi. \quad (38) \end{aligned}$$

We choose λ_0 such that the budget constraint $x_0 = \tilde{X}_0$ is fulfilled. Taking the derivative of the right-hand side of (38) with respect to λ_0 for $t = 0$ gives

$$-\frac{F_{\Psi,0}(\ln \lambda_0 + \ln G_U) - F_{\Psi,0}(\ln \lambda_0 + \ln G_L)}{\lambda_0^2} < 0,$$

therefore the maximum value of (38) is $G_U \mathbb{E}^{\mathbb{P}} \left[e^{-\Psi_{0,T}} \right]$, when $\lambda_0 = -\infty$, and similarly the minimum value is $G_L \mathbb{E}^{\mathbb{P}} \left[e^{-\Psi_{0,T}} \right]$. So, as long as x_0 lies within this range, it is possible to choose a suitable value of λ_0.

Define

$$H_t(a) \equiv H(a, t, r_t, \theta_t) = \mathbb{E}_t^{\mathbb{P}} \left[e^{ia\Psi_{t,T}} \right], \quad (39)$$

where $i := \sqrt{-1}$. Now look at the three pieces of (37). The first is

$$G_U \int_{g_{U,t}}^{\infty} e^{-\psi} f_{\Psi,t}(\psi)\, d\psi = \frac{G_U}{2\pi} \int_{-\infty}^{\infty} \int_{g_{U,t}}^{\infty} e^{-(1+ia)\psi}\, d\psi H_t(a) da = \frac{e^{\Psi_{0,t}}}{2\pi\lambda_0} \int_{-\infty}^{\infty} \frac{e^{-iag_{U,t}}}{1+ia} H_t(a) da,$$

whereas the third is

$$\begin{aligned} \frac{e^{\Psi_{0,t}}}{\lambda_0} \mathbb{P}_t \left(g_{L,t} < \Psi_{t,T} < g_{L,t} \right) &= \frac{e^{\Psi_{0,t}}}{2\pi\lambda_0} \int_{-\infty}^{\infty} \int_{g_{L,t}}^{g_{U,t}} e^{-ia\psi}\, d\psi H_t(a) da \\ &= \frac{e^{\Psi_{0,t}}}{2\pi\lambda_0} \int_{-\infty}^{\infty} \frac{i}{a} \left(e^{-iag_{U,t}} - e^{-iag_{L,t}} \right) H_t(a) da. \end{aligned}$$

Regarding the second term, consider

$$I_t := H_t(i) = \mathbb{E}_t^{\mathbb{P}} \left[e^{-\Psi_{t,T}} \right] = \int_{-\infty}^{\infty} e^{-\psi} f_{\Psi,t}(\psi)\, d\psi,$$

which is finite. Then, we get for the second term

$$\begin{aligned} G_L \left\{ I_t - \int_{g_{L,t}}^{\infty} e^{-\psi} f_{\Psi,t}(\psi)\, d\psi \right\} &= G_L I_t - \frac{G_L}{2\pi} \int_{-\infty}^{\infty} \int_{g_{L,t}}^{\infty} e^{-(1+ia)\psi}\, d\psi H_t(a) da \\ &= G_L I_t - \frac{e^{\Psi_{0,t}}}{2\pi\lambda_0} \int_{-\infty}^{\infty} \frac{e^{-iag_{L,t}}}{1+ia} H_t(a) da. \end{aligned}$$

By putting the three components together, we get for (37)

$$\tilde{X}_t = G_L H_t(i) + \int_{-\infty}^{\infty} \varepsilon(a) N_t(a) H_t(a) da, \tag{40}$$

where

$$N_t(a) := e^{(1+ia)\Psi_{0,t}} \text{ and } \varepsilon(a) := \frac{i}{2\pi\lambda_0 a(1+ia)} e^{-ia\ln\lambda_0}\left(e^{-ia\ln G_U} - e^{-ia\ln G_L}\right),$$

i.e.,

$$d\tilde{X}_t = G_L\, dH_t(a)|_{a=i} + \int_{-\infty}^{\infty} \varepsilon(a) d(HN)_t\, da. \tag{41}$$

It is worth noting that changing to a different utility function only requires altering the function ε in the preceding analysis as H and N depend only on the model. We conclude this part by proving some important properties of the process $\tilde{X}$ including being real, non-negative and convergent to the strategy with no constraints.

Proposition 7. *Define*

$$\chi_T(a) = \frac{e^{-iag_{U,T}} - e^{-iag_{L,T}}}{2\pi a(a-i)} + \frac{e^{iag_{U,T}} - e^{iag_{L,T}}}{2\pi a(a+i)},$$

where $g_{U,T}$ and $g_{L,T}$ are given in (36). Then, we have that:

1. *For each $a > 0$, $\chi_T(a)$ is a real random variable;*
2. $|\chi_T(a)| \leq \pi^{-1} \min\left\{(1+a)\ln(G_U/G_L), 2(a^{-2}+a^{-3})\right\}$*;*
3. $\mathbb{E}_t^{\mathbb{P}}\left[\int_0^\infty \chi_T(a)da\right] = \int_0^\infty \mathbb{E}_t^{\mathbb{P}}\left[\chi_T(a)\right]da = \lambda_0 e^{-\Psi_{0,t}} \int_{-\infty}^{\infty} \varepsilon(a) N_t(a) H_t(a) da$*; and*
4. *$\tilde{X}_t$ is real and non-negative for any t and satisfies*

$$G_L \mathbb{E}_t^{\mathbb{P}}[e^{-\Psi_{t,T}}] \leq \tilde{X}_t \leq G_U \mathbb{E}_t^{\mathbb{P}}[e^{-\Psi_{t,T}}].$$

Proof. See Section EC.2 of the e-companion of the paper. ∎

Lemma 8. *Let Q be a strictly positive, integrable random variable. Then,* $\lim_{\epsilon\to 0} \epsilon^{-1}\mathbb{E}\left[Q\mathbf{1}_{Q<\epsilon}\right] = 0$ *and* $\lim_{n\to\infty} n^{-1}\mathbb{E}\left[Q\mathbf{1}_{Q>n}\right] = 0$.

Proof. See Section EC.2 of the e-companion of the paper. ∎

If we apply Lemma 8 to the random variable $\lambda_0 e^{-\Psi_{0,t}-\Psi_{t,T}}$ with $\epsilon = G_U^{-1}$ and $n = G_L^{-1}$ conditional on the information at time t under the measure $\mathbb{P}$ and let $G_U \to \infty$ and $G_L \to 0$, the first two terms in (35) vanish in the limit and we get that

$$\lim_{G_U\to\infty, G_L\to 0} \tilde{X}_t = \frac{e^{\Psi_{0,t}}}{\lambda_0} = X_t^*,$$

that is, the solution for X when there are no constraints on the strategy.

4.2. Probability hedging in the affine multivariate model setting

In order to proceed further, we need to look at H in more detail by making some explicit model assumptions. More specifically, we assume that the instantaneous return r_t satisfies the equation

$$dr_t = \zeta_r(\mu_r - r_t)dt + \sigma_r^\top dW_t,$$

where σ_r is a vector of n entries and $\zeta_r > 0$ and μ_r are scalars. This model has appeared on various occasions in the relevant literature. For example, Bielecki *et al.* (2000) study asset allocation with Vašíček interest rates and a risky asset driven by uncorrelated Brownian motions; Bajeux-Besnainou *et al.* (2003) concentrate on dynamic portfolio optimisation, where stochastic interest rates and stock prices are correlated, for investors displaying HARA. In addition, as a multivariate generalization of the Ornstein–Uhlenbeck process, we assume that the market price of risk θ_t satisfies

$$d\theta_t = K(\mu_\theta - \theta_t)dt + \Xi dW_t,$$

where $K \equiv (\kappa_{j,k})$ and $\Xi \equiv (\xi_{j,k})$ are $n \times n$ matrices. By affinity of our model choice, it is possible to write H in (39) in the form

$$H(a,t,r_t,\theta_t) = \exp\left(-A(T-t,a) - B(T-t,a)r_t - C(T-t,a)^\top\theta_t - \frac{1}{2}\theta_t^\top D(T-t,a)\theta_t\right) \quad (42)$$

for some scalar functions A and B, vector function C, such that $A(0,\cdot) = B(0,\cdot) = 0$ and $C(0,\cdot) = \mathbf{0}$, and symmetric matrix D to be determined (see Section EC.2 of the paper's e-companion). We also define the vector process

$$E(s,a,\theta) = B(s,a)\sigma_r + \Xi^\top C(s,a) + \Xi^\top D(s,a)\theta. \quad (43)$$

This leads us to the following result. (Similarly to H, we will use, in general, for convenience the notation $E(s,a)$ without explicit mention of other arguments.)

Theorem 9. *The optimally controlled constrained process is given by*

$$\begin{aligned} d\tilde{X}_t &= r_t\tilde{X}_t dt - \Big[G_L H_t(i)E(T-t,i) \\ &\quad + \int_{-\infty}^{\infty} \varepsilon(a)H_t(a)N_t(a)\left[-(1+ia)\theta_t + E(T-t,a)\right] da\Big]^\top (\theta_t dt + dW_t). \end{aligned} \quad (44)$$

The associated optimal asset allocation strategy at time t is given by

$$\begin{aligned} \tilde{\pi}_t^\top\Sigma_t &= -G_L H_t(i)E(T-t,i) + \int_{-\infty}^{\infty} \varepsilon(a)H_t(a)N_t(a)\left[(1+ia)\theta_t - E(T-t,a)\right]^\top da \quad (45) \\ &= \tilde{X}_t\theta_t^\top - G_L H_t(i)[\theta_t + E(T-t,i)]^\top - \int_{-\infty}^{\infty} \varepsilon(a)H_t(a)N_t(a)\left[-ia\theta_t + E(T-t,a)\right]^\top da. \end{aligned}$$

Proof. See Section EC.2 of the e-companion of the paper. ■

So, we have a semi-explicit solution for the optimal investment strategy in the constrained case where we have completed the market by the addition of artificial assets. Now it would be interesting to express as a probability hedge, i.e., to see how $\tilde{\pi}_t$ compares with $\pi_t^*\mathbb{P}_t\left(X_T^* \in [G_L, G_U]\right)$.

Corollary 10. *The optimal constrained strategy is given by*

$$\tilde{\pi}_t^\top\Sigma_t = \mathbb{P}_t\left(X_T^* \in [G_L,G_U]\right)(\pi_t^*)^\top\Sigma_t - G_L H_t(i)E(T-t,i) - \int_{-\infty}^{\infty} \varepsilon(a)H_t(a)N_t(a)E(T-t,a)da. \quad (46)$$

Proof. See Section EC.2 of the e-companion of the paper. ■

As a final remark, the function $\varepsilon(a)$, and hence the integrand in (46) as a whole, has a pole at $a = i$. Working out $2\pi i$ times the residue at this pole (in line with the Residue Theorem), we get $(G_U - G_L)H_t(i)E(T-t,i)$. Therefore, we cannot cancel out the term $G_L H_t(i)E(T-t,i)$ by choosing a different path to integrate over, but we can replace it with a term that involves instead G_U. Comparing with our result in Corollary 5, the term

$$-G_L H_t(i)E(T-t,i) - \int_{-\infty}^{\infty} \varepsilon(a)H_t(a)N_t(a)E(T-t,a)da$$

in (46) represents the adjustment to the optimal strategy when we switch from the one-factor to a general affine multivariate model.

4.3. Application: the constrained strategy in a model with two risky assets

Having presented our general stochastic model setting in the previous section, in order to facilitate our illustration of the constrained strategy, which is the core of this research, we model the risk premium θ_1 corresponding to the genuine risky asset S_1 (with constant volatility σ_1) as a non-central Ornstein–Uhlenbeck process

$$d\theta_{1,t} = \kappa_{1,1}(\mu_{\theta_1} - \theta_{1,t})dt + \xi_{1,1}dW_{1,t} + \xi_{1,2}dW_{2,t},$$

where W_2 is a $\mathbb{P}$-Brownian motion independent of W_1. In order to be able to complete the market and work out our optimal strategy, we enlarge the asset mix by introducing the artificial risky asset S_2 (with constant volatility σ_2). The dynamics of θ_2 is given by

$$d\theta_{2,t} = \kappa_{2,2}(\mu_{\theta_2} - \theta_{2,t})dt + \xi_{2,2}dW_{2,t}.$$

Finally, we recall from (29) that the Radon–Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ takes the form

$$L_t = \exp\left[-\sum_{k=1}^{2}\left(\int_0^t \theta_{k,s}dW_{k,s} + \frac{1}{2}\int_0^t \theta_{k,s}^2 ds\right)\right]. \tag{47}$$

For the sake of exemplification and consistency with our earlier numerical analysis in Section 3, we focus on a model with constant interest rates, however this is not intended to be a restriction. Note that when converting into the real world without the artificial asset, we project the optimal policy from the 2-dimensional space of assets, with the artificial one included, onto the 1-dimensional space of feasible assets and the optimal policy is given by (32). Based on this set-up, we get from (34)

$$\begin{aligned}\Psi_{t,s} \quad = \quad & \ln\frac{X_s^*}{X_t^*} = r\,(s-t) - \ln\frac{L_s}{L_t} = \left(r - \frac{\xi_{1,1}}{2} - \frac{\xi_{2,2}}{2}\right)(s-t) + \frac{\xi_{1,1}}{2(\xi_{1,1}^2+\xi_{1,2}^2)}\left(\theta_{1,s}^2 - \theta_{1,t}^2\right) \\ & + \frac{1}{2\xi_{2,2}}\left(\theta_{2,s}^2 - \theta_{2,t}^2\right) + \left(\frac{1}{2} + \frac{\kappa_{1,1}\xi_{1,1}}{\xi_{1,1}^2+\xi_{1,2}^2}\right)\int_t^s \theta_{1,u}^2 du + \left(\frac{1}{2} + \frac{\kappa_{2,2}}{\xi_{2,2}}\right)\int_t^s \theta_{2,u}^2 du \\ & - \frac{\kappa_{1,1}\mu_{\theta_1}\xi_{1,1}}{\xi_{1,1}^2+\xi_{1,2}^2}\int_t^s \theta_{1,u}du - \frac{\kappa_{2,2}\mu_{\theta_2}}{\xi_{2,2}}\int_t^s \theta_{2,u}du + \frac{\xi_{1,2}}{\sqrt{\xi_{1,1}^2+\xi_{1,2}^2}}\int_t^s \theta_{1,u}dW_{2,u},\end{aligned} \tag{48}$$

which amounts to a multivariate (two-factor) extension of the classical Ornstein–Uhlenbeck driven stochastic volatility model studied in, e.g., Scott (1987), Stein and Stein (1991) and Schöbel and Zhu (1999). Thinking in terms of model tractability, what is convenient about this model is that the conditional distribution of each of $\left(\int_t^s \theta_{k,u}du, \theta_{k,s}\right)$ given $\mathcal{F}_t$ is bivariate normal. We also know from Li and Wu (2019) the conditional Laplace transform of $\int_t^s \theta_{k,u}^2 du$ given $\theta_{k,s}$ and $\int_t^s \theta_{k,u}du$. Based on these, we derive in Section EC.3 of the paper's e-companion our own simulation scheme of the one-factor model $\bar{\Psi}_{t,s}$, which we can use in order to simulate $\bar{\Psi}_{t,s}$ by independence of $\left\{\left(\int_t^s \theta_{k,u}du, \int_t^s \theta_{k,u}^2 du, \theta_{k,s}\right)\right\}_{k=1,2}$.

For ease of comparison with the probability hedge under the basic model of Section 2, we consider the same four long-term investors with utility parameter values $\rho = -0.25$, -1, -4 and -10, respectively, corresponding to a pattern of increasing risk-aversion. As before, each of them invests a total of $x_0 = 10{,}000$ in stocks and a risk-free inflation bond with an investment horizon of 30 years. We focus on a logarithmic utility hedge (see also corresponding lower and upper bounds in Table 3) and aim to see how $\mathbb{P}_t\left(X_T^* \in [G_L, G_U]\right)$, which determines the optimal constrained strategy (see Corollary 10), evolves under the model of this section. To this end, using as our basis the estimation of Gerrard *et al.* (2020), we fix certain parameter values of the traded asset: $\kappa_{1,1} = 0.32111$, that is, a reasonably slow mean-reversion, aiming to make more discernible the particular model feature; in addition, we set $\theta_{1,0} = 0.15625$, $\mu_{\theta_1} = 0.15625$ and $r = 0$ for consistency with our basic one-factor model in the previous sections. (Qualitatively, our results did not vary when we assumed a faster speed of mean-reversion estimate.) We then carry out a sensitivity analysis for varying volatility τ_1 of the risk premium θ_1. We present in Figure 3 our results for the distribution of the probability hedge at each point in time across $\rho = -0.25$, -1, -4 and -10 (top to bottom), and $\xi_{1,1} = \varrho\tau_1$, where $\varrho = 0.03$ and $\tau_1 = 0.05$, 0.1 and 0.2 (left to right).

Comparing with the analysis in Section 3.4, the qualitative behaviour of the term structure of the probability and its characteristics do not exhibit, generally, abnormal changes; especially for risk-averse investors. We rather mostly observe differences in the numerical details which are enabled by the more flexible modelling as the evolution of the probability is now determined not only by the wealth, but also by the risk premium process. While this increases the problem dimensionality, it relaxes potential pressure of parameter time-dependence. More specifically, as in the constant θ case, for the risky investor the mean probability is around half every year. Increasing τ_1 has a boosting effect on the skewness and excess kurtosis; indeed, comparing with the case of constant market risk premium (see the left side of Figure 1), both quantities become more significantly positive and increasing with risk premium uncertainty. Therefore, a more volatile risk premium implies a fatter-tailed distribution to the right for the probability of wealth to remain constrained – which is required in an efficient hedging strategy – especially for our risky investor with a large lower-upper bound range. Although the same upward-shifting effect on the skewness and kurtosis due to increasing τ_1 is observed for more risk-averse investors, this diminishes, i.e., the impact of risk premium uncertainty fades away, with increasing risk-aversion, which quite naturally becomes the primary factor driving the hedge – the mean probability becomes smaller with the lower and upper bound close to each other – as the investors become less willing to take additional risk. All in all, accounting for additional sources of randomness is mainly important for risk-takers with an obvious reflection of their optimal hedging strategy.

4.4. A heuristic strategy

As previously, we assume the same investment horizon of 30 years and four long-term investors. We reconsider the original unconstrained logarithmic utility hedge with an associated optimal investment strategy (31) in risky assets. Following standard human capital theory, pension savers

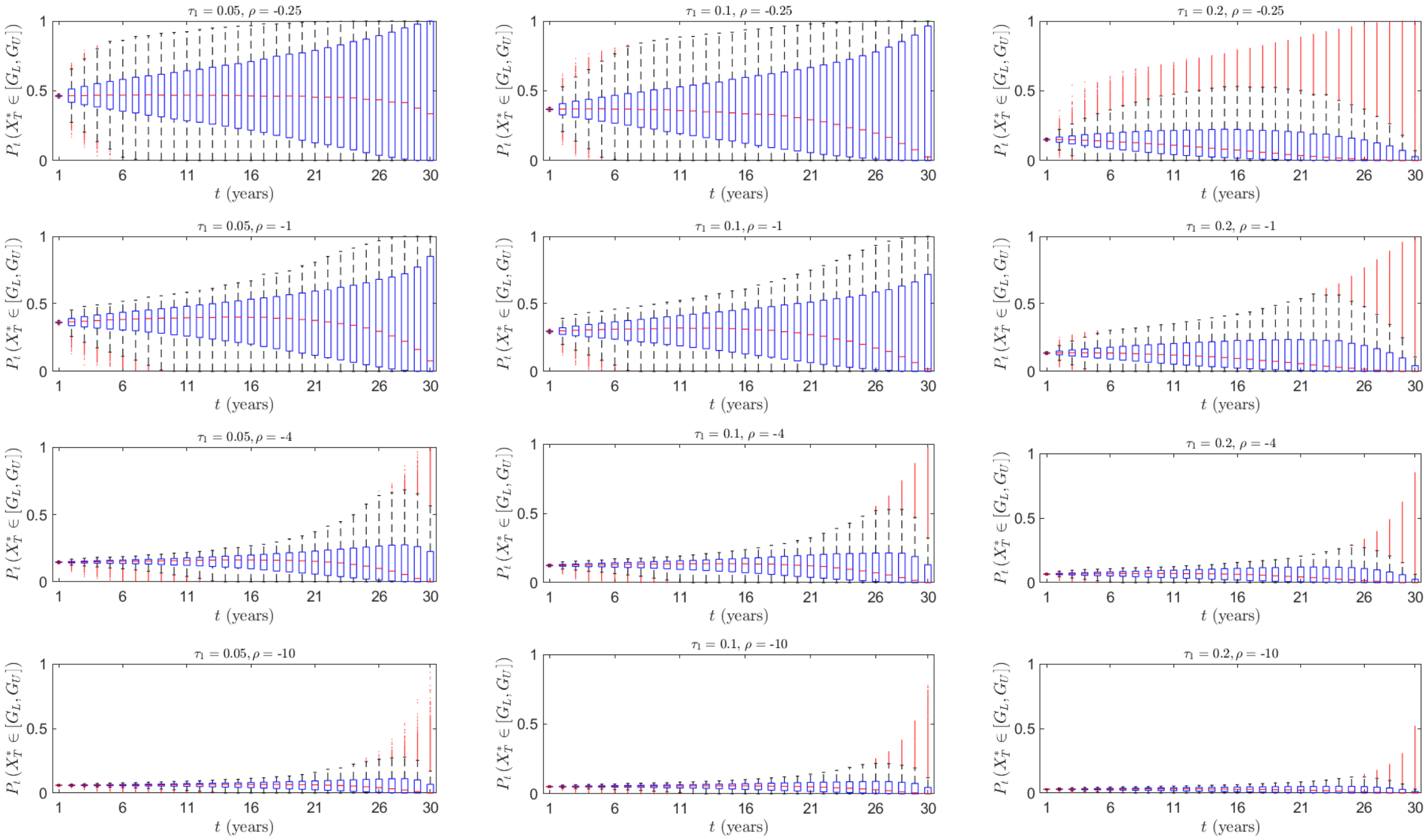

Figure 3: Probability hedge distribution evolution, $\mathbb{P}_t(X_T^* \in [G_L, G_U])$, under logarithmic utility for investors with different level of risk-aversion controlled by parameter ρ, and resulting bounds G_L and G_U as in Table 3, and for varying volatility τ_1 of the stochastic risk premium process θ_1. The probability distribution estimates are based on simulation of the model defined in (48). Other parameter values: $\theta_{1,0} = 0.15625$, $\kappa_{1,1} = 0.32111$, $\mu_{\theta_1} = 0.15625$, $\xi_{1,1} = \varrho\tau_1$, where $\varrho = 0.03$ and $\tau_1 = 0.05$, 0.1 and 0.2 (left to right), $r = 0$, $x_0 = 10{,}000$, $T = 30$ years.

should invest according to their available funds, while many pension schemes do not allow more than 100% investment in risky assets. This is often a legal constraint, but it can also be strategic. Černý and Melicherčík (2020) deal with this type of constraint and present a simple near-optimal strategy that takes this into account. Therefore, we cap the amount in the risky asset by the available wealth and define

$$\varpi_t^* = \min\left(\pi_t^*, X_t^*\right) = \pi_t^* \mathbf{1}_{\{\pi_t^* \leq X_t^*\}} + X_t^* \mathbf{1}_{\{\pi_t^* > X_t^*\}}$$

and the associated adjusted wealth process

$$\mathcal{X}_t^* = X_t^* \mathbf{1}_{\{\pi_t^* \leq X_t^*\}} + \frac{\sigma}{\theta_t} X_t^* \mathbf{1}_{\{\pi_t^* > X_t^*\}}.$$

(A variant of the above could allow for $100\beta\%$, $\beta \in (0,1]$, investment of wealth in the risky asset for a number of years before imposing the capped rule.) The regime-switching in this case depends on the stochastic risk premium process θ; as we have seen, each utility has its own rules for switching. In addition, we experiment here by imposing our upper and lower bounds aiming to improve on the risk:

$$\tilde{\mathcal{X}}_t = \min\left(G_U, \max\left(G_L, \mathcal{X}_t^*\right)\right).$$

We examine the usual four levels of risk-aversion: low with $\rho = -0.25$, -1, -4 and -10 as the highest. We simulate and compare the resulting terminal reward $\tilde{\mathcal{X}}_T$ we have heuristically constructed against our original $\tilde{X}_T$ in terms of certainty equivalence. Table 5 shows that the latter outperforms fairly moderately the heuristic approach for low and medium levels of risk-aversion, while with high risk-aversion the outperformance becomes more noticeable. In addition, as noted in the previous section, the impact of risk premium uncertainty decreases as risk-aversion increases. Although the discrepancies between the two become generally more pronounced with a more volatile risk premium (larger τ), it could be argued that they are not particularly large, consistently with our earlier conclusion from our numerical results when incorporating our bounds-based trade-off. We have also observed that the relative mix of risky assets and bonds in the optimal portfolio shifts progressively towards the risky asset in the medium term, after which there is a substitution towards bonds until it roughly stabilizes by the end of the 30-year period (especially for a more volatile risk premium). This resembles a pattern observed over time in the so-called stochastic lifestyling phenomenon (Cairns *et al.*, 2006, Černý and Melicherčík, 2020).

	Heuristic with bounds CE			Optimal with bounds CE		
τ	0.05	0.1	0.2	0.05	0.1	0.2
$\rho = -0.25$	12,123	12,400	14,776	12,961	13,813	16,126
$\rho = -1$	11,267	11,490	13,183	11,836	12,466	14,319
$\rho = -4$	10,384	10,456	11,037	10,540	10,736	11,396
$\rho = -10$	10,155	10,199	10,406	10,218	10,305	10,547

Table 5: Certainty equivalents of the heuristic and optimal strategies under logarithmic utility for investors with different level of risk-aversion controlled by parameter ρ and for varying volatility τ of the stochastic risk premium process θ. "Optimal with bounds CE" entries correspond to the certainty equivalent (CE) of $\tilde{X}_T$; "heuristic with bounds CE" entries correspond to the CE of $\tilde{\mathcal{X}}_T$. Other parameter values: G_L and G_U as in Table 3, $\sigma = 0.16$, $\theta_0 = 0.15625$, $\kappa = 0.32111$, $\mu_\theta = 0.15625$, $\xi = \varrho\tau$, where $\varrho = 0.03$ and $\tau = 0.05$, 0.1 and 0.2 (left to right), $r = 0$, $x_0 = 10{,}000$, $T = 30$ years.

5. Concluding discussion

Consider a clever practitioner working in the machine room that delivers better future pensions for their long-term saver clients. It is key that professionals implementing new ideas have a hands-on type of intuition of what hedging does to the money under management. In this paper, we discover that a financial hedging strategy that fits in with this goal is possible, providing a better distribution of the terminal wealth than traditional hedging approaches.

More specifically, inspired by Merton's (2014) vision for long-term saving "through clear and meaningful communication and simplicity of choices", we develop a system with double bounds whereby a fund manager offers the savers the option to choose their minimum guarantee for the terminal reward, subject to a budget constraint and an upper bound which is set, by assumption, to be achieved half of the time. We make general assumptions for the utility of the investor and the random driving factors, and study the transition from unhedged to constrained allocation strategies, which we prove that are linked via the probability of landing up within the bounds. This probability determines the optimal investment strategy (the financial hedge) for the long-term saver. We dub this probability hedging and show how the probability measure varies with different underlying utility functions.

We find that, under a constrained strategy, the impact of the fund manager's belief about the underlying utility is minimal, even if this differs from the consumer's unknown risk profile. However, more pivotal are the practical consequences from the particular assumption of a logarithmic utility. We prove that the probability hedging under this assumption takes place under the physical measure. We also show that across investors with different risk preferences, this can result in high lower guarantee as well as best-case outcome and minimal reduction in terms of certainty equivalence compared to a hypothetical correctly matched hedge for the investor. We also investigate the effect of additional sources of randomness, such as a stochastic risk premium. This more flexible modelling enables some changes in the distribution of the probability of wealth to remain constrained, which is the main determinant of the optimal constrained strategy. In particular, we notice a fatter tailed distribution with increasing volatility of risk premium, which is more obvious for risky savers. This impact of the risk premium uncertainty fades away with increasing risk-aversion.

We conclude by mentioning potential future add-ons. Blanchet-Scalliet *et al.* (2008) study the influence of exit time uncertainty on portfolio selection for investors. Unlike studies dedicated to rather non-actuarial institutional investors, an interesting question for a pension fund is the extent to which our theoretical framework can be adapted upon introducing mortality risk. Battocchio *et al.* (2007) study the fund wealth during the accumulation and post-retirement payout phases as well as introduce a demographic dimension. Whilst it is beyond the scope of the current paper how the consumers allocate their wealth after retirement, the introduction of mortality risk during the accumulation phase and its potential effect on the construction of a probability hedge is an interesting avenue which we study as part of a separate research. Finally, the investigation of strategies related to maximization of a utility criterion when constraints are imposed on intermediate dates (see El Karoui *et al.*, 2005) or of the probability of beating a stochastic benchmark (see Browne, 1999a,b) are certainly challenging and additional new problems by themselves. The original contributions rely on basic market model assumptions; inclusion of multiple stochastic factors and American-type guarantees inevitably add to the problem intractability and implementation, but also the ideas themselves to the pension communication challenge.

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E-companion to "On optimal constrained investment strategies for long-term savers in stochastic environments and probability hedging"

EC.1. Supplementary figures

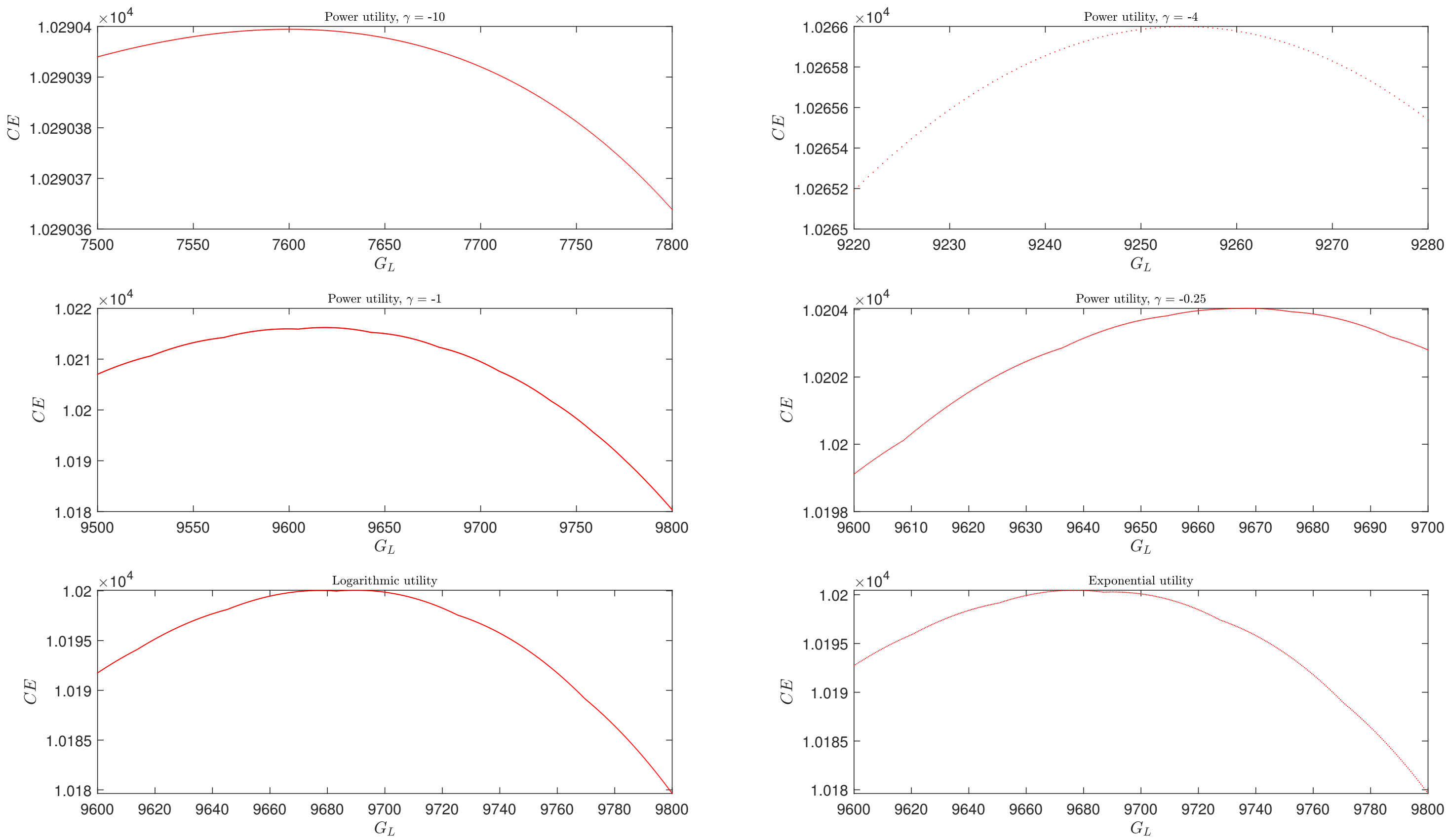

Figure EC.1: Certainty equivalent maximization for different constrained strategies. We exhibit the certainty equivalent CE as a function of the lower bound G_L focusing on the peak which corresponds to the outcome from expression (26), based on the procedure described in Section 2.3, for different driving utilities (power with parameter γ, logarithmic and exponential) as indicated on the plots.

EC.2. Proofs

Proof of Proposition 1. From (5),

$$\begin{aligned} F^{(\alpha)}(x,t) &= \mathbb{S}_t^{(\alpha)}\left[\Upsilon\left(\lambda_0 L_T^{(\theta)}\right) \leq x\right] \\ &= \mathbb{S}_t^{(\alpha)}\left[\theta\left(W_T^{(\alpha)} - W_t^{(\alpha)}\right) \leq \ln\lambda - \ln U'(x) - \theta W_t + \alpha\theta(T-t)\right] \end{aligned}$$

from which (10) follows. It then follows from (10) that

$$\mathbb{S}_t^{(\alpha)}\left(X_T^* \in [G_L, G_U]\right) = F^{(\alpha)}(G_U,t) - F^{(\alpha)}(G_L,t).$$

■

Proof of Proposition 2.

i) We have from (6) that

$$\begin{aligned} X_t^* &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[X_T^*\right] = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\Upsilon\left(\lambda_0 L_T^{(\theta)}\right)\right] \\ &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\Upsilon\left(\lambda e^{-\theta\left(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}}\right)-\theta W_t+\theta^2(T-t)}\right)\right], \end{aligned}$$

from which (11) follows.

ii) From (8),

$$\begin{aligned} \tilde{X}_t &= e_t^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\min\left\{G_U, \max\left\{G_L, \Upsilon\left(\lambda_0 L_T^{(\theta)}\right)\right\}\right\}\right] \\ &= e^{-r(T-t)}\left(G_U\left[1-F^{(\theta)}(G_U,t)\right] + G_L F^{(\theta)}(G_L,t) + \int_{G_L}^{G_U} x f^{(\theta)}(x,t)\,dx\right), \end{aligned}$$

where $f^{(\theta)}(x,t)$ is the corresponding density. (13) follows by straightforward calculus.

■

Proof of Theorem 4. From (13),

$$d\tilde{X}_t = r\tilde{X}_t\,dt - e^{-r(T-t)}\int_{G_L}^{G_U} dx\,dF^{(\theta)}(x,t),$$

where, using Itô calculus,

$$\begin{aligned} dF^{(\theta)}(x,t) &= \phi\left(\frac{\ln\lambda - \ln U'(x) - \theta W_t + \theta^2(T-t)}{\theta\sqrt{T-t}}\right) \\ &\quad \times\left\{\left(\frac{\ln\lambda - \ln U'(x) - \theta W_t}{2\theta(T-t)^{\frac{3}{2}}} - \frac{\theta}{2\sqrt{T-t}}\right)dt - \frac{dW_t}{\sqrt{T-t}}\right. \\ &\quad \left. - \frac{\ln\lambda - \ln U'(x) - \theta W_t + \theta^2(T-t)}{2\theta(T-t)^{\frac{3}{2}}}dt\right\} \\ &= -\frac{1}{\sqrt{T-t}}\phi\left(\frac{\ln\lambda - \ln U'(x) - \theta W_t + \theta^2(T-t)}{\theta\sqrt{T-t}}\right)(\theta\,dt + dW_t). \end{aligned}$$

We conclude that

$$d\tilde{X}_t = r\tilde{X}_t dt + e^{-r(T-t)} I\left(t; W_t, \lambda\right)\left(\theta\,dt + dW_t\right),$$

where

$$\begin{aligned} \mathcal{I}(t;W_t,\lambda) &= \frac{1}{\sqrt{T-t}}\int_{G_L}^{G_U}\phi\left(\frac{\ln\lambda - \ln U'(x) - \theta W_t + \theta^2(T-t)}{\theta\sqrt{T-t}}\right)dx \\ &= -\theta\int_{\mathcal{K}(G_L,t;W_t,\lambda)+\theta\sqrt{T-t}}^{\mathcal{K}(G_U,t;W_t,\lambda)+\theta\sqrt{T-t}} \phi(x)\lambda\mathcal{H}\left(x,t;W_t\right)\Upsilon'\left(\lambda\mathcal{H}\left(x,t;W_t\right)\right)dx \end{aligned}$$

and the second equality follows by a variable change with $\mathcal{H}$ given by (12). Finally, from $\Upsilon\left(U'(x)\right) = x$, we deduce that $\Upsilon'\left(U'(x)\right) = 1/U''(x)$ and

$$-U'(x)\Upsilon'\left(U'(x)\right) = -\frac{U'(x)}{U''(x)} = \frac{1}{A(x)}, \text{ or } -a\Upsilon'(a) = \frac{1}{A\left(\Upsilon(a)\right)},$$

from which (24) follows. Equation (1) then implies (23). ■

Proof of Corollary 5.

Exponential utility. A comparison of (15) and (1) shows that the investment strategy π^* which generates this process is

$$\pi_t^* = \frac{\theta}{\xi\sigma} e^{-r(T-t)}.$$

Then, from (24), and given $U'(x) = e^{-\xi x}, U''(x) = -\xi e^{-\xi x}, A(x) = \xi$, we have that

$$\begin{aligned}
\mathcal{I}(t; W_t, \lambda) &= \frac{\theta}{\xi}\left\{\Phi\left(\mathcal{K}\left(G_U, t; W_t, \lambda\right) + \theta\sqrt{T-t}\right) - \Phi\left(\mathcal{K}\left(G_L, t; W_t, \lambda\right) + \theta\sqrt{T-t}\right)\right\} \\
&= \frac{\theta}{\xi}\mathbb{S}_t^{(\theta)}\left(X_T^* \in [G_L, G_U]\right)
\end{aligned}$$

which follows from (10). This, finally, implies that

$$\tilde{\pi}_t = \frac{\theta}{\xi\sigma} e^{-r(T-t)}\mathbb{S}_t^{(\theta)}\left(X_T^* \in [G_L, G_U]\right) = \pi_t^*\mathbb{S}_t^{(\theta)}\left(X_T^* \in [G_L, G_U]\right).$$

Power utility. From (17), we get that

$$\pi_t^* = \frac{\theta(\eta+1)}{\sigma} X_t^*.$$

Given $U'(x) = x^{-1/(\eta+1)}, U''(x) = -\frac{x^{-(\eta+2)/(\eta+1)}}{\eta+1}, A(x) = \frac{x^{-1}}{\eta+1}, \Upsilon(a) = a^{-(\eta+1)}$, and hence $A \circ \Upsilon(a) = \frac{a^{\eta+1}}{\eta+1}$, we get that

$$\begin{aligned}
\mathcal{I}(t; W_t, \lambda) &= \frac{\theta(\eta+1)e^{\frac{(\eta+1)(\theta W_t - \theta^2(T-t))}{\eta+1}}}{\lambda^{\eta+1}} \int_{\mathcal{K}(G_L, t; W_t, \lambda)+\theta\sqrt{T-t}}^{\mathcal{K}(G_U, t; W_t, \lambda)+\theta\sqrt{T-t}} \frac{\exp\left(-\frac{1}{2}x^2 + \theta(\eta+1)\sqrt{T-t}x\right)}{\sqrt{2\pi}} dx \\
&= \theta(\eta+1)X_t^* e^{r(T-t)}\left\{\Phi\left(\mathcal{K}\left(G_U, t; W_t, \lambda\right) - \eta\theta\sqrt{T-t}\right)\right. \\
&\quad \left. -\Phi\left(\mathcal{K}\left(G_L, t; W_t, \lambda\right) - \eta\theta\sqrt{T-t}\right)\right\},
\end{aligned}$$

implying

$$\tilde{\pi}_t = \pi_t^*\mathbb{S}_t^{(-\eta\theta)}\left(X_T^* \in [G_L, G_U]\right).$$

Logarithmic utility. This follows from the power utility case for $\eta = 0$. ■

Proof of Corollary 6. From (13) and (10), we have that

$$\begin{aligned}
\tilde{X}_t &= e^{-r(T-t)}\left(G_U - \int_{G_L}^{G_U} \Phi\left(\frac{\ln x}{\varrho} + \varpi\right) dx\right) \\
&= e^{-r(T-t)}\left[G_U\left(1 - \Phi\left(\frac{\ln G_U}{\varrho} + \varpi\right)\right) + G_L\Phi\left(\frac{\ln G_L}{\varrho} + \varpi\right) + \frac{1}{\varrho}\int_{G_L}^{G_U} \phi\left(\frac{\ln x}{\varrho} + \varpi\right) dx\right],
\end{aligned}$$

where

$$\varrho := \theta(\eta+1)\sqrt{T-t} \text{ and } \varpi := \frac{\ln\lambda - \theta W_t + \theta^2(T-t)}{\theta\sqrt{T-t}}.$$

In addition from (23),

$$\tilde{\pi}_t = \frac{\theta(\eta+1)e^{-r(T-t)}}{\sigma\varrho}\int_{G_L}^{G_U} \phi\left(\frac{\ln x}{\varrho} + \varpi\right) dx.$$

This implies that

$$\begin{aligned}\tilde{X}_t &= e^{-r(T-t)}\left[G_U\left(1-\Phi\left(\frac{\ln G_U}{\varrho}+\varpi\right)\right)+G_L\Phi\left(\frac{\ln G_L}{\varrho}+\varpi\right)\right]+\frac{\sigma}{\theta(\eta+1)}\tilde{\pi}_t \\ &> \frac{\sigma}{\theta(\eta+1)}\tilde{\pi}_t=\frac{X_t^*}{\pi_t^*}\tilde{\pi}_t,\end{aligned}$$

where the last equality follows from (25). This result encompasses also the special case of the logarithmic utility function with $\eta = 0$. ■

Lemma 11. *Define*

$$f(u)=\int_0^\infty \frac{a\cos au+\sin au}{\pi a(1+a^2)}da.$$

Then,

$$f(u)=\begin{cases}\frac{1}{2}, & \text{if } u>0 \\ e^u-\frac{1}{2}, & \text{if } u<0\end{cases}.$$

Proof. First consider the case $u > 0$. For $\delta > 0$, $\delta \neq 1$, the one-sided Laplace transform of $f(u)$ is given by

$$\begin{aligned}\tilde{f}(\delta) &= \int_0^\infty e^{-\delta u}\int_0^\infty \frac{a\cos au+\sin au}{\pi a(1+a^2)}dadu \\ &= \int_0^\infty \frac{1}{\pi a(1+a^2)}\int_0^\infty e^{-\delta u}\left(\frac{a}{2}\left(e^{iau}+e^{-iau}\right)+\frac{1}{2i}\left(e^{iau}-e^{-iau}\right)\right)duda \\ &= \frac{1+\delta}{\pi}\int_0^\infty \frac{da}{\delta^2-1}\left(\frac{1}{1+a^2}-\frac{1}{\delta^2+a^2}\right)=\frac{1}{2\delta}.\end{aligned}$$

We therefore conclude that $f(u) = \frac{1}{2}$ if $u > 0$. Now consider $u < 0$. In this case,

$$\begin{aligned}\tilde{f}(\delta) &= \int_{-\infty}^0 e^{\delta u}\int_0^\infty \frac{a\cos au+\sin au}{\pi a(1+a^2)}dadu=\int_0^\infty e^{-\delta v}\int_0^\infty \frac{a\cos av-\sin av}{\pi a(1+a^2)}dadv \\ &= \int_0^\infty \frac{1}{\pi a(1+a^2)}\int_0^\infty e^{-\delta v}\left(\frac{a}{2}\left(e^{iav}+e^{-iav}\right)+\frac{1}{2i}\left(e^{-iav}-e^{iav}\right)\right)dvda \\ &= \frac{\delta-1}{\pi}\int_0^\infty \frac{da}{\delta^2-1}\left(\frac{1}{1+a^2}-\frac{1}{\delta^2+a^2}\right)=\frac{1}{\delta+1}-\frac{1}{2\delta}.\end{aligned}$$

This time we conclude that $f(u) = e^u - \frac{1}{2}$ when $u < 0$. ■

Proof of Proposition 7.

1. We have that

$$\begin{aligned}\frac{e^{-iag_{U,T}}-e^{-iag_{L,T}}}{2\pi a(a-i)} &= \frac{1}{2\pi a(a^2+1)}\left\{a\left(\cos ag_{U,T}-\cos ag_{L,T}\right)-ia\left(\sin ag_{U,T}-\sin ag_{L,T}\right)\right. \\ &\quad \left.+i\left(\cos ag_{U,T}-\cos ag_{L,T}\right)+\left(\sin ag_{U,T}-\sin ag_{L,T}\right)\right\}.\end{aligned}$$

   The two imaginary terms are odd functions of a. Therefore, when we add to them the equivalent ones with a replaced by $-a$, we obtain

$$\chi_T(a)=\frac{a\left(\cos ag_{U,T}-\cos ag_{L,T}\right)+\left(\sin ag_{U,T}-\sin ag_{L,T}\right)}{\pi a(a^2+1)}, \quad \text{(EC.1)}$$

   from which the first statement becomes obvious.

2. Now

$$\frac{\sin a g_{U,T} - \sin a g_{L,T}}{a} = \int_{g_{L,T}}^{g_{U,T}} \cos a\theta d\theta, \quad \cos a g_{U,T} - \cos a g_{L,T} = -a \int_{g_{L,T}}^{g_{U,T}} \sin a\theta d\theta.$$

Noting that $g_{U,T} - g_{L,T} = \ln(G_U/G_L)$, this gives us

$$|\chi_T(a)| \leq \frac{(g_{U,T} - g_{L,T})(1+a)}{\pi(1+a^2)} \leq \frac{1+a}{\pi} \ln \frac{G_U}{G_L}. \tag{EC.2}$$

In addition, it is straightforward to see that

$$\left| \frac{a\left(\cos a g_{U,T} - \cos a g_{L,T}\right)}{a(a^2+1)} \right| \leq \frac{2}{a^2} \quad \text{and} \quad \left| \frac{\sin a g_{U,T} - \sin a g_{L,T}}{a(a^2+1)} \right| \leq \frac{2}{a^3}. \tag{EC.3}$$

Then, based on (EC.1) and by combining (EC.2)–(EC.3), we obtain the second statement.

3. From the second statement, we have that

$$\int_0^\infty |\chi_T(a)| da \leq C \left(\int_0^1 (1+a)\, da + \int_1^\infty (a^{-2} + a^{-3})\, da \right) = 3C,$$

where $C = \pi^{-1} \max\{2, \ln(G_U/G_L)\}$. Since the integral is absolutely convergent, it is permissible to interchange the integral sign and the expectation (first equality in the third statement) and the second equality then follows.

4. From (EC.1) and Lemma 11,

$$\int_0^\infty \chi_T(a) da = f(g_{U,T}) - f(g_{L,T}) = \begin{cases} e^{g_{U,T}} - e^{g_{L,T}}, & \text{if } 0 > g_{U,T} > g_{L,T} \\ 1 - e^{g_{L,T}}, & \text{if } g_{U,T} > 0 > g_{L,T} \\ 0, & \text{if } g_{U,T} > g_{L,T} > 0 \end{cases},$$

therefore

$$0 \leq \int_0^\infty \chi_T(a) da \leq e^{g_{U,T}} - e^{g_{L,T}} = \lambda_0 (G_U - G_L) e^{-\Psi_{0,T}}. \tag{EC.4}$$

Then,

$$G_L \mathbb{E}_t^{\mathbb{P}}[e^{-\Psi_{t,T}}] + \frac{e^{\Psi_{0,t}}}{\lambda_0} \int_0^\infty \mathbb{E}_t^{\mathbb{P}} [\chi_T(a)]\, da = G_L \mathbb{E}_t^{\mathbb{P}}[e^{-\Psi_{t,T}}] + \int_{-\infty}^\infty \varepsilon(a) N_t(a) H_t(a) da = \tilde{X}_t$$

from (40), but also from (EC.4)

$$\tilde{X}_t \leq G_U \mathbb{E}_t^{\mathbb{P}}[e^{-\Psi_{t,T}}].$$

■

Proof of Lemma 8. For the first statement,

$$\epsilon^{-1} \mathbb{E}\left[Q \mathbf{1}_{Q<\epsilon}\right] = F_Q(\epsilon) + \epsilon^{-1} \int_0^\epsilon F_Q(dq) \leq 2F_Q(\epsilon).$$

The second statement is true even without the factor of n^{-1} – the sequence $\{\mathbb{E}[Q\mathbf{1}_{\{m \leq Q < m+1\}}]\}_m$ sums to $\mathbb{E}[Q] < \infty$, therefore $\sum_{m=n}^\infty \mathbb{E}[Q\mathbf{1}_{\{m \leq Q < m+1\}}]$ approaches 0 as $n \to \infty$. ■

Proof of Theorem 9. Let

$$Z_t := \mathbb{E}_t^{\mathbb{P}}\left[e^{ia\Psi_{0,T}}\right] = \mathbb{E}_t^{\mathbb{P}}\left[\exp\left(ia\int_0^T r_s\,ds + ia\int_0^T \theta_s^\top\,dW_s + \frac{1}{2}ia\int_0^T \theta_s^\top\theta_s\,ds\right)\right],$$

which is a martingale. Then, given H_t defined in (39), we have that

$$Z_t = \exp\left(ia\int_0^t r_s\,ds + ia\int_0^t \theta_s^\top\,dW_s + \frac{1}{2}ia\int_0^t \theta_s^\top\theta_s\,ds\right)H_t,$$

from which

$$\ln H_t = \ln Z_t - ia\left(\int_0^t r_s\,ds + \int_0^t \theta_s^\top\,dW_s + \frac{1}{2}\int_0^t \theta_s^\top\theta_s\,ds\right). \tag{EC.5}$$

Given the general form of H_t in (42), we have for the last term that

$$\begin{aligned} d(\theta_t^\top D(T-t,a)\theta_t) &= -\theta_t^\top \dot{D}(T-t,a)\theta_t dt - 2\theta_t^\top D(T-t,a)K(\theta_t-\mu_\theta)dt \\ &\quad +2\theta_t^\top D(T-t,a)\Xi\,dW_t + \operatorname{tr}\left(\Xi^\top D(T-t,a)\Xi\right)dt, \end{aligned}$$

with $\dot{D}(s,a)$ denoting the element-wise derivative of $D(s,a)$ with respect to s. Therefore, we get that

$$\begin{aligned} d\ln Z_t &= iar_t dt + ia\theta_t^\top dW_t + \frac{1}{2}ia\theta_t^\top\theta_t dt + \dot{A}(T-t,a)dt + \dot{B}(T-t,a)r_t dt \\ &\quad -B(T-t,a)\left[\zeta_r(\mu_r - r_t)\,dt + \sigma_r^\top dW_t\right] + \dot{C}(T-t,a)^\top\theta_t dt \\ &\quad -C(T-t,a)^\top\left[-K(\theta_t-\mu_\theta)dt + \Xi dW_t\right] + \frac{1}{2}\theta_t^\top\dot{D}(T-t,a)\theta_t dt \\ &\quad +\theta_t^\top D(T-t,a)K(\theta_t-\mu_\theta)dt - \theta_t^\top D(T-t,a)\Xi dW_t - \frac{1}{2}\operatorname{tr}\left(\Xi^\top D(T-t,a)\Xi\right)dt \\ &= \left[ia + \dot{B}(T-t,a) + \zeta_r B(T-t,a)\right]r_t dt + \frac{1}{2}\theta_t^\top\left[ia\mathbf{I} + \dot{D}(T-t,a) + 2D(T-t,a)K\right]\theta_t dt \\ &\quad + \left[\dot{C}(T-t,a)^\top + C(T-t,a)^\top K - \mu_\theta^\top K^\top D(T-t,a)\right]\theta_t dt \\ &\quad + \left[\dot{A}(T-t,a) - \zeta_r\mu_r B(T-t,a) - C(T-t,a)^\top K\mu_\theta - \frac{1}{2}\operatorname{tr}\left(\Xi^\top D(T-t,a)\Xi\right)\right]dt \\ &\quad - \left[-ia\theta_t^\top + B(T-t,a)\sigma_r^\top + C(T-t,a)^\top\Xi + \theta_t^\top D(T-t,a)\Xi\right]dW_t, \end{aligned}$$

where $\mathbf{I}$ denotes the identity matrix and $\dot{A}(s,a)$, $\dot{B}(s,a)$, $\dot{C}(s,a)$ the (element-wise) derivatives of $A(s,a)$, $B(s,a)$, $C(s,a)$ with respect to s. Then,

$$\frac{dZ_t}{Z_t} = d\ln Z_t + \frac{1}{2}\left|\theta_t^\top\left(-ia\mathbf{I} + D(T-t,a)\Xi\right) + B(T-t,a)\sigma_r^\top + C(T-t,a)^\top\Xi\right|^2 dt.$$

Furthermore, from (43),

$$E(T-t,a) = B(T-t,a)\sigma_r + \Xi^\top C(T-t,a) + \Xi^\top D(T-t,a)\theta_t,$$

resulting in

$$\frac{dZ_t}{Z_t} = d\ln Z_t + \frac{1}{2}(E(T-t,a) - ia\theta_t)^\top(E(T-t,a) - ia\theta_t)dt.$$

Since Z is a martingale, we require

$$
\begin{aligned}
0 &= ia + \dot{B}(s,a) + \zeta_r B(s,a), \\
\mathbf{O} &= ia\mathbf{I} + \dot{D}(s,a) + 2D(s,a)K + (-ia\mathbf{I} + D(s,a)\Xi)\left(\Xi^\top D(s,a) - ia\mathbf{I}\right), \\
\mathbf{0} &= \dot{C}(s,a)^\top + C(s,a)^\top K - \mu_\theta^\top K^\top D(s,a) + \left[B(s,a)\sigma_r^\top + C(s,a)^\top \Xi\right]\left[-ia\mathbf{I} + \Xi^\top D(s,a)\right], \\
0 &= \dot{A}(s,a) - \zeta_r \mu_r B(s,a) - C(s,a)^\top K \mu_\theta - \frac{1}{2}\mathrm{tr}\left(\Xi^\top D(s,a)\Xi\right) \\
&\quad + \frac{1}{2}\left[B(s,a)\sigma_r^\top + C(s,a)^\top \Xi\right]\left[\Xi^\top C(s,a) + B(s,a)\sigma_r\right],
\end{aligned}
$$

where $\mathbf{0}$ and $\mathbf{O}$ represent, respectively, the zero vector and the zero matrix. The first equation solves as

$$B(s,a) = -ia\zeta_r^{-1}\left(1 - e^{-\zeta_r s}\right),$$

whereas the others are harder and one has to resort to numerical solutions. Once we have solved the equations, we have

$$dZ_t = -Z_t\left[-ia\theta_t + E(T-t,a)\right]^\top dW_t.$$

Revisiting now (EC.5), we can say that

$$
\begin{aligned}
d\ln H_t &= d\ln Z_t - ia\left(r_t\,dt + \theta_t^\top\,dW_t + \frac{1}{2}\theta_t^\top\theta_t\,dt\right) \\
&= -\left(-ia\theta_t + E\right)^\top dW_t - \frac{1}{2}(-ia\theta_t + E)^\top(-ia\theta_t + E)\,dt - ia\left(r_t\,dt + \theta_t^\top\,dW_t + \frac{1}{2}\theta_t^\top\theta_t\,dt\right) \\
&= -E^\top dW_t - iar_t\,dt + \frac{1}{2}(a^2 - ia)\theta_t^\top\theta_t dt + iaE^\top\theta_t - \frac{1}{2}E^\top E\,dt
\end{aligned}
$$

and, therefore,

$$\frac{dH_t}{H_t} = -E(T-t,a)^\top\,dW_t - iar_t\,dt + \frac{1}{2}(a^2 - ia)\theta_t^\top\theta_t\,dt + iaE(T-t,a)^\top\theta_t\,dt. \tag{EC.6}$$

Finally,

$$
\begin{aligned}
d(HN)_t &= HdN_t + NdH_t + d\langle N,H\rangle_t \\
&= HN\Big\{(1+ia)\left(r_t\,dt + \theta_t^\top dW_t\right) + \frac{1}{2}(1+ia)(2+ia)\theta_t^\top\theta_t dt - iar_t dt \\
&\quad + \frac{1}{2}(a^2 - ia)\theta_t^\top\theta_t dt + iaE^\top\theta_t dt - E^\top dW_t - (1+ia)E^\top\theta_t dt\Big\} \\
&= HN\left\{r_t dt + \left[(1+ia)\theta_t - E\right]^\top(\theta_t dt + dW_t)\right\}.
\end{aligned} \tag{EC.7}
$$

From (EC.6)–(EC.7), we get for $\tilde{X}$ in (41)

$$
\begin{aligned}
d\tilde{X}_t &= G_L H_t(i)\left\{r_t\,dt - E(T-t,i)^\top(\theta_t\,dt + dW_t)\right\} \\
&\quad + \int_{-\infty}^{\infty}\varepsilon(a)H_t(a)N_t(a)\left\{r_t\,dt + \left[(1+ia)\theta_t - E(T-t,a)\right]^\top(\theta_t\,dt + dW_t)\right\},
\end{aligned}
$$

from which (44) follows. Then, a comparison with (33) yields (45). This completes the proof. ■

Proof of Corollary 10. We have that

$$\begin{aligned}\mathbb{P}_t\left(X_T^* \in [G_L, G_U]\right) &= \mathbb{P}_t[g_{L,t} \le \Psi_{t,T} \le g_{U,t}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{a}(e^{-ia\ln G_U} - e^{-ia\ln G_L})e^{-ia\ln\lambda_0 + ia\Psi_{0,t}} H_t(a)\, da \\ &= \lambda_0 e^{-\Psi_{0,t}} \int_{-\infty}^{\infty} (1+ia)\varepsilon(a) N_t(a) H_t(a)\, da.\end{aligned}$$

Therefore,

$$\mathbb{P}_t\left(X_T^* \in [G_L, G_U]\right)(\pi_t^*)^\top \Sigma_t = \left[\int_{-\infty}^{\infty} (1+ia)\theta_t \varepsilon(a) N_t(a) H_t(a)\, da\right]^\top$$

and, from (45), the result (46) follows. ■

EC.3. The conditional behaviour of $\{\theta_u : t \le u \le s\}$ and the simulation of $\bar{\Psi}_{t,s}$

Consider the model of general form

$$\bar{\Psi}_{t,s} = \beta_0\,(s-t) + \beta_1 K_{t,s} + \beta_2 J_{t,s} + \beta_3\left(\theta_s^2 - \theta_t^2\right) + \sqrt{\beta_4 J_{t,s}}\,\mathcal{Z}, \tag{EC.1}$$

where $s > t$, $\mathcal{Z} \sim \mathcal{N}(0,1)$, $K_{t,s} := \int_t^s \theta_u du$, $J_{t,s} := \int_t^s \theta_u^2 du$ and

$$\theta_s = \mu + (\theta_t - \mu)\, e^{-\kappa(s-t)} + \sqrt{\frac{\xi^2}{2\kappa}\left(1 - e^{-2\kappa(s-t)}\right)}\,\tilde{\mathcal{Z}}, \tag{EC.2}$$

where $\tilde{\mathcal{Z}} \sim \mathcal{N}(0,1)$ is independent of $\mathcal{Z}$.

EC.3.1. Conditional distribution of $\int_t^s \theta_u\, du$ given θ_s

The vector $(\theta_s, K_{t,s})^\top$ has a bivariate normal distribution with expectations

$$\begin{pmatrix} m(s) \\ m_K(s) \end{pmatrix} = \begin{pmatrix} y_{t,s} & 1 - y_{t,s} \\ \frac{1}{\kappa}(1 - y_{t,s}) & \frac{1}{\kappa}(1-y_{t,s})(h_{t,s} - 1) \end{pmatrix} \begin{pmatrix} \theta_t \\ \mu \end{pmatrix} \tag{EC.3}$$

and covariance matrix

$$\Sigma = \frac{\xi^2(1 - y_{t,s})}{2\kappa^2} \begin{pmatrix} \kappa(1 + y_{t,s}) & 1 - y_{t,s} \\ 1 - y_{t,s} & \frac{1}{\kappa}(2h_{t,s} - 3 + y_{t,s}) \end{pmatrix}, \tag{EC.4}$$

where $y_{t,s} := \exp(-\kappa(s-t))$ and $h_{t,s} := \kappa(s-t)/(1 - y_{t,s}) = -\ln y_{t,s}/(1 - y_{t,s})$.

It is a standard result that $(K_{t,s}|\,\theta_s)$ is normal with

$$\mathbb{E}_t[K_{t,s}\,|\,\theta_s] = \mu(s-t) + \frac{1 - y_{t,s}}{\kappa(1 + y_{t,s})}(\theta_t + \theta_s - 2\mu)$$

and

$$\mathbb{V}\mathrm{ar}_t[K_{t,s}\,|\,\theta_s] = \frac{\xi^2}{\kappa^3(1 + y_{t,s})}\left[\kappa(1 + y_{t,s})(s-t) - 2(1 - y_{t,s})\right].$$

EC.3.2. Conditional expectation of $\int_t^s \theta_u^2\, du$ given θ_s and $\int_t^s \theta_u\, du$

For any $w \in (t,s)$, the vector $\left(\theta_w, \theta_s, \int_t^s \theta_u\, du\right)^\top$ is a trivariate normal random variable with expectations

$$\begin{pmatrix} m(w) \\ m(s) \\ m_K(s) \end{pmatrix} = \begin{pmatrix} \mu + y_{t,w}(\theta_t - \mu) \\ \mu + y_{t,s}(\theta_t - \mu) \\ \mu(s-t) + \frac{1}{\kappa}\big(1 - y_{t,s}\big)(\theta_t - \mu) \end{pmatrix}$$

and covariance matrix

$$\Sigma = \frac{\xi^2}{2\kappa^2} \begin{pmatrix} \kappa\big(1 - y_{t,w}^2\big) & \kappa y_{w,s}\big(1 - y_{t,w}^2\big) & \big(1 - y_{t,w}\big)\big(2 - y_{w,s} - y_{t,s}\big) \\ \kappa y_{w,s}\big(1 - y_{t,w}^2\big) & \kappa\big(1 - y_{t,s}^2\big) & \big(1 - y_{t,s}\big)^2 \\ \big(1 - y_{t,w}\big)\big(2 - y_{w,s} - y_{t,s}\big) & \big(1 - y_{t,s}\big)^2 & 2(s-t) - \frac{4}{\kappa}(1 - y_{t,s}) + \frac{1}{\kappa}(1 - y_{t,s}^2) \end{pmatrix}.$$

Let's denote by $N(w)$ the inverse of Σ. Then, the joint density of $(\theta_w, \theta_s, K_{t,s})^\top$ takes the form

$$\varkappa \exp\left(-\frac{1}{2}\left[n_{ww}(\theta_w - m(w))^2 + 2n_{ws}(\theta_w - m(w))(\theta_s - m(s)) + 2n_{wk}(\theta_w - m(w))(K_{t,s} - m_K(s)) + \cdots\right]\right),$$

where $\varkappa$ is a constant and the remaining terms are ignored being irrelevant to the aim of working out the conditional distribution of θ_w given both θ_s and $K_{t,s}$. Focusing on the above exponent, we can write that as

$$\frac{1}{2}\left[n_{ww}\theta_w^2 - 2\theta_w\left(n_{ww}m(w) - n_{ws}(\theta_s - m(s)) - n_{wk}(K_{t,s} - m_K(s))\right) + \cdots\right],$$

from which the conditional distribution of θ_w given θ_s and $K_{t,s}$ is normal with variance

$$\mathbb{V}\mathrm{ar}_t\left[\theta_w \middle|\, \theta_s, K_{t,s}\right] = \frac{1}{n_{ww}(w)}$$

and expectation

$$\mathbb{E}_t[\theta_w \mid \theta_s, K_{t,s}] := e(w) := m(w) - \frac{n_{ws}(w)}{n_{ww}(w)}\big(\theta_s - m(s)\big) - \frac{n_{wk}(w)}{n_{ww}(w)}\big(K_{t,s} - m_K(s)\big).$$

Assuming we can invert the covariance matrix, we get

$$\mathbb{E}_t\left[\int_t^s \theta_u^2\, du \,\middle|\, \theta_s, K_{t,s}\right] = \int_t^s \left(\frac{1}{n_{ww}(w)} + e^2(w)\right) dw,$$

where

$$\frac{1}{n_{ww}} = \frac{\xi^2(1-\phi)\big[(1+\phi)(\phi^2 - y_{t,s}^2)h_{t,s} - 4\phi(\phi - y_{t,s})\big]}{2\kappa(1 - y_{t,s})\phi^2\big[(1 + y_{t,s})h_{t,s} - 2\big]}, \tag{EC.5}$$

$$\frac{n_{ws}}{n_{ww}} = \frac{(1-\phi)\big[\phi + y_{t,s} - y_{t,s}(1+\phi)h_{t,s}\big]}{(1 - y_{t,s})\phi\big[(1 + y_{t,s})h_{t,s} - 2\big]}, \tag{EC.6}$$

$$\frac{n_{wk}}{n_{ww}} = -\frac{\kappa(1-\phi)(\phi - y_{t,s})}{(1 - y_{t,s})\phi\big[(1 + y_{t,s})h_{t,s} - 2\big]} \tag{EC.7}$$

and $\phi = \phi(w) := \exp(-\kappa(w-t))$. Noting that only $m(w)$ and $\phi(w)$ depend on w and since the above expressions all involve ϕ, though none w, it is worth changing the variable of integration to get

$$\mathbb{E}_t\left[\int_t^s \theta_u^2\,du \,\middle|\, \theta_s, K_{t,s}\right] = \mathcal{E}_1 + \mathcal{E}_2, \tag{EC.8}$$

where

$$\mathcal{E}_1 := \int_{y_{t,s}}^1 \frac{d\phi}{\kappa\phi n_{ww}} = \frac{\xi^2\left[8 - 5h_{t,s}(1+y_{t,s}) + h_{t,s}^2(1+y_{t,s}^2)\right]}{2\kappa^2\left[(1+y_{t,s})h_{t,s} - 2\right]}$$

and

$$\mathcal{E}_2 := \int_{y_{t,s}}^1 \left(\mu + \phi(\theta_t - \mu) - \frac{n_{ws}}{n_{ww}}\big(\theta_s - m(s)\big) - \frac{n_{wk}}{n_{ww}}\big(K_{t,s} - m_K(s)\big)\right)^2 \frac{d\phi}{\kappa\phi}. \tag{EC.9}$$

Upon substituting (EC.6)–(EC.7) in (EC.9) and defining

$$\begin{aligned}
\mathcal{A} &= \theta_t + \frac{\tilde{\Theta}_s(1 - y_{t,s}h_{t,s}) - \tilde{K}_{t,s}}{(1-y_{t,s})\left[(1+y_{t,s})h_{t,s} - 2\right]}, \quad \mathcal{B} = \frac{-\tilde{\Theta}_s(1-y_{t,s}) + \tilde{K}_{t,s}(1+y_{t,s})}{(1-y_{t,s})\left[(1+y_{t,s})h_{t,s} - 2\right]}, \\
\mathcal{C} &= \frac{y_{t,s}\left(\tilde{\Theta}_s(h_{t,s}-1) - \tilde{K}_{t,s}\right)}{(1-y_{t,s})\left[(1+y_{t,s})h_{t,s} - 2\right]}, \quad \tilde{\Theta}_s = \theta_s - y_{t,s}\theta_t, \quad \tilde{K}_{t,s} = \kappa K_{t,s} - (1-y_{t,s})\theta_t,
\end{aligned}$$

we get the simplified expression

$$\begin{aligned}
\mathcal{E}_2 &= \int_{y_{t,s}}^1 \frac{\left(\mathcal{A}\phi^2 + \mathcal{B}\phi + \mathcal{C}\right)^2}{\kappa\phi^3} d\phi \\
&= \frac{1}{2}\left(\mathcal{A}^2 + \frac{\mathcal{C}^2}{y_{t,s}^2}\right)(1 - y_{t,s}^2) + 2\mathcal{B}\left(\mathcal{A} + \frac{\mathcal{C}}{y_{t,s}}\right)(1 - y_{t,s}) + (\mathcal{B}^2 + 2\mathcal{A}\mathcal{C})h_{t,s}(1 - y_{t,s}).
\end{aligned}$$

EC.3.3. Simulation scheme for $\bar{\Psi}_{t,s}$

The algorithm next is suggested. Assuming that we have already the simulated value $\hat{\theta}_{n\delta}$ at time $t = n\delta$, $\delta > 0$, we proceed to time $s = (n+1)\delta$ as follows, starting with $n = 0$:

1. Obtain a simulated value

$$\hat{\theta}_{(n+1)\delta} = \mu + \left(\hat{\theta}_{n\delta} - \mu\right)e^{-\kappa\delta} + \tilde{\mathcal{Z}}_n\sqrt{\frac{\xi^2}{2\kappa}\left(1 - e^{-2\kappa\delta}\right)},$$

   where $\tilde{\mathcal{Z}}_n \sim \mathcal{N}(0,1)$.
2. Conditional on $\hat{\theta}_{(n+1)\delta}$, obtain a simulated value for $\int_{n\delta}^{(n+1)\delta}\theta_u\,du$ (see Section EC.3.1):

$$\hat{K}_{n\delta,(n+1)\delta} = \mu\delta + \frac{1 - e^{-\kappa\delta}}{\kappa\left(1 + e^{-\kappa\delta}\right)}\left(\hat{\theta}_{n\delta} + \hat{\theta}_{(n+1)\delta} - 2\mu\right) + \bar{\mathcal{Z}}_n\sqrt{\frac{\xi^2}{\kappa^3}\left[\kappa\delta\left(1 + e^{-\kappa\delta}\right) - 2\left(1 - e^{-\kappa\delta}\right)\right]},$$

   where $\bar{\mathcal{Z}}_n \sim \mathcal{N}(0,1)$ is independent of $\tilde{\mathcal{Z}}_n$.
3. Calculate the conditional expectation, $\mathcal{E}_{(n+1)\delta}$, of $\int_{n\delta}^{(n+1)\delta}\theta_u^2\,du$ given $\hat{\theta}_{(n+1)\delta}$ and $\hat{K}_{n\delta,(n+1)\delta}$ using the formula in equation (EC.8).
4. From (EC.1), set

$$\hat{\bar{\Psi}}_{n\delta,(n+1)\delta} = \beta_0\delta + \beta_1\hat{K}_{n\delta,(n+1)\delta} + \beta_2\mathcal{E}_{(n+1)\delta} + \beta_3(\hat{\theta}_{(n+1)\delta}^2 - \hat{\theta}_{n\delta}^2) + \mathcal{Z}_n\sqrt{\beta_4\mathcal{E}_{(n+1)\delta}},$$

   where $\mathcal{Z}_n \sim \mathcal{N}(0,1)$ is independent of both $\tilde{\mathcal{Z}}_n$ and $\bar{\mathcal{Z}}_n$.

We conclude this section by noting the efficiency of the proposed simulation scheme. Although there is some inherent bias due to the approximation of $(J_{t,s} \mid \theta_s, K_{t,s})$ by its expected value, this is negligible over each range $(n\delta, (n+1)\,\delta)$ for sufficiently small δ. In addition, our scheme saves considerably on computing power but also ease of use compared to the, otherwise accurate, method of Li and Wu (2019) coupled with repeated transform inversion or the stylish general simulation scheme of Cui *et al.* (2021) based on Markov chain approximations, especially when it is necessary to generate entire sample trajectories as in our case. For brevity, we do not report here more numerical results which illustrate this, but we can make these available upon request. If further accuracy is needed, it is possible to expand our method by incorporating the conditional variance of $(J_{t,s} \mid \theta_s, K_{t,s})$, which can be derived using an extended approach of the one shown here for the conditional mean; again, in the interest of space, details are currently omitted. Having access to both the conditional mean and variance, one can then implement a moment-matching technique along the lines of Chen *et al.* (2012), or Kyriakou *et al.* (2021a) when higher conditional moments are in hand.