THE CITY UNIVERSITY
CONTROL ENGINEERING CENTRE
DEPARTMENT OF ELECTRICAL, ELECTRONIC AND INFORMATION ENGINEERING
LONDON EC1V 0HB

THESIS SUBMITTED
FOR THE AWARD OF THE Ph.D DEGREE
in CONTROL ENGINEERING

A UNIFIED APPROACH TO DECENTRALISED CONTROL BASED ON THE EXTERIOR ALGEBRA AND ALGEBRAIC GEOMETRY METHODS

B. A. LAIOS , BSc(Hons), MIEEE, MAIAA, AMIEE.

NOVEMBER 1990

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my sincere gratitude to my thesis supervisor and great teacher Dr. Nicos Karcanias. His enthusiasm, expertise and active participation in this project since he has worked as hard as I, have helped me to carry out this programme of research despite those periods of despair well known to research students. My long term association with him not only determined the contents of this work, but has also profoundly influenced the spirit and scientific method on the thesis as a whole. For all this and many more I shall always be grateful.

I would also like to express my thanks to my friend Dr. P. Moukas for many stimulating discussions and for his excellent job in typing and finishing an extremely difficult thesis after being abandoned by two previous secretaries.

I am very grateful to Prof. Peter Roberts, Director of the Control Engineering Centre for his generous moral support, his interest in the progress of my work and sympathetic understanding.

It gives me great pleasure to record my indebtedness to Prof. T.G. Koussiouris who has carefully scrutinised the whole thesis and eliminated numerous errors. He also provided invaluable help through his assistance with the preparation of the final script and the checking of all the mathematical results.

My long term undergraduate and postgraduate studies at City University leading to this thesis have been made possible by the unbounded support of my father Aristotle. Without his undertaking the full range of responsibilities during all those years, my studies could never materialise. In recognition of his tremendous support, I dedicate this thesis to him.

Last, but not least, special thanks to Joan Rivellini for all her unfailing courtesy, help, encouragement and keeping the spirits high at the Centre.

DECLARATION

CONTENTS

ABSTRACT

The aim of the thesis is to provide a unifying framework and tools for the study of a number of control theory problems arising in Decentralised control. These problems are known as frequency assignment problems and they include the decentralised problem of pole assignment by state, output feedback and zero assignment by decentralised squaring-down. It is further shown that decentralised dynamic problems where the dynamic complexity of the controller is fixed , may also be reduced to the same formulation given for the constant Determinantal Assignment Problem(DAP). The unifying mathematical problem studied here is the Decentralised - Determinantal Assignment Problem (D-DAP) which is a special form of the general DAP defined for multivariable systems.

The study of D-DAP involves the use of tools from exterior algebra in an essential way and also tools from classical algebraic geometry, specifically the theory of Grassmann varieties. In this thesis the mathematical framework of D-DAP is fully developed and used for the study of pole, zero assignment problems by decentralised constant controllers and new solvability conditions are given.

The mathematical framework of D-DAP also allows the study of structural properties of Decentralised controllers. In fact new invariants for decentralised control are introduced in the form of Plücker matrices and Decentralised Indices and a new characterisation of fixed modes and fixed zeros is given based on tools from exterior algebra.The classical notion of fixed modes, fixed zeros is extended to almost fixed modes, almost fixed zeros for certain families of systems.

Finally it is shown that the general framework of D-DAP is suitable for the study of dynamic problems such as pole assignment by decentralised classical controllers of the P-D, P-I, P-I-D type.

LIST OF SYMBOLS AND ABBREVIATIONS

Throughout this thesis, the following symbols and abbreviations will be used:

$\mathbb{R}$, $\mathbb{C}$;	the field of real, complex numbers respectively
$\mathbb{R}[s]$;	the ring of polynomials over $\mathbb{R}$
$\mathbb{R}(s)$;	the field of rational functions
$\mathbb{R}^n$, $\mathbb{C}^n$,$\mathbb{R}^n(s)$;	n−dimensional vector spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ respectively
$\mathbb{R}^{n \times n}$;	the set of matrices with elements from $\mathbb{R}$
$\mathcal{R}\{A\}$, $\rho\{A\}$;	the range space, the rank of a linear transformation A
$\mathcal{N}_r\{A\}$, $\mathcal{N}_l\{A\}$;	the right, left null spaces of a linear transformation A
$\mathcal{V}$, $\underline{x}$;	V is a vector space, x is a vector
$\underline{x}_1 \wedge \underline{x}_2, \cdots, \wedge \underline{x}_k$;	The exterior product of k−vectors
$\wedge^p \mathcal{V}$;	the p−exterior power of the vector space V
$C_p(A)$;	the p−th compound matrix of A
AZ	Almost Zero
AFZ	Almost Fixed Zero
CA	Completely Assignable
CPA	Completely Pole Assignable
CZA	Completely Zero Assignable
DAP	Determintal Assignment Problem
DAZ	Decentralised Almost Zero
DC	Decentralised Characteristic
D−DAP	Decentralised Determintal Assignment Problem
D−COC	Decentralised Constant Output Controller

D−DOC	Decentralised Dynamic Output Controller
D−PAP	Decentralised Pole Assignment Problem
D−ZAP	Decentralised Zero Assignment Problem
D−ℝ[s]−GR	Decentralised Polynomial Grassmann Representative
D−SUS	Decentralised Strongly Unstable
DOF	Dynamic Output Feedack
DPM	Decentralised Plucker Matrix
DSF	Decentralised State Feedback
DSD	Decentralised Squaring Down
DSNA	Decentralised Strongly Non Assigned
EUS	Entire Un-stable
FM, FZ	Fixed Mode, Fixed Zero
GCD	Greatest Common Divisor
GPA	Generically Pole Assignable
GR	Grassmann Representative
GZA	Generically Zero Assignable
LEF	Left Echelon Form
LCMFD	Left Coprime Matrix Fraction Description
LNA	Linearly Non Assignable
MFD	Matrix Fraction Description
PAZ	Prime Almost Zero
P−D	Proportional Derivative
P−I	Proportional Integral
P−I−D	Proportional Integral Derivative
PNA	Pole Non Assignable
QPR	Quadratic Plucker Relations
RCMFD	Right Coprime Matrix Fraction Description
RQPR	Reduced Quadratic Plucker Relations
ℝ[s]−GR	Polynomial Grassmann Representative
TFM	Transfer Function Matrix
ZNA	Zero Non Assignable

Chapter 1

INTRODUCTION

For the study of the structural properties of linear systems as well as the analysis and design of control systems various methods have been developed.

The pioneering work of Kalman on the state space description of dynamic systems was followed by Rosenbrock and Popov who extended the classical transfer function method and ideas to the multivariable systems. Rosenbrock's work,[Ros.1], on the algebraic approach based on the polynomial matrix theory provided the framework for the generalisation of the classical frequency response techniques for single input single output (SISO) systems to the multiple input multiple output (MIMO) systems. The state-space and the algebraic methods are only two extremes of a whole spectrum of possible descriptions for linear systems. Although we can work exclusively in the time domain (state-space) or frequency domain (transfer function) we can also transfer results from one framework to the other.

The state-space method is well developed and is suited for the study of system properties such as controllability,observability, minimality, redundancy, etc. The structural characteristics generally known as invariants are well developed and have been used for the solution of control problems such as pole shifting, quadratic regulators, tracking algorithms, state estimators, decoupling controllers, hierarchical controllers etc. The popularity of state-space methods is closely related to the availability of powerful computational tools from numerical linear algebra implemented on digital computers. However when it comes to practice, state-space methods lacked the spectacular success of the classical methods due mainly to the lack of accuracy of the mathematical models; i.e., A, B, C matrices in representing the real world. The sensitivity to uncertainty of the signals and system dynamics as well as the failure to cope with time delays, led many people to take another look to the classical transfer function approach.

At the beginning various control engineers explored the relationship between state-space models and transfer function descriptions. From their work those people established two fundamental methods that are the main ingredients of the transfer function approach The first method known as the algebraic approach treats the system as an operator between rational vector spaces and the basic tools are algebraic (polynomials, polynomial matrices, integral matrices, theory of rings).The other method views the system as a map between spaces of periodic signals and thus its tools are those of complex analysis. The algebraic approaches are most suitable for addressing structural questions related to the system and design problems. The complex variable approaches on the other hand are most suitable for describing quantitative properties and design problems.

The traditional task of control design has been that of selecting the dynamics of a given type control structure that is fully coupled i.e., multivariable controllers,simple diagonal, decentralised, hierarchical controllers e.t.c. The problem of selection of control structure has been overlooked so far and has been simply addressed as a problem of engineering specifications (constraints) . The selection of control structures is an important issue where control theory may also contribute (EPIC project). Techniques so far for the selection of control structures have originated in the chemical and aerospace engineering and they are mostly of empirical nature. The way control theory may contribute is by providing tests, criteria which may exclude bad choices of the control structure . The purpose of this thesis is to develop tools for the selection of control structure and design of controllers after system structure has being decided for problems referred to as frequency assignment.

Of course hybrid approaches combining both approaches have recently emerged and a typical representative of the new philosophies is the algebrogeometric approach which represents a specific school of thought for tackling system design problems collectively known as the algebrogeometric assignment problems. Within this framework two basic approaches have emerged (i) the modern algebraic geometry approach and (ii) the exterior algebra or classical algebrogeometric approach. The first approach considers the plant and the controller as elements of algebraic varieties of an affine space and studies the solvability of pole assignment by state or output feedback, simultaneous stabilisation e.t.c., by using tools from modern algebraic geometry . Important conditions for generic solvability of control problems have been derived within this framework . These conditions may be used as criteria for the selection of inputs,outputs e.t.c., The main disadvantage of the approach is that the nongeneric cases are difficult to handle and that no procedures for computing the controllers have been suggested.

The second approach introduced by [Kar.1] is referred to as Determinantal Assignment Problem (DAP) which has been formulated as a unifying method to all problems of frequency assignment nature. This is based on the idea that determinantal problems are of multilinear nature and thus they may naturally split to a linear problem and a multilinear problem according to the decomposability nature of the multivectors. The solution is thus reduced to the solvability of a set of linear equations characterising the linear problem together with a set of quadratic equations for the multilinear problem of decomposability. Classical algebraic geometry makes use of a projective space instead of an affine to determine the existence of solutions. This approach relies heavily on exterior algebra to provide the necessary tools as well as the new set of invariants, which characterise the solvability of the problem.

The distinct advantages of the DAP approach with respect to the first are that it provides the means for computing the solutions , it can handle both generic and exact solvability investigations and introduces new criteria for the characterisation of solvability of different problems .The computation of solutions is reduced to an optimisation problem of a function with quadratic equality constraints . The development of an algorithmic technique is essential for the method to become a C.A.D. tool for frequency assignment.

The types of problems which have been tackled by DAP so far are pole, zero assignment problems with centralised compensators. The extension of the approach so that it can handle similar design problems for the decentralised control of large scale systems is the main theme of this work. The main characteristic of a decentralised controller is that the controller is not fully coupled and restricted but its structure is partially fixed in our case because of the decentralised assumption. This is not a trivial extension of DAP as far as solvability of problems is concerned since we are now forced to work with varieties in a projective space (subvarieties of Grassmann varieties) for which the theory is not developed. Although in this thesis we deal with decentralised problems arising in large scale systems, the methodology equally applies to general partially fixed structure controller where the matrices are not of the block diagonal structure. An attempt is also made to show that this general framework of DAP, either decentralised or centralised may also be extended to deal with certain dynamic problems such as pole, zero assignment by Proportional plus Integral (P-I) , Proportional plus Derivative (P-D), or Proportional plus Integral plus Derivative (P-I-D) multivariable controllers of general architecture; i.e., decentralised or centralised.

The extension of the DAP framework to the case of D-DAP enhances our understanding about structural issues of decentralised control, introduces new invariants such as the Decentralised Indices and D-Plücker matrices, provide a new characterisation of fixed modes and fixed zeros as well a novel method for their computation and leads to an extension of the exact notion of fixed modes and zeros to those of almost fixed modes, fixed zeros. Those concepts are related to the limitations of feedback design for certain families of systems. Finally new criteria for decentralised pole, zero assignment are derived based on the properties of D-Plucker matrices and relationships of the decentralised groupings of input-output or Forney dynamical order. Note that these conditions are necessary but not sufficient because the key tool for deriving such conditions i.e., the D- Grassmann variety is a mathematical object which has not properly studied in the control literature unlike the centralised case where the literature is quite rich. This is the standard theory of Grassmann variety and its subvarieties in a projective space.

The structure of the thesis is as follows. In chapter 2 we provide a comprehensive introduction to the fundamental algebraic tools which form the backbone of the DAP method. Emphasis is given to the representation theory of multilinear maps and its relationship to compound matrix theory. The existence of the so called Plücker embedding, an essential tool behind the introduction of the new invariants associated with the approach is also clarified. A short introduction to the standard polynomial and rational theory is also given in this chapter. Chapter 3 is a survey of the basic results from decentralised control and in chapter 4 we formulate the Decentralised frequency assignment problems of pole, zero assignment as to a different version of the Decentralised-Determinantal Assignment Problem (D-DAP) . It is also there where it is shown that D-DAP splits naturally into a linear problem of zero assignment of polynomial combinants and a standard multilinear problem for the decomposability of multivectors with additional constraints imposed by the decentralisation structure.

In chapter 5 we examine the implications of decentralised assumptions on the projective invariants associated with DAP as well as the nature of a new variety to be known as Decentralised Grassmann variety. The notion of Decentralisation Index emerges as quite central since it defines the number of additional constraints i.e., zero Plücker coordinates due to the decentralised nature of the problem , the reduced Grassmann representative and the Plücker matrices. A novel technique for the computation of the decentralisation index is also given there.

In chapter 6 we explore the decentralised index and the structure of the decentralised Grassmann representative to provide a novel characterisation of fixed modes, fixed zeros as well as a procedure for the computation of the fixed pole, fixed zero polynomial. Using the notion of almost zeros of a set of polynomials introduced by [Kar.1] we extend the notion fixed modes, fixed zeros to those of almost fixed modes, zeros. Almost zeros are connected to restrictions of frequency mobility and this is expressed by the so called trapping discs. A technique for the evaluation of those discs is also given in chapter 6.

We then proceed in chapter 7 to investigate the properties of Decentralised Grassmann variety and to define new necessary conditions for Decentralised pole assignment by state, output feedback and Decentralised zero assignment for squaring down problems.

An essential part of the work here is the computation of the dimension of the dynamic varieties by using the standard intersection variety theory. The solvability conditions are expressed in terms of the rank properties of the Decentralised Plücker matrix and relationships between the Decentralised input-output groupings, the McMillan degree, and the Forney indices. Finally in this chapter a variety of dynamic problems, decentralised and centralised, are considered from the control point of view using multivariable P-I, P-D, P-I-D controllers and is shown there that these problems may also be tackled within the framework of DAP or D-DAP using the tools of this work. Finally chapter 8 concludes the present work and draws the attention to some future research directions.

Chapter 2

EXTERIOR ALGEBRA AND POLYNOMIAL MATRICES

2.0 Introduction

This chapter introduces the main concepts of exterior algebra and some fundamental properties of polynomial matrices.

Section 1 provides a definition of a multilinear function while 2 describes the exterior powers of a general vector space. Then section 3 specialises the results of the two previous sections to the case of a finite dimensional vector space. Section 4 gives some properties of exterior powers of linear maps, which are also expanded in the following section where the compound matrices and Grassmann products are defined and their properties examined. Section 6 is used to introduce the Plücker co-ordinates, projective spaces, Grassmann manifold and the Grassmann variety. The decomposability of vectors is studied in section 7 which also defines the Quadratic Plücker relationships (QPR). Section 8 provides a break with the geometric approach used so far to return to the study of some algebraic properties of polynomial matrices.

It should be stressed that the above tools from exterior algebra will be used throughout the thesis to define a complete basis-free invariant for any rational vector space which are usually associated with linear dynamic systems. It will be shown later on in chapter 5 that for a rational vector space $\mathcal{V}$, the canonical Grassmann representative and the associated Plücker matrix completely characterise $\mathcal{V}$.

2.1 Multilinear Vectors—Multivectors

Let $\mathcal{V}$ and $\mathcal{U}$ be vector spaces over a field $\mathcal{F}$ of characteristic 0. A *p-linear map* from $\mathcal{V}$ to $\mathcal{U}$ is a map ϕ: $\overset{p}{\underset{1}{\times}}\mathcal{V} \rightarrow \mathcal{U}$ which is linear with respect to each argument, i.e.

$$\phi(\underline{x}_1, \cdots, \lambda\underline{x}_i + \mu\underline{y}_i, \cdots, \underline{x}_p) = \lambda\phi(\underline{x}_1, \cdots, \underline{x}_i, \cdots, \underline{x}_p) + \mu\phi(\underline{x}_1, \cdots, \underline{y}_i, \cdots, \underline{x}_p) \tag{2.1}$$

where λ, $\mu \in \mathcal{F}$ and $\underline{x}_i$, $\underline{y}_i \in \mathcal{V}$. A p-linear map from $\mathcal{V}$ to $\mathcal{F}$ is called a *p-linear function* in $\mathcal{V}$.

A p-linear map ϕ: $\overset{p}{\underset{1}{\times}}\mathcal{V} \rightarrow \mathcal{U}$ is called *skew-symmetric* if for every permutation σ, $\sigma \in S_p$ (S_p is the group of permutations of p objects)

$$\phi(\underline{x}_{\sigma(1)}, \cdots, \underline{x}_{\sigma(p)}) = \text{sign } \sigma \cdot \phi(\underline{x}_1, \cdots, \underline{x}_p) \tag{2.2}$$

where sign σ is the sign of the permutation. Every p-linear map ϕ from $\mathcal{V}$ to $\mathcal{U}$ determines a skew symmetric p-linear map ψ which is given by:

$$\psi = \sum_{\sigma} \text{sign } \sigma \cdot \sigma \cdot \phi \tag{2.3}$$

Example (2.1): Determinants provide an example of multilinearity. For instance, the determinant d(A) of an n×n matrix,A, with entries in $\mathcal{F}$ is a function of the columns of A. Let $A = [\underline{a}_1, \cdots, \underline{a}_n] \in \mathcal{F}^{n \times n}$, then the determinant d(A) is a function d: $\overset{n}{\underset{1}{\times}}\mathcal{F}^n \rightarrow \mathcal{F}$ for which

$$d(\underline{a}_1, \cdots, \lambda \underline{a}_i + \mu \underline{a}_j, \cdots, \underline{a}_n) = \lambda d(\underline{a}_1, \cdots, \underline{a}_i, \cdots, \underline{a}_n) + \mu(\underline{a}_1, \cdots, \underline{a}_j, \cdots, \underline{a}_n)$$

So the determinant is a an n-linear skew symmetric function of $\mathcal{F}^n$. □

Proposition (2.1): Let ϕ be a p-linear map from $\mathcal{V}$ to $\mathcal{U}$. Then the following conditions are equivalent:

(i) ϕ is skew symmetric;

(ii) $\phi(\underline{x}_1, \cdots, \underline{x}_p) = 0$ whenever $\underline{x}_i = \underline{x}_j$ for some pair (i, j): $i \neq j$

(iii) $\phi(\underline{x}_1, \cdots, \underline{x}_p) = 0$ whenever the vectors $\underline{x}_1, \cdots, \underline{x}_p$ are linearly dependent. □

2.2 Exterior Powers of a Vector Space [Gre. 1]

Let $\mathcal{V}$ be an arbitrary vector space and $p \geq 2$ be an integer. Then a vector space $\wedge^p\mathcal{V}$ together with a skew symmetric p-linear map

$$\wedge^p : \overset{p}{\underset{1}{\times}}\mathcal{V} \rightarrow \wedge^p\mathcal{V} \tag{2.4}$$

is called a *p-th exterior power of* $\mathcal{V}$ if the following conditions are satisfied:

(i) The vectors $\wedge^p(\underline{x}_1, \cdots, \underline{x}_p)$ generate $\wedge^p\mathcal{V}$.

(ii) If ψ is any skew symmetric p-linear map of $\overset{p}{\underset{1}{\times}} \mathcal{V}$ into an arbitrary vector space $\mathcal{U}$ then there exists a linear map

$$f: \wedge^p \mathcal{V} \rightarrow \mathcal{U} \text{ such that } \psi = f \circ \wedge^p.$$

It is proved that conditions (i) and (ii) are equivalent to the following condition:

(iii) If ψ is any skew symmetric p-linear map of $\overset{p}{\underset{1}{\times}} \mathcal{V}$ into a vector space $\mathcal{U}$ then there exists a unique linear map f: $\wedge^p \mathcal{V} \rightarrow \mathcal{U}$ such that $\psi = f \circ \wedge^p$ □

The elements of $\wedge^p\mathcal{V}$ are called *p-vectors.* A p-vector of the form $\wedge^p(\underline{x}_1, \cdots, \underline{x}_p)$ is called *decomposable*, and is denoted by $\underline{x}_1 \wedge \cdots \wedge \underline{x}_p$. Condition (i) states that $\wedge^p\mathcal{V}$ is generated by its decomposable elements.

The skew symmetric property of the p-linear map $\wedge^p$ implies that for every permutation $\sigma \in S_p$

$$\underline{x}_{\sigma(1)} \wedge \cdots \wedge \underline{x}_{\sigma(p)} = \text{sign } \sigma \cdot \underline{x}_1 \wedge \cdots \wedge \underline{x}_p \tag{2.5}$$

Now suppose that $\{\underline{x}_1, \cdots, \underline{x}_p\}$ are linearly dependent vectors. Then the skew symmetry of $\wedge^p$ implies that:

$$\underline{x}_1 \wedge \cdots \wedge \underline{x}_p = 0 \tag{2.6}$$

Conversely, p-vectors which satisfy (2.6) are linearly dependent. The vector space $\wedge^p\mathcal{V}$ may always be defined and it is a subspace of the p-th tensorial power of $\mathcal{V}$, $\otimes^p\mathcal{V}$.

2.3 Exterior Algebra Over a Vector Space of Finite Dimension.

The results and definitions given above for general vector spaces will be specialised and discussed in more detail for the case of finite dimensional vector spaces. Suppose that $\mathcal{V}$ is a vector space of dimension n over the field $\mathcal{F}$. The p-th exterior power of $\mathcal{V}$, $\wedge^p\mathcal{V}$ may always be defined; $\wedge^p\mathcal{V}$ is a vector subspace of the p-th tensorial power of $\mathcal{V}$, $\otimes^p\mathcal{V}$. The pair $(\wedge^p\mathcal{V}, \wedge^p)$ is uniquely defined up to an isomorphism. If $\{\underline{e}_i, i = 1, \cdots, n\}$ is a basis of $\mathcal{V}$, then the products:

$$\underline{e}_{i_1}\wedge\underline{e}_{i_2}\wedge\cdots\wedge\underline{e}_{i_p} \quad 1 \le i_1 \le i_2 \le \cdots \le i_p \le n \tag{2.8}$$

span the vector space $\wedge^p\mathcal{V}$. There are $\binom{n}{p}$ choices of distinct indices $i_1, \cdots, i_p$ from 1 to n, and they can be arranged uniquely in increasing order. It can be proved that the elements defined above $\{\underline{e}_{i_1}\wedge, \cdots\wedge\underline{e}_{i_p}\}$ are linearly independent and thus form a basis for $\wedge^p\mathcal{V}$. Clearly

$$\dim \wedge^p\mathcal{V} = \binom{n}{p}, \quad p = 0, 1, \cdots, n \tag{2.9}$$

and $\wedge^p\mathcal{V} = 0$ for $p > n$.

An arbitrary vector space $\wedge^p\mathcal{V}$ is a p-vector and an element of the form $\underline{x}_1\wedge\cdots\wedge\underline{x}_p$ where $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_p \in \mathcal{V}$ is decomposable. Every p-vector $\underline{u}$ of $\wedge^p\mathcal{V}$ can be uniquely represented in the form:

$$\underline{u} = \sum_{<} a_{i_1 i_2,\cdots,i_p} \ \underline{e}_{i_1}\wedge\underline{e}_{i_2}\wedge\cdots\wedge\underline{e}_{i_p} \tag{2.10}$$

where the symbol $<$ indicates that the indices $(i_1, \cdots, i_p)$ are ordered lexicographically $(1 \le i_1 \le i_2 \le \cdots \le i_p \le n)$. The coefficients $a_{i_1 i_2,\cdots,i_p}$ are called the *co-ordinates of the p-vector* $\underline{u}$ *with respect to the basis* $\{\underline{e}_i, i = 1, \cdots, n\}$ of $\mathcal{V}$.

2.4 Exterior Powers of Linear Maps: Determinants

Theorem (2.1) [Bir. 1]: Let $\mathcal{V},\mathcal{U}$ be finite dimensional vector spaces over a field $\mathcal{F}$, and let $G: \mathcal{V}\to\mathcal{U}$ be a linear map. Then, there is a unique homomorphism $\hat{G}: \wedge\mathcal{V}\to\mathcal{U}$ of the exterior algebras such that $\hat{G}(\underline{x}) = G(\underline{x})$ for any $\underline{x} \in \mathcal{V}$. Notice that $\hat{G}$ maps $\wedge^p\mathcal{V}$ to $\wedge^p\mathcal{U}$ for all p. □

The homomorphism $\hat{G}$ is a linear map. The result simply means the following: If G is a linear map of a vector space $\mathcal{V}$ into a vector space $\mathcal{U}$ over $\mathcal{F}$, then to $(\underline{x}_1, \cdots, \underline{x}_p), \in \overset{p}{\underset{1}{\times}}\mathcal{V}$ we may correspond the element $G(\underline{x}_1)\wedge\cdots\wedge G(\underline{x}_p)$ of $\wedge^p\mathcal{U}$. This defines an alternating multilinear map ψ of $\overset{p}{\underset{1}{\times}}\mathcal{V}$ into $\wedge^p\mathcal{U}$. By the definition of the exterior product there exists a unique linear map $\hat{G}$ of: $\wedge^p\mathcal{V}\to\wedge^p\mathcal{U}$ such that

$$\hat{G}(\underline{x}_1, \cdots, \underline{x}_p) = G(\underline{x}_1)\wedge\cdots\wedge G(\underline{x}_p) \tag{2.11}$$

We may write $\wedge^p G$ for $\hat{G}$ and we call it the *p-th exterior power of the linear map* G. Clearly,

$$\wedge^p G(\underline{x}_1, \cdots, \underline{x}_p) = G(\underline{x}_1)\wedge\cdots\wedge G(\underline{x}_p) \tag{2.12}$$

Equation (2.12) defines the linear map $\wedge^p G$ of $\wedge^p \mathcal{V}$ into $\wedge^p \mathcal{U}$. An important property of $\wedge^p G$ is defined below:

Corollary (2.1) [Mar. 1]: Let $F: \mathcal{V} \to \mathcal{U}$ and $G: \mathcal{U} \to \mathcal{W}$ be linear maps of finite dimensional vector spaces over $\mathcal{F}$. Let $H = G \circ F$. Then

$$\wedge^p (G \circ F) = \wedge^p H = \wedge^p G \circ \wedge^p F \qquad \square$$

2.5 Representation Theory of Exterior Powers of Linear Maps [Kar. 1] [Mar. 1]

2.5.1 Definitions and Basic Results

Let $\mathcal{V}$ be an m-dimensional vector space over the field $\mathcal{F}$ and let $\wedge^p \mathcal{V}$, $p \leq m$ be the p-th exterior power of $\mathcal{V}$. If $\{\underline{v}_i, i = 1, \cdots, m\}$ is a basis of $\mathcal{V}$, then $\wedge^p \mathcal{V}$ is spanned by the vectors of the basis $\{\underline{v}_{\underset{\omega}{\wedge}}, \omega = (i_1, \cdots, i_p), 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq m, \underline{v}_{\underset{\omega}{\wedge}} = \underline{v}_{i_1} \wedge \underline{v}_{i_2} \wedge \cdots \wedge \underline{v}_{i_p}\}$. Every vector $\underline{v} \in \wedge^p \mathcal{V}$ may be written as $\underline{v} = \sum_{\omega} a_\omega \underline{v}_{\underset{\omega}{\wedge}}$.

Let $r^p_{\mathcal{V}}$ be the map of $\wedge^p \mathcal{V}$ into $\mathcal{F}^{\binom{m}{p}}$ defined by:

$$r^p_{\mathcal{V}}(\underline{v}) = [\ \cdots, a_w, \cdots\ ] \tag{2.13}$$

Then $r^p_{\mathcal{V}}$ is linear and it is called the *representation map* of $\wedge^p \mathcal{V}$ associated with the basis $\{\underline{v}_i, i = 1, \cdots, m\}$. It can be seen that there is such map associated to every basis of $\mathcal{V}$. The image of $\wedge^p \mathcal{V}$ under this map is called the representation of $\wedge^p \mathcal{V}$ relative to the basis $\{\underline{v}_i, i = 1, \cdots, m\}$ of $\mathcal{V}$. The following result can be easily verified.

Proposition (2.2): The representation of the p-th exterior power of an m-dimensional vector space $\mathcal{V}$ over $\mathcal{F}$, are isomorphisms of $\wedge^{p}\mathcal{V}$ onto $\mathcal{F}^{\binom{m}{p}}$. □

Let $\mathcal{V}$,$\mathcal{U}$ be two vector spaces over the field $\mathcal{F}$ of dimensions m, n, respectively and let h be a linear map of $\mathcal{V}$ into $\mathcal{U}$. The linear map h can be represented with respect to the bases $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$ and $B_{\mathcal{U}} = \{\underline{u}_i,\ i = 1, \cdots, n\}$ of $\mathcal{V}$ and $\mathcal{U}$ by a matrix $H_{\mathcal{V}}^{\mathcal{U}}$ which is defined by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "h"] \arrow[d, "r^{1}_{\mathcal{V}}"'] & \mathcal{U} \arrow[d, "r^{1}_{\mathcal{U}}"] \\
\mathcal{F}^{m} \arrow[r, "H^{\mathcal{V}}_{\mathcal{U}}"'] & \mathcal{F}^{n}
\end{tikzcd}$$

where $r^{1}_{\mathcal{V}}$, $r^{1}_{\mathcal{U}}$ are the representation maps of $\mathcal{V}$ and $\mathcal{U}$ onto $\mathcal{F}^{m}$ and $\mathcal{F}^{n}$ respectively. Because $\mathcal{V}$, $\mathcal{U}$ are isomorphic to $\mathcal{F}^{m}$, $\mathcal{F}^{n}$ respectively, $\mathcal{F}^{m}$, $\mathcal{F}^{n}$ may be used to represent $\mathcal{V}$, $\mathcal{U}$ and the matrix $H^{\mathcal{V}}_{\mathcal{U}}$ to represent the linear map h.

Let $\wedge^{p}\mathcal{V}$, $\wedge^{p}\mathcal{U}$ be the p-th exterior powers of $\mathcal{V}$, $\mathcal{U}$ respectively, where $p \leq \min(m,n)$. Then $h:\mathcal{V}\rightarrow\mathcal{U}$ implies the existence of a linear map $\wedge^{p}h$: $\wedge^{p}\mathcal{V}\rightarrow\wedge^{p}\mathcal{U}$. If we denote by $r^{p}_{\mathcal{V}}$, $r^{p}_{\mathcal{U}}$ the representation maps of $\wedge^{p}\mathcal{V}$, $\wedge^{p}\mathcal{U}$ with respect to the bases $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$ and $B_{\mathcal{U}} = \{\underline{u}_i,\ i = 1, \cdots, n\}$ of $\mathcal{V}$, $\mathcal{U}$ respectively, then applying the representation result for linear maps, we have the following commutative diagram:

$$\begin{tikzcd}
\wedge^{p}\mathcal{V} \arrow[r, "\wedge^{p}h"] \arrow[d, "\underline{v} = \sum_{\omega} a_{\omega}\underline{v}_{\omega}\wedge", "r^{p}_{\mathcal{V}}"'] & \wedge^{p}\mathcal{U} \arrow[d, "\underline{u} = \sum_{\rho} a_{\rho}\underline{u}_{\rho}\wedge"', "r^{p}_{\mathcal{U}}"] \\
\mathcal{F}^{\binom{m}{p}} \arrow[r, "\wedge^{p}H^{\mathcal{V}}_{\mathcal{U}}"] & \mathcal{F}^{\binom{n}{p}}
\end{tikzcd}$$

and thus the matrix $\wedge^{p}H^{\mathcal{V}}_{\mathcal{U}}$ is defined by the equation

$$\wedge^p H_{\mathcal{U}}^{\mathcal{V}} \cdot \begin{bmatrix} \vdots \\ a_\omega \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ b_\rho \\ \vdots \end{bmatrix} \tag{2.14}$$

The description of $\wedge^p H_{\mathcal{U}}^{\mathcal{V}}$ in terms of $H_{\mathcal{U}}^{\mathcal{V}}$ will be defined below and that will establish the links between the present subject and the compound matrix theory.

Let $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$, $B_{\mathcal{U}} = \{\underline{u}_i,\ i = 1, \cdots, n\}$ be bases of $\mathcal{V}$ and $\mathcal{U}$ respectively and let $B_{\mathcal{V}}^p = \{\underline{v}_\omega\wedge = \underline{v}_{i_1}\wedge\cdots\wedge\underline{v}_{i_p},\quad \omega = (i_1, \cdots, i_p),\quad 1 \le i_1 \le\cdots\le i_p \le m\}$, $B_{\mathcal{U}}^p = \{\underline{u}_\rho\wedge = \underline{u}_{j_1}\wedge\cdots\wedge\underline{u}_{j_p},\quad \omega = (j_1, \cdots, j_p),\quad 1 \le j_1 \le \cdots \le j_p \le n\}$ be the induced bases of $\wedge^p\mathcal{V}$ and $\wedge^p\mathcal{U}$ respectively. If

$$h(\underline{v}_i) = \sum_{j=1}^{n} c_{ij}\underline{u}_j,\quad i = 1, \cdots, m,\quad H_{\mathcal{U}}^{\mathcal{V}} = [c_{ij}] \tag{2.15}$$

then for all basis vectors $\underline{v}_\omega \in \wedge^p\mathcal{V}$ we have

$$\wedge^p h(\underline{v}_{i_1}\wedge\cdots\wedge\underline{v}_{i_p}) = \Big(\sum_{j=1}^{n} c_{i_1 j}\underline{u}_j\Big)\wedge\cdots\wedge\Big(\sum_{j=1}^{n} c_{i_p j}\underline{u}_j\Big) \tag{2.16}$$

However,

$$\underline{u}_{k_1}\wedge\cdots\wedge\underline{u}_{k_p} = \operatorname{sign}\begin{pmatrix} j_1, & \cdots, & j_p \\ k_1, & \cdots, & k_p \end{pmatrix} \underline{u}_{j_1}\wedge\cdots\wedge\underline{u}_{j_p} \tag{2.17}$$

Also by the theory of determinants

$$H_{i_1,\cdots,i_p}^{j_1,\cdots,j_p} = \begin{bmatrix} c_{i_1 j_1} & \cdots & c_{i_p j_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 j_p} & \cdots & c_{i_p j_p} \end{bmatrix} =$$

$$= \operatorname{sign}\begin{pmatrix} j_1, & \cdots, & j_p \\ k_1, & \cdots, & k_p \end{pmatrix} c_{i_1 j_1}\cdots c_{i_p j_p} \tag{2.18}$$

Clearly, the quantities of equation (2.18) are the entries of the matrix, $\wedge^p H_{\mathcal{U}}^{\mathcal{V}}$ which represents the linear map $\wedge^p h$: $\wedge^p \mathcal{V} \rightarrow \wedge^p \mathcal{U}$ with respect to the bases $B_{\mathcal{V}}^p$, $B_{\mathcal{U}}^p$.

2.5.2 Compound Matrices and Grassmann Products [Mar. & Min. 1]

The results of the section 2.5.1 may be simplified by introducing some useful notation and definitions on the sequences of integers and on submatrices of a given matrix.

(*i*) Notation

(a) $Q_{p,n}$ denotes the set of strictly increasing sequences of p integers ($1 \leq p \leq n$) chosen from $1, \cdots, n$ e.g. $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$. Thus, the number of the sequences which belong $Q_{p,n}$, is $\binom{n}{p}$. If $\alpha, \beta \in Q_{p,n}$, we say that α precedes β (α<b), if there exists an integer t ($1 \leq t \leq p$) for which $\alpha_1 = \beta_1, \cdots, \alpha_{t-1} = \beta_{t-1}, \alpha_t < \beta_t$, where α_i, β_i denote the elements of α, β respectively, e.g. in the set $Q_{3,8}$, (3,5,8), (4,5,6). This describes the *lexicographic ordering* of the elements of $Q_{p,n}$. The set of sequences $Q_{p,n}$ from now on will be assumed with its sequences lexicographically ordered and the elements of the ordered set $Q_{p,n}$ will be denoted by $Q_{p,n}(t)$, $t = 1, \cdots, \binom{n}{p}$ or simply by ω.

(b) $Q_{p,n}^{\alpha)}$ denotes the subset of $Q_{p,n}$ whose sequences do not contain any of the indices of a given $\alpha \in Q_{p,n}$, e.g. $Q_{2,5}^{\alpha)} = \{(3,4), (3,5), (4,5)\}$ if $\alpha = (1,2)$. This set has $\binom{n-p}{p}$ elements. The elements of $Q_{p,n}^{\alpha)}$ will be denoted by $Q_{p,n}^{\alpha)}(t)$ or simply $\omega_{\alpha)}$.

(c) If $c_1, \cdots, c_n$ are elements of the field $\mathcal{F}$ and $\omega = (i_1, \cdots, i_p)$ is a sequence in $Q_{p,n}$, $1 \leq p \leq n$, then the product $c_{i_1} \cdots c_{i_p}$ will be denoted by c_ω.

(d) Suppose $A = [a_{ij}] \in M_{m,n}(\mathcal{F})$ where $M_{m,n}$ denotes the set of mxn matrices of the field $\mathcal{F}$; let k,p be positive integers satisfying $1 \leq k \leq m$, $1 \leq p \leq n$ and let $\alpha = (i_1, \cdots, i_k) \in Q_{k,m}$ and $\beta = (j_1, \cdots, j_p) \in Q_{p,n}$. Then $A[\alpha|\beta] \in M_{k,p}(\mathcal{F})$ denotes the submatrix of A which contains the rows $i_1, \cdots, i_k$ and the columns $j_1, \cdots, j_p$. We use the notation $A(\alpha|\beta]$ to designate the submatrix of A which excludes rows $i_1, \cdots, i_k$ and includes columns $j_1, \cdots, j_p$. The submatrices $A[\alpha|\beta)$ and $A(\alpha|\beta)$ can be defined similarly.

(ii) *Compound Matrices* [*Mar. & Min.* 1]

Let $A \in \mathcal{F}^{m \times n}$ and $1 \leq p \leq \min(m,n)$, then the *p-th compound matrix* or *p-th adjugate* of A is the $\binom{m}{n} \times \binom{n}{p}$ matrix whose entries are $\det(A[\alpha|\beta])$, $\alpha \in Q_{p,m}$, $\beta \in Q_{p,n}$ arranged lexicographically in α and β. This matrix will be designated by $C_p(A)$. For example, if $A \in \mathcal{F}^{3 \times 3}$ and $p = 2$, the $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$ and

$$C_2(A) = \begin{bmatrix} \det\{A[(1,2)\ (1,2)]\} & \det\{A((1,2)\ (1,3))\} & \det\{A[(1,2)\ (2,3)]\} \\ \det\{A[(1,3)\ (1,2)]\} & \det\{A[(1,3)\ (1,3)]\} & \det\{A[(1,3)\ (2,3)]\} \\ \det\{A[(2,3)\ (1,2)]\} & \det\{A[2,3)\ (1,3)]\} & \det\{A[(2,3)\ (2,3)]\} \end{bmatrix} \tag{2.19}$$

or setting for convenience $\det\{A[\alpha|\beta]\} = a_\beta^\alpha$ we have

$$C_2(A) = \begin{bmatrix} a_{1,2}^{1,2} & a_{1,3}^{1,3} & a_{2,3}^{1,2} \\ a_{1,2}^{1,3} & a_{1,3}^{1,3} & a_{2,3}^{1,3} \\ a_{1,2}^{2,3} & a_{1,3}^{2,3} & a_{2,3}^{2,3} \end{bmatrix} \tag{2.20}$$

Properties of Compound Matrices

(a) If $A \in \mathcal{F}^{n \times n}$, $1 \leq p \leq n$ and also A is non-singular

(i) $(C_p(A))^{-1} = C_p(A^{-1})$ (2.21)

(ii) $C_p(A) = (C_p(A))^*$ (2.22)

where A^* is the conjugate transpose of $A(\mathcal{F} = \mathbb{C})$.

(iii) $C_p(A^T) = (C_p(A))^T$ (2.23)

where A^T is the transpose of A.

(iv) $C_p(\overline{A}) = \overline{C_p(A)}$ (2.24) where $\bar{A}$ is the conjugate of $A(\mathcal{F}=\mathbb{C})$.

(v) $C_p(kA) = k^p C_p(A) \quad \forall\ k \in \mathcal{F}$ (2.25)

(vi) $C_p(I_n) = I_{\binom{n}{p}}$ (2.26)

(vii) Sylvester-Franke Theorem

$$\det\{C_p(A)\} = (\det A)^{\binom{n-1}{p-1}} \tag{2.27}$$

□

(*b*) *Binet-Cauchy Theorem*: If $A \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times k}$ and $1 \leq p \leq \min(m,n,k)$ then:

$$C_p(AB) = C_p(A)C_p(B) \tag{2.28}$$

□

(c) If $A \in \mathcal{F}^{p\times n}$ and the p rows of A are denoted by $\underline{a}_1^T, \cdots, \underline{a}_p^T$ in succession $(1 \leq p \leq n)$, then $C_p(A)$ is an $\binom{n}{p}$-tuple and it is called the *Grassmann product* or *skew symmetric product* of the vectors $\{\underline{a}_1^T, \cdots, \underline{a}_p^T\}$ for reasons which will become apparent later on. The usual notation for this $\binom{n}{p}$-tuple of subdeterminants of A is $\underline{a}_1^T\wedge\cdots\wedge\underline{a}_p^T$ and it denotes a row vector. The Grassmann product of the columns of a matrix $A \in \mathcal{F}^{n\times p}$ $(1 \leq p \leq n)$ may be defined in a similar manner; the product in this case, however, will be an $\binom{n}{p}$-column vector. If $\{\underline{a}_1, \cdots, \underline{a}_p\}$ are the columns of A, in this case, then this $\binom{n}{p}$-tuple of subdeterminants of A will be denoted by $\underline{a}_1\wedge\cdots\wedge\underline{a}_p$. By the properties of determinants, if $\sigma \in S_p$ (where S_p denotes the totality of permutations of 1, $\cdots$, p), then

$$\underline{a}_{\sigma(1)}\wedge\cdots\wedge\underline{a}_{\sigma(p)} = \operatorname{sign}\sigma\ \underline{a}_1\wedge\cdots\wedge\underline{a}_p \tag{2.29}$$

If $B \in \mathcal{F}^{n\times n}$, $A \in \mathcal{F}^{n\times p}$, then by the Binet-Cauchy theorem it follows that:

$$C_p(B)\underline{a}_1\wedge\cdots\wedge\underline{a}_p = B\underline{a}_1\wedge\cdots\wedge B\underline{a}_p \tag{2.30}$$

Grassmann products suitably deployed may greatly reduce the complexity of the expressions in compound matrices. Thus, let $A \in \mathcal{F}^{m \times n}$ and $1 \leq p \leq \min(m,n)$. The matrix A may be written in terms of its rows or columns respectively as

$$A = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_m^T \end{bmatrix} \qquad \text{or } A = [\underline{a}_1, \cdots, \underline{a}_n] \tag{2.31}$$

Let $\omega = \{i_1, \cdots, i_p\} \in Q_{p,m}$ and $\phi = \{j_1, \cdots, j_p\} \in Q_{p,n}$ and let us denote by $\underline{a}_\omega^T \wedge$ the Grassmann product $\underline{a}_{i_1}^T \wedge \cdots \wedge \underline{a}_{i_p}^T$ and by $\underline{a}_\phi \wedge$ the Grassmann product $\underline{a}_{j_1} \wedge \cdots \wedge \underline{a}_{j_p}$. The p-th compound matrix of A may then be expressed in either of the following forms:

$$C(A) = \begin{bmatrix} \vdots \\ \underline{a}_\omega^T \wedge \\ \vdots \end{bmatrix}, \quad \omega \in Q_{p,m} \quad \text{or } C_p(A) = [\ \cdots, \underline{a}_\phi \wedge, \cdots\ ] \tag{2.32}$$

2.5.3 Compound Matrices and Exterior Algebra [Mar. 1] [Kar. 1]

We may now return to the Eqn. (2.18). We first note that the matrix $H_{\mathcal{U}}^{\mathcal{V}}$ defined by (2.15) or by

$$[h(\underline{v}_1), h(\underline{v}_2), \cdots, h(\underline{v}_m)] = [\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n] \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix} \tag{2.33}$$

is the matrix representation of $h: \mathcal{V} \rightarrow \mathcal{U}$ with respect to the bases $B_{\mathcal{V}}$ and $B_{\mathcal{U}}$ of $\mathcal{V}$, $\mathcal{U}$ respectively. Note that $H_{i_1, \cdots, i_p}^{j_1, \cdots, j_p}$ is the p-th order minor of $H_{\mathcal{U}}^{\mathcal{V}}$ that lies on the $\{j_1, \cdots, j_p\}$ rows and $\{i_1, \cdots, i_p\}$ columns. If we define $\underline{h}_\omega \wedge = \underline{h}_{i_1} \wedge \cdots \wedge \underline{h}_{i_p}$, where $\{\underline{h}_{i_1}, \cdots, \underline{h}_{i_p}\}$ are the columns of H that correspond to the indices $(i_1, \cdots, i_p) \in Q_{p,m}$, then Eqn. (2.16) may be written as:

$$\wedge^p h(\underline{v}_\omega \wedge) = [\ \cdots, \underline{u}_\rho \wedge, \cdots\ ]\, \underline{h}_\omega \wedge, \quad \rho \in Q_{p,n} \tag{2.34}$$

Given that the relationship holds for all $\omega \in Q_{p,m}$, $\{\underline{v}_\omega \wedge,\ \omega \in Q_{p,m}\}$ is a basis of $\wedge^p \mathcal{V}$ and $\{\underline{u}_\rho \wedge,$ $\rho \in Q_{p,n}\}$ is a basis of $\wedge^p \mathcal{U}$ we may write:

$$[\ \cdots, \wedge^p h(\underline{v}_\omega \wedge), \cdots\ ] = [\ \cdots, \underline{u}_\rho \wedge, \cdots\ ][\ \cdots, \underline{h}_\omega \wedge, \cdots\ ] = B^p_{\mathcal{U}} \wedge^p H^{\mathcal{V}}_{\mathcal{U}} = B^p_{\mathcal{U}} C_p(H^{\mathcal{V}}_{\mathcal{U}}) \quad (2.35)$$

where $\wedge^p H^{\mathcal{V}}_{\mathcal{U}} = C_p(H^{\mathcal{V}}_{\mathcal{U}})$ is the matrix representation of $\wedge^p h$ with respect to the bases $B^p_{\mathcal{V}}$, $B^p_{\mathcal{U}}$ and it is defined by the p-th compound matrix of $H^{\mathcal{V}}_{\mathcal{U}}$. These considerations lead to the following result.

Theorem (2.2): Let $\mathcal{V}$, $\mathcal{U}$ be two vector spaces over $\mathcal{F}$, with $\dim\mathcal{V} = m$, $\dim\mathcal{U} = n$ and let $h: \mathcal{V} \rightarrow U$ be a linear map of $\mathcal{V}$ into U. Let $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$, $B_{\mathcal{U}} = \{\underline{u}_j,\ j = 1, \cdots, n\}$ be bases of $\mathcal{V}$, $\mathcal{U}$ respectively and let $H^{\mathcal{V}}_{\mathcal{U}}$ be the matrix representation of h with respect to the bases $B_{\mathcal{V}}$, $B_{\mathcal{U}}$. If $\wedge^p h: \wedge^p \mathcal{V} \rightarrow \wedge^p \mathcal{U}$ $1 \leq p \leq \min(n,m)$, is the p-th exterior power of h, then $\wedge^p h$ may be represented with respect to the induced bases $B^p_{\mathcal{V}} = \{\underline{v}_\omega \wedge,\ \omega \in Q_{p,m}\}$, $B^p_{\mathcal{U}} = \{\underline{u}_\rho \wedge,\ \rho \in Q_{p,n}\}$ of $\wedge^p \mathcal{V}$, $\wedge^p \mathcal{U}$ respectively, by the matrix $\wedge^p H^{\mathcal{V}}_{\mathcal{U}} = C_p(H^{\mathcal{V}}_{\mathcal{U}})$, where $C_p(H^{\mathcal{V}}_{\mathcal{U}})$ is the p-th compound matrix of $H^{\mathcal{V}}_{\mathcal{U}}$. □

The above result can be represented by the following commutative diagram:

$$\begin{tikzcd}
B_{\mathcal{V}}, \mathcal{V} \arrow[r, "h"] \arrow[d, "r^1_{\mathcal{V}}"'] & \mathcal{U}, B_{\mathcal{U}} \arrow[d, "r^1_{\mathcal{U}} \Rightarrow r^p_{\mathcal{V}}"] & B^p_{\mathcal{V}}, \wedge^p \mathcal{V} \arrow[r, "\wedge^p h"] \arrow[d] & \wedge^p \mathcal{U}, B^p_{\mathcal{U}} \arrow[d, "r^p_{\mathcal{U}}"] \\
\mathcal{F}^m \arrow[r, "H^{\mathcal{V}}_{\mathcal{U}}"'] & \mathcal{F}^n & \mathcal{F}^{\binom{m}{p}} \arrow[r, "C_p(H^{\mathcal{V}}_{\mathcal{U}})"'] & \mathcal{F}^{\binom{n}{p}}
\end{tikzcd}$$

It has been shown that the pairs of the vector spaces $(\wedge^p \mathcal{V}, \mathcal{F}^{\binom{m}{p}})$ and $(\wedge^p \mathcal{U}, \mathcal{F}^{\binom{n}{p}})$ are isomorphic. In fact, every basis $B_{\mathcal{V}}$ of $\mathcal{V}$ and $B_{\mathcal{U}}$ of $\mathcal{U}$ induces a decomposable basis for $\wedge^p \mathcal{V}$, $\wedge^p \mathcal{U}$ and the corresponding representation maps $r^p_{\mathcal{V}}$, $r^p_{\mathcal{U}}$ define isomorphisms between $\wedge^p \mathcal{V}$, $\mathcal{F}^{\binom{m}{p}}$ and $\wedge^p \mathcal{V}$, $\mathcal{F}^{\binom{n}{p}}$.

The linear map $\wedge^p H^{\mathcal{V}}_{\mathcal{U}} = C_p(H^{\mathcal{V}}_{\mathcal{U}})$: $\mathcal{F}^{\binom{n}{p}} \rightarrow \mathcal{F}^{\binom{n}{p}}$ is induced by the map: $H^{\mathcal{V}}_{\mathcal{U}}$: $\mathcal{F}^m \rightarrow \mathcal{F}^n$ and it is a representation of the linear map $\wedge^p h$: $\wedge^p \mathcal{V} \rightarrow \wedge^p \mathcal{U}$. Thus, it is clear that, as any pair of vector spaces $\mathcal{V}$, $\mathcal{U}$ of finite dimension and their linear map h can be discussed by means of m-tuples, n-tuples and matrices, their p-th exterior powers $\wedge^p \mathcal{V}$, $\wedge^p \mathcal{U}$ and their linear map $\wedge^p h$ may be discussed in terms of $\binom{m}{p}$-tuples, $\binom{n}{p}$-tuples and compound matrices.

Before we close this section we discuss the composition of exterior powers of linear maps and its relationship to the Binet-Cauchy Theorem. Let $\mathcal{V}$, $\mathcal{U}$, $\mathcal{W}$ be vector spaces over the field $\mathcal{F}$, such that $\dim\mathcal{V} = m$, $\dim\mathcal{U} = n$, $\dim\mathcal{W} = k$ and let f, g be two linear maps such that f: $\mathcal{V} \rightarrow \mathcal{U}$, g: $\mathcal{U} \rightarrow \mathcal{W}$. The composite map $h = g \circ f$, h: $\mathcal{V} \rightarrow \mathcal{W}$ may be defined by the following commutative diagram:

$$\begin{tikzcd}
B_{\mathcal{V}}, \mathcal{V} \arrow[r, "f"] \arrow[d, "r^1_{\mathcal{V}}"'] & \mathcal{U},\ B_{\mathcal{U}} \arrow[r, "g"] \arrow[d, "r^1_{\mathcal{U}}"] & \mathcal{W}, B_{\mathcal{W}} \arrow[d, "r^1_{\mathcal{W}}"] \\
\mathcal{F}^m \arrow[r, "F^{\mathcal{V}}_{\mathcal{U}}"'] & \mathcal{F}^n \arrow[r, "G^{\mathcal{U}}_{\mathcal{W}}"'] & \mathcal{F}^k
\end{tikzcd}$$

where $r^1_{\mathcal{V}}$, $r^1_{\mathcal{U}}$, $r^1_{\mathcal{W}}$ are the representation maps of $\mathcal{V}$, $\mathcal{U}$, $\mathcal{W}$ with respect to the bases $B_{\mathcal{V}}$, $B_{\mathcal{U}}$, $B_{\mathcal{W}}$ respectively, and $F^{\mathcal{V}}_{\mathcal{U}}$, $G^{\mathcal{U}}_{\mathcal{W}}$ are the matrix representations of the linear maps f, g, relative to the bases $B_{\mathcal{V}}$, $B_{\mathcal{U}}$, $B_{\mathcal{W}}$ correspondingly. Hence:

$$f = (r^1_{\mathcal{U}})^{-1} \circ F^{\mathcal{V}}_{\mathcal{U}} \circ r^1_{\mathcal{V}}, \quad g = (r^1_{\mathcal{W}})^{-1} \circ G^{\mathcal{U}}_{\mathcal{W}} \circ r^1_{\mathcal{U}}$$

$$h = g \circ f = (r^1_{\mathcal{W}})^{-1} \circ G^{\mathcal{U}}_{\mathcal{W}} \circ r^1_{\mathcal{U}} \circ (r^1_{\mathcal{U}})^{-1} \circ F^{\mathcal{V}}_{\mathcal{U}} \circ r^1_{\mathcal{V}} = (r^1_{\mathcal{W}})^{-1} \circ G^{\mathcal{U}}_{\mathcal{W}} \circ F^{\mathcal{V}}_{\mathcal{U}} \circ r^1_{\mathcal{V}}$$

or

$$r^1_{\mathcal{W}} \circ h \circ (r^1_{\mathcal{V}})^{-1} = G^{\mathcal{U}}_{\mathcal{W}} \circ F^{\mathcal{W}}_{\mathcal{U}} = H^{\mathcal{V}}_{\mathcal{W}} \tag{2.36}$$

which simply means that the matrix representation of the composite map h with respect to the bases $B_{\mathcal{V}}$, $B_{\mathcal{W}}$ is $H^{\mathcal{V}}_{\mathcal{W}} = G^{\mathcal{U}}_{\mathcal{W}} F^{\mathcal{W}}_{\mathcal{U}}$.

The above result may readily be interpreted to the linear map $\wedge^p h = \wedge^p(g \circ f) = \wedge^p g \circ \wedge^p f$ where $\wedge^p f$: $\wedge^p\mathcal{V} \rightarrow \wedge^p\mathcal{U}$, $\wedge^p g$: $\wedge^p\mathcal{U} \rightarrow \wedge^p\mathcal{W}$ and $\wedge^p h$: $\wedge^p\mathcal{V} \rightarrow \wedge^p\mathcal{W}$. Using the induced bases $B^p_{\mathcal{V}}$, $B^p_{\mathcal{U}}$, $B^p_{\mathcal{W}}$ of $\wedge^p\mathcal{V}$, $\wedge^p\mathcal{U}$, $\wedge^p\mathcal{W}$ we have the following commutative diagram:

$$\begin{tikzcd}
B^p_{\mathcal{V}}, \wedge^p \mathcal{V} \arrow[r, "\wedge^p f"] \arrow[d, "r^p_{\mathcal{V}}"'] & \wedge^p \mathcal{U},\ B^p_{\mathcal{V}} \arrow[r, "\wedge^p g"] \arrow[d, "r^p_{\mathcal{U}}"'] & \wedge^p \mathcal{W},\ B^p_{\mathcal{W}} \arrow[d, "r^p_{\mathcal{W}}"'] \\
\mathcal{F}^{\binom{m}{p}} \arrow[r, "C_p(F^{\mathcal{V}}_{\mathcal{U}})"] & \mathcal{F}^{\binom{n}{p}} \arrow[r, "C_p(G^{\mathcal{U}}_{\mathcal{W}})"] & \mathcal{F}^{\binom{k}{p}}
\end{tikzcd}$$

from which by using similar arguments to those in the derivation of equation (2.36) we have that

$$r^p_{\mathcal{V}} \circ \wedge^p h \circ (r^p_{\mathcal{V}})^{-1} = r^p_{\mathcal{V}} \circ \wedge^p g \circ \wedge^p f \circ (r^p_{\mathcal{V}})^{-1} = C_p(G^{\mathcal{U}}_{\mathcal{V}}) C_p(F^{\mathcal{V}}_{\mathcal{U}}) = C_p(H^{\mathcal{V}}_{\mathcal{W}}) \tag{2.37}$$

The above relationship clearly states that the Binet-Cauchy Theorem in its compound matrix form expresses the composition law of the exterior powers of linear maps when matrix representations are considered.

2.6 Plücker Coordinates and Grassmann Variety

2.6.1 Plücker Coordinates [Mar., 1][Gre, 1][Hod & Ped., 1]

Let $\mathcal{V}$ be an m-dimensional subspace of an n-dimensional vector space $\mathcal{U}$ over a field $\mathcal{F}$. The map f: $\mathcal{V} \rightarrow \mathcal{U}$ defined by $f(\underline{x}) = \underline{x}$, $\underline{x} \in \mathcal{V}$ is linear and by Theorem (2.1) there is a unique homomorphism $\bar{f}$: $\wedge\mathcal{V} \rightarrow \wedge\mathcal{U}$ associated with f. Since $\dim\mathcal{V} = m$, $\wedge^m\mathcal{V}$ is a one-dimensional space and it is mapped by $\bar{f}$ onto a one-dimensional subspace of $\wedge^m\mathcal{U}$. Thus if $B_{\mathcal{V}} = \{\underline{v}_i, i = 1, \cdots, m\}$ is a basis of $\mathcal{V}$ then $\wedge^m\mathcal{V}$ is spanned by the element $\underline{v}_i \wedge \cdots \wedge \underline{v}_m$ and $\bar{f}$ maps this element onto

$$\bar{f}(\underline{v}_1 \wedge \cdots \wedge \underline{v}_m) = f(\underline{v}_1) \wedge \cdots \wedge f(\underline{v}_m) = \underline{v}_1 \wedge \cdots \wedge \underline{v}_m \tag{2.38}$$

in $\wedge^m\mathcal{U}$. The vectors $\underline{v}_i$, $i = 1, \cdots, m$ are linearly independent and so $\underline{v}_1 \wedge \cdots \wedge \underline{v}_m$ is a non-zero element of $\wedge^m\mathcal{U}$. In fact the injection map f: $\mathcal{V} \rightarrow \mathcal{U}$ defined by $f(\underline{x}) = \underline{x}$, $\underline{x} \in \mathcal{V}$ induces an injection map $\wedge^m f$: $\wedge^m\mathcal{V} \rightarrow \wedge^m\mathcal{U}$ defined by $\wedge^m f(\underline{x}\wedge) = \underline{x}\wedge$, $\underline{x}\wedge \in \wedge^m\mathcal{V}$. The vector $\underline{v}_1 \wedge \cdots \wedge \underline{v}_m$ spans a one-dimensional subspace of $\wedge^m\mathcal{U}$ which depends only on $\mathcal{V}$. Now let $B_{\mathcal{U}} = \{\underline{u}_j$, $j = 1, \cdots, n\}$ be a basis of $\mathcal{U}$, then using matrix representations we have the following commutative diagram

$$
\begin{tikzcd}
B_{\mathcal{V}}, \mathcal{V} \arrow[r, "f"] \arrow[d, "r^1_{\mathcal{V}}"'] & \mathcal{U}, B_{\mathcal{U}} \arrow[d, "r^1_{\mathcal{U}}"] \\
\mathcal{F}^m \arrow[r, "A = F^{\mathcal{V}}_{\mathcal{U}}"] & \mathcal{F}^n
\end{tikzcd}
$$

where $A = F^{\mathcal{V}}_{\mathcal{U}}$ is the matrix representation of f with respect to $B_{\mathcal{V}}$ and $B_{\mathcal{U}}$. In fact if

$$f(\underline{v}_i) = \underline{v}_i = \sum_{j=1}^{n} a_{ij} \underline{u}_j \tag{2.39}$$

then $A = F^{\mathcal{V}}_{\mathcal{U}}$ is the matrix

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{2.40}$$

The column span of A is a subspace of F^n and it is the representation of $f(\mathcal{V})$ with respect to the bases $B_{\mathcal{U}}$ and $B_{\mathcal{V}}$. The representation of $\underline{v}_i \wedge \cdots \wedge \underline{v}_m$ with respect to the bases $\wedge^m B^m_{\mathcal{U}}, \wedge^m B^m_{\mathcal{V}}$ is defined by the commutative diagram

$$
\begin{tikzcd}
\wedge^m B^m_{\mathcal{V}}, \wedge^m \mathcal{V} \arrow[r, "\wedge^m f"] \arrow[d, "r^m_{\mathcal{V}}"'] & \wedge^m \mathcal{U}, \wedge^m B^m_{\mathcal{U}} \arrow[d, "r^m_{\mathcal{U}}"] \\
\mathcal{F} = \mathcal{F}^{\binom{m}{n}} \arrow[r, "C_m\left(F^{\mathcal{V}}_{\mathcal{U}}\right) = C_m(A)"] & \mathcal{F}^{\binom{m}{n}}
\end{tikzcd}
$$

thus, if $\omega = (i_1, \cdots, i_m) \in Q_{m,n}$, then

$$\underline{v}_1\wedge\cdots\wedge\underline{v}_m = \sum a_\omega\, \underline{u}_\omega\wedge = [\ \cdots,\ \underline{u}_\omega\wedge,\ \cdots\ ]\begin{bmatrix}\vdots\\ a_\omega\\ \vdots\end{bmatrix} \tag{2.41}$$

and hence the matrix

$$C_m(A) = C_m\left(F_{\mathcal{U}}^{\mathcal{V}}\right) = \underline{a}_1\wedge\cdots\wedge\underline{a}_m = \begin{bmatrix}\vdots\\ a_\omega\\ \vdots\end{bmatrix} \in \mathbb{R}^{\binom{n}{m}\times 1} \tag{2.42}$$

is the "matrix" representation of $\wedge^m f$ with respect to $\wedge^m B_{\mathcal{U}}^m$, $\wedge^m B_{\mathcal{V}}^m$. The $\binom{n}{m}$-tuple $(\ \cdots,\ a_\omega,\ \cdots\ )$ are the co-ordinates of the one-dimensional subspace $\wedge^m f(\wedge^m \mathcal{V})$ with respect to the two given bases. If $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$ and $B_{\mathcal{V}'} = \{\underline{v}_i',\ i = 1, \cdots, m\}$ are two bases of $\mathcal{V}$ and $B_{\mathcal{U}}$ is a fixed basis of $\mathcal{U}$ then the matrix representations of f with respect to those two bases $B_{\mathcal{V}}$, $B_{\mathcal{V}'}$ are related by the coordinate transformation $Q_{\mathcal{V}'}^{\mathcal{V}}$

$$F_{\mathcal{U}}^{\mathcal{V}'} = F_{\mathcal{U}}^{\mathcal{V}} Q_{\mathcal{V}'}^{\mathcal{V}},\quad F_{\mathcal{U}}^{\mathcal{V}'},\ F_{\mathcal{U}}^{\mathcal{V}} \in \mathcal{F}^{n\times m},\quad Q_{\mathcal{V}'}^{\mathcal{V}} \in \mathcal{F}^{m\times m} \tag{2.43}$$

and thus

$$C_m\left(F_{\mathcal{U}}^{\mathcal{V}'}\right) = C_m\left(F_{\mathcal{U}}^{\mathcal{V}}\right) C_m\left(Q_{\mathcal{V}'}^{\mathcal{V}}\right) = C_m\left(F_{\mathcal{U}}^{\mathcal{V}}\right) q,\ q = \det\{Q_{\mathcal{U}}^{\mathcal{V}}\} \in \mathcal{F}-\{0\} \tag{2.44}$$

The two vectors $\underline{t} = \underline{v}_1\wedge\cdots\wedge\underline{v}_m$, $\underline{t}' = \underline{v}_1'\wedge\cdots\wedge\underline{v}_m'$ are related by

$$\underline{t}' = [\ \cdots,\ \underline{u}_\omega\wedge,\ \cdots\ ]\begin{bmatrix}\vdots\\ a_\omega'\\ \vdots\end{bmatrix} = [\ \cdots,\ \underline{u}_\omega\wedge,\ \cdots\ ]\begin{bmatrix}\vdots\\ a_\omega\\ \vdots\end{bmatrix} q = q\,\underline{t} \tag{2.45}$$

or

$$a_\omega' = q\, a_\omega, \qquad \omega \in Q_{m,n} \tag{2.46}$$

Definition (2.2): The scalars a_ω of Eqn. (2.42) are called *Plücker coordinates of the subspace* $\mathcal{V}$ relative to the basis $B_{\mathcal{V}}$ of $\mathcal{V}$ and $B_{\mathcal{U}}$ of $\mathcal{U}$. □

Eqn. (2.46) shows that any two sets of Plücker coordinates of $\mathcal{V}$, which correspond to two different bases of $\mathcal{V}$, with respect to the fixed basis $B_{\mathcal{U}}$ of $\mathcal{U}$ differ by a non-zero scaler factor. Hence the ratios of a_ω's are the same as the corresponding ratios of a'_ω's ($a_{\omega_1} = q\, a'_{\omega_1}$, $a_{\omega_2} = q\, a'_{\omega_2}$, $a_{\omega_1}/a_{\omega_2} = a'_{\omega_1}/a'_{\omega_2}$). Therefore, the ratios are uniquely determined by $\mathcal{V}$. Sometimes, the ratios of the a_ω rather than the a'_ω themselves, are called *the Plücker coordinates of* $\mathcal{V}$.

Definition (2.3): The set of all equivalence classes of non-zero vectors in $\mathcal{F}^{p+1}$ as defined above, is called *projective space of dimension p over* $\mathcal{F}$, denoted by $\mathbb{P}^p(\mathcal{F})$. Each equivalence class defines a point of this projective space. If Q is any point of $\mathbb{P}^p(\mathcal{F})$ and if $\underline{x} = (x_0, \cdots, x_p)$ is any vector of the equivalence class which defines Q, then the x_i 's are called *homogeneous coordinates* of Q. □

2.6.2 Grassmann Manifold and Grassmann Variety

If we set $p = \binom{n}{m} - 1 = \dim \wedge^m \mathcal{U} - 1$, then we can easily see that the Plücker coordinates of $\mathcal{V}$, enumerated in lexicographic order, may be considered as the homogeneous coordinates of a point in $\mathbb{P}^p(\mathcal{F})$. However, every point in $\mathbb{P}^p(\mathcal{F})$ does not represent an m-dimensional subspace of $\mathcal{U}$. Elements of $\wedge^m \mathcal{U}$ of the type $q\underline{v}_1 \wedge \cdots \wedge \underline{v}_m$ where $\underline{v}_1, \cdots, \underline{v}_m$ are linearly independent vectors of $\mathcal{V}$ and $q \in \mathcal{F} - \{0\}$ are called *simple* or *decomposable m-vectors.* Decomposable multivectors uniquely define m-dimensional subspaces of $\mathcal{U}$ as it is shown below.

Proposition (2.3): Let $\mathcal{U}$ be an n-dimensional vector space over $\mathcal{F}$ and let $\underline{y}\wedge = \underline{y}_1 \wedge \cdots \wedge \underline{y}_m$, $\underline{z}\wedge = \underline{z}_1 \wedge \cdots \wedge \underline{z}_m$ be two decomposable non-zero elements of $\wedge^m \mathcal{U}$ and let us denote by $\mathcal{V}_y = \mathrm{span}\{\underline{y}_1, \cdots, \underline{y}_m\}$ and $\mathcal{V}_z = \mathrm{span}\{\underline{z}_1, \cdots, \underline{z}_m\}$ the subspaces of $\mathcal{U}$ defined by $\underline{y}\wedge$ and $\underline{z}\wedge$ respectively. Necessary and sufficient condition for $V_y = \mathcal{V}_z$ is

$$\underline{y}\wedge = \underline{y}_1 \wedge \cdots \wedge \underline{y}_m = q\, \underline{z}_1 \wedge \cdots \wedge \underline{z}_m = q\, \underline{z}\wedge, \quad q \in \mathcal{F} - \{0\} \tag{2.47}$$

□

Definition (2.4): Let $\mathcal{U}$ be a vector space over a field $\mathcal{F}$ with $\dim \mathcal{U} = n$. The *Grassmannian* $G(m, \mathcal{U})$ is defined as the set of m-dimensional subspaces $\mathcal{V}$ of $\mathcal{U}$; $G(m, \mathcal{U})$ actually admits the structure of an analytic manifold which is known as the *Grassmann manifold.* □

Example (2.2): To demonstrate that every vector of $\wedge^m\mathcal{U}$ does not define an m-dimensional subspace $\mathcal{V}$ of $\mathcal{U}$, we consider the simple case of $\dim\mathcal{U} = n = 4$, $\dim\mathcal{V} = m = 2$. Let $B_{\mathcal{V}} = \{\underline{v}_1, \underline{v}_2\}$ be a basis of $\mathcal{V}$. Then, as it is well known, we can extend $B_{\mathcal{V}}$ to $B_{\mathcal{U}} = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ which is a basis of $\mathcal{U}$. Thus the induced basis of $\wedge^2\mathcal{U}$ is $\wedge^2\mathcal{B}_{\mathcal{U}} = \{\underline{v}_1\wedge\underline{v}_2, \underline{v}_1\wedge\underline{v}_3, \underline{v}_1\wedge\underline{v}_4, \underline{v}_2\wedge\underline{v}_3, \underline{v}_2\wedge\underline{v}_4, \underline{v}_3\wedge\underline{v}_4\}$ and so

$$\underline{v} = \underline{v}_1\wedge\underline{v}_2 = \sum_{1\leq i\leq j\leq 4} c_{ij}\underline{v}_i\wedge\underline{v}_j \tag{2.48}$$

where c_{ij}, $1 \leq i \leq j \leq 4$ is a set of Plücker coordinates of $\mathcal{V}$.

Clearly, the one-dimensional subspace $q\underline{v}$ of $\wedge^2\mathcal{U}$ consists of decomposable 2-vectors and it is uniquely defined by $\mathcal{V}$. But

$$\underline{v}\wedge\underline{v} = (\underline{v}_1\wedge\underline{v}_2)\wedge(\underline{v}_1\wedge\underline{v}_2) = 0$$

and thus

$$\begin{aligned}
\underline{v}\wedge\underline{v} &= \sum_{1\leq i\leq j\leq 4} c_{ij}\underline{v}_i\wedge\underline{v}_j \;\wedge \sum_{1\leq h\leq k\leq 4} c_{hk}\underline{v}_h\wedge\underline{v}_k = \\
&= (c_{12}\,\underline{v}_1\wedge\underline{v}_2 + c_{13}\,\underline{v}_1\wedge\underline{v}_3 + c_{14}\,\underline{v}_1\wedge\underline{v}_4 + c_{23}\underline{v}_2\wedge\underline{v}_3 + c_{24}\,\underline{v}_2\wedge\underline{v}_4 + \\
&+ c_{34}\,\underline{v}_3\wedge\underline{v}_4)\wedge(c_{12}\underline{v}_1\wedge\underline{v}_2 + c_{13}\,\underline{v}_1\wedge\underline{v}_3 + c_{14}\,\underline{v}_1\wedge\underline{v}_4 + c_{23}\underline{v}_2\wedge\underline{v}_3 + \\
&+ c_{24}\,\underline{v}_2\wedge\underline{v}_4 + c_{34}\,\underline{v}_3\wedge\underline{v}_4) = \\
&= c_{12}c_{34}\underline{v}_1\wedge\underline{v}_2\wedge\underline{v}_3\wedge\underline{v}_4 + c_{13}c_{24}\underline{v}_1\wedge\underline{v}_3\wedge\underline{v}_2\wedge\underline{v}_4 + c_{14}c_{23}\underline{v}_1\wedge\underline{v}_4\wedge\underline{v}_2\wedge\underline{v}_3 + \\
&+ c_{23}c_{14}\underline{v}_2\wedge\underline{v}_3\wedge\underline{v}_1\wedge\underline{v}_4 + c_{24}c_{13}\underline{v}_2\wedge\underline{v}_4\wedge\underline{v}_1\wedge\underline{v}_3 + c_{34}c_{12}\underline{v}_3\wedge\underline{v}_4\wedge\underline{v}_1\wedge\underline{v}_2 = \\
&= 2\,(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23})\,\underline{v}_1\wedge\underline{v}_2\wedge\underline{v}_3\wedge\underline{v}_4 = 0
\end{aligned}$$

Since $\underline{v}_1\wedge\underline{v}_2\wedge\underline{v}_3\wedge\underline{v}_4 \neq 0$ because $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ are linearly independent, we have

$$c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0 \tag{2.49}$$

Clearly, the above condition is a necessary condition for the general 6-tuple $(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34})$ to be the Plücker coordinates of a 2-dimensional subspace $\mathcal{V}$, or in other words to be the coordinates of a decomposable vector. □

Such a condition is known as a *Quadratic Plücker Relationship.* This Quadratic Plücker Relationship defines a hyperspace in the 5-dimensional projective space $\mathbb{P}^5(\mathcal{F})$, which is known as the *Grassmann variety* of this projective space.

In general, those points of $\mathbb{P}^p(\mathcal{F})$, $p = \binom{m}{n} - 1$ which correspond to m-dimensional subspaces $\mathcal{V}$ of the n-dimensional vector space $\mathcal{U}$ must satisfy a set of quadratic relationships of the type (3.49). It will be seen that this set of relationships defines an algebraic variety of the projective space $\mathbb{P}^p(\mathcal{F})$ which is known as the *Grassmann variety of* $\mathbb{P}^p(\mathcal{F})$.

2.7 Decomposability, Grassmann Representatives, and Quadratic Relationships

A number of further results concerning the decomposability of vectors as well as the definition of the Quadratic Plücker Relationships will be considered next. For more details see Marcus [Mar. 1] and Hodge & Pedoe [Hod. & Ped. 1].

Proposition (2.4): Let $\mathcal{U}$ be a vector space over $\mathcal{F}$ with dim $\mathcal{U} = n$ and let $\underline{z} \neq 0 \in \wedge^m\mathcal{U}$, $m < n$. Then $\underline{z}$ is decomposable if and only if there exists a linearly independent set of vectors $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_m$ in $\mathcal{U}$ such that $\underline{u}_i \wedge \underline{z} = 0$ $(i = 1, \cdots, m)$. □

Proposition (2.5): Let $\mathcal{U}$ be a vector space over $\mathcal{F}$ with dim $\mathcal{U} = n$ and let $B_{\mathcal{U}} = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n\}$ be a basis of $\mathcal{U}$. The induced basis of $\wedge^m\mathcal{U}$, $m < n$ is $B^m_{\mathcal{U}} = \{\underline{u}_\omega\wedge,\ \omega \in Q_{m,n}\}$ and thus any $\underline{z} \in \wedge^m\mathcal{U}$ can be written as:

$$\underline{z} = \sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge \tag{2.50}$$

This vector is decomposable if and only if there exists a matrix $A \in \mathcal{F}^{n\times m}$ such that

$$a_\omega = \det(A[\omega|\ 1, 2, \cdots, m]),\ \omega \in Q_{m,n}, \quad \text{or} \quad C_m(A) = [\cdots, a_\omega, \cdots] \tag{2.51}$$

□

Proposition (2.6): Let $\mathcal{U}$ be a vector space over $\mathcal{F}$ with dim $\mathcal{U} = n$, then any vector of $\wedge^{n-1}\mathcal{U}$ is decomposable. □

Proposition (2.7): Let $\mathcal{V} \in G(m,\mathcal{U})$, dim $\mathcal{U} = n$, then any non-zero decomposable element

$\underline{x}_1\wedge\cdots\wedge\underline{x}_m$, $x_i \in \mathcal{V}$, $i \in \underline{m}$ is called *Grassmann representative* for $\mathcal{V}$ [Mar., 1]. The Grassmann representatives all differ only by non-zero scalar factors so that we shall denote any of them by $\underline{g}(\mathcal{V})$. □

The characterisation of a vector space $\mathcal{V}$ of $G(m,\mathcal{U})$ by its Grassmann representative $\underline{g}(\mathcal{V})$, provides the means for the definition of the classical Plücker embedding [Hod., & Ped., 1]. The elements, $\{a_\omega\}$ of (2.51) may be regarded as the homogeneous co-ordinates of a point in the projective space $\mathbb{P}^v(\mathcal{F})$ where $v = \binom{n}{m} - 1$. Then this point depends only on the subspace $\mathcal{V}$ and not on the choice of basis. What we have constructed so far is a well defined projective mapping p: $G(m,n) \rightarrow \mathbb{P}^v(\mathcal{F})$ with $p(\mathcal{V}) = \underline{g}(\mathcal{V})$. The co-ordinates of the point $\{a_\omega\}$ in $\mathbb{P}^v(\mathcal{F})$ are called the *Plücker co-ordinates* of $\mathcal{V}$ and the mapping p is the *Plücker embedding of the Grassmannian $G(m,\mathcal{U})$ into the projective space $\mathbb{P}^v(\mathcal{F})$*. The Plücker image of the Grassmannian $G(m,\mathcal{U})$ in $\mathbb{P}^v(\mathcal{F})$ is an algebraic variety known as the *Grassmann variety* which will be denoted by $\Omega(m,n)$.

Theorem (2.3): Let $(\cdots, c_\omega, \cdots)$, $\omega \in Q_{m,n}$ be a system of co-ordinates in $\mathbb{P}^v(\mathcal{F})$ where $v = \binom{n}{m}-1$. The Grassmann variety $\Omega(m,n)$ is an algebraic variety in $\mathbb{P}^v(\mathcal{F})$ which is defined by the equations

$$\sum_{k=1}^{m+1}(-1)^{k-1}c_{i_1,\cdots,i_{m-1}}c_{j_1,\cdots,j_{k-1},j_{k+1},\cdots,j_{m+1}} = 0 \tag{2.52}$$

where $1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n$ and $1 \leq j_1 < j_2 < \cdots < i_{m+1} \leq n$. □

The set of the quadratic equations defined by Eqn. (2.52) is known as the set of *Quadratic Plücker Relations* (*QPR*) [Hod., & Ped. 1].

Theorem (2.4) [Hod. & Ped.]: The Grassmann variety $\Omega(m,n)$ is an irreducible algebraic variety with dimension $m(n-m)$, which lies in $\mathbb{P}^v(\mathcal{F})$ and not in a space of lower dimension. □

Corollary (2.2): The co-ordinates of a generic point of the Grassmann variety $\Omega(m,n)$ can be expressed rationally in terms of $m(n-m)$ independent indeterminates. □

Corollary (2.3): The Grassmann variety $\Omega(m,n)$ is projective equivalent to the Grassmann variety $\Omega(n-m,n)$. In fact $\dim \Omega(n-m, n) = (n-m)\cdot(n-n+m) = m(n-m) = \dim \Omega(n,m)$ and $\Omega(n-m,n)$ lies in the same projective space as $\Omega(m,n)$ since $\binom{n}{n-m} = \binom{n}{m}$. □

Corollary (2.4): If m=1, then $\Omega(1,n) = \mathbb{P}^{n-1}(\mathcal{F})$, because $\dim \Omega(1,n) = n - 1$. Also if $m = n - 1$ then $\Omega(n-1, n) = \mathbb{P}^{n-1}(\mathcal{F})$. So in these cases the Plücker embedding is bijective, or in other words every vector of $\wedge^{n-1}\mathcal{U}$ is decomposable. □

A useful parametrisation of the Quadratic Plücker Relations (QPR) in terms of a minimal set of algebraically independent quadratics have been obtained in [Gia. &, 1]. Their results rely heavily on the following theorem.

Theorem (2.5) [Hod. & Ped., 1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathcal{F}^\sigma$, $\sigma = \binom{p}{q}$, be a decomposable vector satisfying the set of QPRs and let $p_{a_1 \cdots, a_q}$ be a non-zero co-ordinate of $\underline{k}$. If we define by:

$$h_{ij} = p_{a_1, \cdots, a_{i-1}, j, a_{i+1}, \cdots, a_q}, \quad i \in \underline{q}, j \in \underline{p} \tag{2.53}$$

then $C_q(H) = \underline{k}$ where $H = [h_{ij}]$ □

Corollary (2.5) [Hod. & Ped., 1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathcal{F}^\sigma$, $\sigma = \binom{p}{q}$, be a decomposable vector and let the first co-ordinate of $\underline{k}$ be non-zero. The H matrix defined by Theorem (2.5) has the form

$$H = [p_a \, I_q, \, X^T]^T \in \mathcal{F}^{p \times q}, \text{ where } p_a = p_{1,2,\cdots,q} \neq 0 \tag{2.54}$$

or in a more detailed form, is expressed as in (2.55), where by H_0 we denote the matrix that corresponds to the first non-zero Plücker co-ordinate

$$H_0 = \begin{bmatrix} p_{1,2,\cdots,q} & 0 & & 0 \\ 0 & p_{1,2,\cdots,q} & & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & p_{1,2,\cdots,q} \\ & & & \\ (-1)^{q-1} p_{2,\cdots,q,q+1} & (-1)^{q-2} p_{1,3,\cdots,q,q+1} & & p_{1,2,\cdots,q-1,q+1} \\ & \vdots & & \\ (-1)^{q-1} p_{2,\cdots,q,p} & (-1)^{q-2} p_{1,3,\cdots,q,p} & & p_{1,2,\cdots,q-1,p} \end{bmatrix} \tag{2.55}$$

□

Example (2.4): Let $[p_0,p_1,p_2,p_3,p_4,p_5]^T$ be a point of the Grassmann variety $\Omega(2,4)$ of the projective space $\mathbb{P}^5$. A basis of the subspace $\mathcal{V}$ whose Plücker co-ordinates of the given point is:

$$H_0 = \begin{bmatrix} p_0 & 0 \\ 0 & p_0 \\ -p_3 & p_1 \\ -p_4 & p_2 \end{bmatrix} \tag{2.56}$$

under the assumption $p_0 \neq 0$. If we assume that $p_2 \neq 0$ then a basis of the same subspace is given by:

$$H_2 = \begin{bmatrix} p_2 & 0 \\ p_4 & p_0 \\ p_5 & p_1 \\ 0 & p_2 \end{bmatrix} \tag{2.57}$$

It can be seen that $H_2 = H_0\,Q$, where

$$Q = \begin{bmatrix} \frac{p_2}{p_0} & 0 \\ \frac{p_4}{p_0} & 1 \end{bmatrix}, \ \det Q \neq 0 \tag{2.58}$$

Note that $C_2(H_0) = [p_0^2, p_0p_1, p_0p_2, p_0p_3, p_0p_4, p_1p_4-p_2p_3]$ which implies that $p_5 = (p_1p_4-p_2p_3)\,/\,p_0$, or equivalently $p_0p_5-p_1p_4+p_2p_3 = 0$ □

The above example suggests a method for writing down an independent set of QRPs which completely describe $\Omega(q,p)$; such a set will be referred to as the *Reduced Quadratic Plücker Relations* (*RQPR*). We know that $\dim\Omega(q,p) = q(p-q)$, so it is clear that the number of RQPR is $\binom{p}{q}-q(p-q)-1$.

Proposition (2.8) [Gia, 1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathcal{T}^\sigma$, $\sigma = \binom{p}{q}$ be a decomposable vector and let the first co-ordinate of $\underline{k}$ be non-zero. If H_0 is the matrix which is defined by Corollary (2.5), then the equation

$$C_q(H_0) = [\cdots, p_\omega, \cdots]^T \, p_{1,2,\cdots,q}^{q-1} \tag{2.59}$$

defines a set of RQPR with respect to $p_{1,2,\cdots,q}$ co-ordinate. □

A similar procedure can be applied for any non-zero co-ordinates of $\underline{k}$.

Example (2.5): Let $\underline{k} = [p_0, p_1, p_2, \cdots, p_9]$ be a point of the Grassmann variety $\Omega(3,5)$ of the projective space $\mathbb{P}^9$. We can reconstruct H_0 according to the corollary (2.5) assuming $p_0 \neq 0$ as follows:

$$H_0 = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \\ p_6 & -p_3 & p_1 \\ p_7 & -p_4 & p_2 \end{bmatrix} \tag{2.60}$$

The number of RQPR in this case is $\binom{5}{3} - 3(5-3)-1 = 3$. Equation (2.52) gives the following set of equations:

$$\begin{gathered} p_0^3 = p_0p_0^2, \quad p_0^2p_1 = p_1p_0^2, \quad p_0^2p_2 = p_2p_0^2, \quad p_0^2p_3 = p_3p_0^2 \\ p_0^2p_4 = p_4p_0^2, \quad p_0^2p_6 = p_6p_0^2, \quad p_0^2p_7 = p_7p_0^2 \end{gathered} \tag{2.61}$$

$$\begin{gathered} p_0(p_1p_4 - p_2p_3) = p_5p_0^2, \quad p_0(p_1p_7 - p_2p_6) = p_8p_0^2 \\ p_0(p_3p_7 - p_4p_6) = p_9p_0^2 \end{gathered} \tag{2.62}$$

It is clear that the set (2.61) is trivial, whereas the set (2.52) obtained after the correlation of p_0 is a minimal set. □

2.8. Polynomial Matrices: General Properties

Some of the basic properties of polynomial matrices related to invariants and canonical forms under different types of unimodular equivalence are summarised below. These properties, also hold true over any Principal Ideal Domain (PID).

Definition (2.5): A non-singular square polynomial matrix $U(s) \in \mathbb{R}^{p\times p}[s]$ whose determinant is not a function of s is called *unimodular matrix* (i.e. $\det U(s) = c \in \mathbb{R}-\{0\}$). □

Note that unimodular matrices represent products of elementary row, column operations on polynomial matrices. In fact, post-multiplication by a unimodular matrix corresponds to products of elementary column operations, while pre-multiplication is equivalent to products of elementary row operations. By elementary operations we can reduce polynomial matrices to several "canonical" forms.

Theorem (2.6): *Column Hermite Form* [Kai. 1]: Any polynomial matrix $M(s) \in \mathbb{R}^{p\times q}[s]$, $\rho\{M(s)\} = t$ with $t \leq \min\{p,q\}$ can be reduced by elementary row operations (i.e. by premultiplication by a unimodular matrix) to a (lower or upper) quasi-triangular form in which

(i) the last $p-t$ rows are identically zero;

(ii) in column j, $1 \leq j \leq t$, the diagonal element is monic and of higher degree than any (non-zero) element above it;

(iii) in column j, $1 \leq j \leq t$, if the diagonal element is unity, then all elements above it are zero; and

(iv) no particular statements can be made about the elements in the last $q-t$ columns and the first t rows □

Remark (2.1): By interchanging the roles of rows and columns a similar *row-Hermite form* can be obtained. □

Theorem (2.7) *Smith Form* [Kai. 1]: For any polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$, $\rho\{M(s)\} = t$ with $t \leq \min\{p,q\}$, we can find elementary row and column operations or corresponding unimodular matrices $U(s) \in \mathbb{R}^{p \times p}[s]$, $V(s) \in \mathbb{R}^{q \times q}[s]$, such that

$$U(s)\, M(s)\, V(s) = S(s) \tag{2.63}$$

where

$$S(s) = \begin{array}{cc} \left[\begin{array}{cccccc} f_1(s) & & & 0 & & \\ & \ddots & & & \ddots & \\ & & f_t(s) & & & 0 \\ & & & & & \\ 0 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 0 \end{array}\right] & \begin{array}{c} \uparrow \\ t \\ \downarrow \\ \\ \uparrow \\ p-t \\ \downarrow \end{array} \\ \begin{array}{cc} \leftarrow \; t \; \rightarrow & \qquad \leftarrow \; q-t \; \rightarrow \end{array} & \end{array} \tag{2.64}$$

and the set $\{f_i(s),\ i = 1, \cdots, t\}$ is uniquely defined modulo $c \in \mathbb{R}$ $(c \neq 0)$ and they satisfy the divisibility conditions

$$f_i(s) \;/\; f_{i+1}(s), \quad i = 1, \cdots, t-1 \tag{2.65}$$

If $D_i(s)$ denote the greatest common divisor of all i^{th}-order minors of $M(s)$, then the set of $f_i(s)$ polynomials is defined by the *Smith Algorithm* i.e.

$$f_i(s) = D_i(s) \;/\; D_{i-1}(s), \quad D_0(s) = 1, \quad i = 1, 2, \cdots, t \tag{2.66}$$

□

The matrix $S(s)$ is called the *Smith form of* $M(s)$. The $\{D_i(s),\ i = 1, \cdots, t\}$ are called the *determinant divisors of* $M(s)$ and $\{f_i(s),\ i = 1, \cdots, t\}$ the *invariant polynomials of* $M(s)$.

Definition (2.6) [Kai. 1]: A square polynomial matrix $Q(s) \in \mathbb{R}^{q\times q}[s]$ is said to be a *right divisor* (*R.D*) of the polynomial matrix $M(s) \in \mathbb{R}^{p\times q}[s]$, with $p \geq q$, if and only if there exists a polynomial matrix $M_1(s) \in \mathbb{R}^{p\times q}[s]$, such that

$$M(s) = M_1(s)\, Q(s) \tag{2.67}$$

Let $Q_*(s)$ be a R.D. of M(s). Then $Q_*(s)$ is said to be a *greatest right divisor* (*G.R.D*) of M(s) if and only if deg $\{\det Q_*(s)\} \geq$ deg $\{Q(s)\}$ for every R.D. Q(s) of M(s). □

Remark (2.4): Greatest right divisors of polynomial matrices are not unique. They differ only by unimodular (left) factors. □

Definition (2.7) [Kai. 1]: A polynomial matrix $M(s) \in \mathbb{R}^{p\times q}[s]$, $p \geq q$, $\rho\,\{M(s)\} = q$ is said to be *irreducible or least degree* if and only if one of the following equivalent conditions are satisfied:

(i) all the G.R.D. of M(s) are unimodular matrices;

(ii) the Smith Form of M(s) is $[I_q,\, 0]^T$;

(iii) the greatest common divisor of all q-order minors of M(s) is 1;

(iv) $\rho\{M(s)\} = q$, for every $s \in C$. □

Definition (2.8) [Kai. 1]: A square polynomial matrix $Q(s) \in \mathbb{R}^{q\times q}[s]$ is said to be a *greatest common right divisor* (*G.C.R.D*) of the two polynomial matrices $M_1(s) \in \mathbb{R}^{p\times q}[s]$, $M_2(s) \in \mathbb{R}^{m\times q}[s]$ if and only if Q(s) satisfies the following properties:

(i) Q(s) is a common right divisor of $M_1(s)$ and $M_2(s)$;

(ii) if $Q'(s) \in \mathbb{R}^{q\times q}[s]$ is any other common right divisor of $M_1(s)$ and $M_2(s)$, then $Q'(s)$ is a right divisor of Q(s), or in other words deg $\left\{\det\,\{Q(s)\}\right\} \geq$ deg $\{\det Q'(s)\}$. □

Remark (2.3): Greatest common divisors of two polynomial matrices are not unique. They differ only by unimodular (left) factors. □

Definition (2.9) [Kai. 1]: Two polynomial matrices $M_1(s) \in \mathbb{R}^{p\times q}[s]$, $M_2(s) \in \mathbb{R}^{m\times q}[s]$ with $\rho\left\{\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}\right\} = q$ are said to be *relatively right prime or right coprime* if and only if one of the following equivalent conditions is satisfied:

(i) all G.C.R.D of $M_1(s)$ and $M_2(s)$ are unimodular matrices;

(ii) the Smith form of $\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ is $\begin{bmatrix} I_q \\ 0 \end{bmatrix}$;

(iii) the greatest common divisor of all q-order minors of $\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ is 1;

(iv) $\rho\left\{\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}\right\} = q$, for every $s \in \mathbb{C}$. □

Left divisors (L.D.), Greatest Left Divisors (G.L.D.) and Greatest Common Left Divisors (G.C.L.D) can be defined with the obvious changes. For convenience, we shall henceforth talk only of right divisors.

Remark (2.4): Let $G(s)=N_r(s)D_r^{-1}(s)=D_l^{-1}(s)N_l(s)$. A right MFD (left MFD) $\{D_r(s), N_r(s)\}$ ($\{D_l(s), N_l(s)\}$) of a transfer function matrix G(s) is called a *right coprime MFD* (a left co-prime MFD), if and only if the matrices $D_r(s)$, $N_r(s)$ ($D_l(s)$, $N_l(s)$) are right coprime (left coprime). □

Let $M(s) \in \mathbb{R}^{p\times q}[s]$, $p \geq q$ be a polynomial matrix with $\rho\{M(s)\} = q$ and let us write it in terms of its q column polynomial vectors as $M(s) = [\underline{m}_1(s), \cdots, \underline{m}_q(s)]$ where $\underline{m}_i(s) = [m_{1i}(s), \cdots, m_{pi}(s)]^T$, $i = 1, \cdots, q$. Then we may define [Ros., 1], [Wol., 1]:

Definition (2.10)

(i) The *degree of the polynomial vector* $\underline{m}_i(s)$ is the highest degree occurring among the degrees of its polynomial elements $m_{ji}(s)$, i.e.

$$\deg \underline{m}_i(s) = \max_{j=1,\cdots,p} \{\deg m_{ji}(s)\} \quad i = 1, \cdots, q \tag{2.68}$$

(ii) The *complexity c of* M(s) is the sum of the degree of its column polynomial vectors, i.e.

$$c = \sum_{i=1}^{q} \deg \{\underline{m}_i(s)\} \tag{2.69}$$

(iii) The *degree d of* M(s) is the highest degree occurring among the degrees of all its q-order minors.

□

Since a q-order minor of M(s) is a sum of products of polynomials one from each column, the maximum degree occurring among all the q-order minors of M(s), i.e. its degree d can not exceed its complexity c, i.e., we have [Ros. 1] [Wol. 1] c ≥ d. Let now that $g_i = \deg \{\underline{m}_i(s)\}$, $i = 1, \cdots, q$, and write

$$\underline{m}_i(s) = \underline{m}_i^0 + \underline{m}_i^1 s + \cdots + \underline{m}_i^{g_i} s^{g_i} = \sum_{k=0}^{g_i} \underline{m}_i^k s^k, \quad i = 1, \cdots, q \tag{2.70}$$

Then M(s) can be written as

$$M(s) = [\underline{m}_1(s), \cdots, \underline{m}_q(s)] = [\underline{m}_1^{g_1}, \cdots, \underline{m}_q^{g_q}] \begin{bmatrix} s^{g_1} & & 0 \\ & \ddots & \\ 0 & & s^{g_q} \end{bmatrix} + M_b Z(s) \tag{2.71}$$

where $M_b \in \mathbb{R}^{p \times c} (c = \sum_{i=1}^{q} g_i)$, and

$$Z(s) = \begin{bmatrix} \underline{e}_{g_1}(s) & & \\ & \ddots & \\ & & \underline{e}_{g_q}(s) \end{bmatrix} \in \mathbb{R}^{c \times q}[s] \quad \text{where } \underline{e}_{g_i}(s) = [1, s, \cdots, s^{g_i - 1}]^T \tag{2.72}$$

The matrix $[\underline{m}_1^{g_1}, \cdots, \underline{m}_q^{g_q}] = M_a \in \mathbb{R}^{p \times q}$ is called the *highest (column) degree coefficient matrix* of M(s).

Definition (2.11) [Kai. 1]: A polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$ is said to be *column proper or column reduced* if the matrix M_a has full rank q. □

Proposition (2.9) [Ros. 1]: A polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$ is column reduced if its complexity c is equal to its degree d. □

Proposition (2.10) [Ros. 1]: Let $M(s) \in \mathbb{R}^{p \times q}[s]$ be a polynomial matrix which is not column reduced. Then there always exists a unimodular matrix $U(s) \in \mathbb{R}^{q \times q}[s]$, $\det\{U(s)\} \in \mathbb{R} - \{0\}$, such that the polynomial matrix $M'(s) = M(s)\,U(s)$ is column reduced. □

2.8.1 The algebraic Structure of Rational Vector Spaces

Let $G(s) \in \mathbb{R}^{m \times l}[s]$, $m \geq l$, $\rho\{G(s)\} = l$ be a matrix. Let us also denote by $\mathcal{V}_G$ the set of all linear combinations of the columns of G(s) with multipliers in $\mathbb{R}(s)$, i.e. if $G(s) = [\underline{g}_1(s), \cdots, \underline{g}_l(s)]$, then $\mathcal{V}_G = \text{span}_{\mathbb{R}(s)}\{\underline{g}_1(s), \cdots, \underline{g}_l(s)\}$. Clearly $\mathcal{V}_G$ is a linear vector space over $\mathbb{R}(s)$ and $\dim \mathbb{V}_G = l$, and it is called the *rational vector space generated by G(s)*.

From any rational basis matrix G(s) of $\mathcal{V}_G$ we can generate a polynomial basis of $\mathcal{V}_G$ by means of a right MFD of G(s), i.e. if $G(s) = N(s)\,D^{-1}(s)$ with $N(s) \in \mathbb{R}^{m \times l}[s]$, $D(s) \in \mathbb{R}^{l \times l}[s]$, $\det\{D(s)\} \not\equiv 0$, then clearly the columns of N(s) define a polynomial basis of $\mathcal{V}_G$. More precisely, if $N(s) = [\underline{n}_1(s), \cdots, \underline{n}_l(s)]$ then $\text{span}_{\mathbb{R}(s)}\{\underline{n}_1(s), \cdots, \underline{n}_l(s)\} = \mathcal{V}_G$ and $\text{span}_{\mathbb{R}[s]}\{\underline{n}_1(s), \cdots, \underline{n}_l(s)\} = \mathcal{M}_N$ where $\mathcal{M}_N$ denotes the set of all linear combinations of the columns of N(s) with multipliers in $\mathbb{R}[s]$. The set $\mathcal{M}_N$ is a free $\mathbb{R}[s]-$module [Bir. 1] and it is called the *polynomial module generated by* N(s). Some of the important properties of such $\mathbb{R}[s]$ modules are summarised without a proof below [Bir. 1].

Proposition (2.11): Let $\mathcal{M}_{N_1}$, $\mathcal{M}_{N_2}$ be the polynomial modules generated by the polynomial matrices $N_1(s)$, $N_2(s) \in \mathbb{R}^{m \times l}$, with $\rho\{N_1(s)\} = \rho\{N_2(s)\} = l$. If $N_1(s) = N_2(s)\,Q(s)$, where $Q(s) \in \mathbb{R}^{l \times l}[s]$, $\det\{Q(s)\} \not\equiv 0$, then $\mathcal{M}_{N_1} \subseteq \mathcal{M}_{N_2}$. □

Proposition (2.12): Let $N_1(s)$, $N_2(s) \in \mathbb{R}^{m \times l}[s]$ be two polynomial bases of the same polynomial module $\mathcal{M}_N$. Then, there exists a unimodular matrix $Q(s) \in \mathbb{R}^{l \times l}$, $\det\{Q(s)\} = c \in \mathbb{R} - \{0\}$ such that $N_1(s) = N_2(s)\,Q(s)$. □

Thus, unimodular matrices represent co-ordinate transformations of a polynomial module.

Proposition (2.13): Let $N(s) \in \mathbb{R}^{m \times l}[s]$ be a basis of the polynomial module $\mathcal{M}_N$. Then the degree of N(s) is an invariant of $\mathcal{M}_N$, or in other words if $N_1(s) \in \mathbb{R}^{m \times l}[s]$ is any other basis of $\mathcal{M}_N$ then

$\deg\{N(s)\} = \deg\{N_1(s)\}$. □

Proposition (2.14): Let $N_1(s)$, $N_2(s) \in \mathbb{R}^{m\times l}[s]$, $m \geq l$, $\rho\{N_1(s)\} = l$, $\rho\{N_2(s)\} = l$ and let $d_1 = \deg \{N_1(s)\}$, $d_2 = \{N_2(s)\}$. If $N_1(s) = N_2(s)\, Q(s)$, $Q(s) \in \mathbb{R}^{l\times l}[s]$, deg $\{\det Q(s)\} = q \geq 1$. then

(i) $d_1 = d_2 + q$

(ii) $\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2}$

where $\mathcal{M}_{N_1}$, $\mathcal{M}_{N_2}$ are the polynomial modules generated by the polynomial matrices $N_1(s)$, $N_2(s)$, respectively. □

Clearly, the above conditions represent the extraction of a right divisor $Q(s)$ of the polynomial matrix $N_1(s)$. This observation leads us to the following conclusions: Let $N_1(s) \in \mathbb{R}^{m\times l}[s]$, $m \geq l$, $\rho\{N(s)\} = l$ be a polynomial matrix which can be written in terms of its columns as $N_1(s) = [\underline{n}_1^1(s), \cdots, \underline{n}_l^1(s)]$. Let us assume that $N_1(s)$ is not irreducible and let $\mathcal{V} = \text{span}_{\mathbb{R}(s)}\{\underline{n}_1^1(s), \cdots, \underline{n}_l^1(s)\}$, $\mathcal{M}_{N_1} = \text{span}_{\mathbb{R}[s]}\{\underline{n}_1^1(s), \cdots, \underline{n}_l^1(s)\}$ be the rational vector space $\mathcal{V}$ and the polynomial module $\mathcal{M}_{N_1}$ spanned be its columns. Then, if $Q_i(s)$, $i = 1, 2, \cdots$ are right divisors of $N_1(s)$, i.e.

$$N_1(s) = N_{i+1}(s)\, Q_i(s),\ i = 1, 2, \cdots \tag{2.73}$$

and the deg $\{\det Q_i(s)\} = q_i \geq 1$ are such that $q_1 \leq q_2 \leq q_3 \leq \cdots$, and if $Q_i(s)$ divides $Q_{i+1}(s)$, then

$$\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2} \subset \mathcal{M}_{N_3} \subset \cdots \tag{2.74}$$

and

$$\deg\{N_1(s)\} \geq \deg\{N_2(s)\} \geq \deg\{N_3(s)\} \geq \cdots \tag{2.75}$$

Moreover, if $Q_G(s)$ is a greatest right divisor of $N_1(s)$ so that $N_1(s) = N(s)\, Q_G(s)$, then

$$\mathcal{M}_{N_1} \subset \mathcal{M}_N \text{ and } \deg\{N_i(s)\} \geq \deg\{N(s)\} \tag{2.76}$$

The polynomial module $\mathcal{M}_N$ is the *maximal* $\mathbb{R}[s]$*-module* of the rational vector space $\mathcal{V}$ and all its polynomial bases are least degree, or irreducible polynomial matrices. In other words, if we consider the set of all polynomial vectors in $\mathcal{V}$ then this set coincides with the module $\mathcal{M}_N$ defined above.

Definition (2.12) [For. 1]: A polynomial matrix $N(s) \in \mathbb{R}^{m\times l}[s]$, $m \geq l$ and $\rho\{N(s)\} = l$ is said to be *minimal basis* of the rational vector space $\mathcal{V}$, $\mathcal{V}$ = col sp $\{N(s)\}$, if:

(i) N(s) is least degree

(ii) N(s) is column reduced. □

Remark (2.7): Let $N_1(s) \in \mathbb{R}^{m\times l}[s]$, $m \geq l$, $\rho\{N_1(s)\} = l$. If $N(s)$, $N^*(s) \in \mathbb{R}^{m\times l}[s]$ are two minimal bases of the rational vector space $\mathcal{V}$ spanned be the columns of N(s), the $N(s) = N^*(s)\ Q(s)$, where Q(s) is an $\mathbb{R}[s]$-unimodular matrix □

Theorem (2.8) [For, 1]: Let $N(s) = [\underline{n}_1(s), \cdots, \underline{n}_l(s)] \in \mathbb{R}^{m\times l}(s)$, $m \geq l$, $\rho\{N(s)\} = l$ be a minimal basis of a rational vector space $\mathcal{V}_N$ = col $\mathrm{sp}_{\mathbb{R}(s)}\{N(s)\}$ and let $\delta_i = \deg \underline{n}_i(s)$, $i = 1, \cdots, l$. The degrees $\{\delta_i, i = 1, \cdots, l\}$ are invariants of $\mathcal{V}_N$. □

Forney has defined the indices $\{\delta_i, i = 1, \cdots, l\}$ as the *invariant dynamical indices* of $\mathcal{V}_N$, and their sum $\delta = \sum_{i=1}^{l}\delta_i$ as the *invariant dynamical order* of $\mathcal{V}_N$. The set $\{\delta_i, i = 1, \cdots, l\}$ does not define a complete [Bir. & McL. 1] set of invariants for $\mathcal{V}_N$. A complete invariant is defined by the "echelon form" minimal basis of $\mathcal{V}_N$ [For, 1].

2.8.2 Further Properties of Rational Matrices

Some further results on the properties and structure of rational matrices related to MFDs and minimality of realisations are summarised here.

Proposition (2.15) [Kai. 1]: Let $G(s) \in \mathbb{R}^{m\times l}(s)$, $\rho\{G(s)\}$ = min $\{m, l\}$ be a rational matrix and let $\{D_r(s), N_r(s)\}$ be a right MFD of G(s), i.e. $G(s) = N_r(s)\ D_r^{-1}(s)$. Then any realisation of G(s) with order equal to the degree of the determinant of the denominator matrix (i.e. $n = \deg\{\det D_r(s)\}$) will be minimal (or equivalently, observable and controllable), if and only if the MFD is coprime □

Proposition (2.16) [Kai. 1]: Suppose $\{N_i(s)\ D_i^{-1}(s), i = 1, 2\}$ are two coprime MFDs of the rational matrix $G(s) \in \mathbb{R}^{m\times l}$, $\rho\{G(s)\} = \min\{m, l\}$. Then there exists a unimodular matrix $Q(s) \in \mathbb{R}^{m\times l}[s]$, such that $D_1(s) = D_2(s)\ Q(s)$ and $N_1(s) = N_2(s)\ Q(s)$. □

Proposition (2.17) [Kai. 1]: If $\{D(s), N(s)\}$ is any MFD of $G(s) \in \mathbb{R}^{m \times l}(s)$ with $\rho\{G(s)\} = \min\{m, l\}$ and $\{\overline{D}(s), \overline{N}(s)\}$ is a coprime MFD of G(s), then there exists a polynomial matrix $R(s) \in \mathbb{R}^{l \times l}[s]$, not necessarily unimodular, such that $D(s) = \overline{D}(s)\, R(s)$ and $N(s) = \overline{N}(s)\, R(s)$. □

Proposition (2.18) [Kai. 1]: The determinantal degree of the denominator matrix of any right CMFD of $G(s) \in \mathbb{R}^{m \times l}(s)$ with $\rho\{G(s)\} = \min\{m, l\}$ is equal to the determinantal degree of the denominator matrix of any left CMFD of G(s). □

The most important tool in the study of the properties of rational matrices is the Smith-McMillan form which is defined next [Ros, 2], [Kai. 1].

Theorem (2.9) (*Smith-MacMillan*): Let $G(s) \in \mathbb{R}^{m \times l}(s)$, $\rho\{G(s)\} = t \leq \min\{m, l\}$. Then there always exist unimodular matrices $Q_1(s) \in \mathbb{R}^{m \times m}[s]$, $Q_2(s) \in \mathbb{R}^{l \times l}[s]$ such that

$$Q_1(s)\; G(s)\; Q_2(s) = M(s) \tag{2.77}$$

where $M(s) \in \mathbb{R}^{m \times l}(s)$ is defined by

$$M(s) = \begin{array}{c} \left[\begin{array}{ccc|ccc} \frac{\epsilon_1(s)}{\psi_1(s)} & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \frac{\epsilon_t(s)}{\psi_t(s)} & & & 0 \\ 0 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 0 \end{array}\right] \begin{array}{l} \uparrow \\ t \\ \downarrow \\ \uparrow \\ m-t \\ \downarrow \end{array} \\ \leftarrow\; t \;\rightarrow \qquad \leftarrow\; l-t \;\rightarrow \end{array} \tag{2.78}$$

The pairs of monic polynomials $\{\epsilon_i(s), \psi_i(s)\}$ are co-prime $i = 1, \cdots, t$, uniquely defined and satisfy the division properties: $\psi_{i+1}(s)/\psi_i(s)$, $i = 1, \cdots, t\text{-}1$, $\epsilon_i(s)/\epsilon_{i+1}(s)$, $i = 1, 2, \cdots, t-1$. If D(s) is the monic least common multiple of the denominators of the elements of G(s), then $D(s) = \psi(s)$. □

The sum of the deg $\psi_i(s)$, $i = 1, \cdots, t$ is called the *MacMillan degree* of G(s) and for strictly proper G(s), it is equal to the determinantal degree of the denominator matrix of any co-prime MFD of G(s).

Proposition (2.19) [Kai. 1]:

(i) The (right or left) numerators of coprime MFDs of G(s) all have the same Smith form.

(ii) The denominators of coprime MFDs of G(s) all have the same non-unity invariant polynomials □

Let $G(s) \in \mathbb{R}^{m \times l}(s)$ be a rational transfer function matrix. We say that G(s) is *proper* if $\lim_{s \to \infty} G(s) < \infty$ and that G(s) is *strictly proper* if $\lim_{s \to \infty} G(s) = 0$. An important result characterising these properties of a G(s) are given below.

Proposition (2.20) [Kai. 1]:

(i) If $G(s) \in \mathbb{R}^{m \times l}(s)$ is a *strictly proper* (proper) rational transfer function matrix and $G(s) = N(s)\, D^{-1}(s)$, then every column of N(s) has degree strictly less than (less than or equal to) that of the corresponding column of D(s).

(ii) If D(s) is column reduced, then $G(s) = N(s)\, D^{-1}(s)$ is strictly proper (proper) if and only if each column of N(s) has degree less than (less than or equal to) the degree of the corresponding column of D(s) □

2.8.3 Poles and Zeros of Rational Matrices

The Smith-MacMillan form of a rational matrix provides the means for a natural extension of the definition of poles and zeros [Ros, 2], [MacF. & Kar., 1] from the scalar to the matrix case

Definition (2.13): Let $G(s) \in \mathbb{R}^{m \times l}(s)$. Then,

(i) The *zeros* of G(s) are defined as the roots of the numerator polynomials $\{\epsilon_i(s)\}$ of the Smith-MacMillan form.

(ii) The *poles* of G(s) are defined as the roots of the denominator polynomials $\{\psi_i(s)\}$ of the Smith-MacMillan form. □

The polynomials defined by

$$z(s) = \prod_{i=1}^{t} \epsilon_i(s), \qquad p(s) = \prod_{i=1}^{t} \psi_i(s) \tag{2.79}$$

are referred to as the *zero* , *pole polynomial* respectively of G(s). From the results of the previous section we have the following alternative characterisation of poles and zeros.

Proposition (2.22) [Kai. 1]: Let $G(s) \in \mathbb{R}^{m \times l}(s)$ and let $G(s) = D_L(s)^{-1}N_L(s) = N_R(s)\, D_R(s)^{-1}$ be left, right coprime MFDs. Then,

(i) The pole polynomial p(s) of G(s) is given by $p(s) = \det\{D_L(s)\} = c \det\{D_R(s)\}$, $c \in \mathbb{R} \neq 0$.

(ii) The zero polynomial z(s) of G(s) is given by the product of the invariant polynomials of $N_L(s)$, or equivalently $N_R(s)$. □

Consider now a $G(s) \in \mathbb{R}^{m \times l}(s)$, $m \geq l$, $\rho\{G(s)\} = l$ and let {D(s), N(s)} be a right coprime MFD pair. If Z(s) is greatest right divisor of N(s), we may write

$$G(s) = N(s)\, D(s)^{-1} = \overline{N}(s)\, Z(s)\, D(s)^{-1} \tag{2.80}$$

where $\overline{N}(s)$ is a least degree basis matrix for $\mathfrak{X}_G$=col−span$_{\mathbb{R}(s)}$$\{G(s)\}$. Using the above factorisation of G(s) we have $p(s) = \det\{D(s)\}$, $z(s) = \det\{Z(s)\}$ and thus

$$C_l(N(s)) = C_l(\overline{N}(s)\, Z(s)) = C_l(\overline{N}(s))z(s) \in \mathbb{R}^{\binom{m}{l} \times 1}[s] \tag{2.81}$$

Clearly, eqn (2.81) implies:

Remark (2.5): If N(s) is a numerator of a coprime MFD of G(s), then the zero polynomial z(s), is the greatest common divisor of the polynomial entries of $C_l(N(s))$ □

Let us suppose now that G(s) is a rational transfer function matrix with $G(s) \in \mathbb{R}^{m \times l}(s)$, $\rho\{G(s)\} = \min\{m, l\}$ and that $\{D_L(s), N_L(s)\}$, $\{D_R(s), N_R(s)\}$ are left and right MFDs of G(s), respectively, not necessarily coprime. Then by Proposition (2.17) the corresponding realisations of G(s) corresponding to those two MFDs are not minimal. It can be proved [Ros. 1] that:

(i) the (Smith) zeros of $[D_L(s),\ N_L(s)]$ correspond to uncontrollable modes of the equivalent state space realisation and they are termed *input-decoupling* (i.d.) zeros of the MFD

(ii) the (Smith) zeros of $\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}$ correspond to unobservable modes of the equivalent state space realisation and they are termed the *output-decoupling* (o.d.) zeros of the MFD.

To distinguish the transfer function zeros from the decoupling zeros, we often call the transfer function zeros *transmission zeros.*

Proposition (2.23): Let $G(s) \in \mathbb{R}^{m \times l}(s)$ with $\rho\{G(s)\} = \min\{m, l\}$ and let $\{D_L(s),\ N_L(s)\}$, $\{D_R(s),\ N_R(s)\}$ be left and right MFDs of G(s), respectively, not necessarily coprime. Then

$$C_m[D_L(s)\ N_L(s)] = C_m\left\{T_L(s)\right\} z_{i.d.}(s),\quad C_l\left\{\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}\right\} = C_l\left\{T_R(s)\right\} z_{o.d.}(s) \tag{2.82}$$

□

Chapter 3

DECENTRALISED CONTROL OF LARGE SCALE DYNAMICAL SYSTEMS

3.0 Introduction

During the last decade, a great number of researchers in the field of automatic control engineering have concerned themselves with the mathematical analysis and synthesis of complex dynamic systems which are also called integrated systems, interconnected systems, decentralised systems, hierarchical systems or large-scale systems. Most of the references up to 1980 dealing with the above topics can be found in the books of [Wynn.1], [Sae. 1], [Mic. & Mil. 1], [Sil.1] and in the special issue of the Transactions in Automatic Control published by IEEE [IEEE.1].

Decentralised control systems are defined to be large dynamic systems with several automatic controllers each operating on the system with partial information on the states of the controlled system. This definition due to [McFa. 1] suggests that there are certain architectural constraints on the controlled system as well as on the feedback matrix of the resulting closed-loop system. With the constraints imposed, controllability of the systems does no longer imply stabilisability.

Given a large-scale interconnected system, a useful assumption must therefore be made that no control agent, such as a supervisory controller, possesses the complete information which describes mathematically the controlled system and the environment in which the system is to operate. Since each local automatic controller has access to a different set of the state variables of the whole controlled system state, it is possible for the system to become unstable in the absence of communication among the autonomous controllers. Therefore, when considering the decentralised control of a large-scale system using autonomous controllers the properties of controllability and observability of the controlled system are far from sufficient for its automation, contrary to the situation in a centralised framework suggested by the classical multivariable control theory.

Hence the main objective of this chapter is to give the definition of fixed modes in decentralised control of large-scale systems as well as their different characterisations with respect to different mathematical descriptions of the controlled systems. Also, a short survey of the various methods appeared in the literature to avoid these decentralised fixed modes, is given. But fixed modes are not the only theme of this thesis and because we are mainly interested in the control problems associated with the automation of large scale systems, we also review some known results associated with the controllability (pole assignability) or stabilisability of a decentralised dynamic system using different structures of automatic controllers such as decentralised constant state feedback, and decentralised dynamic output feedback etc.

3.1 Fixed Modes of Decentralised Systems.

The concept of decentralised fixed modes introduced by [Wan. and Dav. 1] plays a critical role in many control problems of large-scale systems such as the control of large space structures (LSS) which is the current hottest topic of research. For example, the stabilisation problem or the controllability problem as defined by [Won. 1] depends on the properties of these modes. The presence of fixed modes in the right half plane of the classic root-locus diagram, shows that decentralised stabilisation is impossible while the presence of any sort of fixed modes does not allow arbitrary frequency assignment under decentralised control.

Thus, the characterisation and determination of decentralised fixed modes has received much attention by the control engineers in the last years. A detailed mathematical review of the most important results by previous researchers is as follows:

3.1.1 Definition of Decentralised Fixed Modes Using The State-Space Description Of Dynamic Systems.

The mathematical description to be considered is a k-channel linear dynamic system given by the state-space description (SSD):

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + \sum_{i=1}^{k} B_i \underline{u}_i(t) \\ \underline{y}_i(t) &= C_i \underline{x}(t); \quad i = 1,2, \cdots ,k \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times l_i}$, $C_i \in \mathbb{R}^{m_i \times n}$, $i \in k = \{1, 2, ..., k\}$, $\underline{u}_i(t)$ and $\underline{y}_i(t)$ are the vectors of control inputs and measured output variables associated with channel i.

We shall assume that (3.1) is controllable from all controls $\underline{u}_i(t)$, similarly jointly observable but not necessarily controllable from any single control $\underline{u}_i(t)$ and similarly not necessarily observable from any single output $\underline{y}_i(t)$. Of course, if $[C_i, A, B_i]$ is complete that is if (C_i, A) is observable and (A, B_i) controllable for some i, the problem is trivial. Also, the above hypothesis for joint controllability and observability does not imply that the integrated system, Σ, where $\Sigma = \{C_i, A, B_i : k\}$ is controllable or observable through a simple subsystem. This result is proved by [Won. 1] within the framework of geometric approach.

A controller for the above integrated system is called decentralised if each autonomous controller $\underline{u}_i(t)$ depends only on local measurements, $\underline{y}_i(t)$ for its autonomous operation. A generalised description of the local autonomous controller for the i^{th}channel is a linear dynamic system, Σ_i, whose inputs include the measure of local output of the system, $\underline{y}_i(t)$, as well as command inputs, $\underline{v}_i(t)$, from other echelons in the controller, Σ_i, is used as the control input, $\underline{u}_i(t)$ for the subsystem, i. Also the i^{th} controller admits a dynamic mathematical description of the form:

$$\begin{aligned} \dot{\underline{z}}_i(t) &= H_i\underline{z}_i(t) + L_i\underline{y}_i(t) + R_i\underline{v}_i(t) \\ \underline{u}_i(t) &= M_i\underline{z}_i(t) + F_i\underline{y}_i(t) + G_i\underline{v}_i(t) \end{aligned} \tag{3.2}$$

In the case of autonomous local controller with constant feedback of Decentralised-Constant Output Feedback (D-COF). We have that

$$\underline{u}_i(t) = F_i\, \underline{y}_i(t) + \underline{v}_i(t), \quad F_i \in \mathbb{R}^{l_i \times m_i} \tag{3.3}$$

If the local autonomous controllers (3.3) are applied to the controlled system (3.1), then the mathematical description of the integrated system is:

$$\begin{aligned} \dot{\underline{x}}(t) &= (A + \sum_{j=1}^{k} B_j F_j C_j)\, \underline{x}(t) + \sum_{i=1}^{k} B_i\, \underline{v}_i(t) \\ \underline{y}_i(t) &= C_i\underline{x}(t), \qquad i \in k \end{aligned} \tag{3.4}$$

Definition (3.1) [Wan & Dav. 1]: The fixed modes of the integrated system (3.1) under decentralised constant output feedback (D-COF) are the roots of the fixed polynomials $\theta_D(s)$ defined by:

$$\theta_D(s;\, A,\, B,\, C|F_D) = \text{g.c.d. } |sI_n - A_F| = \text{g.c.d.}\{\det(s\, I_n - A - BFC) | F_D \in \mathfrak{F}\} \tag{3.5}$$

where $A_F = A + \sum_{i=1}^{k} B_i F_i C_i$, $\forall F_i \in \mathcal{F}$ = vector space of all block diagonal matrices, $B = [B_1, B_2, \cdots, B_k]$, $B_i \in \mathbb{R}^{n \times l_i}$ and $C^T = [C_1, C_2, \cdots, C_k]$, $C_i \in \mathbb{R}^{m_i \times n}$.

Definition (3.2) [Wan & Dav. 1]: Given the system (3.1), let F_D given by :

$$F_D = \text{block-diag}\{F_1, F_2, \cdots, F_k\}, \quad F_i \in \mathbb{R}^{l_i \times m_i}$$

The *decentralised fixed modes* of (3.1) with respect to F are given by:

$$\Phi_D = \Lambda\,(A, B, C; F_D) \triangleq \bigcap_{F_D \in \mathcal{F}} \sigma\,(A + BF_D C) \tag{3.6}$$

where (i) $\sigma(A + BF_D C)$ denotes the spectrum (eigenvalues) of the matrix $A + BF_D C$ and (ii) the intersection is indexed over all appropriately partitioned block diagonal matrices F_D.

Definition (3.3): Consider the system $\Sigma(A, B, C)$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{m \times n}$ and B and C are partitioned as $B = [B_1, B_2, \cdots, B_k]$, $B_i \in \mathbb{R}^{n \times l_i}$, $C^T = [C_1, C_2, \cdots, C_k]$, $C_i \in \mathbb{R}^{m_i \times n}$, respectively. The *Centralised Fixed Modes* of $\Sigma(A, B, C)$ under $\forall\ F_c \in \mathbb{R}^{l \times m}$ are given by the roots of $\theta_c(s)$ where:

$$\theta_c(s; A, B, C \mid F_c) = \text{g.c.d.}\left\{\det(sI_n - A - BFC) \mid F_c \in \mathbb{R}^{l \times m}\right\} \tag{3.7}$$

or

$$\Phi_c(A,B,C; F_c) \triangleq \bigcap_{F_c \in \mathbb{R}^{l \times m}} \sigma\,(A + BF_c C) \tag{3.8}$$

Remark (3.1): The roots of the centralised fixed polynomial $\theta_c(s)$ are the uncontrollable or unobservable characteristic values of A and $\theta_c(s)$ obviously divides $\theta_D(s)$.

Remark (3.2): Closely related to the pole assignment problem, is the problem of determining whether the whole system, Σ, can be stabilised with decentralised control. A system with no decentralised fixed modes or with only stable decentralised fixed modes in general, may not be stabilisable by a feedback law of the form (3.3). However, stabilisation can be achieved with dynamic feedback of the form:

$$\underline{u} = G\underline{z} + H\underline{y}$$
$$\dot{\underline{z}} = D\underline{z} + K\underline{y} \tag{3.9}$$

where:

$$\begin{aligned} D &= \mathrm{diag}(D_1, D_2, \cdots, D_k), \; D_i \in \mathbb{R}^{r_i \times r_i} \\ K &= \mathrm{diag}(K_1, K_2, \cdots, K_k), \; K_i \in \mathbb{R}^{r_i \times m_i} \\ G &= \mathrm{diag}(G_1, G_2, \cdots, G_k), \; G_i \in \mathbb{R}^{l_i \times r_i} \\ H &= \mathrm{diag}(H_1, H_2, \cdots, H_k), \; H_i \in \mathbb{R}^{l_i \times m_i} \end{aligned} \tag{3.10}$$

The relation between static controllers (D-COC) described by (3.3) and dynamic controllers of Decentralised-Dynamic Output Feedback as described by (3.9) is clarified by the following theorem.

Theorem (3.1) [Wan & Dav. 1]: (a) For any decentralised feedback law of the form (9), the decentralised fixed polynomial $\theta_D(s)$ is a factor of the characteristic polynomial of the closed loop system defined (1) and (9). (b) For every open subset S of the complex plane, there exists a decentralised dynamic output feedback (D-DOF) law of the form (9), such that the characteristic polynomial of the closed-loop system (1) and (9) has the form $\theta_D(s)\cdot\theta(s)$, where the zeros of $\theta(s)$ are contained in S.

Remark 3.3: The above theorem is a pure existence statement which says that the fixed modes cannot be moved and the non-fixed modes can be shifted into arbitrary positions by the decentralised dynamic output feedback.

3.1.2 Computations of Fixed Modes

The following technique has been suggested by [Dav & Özg. 1] and shows that the decentralised fixed modes of a system may be calculated in a very simple manner The main steps of the technique are as follows:

Step 1: Calculate the eigenvalues of A.

Step 2: Select arbitrary matrices F_i, $i=1,2,\cdots,k$ possibly using a pseudorandom number generator.

Step 3: Compute the eigenvalues of $A + \sum_{i=1}^{k} B_i F_i C_i$.

Step 4: The decentralised fixed modes are contained in those eigenvalues of $A + \sum_{i=1}^{k} B_i F_i C_i$ which are common with the eigenvalues of A.

Step 5: If in doubt as to which the decentralised fixed modes of (1) are, choose new arbitrary matrices F_i, in step 2 and repeat steps 3 and 4.

3.1.3 Fixed Modes of an Elastic Spacecraft System

An example illustrating the calculation of fixed modes is given by [West *et al*, 1] where it is attempted to control a large scale dynamical system, the spacecraft of order 100, using decentralised actuators and sensors. It is also shown there, that the decentralised fixed modes are identical to the centralised fixed modes a result already obtained mathematically by Saeks [Sae.1]. The physical meaning of the fixed modes result when a sensor or an actuator is located at a node of an elastic mode. This phenomenon was well known to the early pioneers of rocketry, since the crucial location of sensors and actuators in an open loop unstable vehicle such as a rocket have resulted in spectacular catastrophes. Another conclusion of the above mentioned papers is that a solution of the decentralised control problem exists if and only if a solution exists for the centralised control problem. Also, due to the uncertainty of locating sensor and actuators at the same position such as the nodes of the system's elastic modes, there are actually no "precise fixed modes" but only "approximate". Another interesting result of the above application of decentralised control is the elimination of the "spillover phenomena" usually associated with the excitation of unmodeled high-frequency elastic modes of the large space structure. The "spillover phenomena" are very common in large scale systems such as aerospace systems, where the practice of using centralised (multivariable) control leads to the stabilisation and control of only a subset of the eigenvalues of the controlled system.

3.1.4 State-Space Characterisation of Decentralised Fixed Modes

The following six theorems characterise the fixed modes of a decentralised system and were obtained by Davison and Özgüner [Dav. and Özg. 1].

Theorem 3.2 [Dav. and Özg. 1]: Given a k-control agent decentralised system with k=2, then $\lambda \in$ sp(A) is NOT a decentralised fixed mode of the system (3.1) for k=1, 2 if and only if the following conditions all hold:

(i) λ is NOT a decentralised fixed mode of

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B_1\underline{u}_1 + B_2\underline{u}_2 \\ \underline{y}_1 &= C_1\underline{x} \\ \underline{y}_2 &= C_2\underline{x} \end{aligned} \tag{3.11}$$

(ii) $$\text{rank}\begin{bmatrix} A-\lambda I & B_1 \\ C_1 & 0 \end{bmatrix} \geq n \tag{3.12}$$

(iii) $$\text{rank}\begin{bmatrix} A-\lambda I & B_2 \\ C_2 & 0 \end{bmatrix} \geq n \tag{3.13}$$

In the case where A is diagonal then with two control agents we have:

$$\dot{\underline{x}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \underline{x} + \begin{bmatrix} B_1^* \\ \overline{B}_1 \end{bmatrix} \underline{u}_1 + \begin{bmatrix} B_2^* \\ \overline{B}_2 \end{bmatrix} \underline{u}_2$$

$$\underline{y}_1 = \begin{bmatrix} C_1^* & \overline{C}_1 \end{bmatrix} \underline{x} \tag{3.14}$$

$$\underline{y}_2 = \begin{bmatrix} C_2^* & \overline{C}_1 \end{bmatrix} \underline{x}$$

where $\underline{u}_1 \in \mathbb{R}^{l_1}$, $\underline{u}_2 \in \mathbb{R}^{l_2}$, $\underline{y}_1 \in \mathbb{R}^{m_1}$, $\underline{y}_2 \in \mathbb{R}^{m_2}$, $\lambda_1 \in C$, B_1^*, B_2^* are l_1, l_2 row vectors respectively and C_1^*, C_2^* are m_1, m_2 column vectors respectively.

Let

$$\begin{aligned} \overline{B}_1 &= \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_{l_1}^1 \end{bmatrix}, & \overline{B}_2 &= \begin{bmatrix} b_1^2 & b_2^2 & \cdots & b_{l_2}^2 \end{bmatrix} \\ \overline{C}_1^T &= \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_{m_1}^1 \end{bmatrix}, & \overline{C}_2^T &= \begin{bmatrix} c_1^2 & c_2^2 & \cdots & c_{m_2}^2 \end{bmatrix} \end{aligned} \tag{3.15}$$

and assume that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all distinct and occur in complex conjugate pairs.

Theorem 3.3 [Dav. and Özg. 1]: In system (3.14) λ_1 is NOT a decentralised fixed mode if and only if the following conditions are all satisfied:

(i) $$\det\begin{bmatrix} B_1^* & B_2^* \end{bmatrix} \neq 0 \quad \text{and} \quad \det\begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} \neq 0 \tag{3.16}$$

(ii) The condition $B_1^* = 0$ and $C_2^* = 0$, and λ_1 is a transmission zero of

$$\left\{ c_i^2, \begin{bmatrix} \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, b_j^1 \right\} \ \forall\, i \in [1, 2, \cdots, r_2], \ \forall\, j \in [1, 2, \cdots, m_1] \tag{3.17}$$

does *not* hold.

(iii) The condition $B_2^* = 0$ and $C_2^* = 0$ and λ_1 is a transmission zero of

$$\left\{ c_i^1, \begin{bmatrix} \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, b_j^2 \right\} \ \forall\, i \in [1, 2, \cdots, r_1], \ \forall\, j \in [1, 2, \cdots, m_2] \tag{3.18}$$

does *not* hold.

Theorem 3.4 [Dav. and Özg, 1]: Given the k-control agent decentralised system with $k \geq 3$, then $\lambda \in$ sp(A) is *not* a decentralised fixed mode of (3.1) if and only if λ *not* a decentralised fixed mode of any of the following k-1 control agent systems for (3.1).

(1) $$\left\{ \begin{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ C_3 \\ \vdots \\ C_k \end{bmatrix}, \ A, \ \left(\left(B_1, \ B_2 \right), \ B_3, \ \cdots, \ B_k \right) \right\}$$

(2) $$\left\{ \begin{bmatrix} C_1 \\ \begin{pmatrix} C_2 \\ C_3 \end{pmatrix} \\ \vdots \\ C_k \end{bmatrix}, \ A, \ \left(B_1, \ \left(B_2, \ B_3 \right), \ \cdots, \ B_k \right) \right\}$$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

(k−2) $$\left\{ \begin{bmatrix} C_1 \\ \vdots \\ \begin{pmatrix} C_{k-2} \\ C_{k-1} \end{pmatrix} \\ C_k \end{bmatrix}, \ A, \ \left(B_1, \ \cdots, \ \left(B_{k-2}, \ B_{k-1} \right), \ B_k \right) \right\}$$

(k−1) $$\left\{ \begin{bmatrix} C_1 \\ \vdots \\ C_{k-2} \\ \begin{pmatrix} C_{k-1} \\ C_k \end{pmatrix} \end{bmatrix}, \ A, \ \left(B_1, \ \cdots, \ B_{k-2}, \ \left(B_{k-1}, \ B_k \right) \right) \right\}$$

(k) $$\left\{ \begin{bmatrix} C_1 \\ \vdots \\ C_{k-3} \\ \begin{pmatrix} C_{k-2} \\ C_k \end{pmatrix} \\ C_{k-1} \end{bmatrix}, \ A, \ \left(B_1, \ \cdots, \ B_{k-3}, \ \left(B_{k-2}, \ B_k \right), \ B_{k-1} \right) \right\}$$

Consider, now, the systems consisting of a number of arbitrary interconnected subsystems.

$$\dot{\underline{x}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \underline{u}_1 + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \underline{u}_2$$

$$\underline{y}_1 = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} \underline{x} \tag{3.19}$$

$$\underline{y}_2 = \begin{bmatrix} 0 & I_{n_2} \end{bmatrix} \underline{x}$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$.

Theorem (3.5) [Dav. and Özg. 1]: The system (3.19) with two control agents, has no decentralised fixed modes if and only if

(i) (A_i, B_i) is controllable for i=1,2

(ii) $\left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right\}$ is controllable $\quad$ (3.20)

Consider now the k-agent interconnected system

$$\dot{\underline{x}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1k} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2k} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{k1} & A_{k2} & A_{k3} & \cdots & A_{kk} \end{bmatrix} \underline{x}$$

$$+ \begin{bmatrix} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underline{u}_1 + \begin{bmatrix} 0 \\ B_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underline{u}_2 + \cdots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ B_k \end{bmatrix} \underline{u}_k$$

$$
\begin{aligned}
\underline{y}_1 &= [\, I_{n_1}, 0, 0, \cdots, 0]\, \underline{x} \\
\underline{y}_2 &= [\, 0, I_{n_2}, 0, \cdots, 0]\, \underline{x} \\
&\ \ \vdots \\
\underline{y}_k &= [\, 0, 0, 0, \cdots, I_{n_k}]\, \underline{x}
\end{aligned}
\tag{3.21}
$$

System (3.21) has decentralised fixed modes if

(i) (A_{ii}, B_i) is controllable for all $i=1, 2, \cdots, k$

(ii) $\left\{ \begin{bmatrix} A_{i_1 i_1} & A_{i_1 i_2} & \cdots & A_{i_1 i_g} \\ A_{i_2 i_1} & A_{i_2 i_2} & \cdots & A_{i_2 i_g} \\ \vdots & \vdots & \vdots & \vdots \\ A_{i_g i_1} & A_{i_g i_2} & \cdots & A_{i_g i_g} \end{bmatrix}, \begin{bmatrix} B_{i_1} & 0 & \cdots & 0 \\ 0 & B_{i_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{i_g} \end{bmatrix} \right\}$ is controllable where

$i_1 = 1, 2, \cdots, k-g+1;$

$i_2 = i_1+1, i_1+2, \cdots, k-g+2;$

$i_3 = i_2+1, i_2+2, \cdots, k-g+3;$

where $g=2, 3, \cdots, k$

□

The above results characterise the behaviour of large scale systems in which local state feedback is allowed. It implies that given two interconnected subsystems in which each subsystem is controllable and in which the total integrated system is jointly controllable, can always be stabilised using local state feedback with dynamic compensation.

In the case of output feedback we have the following system

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1k} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2k} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{k1} & A_{k2} & A_{k3} & \cdots & A_{kk} \end{bmatrix} \underline{x} +$$

$$+ \begin{bmatrix} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underline{u}_1 + \begin{bmatrix} 0 \\ B_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underline{u}_2 + \cdots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ B_k \end{bmatrix} \underline{u}_k$$

$$\begin{aligned} y_1 &= [\ C_1, 0, 0, \cdots, 0]\ \underline{x} \\ \underline{y}_2 &= [\ 0, C_2, 0, \cdots, 0]\ \underline{x} \\ &\ \vdots \\ \underline{y}_k &= [\ 0, 0, 0, \cdots, C_k]\ \underline{x} \end{aligned} \tag{3.22}$$

where $A_{ij}=B_{ij}K_{ij}C_{ij}$, $i=1, 2, \cdots, k$, $j=1, 2, \cdots, k$, $i\neq j$ where K_{ij} is the interconnection gain and B_{ij} and C_{ij} are arbitrary.

Theorem (3.6) [Dav. and Özg, 1]: Given the above system (3.22), assume that (C_i, A_{ij}, B_i) is controllable and observable, then this implies that (3.22) has no decentralised fixed modes, for almost all interconnection gains K_{ij}. □

If it is assumed that (3.22) has the following special structure

$$A_{ij} = B_i\ K_{ij}\ C_j \qquad i\neq j \tag{3.23}$$

the following result is due to [Dav. 3] and [Sae. 2]:

Theorem 3.7: Given the system (3.22) with the structure (3.23) then necessary and sufficient conditions for the system 3.22 to have no decentralised fixed modes are that (C_i, A_{ij}, B_j) be controllable and observable. □

It should be mentioned that the above results rely heavily on the following characterisation of decentralised fixed modes obtained by Anderson and Clements [And. & Cle. 1].

Proposition (3.1) [And. & Cle. 1]: Consider the system (3.1). Then a necessary and sufficient condition for $\lambda \in sp(A)$ to be a decentralised fixed mode of (3.1) is that for some partition of the set $\{1, 2, \cdots, k\}$ into disjoint sets $\{i_1, i_2, \cdots, i_g\}$ and $\{i_{g+1}, i_{g+2}, \cdots, i_k\}$, then

$$\text{rank} \begin{bmatrix} A-\lambda I & B_{i1} & B_{i2} & \cdots & B_{ig} \\ C_{i_{g+1}} & 0 & 0 & \cdots & 0 \\ C_{i_{g+2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ C_{i_k} & 0 & 0 & \cdots & 0 \end{bmatrix} < n \tag{3.24}$$

□

The above proposition avoids the computation method suggested by Davison where for several controllers chosen randomly, one computes the resulting closed-loop frequencies. Then if certain frequencies are common to all controllers with probability one, these are fixed modes. The above result of Anderson and Clements has rich geometric properties since it is helpful in identifying the parts of the overall system which may be held responsible for the fixed modes.

3.2 Frequency Domain Characterisation of Fixed Modes

Although the definition of multivariable poles is unique, various definitions of multivariable zeros have been proposed some of which are overlapping in some definitions [Ros. 1], [McFar & Kar. 1]. According to Davison and Wang [Dav. & Wan. 4], for a linear multivariable system

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + B\,\underline{u} \\ \underline{y} &= C\,\underline{x} + D\,\underline{u} \end{aligned} \tag{3.25}$$

where $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^m$, $\underline{y} \in \mathbb{R}^l$, the transmission zeros of (A, B, C, D) are defined as those values of λ which satisfy

$$\operatorname{rank} \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} < n + \min(m,l) \tag{3.26}$$

The Smith zeros of Rosenbrock's system matrix commonly called *invariant zeros*, do not coincide with transmission zeros, generally. Mac Farlane and Karcanias discussed the relation between transmission zeros defined as the zeros of the transfer matrix and the invariant zeros. The invariant zeros include the transmission zeros of Mac Farlane and Karcanias plus some of the decoupling zeros. However, when the system is complete, [Won, 1], that is {C, A} is observable and (A, B) is controllable, the sets of invariant zeros and transmission zeros are the same.

The observation that a characterisation of decentralised fixed modes may be obtained in terms the transmission zeros of certain subsystems has been made by Fessas [Fes. 1]. A recent result due to Davison and Wang [Dav. & Wan. 5] is provided by the following system

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + B\,\underline{u} \qquad & \dot{\underline{x}} &= A\,\underline{x} + [B_1, B_2, \cdots, B_l]\,\underline{u} \\ \underline{y} &= C\,\underline{x} & \underline{y} &= [C_1, C_2, \cdots, C_m]^T\,\underline{x} \end{aligned} \tag{3.27}$$

where $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^l$, $\underline{y} \in \mathbb{R}^m$. Assume, the following controller, $\underline{u} = K(s)\underline{y}$, whose structure is constrained by

$$K = \left\{ (i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k) \right\} \tag{3.28}$$

where $i \in [1, 2, \cdots, l]$, $j_p \in [1, 2, \cdots, m]$, $p = 1, 2, \cdots, k$, i.e. all elements of K(s), except for the (i_p, j_p) elements corresponding to the ${i_p}^{th}$ row and ${j_p}^{th}$ column of K(s) are considered to be zero. Then, system (3.27) with a dynamic output feedback (DOF) controller whose architecture is as (3.28), has a decentralised fixed mode at $\lambda \in sp(A)$ with respect to K(s) if and only if λ is a transmission zero of all the following subsystems

$$\left\{ \begin{bmatrix} C^T_{jp_1} \\ C^T_{jp_2} \\ \vdots \\ C^T_{jp_s} \end{bmatrix}, A, \left(B_{ip_1}, B_{ip_2}, \cdots, B_{ip_s} \right) \right\} \tag{3.29}$$

$p_1=1, 2, \cdots, k+1-s$, $p_2=p_1+1, p_1+2, \cdots, k+2-s$, $p_s= p_{s-1}+1, p_{s-1}+2, \cdots, k$ for $s = 1, 2, \cdots, \min(m,l)$ □

It is clear that the above subsystem correspond to the subsystems associated with all nonsingular 1×1, 2×2, $\cdots$, $\min(l,m)\times\min(l,m)$ matrices of the controller matrix K(s).

From the above theorem, the following two corollaries follow immediately.

Corollary (3.1): Consider the plant (3.27), subject to the controller structure (3.28); then

i) If any of the subsystems (3.29) have transmission zeros which are disjoint from the eigenvalues of A, this implies that the system (3.27) has no decentralised fixed mode with respect to K.

ii) If any of the subsystems (3.29) are minimum phase, this implies that any decentralised fixed modes with respect to K of the plant are stable.

iii) Let's $m = l$, then if K(s) is nonsingular, this implies that a necessary condition for $\lambda \in sp(A)$ to be a decentralised fixed mode with respect to K is that λ be a transmission zero of (3.27).

Corollary 3.2: Given the system (3.27) assume that $\lambda \in sp(A)$ is a decentralised fixed mode with respect to the controller structure (3.28) then λ is a decentralised fixed mode of the system with respect to a new controller structure $\left\{k, (i_{k+1} j_{k+1})\right\}$ if and only if is a transmission zero of all the following subsystems

$$\left\{ C_{j_{p+1}}, A, B_{i_{p+1}} \right\} \tag{3.30}$$

and

$$\left\{ \begin{bmatrix} C^T_{jp_1} \\ C^T_{jp_2} \\ \vdots \\ C^T_{jp_s} \\ C^T_{jp_{k+1}} \end{bmatrix}, A, \left(B_{ip_1}, B_{ip_2}, \cdots, B_{ip_s}, B_{ip_{k+1}} \right) \right\} \tag{3.31}$$

$p_1 = 1, 2, \cdots, k+1-s, \quad p_2 = p_1+1, p_1+2, \cdots, k+2-s, \quad p_s = p_{s-1}+1, p_{s-1}+2, \cdots, k$ for $s = 1, 2, \cdots, \min(m,l)-1$ □

The above is a characterisation of decentralised fixed modes, theorem 3.8 is a necessary and sufficient condition for a pole to be a decentralised fixed mode under certain controller structure, provided that the system is controllable and observable. Theorem 3.8 also gives a way of choosing a new controller structure, under which the system has no decentralised fixed mode.

A sufficient condition which can be used to verify that the system $\{C, A, B; K\}$ where $K = \text{block diag}\{K_1, \cdots, K_N\}$, $K \in \mathbb{R}^{l_i \times m_i}$ has no centralised fixed mode under K, is given by the following corollary.

Corollary (3.3): If there exists a subsystem $\{C_i, A, B_i; K_i^*\}$, $K_i^* \in \mathbb{R}^{r_i \times r_i}$ which has no transmission zeros for all $\lambda \in sp(A)$, that is

$$\text{rank}\begin{bmatrix} A-\lambda I & B_i \\ C_i & 0 \end{bmatrix} = n + \sum_{i=1}^{N} r_i, \ \forall \lambda \in sp(A) \tag{3.32}$$

then the system $\{C, A, B; K\}$ has no decentralised fixed modes under controller structure, K. □

The above results can easily be extended to the frequency domain. The transfer function matrix corresponding to the SSD (3.27) is

$$G(s) = C\,(sI-A)^{-1}B \tag{3.33}$$

and the characteristic polynomial p(s) of (3.33) is

$$p(s) = \det\,(sI-A) \tag{3.34}$$

Consider now the Rosenbrock system matrix

$$P(s) = \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \tag{3.35}$$

where A, B, C are $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, $\mathbb{R}^{m\times n}$ matrices. But

$$\begin{bmatrix} I_n & 0 \\ C(sI-A)^{-1} & I_m \end{bmatrix}\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} sI-A & B \\ 0 & C(sI-A)^{-1}B \end{bmatrix}\begin{bmatrix} sI-A & B \\ 0 & G(s) \end{bmatrix}$$

hence, for a square system, m=l, we have

$$\det\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} sI-A & B \\ 0 & G(s) \end{bmatrix} = \det\,(sI-A).\ \det G(s) = p(s)\det G(s) \tag{3.36}$$

From the above relation, and the usual definition of the transmission zeros, we have:

Lemma (3.1) [Dav. & Wan. 5]: For a given SSD of a decentralised dynamical system (C, A, B; K) the transmission zeros (T.Z) of C, A, B , TZ(C, A, B) are:

$$T.Z(C,\ A,\ B) = \left\{\lambda\colon \lambda\in\mathbb{C};\ p(s)\ \det G(s)\big|_{s=\lambda}=0\right\} \tag{3.37}$$

Again, corollary (3.3) is a sufficient condition that the system (3.27) has no decentralised fixed mode where now the rank test of a certain subsystem is replaced by the condition that the characteristic value of the system A should not be a root of (3.37)

A more complete characterisation of fixed modes in the frequency domain has been obtained by Anderson [And. 2] who gives the necessary and sufficient conditions on the transfer function matrix for the existence of fixed modes.

3.3 Fixed Modes of Transfer Matrix Descriptions

3.3.1 Decentralised Fixed Modes Using Constant Output Controllers.

Consider the multivariable (Multiple Input-Multiple Output) linear time-invariant dynamic system described by

$$\begin{bmatrix} A_1(s) & A_2(s) & \cdots & A_m(s) \end{bmatrix} \begin{bmatrix} \underline{y}_1(s) \\ \underline{y}_2(s) \\ \vdots \\ \underline{y}_m(s) \end{bmatrix} = \begin{bmatrix} B_1(s) & B_2(s) & \cdots & B_m(s) \end{bmatrix} \begin{bmatrix} \underline{u}_1(s) \\ \underline{u}_2(s) \\ \vdots \\ \underline{u}_m(s) \end{bmatrix} \tag{3.38}$$

or

$$A(s)\ \underline{y}(s) = B(s)\ \underline{u}(s) \tag{3.39}$$

or

$$\underline{y}(s) = A^{-1}(s)\ B(s)\ \underline{u}(s) = G(s)\ \underline{u}(s) \tag{3.40}$$

where each $\underline{y}_i(s)$, $\underline{u}_i(s)$ is a vector and the $A_i(s)$, $B_j(s)$ are polynomial matrices with no polynomial left divisor of $A_1(s), \cdots, A_m(s)$, $B_1(s), \cdots, B_m(s)$ with non-constant determinant and $G(s)$ is the rational transfer function matrix description of the linear multivariable dynamical system. By assuming a decentralised constant output controllers of the form

$$\underline{u}_1 = -K_i\ \underline{y}_1 + \underline{v}_i \quad , \quad i = 1, 2, \cdots, m \tag{3.41}$$

the closed-loop system is given by

$$\underline{y}(s) = A_k^{-1}(s)\ B(s)\ \underline{v}(s) \tag{3.42}$$

where

$$A_k(s) = [A_1(s) + B_1(s)K_1,\ A_2(s) + B_2(s)K_2,\ \cdots,\ A_m(s)+B_m(s)K_m] \tag{3.43}$$

Definition (3.4): The linear multivariable system (3.39) is said to have a decentralised fixed mode at s_0 under feedback controller structure (3.31) if and only if for all constant K_i of appropriate dimension

$$\det\ [A_1(s_0) + B_1(s_0)K_1,\ A_2(s_0) + B_2(s_0)K_2,\ \cdots,\ A_m(s_0) + B_m(s_0)K_m] = 0 \tag{3.44}$$

A necessary and sufficient condition for existence of a fixed mode at s_0, independently of K_i, is provided by the following theorem.

Theorem (3.8) [And. 2]: The dynamical system (3.40) with matrix fraction description $A^{-1}(s)\ B(s)$ has a fixed mode at s_0 under Decentralised constant output controllers (D-COC) if and only if there exists a non-empty subset $\{i_1, i_2, \cdots, i_\alpha\}$ of $\{1, 2, \cdots, m\}$, for which

$$\operatorname{rank}\begin{bmatrix} A_{i_1}(s_0) & A_{i_2}(s_0) & \cdots & A_{i_\alpha}(s_0) & B_{i_1}(s_0) & B_{i_2}(s_0) & \cdots & B_{i_\alpha}(s_0) \end{bmatrix} <$$

$$< \sum_{j=1}^{\alpha} (\text{number of columns of } A_{i_j}) \tag{3.45}$$

3.3.2 Decentralised Dynamic Output Controllers (D-DOC)

Consider the use of decentralised dynamic controllers of the form

$$\underline{u}_i(s) = K_i(s)\ \underline{y}_i(s) + \underline{v}_i(s) \tag{3.46}$$

$$= B_i^*(s)\ A_i^{*-1}(s)\ \underline{y}_i(s) + \underline{v}_i(s) \tag{3.47}$$

Then, the closed-loop system has a transfer function matrix

$$\begin{bmatrix} A_1^*(s) \\ A_2^*(s) \\ \vdots \\ A_m^*(s) \end{bmatrix} [A_1(s)\ A_1^*(s) + B_1(s)\ B_1^*(s)\ \cdots\ A_m(s)A_m^*(s) + B_m(s)\ B_m^*(s)]^{-1}B(s) \tag{3.48}$$

Definition (3.5): The dynamical linear system (3.38) is said to have a decentralised fixed mode at s_0 under D-DOC of the form $B_i^*(s)\ A_1^{*-1}(s)$ if and only if

$$\det\left[A_1(s_0)\ A_1^*(s_0)+B_1(s_0)\ B_1^*(s_0)\ \cdots\ A_m(s_0)A_m^*(s_0)+B_m(s_0)\ B_m^*(s_0)\right] = 0 \tag{3.49}$$

Theorem (3.9) [And. & Cle. 1]: Consider the linear multivariable dynamic system whose MFD is $A^{-1}(s)\ B(s)$. Suppose the above system has a fixed mode at s_0. Then, the closed-loop system using D-DOC of the form (3.46) has a fixed mode at s_0. □

Another transfer function test for the existence of a decentralised fixed mode is provided by the following theorem [And. 2]:

Theorem (3.10): Consider a transfer function matrix

$$W(s) = \begin{bmatrix} W_{11}(s) & W_{12}(s) \\ W_{21}(s) & W_{22}(s) \end{bmatrix} \tag{3.50}$$

Let $\alpha(s)$ be the characteristic polynomial of $W(s)$ and suppose $W_{11}(s)$ has β rows and $W_{21}(s)$ has $\overline{\beta}$ rows. Then under a control structure of the form $u_j = K_j y_j + v_j$, $j = 1, 2$ the following conditions are equivalent.

(i) With $[A_1(s)\ \ A_2(s)]^{-1}[B_1(s)\ \ B_2(s)]$ a left coprime MFD of $W(s)$

$$\operatorname{rank}\left[A_1(s_0)\ \ B_1(s_0)\right] < \beta \tag{3.51}$$

and the fixed mode has degree k where the degree of the fixed mode s_0 is defined as the largest positive integer k such that all $\beta\times\beta$ minors of $[A_1(s)\ \ B_1(s)]$ have a zero at s_0 of order at least k.

(ii) Suppose $\alpha(s_0)$ has a zero of order k. Let $\epsilon\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}$ be the number of zeros at s_0 of $W\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}$ which represent the minor formed from rows $(i_1, i_2, \cdots, i_p)$ and columns $(l_1, l_2, \cdots, l_p)$. Let $\epsilon\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} \equiv 0$ corresponding to no poles or zeros, $\epsilon\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}$ < 0 corresponding to there being negative number of poles at s_0 and $\epsilon\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = \infty$ corresponding to a minor which is identically zero. Let $m = \beta + \overline{\beta}$ and

$$\delta_r\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = \left|\left\{i'_1, \cdots, i'_{m-p}\right\} \cap \left\{1, 2, \cdots, \beta\right\}\right| \tag{3.52}$$

where

$$\left\{i_1, \cdots, i_p\right\} \cup \left\{i'_1, \cdots, i'_{m-p}\right\} = \left\{1, 2, \cdots, m\right\} \tag{3.53}$$

and

$$\delta_c\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = \left|\left\{l_1, \cdots, l_p\right\} \cap \left\{1, 2, \cdots, \gamma\right\}\right| \tag{3.54}$$

Then, there exists $0 < \lambda < k$ such that whenever $\delta_r + \delta_c \geq \beta$

$$\epsilon\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} \geq (\lambda - k) + (\delta_r + \delta_c - \beta) \tag{3.55}$$

for all minors of k(s). □

The above result says that there is a fixed mode of degree k if and only if certain minors have s_0 as a zero of certain minimum order, or a pole of limited multiplicity, while at the same time, s_0 must be a pole of W(s). Also, the quantity δ_r computes the number of rows in the first β rows of W(s) which are not in the minor under scrutiny, while d_c computes the number of columns in the first γ columns of W(s) which are in the minor under scrutiny and of the quantity $\delta_r + \delta_c - \beta$ is associated with the position of the minor.

We also mention the transfer function characterisation proposed by Davison and Özgumer [Dav. & Özg. 1], Vidyasagar and Viswanadham [Vid. 1] that express the above result of Anderson in terms of the greatest common divisor of certain minors of the transfer function matrix and its characteristic polynomial.

3.4 Structural Fixed Modes.

The concept of structural fixed modes has been introduced recently by Sezer and Šiljak [Sez. & Šil. 1]. The motivation for this introduction was the ill-posed numerical problem associated with the computations of the rank of the relevant matrices which characterised all the existence tests for fixed modes. Decentralised fixed modes may originate from two distinct sources. It is either a consequence of a perfect matching of system parameters in which case a slight change of the parameters can eliminate the mode or it is due to special geometric structure of the system. In the latter case, no matter how much a parameter is changed, the mode remains fixed. Hence to control a decentralised dynamic system, it is necessary to change the structure of the system or the architecture of the controllers.

The main result of Sezer and Šiljak for structural fixed modes is as follows:

Theorem (3.11): Consider the system, S

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \sum_{i=1}^{N} B_i \underline{u}_i \\ \underline{y}_i &= C\underline{x}_i; \; i = 1, 2, \cdots, N \end{aligned} \tag{3.56}$$

where $\underline{x}(t) \in \mathbb{R}^n$, $\underline{u}_i(t) \in \mathbb{R}^{l_i}$, $\underline{y}_i \in \mathbb{R}^{m_i}$. When decentralised output controllers (D-DOC)

$$\underline{u}_i = K_i \underline{y}_i, \; i = 1, 2, \cdots, N \tag{3.57}$$

are applied to (3.56), then the system has structural fixed modes with D-DOC if and only if either of the two following conditions holds:

(i) If there is a partition of $N^* = \{1, 2, \cdots, N\}$ into disjoint subsets $N_k = \{i_1, i_2, \cdots, i_k\}$, $N^* - N_k = \{i_{k+1}, i_{k+2}, \cdots, i_N\}$ and a permutation matrix P such that:

$$P^T A P = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$P^T B^{N_k} = \begin{bmatrix} B_1^{N_k} \\ B_1^{N_k} \\ B_1^{N_k} \end{bmatrix}, \quad P^T B^{N^*-N_k} = \begin{bmatrix} 0 \\ 0 \\ B_1^{N^*-N_k} \end{bmatrix} \tag{3.58}$$

$$C^{N_k} P = \begin{bmatrix} C_1^{N_k} & 0 & 0 \end{bmatrix}$$

$$C^{N-N_k} P = \begin{bmatrix} C_1^{N^*-N_k} & C_2^{N^*-N_k} & C_3^{N^*-N_k} \end{bmatrix}$$

where $B^{N_k} = [B_{i_1}, B_{i_2}, \cdots, B_{i_k}]$, $C^{N_k} = [C_{i_1}^T, C_{i_2}^T, \cdots, C_{i_k}^T]$.

(i) There exists a $N_k \subset N^*$ such that

$$\overline{\rho}\begin{bmatrix} A & B^{N_k} \\ C^{N^*-N_k} & 0 \end{bmatrix} < n \tag{3.59}$$

where $\overline{\rho}$ = generic rank = $\max\limits_{d \in \mathbb{R}^\nu} \{\rho[\overline{M}(d)]\}$ and where ν is the number of non-zero arbitrary entries of a structured matrix, $\overline{M}$. $\overline{M}(d)$ is obtained by replacing the arbitrary non-zero entries of $\overline{M}$ by the corresponding components of $d \in \mathbb{R}^\nu$.

3.4.1 How to Avoid Fixed Modes in Decentralised Control

When a decomposition of the system is not imposed it is possible to choose a certain structure of the system which has no fixed modes with a decentralised control.

Problem Statement: Suppose we are given a multivariable linear dynamic system described by

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \\ \underline{x} &\in \mathbb{R}^n, \ \underline{u} \in \mathbb{R}^l, \ \underline{y} \in \mathbb{R}^m \end{aligned} \tag{3.60}$$

Under what conditions (3.60) can be decomposed in N subsystems S_k with l_k and m_k inputs and outputs such that the system has no fixed modes with respect to decentralised feedback law (3.57)

In the case of structural fixed modes, system (3.60) can be put in form (3.58) in which case the inputs are partitioned into two sets U_{S_k} and $U_{S\text{-}S_k}$ while the outputs into Y_{S_k} and $Y_{S\text{-}S_k}$ of dimensions l_{S_k} and $l_{S\text{-}S_k}$ and m_{S_k} and $m_{S\text{-}S_k}$ respectively. In that case, fixed modes are avoided when

$$\begin{aligned} &\sum_{i=1}^{N} l_i = l, \ \sum_{i=1}^{N} m_i = m, \ N \geq 2. \\ &\sum_{i=1}^{r} l_i \neq l_{S_k} \ \text{OR} \ \sum_{i=1}^{r} m_i \neq m_{S_k}, \ 1 < r < N \end{aligned} \tag{3.61}$$

Surely, the above conditions are necessary to destroy the particular structure of matrices B and C in the form (3.58).

3.4.2 How to Eliminate Fixed Modes in Decentralised Control

According to the basic concept of a system, the decomposition is always determined by physical constraints. These constraints may be geographical as in the case of telecommunication systems, power generation systems or functional as in the case of an automatic spacecraft where the navigation, guidance, communication, propulsion and payload operation are clearly distinguished. In those cases, if the decomposed system has fixed modes, it is critical for the operation of the integrated system to find a way of elimination of those undesirable modes.

In the case of undesirable fixed modes several methods for their elimination have been proposed in the control literature. Based upon the information exchange between subsystems, Armantano and Singh [Arm. 1] characterise fixed modes by means of block-diagonally dominant matrices. Their procedure is based upon the following description of the system.

Let

$$S_i : \qquad \dot{\underline{x}}_i = A_{ii}\underline{x}_i + B_i\underline{u}_i + \sum_{\substack{i=1 \\ j\neq 1}}^{N} A_{ij}\underline{x}_j, \quad i = 1, 2, \cdots, N$$
$$\underline{y}_i = C_i\underline{x}_i, \quad \underline{x}_i \in \mathbb{R}^{n_i}, \quad \underline{u}_i \in \mathbb{R}^{l_i}, \quad \underline{y}_i \in \mathbb{R}^{m_i} \tag{3.62}$$

be the description of the interconnected subsystems S_i that form the system Σ. Let

$$\begin{aligned} A &= \left\{A_{ij}, \ i = 1, 2, \cdots, N; \ j = 1, 2, \cdots, N\right\} \in \mathbb{R}^{n \times n} \\ B &= \text{block-diag}\left\{B_1, B_2, \cdots, B_N\right\} \in \mathbb{R}^{n \times l} \\ C &= \text{block-diag}\left\{C_1, C_2, \cdots, C_N\right\} \in \mathbb{R}^{m \times n} \end{aligned} \tag{3.63}$$

Then (3.62) can be rewritten as

$$\Sigma: \qquad \begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{3.64}$$

By applying the decentralised feedback control

$$\underline{u}_i = K_{ii}\underline{y}_i \tag{3.65}$$

or

$$\underline{u} = K\underline{y}; \quad K = \text{block-diag}[\, K_{11}, \cdots, K_{NN}] \tag{3.66}$$

the closed-loop matrix A + BKC is written

$$A + BKC = \begin{bmatrix} \widehat{A}_{11} & A_{12} & \cdots & A_{1N} \\ A_2 & \widehat{A}_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & \widehat{A}_{NN} \end{bmatrix} \tag{3.67}$$

where $\hat{A}_{ii} = A_{ii} + B_i K_{ii} C_i$, $i = 1, 2, \cdots, N$. Then if the diagonal submatrices $\hat{A}_{ii}$ are non-singular and if

$$\left\| \hat{A}_{ii}^{-1} \right\|^{-1} > \sum_{\substack{j=1 \\ j \neq 1}}^{N} \left\| A_{ij} \right\|, \ \forall \ i \in \underline{N} \tag{3.68}$$

the closed-loop matrix is strictly block diagonal dominant.

Theorem (3.12) [Arm. 1]: If the closed loop matrix under decentralised control, A + BKC is strictly block-diagonally dominant, then A + BKC is non-singular.

Corollary (3.4) [Arm. 1]: If $\lambda \in \mathcal{F}$ is a decentralised fixed mode, then

$$\left\| (\hat{A}_{ii} - \lambda I_i)^{-1} \right\|^{-1} \leq \sum_{\substack{j=1 \\ j \neq 1}}^{N} \left\| A_{ij} \right\|, \ \forall \ k_{ii} \in \mathbb{R}^{l_i \times m_i} \tag{3.69}$$

for at least one i, $i \in \underline{N} = \{1, 2, \cdots, N\}$.

Based upon the corollary, Armentano and Singh, derive a technique to determine the minimum crosstalk between controllers when there is information exchange between the decentralised controllers, to eliminate fixed modes.

Similar procedures have been obtained by Locatelli *et al* [Loc. 1] using graph theory, Senning [Sen. 1] using optimisation techniques and Groumpos and Lopanor [Gro. 1] using a hierarchical approach to stabilise the unstable fixed modes using a global controller. Another interesting application which is very innovative indeed in decentralised control is that by Trave *et al* [Tra. 1]. They use the classic practical approach of harmonic (vibrational) control to cancel the unstable fixed modes. The principle of harmonic control consists in the introduction of harmonic functions (periodic vibrations) on the parameters of the system matrix A such that we obtain a time-varying system.

$$\dot{\underline{x}} = [A + H(t)] \, \underline{x} \tag{3.70}$$

where

$$\underline{h}_{ij}(t) = a_{ij}\sin w_{ij} \tag{3.71}$$

Application of harmonic control results in a time-varying feedback matrix and clearly resembles the result of Anderson and Moore [And. 3] who have showed that time-varying feedback laws can eliminated fixed modes under certain structural conditions. Also, in the case of scalar sub-systems Purviance and Tylee [Pur. 1] have showed that a sinusoidal feedback can eliminate fixed modes in case of decentralised control with better performance than a time-varying control law.

3.5 Mathematical Analysis of the Stabilisation and Pole Assignment Techniques Using Decentralised Controllers

The mathematical description of the problem of stabilising decentralised control systems using automatic controllers, each operating with partial information on the states of the subsystems is as follows :

3.5.1 The state-space description of Decentralised Systems

Consider a linear invariant dynamical system of high-order with k local controllers described by

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \sum_{i=1}^{k} B_i\underline{u}_i \\ \underline{y}_i &= C_i\underline{x} \end{aligned} \tag{3.72}$$

where x is the n-dimensional integrated state vector, $\underline{u}_i$ is the l_i-dimensional control vector (Actuator output) where i = 1, 2, $\cdots$, k.

Since the system is decentralised then

$$\underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_k \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_k \end{bmatrix}$$

where $u_i(t) \in \mathbb{R}^{l_i}$, $\underline{y}(t) \in \mathbb{R}^{m_i}$, $i = 1, 2, \cdots, k$ and $\sum_{i=1}^{k} l_i = l$, $\sum_{i=1}^{k} m_i = m$.

3.5.2 Architecture of Decentralised Automatic Controllers

We distinguish between two categories of controllers. (1) Static and (2) Dynamic controllers.

(1) **Static Controllers:** This kind of controllers generate control commands proportional to the current information obtained by the sensors, that is

$$\underline{u}(t) = F(t)\,\underline{y}(t) + \underline{v}(t) \tag{3.74}$$

where

$$\underline{u}(t)^T = [\underline{u}_1(t)^T, \underline{u}_2(t)^T, \cdots, \underline{u}_k(t)^T] \tag{3.75}$$

$$\underline{y}(t)^T = [\underline{y}_1(t)^T, \underline{y}_2(t)^T, \cdots, \underline{y}_k(t)^T] \tag{3.76}$$

$$\underline{v}(t)^T = [\underline{v}_1(t)^T, \underline{v}_2(t)^T, \cdots, \underline{v}_k(t)^T] \tag{3.77}$$

(2) **Dynamic Controllers** have the following dynamic structure

$$\dot{\underline{z}}(t) = L\,\underline{z}(t) + M\,\underline{y}(t) \tag{3.78}$$

$$\underline{u}(t) = K\,\underline{z}(t) + \underline{v}(t) \tag{3.79}$$

or

$$\begin{bmatrix} \dot{\underline{z}}(t) \\ \underline{u}(t) \end{bmatrix} = \begin{bmatrix} L & M \\ K & 0 \end{bmatrix} \begin{bmatrix} \underline{z}(t) \\ \underline{y}(t) \end{bmatrix} \tag{3.80}$$

where $\underline{z}(t)$ is the controller state vector and where

$$\underline{z}(t)^T = [\underline{z}_1(t)^T, \underline{z}_2(t)^T, \cdots, \underline{z}_k(t)^T]$$
$$\dim \underline{z}(t) = r, \quad \dim \underline{z}_i(t) = r_i, \quad r = \sum_{i=1}^{k} r_i \tag{3.81}$$

3.5.3. Stabilisability of Dynamic Systems Using Decentralised Controllers

When the controller (3.80) is connected to the dynamic system (3.72), the system description of the coupled system is

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{z}} \end{bmatrix} = \begin{bmatrix} A & BK \\ MC & L \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{v} \tag{3.82}$$

When the controller is static, then (3.82) reduces to

$$\dot{\underline{x}} = (A + BFC)\,\underline{x} + B\,\underline{v} \tag{3.83}$$

Statement of the Problem: The fundamental problem of decentralised control is whether there exist matrices L, M, K, F with the required block diagonal structure, such that the closed-loop system (3.82) has preassigned eigenvalues (pole assignment) or is asymptotically stable (D-stabilisability).

When no structural constraints are imposed on F and C, it is well known that the pole assignability of (3.83) is equivalent to the controllability of A and B. The question of stabilisability of (A, B) is equivalent to the controllability of the unstable modes of A. It is obvious that pole assignment or stabilisability of classical centralised methods is a necessary condition for the controllability property to hold in the decentralised case; however, the condition is not sufficient since in decentralised control the matrices F, C are constrained to be of particular structure.

3.5.4 Decentralised-Static Output Controllers (D-SOC)

We first investigate the question which eigenvalues of the system matrix, A, can be shifted by decentralised static output controllers (D-SOC). To do so, we need some concepts about the controllability or better the uncontrollability for the case of centralised control systems. The main concept is that of fixed mode of a dynamic system which we defined already in section 3.1.

Under static output control, the centralised fixed modes of the dynamic system are those eigenvalues of A which are invariant with respect to the arbitrary static controller, K. In the case of the state-space description of the dynamic system, the fixed modes are completely determined by those eigenvalues for which the following two rank tests fail to satisfy

$$\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \neq n, \quad \text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} \neq n \tag{3.84}$$

Hence the interplay between the uncontrollable and unobservable subspaces isolates the centralised fixed modes.

In the case of decentralised static output controllers of the form

$$\underline{u}_i = F_i \underline{y}_i \tag{3.85}$$

the closed loop system matrix is

$$A_D = A + \sum_{i=1}^{k} B_i F_i C_i \tag{3.86}$$

which is an explicit expression of the classical centralised system matrix, A + BFC, in the case F is a block-diagonal matrix. Then, necessary and sufficient conditions for a characteristic value to be a fixed mode under decentralised static control of the form (3.85) are given by Anderson and Clements [And. & Cle. 1] based on the rank criterion of a special matrix.

3.5.5 Decentralised-Dynamic Output Controllers

Ii is well known that for centralised control, the arbitrary eigenvalue assignment can be achieved using dynamic output controllers. Indeed, the theory states that those eigenvalues which can be moved by static controllers can be arbitrarily placed by dynamic controllers, whereas those eigenvalues which are fixed under all static controllers are also fixed by dynamic controllers. This characterisation of the pole assignment properties using dynamic controllers in terms of the pole assignment properties using static controllers has been helpful in the synthesis of centralised controllers. This particular approach consists of using static controllers to make the system controllable from a single input channel and observable from a single output channel. Then the dynamic controller for arbitrary pole assignment can be considered for the system with a single input - single output. The above approach for synthesis of centralised controllers has been extended by Corfmat and Morse [Cor. & Mor. 1] and Fessas [Fes. 1] to the synthesis of decentralised controllers. For this type of synthesis, the idea of strongly connected subsystems dominates the whole system engineering process.

3.5.6 Hierarchical System Decomposition into Strongly Connected Subsystems

According to Corfmat and Morse [Cor. & Mor. 1] the decentralised control of dynamic systems can be solved after the system has been decomposed into strongly connected subsystems which can be dealt with as completely independent thus forming a hierarchy of controllers. Although such a decomposition is not always possible, it is useful to know when it is possible to do the decomposition. Then it is assured that a fixed mode of the centralised system may be, with respect to the relevant subsystem either uncontrollable or unobservable and hence a fixed mode with respect to centralised output feedback in the subsystem or only a fixed mode with respect to the decentralised control of the subsystem.

The approach of Corfmat and Morse is based on the following topological terms

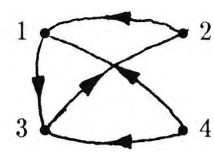
Definition (3.6): The graph of the linear dynamic system $\Sigma = \{C_i, A, B_i; \underline{k}\}$ described by (1) is defined as a pair $(\underline{k}, \Gamma)$ consisting of k nodes labelled $1, 2, \cdots, k$, with node i representing the i^{th} channel of Σ and a function Γ mapping $\underline{k}$ into its power set according to the rule $\Gamma(j) = \{i : H_{ij} \neq 0\}$; $H_{ij} = C_i(\lambda I - A)^{-1}B_j$ is the transfer function from input u_j of Σ to system output y_i. This function is geometrically represented by the directed arc drawn from node j to node i just in case u_i influences y_i in the sense that $H_{ij} \neq 0$. Hence, the graph of Σ is the usual signal flow graph between the channels of Σ.

For example, the system matrix

$$G(s) = \begin{bmatrix} g_{11} & g_{12} & 0 & g_{14} \\ 0 & g_{22} & g_{23} & 0 \\ g_{31} & 0 & 0 & g_{34} \end{bmatrix} \tag{3.87}$$

can be represented by the following system graph

G_{Σ}:



(3.88)

It is seen from above that the graphs of Σ are directed finite graphs with no loops or parallel arcs.

Intuitively, it may be expected that if (A_D, B_j) is controllable for some feedback F where $A_D = A + \sum_{i=1}^{k} B_i F_i C_i$, then each mode of the system graph G_{Σ} must be reached from node j along a directed path in G_{Σ}. Hence a necessary condition for the decentralised controllability of an interconnected system is the following proposition:

Proposition (3.2): If some $j \in \underline{k}$ and $F_1, F_2, \cdots, F_k$, the resulting closed loop system is controllable from $\underline{u}_i$, i.e. the pair

$$(A_D, B_j) = (A + \sum_{i=1}^{k} B_i F_i C_i, B_j) \tag{3.89}$$

is controllable, then for each $i \in \underline{k}$ with $i \neq j$, there exists a directed path in G_{Σ} from node j to node i.

□

Applying the above-mentioned proposition to a two input - two output system, it is seen that the conditions of proposition 1 are equivalent to $G_{21}(s) \neq 0$ and $G_{12}(s) \neq 0$. But this is only a necessary condition since another condition is the completeness of the interconnection. (A system (C, A, B) is complete if (C, A) is observable and (A, B) is controllable.) Hence the generalisation to the MIMO case, we introduce the following concept of complementary subsystems [Corfmat & Morse, 1976].

Consider a decentralised system $\Sigma = \{C_i, A, B_i : \underline{k}\}$ determined by the SSD (3.72). If $\underline{s}$ is a nonempty subset of $\underline{k}$, with elements $i_1, i_2, \cdots, i_p$ ordered so that $i_1 < i_2 < \cdots < i_p$, then we define B_s and C_s so that

$$B_{\underline{s}} = [\, B_{i_1}, B_{i_2}, \cdots, B_{i_p}], \qquad C_{\underline{s}}^T = [\, C_{i_1}, C_{i_2}, \cdots, C_{i_p}] \tag{3.90}$$

Hence, $(C_{\underline{s}_2}, A, B_{\underline{s}_1})$ is a subsystem of Σ with the inputs u_i, $i \in \underline{s}_1$, and outputs y_j, $j \in \underline{s}_2$, to which we associate the transfer matrix

$$G_{\underline{s}_2, \underline{s}_1} = C_{\underline{s}_2} (sI - A)^{-1} B_{\underline{s}_1} \tag{3.91}$$

Definition (3.7): If Σ is a decentralised system described by (1) and $\underline{s}_2$ a proper subset of $\underline{k}$, then the subsystem (C_{s_2}, A, B_{s_1}) with $\underline{s}_2 = \underline{k} - \underline{s}_1$ is called the *Complementary Subsystem of* Σ.

□

We have seen earlier that for a given system Σ, and some fixed $j \in \underline{k}$, it is possible to find some decentralised static output controller, F_i such that (C_j, A_F, B_j) is a controllable, observable system. If an F with this specialised structure exists, F is a block diagonal matrix, then it turns out that such F exists for each and every $j \in \underline{k}$ and when this is so we say that Σ can be made controllable and observable through a single controller with decentralised static output feedback. But, the proposition is only a necessary condition hence we have the following theorem due to Corfmat and Morse.

Theorem (3.13): Let Σ be a decentralised control system described by (3.72). Then there exists a decentralised static output controller of the form $u_i = F_i y_i$, $i \in \underline{k}$, such that the resulting closed-loop system is controllable and observable from a single controller $j \in \underline{k}$, if and only if each complementary subsystem of Σ which contains input channel j, is complete.

□

Using the concept of fixed mode, Wang and Davidson and Corfmat and Morse showed that the eigenvalues of a decentralised system can be placed in an arbitrary region of the complex plane using D-dynamic controllers such as those of (3.80) if and only if the fixed modes lie in that region. According to Corfmat and Morse, the fixed modes represent the set of eigenvalues of the system A which cannot be moved by any family of decentralised dynamic controllers while all the remaining eigenvalues of the system A, can be arbitrarily placed by an appropriate choice of D-DOC. This can be represented by the following theorem due to Corfmat and Morse [Cor. & Mor. 1]

Theorem (3.14): Let $\Sigma = \{C_i, A, B_i; \underline{k}\}$ and j satisfy the hypotheses of Theorem 1, and write $\alpha_D(\lambda)$ for the uncontrollable polynomial of (A_D, B_j). Then

(i) For all F $\in \mathcal{F}$ where $\mathcal{F}$ the space of diagonal matrices

$$\forall s \in \underline{k}(j),\ \ \rho\ (C_{\underline{k}-s}, A, B_s)\ \text{ divides }\ \alpha_D(\lambda)$$

where $\rho(\cdots)$ is the remnant polynomial of SSD (A ,B ,C) defined as the product of the transmission polynomials of the transfer matrix $C(\lambda I - A)^{-1}B$, in the case where the transfer matrix rank is less than the number of these polynomials. When the rank transfer matrix is greater than the number of the transmission polynomials then $\rho(L, A, B) = 1$ which is the condition for the system to be complete (both controllable and observable).

(ii) If in addition, for each j $\in \underline{k}$ with i $\neq$ j there is a path in the system graph G_Σ, from node j to node i, then the uncontrollable polynomial, $\alpha_D(\lambda)$, is independent of the static decentralised controller, F, where F $\in \mathcal{F}_j$ and $\alpha_D(\lambda)$ divides $\prod_{s \in \underline{k}(j)} \rho\ (C_{\underline{k}-\underline{s}}, A, B_{\underline{s}})$ where $\underline{k}$(j) denotes the class of all proper subsets of $\underline{k}$ which contain j.

The above results are valid for so-called interconnected subsystems, where the subsystems are interconnected only through their inputs and outputs.

Consider the subsystems

$$\begin{aligned} \dot{\underline{x}}_i &= A_i\underline{x}_i + B_i\underline{u}_i \\ \underline{y}_i &= C_i\underline{x}_i + D_i\underline{u}_i, \quad i = 1, 2, \cdots, k \end{aligned} \tag{3.92}$$

which are interconnected according to the rule

$$\underline{u}_i = \sum_{\substack{j=1 \\ j\neq i}}^{k} F_{ij}\underline{y}_j - \underline{v}_i, \quad i = 1, 2, \cdots, k$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times l_i}$, $C_i \in \mathbb{R}^{m_i \times n_i}$, $F_{ij} \in \mathbb{R}^{l_i \times m_i}$.

These systems may be written in the form (3.1) where

$$A = \begin{bmatrix} A_1 & B_1F_{12}C_2 & \cdots & B_1F_{1k}C_k \\ B_2F_{21}C_1 & A_2 & \cdots & B_2F_{2k}C_k \\ \vdots & \vdots & \vdots & \vdots \\ B_kF_{k1}C_1 & \cdots & B_kF_{k,k-1}C_{k-1} & A_k \end{bmatrix} \tag{3.93}$$

$$B = \text{diag}\,[B_1, B_2, \cdots, B_k] \tag{3.94}$$

$$C = \text{diag}\,[C_1, C_2, \cdots, C_k] \qquad (3.94)$$

The composite SSD for (3.92) is

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u}, \quad \underline{u} = F\underline{y} + \underline{v} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \tag{3.95}$$

where $\underline{x} = \text{col}(\underline{x}_i)$, $\underline{u} = \text{col}(\underline{u}_i)$, $A = \text{diag}\,[A_1, A_2, \cdots, A_k]$, $B = \text{diag}\,[B_1, B_2, \cdots, B_k]$, $C = \text{diag}\,[C_1, C_2, \cdots, C_k]$ and $F = \text{mat}\,[F_{ij}]$.

By combining (3.95) into a single equation for the overall integrated system, we obtain the integrated SSD.

$$\begin{aligned} \dot{\underline{x}} &= A_D\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{3.96}$$

where

$$A_D = A + BFC \tag{3.97}$$

Since the above integrated form of the state-space description (SSD)

$$B\underline{u} = \sum_{i=1}^{k} \bar{B}_i\underline{u}_i \tag{3.98}$$

and

$$\underline{y}_i = \tilde{C}_i \underline{x}, \qquad i = 1, 2, \cdots, k$$

where

$$\tilde{B}_i = \text{col}\,[0, 0, \cdots, B_i, 0, \cdots, 0] \qquad (3.100)$$
$$\tilde{C}_i = \text{col}\,[0, 0, \cdots, C_i, 0, \cdots, 0]$$

we can see that (3.96) decomposes into the form of the decentralised controllers of (3.72). Using the above special forms for the interconnected subsystems Saeks [Sae. 2] obtained the following theorem:

Theorem (3.15): For the interconnected systems of the form (36) we have that

$$\Phi_C(A, B, C; F_C) = \Phi_D(A, B, C; F_D) = \bigcup_i \Phi_C(A_i, B_i, C_i) \qquad (3.101)$$

□

Thus, in order to show that the decentralised controllers can place the eigenvalues of the system in prespecified locations, if and only if there exist centralised controllers which can place the eigenvalues of those positions, Saeks establishes that the set of fixed modes of the centralised, decentralised and those of the interconnected subsystems are the same.

However, although the above theorem says that the fixed modes of the centralised controlled system and the decentralised controlled system coincide. it does not imply that local output feedback, complex or real, will assign an arbitrary set of characteristic values. Indeed, Desoer and Chan [Des. & Cha. 1] have shown that local dynamic output controllers or local constant state controllers are sufficient for stabilisation purposes of dynamical systems, but not sufficient for arbitrary eigenvalue assignment.

In order to arbitrarily assign the eigenvalues of the overall system, the feedback matrix should be block-diagonal instead of diagonal. In this case, the controllers measure the state or the outputs of certain subsystems called *strongly connected* and whose outputs are fed into each component of the strongly connected subsystems. This again stresses the importance of the hierarchical decomposition of the overall dynamic system into strongly connected subsystems.

One approach to identifying the strongly connected subsystems which constitute the different levels of a hierarchical system with a minimum number of components per level, uses the system adjacency matrix, A. Every component in a strongly connected subsystem affects the input of every other component in the subsystem either by direct connection or indirectly through other components. Hence, the basic steps for hierarchical decomposition of the system is first to identify the strongly connected subsystems and then to place them on the appropriate level of the hierarchy.

The two main techniques in decomposing a large-scale dynamic system into a multilevel hierarchy of strongly connected subsystems are the arithmetic and the geometric methods. The arithmetic method uses sparse matrix techniques manipulated on a computer, while the geometric method involves the adjacency graph of an interconnected dynamical system. The adjacency graph is a generalised polyhedron which has a vertex for each system component. An edge connects two vertices of the adjacency graph, say i, j if the output of component j directly feeds the input of component i.

The two components lie in the same strongly connected subsystem if and only if the two components lie in a common directed circuit. By a common directed circuit we mean a loop in which all edges are oriented in the same direction around the loop.

The importance of the above hierarchical structure of strongly connected subsystems lies in the simplicity of analysis and synthesis of decentralised controllers. In this case the analysis of a complex dynamic system is based on the independent analysis of the strongly connected subsystems of the system. Then certain results about the motion of the dynamic system can be deduced from the study of these subsystems. For example, the integrated system consisting of a set of independent strongly connected subsystems is stable if and only if each subsystem is stable. The main results of the interconnected subsystems using decentralised controllers can be summarised in the following way:

Theorem (3.16): Let $\Sigma = \{C_i, A, B_i : \underline{k}\}$ be a jointly controllable singly observable system. Then with decentralised control, the closed-loop spectrum of Σ can be

(i) freely assigned if and only if the sum of the dimensions of Σ's strongly connected subsystems is equal to the dimension of Σ and spectrum assignment is possible for each strongly connected subsystem.

(ii) stabilised if and only if the complement of the disjoint union of the spectra of Σ's strongly connected subsystems in the spectrum of Σ is stable and stabilisation is possible for each strongly connected subsystem.

Chapter 4

THE DETERMINANTAL FREQUENCY ASSIGNMENT PROBLEM OF DECENTRALISED CONTROL SYSTEMS

4.0 Introduction

The purpose of this chapter is to provide a unifying framework for the study of problems of pole assignment by decentralised state, output feedback and for the problem of zero assignment under decentralised squaring down. An abstract mathematical problem is formulated, that is the decentralised determinantal assignment problem that unifies all decentralised frequency assignment problems. This problem is shown that may be reduced to the study of a linear problem of zero assignment of polynomial combinants [Kar. Gia. Hub. 1] and a multilinear problem of decomposability of multivectors under certain constraints. The present approach is based on recent results [Kar. & Gia. 1] on the unifying study of frequency assignment problems of centralised systems. Thus, the present approach is an extension of the approach developed in [Kar. & Gia. 1] for centralised systems to the case of decentralised control. The centralised case is briefly discussed first and then the case of decentralised control is considered.

4.1 Mathematical Description of Centralised Control Problems.

In general, we consider a linear time-invariant system described by the following state-space description (SSD).

$$\begin{aligned} S(A, B, C, D):\ \dot{\underline{x}}(t) &= A\,\underline{x} + B\,\underline{u}(t) \\ \underline{y}(t) &= C\,\underline{x} + D\,\underline{u}(t) \end{aligned} \tag{4.1}$$

where $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^m$, $\underline{u} \in \mathbb{R}^l$ are the state, the input, and the output vectors respectively and

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{m \times l}$, $D \in \mathbb{R}^{m \times l}$ are real constant matrices.

The operational method based on the Laplace transformation has been extensively used for the synthesis of classical controllers and represents one of the most convenient and powerful mathematical tools for the treatment of a wide variety of mathematical problems. Hence, if we take the Laplace transform of (4.1) with zero initial conditions, the result is another mathematical description known as the *transfer-function matrix description*. Equations (4.1) then lead to the input-output description

$$\underline{y}(s) = G(s)\,\underline{u}(s) \tag{4.2}$$

where s is the complex variable of the Laplace transform and $\underline{y}(s)$ and $\underline{u}(s)$ are the Laplace transforms of the output $\underline{y}(t)$ and input $\underline{u}(t)$ of the control system.

The transfer function matrix, G(s), may be represented by left, right coprime matrix descriptions (MFD) as:

$$G(s) = D_L^{-1}(s)\, N_L(s) = N_R(s)\, D_R^{-1}(s) \tag{4.3}$$

where $N_L(s)$, $N_R(s) \in \mathbb{R}^{m \times l}[s]$, $D_L(s) \in \mathbb{R}^{m \times m}[s]$ and $D_R(s) \in \mathbb{R}^{l \times l}[s]$ are polynomial matrices with real coefficients.

We next define the main frequency assignment problems in control theory for the case of centralised control.

Problem Statement 1: **Pole Assignment by Centralised State Feedback Controllers**

We are given (4.1) and a symmetric set of desired complex numbers $\{\lambda_i\}$ with respect to the real axis. Find a real 1xn matrix $L \in \mathbb{R}^{l \times n}$, such that the eigenvalues of A + BL are precisely those of the symmetric set $\{\lambda_i\}$.

Clearly, when the state feedback controller $\underline{u} = L\underline{x} + \underline{v}$ is applied to (4.1) then:

$$\dot{\underline{x}} = (A + BL)\,\underline{x} + B\underline{v} \tag{4.4}$$

and the characteristic polynomial of the closed loop system is:

$$P_L(s) = \det [\, sI - A, \; -BL \,] = \det [\, A_L(s) \, \tilde{L} \,] \tag{4.5}$$

where

$$A_L(s) = [\, sI - A, \; -B \,], \;\; \tilde{L} = \begin{bmatrix} I_n \\ L \end{bmatrix} \tag{4.6}$$

For the problem of pole assignment under state feedback, we desire that:

$$\det [\, A_L(s) \, \tilde{L} \,] = a(s) \tag{4.7}$$

where a(s) is the desired polynomial whose roots are $\{\lambda_i\}$. □

Problem Statement 2: Pole Assignment by Constant Output Feedback

Given a symmetric set of desired complex numbers $\{\lambda_i\}$, $i = 1, 2, \cdots, n$ find a real $l \times m$ matrix F, such that the eigenvalues of $A + BFC$ are given by the set $\{\lambda_i\}$.

Under the feedback control law of the following structure:

$$\underline{u} = F\underline{y} + \underline{u}_c \tag{4.8}$$

where $\underline{u}_c$ are the command inputs, the state-space description of (4.1), the so called *closed loop form* is given by:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} = A\underline{x} + B(F\underline{y} + \underline{u}_c) = (A + BFC)\,\underline{x} + B\underline{u}_c \\ \underline{y} &= C\underline{x} \end{aligned} \tag{4.9}$$

If instead of (4.1) we use the transfer function description, then the closed loop transfer function under constant output feedback is given by:

$$T(s) = [\ I_m + G(s)\ F\ ]^{-1} \cdot G(s) = G(s)\ [\ I_l + FG(s)\ ]^{-1} \tag{4.10}$$

Also, by using expressions (4.3) for the description of the transfer function matrix G(s), we have the following expressions for the closed-loop transfer function:

$$T(s) = [\ D_L(s) + N_L(s)\ F\ ]^{-1} N_L(s) = N_R(s)\ [\ D_R(s) + F\ N_R(s)\ ]^{-1} \tag{4.11}$$

The closed-loop polynomial $P_F(s)$ is given by:

$$P_F(s) = \det [\ D_L(s) + N_L(s)\ F\ ] = \det [\ D_R(s) + F\ N_R(s)\ ] \tag{4.12}$$

By defining the following matrices:

$$G_L(s) = [\ D_L(s),\ N_L(s)\ ], \qquad G_R(s) = \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}$$
$$F_L = \begin{bmatrix} I_m \\ F \end{bmatrix}, \qquad F_R = [\ I_1 \quad F\ ] \tag{4.13}$$

then the closed-loop characteristic polynomial is

$$P_F(s) = \det [\ G_L(s)\ F_L] = \det [\ F_R\ G_R(s)\ ] \tag{4.14}$$

Then the pole assignment problem by constant output feedback is reduced to choosing an $F \in \mathbb{R}^{l \times m}$ such that if a(s) is the polynomial whose roots are$\{\lambda_i\}$, then:

$$P_F(s) = a(s) \tag{4.15}$$

□

Problem Statement 3: Zero Assignment Rectangular (Non-Square) Linear System

Standard linear multivariable feedback control schemes consider feedback loops between a selected set of measured output variables and an equal number of independent control inputs. If the number of outputs is greater than the number of inputs, it is well known fact that independent control of all outputs is not possible to achieve. Thus, in such cases of "non-square" plants we always have to decide which is the desirable subset of measurements, which will serve as variables to be controlled.

It has been established by Kouvaritakis and MacFarlane [Kou. & McF. 1] that squaring-down procedures which are necessary in the case of the number of measured variables exceeds the number of outputs, generate multivariable zeros which affect the dynamic performance of the system.

The problem of combining the outputs of the system together to create a new set of controlled outputs whose number is equal to the number of commanded inputs has been called "squaring-down" by the above authors. The general situation of the squaring down procedure is to insert a post compensator, K after the system transfer function, G(s). Then, the feed forward transfer function is given by:

$$\tilde{G}(s) = K\,G(s) \tag{4.16}$$

$$= K\,N_R(s)\,D_R^{-1}(s) \tag{4.17}$$

As it can be seen from (4.17), where we make the use of right coprime fraction description of the transfer function matrix G(s), the controlled system has a new set of zeros given by:

$$Z_K(s) = \det [\ K\,N_R(s)\ ] \tag{4.18}$$

It is clear that (4.17) represents a matrix fraction description, which is not necessarily coprime. For a generic G(s), we have no zeros; however, $\tilde{G}(s)$, as a square transfer function matrix has always zeros. Clearly, "squaring-down" introduces new zeros. As it has been shown in Chapter 2, an m×l polynomial matrix may always be factorised as $N_R(s) = \overline{N}(s)\,Z(s)$, where $\overline{N}(s)$ is an m×l least degree matrix and Z(s) an l×l greatest right divisor. Hence, (4.18) yields

$$z_K(s) = \det [\ K\,N_R(s)\ ] = \det [\ Z(s)\ ] \det [\ K\,\overline{N}(s)\ ] = z(s) \det [\ K\,\overline{N}(s)\ ] \tag{4.20}$$

where z(s) is the zero polynomial of G(s) and $\overline{N}(s)$ is an m×l polynomial matrix having no zeros. Clearly, z(s) divides the zero polynomials of all "squared down" systems and thus the newly introduced zeros are those defined by det [K $\overline{N}(s)$]. The zero assignment problem may thus be expressed as:

Problem Statement: Find a constant post compensator $K \in \mathbb{R}^{l \times m}$ such that:

$$z_f(s) = \det [\ K\ \overline{N}(s)\ ] = a(s) \tag{4.21}$$

where a(s) is the desired arbitrary polynomial to be assigned.

4.1.1 The Determinantal Assignment Problem Using Centralised Controllers

The three fundamental control problems listed above are special cases of a more general unifying abstract mathematical problem. This is the so-called *determinantal assignment problem* (DAP) defined by Karkanias and Giannakopoulos [Kar. & Gia. 1] and formulated as follows:

Let $M(s) \in \mathbb{R}^{p \times r}[s]$ be a given real polynomial matrix with rank r and $r < p$. Also, assume a real constant matrix $H \in \mathbb{R}^{r \times p}$ with rank r, i.e. $\rho\{H\} = r$. Then the determinantal assignment problem is to find H such that:

$$f_M(s, H) = \det \{H\ M(s)\} = a(s) \tag{4.22}$$

where a(s) is an arbitrary polynomial.

If we write:

$$H = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \\ \vdots \\ \underline{h}_r^T \end{bmatrix} \text{ and } M(s) = [\ \underline{m}_1(s), \underline{m}_2(s), \cdots, \underline{m}_r(s)\ ] \tag{4.23}$$

$$C_r(H) = \underline{h}_1^T \wedge \underline{h}_2^T \wedge \cdots \wedge \underline{h}_r^T = \underline{h}^T \wedge \in \mathbb{R}^{1 \times \sigma} \tag{4.24}$$

$$C_r(M(s)) = \underline{m}_1(s) \wedge \underline{m}_2(s) \wedge \cdots \wedge \underline{m}_r(s) = \underline{m}(s) \wedge \in \mathbb{R}^{\sigma \times 1}[s] \tag{4.25}$$

where $\sigma = \binom{p}{r}$ and $C_r(\ )$ is the r-th compound matrix of $(\cdot)$.

If we apply the Binet-Cauchy theorem to (4.22) we have:

$$f_M(s, H) = \det [\ H\ M(s)\ ] = C_r[\ H\ M(s)\ ] = C_r(H)\ C_r(M(s)) = \langle \underline{h}^T \wedge, \underline{m}(s)\wedge \rangle \tag{4.27}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Let also h_w, $m_w(s)$, $\omega = (i_1, \cdots, i_r) \in Q_{r,p}$ denote the co-ordinates of the multivectors $\underline{h}^T\wedge$, $\underline{m}(s)\wedge$ respectively, then:

$$f_M(s, H) = \sum_{w \in Q_{r,p}} h_w\ m_w(s) = f_M(s, h_w) \tag{4.27}$$

The coefficients a_j of $f_M(s, H)$ may be seen either as multilinear skew symmetric functions $a_j(h_{ik})$ of the entries h_{ik} of H (h_w is the rxr minor or H which corresponds to the w set of columns of H) or as linear functions $a_j(h_w)$ of the coordinates h_w of the multivector associated with H. This suggests how the study of multilinear skew symmetric functions $a_j(h_{ik})$ can be referred to the simpler study of linear functions. so the DAP may be reduced to the following problems.

(i) **Linear Problem:** Set $\sigma = \binom{p}{r}$, $\underline{h}^T\wedge = \underline{k}^T \in \mathbb{R}^{1\times\sigma}$, $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^{\sigma\times 1}[s]$ and assume $\underline{k}$ to be free. Find the conditions under which vectors $\underline{k}$ exist such that the polynomial f(s, $\underline{k}$) has a given set of zeros, where:

$$f(s,\underline{k}) = \underline{k}^T \underline{p}(s) = \sum_{i=r} k_i\ p_i(s) \tag{4.28}$$

(ii) **Multilinear Problem:** Assume that the linear problem has a solution and $\mathcal{K}$ is the family of vectors $\underline{k}$ for which f(s, $\underline{k}$) has a given set of zeros. Find whether there exists $\underline{k} \in \mathcal{K}$ which is decomposable. If such a vector exists, determine an $H \in \mathbb{R}^{r\times p}$ such that $\underline{h}\wedge = \underline{k}^T$. □

4.2 Mathematical Description of Decentralised Control Problems

The decentralised assumption in the study of the automatic control of linear systems implies a special structure for the feedback matrices involved in various frequency (pole-zero) assignment problems discussed before. This special structure characterises certain controllers under autonomous operation in which case the feedback matrices assume a diagonal, or block diagonal form. Under this autonomous decentralised operation the sensors measure local variables used by the controllers to drive local actuators. In the case of autonomous controllers the determinantal frequency assignment problems may be formulated as follows:

Problem Statement 1: **Pole Assignment by Decentralised State Feedback**

When the control system under consideration has a decentralised structure, then the control law is based on a number of autonomous (block-diagonal) controllers such that the control action, $\underline{u}(t)$ consists of N control inputs such that:

$$\underline{u}(t) = [\, \underline{u}_1(t)^T, \underline{u}_2(t)^T, \cdots, \underline{u}_i(t)^T]^T, \ \underline{u}_i(t) \in \mathbb{R}^{l_i} \tag{4.29}$$

and N partial output information available at each controller

$$\underline{y}(t) = [\, \underline{y}_1(t)^T, \underline{y}_2(t)^T, \cdots, \underline{y}_i(t)^T]^T, \ \underline{y}_i(t) \in \mathbb{R}^{m_i} \tag{4.30}$$

In this case $\underline{u}_i(t)$, $\underline{y}_i(t)$, i = 1, 2, $\cdots$, N represent the output, input signals available by and at the i^{th} local controller.

Let the actuator matrix, B, be partitioned as:

$$B = [\, B_1, B_2, \cdots, B_N \,] \tag{4.31}$$

where $B_i \in \mathbb{R}^{n \times l_i}$, and the sensor matrix, C, as:

$$C = [\, C_1^T, C_2^T, \cdots, C_N^T \,]^T \tag{4.32}$$

where $C_i \in \mathbb{R}^{m_i \times n}$. Then the decentralised control system obtained from the centralised with partitioned inputs and outputs can be described as:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x} + \sum_{i=1}^{N} B_i\, \underline{u}_i(t) \\ \underline{y}_i(t) &= C_i\, \underline{x}(t), \ i = 1, 2, \cdots, N. \end{aligned} \tag{4.33}$$

In the case of autonomous decentralised controllers we have that

$$\underline{u}_1 = L_1\,\underline{x}_1,\ \underline{u}_2 = L_2\,\underline{x}_2,\ \cdots,\ \underline{u}_N = L_N\,\underline{x}_N$$

where $L_i \in \mathbb{R}^{l_i \times n_i}$ or

$$\begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_N \end{bmatrix} = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & L_N \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_N \end{bmatrix} \tag{4.34}$$

or

$$\underline{u} = L_D\,\underline{x} \tag{4.35}$$

when L_D = block-diagonal $[\ L_1,\ L_2,\ \cdots,\ L_N\ ]$

Problem Statement: Find $L_D \in \mathbb{R}^{1 \times n}$, such that the closed-loop equation under the control structure (4.34), given by:

$$\dot{\underline{x}} = (A + B\,L_D)\,\underline{x} + B\,\underline{u} \tag{4.36}$$

has a characteristic polynomial given by:

$$P_L(s,\ L_D) = \det[\ sI - A,\ \ - B\,L_D\ ] = \det[\ sI - A,\ -B\ ] \begin{bmatrix} I_n \\ L_D \end{bmatrix} \tag{4.37}$$

whose roots are those of the given polynomial polynomial a(s). □

Problem Statement 2: **Pole Assignment by Decentralised Constant Output Feedback**

In the case where local constant output feedback is allowed for control, then the control inputs may be written as $\underline{u}_i = K_i \underline{y}_i$ where $K_i \in \mathbb{R}^{l_i \times m_i}$ and hence:

$$\begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_N \end{bmatrix} = \begin{bmatrix} K_1 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & K_N \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_N \end{bmatrix} \tag{4.38}$$

or

$$\underline{u} = K_D \, \underline{y} \tag{4.39}$$

The closed-loop transfer function (4.11) under the decentralised control law (4.30) becomes:

$$P_K(s, K_D) = \det [\, D_l(s) + N_l(s) K_D \,] = \det [\, D_r(s) + K_D N_r(s) \,] \tag{4.40}$$

where $(N_l(s),\ D_l(s))$, $(N_r(s),\ D_r(s))$ are left, right coprime factorisations of the rational transfer function matrix G(s).

By defining the matrices:

$$T_l(s) = [\, D_l(s), N_l(s) \,] \in \mathbb{R}^{m \times (m+l)}[s] \tag{4.41}$$

$$T_r(s) = \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \in \mathbb{R}^{(m+l) \times m}[s] \tag{4.42}$$

$$K_D^l = \begin{bmatrix} I_m \\ K_D \end{bmatrix} \in \mathbb{R}^{(m+l) \times m},\ K_D^r = [\, I_l \ \ K_D \,] \in \mathbb{R}^{l \times (m+l)} \tag{4.43}$$

we have that

$$P_K(s, K_D) = \det [\, T_l(s) K_D^l \,] = \det [\, K_D^r T_r(s) \,] \tag{4.44}$$

Problem Statement: Find the decentralised controller K_D of appropriate dimensions such that the closed-loop pole polynomial, $P_K(s, K_D)$, given by (4.44), is equal to a given arbitrary polynomial a(s).□

Problem Statement 3: **Decentralised Squaring Down Problem**

The general squaring-down problem in the decentralised case can be represented by the following diagram:

$$\underline{u}(s) \Rightarrow \boxed{G(s)} \Rightarrow \boxed{\begin{matrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_N \end{matrix}} \Rightarrow \underline{c}(s)$$

The decentralised squared-down transfer function matrix G(s) is given by:

$$\tilde{G}(s) = F_D G(s) \tag{4.45}$$

where

$$F_D = \text{block-diag} [\ F_1, F_2, \cdots, F_N\ ],\ \ f_i \in \mathbb{R}^{l_i \times m_i} \tag{4.46}$$

Let [$N_R(s)$, $D_R(s)$] be a right fractional representation of G(s), i.e.

$$G(s) = N_R(s)\ D_R^{-1}(s) \tag{4.47}$$

Then, by substituting G(s) in (4.45) by (4.47) we have that:

$$\tilde{G}(s) = F_D\ N_R(s)\ D_R^{-1}(s) \tag{4.48}$$

The zero polynomial for the decentralised case, $z_D(s)$ may be written as:

$$\bar{z}(s, F_D) = \det [F_D \;\; N_R(s)] \tag{4.49}$$

Also, if $Z_R(s)$ is a greatest right divisor of $N_R(s)$ then:

$$N_R(s) = \overline{N}(s) \, Z_R(s) \tag{4.50}$$

Using (4.50) in (4.49) we get

$$z(s, F_D) = \det [F_D \;\; \overline{N}(s) Z_R(s)] = \det [F_D \;\; \overline{N}(s)] \, z(s) \tag{4.51}$$

where $z(s) = \det [Z_R(s)]$ is the zero polynomial of the transfer function matrix G(s).

Therefore, the general problem of decentralised zero assignment (D-ZAP) is to find F_D of a special structure (block-diagonal) such that G(s) has a given zero structure. This problem of choosing the controller F_D to assign zeros of the system is of extreme importance in the synthesis of large scale dynamic systems and may be thought of as the generalisation of Rosenbrock's zero assignment problem [Ros. 1]

Problem Statement of D-ZAP: Find F_D, where F_D is of the special structural form (4.46), such that:

$$z_D(s, F_D) = \det [F_D \overline{N}(s)] = a(s) \tag{4.52}$$

where a(s) is the desired polynomial, whose zeros are located in a desirable region of the complex plane.

□

4.3 The Decentralised-Determinantal Assignment Problem (D-DAP)

All problems studied so far, are special cases of a more general problem which may be defined as follows:

Let $M(s) \in \mathbb{R}^{r \times p}[s]$ be a polynomial matrix, $p < r$, and let $\rho\{M(s)\} = p$. Consider the following families of matrices:

(i) Let $H_i \in \mathbb{R}^{p_i \times r_i}$, $i = 1, 2, \cdots, N$. Then we define by H_D the matrix

$$H_D = \text{block-diag}\,\{H_1, \cdots, H_N\} \in \mathbb{R}^{p \times r} \tag{4.53}$$

(ii) Let $\overline{H}_i \in \mathbb{R}^{p_i \times r_i}$, $i = 1, 2, \cdots, N$. Then we define by $\tilde{H}_D$ the matrix

$$\tilde{H}_D = [\, I_p, \overline{H}_D] \in \mathbb{R}^{p \times r} \tag{4.54}$$

where

$$\overline{H}_D = \text{block-diag}\,\{\, \overline{H}_1, \cdots, \overline{H}_N \,\} \in \mathbb{R}^{p \times (r-p)} \tag{4.55}$$

Two types of decentralised assignment problems may be defined on M(s), according to whether matrices of H_D, or $\tilde{H}_D$ type are considered. In the following the problem defined on $\tilde{H}_D$ type matrices is considered. The formulation of H_D type matrices is similar.

4.3.1 Decentralised Determinantal Assignment Problem (D−DAP)

Find an $\tilde{H}_D$ type matrix such that the polynomial

$$f_M(s, \tilde{H}_D) = \det [\, \overline{H}_D M(s) \,] \tag{4.56}$$

has a given set of zeros, or

$$f_M(s, \tilde{H}_D) = \det [\, \tilde{H}_D M(s) \,] = a(s) \tag{4.57}$$

where a(s) is a desired polynomial.

If we denote by $\underline{h}_i^T$, $i \in \underline{p}$, the rows of $\tilde{H}_D$ and by $\underline{m}_i(s) \in \underline{p}$ the columns of M(s), then:

$$C_p(\tilde{H}_D) = \underline{h}_1^T \wedge \cdots \wedge \underline{h}_p^T = \underline{h}_D^T \wedge \in \mathbb{R}^{1 \times \binom{r}{p}} \tag{4.58}$$

the columns of M(s), then

$$C_p(M(s)) = \underline{m}_1(s)\wedge\cdots\wedge\underline{m}_p(s) = \underline{m}(s)\wedge \in \mathbb{R}^{\binom{r}{p}\times 1} = \underline{p}(s) \tag{4.59}$$

where $C_p(.)$ is the p^{th} compound matrix of (.). Then, by the Binet-Cauchy theorem:

$$f_M(s, \bar{H}_D) = C_p(\bar{H}_D)\cdot C_p(M(s)) = <\underline{h}_D\wedge, \underline{m}(s)\wedge> = a(s) \tag{4.60}$$

where $<\,,\,>$ is the scalar product.

The above general problem may be reduced to two problems:

(i) **Linear Subproblem of DDAP:** Assume $\underline{h}_D^T\wedge = \underline{k}_D^T = [\ k_1, \cdots, k_{i-1}, k_i, k_{i+1}, \cdots]^T$ where the non-zero elements in $\underline{k}_D$ are arbitrary. Find the conditions under which vector $\underline{k}_D$ exists such that the polynomial f(s, $\underline{k}_D$) has a given set of zeros, i.e.

$$f(s, \underline{k}_D) = \underline{k}_D^T\underline{p}(s) = \underline{k}_D^T\cdot\underline{m}(s)\wedge = a(s) \tag{4.61}$$

(ii) **Multilinear Subproblem of DDAP:** Let $\bar{\underline{p}}(s)$ be a subvector of $\underline{p}(s)$ defined by those co-ordinates which do not correspond to the zero co-ordinates of $\underline{k}_D^T$. Assume that K represents the family of solution vectors for which (4.61) has a given set of zeros. Find whether there exists $\underline{k}_D \in K$, which is decomposable and the solution of the exterior equation $\underline{h}\wedge^T = \underline{k}_D$ has the decentralised structure (4.54). □

A polynomial such as f(s, $\underline{k}_D$) in (4.61) is known as a *Polynomial Combinant*; it is generated by the polynomial vector $\underline{p}(s) = \underline{m}(s)\wedge$ and its structure and properties play a key role in the study of assignment problems in decentralised control. The main difference between centralised control studied by Karkanias *et al* [Kar.3] and decentralised control of the present investigation are the additional conditions imposed by the decentralised matrix of the problem. Such conditions imply that certain co-ordinates of the controller-parameter vector $\underline{k}$ in (4.61) are zero and hence only a subset of the elements $p_i(s)$, of $\underline{p}(s)$ are essential for the study or the zero distribution of f(s, $\underline{k}_D$).

For the centralised case, it is known that $\underline{h}\wedge = C_r(H) = \underline{h}_1\wedge\underline{h}_2\wedge\cdots\wedge\underline{h}_r = \underline{k}^T$, $\underline{h}\wedge \in \mathbb{R}^{\binom{r}{p}}$ has a solution, if and only if $\underline{k}$ is a decomposable vector of $\mathbb{R}^{\binom{r}{p}}$, or when $\underline{k}$ corresponds to a point of the Grassmann variety $\Omega(r, p)$ of the projective space $\mathbb{P}^{\binom{r}{p}-1}$.

The decomposability of the parameter vector $\underline{k}$ implies that $\underline{k}$ is not a free vector of $\mathbb{R}^{\binom{r}{p}}$, but its co-ordinates satisfy the set of Quadratic Plücker Relationships(QPR), which define the Grassmann variety $\Omega(r, p)$. As it will be shown later in the thesis, the decentralised assumption for the autonomous operation of a linear dynamic system implies that some of the co-ordinates of the decentralised parameter vector $\underline{k}$ are zero; hence in the study of Decentralised-Determinantal Assignment Problems (D-DAP) the vector $\underline{k}$ is not free but corresponds to a point of a subvariety $\Omega_D(r, p)$ of $\Omega(r, p)$, this is defined by the set of Quadratic Plücker Relationships and the fixed zeros in $\underline{k}$. In the following, we shall restrict ourselves to the study of the linear sub-problem of D-DAP and we shall investigate the conditions under which $f(s, \underline{k}_D)$ has fixed, or almost fixed zeros for all vectors $\underline{k}$ generated by the decentralised control matrix $\tilde{H}_D$.

Chapter 5

INDICES OF DECENTRALISATION AND DECENTRALISED GRASSMAN INVARIANTS

5.0 Introduction

In the previous chapter where the various frequency assignment problems have been discussed, it has been demonstrated that a number of important polynomial matrices emerge. Such matrices are

$$[sI-A, \quad B], \begin{bmatrix} sI-A \\ -C \end{bmatrix}, [D_l(s), \quad N_l(s)], \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}, \quad N_r(s) \tag{5.1}$$

The above polynomial matrices form the cornerstone of the polynomial system theory developed by such authors as Rosenbrock [Ros. 1], Wolovich [Wol. 1, Wol. 2], Forney [For. 1], Kucera [Kuc. 1] and Callier and Desoer [Cal. and Des. 1].

From the mathematical point of view, the polynomial matrices (5.1) define bases of rational vector spaces which play a fundamental role in the theory developed by the above authors. These spaces are

$$\begin{aligned} \mathfrak{X}_{A,B} &= \text{row-span } [sI-A, \quad B] \\ \mathfrak{X}_{A,C} &= \text{col-span} \begin{bmatrix} sI-A \\ -C \end{bmatrix} \\ \mathfrak{X}_l &= \text{row-span } [D_l, \quad N_l] \\ \mathfrak{X}_r &= \text{col-span} \begin{bmatrix} D_r \\ N_r \end{bmatrix} \end{aligned} \tag{5.2}$$

$\mathfrak{X}_C$ = col-span$\{N_r(s)\}$

The classical theory of Rosenbrock, Wolovich and Forney deals with the study of polynomial matrices and rational vector spaces. One method to obtain the coprime polynomial matrices on which the algebraic approach heavily relied upon is to obtain polynomial matrices of special structure the so-called *polynomial echelon form*. This depends on the existence of a minimal polynomial basis and according to Forney, the column-degrees of all minimal bases for a given linear vector space over the rational functions are invariant.

For the study of the determinantal frequency assignment problems this description of invariants is not very convenient. Its alternative set of invariants for rational vector spaces have been recently introduced by Karcanias and Giannakopoulos [Kar. & Gia. 1] in terms of the canonical Grassmann representatives, or the Plücker matrix of a rational vector space.

The new invariants are naturally connected to the structure of the determinantal frequency assignment problems. For the case of decentralised control, the nature of the partially fixed structure of the controllers imposes some restrictions in the Grassmann and Plücker matrix invariants of the various problems. The aim of this chapter is first to review the standard results of Grassmann representation and Plücker matrices of the various rational vector spaces emerging in decentralised control of linear dynamic systems. As a result, the decentralised Grassmann, Plücker invariants emerge as the important natural tools for the study of decentralised control of large scale dynamic systems.

This chapter is organised as follows: In section 5.1 we review the results of Karcanias *et al* [Kar. 3] on Grassmann representatives of rational vector spaces and Plücker matrices of the various rational vector spaces used in decentralised control are examined in section 5.2. first and then the results are specialised for the various decentralised control problems. An essential part of this study is the definition of the decentralised set of indices for the various cases. We then proceed in section 5.3 to derive the Decentralised Plücker matrices associated with the abstract decentralised problems.

5.1 Grassmann Invariants and Their Use in Pole-Zero Assignment Problems Using Centralised Controllers.

In this section we present some of the basic results concerning the study of the centralised determinantal assignment problem [Kar. & Gia. 1] using multivariable controllers. These results provide the starting point for the study of decentralised determinantal assignment problems.

Let $\mathcal{M}(s)=\{M(s)\colon M(s) \in \mathbb{R}^{p\times r}(s),\ r<p,\ \rho\{M(s)\}=r$. The matrices $M(s)$, $M'(s) \in \mathcal{M}(s)$ will be called *column-equivalent over* $\mathbb{R}(s)$ and shall be denoted by $M(s)\mathcal{E}M'(s)$, if there exists $Q(s) \in \mathbb{R}^{r\times r}(s)$, let $\det Q(s) = c \neq 0$, such that

$$M'(s) = M(s)\, Q(s). \tag{5.3}$$

The above equation, (5.3), defines an equivalence relationship, $\mathcal{E}$, on $\mathcal{M}(s)$. $\mathcal{E}(M)$ will denote the equivalence class (orbit) of $M(s) \in \mathcal{M}(s)$ and $M(s)/\mathcal{E}$ will denote the set of equivalence classes (quotient orbit) of $M(s)$ under $\mathcal{E}$. Each orbit partitions $M(s)$ and each orbit $\mathcal{E}(M)$ is a rational vector space $\mathfrak{X}_m$=col-span$\{M(s)\}$. The vector space has dimension r, $\dim \mathfrak{X}_m=r$, $\mathfrak{X}\subset\mathbb{R}^p(s)$ and hence $M(s)/\mathcal{E}$ is the Grassmannian $\mathcal{G}(r,\mathbb{R}^p(s))$. The set of invariants which characterise the rational vector space $\mathfrak{X}_m$ is based on the following result.

Lemma (5.1) [Hod. & Ped. 1]: Let $\mathcal{V}$ be a linear vector space over a field $\mathcal{F}$, $\dim\mathcal{V}=p$ and let $\underline{x}\wedge = \underline{x}_1\wedge\cdots\wedge\underline{x}_r = \underline{z}_1\wedge\cdots\wedge\underline{z}_r$ be two decomposable non-zero elements of $\wedge^r\mathcal{V}$. Let us further denote by $\mathcal{V}_x = \text{span}\ \{\underline{x}_1, \cdots, \underline{x}_r\}$, $\mathcal{V}_z = \text{span}\ \{\underline{z}_1, \cdots, \underline{z}_r\}$ the subspaces of $\mathcal{V}$ defined by $\underline{x}\wedge$, $\underline{z}\wedge$, respectively. The necessary and sufficient condition for $\mathcal{V}_x = \mathcal{V}_z$ is that

$$\underline{x}_1\wedge\cdots\underline{x}\wedge = c\cdot\underline{z}_1\wedge\cdots\wedge\underline{z}_r,\quad c \in\mathcal{F} -\{0\} \tag{5.4}$$

□

Let $M(s) \in\mathcal{M}(s)$, $\underline{m}_i(s) \in\mathbb{R}^{p\times 1}(s)$, $i \in \underline{r}$ be the column vectors of $M(s)$, $\underline{m}(s)\wedge = \underline{m}_1(s)\wedge\cdots\wedge\underline{m}_r(s)$ and let $\mathfrak{X}_m$ = col-span$_{\mathbb{R}(s)}\{M(s)\}$. Then using the above Lemma the following result may be obtained.

Proposition (5.1) [Kar & Gia. 1]: Let M(s), $M'(s) \in \mathcal{M}(s)$. Then $M(s) \in M'(s)$ if and only if either of the two equivalent conditions hold true

$$\mathfrak{X}_m = \mathfrak{X}_{m'}$$

There exists $q(s) \in \mathbb{R}(s)$, $q(s) \neq 0$, such that

$$\underline{m}'(s)\wedge \ = q(s)\cdot\underline{m}(s)\wedge$$

□

The $\sigma = \binom{p}{r}$ co-ordinates of the polynomial vector $\underline{m}(s)$, are the co-ordinates of the subspace $\wedge^r \mathfrak{X}_m$ and are known as the Plücker co-ordinates, $m_\omega(s)$. Then

$$\underline{m}(s)\wedge = [\cdots, m_\omega(s), \cdots]^T, \ \ m_\omega(s) \in \mathbb{R}(s), \ \ \omega = (i_1, \cdots, i_r) \in Q_{r,p} \tag{5.5}$$

If $\bar{\sigma} = \binom{p}{r} - 1$, then the Plücker co-ordinates of $\mathfrak{X}_m$ can be considered as the homogeneous co-ordinates of a point in the projective space $P_{\bar{\sigma}}(\mathbb{R}(s))$. However, not every point in $P_{\bar{\sigma}}(\mathbb{R}(s))$ represents an r-dimensional subspace of $\mathbb{R}^p(s)$; the points of $P_{\bar{\sigma}}(\mathbb{R}(s))$ which characterise r-dimensional subspaces of $\mathbb{R}^p(s)$ are those associated with decomposable vectors of $\mathbb{R}^\sigma(s)$. Hence, only decomposable vectors of $\mathbb{R}^\sigma(s)$ uniquely define r-dimensional subspaces of $\mathbb{R}^p(s)$. By associating to every $\mathfrak{X}_m \in \mathcal{G}(r, \mathbb{R}^p(s))$ its Plücker co-ordinates $\{m_\omega(s)\}$ the map $p: \mathcal{G}(r, \mathbb{R}^p(s)) \to P_{\bar{\sigma}}(s)$ is defined and is known as *Plücker embedding* of $\mathcal{G}(r, \mathbb{R}^p(s))$ in the projective space $P_{\bar{\sigma}}(\mathbb{R}(s))$. The Plücker image of the Grassmannian space $\mathcal{G}(r, \mathbb{R}^p(s))$ is an algebraic variety known as the *Grassmann variety of the projective space.*

If $\mathfrak{X}_m \in \mathcal{G}(r, \mathbb{R}^p(s))$, then any non-zero decomposable multivector $\underline{m}(s)\wedge = \underline{m}_1(s)\wedge\cdots\wedge\underline{m}_r(s)$, $m_i(s) \in \mathfrak{X}_{\underline{m}}$ is called a *rational Grassmann Representative* ($\mathbb{R}(s)-\mathbb{GR}$) of $\mathfrak{X}_m$ and it is the exterior product of the columns of a matrix $M(s) = [\underline{m}_1(s), \underline{m}_2(s), \cdots, \underline{m}_r(s)] \in \mathcal{M}(s)$; the $\mathbb{R}(s)-\mathbb{GR}$ of $\mathfrak{X}_m$ all differ only by a non-zero scalar factor $q(s) \in \mathbb{R}(s)$. If $\underline{m}(s)$ has its elements from the ring of polynomials $\mathbb{R}[s]$, it will be called a polynomial-$\mathbb{GR}$ ($\mathbb{R}[s]-\mathbb{GR}$). If $\underline{m}(s)\wedge = [\cdots, m_i(s), \cdots]^T \in \mathbb{R}^\sigma[s]$, then we may write:

$$\underline{m}(s)\wedge \ = \sum_{i=0}^{d} \underline{P}_i s^i = P_d \underline{e}_d(s) \tag{5.6}$$

where $P_d=[\underline{P}_0, \underline{P}_1, \cdots, \underline{P}_d]\in\mathbb{R}^{\sigma\times(d+1)}$ and it will be called the basis matrix, $\underline{e}_d(s)=[1, s, \cdots, s^d]^T$ and $d=\max\{\deg m_i(s)\}$ denotes the degree of $\underline{m}(s)\wedge$. An $[\mathbb{R}[s]-G\mathbb{R}$ will be called reduced, if the polynomials $\{m_j(s), j\in\sigma\}$ are coprime and it will be called monic if $\|P_d\|=1$.

Proposition (5.2) [Kar. & Gia. 1]: Let $M(s)=[\underline{m}_1(s), \cdots, \underline{m}_r(s)]\in\mathcal{M}(s)$, $\underline{m}(s)\wedge=\underline{m}_1(s)\wedge\cdots\wedge\underline{m}_r(s)$, $\mathcal{S}(M)=\mathfrak{X}_m$ and let p(s) and z(s) be the monic pole and zero polynomials respectively of M(s). There exists a reduced and monic $\mathbb{R}[s]-G\mathbb{R}$, $\underline{\tilde{n}}(s)\wedge$ of such that $\underline{m}(s)\wedge$ may be uniquely factorised as

$$\underline{m}(s)\wedge=c\,\frac{z(s)}{p(s)}\,\underline{\tilde{n}}(s)\wedge,\quad c\in\mathbb{R} \tag{5.7}$$

□

The reduced and monic $\mathbb{R}[s]-G\mathbb{R}$ is defined as the *canonical polynomial Grassmann representative* $(C-\mathbb{R}[s]-G\mathbb{R})$ of $\mathfrak{X}_m$ and shall be denoted by $g(\mathfrak{X}_m)$. Because $g(\mathfrak{X}_m)=\underline{n}(s)\wedge$, where $\tilde{N}(s)=[\cdots, \tilde{n}(s),\cdots]$ is a basis of the maximal $\mathbb{R}[s]-$module of $\mathfrak{X}_m$, we have the following.

Remark (5.1) [Kar. & Gia. 1]: $g(\mathfrak{X}_m)$ is a decomposable vector of $\mathbb{R}^\sigma[s]$, $\sigma=\binom{p}{r}$ and $\deg\{g(\mathfrak{X}_m)\}=\delta$ where δ is the Forney dynamical order [For. 1]. $g(\mathfrak{X}_m)$ is uniquely defined module $c\in\mathbb{R}$ and may be represented by

$$g(\mathfrak{X}_m) = P_\delta\underline{e}_\delta(s),\ P_\delta\in\mathbb{R}^{\sigma\times(\delta+1)},\ \underline{e}_\delta(s)=[1, s, \cdots, s^\delta]^T \tag{5.8}$$

The basis matrix P_δ of $g(\mathfrak{X}_m)$ will be referred to as the *Plücker Matrix* of $\mathfrak{X}_m$.

Theorem (5.1) [Kar. & Gia. 1]: $g(\mathfrak{X}_m)$ or the Plücker matrix P_δ is a complete invariant for matrices $M(s)\in\mathcal{M}(s)$ under $\mathbb{R}[s]$-column equivalence. □

Hence a subspace $\mathfrak{X}_m\in G(r, \mathbb{R}^p(s))$ is uniquely characterised by the decomposable reduced and monic vector $g(\mathfrak{X}_m)\in\mathbb{R}^\sigma[s]$ or by the Plücker matrix P_δ is a complete or basis free, invariant for subspaces $\mathfrak{X}_m\in G(r, \mathbb{R}^p(s))$.

When a linear multivariable system is described by the transfer function matrix G(s), where $G(s) \in \mathbb{R}^{m\times l}(s)$, $m>l$, $\mathrm{rank}_{\mathbb{R}(s)}\{G(s)\}=l$ or by a right coprime fractional representation $G(s) = N_r(s)D_r^{-1}(s)$, we may associate the following rational vector spaces:

$$\mathfrak{X}_c = \text{column-span}_{\mathbb{R}(s)}\{G(s)\}$$

$$\mathfrak{X}_f = \text{column-span}_{\mathbb{R}(s)}\left\{\begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \tag{5.9}$$

Then $\mathfrak{X}_c$ characterises the family of systems $G'(s)$ derived from G(s) under a rational full rank pre-compensation, that is $G'(s)=G(s)Q(s)$, $Q(s) \in \mathbb{R}^{l\times l}(s)$ and not the particular G(s); $\mathfrak{X}_c$ will be referred to as the *precompensator space* of the system. On the other hand, $\mathfrak{X}_f$ is a space characterising the particular G(s) and will be referred to as the *right rational fractional representation space* of the system. The canonical polynomial Grassmann representatives, $(C-\mathbb{R}[s]-G\mathbb{R})$, of $\mathfrak{X}_c$ and $\mathfrak{X}_f$ will be denoted by $g(\mathfrak{X}_c)$ and $g(\mathfrak{X}_f)$.

Remark (5.2) [Kar. & Gia. 1]: $g(\mathfrak{X}_c)=P_\delta^c \underline{e}_\delta(s)$ where $P_\delta^c \in \mathbb{R}^{\binom{m}{l}\times(\delta+1)}$ and δ is the Forney order of $\mathfrak{X}_c \cdot g(\mathfrak{X}_f)=P_v^f \underline{e}_v(s)$ where $P_v^f \in \mathbb{R}^{\binom{m+l}{l}\times(v+1)}$ and v is the McMillan degree of G(s).

If $G(s)=N_{rc}(s)D_{rc}^{-1}(s)=\widehat{N}_r(s)\widehat{D}_r^{-1}(s)$ where $(N_r(s),D_r(s))$ right coprime, but $(\widehat{N}_r(s), \widehat{D}_r(s))$ not necessarily so, then if

$$N_{rc}(s) = [\underline{n}_1(s), \cdots, \underline{n}_l(s)], \quad g(\mathfrak{X}_c) = \underline{p}^c(s) \in \mathbb{R}^{\binom{m}{l}}(s)$$

and

$$\widehat{T}_r(s) = [\widehat{\underline{t}}_1(s), \cdots, \widehat{\underline{t}}_l(s)] = [\widehat{D}_r(s)^T\ \widehat{N}_r(s)^T]^T, \quad g(\mathfrak{X}_f) = \underline{p}^f(s) \in \mathbb{R}^{\binom{m+l}{l}}(s)$$

then by proposition (5.2) we have that

$$\underline{n}(s)\wedge = \underline{n}_1 \wedge \cdots \wedge \underline{n}_l(s) = c\cdot z(s)\cdot \underline{p}^c(s) = c\cdot z(s)\cdot P_\delta^c \underline{e}_\delta(s)$$

$$\hat{t}(s)\wedge = \hat{\underline{t}}_1(s)\cdots\wedge\hat{\underline{t}}_l(s) = \hat{c}\cdot\hat{z}(s)\cdot p(s) = \hat{c}\cdot\hat{z}(s)\cdot P_r^c\underline{e}(s) \tag{5.10}$$

where $\hat{z}(s)$ is the zero polynomial of G(s) and z(s) the output decoupling zero polynomial of the state-space description associated with the non-coprime fractional representation $\hat{N}_r(s)\ \hat{D}_r^{-1}(s)$.

Using Plücker matrices, the determinantal frequency assignment problems as defined in chapter 4 section 2 may be expressed as follows:

(i) Determinantal zero assignment problem: Let $\hat{z}(s)$ be the zero polynomial G(s), $\underline{k}^T\wedge\in\mathbb{R}^{1\times r}$, $r=\binom{m}{l}$, be the exterior product of the squaring down compensator k and let $\underline{k}_p$ be an arbitrary vector of $\mathbb{R}^r$. For a given $\underline{k}_p\in\mathbb{R}^r$ investigate the zero distribution properties of the polynomial

$$Z(s, \underline{k}_p) = c\cdot\hat{z}(s)\cdot\underline{k}_p^T\ P_\delta^c\underline{e}_\delta(s),\quad c\in\mathbb{R}-\{0\} \tag{5.11}$$

Also, if $\underline{k}_p$ assigns the zeros of z(s) at given locations in C, determine whether there exists K such that $\underline{k}^T\wedge = \underline{k}_p^T$.

(ii) Determinantal Pole Assignment Problem: Let $f^T\wedge\in\mathbb{R}^{1\times\upsilon}$, $\upsilon=\binom{m+l}{l}$, be the exterior product of the composite feedback matrix $F_r=[I_l\ \ F]\in\mathbb{R}^{l\times(m+l)}$ and let $\underline{f}_p$ be an arbitrary vector of $\mathbb{R}^\upsilon$. Then for $\underline{f}_p\in\mathbb{R}^\upsilon$, investigate the zero properties of the polynomial

$$p(s, \underline{f}_p) = c'\cdot\underline{f}_p^T\ P_\upsilon^f\ \underline{e}_\upsilon(s),\quad c'\in\mathbb{R}-\{0\} \tag{5.12}$$

Also, if $\underline{f}_p$ assigns the zeros of p(s) at given locations in C, determine whether there exists F such that $\underline{f}^T\wedge = \underline{f}_p^T$.

If the zero polynomials of G(s) and KG(s) are $\bar{z}(s)$, $z_k(s)$ respectively, then

$$z_k(s) = \bar{z}(s)\,\overline{z}(s,\underline{k}_p) \tag{5.13}$$

where

$$\overline{z}(s,\underline{k}_p) = \underline{k}_p^T\, P_\delta^c\, \underline{e}_\delta(s), \;\; \underline{k}_p \in G(P_{r-1}) \tag{5.14}$$

The maximal degree of $\overline{z}(s,\underline{k}_p)$ is equal to the Forney degree δ of $\mathfrak{X}_f$. The problem of finding $\underline{k}_p \in \mathbb{R}^r$, $r=\binom{m}{l}$ such that

$$\overline{z}(s,\underline{k}_p) = a(s) = \sum_{i=0}^{\delta} a_i s^i \tag{5.15}$$

where $a_i \in \mathbb{R}$ will be referred to as *the linear determinantal zero assignment problem* (LDZAP). The solution of this problem is reduced to the study of the equation

$$(P_\delta^c)^T \underline{k}_p = \underline{a}, \quad \underline{a} = [a_0, \cdots, a_\delta]^T \in \mathbb{R}^{\delta+1} \tag{5.16}$$

A system for which equation (5.16) has a solution $\underline{k}_p \in \mathbb{R}^T$ for all $\underline{a} \in \mathbb{R}^{\delta+1}$ will be called *linearly zero assignable* (LZA). Furthermore, if for every $\underline{a}$ equation (5.16) has a solution $\underline{k}_p$ where $\underline{k}_p$ is also decomposable, then the system will be called *completely zero assignable* (CZA).

The linear determinantal pole assignment problem (LDPAP) may be defined in a similar manner. Then the solution of LDPAP is reduced to the study of the equation

$$(P_v^f)^T \underline{f}_p = \underline{b}, \quad \underline{b} \in \mathbb{R}^{(v+1)} \underline{f}_p \in \mathbb{R}^v \tag{5.17}$$

where

$$\underline{b} = [b_0, \cdots, b_\upsilon]\ \underline{e}_\upsilon(s) = \underline{b}^T\underline{e}_\upsilon(s) \tag{5.18}$$

Similarly, we may define *linearly pole assignable* (LPA) systems and *completely pole assignable* (CPA) systems.

The dynamic system will be called *generically pole assignable* (GPA) if the set of all coefficient vectors $\underline{b}$ for which the pole assignment problem is solvable is open and dense in $\mathbb{R}^{\upsilon+1}$ and its complement has measure zero. This ensures that the system is pole assignable for "almost all" $\underline{a}\in\mathbb{R}^{\upsilon+1}$. More details concerning the notion of genericity can be found in Willems & Hesselink [Will. & Hes. 1].

A system for which all decomposable vectors $\underline{k}_p$ are such that $\overline{z}(s,\underline{k}_p)\neq c$, $c\in\mathbb{R}$ will be called *strongly zero non-assignable* (SZNA), and similarly a system for which there is no decomposable vector $\underline{f}_p$ such that $f(s,\underline{f}_p)=c$, $c\in\mathbb{R}-\{0\}$ will be called *strongly pole non-assignable* (SPNA).

The following theorems of Karkanias and Giannakopulos [Kar. & Gia. 1] provide some useful conditions concerning the pole-zero assignability property of a system.

Theorem (5.2): Let $P_\delta^c=[\underline{p}_0, \cdots, \underline{p}_\delta] = [\underline{p}_0, \tilde{P}_c] \in\mathbb{R}^{r\times(\delta+1)}$ be the Plücker matrix of the vector space $\mathfrak{B}_c$ associated with G(s) and let $\pi_c=\text{rank } P_\delta^c$ and $\tilde{\pi}_c=\text{rank}\{\tilde{P}_c\}$.

1) G(s) is LZA if and only if $\pi_c=\delta+1$ or $\binom{m}{l}\geq\delta+1$ and $\pi_c=\delta+1$
2) Necessary conditions for G(s) to be CZA are $\binom{m}{l}\geq\delta+1$ and $\pi_c=\delta+1$
3) Sufficient conditions for G(s) to be SZNA are $\binom{m}{l}\leq\delta$ and $\tilde{\pi}_c=\binom{m}{l}$ □

The corresponding result for the pole assignment problem may be stated as follows:

Theorem (5.3) [Kar. & Gia. 1]: Let $P_{\upsilon}^{f}=[\underline{p}_0, \cdots, \underline{p}_{\upsilon}]=[\underline{p}_0, \tilde{P}_f]\in\mathbb{R}^{\upsilon\times(\upsilon+1)}$ be the Plücker matrix of the vector space $\mathfrak{G}$ associated with G(s) and let $\pi_f=\mathrm{rank}\{P_{\upsilon}^{f}\}$ and $\tilde{\pi}_f=\mathrm{rank}\{\tilde{P}_f\}$

1) G(s) is LPA if and only if $\pi_f=\upsilon+1$ or $\binom{m+l}{l}\geq\upsilon+1$ and $\pi_f=\upsilon+1$
2) Necessary conditions for G(s) to be CPA are $\binom{m+l}{l}\geq\upsilon+1$ and $\tilde{\pi}_f=\upsilon+1$
3) Sufficient conditions for G(s) to be SPNA are $\binom{m+l}{l}\leq\upsilon$ and $\tilde{\pi}_f=\binom{m+l}{l}$ □

The case of generically pole assignable systems is characterised by the following result

Proposition (5.3) [Kar. & Gia. 1]: A necessary condition for a system to be generically pole assignable is $\mathrm{rank}\{P_{\upsilon}^{f}\}=\upsilon+1$. □

The above results when applied to the centralised control problems provide the following Plücker matrices.

(i) Pole assignment using centralised-constant state controllers

If we represent by $\underline{b}^T(s)\wedge$ the exterior product of the rows or the matrix pencil $B(s)=[sI-A,\ B]\in\mathbb{R}^{n\times(n+1)}[s]$ then

$$\underline{b}^T(s)\wedge= C_n\Big\{[sI-A,\ B]\Big\} = \underline{e}_n^T(s)\ P_n^T \tag{5.19}$$

where $C\{\cdot\}$ is the n^{th} compound matrix of $\{\cdot\}$ and P_n the Plücker matrix. If $g(\mathcal{V})$ is the canonical polynomial Grassmann representative of the rational vector space

$$\mathcal{V}_B=\text{column-span}_{\mathbb{R}(s)}\{B^T(s)\} \tag{5.20}$$

then $\underline{b}(s)\wedge$ may be uniquely factorised as

$$\underline{b}^T(s)\wedge= c\cdot z_B(s)\ \underline{g}(\mathcal{V}) =c\cdot z_B(s)\ P(A,B)\ \underline{e}_n(s) \tag{5.21}$$

where the roots of $z_B(s)$ correspond to the uncontrollable modes of the system and P(A,B) is the controllability Plücker matrix. If the system is controllable, then

$$\underline{b}^T(s)\wedge = \tilde{c}\ g(\mathcal{V}_B) = \tilde{c}\ P(A,B)\ \underline{e}_n(s) \tag{5.22}$$

where $P(A,B) = P_n \in \mathbb{R}^{\binom{m+l}{n}\times(n+1)}$.

If we assume

$$\underline{b}^T(s)\wedge = \underline{e}_n^T(s)\ P(A,B) \tag{5.23}$$

where

$$P(A,B) = \left[\begin{array}{c|ccc} b_0 & a_0^2 & \cdots & a_0^p \\ b_1 & a_1^2 & \cdots & a_1^p \\ \vdots & \vdots & \cdots & \vdots \\ b_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^p \\ \hline 1 & 0 & \cdots & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underline{b} & \hat{P}(A,B) \\ \hline 1 & 0 \end{array}\right] \tag{5.24}$$

$[\underline{b}^T,\ 1] = [b_0,\ \cdots,\ b_{n-1}]$ is the coefficient vector of the polynomial $\det\{sI-A\}$, $p=\binom{n+l}{n}$, $a\in\mathbb{R}^{n+1}$, then P(A,B) is called the *reduced controllability Plücker matrix.*

For the case of pole assignment by constant state controller (CSC), the equation

$$P(A,B)\ \underline{k} = \underline{a},\ \underline{a} \in \mathbb{R}^{n+1},\ \underline{k} \in \mathbb{R}^{\binom{n+l}{n}} \tag{5.25}$$

takes the form

$$\begin{bmatrix} \underline{b} & \hat{P}(A,B) \\ 1 & \underline{0}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{\underline{k}} \end{bmatrix} = \begin{bmatrix} \hat{\underline{a}} \\ 1 \end{bmatrix} \tag{5.26}$$

or

$$\hat{P}(A,B)\,\hat{\underline{k}} = \hat{\underline{a}} - \underline{b} \tag{5.27}$$

where $[1, \hat{\underline{k}}]^T$ is the exterior product of the columns of $[I_n, L^T]^T$, $L \in \mathbb{R}^{l \times n}$ and $[\underline{a}^T, 1] = [a_0, a_1, \cdots, a_{n-1}\ \ 1]$ is the coefficient vector of the polynomial to be assigned. For details see Chapter 4 and equations (4.4)-(4.7).

(ii) **Pole assignment using centralised constant output controllers**

Let $G(s) \in \mathbb{R}^{m \times l}(s)$, $\text{rank}_{\mathbb{R}(s)}\{G(s)\} = \min\{m,l\}$ and let $[D_{rc}(s), N_{rc}(s)]$ be a right coprime fractional representation of $G(s)$ i.e. $G(s) = N_{rc}(s)D_{rc}(s)^{-1}$, where $N_{rc}(s) \in \mathbb{R}^{m \times l}[s]$, $D_{rc}(s) \in \mathbb{R}^{l \times l}[s]$ are coprime and $D_{rc}(s)$ is column-reduced i.e. degdet $D_{rc}(s) = \sum_{i=1}^{l} \delta_{ci} D_{rc}(s)$ where δ_{ci} is the degree of the i^{th} column of D(s). If

$$\underline{g}(\mathcal{V}_R) = c\, C_l\Big\{[D^T(s)\ N^T(s)]^T\Big\},\ c \in \mathbb{R} - \{0\} \tag{5.28}$$

is the canonical polynomial Grassmann representative of $\mathcal{V}_R =$ col-span$\Big\{[D_{rc}^T(s),\ N_{rc}^T(s)]^T\Big\}$ then the first entry of $\underline{g}(\mathcal{V}_R)$ is c det$\{D_{rc}(s)\}$ and by assuming that G(s) is strictly proper transfer function matrix then all other co-ordinates of $g(\mathcal{V}_R)$ have degrees strictly less than the degree of det $D_{rc}(s)$. Thus we may write

$$\underline{g}(\mathcal{V}_R) = P(T_R)\,\underline{e}_n(s) \tag{5.29}$$

where

$$P(T_R) = \begin{bmatrix} P_0 & P_1 & \cdots & P_{n-1} & 1 \\ P_0^2 & P^2 & \cdots & P_{n-1}^2 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ P_0^q & P_1^q & \cdots & P_{n-1}^q & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \underline{p}^T & 1 \\ \hat{P}(T_R) & 0 \end{bmatrix} \tag{5.30}$$

and $P(T_R) \in \mathbb{R}^{q\times(n+1)}$ is the Plücker matrix of $\mathcal{V}_R$, $q = \binom{m+l}{l}$ while $\hat{P}(T_R) \in \mathbb{R}^{(q-1)\times n}$ will be referred to as the *reduced Plücker matrix* of the vector space $\mathcal{V}_R$. Also, the first row of $P(T_R)$ represents the characteristic polynomial of the open-loop system i.e.

$$p_0 + p_1 s + p_2 s^2 + \cdots + p_{n-1} s^{n-1} + s^n = [\underline{p}^T, 1]\,\underline{e}(s) = \tfrac{1}{c} \det \left\{D_{rc}(s)\right\} \tag{5.31}$$

then the linear problem of pole assignment by centralised constant output controllers formulated as:

$$P^T(T_R)\,\underline{k} = \underline{a},\ \underline{a} \in \mathbb{R}^{n+1},\ \underline{k} \in \mathbb{R}^q,\ q = \binom{m+l}{l} \tag{5.32}$$

where the roots of the polynomial $a(s) = \underline{a}^T \underline{e}_n(s)$ are the poles to be assigned, may be formulated as

$$\begin{bmatrix} \underline{p} & \hat{P}(T_R)^T \\ L & \underline{0}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{\underline{k}} \end{bmatrix} = \begin{bmatrix} \hat{\underline{a}} \\ 1 \end{bmatrix} \tag{5.33}$$

where $[1, \hat{\underline{k}}]$ is the exterior product of the rows of $[I_L, F]$ and $[\underline{a}, 1] = [a_0, a_1, \cdots, 1]$ are the coefficients of the desired polynomial. The matrix $[I, F]$ has already been defined in Chapter 4 Problem 2. Again a necessary condition for generic assignability by centralised controllers is provided by Proposition 5.3.

(iii) **Zero assignment by centralised squaring down controllers**

We have seen in Chapter 4, Problem 3, that the zero polynomial of the squared down system is given by

$$z_k(s) = \det \{K N_R(s)\} \tag{5.34}$$

If $z(s) \in \mathbb{R}^{l \times l}[s]$ is the greatest right divisor of $N_R(s)$ then

$$N_R(s) = \bar{N}(s)\ Z(s) \tag{5.35}$$

where $\bar{N}(s) \in \mathbb{R}^{m \times l}[s]$ is a minimal basis of the rational vector space $\mathfrak{S}_c$ spanned by the columns of $N_R(s)$. Then, equation (5.34) is written

$$z_k(s) = \det \{K\bar{N}(s)\ Z(s)\}$$

and by the Binet-Cauchy theorem equation (5.36) yields

$$z_k(s) = C_l(k)\ C_l\{\bar{N}(s)\}\ z(s) = z_f(s)\ z(s)$$

where $z(s) = \det Z(s)$ is the zero polynomial of G(s). Since $\bar{N}(s)$ is a minimal basis of $\mathfrak{S}_c$, then $C_l\{N(s)\}$ may be written as

$$C_l\{\bar{N}(s)\} = c\ g(\mathfrak{S}_c) = c\ P_\delta\ \underline{e}_\delta(s),\ \ c \in \mathbb{R}-\{0\} \tag{5.36}$$

where $\underline{g}(\mathfrak{S}_c)$ is the canonical polynomial Grassmann representative of $\mathfrak{S}_c$ and $P_\delta \in \mathbb{R}^{p\times(\delta+1)}$, $p = \binom{m}{l}$ the Plücker matrix while δ is the dynamical order of the rational vector space $\mathfrak{S}_c$.

Again, a necessary condition for the generic zero assignment (GZA) problem specified by the already familiar equation

$$P_\delta^T \underline{k} = \underline{a},\ \underline{a} \in \mathbb{R}^{\delta+1},\ k \in \mathbb{R}^p,\ p = \binom{m}{l} \tag{5.37}$$

is provided by the proposition (5.3) that is rank $\{P_\delta\} = \delta+1$.

5.2 Decentralisation Indices for Decentralised Control Systems

We have seen in chapter 4 that the controllers in the case of decentralised control assume the following structure:

$$K = \begin{bmatrix} I_r \\ K_D \end{bmatrix} \in \mathbb{R}^{p\times r}, \quad K_D = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & K_N \end{bmatrix} \in \mathbb{R}^{\upsilon\times r}, \quad \upsilon = p-r \tag{5.38}$$

and

$$K_i = \{K_1, K_2, \cdots, K_N\} \in \mathbb{R}^{\upsilon_i\times r_i}$$

To investigate the above structure, we introduce the following definitions:

Definition (5.1): Let $Q_{r,p}$ denote the set of strictly increasing sequences of r integers $(1 \le r \le p)$ chosen from 1, 2, $\cdots$, p. $D_{r,p}$ denotes the set of sequences of r distinct integers chosen form 1, 2, $\cdots$, p and $S_{r,p}$ the totality of r integers chosen form 1, 2, $\cdots$, p. Thus, $D_{r,p}$ is obtained from $Q_{r,p}$ by associating with each sequence $\omega \in Q_{r,p}$ the r! sequences obtained by reordering the integers in all possible ways. $Q_{r,p}$ has $\binom{p}{r} = \frac{p!}{r!(p-r)!}$ sequences in it, $D_{r,p}$ has $\binom{p}{r} = \frac{p!}{(p-r)!}$ sequences in it and $S_{r,p}$ has p^rsequences in it.

Definition (5.2): Let $K \in \mathbb{R}^{p\times r}$, $r<p$, rank$\{K\} = r$ and $\omega = \{i_1, i_2, \cdots, i_r\} \in Q_{r,p}$ and let (i_j) be the subset of the integers from 1, 2, $\cdots$, r which correspond to the non-zero elements of K matrix in the i_j row

(i) We define as the index ω in K the set

$$I(\omega,K) = \{(i_1), \cdots, (i_r)\} \tag{5.39}$$

The sequence ω will be called *complete* in K, if for all $j \in \underline{r}$, $\{i_j\} \neq \emptyset$; otherwise it will be called *noncomplete* in K.

(ii) For a $\omega \in Q_{r,p}$ which is complete in K we define as the *span of* ω *in* K the set of sequences

$$sp(\omega,K) = \{z: z=(j_1,j_2,\cdots j_r) \in S_{r,p} | \, j_1 \in (i_1),\cdots,j_r \in (i_r)\} \tag{5.40}$$

(iii) For $\omega \in Q_{r,p}$ which is complete in K we define as the *basis of* ω *in* K the subset of sequences of span(ω,K) defined by

$$b(\omega,K) = \{v:v=(k_1,\cdots,k_r) \in D_{r,p} \text{ and } v \in span(\omega,K)\} \tag{5.41}$$

(iv) A sequence $\omega \in Q_{r,p}$ will be called *degenerate* in K if it is noncomplete in K, or if it is complete in K and $b(\omega,K) = \emptyset$. If ω is complete in K and $b(\omega,K) \neq \emptyset$, then it will be called *nondegenerate* in K.

(v) If ω is nondegenerate in K and $v = (k_1, k_2, \cdots, k_v) \in b(\omega,K)$, we define $\{v\}$ as the *representation of* v *in* ω the set of ordered pairs

$$\{v\} = \{(i_1, k_1), (i_2, k_2), \cdots, (i_r,k_r)\} \tag{5.42}$$

the set of representations of all $v \in b(\omega,K)$ is denoted by

$$\{b(\omega,K)\} = \{v : v \in b(\omega,K)\} \tag{5.43}$$

and will be called *representation of* ω *in* K.

□

Using the Laplace expansion theorem for determinants the following result may be readily established.

Proposition (5.4): Let $K = K^{1,2,\cdots,r}_{i_1,i_2,\cdots,i_r} = K_{\omega]}$ be the minor of K which corresponds to the set {1, 2 , $\cdots$, r} columns and the set $\omega = \{i_1, i_2, \cdots, i_r\}$ rows then

(i) $K_{\omega]} = \emptyset$, if ω is degenerate in K

(ii) $K_{\omega]} = \text{sign}(p)\, a_{i_1k_1}\, a_{i_2k_2} \cdots a_{i_rk_r}$, if ω is nondegenerate in K and where a_{ij} are the elements of K. □

The following example illustrates the above definitions and proposition.

Example (5.1): Let K be defined as

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ l_1 & 0 & 0 \\ 0 & l_1 & l_2 \end{bmatrix} \in \mathbb{R}^{5\times 3} \tag{5.44}$$

The set of $Q_{3,5}$ sequences defining the $\underline{g}(K)$ multivector and the corresponding bases of the sequences $\omega \in Q_{3,5}$ in K are:

$$\begin{aligned}
I(1,2,3) &= \{(1),(2),(3)\} &&\rightarrow b(1,2,3) = \{(1,2,3)\} \\
I(1,2,4) &= \{(1),(2),(1)\} &&\rightarrow b(1,2,4) = \emptyset \\
I(1,2,5) &= \{(1),(2),(2,3)\} &&\rightarrow b(1,2,5) = \{(1,2,3)\} \\
I(1,3,4) &= \{(1),(3),(1)\} &&\rightarrow b(1,3,4) = \emptyset \\
I(1,3,5) &= \{(1),(3),(2,3)\} &&\rightarrow b(1,3,5) = \{(1,3,2)\} \\
I(1,4,5) &= \{(1),(1),(2,3)\} &&\rightarrow b(1,4,5) = \emptyset \\
I(2,3,4) &= \{(2),(3),(1)\} &&\rightarrow b(2,3,4) = \{(2,3,1)\} \\
I(2,3,5) &= \{(2),(3),(2,3)\} &&\rightarrow b(2,3,5) = \emptyset \\
I(2,4,5) &= \{(2),(1),(2,3)\} &&\rightarrow b(2,4,5) = \{(2,1,3)\} \\
I(3,4,5) &= \{(3),(1),(2,3)\} &&\rightarrow b(3,4,5) = \{(3,1,2)\}
\end{aligned} \tag{5.45}$$

By inspection of the bases of $\omega \in Q_{3,5}$ sequences the representations of $b(\omega,K)$ are defined and from these the minors. Hence,

$$\begin{aligned}
\{b(1,2,3)\} &= \{(1,1),(2,2),(3,3)\} &&\rightarrow k_{123} = 1 \\
(1,2,4) &: \text{degenerate} &&\rightarrow k_{124} = 0 \\
\{b(1,2,5)\} &= \{(1,1),(2,2),(5,3)\} &&\rightarrow k_{125} = l_3
\end{aligned}$$

$$\begin{array}{lll} \{1,3,4\} : \text{degenerate} & \rightarrow & k_{134} = 0 \\ \{b(1,3,5)\} = \{(1,1),(3,3),(5,2)\} & \rightarrow & k_{135} = -l_2 \\ (1,4,5) : \text{degenerate} & \rightarrow & k_{145} = 0 \\ \{b(2,3,4)\} = \{(2,2),(3,3),(4,1)\} & \rightarrow & k_{234} = l_1 \\ (2,3,5) : \text{degenerate} & \rightarrow & k_{235} = 0 \\ \{b(2,4,5)\} = \{(2,2),(4,1),(5,3)\} & \rightarrow & k_{245} = l_1 l_3 \\ \{b(3,4,5)\} = \{(3,3),(4,1),(5,2)\} & \rightarrow & k_{345} = l_1 l_2 \end{array} \tag{5.46}$$

The zero entries in the multivector $\underline{g}(K)$ are those which correspond to the degenerate sequences $\omega \in Q_{3,5}$ in K and they are

$$\{k_{124}, k_{134}, k_{145}, k_{235}\} \tag{5.47}$$

An alternative procedure for finding the location of zeros in $\underline{g}(K)$ is by using the procedure for writing down the set of Reduced Quadratic Plücker Relations [Kar. & Gia., 1]. The procedure may be illustrated by the same example as above.

Example (5.2): Let $K \in \mathbb{R}^{5\times 3}$ be the matrix with structure as in Example (5.1) and let $\underline{g}(K) = [k_{123}, k_{124}, k_{125}, k_{134}, k_{135}, k_{145}, k_{234}, k_{235}, k_{245}, k_{345}]$ be a decomposable vector with $k_{123} = 1$. The matrix K which corresponds to $\underline{g}(K)$ is then defined by

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_{234} & -k_{134} & k_{124} \\ k_{235} & -k_{135} & k_{125} \end{bmatrix} \tag{5.48}$$

For the decentralised case we must have

$$k_{134} = k_{124} = k_{235} = 0 \tag{5.49}$$

The set of Reduced Quadratic Plücker Relations may then be expressed by the nontrivial relations in the equation

$$C_3(K) = \underline{g}(K) \tag{5.50}$$

that is

$$\begin{aligned}&k_{123} = k_{123} = 1,\ \ k_{124} = k_{124},\ \ k_{125} = k_{125},\ \ k_{134} = k_{134},\ k_{135} = k_{135},\\ &k_{145} = -k_{125}k_{134} + k_{124}k_{135},\ \ k_{234} = k_{234},\ \ k_{235} = k_{235},\\ &k_{245} = -k_{234}k_{125} + k_{124}k_{235},\ \ k_{345} = -k_{234}k_{135} + k_{134}k_{235}\end{aligned} \tag{5.51}$$

□

From the set of conditions (5.49) it follows that

$$k_{145} = -k_{123}\cdot 0 + 0\cdot k_{135} = 0 \rightarrow k_{145} = 0 \tag{5.52}$$

and hence we have the conditions

$$\{k_{134} = 0,\ \ k_{124} = 0,\ \ k_{235} = 0,\ \ k_{145} = 0\} \tag{5.53}$$

which are similar to the conditions of (5.47).

The conditions for decentralisation and decomposability thus become

$$\begin{aligned}&k_{123} = k_{123} = 1,\ \ k_{124} = k_{124} = 0,\ \ k_{125} = k_{125},\ \ k_{134} = k_{134} = 0,\\ &k_{135} = k_{135},\ \ k_{145} = 0,\ \ k_{145} = 0,\ \ k_{234} = k_{234},\ \ k_{235} = k_{235} = 0,\\ &k_{245} = -k_{125}k_{234},\ \ k_{345} = -k_{234}k_{135}\end{aligned} \tag{5.54}$$

The above example demonstrates an alternative procedure for deriving the location of zeros in $\overline{k}_D\wedge$ or $\overline{l}_D\wedge$ we first note:

Lemma (5.2) [Kar. & Gia. 1]: Let $k = [\cdots, k_\omega, \cdots]^T \in \mathbb{R}^\sigma$, $\sigma = (P_r)$ be a decomposable vector and let $k_{1,2,\cdots,r} = 1$. A basis matrix for the subspace characterised by k may be defined by

$$K = \begin{bmatrix} I_r \\ R \end{bmatrix} \in \mathbb{R}^{p\times r},\quad R \in \mathbb{R}^{(p-r)\times r} \tag{5.55}$$

where

$$\mathrm{R} = \begin{bmatrix} (-1)^{r-1}\mathrm{k}_{2,\cdots,r,r+1} & (-1)^{r-2}\mathrm{k}_{1,3,\cdots,r,r+1} & \cdots & \mathrm{k}_{1,2,\cdots,r-1,r+1} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{r-1}\mathrm{k}_{2,\cdots,r,p} & (-1)^{r-2}\mathrm{k}_{1,3,\cdots,r,p} & \cdots & \mathrm{k}_{1,2,\cdots,r-1,p} \end{bmatrix} \tag{5.56}$$

□

Proposition (5.5) [Kar. & Gia. 1]: A set of Reduced Quadratic Plücker relations (RQPR) for points of $\mathbb{R}^{\sigma}$ with the first Plücker coordinate $\mathrm{k}_{1,2,\cdots,r}=1$ is given by

$$\underline{\mathrm{k}} = \begin{bmatrix} 1 \\ \vdots \\ \mathrm{k}_r \\ \vdots \end{bmatrix} = \mathrm{C}_r\left\{\begin{bmatrix} \mathrm{I}_r \\ \mathrm{R} \end{bmatrix}\right\} \tag{5.57}$$

□

Based upon the above results of algebraic geometry we may formulate now the alternative procedure for the computation of fixed zeros in the decomposable vector $\underline{\mathrm{k}}$ characterising a decentralised control structure

Proposition (5.6): The zero coordinates of the decomposable vector $\underline{\mathrm{k}} = [\cdots, \mathrm{k}_{\omega}, \cdots]^{\mathrm{T}} \in \mathbb{R}^{\sigma}$, $\sigma = \binom{p}{r}$ which characterises a decentralised control system are given by

(i) The decentralisation assumption implies that a set of Plücker coordinates in the expression of R must be set equal to zero.

(ii) An extra set of zero Plücker coordinates is obtained by introducing those zero coordinates defined in (i) into the set of reduced quadratic Plücker relations. The above procedure yields some extra zero Plücker coordinates.

The two procedures defined above yield the number of fixed zeros in the multivector $\mathrm{C}_v(\mathrm{K}) = \underline{\mathrm{k}}_{\mathrm{D}} \wedge$ where K is defined by (5.38). The following Theorem summarises the results so far the case of decentralised control systems.

Theorem (5.4): Let $K \in \mathbb{R}^{p \times r}$ be the decentralised control matrix

$$K = \begin{bmatrix} I_r \\ K_D \end{bmatrix}, \; K_D = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & K_N \end{bmatrix} \in \mathbb{R}^{r \times r}, \; r = p - r \tag{5.58}$$

and $K_i \in \mathbb{R}^{r_i \times r_i}$ and let $\Phi = \{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_m}\}$ be the set of degenerate in K sequences of $Q_{r,p}$. For generic values in the elements of the submatrices K_i in K_D, the set of zero Plücker coordinates in the decomposable vector $\underline{k} \wedge = C_r(K) = [1, \cdots, k_\omega, \cdots]^T \in \mathbb{R}^\sigma$, $\sigma = \binom{p}{r}$ is given by

$$\Theta = \left\{ k_{\omega_{i_1}}, k_{\omega_{i_2}}, \cdots, k_{\omega_{i_\mu}} \right\} \tag{5.59}$$

The proof of the above result follows immediately from proposition (5.4).

The set of degenerate in K sequences of $Q_{r,p}$, $\Phi = \{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_m}\}$ which define the fixed zeros in $\underline{k} \wedge$ will be called the set of *decentralisation indices* (DI) of the system. Since the set Φ characterises the structure of decentralisation, its importance in decentralised control will be examined next.

5.3 Decentralised Plücker matrices and the Decentralised Grassmann Subvariety

We have seen in Chapter 4 that the decentralised determinantal pole assignment problems may be characterised by the following multivectors

(i) Decentralised Constant State Controllers (D-CSC)

The determinantal polynomial equation is given by

$$P_L(s,L_D) = \det\left\{C(s)\ \tilde{L}_D\right\} = \underline{c}(s)^T\wedge\cdot\tilde{\underline{l}}_D \tag{5.60}$$

where

$$C(s)=[sI-A,\ -B]$$

$$\tilde{L}_D^T=[I_n\ L_D] \tag{5.61}$$

or

$$L_D=\text{block-diag}\{L_1,\ L_2,\ \cdots,\ L_N\},\quad L_i\in\mathbb{R}^{n_i\times l_i} \tag{5.62}$$

$\underline{c}(s)^T\wedge$:= exterior product of the rows of (5.61)

$\tilde{\underline{l}}_D$:= exterior product of the columns of (5.62)

(ii) **Decentralised Constant Output Controllers (D-COC)**

Again as shown in Chapter 4, the determinantal polynomial equation is

$$P_K(s,K_D) = \det\left\{T_l(s)\ \tilde{K}_D^l\right\}=\det\left\{\tilde{K}_D^r T_r(s)\right\} \tag{5.63}$$

$$P_K(s,K_D) = \left((\underline{t}_l^T(s))\wedge,\ \tilde{\underline{k}}_D^l\right) = (\tilde{\underline{k}}^r)^T\wedge\cdot\underline{t}_r(s)\wedge \tag{5.64}$$

where $(.)^T$ is the transpose of (.).

Here the multivectors $\underline{t}_l^T(s)\wedge$ and $\underline{l}(s)^T\wedge$ are polynomial Grassmann representatives

(R[s]−GR) of the rational vector spaces defined by G(s) and $T_l(s)=[D_l(s)\ N_l(s)]$. The coprimeness hypothesis of $T_l(s)=D_l^{-1}(s)\ N_l(s)$ implies that the multivector $\underline{t}_l(s)^T\wedge$ is reduced and the controllability hypothesis also implies that $\underline{l}(s)^T\wedge$ is reduced.

If P_C and P_L are the controllability, left fractional Plücker matrices then

$$\underline{t}_l^T(s)\wedge = \underline{e}_n^T(s)\ P_L \tag{5.65}$$

$$\underline{c}^T(s)\wedge = \underline{e}_n^T(s) P_C \tag{5.66}$$

where

$$\underline{e}_n^T(s) \quad = [1, s, \cdots, s^n] \tag{5.67}$$

$$P_L \in \mathbb{R}^{(n+1)\times r}, \quad r=\binom{m+l}{m} \tag{5.68}$$

$$P_C \in \mathbb{R}^{(n+1)\times \rho}, \quad \rho=\binom{n+l}{n} \tag{5.69}$$

Hence, from (5.60) and (5.63) we have

$$P_L(s, L_D) = \underline{e}_n^T(s)\ P_C \begin{bmatrix} 1 \\ \underline{l}_D \end{bmatrix} \tag{5.70}$$

$$P_K(s, K_D) = \underline{e}_n^T(s)\ P_L \begin{bmatrix} 1 \\ \underline{k}_D \end{bmatrix} \tag{5.71}$$

and for pole assignment, we have that

$$P_L(s,L_D) = \Phi_L(s) \tag{5.72}$$

$$P_K(s,L_D) = \Phi_K(s) \tag{5.73}$$

where $\Phi_L(s)$ and $\Phi_K(s)$ are the arbitrary desired polynomials.

Let $\Phi_{K}(s) = \underline{e}_n^T(s)\,[a_0, a_1, \cdots, a_{n-1}, 1]^T \in \mathbb{R}[s]$ be the desired polynomial. Then, equations (5.71) and (5.73) are reduced to the following two subproblems.

(i) Determine whether there exists a solution for some arbitrary $\underline{k} \in \mathbb{R}^n$ and any vector $\underline{a} \in \mathbb{R}^n$ of the equation

$$P_L \begin{bmatrix} 1 \\ \underline{k}_D \end{bmatrix} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{a} \\ 1 \end{bmatrix} \tag{5.74}$$

(ii) Let $\mathcal{K}(\underline{a})$ be the family of solutions of (5.74) for a given $\underline{a}$. Then, the decentralised pole assignment problem (DPAP) is defined by the following statement.

DPAP Statement: Determine whether there exists a decomposable vector $[1\ \underline{k}_D]^T \in \mathcal{K}(\underline{a})$ which satisfies also the decentralised assumptions i.e. it has a set of zero co-ordinates at the locations of the decentralisation indices, $\Phi = \left\{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_\mu}\right\}$.

We, therefore, conclude that the study of the pole assignment using decentralised control is reduced to the investigation of the properties of the Plücker matrices P_C and P_L. The Plücker matrix P_C is the object of study for the case of state feedback while P_L assumes the central interest for the case of decentralised control using constant output feedback. Another important concept for the case of decentralised control is the motion of Φ-decentralised subvariety, $\Psi(\sigma,\Phi)$ introduced by the following proposition.

Proposition (5.7): Let $\Phi = \{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_\mu}\}$ be the subset of degenerate sequences of $Q_{r,p}$. The set of equations describing the Φ-decentralised subvariety of $\Omega(r,p)$, $\Psi(\sigma,\Phi)$, where $\sigma=\binom{p}{r}$ is given by the set of nontrivial equations defined by

$$R^{\sigma} = \begin{bmatrix} 1 \\ \vdots \\ k_{\omega} \\ \vdots \end{bmatrix} = C_r \begin{bmatrix} I_r \\ R \end{bmatrix} \tag{5.75}$$

$$k_{\omega_{i_1}} = k_{\omega_{i_2}} = \cdots = k_{\omega_{i_\mu}} = 0 \tag{5.76}$$

where R is a matrix which is a function of $\{k_\omega\}$ and is given by

$$R = \begin{bmatrix} (-1)^{r-1}k_{2,\cdots,r,r+1} & (-1)^{r-2}k_{1,3,\cdots,r,r+1} & \cdots & k_{1,2,\cdots,r-1,r+1} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{r-1}k_{2,\cdots,r,p} & (-1)^{r-2}k_{1,3,\cdots,r,p} & \cdots & k_{1,2,\cdots,r-1,p} \end{bmatrix} \tag{5.77}$$

□

It is then known [Kar. & Gia., 1] that the solution of the linear subproblem defined by (5.74) is a linear variety of $P^\sigma(R)$. The decentralised decomposable vectors $[1 \;\; \underline{k}_D]^T$ also define a variety of $P^\sigma(R)$; such a variety has been defined above by equations (5.75) and (5.76) and been called Φ-*decentralised subvariety* of the Grassmann variety Ω(r,p). (r=m, p=m+l, $\sigma=\binom{m+l}{l}$ of the projective space $P^\sigma(\mathbb{R})$. Also, the existence of the decomposable vector

$$\begin{bmatrix} 1 \\ k_D \end{bmatrix} \in K(\underline{a}) \tag{5.78}$$

implies that $K(\underline{a})$ and $\Psi(\sigma,\Phi)$ must intersect at real points.

We may therefore summarise the above observations by the following result.

Proposition (5.8): Necessary and sufficient conditions for the existence of a decentralised feedback controller that will assign arbitrary the poles of the closed-loop system is

$$K(\underline{a}) \bigcap \Psi(\sigma,\Phi) \neq 0 \tag{5.79}$$

and that the set of common points contains at least one real point.

The set of equations defining $\Psi(\sigma,\Phi)$ can be demonstrated by the following example.

Example (5.3): Let

$$\underline{k} = [k_{123}, k_{124}, k_{125}, k_{134}, k_{135}, k_{145}, k_{234}, k_{235}, k_{245}, k_{345}]^T$$

be the co-ordinates of a point in $\mathbb{R}^{\binom{5}{3}} = \mathbb{R}^{10}$.

Let us also assume that the set

$$\Phi = \Big\{(1,3,4),\ (1,2,4),\ (2,3,5),\ (1,4,5)\Big\}.$$

Then the set of equations describing the $(10,\Phi)$ variety are

$$\begin{aligned}
&k_{134} = k_{124} = k_{235} = k_{145} = 0\\
&k_{123} = k_{123} = 1,\ \ k_{124} = k_{124} = 0,\ \ k_{125} = k_{125},\\
&k_{134} = k_{134} = 0,\ \ k_{135} = k_{135},\ \ k_{145} = 0,\ \ k_{234} = k_{234},\\
&k_{235} = k_{235} = 0,\ \ k_{245} = -k_{125}k_{234},\ \ k_{345} = -k_{234}k_{135}
\end{aligned}$$

from the above equations, the nontrivial set

$$\begin{aligned}
&k_{123} = 1,\ k_{124} = k_{134} = k_{145} = k_{235} = 0\\
&k_{245} = -k_{125}k_{234},\ \ k_{345} = -k_{234}k_{135}
\end{aligned}$$

describes the decentralised subvariety of $\Psi(10,\Phi)$.

□

Note that if a point of $P_{\sigma-1}$ satisfies the equations of $\Psi(\sigma,\Phi)$ defined by (5.75) and (5.76) then a decentralised controller is defined by

$$K = \begin{bmatrix} I_r \\ R(\Phi) \end{bmatrix} \tag{5.80}$$

where $R(\Phi)$ is the matrix defined by (5.77) with the conditions $k_{\omega_{i_1}} = k_{\omega_{i_2}} = \cdots = k_{\omega_{i_\mu}} = 0$ imposed on its structure.

Example (5.4): The set of equations describing $(10,\Phi)$ where $\Phi = \Big\{(1,3,4),\ (1,2,4),\ (2,3,5),\ (1,4,5)\Big\}$ is given by

$$\begin{aligned}
&k_{123} = 1,\ \ k_{124} = k_{134} = k_{145} = k_{235} = 0\\
&k_{245} = -k_{125}k_{234},\ \ k_{345} = -k_{234}k_{135}
\end{aligned}$$

The set of the above conditions may be linearised in the following ways. Let

i) $k_{125} = a$, $k_{135} = b$

ii) $k_{234} = \gamma$

These two linearisation schemes yield the following structure for the controllers

$$K_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_{234} & 0 & 0 \\ 0 & -b & a \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & 0 & 0 \\ 0 & -k_{135} & k_{125} \end{bmatrix}$$

For K_1 and K_2 the corresponding multivectors are

$\underline{k}_1 = [1, 0, a, 0, b, 0, k_{234}, 0, k_{245}, k_{345}]^T$

$\underline{k}_2 = [1, 0, k_{125}, 0, k_{135}, 0, \gamma, 0, k_{245}, k_{345}]^T$

and for decomposability of $\underline{k}_1$ and $\underline{k}_2$ the following linear conditions must hold true

$\underline{k}_1$: $k_{245} = ak_{234}$, $k_{345} = bk_{234}$

$\underline{k}_2$: $k_{245} = -k_{125}$, $k_{345} = -\gamma k_{135}$

Attention is now focused on the derivation of necessary conditions for pole assignability by decentralised feedback. The fixed zeros of the set Φ are used to define the decentralised Plücker matrices.

Given that G(s) has been assumed strictly proper, the first column of P_L is defined by the coefficients of the pole polynomial $\left\{\det D_l(s)\right\}$ and hence we may write

$$P_l\begin{bmatrix} 1 \\ \underline{k}_D \end{bmatrix} =$$

$$= \begin{bmatrix} \gamma_0 & a_0^2 & \cdots & a_0^\sigma \\ \gamma_1 & a_1^2 & \cdots & a_1^\sigma \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^\sigma \\ & & & \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \underline{k}_D \end{bmatrix} =$$

$$= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ \\ 1 \end{bmatrix} \tag{5.81}$$

or

$$\begin{bmatrix} \underline{\gamma} & \tilde{P}_l \\ & \\ 1 & \underline{0}^T \end{bmatrix} \begin{bmatrix} 1 \\ \\ \underline{k}_D \end{bmatrix} = \begin{bmatrix} \underline{a} \\ \\ 1 \end{bmatrix} \tag{5.82}$$

where $P_l \in \mathbb{R}^{(n+1)\times\sigma}$, $\tilde{P}_l \in \mathbb{R}^{n\times(\sigma-1)}$ and $\sigma = \binom{m+1}{m}$. From the above conditions it is readily shown that the following condition has to be satisfied

$$\tilde{P}_l \underline{k}_D = \underline{a} - \underline{\gamma} = \underline{b} \tag{5.83}$$

The matrix $\tilde{P}_l(\tilde{P}_C)$ is known as the *reduced left fractional Plücker matrix* (*reduced controllability Plücker matrix*). The decentralised assumption implies that there is a set of sequences $\left\{\Phi = \{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_\mu}\}\right\}$ of $Q_{m,m+l}$ for which the corresponding co-ordinates in $\underline{k}_D$ are zero i.e.

$$\underline{k}_D = [k_{\omega_{i_1}}, \cdots, k_{\omega_{i_1-1}}, \underset{\substack{\downarrow \\ \omega_{i_1}}}{0}, k_{\omega_{i_1+1}}, \cdots, k_{\omega_{i_\mu-1}}, \underset{\substack{\downarrow \\ \omega_{i_\mu}}}{0}, k_{\omega_{i_\mu+1}}, \cdots, k_{\omega_\sigma}] \tag{5.84}$$

The vector defined by $\underline{k}$ by dropping the zero co-ordinates will be denoted

$$\tilde{\underline{k}}_D = [k_{\omega_2}, \cdots, k_{\omega_{i_1-1}}, k_{\omega_{i_1+1}}, \cdots, k_{\omega_{i_\mu-1}}, k_{\omega_{i_\mu+1}}, \cdots, k_{\omega_\sigma}] \tag{5.85}$$

The existence of the fixed zeros in $\underline{k}_D$ (defined by the decentralisation indices Φ) implies that equation (5.83) may be reduced to

$$\hat{P}_l \, \hat{\underline{k}}_D = \underline{a} - \underline{\gamma} = \underline{b} \tag{5.86}$$

where $\hat{P}_l \in \mathbb{R}^{(n\times(\sigma-\mu-1))}$ is the matrix obtained from $\tilde{P}_l$ by dropping the set of columns which correspond to the set of decentralisation indices $\Phi = \left\{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_\mu}\right\}$. $\hat{P}_l$ will be referred to as the *Reduced decentralised left fractional Plücker Matrix* (Reduced decentralised controllability Plücker matrix).

A vector of the type $\tilde{\underline{k}}_D \in \mathbb{R}^{\sigma-\mu-1}$, $\sigma = \binom{m+l}{m}$, may always be thought as a subvector of a general vector $[1\ \underline{k}_D]^T \in \mathbb{R}^{\sigma}$ which may be obtained from $[1\ \underline{k}_D]$ by dropping out the co-ordinates $\left\{k_{\omega_{i_1}} = 1, k_{\omega_{i_1}} = 0, \cdots, k_{\omega_{i_\mu}} = 0\right\}$; the co-ordinates of $\tilde{\underline{k}}_D$ are characterised by the set of sequences $\tilde{\Phi} = \left\{\omega_2, \cdots, \omega_{i_1-1}, \omega_{i_1+1}, \cdots, \omega_{i_\mu-1}, \omega_{i_\mu+1}, \cdots, \omega_\sigma\right\}$ which is a complementary set to that of $\Phi = \left\{\omega_1, \omega_{i_1}, \cdots, \omega_{i_\mu}\right\}$ $(\omega_1 = (1, 2, \cdots, m))$. Vectors of the $\tilde{k}_D$ type with co-ordinates parametrised by the set Φ will be called *Φ-indexed vectors* of $\mathbb{R}^{\sigma-\mu-1}$.

Note that every vector of $\mathbb{R}^{\sigma-\mu-1}$ may become Φ-indexed by enumerating its co-ordinates by the set of indices. The vector $[1\ \underline{k}_D]$ is decomposable and its co-ordinates satisfy conditions (5.77). By inserting $k_{\omega_{i_1}} = k_{\omega_{i_2}} = \cdots = k_{\omega_{i_\mu}} = 0$ into $[1\ \underline{k}_D] = C_r\begin{bmatrix} I_r \\ R \end{bmatrix}$ and by dropping out the obvious identities, a set of nontrivial conditions on the co-ordinates of $\tilde{\underline{k}}_D$ is obtained, such a set of relations is defined as the set of *Φ-decentralised Reduced Plücker Relations* (Φ-DRPR). A Φ-indexed vector $\underline{k}_D$ which satisfies the set Φ-DRPR will be called *Φ-decomposable.*

Proposition (5.9): Let $[1\ \underline{k}_D^T]^T = [1, k_{\omega_2}, \cdots, k_{\omega_\sigma}] \in \mathbb{R}^{\sigma}$, $\sigma = \binom{p}{r}$, $\Phi = \left\{\omega_{i_1}, \cdots, \omega_{i_\mu}\right\}$ be a set of sequences of $Q_{r,p}$, $k_{\omega_{i_1}} = \cdots = k_{\omega_{i_\mu}} = 0$ and let $\tilde{\underline{k}}_D$ be the Φ-indexed vector obtained from $[1\ \underline{k}_D^T]^T$ by dropping the co-ordinates $\left\{k_{\omega_1} = 1, k_{\omega_{i_1}} = \cdots = k_{\omega_{i_\mu}} = 0\right\}$. Then $[1\ \underline{k}_D^T]^T$ is decomposable if and only if, $\tilde{\underline{k}}_D$ is Φ-decomposable.

□

The above result implies that the set of Φ-DRPR may be used instead of the set RQPR. The investigation of the decomposability of $[1\ \underline{k}_D^T]^T$ with decentralisation indices Φ, may thus be reduced to a study of Φ-decomposability of the smaller dimension vector $\tilde{\underline{k}}_D$.

When Φ-decomposability of $\tilde{\underline{k}}_D$ is established, decomposability of the vector $[1\ \underline{k}_D^T]^T$ follows immediately and reconstruction of the $[I_r\ \ R]^T$ matrix associated with $[1\ \underline{k}_D^T]^T$ is achieved in the way described before. If $\tilde{\underline{k}}_D$ is a Φ-indexed vector, then the vector $[1\ \underline{k}_D^T]^T$ obtained by inserting in the appropriate order of the set of co-ordinates $\left\{k_{\omega_1} = 1, k_{\omega_i} = 0, \cdots, k_{\omega_{i_\mu}} = 0\right\}$ will be referred to as the *Φ-completed vector* of the Φ-indexed vector $\tilde{\underline{k}}_D$. Proposition (5.9) then states that a Φ-indexed vector is Φ-decomposable if and only if the Φ-completed vector is decomposable.

Example (5.5): For the vector $\underline{k}$ of Example (5.1) where $\underline{k} = [k_{123} = 1,\ k_{124} = 0, k_{125}.k_{134} = 0, k_{135},k_{145} = 0, k_{234}.k_{235} = 0, k_{245}, k_{345}]^T \in \mathbb{R}^{10}$, the Φ-indexed vector of $\mathbb{R}^5$ is $\tilde{\underline{k}} = [k_{125}, k_{135}, k_{234}, k_{245}, k_{345}]^T \in \mathbb{R}^5$ and the set of Φ-DRPR is $k_{245} = -k_{125}k_{234},\ k_{345} = -k_{234},k_{135}$ the vector $\underline{k}$ is the Φ-completed vector of $\mathbb{R}^{10}$ which is derived from the Φ-indexed $\tilde{\underline{k}}_D$ of $\mathbb{R}^5$.

Within the framework of exterior algebra, the problems of decentralised pole assignment by state, or output controllers may be formulated as follows:

(i) **Decentralised Pole Assignment by state controllers:** Let $\Phi = \left\{\omega_{i_1}, \cdots, \omega_{i_\mu}\right\}$ be the set of indices of decentralisation, $\hat{P}_c$ be the reduced decentralised controllability Plücker matrix and $\underline{a}$, $\underline{\gamma}$ be the reduced coefficient vectors of the closed-loop and open-loop polynomial, respectively.

(a) Determine whether there exists a solution of the equation

$$\hat{P}_c\, \hat{l}_D = \underline{a} - \underline{\gamma} = \underline{b} \quad \underline{a} \in \mathbb{R}^n \tag{5.87}$$

(b) Let $\hat{L}(\underline{a})$ be the family of solutions of (5.87). Index according to Φ every vector $\hat{\underline{l}}_D \in \hat{L}(\underline{a})$ and determine whether there exists a Φ-indexed vector $\hat{\underline{l}}_D \in \hat{L}(\underline{a})$ which satisfies the Φ-decentralised reduced Plücker relations.

(ii) **Decentralised Pole Assignment by output feedback:** Let $\Phi = \left\{\omega_{i_1}, \cdots, \omega_{i}\right\}$ be the set of indices of decentralisation, $\hat{P}_l$ be the reduced decentralised left fractional Plücker matrix, $\underline{a}$, $\underline{\gamma}$ be the reduced coefficient vectors of the closed-loop and open-loop polynomial, respectively.

(a) Determine whether there exists a solution of the equation

$$\hat{P}_l\, \hat{\underline{k}}_D = \underline{a} - \underline{\gamma} = \underline{b} \quad \text{for V } \underline{a} \in \mathbb{R}^n \tag{5.88}$$

(b) Let $\hat{K}(\underline{a})$ be the family of solutions of (5.88). Index according to Φ every vector $\hat{\underline{k}}_D \in \hat{K}(\underline{a})$ and determine whether there exists a Φ-indexed vector $\hat{\underline{k}}_D \in \hat{K}(\underline{a})$ which satisfies the Φ-indexed reduced Plücker relations.

Chapter 6

FIXED, ALMOST FIXED MODES AND ZEROS IN DECENTRALISED CONTROL

6.0 Introduction

The objective of this chapter is to provide a new characterisation within the exterior algebra framework of fixed modes and zeros of linear dynamic systems, extend these concepts to the case of "almost fixed" and finally apply these concepts to the case of Decentralised control. The desire to extend the above concepts of fixed modes,zeros to those of almost fixed, stems from the critical importance of multivariable poles and zeros in the analysis and synthesis of automatic systems and their mathematical characterisation by the common divisor of a set of polynomials. Because, the usual definitions of multivariable zeros involves the rank test of a special matrix [Ros. 1; McFar & Kar. 1] as well as the uncertainty involved in the mathematical description of dynamic systems, the precise concepts of modes and zeros are of little relevance from a computational and practical point of view. The same observation may be extended to the classical concepts of controllability and observability.

The problem of restoring the various concepts of exact modes and zeros for multivariable systems with uncertainty in the parameters has been studied by Karcanias *et al* [Kar., 1] using the classical centralised control approach. Their main achievement is the extension of the concept of exact zero of a set of polynomials to that of "almost zero". Since the dynamical interpretation of an exact zero involves the minimisation of a certain function, the generalisation of an "almost zero" is again based on the minimum of a certain measure function defined on a set of polynomials.

In section 6.2 fixed modes and zeros for decentralised control are introduced as a natural vehicle of unifying the frequency assignment problems of centralised control to the decentralised case. Following the definition and discussion of almost zeros of a set of polynomials in section 6.2, we extend the notions of fixed modes and fixed zeros to those of almost fixed modes and almost fixed zeros in section 6.4. In section 6.3 we study the invariance properties and provide a new characterisation for the greatest common divisor of a set of polynomials. Using the decentralised Grassmann representative we show that for special families of decentralised systems, for which arbitrary assignment is not possible, the family of decentralised strongly nonassigned systems the concept of almost fixed zero is introduced. A careful examination of the properties of almost fixed zeros shows that for all decentralised controllers, the corresponding decentralised polynomial combinant has at least one zero in a finite radius disk. The general properties and computational aspects of the radii of the "trapping disks" are discussed in section 6.5 and a new necessary condition for stabilisability is given in terms of the almost fixed zeros. New criteria for fixed, almost fixed modes and fixed, almost fixed zeros are given for the case of decentralised control in section 6.7 while in section 6.6 we provide quick tests for the existence of fixed modes and fixed zeros.

6.1 The Decentralised $\mathbb{R}[s]$ - Grassmann Representative, Fixed Modes, Zeros.

The study of D*,2DAP is intimately related to the investigation of the properties of $f_M(s, H)$, when H $\in \mathcal{H}^{\nu}_{r,p}$, or $\mathcal{H}^{\nu}_{q,p}$. Such an investigation provides a new characterisation of fixed modes under DOF(DSF) and fixed zeros under DSD and yields new necessary conditions for the solvability of pole assignment under DOF(DSF), zero assignment under DSD. The nature of fixed zeros of $f_M(s, H)$ for $\mathcal{H}^{\nu}_{r,p}$, or $\mathcal{H}^{\nu}_{q,p}$ will be investigated in this section; this investigation leads to a new criterion for the computation of the fixed zero polynomial $f_M(s, H)$.

Let $M(s) \in \mathbb{R}^{p\times q}[s]$, $p < q$, $\rho_{\mathbb{R}(s)}\{M(s)\} = p$, $r = q - p$, $\mathcal{I}^{\nu}_p$, $\mathcal{I}^{\nu}_r$ be given ν-partitions of p, r respectively, $\mathcal{H}^{\nu}_{r,p}$ the corresponding set of matrices and $\mathfrak{D}(\mathcal{H}^{\nu}_{r,p})$ its decentralisation characteristic. If M(s) has zeros, then we may write $M(s) = Z(s)\, M'(s)$, where $Z(s) \in \mathbb{R}^{p\times p}[s]$ is a greatest left divisor, $|Z(s)| = z(s)$ is the zero polynomial of M(s) and $M'(s) \in \mathbb{R}^{p\times q}[s]$ is a full rank matrix with no zeros; such a factorisation of M(s) will be referred to as a *prime factorisation.* The multivector $\underline{m}^T(s)\wedge$ $\mathbb{R}^{1\times\sigma}[s]$, $\sigma = \binom{q}{p}$ is known as an $\mathbb{R}[s]$-*Grassmann Representative* ($\mathbb{R}[s]$-GR).

Proposition (6.1) [Kar.5]: Let $M(s) = Z(s)\, M'(s)$ be a prime factorisation and let $\mathcal{X}_M = \text{row-span}_{\mathbb{R}(s)}\{M(s)\}$.

(i) If $\underline{m}'^T(s)\wedge$ is the multivector of $M'(s)$, then

$$\underline{m}^T(s)\wedge = z(s)\cdot\underline{m}'^T(s)\wedge \tag{6.1}$$

is also a prime factorisation of $\underline{m}^T(s)\wedge$.

(ii) $\underline{m}(s)^T\wedge$ is a complete invariant of $\mathfrak{X}_M$ modulo $t(s) \in \mathbb{R}[s]$ and $m'^T(s)\wedge$ is a complete invariant of $\mathfrak{X}_M$ modulo $c \in \mathbb{R}$.

(iii) If δ is the Forney dynamical order of $\mathfrak{X}_M$, then $\deg \underline{m}'^T(s)\wedge = \delta$.

□

The coprime polynomial multivector $\underline{m}'^T(s)\wedge$ is also an $\mathbb{R}[s]$-GR of $\mathfrak{X}_M$ and has been called a *canonical-*$\mathbb{R}[s]$*-GR* (C-$\mathbb{R}[s]$-GR).

Definition (6.1): Let $\{m_{\omega_i}(s),\ \omega_i \in Q_{p,q},\ i \in \binom{q}{p}\}$ be the lexicographically ordered set of the Plücker co-ordinates of $\underline{m}(s)^T\wedge$ and let $\mathfrak{D}(\mathcal{H}^{\nu}_{r,p}) = \{m_{i_j} \in Q_{p,q},\ j \in \mu\}$ be the decentralisation characteristic of $\mathcal{H}^{\nu}_{r,p}$. The subvector of $\underline{m}^T(s)\wedge$ obtained by dropping the Plücker co-ordinates that correspond to $\mathfrak{D}(\mathcal{H}^{\nu}_{r,p})$, i.e.

$$\hat{\underline{m}}^T(s)\wedge = [m_{\omega_i}(s),\ \cdots,\ m_{\omega_{i_1-1}}(s),\ m_{\omega_{i_1+1}}(s),\ \cdots,\ m_{\omega_{i_\mu-1}}(s),\ m_{\omega_{i_\mu+1}}(s),\ \cdots$$

$$\cdots,\ m_{\omega_{i_\sigma}}(s)] \in \mathbb{R}^{1\times(\sigma-\mu)}[s] \tag{6.2}$$

will be called the $\mathcal{H}^{\nu}_{r,p}$*-decentralised-*$\mathbb{R}[s]$*-GR* ($\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR) of M(s). □

The $\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR of M′(s), $\hat{\underline{m}}'(s)\wedge$, is defined in a similar manner. Clearly, by eqn. (6.1) we have that

$$\hat{\underline{m}}^T(s)\wedge = z(s)\cdot\hat{\underline{m}}'^T(s)\wedge \tag{6.3}$$

Note that although $\underline{m}'^T(s)\wedge$ is coprime, the corresponding $\hat{\underline{m}}'^T(s)\wedge$ vector is not necessarily coprime. Thus, let $d(s) \in \mathbb{R}[s]$ be a greatest left divisor of $\hat{\underline{m}}'^T(s)\wedge$. We may write $\hat{\underline{m}}'^T(s)\wedge = d(s)\cdot\bar{\underline{m}}^T(s)\wedge$ and thus

$$\hat{\underline{m}}^T(s)\wedge = z(s)\cdot d(s)\cdot\bar{\underline{m}}^T(s)\wedge \tag{6.4}$$

$\bar{\underline{m}}^T(s)\wedge$ will be called a *canonical* $\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR (C-$\mathcal{H}^{\nu}_{r,p}$-D$\mathbb{R}[s]$-GR) and d(s) an $\mathcal{H}^{\nu}_{r,p}$-*decentralisation polynomial* ($\mathcal{H}^{\nu}_{r,p}$-DP) of M(s).

Proposition (6.2): The C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR $\bar{\underline{m}}^T\wedge$ and $\mathcal{H}^{\nu}_{r,p}$-DP of M(s) are invariant modulo $c \in \mathbb{R}$ of all $\mathcal{H}^{\nu}_{r,p}$ type D-DAPs defined on polynomial basis matrices $\overline{M}(s)$ of $\mathfrak{X}_M$. □

This result readily follows from the invariance of C-$\mathbb{R}$[s]-GR, $\underline{m}'^{T}(s)\wedge$. Note that different $\mathfrak{X}_M$ rational vector spaces may have the same $\tilde{\underline{m}}^{T}(s)\wedge$ and d(s) (may have the same subset of Plücker co-ordinates apart from those defined by the $\mathfrak{D}(\mathcal{H}^{\nu}_{r,p})$ sequences).

In the following we shall denote $\tilde{\underline{m}}^{T}(s)\wedge \triangleq \underline{p}^{T}(s) \in \mathbb{R}^{1\times(\sigma-\mu)}[s]$. If ψ = deg d(s), then deg $\underline{p}(s) = \delta - \psi \triangleq \delta'$ and $\underline{p}(s) = P\,\underline{e}_{\delta'}(s)$, where $P \in \mathbb{R}^{(\sigma-\mu)\times(\delta'+1)}$ and $\underline{e}_{\delta'}(s) = [1, s, \cdots, s^{\delta'}]^{T}$. The real matrix P, uniquely defines $\underline{p}(s)$ and will be referred to as the $\mathcal{H}^{\nu}_{r,p}$-*decentralised Plücker matrix* ($\mathcal{H}^{\nu}_{r,p}$-DPM) of M(s). By Proposition (6.2) we have:

Remark (6.1): The $\mathcal{H}^{\nu}_{r,p}$-DPM P is invariant (mod c, c $\in \mathbb{R}$) for all $\mathcal{H}^{\nu}_{r,p}$ type D–DAPs defined on polynomial basis matrices $\overline{M}(s)$ of $\mathfrak{X}_M$. □

If $\hat{\underline{k}} = \hat{\underline{h}}\wedge$ is the reduced vector of $\mathcal{H} \in \mathcal{H}^{\nu}_{r,p}$, then the combinant $f_M(s, H)$ may be expressed as

$$f_M(s, H) = <\underline{m}(s)\wedge, \underline{h}\wedge> = z(s)\, d(s) <\tilde{\underline{m}}(s)\wedge, \hat{\underline{h}}\wedge> = z(s)\, d(s)\, f^{*}_M(s, H) \tag{6.5}$$

$f^{*}_M(s, H) \triangleq \underline{p}(s)^{T}\, \hat{\underline{k}}$ will be referred to as the $\mathcal{H}^{\nu}_{r,p}$-*canonical combinant* and it is completely defined by the $\mathcal{H}^{\nu}_{r,p}$-DPM P. The factorisation (6.5) is crucial in the study of fixed zeros of $f_M(s, H)$, as well as in the investigation of the zero location of the nonfixed zeros.

Definition (6.2): The $\mathcal{H}^{\nu}_{r,p}$-type D–DAP defined on M(s) has a *fixed zero* at s = $\lambda \in \mathbb{C}$, if for $\forall$ H $\in \mathcal{H}^{\nu}_{r,p}$ the combinant $f_M(s, H)$ has a fixed zero at s = λ. □

The characterisation of the set of FZs of D–DAP is considered next. To establish the main result of this section we need the following lemma.

Lemma (6.1) : The Grassmann variety Ω(p,q) lies in the projective space $\mathbb{P}^{\sigma-1}$, $\sigma = \binom{q}{p}$ and not in a space of lower dimension. □

Theorem (6.1): Necessary and sufficient condition for s = $\lambda \in \mathbb{C}$ to be a FZ of the $\mathcal{H}^{\nu}_{r,p}$-D–DAP defined on M(s), is that λ is a zero of f(s) = z(s) d(s).

Proof: By eqn.(6.5) it follows that every zero of z(s) d(s) is a zero of $f_M(s, H)$, $\forall$ H $\in \mathcal{H}^{\nu}_{r,p}$ and this proves the sufficiency. To prove the necessity, consider the combinant $f^{*}_M(s, H)$ generated by the coprime polynomial vector $\underline{p}(s)$ and assume that $f^{*}_M(s, H)$ has a fixed zero λ for $\forall\, \hat{\underline{k}} \in \mathbb{R}^{\sigma-\mu}$. By the coprimeness assumption for $\underline{p}(s)$, $\underline{p}(\lambda) = P\,\underline{e}_{\delta'}(\lambda) \neq 0$ and thus

$$f^*_M(s, H) = \hat{\underline{k}}^T P\,\underline{e}_{\delta'}(\lambda) = 0, \quad \hat{\underline{k}} \in \mathbb{R}^{\sigma-\mu} \tag{6.6}$$

implies that $\hat{\underline{k}}^T \in \mathcal{N}_l\{\underline{p}(\lambda)\}$, where dim $\mathcal{N}_l\{\underline{p}(\lambda)\} = \sigma - \mu - 1$. By expanding the reduced vector $\hat{\underline{k}}$ to the $\mathcal{D}(\mathcal{H}^{\nu}_{r,p})$-structured vector $\underline{k}$ (by reintroducing the set of zero co-ordinates that correspond to $\mathcal{D}(\mathcal{H}^{\nu}_{r,p})$), it follows that $\underline{k}$ must satisfy the set of QPRs defining the Grassmann variety Ω(p,q), as well as the conditions (6.6), which define a linear space of $\mathbb{P}^{\sigma-1}$; since those two properties must hold true for all $\mathcal{D}(\mathcal{H}^{\nu}_{r,p})$-structured vectors $\underline{k}$, it follows that Ω(p,q) is contained in a linear space of $\mathbb{P}^{\sigma-1}$ of lower dimension than $\mathbb{P}^{\sigma-1}$ which contradicts the fundamental property of Ω(p,q) established by Lemma (6.1). □

The polynomial $f(s) \triangleq z(s)\,d(s)$ will be referred to as the $\mathcal{H}^{\nu}_{r,p}$-*fixed polynomial* ($\mathcal{H}^{\nu}_{r,p}$-FP). Given that for any $\underline{k} \in \mathbb{R}^{\sigma}|^{\nu}_{r,p}$ the first co-ordinate is always equal to 1 we have:

Remark (6.2): The $\mathcal{H}^{\nu}_{r,p}$-FP f(s) of the $\mathcal{H}^{\nu}_{r,p}$-D-DAP defined on $M(s) = [M_1(s), M_2(s)] \in \mathbb{R}^{p\times q}[s]$, $M_1(s) \in \mathbb{R}^{p\times p}[s]$, divides $|M_1(s)|$. □

The $\mathcal{H}^{\nu}_{r,p}$-fixed polynomial is defined in a similar manner; however, the property described by Remark (6.2) is not any longer valid. Our attention is focused next on the zero distribution properties of the $\mathcal{H}^{\nu}_{r,p}$-canonical combinant $f^*_M(s, H)$.

6.2 Almost Zeros of a Set of Polynomials

As it has already been stated, due to parameter uncertainty or round off computational errors, the notion of exact zeros is of little relevance to engineering system models. This led to the definition of the extended notion of almost zeros.

6.2.1 Almost Zero Equivalence [Kar.3]

Let $\mathcal{P} = \{p_i(s)\colon p_i(s) = a^i_0 + a^i_1 s + a^i_2 s^2 + \cdots + a^i_{d_i-1} s^{d_i-1} + a^i_{d_i} s^{d_i}\}$ be a set of polynomials and let $d = \max\{d_i,\ i \in \underline{m}\}$. We may always associate a polynomial vector $\underline{p}(s)$, $p(s) \in \mathbb{R}^m(s)$, where

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_k(s) \\ \vdots \\ p_m(s) \end{bmatrix} =$$

$$= \begin{bmatrix} a_0^1 & a_1^1 & \cdots & a_{d_1}^1 & 0 & & & \cdots & 0 \\ a_0^2 & & & & a_{d_2}^2 & 0 & & \cdots & 0 \\ & & & & & & & & \\ a_0^k & & & & & & & & a_{d_K}^k \\ & & & & & & & & \\ a_0^m & & & & & & a_{d_m}^m & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ \\ \\ s^d \end{bmatrix}$$

$$= P_d \, \underline{e}_d(s), \quad P_d \in \mathbb{R}^{m \times (d+1)} \tag{6.7}$$

The polynomial vector $\underline{p}(s)$ is characterised by the matrix $P_d \in \mathbb{R}^{m \times (d+1)}$, which is defined as the basis matrix of $\underline{p}(s)$ and the polynomial vector $\underline{e}_d$, $\underline{e}_d(s) \in \mathbb{R}^{(d+1)}(s)$.

When $s \in \mathbb{C}$, $\underline{p}(s)$ defines a vector valued analytic function with domain $\mathbb{C}$ and co-domain $\mathbb{C}^m$; the norm of $\underline{p}(s)$ is defined as a positive definite real function with domain $\mathbb{C}$ as

$$\left\| \underline{p}(s) \right\| = \sqrt{(\underline{p}^T(s^*) \, \underline{p}(s))} = \sqrt{(\underline{e}_d^T(s^*) \, P_d^T \, P_d \, \underline{e}_d(s))} \tag{6.8}$$

where s^* is the complex conjugate of s and $(\, . \,)^T$ denotes transposition. Note, that if $q(s) = s+a$ is a common factor of the polynomial $p_i(s)$, $i \in \underline{m}$, then for all $i \in \underline{m}$ $p_i(-\alpha) = 0$, $p(-\alpha) = \underline{0}$ and thus $\left\| \underline{p}(-\alpha) \right\| = 0$. This observation leads to the following definition.

Definition (6.3): Let $\mathcal{P}$ be a set of polynomials of $\mathbb{R}(s)$, $p(s)$ be the associated polynomial vector and let $\phi(\sigma,w) \triangleq \left\| \underline{p}(s) \right\|$, where $s = \sigma + jw \in \mathbb{C}$. An ordered pair (z_k, ϵ_k), $z_k \in \mathbb{C}$, $\epsilon_k \in \mathbb{R}$ and $\epsilon_k \geq 0$, defines an almost zero of $\mathcal{P}$ at $s = z_k$ and of order ϵ_k, if $\phi(\sigma,w)$ has a minimum at $s = z_k$ with value ϵ_k.

From the set $\mathcal{Z} = \{(z_k, \epsilon_k), k \in \underline{r}\}$ of almost zeros of $\mathcal{P}$, the element $(\tilde{z}, \tilde{\epsilon})$ for which $\tilde{\epsilon} = \inf \{\epsilon_k, k \in \underline{r}\}$ is defined as the *prime almost zero of* $\mathcal{P}$.

It is clear that if $\mathcal{P}$ has an exact zero, then the corresponding ϵ is zero. The magnitude of ϵ at an almost zero s = z provides an indication of how well z may be considered as an appropriate zero of $\mathcal{P}$; we could note however, that ϵ depends on the scaling of the polynomials $\underline{p}_i(s)$ in $\mathcal{P}$ by a constant c, $c \in \mathbb{R}-\{0\}$.

The set $\mathcal{P}$ may be standardised in various ways. We shall adopt the following standardisation: Let $\underline{p}(s)$ be written as

$$\underline{p}(s) = \underline{p}_0 + p_1 s + \underline{p}_{d-1}s^{d-1} + \underline{p}_d s^d, \quad \underline{p}_i \in \mathbb{R}^m;\ i = 0, 1, \cdots, d \tag{6.9}$$

The polynomial vector $\underline{p}(s)$ will be said to be monic if $\|\underline{p}_d\| = 1$. In the case where $\|\underline{p}_d\| \neq 1$, $\underline{p}(s)$ may become monic by dividing all $p_i(s)$ by $\|\underline{p}_d\|$.

The polynomial vector $\underline{p}(s)$ may also be expressed around $s = \alpha$, $\alpha \in \mathbb{C}$ by a Taylor type expansion as

$$\underline{p}(w) = \underline{b}_0 + w\underline{b}_1 + \cdots + w^{d-1}\underline{b}_{d-1} + w^d\underline{b}_d = [B_d]\,\underline{e}(w) \tag{6.10}$$

$$w = s-\alpha, \quad b_i \in \mathbb{C}^m$$

$$\underline{b}_0 = \underline{p}(\alpha) \ \text{ and } \underline{b}_i = \frac{1}{i!}\left[\frac{d^i\{\underline{p}(s)\}}{d\, s^i}\right]_{s=a} \quad i = 1, 2, \cdots, d+1 \tag{6.11}$$

For all $\alpha \in \mathbb{C}$ the highest vector coefficient $\underline{b}_d$ in (6.10) is equal to the highest vector coefficient $\underline{p}_d$ of equation (6.9).

If $s = \alpha$ is an almost zero of $\mathcal{P}$ then the corresponding order ϵ is equal to $\|\underline{b}_0\|$ of equation (6.11).

6.2.1.1 The notion of Normal Equivalence.

Let $\mathcal{P}'$ $\mathcal{P}''$, be two sets of polynomials in $\mathbb{R}(s)$ and let $\underline{p}'(s) = P'\underline{e}(s)$, $p''(s) = P''\underline{e}(s)$ be their respective polynomial representation, where $P' \in \mathbb{R}^{r\times(d+1)}$, $P'' \in \mathbb{R}^{k\times(d+1)}$ are the corresponding basis matrices. The sets $\mathcal{P}'$, $\mathcal{P}''$, will be said to be *normally equivalent* (NE), and shall be denoted as $P' \mathcal{E} P''$, if there exists an orthogonal matrix Q such that

$$P' = Q \begin{bmatrix} P'' \\ O_{r-k} \end{bmatrix}, \quad Q \in \mathbb{R}^{r \times r}, \quad \text{if } r \geq k$$

or

$$\begin{bmatrix} P' \\ O_{k-r} \end{bmatrix} = Q\ P'', \ Q \in \mathbb{R}^{k \times k}, \text{ if } k \geq r \tag{6.12}$$

□

It should be noted that such an equivalence is defined between sets of polynomials of $\mathbb{R}(s)$ which have the same degree, but not necessarily the same number of polynomials.

Proposition (6.3): The almost zero structure of a set of polynomials $\mathcal{P}$ which is defined by $\|\underline{p}(s)\|$, is invariant under normal equivalence. □

An important consequence of this proposition is that the almost zero structure of $\mathcal{P}$ may be studied on any set $\mathcal{P}'$ of the normal equivalence class, $\mathcal{E}(\mathcal{P})$, of $\mathcal{P}$.

6.2.1.2 Singular Value Decomposition of the Basis Matrix

Let $P \in \mathbb{R}^{m \times n}$ and rank $(P) = \rho \leq \min(m,n)$. Then there exist two orthogonal matrices Q, R, of order m, n respectively such that

$$P = Q \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} R^T = [Y, Y'] \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U^T \\ U'^T \end{bmatrix} = Y\ \Gamma\ U^T \tag{6.13}$$

where $\Gamma = \text{diag}\ \{\gamma_1, \cdots, \gamma_\rho\}$, $\gamma_i > 0$ and $Y^T Y = I_\rho = U^T U$.

The set of γ_i^2, $i \in \rho$, are common positive eigenvalues of $P^T P$ and $P\ P^T$, the columns of Y are the eigenvectors of $P^T P$ corresponding to γ_i^2 and the columns of U are the eigenvectors of $P^T P$ corresponding to γ_i^2.

The above result defines the singular value decomposition (SVD) of P. In the case where the γ_i are distinct, then Y and U are uniquely defined.

Thus, for example, it can be shown that P and $\Gamma\ U^T$ have the same almost zero structure.

6.2.2 The Location of Prime Almost Zeros: The Prime Disc

If $\overline{\gamma}$ and γ are the maximum and minimum singular values of P respectively, then the ratio $\overline{\gamma}/\gamma = \theta$ is defined as *the condition number of P.* It is important to know that the general shaping of the almost zeros of a set of polynomials depends on θ and the maximum degree d of the polynomials.

Let $\mathcal{P}$ be a set of polynomials of R(s), $P_d \in \mathbb{R}^{m\times(d+1)}$ be the basis matrix of $\mathcal{P}$ and let $\overline{\gamma}$, γ be the maximal, minimal singular values of P_d, respectively. Then

$$\gamma \|\underline{e}_d(s)\| < \Phi(\sigma, w) < \overline{\gamma}\,\|\underline{e}_d(s)\| \tag{6.14}$$

So if (z, ϵ^2) is a minimum of $\|\underline{p}(s)\|^2$, $s \in \mathbb{C}$, $z \in \mathbb{C}$, $\epsilon > 0$, then (z, ϵ) is a minimum of $\|\underline{p}(s)\|$ and *vice versa.* Thus the prime almost zero of $\mathcal{P}$ is always within the circle centred at the origin of the complex plane and with radius ρ, defined as the *unique positive real solution* of the equation

$$1 + r^2 + \cdots + r^{2d} = \overline{\gamma}^2/\gamma^2 = \theta^2 \tag{6.15}$$

The disc $[0, \rho]$ within which the prime almost zero lies, will be referred to as the prime disc of $\mathcal{P}$. The radius ρ of the disc is defined by the degree d of the condition number θ of $\mathcal{P}$. Clearly ρ is an invariant of $\mathcal{E}(\mathcal{P})$. The following general results may be stated for the radius ρ.

If d is the degree and θ is the condition number of $\mathcal{P}$ then the radius $\rho = f(d, \theta)$ of the prime disc is a uniquely defined function of d and θ and it has the following properties:

(i) the radius ρ is invariant under the scaling of the polynomials of $\mathcal{P}$ by the same non-zero constant c.

(ii) the radius ρ is monotonically decreasing function of d and $1/\theta$.

(iii) the radius ρ is within the following intervals:

- (a) if $d+1 > \theta^2$, then $0 < \rho < 1$
- (b) if $d+1 > \theta^2$, then $1 < \rho < \theta^{1/2}$
- (c) if $d+1 = \theta^2$, then $\rho = 1$.

Thus scaling all polynomials of $\mathcal{P}$ by the same non-zero constant c implies multiplication of all singular values of P_d by $|c|$ and thus does not affect the condition number θ; because ρ is a function of θ and d only, the invariance of ρ follows.

6.2.2.1 Conditioned Sets of Polynomials

The conditioning of the polynomials plays an important role in determining the position of the prime almost zero. In fact, the prime almost zero is always in the vicinity of the origin of the complex plane. The uncertainty in its exact position is measured by the radius of the prime disc. Well conditioned sets of polynomials $\mathcal{P}$ (i.e. $\theta \simeq 1$) have a very small radius prime disc, even for very small values of the degree d. Badly conditioned sets of polynomials $\mathcal{P}$ (i.e. $\theta >> 1$) have very large radius disc, even for large values of the degree d .

Thus, necessary, but not sufficient condition, for the prime almost zero of a set $\mathcal{P}$ to be away from the origin of the complex plane, is that $\mathcal{P}$ is badly conditioned and its degree is relatively small.

Examples demonstrating the effects of the condition number on the distribution of almost zeros will be given later.

6.2.3 Computing the Almost Zero

The results on the prime disc also establish the existence of at least one almost zero for every set of polynomials.

Let $\mathcal{P}$ be a set of polynomials of $\mathbb{R}[s]$, $\underline{p}(s) = P_d\, \underline{e}_d(s)$ be the polynomial vector associated with $\mathcal{P}$ and $P_d \in \mathbb{R}^{m\times(d+1)}$ be the corresponding basis matrix. Necessary conditions for $z \in \mathbb{C}$ to be an almost zero of $\mathcal{P}$ are

$$\underline{e}_d^T(z^*)\, \Delta^T\, P_d^T\, P_d\, \underline{e}_d(z) = 0$$

and (6.16)

$$\underline{e}_d^T(z^*)\, P_d^T\, P_d\, \Delta\, \underline{e}_d(z) = 0$$

where

$$\Delta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & d & 0 \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)} \tag{6.17}$$

In fact it can be seen that the two conditions (6.16) are not independent, the first is just the complex conjugate of the second and vice versa. For the case where $z \in \mathbb{R}$, the two conditions coincide. These conditions are necessary but not sufficient.

Sufficient conditions for a solution $z = \sigma + jw \in \mathbb{C}$ of conditions (6.16) to be almost zero are:

$$\underline{e}_d^T(z^*) \{ (\Delta^T)^2 P_d^T + 2\Delta^T P_d^T P_d \Delta + P_d^T P_d \Delta^2 \} \underline{e}_d(z) > 0 \tag{6.18}$$

and

$$2 \{ \underline{e}_d^T(z^*) \Delta^T P_d^T P_d \Delta \underline{e}_d(z) \}^2 > \{ \underline{e}_d^T(z^*) (\Delta^T)^2 P_d^T P_d \underline{e}_d(z) \}^2 + $$
$$+ \{ \underline{e}_d^T(z_*) P_d^T P_d \Delta^2 \underline{e}_d(z) \}^2 \tag{6.19}$$

These results may be used for the analytic computations of the almost zeros. In practice, such analytic computations are tedious, therefore a numerical technique, of the hill climbing type, has to be used for the computation of almost zeros. A programme was developed that plots the norm of each of the polynomial vectors $\underline{p}(s)$ as well as the surface of $\|p(s)\|$ over a selected rectangle of the complex plane where $\Phi(\sigma, w)$ has a minimum. A standard minimisation routine is then used to compute the almost zero . In the case where we are interested in computing the prime almost zero, we compute the radius of the prime disc and then we select a point inside the prime disc as the initial guess of the hill climbing algorithm[Gia.1].

Example (6.1): Let the set of polynomials $\mathcal{P}$ be defined by the polynomial vector $\underline{p}(s)$

$$\underline{p}(s) = \begin{bmatrix} 1.2 + 3.1s + 2s^2 + s^3 \\ 3 + 2.2s + s^2 + 2.1s^3 \end{bmatrix} = \begin{bmatrix} 1.2 & 3.1 & 2 & 1 \\ 3 & 2.2 & 1 & 2.1 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

Again using the same computer programmes we perform the following calculations:

The singular values of P are

$$\overline{\gamma} = 4.7836059 \qquad \underline{\gamma} = 0.5358303$$

The radius of the prime disc is

$$1 + r^2 + r^4 = (\overline{\gamma} / \underline{\gamma})^2 = 79.699619$$

so

$$\rho = 2.895749$$

The almost zero is located at

$$(0.002234, \quad 1.038991)$$

and its norm

$$\epsilon = 9.019537$$

Example (6.2): Let the set of polynomials be defined by the polynomial vector $\underline{p}(s)$

$$\underline{p}(s) = \begin{bmatrix} 1.1 + 2s + s^2 + 3s^3 \\ -.8 + 2.5s + s^3 \\ 1.7 + 2s + s^2 + .5s \end{bmatrix} = \begin{bmatrix} 1.1 & 2 & 1 & 3 \\ -.8 & 2.5 & 0 & 1.0 \\ 1.7 & 2.0 & 1.0 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

The singular values of P are:

$$\overline{\gamma} = 3.425584 \qquad \gamma = 1.55466 \qquad \gamma_0 = 1.86547$$

these singular values have been calculated, using a computer package such as MATLAB and then we can use the same computer programmes,[Gia.1], to find the radius of the prime disc.

$$1 + r^2 + r^4 + r^6 = (\overline{\gamma} / \gamma)^2 = 3.3715116$$

so

$$\rho = 0.98204$$

The almost zero is found, using the computer programme to be located at

$$(-0.029640, \quad 0.834804)$$

Now we scale the polynomials and observe the effect on the almost zero and the plot of the vector norm.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.1 & 2 & 1 & 3 \\ -.8 & 2.5 & 0 & 1.0 \\ 1.7 & 2.0 & 1.0 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

the almost zero is at

$$(-0.010193, \ 1.093143)$$

Example (6.3): This example is to verify proposition (6.1). Let $\underline{p}(s)$ be a polynomial vector with P as its basis matrix. Y and U are defined as the eigenvector matrices of P P^T and P^T P respectively. The purpose is the show that P and Γ U^T have the same almost zero pattern.

$$\underline{p}(s) = \begin{bmatrix} 1 + 2s + s^2 \\ 3 + 5s \\ 2 + 6s + s^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 5 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$$

The singular values are:

$$\overline{\gamma} = 7.4070200 \qquad \gamma_0 = 5.1120706 \qquad \gamma = 0.0528189$$

$$1 + r^2 + r^4 = (\overline{\gamma} / \gamma)^2 = 19665.651$$

So the radius of the disc is

$$\rho = 11.82081.$$

$$P = \begin{bmatrix} 6 & 8 & 15 \\ 8 & 34 & 11 \\ 15 & 11 & 41 \end{bmatrix} \qquad P^T P = \begin{bmatrix} 14 & 14 & 18 \\ 14 & 40 & 8 \\ 18 & 8 & 27 \end{bmatrix}$$

$$Y = \begin{bmatrix} .94 & .33 & -.07 \\ -.12 & .54 & .84 \\ -.31 & .78 & -.54 \end{bmatrix} \qquad U = \begin{bmatrix} .84 & .47 & .26 \\ -.19 & .72 & -.67 \\ -.5 & .51 & .7 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} .05 & 0 & 0 \\ 0 & 7.41 & 0 \\ 0 & 0 & 5.11 \end{bmatrix} \qquad \Gamma\, U^T = \begin{bmatrix} .04 & -.1 & -.03 \\ 3.48 & 5.34 & 3.78 \\ 1.33 & -3.42 & 3.50 \end{bmatrix}$$

The almost zero of $\mathcal{P}$ is at:

$$\sigma = -0.190770 \qquad w = 0 \qquad \epsilon = 11.349000$$

The almost zero of $\Gamma\, U^T$ is at:

$$\sigma = -0.1904 \qquad w = 0 \qquad \epsilon = 11.219206$$

Thus we note that P and $\Gamma\, U^T$ have the same almost zero structure.

It can also be shown that the scaling of both matrices by the same factor C where $C \in \mathbb{R} - \{0\}$ does not affect the position of the almost zero, however, the new norm becomes the product of the factor C and the previous norm.

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} P = \begin{bmatrix} 10 & 20 & 10 \\ 30 & 0 & 50 \\ 20 & 60 & 10 \end{bmatrix}$$

almost zero at $z = -0.191024 + j0$.

However, the scaling of the two matrices by different factors results in different almost zero distributions.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} P = \begin{bmatrix} 1 & 2 & 1 \\ 15 & 0 & 25 \\ 6 & 18 & 3 \end{bmatrix}$$

almost zero $z = -0.082927 - j0.602197 \quad \epsilon = 67.888891$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Gamma\, U^T = \begin{bmatrix} .04 & -.1 & -.3 \\ 17.4 & 26.7 & 18.9 \\ 3.99 & -10.26 & 10.74 \end{bmatrix}$$

almost zero at $z = -0.339018 - j0.298271 \quad \epsilon = 82.678486$. □

6.2.4 The Scaling of the Polynomials

The foregoing example reveals that the almost zero structure is not invariant under scaling. The sets of polynomials $\mathcal{P} = \{p_i(s),\ i \in \underline{m}\}$ and $\mathcal{P}' = \{k_i\, p_i(s),\ k_i \in \mathbb{R} - \{0\},\ i \in \underline{m},\ k_i \neq k_j$ for at least two $i, j \in \underline{m}\}$ do not belong to the same normal equivalence class. If $k_i = k$ for all $i \in \underline{m}$, then $\Phi(\sigma, w) = |k|\ \Phi(\sigma, w)$ and $\mathcal{P}$, $\mathcal{P}'$ have the same almost zero distribution. If (z_i, ϵ_i), (z'_i, ϵ'_i) are almost zeros of $\mathcal{P}$, $\mathcal{P}'$, respectively for which $z_i = z'_i$, then $\epsilon'_i = |k|\ \epsilon_i$.

6.2.5 The Pinning of Zeros of the Combinants of P by the Almost Zeros

Let $\mathcal{P} = \{ p_i(s),\ p_i(s) \in \mathbb{R}[s],\ i \in \underline{m} \}$, $\underline{p}(s) = P_d\, \underline{e}_d(s)$ be a polynomial vector representative of $\mathcal{P}$, $P_d \in \mathbb{R}^{m\times(d+1)}$, and let $\underline{k} \in \mathbb{R}^{m\times 1}$. The polynomial

$$f(s, \underline{k}) = \underline{k}^T\, \underline{P}_d\, \underline{e}_d(s) = \sum_{i=1}^{m} k_i\, p_i(s) \tag{6.20}$$

is defined as a combinant of $\mathcal{P}$.

The zeros of all combinants $f(s, c\underline{k})$, $c \in \mathbb{R} - \{0\}$, are the same and thus, whenever we are interested in the properties of zeros of combinants we may assume the $\|k\| = 1$. If $s = z$ is an exact zero of $\mathcal{P}$, then $s = z$ is also an exact zero of $f(s, \underline{k})$ for all vectors $\underline{k}$. Under certain conditions, if $\mathcal{P}$ has an almost zero at $s = z$, then $f(s, \underline{k})$ has a zero in the vicinity of z for all choices of the parameter vector $\underline{k}$.

6.3 An Algorithmic Procedure for the Computation of Fixed Mode, Zero Polynomials

Using the concept of extended-$\mathbb{R}$-equivalence (E-$\mathbb{R}$-E) and the shifting operations on real matrices introduced by Karcanias a systematic new method for the computations of the greatest common divisor(g.c.d) of a set of polynomials is suggested which may be applied for the computation of the fixed mode, zero polynomials

6.3-1. Extended-R-Equivalence

Let $\mathcal{P}_{m\times d} = \{ p_i(s) : p_i(s)\in\mathbb{R}[s]\ ,\ i\in m\ ,\ d_i = \deg p_i(s)\ ,\ \ d = \max\ (d_i)\ \}$ be a set of polynomials of $\mathbb{R}[s]$; m , d will be referred to as the dimension and the degree of $\mathcal{P}_{m,d}$ respectively . Let us define the sets $\{\mathcal{P}_d\} = \{\mathcal{P}_{m_i,d}\ ,\ m_i\in Z^+,\ d\in Z^+\}$ and $<\mathcal{P}_d> = \{ P_{m_i,d'}\ ,\ m_i\in Z^+\ ,\ d'<d,\ d\in Z^+ \}$. Given a set of polynomials we define the polynomial vector

$$\underline{p}_m^T(s) = [\ p_1(s), \ldots , p_m(s)\ ] = P_m \underline{e}_d(s)\ ,\ \underline{e}_d^T(s) = [\ 1\ \ s\ \ s^2\ \ldots\ s^d\ ]\ ,\ P_m \in \mathbb{R}^{m \times (d+1)} \tag{6.21}$$

will be referred to as the *vector representative* (v,r) and the matrix P_m as the *basis matrix* (b,m) of $\mathcal{P}_{m,d}$. The basis matrix , P_m , may also be expressed as

$$P_m = [\ O_{m,c}\ ;\ \underline{p}_{c+1} \cdots \quad \underline{p}_{d+1}\ ]\ ,\ \underline{p}_i \in \mathbb{R}^m\ ,\ i \in \mathrm{m}\ ,\ \underline{p}_{c+1} \neq 0 \tag{6.22}$$

where $c \in \{0,1,2, \ldots \}$. The integer c will be referred to as the *order* of $\mathcal{P}_{m,d}$. The set $\mathcal{P}_{m,d}$ will be called *proper* if c=0 otherwise it will be called *nonproper*.

Let $\mathcal{P}_{m,d} \in \{\ \mathcal{P}_d\ \}$, $m^* \in \mathbb{Z}^+$, $m^* \geq m$. An m^**-description* , $\mathcal{P}_{m,d}$ is defined by expanding $\mathcal{P}_{m,d}$ by m^*-m zeros and the corresponding basis matrix P_{m*} will be referred to as an m^*-extension of P_m and has the form

$$P_{m*} = \begin{bmatrix} P_m \\ \\ O_{m*-m,d+1} \end{bmatrix} \in \mathbb{R}^{m* \times (d+1)} \tag{6.23}$$

Definition (6.4): Let $\mathcal{P}^1_{m_1,d}, \mathcal{P}^2_{m_2,d} \in \{\ \mathcal{P}_d\ \}$, $m^* = \max\ (m_1,m_2)$ and let $P^i_{m^*}$, i= 1, 2 be the basis matrices. The sets $\mathcal{P}^1_{m_1,d}$, $\mathcal{P}^2_{m_2,d}$ will be called *Extended - $\mathbb{R}$ - Equivalent (E-$\mathbb{R}$-E)* and will be denoted by $\mathcal{P}^1_{m_1,d} \quad \mathcal{E}\ \mathcal{P}^2_{m_2,d}$ if there is exists a matrix $Q \in \mathbb{R}^{m^*,m^*}$, $|Q| \neq 0$, such that

$$P^2_{m^*} = Q\ P^1_{m^*} \tag{6.24}$$

□

It can be shown that $\mathcal{E}(\mathcal{E}_n)$ is an equivalence relation on $\{\ \mathcal{P}_d\ \}$ and the corresponding equivalence class of $\mathcal{P}_{m,d} \in \{ \mathcal{P}_d \}$ will be denoted by $\mathcal{E}(\mathcal{P}_{m,d})$.

6.3-2: Hermite forms

By performing elementary operations a square matrix can be transformed to a canonical form such as the *triangular form.*

Let $P_m \in \mathbb{R}^{m,d+1}$ and $\rho(P_m) = r \leq \min(m,d+1)$. There exists $Q \in \mathbb{R}^{m,m}$, $|Q| \neq 0$ such that

$$P_m^H = QP_m = \begin{bmatrix} P_r^H \\ \\ Q_{m-r,d+1} \end{bmatrix}, \quad P_r^H \in \mathbb{R}^{r \times d+1} \tag{6.25}$$

where $P_r^H = (h_{ij})$ is a matrix known as the *left-echelon-form* (*LEF*) of P_m having the following structure

$$P_r^H = \begin{array}{c} \begin{array}{ccccccccccccc} & & n_1 & & & & n_2 & & & n_r & & & \end{array} \\ \begin{bmatrix} 0 \dots 0 & 1 & * \dots * & 0 & * \dots * & 0 & * \dots * \\ 0 \dots 0 & 0 & 0 \dots 0 & 1 & * \dots * & 0 & * \dots * \\ 0 \dots 0 & 0 & 0 \dots 0 & 0 & * \dots * & 0 & * \dots * \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 \dots 0 & 0 & 0 \dots 0 & 0 & 0 \dots 0 & 1 & * \dots * \end{bmatrix} \end{array} \tag{6.26}$$

The properties associated with the LEF are as follows

(i) $\rho(P_r^H) = r$

(ii) There is a sequence of integers $n_1, n_2, \dots, n_r, 1 \leq n_1 < n_2 < \dots < n_r \leq d+1$ such that $h_{ij}=0$, $j=1, \dots, n_i - 1$, $h_{tn_i} = 0$, $t=1, \dots, i-1, i+1, \dots, r$.

(iii) The rest of $h_{ij} \in \mathbb{R}$ and they are uniquely defined.

□

Example (6.4) : Let

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 6 & 2 & 0 & 0 & 4 \\ 0 & 1 & 3 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 2 & 1 & 2 \end{bmatrix}$$

By performing the following row operations

(i) Interchange the first and the third rows

(ii) Substract two times the first row from the second row

(iii) Substract the first row from the fourth row

we transform the A matrix into the row equivalent matrix

$$B=\begin{bmatrix} 0 & 1 & 3 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Finally after further elementary operations we reduce the B matrix into the following Hermite form

$$H=\begin{bmatrix} 0 & 1 & 3 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this example it can seen that $n_1 = 2$, $n_2 = 5$, $n_3 = 6$, and $r = 3$

□

It also has been shown by Karcanias that P_r^H is a complete invariant of $\mathcal{E}(\mathcal{P}_{m,d})$. An equivalent complete invariant for $\mathcal{E}(\mathcal{P}_{m,d})$ is defined by the space $\mathcal{R}(P_m)$ = row-span $\{\mathcal{P}_m\}$. Also the set of polynomials $\mathcal{P}^H_{r,d} \in \{\mathcal{P}_d\}$ defined by the basis matrix(b.m) P_r^H (the LEF) is a canonical form for the $\mathcal{E}(\mathcal{P}_{m,d})$.

Another important function defined on any $\mathcal{P}_{m,d} \in \{\mathcal{P}_d\}$ is the g.c.d. which is also an invariant of $\mathcal{E}(\mathcal{P}_{m,d})$. For the computation of the g.c.d the canonical set $\mathcal{P}^H_{r,d}$ will be used because this set happens to have the least dimension and the simplest structure . Then the vector representative of $\mathcal{P}^H_{r,d}$ has the following form

$$\underline{p}_r^H(s) = P_r^H \, \underline{e}_d(s) = s^{n_i - 1} \hat{p}_r^H(s) \tag{6.27}$$

where

$$P_r^H = \begin{bmatrix} P^0 & \vdots & P^1 & \vdots & P^2 \end{bmatrix} \tag{6.28}$$

$$P^0 = \begin{bmatrix} 0 & \dots & 1 & a^1_{11} & \dots & a^1_{1k_1} & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} a^1_{1k_1} & \dots & a^{r-1}_{1k_{r-1}} & 0 \\ a^2_{21} & \dots & a^{r-1}_{2k_{r-1}} & 0 \\ \vdots & \dots & \vdots & \dots \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} a^1_{1k_1} & \dots & a^1_{1k_r} \\ a^r_{21} & \dots & a^r_{2k_r} \\ \vdots & \dots & \vdots \\ a^r_{r1} & \dots & a^r_{rk_r} \end{bmatrix}$$

Remark 6.3: Let $\phi(s)$, $\psi(s)$ be the g.c.d.s of $\mathcal{P}_{m,d}, \mathcal{P}^*_{r,k}$ where $\mathcal{P}^*_{r,k} = \{t_i(s),\ i \in r\}$ and $t_i(s)$ are the coordinates of $\hat{p}^H_r(s)$. Then

$$\phi(s) = c\, s^{n_1 - 1}\, \psi(s), \quad \text{and } c \in \mathbb{R}\text{-}\{0\} \text{ and } \psi(0) \neq 0. \tag{6.29}$$

□

Proposition 6.4: Let $\mathcal{P}_{m,d} \in \{\mathcal{P}_d\}$ be a proper set of a minimal degree δ

(i) If $\phi(s)$ is a g.c.d of $\mathcal{P}_{m,d}$, then deg $\phi(s) \leq \delta$

(ii) If $\delta = 0$, then $\mathcal{P}_{m,d}$ is coprime

(iii) If P_m is a basis matrix of $\mathcal{P}_{m,d}$ and $\rho(P_m) = \delta + 1$, then $\mathcal{P}_m$ is coprime .

□

Definition 6.4 : (i) Let $\underline{a}^\top \in \mathbb{R}^{1\times k}$ be a vector of the following structure

$$\underline{a}^\top = [\ 0 \ \dots\dots\dots\ 0\ a_0\ a_1 \ \dots\dots\dots\ a_{k'}\ ]\ ,\quad a_0 \neq 0 \tag{6.30}$$

and let the number of zero elements be σ-1, $\sigma \in \mathbb{Z}^+$. σ will be called the *index* of $\underline{a}$. Let $A \in \mathbb{R}^{m\times k}$, $\underline{a}_i^\top$ be the ith row of A and let σ_i be the corresponding index.Then the matrix A will be referred to as an $(\sigma_1, \sigma_2, \dots, \sigma_m)$-*indexed matrix.*

(ii) On an indexed vector $\underline{a}^{\mathsf{T}} \in \mathbb{R}^{1\times k}$ we define the shifting operation $\square \mathrm{a}^{\mathsf{T}} = \tilde{\mathrm{a}}'$ where

$$\tilde{\underline{a}}^{\mathsf{T}} = [\, a_0\ a_1 \,\dots\dots\, a_{k'}\ ,\ 0 \,\dots\dots\dots\, 0\,] \in \mathbb{R}^{1\times k} \tag{6.31}$$

where again the number of zero elements are σ-1. Similarly we may define the shifting operation $\square$ for the rows of the matrix A

(iii) Let $\mathcal{P}_{m,d} \in \{\mathcal{P}_d\}$ and P_m the corresponding basis matrix . The shifting operation may be defined on $\{\mathcal{P}_d\}$ by $\square\mathcal{P}_{m,d} = \widetilde{\mathcal{P}}_{m,d} \in <\mathcal{P}_d>$ where the basis matrix of $\widetilde{\mathcal{P}}_{m,d}$ is defined by $\widetilde{P} = \square P_m$. The set $\widetilde{\mathcal{P}}_{m,d}$ will be referred to as the *shifted set* of $\mathcal{P}_{m,d}$.

$\square$

Theorem 6.2: Let $\mathcal{P}_{m,d} \in \{\mathcal{P}_m\}$ be $(\sigma_1, \sigma_2, \dots, \sigma_m)$- indexed, $\widetilde{\mathcal{P}}_{m,d} = \square\mathcal{P}_{m,d}$ and let $\phi(s)$, $\psi(s)$ be the g.c.d.s of $\mathcal{P}_{m,d}$, $\widetilde{P}_{m,d}$ respectively . Then

$$\phi(s) = c\, s^{\sigma_1 - 1}\, \psi(s)\ ,\ \ c \in \mathbb{R}\text{-}\{0\},\ \ \phi(0) \neq 0 \tag{6.32}$$

Proof: The vector representatives $\underline{p}(s)$ and $\underline{\tilde{p}}_m(s)$ of $\mathcal{P}_{m,d}$ and $\widetilde{\mathcal{P}}_{m,d}$ respectively are related by

$$\underline{p}_m(s) = \mathrm{diag}\,\{\, s^{\sigma_1 - 1}, \dots, s^{\sigma_m - 1} \,\}\ \underline{\tilde{p}}_m(s) = D(s, \sigma_i)\ \underline{\tilde{p}}_m(s) \tag{6.33}$$

If $\mathcal{P}_{m,d}$ is non proper $\sigma_i > 1$ then $s^{\sigma_1 - 1}$ divides $\phi(s)$; furthermore, by the shifting operation $\underline{\tilde{p}}(0) \neq 0$, i.e. $\phi(0) \neq 0$. Let $z \in \mathbb{C}$ - $\{0\}$ be a zero of $\phi(s)$ of multiplicity τ. Then, from (6.33) and given that $|D(z, \sigma_i)| \neq 0$, we have $\underline{\tilde{p}}_m(z) = 0$. Let us denote

$$\underline{p}^{(j)}(s) = \frac{d^j}{ds^j}\, \underline{p}(s) \tag{6.34}$$

and

$$D^{(j)}(s, \sigma_i) = \mathrm{diag}\,\{\, s^{\sigma_1 + j - 1}, \cdots, s^{\sigma_m + j - 1} \} \tag{6.35}$$

Then by (6.33)

$$\underline{p}_m^{(j)}(s) = D^{(j)}(s, \sigma_i)\, \tilde{\underline{p}}_m^{(j)}(s), j = 1, \cdots \tag{6.36}$$

Since $|D^{(j)}(z, \sigma_i)| \neq 0$, then for s = z and j = 1, ..., $\tau-1$, (6.36) implies that $\underline{p}_m^{(j)}(z) = 0 = D^{(j)}(z, \sigma_i)\, \tilde{\underline{p}}_m^{(j)}(z)$ and thus $\tilde{\underline{p}}_m^{(j)}(z) = 0$ for j = 1, ..., $\tau-1$. Therefore an elementary divisor $(s-z)^\tau$ of $\phi(s)$ is also a factor in $\tilde{\phi}(s)$ with degree at least τ. By considering (6.36), (6.33) for the zeros of $\tilde{\phi}(s)$, it is readily seen that $\tilde{\phi}(s)$ divides $\phi(s)$ and this completes the proof. □

6.3-3 : A new algorithm for the computation of the g.c.d. of $\mathcal{P}_{m,d}$

Let $\mathcal{P}_{m,d} \in \{\mathcal{P}_d\}$, $\phi(s)$ be a g,c.d. of the $\mathcal{P}_{m,d}$, $P_m \in \mathbb{R}^{m\times d+1}$ be a basis matrix of $\mathcal{P}_{m,d}, \rho(P_m)$ = r and let $c \geq 0$ be the order of $\mathcal{P}_{m,d}$. The computation of $\phi(s)$ is according to the following method .

(i) **Non - proper sets:** If $c \geq 1$, then $P_m = [\ 0_{m,c}, \overline{P}_m\ ]$, s^c is an elementary divisor of $\phi(s)$ at s=0 and we may write $\phi(s) = s^c\ \overline{\phi}(s)$, where $\overline{\phi}(s)$ is a g.c.d. of the proper set defined by the basis matrix $\overline{P}_m$.

(ii) **Proper sets:** If c=0 the first column of P_m is non-zero and $\phi(0) \neq 0$. To compute $\phi(s)$ for the case of proper sets we apply the following algorithm .

Step 1: If r = d+1 , then the set $\mathcal{P}_{m,d}$ is coprime

Step 2: If r< d+1 , we distinguish the following two cases

(a) If r=1 , then any non-zero polynomial in $\mathcal{P}_{m,d}$ defines the g.c.d.

(b) If r>1 , we define a maximal, linearly independent set of r vectors amongst the rows of P_m and we denote by $P_r \in \mathbb{R}^{r\times d+1}$ the corresponding submatrix of P_m The set $\mathcal{P}_{r,d} \subseteq \mathcal{P}_{m,d}$ defined by the basis matrix P_r will be called a normal subset of $\mathcal{P}_{m,d}$

Step 3: Let r>1 and $\mathcal{P}_{r,d}$ be a normal subset of $\mathcal{P}_{m,d}$ with b.m P_r Compute the LEF $P_r^{\geq}$ or P_r and thus the canonical set $\mathcal{P}_{r,d}^{H}$. Let t(s)= $1+a_1s+ \ldots +a_\delta s^\delta$ be a minimal degree polynomial of $\square\mathcal{P}_{r,d}^{H}$. Then

(a) If $\delta = 0$, then the set $\mathcal{P}_{m,d}$ is coprime ;

(b) If $\delta \geq 1$, then $\phi(s)$ divides t(s); to compute $\phi(s)$ we distinguish the following two cases :

(i) If δ = 1,2, then compute the zeros of t(s) and test whether or not they are zeros of a v.r of $\mathcal{P}_{m,d}$.

(ii) If $\delta \geq 3$, then by the combined action of E.R.E transformations and shifting we obtain a δ-reduced set $\mathcal{P}_{r,\delta}^{*}$ of $\mathcal{P}_{r,d}^{\geq}$ having t(s) as a first element; $\mathcal{P}_{r,\delta}^{*}$ and $\mathcal{P}_{m,d}$ have the same g.c.d.

Step 4: $\mathcal{P}^*_{r,\delta}$ is proper and thus for the computation of $\phi(s)$ repeat the above steps .

□

The above algorithm may be illustrated by the following example

Example (6.4): Let $\mathcal{P}_{3,4} = p(s) = [\ p_1(s)\ p_2(s)\ p_3(s)\ ]^T$

where

$$p_1(s) = s^4 + s^3 - s - 1,$$
$$p_2(s) = s^3 + 3s^2 - s - 3$$
$$p_3(s) = s^4 - 1$$

In the following by $P_1 \xrightarrow{E} P_2$, $P_1' \xrightarrow{\oplus} P_2'$ we mean that P_2, P_2' are obtained from P_1 , P_1' by elementary row operations or shifting, respectively. A b.m. of $\mathcal{P}_{3,4}$ is defined by

$$P_3 = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \\ -3 & -1 & 3 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \rho(P_{3,4}) = 3 = r$$

Since $r=3<4+1=5$ we compute the LEF P_3^H

$$P_3 \xrightarrow{E} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -3 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} = P_3^H$$

The minimal degree polynomial in $\square\mathcal{P}^H_{3,4}$ is $p(s) = s^2-1$. Clearly ± 1 are the roots of $p_i(s)$, i=1,2,3 and thus $\phi(s) = s^2-1$. Alternatively , from $\mathcal{P}^H_{3,4}$ we compute a δ-reduced set as

$$P_3^H = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\oplus}$$

$$\xrightarrow{\oplus} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathrm{E}} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{P}_3'$$

P_3' is the b.m. of the reduced set and $\delta(s) = -1+s^2 = \phi(s)$.

It should be stressed that the above algorithm may be used on the reduced Decentralised Grassmann representative for the computation of the fixed zero polynomials.

6.4 Almost Fixed Modes and Zeros in Decentralised Control

A frequency $\lambda \in \mathbb{C}$ is FZ of M(s), if λ is a zero of all combinants f (s,H), $H \in \mathcal{H}^{\nu}_{r,p}$. An obvious extension of FZ notion may be introduced by relaxing the exact notion in the sense that all combinants $f_M(s,H)$, $H \in \mathcal{H}^{\nu}_{r,p}$ have at least one zero in a finite radius disc centred at λ. It will be shown that under certain general assumptions on M(s), such phenomena occur in the study of $\mathcal{H}^{\nu}_{r,p}$-D-DAP. Central to our analysis here are the zero distribution properties of an $\mathbb{R}$[s]-combinant introduced recently [Kar.3]. The key role to our present investigation is played by the $\mathcal{H}^{\nu}_{r,p}$-DPM, P, which generates the canonical combinant $f^*_M(s,H)$.

If $\underline{p}(s) \in \mathbb{R}^{(\sigma-\mu)}[s]$, deg $\underline{p}(s) = \delta'$ is the C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}$[s]-GR of $\mathfrak{S}_M$, then

$$f_M(s,\, H) = f(s)\cdot\hat{\underline{k}}^{\mathrm{T}}\ \underline{p}(s) = f(s)\cdot\hat{\underline{k}}_{\mathrm{T}}\ \mathrm{P}\ \underline{e}_{\delta'}(s) = f(s)\cdot f^*_M(s,\, H) \tag{6.37}$$

where f(s) is the $\mathcal{H}^{\nu}_{r,p}$-FP and $\mathrm{P} \in \mathbb{R}^{(\sigma-\mu)\times(\delta'+1)}$ is the $\mathcal{H}^{\nu}_{r,p}$-DPM of $\mathfrak{S}_M$. $f^*_M(s,H)$ generates the nonfixed zeros of D-DAP, and their assignability is examined next. Let $a(s) \in \mathbb{R}[s]$ be the polynomial to be assigned; then, max deg $a(s) = \delta'$, $\underline{a}(s) = [a_0, \cdots, a_\delta]\,\underline{e}_{\delta'}(s) = \underline{a}^{\mathrm{T}}_{\delta'}\,\underline{e}_{\delta'}(s)$ and the problem of finding $H \in \mathcal{H}^{\nu}_{r,p}$ such that $f^*_M(s,\, H) = a(s)$ is reduced to the solution of

$$\mathrm{P}^{\mathrm{T}}\ \hat{\underline{k}} = \underline{a}_{\delta'},\ \ \mathrm{P}^{\mathrm{T}} \in \mathbb{R}^{(\delta'+1)\times(\sigma-\mu)},\ \ \underline{a}_{\delta'} \in \mathbb{R}^{\delta'+1},\ \sigma = \binom{q}{p} \tag{6.38}$$

where $\hat{\underline{k}}$ is the reduced vector of an $H \in \mathcal{H}^{\nu}_{r,p}$. Throughout the rest of the chapter it will be assumed that $\hat{k}$ is free; that is, we shall ignore the restricted decomposability conditions on $\hat{k}$.

Definition 6.5: The $M(s) \in \mathbb{R}^{p\times q}[s]$, $p < q$, $\rho_{\mathbb{R}[s]}\{M(s)\} = p$ will be called $\mathcal{H}^{\nu}_{r,p}$-*linearly assignable* ($\mathcal{H}^{\nu}_{r,p}$-LA), if eqn. (6.38) has a solution for all $\underline{a}_{\delta'}$; otherwise, it will be called $\mathcal{H}^{\nu}_{r,p}$-*linearly nonassignable* ($\mathcal{H}^{\nu}_{r,p}$-LNA). M(s) will be called $\mathcal{H}^{\nu}_{r,p}$-*restricted assignable* ($\mathcal{H}^{\nu}_{r,p}$-RA) if it is $\mathcal{H}^{\nu}_{r,p}$-LA and the multilinear subproblem of D-DAP has a solution for at least one solution of eqn. (6.38). If M(s) is $\mathcal{H}^{\nu}_{r,p}$-RA and $f(s) = c$, $c \in \mathbb{R}-\{0\}$, then it will be called $\mathcal{H}^{\nu}_{r,p}$-*completely assignable* ($\mathcal{H}^{\nu}_{r,p}$-CA). Finally, M(s) will be called $\mathcal{H}^{\nu}_{r,p}$-*strongly nonassignable* ($\mathcal{H}^{\nu}_{r,p}$-SNA), if for all reduced vectors $\hat{\underline{k}}$ of $\mathcal{H}^{\nu}_{r,p}$, $f^{*}_{M}(s, H) \neq c$, $c \in \mathbb{R}-\{0\}$. □

Note that if M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA, then there is no $H \in \mathcal{H}^{\nu}_{r,p}$ such that $f_{M}(s, H)$ has all its zeros at infinity. Similar definitions may be given for the $\mathcal{H}^{\nu}_{r,p}$-D-DAP. The above definitions are within the framework of those given for the centralised case [Kar.1]. Following the results in [Kar.1] we have:

Proposition 6.5: Let $P = [\underline{p}_0, \underline{p}_1, \cdots, \underline{p}_{\delta'}] = [\underline{p}_0, P']$ be the $\mathcal{H}^{\nu}_{r,p}$-DPM of $\mathcal{G}_M$ and let $\pi = \rho\{P\}$, $\pi' = \rho\{P'\}$.

(i) M(s) is $\mathcal{H}^{\nu}_{r,p}$-LA if an only if $\pi = \delta' + 1$.

(ii) Necessary condition for M(s) to be $\mathcal{H}^{\nu}_{r,p}$-RA is that $\pi = \delta' + 1$.

(iii) If $\binom{q}{p} - \mu \leq \delta'$, then M(s) is $\mathcal{H}^{\nu}_{r,p}$-LNA and thus it is not $\mathcal{H}^{\nu}_{r,p}$-RA.

(iv) If $\binom{q}{p} - \mu \leq \delta'$ and $\pi' = \binom{q}{p} - \mu$, then M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA. □

Note that the zero assignment properties characterised by the above result in terms of the $\mathcal{H}^{\nu}_{r,p}$-DPM, refer to families of M(s) matrices having the same P matrix and not to a particular matrix. The zero distribution properties of the non-fixed zeros of the $\mathcal{H}^{\nu}_{r,p}$-D-DAP are examined next. Of particular interest is the study of "almost fixed zero" phenomena.

Definition 6.6: The $\mathcal{H}^{\nu}_{r,p}$-D-DAP defined on M(s) has an *almost fixed zero* (AFZ) at $s_0 \in \mathbb{C}$ of order R, $0 \leq R < \infty$, and shall be denoted by (s_0, R), if for $\forall\ H \in \mathcal{H}^{\nu}_{r,p}$, $f_{M}(s,H)$ has at least one zero in the disk $D[s_0, R] = \{s: |s-s_0| \leq R\}$. □

The above definition of AFZs clearly covers the case of FZs, since then $R = 0$ and the disk becomes a point. Those AFZs for which $0 < R < \infty$ will be referred to as *essential*. The search for essential AFZs is connected to the study of $f^{*}_{M}(s,H)$ canonical combinant[Kar.6].

Lemma 6.2 : Let $\underline{p}(s) \in \mathbb{R}^m[s]$, $s_0 \in \mathbb{C}$, $w = s - s_0$, $\deg \underline{p}(s) = d$ and let

$$\underline{\tilde{p}}(w) = \underline{q}_0 + w\underline{q}_1 + \cdots + w^d \underline{q}_d, \quad q_i \in \mathbb{C}^m \tag{6.39}$$

be the Taylor expansion of $\underline{p}(s)$ at $s = s_0$. For every $\underline{k} = \mathbb{R}^m$, the combinant $f(s,\underline{k}) = \underline{k}^T \underline{p}(s)$ has at least one zero in the minimal radius disk $D[s_0, R(s_0,\underline{k})]$, where

$$R(s_0, \underline{k}) = \min \left\{ \left[\binom{d}{i} \frac{|\underline{k}^T \underline{q}_0|}{|\underline{k}^T \underline{q}_i|} \right]^{1/i}, \ i \in \underline{d} \right\} \tag{6.40}$$

□

The above result provides the means for the investigation of the AFZ phenomena in the $\mathcal{H}^{\nu}_{r,p}$-D-DAP.

Theorem 6.3: The $\mathcal{H}^{\nu}_{r,p}$-D-DAP defined on M(s) has an AFZ (s_0, R_0) for $\forall\ s_0 \in \mathbb{C}$, if and only if M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA.

Proof: Let $\underline{p}(s) = [\underline{p}_0, \underline{p}_1, \cdots, \underline{p}_{\delta'}] \underline{e}_{\delta'}(s) = [\underline{p}_0, P'] \underline{e}_{\delta'}(s)$ be the C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}$[s]-GR and let $f^*_M(s, H) = \hat{\underline{k}}^T \underline{p}(s)$ be the corresponding canonical combinant. If $\underline{\tilde{p}}(w) = [\underline{q}_0, \underline{q}_1, \cdots, \underline{q}_{\delta'}] \underline{e}_{\delta'}(w) = [\underline{q}_0, Q'] \underline{e}_{\delta'}(w) = Q\, \underline{e}_{\delta'}(w)$, $w = w - s_0$, is the Taylor expansion at $s = s_0$ of $\underline{p}(s)$, then

$$\underline{q}_0 = P\, \underline{e}_{\delta'}(s_0), \quad \underline{q}_i = P \left[\frac{1}{i!} \frac{d^i}{d s^i} \underline{e}_{\delta'}(s) \right]_{s=s_0}, \quad i \in \underline{\delta'} \tag{6.41}$$

and thus $Q = P\, R$, where R is a full rank, lower triangular matrix. Thus,

$$f^*_M(s, H) = \hat{\underline{k}}^T P\, \underline{e}_{\delta'}(s) = \hat{\underline{k}}^T Q\, R^{-1} \underline{e}_{\delta'}(w), \quad w = s - s_0 \tag{6.42}$$

where R^{-1} is also lower triangular.

If $R(s_0, \hat{\underline{k}})$ is finite for $\forall\ \hat{\underline{k}}$ reduced vector from $\mathcal{H}^{\nu}_{r,p}$, then clearly M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA, since otherwise, i.e. $f^*_M(s.H) = c$, $c \in \mathbb{R}$ for some H, the corresponding radius could become infinite. Thus, if (s_0, R_0) is an AFZ, then M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA.

Assume now that M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA. By Lemma (6.2), it follows that a necessary condition for the radius $R(s_0, \hat{\underline{k}})$ to become infinite for some $\hat{\underline{k}}$, is that $\hat{\underline{k}}^T \underline{q}_i = 0, \ \forall \ i \in \underline{\delta}'$. If such a vector $\hat{\underline{k}}$ exists, then $f^*_M(s,H) = \hat{\underline{k}}^T \underline{q}_0$ constant and this contradicts the assumption that M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA. Thus, for $\forall$ H $\in \mathcal{H}^{\nu}_{r,p}$ the corresponding radius is finite and $R_{0,m} = \max\{R(s_0, \hat{\underline{k}}, \forall \hat{\underline{k}}\}$ exists and it is finite.

This result establishes that if M(s) is $\mathcal{H}^{\nu}_{r,p}$-SNA, then $\forall \ s_0 \in \mathbb{C}$ is AFZ. The radius

$$R_{0,m} = \max\{R(s_0, \hat{\underline{k}}), \ \forall \ \hat{\underline{k}} \text{ reduced from } \mathcal{H}^{\nu}_{r,p}\} \tag{6.43}$$

defines the minimal disk that contains at least one zero for all $f^*_M(s, H)$ combinants. Thus, for $\mathcal{H}^{\nu}_{r,p}$-SNA, M(s) matrices the mobility of every $s_0 \in \mathbb{C}$ as a zero of $f^*_M(s,H)$ is always restricted within the disk $D[s_0, R_{0,m}]$. An estimation of upper bounds for $D[s_0, R_{0,m}]$ is given by the following result.

Theorem 6.4 : Let $\underline{p}(s) \in \mathbb{R}^{(\sigma-\mu)}[s]$ be the C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}$[s]-GR of an $\mathcal{H}^{\nu}_{r,p}$-SNA M(s), $s_0 \in \mathbb{C}$, $\underline{\bar{p}}(w) = \underline{q}_0 + w\underline{q}_1 + \cdots + w^{\delta'}\underline{q}_{\delta'}$, $w = s - s_0$ be the Taylor expansion of $\underline{p}(s)$ at $s = s_0$ and let $\chi \in \mathbb{R}^+$. A sufficient condition for

$$R_{0,m} \le \frac{\chi}{2^{1/\delta'} - 1} \triangleq R(s_0, \chi) \tag{6.44}$$

is that the matrix $Q(s_0, \chi)$ defined by

$$Q(s_0, \chi) \triangleq \underline{q}_{\delta'} \underline{q}^{*T}_{\delta'} \chi^{2\delta'} + \cdots + \underline{q}_1 \underline{q}^{*2}_{\delta'} \chi^2 - \underline{q}_0 \underline{q}^{*T}_0 \tag{6.45}$$

to be positive semidefinite. □

This result is an adaptation of a result from [Kar.3] and may be used in two different ways

(i) $\chi \in \mathbb{R}^+$ is given, then positive semidefiniteness of $Q(s_0, \chi)$ implies the existence of an $R(s_0, \chi)$ upper bound for $R_{0,m}$.

(ii) Find the minimum $\chi \in \mathbb{R}^+$ for which $Q(s_0, \chi)$ is positive semidefinite. In this case the smallest $R(s_0, \chi)$ type upper bound is sought.

Theorem (6.4) is valid for all points $s_0 \in \mathbb{C}$. The size of the radius $R_{0,m}$ may vary for the different points of $\mathbb{C}$. Recent work [Kar.5] has shown that there exist certain points of $\mathbb{C}$, defined as "almost zeros" for which certain upper bounds of the zero radius are minimised. A summary of the results from [Kar.5] applied to our case is given next.

Definition 6.7: Let $\underline{p}(s) \in \mathbb{R}^{(\sigma-\mu)}[s]$ be the C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR of M(s), $s = \sigma + j\omega$ be the complex variable and let

$$\left\|\underline{p}(s)\right\| \triangleq \phi(\sigma, \omega) \triangleq \sqrt{\underline{p}(s^*)^T \underline{p}(s)}, \ s^* = \sigma - j\omega \tag{6.46}$$

A $z \in \mathbb{C}$ will be called an $\mathcal{H}^{\nu}_{r,p}$-*decentralised almost zero* ($\mathcal{H}^{\nu}_{r,p}$-DAZ) of M(s), if z is a minimum of $\phi(\sigma, \omega)$. The global minimum of $\phi(\sigma, \omega)$ is defined as the *prime* $\mathcal{H}^{\nu}_{r,p}$-DAZ ($\mathcal{H}^{\nu}_{r,p}$-DAZ) of M(s). □

The distribution of $\mathcal{H}^{\nu}_{r,p}$-DAZs in $\mathbb{C}$ and their computational aspects are discussed in [Kar.3]. Note that this new notion is a straightforward extension of the algebraic exact notion. Since it is defined by the invariant $\underline{p}(s)$, we have:

Remark (6.3): The set $\mathcal{H}^{\nu}_{r,p}$-DAZs is invariant for all $\mathcal{H}^{\nu}_{r,p}$-D–DAPs defined on the basis matrices M(s) of a given rational vector space $\mathfrak{S}_M$. □

The importance of $\mathcal{H}^{\nu}_{r,p}$-DAZs as "strong poles" of attraction for the zeros of $f^*_M(s, H)$ combinants is highlighted by the properties of certain families of upper bounds of the $R(s_0, \underline{k})$ radii.

Lemma (6.3) : A family of upper bounds for the minimal radius disk $R(s_0, \underline{k})$ of Lemma (6.1) may be defined for the following families of vectors $k \in \mathbb{R}^m$.

(i) If $\underline{k}^T \underline{p}_d \neq 0$, then

$$R_d(s_0, \underline{k}) = \left[\frac{\|\underline{q}_0\|}{\left|\underline{k}^T \underline{p}_d\right|}\right]^{1/d} \tag{6.47}$$

(ii) If $\underline{k}^T \underline{p}_d = \underline{k}^T \underline{p}_{d-1} = \cdots = \underline{k}^T \underline{p}_{i+1} = 0$ and $\underline{k}^T \underline{p}_i \neq 0$, then

$$R_i(s_0, \underline{k}) = \left[\binom{d}{i} \frac{\|\underline{q}_0\|}{\left|\underline{k}^T \underline{p}_i\right|}\right]^{1/i} \tag{6.48}$$

where $\underline{p}_i$ denote the coefficient vectors of $\underline{p}(s)$. □

The importance of $\mathcal{H}^{\nu}_{r,p}$-DAZs is then indicated by the following result [Kar.5].

Proposition 6.6: Let $\underline{p}(s) = \underline{p}_0 + s\underline{p}_1 + \cdots + s^{\delta'}\underline{p}_{\delta'} \in \mathbb{R}^{(\sigma-\mu)}[s]$ be the C-$\mathcal{H}^{\nu}_{r,p}$-D-$\mathbb{R}[s]$-GR of a $\mathcal{H}^{\nu}_{r,p}$-SNA M(s) and let z$'$, z be an $\mathcal{H}^{\nu}_{r,p}$-DAZ, the $\mathcal{H}^{\nu}_{r,p}$-PDAZ of M(s), respectively. For all $\underline{k} \in \mathbb{R}^{(\sigma-\mu)}$ such that $\underline{k}^T \underline{p}_{\delta'} = \cdots = \underline{k}^T \underline{p}_{i+1} = 0$, and $\underline{k}^T \underline{p}_i \neq 0$ and for all i, i = δ', $\delta'-1$, $\cdots$, 1, we have:

(i) $R_i(z', \underline{k}) < R_i(s_0, \underline{k})$, for all $s_0 \in \mathbb{C}$: $|s_0 - z'| < \epsilon$.

(ii) $R_i(z, \underline{k}) < R_i(s_0, \underline{k})$, for all $s_0 \in \mathbb{C}$.

□

Thus, for the general families of $\underline{k}$ vectors defined above, the radii $R_i(s_0, \underline{k})$ are locally minimised when s_0 is an $\mathcal{H}^{\nu}_{r,p}$-DAZ and are globally minimised when s_0 is the $\mathcal{H}^{\nu}_{r,p}$-PDAZ. This result indicates that the $\mathcal{H}^{\nu}_{r,p}$-DAZs act as strong poles of attraction for the zeros of $f^*_M(s, H)$.

For $\mathcal{H}^{\nu}_{r,p}$-SNA matrices M(s), the mobility of the zeros of $f^*_M(s, H)$ is restricted within finite radius disks, the radii of which may be estimated by Proposition (6.6); we shall refer to such disks as *zero trapping disks*. If for some $s_0 \in \mathbb{C}^+$, the disk d[s_0, R(s_0, χ)] lies entirely in $\mathbb{C}^+$, then s_0 will be called *entirely unstable* (EUS). If the $\mathcal{H}^{\nu}_{r,p}$-D-DAP defined on M(s) has at least one EUS point s_0, then M(s) will be called *$\mathcal{H}^{\nu}_{r,p}$-decentralised strongly unstable* ($\mathcal{H}^{\nu}_{r,p}$-D−SUS) and, clearly, $f^*_M(s, H)$ cannot be stabilised under any H $\in \mathcal{H}^{\nu}_{r,p}$. Theorem (6.4) may be used to provide the following test for the above property.

Corollary 6.1: A sufficient condition for M(s) to be $\mathcal{H}^{\nu}_{r,p}$-D−SUS is that for some $s_0 \in \mathbb{C}^+$ the matrix Q(s_0, χ) is positive semidefinite for $\chi = (2^{1/\delta'} - 1)\cdot \mathrm{Re}(s_0)$.

Proof: If for some s_0 and $\chi = (2^{1/\delta'} - 1)\cdot \mathrm{Re}(s_0)$, Q($s_0$, χ) is positive semidefinite, the predicted upper bound is R(s_0, χ) = Re(s_0) and the disk lies entirely in $\mathbb{C}^+$.

□

6.5 The Computation of an Upper Bound for the Minimal Radius Disc

It has been established that the almost zeros of $\mathcal{P}$ act as poles of attraction for the zeros of the $\underline{k}$-combinant $f(s, \underline{k})$ of $\mathcal{P}$. The radius $R(s_0, \underline{k})$ of the disc $D(s_0, \underline{k})$ within which an exact zero of $f(s, \underline{k})$ may always be found will be referred to as the zero radius at $s = s_0$.

Let $\mathcal{P}$ be a set of polynomials, $\underline{p}(s) = \underline{p}_0 + s\underline{p}_1 + \cdots + s^d\underline{p}_d$ be a polynomial vector representative of $\mathcal{P}$, $\underline{p}(w) = \underline{b}_0 + w\underline{b}_1 + \cdots + w^d\underline{b}_d$, $w = s - s_0$, be the Taylor expansion of $\underline{p}(s)$ at $s = s_0$, $s_0 \in \mathbb{C}$, let θ_i be the angle of $\underline{k} \in \mathbb{R}^m$ with the $\underline{p}_i$ vector. Sufficient condition for the existence of $\alpha \in \mathbb{R}^+$ and a family of $\underline{k}$ vectors such that $R(s_0, \underline{k}) \leq \alpha$ are:

(i) if $\underline{k}^T p_d \neq 0$ then [1]

$$\alpha \geq \left[\frac{\|\underline{b}_0\|}{\|\underline{p}_d\|}\right]^{1/d} \tag{6.49}$$

$$|\cos\theta_d| \geq \left[\frac{\|\underline{b}_0\|}{\|\underline{p}_d\|}\right] \alpha^d = \cos\overline{\theta}_d(s_0, \alpha) \tag{6.50}$$

(ii) $\underline{k}^T\underline{p}_d = \underline{k}^T\underline{p}_{d-1} = \cdots = \underline{k}^T\underline{p}_{i+1} = 0$, and $\underline{k}^T\underline{p}_i \neq 0$ then

$$\alpha \geq \left[\binom{d}{i}\frac{\|\underline{b}_0\|}{\|\underline{p}_i\|}\right]^{1/i} \tag{6.51}$$

$$|\cos\theta_i| \geq \binom{d}{i}\frac{\|\underline{b}_0\|}{\|\underline{p}_i\|}\alpha^i = \cos\overline{\theta}_i(s_0, \alpha) \tag{6.52}$$

The above conditions for $\underline{k}$ are sufficient but not necessary for $R(s_0, \underline{k})$ to be less than or equal to some α satisfying conditions (6.49), or (6.51); this is due to the fact that the $R_i(s_0, \underline{k})$ upper bounds have been used. The set of conditions (6.50) and (6.51) define families $k_i(s_0, \alpha)$ of parameter vector $\underline{k}$ where $k_i(s_0, \alpha) = \left\{\underline{k}: \underline{k}^T \underline{p}_j = 0,\ j = i+1, \cdots, d, \underline{k}^T \underline{p}_i \neq 0,\ |\cos\theta_i| \geq \cos\overline{\theta}_i(s_0, \alpha)\right\}$ and $i = d, d-1, \cdots, i$, such that for all $\underline{k} \in k_i(s_0, \alpha)$, $R(s_0, \underline{k}) \leq \alpha$ the following result can be stated.

Let $z \in \mathbb{C}$ be an almost zero of $\mathcal{P}$, $s_0 \in \mathbb{C}$ and let $\alpha \in \mathbb{R}^+$, where

$$\alpha \geq \max\left\{\left[\binom{d}{i}\frac{\|\underline{b}_0(s_0)\|}{\|\underline{p}_i\|}\right]^{1/i},\ \text{for all } |s_0 - z| < \epsilon\right\} \tag{6.53}$$

for all α satisfying (6.53) and for all i for which the families $k_i(s_0, \alpha)$ are defined then $k_i(s_0, \alpha) \subseteq k_i(z, \alpha)$ with equality holding when $s_0 = z$.

Now let $\underline{p}(w) = \underline{b}_0 + w\underline{b}_1 + \cdots + w^d \underline{b}_d$, $w = s - s_0$, be the Taylor expansion of the polynomial vector representative $\underline{p}(s)$ of the set of polynomials $\mathcal{P}$ at $s = s_0$, $s_0 \in \mathbb{C}$ and let θ'_1 be the angles of $\underline{k} \in \mathbb{R}^m$ and $\underline{b}_i$.

For all $\alpha \in \mathbb{R}^+$ and all $\underline{k}$ such that

$$\alpha \leq d\,\frac{\|\underline{b}_0\|}{\|\underline{b}_1\|} \text{ and } |\cos\theta'_1| \geq \alpha\,\frac{\|\underline{b}_i\|}{d\,\|\underline{b}_0\|} \tag{6.54}$$

then $R(s_0, \underline{k}) \leq \alpha$.

The properties of the zero radius discussed so far depend on the particular choice of the parameter vector $\underline{k}$ or on some family of parameter vectors. Now we want to find an upper bound for the zero radius, which is independent of the parameter vector $\underline{k}$.

Let $\underline{p}(w) = \underline{b}_0 + w\underline{b}_1 + \cdots + w^d \underline{b}_d$, $w = s - s_0$, be the Taylor expansion of the polynomial vector representative $\underline{p}(s)$ of the set of polynomials $\mathcal{P}$ at $s = s_0$, $s_0 \in \mathbb{C}$ and let $x \in \mathbb{R}^+$. A sufficient condition for $R(s_0, \underline{k}) \leq x/(2^{1/d} - 1)$ for all $\underline{k} \in \mathbb{R}^m$, is that the matrix $B(s_0, x)$ is positive semidefinite, where

$$B(s_0, x) = \underline{b}_d\, \underline{b}_d^{*T} x^{2d} + \cdots + \underline{b}_1\, \underline{b}_1^{*T} x^2 - \underline{b}_0\, \underline{b}_0^{*T} \tag{6.55}$$

This result can be used in two different ways:

(i) If $x \in \mathbb{R}^+$ is fixed, then the positive semidefiniteness of $B(s_0, x)$ implies the existence of an upper bound $R(s_0, x)$ of $R(s_0, \underline{k})$ which is independent of $\underline{k}$; $R(s_0, x)$ of $R(s_0, \underline{k})$ which is independent of $\underline{k}$; $R(s_0, x)$ is then given by [Kar.3]:

$$R(s_0, x) = \frac{x}{2^{1/d} - 1} \tag{6.56}$$

(ii) find the minimum positive x for which $B(s_0, x)$ is positive semidefinite. In this case the smallest of the (6.56) type upper bounds for $R(s_0, \underline{k})$ which are independent of $\underline{k}$ is defined.

The above ideas are illustrated in the following example.

Example (6.4): Let

$$\underline{p}(s) = \begin{bmatrix} s + 1.1 \\ s^2 + s \end{bmatrix}$$

Using the optimisation programme,[Gia.1], we find an almost zero at $s = -1$. The Taylor expansion of $\underline{p}(s)$ at

$$\underline{p}(w) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ -1 \end{bmatrix} + w^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{b}_0 + \underline{b}_1 w + \underline{b}_2 w^2, \quad w = s+1$$

Thus we may compute $B(-1, x)$ as follows

$$B(-1, x) = \underline{b}_2\, \underline{b}_2^{*T} + \underline{b}_1\, \underline{b}_1^{*T} x^2 - \underline{b}_0\, \underline{b}_0^{*\,Ty} = \begin{bmatrix} 1 & -x^2 \\ -x^2 & x^4 + x^2 \end{bmatrix} x \in \mathbb{R}^+$$

(i) Assume that we wish to test whether $R(-1, \underline{k}) \leq 1$ for all $\underline{k}$. Then, the equation $x/(2^{1/2}-1) = 1$ gives that $x = 0.41$; it may be readily verified that $B(-1, 0.41)$ is positive definite and thus $R(-1, \underline{k}) \leq 1$ for all $\underline{k}$.

(ii) Since $R(-1, \underline{k}) \leq 1$, we may search for smaller bounds for $R(-1, \underline{k})$.

Using again the optimisation programme, we find the minimum positive x for which $B(s_0, x)$ is positive semidefinite and this gives a minimal , independent $\underline{k}$ upper bound for $R(-1, \underline{k})$ which is $R^* = 0.71$.

The family of $\underline{k}$-combinants of $\mathcal{P}$ for the above example is defined by

$$f(s, \underline{k}) = k\,(s+1.1) + s\,(s+1)$$

Finding the roots of $f(s, \underline{k})$ as a function of the scaler parameter k, is equivalent to a single-input-output root locus problem with a transfer function

$$g(s) = \frac{k\,(s+1.1)}{s\,(s+1)}$$

From the root locus associated with g(s) , the minimal radius disc $D(-1, R_{min})$ which contains at least one zero for all values of k, may be computed graphically. $R_{min} \simeq 0.43$ and thus $R^* = 0.71$ is a good estimate for the minimal radius. □

6.5.1 Use of the Sensitivity to Scaling for Improved Bounds of the Zero Trapping Region

We know that each disc has an almost zero at its centre, and each disc contains at least a zero, so now the objective is to reduce the uncertainty in the location of the zeros. We apply different scaling and for each one we find the almost zero i.e., the position of the centre of the disc, and its radius. Thus we get a number of overlapping circles and the region common to all the circles defines the most likely place where a zero can be found.

An example is given to illustrate this technique. It shows how we get a considerable reduction in the zero trapping region. For the calculations, various computer programmes have been used.

Example (6.5): Let the basis matrix be:

$$P = \begin{bmatrix} .6 & 1.6 & 1 \\ 2.5 & 3 & 2 \end{bmatrix}$$

So

$$\underline{p}(s) = \begin{bmatrix} .6 + 1.6s + s^2 \\ 2.5 + 3s + 2s^2 \end{bmatrix}$$

and

$$s_0 = -.75 - j\ .74$$

This gives the vectors:

$$\underline{b}_0 = \begin{bmatrix} -.58 - j\ .07 \\ .29 \end{bmatrix}, \quad \underline{b}_1 = \begin{bmatrix} .1 + j\ 1.48 \\ -j\ 2.96 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore $x = 0.41$, and $R(s_0, x) = 0.41 / (2^{1/2} - 1) = 0.9898$. Using the scaling (3, 1) gives

$$\underline{p}(s) = \begin{bmatrix} 1.8 + 4.8s + 3s^2 \\ 2.5 + \ \ 3s + 2s^2 \end{bmatrix}$$

and

$$s_0 = -.76 - j\ .43$$

$$\underline{b}_0 = \begin{bmatrix} -.68 - j\ .11 \\ .29 \end{bmatrix}, \quad \underline{b}_1 = \begin{bmatrix} .24 + j\ 2.58 \\ -.04 - j\ 2.96 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, $x = 0.32$, and $R(s_0, x) = 0.32 / (2^{1/2} - 1) = 0.7725$. Using the scaling (1, 3) gives

$$\underline{p}(s) = \begin{bmatrix} .6 + 1.6s + s^2 \\ 7.5 + 9s + 6s^2 \end{bmatrix} \text{ and } s_0 = -.75 - j\ .82$$

$$\underline{b}_0 = \begin{bmatrix} -.71 - j\ .01 \\ .09 \end{bmatrix}, \quad \underline{b}_1 = \begin{bmatrix} .1 - j\ 1.64 \\ -j\ 9.84 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Therefore, $x = 0.501$ and $R(s_0, x) = 0.501 / (2^{1/2} - 1) = 1.2095$.

6.6 Quick tests for existence of fixed modes and fixed zeros

Assume that

$$P_d \underline{e}_d(s) = \underline{g}_R(s)$$

is the reduced Grassmann representative, i.e. we have dropped from the Grassmann representative the co-ordinates corresponding to zeros in the compensator matrix K.

Proposition 6.8: Let $\underline{\hat{P}}_\delta$ be the reduced Plücker matrix corresponding to the Grassmann representative, i.e. we define $\hat{P}_\delta$ a submatrix of P_δ by dropping those rows which correspond to indices of decentralisation. Then, if

$$\mathcal{N}_r(\hat{P}_\delta) = \{0\} \qquad \text{there exist fixed zeros in D−DAP} \qquad (6.57)$$

$$\mathcal{N}_r(\hat{P}_\delta) \neq \{0\} \qquad \text{we may or may not have zeros in D−DAP} \qquad (6.58)$$

□

For the case of zero assignment we investigate the effect of decentralisation structure on the degree of the Grassmannian $\underline{g}_R(s)$. Then, if

$$\theta[\underline{g}_R(s)] < \theta[\underline{g}(s)] \qquad (6.59)$$

then no matter how we select the controllers, we assign zeros at infinity.

Example 6.6 : Investigate the fixed zero structure aspects of the system described by the transfer function matrix, G(s), where

$$G(s) = \begin{bmatrix} 1 & s^2+1 \\ s & 0 \\ s+1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} (s+2)^2 & 1 \\ 0 & (s+3)(s+2) \end{bmatrix}^{-1} = N(s)D(s)^{-1}$$

under the following decentralised squaring down structures

$$K_1 = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & k_3 & k_4 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} k_1 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} k_1 & k_2 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

The Grassmann representative of G(s) is given by

$$C_2^T\{G(s)\} = [\det a_1, \det a_2, \det a_3, \det a_4, \det a_5, \ \det a_6]$$

where

$$\det a_1 = \begin{vmatrix} 1 & s^2+1 \\ s & 0 \end{vmatrix} = -s\,(s^2+1) = -(s^3+s)$$

$$\det a_2 = \begin{vmatrix} 1 & s^2+1 \\ s+1 & 1 \end{vmatrix} = 1 - (s+1)\,(s^2+1) = -(s^3 + s^2 + s)$$

$$\det a_3 = \begin{vmatrix} 1 & s^2+1 \\ 0 & s+1 \end{vmatrix} = s + 1$$

$$\det a_4 = \begin{vmatrix} s & 0 \\ s+1 & 1 \end{vmatrix} = s - 0 = s$$

$$\det a_5 = \begin{vmatrix} s & 0 \\ 0 & s+1 \end{vmatrix} = s(s+1)$$

$$\det a_6 = \begin{vmatrix} s+1 & 1 \\ 0 & s+1 \end{vmatrix} = (s+1)^2 = s^2 + 2s + 1$$

The $\underline{g}(s)$ of N(s) is

$$\underline{g}^T(s) = [-(s^3+s) \quad -(s^3+s^2+s) \quad s+1 \quad s \quad s^2+s \quad s^2+2s+1]$$

The Plücker matrix corresponding to the above polynomial vector is

$$P_3 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

Now let us examine the multi-vectors which correspond to the various architectures of the decentralised squaring down controllers

(i) Decentralised Controller $K_1 = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & k_3 & k_4 \end{bmatrix}$

The multi-vector corresponding to the above structure is

$$\underline{k}_1^T = [k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}]$$

where

$$k_{12} = \begin{vmatrix} k_1 & 0 \\ 0 & k_2 \end{vmatrix}, \ k_{13} = \begin{vmatrix} k_1 & 0 \\ 0 & k_3 \end{vmatrix}, \ k_{14} = \begin{vmatrix} k_1 & 0 \\ 0 & k_4 \end{vmatrix},$$

$$k_{23} = \begin{vmatrix} 0 & 0 \\ k_2 & k_3 \end{vmatrix}, \quad k_{24} = \begin{vmatrix} 0 & 0 \\ k_2 & k_4 \end{vmatrix}, \quad k_{34} = \begin{vmatrix} 0 & 0 \\ k_3 & k_4 \end{vmatrix}$$

or

$$\underline{k}_1^T = [k_1k_2, \ k_1k_3, \ k_1k_4, \ 0, \ 0, \ 0]$$

The reduced Plücker matrix due to the above decentralised structure is defined by

$$\underline{k}^T P_\delta \, \underline{e}_\delta(s) =$$

$$= [k_1k_2, \ k_1k_3, \ k_1k_4, \ 0, \ 0, \ 0] \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

and is given by

$$\hat{P}^R_{1\delta} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Clearly, the reduced Plücker matrix corresponding to the decentralised structure has full rank and thus we have no fixed zeros. Clearly the reduced Grassmann representative is defined by

$$\underline{g}_1(s) = \begin{bmatrix} -s^3 - s \\ -s^3 - s^2 - s \\ s+1 \end{bmatrix}$$

and it is a prime vector.

(ii) Decentralised Controller $K_2 = \begin{bmatrix} k_1 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix}$

The multi-vector corresponding to the above structure is

$$\underline{k}_2^T = [k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}]$$

where

$$k_{12} = \begin{vmatrix} k_1 & k_2 \\ 0 & 0 \end{vmatrix}, \ k_{13} = \begin{vmatrix} k_1 & 0 \\ 0 & k_3 \end{vmatrix}, \ k_{23} = \begin{vmatrix} k_1 & 0 \\ 0 & k_4 \end{vmatrix},$$

$$k_{23} = \begin{vmatrix} k_2 & 0 \\ 0 & k_3 \end{vmatrix}, \ k_{24} = \begin{vmatrix} k_2 & 0 \\ 0 & k_4 \end{vmatrix}, \ k_{34} = \begin{vmatrix} 0 & 0 \\ k_3 & k_4 \end{vmatrix}$$

or

$$\underline{k}_2^T = [0, \ k_1k_3, \ k_1k_4, \ k_2k_3, \ k_2k_4, \ 0]$$

The reduced Plücker matrix corresponding to K_2 is

$$\hat{P}^R_{2\delta} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and the Grassmann representative is

$$\underline{g}(s) = \begin{bmatrix} -s^3 - s^2 - s \\ s+1 \\ s \\ s(s+1) \end{bmatrix}$$

which has full rank and thus the corresponding system has no fixed zeros.

(iii) Decentralised controller

$$K_3 = \begin{bmatrix} k_1 & k_2 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

The corresponding multivector is defined by

$$\underline{k}_3 = [\, k_{12} \,,\, k_{13} \,,\, k_{14} \,,\, k_{23} \,,\, k_{24} \,,\, k_{34} \,]$$

where

$$k_{12} = \begin{vmatrix} k_1 & k_2 \\ 0 & 0 \end{vmatrix}, \quad k_{13} = \begin{vmatrix} k_1 & k_3 \\ 0 & 0 \end{vmatrix}, \quad k_{14} = \begin{vmatrix} k_1 & 0 \\ 0 & k_4 \end{vmatrix},$$

$$k_{23} = \begin{vmatrix} k_2 & k_3 \\ 0 & 0 \end{vmatrix}, \quad k_{24} = \begin{vmatrix} k_1 & 0 \\ 0 & k_4 \end{vmatrix}, \quad k_{34} = \begin{vmatrix} k_3 & 0 \\ 0 & k_4 \end{vmatrix}$$

and thus

$$\underline{k}_3 = [\, 0 \quad 0 \quad k_1k_4 \quad 0 \quad k_2k_4 \quad k_3k_4 \,]$$

The corresponding reduced Grassmann representative is given by

$$\underline{g}_3(s) = [\, s+1 \quad s(s+1) \quad (s+1)^2 \,]^T$$

and thus clearly we have a fixed finite zero fo this control structure at $s = -1$. Also we should note that the centralised combinant has degree 3, whereas the corresponding decentralised is

$$\underline{k}_3^T P_\delta \underline{e}_\delta(s) = k_1 k_4 (s+1) + k_2 k_4 s(s+1) + k_3 k_4 (s+1)^2$$

and has degree 2. Thus, apart from finite fixed zero we also have a fixed zero at $s = \infty$ for this particular structure.

6.7 The Decentralised Pole, Zero Assignment Problems: Fixed and Almost Fixed Modes, Zeros

We may specialise the results on $\mathcal{H}^{\nu}_{r,p}$−D-DAP, $\mathcal{H}^{\nu}_{q,p}$−D-DAP to the cases of D-PAP by DOF (DSF), D-ZAP by DSD respectively, since the control theory problems are special cases of the corresponding abstract formulations.

Given that DOF covers also the DSF case, the D-PAP by DSF will be examined. Note that the essential difference between D-PAP and D-ZAP lies in the interpretation of the zeros of the corresponding combinants; thus, in the first case they express closed−loop poles, while in the second zeros of squared down systems.

The term *mode* (pole) will be used for a zero of the combinant of D-PAP, while the term *zero* will be retained in the cases of D-ZAP; all other terms and definitions will be used with the same meaning throughout this section. Note that for the D-PAP, D-ZAP the role of M(s) is played by the matrices $T_l(s)$, $N_r(s)$ respectively.

The $\mathbb{R}[s]$−GRs of $T_l(s)$, $N_r(s)$ are defined by

$$C_m(T_l(s)) \triangleq \underline{p}_1(s)^T \in \mathbb{R}^{1\times\tau}[s],\ \tau = \binom{m+l}{m},\ \deg \underline{p}_1(s) = \upsilon \tag{6.60}$$

$$C_l(N_r(s)) \triangleq \underline{p}_g(s) \in \mathbb{R}^{\rho}[s],\quad \rho = \binom{m}{m},\ \deg \underline{p}_g(s) = d \tag{6.61}$$

where υ is the McMillan degree of G(s) and d is the degree of $N_r(s)$. The families of decentralised output feedbacks, squaring down compensators will be denoted by

$$\mathcal{K}^{\nu}_{l,m} = \{\tilde{K}_d = \begin{bmatrix} I_m \\ K_d \end{bmatrix}\}, \quad \mathcal{F}^{\nu}_{l,m} = \{F_d\} \tag{6.62}$$

where K_d, F_d are block diagonal matrices of fixed dimension blocks which are defined by

$$u = \text{block-diag}\ \{\ K_1, \cdots, K_\nu\}\ y = K_d\ y\ ;\ K_i \in \mathbb{R}^{l_i \times m_i};\ i \in \nu$$

and

$$z = \text{block-diag}\ \{\ F_1, \cdots, F_\nu\}\ y = F_d\ y\ ;\ F_i \in \mathbb{R}^{l_i \times m_i};\ i \in \nu$$

respectively.

The D-PAP by DOF and the D-ZAP by DSD will be denoted in short by $\mathcal{K}^{\nu}_{l,m}$−D-PAP and $\mathcal{F}^{\nu}_{l,m}$−D-ZAP respectively. Let $\mathfrak{D}(\mathcal{K}^{\nu}_{l,m})$, $\mathfrak{D}(\mathcal{F}^{\nu}_{l,m})$ be the decentralisation characteristics of the above sets and let $\tilde{p}_l(s)$, $\underline{\tilde{p}}_g(s)$ denote the $\mathcal{K}^{\nu}_{l,m}$−, $\mathcal{F}^{\nu}_{l,m}$−D−$\mathbb{R}$[s]−GRs respectively, Then,

Corollary 6.2: (i) The fixed pole polynomial, $f_p(s)$, of $\mathcal{K}^{\nu}_{l,m}$−D-PAP is given by the zero polynomial $\underline{\tilde{p}}_l(s)$. (ii) The fixed zero polynomial, $f_z(s)$, of $\mathcal{F}^{\nu}_{l,m}$−D-ZAP is given by the zero polynomial of $\underline{\tilde{p}}_g(s)$. □

If $C_m(\tilde{K}_d) = \underline{k} \in \mathbb{R}^{\tau}$, $K_d \in \mathcal{K}^{\nu}_{l,m}$, $C_l(F_d) = \underline{f}^T \in \mathbb{R}^{1\times\rho}$, $F_d \in \mathcal{F}^{\nu}_{l,m}$ and $\hat{\underline{k}}$, $\hat{\underline{f}}^T$ are the corresponding reduced vectors, then the pole, zero combinants may be expressed as

$$P_K(s, K_d) = f_p(s)\cdot\hat{\underline{k}}^T\ \underline{\tilde{p}}_l(s) = f_p(s)\ p^*_K(s, K_d) \tag{6.63}$$

$$z_F(s, F_d) = f_z(s)\cdot\hat{\underline{f}}^T\ \underline{\tilde{p}}_g(s) = f_z(s)\ z^*_F(s, F_d) \tag{6.64}$$

where $\underline{\tilde{p}}(s) \in \mathbb{R}^{(\tau-\upsilon)}[s]$, $\underline{\tilde{p}}_g(s) \in \mathbb{R}^{(\rho-\mu')}[s]$ are the C−$\mathcal{K}^{\nu}_{l,m}$ −D−$\mathbb{R}$[s]−GR, C−$\mathcal{F}^{\nu}_{l,m}$−D−$\mathbb{R}$[s]−GR respectively and μ, m$'$ denote the number of elements in $\mathfrak{D}(\mathcal{K}^{\nu}_{l,m})$, D($\mathcal{F}^{\nu}_{l,m}$) correspondingly; $p^*_K(s, K_d)$, $z^*_F(s, F_d)$ represent the corresponding canonical combinants. If deg $\underline{\tilde{p}}(s) = \tilde{\upsilon}$, deg $\underline{\tilde{p}}_g = \tilde{\delta}$, then

$$\underline{\tilde{p}}_1(s) = P_d(T_l)\ \underline{e}_{\tilde{\upsilon}}(s),\ \ \underline{p}_g(s) = P_d(G)\ \underline{e}_{\tilde{\delta}}(s) \tag{6.65}$$

where $P_d(T_l) \in \mathbb{R}^{(\tau-\mu)(\tilde{\upsilon}+1)}$, $P_d(G) \in \mathbb{R}^{(\rho-\mu')(\tilde{\delta}+1)}$ are the $\mathcal{K}^{\nu}_{l,m}-$, $\mathcal{F}^{\nu}_{l,m}-$*Decentralised Plücker Matrices* respectively. Using $P_d(T_l)$, $P_d(G)$ we may state the following results for the solvability of $\mathcal{K}^{\nu}_{l,m}-$D-PAP, $\mathcal{F}^{\nu}_{l,m}-$D-ZAP.

Corollary 6.3:

(a) Necessary conditions for G(s) to be completely pole assignable by DOF are:

(i) $\underline{p}_l(s)$ is coprime, (ii) $\rho\{P_d(T_l)\} = \tilde{\upsilon} + 1$.

(b) Necessary conditions for G(s) to be completely zero assignable by DSD are:

(i) $\underline{p}_g(s)$ is coprime, (ii) $\rho\{P_d(G)\} = \tilde{\delta} + 1$. □

The MFD representation $T_l(s)$ is assumed coprime and thus noncoprimeness of $\underline{p}_l(s)$ may be the result of decentralisation. For the zero assignment problem, however, noncoprimeness of $\underline{p}_g(s)$ may be the result of zeros present in G(s) as well as of the decentralisation.

For families of pole, zero nonassignable systems almost fixed mode, almost fixed zero phenomena occur. Thus, we have:

Corollary 6.4: Let $P_d(T_l) = [\underline{p}_0, P^*_d(T_l)]$, $P_d(G) = [\underline{p}_0, P^*_d(G)]$. Then,

(i) If $\mathcal{N}_r\{P^*_d(T_l)\} = \{0\}$, then for $\forall\ \lambda_0 \in \mathbb{C}$, the closed loop pole polynomial has a λ_0-almost fixed mode for all DOF K_d.

(ii) If $\mathcal{N}_r\{P^*_d(G)\} = \{0\}$, then for $\forall\ z_0 \in \mathbb{C}$, the zero polynomial of the squared down system has a z_0-almost fixed zero for all DSD compensators F_d □

The presence of entire unstable λ_0-almost fixed modes, z_0-almost fixed zeros implies that the system cannot be stabilised by DOF, cannot be made minimum phase by DSD respectively. Thus, we have:

Corollary 6.5: Let $\mathcal{N}_r\{P^*_d(T_l)\} = \{0\}$, and $\mathcal{N}_r\{P^*_d(G)\} = \{0\}$. Then,

(i) Necessary conditions for G(s) to be stabilisable under DOF are that G(s) has no unstable fixed mode and no entirely unstable almost fixed mode.

(ii) Necessary conditions for G(s) to be minimum phase under DSD are that G(s) has not right half plane fixed zero and no entire unstable almost fixed zero.

□

6.8 Conclusions

The study of the abstract Decentralised determinantal assignment problem provides a unified framework for the study of large scale systems. Central to this approach are the concepts of Decentralised Grassmann representative and Decentralised Plucker matrix. The computation of fixed , almost fixed modes via the exterior algebra tools , advocated in the present thesis illustrates the mechanism responsible for the generation of non-fixed modes and zeros and also leads to new necessary conditions for complete pole, zero assignability. The concepts of almost fixed modes and zeros is a natural extension of the fixed mode, zero concept and their presence imposes considerable obstacles in the stabilisation of pole, zero combinants. Testing the non-existence of entirely unstable almost fixed modes, zeros is not an easy process and further work is needed along these lines.

Chapter 7

A NEW ALGEBROGEOMETRIC FRAMEWORK AND CONDITIONS FOR DECENTRALISED ASSIGNMENT PROBLEMS

7.0 Introduction

In this chapter the problem of pole assignment by Decentralised output feedback is first considered as a problem of intersection of the Decentralised Grassmann variety and a linear variety associated with the assignment of a given polynomial.

A new necessary and sufficient condition for the existence of a decentralised complex, constant controller is derived which clearly provides a new necessary condition for the existence of a real decentralised controller.

The methodology developed may also be applied to a variety of further D−DAP problems. The essential part of the approach is the computation of the dimension of the corresponding Decentralised Grassmann variety which is demonstrated here for the case of Decentralised Output feedback.

Another topic covered in this chapter is the extension of the framework developed for the constant D−DAP to dynamic compensation problems associated with the assignment of frequencies. It is shown that the centralised or decentralised problems of pole, zero assignment with controllers of the P−I, P−D, P−I−D or restricted McMillan degree type may be reduced to equivalent constant DAP or D−DAP problems and this demonstrates the broadness of the DAP approach.

7.1 Algebraic Varieties

Let $P^n(F)$ be a projective space over a field F with dim $P^n(F)=n$. If we choose an allowable co-ordinate system in $P^n(F)$ then the set of points whose coordinates satisfy the equations

$$f_i(x_1,x_2,\cdots,x_n)=0\ ;\ i=1,2,3,\cdots,r \tag{7.1}$$

where $f_i(x_1, x_2, \cdots, x)$ are homogeneous polynomials over F is called an *Algebraic Variety,* V. It may happen, of course that there are no points satisfying the equations. While this is of no general interest, the varieties considered here will be assumed to have at least one point satisfying (7.1).

Clearly an algebraic variety in $P^n(F)$ has been defined in a particular coordinate system. It can be shown that if an aggregate of points in $P^n(F)$ forms an algebraic variety in one coordinate system, then they form an algebraic variety in any other allowable coordinate system although the equations in the two systems, may be different.

If V_1 and V_2 are two varieties given by the equations

$$f_i(x_1,x_2,\cdots,x_i)=0\ ;\quad i=1,2,\cdots,r_1 \tag{7.2$_a$}$$

$$g_j(x_1,x_2,\cdots,x_i)=0;\ j=1,2,\cdots,r_2 \tag{7.2$_b$}$$

respectively, the points common to V_1 and V_2 satisfy both sets of equation simultaneously and therefore define a third algebraic variety which is called the *intersection* of V_1 and V_2 and is denoted by $V_1 \cap V_2$.

The points which satisfy the set of equations

$$f_i(x_1,x_2,\cdots,x_i)\,g_j(x_1,x_2,\cdots,x_n)=0;\ i=1,2,\cdots,r_1;\quad j=1,2,\cdots,r_2 \tag{7.3}$$

define another algebraic variety which is called the *sum of* V_1 and V_2 and is symbolised V_1+V_2.

If every solution of the equation of 7.2_a satisfies equation 7.2_b then V_1 is said to be *contained* in V_2 or V_1 is said to be a *subvariety* of V_2 and this is denoted by $V_1 \subset V_2$.

A variety, V, is said to be *reducible* if it can be expressed as the sum of two algebraic varieties each distinct from V; in other words, necessary and sufficient condition for the reducibility of a variety, V, is the existence of a product of two forms $f(x_1, x_2, \cdots, x_n)$ and $g(x_1, x_2, \cdots, x_n)$ which vanishes at all points of V without either form having this property. A variety which is not reducible is said to be *irreducible.*

Let V be an irreducible algebraic variety in $P^n(F)$ and let us suppose that V is not contained in $x_0 = 0$ and that $(s_1, s_2, \cdots, s_n)$ are the non homogeneous co-ordinates of a generic point. Then there exists an integer $0 < d < n$ such that

(i) a set of points P_i say $P_{i_1}, P_{i_2}, \cdots, P_{i_d}$ is algebraically independent over F, but
(ii) any set of $(d+1)$ - points P_i is algebraically dependent over F.

Hence, if $d<n$ then there exist non zero polynomials

$$g_j(x_1, x_2, \cdots, x_d, x_{d+1}) = 0 \quad \text{in} \quad F[x_1, x_2, \cdots, x_d, x_{d+1}]$$

which are such that

$$g_j(P_{i_1}, \cdots, P_{i_d}, P_{i_j}) = 0\,; \qquad j = d+1, \cdots, n$$

where $i_1, \cdots, i_d, i_{d+1}, \cdots, i_n$ is a rearrangement of $1, \cdots, n$. In none of the polynomials $g_j(x_1, \cdots, x_d, x_{d+1}) = 0$ can all terms containing x_{d+1} be absent, for this would imply the algebraic dependence of $P_{i_1}, \cdots, P_{i_d}$. The integer d with $0<d<n$ is called the *dimension of the variety* V; i.e. $\dim V = d$.

The dimension of an irreducible variety V and the dimension of the projective space $P^n(F)$ are such that

$$n_{eqs} + d = n$$

where n_{eqs} denotes the minimal number of equations which define the variety V.

Thus V is an irreducible variety of dimension $n-1$ if and only if it is defined by a single irreducible equation $f(x_0, x_1, \cdots, x_n) = 0$. Such a variety is called *primal* when the equation is linear so that the variety is a linear space which is called a *prime*. In general, the subspaces P^{n-1} of a projective space P^n are called *primes* or *tangential spaces.*

An irreducible variety of dimension d is a d–linear space if and only if it is defined by n–d linearly independent linear equations.

7.1.1 Intersection of Varieties

The following result on the intersection of varieties play a crucial role in the solution of the D–DAP problem.

Lemma 7.1: Let V_1 and V_2 be two irreducible varieties of dimension d_1 and d_2 respectively in $P^n(F)$, where F is an algebraically closed field, then $V_1 \bigcap V_2$ is generically empty if $d_1+d_2 < n$. If $d_1+d_2 \geq n$ the intersection $V_1 \bigcap V_2$ does always exist. □

It is clear that if V_1 is a generic linear space of dimension n–d then it meets an irreducible variety V_2 of dimension d in a finite number of points; this number g of points is called the *order* of the variety V_2. Also note that if V_2 is also linear then V_1 and V_2 meet in an only one projective point and thus g=1.

Generally it can be proved that the order of a variety V is equal to the degree of the homogeneous polynomial defining V when they are in a special form called the *Cayley–form* .

7.2 Dimension of Decentralised Varieties $H^{v}_{r,p}$, $H^{v}_{q,p}$

In computing the dimension of $\mathcal{H}^{v}_{r,p}$. we will be using the theory of the dimension of Grassmann variety Ω(r,p) which is a follows

Let the matrix

$$H^{v}_{r,p} = \begin{bmatrix} & I_p & & \\ \begin{array}{c} \leftarrow p_1 \rightarrow \\ r_1 \updownarrow \begin{bmatrix} K_1 \end{bmatrix} \end{array} & & & \\ & \begin{array}{c} \leftarrow p_2 \rightarrow \\ r_2 \updownarrow \begin{bmatrix} K_2 \end{bmatrix} \end{array} & & \\ & & \ddots & \\ & & & \begin{array}{c} \leftarrow p_s \rightarrow \\ r_s \updownarrow \begin{bmatrix} K_s \end{bmatrix} \end{array} \end{bmatrix}$$

where $K_s \in \mathbb{R}^{r_s \times p_s}$, $r_s \in I^{v}_{r} = \{ r_s,\ s \in \underline{v} \}$, $p_s \in I^{v}_{p} = \{ p_s,\ s \in \underline{v} \}$. Let H be a generic element of $\mathcal{H}^{v}_{r,p}$, and let

$$\underline{h} = \underline{k} = [k_{w_1}, k_{w_2}, \ldots, k_{w_{i-1}}, 0, k_{w_{i+1}}, \ldots, k_{w_{i-1}}, 0, k_{w_{i+1}}, \ldots, k_w]^T \in \mathbb{R}^{\sigma}$$

where $w_{ij} \in Q_{r,p}$.

The set $\mathcal{D}(\mathcal{H}^{v}_{r,p}) = \{ w_{i_j} \in Q_{r,p},\ j \in m,\ k_{w_{i_j}} = 0 \}$ has been defined as the *Decentralisation Characteristic(DC)* of $H \in \mathcal{H}^{v}_{r,p}$

The subvector of $\underline{k}$ obtained by dropping the Plücker co-ordinates that correspond to the decentralised characteristic $D(\mathcal{H}^{v}_{r,p})$, i.e.

$$\underline{h} = \underline{k} = [k_{w_i}, \ldots, k_{w_{i-1}}, k_{w_{i+1}}, \ldots, k_w]^T \in \mathbb{R}^{r-\mu} \tag{7.4}$$

will be referred to as the *reduced vector* of $H \in \mathcal{H}^{v}_{r,p}$. Also, the point $(\ldots, w_{ij}, \ldots)$ corresponds to a generic $\mathcal{H}^{v}_{r,p}$ of $\mathcal{P}^{\sigma-1}(\mathbb{R})$ and on the Grassmann variety Ω(r,p).

Lemma 7.1: $\mathcal{H}^{v}_{r,p}$ is a subvariety of the Grassmann variety. □

To find the dimension of $\mathcal{H}^{v}_{r,p}$, we first normalise the co-ordinates of a generic point. From equation (7-4) none of the w_{ij} is zero. We normalise the co-ordinates so that $k_{w_1} = 1$. If we define N points $B^i = (b^i_1, \ldots, b^i_p)$, $i = 1, \ldots, r$ where $b^i_j = k_{w_{ij}}$ $(i = 1, \ldots, r;\ j = 1, \ldots, p)$ where r points determine a basis for the $\mathcal{H}^{v}_{r,p}$ with Grassmann co-ordinates $(\ldots, w_{ij}, \ldots)$. Since $B^1, B^2, \ldots, B^r$ form a basis for the generic $\mathcal{H}^{v}_{r,p}$ and $b^i_j = k_{w_{ij}}$ all Grassmann co-ordinates of this $\mathcal{H}^{v}_{r,p}$ can be expressed integrally in terms of the co-ordinates $k_{w_{ij}}$ $(i = 1, \ldots, r;\ j = 1, \ldots, p)$ where $k_{w_1} = 1$. When $j = 1, \ldots, i-1, i+1, \ldots, r$ the corresponding co-ordinate is zero. Hence, the Grassmann co-ordinate of a generic $\mathcal{H}^{v}_{r,p}$ can be expressed integrally in terms of $\sum_{s=1}^{v} r_s p_s$ co-ordinates $k_{w_{ij}}$ $(i = 1, \ldots, r;\ j = r+1, \ldots, p)$ which are independent. This follows from the fact that there exists an $\mathcal{H}^{v}_{r,p}$ determined by r+1 points

$$
\begin{aligned}
B^1 &= (1, 0, \ldots, 0,\ b^1_{r+1}, \ldots, b^1_p) \\
B^2 &= (0, 1, \ldots, 0,\ b^2_{r+1}, \ldots, b^2_p) \\
&\vdots \\
B^r &= (0, 0, \ldots, 1,\ b^r_{r+1}, \ldots, b^r_p)
\end{aligned}
\tag{7-5}
$$

which the b^i_j which appears in (7-2.3) are independent co-ordinates. The Grassmann co-ordinates of this $\mathcal{H}'^{v}_{r,p}$ are given by the equations $k'_{w_{ij}} = b^i_j$ and since we have proved above that the dimension of a generic point on $\Omega(r,p)$ is at most $\sum_{s=1}^{v} r_s p_s$, the construction of a point on $\Omega(r,p)$ with dimension $\sum_{s=1}^{v} r_s p_s$ over $\mathbb{R}$ proves that the dimension of a generic point is $\sum_{s=1}^{v} r_s p_s$. Hence we have the result

Theorem 7.1: The subvariety $\mathcal{H}^{v}_{r,p}$ is irreducible and of dimension $\sum_{s=1}^{v} r_s p_s$

Proof: By theorem 2.4, the Grassmann variety is an irreducible variety and we know that a subvariety of an irreducible variety is irreducible itself. □

7.3 Conditions for the Solvability of Decentralised Pole-Assignment Problems Using Constant (Proportional) Controllers

Consider the D-PAP defined in Chapter 4 and let us denote by $\underline{s}_i$, $i \in \underline{m}$ the columns of K^l_D and by $\underline{t}_i{}^T(s)$, $i \in \underline{m}$ the rows of $T_l(s)$; then by the Binet-Cauchy theorem, the decentralised closed loop pole polynomial is expressed as

$$p_K(s, K_D) = <\underline{s}\wedge, \underline{t}(s)\wedge> \tag{7.6}$$

We may reduce the D-PAP to the following two problems

(i) **Linear Subproblem:** Let $\sigma = \binom{m+l}{m}$, $\underline{s}\wedge = \underline{k} \in \mathbb{R}^{\sigma}$, $\underline{t}(s)^T\wedge = \underline{p}_l(s)^T \in \mathbb{R}^{1\times\sigma}[s]$ and assume that $\underline{k}$ is free. Find conditions under which there exist vectors $\underline{k}$ such that the polynomial $p_K(s,\underline{K}_D)$

$$\underline{p}_K(s,K_D) = k\ \underline{P}_l(s)^T \tag{7.7}$$

has a given set of zeros Z.

(ii) **Multilinear Subproblem:** Assume that the linear problem has a solution and that k(z) is the family of vectors $\underline{k}$ for which $\underline{p}_K(s, K_D)$ has a given set Z of zeros. Determine whether there exist a $\underline{k} \in K(Z)$ such that

$$\underline{k} = C_m[I_m, K_D]$$

where $[I_m, K_D] \in \mathbb{R}^{m_i\times(m_i\times l_i)}$

□

The key idea behind the present approach is the investigation of complex intersections of the linear variety defined by the linear subproblem with the Grassmann variety. The advantage of the approach lies in its algorithmic nature. In terms of the new invariant, the decentralised Plücker matrix, stronger necessary conditions for complex and for generic pole assignability are given.

Let $\underline{a}(s) \in \mathbb{R}[s]$ be the polynomial to be assigned. Then if $\deg \underline{a}(s) = \tilde{\upsilon}$, $\underline{a}(s) = [a_0, \ldots, a_{\tilde{\upsilon}}]e_{\upsilon}(s)$ and the problem of finding $K_D \in \mathcal{K}^{\upsilon}_{e,m}$ such that $P^*_k(s, K_D) = a(s)$ is reduced to the solution of

$$P_d(T_l)^T\underline{k} = \underline{a}_{\tilde{\upsilon}}, \qquad P_D(T_l)^T \in \mathbb{R}^{(\tilde{\upsilon}-1)\times(\sigma-\mu)},\ \underline{k} \in \mathbb{R}^{(\sigma-\mu)} \tag{7.8}$$

where $\underline{k}$ is the corresponding reduced vector. Thus, the following results are readily established.

Proposition 7.1: A necessary condition for G(s) to be CPA under Decentralised constant output feedback is that

(i) $\rho\,[P_D(T_l)] = \bar{v} + 1$

(ii) $p_l(s)$ is co-prime □

If the linear subproblem has always a solution, then D-PAP is completely solvable if and only if among the solutions of the linear subproblem there exists at least one vector $\underline{k} = [1, \underline{k}^T]^T$ which belongs to the Grassmann variety $\Omega(r,p)$ of the projective space $P^{r-1}(\mathbb{R})$. Since the family of the solutions of the linear subproblems defined by equation (7.5) is a $(\sigma-\mu-n)$ linear space of the projective space $P^{r-1}(\mathbb{R})$, then the solution of D-PAP is reduced to a classical problem of intersection of irreducible varieties.

Theorem 7.2: Let $P_D(T_l) = [\underline{p}_0, P_D^*(T_l)] = [\underline{p}_0(T_l), \underline{p}_1(T_l), \ldots, \underline{p}_{\bar{v}}(T_l)]$ be the $\mathcal{K}_{l,m}^{v}$ DPM. Let G(s) $\in \mathbb{R}^{m \times l}(s)$ be the strictly proper transfer function matrix. Then if the system has no fixed modes, we have:

(a) A sufficient condition for G(s) to be GPA by a complex output feedback, $K_i \in \mathbb{C}^{l_i \times m_i}$, is that

$$\sum_{i=1}^{v} l_i m_i \geq \bar{v} \text{ and } \rho\{\underline{P}_D^*(T_l)\} = \bar{v}$$

(b) A sufficient condition for G(s) to be CPA by a complex output feedback, $K_i \in \mathbb{C}^{l_i \times m_i}$ is that

$$\sum_{i=1}^{v} l_i m_i \geq \bar{v}+1 \text{ and } \rho\{\underline{P}_D^*(T_l)\} = \bar{v}$$

(c) A necessary condition for G(s) to be CPA or GPA by a real output feedback $K_i \in \mathbb{R}^{l_i \times m_i}$, $i \in \underline{v}$ is that

$$\sum_{i=1}^{v} l_i m_i \geq \bar{v}$$

Proof:

(a) If $\rho\{\underline{P}_D^*(T_l)\} = \bar{v}$, then the linear subproblem is always solvable. Let $\mathcal{K}$ be the family of the vectors $\underline{k} = [I, \underline{k}^T]^T$ for which $\underline{P}_D(T_l)^T\underline{k} = \underline{a}\bar{v}$. Then $\underline{k}$ is $(\sigma-n-\mu)$ linear space of the projective space $\mathbb{P}^{\sigma-1}(s)$ which is an irreducible variety of the same projective space. If

$\sum_{i=1}^{\upsilon} m_i l_i \geq \upsilon$ then

$$\dim \Omega(r,p) + \dim \underline{k} = \sum_{i=1}^{\upsilon} m_i l_i + \sigma - n - \mu > \sigma - 1$$

and also by Lemma 7.1 the two varieties k and Ω(r,p) always intersect. Note that in this case complete pole assignability cannot be guaranteed due to the extra condition on $\underline{k}$, that the first co-ordinate is non-zero, however, it can be readily verified that the set of all coefficient vectors $\underline{a}$(s) which cannot be assigned by such vectors $\underline{k}$ lie in a hyperplane.

(b) The extra condition on k, that the first co-ordinate is non-zero can be expressed in terms of a linear equation $k_0 = c$, $c \in \mathbb{R} - \{0\}$ and so the linear variety k in this case has dimension $\sigma - n - m - 1$. So, if $\sum_{i=1}^{\upsilon} m_i k_i \geq \tilde{\upsilon} + 1$, then the varieties k and Ω(r,p) always intersect.

(c) Clearly, $\sum_{i=1}^{\upsilon} m_i l_i \geq \tilde{\upsilon}$ is a necessary condition for G(s) to be CPA or GPA by a real output feedback because otherwise there is not even a complex feedback which solves D-PAP.

□

Corollary 7.1: Necessary conditions for the strictly proper G(s) to be CPA or GPA by real output feedback are that $\sum_{i=1}^{\upsilon} m_i l_i \geq \tilde{\upsilon}$ and $\rho\ \{\underline{P}^*_D(T_l)\} = \tilde{\upsilon}$. If the system is proper the the above conditions hold true for the $\tilde{\upsilon}+1$ instead of $\tilde{\upsilon}$.

Example 7.1: Let

$$G(s) = \frac{1}{s^3+s^2-1}\begin{bmatrix} s+1 & s(s+1) \\ s(s+1) & 1 \end{bmatrix}$$

be the transfer matrix of a system. A right co-prime MFD of G(s) is given by

$$G(s) = \begin{bmatrix} 0 & -(s+1) \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -s & 1 \\ 1 & -s(s+1) \end{bmatrix}^{-1} = N_r(s) D_r^{-1}(s)$$

$$C_2\begin{Bmatrix} D_r(s) \\ N_r(s) \end{Bmatrix} = \begin{bmatrix} -s & 1 \\ 1 & -s(s+1) \\ 0 & -(s+1) \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s^3+s^2-1 \\ s^2+s \\ 1 \\ -(s+1) \\ -(s^2+s) \\ -(s+1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix} = P_3 \underline{e}_3(s)$$

It can be easily verified that $\rho\{P_3\}=4 = \tilde{\upsilon}+1$

Now consider a decentralised output feedback of the form

$$k = \begin{bmatrix} 1 & 0 & k_1 & 0 \\ 0 & 1 & 0 & k_2 \end{bmatrix}$$

Since $m_1=1$, $m_2=1$, $l_1=1$, $l_2=1$ and the condition $\sum_{i=1}^{\upsilon} m_i l_i \geq \upsilon$ is not satisfied here since $2>3$. So, the above system is not pole assignable.

□

Example 7.2: Consider the dynamic system described by the polynomial matrix $M(s) \in \mathbb{R}^{5\times 2}$

$$M(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \\ 25 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

The associated Plücker matrix has dimension $\sigma\times(\tilde{\upsilon}+1)$ where $\sigma=\binom{m}{l}$ and $\tilde{\upsilon}$ is the Forney's dynamical order of M(s). Hence, $\sigma=\binom{5}{2}=10$, $\tilde{\upsilon}+1=2+1=3$ and $P_{\tilde{\upsilon}} \in \mathbb{R}^{10\times 3}$. The Plücker matrix of M(s) is given by

$$C_2\Big(M(s)\Big) = C_2\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \\ 25 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s^2+3s+2 \\ 0 \\ s^2+s \\ -25s^2+45 \\ 0 \\ 0 \\ 25 \\ 25 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = P_2\underline{e}_2(s)$$

Clearly, rank of P_2 is $d=\tilde{v}+1$ and so $M(s)$ is linearly pole assignable

Now consider a decentralised output feedback of the form

$$k = \begin{bmatrix} 1 & 0 & k_1 & 0 & 0 \\ 0 & 1 & 0 & k_2 & k_3 \end{bmatrix}$$

applied on the above system and so we have

$$\sum_{i=1}^{3} m_i l_i = m_1 l_1 + m_2 l_2 + m_3 l_3 = 1+1+1 = 3$$

and the condition for CPA system is satisfied. In fact, if

$$\underline{k} = [1, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9] \in \mathbb{R}^{10}$$

is a solution of D–PAP, then it has to satisfy the linear equations

$$\underline{P}_2{}^T\, \underline{k} = \underline{a}$$

and a set of reduced QPR. We therefore have

$$\begin{aligned} \underline{P}_2{}^T\, \underline{k} &= \underline{a}, \\ k_5 - k_1k_4 + k_2k_3 &= 0 \\ k_8 - k_1k_7 + k_2k_6 &= 0 \\ k_9 - k_3k_7 + k_4k_6 &= 0 \end{aligned}$$

or, equivalently

$$k_1 \in \mathbb{R} \qquad k_2 \in \mathbb{R}-\{0\} \qquad k_3 = a_0-2 \qquad k_4 \in \mathbb{R} \qquad k_5 = k_1k_4 + k_2(a_0-2)$$

$$k_6 = \frac{k_4(k_1-2) - \frac{k_2}{2}(k_1-1) + \frac{k_1}{2}(a_2-1) + \frac{1}{2}(a_0-a_1+1)}{k_2} \qquad k_7 = \frac{1}{2}(2k_4-k_2+a_2-1)$$

$$k_8 = \frac{1}{2}(4k_4-k_2+a_1-a_0-1) \qquad k_9 = k_3k_7 - k_4k_6 \qquad k_2 \neq 0$$

Thus, any polynomial $a(s) = a_0 + a_1s + a_2s^2$ can be assigned by a decentralised real feedback controller. □

7.4 Determinantal Assignment Problems Using Multivariable Dynamic Controllers

Consider the general multivariable system with unity feedback whose plant transfer function matrix(TFM) is $G(s)\in\mathbb{R}^{m\times l}(s)$, and with forward controller TFM $C(s)\in\mathbb{R}^{l\times m}(s)$ and assume the following coprime MFD as

$$G(s)= A_1^{-1}B_1 = B_2A_2^{-1}, \qquad C(s) = D_1^{-1}N_1 = N_2D_2^{-1} \tag{7.9}$$

Under unity feedback the characteristic pole polynomial of the closed–loop system is

$$f(s) = |F_1(s)| = |F_2(s)| \tag{7.10}$$

where

$$F_1(s) = A_1(s)D_2(s) + B_1(s)N_2(s) \tag{7.11}$$

$$F_2(s) = D_1(s)A_2(s) + N_1(s)B_2(s) \tag{7.12}$$

Thus we may write

$$f(s) = \left| [D_1(s) \;\; N_1(s)] \begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix} \right| \tag{7.13}$$

$$= \left| [A_1(s) \;\; B_1(s)] \begin{bmatrix} D_2(s) \\ N_2(s) \end{bmatrix} \right| \tag{7.14}$$

Remark 7.1: The general feedback structure covers all the cases $m \geq l$ or $m \leq l$. In fact

(i) If $l \geq m$, then $C(s)$ may be interpreted as a *precompensator*

(ii) If $l \leq m$, then $C(s)$ may be interpreted as a *feedback compensator*

□

Let the multivariable controller be given by

$$\begin{aligned} C(s) &= K_o + s^{-1}K_1 + sK_2 \in \mathbb{R}^{l \times m}[s], \qquad K_o, K_1, K_2 \in \mathbb{R}^{l \times m} \\ &= [sI_l]^{-1}\,[s^2K_2 + sK_0 + K_1] \\ &= [s^2K_2 + sK_0 + K_1][sI_m]^{-1} \end{aligned} \tag{7.15}$$

Remark 7.2: Left MFD coprime iff $\rho[0, K_1] = l$

Right MFD coprime iff $\rho\begin{bmatrix} K_1 \\ 0 \end{bmatrix} = m$

Thus, if $l \leq m$ use left MFD ; if $l \geq m$ use right MFD

7.4.1 Multivariable P–I controller

$$C(s)=K_0+s^{-1}K_1 = [sI_l]^{-1}[sK_0+K_1] = [sK_0+K_1][sI_m]^{-1} \tag{7.15}$$

(i) $l \leq m$: $\rho(K_1) = l$ for coprimeness of L–MFD

$$f(s) = \left|\begin{pmatrix} sI_l & sK_o+K_1 \end{pmatrix}\begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix}\right| = \left|\begin{bmatrix} I_l & K_0 & K_1 \end{bmatrix}\begin{bmatrix} sA_2(s) \\ sB_2(s) \\ B_2(s) \end{bmatrix}\right| = |F_l\ T_2^{\alpha}[s]| \tag{7.16}$$

$$F_l = [I_l\ K_0\ K_1] \in \mathbb{R}^{l\times(l+2m)} \tag{7.17}$$

$$T_2^{\alpha}(s) = \begin{bmatrix} sA_2(s) \\ sB_2(s) \\ B_2(s) \end{bmatrix} \in \mathbb{R}^{(l+2m)\times l}[s] \tag{7.18}$$

(ii) $l \geq m$: $\rho(K_1) = m$ for coprimeness of R–MFD

$$f(s) = \left|[A_1(s)\ B_1(s)]\begin{bmatrix} sI_m \\ sK_0+K_1 \end{bmatrix}\right| = |s\,A_1(s) + sB_1(s)K_o + B_1(s)K_1| =$$

$$= \left|[sA_1(s)\ \ sB_1(s)\ \ B_1(s)\ ]\begin{bmatrix} I_m \\ K_0 \\ K_1 \end{bmatrix}\right| = |\ T_1^{\alpha}(s)\ F_r| \tag{7.19}$$

$$F_r = \begin{bmatrix} I_m \\ K_0 \\ K_1 \end{bmatrix} \tag{7.20}$$

$$T_1^{\alpha}(s) = [sA_1(s)\ \ sB_1(s)\ \ B_1(s)\ ] \in \mathbb{R}^{m\times(m+2l)}[s] \tag{7.21}$$

Conclusions: Using multivariable dynamic P–I controllers the DAP is reduced to a constant DAP problem.

$$l \leq m: \qquad f(s) = \left|\begin{pmatrix} I_l & K_0 & K_1 \end{pmatrix}\begin{bmatrix} sA_2(s) \\ sB_2(s) \\ B_2(s) \end{bmatrix}\right| \tag{7.22}$$

$$l \geq m : \qquad f(s) = \left| \begin{bmatrix} sA_1(s) & sB_1(s) & B_1(s) \end{bmatrix} \begin{bmatrix} I_m \\ K_0 \\ K_1 \end{bmatrix} \right| \tag{7.23}$$

New invariants and solvability conditions are introduced by the Plucker matrices of $T_2^{\alpha}(s)$, $T_1^{\alpha}(s)$.

7.4.2 Multivariable P–D Controllers

Let $C(s) = K_o + sK_2 = [\ I_l]^{-1} [\ sK_2 + K_o\ ] = [\ sK_2 + K_o\ ]\ [\ I_m\ ]^{-1}$ (7.24)

The above MFD are always C– coprime but not necessarily coprime at $s = \infty$

Remark 7.3 : If we denote by

$$C_l(s) = [\ I_l,\ sK_2 + K_0\ ] \in \mathbb{R}^{l \times (m+l)}, \quad C_r(s) = \begin{bmatrix} sK_2 + K_0 \\ I_m \end{bmatrix} \in \mathbb{R}^{(m \times l) \times m}(s) \tag{7.25}$$

(i) L–MFD is coprime at infinity if and only if either of the following equivalent conditions hold true

(a) $C_l(s)$ has no zeros at $s = \infty$

(b) $C_l(s)$ as a pencil has no nonlinear ∞–ed

(ii) R–MFD is coprime at infinity if and only if either the following equivalent conditions hold true:

(a) $C_r(s)$ has no zeros at $s = \infty$

(b) $C_r(s)$ as a pencil has no nonlinear ∞–ed

□

Remark 7.4 : (i) If $\rho(K_2) = l$, then L–MFD is coprime at ∞

(ii) If $\rho(K_2) = m$ then R– MFD is coprime at ∞

□

Remark 7.5: Using matrix pencil theory we can readily characterise the conditions for coprimeness at s $= \infty$

□

Remark 7.6: Use of P–D feedback may result in a singular closed loop system. Conditions are defined on (K_o, K_2) such that we guarantee internal properness.

□

Again we may distinguish the following two cases

(i) $l \leq m$ (feedback compensation)

$$f(s) = \left|[I_l,\ sK_2+K_0][A_2(s),\ B_2(s)]^T\right| = \left|A_2(s)+sK_2B_2(s)+K_0B_2(s)\right|$$

$$= \left|[I_l, K_0, K_2][A_2(s), B_2(s), sB_2(s)]^T\right| = \left|F_l\ T_2^{\beta}(s)\right| \tag{7.26}$$

where

$$F_l = [I_l, K_0, K_2] \in \mathbb{R}^{l \times (l+2m)} \tag{7.27}$$

$$T_2^{\beta}(s) = \begin{array}{c} \left[\begin{array}{c} A_2(s) \\ B_2(s) \\ sB_2(s) \end{array}\right] \\ \leftarrow l \rightarrow \end{array} \begin{array}{c} \updownarrow l \\ \updownarrow m \\ \updownarrow m \end{array} \tag{7.28}$$

(ii) $l \geq m$ (Precompensation Configuration)

$$f(s) = \left|[A_1(s), B_1(s)][I_m,\ sK_2+K_0]^T\right| = \left|A_1(s)+sB_1(s)K_2+B_1(s)K_0\right|$$

$$= \left|[A_1(s),\ B_1(s),\ sB_1(s)][I_m,\ K_0, K_2]^T\right| = \left|T_1^{\beta}(s)\ F_r\right| \tag{7.29}$$

where

$$T_1^{\beta}(s) = \begin{array}{c} \left[\begin{array}{ccc} A_1(s) & B_1(s) & sB_1(s) \end{array}\right] \\ \begin{array}{ccc} \leftarrow m \rightarrow & \leftarrow l \rightarrow & \leftarrow l \rightarrow \end{array} \end{array} \updownarrow m \ \in \mathbb{R}^{m \times (m+2l)} \tag{7.30}$$

$$F_r = [I_m,\ K_0,\ K_2]^T \in \mathbb{R}^{(m+2l)\times m} \tag{7.31}$$

Conclusion: Dynamic P–D Determinantal Assignment Problem(DAP) is reduced to a constant DAP problem:

$l \le m$: $$f(s) = \left| I_l A_2(s) + K_o B_2(s) + K_2 s B_2(s) \right| \tag{7.32}$$

$l \ge m$: $$f(s) = \left| A_1(s) I_m + B_1(s) K_o + s B_1(s) K_2 \right| \tag{7.33}$$

Remark 7.7 : Additional conditions for internal properness are also needed on the top of the solvability conditions for the constant DAP problems.

7.4.3 Multivariable P–I–D Controllers

Let

$$\begin{aligned} C(s) &= K_o + s^{-1} K_1 + s\, K_2 = \\ &= [sI_l]^{-1}\, [s^2\, K_2 + s\, K_o + K_1] = [\, s^2\, K_2 + s\, K_o + K_1\,]\, [\, s\, I_m\,]^{-1} \end{aligned} \tag{7.34}$$

Remark 7.8 : Conditions for coprimeness at $s = \infty$ as well as in $\mathbb{C}$ and internal properness of the feedback are needed.

(i) $l \le m$ (feedback configuration)

$$f(s) = \left| sA_2(s) + s^2\, K_2\, B_2(s) + sK_o B_2(s) + K_1 B_2(s) \right| \tag{7.35}$$

$$= \left| \, [\, I_l\ K_o\ K_1\ K_2\,]\, [\, sA_2(s)\ \ sB_2(s)\ \ B_2(s)\ \ s^2 B_2(s)\,]^T \right|$$

$$= \left| \, \tilde{F}\ T_2^{\gamma}(s)\ \right| \tag{7.36}$$

where

$$\tilde{F}_l = [\ I_l\ K_o\ K_1\ K_2\ ] \in \mathbb{R}^{l\times(l+3m)} \tag{7.37}$$

$$T_2^{\gamma}(s) = [\ sA_2(s)\ \ sB_2(s)\ \ B_2(s)\ \ s^2B_2(s)\ ]^T \in \mathbb{R}^{(l+3m)\times l} \tag{7.38}$$

(ii) $\quad l \geq m$ (Precompensation configuration)

$$f(s) = \left|\ sA_1(s)\ \ s^2B_1(s)K_2 + sB_1(s)K_o + B_1(s)K_1\ \right|$$

$$= \left|\ [\ sA_1(s)\ \ sB_1(s)\ \ B_1(s)\ \ s^2B_1(s)\ ]\ [\ I_m\ \ K_o\ \ K_1\ \ K_2\ ]^T\ \right|$$

$$= \left|\ T_1^{\gamma}(s)\ \ \tilde{F}_r\ \right| \tag{7.39}$$

where

$$T_1^{\gamma}(s) = [\ sA_1(s)\ \ sB_1(s)\ \ B_1(s)\ \ s^2B_1(s)\ ] \in \mathbb{R}^{m\times(m+3l)}[\ s\ ] \tag{7.40}$$

$$\tilde{F}_r = [\ I_m\ \ K_o\ \ K_1\ \ K_2\ ]^T \in \mathbb{R}^{(m+3l)\times m} \tag{7.41}$$

Conclusion: The dynamic P–I–D Determinantal Assignment Problem (DAP) is reduced to a constant DAP problem

$l \leq m$:
$$f(s) = \left|\ [\ I_l\ K_o\ K_1\ K_2\ ]\ [\ sA_2(s)\ \ sB_2(s)\ \ B_2(s)\ \ s^2B_2(s)\ ]^T\right| \tag{7.42}$$

$l \geq m$:
$$f(s) = \left|[\ sA_1(s)\ \ sB_1(s)\ \ B_1(s)\ \ s^2B_1(s)\ ]\ [\ I_m\ \ K_o\ \ K_1\ \ K_2\ ]^T\right| \tag{7.43}$$

7.5 Multivariable Controllers of Bounded Dynamic Order

Consider the multivariable system $G(s) \in \mathbb{R}^{l\times m}(s)$ such that

$$C(s) = D_1^{-1}(s)N_1(s) = N_2(s)D_2^{-1}(s) \tag{7.44}$$

Assume that MFD's are coprime and reduced and let

$$T_l(s) = [\ D_1(s)\ \ N_1(s)\ ] = T_\mu s^\mu + T_{\mu-1}s^{\mu-1} + \cdots + T_o) \tag{7.45}$$

$$T_r(s) = [\ D_2(s)\ N_2(s)\ ] = \widetilde{T}_\nu s^\nu + \widetilde{T}_{\nu-1}s^{\nu-1} + \cdots + \widetilde{T}_o \tag{7.46}$$

where μ is the observability index of the minimal system i.e., the maximal of the observability indices and ν is the controllability index i.e., the maximal of the controllability indices.

The above descriptions do not guarantee properness of the resulting transfer functions therefore additional conditions are needed to make sure we have a proper controller. Although these descriptions cannot also guarantee a fixed McMillan degree compensator, they define an upper bound limit for it.

Lemma 7.3 : The extended McMillan degree $\delta^*_M(C)$ of C(s) i.e., total number of finite and infinite poles is

$$\delta^*_M = \partial_M[\ C_l(\ T_l(s))\ ] = \partial_M\ [\ C_m(T_r(s))\ ]$$

where $\partial_M[\ .\ ]$ denotes the matrix degree defined as the maximum of degrees of maximal order minors.

□

The above result [Kar.7] holds for a general system and leads to the following proposition

Proposition 7.2 : A dynamic system described by (7.43), (7.44) has maximum extended McMillan degree

(i) $l\mu$ if (7.43) system description is used
(ii) $m\nu$ if (7.44) system description is used

□

Using the descriptions (7.43) and (7.44) we may consider the pole assignment problem as

$$f(s) = |[\ D_1(s)\ N_1(s)\ ][\ A_2(s)\ B_2(s)\ ]^T| = |[\ A_1(s)\ B_1(s)\ ][\ D_2(s)\ N_2(s)\ ]^T| \tag{7.47}$$

(i) $l \le m$ i.e. feedback configuration

$$f(s) = \left|\ [\ D_1(s)\ N_1(s)\ ]\ [\ A_2(s)\ B_2(s)\ ]^T\ \right| = \left|(\ T_\mu s^\mu + \cdots + T_o)\ (\ P_2(s)\ )\ \right| =$$

$$= \left|\ [\ T_\mu\ T_{\mu-1}\ \cdots\ T_o\ ]\ [\ s^\mu P_2(s)\ \ s^{\mu-1}P_2(s)\ \cdots\ P_2\ ]^T\ \right| =$$

$$= \left| \; F_1^{\mu} \; P_2^{\mu}(s) \; \right| \tag{7.48}$$

where

$$P_2(s) = [\; A_2(s) \; B_2(s) \;]^T \;, \qquad F_1^{\mu} = [\; T_{\mu} \; T_{\mu-1} \; \cdots \; T_o \;] \tag{7.49}$$

$$P_2^{\mu}(s) = [\; s^{\mu} P_2(s) \;\; s^{\mu-1} P_2(s) \;\; \cdots \;\; P_2 \;(s)]^T \tag{7.50}$$

clearly this problem is reduced to a constant Determinantal Assignment Problem (DAP).

(ii) $l \geq m$ i.e., Precompensation Configuration

$$f(s) = \left| \; [\; A_1(s) \; B_1(s) \;] \; [\; D_2(s) \; N_2(s) \;]^T \right|$$

$$= \left| \; P_1(s) \; (\; \tilde{T}_{\nu} \; s^{\nu} \; + \; \cdots \; + \; \tilde{T}_o \;) \; \right|$$

$$= \left| \; [\; s^{\nu} P_1(s) \;\; s^{\nu-1} P_1(s) \;\; \cdots \;\; P_1(s) \;] \; [\; \tilde{T}_{\nu} \;\; \tilde{T}_{\nu-1} \;\; \cdots \;\; \tilde{T}_o \;]^T \right|$$

$$= \left| \; P_1^{\nu}(s) \;\; F_2^{\nu} \; \right| \tag{7.51}$$

where

$$P_1(s) = [\; A_1(s) \;\; B_1(s) \;] \qquad P_1^{\nu}(s) = [\; s^{\nu} P_1(s) \;\; s^{\nu-1} P_1(s) \;\; \cdots \;\; P_1(s) \;]$$

$$F_2^{\nu} = \; [\; \tilde{T}_{\nu} \;\; \tilde{T}_{\nu-1} \;\; \cdots \;\; \tilde{T}_o \;]^T \tag{7.52}$$

Again this is a problem of a constant DAP type.

7.6 Decentralised Control Problems

We consider the pole–assignment problems for the decentralised case and we shall show that the fixed dynamics control problems are reduced to constant versions of the decentralised DAP

Assume the $\mathbb{R}[s]$–coprime MFD's

$$G(s) = A^{-1}B = \tilde{B}\tilde{A}^{-1} \qquad m \times l \tag{7.53}$$

$$C(s) = D^{-1}N = \tilde{N}\tilde{D}^{-1} \qquad l \times m \tag{7.54}$$

and that C(s) is decentralised, i.e.

$$C(s) = \begin{bmatrix} C_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_k(s) \end{bmatrix} \tag{7.55}$$

If $C_i(s) = D_i^{-1}N_i = \tilde{N}_i\tilde{D}_i^{-1} \in \mathbb{R}^{l_i \times m_i}(s)$ are the particular controllers, where the MFD's are $\mathbb{R}[s]$ co-prime. Then

$$C(s) = \begin{bmatrix} \tilde{N}_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{N}_k(s) \end{bmatrix} \begin{bmatrix} \tilde{D}_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{D}_k(s) \end{bmatrix}^{-1} = \tilde{N}\,\tilde{D}^{-1} \tag{7.56}$$

$$= \begin{bmatrix} D_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_k(s) \end{bmatrix}^{-1} \begin{bmatrix} N_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_k(s) \end{bmatrix} = D^{-1}N$$

Remark 7.9: The MFD's in (7.56), are co-prime iff t he $C_i(s) = D_i^{-1}N_i = \tilde{N}_i\tilde{D}_i^{-1}$ MFDs are co-prime.

□

7.6.1. Expression of Decentralised Closed-Loop Pole Polynomial

$$f(s) = \left| \underbrace{\begin{bmatrix} D(s) & N(s) \end{bmatrix}}_{\leftarrow\, l+m\, \rightarrow} \updownarrow \underbrace{\begin{bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \end{bmatrix}}_{\leftarrow l \rightarrow} \begin{matrix} \uparrow \\ l+m \\ \downarrow \end{matrix} \right| \tag{7.57}$$

or

$$f(s) = \left| \underbrace{\begin{bmatrix} A(s) & B(s) \end{bmatrix}}_{\leftarrow\, m+l\, \rightarrow} \updownarrow m \underbrace{\begin{bmatrix} \tilde{D}(s) \\ \tilde{N}(s) \end{bmatrix}}_{\leftarrow m \rightarrow} \begin{matrix} \uparrow \\ m+l \\ \downarrow \end{matrix} \right| \tag{7.58}$$

Taking into account the special form of $(\tilde{N}, \tilde{D})$, (N, D), the above conditions may be rewritten as

$$f(s) = \left| [A_1(s), \cdots, A_k(s) \vdots B_1(s), \cdots, B_k(s)] \begin{bmatrix} \tilde{D}_1(s) & & 0 \\ & \ddots & \\ 0 & & \tilde{D}_k(s) \\ \cdots & \cdots & \cdots \\ \tilde{N}_1(s) & & 0 \\ & \ddots & \\ 0 & & \tilde{N}_k \end{bmatrix} \right|$$

or

$$f(s) = \left| \underbrace{[A_1(s), \ B_1(s)}_{m_1+l_1} \vdots \ \cdots \ \vdots A_k(s), \ B_k(s)]\updownarrow m \begin{bmatrix} \begin{matrix} \tilde{D}_1(s) \\ \tilde{N}_1(s) \end{matrix} \updownarrow m_i+l_i & & 0 \\ \leftarrow m_i \rightarrow & & \\ & \ddots & \\ & & \tilde{D}_k(s) \\ 0 & & \tilde{N}_k(s) \end{bmatrix} \right|$$

$$= \left| [P_1(s), \cdots, P_k(s)] \begin{bmatrix} \tilde{T}_1(s) & & 0 \\ & \ddots & \\ 0 & & \tilde{T}_k(s) \end{bmatrix} \right|$$

or decentralised formula I

$$f(s) = \left| \begin{bmatrix} P_1(s) \ \tilde{T}_1(s) \vdots & \cdots & \vdots P_k(s) \ \tilde{T}_k(s) \end{bmatrix} \right| \tag{7.58}$$

here

$$P_i(s) = [A_i(s), B_i(s)] \in \mathbb{R}^{m \times (m_i + l_i)}[s] \tag{7.59}$$

$$\tilde{T}_i(s) = \begin{bmatrix} \tilde{D}_i(s) \\ \tilde{N}_i(s) \end{bmatrix} \in \mathbb{R}^{(m_i + l_i) \times m_i} \tag{7.60}$$

A similar formalisation may be given for other expression of f(s), i.e.

$$f(s) =$$

$$= \left| \begin{bmatrix} \underbrace{D_1(s) \quad N_1(s)}_{\leftarrow l_1+m_1 \rightarrow} & \vdots & & & \\ & & \ddots & & \\ & & & \vdots & \overbrace{D_k(s) \quad N_k(s)}^{\leftarrow l_k+m_k \rightarrow} \end{bmatrix} \begin{bmatrix} \tilde{A}_1(s) \\ \tilde{B}_1(s) \\ \cdots \\ \vdots \\ \cdots \\ \tilde{A}_k(s) \\ \tilde{B}_k(s) \end{bmatrix} \right|$$

or Decentralised Formula II

$$f(s) = \left\| \begin{bmatrix} T_1(s) & \tilde{P}_1(s) \\ & \\ & \\ T_k(s) & \tilde{P}_k(s) \end{bmatrix} \right\| \tag{7.61}$$

where

$$\tilde{P}_i(s) = \begin{bmatrix} \tilde{A}_i(s) \\ \tilde{B}_i(s) \end{bmatrix} \in \mathbb{R}^{(l_i+m_i)\times l_i}[s] \tag{7.62}$$

$$T_i(s) = [D_i(s), N_i(s)] \in \mathbb{R}^{l_i \times (l_i+m_i)}[s] \tag{7.63}$$

Remark 7.10 : The set of $\{P_i(s),\ \ i \in \underline{k}\}$, $\{\tilde{P}_i(s), i \in \underline{k}\}$ are system invariants modulo left, right $\mathbb{R}[s]$–equivalent, respectively.

□

7.6.2 Transformation of Decentralised Dynamic Control Problems to Constant Decentralised Problems

From decentralised formulae (I), (II) it follows that the nature of each channel controller may be treated individually i.e. one controller may be constant, the other P-I, P-D, or any other type. Central to this analysis are the channel matrices defined by:

(a) Right i-th Channel Matrix (from formula I)

$$Q_i(s) = P_i(s)\,\tilde{T}_i(s) \in \mathbb{R}^{m \times m_i}[s] \tag{7.64}$$

$$P_i(s) = [A_i(s),\ B_i(s)\ ] \in \mathbb{R}^{(m_i+l_i)\times m_i}[s]$$

$$\tilde{T}_i(s) = \begin{bmatrix} \tilde{D}_i(s) \\ \tilde{N}_i(s) \end{bmatrix} \in \mathbb{R}^{(m_i+l_i)\times m_i}[s] \tag{7.65}$$

(b) Left i-th Channel Matrix (from formula II)

$$Q'_i(s) = T_i(s)\,\tilde{P}_i(s) \in \mathbb{R}^{l_i \times l}[s] \tag{7.66}$$

$$\tilde{P}_i(s) = \begin{bmatrix} \tilde{A}_i(s) \\ \tilde{B}_i(s) \end{bmatrix} \in \mathbb{R}^{(l_i+m_i)\times l}[s], \qquad T_i(s) = [D_i(s),\ N_i(s)] \in \mathbb{R}^{l_i \times (l_i+m_i)}[s] \tag{7.67}$$

All previous results on transforming dynamic problems to constant DAP problems are applicable here in the following way.

Case (1): Constant Controllers

$$\tilde{T}_i(s) = \begin{bmatrix} I \\ \tilde{N}_i \end{bmatrix}, \qquad T_i(s) = [I,\ N_i] \tag{7.68}$$

Thus,

$$Q_i(s) = [A_i(s),\ B_i(s)]\begin{bmatrix} I \\ \cdots \\ \tilde{N}_i \end{bmatrix} = P_i(s)\tilde{F}_i^D \tag{7.69}$$

$$Q'_i(s) = [I,\ N_i]\begin{bmatrix} \tilde{A}_i(s) \\ \tilde{B}_i(s) \end{bmatrix} = F_i^D\tilde{P}_i(s) \tag{7.70}$$

Case (2): P–I Controllers

$$\tilde{T}_i(s) = \begin{bmatrix} sI_{m_i} \\ sK_{o_i}+K_{1_i} \end{bmatrix}, \qquad T_i(s) = [sI_{l_i},\ sK_{o_i}+K_{1_i}]$$

$$Q_i(s) = [A_i(s),\ B_i(s)] \begin{bmatrix} sI_{m_i} \\ sK_{o_i}+K_{1_i} \end{bmatrix} = sA_i(s)+sB_i(s)K_{o_i}+B_i(s)K_{1_i} =$$

$$= [sA_i(s),\ sB_i(s),\ B_i(s)] \begin{bmatrix} I_{m_i} \\ K_{o_i} \\ K_{1_i} \end{bmatrix} = P_i^{\alpha}(s)\ \tilde{F}_{m_i} \tag{7.71}$$

$$Q'(s) = [sI,\ sK_{o_i}+K_{1_i}] \begin{bmatrix} \tilde{A}_i(y) \\ \tilde{B}_i(s) \end{bmatrix} = s\tilde{A}_i(s)+K_{o_i}\tilde{B}_i(s)\cdot s+K_{1_i}\tilde{B}_i(s) =$$

$$= \begin{bmatrix} \underset{\leftarrow l \rightarrow}{I_{l_i}} & \underset{\leftarrow m \rightarrow}{K_{o_i}} & \underset{\leftarrow m \rightarrow}{K_{1_i}} \end{bmatrix} \begin{bmatrix} s\tilde{A}_i(s) \\ s\tilde{B}_i(s) \\ \tilde{B}_i(s) \end{bmatrix} = F_{l_i}^{\alpha}\tilde{P}_i^{\alpha}(s) \tag{7.72}$$

Note 7.1: α refers to P–I, o to constant controller

Remark 7.11 : Similar analysis applies to the other forms of controller. We may use the previous case results (P–D, P–I–D, etc.).

□

Summary: Let (τ_i) denote the type of controller that corresponds to i–th channel. Then the matrices $Q_i(s)$, $Q_i'(s)$ may be factorised as

$$Q_i(s) = P_i(s)\ \tilde{T}_i(s) = P_i^{(\tau_i)}\tilde{T}_i(s) = P_i^{(\tau_i)}(s)\cdot\tilde{F}_i^{(\tau_i)} \tag{7.73}$$

$$Q'_i(s) = T_i(s)\ \tilde{P}_i(s) = F_i^{(\tau_i)}\cdot\tilde{P}_i^{(\tau_i)}(s) \tag{7.74}$$

where $P_i^{(\tau_i)}(s)$, $\tilde{P}_i^{(\tau_i)}(s)$ are the i–th channel polynomial matrices defined from the (τ_i)–type and $P_i(s)$, $\tilde{P}_i(s)$ and $\tilde{F}_i^{(\tau_i)}$, $F_i^{(\tau_i)}$ are constant design parameter matrices defined from the (τ_i)–type of controller.

From Decentralised formulae we have:

$$f(s) = \left| \begin{bmatrix} F_1^{(\tau_1)} & & & & 0 \\ & \ddots & & & \\ & & F_i^{(\tau_i)} & & \\ & & & \ddots & \\ 0 & & & & F_k^{(\tau_k)} \end{bmatrix} \begin{bmatrix} \tilde{P}_1^{(\tau_1)}(s) \\ \vdots \\ \tilde{P}_i^{(\tau_i)}(s) \\ \vdots \\ \tilde{P}_k^{(\tau_k)}(s) \end{bmatrix} \right| = \left| K^* \tilde{P}^*(s) \right| \tag{7.75}$$

or

$$f(s) = \left| [P_1^{(\tau_1)}(s), \cdots, P_i^{(\tau_i)}(s), \cdots, P_k^{(\tau_k)}(s)] \begin{bmatrix} \tilde{F}_1^{(\tau_1)} & & & & 0 \\ & \ddots & & & \\ & & \tilde{F}_i^{(\tau_i)} & & \\ & & & \ddots & \\ 0 & & & & \tilde{F}_k^{(\tau_k)} \end{bmatrix} \right| =$$

$$= \left| P^*(s)\ \tilde{K}^* \right| \tag{7.76}$$

which are both in the form of decentralised constant DAP.

Assume P–I decentralised controllers, then from the second formula we have:

$$f(s) = \left| \begin{bmatrix} F_{l_1} & & 0 \\ & \ddots & \\ 0 & & F_{l_n} \end{bmatrix} \begin{bmatrix} \tilde{P}_1^{\alpha} \\ \vdots \\ \tilde{P}_n^{\alpha} \end{bmatrix} \right| \tag{7.77}$$

where

$$F_{l_i} = [I_{l_i}, K_{o_i}, K_{1_i}] \in \mathbb{R}^{l_i \times (l_i + 2m_i)} \tag{7.78}$$

$$\tilde{P}_i^{\alpha}(s) \in \mathbb{R}^{(l_i + 2m_i) \times l}[s] \tag{7.79}$$

Conditions, necessary , for solvability, fixed modes etc. may be discussed within this framework.

7.7 Conclusions

Although the algebrogeometric tools have been applied only in the case of Decentralised pole assignment by constant output feedback, the results readily extend to the case of zero assignment. It is worth pointing out that the machinery based on Schubert varieties and calculus which has been used in the centralised case for the derivation of conditions for existence of real solutions does not apply here in a straightforward manner. Further work is needed for the derivation of sufficient conditions for the existence of real solutions.

CONCLUSIONS

An Exterior Algebra and classical Algebraic Geometry framework has been developed which handles both generic and exact problems arising in Decentralised control of large scale systems. The same philosophy provides also a computational framework for the evaluation of various control architectures as possible solutions to control problems.

Dynamic problems of frequency assignment may also be reduced to the same formulation of a constant DAP or D-DAP and this demonstrates the generality of the approach as far as variety of problems that may be handled. Although Decentralised control problems have been considered here, the special machinery developed for D-DAP is also suitable for simple type of control where simple refers to a control scheme with sparse structure. In the latter case the technique may also be used for the development of diagnostics for the suitability of certain control structures.

Although Exterior Algebraic tools create no problems as far as D-DAP , the Algebraic Geometry tools are underdeveloped for the specific application and this is reflected in the fact that necessary and sufficient conditions for the frequency assignment problems by complex Decentralised controllers may be derived, but there are no sufficient conditions for constant controllers as in the centralised case. The advantages and disadvantages of the present framework are considered next together with a set of tasks for further research.

The root of the difficulties in deriving sufficient conditions for the existence of solutions lies in that there is no theory of Schubert varieties of the Decentralised Grassmann variety. Because of that it is difficult to establish sufficient conditions by deploying the order of such varieties which is the basis of the technique used for the centralised case.

The development of generic solvability conditions heavily relies on further work on this mathematical area and this is an important task for future research. The investigation of tools which may provide an alternative working environment to that given by the Schubert varieties is also worth examining. This research plan may contribute to the derivation of alternative sufficient conditions.

In summary control problems with partially fixed control structure may be handled with some difficulty when we address the question of sufficient conditions but necessary conditions are readily derived since computation of dimensions of corresponding varieties can cause no problem. These necessary conditions characterise general features of the Decentralised control scheme and may be used in a negative way that is for the exclusion of bad control structure options.

The exterior algebra tools such as the Decentralisation Index, Decentralised Plücker matrices provide alternative necessary conditions for the solvability of Decentralised frequency assignment problems. In fact in this way they may be used as complementary tests for the selection of control structure. It should also be mentioned that the algebraic-geometry based criteria are parameter independent since they are defined by the Decentralisation groupings, the McMillan degree and the Forney order whereas exterior algebra based tools explicitly depend on model parameter as well as the decentralised groupings.

The Decentralised framework described by D-DAP has been shown to be suitable for studying problems of fixed dynamic complexity of sparse decentralised structure. Problems such as Decentralised pole assignment by P-I, P-D, P-I-D controllers is reduced to the D-DAP framework. Necessary conditions for solvability of such problems may be derived using similar procedures for the constant case. There are some technicalities associated with causality of the resulting control scheme but the procedure in general is quite similar. Such tests may also be used for the evaluation of control structures. Extension of the approach to that area is also important and it is left for future work.

The new characterisation of fixed modes ,zeros and the extension of the notion to that of almost fixed mode,zeros has two distinct advantages. First it is connected to the general computational framework of the D-DAP approach and second it provides an interesting dynamic characterisation of the almost fixed modes as modes with restricted mobility under feedback compensation.

The technique for the evaluation of the trapping discs may be easily implemented and it also leads to narrow bounds for frequency mobility . An important problem in this area which has not been addressed before is the behaviour of almost fixed modes, zeros under dynamic Decentralised schemes. Preliminary results in this area although not reported in the thesis indicate an increase in the dimension of the trapping discs but no significant influence on the location of almost fixed modes, zeros. Deriving results in this area is important since they may be used to qualify the minimal order dynamics for which trapping discs may become infinite and thus permit unbounded mobility of modes, zeros under compensation. The key notion of almost zero as it has been introduced in [Kar.4] needs some improvement as far as defining the location of almost zero. It should be noted that this definition depends on scaling of the polynomials and needs some improvement.This is also a task for future research.

For the concepts and tools to be useful in practice for the evaluation of control structures there is a need for the development of the computational framework of D-DAP. Work in this area has already progressed for the centralised case (Ph.D thesis M.Mitrouli) and the basic algorithms for the exterior algebra computations is addressed there and may also be used for D-DAP. The additional feature here is the computation of the Decentralisation Index which has been described in the appropriate chapter.

The computation of solutions of DAP may be reduced to an optimisation problem where the linear part defines a performance index to be minimised and the quadratic Plucker relations define the equality constraints. The same philosophy and algorithm may also be applied to the D-DAP. In fact the additional characteristics are those defined by the Decentralised constraints. This computational method has the potential to develop to a C.A.D. technique for constant or simple dynamic decentralised controllers for large scale systems. In fact when the necessary conditions are satisfied the algorithm may be used for the computation of solutions despite the fact that sufficient solutions may not exist. In this design approach the optimisation algorithm is of high importance and the resulting solutions are the appropriate ones. Two important tasks here are the selection of the algorithm which may perform well with arbitrary initial conditions, its convergence and second the interpretation of the approximate frequency assignment solutions. Those two problems should not be underestimated and need to be seriously addressed in order for the D-DAP methodology to form the basis of a design technique for Decentralised sparse, simple dynamic order controllers. Its ability to tackle all the latter cases indicates that there is a lot of potential in that area.

D-DAP as well as DAP address the problem of frequency assignment and also stabilisation in an indirect manner. Addressing the problem of stabilisation within DAP is difficult since the coefficient vectors of stable polynomials do not have easily handled properties by the present algebrogeometric tools. The exterior algebra tools however may also be used for the decentralised stabilisation and this is a topic under investigation.

In summary D-DAP was shown to be a useful methodology for addressing issues associated to the selection of control structures and synthesis of Decentralised controllers. It also has the potential to form the basis of a C.A.D technique which may handle arbitrary sparse structure and limitations in dynamic order of the controllers. For the time being such techniques do not exist....

REFERENCES

[And.1] ANDERSON, B.D.O and D.J. CLEMENTS: "Algebraic characterisation of fixed modes in decentralised control" *Automatica*, Vol. 17, pp. 703-712, 1981.

[And.2] ANDERSON, B.D.O : "Transfer function matrix description of decentralised fixed modes " *IEEE Trans. Autom. Contr.*, Vol. 27, pp. 1176-1182, 1982.

[And.3] ANDERSON, B.D.O and J.B. MOORE : "Time - varying feedback laws for decentralised control" *IEEE Trans. Autom. Contr.*, Vol. 26, pp. 1133-1139, 1981.

[Aok.1] AOKI, M : "On feedback stabilisability of Decentralised Dynamic systems" *Automatica*, Vol. 8, pp. 163-173, 1972.

[Arm.1] ARMENTANO, V and M.G.SINGH: "A procedure to eliminate fixed modes with reduced information pattern" *IEEE Trans. Autom. Contr.*, Vol. 27, pp.258-260, 1982.

[Ath.1] ATHANS, M: "Special Issue on Large Scale Systems" *IEEE Trans. Autom. Contr.*, Vol. 23, April 1982.

[Ath.2] ATHANS, M: "Advances and open problems on the control of Large Scale Systems " *Proceed. of Sev. IFAC Congr.*, Finland, 1978.

[Ath.3] ATHANS, M: "The expert team of experts approach to Command and Control organisations" *IEEE Contr. Mag.*, 1982.

[Bir.1] BIRKHOFF, G and S. MACLANE : "A survey of Modern Algebra", Mac-Millan, N.Y. 1965.

[Cal.1] CALLIER, F.M and C.A. DESOER: "Multivariable Feedback Systems" , Springler-Verlag, N.Y., 1982.

[Che.1] CHEN, C.T: "Linear System theory and design", H.R.W, N.Y, 1984.

[Cor.1] CORFMAT, J.P and A.S. MORSE: "Decentralised control of linear multivariable systems", *Automatica*, Vol. 12, pp. 479-495, 1976.

[Dav.1] DAVISON, E.J. and U. OZGUNER: "Characterisation of Decentralised fixed modes for interconnected systems" *Automatica*, Vol. 12, pp. 169-182, 1983.

[Dav.2] DAVISON, E.J: "Connectability and Structural Controllability of Composite Systems", *Automatica*, Vol. 13, pp. 109-123, 1977.

[Dav.3] DAVISON, E,J: "The robust Decentralised control of a servomechanism problem for composite systems with input -output interconnections", *IEEE Trans. Autom. Contr.*, Vol. 24, pp. 325-328, 1979.

[Dav.4] DAVISON, E.J and S.H. WANG: "Properties and calculation of transmission zeros", *Automatica* , Vol. 10, pp. 643- 650, 1974.

[Dav.5] DAVISON, E.J and S.H. WANG: "A characterisation of decentralised fixed modes in terms of transmission zeros", *IEEE Trans. Autom. Contr.*, Vol. 30, pp. 81-82, 1985.

[For.1] FORNEY, G.D: "Minimal bases of rational vector spaces with applications to multivariable linear systems", *SIAM J. Contr. and Optim.*, Vol. 13, pp. 493-520, 1975.

[Gan.1] GANTMACHER, F.R: "The theory of matrices" , Chelsea, N.Y, 1959.

[Gia.1] GIANAKOPOULOS, C: "Frequency assignment problems of linear multivariable problems", Ph.D Thesis, City University, London, 1984.

[Gre.1] GREUB, W.H: "Multilinear Algebra", Springler-Verlag, N.Y, 1967.

[Gro.1] GROUMPOS, P.P and K.A. LOPARO: "Structural control of large scale systems", *IEEE C.D.C. Conf*, 1980.

[Fes.1] FESSAS. P: "Decentralised Control of linear dynamical systems via polynomial matrix methods", *Int. Journ. Contr.*, Vol. 32, pp. 127-147, 1980.

[Fes.2] FESSAS, P: " A note on "An example in Decentralised Control systems", *IEEE Trans. Autom. Contr.*, Vol. 24, pp. 669, 1979.

[Hen.1] HENRICI, P: "Applied and Computational Complex Analysis", J. Wiley, N.Y, 1974.

[Her.1] HERMANN, R and C.F. MARTIN: "Applications of algebraic geometry to systems theory", *IEEE Trans. Autom. Contr.*, Vol. 22, pp.19-27, 1977.

[Ho. 1] HO, Y.C and A.S. MITTER: "Directions in Large Scale Systems.Many person optimisation and Decentralised Control", Plenum, N.Y, 1976.

[Hod.1] HODGE W.N.D and D. PEDOE: "Methods of Algebraic Geometry", Cambridge University Press, 1952.

[Ike.1] IKEDA, M and D.D SILJAK: "Counterexamples to Fessas' conjecture", *IEEE Trans. Autom. Contr.*, Vol. 24, pp. 670, 1979.

[Kai.1] KAILATH, T: "Linear Systems", Prentice Hall, 1980.

[Kar.1] KARCANIAS,N: "Geometric theory of zeros and its use in feedback analysis", Ph.D Thesis, University of Manchester, 1975.

[Kar.2] KARCANIAS, N: "The use of Grassmann products in the computation of the invariant zero polynomial of square linear proper systems", *Int. Jour. Contr.*, Vol. 28, pp. 361-381, 1978.

[Kar.3] KARCANIAS, N and B.KOUVARITAKIS: "The output zeroing problem and its relationship to the invariant zero structure", *Int. Jour. Contr.*, Vol. 30, pp. 395-415, 1979

[Kar.4] KARCANIAS,N., C. GIANAKOPOULOS,and M.HUBBARD:"Almost zeros of a set of polynomials", *Int. Jour. Contr.*, Vol. 38, pp. 1213-1238, 1983.

[Kar.5] KARCANIAS,N and C.GIANAKOPOULOS: "Grassmann invariants, almost zeros and the determinantal zero, pole assignment problems of linear multivariable systems", *Int. Jour. Contr.*, Vol. 40, pp. 673-698, 1984.

[Kar.6] KARCANIAS, N and C.GIANAKOPOULOS: "Pole assignment of strictly proper and proper linear systems by constant output feedback", *Int. Jour. Contr.*, Vol. 42, pp. 543-565, 1985.

[Kar.7] KARCANIAS. N : Personal Communication

[Kar.8] KARCANIAS,N., LAIOS.B and C.GIANAKOPOULOS: "Decentralised Determinantal assignment problems: fixed and almost fixed modes and zeros", *Int. Jour. Contr.*, Vol.48, pp. 129−147, 1988.

[Ken.1] KENDIG, K: "Elementary Algebraic geometry", Springer- Verlag, N.Y, 1977.

[Kim.1] KIMURA, H: "Pole assignment by output feedback", *IEEE Trans. Autom. Contr.*, Vol. 20, pp. 509-516, 1975.

[Kou.1] KOUVARITAKIS,B and A.G.J.MACFARLANE: "Geometric approach to analysis and synthesis of system zeros", *Int. Jour. Contr.*, Vol. 23, pp. 149-181, 1976.

[Kou.2] KOUVARITAKIS,B and J.M.EDMUNDS: "The characteristic frequency and characteristic gain design method for multivariable feedback systems", *Proc. of the Sympos. on altern. for linear multivariable control*, pp. 229-246, 1978.

[Kuc.1] KUCERA, V: "Discrete Linear Control", Wiley, 1979.

[Loc.1] LOCATELLI,V and T.TARANTINI: "Pole assignment : Role and choice of the underlying information pattern", *Ricerc. di Autom.*, Vol. 8, pp. 107-126, 1977.

[Mac.1] MACFARLANE,A.G.J. and N.KARCANIAS: "Poles and zeros of linear multivariable systems: a survey of algebraic, geometric and complex variable theory", *Int.Jour. Contr.*, Vol. 24, pp. 33-74, 1976.

[Mac.2] MACFARLANE,A.G.J and N.KARCANIAS: "Relationships between state-space and frequency response concepts", *Proc. of the 7th IFAC cong.*, pp.1771-1781, 1978.

[Mar.1] MARCUS,M: "Finite dimensional Multilinear algebra", Marcel and Dekker, N.Y. 1973.

[Mar.2] MARCUS,M and H.MINC: "A survey of Matrix theory and matrix inequalities", Allyn and Bacon, Boston, 1964.

[Mars.1] MARDEN,M: "The geometry of zeros of a polynomial of a complex variable", *Amer. Math. Soc. Publ.*, 1949.

[Mart.1] MARTIN,C.F and R.HERMANN: "Applications of algebraic geometry to systems theory: the Macmillan degree and the Kronecker indices as topological and holomorphic invariants", *SIAM Jour. Contr. and Optim.*, Vol. 16, pp. 343-355, 1978.

[Mic.1] MICHELL, A.N and R.K.MILLER: "Qualitative analysis of large scale dynamical systems", Academic Press, N.Y. 1977.

[Ros.1] ROSENBROCK,H.H: "State-space and Multivariable Theory", Nelson, London, 1970

[Ros.2] ROSENBROCK, H.H and A.C.PUGH: "Contributions to a Hierarchical theory of systems", *Int. Jour. Contr.*,Vol. 19, pp. 845-867, 1974.

[Sae.1] SAEKS, R: "Large Scale Dynamical Systems", Point Lobos Press, L.A., 1976.

[Sae.2] SAEKS, R: "On the decentralised control of interconnected dynamical systems", *IEEE Trans. Autom. Contr.*, Vol. 24, pp. 269-271, 1979.

[Sen.1] SENNING, M.F: "Feasibly Decentralised control", Ph.D Thesis, ETH, Zurich, 1979.

[Sez.1] SEZER, M.E and O.HUSEYIN: "Stabilisation of linear time invariant interconnected systems using local state feedback", *IEEE Trans. Syst. Man Cybern.*, Vol. 8, pp. 751-756, 1978.

[Sez.2] SEZER, M.E and D.D. SILJAK: "Structurally fixed modes", *Systems and Control Letters*, Vol.1, pp.60-64, 1981.

[Sez.3] SEZER, M.E and D.D. SILJAK: "On structural decomposition and stabilisation of Large Scale Systems", *IEEE Trans. Autom. Contr.*, Vol. 26, pp. 439-444, 1981.

[Sil.1] SILJAK, D.D: "Large Scale Dynamic Systems", North Holland, 1978.

[Tra.1] TRAVE, L., A.M. TARAS and A. TITLI: "An application of vibrational control to cancel unstable decentralised fixed modes", *IEEE Trans. Autom. Contr.*, Vol. 30, pp. 283-286, 1985.

[Vid.1]VIDYASAGAR,M and N.VISWANADAM: "Algebraic characterisation of decentralised fixed modes" *Systems and Control Letters*,Vol.3, pp. 69-72, 1983.

[Wal.1] WALKER, R.U: "Algebraic curves" Springer-Verlag, N.Y., 1979.

[Wan.1] WANG, S.H. and E.J.DAVISON: "On the stabilisation of Decentralised control systems", *IEEE Trans. Autom. Contr.*, Vol. 18, pp. 473-478, 1973.

[Wes.1] WEST-VUKOVICH G.S, E.J.DAVISON and P.C.HUGHES: "The Decentralised control of large flexible space structures", *IEEE Trans.Autom.Cont.*, Vol.29, pp. 866-880,1984.

[Wil.1] WILLEMS,J.C. and W.H.HESSELINK: "Generic properties of the pole placement problem" Proc. 7th IFAC Congress ,pp. 1725-1731, 1978.

[Wol.1] WOLOVICH,W.A: "Linear Multivariable Systems" , Springer-Verlag, N.Y., 1974.

[Wol.2] WOLOVICH, W.A. and P.ANTSAKLIS: "The Diofantine equations with applications" *SIAM Jour. Contr. Optim.*, pp. 777-787, 1984.

[Won.1] WONHAM, W.M: "Linear Multivariable Control: A Geometric approach", Springer-Verlag, N.Y., 1979.

[Wym.1] WYMORE, A.W: "A Mathematical theory of Systems Engineering", Wiley, N.Y., 1967