Extensions of Integrable Quantum Field Theories Based on Lorentzian Kac-Moody Algebras

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Quantum field theory arose out
of our need to describe the
ephemeral nature of life
- Anthony Zee

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Declaration

This work was carried out while studying for the degree of Doctor of Philosophy at City, University of London. I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgment, the work presented is entirely my own. Powers of discretion are granted to the Director of Library Services to allow this thesis to be copied in whole or in part without further reference to the author. This permission covers only single copies made for study purposes, subject to normal conditions of acknowledgement.

Abstract

In this thesis, we develop a framework to study n-extensions of Kac-Moody algebras, and use the resulting Lorentzian algebras to study Lorentzian extension of Toda field theories and their integrability. We begin our discourse by providing context and motivation for the study of these ideas, mainly through illustrating the unmatched historical successes of quantum field theory in science and the role of symmetry algebras and integrability in mathematical physics.

Continuing, we develop a new framework to extend finite, $\mathfrak{g}_f$ Kac-Moody algebras, through their simple root structure n times to what we name n-extended Lorentzian Kac-Moody algebras, $\mathfrak{g}_{-n}$. Using constants from the Casimir operators of $\mathfrak{g}_{-n}$, we find a novel type of decomposition of $\mathfrak{g}_{-n}$. We derive conditions in which these decompositions are possible, and tabulate all possible decompositions of $\mathfrak{g}_{-n}$.

Applying the methods we developed in the construction of $\mathfrak{g}_{-n}$, we build *Lorentzian* Toda field theories as extensions of Toda field theories based on $\mathfrak{g}_f$. We find that on each subsequent addition of simple roots in the n-extension procedure results in the Lorentzian Toda field theory alternating between conformal and massive behaviour. We calculate mass ratios for the massive theories, and using the Painlevé test, we find that some of these Lorentzian models can not be integrable.

Examining another class of Lorentzian Toda field, which we name the *null root* models, we show that these pass the Painlevé test. Furthermore, we show that these models can also possess the more restrictive Painlevé property, showing this explicitly for a spin-3 rank-2 example, meaning that this example is integrable. The procedure used is generalizable, and we therefore conclude that more models from the null root class of Lorentzian Toda field theories are very likely to also be integrable models.

Chapter 1

Introduction

Quantum field theories are the most successful theories within any and all scientific disciplines. For example, quantum electro dynamics (QED) is by far the precisest theory in existence, with its electromagnetic fine-structure constant having been experimentally confirmed to precisions at the scale of 10^{-11} [1], through a diverse array of experimental methods with increasing accuracy over the years [1, 2, 3, 4, 5, 6]. In addition to QED, quantum field theory also underpins the electroweak (EW) and strong force governed quantum chromodynamics (QCD) sectors of the standard model of particle physics [7, 8, 9], and, along with General Relativity [10], it is a corner-stone of most descriptions of physics beyond the standard model.

Such physical theories aim to describe phenomena that the standard model itself falls short of explaining. For one, the standard model cannot explain observations such as the asymmetry between matter and anti-matter, neutrino oscillation, the strong CP problem, the nature of dark matter and dark energy, or even its choice of its own parameters [11]. Additionally, at a fundamental level the standard model omits the gravitational description of the universe, and as the most successful theory of gravitation, there have been many attempts to combine General Relativity with quantum field theories, hoping to create a unified theory that can answer questions each theory alone cannot.

The most widely studied physical theory that aims to combine relativity with quantum field theory is string theory [12, 13, 14, 15, 16] - a theory of quantum gravity that introduces up to 11-dimensios of spacetime in its formulation known as M-theory [17]. M-theory is of particular interest as through dualities the other five categories of string theories can be recovered, making it essential in unifying the mathematical descriptions of string theory. As it exists in one extra dimension than the maximum number of dimensions any other type of string theory may possess, it correspondingly has a larger symmetry algebra known as E_{11} [18] that is larger than the E_{10} symmetry algebra that certain 2-dimensional reductions of M-theory are known to be invariant under [19].

As we have been alluding to, these symmetry groups and algebras are invaluable in the construction of all the field theories mentioned so far. In particular, Lie groups, Lie algebras, and the latter's generalizations: Kac-Moody algebras [20], which we shall often denote by $\mathfrak{g}$, describe and contain information about the symmetries the fields in these field theories obey. While E_{10} and E_{11} are both Kac-Moody algebras, they belong to different categories, E_{10} is a hyperbolic $\mathfrak{g}$ [21], however E_{11} does not fall into this hyperbolic category and is known as a *Lorentzian* Kac-Moody algebra, $\mathfrak{g}$ [22]. Motivated partially by its appearance in string theory and the fact that Lorentzian Kac-Moody algebras are less well-defined in relation to the hyperbolics, efforts have been made to study a larger class of Kac-Moody algebras that are also Lorentzian and include E_{11} [23, 24], and highlighted the potential of extending Kac-Moody algebras on the level of their Dynkin diagrams through the addition of extra nodes on the diagrams in well-defined positions defining what is known as the n-extended Lorentzian Kac-Moody algebras $\mathfrak{g}_{-n}$ [25], allowing E_{10} to be extended to E_{11}, for example see [24] and sections 2.3.3-2.4 for a further and more precise definition. Notationally, starting from finite Kac-Moody algebras, $\mathfrak{g}_f$ and their affine extension, $\mathfrak{g}_a$, both of which we shall elucidate in more detail shortly, this extension procedure follows the pattern

$$\mathfrak{g}_f \to \mathfrak{g}_a \to \mathfrak{g}_{-1} \to \mathfrak{g}_{-2} \to \cdots \to \mathfrak{g}_{-(n-1)} \to \mathfrak{g}_{-n}$$

By construction $\mathfrak{g}_{-n}$ contains $\mathfrak{g}_{-(n-1)}$ in general, it then follows that E_{11} contains E_{10}. A more surprising result to find that E_{10} contains every simply laced hyperbolic Kac-Moody algebra as its subalgebras [26], giving E_{10} the richest structure of any hyperbolic $\mathfrak{g}$, again highlighting the importance of better understanding E_{10} and E_{11}, in addition to the construction and decompositions of $\mathfrak{g}_{-n}$ in general.

From a more physical, albeit heuristic perspective, E_{10} and E_{11} both contain the $SU(3) \times SU(2) \times U(1)$ symmetry group of the combined QCD, QED and EW sections of the standard model, adding to the interest of studying E_{10} and E_{11} in relation to theories which could reduce to the standard model in certain situations. On a conceptual level, this is what some string theories hope to achieve, although currently none have come close. However, utilizing the symmetries provided by E_{11} has led to particular success in treating M-theory as gauged supergravity theories though the embedding tensor in dimensions $D \geq 4$ [27, 28, 29, 30]. Emphasizing that most of the successes of E_{11} have come from its physical applications, whereas purely algebraically, even in simple terms of its full classification, let alone its representation theory, much is still to be discovered.

Even E_{10} and the hyperbolic algebras are not well understood compared to the vast data and knowledge available for the aforementioned finite Kac-Moody algebras, $\mathfrak{g}_f$, and their so-called *affine* extensions, $\mathfrak{g}_a$, which have been completely classified with much of their representation theory known [31, 32]. The hyperbolic Kac-Moody algebras are however better understood than the Lorentzians on the level of classification, for example it is known that E_{10} belongs to a class of 238 hyperbolic $\mathfrak{g}$ between ranks 3-10 [33, 34], whereas Lorentzian $\mathfrak{g}$ are less well classified with E_{11} belonging to an infinitely large class that can exist at arbitrarily large ranks, unlike hyperbolics whose ranks cannot exceed the rank-10 of E_{10}.

We do, however, have some insights into E_{10} and E_{11} at deeper levels than merely their classification. For example, for hyperbolic Kac-Moody algebras some root multiplicities are known for certain algebras [33, 35], for E_{10} and the Lorentzian E_{11} they are only known at low level representations of level 18 and 10, respectively [22, 36]. Additionally, Weyl groups have not been identified for any Lorentzian algebras, but have been identified for a few hyperbolic $\mathfrak{g}$ [37, 38], giving their Weyl groups potential to be studied as mathematical objects in isolation, as well as in relation to their uses in classical and quantum field theories.

Weyl groups of $\mathfrak{g}$ are of particular interest when constructing Calogero-Moser-Sutherland systems [39, 40], this can be seen from writing their general Hamiltonian as

$$\mathcal{H}_{\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16}\sum_{\alpha\in\Delta}(\alpha\cdot q)^2 + \frac{1}{2}\sum_{\alpha\in\Delta} g_\alpha V(\alpha\cdot q) \quad m, g_\alpha \in \mathbb{R}, \tag{1.1}$$

which describes n particles moving on a line with conjugate momenta p assembled into canonical coordinates $p, q \in \mathbb{R}^n$. $V(x)$ is the potential term of the system that categories various behaviours of the system: for example, the CMS models are taken with $V(x) = 1/\sin^2(x)$, $V(x) = 1/\sinh^2(x)$ or $V(x) = 1/^2(x)$, whereas the Calogero model has $V(x) = 1/x^2$. α are roots of the root system Δ of $\mathfrak{g}$, and to sum over this root system, like we do in $\mathcal{H}_{\text{CMS}}$, we require the Weyl group, which acts on simple roots of the Kac-Moody algebra by reflecting them in hyperplanes perpendicular to the simple roots. If we were to not sum over all Δ then H_{CMS} would not be invariant under the action of the Weyl group and the integrable features of the model would be lost. For $\mathfrak{g}_f$, the Weyl group action will terminate, giving a finite root system, Δ. Whereas for $\mathfrak{g}_a$, hyperbolic and Lorentzian Kac-Moody algebras Δ will be infinite in size.

As the Weyl group is known for $\mathfrak{g}_f$, and $\mathfrak{g}_a$, this process of obtaining and summing over the entire root space is possible, however for hyperbolic Kac-Moody algebras this can only be conducted for the known Weyl groups in [37, 38], and the first non-supersymmetric hyperbolic Calogero model has only been constructed recently for one of these known hyperbolic Weyl groups, AE_3 [41], which is an extension of the modular group $\text{PSL}(2,\mathbb{Z})$. However, as mentioned above, no known Weyl groups exist for Lorentzian Kac-Moody algebras, the possibility of constructing a Lorentzian Calogero-Moser-Sutherland model will require more mathematical development.

Asides from Calogero-Moser-Sutherland models, Kac-Moody algebras have found many uses within both classical and quantum field theories. Out of all field theories that are built from $\mathfrak{g}$, Toda field theories [42, 43, 44] are easily the most comprehensive, well understood, and arguably retain this status when compared amongst all classical and quantum integrable field theories. The Lagrangian of classical Toda field theory, based on the Kac-Moody algebra $\mathfrak{g}$, lives in two spacetime dimensions and may be written as

$$\mathscr{L}_{\mathfrak{g}} = \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi - \frac{g}{\beta^2}\sum_{i=n}^{r} e^{\beta\alpha_i\cdot\phi} \tag{1.2}$$

the constants $g \in \mathbb{R}$ and $\beta \in \mathbb{R}$, or $\beta \in i\mathbb{R}$ which gives non-trivial solutions to the Yang-Baxter equation [45]. The simple roots α_i, $i \in \{1, \cdots, r\}$ act as a basis of the root space, Δ of a rank-r $\mathfrak{g}$. In (1.2), α_i are associated to r scalar fields, $\phi(x,t)$ of the theory, which posses as many components as the rank of $\mathfrak{g}$, so that the roots are also represented in $\mathbb{R}^n$, rather than in the complex plane. We then have $\phi^a(x,t)$ for $a \in \{1, \cdots, r\}$. Folded $\mathfrak{g}$ have also been constructed and studied, whereby the field components are identified in less general methods than with non-folded algebras [46].

Since the first discrete Toda models were discovered and studied [42], many versions of both the discrete and the continuous $\mathscr{L}_{\boldsymbol{g}}$ theory have been studied. For instance, taking $\mathscr{L}_{\boldsymbol{g}}$ with $n = 1$ and α being the simple roots of $\mathfrak{g}_f$ with rank-r corresponds with the conformal Toda field theories [47, 48]. In a similar way to which we can extend the finite Kac-Moody algebra $\mathfrak{g}_f$ to its affine counterpart, $\mathfrak{g}_a$, we may also alter the field content of $\mathscr{L}_{\boldsymbol{g}_f}$ to $\mathscr{L}_{\boldsymbol{g}_a}$ through taking $n = 0$, by adding to the simple root system an affine root, taken to be the negative of the highest root. This affine root is denoted as α_0, and associated to ϕ_0, which acts as to perturb the potential of the $\mathscr{L}_{\boldsymbol{g}_f}$ into the new affine theory. The $\mathscr{L}_{\boldsymbol{g}_a}$ does not have conformal symmetry and possesses r massive fields, and importantly both $\mathscr{L}_{\boldsymbol{g}_a}$ and $\mathscr{L}_{\boldsymbol{g}_f}$ have the field components set as $a = r$.

$\mathscr{L}_{\boldsymbol{g}_a}$ theories may be further perturbed with two additional fields, η, ζ, to the affine theory, forming the theories known as conformal affine Toda field theories [49, 50], $\mathscr{L}_{\boldsymbol{g}_{ca}}$. As the name suggests, these are also conformal Toda field theories and hence do not contain any massive fields, unlike $\mathscr{L}_{\boldsymbol{g}_a}$. These extensions from finite to conformal affine through successive perturbation of the field contents may be summarized as the following:

$$\underset{\mathscr{L}_{\mathfrak{g}_f}}{\text{CFT}} \quad \xrightarrow{\phi_0} \quad \underset{\mathscr{L}_{\mathfrak{g}_a}}{\text{Massive}} \xrightarrow{\eta,\zeta} \quad \underset{\mathscr{L}_{\mathfrak{g}_{ca}}}{\text{CFT}}$$

Extension procedures similar to those that result in the n-extended Lorentzian Kac-Moody algebras, $\mathfrak{g}_{-n}$, may also be applied to Toda field theories. This leads to a similar extension pattern of alternating conformal and massive theories on each successive field extension and perturbation of the pervious field content [51]. The underlying algebras of these perturbed Lorentzian theories is not quite that of the full $\mathfrak{g}_{-n}$, in fact the simple root content of a massive Lorentzian Toda theory is found to not be a Kac-Moody algebra and is denoted as $\mathring{\boldsymbol{g}}_{-(2n)}$, with the analogy to the highest root, α_0 in finite and affine theories being α_{-2n} for these Lorentzian n-extensions, where α_{-2n} is associated to the field we perturb with, denoted as ϕ_{-2n}. The extension starting from $\mathscr{L}_{\boldsymbol{g}_f}$ to $n = 2$ may be summarized as the following:

$$\underset{\mathscr{L}_{g_f}}{\text{CFT}} \quad \xrightarrow{\phi_0} \quad \underset{\mathscr{L}_{g_a}}{\text{Massive}} \quad \xrightarrow{\phi_{-1}} \quad \underset{\mathscr{L}_{g_{-1}}}{\text{CFT}} \quad \xrightarrow{\phi_{-2}} \quad \underset{\mathscr{L}_{\mathring{g}_{-2}}}{\text{Massive}}.$$

However, in general, it is possible to construct Lorentzian Toda field theories with any integer value of the extension constant n.

Toda field theories are of remarkable interest in relation to $\mathfrak{g}_{-n}$ algebras discussed above, due to their rich history and comprehension as physical theories when based on $\mathfrak{g}_f$, $\mathfrak{g}_a$ algebras. For example, and regarding affine Toda theories in particular, one of these outstanding facts is that the quantum scattering matrices can be constructed to all orders in perturbation theory, through use of the so-called bootstrap approach [52, 53, 54, 55, 56, 57]. This arises from the fact that Toda field theories, as well as Calogero-Moser-Sutherland systems, are *integrable theories* when the underlying Kac-Moody algebra is either $\mathfrak{g}_f$ or $\mathfrak{g}_a$. On the classical level, integrable theories are those consisting of nonlinear differential equations that can, at least in principle, be solved analytically. In comparison, most nonlinear differential equations have more unpredictable behaviour and can only be solved approximately rather than exactly, as is the case for integrable systems. For field theories, this means that we are mainly dealing with partial differential equations with an infinite phase space, so that integrability is uncovered and utilized through a variety of techniques related to the properties of the solutions [58]. Conversely, on the quantum level, the integrability of a theory can be attributed to the feature that an n-particle S-matrix can be factorised into 2-particle S-matrices.

For discrete and continuous formulations of Toda field theories, their integrability has been proven through the construction of Lax pairs [59], zero-curvature conditions [49, 50] and the Painlevé test [60]. Some efforts had previously been made to investigate the integrability of Toda field theories based on hyperbolic $\mathfrak{g}$ through conducting the Painlevé test, which showed that hyperbolic Toda field theories do not have the possibility of being integrable [21]. However, these results do not rule out the possibility of some categories of Lorentzian Toda field theories being integrable, and no such investigations had previously been undertaken.

The fact that Toda theories based on simply laced Kac-Moody algebras have exact scattering matrices is due to their classical mass ratios [61] being preserved to all orders of perturbation theory [52, 53, 54, 55, 62]. These results rely essentially on the theory's integrability, but it must be noted that although a theory retains special status for being integrable, the framework for Toda theories is unique in its ability to find exact S-matrices for the range of models that $\mathfrak{g}_a$ supplies. Such results are also possible for theories based on non-simply laced $\mathfrak{g}$ [63, 64], however, as the masses will have different renormalization factors the results required additional algebraic machinery and utilized properties of q-deformed Coxeter elements [63]. In both cases, it is still the case that the root system of $\mathfrak{g}$ that provides the underlying structure of the systems, in the latter $\mathfrak{g}$ is the dual affine algebra, which physically corresponds to very strong or very weak coupling in the classical limit. As mentioned above, the coupling constant β of (1.2) may be either real or imaginary, and when $\beta \in \mathbb{C}$ the Yang-Baxter equation is not trivially solved as in the $\beta \in \mathbb{R}$ scenario due to quantum Kac-Moody algebras symmetries, in turn allowing the S-matrices to still factor into an exactly solvable form [45].

In this thesis, we will examine the integrability of various Toda field theories based on Lorentzian Kac-Moody algebras. To do so, in chapter 2, we will build up a definition of a new class of Lorentzian Kac-Moody algebras starting from $\mathfrak{g}_f$, extending to $\mathfrak{g}_a$, and continuing these extensions in what we name as an n-extension procedure, resulting in n-extended Lorentzian Kac-Moody algebras, $\mathfrak{g}_{-n}$, which are a new class of Lorentzian Kac-Moody algebras. We shall see how these $\mathfrak{g}_{-n}$ algebras decompose in a very natural process according to their three-dimensional principal subalgebras. These decompositions are, to our knowledge, a completely novel type of decomposition, which are deeply connected to the Casimir operators of the decomposing $\mathfrak{g}_{-n}$ and their respective eigenvalues. In chapter 3 we motivate Toda field theories through lattice, finite, affine and conformal affine to see the alternating patterns between massive and conformal field theories, patterns which we see continuing when applying some perturbations on the $\mathfrak{g}_{-n}$ framework. We focus further analysis on the massive models, based on perturbed $\mathfrak{g}_{-n}$ algebras, examining their mass ratios and relevant eigenvalue spectra of the underlying algebras. For both the massive and CFT theories based on perturbed $\mathfrak{g}_{-n}$, we show that these theories can not be integrable through failure of the Painlevé test, even though the $\mathfrak{g}_a$ theories we extended from are integrable.

In chapter 4 we continue with our integrability analysis on Lorentzian generalised Cartan matrices in the form of the Painlevé test and the Painlevé property. The latter being more rigorous from the former, and giving very strong evidence for the integrability of these new models we will analyse. These integrable Lorentzian Toda field theories differ from those in chapter 3 through being *purely* Lorentzian, in the sense that we have not started with a finite or affine Kac-Moody algebra and extended it, instead we have started the construction with a purely Lorentzian lattice and formed a generalized Cartan matrix from there. This uncovers a new class of 2-dimensional Toda field theories, infinite in number, which both pass the Painlevé test and possess the Painlevé property. Although the analysis here is limited to 2-dimensional generalized Cartan matrices, the procedure in chapter 4 may be trivially generalized to higher dimension Cartan matrices. To conclude the thesis, chapter 5 gives a summary of all key results, along with current status of relevant fields and an outlook for future research.

Chapter 2

n-Extended Lorentzian Kac-Moody Algebras

Symmetry algebras and groups are ubiquitous in modern physics. Since the 1970s and before, the finite symmetry algebras have played an essential role in the formulation of fundamental theories in particle physics and the standard model. It was later recognized that infinite dimensional symmetry algebras, namely Kac-Moody algebras [20], are essential in many theories of physics beyond the standard model, especially in conformal field theories and string theories [65, 66, 67]. In particular, the E_{10} [21, 68] and E_{11} [18, 69] infinite dimensional Kac-Moody algebras have been integral in the formation of type II superstring theory and M-theory [14, 28, 29].

The algebra E_{10} is known as a *hyperbolic* algebra, which belongs to a set of Kac-Moody algebras that have been completely classified in terms of their Dynkin diagrams, e.g. [33] for explicit results from ranks 3-10. Conversely, E_{11} is not hyperbolic and belongs to a larger class of algebras that have not been fully classified, and are known as the *Lorentzian* Kac-Moody algebras. The hyperbolic and Lorentzian algebras lie in contrast to the finite and affine Kac-Moody algebras, which have been completely classified with much of their representation theory also understood [32].

Motivated partially through the above physical interest in Lorentzian algebras, but also from the limited knowledge of the class of Kac-Moody algebras that E_{11} belongs to, the authors of [24] studied a particular set of Kac-Moody algebra which could be found through extending an affine Dynkin diagram by attaching additional nodes. This procedure results in what is known as over-extended and very-extended diagrams, depending on whether one or two nodes have been attached onto the affine diagram, respectively. In this chapter, we build from the knowledge of finite and affine Kac-Moody algebras and focus on generalizing this procedure to n nodes of extension, studying the behaviour of the resulting *n-extended Lorentzian Kac-Moody algebra*. One such way of understanding these algebras is through their principal $SO(3)$-subalgebras, and for hyperbolic and Lorentzian algebras, their analogous principal $SO(1,2)$-subalgebras. From these, we shall better understand how the n-extended algebras decompose into subalgebras.

2.1 Finite Lie Algebras

Before defining Kac-Moody algebras explicitly in section 2.3, we introduce the concept of finite Lie algebras in this section, and affine Lie algebras in the following. The majority of algebras that this thesis encounters will have finite Lie algebras at the core of their structure, which we will build on through the extension procedures detailed in the subsequent sections of this chapter. As mentioned previously, the finite semisimple Lie algebras are fully classified in terms of both structure and representation theory [32], and are the least complex subclass of Kac-Moody algebras, hence, here they will provide motivation for the study of the n-extended Kac-Moody algebras that this chapter focuses on.

Finite Lie algebras, $\mathfrak{g}_f$ are composed of a finite dimensional vector space over a field, endowed with bilinear operation which takes in pairs of vectors from its vector space. This bilinear map $\mathfrak{g}_f \times \mathfrak{g}_f \to \mathfrak{g}_f$ is denoted as the Lie bracket, $[\cdot, \cdot]$ satisfying the following axioms:

(i) [X,X] = 0

(ii) [X,[X,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 (Jacobi Identity)

(iii) [X,Y] = -[Y,X]

for all X, Y, Z in our vector space, and where (iii) is implies (i) as $Y \to X$, as long as the underlying field's characteristic is not 2.

To better understand the structure of Lie algebras, we introduce the Cartan subalgebra of $\mathfrak{g}_f$, denoted as $\mathfrak{h}_f$ for finite Lie algebras. For $\mathfrak{g}_f$ semisimple, $\mathfrak{h}_f$ is defined as being the maximum Abelian subalgebra of $\mathfrak{g}_f$, which contains elements $H_i \in \mathfrak{h}_f$ for $i \in \{1, \cdots, r\}$, where r is the rank of the Lie algebra. There are multiple ways of constructing the entire $\mathfrak{g}_f$ from $\mathfrak{h}_f$ [32], here we present the Chevalley basis obeying the Serre relations as it can also be used in the construction of a more general Kac-Moody algebra in section 2.3. As well as H_i, the Chevalley generators include E_i and F_i and obey the following

$$\begin{aligned} &[H_i, H_j] = 0, \\ &[H_i, E_j] = K_{ij} E_j, \\ &[H_i, F_j] = -K_{ij} F_j, \\ &[E_i, F_j] = \delta_{ij} H_i, \end{aligned} \tag{2.1}$$

along with $[F_i, \cdots, [F_i, F_j], \cdots] = 0$ and $[E_i, \cdots, [E_i, E_j], \cdots] = 0$. The generators E_i and F_i are associated to a triangular decomposition of the Lie algebra with E_i being the upper half, F_i the lower and H_i the centre. As such, E_i and F_i are known as the step generators, or step operators and are one-to-one related to a special set of vectors within the algebra known as the roots, α, which all together form the set known as the root lattice $\Lambda_\alpha \ni \alpha$. The step operators act as raising and lowering the root vectors, and together completely describe the algebra.

The matrix K_{ij} is unique for a given algebra up to isomorphism and is called the Cartan matrix of the Lie algebra. We may always find a special basis of $\mathfrak{g}_\alpha$, composed of the simple roots of the system, denoted as α_i such that

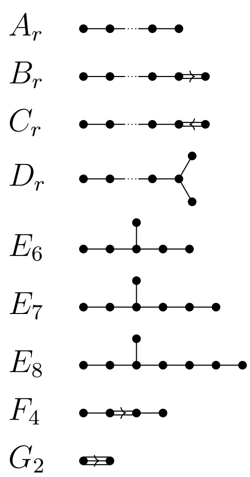
$$\alpha_i \cdot \alpha_j^\vee = K_{ij}, \quad \text{where } i, j \in \{1, \cdots, r\}. \tag{2.2}$$

Where $\alpha_i^\vee = \frac{2\alpha_i}{\alpha_i \cdot \alpha_i}$ is the dual root in this Chevalley basis.[1] As K completely describes the structure of the algebra, we may construct the adjacency matrix

$$2\delta_{ij} - K_{ij} \text{ for } i, j \in \{1, \cdots, r\}, \tag{2.3}$$

resulting in the interpretation of an undirected graph defined as the Dynkin diagram of the Lie algebra. In this Chevalley basis, and until otherwise specified, each of the r positive diagonals in this adjacency matrix are associated to a node on the diagram, with the negative off-diagonal components corresponding to the connections between those nodes. For example, -1 corresponds to one line connecting the nodes, -2 would be two lines and 0 would be no connection between those nodes. If K and therefore the adjacency matrix is not symmetric, then we draw an arrow between two nodes corresponding to the more negative to less negative off-diagonal component from K, as we see in the B_r, C_r, F_4 and G_2 algebras we will see shortly. In these cases, the roots of the algebra clearly have different absolute lengths that have been absorbed into the normalization of Chevalley basis definition.

The process of forming root systems and writing down the Dynkin diagrams of the semisimple Lie algebras has lead to the complete structural classification of all these finite Lie algebras [31, 70]. Their Dynkin diagrams are as follows



Dynkin diagrams of the finite Kac-Moody algebras, $\mathfrak{g}_f$

where each node of the Dynkin diagram corresponds to a simple root, α_i of the corresponding Lie algebra. The A_r, B_r, C_r and D_r series are the complex $\mathfrak{sl}_{r+1}, \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}$

[1]For the simply-laced cases that we shall see shortly $\alpha^\vee = \alpha$.

and $\mathfrak{so}_{2r}$ Lie algebras, respectively. On the level of these diagrams, $\mathfrak{g}_f$ may be identified through the defining characteristic that the deletion of *any one node* leaves a set of connected Dynkin diagrams that are also of type $\mathfrak{g}_f$. They are the complexifications of the infinitesimal expansions around the identity of the Lie groups that go by the same names. The remaining five finite Lie algebras are the exceptional Lie algebras, in particular E_6, E_7 and E_8, and their extensions will be of particular interest to us in this chapter and beyond.

Starting from the Dynkin diagram of the system and therefore the Cartan matrix of the system, we can reconstruct the entire root system of the Lie algebra by taking reflection of the roots in hyperplanes orthogonal to the simple roots. This set of reflections forms a group that is clearly a subgroup of the isometry group of the algebra's root system, and is known as the *Weyl group* of the algebra. The Weyl group acts as

$$w_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \tag{2.4}$$

on a vector v in the root space, defined by

$$\{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in H\}, \text{where } \alpha : H \to \mathbb{C}, \tag{2.5}$$

reflecting it about the hyperplane orthogonal to the roots α. Starting with the simple roots, repeated action of the Weyl group will recover the entire root space. For the finite Lie algebras this repeated action will cycle about a finite number of roots, whereas for the majority of Lie algebras considered in this thesis the action of the Weyl group will not terminate and hence will give an infinite number of roots. This is the case for the affine Lie algebras, with their corresponding affine Weyl groups, which we shall meet in the next section and the Lorentzian Kac-Moody algebras that are the focus of this chapter and built upon in the remainder of this thesis.

2.2 Affine Lie Algebras

In this section we will look at two different equivalent methods to extend the finite Lie algebras examined previously into a class of infinite dimensional algebras called *affine* Lie algebras, $\mathfrak{g}_a$ the first of which will look at a purely algebraic extension, whereas the second will use only the language of Dynkin diagrams and Cartan matrices introduced in the previous section. The latter method will be most relevant to our extension procedure in following sections, whereas the former is more optional for the interested reader.

2.2.1 Central Extensions of the Loop Algebra

The first method we describe to construct $\mathfrak{g}_a$ in this subsection largely follows [71], and may be summarised as a central extension of $\mathfrak{g}_f$ tensored with its loop algebra of Laurent polynomials. We will break this definition down to better understand its meaning. The central extension of $\mathfrak{g}$ may be written as the short exact sequence

$$0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{2.6}$$

where $\mathfrak{a}$ is a finite Abelian Lie algebra that is the centre of $\tilde{\mathfrak{g}}_f$. Where the centre of a Lie algebra is defined to be the set of all $x \in \mathfrak{g}$ such that $[x, y] = 0$ for any and all $y \in \mathfrak{g}$. This construction arises from the bilinear map, $\sigma : \mathfrak{g}_f \times \mathfrak{g}_f \to \mathfrak{a}$ where the action of σ satisfies the Jacobi identity for all elements of $\mathfrak{a}$. A Lie algebra's structure satisfying the axioms in section 2.1 may be constructed for $\mathfrak{g}_f \bigoplus \mathfrak{a}$ by action of the Lie bracket

$$[(a, X), (b, Y)] = ([a, b], \sigma(a, b)), \tag{2.7}$$

for elements $a, b \in \mathfrak{a}$ and $X, Y \in \mathfrak{g}_f$.

The next element we need for this description of $\mathfrak{g}_a$ is the loop algebra of $\mathfrak{g}_f$. The loop algebra is an infinite dimensional Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}_f \otimes \mathbb{C}^\infty(S^1), \tag{2.8}$$

where $\mathbb{C}^\infty(S^1)$ is the algebra of infinitely differentiable functions on the circle manifold S^1, meaning $\tilde{\mathfrak{g}}$ can be thought of as the parameterisation of loops in $\mathfrak{g}_f$. The Lie bracket of $\tilde{\mathfrak{g}}$ takes the form

$$[X \otimes X', Y \otimes Y'] = [X, Y] \otimes X'Y', \tag{2.9}$$

where $X', Y' \in \mathbb{C}^\infty(S^1)$.

For the construction of $\mathfrak{g}_a$, we use the Laurent polynomials of the form $\sum_i p_i t^i$ for independent variables t^i and $p_k \in \mathbb{C}$, which form a ring denoted $\mathbb{C}[t, t^{-1}]$ to form the loop algebra $\mathfrak{g}_f \otimes \mathbb{C}[t, t^{-1}]$. Similarly to equation (2.8) this gives us an infinite dimensional Lie algebra of vector fields in $\mathfrak{g}_f$ on a circle, but differs in allowing us a useful parametrisation in the single independent variable t from the Laurent polynomials. Putting this definition together with the knowledge that led us to the sequence (2.6), we may for the central extension

$$0 \to \mathbb{C} \cdot \mathfrak{c} \to \mathfrak{g}' \to \tilde{\mathfrak{g}} \to 0 \tag{2.10}$$

where now $\tilde{\mathfrak{g}} = \mathfrak{g}_f \otimes \mathbb{C}[t, t^{-1}]$, and $\mathfrak{c}$ is a central element such that $\mathfrak{g}' = \tilde{\mathfrak{g}} \oplus \mathbb{C} \cdot \mathfrak{c}$. Finally, $\mathfrak{g}_a$ is formed through adjoining a final basis element to the algebra, that acts on $\mathfrak{g}'$ through a derivation, d,

$$d([X, Y]) = [dX, Y] + [X, dY], \quad [d, X] = d(X) \quad \forall X, Y \in \mathfrak{g}_f \tag{2.11}$$

such that $\mathfrak{g}_a = \mathfrak{g}_f \oplus \mathbb{C}d$. This general affine Lie algebra is hence an infinite-dimensional extension of a corresponding finite Lie algebra. More information regarding this algebraic construction, including the exact form of the modified Lie bracket and details of the representation theory of $\mathfrak{g}_a$ can be found in [32] and [71]. However, this information is not pertinent for future discussion in this chapter, so we now turn our concentration to extensions of $\mathfrak{g}_f$ to $\mathfrak{g}_a$ at the level of the Dynkin diagram.

2.2.2 Extensions of the Finite Dynkin Diagram

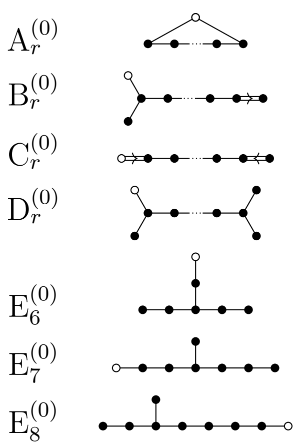
The algebraic extension of $\mathfrak{g}_f$ in the previous subsection provides one useful way to illustrate how a finite Lie algebra can be extended into an infinite Lie algebra. However, for the majority of this chapter and subsequent ones the simple root basis of the algebra will become increasingly important for us, and thus finding an extension procedure that focuses on the root space of the algebra would be more direct and concise for our purposes.

Extending the Dynkin diagrams of $\mathfrak{g}_f$ does exactly this, providing an equivalent way to the previous subsections methods. To create an affine extension from the Dynkin diagram's perspective we start with the set of simple roots, α_i from $\mathfrak{g}_f$. From the set of α_i in whatever representation for the roots we choose, we may always construct the highest root

$$\theta := \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{N}, \tag{2.12}$$

where n_i are the *Kac labels* [31] for the given $\mathfrak{g}_f$. This construction of the highest root guarantees that taking the set $\alpha_i^{(0)} \in \{\alpha_1, \cdots, \alpha_r, -\theta\}$ results in the basis of simple roots for $\mathfrak{g}_a$. Where we have adopted the convention of a superscript to denote the extended nature of the algebra, starting from zero in which we have the algebra extended to an affine level only. From $\alpha_i^{(0)}$ we may form the affine Cartan matrix $K_{ij}^{(0)} := \alpha_i^{(0)} \cdot \alpha_j^{(0)}$ where now $i, j \in \{1, \cdots, r+1\}$ for the rank $r+1$ affine algebra, based on the rank r finite algebra. Hence, the corresponding affine Dynkin diagram can be written down as defined through the adjacency matrix given in equation (2.3).

Following this procedure for all the finite Lie algebras in section 2.1 results in the following Dynkin diagrams



Dynkin diagrams of the affine Kac-Moody algebras, $\mathfrak{g}_a$

for all the affine Lie algebras bases off the finite semisimple algebras, where the empty nodes represent the affine nodes from the extension. Similarly to the definition of the $\mathfrak{g}_f$ diagrams, those of $\mathfrak{g}_a$ can be understood as there being *at least one* node whose deletion leads to a set of Dynkin diagrams of $\mathfrak{g}_f$ type. On the level of the root lattice, if we were to apply Weyl transformation on a simple root system of $\mathfrak{g}_a$ we would find that the generated root system does not close like it did for $\mathfrak{g}_f$, and we would generate the root lattice Λ_A, containing an infinite amount of roots. To recover the full root lattice for a $\mathfrak{g}_a$ in general, one must use the affine Weyl group, which utilizes certain orbits of Coxeter elements of the algebra to order the infinite number of roots. However, we spare more details of affine Weyl groups and refer the reader to [72], for example. The feature of possessing an infinite number of roots is characteristic of infinite dimensional Lie algebras, and is by definition something that we shall continue to see in the next section for most of the Kac-Moody algebras we encounter.

2.3 Kac-Moody algebras with Extended, Over-Extended and Very-Extended Root and Weight Lattices

Kac-Moody algebras [20], are more general constructions of all $\mathfrak{g}_f$ and $\mathfrak{g}_a$ we have encountered so far, in that both the affine and finite Lie algebras are subclasses of Kac-Moody algebras. To construct Kac-Moody algebras, we generalize the definition of the rank-r Cartan matrix, K, and therefore also the associated Dynkin diagram. Identically to before, a rank-r Dynkin diagram is defined as an undirected graph with adjacency matrix $2\delta_{ij} - K_{ij}$ for $i, j \in \{1, \cdots, r\}$, where until stated otherwise we assume that $K_{ii} = 2$ and $K_{i \neq j} < 0$.[2]

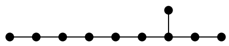
This starting point clearly differs greatly from that of section 2.1 and 2.2, as we may now take any valid Dynkin diagram, along with K and the associated root system, as our starting point and from here we can construct the Chevalley generators in a basis using the simple roots of the algebra, such that they obey the Serre relations of equation (2.1) along with $[F_i, \cdots, [F_i, F_j], \cdots] = 0$ and $[E_i, \cdots, [E_i, E_j], \cdots] = 0$. Any remaining generators may be constructed through combinations of commutators in these Serre relations.

Although this process of finding generators can be done in theory, in practice it has not been carried out for completely general Kac-Moody algebras, and has only been undertaken for the finite and affine examples we have seen previously [71]. Due to the inability to construct all the generators for a generalized Kac-Moody algebra, the structure at the level of the Dynkin diagram and root lattice becomes increasingly important in both classifying and further analysing the algebra. Hence, it is at the root lattice level we shall try and best understand further extension of the Kac-Moody algebras we encounter.

[2] This is certainly not the case for all generalized Cartan matrices, and we will see cases within this chapter and beyond where we have other. We hold this convention for this section as to illustrate the connection to $\mathfrak{g}_f$ and $\mathfrak{g}_a$.

2.3.1 Hyperbolic Kac-Moody Algebras and Over-Extended Root Lattices

Going beyond the affine extensions on the Dynkin diagram we saw in section 2.2, we may continue to add additional nodes to the diagram. This process results in what is known as the *over-extended* Lie algebras [24], where the affine extension is known as *extended*. For example, taking the $E_8^{(0)}$ algebra and adding another node to the affine node results in

$E_8^{(1)}$:  (2.13)

where the superscript denotes that this is an extended root system, and we can see that the additional node has been attached to the long leg of the $E_8^{(0)}$ diagram on its affine node. More generally, this single extension procedure on the affine algebras results in *hyperbolic* Kac-Moody algebras, where the additional node is always attached to the node from the affine extension. One way of defining the hyperbolic Kac-Moody algebras is through their Dynkin diagrams, in that they are a *connected diagram, where the deletion of any single node leaves a set of connected Dynkin diagrams, such that each diagram is of* $\mathfrak{g}_f$*, apart from at most one which is a* $\mathfrak{g}_a$. The hyperbolics have been completely classified, with there being an infinite amount of rank-2 hyperbolic, and 238 between ranks 3-10 [33, 34]. No hyperbolic Kac-Moody algebras exist with ranks greater than 10.

To construct the root lattice of an over-extended Kac-Moody algebra we largely follow the conventions of [24, 65, 73], starting with the root lattice, Λ_f, corresponding to the rank-r $\mathfrak{g}_f$ we are wanting to extend. We combine Λ_f with the 2-dimensional self-dual Lorentzian lattice that we denote $\Pi^{(1,1)}$, with the inner product

$$z \cdot w = -z^+ w^- - z^- w^+, \quad z, w \in \Pi^{(1,1)}, \tag{2.14}$$

where $z = (z^+, z^-)$ and $w = (w^+, w^-)$. Other conventions for inner products have been adopted to form over-extended Lorentzian root lattices [36, 74], but here we find the $\Pi^{(1,1)}$ most natural to reach the larger extensions we shall see shortly for the construction of very-extended and the n-extended algebras of the next section.

To help us construct the extended Kac-Moody algebras we define two primitive null vectors $k, \bar{k} \in \Pi^{(1,1)}$ as $k = (1, 0)$ and $\bar{k} = (0, -1)$ with the property that

$$k \cdot k = \bar{k} \cdot \bar{k} = 0, \quad k \cdot \bar{k} = 1, \tag{2.15}$$

and we will also use the combination $\pm(k + \bar{k})$ to form two vectors of length 2. To obtain the extended affine root lattice we again take θ from equation (2.12), and combine this with k to define $\alpha_0 = k - \theta$, giving

$$\{\alpha_0, \alpha_1, \cdots, \alpha_r\} \in \Lambda_{\mathfrak{g}_f} \oplus \Pi^{(1,1)}. \tag{2.16}$$

Defining the extended affine root lattice as $\Lambda_{\mathfrak{g}_0} := \Lambda_{\mathfrak{g}_f} \oplus \Pi^{(1,1)}$, we may continue to add an additional simple root, $\alpha_{-1} = -(k + \bar{k})$ to form

$$\{\alpha_{-1}, \alpha_0, \alpha_1, \cdots, \alpha_r\} \in \Lambda_{\mathfrak{g}_{-1}}, \tag{2.17}$$

where the over-extended root lattice, $\Lambda_{\mathfrak{g}_{-1}} := \Lambda_{\mathfrak{g}_f} \oplus \Pi^{(1,1)}$ has similar structure to $\Lambda_{\mathfrak{g}_0}$ with the addition of one extra root, α_{-1}.

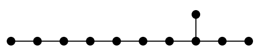
This construction allows us to take any finite Lie algebra's root lattice and extend twice, transitioning the finite Lie algebra through to an affine Lie algebra after the first extension, and resulting in a hyperbolic Kac-Moody algebra after the over-extension, like we saw in the (2.13) hyperbolic example. In the next subsection and sections we will see the results of continuing this process beyond the initial extensions through adding more $\Pi^{(1,1)}$ lattices and null vectors, but first we introduce the main class of algebras studied in this thesis, *Lorentzian* Kac-Moody algebras.

2.3.2 Lorentzian Kac-Moody Algebras and Very-Extended Root Lattices

To define Lorentzian Kac-Moody algebras, we take note of the definition given in [24], stating that Dynkin diagrams of Lorentzian type must be *connected diagrams* ***with at least one node*** *whose deletion gives a set of Dynkin diagrams, each being of a* $\mathfrak{g}_f$*, with at most one that is a* $\mathfrak{g}_a$. This definition is a superset of the hyperbolic algebras defined above, distinct by the caveat that they only need one node to be deleted creating the $\mathfrak{g}_f$ and one $\mathfrak{g}_a$, whereas the hyperbolics must have this property for every node in their Dynkin diagram.

Equivalently, on the level of their Cartan matrices, Lorentzian Kac-Moody algebras have non-singular, non-degenerate and indefinite K such that exactly one eigenvalue is negative - however, we must clarify that this is not a defining characteristic of Lorentzian Kac-Moody algebras like the definition on the level of the Dynkin diagram. One would find that hyperbolic Kac-Moody algebras also have a Lorentzian K, so we make this distinction clear here to untangle any confusing language found in other parts of the literature.

An example of a Lorentzian algebra with Lorentzian Dynkin diagram and Cartan matrix as defined above is

$E_8^{(2)}$:  (2.18)

otherwise known as E_{11} in literature [18]. This Kac-Moody algebra may be formed through an extension procedure resulting in a *very-extended* root lattice, analogous to the over-extension that resulted in $E_8^{(1)}$ above. To achieve this very-extended root lattice we take the over-extended lattice $\Lambda_{\mathfrak{g}_{-1}}$ and compose an addition $\Pi^{(1,1)}$ to it, forming $\Lambda_{\mathfrak{g}_{-2}} := \Lambda_{\mathfrak{g}_{-1}} \oplus \Pi^{(1,1)}$. Within the second Lorentzian lattice associated to $\Lambda_{\mathfrak{g}_{-2}}$ we define the primitive vectors as $l, \bar{l} \in \Pi^{(1,1)}$, allowing us to form the very-extended simple root $\alpha_{-2} = k - (l + \bar{l})$, and we can see both from the $E_8^{(2)}$ example and the general form of α_{-2} that this extended node attaches only to the previous extended node, so that we are beginning to build a tail of extended nodes onto the original Dynkin diagram, extending from the affine node corresponding to α_0. The very-extended root lattice is thus

$$\{\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \cdots, \alpha_r\} \in \Lambda_{\mathfrak{g}_{-2}}. \tag{2.19}$$

The results for the extended affine, over-extended and very-extended simple root systems and root lattices are summarized below:

algebra	root lattice structure	added root	Dynkin diagram	expl.
$\boldsymbol{g}_0 \equiv \boldsymbol{g}_a$	$\Lambda_{\boldsymbol{g}_0} = \Lambda_{\boldsymbol{g}_f} \oplus \Pi^{1,1}$	$\alpha_0 = k - \theta$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\bullet}$	$E_8^{(0)}$
$\boldsymbol{g}_{-1}$	$\Lambda_{\boldsymbol{g}_{-1}} = \Lambda_{\boldsymbol{g}_f} \oplus \Pi^{1,1}$	$\alpha_{-1} = -\left(k + \bar{k}\right)$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\circ} - \underset{\alpha_{-1}}{\bullet}$	$E_8^{(1)} \equiv E_{10}$
$\boldsymbol{g}_{-2}$	$\Lambda_{\boldsymbol{g}_{-2}} = \Lambda_{\boldsymbol{g}_{-1}} \oplus \Pi^{1,1}$	$\alpha_{-2} = k - \left(\ell + \bar{\ell}\right)$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\circ} - \underset{\alpha_{-1}}{\circ} - \underset{\alpha_{-2}}{\bullet}$	$E_8^{(2)} \equiv E_{11}$

Table 2.1: Extended, over-extended and very-extended Lie algebras, root lattices, extensions and partial Dynkin diagrams. We identify $\boldsymbol{g}_0 \equiv \boldsymbol{g}_a$ notationally as to link with the notation of the affine extened root α_0.

2.3.3 Fundamental Weights

The *fundamental weights*, λ_i of a Lie algebra with Cartan matrix as in equation (2.2) are defined to be

$$2\frac{\lambda_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \delta_{ij}, \tag{2.20}$$

where there is a λ_i for each simple root, α_i in the algebra. λ_i will become useful to us in future sections of this chapter when we examine the decomposition of Lorentzian Kac-Moody algebras, so we begin to motivate them here. The fundamental weights associated to all $\mathfrak{g}_f$, which we denote as λ_i^f, and can be found for instance in [31]. So analogously to our description of α_i for the extended algebras, we focus on modifying λ_i^f for various levels of extension. For over-extended and very-extended algebras, we have that

$$\lambda_i^o = \lambda_i^f + n_i\lambda_0^o, \quad \lambda_0^o = \bar{k} - k, \quad \lambda_{-1}^o = -k, \tag{2.21}$$

$$\lambda_i^v = \lambda_i^f + n_i\lambda_0^v \quad \lambda_0^v = \bar{k} - k + \frac{\ell + \bar{\ell}}{2}, \quad \lambda_{-1}^v = -k, \quad \lambda_{-2}^v = -\frac{\ell + \bar{\ell}}{2}, \tag{2.22}$$

respectively, with $i = 1, \ldots, r$. Where we are using the Lorentzian inner product defined in equation (2.14) such that these fundamental weights obey equation (2.20), giving

$$\lambda_i^o \cdot \alpha_j = \delta_{ij}, \qquad i, j = -1, 0, 1, \ldots, r, \tag{2.23}$$

$$\lambda_i^v \cdot \alpha_j = \delta_{ij}, \qquad i, j = -2, -1, 0, 1, \ldots, r. \tag{2.24}$$

With $\alpha_i \in \Lambda_{\boldsymbol{g}_{-1}}$ for the former, and $\alpha_i \in \Lambda_{\boldsymbol{g}_{-2}}$ for the latter. We also find it useful to define the *Weyl vectors*, ρ, building on that from [65] as the sum over all fundamental weights

$$\rho^o = \sum_{j=-1}^{r} \lambda_j = \rho^f + h\bar{k} - (1+h)k, \tag{2.25}$$

$$\rho^v = \sum_{j=-2}^{r} \lambda_j = \rho^f + h\bar{k} - (1+h)k - (1-h)\frac{\ell+\bar{\ell}}{2}, \tag{2.26}$$

ρ^f is the Weyl vector for the finite Lie algebras we have extended from, and h denotes the *Coxeter number*, which may be calculated for all $\mathfrak{g}_f$ as $h = 1 + \sum_i^r n_i$ [31], where the n_i are the Kac labels introduced in (2.12).

2.4 n-extended Lorentzian Kac-Moody Algebras

After collecting the preliminaries in the previous section, we move on to the natural question of what do root systems look like beyond very-extended extensions? It turns out that we can continue this expansion pattern on the level of the Cartan matrix and Dynkin diagram in a canonical way to define $\mathfrak{g}_{-n}$ as an *n-extended* Lorentzian Kac-Moody algebra, a new class of Lorentzian Kac-Moody algebras that we studied in detail within [25], which we shall closely follow for the remainder of this chapter. We form the root lattice of $\mathfrak{g}_{-n}$ through n repeated additions of the Lorentzian self-dual lattice $\Pi^{(1,1)}$ to form

$$\Lambda_{\boldsymbol{g}_{-n}} = \Lambda_{\boldsymbol{g}} \oplus \Pi_1^{(1,1)} \oplus \ldots \oplus \Pi_n^{(1,1)}. \tag{2.27}$$

We generalize the $k, \bar{k}$ and $l, \bar{l}$ vectors from section 2.3, so that $k_i, \bar{k}_i \in \Pi_i^{(1,1)}$. Like before, the $k_i, \bar{k}_i$ null-vectors have the properties that $k_i \cdot k_i = \bar{k}_i \cdot \bar{k}_i = 0$, $k_i \cdot \bar{k}_i = 1$ such that we can form the two vectors $\pm\left(k_i + \bar{k}_i\right)$ of length 2 in each corresponding $\Pi_i^{(1,1)}$.

The simple root system of an n-extended Lorentzian Kac-Moody algebra extended from a rank-r $\mathfrak{g}_f$ has a simple root system consisting of r simple roots from $\mathfrak{g}_f$ and n extended roots α_{-i}, $i = 1, \ldots, n$

$$\alpha^{(n)} := \left\{\alpha_1, \ldots, \alpha_r, \alpha_0 = k_1 - \theta, \alpha_{-1} = -\left(k_1 + \bar{k}_1\right), \ldots, \alpha_{-j} = k_{j-1} - \left(k_j + \bar{k}_j\right)\right\} \tag{2.28}$$

for $j = 2, \ldots, n$. Giving the n-extended algebra a total rank of $r + n + 1$, with the same number of simple roots. The fundamental weights are calculated through the orthogonality relations with Lorentzian inner product in the n-extended sections

$$\lambda_i^{(n)} \cdot \alpha_j^{(n)} = \delta_{ij}, \qquad i, j = -n, 0, 1, \ldots, r. \tag{2.29}$$

Rearranging the identity for simply laced algebras $\lambda_i^{(n)} \cdot \lambda_i^{(n)} = K_{ij}^{-1}$ we find that

$$\lambda_i^{(n)} = \sum_{j=-n}^{r} K_{ij}^{-1}\alpha_j^{(n)}, \tag{2.30}$$

allowing us to construct the $n+r+1$ fundamental weights for the n-extended algebra as

$$\begin{aligned}
\lambda_i^{(n)} &= \lambda_i^f + n_i\lambda_0^{(n)}, \qquad i = 1,\dots,r, && (2.31)\\
\lambda_0^{(n)} &= \bar{k}_1 - k_1 + \frac{1}{n}\sum_{i=2}^{n}\left[k_i + (n+1-i)\bar{k}_i\right], && (2.32)\\
\lambda_{-1}^{(n)} &= -k_1, && (2.33)\\
\lambda_{-2}^{(n)} &= -\frac{1}{n}\sum_{i=2}^{n}\left[k_i + (n+1-i)\bar{k}_i\right], && (2.34)\\
\lambda_{-3}^{(n)} &= \frac{1}{n}(n-2)(k_2-\bar{k}_2) - \frac{2}{n}\sum_{i=3}^{n}\left[k_i + (n+1-i)\bar{k}_i\right], && (2.35)\\
\lambda_{-4}^{(n)} &= \frac{1}{n}(n-3)(k_2-\bar{k}_2+k_3-2\bar{k}_3) - \frac{3}{n}\sum_{i=4}^{n}\left[k_i + (n+1-i)\bar{k}_i\right], && (2.36)\\
&\ \vdots\\
\lambda_{-\ell}^{(n)} &= \frac{1}{n}(n+1-\ell)\sum_{i=2}^{\ell-1}\left[k_i + (1-i)\bar{k}_i\right] + \frac{(1-\ell)}{n}\sum_{i=\ell}^{n}\left[k_i + (n+1-i)\bar{k}_i\right],\\
&= \frac{(1-\ell)}{n}\sum_{i=2}^{n}\left[k_i + (1-i)\bar{k}_i\right] + \sum_{i=2}^{\ell-1}\left[k_i + (1-i)\bar{k}_i\right] + (1-\ell)\sum_{i=\ell}^{n}\bar{k}_i, && (2.37)
\end{aligned}$$

all belonging to the n-extended weight lattice of the n-extended algebra $\mathfrak{g}_{-n}$. Summing up these weights we derive the Weyl vector for the n-extended system

$$\begin{aligned}
\rho^{(n)} &= \sum_{j=-n}^{r}\lambda_j && (2.38)\\
&= \rho^f + h\bar{k}_1 - (1+h)k_1 + \sum_{i=2}^{n}\left[\left(\frac{h}{n} + \frac{n+1-2i}{2}\right)k_i + \frac{(n+1-i)(2h+n(1-i))}{2n}\bar{k}_i\right].
\end{aligned}$$

Noting that the above reproduces equations (2.25) and (2.26) for $n=1$ and $n=2$, respectively.

The general expression for $\rho^{(n)}$ allows us to construct a generalization of the Freudenthal-de Vries strange formula by computing $(\rho^{(n)})^2$. For a semisimple finite Lie algebra $\mathfrak{g}_f$ with rank r it is well known, [75, 32] and references within, to be

$$\left(\rho^f\right)^2 = \frac{h}{12}\dim\boldsymbol{g} = \frac{h(h+1)r}{12}. \tag{2.39}$$

Thus, using equation (2.38) we may directly calculate that, for $n \geq 1$,

$$\rho^{(n)}\cdot\rho^{(n)} = \frac{h(h+1)r + n(n^2-1)}{12} - \frac{h(h+n)(1+n)}{n}, \tag{2.40}$$

for the n-extended algebras. Which agrees with that found for the $n = 1$ over-extended case found in [24].

2.5 Principal $SO(3)$ and $SO(1, 2)$ Subalgebras

Unlike the hyperbolic Kac-Moody algebras that have been completely classified [33], the Lorentzian Kac-Moody algebras can exist above rank-10 and are much greater in number. As a result, attempts have been made to place the Lorentzians into subcategories through finding properties unique to some Lorentzians but not others. One such property that has been very useful for the finite and affine Kac-Moody algebras has been the study of a principal $SO(3)$-subalgebra [76] within $\mathfrak{g}_f$ or $\mathfrak{g}_a$, especially in relation to how those algebras decompose into subalgebras, a feature that we will be particularly interested in for our n-extended algebras in the proceeding sections in this chapter. The principal $SO(3)$-subalgebra has also proven useful in the study of integrable systems based on $\mathfrak{g}_f$ and $\mathfrak{g}_a$ [61, 77]. Therefore, for both of these reasons, we are motivated to better understand the structure of $SO(3)$ algebras within our $\mathfrak{g}_{-n}$, and if a $SO(3)$ does not exist, then we would like to know if anything analogous does.

In relations to the generators of the Chevalley basis for $\mathfrak{g}_f$ or $\mathfrak{g}_a$ that obey the Serre relations of equation (2.1), the H_i, E_i, F_i form the principal $SO(3)$-subalgebra generators

$$J_3 = \sum_{i=1}^{r} D_i H_i, \quad J_+ = \sum_{i=1}^{r} n_i E_i, \quad J_- = \sum_{i=1}^{r} n_i F_i, \quad , \tag{2.41}$$

all quantities are as before, except D_i, which we have defined to be

$$D_i := \sum_{j=1}^{r} K_{ji}^{-1}, \tag{2.42}$$

and we shall see that these constants will play an important role throughout this thesis. The $SO(3)$ generators in equation (2.41) satisfy

$$[J_+, J_-] = J_3, \quad [J_3, J_\pm] = \pm J_\pm. \tag{2.43}$$

The Hermiticity properties $E_i^\dagger = F_i$, $H_i^\dagger = H_i$ are inherited by the generators J_+ and J_- as $J_+^\dagger = J_-$ when $n_i \in \mathbb{R}$. Importantly, the $SO(3)$-commutation relation $[J_+, J_-] = J_3$ is satisfied when the Kac labels $n_i = \sqrt{D_i}$. However, we shall see shortly that this is not always possible for Lorentzian Kac-Moody algebras, meaning one must expand the definition of these principal subalgebras.

For Lorentzian Kac-Moody algebras, in searching for a three-dimensional principal subalgebra analogous to $SO(3)$, we find that instead we have a principal $SO(1, 2)$-subalgebra [24, 73]. The generators of $SO(1, 2)$ are

$$\hat{J}_3 = -\sum_{i=1}^{r} \hat{D}_i H_i, \quad \hat{J}_+ = \sum_{i=1}^{r} p_i E_i, \quad \hat{J}_- = \sum_{i=1}^{r} q_i F_i, \quad \hat{D}_i := \sum_{j=1}^{r} K^{-1}_{ji}, \tag{2.44}$$

and obey

$$\left[\hat{J}_+, \hat{J}_-\right] = -\hat{J}_3, \quad \left[\hat{J}_3, \hat{J}_\pm\right] = \pm \hat{J}_\pm. \tag{2.45}$$

The generators in equation (2.44) are Hermitian when $p_i q_i = |p_i|^2 = -\hat{D}_i$, meaning *it is a necessary and sufficient for the existence of a $SO(3)$-principal subalgebra or a $SO(1,2)$-principal subalgebra that $D_i > 0$ or $\hat{D}_i < 0$ for all i, respectively.*

For n-extended algebras $\mathfrak{g}_{-n}$, we find an additional necessary condition such that $\mathfrak{g}_{-n}$ possesses both a $SO(3)$ and $SO(1,2)$ subalgebra, namely that

$$\exists k \in S = \{-n, \ldots, 0, 1, \ldots, r\} \text{ where } D_k = \sum_{j=-n}^{r} K^{-1}_{jk} = 0 \tag{2.46}$$

for *at least one* $k \in S$. We may then decompose the index set S as $\tilde{S} = S \backslash \{k\} = S_1 \cup S_2$, such that $K_{ij} = 0$ for all $i \in S_1$, $j \in S_2$ and $K_{i'k} \neq 0$, $K_{j'k} \neq 0$ for two specific $i' \in S_1$ and $j' \in S_2$. Thus removing the node k from the connected Dynkin diagram $\mathfrak{g}_{-n}$ will decompose it into two connected diagrams such that two generators indexed by $i \in S_1$ and $j \in S_2$ will commute. Thus when $D_i > 0$ for $i \in S_1$ and $D_j < 0$ for $j \in S_2$ we can formulate two commuting principal subalgebras with generators $\{J_3, J_\pm\}$ and $\{\hat{J}_3, \hat{J}_\pm\}$. For instance, we have

$$\left[J_3, \hat{J}_+\right] = \sum\nolimits_{i \in S_1, j \in S_2} D_i \sqrt{-\hat{D}_j}\,[H_i, E_j] = \sum\nolimits_{i \in S_1, j \in S_2} D_i \sqrt{-\hat{D}_j} K_{ij} E_j = 0, \tag{2.47}$$

and similarly for the other generators. This commuting structure extends to the $SO(3)$ and $SO(1,2)$ Casimir operators

$$C = J_3 J_3 - J_+ J_- - J_- J_+, \quad \text{and} \quad \hat{C} = \hat{J}_3 \hat{J}_3 - \hat{J}_+ \hat{J}_- - \hat{J}_- \hat{J}_+, \tag{2.48}$$

respectively. So that we have $SO(3) \oplus SO(1,2)$ with $[C, \hat{C}] = 0$.

From the identity $\lambda_i^{(n)} \cdot \lambda_i^{(n)} = K^{-1}_{ij}$, which led us to our definition of the fundamental weights in equation (2.30), we can see that the fundamental weights, $\lambda_i^{(1)}$, of hyperbolic algebras $\mathfrak{g}_{-1}$ all lie in the forwards lightcone in the weight space - in other words, any two weight vectors will be timelike separated with respects to their hyperbolic inner product. Just as λ_i^f belonging to $\mathfrak{g}_f$ can also all be found in this area of their lightcone. However, the cases in which we have a decomposing index set, as detailed above, can be understood through there existing one lightlike weight in weight space that separates the two sections of weights corresponding to the principal $SO(3)$-subalgebras and those of the $SO(1,2)$-subalgebra.

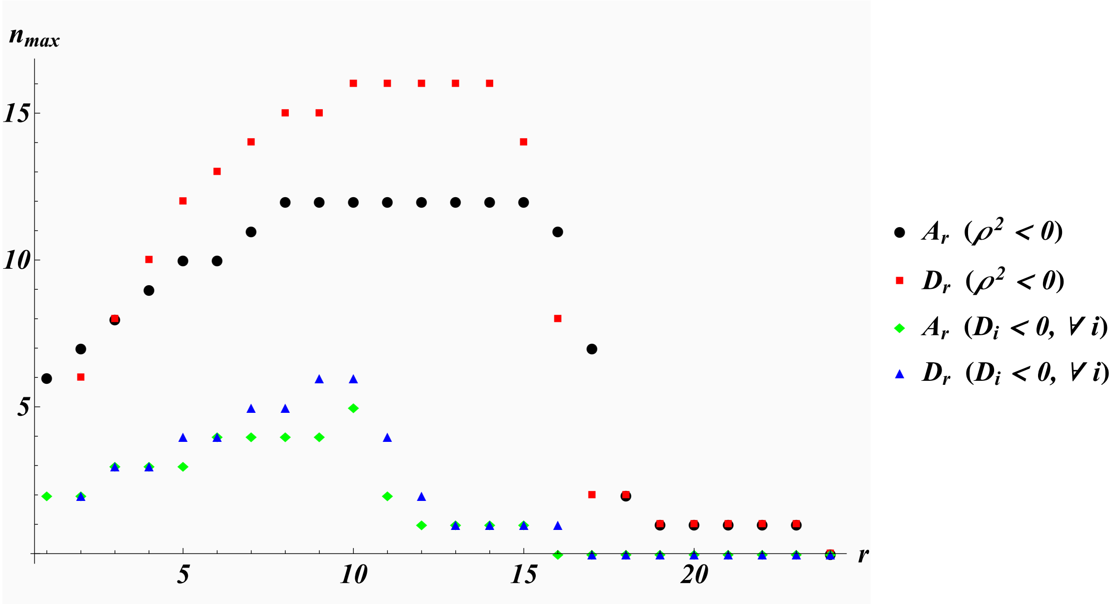


Figure 2.1: Maximum values of n for $\boldsymbol{g}_{-n}$ with rank r to possess a SO(1,2)-principal subalgebra from the necessary condition $\rho^2 < 0$ versus the necessary and sufficient condition $D_i < 0$, $\forall i$.

In cases in which there is not this separation by a light-like weight vector, the weight vectors are not segregated and therefore do not form a principal $SO(1,2)$-subalgebra, which is the situation with most general Lorentzian Kac-Moody algebras. We, however, concentrate on these decomposing exceptions for the remaining sections of this chapter. To do this we compute the inner products of the generators in the adjoint representation, as carried out for instance in [73], giving

$$(J_3, J_3) = \rho^{(n)} \cdot \rho^{(n)} > 0, \qquad \text{and} \qquad (J_\pm, J_\pm) = \rho^{(n)} \cdot \rho^{(n)} > 0, \tag{2.49}$$

$$(\hat{J}_3, \hat{J}_3) = \rho^{(n)} \cdot \rho^{(n)} < 0, \qquad \text{and} \qquad (\hat{J}_\pm, \hat{J}_\pm) = -\rho^{(n)} \cdot \rho^{(n)} > 0. \tag{2.50}$$

This allows us to use the signatures of $\rho^{(n)} \cdot \rho^{(n)}$ to give a necessary condition of the existence of the principal subalgebras. Hence, using the generalised Freudenthal-de Vries strange formula in equation (2.40), we may determine upper bounds, $n_{\max}$, for $\mathfrak{g}_{-n}$ based on semisimple $\mathfrak{g}_f$ of rank-r, to possess a principal $SO(1,2)$-subalgebra in accordance with the inequalities in (2.50). For the exceptional series of semisimple $\mathfrak{g}_f$ we calculate that

$$E_6 : n_{\max} = 23, \qquad E_7 : n_{\max} = 17, \qquad E_8 : n_{\max} = 14. \tag{2.51}$$

For the A_r and D_r algebras, we present the results in figure 2.1 for different values of r.

From figure 2.1 we can see the upper limits for A_r and D_r such that no n-extended algebras may possess a principal $SO(1,2)$-subalgebra are at $r \geq 24$. This has also been shown for the over-extended, $n = 1$ case in [65]. At $r < 24$ it is possible for a $SO(1,2)$-subalgebra to exist, however cross-checking with $\rho^2 < 0$ from equation (2.50) implies that no such principal subalgebra exists for A_r with $n > 12$ or D_r with $n > 16$.[3]

[3]For the over and very extended cases our results differ mildly in one case from a typo in [24], where

We note that the criteria in equation (2.50) acts as a guide in narrowing down the possibilities of finding a $\mathfrak{g}_{-n}$ with a principal $SO(1,2)$-subalgebra, however is only a necessary, but not sufficient condition in finding such a subalgebra. In the next section, we therefore directly solve for the $D_i^{(n)}$ values, unique to a given $\mathfrak{g}_{-n}$ and its corresponding K, so that we can classify the subalgebras, whenever they exist, according to equations (2.42) and (2.44).

2.6 Expansion Coefficients of the Diagonal Principal Subalgebra Generator

The general construction of the fundamental weights of $\mathfrak{g}_{-n}$ that we collected in equations (2.42) and (2.44) allows us to calculate the $D_i^{(n)}$ coefficients directly for any n-extended Lorentzian algebra of our choosing. In this section, we focus on $\mathfrak{g}_f$ as simply laced, meaning that all the corresponding simple roots of the finite algebra will be of length 2. In this scenario, the inverse Cartan takes the form $\lambda_i^{(n)} \cdot \lambda_i^{(n)} = K_{ij}^{-1}$, and therefore equation (2.42) is equivalent to

$$D_k^{(n)} = \sum_{j=-n}^{r} K_{kj}^{-1} = \rho^{(n)} \cdot \lambda_k^{(n)}, \qquad k = -n, \ldots, -1, 0, 1, \ldots, r. \tag{2.52}$$

Hence, we may calculate $D_k^{(n)}$ through the Lorentzian inner product of the Weyl vectors (2.38) with the weight vectors in (2.31) (2.37), or directly through the inverse of K. In doing so, we hope to uncover classifications of $\mathfrak{g}_{-n}$ through their 3-dimensional principal subalgebras, $SO(3)$ and $SO(1,2)$, in a more explicit way than the analysis of (2.50) allowed.

Using equation (2.52) we may derive a general formula for the expansion coefficients of $\mathfrak{g}_{-n}$

$$D_i^{(n)} = D_i^f + n_i D_0^{(n)}, \quad \text{and} \quad D_{-j}^{(n)} = (n-j+1)\left(\frac{j-1}{2} - \frac{h}{n}\right), \quad i = 1, \ldots, r; j = 0, \ldots, n, \tag{2.53}$$

for the extended part of the extended semisimple $\mathfrak{g}_f$. We have abbreviated $D_i^f := \rho^f \cdot \lambda_i^f$ from also using equation (2.52). For the over-extended and very-extended algebras the expressions in (2.53) become

$$D_{-1}^o = -h, \qquad D_0^o = -(2h+1), \qquad D_i^o = D_i^f + n_i D_0^o, \tag{2.54}$$

$$D_{-2}^v = \frac{1}{2}(1-h), \qquad D_{-1}^v = -h, \qquad D_0^v = -\frac{3}{2}(h+1), \qquad D_i^v = D_i^f + n_i D_0^v. \tag{2.55}$$

For a given $\mathfrak{g}_f$, the Weyl vectors ρ^f, Coxeter numbers h and Kac labels n_i are algebra specific and well known, see e.g [31]. We list them here for convenience and our reference

it was stated that also the over extended $A_{16}^{(1)}$ possess a $SO(1,2)$-principal subalgebras.

in table 2.2.

	Kac labels n_i	exponents e_i	Coxeter number h
A_r	$1, \ldots, 1$	$1, 2, 3, \ldots, r$	$r+1$
D_r	$1, 2, 2, \ldots, 2, 1, 1$	$1, 3, 5, \ldots, 2r-5, 2r-3, r-1$	$2r-2$
E_6	$1, 2, 2, 3, 2, 1$	$1, 4, 5, 7, 8, 11$	12
E_7	$2, 2, 3, 4, 3, 2, 1$	$1, 5, 7, 9, 11, 13, 17$	18
E_8	$2, 3, 4, 6, 5, 4, 3, 2$	$1, 7, 11, 13, 17, 19, 23, 29$	30

Table 2.2: Kac labels, exponents and Coxeter number for the simply laced Lie algebras.

For the above $\mathfrak{g}_f$ cases, the Weyl vectors are known in terms of the simple roots

$$A_r \quad : \quad \rho^f = \sum_{i=1}^{r} \frac{i}{2}(r-i+1)\alpha_i \,, \tag{2.56}$$

$$D_r \quad : \quad \rho^f = \sum_{i=1}^{r-2} \left[ir - \frac{i(i+1)}{2} \right] \alpha_i + \frac{r(r-1)}{4}(\alpha_{r-1} + \alpha_r), \tag{2.57}$$

$$E_6 \quad : \quad \rho^f = (8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6, \tag{2.58}$$

$$E_7 \quad : \quad \rho^f = \frac{1}{2}(34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7), \tag{2.59}$$

$$E_8 \quad : \quad \rho^f = 46\alpha_1 + 68\alpha_2 + 91\alpha_3 + 135\alpha_4 + 110\alpha_5 + 84\alpha_6 + 57\alpha_7 + 29\alpha_8. \tag{2.60}$$

This allows us to calculate each D_i^f from direct application of equation (2.53), giving

$$\begin{aligned}
A_r \quad &: \quad D_i^f = \frac{i}{2}(r-i+1), \quad i = 1, \ldots, r \\
D_r \quad &: \quad D_{r-1}^f = D_r^f = \frac{r(r-1)}{4}, \; D_j^f = jr - \frac{j(j+1)}{2}, \quad j = 1, \ldots, r-2 \\
E_6 \quad &: \quad D_1^f = 8, \; D_2^f = 11, \; D_3^f = 15, \; D_4^f = 21, \; D_5^f = 15, \; D_6^f = 8, \\
E_7 \quad &: \quad D_1^f = 17, \; D_2^f = \frac{49}{2}, \; D_3^f = 33, \; D_4^f = 48, \; D_5^f = \frac{75}{2}, \; D_6^f = 26, \\
&\quad\;\; D_7^f = \frac{27}{2}, \\
E_8 \quad &: \quad D_1^f = 46, \; D_2^f = 68, \; D_3^f = 91, \; D_4^f = 135, \; D_5^f = 110, \; D_6^f = 84, \\
&\quad\;\; D_7^f = 57, \; D_8^f = 29.
\end{aligned} \tag{2.61}$$

Evidently all constants D_i^f for all semisimple Lie algebras are positive, and hence a principal $SO(3)$-subalgebra may be obtained for each $\mathfrak{g}_f$, as is obviously expected. Continuing, for the over-extended algebras we therefore obtain

$$A_r^{(1)} \;:\; D^o_{-1} = -(r+1), \quad D^o_0 = -(2r+3), \quad D^o_i = \frac{i}{2}(r-i+1) - (2r+3) \tag{2.62}$$

$$D_r^{(1)} \;:\; D^o_{-1} = 2-2r,\; D^o_0 = 3-4r,\; D^o_1 = 2-3r,\; D^o_j = (j-8)r - \frac{j(j+1)}{2} + 6, \tag{2.63}$$
$$D^o_{r-1} = D^o_r = \frac{r(r+1)}{4} + 3 - 4r,$$

$$E_6^{(1)} \;:\; D^o_{-1} = -12,\; D^o_0 = -25,\; D^o_1 = -17,\; D^o_2 = -89,\; D^o_3 = -110,\; D^o_4 = -154, \tag{2.64}$$
$$D^o_5 = -185,\; D^o_6 = -267,$$

$$E_7^{(1)} \;:\; D^o_{-1} = -18,\; D^o_0 = -37,\; D^o_1 = -57,\; D^o_2 = -\frac{99}{2},\; D^o_3 = -78,\; D^o_4 = -100, \tag{2.65}$$
$$D^o_5 = -\frac{147}{2},\; D^o_6 = -48,\; D^o_7 = -\frac{47}{2},$$

$$E_8^{(1)} \;:\; D^o_{-1} = -30,\; D^o_0 = -61,\; D^o_1 = -15,\; D^o_2 = -359,\; D^o_3 = -580,\; D^o_4 = -658, \tag{2.66}$$
$$D^o_5 = -927,\; D^o_6 = -1075,\; D^o_7 = -1346,\; D^o_8 = -1740,$$

with $i = 1, \dots, n$, $j = 2, \dots, n-2$, noting that we begin to see negative values for D^o_i in these extended sections of $\mathfrak{g}_{-1}$. For the very-extended algebras, we similarly compute

$$A_r^{(2)} \;:\; D^v_{-2} = -\frac{r}{2},\; D^v_{-1} = -(r+1),\; D^v_0 = -\frac{3}{2}(r+2), \tag{2.67}$$
$$D^v_i = \frac{r}{2}(i-3) + \frac{i}{2}(1-i) - 3$$

$$D_r^{(2)} \;:\; D^v_{-2} = \frac{3}{2} - r,\; D^v_{-1} = 2-2r,\; D^v_0 = \frac{3}{2} - 3r,\; D^v_1 = \frac{1}{2} - 2r, \tag{2.68}$$
$$D^v_j = (j-6)r - \frac{j(j+1)}{2} + 3,\; D_{r-1} = D_r = \frac{r(r+1)}{4} + \frac{3}{2} - 3r,$$

$$E_6^{(2)} \;:\; D^v_{-2} = -\frac{11}{2},\; D^v_{-1} = -12,\; D^v_0 = -\frac{39}{2},\; D^v_1 = D^v_6 = -\frac{23}{2},\; D^v_2 = -28, \tag{2.69}$$
$$D^v_3 = D^v_5 = -24,\; D^v_4 = -\frac{75}{2},$$

$$E_7^{(2)} \;:\; D^v_{-2} = -\frac{17}{2},\; D^v_{-1} = -18,\; D^v_0 = -\frac{57}{2},\; D^v_1 = -40,\; D^v_2 = -\frac{65}{2}, D^v_3 = -\frac{105}{2} \tag{2.70}$$
$$D^v_4 = -66,\; D^v_5 = -48,\; D^v_6 = -31,\; D^v_7 = -15,$$

$$E_8^{(2)} \;:\; D^v_{-2} = -\frac{29}{2},\; D^v_{-1} = -30,\; D^v_0 = -\frac{93}{2},\; D^v_1 = -47,\; D^v_2 = -\frac{143}{2}, \tag{2.71}$$
$$D^v_3 = -95\; D^v_4 = -144,\; D^v_5 = -\frac{245}{2}x,\; D^v_6 = -102,\; D^v_7 = -\frac{165}{2},\;\; D^v_8 = -64.$$

with $i = 1, \dots, n$, $j = 2, \dots, n-2$.

From these expressions we find directly the maximal value of n for $\mathfrak{g}_{-n}$ with rank-r to possess a principal $SO(1,2)$-subalgebra from the necessary and sufficient condition $D_i < 0$, $\forall i$. For the exceptional Lie algebras we obtain

$$E_6 : n_{\max} = 5, \qquad E_7 : n_{\max} = 6, \qquad E_8 : n_{\max} = 7. \tag{2.72}$$

For A_r and D_r the results are reported in figure 2.1. Comparing these exact values to those resulting from the analysis of the necessary condition $\rho^2 < 0$ shows consistency, but also that the latter values are more restrictive. In the next section, we shall examine the even rarer set of examples in which we may find both a principal $SO(3)$ and $SO(1,2)$-subalgebra contained within an n-extended Lorentzian Kac-Moody algebra.

2.7 Direct Decomposition of n-Extended Lorentzian Kac-Moody Algebras

As argued in section 2.5, when we find a $\mathfrak{g}_{-n}$ such that one of its constants $D_i^{(n)} = 0$, there is the possibility that we may simultaneously find a principal $SO(3)$ and $SO(1,2)$-subalgebra. This requires, however, that the $D_i^{(n)}$ for i belonging to the two separate index sets S_1 and S_2 are of definite sign. If no such separation of positive and negative index sets exists then the algebra can not be decomposed further using the principal 3-dimensional subalgebras.

To identify whether we have simultaneous decomposition into both principal $SO(3)$ and $SO(1,2)$-subalgebras or not for a given $\mathfrak{g}_{-n}$ based on semisimple $\mathfrak{g}_f$, we set our solutions of equation (2.53) to zero and solve for values of n, i, j. We can see that the only meaningful solutions occur when $n, i \in \mathbb{N}$ and $i \leq n$, $j \leq n$, as anything outside these bounds clearly can not exist on a Dynkin diagram for $\mathfrak{g}_{-n}$.

First looking at the extended sections of the Dynkin diagram, from equation (2.53) we find that

$$D_{-j}^{(n)} = 0, \qquad \text{for } j = 1 + \frac{2h}{n}. \tag{2.73}$$

Due to the condition that $j \leq n$ there are a finite number of solutions. We calculated all the solutions for the $A_r^{(n)}$ and $D_r^{(n)}$ series to be

$$A_r^{(n)} \; : \; D_i^{(n)} = 0 \text{ for } (n, r, j) = (3, 2, 3), (4, 1, 2), (4, 3, 2), (4, 5, 4), (5, 4, 3), \ldots \tag{2.74}$$

$$D_r^{(n)} \; : \; D_i^{(n)} = 0 \text{ for } (n, r, j) = (4, 4, 4), (5, 6, 5), (6, 4, 3), (6, 7, 5), (7, 8, 5), \ldots \tag{2.75}$$

which we found using the Coxeter numbers we collected in table 2.2. For the $\mathfrak{g}_{-n}$ extended from the exceptional simply-laced $\mathfrak{g}_f$ we have

$$E_6^{(n)} = E_6^{(j-1)} \diamond L \diamond A_{n-j} \text{ for } (n, j) = (6, 5), (8, 4), (12, 3), (24, 2), \tag{2.76}$$

$$E_7^{(n)} = E_7^{(j-1)} \diamond L \diamond A_{n-j} \text{ for } (n, j) = (9, 5), (12, 4), (18, 3), (36, 2), \tag{2.77}$$

$$E_8^{(n)} = E_8^{(j-1)} \diamond L \diamond A_{n-j} \text{ for } (n, j) = (10, 7), (12, 6), (15, 5), (20, 4), (30, 3), (60, 2). \tag{2.78}$$

We denote, L as the Lorentzian root corresponding to the node that needs to be deleted.

For the parts of the Dynkin diagrams corresponding to semisimple Lie algebras also the expressions for ρ^f need to be treated case-by-case. We find

$$A_r^{(n)} \;:\; D_i^{(n)} = 0 \qquad \text{for} \;\; i = \frac{r+1}{2} \pm \frac{1}{2}\sqrt{r^2 - 6r - 4n - 11 - \frac{8(1+r)}{n}}, \tag{2.79}$$

$$D_r^{(n)} \;:\; D_i^{(n)} = 0 \qquad \text{for} \;\; i = \frac{r-1}{2} \pm \sqrt{r^2 - 9r - 2n + \frac{25}{4} + \frac{8(1-r)}{n}}. \tag{2.80}$$

For the over and very extended algebras the only solutions are

$$A_r^{(1)} \;:\; r = 16, i = 7, 10;\;\; r = 18, i = 6, 13;\;\; r = 26, j = 5, 22, \tag{2.81}$$

$$A_r^{(2)} \;:\; r = 12, i = 6, 7;\;\; r = 13, i = 5, 9;\;\; r = 18, j = 4, 15, \tag{2.82}$$

$$D_r^{(1)} \;:\; r = 17, j = 13;\;\; r = 18, j = 12;\;\; r = 20, j = 11;\;\; r = 39, j = 9,$$

$$D_r^{(2)} \;:\; r = 13, j = 10;\;\; r = 14, j = 9;\;\; r = 25, j = 7.$$

The exceptional E-series has no solutions on this section of the Dynkin diagram. The remaining solutions are presented in tables 2.3 and 2.4 for the A and D series, respectively.

We note that not all of the results from tables 2.3 and 2.4 fall into the $A_r^{(n)}$ and $D_r^{(n)}$ n-extended series, so we introduce the notation $\hat{A}_r^{(n,m)}$ labelling an A_r-Dynkin diagram with n roots successively attached to the m-th node in form of an A_n-algebra. The special case of the n-extended symmetric Dynkin diagram with n roots attached to the middle node of A_r we denote by $\hat{A}_r^{(n)}$.

Some of the $\hat{A}_r^{(n,m)}$-algebras are equivalent to the n-extended versions of the E-series. We have $\hat{A}_5^{(n+2,3)} \equiv E_6^{(n-2)}$, $\hat{A}_n^{(1,4)} \equiv E_7^{(n-7)}$ and $\hat{A}_n^{(1,3)} \equiv E_8^{(n-8)}$. We also have the symmetries $\hat{A}_r^{(n,m)} = \hat{A}_r^{(n,r+1-m)} = \hat{A}_{n+m}^{(r+1-m,m)} = \hat{A}_{r+n+1-m}^{(m-1,m)}$. In the resulting decomposition we also encounter algebras that decompose further by possessing Lorentzian roots on their extended legs of the corresponding Dynkin diagrams. We mark them in bold in tables 2.3 and 2.4. The precise way in which they decompose is reported in following sections in tables 2.6 and 2.7.

$A_{16}^{(1)} = \hat{A}_{13}^{(1)} \diamond L^2 \diamond A_2$	$A_{18}^{(1)} = \hat{A}_{11}^{(1)} \diamond L^2 \diamond A_6$	$A_{19}^{(1)} = \hat{A}_{11}^{(1)} \diamond L^2 \diamond A_{16}$
$A_{12}^{(2)} = \hat{A}_{11}^{(2)} \diamond L^2$	$A_{13}^{(2)} = \hat{A}_9^{(2)} \diamond L^2 \diamond A_3$	$A_{18}^{(2)} = \hat{A}_7^{(2)} \diamond L^2 \diamond A_{10}$
$A_{12}^{(3)} = \hat{A}_{11}^{(3)} \diamond L$	$A_{14}^{(3)} = \hat{A}_7^{(3)} \diamond L^2 \diamond A_6$	$A_{38}^{(3)} = E_6^{(1)} \diamond L^2 \diamond A_{32}$
$A_{11}^{(4)} = \hat{A}_9^{(4)} \diamond L^2 \diamond A_1$	$A_{13}^{(4)} = \hat{A}_7^{(4)} \diamond L^2 \diamond A_5$	$A_{27}^{(4)} = E_6^{(2)} \diamond L^2 \diamond A_{21}$
$A_{24}^{(5)} = E_6^{(3)} \diamond L^2 \diamond A_{18}$	$A_{11}^{(6)} = \boldsymbol{\hat{A}_9^{(6)}} \diamond L^2 \diamond A_1$	$A_{23}^{(6)} = E_6^{(4)} \diamond L^2 \diamond A_{17}$
$A_{13}^{(7)} = \boldsymbol{\hat{A}_7^{(7)}} \diamond L^2 \diamond A_5$	$A_{11}^{(8)} = \boldsymbol{\hat{A}_{11}^{(8)}} \diamond L$	$A_{23}^{(8)} = \boldsymbol{E_6^{(6)}} \diamond L^2 \diamond A_{17}$
$A_{14}^{(10)} = \boldsymbol{\hat{A}_7^{(10)}} \diamond L^2 \diamond A_6$	$A_{24}^{(10)} = \boldsymbol{E_6^{(8)}} \diamond L^2 \diamond A_{18}$	$A_{12}^{(13)} = \boldsymbol{\hat{A}_{11}^{(13)}} \diamond L^2$
$A_{13}^{(14)} = \boldsymbol{\hat{A}_9^{(14)}} \diamond L^2 \diamond A_3$	$A_{27}^{(14)} = \boldsymbol{E_6^{(12)}} \diamond L^2 \diamond A_{21}$	$A_{18}^{(19)} = \boldsymbol{\hat{A}_7^{(19)}} \diamond L^2 \diamond A_{10}$
$A_{38}^{(26)} = \boldsymbol{E_6^{(24)}} \diamond L^2 \diamond A_{32}$	$A_{16}^{(34)} = \boldsymbol{\hat{A}_{13}^{(34)}} \diamond L^2 \diamond A_2$	$A_{18}^{(38)} = \boldsymbol{\hat{A}_{11}^{(38)}} \diamond L^2 \diamond A_6$
$A_{26}^{(54)} = \boldsymbol{\hat{A}_9^{(54)}} \diamond L^2 \diamond A_{16}$		

Table 2.3: Decomposition of the n-extended algebras $A_r^{(n)}$.

$D_{17}^{(1)} = E_8^{(5)} \diamond L \diamond D_4$	$D_{18}^{(1)} = E_8^{(4)} \diamond L \diamond D_6$	$D_{20}^{(1)} = E_8^{(3)} \diamond L \diamond D_9$
$D_{39}^{(1)} = E_8^{(1)} \diamond L \diamond D_{30}$	$D_{13}^{(2)} = E_7^{(4)} \diamond L \diamond A_3$	$D_{14}^{(2)} = E_7^{(3)} \diamond L \diamond D_5$
$D_{25}^{(2)} = E_7^{(1)} \diamond L \diamond D_{18}$	$D_{13}^{(3)} = \hat{A}_{10}^{(1,5)} \diamond L \diamond D_5$	$D_{16}^{(3)} = \hat{A}_9^{(1,5)} \diamond L \diamond D_9$
$D_{11}^{(4)} = \hat{A}_{13}^{(1,6)} \diamond L^2$	$D_{12}^{(4)} = \hat{A}_{11}^{(1,6)} \diamond L \diamond D_4$	$D_{14}^{(4)} = \hat{A}_{10}^{(1,6)} \diamond L \diamond D_7$
$D_{21}^{(4)} = E_7^{(2)} \diamond L \diamond D_{15}$	$D_{11}^{(5)} = \hat{A}_{13}^{(1,7)} \diamond L \diamond A_1^{(2)}$	$D_{81}^{(5)} = E_8^{(1)} \diamond L \diamond D_{76}$
$D_{13}^{(6)} = \hat{A}_{12}^{(1,5)} \diamond L \diamond D_6$	$D_{52}^{(6)} = E_8^{(2)} \diamond L \diamond D_{47}$	$D_{43}^{(7)} = E_8^{(3)} \diamond L \diamond D_{38}$
$D_{11}^{(8)} = \boldsymbol{\hat{A}_{16}^{(1,7)}} \diamond L \diamond A_1^{(2)}$	$D_{13}^{(8)} = \boldsymbol{\hat{A}_{14}^{(1,5)}} \diamond L \diamond D_6$	$D_{17}^{(8)} = E_7^{(6)} \diamond L \diamond D_{11}$
$D_{39}^{(8)} = E_8^{(4)} \diamond L \diamond D_{34}$	$D_{37}^{(9)} = E_8^{(5)} \diamond L \diamond D_{32}$	$D_{11}^{(10)} = \boldsymbol{\hat{A}_{19}^{(1,8)}} \diamond L^2$
$D_{36}^{(10)} = E_8^{(6)} \diamond L \diamond D_{31}$	$D_{12}^{(11)} = \boldsymbol{\hat{A}_{18}^{(1,6)}} \diamond L \diamond D_4$	$D_{14}^{(13)} = \boldsymbol{\hat{A}_{19}^{(1,5)}} \diamond L \diamond D_7$
$D_{36}^{(14)} = \boldsymbol{E_8^{(10)}} \diamond L \diamond D_{31}$	$D_{13}^{(16)} = \boldsymbol{\hat{A}_{23}^{(1,6)}} \diamond L \diamond D_5$	$D_{37}^{(16)} = \boldsymbol{E_8^{(12)}} \diamond L \diamond D_{32}$
$D_{39}^{(19)} = \boldsymbol{E_8^{(15)}} \diamond L \diamond D_{34}$	$D_{16}^{(20)} = \boldsymbol{\hat{A}_{26}^{(1,5)}} \diamond L \diamond D_9$	$D_{21}^{(20)} = \boldsymbol{E_7^{(18)}} \diamond L \diamond D_{15}$
$D_{13}^{(24)} = \boldsymbol{\hat{A}_{33}^{(1,8)}} \diamond L \diamond A_3$	$D_{43}^{(24)} = \boldsymbol{E_8^{(20)}} \diamond L \diamond D_{38}$	$D_{14}^{(26)} = \boldsymbol{\hat{A}_{34}^{(1,7)}} \diamond L \diamond D_5$
$D_{52}^{(34)} = \boldsymbol{E_8^{(30)}} \diamond L \diamond D_{47}$	$D_{25}^{(48)} = \boldsymbol{\hat{A}_{54}^{(1,5)}} \diamond L \diamond D_{18}$	$D_{17}^{(64)} = \boldsymbol{\hat{A}_{76}^{(1,11)}} \diamond L \diamond D_4$
$D_{81}^{(64)} = \boldsymbol{E_8^{(60)}} \diamond L \diamond D_{76}$	$D_{18}^{(68)} = \boldsymbol{\hat{A}_{79}^{(1,10)}} \diamond L \diamond D_6$	$D_{20}^{(76)} = \boldsymbol{\hat{A}_{86}^{(1,9)}} \diamond L \diamond D_9$
$D_{39}^{(152)} = \boldsymbol{\hat{A}_{160}^{(1,7)}} \diamond L \diamond D_{30}$		

Table 2.4: Decomposition of the n-extended algebras $D_r^{(n)}$.

2.7.1 Reduced System Versus n-Extended Versions

We would now like to know how to express quantities of the full n-extended lattice such as roots, weights, Weyl vectors and determinants of the Cartan matrix in terms of those same quantities obtained from the reduced lattices and vice versa. We again follow here largely the reasoning presented in [25, 24], however, with the key difference that the node to be removed from the full n-extended Dynkin diagram is not identified as the one that decomposes the system into finite and affine diagrams, but rather the node ℓ for which $D_\ell^{(n)} = 0$. The former node might in fact not even exist for the cases considered here. Moreover, these two types of nodes are always different. Our construction applies to all n-extended lattices.

Keeping consistency with previous sections, we denote roots and weights related to the full n-extended lattice as α_i and λ_i for $i \in S = \{-n, \dots, 0, 1, \dots, r\}$, respectively. We denote roots and weights related to the reduced system as $\tilde{\alpha}_i, \tilde{\lambda}_i$ for $i \in \tilde{S} = S \backslash \{\ell\} = S_1 \cup S_2$. The root related to the node ℓ can then be expressed as

$$\alpha_\ell = x - \nu, \qquad \text{with } \nu := -\sum\nolimits_{i \in \tilde{S}} K_{\ell i} \tilde{\lambda}_i, \tag{2.83}$$

where the vector x is defined by the orthogonality properties $x \cdot \tilde{\alpha}_i = x \cdot \nu = 0$. Consequently, we have $K_{\ell\ell} = \alpha_\ell^2 = 2 = \nu^2 + x^2$ and the fundamental weights can be expressed as

$$\lambda_\ell = \frac{x}{x^2}, \qquad \lambda_i = \tilde{\lambda}_i + \left(\nu \cdot \tilde{\lambda}_i\right) \lambda_\ell. \tag{2.84}$$

We now sum up the fundamental weights to construct the Weyl vector then yields a relation between the Weyl vectors in the two respective systems

$$\rho = \sum\nolimits_{i \in S} \lambda_i = \lambda_\ell + \sum\nolimits_{i \in \tilde{S}} \lambda_i = \tilde{\rho} + (1 + \nu \cdot \tilde{\rho}) \, \lambda_\ell . \tag{2.85}$$

Next, to relate the determinants of the Cartan matrices for the two systems we employ Cauchy's expansion theorem for bordered matrices, see for example [78], we have

$$\det K = K_{\ell\ell} \det \tilde{K} - \sum\nolimits_{i,j \in \tilde{S}} K_{\ell i} (\mathrm{adj}\ \tilde{K})_{ij} K_{j\ell}, \tag{2.86}$$

where adj $\tilde{K}$ denotes the adjugate matrix of $\tilde{K}$, i.e. the transpose of its cofactor matrix. Recalling that $(\mathrm{adj}\ \tilde{K})_{ij} = \tilde{K}^{-1}_{ij} \det \tilde{K}$, $\tilde{K}^{-1}_{ij} = \lambda_i \cdot \lambda_j$ and $K_{\ell\ell} = 2$, relation (2.86) can be re-expressed as

$$\det K = \left(2 - \nu^2\right) \det \tilde{K}. \tag{2.87}$$

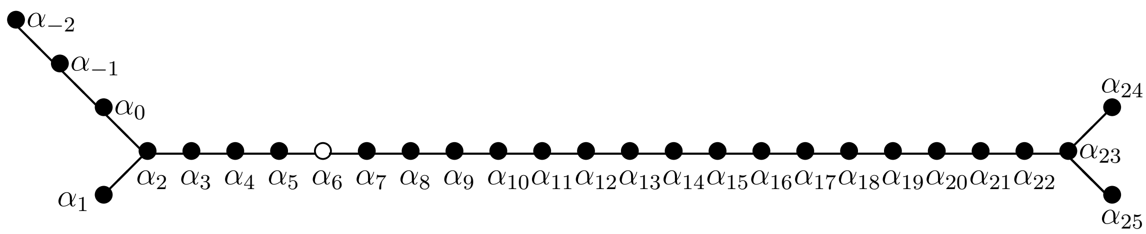
To illustrate the working of this formula and at the same time to check our expressions from above for consistency, we present explicitly two examples from the tables 2.3 and 2.4.

Example $D_{17}^{(1)} = E_8^{(5)} \diamond L \diamond D_4$: With $\nu = \lambda_1^{D_4} + \lambda_{-5}^{E_8^{(5)}}$, $\left(\lambda_1^{D_4}\right)^2 = 1$, $\left(\lambda_{-5}^{E_8^{(5)}}\right)^2 = 4/5$ we compute $\nu^2 = 9/5$. Furthermore we calculate the determinants $\det K_{D_{17}^{(1)}} = -4$, $\det K_{E_8^{(5)}} = -5$, $\det K_{D_4} = 4$ and hence confirm formula (2.87).

Example $D_{25}^{(2)} = E_7^{(1)} \diamond L \diamond D_{18}$: With $\nu = \lambda_1^{D_8} + \lambda_{-1}^{E_7^{(1)}}$, $\left(\lambda_1^{D_8}\right)^2 = 1$, $\left(\lambda_{-1}^{E_7^{(1)}}\right)^2 = 0$ we compute $\nu^2 = 1$. We also calculate the determinants $\det K_{D_{25}^{(2)}} = -8$, $\det K_{E_7^{(1)}} = -2$, $\det K_{D_{18}} = 4$ and hence confirming once more formula (2.87). We shall examine this example, along with its Dynkin diagram, in greater detail in the following subsection.

2.7.2 Decomposition of the Very-Extended D_{25}-Algebra aka k_{28}

Continuing to examine the very-extended $D_{25}^{(2)}$ example above, we write down its Dynkin diagram as:



$D_{25}^{(2)}$-Dynkin diagram on the root lattice for $D_{25} \oplus \Pi^{1,1} \oplus \Pi^{1,1}$

Noting that the construction of this diagram, or equivalently the construction of the very-extended Cartan matrix of a $\mathfrak{g}_{-n}$ could have been carried out with alternative methods. We however, stick to the construction methods based on the α_i we have detailed in section 2.4 with the corresponding root space is constructed as indicated in (2.28), to retain consistency within this chapter. However, as we shall see shortly, this is not the most convenient representation

The $D_{25}^{(2)}$ algebra in this example belongs to a special class of hyperbolic Kac-Moody algebras studied by Gaberdiel, Olive and West in [24]. It possesses at least one node that when removed leaves a set of disconnected Dynkin diagrams of finite type, with at most one being of affine type. We may check this definition by removing the hollow node in the above Dynkin diagram corresponding to the root labelled by α_6, we are left with a disconnected diagram of which one corresponds to the finite dimensional D_{19}-algebra and the other to the affine $E_7^{(0)}$-algebra.

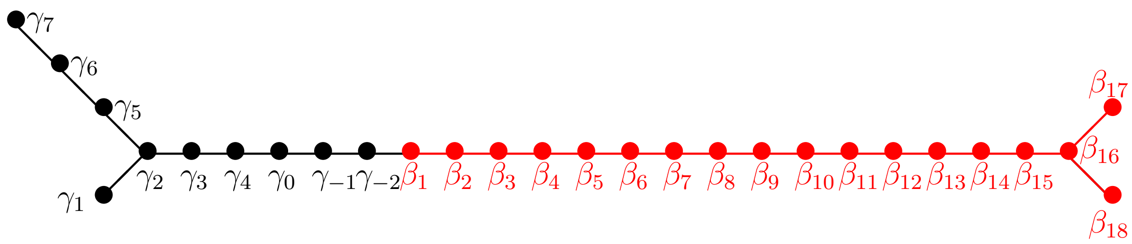
Here, we are especially interested in the construction of the reduced Dynkin diagram from the decomposition corresponding to $E_7^{(2)} \diamond D_{18}$. To see this decomposition more clearly, instead of the representation (2.28), we may also represent the roots as

$$\beta_1 \ : \ = \alpha_8 + \ell, \qquad \beta_i := \alpha_{i+7},\ i = 2, \ldots, 18, \tag{2.88}$$

$$\gamma_i \ : \ = \alpha_i,\ i = 1, \ldots, 4,\ \gamma_5 := \alpha_0,\ \gamma_6 := \alpha_{-1},\ \gamma_7 := \alpha_{-2},\ \gamma_0 := \alpha_5 - \bar{k}, \tag{2.89}$$

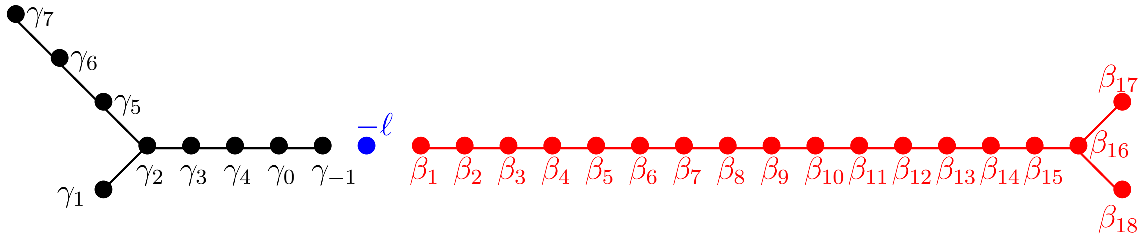
$$\gamma_{-1} \ : \ = -(k + \bar{k}) - \ell,\ \gamma_{-2} := -(\ell + \bar{\ell}). \tag{2.90}$$

Using the standard rules for the construction of Dynkin diagrams, we obtain the same diagram as above:



$E_7^{(2)} \diamond D_{18}$-Dynkin diagram on the root lattice for $E_7^{(0)} \oplus \Pi^{1,1} \oplus \Pi^{1,1} \oplus D_{18}$

This construction of the Dynkin diagram differs from the previous in that we have not used the standard representation for the over-extended and very-extended roots, but rather we have linked the very-extended root γ_{-2} of $E_7^{(2)}$ with a simple root β_1 of the semisimple Lie algebra D_{18}. Deleting $\bar{\ell}$ now has the effect that the two links connecting γ_{-2} are severed so that this algebra decomposes into $E_7^{(1)} \oplus \Pi^{1,1} \oplus D_{18}$. Thus $\gamma_{-2} = -\ell$ has becomes a separate disconnected root of zero length $\gamma_{-2} \cdot \gamma_{-2} = \ell^2 = 0$. In addition, we obtain two separate disconnected Dynkin diagrams for the over extended algebra $E_7^{(1)}$ and the semisimple Lie algebra D_{18}:



Reduced Dynkin diagram illustrating the decomposition $D_{25}^{(2)} = E_7^{(1)} \diamond L \diamond D_{18}$

Clearly we have that the root $\alpha_\ell = \alpha_7$ for which $D_\ell = 0$ is different from the root α_6 that need to be chosen for the very extended root lattice to reduce to a $\mathfrak{g}_a$ and a $\mathfrak{g}_f$ Kac-Moody algebra. However, a n-extended construction of simple roots may still be found such that we may form the two decomposing algebras separately and extend and combine them into the total algebra.

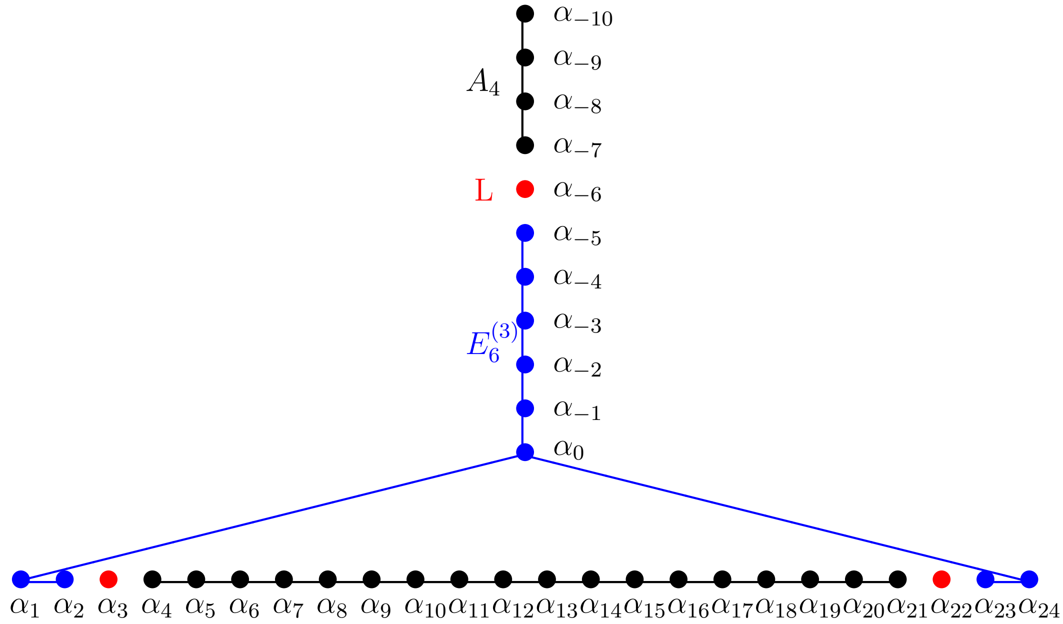
2.7.3 Double and Triple Decomposition

The previous section's example illustrated in more detail how a choice of roots may be found such that a given $\mathfrak{g}_{-n}$ that contains a $D_\ell = 0$ may decompose into two algebras with one ℓ. However, as we have already seen in tables 2.3 and 2.4, it is also possible for n-extended algebras to decompose at two or three nodes, say ℓ, ℓ' and ℓ'', for which $D_\ell = D_{\ell'} = D_{\ell''} = 0$. In this subsection, we give an illustrative example of decomposing $\mathfrak{g}_{-n}$ from table 2.3 and another from table 2.4, both of which decompose on the semisimple and extended parts of their associated Dynkin diagrams.

Example $A_{24}^{(10)} = E_6^{(3)} \diamond L \diamond A_4 \diamond L^2 \diamond A_{18}$. The first example from table 2.3 gives the triple decomposition:

$$\begin{array}{ccccc} & & E_6^{(8)} \diamond L^2 \diamond A_{18} & & \\ & \nearrow & & \searrow & \\ A_{24}^{(10)} & & & & E_6^{(3)} \diamond L \diamond A_4 \diamond L^2 \diamond A_{18} \\ & \searrow & & \nearrow & \\ & & A_{24}^{(5)} \diamond L \diamond A_4 & & \end{array}$$

which on the level of the Dynkin gives the following:



Reduced Dynkin diagram of $A_{24}^{(10)} = E_6^{(3)} \diamond L \diamond A_4 \diamond L^2 \diamond A_{18}$

Example $D_{36}^{(14)} = A_3 \diamond L \diamond E_6^{(6)} \diamond L \diamond D_{31}$. This example from table 2.4 decomposes as

$$\begin{array}{ccccc} & & D_{36}^{(10)} \diamond L \diamond A_3 & & \\ & \nearrow & & \searrow & \\ D_{36}^{(14)} & & & & A_3 \diamond L \diamond E_6^{(6)} \diamond L \diamond D_{31} \\ & \searrow & & \nearrow & \\ & & E_8^{(10)} \diamond L \diamond D_{31} & & \end{array}$$

Further examples can be obtained from tables 2.3 and 2.4 for the algebras with bold entries.

2.8 Roots, Weights, Weyl Vectors and Decomposition of the $\hat{A}_r^{(n,m)}$-Algebras

Until this point we have mainly covered $\mathfrak{g}_{-n}$ decomposition, but since the $\hat{A}_r^{(n,m)}$-algebras occur naturally in the decomposition of the n-extended Lorentzian Kac-Moody algebras, we shall now discuss them in more detail for completeness. The Dynkin diagrams associated to $\hat{A}_r^{(n,m)}$ are equivalent to those arising in the description of the so-called $T_{p,q,r}$-singularities [79], with the identification $\hat{A}_{p+q+1}^{(r,p+1)} \equiv T_{p,q,r}$.

We would like to understand the simple root structure, and hence the basis for the underlying root lattice, of the $\hat{A}_r^{(n,m)}$-algebras, to put them in the context of the n-extended constructions of section 2.4. To do so, we represent the simple $\hat{A}_r^{(n,m)}$-roots in terms of the r simple roots α_i of the semisimple Lie algebra $\mathfrak{g}_f$ and the Lorentzian roots, with the m^{th} root modified similarly as the affine root for the n-extended algebras $\alpha_m \to \alpha_m + k_1$. Thus the $r+n$ simple $\hat{A}_r^{(n,m)}$-roots are represented as

$$\hat{\alpha} = \left\{\alpha_1, \ldots, \alpha_{m-1}, \alpha_m + k_1, \alpha_{m+1}, \ldots, \alpha_r, \alpha_{-1} = -k_1 - \bar{k}_1, \ldots, \alpha_{-j} = k_{j-1} - k_j - \bar{k}_j\right\}, \tag{2.91}$$

with $j = 2, \ldots, n$. Using the orthogonality relation

$$\lambda_i^{(n,m)} \cdot \hat{\alpha}_j = \delta_{ij}, \qquad i, j = -n, \ldots, -1, 1, \ldots, r, \tag{2.92}$$

together with $\lambda_i^{(n,m)} = \sum_{j=1}^{n+r} \hat{K}_{ij}^{-1} \hat{\alpha}_j$, $\hat{K}_{ij}^{-1} = \lambda_i^{(n,m)} \cdot \lambda_i^{(n,m)}$, we can construct the $n+r$ fundamental weights. We shall focus here on the case for which the extension is attached onto the middle node $\hat{A}_{r=2\ell+1}^{(n,\ell+1)}$, so that $m = h/2$, and refer to them as $\hat{A}_r^{(n)}$. We find in this case the fundamental weights

$$\hat{\lambda}_i^{(n)} = \lambda_i^f + \frac{2n}{nh - 4(n+1)} \min(i, h-i) \left(\lambda_0^{(n)} - \lambda_{h/2}^f\right), \qquad i = 1, \ldots, r, \tag{2.93}$$

$$\hat{\lambda}_{-j}^{(n)} = \lambda_{-j}^{(n)} + \frac{4(n-j+1)}{nh - 4(n+1)} \left(\lambda_0^{(n)} - \lambda_{h/2}^f\right), \qquad j = 1, \ldots, n, \tag{2.94}$$

where λ_i^f are the fundamental weights of A_r and $\lambda_0^{(n)}$, $\lambda_{-j}^{(n)}$ are fundamental weights for the n-extended Lorentzian Kac-Moody algebras as determined above in equations (2.32), (2.37). The Weyl vector results therefore to

$$\hat{\rho}^{(n)} = \sum_{j=-n, j\neq 0}^{r} \hat{\lambda}_i^{(n)} = \rho^{(n)} - h\lambda_0^{(n)} + \frac{n(h^2 + 4n + 4)}{2n(h-4) - 8} \left(\lambda_0^{(n)} - \lambda_m^f\right). \tag{2.95}$$

Next we calculate the expansion coefficients of the three-dimensional principal subalgebras as

$$\hat{D}_i^{(n)} = \hat{\rho}^{(n)} \cdot \hat{\lambda}_i^{(n)} = \frac{n(4 + 4n + h^2)}{16 + 4n(4-h)} \min(i, h-i) + \frac{i}{2}(h-i), \qquad i = 1, \ldots, r, \tag{2.96}$$

$$\hat{D}_{-j}^{(n)} = \hat{\rho}^{(n)} \cdot \hat{\lambda}_{-j}^{(n)} = \frac{(j-n-1)[h^2 + 4j(1+n) + nh(1-j)]}{2n(h-4) - 8}, \qquad j = 1, \ldots, n. \tag{2.97}$$

As before, and for the same reasons as for the n-extended algebras, when the $\hat{D}_i^{(n)} = 0$ for certain values of i the $\hat{A}_r^{(n,m)}$-algebra decomposes. Solving for the condition of $\hat{D}_i^{(n)}$ vanishing, we find that

$$\hat{D}_i^{(n)} = 0, \quad \text{for } i = \frac{n(4n + 4 + h^2)}{2n(h-4) - 8}, h - \frac{n(4n + 4 + h^2)}{2n(h-4) - 8}, \tag{2.98}$$

$$\hat{D}_{-j}^{(n)} = 0, \quad \text{for } j = \frac{h(h+n)}{n(h-4) - 4}, \tag{2.99}$$

Thus the only meaningful solutions, i.e. those for which $i, \in \mathbb{N}$, $i \leq r$, to (2.98) give rise to the decompositions on the leg of the Dynkin diagram corresponding to the A_r-diagram as listed in table 2.5 below.

$\hat{A}_{13}^{(2)} = \hat{A}_{11}^{(2)} \diamond L^2$	$\hat{A}_{13}^{(4)} = \hat{A}_{9}^{(4)} \diamond L^2 \diamond A_1^2$	$\hat{A}_{13}^{(6)} = \hat{\boldsymbol{A}}_{\boldsymbol{9}}^{(\boldsymbol{6})} \diamond L^2 \diamond A_1^2$
$\hat{A}_{13}^{(13)} = \hat{\boldsymbol{A}}_{\boldsymbol{11}}^{(\boldsymbol{13})} \diamond L^2$	$\hat{A}_{17}^{(2)} = \hat{A}_{9}^{(2)} \diamond L^2 \diamond A_3^2$	$\hat{A}_{17}^{(14)} = \hat{\boldsymbol{A}}_{\boldsymbol{9}}^{(\boldsymbol{14})} \diamond L^2 \diamond A_3^2$
$\hat{A}_{19}^{(1)} = \hat{A}_{13}^{(1)} \diamond L^2 \diamond A_2^2$	$\hat{A}_{19}^{(4)} = \hat{A}_{7}^{(4)} \diamond L^2 \diamond A_5^2$	$\hat{A}_{19}^{(7)} = \hat{\boldsymbol{A}}_{\boldsymbol{7}}^{(\boldsymbol{7})} \diamond L^2 \diamond A_5^2$
$\hat{A}_{19}^{(34)} = \hat{\boldsymbol{A}}_{\boldsymbol{13}}^{(\boldsymbol{34})} \diamond L^2 \diamond A_2^2$	$\hat{A}_{21}^{(3)} = \hat{A}_{7}^{(3)} \diamond L^2 \diamond A_6^2$	$\hat{A}_{21}^{(10)} = \hat{\boldsymbol{A}}_{\boldsymbol{7}}^{(\boldsymbol{10})} \diamond L^2 \diamond A_6^2$
$\hat{A}_{25}^{(1)} = \hat{A}_{11}^{(1)} \diamond L^2 \diamond A_6^2$	$\hat{A}_{25}^{(38)} = \hat{\boldsymbol{A}}_{\boldsymbol{11}}^{(\boldsymbol{38})} \diamond L^2 \diamond A_6^2$	$\hat{A}_{29}^{(2)} = \hat{A}_{7}^{(2)} \diamond L^2 \diamond A_{10}^2$
$\hat{A}_{29}^{(19)} = \hat{\boldsymbol{A}}_{\boldsymbol{7}}^{(\boldsymbol{19})} \diamond L^2 \diamond A_{10}^2$	$\hat{A}_{41}^{(6)} = E_6^{(4)} \diamond L^2 \diamond A_{17}^2$	$\hat{A}_{41}^{(8)} = \boldsymbol{E}_{\boldsymbol{6}}^{(\boldsymbol{6})} \diamond L^2 \diamond A_{17}^2$
$\hat{A}_{43}^{(1)} = \hat{A}_{9}^{(1)} \diamond L^2 \diamond A_{16}^2$	$\hat{A}_{43}^{(5)} = E_6^{(3)} \diamond L^2 \diamond A_{18}^2$	$\hat{A}_{43}^{(54)} = \hat{A}_{7}^{(54)} \diamond L^2 \diamond A_{16}^2$
$\hat{A}_{43}^{(10)} = \boldsymbol{E}_{\boldsymbol{6}}^{(\boldsymbol{8})} \diamond L^2 \diamond A_{18}^2$	$\hat{A}_{49}^{(4)} = E_6^{(2)} \diamond L^2 \diamond A_{21}^2$	$\hat{A}_{49}^{(14)} = \boldsymbol{E}_{\boldsymbol{6}}^{(\boldsymbol{12})} \diamond L^2 \diamond A_{21}^2$
$\hat{A}_{71}^{(3)} = E_6^{(1)} \diamond L^2 \diamond A_{32}^2$	$\hat{A}_{71}^{(26)} = \boldsymbol{E}_{\boldsymbol{6}}^{(\boldsymbol{24})} \diamond L^2 \diamond A_{32}^2$	

Table 2.5: Decomposition of the algebras $A_r^{(n)}$ on the A_r-leg of the Dynkin diagram.

Whereas on the extended part of the Dynkin diagram we have $j \in \mathbb{N}$, $j \leq n$, and find that the solutions to equation (2.98) as listed in table 2.6:

$\hat{A}_{7}^{(7)} = \hat{A}_{7}^{(4)} \diamond L \diamond A_2$	$\hat{A}_{7}^{(10)} = \hat{A}_{7}^{(3)} \diamond L \diamond A_6$	$\hat{A}_{7}^{(19)} = \hat{A}_{7}^{(2)} \diamond L \diamond A_{16}$
$\hat{A}_{9}^{(6)} = \hat{A}_{9}^{(4)} \diamond L \diamond A_1$	$\hat{A}_{9}^{(14)} = \hat{A}_{9}^{(2)} \diamond L \diamond A_{11}$	$\hat{A}_{9}^{(54)} = \hat{A}_{9}^{(1)} \diamond L \diamond A_{52}$
$\hat{A}_{11}^{(8)} = \hat{A}_{11}^{(3)} \diamond L \diamond A_3$	$\hat{A}_{11}^{(13)} = \hat{A}_{11}^{(2)} \diamond L \diamond A_{10}$	$\hat{A}_{11}^{(38)} = \hat{A}_{11}^{(1)} \diamond L \diamond A_{36}$
$\hat{A}_{13}^{(6)} = \hat{A}_{13}^{(4)} \diamond L \diamond A_1$	$\hat{A}_{13}^{(13)} = \hat{A}_{13}^{(2)} \diamond L \diamond A_{10}$	$\hat{A}_{13}^{(34)} = \hat{A}_{13}^{(1)} \diamond L \diamond A_{32}$
$\hat{A}_{15}^{(33)} = \hat{A}_{15}^{(1)} \diamond L \diamond A_{31}$	$\hat{A}_{17}^{(14)} = \hat{A}_{17}^{(2)} \diamond L \diamond A_{11}$	$\hat{A}_{19}^{(7)} = \hat{A}_{19}^{(4)} \diamond L \diamond A_2$
$\hat{A}_{19}^{(34)} = \hat{A}_{19}^{(4)} \diamond L \diamond A_2$	$\hat{A}_{21}^{(10)} = \hat{A}_{21}^{(3)} \diamond L \diamond A_6$	$\hat{A}_{25}^{(38)} = \hat{A}_{25}^{(1)} \diamond L \diamond A_{36}$
$\hat{A}_{29}^{(19)} = \hat{A}_{29}^{(2)} \diamond L \diamond A_{16}$	$\hat{A}_{31}^{(13)} = \hat{A}_{31}^{(3)} \diamond L \diamond A_9$	$\hat{A}_{31}^{(43)} = \hat{A}_{31}^{(1)} \diamond L \diamond A_{41}$
$\hat{A}_{41}^{(8)} = \hat{A}_{41}^{(6)} \diamond L \diamond A_1$	$\hat{A}_{43}^{(10)} = \hat{A}_{43}^{(5)} \diamond L \diamond A_4$	$\hat{A}_{43}^{(54)} = \hat{A}_{43}^{(1)} \diamond L \diamond A_{52}$
$\hat{A}_{49}^{(14)} = \hat{A}_{49}^{(4)} \diamond L \diamond A_9$	$\hat{A}_{71}^{(26)} = \hat{A}_{71}^{(3)} \diamond L \diamond A_{22}$	$\hat{A}_{79}^{(89)} = \hat{A}_{79}^{(1)} \diamond L \diamond A_{87}$
$\hat{A}_{111}^{(13)} = \hat{A}_{111}^{(9)} \diamond L \diamond A_3$	$\hat{A}_{127}^{(19)} = \hat{A}_{127}^{(7)} \diamond L \diamond A_{11}$	$\hat{A}_{239}^{(49)} = \hat{A}_{239}^{(5)} \diamond L \diamond A_{43}$

Table 2.6: Decomposition of the algebras $A_r^{(n)}$ on the extended leg of the Dynkin diagram.

To conclude our study of the decomposition of $\hat{A}_r^{(n,m)}$-algebras, we present the algebras in general. We choose to not give their description of simple roots, weights and Weyl vectors as we did for $\mathfrak{g}_{-n}$ in section 2.4, as we will not require these quantities in this thesis. We instead list the decompositions for the $\hat{A}_r^{(n,m)}$-algebras that appeared in table 2.4 for completeness of the description of the $\mathfrak{g}_{-n}$ decomposition description. These are listed in table 2.7 below:

$\hat{A}_{14}^{(1,5)} = \hat{A}_{12}^{(1,5)} \diamond L \diamond A_1$	$\hat{A}_{16}^{(1,7)} = \hat{A}_{13}^{(1)} \diamond L \diamond A_2$	$\hat{A}_{19}^{(1,8)} = \hat{A}_{13}^{(1,6)} \diamond L \diamond A_5$
$\hat{A}_{18}^{(1,6)} = \hat{A}_{11}^{(1)} \diamond L \diamond A_6$	$\hat{A}_{19}^{(1,5)} = \hat{A}_{10}^{(1,5)} \diamond L \diamond A_8$	$\hat{A}_{23}^{(1,6)} = \hat{A}_{10}^{(1,5)} \diamond L \diamond A_{12}$
$\hat{A}_{26}^{(1,5)} = \hat{A}_{9}^{(1)} \diamond L \diamond A_{16}$	$\hat{A}_{33}^{(1,8)} = E_7^{(4)} \diamond L \diamond A_{21}$	$\hat{A}_{34}^{(1,7)} = E_7^{(3)} \diamond L \diamond A_{23}$
$\hat{A}_{54}^{(1,5)} = E_7^{(1)} \diamond L \diamond A_{45}$	$\hat{A}_{76}^{(1,11)} = E_8^{(5)} \diamond L \diamond A_{62}$	$\hat{A}_{79}^{(1,10)} = E_8^{(4)} \diamond L \diamond A_{66}$
$\hat{A}_{86}^{(1,9)} = E_8^{(3)} \diamond L \diamond A_{74}$	$\hat{A}_{160}^{(1,7)} = E_8^{(2)} \diamond L \diamond A_{150}$	

Table 2.7: Decomposition of the algebras $\hat{A}_r^{(n,m)}$ that occur in table 2.4.

Any remaining $\hat{A}_r^{(n,m)}$-algebras that appear in table 2.4 and are not reported in table 2.7 do not decompose. Analogously to the examples discussed in the previous section, we may also find double decompositions for the $\hat{A}_r^{(n,m)}$-algebras. We give one example below:

Example $D_{14}^{(26)} = A_{23} \diamond L \diamond E_7^{(3)} \diamond L \diamond D_5$

$$
\begin{array}{ccccc}
 & & \hat{A}_{34}^{(1,7)} \diamond L \diamond D_5 & & \\
 & \nearrow & & \searrow & \\
D_{14}^{(26)} & & & & A_{23} \diamond L \diamond E_7^{(3)} \diamond L \diamond D_5 \\
 & \searrow & & \nearrow & \\
 & & D_{14}^{(2)} \diamond L \diamond A_{23} & &
\end{array}
$$

as seen from table 2.4, 2.7 and (2.75).

2.9 Summary

In this chapter we defined a new class of Kac-Moody algebras referred to as the n-extended Lorentzian algebras, $\mathfrak{g}_{-n}$, and investigated their structural properties. To define $\mathfrak{g}_{-n}$, we began through the motivation of finite Kac-Moody algebra, $\mathfrak{g}_f$ and studied their key properties before extending them in a canonical way to an affine Kac-Moody algebra, $\mathfrak{g}_a$. This extension procedure was continued after arriving at $\mathfrak{g}_a$ and generalized beyond the previously studied over-extended, $\mathfrak{g}_{-1}$ and very-extended $\mathfrak{g}_{-2}$ algebras [24], leading us to a new definition consistent for $\mathfrak{g}_{-n}$.

For the corresponding Dynkin diagrams of $\mathfrak{g}_{-n}$ we constructed the root and weight lattices, and provided general equations for all the n-extended simple roots, $\alpha_i^{(n)}$ and fundamental weights $\lambda_i^{(n)}$. From $\lambda_i^{(n)}$ we constructed $\mathfrak{g}_{-n}$'s Weyl vector, $\rho^{(n)}$, for all values of n, and from $\rho^{(n)}\cdot\rho^{(n)}$ we found a generalized Freudenthal-de Vries strange formula, which led us to a necessary condition for $\mathfrak{g}_{-n}$ to possess a principal $SO(1,2)$-subalgebra. From the inner products of the Weyl vector $\rho^{(n)}$ and the fundamental weights, $\lambda_i^{(n)}$ we compute the expansion coefficients $D_i^{(n)}$ for the J_3-generator of the principal $SO(1,2)$ or $SO(3)$ subalgebra. Certain properties of these $SO(3)$ and $SO(1,2)$-subalgebras will be revisited in the following chapter, when we will be concerned with their Casimir eigenvalues in relation to the exponents of the $\mathfrak{g}_{-n}$ and other Lorentzian Kac-Moody algebras that we come across.

When the $D_i^{(n)}$ constants of the J_3-generator vanish, we found that $\mathfrak{g}_{-n}$ decomposed into reduced Dynkin diagrams. For the reduced diagrams we analysed in detail whether $D_i > 0$ or $\hat{D}_i < 0$ for all i, which constitutes a necessary and sufficient condition for the existence of a principal $SO(3)$-subalgebra or a principal $SO(1,2)$-subalgebra, respectively.

We derived explicit formulae to find relevant quantities related to the decomposition of $\mathfrak{g}_{-n}$ on both the side of the reduced components, and on the side of $\mathfrak{g}_{-n}$ itself. Complete lists are provided for *all* decompositions of n-extended Lorentzian Kac-Moody algebras $\mathfrak{g}_{-n}$ and have been verified through explicit calculation to high n. A similarly detailed analysis is presented for the $A_r^{(n)}$-algebras, but for $\hat{A}_r^{(n,m)} \neq A_r^{(n)}$ we only report the decomposition for the cases appearing in the decomposition of $\mathfrak{g}_{-n}$.

In addition to the role we discussed of $\mathfrak{g}_{-n}$ in string theory, these n-extended algebras and similar extended constructions utilizing the Lorentzian $\Pi^{(1,1)}$ lattice are also relevant within the context of classical and quantum integrable systems. Many of these integrable systems, such as the Toda field theories, or the Calogero-Moser-Sutherland systems can be described using the root systems and hence starting from the Dynkin diagrams considered in this chapter. Toda field theories based on $\mathfrak{g}_{-n}$ and other systems with Lorentzian $\Pi^{(1,1)}$ lattices will be considered in the following chapter.

Chapter 3

Lorentzian Toda Field Theories

Many versions of Toda field theories have been studied since the initial discovery of the discrete theory non-linear lattice in the 1975 [42]. For instance, continuous Toda field theories can be constructed as integrable conformal field theories, as we shall see in section 3.1.2 of this chapter. The conformal Toda field theories are based on the finite Kac-Moody algebras, $\mathfrak{g}_f$ we introduced in chapter 2, with fields associated to the simple roots of $\mathfrak{g}_f$. These theories may be extended to non-conformal, massive-theories by expanding their potential about a vector which turns out to be precisely that needed to extend $\mathfrak{g}_f$ to the affine algebra $\mathfrak{g}_a$ - such theories are known as affine Toda field theories accordingly. The Toda theories based on $\mathfrak{g}_a$ may be extended again by the introduction of two extra fields, again introducing conformal behaviour to the so-called conformal affine Toda theories [80, 81].

We find that this expansion procedure can be continued through following roughly the n-extension procedure described in the previous chapter, allowing us to construct Lorentzian Toda field theories based around $\mathfrak{g}_{-n}$ algebras [51]. Analogously to the behaviour observed from finite, affine and conformal affine Toad theories, we note that the Lorentzian theories we discover also follow the pattern of oscillating between conformal and massive theories as we add subsequent fields that perturb the potential of these theories in a well-defined and systematic manner. Unlike the theories bases on $\mathfrak{g}_f$ alone, and the affine and conformal affine theories perturbed from $\mathfrak{g}_f$, we find that the theories based on Lorentzian $\mathfrak{g}_{-n}$ are non-integrable, as confirmed through conducting the Painlevé test.

Even though these theories are found to be non-integrable, we shall see that they still possess interesting properties conserved from the integrable theories which they are perturbed from. For example, the perturbed E_8 theory that results in what we will denote as the $\left(\mathring{\boldsymbol{E}}_8\right)_{-2n}$-Toda field theories, have an almost stable noncrystallographic H_4 compound, meaning that the first four masses are almost identical in all such theories, include the integrable affine $\left(\mathring{\boldsymbol{E}}_8\right)_0$ theory that we shall study in its own right. Before presenting these results, we shall now motivate discrete and continuous classical Toda theories, building upon these classical results to discover Lorentzian Toda field theories, uncover the status of their integrability, and finally examine their mass ratios.

3.1 Toda Field Theories

Toda field theories are some of the best understood field theories in mathematical physics. The integrability of their classical field theory on a discrete lattice has been known for some time [42], and has encouraged much subsequent research into the continuous Toda field theories that this section will mainly focus on. However, before moving onto the continuous classical versions of the theory we will briefly look at the discretized versions to give context to where many of the techniques used throughout the studies of Toda field theories originated from.

3.1.1 Lattice Toda Field Theory

For the description in this subsection, we mainly follow the conventions in [60], writing the Hamiltonians of Toda field theories for N particles on a lattice as

$$\mathcal{H}(p,q) = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + e^{q_j - q_{j+1}}\right), \tag{3.1}$$

along with the lattice boundary condition that $q_{N+1} = q_1$. $\mathcal{H}$ and the boundary condition are in terms of the coordinates and conjugate momenta (q_j, p_j). To illustrate the integrability of this model, and also to aid with our discussion of the Painlevé test later in this chapter in section 3.6, we note that it is possible to write (q_j, p_j) as terms of the variables a_j, b_j as

$$a_j = \frac{1}{2} e^{\frac{1}{2}(q_j - q_{j+1})}, \quad b_j = \frac{1}{2} p_j, \tag{3.2}$$

for $j \in \{1, \cdots, N\}$. Allowing us to write the equations of motion for the Toda lattice as

$$\partial_t a_j = a_j(b_j - b_{j+1}), \quad \partial_t b_j = 2(a_{j-1}^2 - a_j^2). \tag{3.3}$$

This construction allows us to directly write down the Lax pair of the system

$$L = \begin{pmatrix} b_1 & a_1 & \dots & & a_N \\ a_1 & b_2 & & & \\ \vdots & & \ddots & & \\ & & & b_{N-1} & a_{N-1} \\ a_N & & & a_{N-1} & b_N \end{pmatrix}, \; A = \begin{pmatrix} 0 & -a_1 & \dots & & a_N \\ a_1 & 0 & & & \\ \vdots & & \ddots & & \\ & & & 0 & -a_{N-1} \\ -a_N & & & a_{N-1} & 0 \end{pmatrix}, \tag{3.4}$$

satisfying

$$\partial_t L = [A, L] = AL - LA. \tag{3.5}$$

Where $\partial_t L^k = [A, L^k]$ for all $k \in \mathbb{Z}$, meaning that (3.5) implies $Tr(L^k)$ and may be found as its first integrals, which can be shown to be independent and in involution. Hence, showing that this discrete Toda theory is integrable. From this point, many more solutions and behaviours of Lattice Toda can be found [82], that we shall not go into here.

More details of the integrability from Lax pairs can be found in [83]. In the next subsection we examine the continuos Toda theory, which we shall also find to be integrable. However, for continuous Toda we shall see that the integrability comes from the continuous conformal symmetry that the theory admits.

3.1.2 Conformal Toda Field Theory

As we saw in the introduction and rewrite here for reference, the classical Lagrangian of continuous Toda field theories may be taken to be

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{g}{\beta^2}\sum_{i=1}^{r} e^{\beta \alpha_i \cdot \phi}, \tag{3.6}$$

where we have r bosonic fields ϕ^a with $a \in \{1, \cdots, r\}$ that match up to the r simple roots of a rank-r finite Kac-Moody algebra, $\mathfrak{g}_f$, see section 2.1, and where g and β are constants. For this reason, we shall sometimes denote (3.6) as $\mathscr{L}_{\mathfrak{g}_f}$. The equations of motion from equation (3.6) are found to be

$$\partial_\mu \partial^\mu \phi^a + \frac{g}{\beta}\sum_{i=1}^{r} \alpha_i^a e^{\beta \alpha_i \cdot \phi} = 0 \tag{3.7}$$

through using the Euler-Lagrange equation, $\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^a)} = \frac{\partial \mathscr{L}}{\partial \phi^a}$, again with $a \in \{1, \cdots, r\}$.[1] We also shall write $\partial_\mu \partial^\mu := \partial^2$ in the discourse below.

Other properties of this and related Lagrangians will be examined throughout this chapter, but for now we would like to concentrate on the conformal properties. To do so we follow the reasoning of [48], in our notation for the equations of motion in (3.7), noting that each field ϕ_i are invariant under the transformation

$$\phi_i \to \phi_i + \frac{1}{\beta}\lambda_i \ln f^+ f^-, \tag{3.8}$$

where $f^\pm$ are written as arbitrary functions of ϕ_i and λ_i, as we saw in chapter 2, are the fundamental weights of the algebra with identity $\lambda_i \cdot \alpha_j = \delta_{ij}$ relative to the simple roots of $\mathfrak{g}_f$. This gives

$$\delta_f \phi_i = f^\mu \partial_\mu \phi_i \frac{1}{\beta}\lambda_i \partial_\mu f^\mu, \tag{3.9}$$

for an infinitesimal transform of the type in equation (3.8). We may then find the Noether current to be expressed in terms of the conformally improved energy-momentum tensor

$$J_f^\mu = \Theta^{\mu\nu} f_\nu. \tag{3.10}$$

[1] We shall rewrite the form of this Lagrangian later in this chapter. Also, see chapter 4 and equation (4.11) for alternate forms and uses of this Lagrangian.

A necessary condition for this theory to be valid and conformal would be to see that $\Theta_{\mu\nu}$ is conserved and traceless. Expressing $\Theta^{\mu\nu}$ in terms of the non-conformally improved energy-momentum tensor, $T_{\mu\nu}$ we find that

$$\begin{aligned} T^{\mu\nu} &= \sum_i \partial_\mu \phi_i \partial_\nu \phi_i - g_{\mu\nu}\mathscr{L} \\ \Theta^{\mu\nu} &= T^{\mu\nu} - \sum_i \gamma_i (\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)\phi_i, \end{aligned} \tag{3.11}$$

where γ_i are arbitrary constants. A direct calculation shows that

$$\partial^\mu T_{\mu\nu} = \partial^\mu \Theta_{\mu\nu} = 0, \tag{3.12}$$

as expected. We may choose $\gamma_i = \frac{2}{\beta}\lambda_i$, leaving the trace of the conformally improved energy-momentum tensor as

$$\Theta^\mu{}_\mu = \sum_i \big(\frac{2g}{\beta^2} e^{\beta\alpha_i \cdot \phi} + \gamma_i \partial^2 \phi_i\big) = 0, \tag{3.13}$$

where the last equality can be seen by substituting in equation (3.7), and hence illustrating the conformal invariance of the classical Toda field theory described by equation (3.6) based on a finite Kac-Moody algebra $\mathfrak{g}_f$. A similar calculation may be undertaken to show the conformal invariance of a quantum Toda field theory based on $\mathfrak{g}_f$, however as our focus shall be on the classical theories for the remainder of this chapter we only refer the interested reader to [48] for a continuation of this discussion.

The above calculations will become useful in analysing the behaviour of the Lorentzian Toda field theories that we uncover in section 3.5 and beyond. There we shall see conformal behaviour at certain levels of perturbation in Lorentzian Toda field theories. However, at other levels of perturbation in Lorentzian Toda's we do not see conformal behaviour, and we in fact will find these theories to be massive - this is similar to the behaviour that we shall see in the following subsection on Affine Toda field theories.

3.2 Affine Toda Field Theories

Just as we saw a natural connection of finite Kac-Moody algebras, $\mathfrak{g}_f$ to affine Kac-Moody algebras $\mathfrak{g}_a$ in chapter 2, here we find there is a direct connection of conformal Toda field theories based on $\mathfrak{g}_f$ to another Toda field theory based on the $\mathfrak{g}_a$ extended from $\mathfrak{g}_f$. For this reason, such theories are named *affine* Toda field theories. The most striking difference between $\mathfrak{g}_f$ and $\mathfrak{g}_a$ classical Toda theories is that the conformal invariance we found in the previous subsection vanishes, and as we shall see in the proceeding subsection, the theory becomes massive. We discuss the perturbation of finite to affine theories here in detail, as the results will become increasingly important through analogy to the Lorentzian

Toda theories we introduce in section 3.5 and beyond.

3.2.1 Conformal to Affine Toda Field Theory

Starting from the Lagrangian (3.6)[2], associated to rank-r $\mathfrak{g}_f$ with simple roots α_i, we also have r associated real scalar fields Φ_i for $i \in \{1, \cdots, r\}$. We may rewrite our conformal equations of motion from equation (3.7) through using the connection of α_i to their Cartan matrix, K as $\alpha_i \cdot \alpha_j = K_{ij}$, and identify $\Phi_i = \alpha_i \cdot \phi$, resulting in

$$\partial_\mu \partial^\mu \Phi_j + \frac{g}{\beta} \sum_{i=1}^{r} K_{ji} e^{\beta \Phi_i} = 0 \tag{3.14}$$

after acting on both sides of (3.7) with α_j from the left. As K_{ij} will always be invertible for any $\mathfrak{g}_f$ we choose, we can see that the minimum of the potential of $\mathscr{L}_{\mathfrak{g}_f}$ occurs as $e^{\beta\Phi_i} \to \infty$ for each α_i and hence Φ_i. This is in line with the conformal behaviour we proved in equation (3.13) for $\mathscr{L}_{\mathfrak{g}_f}$. For completeness, we note that from equation (3.8) we learnt we may apply a conformal transformation to the fields which was invariant on the equations of motion, using this shift in the fields with $f^+ f^- = \ln \frac{2}{\alpha_i}$ connects equation (3.14) to (3.7) for simply laced algebras.

We would now like to perturb the conformal theory, $\mathscr{L}_{\mathfrak{g}_f}$, with the aim of moving the $e^{\beta\Phi_i} \to \infty$ behaviour of the potential to a location of finite value. Following [84], this is achieved through choosing the shift in the perturbed potential to be

$$\delta V(\phi) = \frac{\epsilon g}{\beta^2} e^{\beta \alpha_0 \cdot \phi}, \tag{3.15}$$

where the perturbation parameter, ϵ, taken in the limit $\epsilon \to 0$, and where α_0 is the affine root that we obtained from extending the Dynkin diagram of $\mathfrak{g}_f$ via the definition of the highest root in equation (2.12). Naming the new minimum of the potential $\phi^{(0)}$, we have

$$\sum_{i=1}^{r} \alpha_i e^{\beta \alpha_i \cdot \phi^{(0)}} = -\varepsilon \alpha_0 e^{\beta \alpha_0 \cdot \phi^{(0)}}. \tag{3.16}$$

at the minimum of the perturbed potential, where we find the new potential $\mathcal{V}_{\mathfrak{g}_0}$ through the equations $\partial \mathcal{V}_{\mathfrak{g}_0} / \partial \phi^a |_{\phi^{(0)}} = 0$, $a = 1, \ldots, r$. Again using $\lambda_i \cdot \alpha_j = \delta_{ij}$ and acting on (3.16) from the left with λ_j we find

$$e^{\beta \alpha_i \cdot \phi^{(0)}} = -\varepsilon \lambda_i \cdot \alpha_0 e^{\beta \alpha_0 \cdot \phi^{(0)}}, \qquad i \in \{1, \ldots, r\}. \tag{3.17}$$

Using this identity, we expand $\mathcal{V}_{\mathfrak{g}_0}$ around the minima of the vacuum at $\phi^{(0)}$ with $\tilde{\phi}$, resulting in

[2] The Lagrangian may take other forms, see [80], but the one we use here is most concise for our framework of simple roots that we developed in chapter 2.

$$\mathcal{V}_{\mathfrak{g}_0}(\phi^{(0)}+\tilde{\phi}) = \varepsilon\frac{g}{\beta^2}e^{\beta\alpha_0\cdot\phi^{(0)}}\left[e^{\beta\alpha_0\cdot\tilde{\phi}} - \sum_{i=1}^{r}\lambda_i\cdot\alpha_0 e^{\beta\alpha_i\cdot\tilde{\phi}}\right] \tag{3.18}$$

and through making the identification that $m^2 = \varepsilon g e^{\beta\alpha_0\cdot\phi^{(0)}}$, $n_0 = 1$ and n_i for $i \in \{1, , r\}$ the Kac labels of $\mathfrak{g}_f$, we find

$$\mathcal{V}_{\mathfrak{g}_0}(\phi^{(0)}+\tilde{\phi}) = \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i e^{\beta\alpha_i\cdot\tilde{\phi}}. \tag{3.19}$$

For this perturbed theory, we write the Lagrangian as

$$\mathscr{L}_{\mathfrak{g}_a} = \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi - \frac{g}{\beta^2}\sum_{i=0}^{r} e^{\beta\alpha_i\cdot\phi}. \tag{3.20}$$

Using equation (3.19), the expansion of potential for $\mathscr{L}_{\mathfrak{g}_a}$ is

$$\mathcal{V}_{\mathfrak{g}_a} = \frac{m}{\beta^2}\sum_{i=1}^{r} n_i + \frac{m^2}{2}\sum_{i=1}^{r} n_i\alpha_i^a\alpha_i^b\phi^a\phi^b + \frac{\beta m^2}{6}\sum_{i=1}^{r} n_i\alpha_i^a\alpha_i^b\alpha_i^c\phi^a\phi^b\phi^c + \cdots \tag{3.21}$$

From this potential, we may read off the mass matrix, $(M^2)^{ab}$ from the squared powers of ϕ as

$$(M^2)^{ab} = m^2\sum_{i=1}^{r} n_i\alpha_i^a\alpha_i^b\phi^a\phi^b, \tag{3.22}$$

three point and higher couplings clearly may also be read off from $\mathcal{V}_{\mathfrak{g}_a}$.

A particularly interesting result was proven in [61] regarding the relation of this mass spectrum and couplings to a certain eigenvector, known as the *Perron-Frobenius* vector, of the Cartan matrix, K. This result was particularly interesting as it explained the fact that the mass ratios of the theory are independent of the symmetry breaking and are even preserved in the quantum theory for both the continuous [84, 54], and the discrete theory [85]. A result which we expand upon and apply to hyperbolic and Lorentzian algebras in Appendix A.

This perturbation procedure has taken us from a conformal field theory to a massive one, and importantly for the $\mathscr{L}_{\mathfrak{g}_f} \to \mathscr{L}_{\mathfrak{g}_a}$, we do not lose integrability. The integrability of $\mathscr{L}_{\mathfrak{g}_a}$ is seen most strikingly through the ability to write down exact S-matrices of the theory, such an exposition is given in detail for all $\mathfrak{g}_a$ within [84, 55].

In the following section, we shall see that $\mathscr{L}_{\mathfrak{g}_a}$ may be modified again to regain the conformal symmetry through adding certain extra fields to the theory. This analogy of going from conformal to massive, to conformal theory through the subsequent addition of extra fields will be a theme of this chapter, and will motivate the behaviour we see for Lorentzian Toda field theories later.

3.3 Conformal Affine Toda Field Theories

Modifying the Lagrangian of conformal affine Toda theory from the form used in [80, 81] to align with our notational conventions in this chapter, gives us

$$\mathscr{L}_{\mathfrak{g}_{ca}} = \partial_+\Phi\partial_-\Phi + \frac{g}{\beta}\sum_{i=0}^{r} e^{\beta\alpha_i\cdot\Phi} \tag{3.23}$$

which resembles that of $\mathscr{L}_{\mathfrak{g}_a}$ based on $\mathfrak{g}_a$, in (3.20), with the use of lightcone coordinates that we shall name (u, v) to distinguish from the spacetime coordinates (x, t), where $u^\pm = u \pm v$ and $\partial_\pm = \frac{1}{2}(\partial_u \pm \partial_v)$. However, this equation differs from $\mathscr{L}_{\mathfrak{g}_a}$ as we take the theories fields, Φ as

$$\Phi = \phi H + \zeta C + \eta T_3 \tag{3.24}$$

where the two fields η and ζ are added in addition to the $r+1$ ϕ fields from the $\mathscr{L}_{\mathfrak{g}_a}$ theory. H, C, D are the generator of the affine Cartan subalgebra of $\mathfrak{g}_a$, and $T_3 = H + hD$ for simply laced algebras [71]. To see the integrability of $\mathscr{L}_{\mathfrak{g}_{ca}}$ we may write down the zero-curvature condition for the associated linear system as

$$\partial_+ A_- - \partial_+ A_- + [A_+, A_-] = 0, \tag{3.25}$$

with

$$\begin{aligned} A_+ &= \partial_+\Phi + e^{ad\Phi}\varepsilon_+, \quad & A_- &= -\partial_-\Phi + e^{-ad\Phi}\varepsilon_- \\ \varepsilon_+ &= \sum_{i=0}^{r} E_{\alpha_i}, & \varepsilon_- &= \sum_{i=0}^{r} F_{\alpha_i}, \end{aligned} \tag{3.26}$$

where $E_{\alpha_i}, F_{\alpha_i}$ are the raising and lowering step operators for $\mathfrak{g}_a$, respectively. The ability to find this zero-curvature condition in equation (3.25) illustrates the integrability of conformal affine Toda field theory, with more details and derivation of these calculations available within [49, 50].

To better understand the conformal invariance of $\mathscr{L}_{\mathfrak{g}_{ca}}$, following [86] and references within, we rewrite equation (3.23) in component form as

$$\mathscr{L}_{\mathfrak{g}_{ca}} = \frac{\hat{\beta}^2}{2}\Big(\partial_\mu\phi\cdot\partial^\mu\phi + \partial_\mu\phi\cdot\partial^\mu\eta + h\partial_\mu\eta\cdot\partial^\mu\zeta\Big) - \sum_{i=0}^{r}\hat{g}_i e^{\beta\phi_i+\eta} \tag{3.27}$$

where h is the Coxeter number of $\mathfrak{g}_a$ introduced in section 2.3.3, and the coupling constants $\hat{g}_i$ and $\hat{\beta}$ are distinct to the coupling constants in $\mathscr{L}_{\mathfrak{g}_a}$ and equation (3.23). We have also absorbed in the identification used in the previous section that $\Phi_i = \alpha_i \cdot \phi$ associated to the affine Cartan matrix $K_{ij} = \frac{2\alpha_i\cdot\alpha_j}{\alpha_i^2}$ of $\mathfrak{g}_a$. The corresponding equations of motion may be written as

$$\begin{aligned}
\partial_+\partial_-\phi &= \frac{1}{\hat{\beta}}(\sum_{i=0}^{r}\hat{g}_i\alpha_i^a e^{\beta\hat{\phi}_i})e^{\hat{\beta}\eta} \quad i\in\{0,\cdots,r\}, &(3.28)\\
\partial_+\partial_-\eta &= 0, &(3.29)\\
\partial_+\partial_-\zeta &= \frac{\hat{g}_0}{\hat{\beta}}e^{-\beta\hat{\phi}_i}e^{\hat{\beta}\eta}, &(3.30)
\end{aligned}$$

where we have assumed that $(\alpha_0)^2=2$ when calculating $\partial_\mu\frac{\partial\mathscr{L}_{\mathfrak{g}ca}}{\partial(\partial_\mu\phi^a)}=\frac{\partial\mathscr{L}}{\partial\phi^a}$. These equations of motion are invariant under the conformal transformations

$$\begin{aligned}
e^{-\phi^a(u,v)} &\to e^{-\tilde{\phi}^a(\tilde{u},\tilde{v})}=e^{-\phi^a(u,v)} &(3.31)\\
e^{-\zeta(u,v)} &\to e^{-\tilde{\zeta}(\tilde{u},\tilde{v})}=\left(\frac{df_1}{du}\right)^{x\in\mathbb{R}}\left(\frac{df_2}{dv}\right)^{x\in\mathbb{R}}e^{-\zeta(u,v)} &(3.32)\\
e^{-\eta(u,v)} &\to e^{-\tilde{\eta}(\tilde{u},\tilde{v})}=\left(\frac{df_1}{du}\right)^{\frac{1}{\beta}}\left(\frac{df_2}{dv}\right)^{\frac{1}{\beta}}e^{-\eta(u,v)} &(3.33)
\end{aligned}$$

for f_1,f_2 analytic functions, showing that e^ϕ transform as scalars, and for $x=0$ e^ζ transforms in the same manner. The equations of motion, along with the potential of $\mathscr{L}_{\mathfrak{g}ca}$ are also invariant under the field transformation

$$\phi\to\phi+\chi\rho,\quad \eta\to\eta-\chi,\quad \zeta\to\zeta+c\chi, \tag{3.34}$$

for a transform by the arbitrary function χ, with $\partial_+\partial_-\chi=0$, c an arbitrary constant, and ρ is the Weyl vector, as defined in section 2.3.3. To confirm that $\mathscr{L}_{\mathfrak{g}ca}$ is indeed a conformal theory we first calculate the non-conformally improved energy-momentum tensor for simply laced algebras with normalized root systems of $\alpha_i^2=2$ as

$$\begin{aligned}
\Theta_{\mu\nu} &= \frac{\hat{\beta}^2}{2}\sum_{i,j=0}^{r}\frac{2}{\alpha_i^2}K_{ij}\left[\partial_\mu\phi_i\partial_\nu\phi_j-\frac{1}{2}g_{\mu\nu}\partial_\mu\phi_i\partial^\mu\phi_j\right]\\
&+\frac{\hat{\beta}^2}{2}\sum_{i=0}^{r}\frac{2}{\alpha_i^2}\left[\partial_\mu\phi_i\partial_\nu\eta+\partial_\nu\phi_i\partial_\mu\eta-g_{\mu\nu}\partial_\mu\phi_i\partial^\mu\eta\right]\\
&+\hat{\beta}^2\frac{h}{2}\left[\partial_\mu\eta\partial_\nu\zeta+\partial_\nu\eta\partial_\mu\zeta-g_{\mu\nu}\partial_\mu\eta\partial^\mu\zeta\right]\\
&+g_{\mu\nu}\sum_{i=0}^{r}\hat{g}_i e^{\beta\phi_i+\eta}
\end{aligned} \tag{3.35}$$

allowing us to write the full conformally improved energy-momentum tensor of [49, 86] for our cases as

$$\Theta_{\mu\nu}^{\mathrm{CAT}}=\Theta_{\mu\nu}-\hat{\beta}\left(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2\right)\left(\sum_{i=0}^{r}\phi_i+h\zeta\right), \tag{3.36}$$

which has a vanishing trace, and hence is the energy-momentum tensor of a conformal theory.

$\mathscr{L}_{\mathfrak{g}_{ca}}$ is thus a conformal theory that is also an integrable theory. It was obtained from the massive, non-conformal, affine $\mathscr{L}_{\mathfrak{g}_a}$ theory through the addition of two extra fields η and ζ. Recalling the previous sections, $\mathscr{L}_{\mathfrak{g}_a}$ was in turn obtained from the extension of another conformal theory with one less field, $\mathscr{L}_{\mathfrak{g}_f}$, based on a finite Kac-Moody algebra $\mathfrak{g}_f$. We have therefore seen the pattern that the extensions procedures of Kac-Moody algebras, and the inclusion of additional fields in terms of the associated Toda field theories, allowed us to alternate between conformal and massive models. This can be summarized by the schematic below for the Lagrangians of the Toda field theories:

$$\overset{\text{CFT}}{\mathscr{L}_{\mathfrak{g}_f}} \quad \xrightarrow{\phi_0} \quad \overset{\text{Massive}}{\mathscr{L}_{\mathfrak{g}_a}} \xrightarrow{\eta,\zeta} \quad \overset{\text{CFT}}{\mathscr{L}_{\mathfrak{g}_{ca}}}$$

We shall see a continuation of this pattern when we extend the base algebra of the Toda field theory past $\mathfrak{g}_a$. This will be achieved through use of the n-extended Lorentzian Kac-Moody algebra construction of chapter 2 to construct new field theories, which we name *Lorentzian* Toda field theories, through carefully chosen perturbations from additional fields associated to simple roots of those extended algebras. Before seeing this, in the next section we shall briefly go over a more compact form for our simple roots in terms of matrices, allowing us to construct the Lorentzian Toda field theories in a more concise notation.

3.4 Lorentzian Matrix Products

In this section, before constructing our Lorentzian Toda field theories we rewrite the n-extended root formation of section 2.4 in terms of matrices to aid within the formulation of Lagrangians, energy-momentum tensors and the integrable status of these theories later in this chapter. Using the Lorentzian inner product from equation (2.14) for a 1-extended root lattice for a $\mathfrak{g}_{-1}$ in accordance with equation (2.27), in component form we have

$$x \cdot y := \sum_{\beta=1}^{\ell} x_\beta y_\beta - \sum_{\beta=1}^{m} \left(x_{\ell+2\beta-1} y_{\ell+2\beta} + x_{\ell+2\beta} y_{\ell+2\beta-1} \right). \tag{3.37}$$

for vectors $x = (x_1, \ldots, x_{\ell+2m})$ and $y = (y_1, \ldots, y_{\ell+2m})$. Then this inner-product definition can be extended naturally to matrix multiplication for a general $N \times (\ell + 2m)$-matrix A and another, $(\ell + 2m) \times N$-matrix B, as

$$(AB)_{ij} := \sum_{\beta=1}^{\ell} A_{i\beta} B_{\beta j} - \sum_{\beta=1}^{m} \left[A_{i(\ell+2\beta-1)} B_{(\ell+2\beta)j} + A_{i(\ell+2\beta)} B_{(\ell+2\beta-1)j} \right], \quad i, j = 1, \ldots, N. \tag{3.38}$$

Illustrating the 1-extended further, we take $N = r+n+1$ to define a $(r+n+1) \times (\ell + 2m)$-matrix M with rows comprised of $r+n+1$ root vectors $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^{\ell+2m})^T$ of dimension

$\ell+2m$, i.e. $M_{i\beta} := \alpha_i^\beta$. The inverse of such a matrix, $M_{i\beta}$ may be found when $r+n+1 \leq \ell+2m$, such that a right inverse can be constructed. To construct this right inverse we utilize the fundamental weights, λ_i, of section 2.3.3 and define the matrix Λ to be composed of λ_i^β running along the columns of Λ for $\beta \in \{1, \cdots, \ell+2m\}$ being the components of each λ_i. Hence, for the components of Λ we have $\Lambda_{\beta i} := \lambda_i^\beta$. Or more precisely, with

$$M_{i\beta} := \alpha_i^\beta, \qquad \Lambda_{\beta i} := \lambda_i^\beta, \qquad i = 1, \ldots, r+n+1;\ \beta = 1, \ldots, \ell, \tag{3.39}$$

we obtain

$$(M\Lambda)_{ij} = \alpha_i \cdot \lambda_j = \delta_{ij} = \lambda_i \cdot \alpha_j = \left(\Lambda^T M^T\right)_{ij}. \tag{3.40}$$

Using Λ, Λ^T and M, M^T we may write a symmetric Cartan matrix, K and its inverse, K^{-1} as

$$\left(MM^T\right)_{ij} = \alpha_i \cdot \alpha_j = K_{ij}, \quad \text{and} \quad \left(\Lambda\Lambda^T\right)_{ij} = \lambda_i \cdot \lambda_j = K_{ij}^{-1}. \tag{3.41}$$

As outlined in section 2.1, a general Cartan matrix is defined as $2\alpha_i \cdot \alpha_j/\alpha_j^2$, but may be simplified to $\alpha_i \cdot \alpha_j$ when taking the length of the roots to be 2 for symmetric K, as we did within our discussions of conformal affine Toad theory in the equations of the previous section. Here however, we shall encounter roots of length zero in the sense that $\alpha_i \cdot \alpha_i = 0$, so we will find it necessary to adopt the symmetric K convention.

In chapter 2 we recall that the constants D_i arose when examining the three-dimensional principal subalgebras $SO(3)$ and $SO(1,2)$ of $\mathfrak{g}_{-n}$, and were especially important in relation to finding their possible decomposition according to their respective $SO(3)$ and $SO(1,2)$ subalgebras [25]. In this chapter, we shall again use equation (2.52) to define the D_i constants into a diagonal matrix $D := diag(D_r, \ldots, D_{-n})$, which we shall use to define the *Painlevé matrix* as

$$P := 2DK, \tag{3.42}$$

also formed with the matrix inner product defined by equation (3.38), to help probe the integrability of the Lorentzian Toda field theories we start to uncover in the next section.

Moreover, for the remainder of this chapter, all matrix products will be taken to satisfy equation (3.38) unless otherwise specified.

3.5 Perturbed $\mathscr{L}_{\mathfrak{g}_{-1}}$-Lorentzian Toda Field Theory

We are now in the position where we can start to form Toda field theories from the n-extended Kac-Moody algebra construction. Illustrating with the most simple case of a 1-extended algebra, $\mathfrak{g}_{-1}$, we start with the associated root lattice $\Lambda_{\mathfrak{g}_{-1}}$ defined in equation (2.17), with affine root $\alpha_0 = k - \theta$ defined via the highest root of the system, θ, and where k is that of equation (2.15). The 1-extended root is also still defined as $\alpha_{-1} = -(k + \bar{k})$. Extending the schematic in section 3.3 for the Lagrangians of the perturbed theories beyond affine we have

$$\overset{\text{CFT}}{\mathscr{L}_{g_f}} \overset{\phi_0}{\longrightarrow} \overset{\text{Massive}}{\mathscr{L}_{g_a}} \overset{\phi_{-1}}{\longrightarrow} \overset{\text{CFT}}{\mathscr{L}_{g_{-1}}} \overset{\phi_{-2}}{\longrightarrow} \overset{\text{Massive}}{\mathscr{L}_{\mathring{g}_{-2}}}, \tag{3.43}$$

where all the perturbed fields are associated to the extra highest root in the given Lagrangian, through the association $\alpha_i \cdot \phi = \Phi_i$ that we have seen before. Up to $\mathscr{L}_{\mathfrak{g}_{-1}}$, the pattern in (3.43) is akin to that of the previous chapters from finite, to affine to conformal affine, whereas the extra perturbation to $\mathscr{L}_{\mathring{g}_{-2}}$ was not observed until [25]. We note at this stage that $\mathring{\mathfrak{g}}_{-2}$ is slightly different to $\mathfrak{g}_{-2}$ of chapter 2, this difference will be elaborated on later in this section, but we shall first focus on $\mathscr{L}_{\mathfrak{g}_{-1}}$ before perturbing it again to reach this new theory.

3.5.1 $\mathscr{L}_{\mathfrak{g}_{-1}}$ Conformal Toda Field Theory

Referring to equations (3.6) and (3.20), we write the Lagrangian for $\mathscr{L}_{\mathfrak{g}_{-n}}$ as

$$\mathscr{L}_{\mathfrak{g}_a} = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi - \frac{g}{\beta^2}\sum_{i=-1}^{r} e^{\beta\alpha_i\cdot\phi}, \tag{3.44}$$

its classical equations of motion can be written as

$$\Box\phi^a + \frac{g}{\beta^2}\sum_{i=-1}^{r}\alpha_i^a e^{\beta\alpha_i\cdot\phi} = 0, \qquad a = 1,\ldots,\ell+2, \tag{3.45}$$

obtained from applying the Euler-Lagrange equation, $\partial_\mu\left[\partial\mathscr{L}/\partial(\partial_\mu\phi^a)\right] = \partial\mathscr{L}/\phi^a$. In terms of the affine Toda field theories potential, $\mathcal{V}_{\mathfrak{g}_a}$, the potential of this theory may be obtained through the perturbation, akin that of section 3.2.1 that took us from finite conformal Toda to the affine theory, as $\mathcal{V}_{\mathfrak{g}_{-1}} = \mathcal{V}_{\mathfrak{g}_a} + \delta\mathcal{V}_{\mathfrak{g}_a}$, where $\delta\mathcal{V}_{\mathfrak{g}_a}$ corresponds to the term in the sum related to α_{-1}.

We require that the additional term in the potential, $\delta\mathcal{V}_{\mathfrak{g}_a}$, does not spoil the vacuum which $\mathcal{V}_{\mathfrak{g}_a}$ possesses, which can be expanded around. This can be viewed on one level as the fields vanishing for $\alpha_i \cdot \phi \to -\infty$, but may also be seen through examining the trace of the energy-momentum tensor as we have done in the previous subsections within this chapter. To examine the energy-momentum tensor of $\mathscr{L}_{\mathfrak{g}_{-1}}$ we deform the fields as $\Phi_i := \alpha_i \cdot \phi - \beta^{-1}\ln(2\alpha_i^{-2})$ to write the Lagrangian, similar to how we did in equation (3.14), as

$$\Box\Phi_j + \frac{g}{\beta^2}\sum_{i=-1}^{r} K_{ji}e^{\beta\Phi_i} = 0, \tag{3.46}$$

with symmetric K_{ij}. Defining φ_i for $i \in \{-1,0,1,,r\}$ as $\Phi_i = (M\varphi)_i$ where we use M as defined in equation (3.39), we rewrite the equations of motion in a new form as

$$\Box\varphi_\alpha + \frac{g}{\beta^2}\sum_{i=-1}^{r}\left(M^T\right)_{\alpha i} e^{\beta(M\varphi)_i} = 0. \tag{3.47}$$

Following the methods to obtain the trace of the energy-momentum tensor, $\Theta^\mu{}_\mu$ used in sections 3.1.2 and 3.3, we obtain

$$\Theta^\mu{}_\mu = \sum_{i=-1}^{r}\left(\frac{2g}{\beta^2}e^{\beta(M\varphi)_i} + \gamma_i\Box\varphi_i\right). \tag{3.48}$$

Hence, as long as we can find an inverse matrix, M^{-1}, we can form $\gamma_i = 2\beta^{-1}\sum_k M^{-1}_{ik}$, meaning the trace of the conformally improved energy-momentum tensor vanishes and the theory is shown to be conformal. This means that in the expansion of the potential, as done in equation (3.21) for the affine theory, the corresponding mass matrix would not be invertible, explaining on another level why massive particles would not be found within this theory.

3.5.2 $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2}}$ Massive Toda Field Theory

We have found that $\mathscr{L}_{\mathfrak{g}_{-1}}$ is a conformal theory that may be studied in its own right, however the remainder of this chapter we focus on the massive theory that results from an additional perturbation from it, that we name $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2}}$. This new theory is obtained through perturbing $\mathcal{V}_{g_{-1}}$ so that

$$\mathcal{V}_{\mathring{\mathfrak{g}}_{-2}}(\phi) := \mathcal{V}_{\mathfrak{g}_{-1}}(\phi) + \delta\mathcal{V}_{\mathfrak{g}_{-1}}(\phi) = \mathcal{V}_{\mathfrak{g}_{-1}}(\phi) + \varepsilon\frac{g}{\beta^2}e^{\beta\alpha_{-2}\cdot\phi}. \tag{3.49}$$

The vacuum $\phi^{(0)}$ for the new potential $\mathcal{V}_{\mathring{g}_{-2}}$ computed from the equations $\partial\mathcal{V}_{\mathring{g}_{-2}}/\partial\phi^a\Big|_{\phi^{(0)}} = 0$, $a = 1,\ldots,r+2$, leads to the constraint

$$\sum_{i=-1}^{r}\alpha_i e^{\beta\alpha_i\cdot\phi^{(0)}} = -\varepsilon\alpha_{-2}e^{\beta\alpha_{-2}\cdot\phi^{(0)}}. \tag{3.50}$$

Multiplying with the fundamental weights λ_j and using the orthogonality relation $\alpha_i\cdot\lambda_j = \delta_{ij}$ yields the relations

$$e^{\beta\alpha_i\cdot\phi^{(0)}} = -\varepsilon\lambda_i\cdot\alpha_{-2}e^{\beta\alpha_{-2}\cdot\phi^{(0)}}, \qquad i = -1,0,1,\ldots,r. \tag{3.51}$$

Expanding now the potential $\mathcal{V}_{\mathring{g}_{-2}}(\phi)$ around the vacuum we obtain with (3.51)

$$\mathcal{V}_{\mathring{g}_{-2}}(\phi^{(0)}+\tilde{\phi}) = \varepsilon\frac{g}{\beta^2}e^{\beta\alpha_{-2}\cdot\phi^{(0)}}\left[e^{\beta\alpha_{-2}\cdot\tilde{\phi}} - \sum_{i=-1}^{r}\lambda_i\cdot\alpha_{-2}e^{\beta\alpha_i\cdot\tilde{\phi}}\right] = \frac{m^2}{\beta^2}\sum_{i=-2}^{r}\hat{n}_i e^{\beta\alpha_i\cdot\tilde{\phi}}, \tag{3.52}$$

where $m^2 = \varepsilon g e^{\beta\alpha_{-2}\cdot\phi^{(0)}}$, $\hat{n}_{-2} = 1$ and $\hat{n}_i = -\lambda_i \cdot \alpha_{-2}$. We now make the choice $\alpha_{-2} = \bar{k}$, so that with the realizations of the fundamental weights for $\boldsymbol{g}_{-1}$ as in [24, 25], and in accordance with the choices that we made in chapter 2

$$\lambda_i = \lambda_i^f + n_i \lambda_0^o, \quad \lambda_0 = \bar{k} - k, \quad \lambda_{-1} = -k, \quad \text{with } i = 1, \ldots, r, \tag{3.53}$$

and λ_i denoting the fundamental weights of $\boldsymbol{g}$, we compute $\hat{n}_{-1} = 1$, $\hat{n}_0 = 1$ and $\hat{n}_i = n_i$ the Kac labels as before. Here we make the distinction between $\mathring{\mathfrak{g}}_{-2}$ and the regular n-extended Lorentzian Kac-Moody algebras, $\mathfrak{g}_{-n}$ defined in chapter 2. We have $\alpha_{-2} \in \Lambda_{\mathfrak{g}_f} \oplus \Pi^{1,1}$ connecting in an almost identical way as the root $k - \left(\ell + \bar{\ell}\right)$ to all the other simple roots with $\alpha_{-2} \cdot \alpha_{-1} = -1$, $\alpha_{-2} \cdot \alpha_i = 0$, $i = 1, \ldots, r$. However, this root also connects to the affine root $\alpha_{-2} \cdot \alpha_0 = 1$, has length zero, i.e. $\alpha_{-2}^2 = 0 \neq 2$, and is defined in a smaller representation space than the standard α_{-2} root. Hence, $\Lambda_{\mathring{\mathfrak{g}}_{-2}}$ can not be viewed as a lattice related to a Kac-Moody algebra, and we refer to it therefore as a root lattice to an *almost over extended algebra.*

To obtain the mass matrix of the $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2}}$ theory, we expand equation (3.52) about zero and obtain

$$\frac{m^2}{\beta^2}\left(\hat{n}_{-2} + \hat{n}_{-1} + \hat{n}_0 + \sum\nolimits_{i=1}^{r} n_i\right) = \frac{m^2}{\beta^2}(2 + h), \tag{3.54}$$

as constant term in the potential, with h being the Coxeter number of $\mathfrak{g}_f$. The choice of α_{-2} has the property

$$\sum\nolimits_{i=-2}^{r} \hat{n}_i \alpha_i = 0, \tag{3.55}$$

which is crucial to ensure the linear terms within the expansion of the potential to vanish. The resulting mass matrix must be square and is obtained to be

$$M^2 = m^2 \sum_{i=-2}^{r} \hat{n}_i \begin{pmatrix} \alpha_i^1 \alpha_i^1 & \ldots & \alpha_i^1 \alpha_i^r & -\alpha_i^1 \alpha_i^{r+2} & -\alpha_i^1 \alpha_i^{r+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_i^1 \alpha_i^r & \ldots & \alpha_i^r \alpha_i^r & -\alpha_i^r \alpha_i^{r+2} & -\alpha_i^r \alpha_i^{r+1} \\ -\alpha_i^1 \alpha_i^{r+2} & \ldots & -\alpha_i^r \alpha_i^{r+2} & \alpha_i^{r+2} \alpha_i^{r+2} & \alpha_i^{r+1} \alpha_i^{r+2} \\ -\alpha_i^1 \alpha_i^{r+1} & \ldots & -\alpha_i^r \alpha_i^{r+1} & \alpha_i^{r+1} \alpha_i^{r+2} & \alpha_i^{r+1} \alpha_i^{r+1} \end{pmatrix}, \tag{3.56}$$

in analogy to the affine Toda theories in equation (3.22). For physical results, equation (3.56) must yield real and positive eigenvalues, which shall be examined for concrete examples in section 3.7. Before looking at specific examples of Lorentzian Toda field theories, in the next section we shall establish a framework to allow insight to the integrability of these models.

3.6 Painlevé Integrability Test

We have already come across several techniques to establish the integrability of field theories. In section 3.3 we used the zero curvature condition to show the integrability of conformal affine Todas, and in section 3.1.1 we used a Lax pair to prove the integrability of the discrete Toda theory on a lattice. Here we shall use a different technique to determine the integrability status of the Lorentzian Todas outlined in the previous section, a technique known as the *Painlevé test* [87, 88, 89, 90]. In chapter 4 we will go into much greater detail regarding the concept of Painlevé integrability, so here we only emphasize the most salient feature: that it provides a necessary but not sufficient condition for the theory being tested to be integrable. In other words, passing the test does not prove integrability, but failing the test shows the theory is certainly not integrable. However, no counter examples are known.

To conduct the Painlevé test on a subset of perturbed $\mathscr{L}_{\mathfrak{g}_{-n}}$, we largely follow and generalize the reasoning of [21, 60] to determine whether these theories can be integrable or not. We start by transforming the equations of motion in (3.47) into light-cone coordinates so that $\Box = \partial_{-}\partial_{+}$. Denoting ∂_{-} by an overdot and ∂_{+} by an overdash, e.g. $\partial_{-}\varphi =: \dot{\varphi}$ and $\partial_{+}\varphi =: \acute{\varphi}$. For further convenience we set $g = \beta = 1$. We write down the second order equation of motion into two separated first order equations, which can be achieved by introduce two quantities, this can be thought of as being akin, but not equal, to canonical variables, as

$$P_{\alpha} = \dot{\varphi}_{\alpha}, \qquad Q_i = e^{(M\varphi)_i}, \qquad \alpha = 1, \ldots, \ell + 2m, i = 1, \ldots, r + n + 1. \tag{3.57}$$

Differentiating these quantities with respect to each light-cone coordinate we obtain

$$\acute{P}_{\alpha} = \Box\varphi_{\alpha} = -\sum_{i=-n}^{r} \left(M^T\right)_{\alpha i} Q_i, \qquad \dot{Q}_i = Q_i \left(MP\right)_i. \tag{3.58}$$

We now expand P_{α} and Q_i, making the standard assumption that both quantities possess no movable critical singularities in some field $\phi(x_{-}, x_{+}) \to 0$, whose leading order is determined by some positive integers $n_p, n_q > 0$

$$Q_i = \sum_{k=0}^{\infty} a_i^{(k)} \phi^{k-n_q}, \qquad P_{\alpha} = \sum_{k=0}^{\infty} b_{\alpha}^{(k)} \phi^{k-n_p}. \tag{3.59}$$

naming this procedure a *Painlevé expansion*. Differentiating these Painlevé expansions, we obtain

$$\dot{Q}_i = \sum_{k=0}^{\infty} (k - n_q) a_i^{(k)} \phi^{k-n_q-1} \dot{\phi}\,, \qquad \acute{P}_{\alpha} = \sum_{k=0}^{\infty} (k - n_p) b_{\alpha}^{(k)} \phi^{k-n_p-1} \acute{\phi}. \tag{3.60}$$

Substituting next the expansions (3.59) and (3.60) into (3.58) and balancing the powers we obtain

$$
\begin{aligned}
(k-n_p)\acute{\phi} b_\alpha^{(k)} &= -\sum_{i=-n}^{r} \left(M^T\right)_{\alpha i} a_i^{(k)}, && (3.61)\\
(k-n_q)\dot{\phi} a_i^{(k)} &= \sum_{m=0}^{k} a_i^{(k-m)} \left(M b^{(m)}\right)_i, && (3.62)
\end{aligned}
$$

with $n_q = n_p + 1$. At this point we have to distinguish between two cases i) when the Cartan matrix is invertible, i.e. when we have a massive Toda theory and ii) when it is not, as will always be with a conformal Toda theory. We shall tackle the former case first in the following subsection.

3.6.1 Invertible Cartan matrix

For $k = 0$ we can solve the equations (3.61) and (3.62) for the leading order coefficient functions when the Cartan matrix is invertible

$$
a_i^{(0)} = -n_p n_q \dot{\phi}\acute{\phi} D_i, \qquad b_\alpha^{(0)} = -n_q \dot{\phi} \sum_{i=-n}^{r} \left(M^T\right)_{\alpha i} D_i, \tag{3.63}
$$

where the D_i are the constants we have seen several times before, as defined in equations (2.42), (2.52) and (2.53).

Next, we factor out the terms in the sum for $m = 0$ and $m = k$ from equation (3.62). Using also $\left(M b^{(0)}\right)_i = -n_q \dot{\phi}$, we re-write (3.61) and (3.62) as

$$
\begin{aligned}
k\dot{\phi} a_i^{(k)} + n_p n_q \dot{\phi}\acute{\phi} D_i \left(M b^{(k)}\right)_i &= \sum_{m=1}^{k-1} a_i^{(k-m)} \left(M b^{(m)}\right)_i, && (3.64)\\
\sum_{i=-n}^{r} \left(M^T\right)_{\alpha i} a_i^{(k)} + (k-n_p)\acute{\phi} b_\alpha^{(k)} &= 0. && (3.65)
\end{aligned}
$$

These equations, (3.64) and (3.65), can be converted into matrix form

$$
T^{(k)} X^{(k)} = Y^{(k)}, \tag{3.66}
$$

when defining the $N + M = (r + n + 1) + (\ell + 2m)$ dimensional column vectors

$$
\begin{aligned}
X^{(k)} &= (a_1^{(k)}, \cdots, a_M^{(k)}, b_1^{(k)}, \cdots, b_N^{(k)})^T, && (3.67)\\
Y^{(k)} &= \sum_{m=1}^{k-1} (a_1^{(k-m)} \left(M b^{(m)}\right)_1, \cdots, a_M^{(k-m)} \left(M b^{(m)}\right)_M, 0, \cdots, 0)^T, && (3.68)
\end{aligned}
$$

together with the $(M + N) \times (M + N)$-matrix

$$T^{(k)} = \begin{pmatrix} A^{(k)}_{M\times M} & B^{(k)}_{M\times N} \\ C^{(k)}_{N\times M} & E^{(k)}_{N\times N} \end{pmatrix}. \tag{3.69}$$

The block matrices in T have entries

$$A^{(k)}_{ij} = k\dot{\phi}\delta_{ij}, \quad B^{(k)}_{i\alpha} = n_p n_q \dot{\phi}\acute{\phi} D_i M_{i\alpha}, \quad C^{(k)}_{\alpha i} = \left(M^T\right)_{\alpha i}, \quad E^{(k)}_{\alpha\beta} = (k-n_p)\acute{\phi}\delta_{\alpha\beta}. \tag{3.70}$$

In this current formulation of the Painlevé integrability test, equation (3.66) is the most pertinent equation. It is a recursive equation that may in principle be solved iteratively at each level k for the coefficient functions contained in $X^{(k)}$ as long as the matrix $T^{(k)}$ is invertible. Whenever this is not the case one a free parameter is introduced, otherwise known as a *resonance* in the Painlevé integrability test parlance, into the set of equations. When there are enough resonances in the system as boundary conditions or integration constants, the system is passing the test and is said to be integrable. We go into more detail regarding resonances and their role and importance for finding integrability of a given theory in chapter 4.

As equation (3.66) is central in discovering the integrability of our models, let us therefore compute the determinant of $T^{(k)}$. Using the identity

$$\det\begin{pmatrix} A & B \\ C & E \end{pmatrix} = \det\begin{pmatrix} A & B \\ C & E \end{pmatrix}\det\begin{pmatrix} I & 0 \\ -E^{-1}C & I \end{pmatrix} = \det(A - BE^{-1}C)\det(E), \tag{3.71}$$

we obtain

$$\det T^{(k)} = (k-n_p)^{r+n}\acute{\phi}^{r+n+1}\dot{\phi}\det\left[k(k-n_p)I - n_p(n_p+1)DK\right]. \tag{3.72}$$

Apart from the pre-factor that we have included for generality, for $n = n_p = 1$, this reduces to the expression previously obtained in [21] for the subclass of hyperbolic Kac-Moody algebras. Taking now $n_p = 1$, the matrix in the determinant becomes the *Painlevé matrix* and the last factor in (3.72) can be read as the characteristic equation for the matrix $P = 2DK$ with eigenvalues $k(k-1)$. Thus we have found that also for the $\mathscr{L}_{g-n}$ Lorentzian Toda theories the integrability test can be reduced to an eigenvalue problem for P.

Before considering the case of a non-invertible Cartan matrix we shall briefly look into the connection of the Painlevé matrix to the factors appearing in the adjoint action of the three-dimensional principal subalgebras that we first encountered in section 2.5.

3.6.2 Connection to Casimir Eigenvalues of the Principal $SO(1,2)$-Subalgebra

For hyperbolic algebras, Nicolai and Olive noticed in [73] that this matrix also emerges from the adjoint action of the principal subalgebra, $SO(1,2)$ that we considered in section 2.5. Earlier, we saw the principal $SO(3)$ and $SO(1,2)$-subalgebras in relation to their expansion coefficients $D_i^{(n)}$ for the J_3-generator of the principal $SO(1,2)$ or $SO(3)$ subalgebra, whereas here the $D_i^{(n)}$ contribute as the Casimir operator

$$\begin{aligned} \mathcal{Q} &= J_3 J_3 - J_+ J_- - J_- J_+ \\ &= J_3 \left(J_3 - 1\right) - 2J_+ J_- = J_3 \left(J_3 + 1\right) - 2J_- J_+, \end{aligned} \tag{3.73}$$

on the Cartan subalgebra for these three-dimensional principal subalgebras, and that in fact the eigenvalues are identical to the Casimir eigenvalues, and where J_+, J_- are defined as in section 2.5. Following [73], this may be seen through first constructing the adjoint action on $\mathcal{Q}$ as

$$\mathrm{ad}_{\mathcal{Q}}(x) := [J_3,[J_3,x]] - [J_+,[J_-,x]] - [J_-,[J_+,x]] \tag{3.74}$$

for $x \in \mathfrak{g}_{-n}$. We may always perform an expansion on the elements of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{-n}$ through the linear combination $\sum_j c_i h_j$, where c_j are expansion coefficients and $h_j \in \mathfrak{h}$, where substituting this into the above gives

$$\mathrm{ad}_{\mathcal{Q}}\left(\sum_j c_j h_j\right) = -2\left[J^-, \left[J^+, \sum_j c_j h_j\right]\right] = 2\sum_{i,j} c_i D_j K_{ij} h_j. \tag{3.75}$$

Now setting $c_j = D_j$ and using the identity $\sum_j K_{ij} D_j = -1$ for all i, we obtain

$$\mathrm{ad}_{\mathcal{Q}}\left(J_3\right) = -2\sum_{i,j} D_i K_{ij} D_j h_j = -\sum_{i,j} P_{ij} D_j h_j = +2J_3. \tag{3.76}$$

Illustrating the importance of P both in the algebraic structure of $\mathfrak{g}_{-n}$ as well as the context of the Painlevé test. We have constructed here the $SO(1,2)$ algebra case, but we note that in generalised cases a principal $SO(1,2)$-subalgebra does not always exist, as we explicitly argued for in chapter 2 for many cases, so that it needs to be replaced in part by a principal $SO(3)$-subalgebra.

3.6.3 Non-Invertible Cartan Matrix

When the Cartan matrix is not invertible we can not derive (3.63) from the equations (3.61) and (3.62). As a specific theory that involves a non-invertible Cartan matrix let us now consider the affine Toda theory $\mathscr{L}_{\mathfrak{g}_a}$-theory, that we are now familiar with. We know from section 3.3 that conformal affine Toda theories are integrable, Lax pairs have been found for non-conformal classical affine Toda field theories [91], and, as mentioned previously, for the quantum affine cases exact S- matrices have been found to factorise into two particle S-matrices - all as a consequence of the integrability of these affine Toda field theories [84]. However, let us also see how the Painlevé test can be implemented, since the same line of argumentation can then also be applied to some extended theories we consider below. Using the fact that $K_{ij} = \tilde{K}_{ij}$ for $i,j = 1,\ldots,r$ with $\tilde{K}$ denoting the invertible Cartan matrix of $\mathfrak{g}_f$ in this specific line of argument, we can split off the last row and the last column from K. Then it is easily seen that (3.63) is replaced by

$$a_i^{(0)} = -n_p n_q \dot{\phi}\acute{\phi}\tilde{D}_i + n_i a_0^{(0)}, \qquad b_\alpha^{(0)} = -n_q \dot{\phi} \sum_{i=1}^{r} \left(M^T\right)_{\alpha i} \tilde{D}_i, \tag{3.77}$$

where $\tilde{D}_i := \sum_{j=1}^r \tilde{K}_{ij}^{-1}$ and the n_i denote the Kac labels as defined after equation (2.12). Following now the same steps as in the previous subsection we derive the matrix T with block matrices

$$A_{ij}^{(k)} = k\dot{\phi}\delta_{ij}, \quad B_{i\alpha}^{(k)} = n_p n_q \dot{\phi}\acute{\phi} D_i M_{i\alpha} - n_i M_{i\alpha} a_0^{(0)}, \quad C_{\alpha i}^{(k)} = \left(M^T\right)_{\alpha i}, \quad E_{\alpha\beta}^{(k)} = (k-n_p)\acute{\phi}\delta_{\alpha\beta}, \tag{3.78}$$

where we defined $D_0 := 0$. Taking now $a_0^{(0)} = 0$, we notice that the only non-vanishing entry in the 0-row of $T^{(k)}$ is $T_{00}^{(k)} = A_{00}^{(k)} = k\dot{\phi}$. We can then expand $\det T^{(k)}$ with respect to the first row and derive

$$\det T^{(k)} = k(k-n_p)^{r+1} \acute{\phi}^{r+2} \dot{\phi}^2 \det \left[k(k-n_p) I_{r\times r} - n_p(n_p+1)\tilde{D}\tilde{K} \right], \tag{3.79}$$

with $\tilde{D}$, $\tilde{K}$ belonging to $\mathfrak{g}$. Thus we have reduced the Painlevé test for the $\mathscr{L}_{\mathfrak{g}_a}$-theory to an eigenvalue problem for the matrix $n_p(n_p+1)\tilde{D}\tilde{K}$ associated to $\mathfrak{g}_f$.

Thus we conclude that the integrability properties of the $\mathscr{L}_{\mathfrak{g}_f}$-theory are inherited by the $\mathscr{L}_{\mathfrak{g}_a}$-theory, that is when $\mathscr{L}_{\mathfrak{g}_f}$ is (non)integrable so is $\mathscr{L}_{\mathfrak{g}_a}$.

For simplicity we derived here the eigenvalue equation (3.72) for symmetric Cartan matrices. We may repeat the same line of argumentation by replacing in M^T roots by coroots, $\alpha_i \to \hat{\alpha}_i = 2\alpha_i/\alpha_i^2$ when $\alpha_i^2 \neq 0$. Then it is easily seen that (3.72) generalizes to the nonsymmetric case for which the Cartan matrix is defined as $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$ when $\alpha_j^2 \neq 0$ and remains $K_{ij} = \alpha_i \cdot \alpha_j$ when $\alpha_j^2 = 0$.

3.6.4 Characteristic Equation of P

We will now continue to present the results of [51], and keep $n_p = 1$ to analyse the characteristic equation for the Painlevé matrix P as defined in (3.42)

$$\det\left[k(k-1)I - P\right] = 0, \tag{3.80}$$

in some more detail. As argued in the previous subsection, for any version of the Lorentzian Toda field theories to be integrable the eigenvalues of the Painlevé matrix must be integer valued and factorize as $k(k-1)$ with $k \in \mathbb{N}$. In particular, this means when the eigenvalues are negative, the theory is not integrable. These cases can be identified easily. We need to argue differently depending on whether the matrix D is positive or negative definite, semi-definite or indefinite.

Denoting by *ind*$A = e_p - e_n$ the *index of the matrix* A, defined as the difference between the positive and negative eigenvalues of A, e_p and e_n, respectively, we have the relation

$$ind(\pm 2DK) = ind(K), \tag{3.81}$$

where the +sign holds for D positive definite and the −sign for D negative definite.

To prove this relation we first note that the matrix $\sqrt{\pm D}K\sqrt{\pm D}$ has the same eigenvalues as $\pm DK$. Here $\sqrt{\pm D}$ is the positive square root with the sign depending on whether D is positive or negative definite. Next we invoke Sylvester's theorem, see e.g. Theorem 12.3 in [92], which states that two symmetric square matrices A and B that are congruent to each other, i.e. $A = QBQ^T$ for some nonsingular matrix Q, have the same index. Applied to the above this means that $ind(\sqrt{\pm D}K\sqrt{\pm D}) = ind(K)$, since $\sqrt{\pm D}^T = \sqrt{\pm D}$. Therefore with $ind(\sqrt{\pm D}K\sqrt{\pm D}) = ind(\pm DK)$ we obtain (3.81).

When D is semi-definite we can define a reduced D-matrix as $\hat{D}$ by setting the positive or negative entries to zero and use a reduced version of (3.81) as $ind(\pm 2\hat{D}K) = ind(K)$.

Since a necessary condition for passing the Painlevé test is that all eigenvalues of $2DK$ are positive, i.e. $ind(2DK) = \ell$ with ℓ denoting the rank of K, the relation (3.81) implies that $ind(\pm K) = \ell$. This means only Lorentzian Toda field theories based on positive or negative definite Cartan matrix can pass the Painlevé test. In turn this means that those theories built from non-definite Cartan matrices that were extended from definite ones through our n-extended Lorentzian root construction can not be integrable, and hence theories perturbed around $\mathscr{L}_{\mathfrak{g}_{-1}}$ or more generally $\mathscr{L}_{\mathfrak{g}_{-n}}$ are not integrable.

This however does not rule out all constructions of Lorentzian Toda field theories to be non-integrable. In chapter 4 we examine the cases in which we do not n-extend a $\mathfrak{g}_f$ algebra, constructing generalized Cartan matrices that we do believe to be integrable in the sense that they both pass the Painlevé test, and also contain enough conserved quantities to possess the *Painlevé property*. We save the details for chapter 4, but the essence of the Painlevé property is that through examining the Painlevé equations (3.61) and (3.62) we uncover enough free parameters to be in involution to declare certain Toda field theories based on Lorentzian Kac-Moody algebras to be integrable, analogous to the sense of Liouville integrability of r real degree of freedom with r analytic single valued global integrals of motion in involution.

3.7 Constructions of Lorentzian Toda field theory

We will now construct and illustrate various types of Toda field theories based on different versions of root systems corresponding to Lorentzian Kac-Moody algebras and their extensions. We will encounter conformally invariant and massive models, and shall concentrate on the construction of non-integrable Lorentzian Toda field theories, which have particular interest as examples of perturbed integrable models that become non-integrable after that perturbation. The results and presentation in this section will very closely follow the corresponding section in [51].

3.7.1 $\mathscr{L}_{\mathring{\mathfrak{g}}_{-n}}$-extended Lorentzian Toda field theory

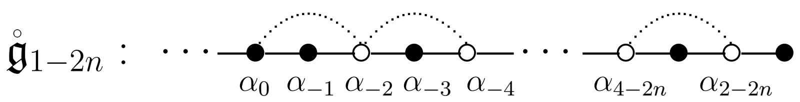
This first type of theories is a series constituting an infinite extension of the perturbed $\mathscr{L}_{\mathfrak{g}_{-1}}$-theory that we introduced in section 3.5. The theories in this series come in one of two variants: The $\mathscr{L}_{\mathring{\mathfrak{g}}_{-n}}$-Lorentzian Toda field theories for odd n are conformally invariant, and those for which n is even are massive, continuing the pattern we observed at the beginning of section 3.5. As a construction principle, we extend the one previously used for the perturbation of the $\mathscr{L}_{\mathfrak{g}_{-1}}$-theory and build the roots as follows. For the massless $\mathscr{L}_{\mathring{\mathfrak{g}}_{-(2n-1)}}$-theories we have the $r+2n$ roots

$$\begin{aligned} &\alpha_i \equiv \text{simple roots of } \mathfrak{g}_f && \text{for } j=1,\ldots,r, \\ &\alpha_{-(2i-2)} = k_i - \textstyle\sum_{j=-(2i-3)}^{r} n_j\alpha_j && \text{for } i=1,\ldots,n, \\ &\alpha_{-(2i-1)} = -(k_i+\bar{k}_i) && \text{for } i=1,\ldots,n. \end{aligned} \tag{3.82}$$

We notice that the roots $\alpha_{-(2i-2)}$ have length zero for $i=2,\ldots,n$, have a standard inner product equal to -1 with nearest neighbour roots on the Dynkin diagram and a more unusual inner product equal to 1 for next to nearest neighbours. The roots $\alpha_{-(2i-1)}$ have length 2 for $i=1,\ldots,n$. Thus we have the inner products

$$\alpha^2_{-(2i-2)}=0, \quad \alpha^2_{-(2j-1)}=2, \quad \alpha_{-k}\cdot\alpha_{-(k+1)}=-1, \quad \alpha_{-2l}\cdot\alpha_{-(2l+2)}=1, \tag{3.83}$$

for $i=2,\ldots,n$, $j=1,\ldots,n$, $k=1,\ldots,2n-2$ and $l=0,1,\ldots,n-2$. At each affine root α_0 the Dynkin diagram is extended by the following segment:



We used here the standard conventions for drawing Dynkin diagrams related to semi-simple Lie algebras in which vertices with bullets indicate roots of length 2 and single line links between two vertices correspond to inner products of -1 between the two corresponding roots. We increase the set of rules by indicating roots of length 0 with an empty circles and inner products of 1 by dotted links between two vertices correspond to the roots. Such type of zero length roots and inner products equal to 1 are not entirely unusual and also occur in the context of Lie superalgebras and of their affine extensions [93], in which the definition of the Cartan matrix is generalized to exclude any normalisation by any roots of zero length, and where exceptions are made within the corresponding Serre relations in forming the generator structure of superalgebra. Here however, we are less concerned with the generator construction of these almost n-extended algebras $\mathring{\mathfrak{g}}_n$, and instead continue to concentrate on what is required to form valid Toda field theories from them.

For the $\mathring{\mathfrak{g}}_{1-2n}$ algebra, the corresponding Cartan matrix is

$$K_{\mathring{\boldsymbol{g}}_{-(2n-1)}} = \left(\begin{array}{ccc|cccc} & & & q_1 & 0 & \cdots & 0 \\ & K_{\boldsymbol{g}} & & \vdots & \vdots & & \vdots \\ & & & q_r & 0 & \cdots & 0 \\ q_1 & \cdots & q_r & 2 & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & & & \\ \vdots & & \vdots & 1 & & \hat{K}_{2n-1} & \\ 0 & \cdots & 0 & 0 & & & \end{array} \right) \tag{3.84}$$

with $q_s := \alpha_0 \cdot \alpha_s$, $s = 1, \dots, r$, and $(2n-1) \times (2n-1)$ matrix $\hat{K}_{2n-1}$ with entries

$$\hat{K}_{2i-1,2i-1} = 2, \quad \hat{K}_{2j,2j} = 0, \quad \hat{K}_{k,k+1} = -1, \quad \hat{K}_{2l,2l+2} = 1, \tag{3.85}$$

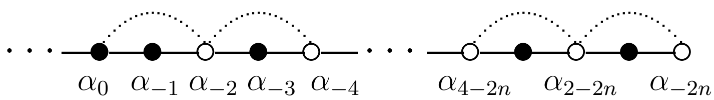
for $i = 1, \dots, n$, $j = 1, \dots, n-1$, $k = 1, \dots, 2n-2$ and $l = 1, \dots, n-2$.

Taking the same roots and adding one root at the end of the Dynkin diagram as the negative highest root, designed to make the linear term in the potential vanish, we obtain the massive $\mathring{\mathcal{L}}_{\boldsymbol{g}_{-(2n)}}$-theory based on $r + 2n + 1$ roots

$$\begin{array}{ll} \alpha_i \equiv \text{simple roots of } \boldsymbol{g} & \text{for } j = 1, \dots, r, \\ \alpha_{-(2i-2)} = k_i - \sum_{j=-(2i-3)}^{r} n_j \alpha_j & \text{for } i = 1, \dots, n, \\ \alpha_{-(2i-1)} = -(k_i + \bar{k}_i) & \text{for } i = 1, \dots, n, \\ \alpha_{-(2n)} = -\sum_{j=-(2n-1)}^{r} n_j \alpha_j . & \end{array} \tag{3.86}$$

Now at each affine root α_0 the Dynkin diagram is extended by the segment:

$\mathring{\mathfrak{g}}_{-2n}$:



With the corresponding Cartan matrix given by

$$K_{\mathring{\boldsymbol{g}}_{-2n}} = \left(\begin{array}{ccc|cccc} & & & q_1 & 0 & \cdots & 0 \\ & K_{\boldsymbol{g}} & & \vdots & \vdots & & \vdots \\ & & & q_r & 0 & \cdots & 0 \\ q_1 & \cdots & q_r & 2 & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & & & \\ \vdots & & \vdots & 1 & & \hat{K}_{2n} & \\ 0 & \cdots & 0 & 0 & & & \end{array} \right) \tag{3.87}$$

where the entries of the $(2n) \times (2n)$ matrix $\hat{K}_{2n}$ are defined as in (3.85) with $i = 1, \dots, n$, $j = 1, \dots, n$, $k = 1, \dots, 2n-1$ and $l = 1, \dots, n-1$.

For the Toda field theories constructed from these root systems it follows from section 3.6, in particular subsection 3.6.4 that the Painlevé integrability test is entirely reduced to an eigenvalue problem for the Painlevé matrix P, which must factor as $n(n-1)$ with n being an integer. This is done through directly reducing the test for the $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2n}}$-extended Lorentzian Toda field theory to the eigenvalue problem for $2D_{\mathring{\mathfrak{g}}_{-(2n-1)}}K_{\mathring{\mathfrak{g}}_{-(2n-1)}}$. For the semi-simple Lie algebras these integer have been identified as the exponents related to properties of the Casimir operator of the principal subalgebra on one hand, as we illustrated in section 3.6.2 from work in [73], and on the other as labelling the spins of conserved W-algebra currents [44].

We will now study the $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2n}}$-extended Lorentzian Toda field theories for some concrete algebras extended from $\mathfrak{g}_f$, and hence $\mathfrak{g}_a$, in more detail in the following segments.

$(\mathring{\boldsymbol{A}}_{2n})_{-2}$-Lorentzian Toda field theories

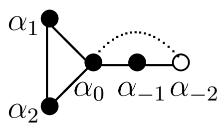
We start with the most simple system in the $(\mathring{\boldsymbol{A}}_{2n})_{-2}$ series, the $(\mathring{\boldsymbol{A}}_{2})_{-2}$-Lorentzian Toda field theory. We represent the $(\mathring{\boldsymbol{A}}_{2})_{-2}$ roots (3.86) on a four dimensional lattice as

$$\alpha_1 = \left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}; 0, 0\right), \quad \alpha_2 = \left(0, \sqrt{2}; 0, 0\right), \quad \alpha_0 = \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}} - \sqrt{2}; 1, 0\right) \tag{3.88}$$

$$\alpha_{-1} = (0, 0; -1, 1), \quad \alpha_{-2} = (0, 0; 0, -1). \tag{3.89}$$

The analogue of the affine root is $\alpha_{-2} = -\sum_{j=-1}^{2} n_j \alpha_j$ with all Kac labels $n_j = 1$. It is easily checked that indeed the roots $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2$ have length 2 and the root α_{-2} has length 0. The Dynkin diagram drawn with the standard rules augmented with the set of rules as stated at the end of the previous subsection, is as follows:

$\mathring{\boldsymbol{A}}_{-2}$:



The eigenvalues of the Cartan matrix $K_{(\mathring{\boldsymbol{A}}_2)_{-1}}$ are $(3.48119, 3., 1.68889, -0.170086)$, with exactly one negative eigenvalue as we expect for a Lorentzian Kac-Moody algebra by definition. The mass matrix (3.56) for this root system is computed to

$$M^2 = \frac{1}{2}m^2 \begin{pmatrix} 3 & 0 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 3 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 2 & -1 \\ \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} & -1 & 2 \end{pmatrix}, \tag{3.90}$$

with positive, that is physical, eigenvalues $(4.1701, 3, 2.3111, 0.51880)$ for $m = \sqrt{2}$. The matrix $D_{(\mathring{\boldsymbol{A}}_2)_{-1}}$ as we defined in (2.53) is negative definite with $D_1 = D_2 = -6$, $D_3 = -7$ and $D_4 = -3$. The eigenvalues of the Painlevé matrix P are $(-42, -36, -12, 2)$ and the relation (3.81) is confirmed as $ind(-2DK) = ind(K) = 2$. The theory fails the Painlevé

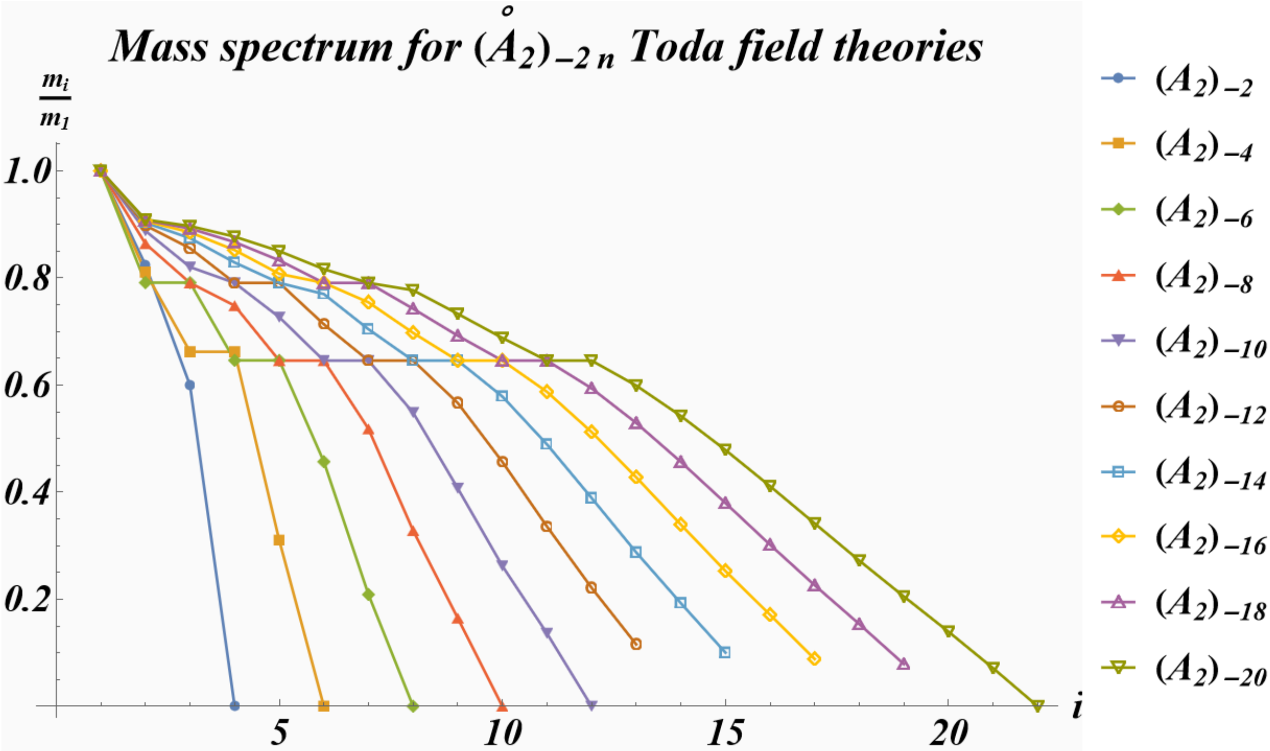


Figure 3.1: Mass ratios for the $r + 2n$ particles in the $(\mathring{\boldsymbol{A}}_{2n})_{-2n}$-Toda field theories extended from the $(\mathring{\boldsymbol{A}}_{2n})_{-2}$ theory.

test and is therefore not integrable, also as expected from the results from section 3.6.4.

Extending the $(\mathring{\boldsymbol{A}}_2)_{-2}$ theory to an $(\mathring{\boldsymbol{A}}_{2n})_{-2}$ through the procedure described in the previous subsection allows us to calculate the mass matrix again using equation (3.56). We plot the results of the first few $(\mathring{\boldsymbol{A}}_{2n})_{-2}$ theories mass ratios results in figure 3.1.

$(\mathring{\boldsymbol{E}}_8)_{-2n}$-Lorentzian Toda field theories

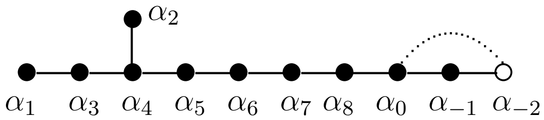
The first member of the $(\mathring{\boldsymbol{E}}_8)_{-2n}$-series is the $(\mathring{\boldsymbol{E}}_8)_0$-theory corresponding to the well studied affine Toda field theories, which describes the scaling limit of the Ising model at critical temperature in magnetic field [53]. The next member is the $(\mathring{\boldsymbol{E}}_8)_{-2}$-theory for which we represent the roots (3.86) on a ten dimensional root lattice as

$$\begin{aligned}
&\alpha_1 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}; 0, 0\right), \qquad && \alpha_2 = (1,1,0,0,0,0,0,0;0,0), \\
&\alpha_3 = (-1,1,0,0,0,0,0,0;0,0), && \alpha_4 = (0,-1,1,0,0,0,0,0;0,0), \\
&\alpha_5 = (0,0,-1,1,0,0,0,0;0,0), && \alpha_6 = (0,0,0,-1,1,0,0,0;0,0), \\
&\alpha_7 = (0,0,0,0,-1,1,0,0;0,0), && \alpha_8 = (0,0,0,0,0,-1,1,0;0,0), \\
&\alpha_0 = (0,0,0,0,0,0,-1,-1;1,0), && \alpha_{-1} = (0,0,0,0,0,0,0,0;-1,1), \\
&\alpha_{-2} = (0,0,0,0,0,0,0,0;0,-1).
\end{aligned} \tag{3.91}$$

We have constructed the analogue of the affine root as $\alpha_{-2} = -\sum_{j=-1}^{8} n_j \alpha_j$ with Kac labels $n = (2,3,4,6,5,4,3,2,1,1,1)$, using those for affine E_8 in [31] as a reference. Using the Lorentzian inner product we compute for the extended part $\alpha_{-2}^2 = 0$, $\alpha_{-1}^2 = 2$,

$\alpha_{-2} \cdot \alpha_{-1} = -1$, $\alpha_{-1} \cdot \alpha_0 = -1$, $\alpha_{-2} \cdot \alpha_0 = 1$. The Dynkin diagram drawn with the standard rules augmented with the set of rules as stated at the end of subsection 3.7.1 is therefore:

$\left(\mathring{\boldsymbol{E}}_{8}\right)_{-2}$:



The conformal part of the theory is the $(\boldsymbol{E}_8)_{-1}$-theory, aka $\boldsymbol{E}_{10}$, whose Cartan matrix has exactly one negative eigenvalue with all other eigenvalues being positive. The Cartan matrix of $(\mathring{\boldsymbol{E}}_8)_{-2}$ has a zero eigenvalue, one negative eigenvalue with the remaining ones being positive. The mass squared matrix (3.56) for the $(\mathring{\boldsymbol{E}}_8)_{-2}$-theory is computed to be

$$M^2 = \frac{1}{2}m^2 \begin{pmatrix} 15 & -3 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ -3 & 27 & -11 & 1 & 1 & 1 & 1 & -1 & 0 & 0 \\ -1 & -11 & 23 & -9 & 1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & -9 & 19 & -7 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & -7 & 15 & -5 & 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & -5 & 11 & -3 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & -3 & 7 & 1 & 0 & 2 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & 4 \end{pmatrix}. \tag{3.92}$$

The ten eigenvalues $(19.4794, 12.8905, 8.8224, 7.4524, 5.1100, 3.7371, 3.0181, 2.1237, 1.1227, 0.2437)$ of M^2 are all positive, thus leading to a physically well-defined classical mass spectrum. We may set here $m = 1$, as only mass ratios will be relevant. Similarly, we compute the masses for the other members of the $(\mathring{\boldsymbol{E}}_8)_{-2n}$-series, which all posses well-defined spectra. We present our results for the first members of the series in figure 3.3 above.

We observe the interesting feature that when comparing the masses with those of standard E_8-affine Toda field theory, four masses are especially stable and remain almost all identical irrespective of the value of n. These masses can be identified when recalling that folding the E_8-affine Toda field theory [46] leads to a grouping of the eight masses in the E_8-theory [53] as two copies of four masses attributed to a theory based on the root space of noncrystallographic type H_4. One set is obtained from the other by a multiplication of the golden ration $\phi = (1 + \sqrt{5})/2$. Normalizing the E_8- masses so that the largest takes on the value 1, we have

$$m_1 = 1, \qquad m_2 = 2\sin(4\theta), \qquad m_3 = \frac{\cos\theta}{\phi\cos(4\theta)}, \qquad m_4 = \frac{1}{2\phi\cos(4\theta)}, \tag{3.93}$$

$$m_5 = \phi^{-1}m_1, \quad m_6 = \phi^{-1}m_2, \quad m_7 = \phi^{-1}m_3, \quad m_8 = \phi^{-1}m_4, \tag{3.94}$$

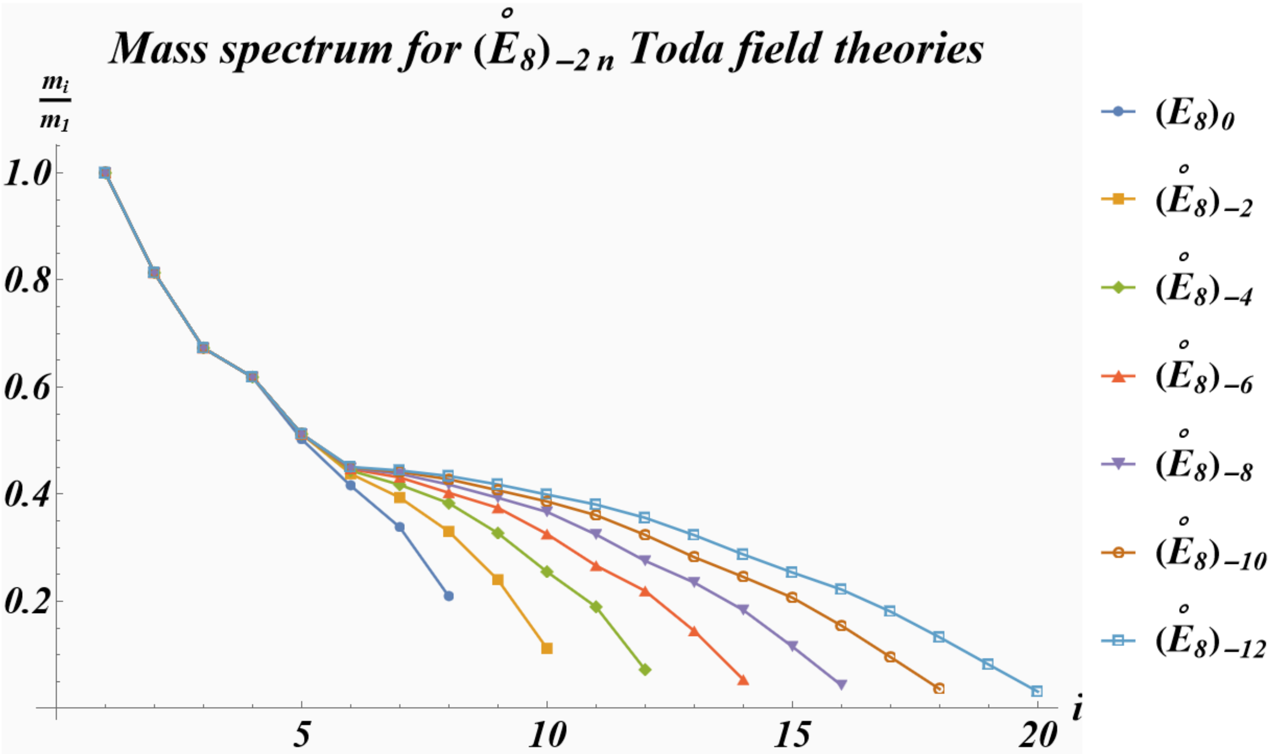


Figure 3.2: Mass ratios for the $r + 2n$ particles in the $\left(\mathring{\boldsymbol{E}}_8\right)_{-2n}$-Toda field theories with almost stable noncrystallographic H_4 compound.

with $\theta = \pi/30$. We observe in figure 2.1 that the four H_4 masses in (3.93) are almost identical in all $(\mathring{\boldsymbol{E}}_8)_{-2n}$-theories.

However, none of these theories, apart from $(\mathring{\boldsymbol{E}}_8)_0$, passes the Painlevé integrability test. In all other cases the eigenvalues of the matrix $2D_{\mathring{\boldsymbol{g}}_{-(2n-1)}} K_{\mathring{\boldsymbol{g}}_{-(2n-1)}}$ are all non integer valued and sometimes negative. We find that $D_{\mathring{\mathfrak{g}}_1} \equiv D_{\boldsymbol{E}_8}$ is positive definite, as is expected for the semi-simple case. We confirm in this case the relation (3.81) as $ind\,(2D_{\boldsymbol{E}_8} K_{\boldsymbol{E}_8}) = ind\,(K_{\boldsymbol{E}_8}) = 8$. Moreover the eigenvalues factorize into $s_i(s_i + 1)$ with $s_i = 1$, 7, 11, 13, 17, 19, 23, 29, corresponding to the 8 exponents of E_8.

In contrast, the matrices $D_{\mathring{\mathfrak{g}}_{-(2n-1)}}$ are negative definite for all values of $n \geq 1$. The $8 + 2n$ eigenvalues for $2D_{(\mathring{\boldsymbol{E}}_8)_{-(2n-1)}} K_{(\mathring{\boldsymbol{E}}_8)_{-(2n-1)}}$ for $n = 1, 2, \ldots$ separate into $8 + n$ negative and n positive eigenvalues. The relation (3.81) is confirmed as

$$ind\left(-2D_{\mathring{\mathfrak{g}}_{-(2n-1)}} K_{\mathring{\mathfrak{g}}_{-(2n-1)}}\right) = ind\left(K_{\mathring{\mathfrak{g}}_{-(2n-1)}}\right) = 8, \quad \text{for } n = 1, 2, \ldots \tag{3.95}$$

Surprisingly the index of K is preserved for all values of n. To explain this, we list here

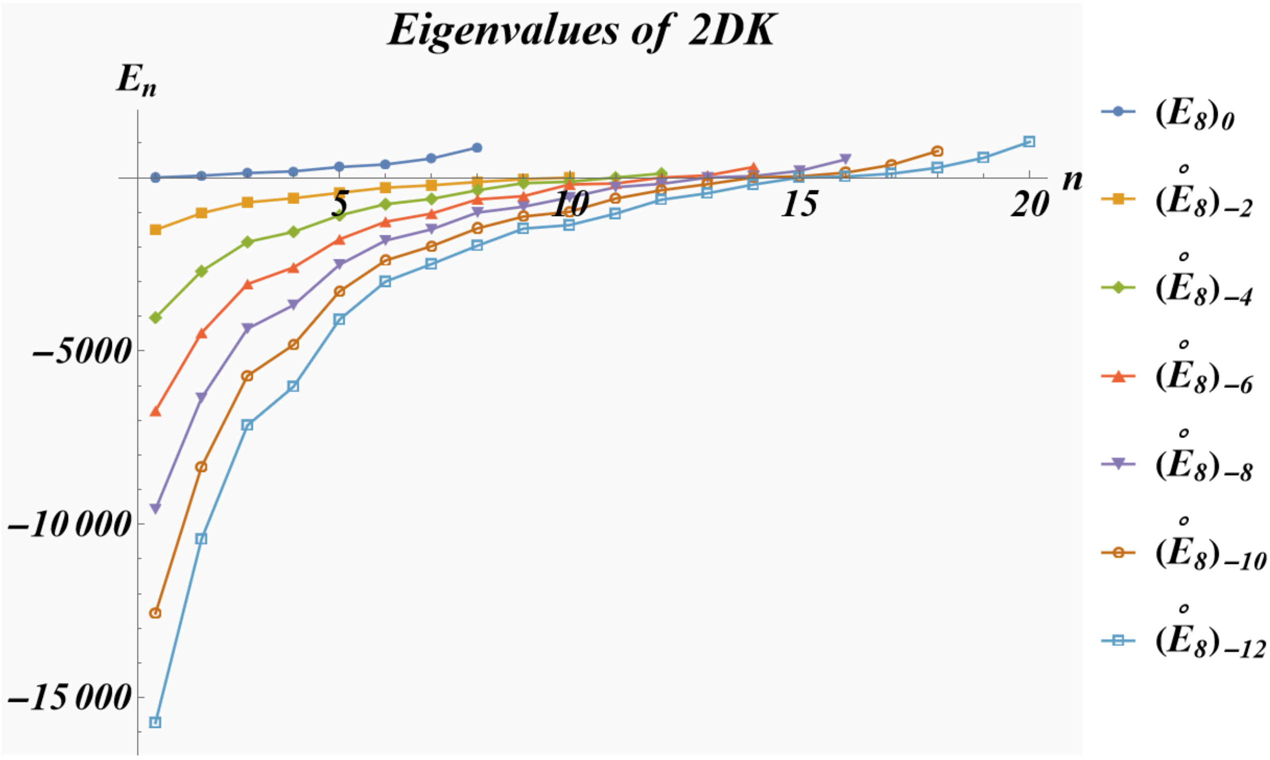


Figure 3.3: Eigenvalue spectra for the Painlevé matrix $2D_{\mathring{\boldsymbol{g}}_{-(2n-1)}}K_{\mathring{\boldsymbol{g}}_{-(2n-1)}}$ of the $\left(\mathring{\boldsymbol{E}}_8\right)_{-2n}$-Toda field theories.

the first characteristic polynomials $\det(K-\lambda\mathbb{I})=0$ for the Cartan matrix $K_{\mathring{\mathfrak{g}}_{-(2n-1)}}$

$$ch\left(K_{\boldsymbol{E}_8}\right) = \lambda^8 - 16\lambda^7 + 105\lambda^6 - 364\lambda^5 + 714\lambda^4 - 784\lambda^3 + 440\lambda^2 - 96\lambda + 1, \tag{3.96}$$

$$ch\left(K_{(\mathring{\boldsymbol{E}}_8)_{-1}}\right) = \lambda^{10} - 20\lambda^9 + 171\lambda^8 - 816\lambda^7 + 2379\lambda^6 - 4356\lambda^5 + 4949\lambda^4 - 3304\lambda^3 + 1140\lambda^2 - 144\lambda - 1, \tag{3.97}$$

$$ch\left(K_{(\mathring{\boldsymbol{E}}_8)_{-2}}\right) = \lambda^{12} - 22\lambda^{11} + 208\lambda^{10} - 1100\lambda^9 + 3531\lambda^8 - 6892\lambda^7 + 7356\lambda^6 - 1914\lambda^5 - 4872\lambda^4 + 5944\lambda^3 - 2626\lambda^2 + 388\lambda + 1, \tag{3.98}$$

$$ch\left(K_{(\mathring{\boldsymbol{E}}_8)_{-4}}\right) = \lambda^{14} - 24\lambda^{13} + 249\lambda^{12} - 1450\lambda^{11} + 5103\lambda^{10} - 10576\lambda^9 + 9896\lambda^8 + 7088\lambda^7 - 31796\lambda^6 + 37074\lambda^5 - 17467\lambda^4 - 520\lambda^3 + 3050\lambda^2 - 636\lambda - 1. \tag{3.99}$$

Where to be completely clear, the λ here is the expansion variable from the characteristic equation, not the fundamental weights as we denoted λ in chapter 2. We observe that in each polynomial of the general form $\sum_{i=1}^{8+2n} a_i\lambda^i$, the sequence of coefficients a_i has exactly $8+n$ sign changes. Thus according to Descartes' rule of signs, see e.g. [94], we have exactly $8+n$ positive real eigenvalues confirming the observation above. The factorization of these eigenvalues into $s_i(s_i+1)$ leads to the form $s_i = 1/2+\lambda_i$ with $\lambda_i \in \mathbb{R}$ and $s_i = \kappa_i$ with $\kappa_i \in \mathbb{R}$, for the negative and positive eigenvalues, respectively.

We depict the eigenvalue spectra for some $\mathscr{L}_{(\mathring{E}_8)_{-2n}}$-extended Lorentzian Toda field theory in figure 3.3 on the previous page, and, as we can partially see from the figure, most of the eigenvalues of the Painlevé matrix are negative or non integer valued. Hence, the $\mathscr{L}_{(\mathring{E}_8)_{-2n}}$-extended Lorentzian Toda field theory fail the Painlevé test and are therefore not integrable.

In the next segment, we shall examine another option in the construction of Lorentzian Toda field theories, that will use a different formalism of simple roots and their root lattices to that we have encountered so far in chapter 2, or that we have used to construct other Lorentzian Toda field theories up top this point.

3.7.2 $\mathcal{L}_{(\mathring{\mathfrak{g}}_1)_{-2n}\diamond(\mathring{\mathfrak{g}}_2)_{-2m}}$-extended Lorentzian Toda field theory

This construction is based on a generalization of what is referred to in [24] as the symmetric fusion of two finite semisimple Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by means of some Lorentzian roots in $\Pi^{1,1}$. Here we consider a root lattice of the form

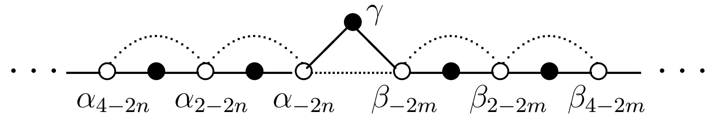
$$\Lambda_{(\mathring{\mathfrak{g}}_1)_{1-2n}\diamond(\mathring{\mathfrak{g}}_2)_{1-2m}} = \Lambda_{(\mathring{\mathfrak{g}}_1)_{1-2n}} \oplus \Pi^{1,1} \oplus \Lambda_{(\mathring{\mathfrak{g}}_2)_{1-2m}}. \tag{3.100}$$

It is comprised of the $r_1 + 2n$ roots α_i with $i = 1 - 2n, \ldots, r_1$ of $(\mathring{\mathfrak{g}}_1)_{1-2n}$, the $r_2 + 2m$ roots β_i with $i = 1 - 2m, \ldots, r_2$ of $(\mathring{\mathfrak{g}}_2)_{-2m}$ and two modified roots

$$\alpha_{-2n} = k_{n+1} - \sum\nolimits_{j=1-2n}^{r_1} n_j\alpha_j, \qquad \beta_{-2m} = \bar{k}_{n+1} - \sum\nolimits_{j=-1-2m}^{r_1} n_j\beta_j$$

with $k_{n+1}, \bar{k}_{n+1} \in \Pi^{1,1}$. The Lorentzian roots used in the construction of the α and β roots are unrelated with mutual inner products equal to zero. They are labeled by $k_i, \bar{k}_i$, $i = 1, \ldots, n$ and $\ell_i, \bar{\ell}_i$, $i = 1, \ldots, m$, respectively. For $n = m = 0$ this construction coincides with the one in [24] apart from a change of sign in the definition of β_0 where we added $\bar{k}$ instead of $-\bar{k}$ used in [24]. We explain the reason for our preferred choice below. The massive version is then constructed by adding a root $\gamma = -(k_{n+1} + \bar{k}_{n+1})$. Using the rules as stated above, the part of the Dynkin diagram where the $(\mathring{\mathfrak{g}}_1)_{-2n}$ and $(\mathring{\mathfrak{g}}_2)_{-2m}$ for $n \geq 1$, $m \geq 1$ are joined is:

$(\mathring{\mathfrak{g}}_1)_{-2n} \diamond (\mathring{\mathfrak{g}}_2)_{-2m}$:



The corresponding Cartan matrix is simply linking up the two affine Cartan matrices

$K_{(\mathring{\mathfrak{g}}_1)_{-2n}}$ and $K_{(\mathring{\mathfrak{g}}_2)_{-2m}}$ as

$$K_{(\mathring{\mathfrak{g}}_1)_{-2n}\diamond(\mathring{\mathfrak{g}}_2)_{-2m}} = \left(\begin{array}{ccccccccc} & & & q_1 & & 0 & 0 & \cdots & 0 \\ & K_{(\mathring{\mathfrak{g}}_1)_{1-2n}} & & \vdots & & \vdots & \vdots & & \vdots \\ & & & q_{r_1} & & 0 & 0 & & \vdots \\ q_1 & \cdots & q_{r_1+2n} & 0 & -1 & 1 & 0 & \cdots & 0 \\ & & & -1 & 2 & -1 & & & \\ 0 & \cdots & 0 & 1 & -1 & 0 & p_{r_2+2m} & \cdots & p_1 \\ \vdots & & \vdots & 0 & & p_{r_2} & & & \\ \vdots & & \vdots & \vdots & & \vdots & & K_{(\mathring{\mathfrak{g}}_2)_{1-2m}} & \\ 0 & \cdots & 0 & 0 & & p_1 & & & \end{array} \right), \qquad (3.101)$$

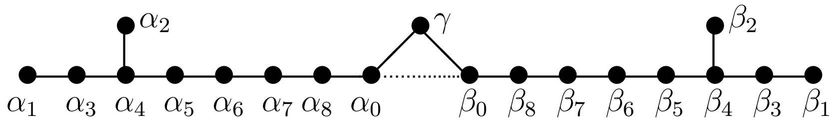
where $q_s := \alpha_{-2n} \cdot \alpha_s$, $s = 1, \ldots, r_1 + 2n$ and $p_s := \beta_{-2m} \cdot \beta_s$, $s = 1, \ldots, r_2 + 2m$.

We present now some examples for Lorentzian Toda field theories build from concrete algebras of this type of construction.

$(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$-Lorentzian Toda field theories

We start with $(\mathring{\boldsymbol{E}}_8)_0 \diamond (\mathring{\boldsymbol{E}}_8)_0$, where $(\mathring{\boldsymbol{E}}_8)_0 \diamond (\mathring{\boldsymbol{E}}_8)_0 \equiv (\boldsymbol{E}_8)_0 \diamond (\boldsymbol{E}_8)_0$ in our notation, and take the same representation for the eight simple roots α_i, $i = 1, \ldots 8$ as defined in (3.91), but we enlarge the representation space from 10 to 18 dimensions by adding 8 zero entries in the vector representation of the root space. The modified affine root $\alpha_0 = k - \sum_{j=1}^{8} n_j \alpha_j$ takes on the same form as in (3.91). Next we construct the roots for the second set of simple roots as $\beta_i^{j+10} = \alpha_i^j$, $i, j = 1, \ldots 8$, and with all remaining entries 0. The second modified affine root is constructed as $\beta_0 = \bar{k} - \sum_{j=1}^{8} n_j \beta_j$. The additional root $\gamma = -(k + \bar{k})$ has therefore non-vanishing entries $\gamma^9 = -\gamma^{10} = -1$. The Dynkin diagram becomes in this case

$(\boldsymbol{E}_8)_0 \diamond (\boldsymbol{E}_8)_0$:



And similarly we construct the Cartan matrix for the other members of the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$-series.

With a well defined root system and vanishing linear term we can compute the mass squared matrix as defined in equation (3.56). Once again we find that all eigenvalues of the mass squared matrix are positive. Taking the normalized square root of these eigenvalues, we depict the classical mass spectra for the first seven members of the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$-series in figure 3.4 above.

We note that all mass spectra in figure 3.4 are non-degenerate. Even though it may appear from the figure that some of the heaviest particles have the same mass, there is in fact always at least a very small difference not visible on the scale used in the figure. For

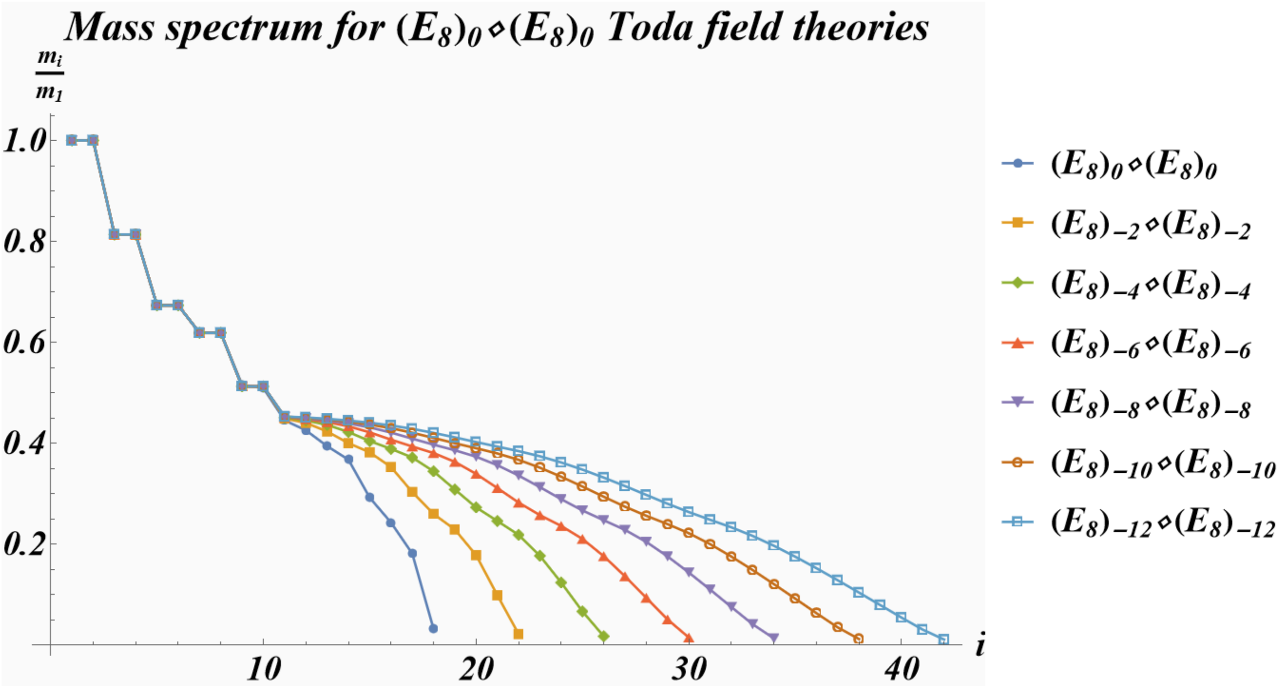


Figure 3.4: Mass ratios for the $18 + 2n$ particles in the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$-Lorentzian Toda field theory.

the lighter particles in the spectrum the difference becomes more apparent. Splitting the particles into sets belonging to the left and right set of roots, α and β, respectively, and comparing with the mass spectrum of the affine $(\boldsymbol{E}_8)_0$-theory, we observe that the mass spectrum of five heaviest particles is almost identical to the masses in the left and right set of roots.

Next we consider the eigenvalues of the Painlevé matrix. First we notice that the diagonal matrix $D_{(\boldsymbol{E}_8)_0 \diamond (\boldsymbol{E}_8)_0}$ is positive definite and that the relation (3.81) holds with $ind\left(K_{(\boldsymbol{E}_8)_0 \diamond (\boldsymbol{E}_8)_0}\right) = 16$. It is these eigenvalue spectrum that motivates the choice for the sign in front of the Lorentzian roots in the definition of β_0. Choosing $-\bar{k}$ instead of $\bar{k}$ will not affect the mass spectrum, but it will reverse the sign in signature of the eigenvalues of $2DK$. However, this theory does not pass the Painlevé integrability test as the eigenvalues of the matrix $2DK$ are all non integer valued.

In contrast, for $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$ with $n \geq 1$ the D-matrix is semi-definite with the four central diagonal entries $D_{(9+n\pm 1)(9+n\pm 1)}$, $D_{(9+n\pm 2)(9+n\pm 2)}$ being positive and the remaining negative. Defining a reduced D-matrix as $\hat{D}$ by setting the positive entries to zero we find a reduced version of (3.81) as $ind(-2\hat{D}K) = ind(K) = 16$. None of the theories in this series passes the Painlevé integrability test as the eigenvalues of the matrix $2DK$ are not only all non integer valued or negative, but in addition some of the eigenvalues occur in complex conjugate pairs. We depict the real eigenvalues in figure 3.5.

We observe that the "almost degeneracy" is roughly preserved for the six heaviest particles. Before concluding this chapter, we would now like to briefly present the results for the simplest $(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$-series of Lorentzian Toda field theories.

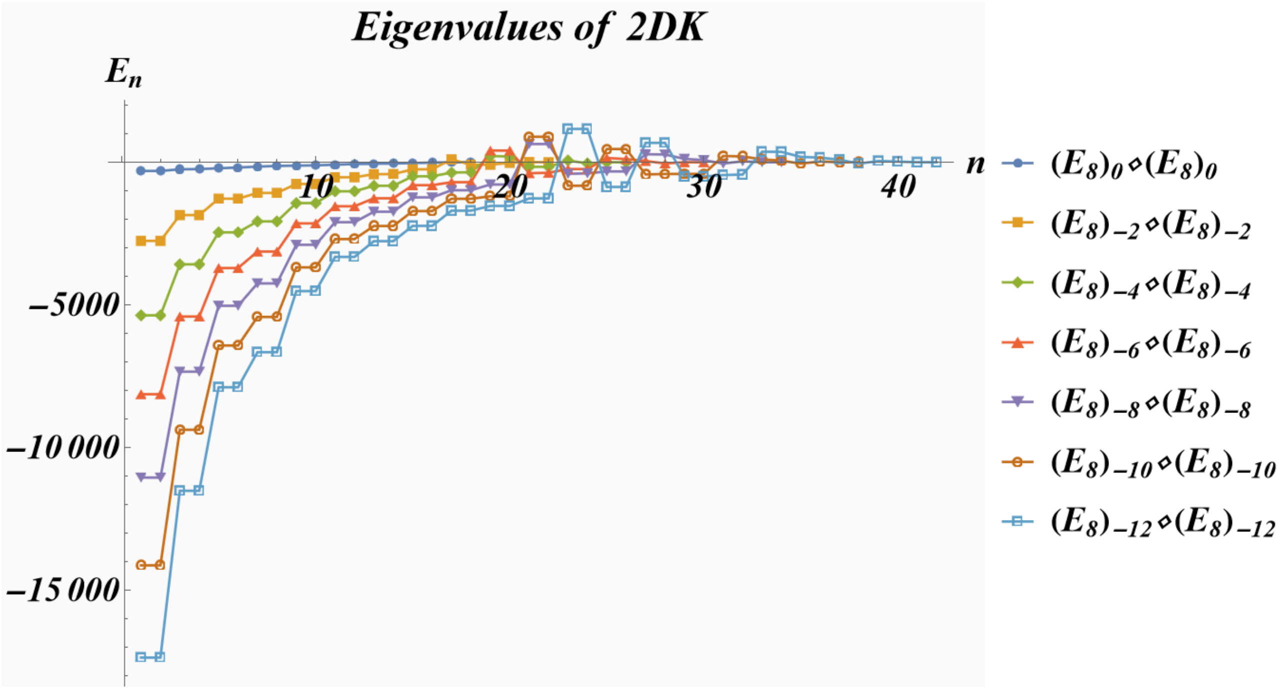
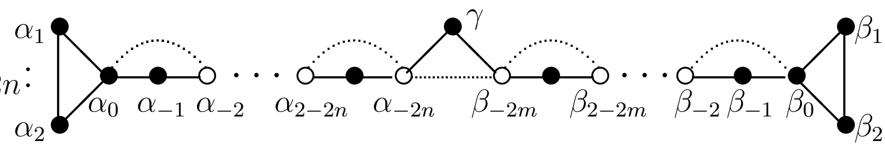


Figure 3.5: Real part of the eigenvalue spectra for the $2DK$-matrix for the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$ Lorentzian Toda field theory, where n is the index of the eigenvalue and E_n represents the eigenvalue value at that index.

$(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$-Lorentzian Toda field theories

Before concluding this chapter, and for completeness, we would like to briefly present the mass ratio and $2DK$ eigenvalue results for the $(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$ theories. The Dynkin diagram for these theories takes the form

$(\boldsymbol{A}_2)_{-2n} \diamond (\boldsymbol{A}_2)_{-2n}$:



where clearly the above represents diagrams with $n > 2$, but the structure can be inferred for $n = 0, 1$ also, e.g. for $n = 1$ with the removal of the α_{-2} and β_{-2} roots from the above diagram.

As done in the previous couple of segments for the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$ and $(\mathring{\boldsymbol{E}}_8)_{-2n}$-Lorentzian Toda field theories, we use the mass matrix in equation (3.56), to calculate and plot the mass ratios that we present in figure 3.6.

We notice from figure 3.6 that the mass ratios also deviate from the massive theory based on $\mathfrak{g}_a$, unlike the stable noncrystallographic part we saw for the $(\mathring{\boldsymbol{E}}_8)_{-2n}$-Lorentzian Toda field theories. Giving behaviour similar to that of which we saw in the $(\mathring{\boldsymbol{E}}_8)_{-2n} \diamond (\mathring{\boldsymbol{E}}_8)_{-2n}$ with $n \geq 1$, providing further confirmation that this disparity is a feature of these symmetrically fused models, or rather that it was unique to those rather special Lorentzian Toda field theories with almost stable noncrystallographic H_4 compound.

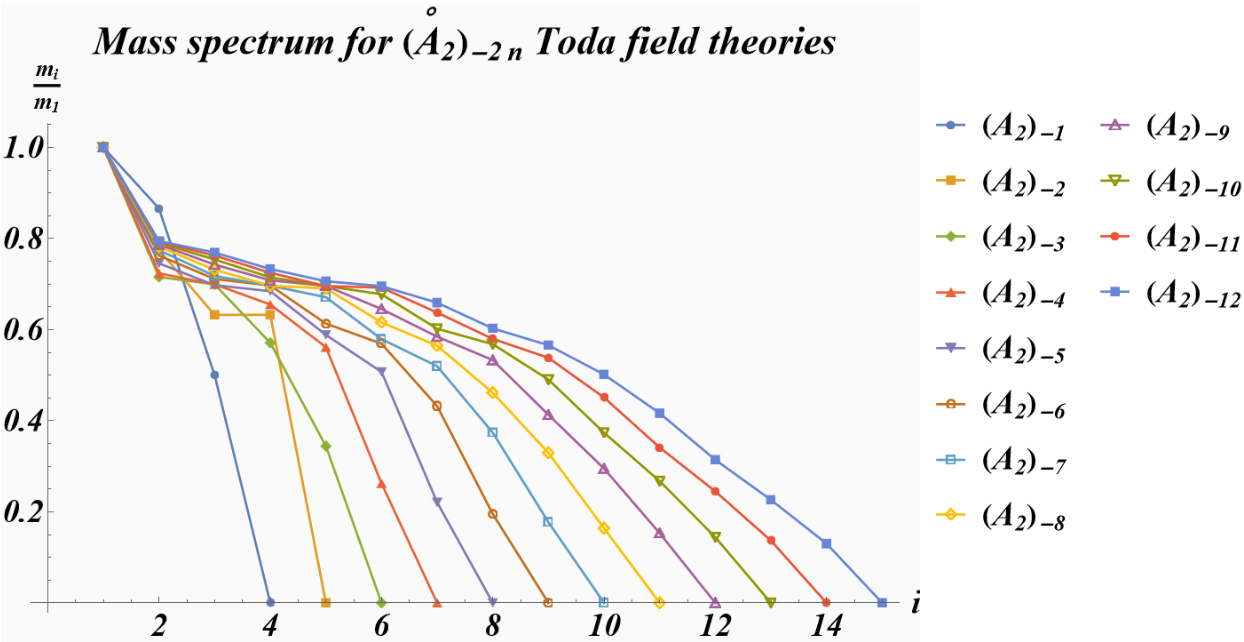


Figure 3.6: Mass ratios for the $7+2n$ particles in the $(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$-Lorentzian Toda field theories for $n<7$.

From figure 3.7 we observe that the eigenvalues of the $2DK$ Painlevé matrix oscillate in sign. Meaning that they also do not pass the Painlevé test, and hence are not integrable theories like their $\mathfrak{g}_f$ and $\mathfrak{g}_a$ based counterparts. As previously mentioned, the following chapter will focus on discovering Lorentzian Toda field theories that we will find to not only pass the Painlevé test, but to also possess the Painlevé property - providing very strong evidence that new integrable models may be found in certain Lorentzian Toda field theories.

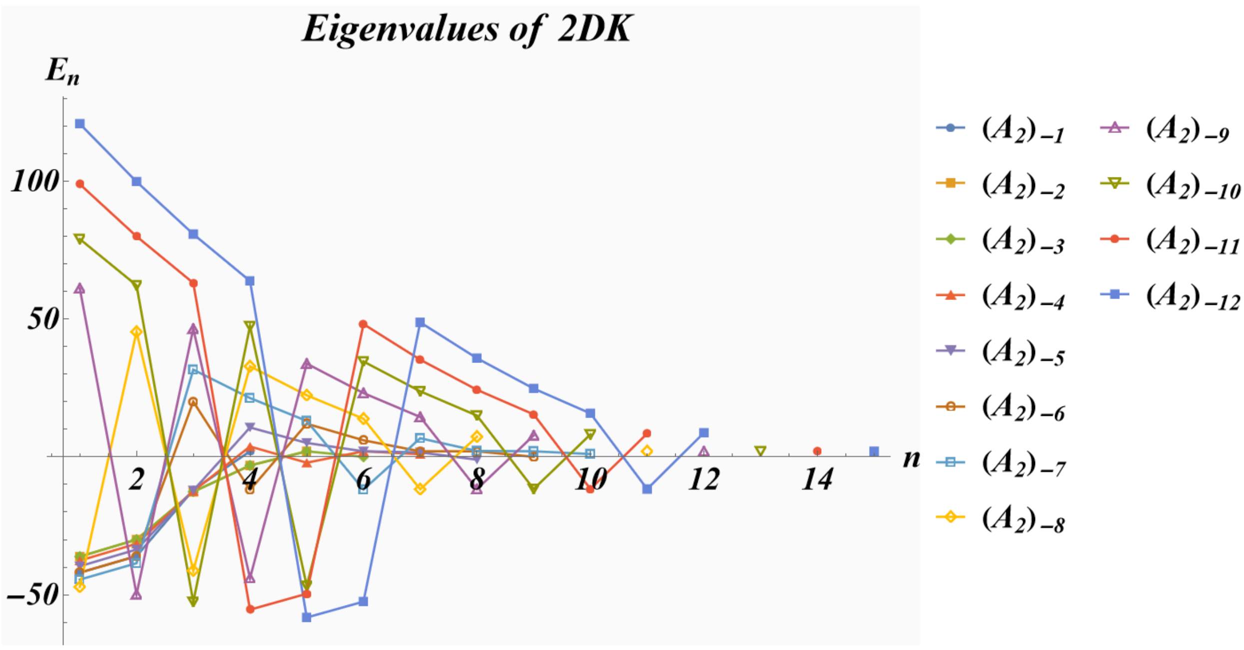


Figure 3.7: Real part of the eigenvalue spectra for the $2DK$-matrix for the $(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$-Lorentzian Toda field theories for $n<7$.

This procedure is completely general and could be carried out for any $(\mathring{\boldsymbol{A}}_r)_{-2n} \diamond (\mathring{\boldsymbol{A}}_r)_{-2n}$-Lorentzian Toda field theories, we choose the simplest $(\mathring{\boldsymbol{A}}_2)_{-2n} \diamond (\mathring{\boldsymbol{A}}_2)_{-2n}$ theories to present here for as an arbitrary example from this series.

3.8 Summary

We began this chapter by presenting classical Toda theories, including the discrete Toda lattice, the classical finite and affine Toda field theories, and the conformal affine Toda theory. We noticed a pattern in the continuous classical integrable Toda field theories that as we extended the base algebra, and hence the fields in the theory associated to vectors in those base algebras, the theories alternated between massive and massless theories with either non-conformal or conformal behaviour, respectively. We presented these theories, along with proofs of their integrability, as motivation for further extending Toda field theories and their associated Kac-Moody algebras, in ways akin to the extension procedure we developed in chapter 2.

In this chapter's development of Lorentzian Toda field theories, we introduced various types of construction principles for massless, conformal theories and massive theories, in which the simple roots of these were defined on Lorentzian lattices. We utilized the Painlevé integrability test to establish that the Lorentzian theories presented here can not be integrable; however, we still find valid theories and calculate their mass ratios. Some of these theories maintain part of their integrable counterparts structure, for example the $(\mathring{\boldsymbol{E}}_8)_{-2n}$-theories contain the four masses of the noncrystallographic H_4-theory obtained by folding the integrable affine $\boldsymbol{E}_8$-theory. Remarkably, these masses are only slightly changed for all values of n, so that we may view this feature as a remnant that survives the perturbation of the integrable system.

In contrast to this chapter, the following focuses on tracking down any Lorentzian Toda field theories that can possibly be integrable. To do so, we will have to move away from the n-extended construction and theories perturbed around these. We shall still use the Painlevé integrability test, but as this only provides a necessary and not sufficient criteria for integrability, we shall also find Lorentzian Toda field theories that possess the Painlevé property, providing the sufficient criteria of these theories' integrability.

Chapter 4

Painlevé Integrability of Lorentzian Toda Systems

Integrability is the essential feature behind Toda field theory's major successes. Recall in chapters 1 and 3 we discussed many of these features including, for one, that Toda field theories based on $\mathfrak{g}_a$ have exact scattering matrices, meaning that they are solved exactly as their classical mass ratios are preserved to all orders of perturbation theory [52, 53, 54, 55, 62] - a feature that is as rare as it is desirable amongst all quantum and classical field theories. Identifying field theories as integrable is therefore important in terms of their potential, and to determine what direction should be taken when further analysing their properties.

Many techniques have been developed to determine the integrability of physical systems. For Toda field theories in particular, in section 3.1.1 we used the Lax pairs in equation (3.4) to prove the integrability of discrete Toda field, and in section 3.3 we utilized the zero-curvature condition to show the integrability of conformal affine Toda field theories. In 3.6 we introduced the Painlevé test, which was effective in showing the non-integrability of Lorentzian Toda field theories based on certain specific perturbations of n-extended Kac-Moody algebras.

In this chapter, we again use the Painlevé test, but now as a tool to identify new integrable Toda field theories. We find several categories of theories that pass the test, and that the non-trivial theories must be based on Lorentzian lattices as they possess a simple root whose squared norm is zero, hence we name them *null root* models. These models are then shown to possess the Painlevé property - a property of the theory that means its equations of motion have a general solution which can be shown to have no movable critical singularities near any non-characteristic manifold [95] - whereby possession of the Painlevé property means that a theory is integrable. There are many ways to determine whether a theory possesses the Painlevé property [90, 95, 96, 97]. We use the WTC method [89], in doing so we show that rank-2 null root models have the Painlevé property, and have strong reasons to believe that higher spin and rank null root may also be shown to be integrable in similar ways to the spin-3 rank-2 examples given in section 4.5.2 below.

4.1 Painlevé Integrability of Toda Field Theories

For partial differential equations (PDEs), singularities occur on an analytic hypersurface S with codimension 1 relative to the entire phase space. The equation

$$\phi(\boldsymbol{z}) = 0, \quad \boldsymbol{z} = \{z_1, \cdots, z_{2N}\} \in \mathbb{C}^{2N} \tag{4.1}$$

describes the *singular manifold* where the singularities are contained in N complex dimensions. Recall that in section 3.6, ϕ is the manifold that we assumed our Painlevé expansion solutions to be expanded around, in this chapter we will also take light-cone coordinates such that $\boldsymbol{z} = \{u, v\} \in \mathbb{C}^2$ for all the examples considered below. Sometimes ϕ is described as non-characteristic manifold and S as a non-characteristic hypersurface in $\mathbb{C}^{2N}$, this emphasises the fact that on a characteristic manifold we can not use Cauchy's existence theorem, which is essential in the process of proving that we have a unique solution for the given initial value problem. Evidently, this means that it is essential for ϕ to be non-characteristic, so that we can solve the initial value problem on our manifold.

Following from these definitions, the Painlevé property for a PDE may be defined as a PDE with a general solution that has no moveable critical singularities near any non-characteristic manifold, or equivalently following the definition of [95]: If S is an analytic non-characteristic complex hypersurface in $\mathbb{C}^{2N}$, then every solution of the PDE which is analytic on $\mathbb{C}^{2N}\backslash S$ is meromorphic on $\mathbb{C}^{2N}$, the solution is then said to possess the Painlevé property and the PDE is Painlevé integrable. Where the distinction between 'integrability' and 'Painlevé integrability' is only that the integrability has been shown through the Painlevé method in contrast to other techniques outlined when introducing the classical integrable Toda field theories in chapter 3.

In searching for new integrable models in the proceeding sections of this chapter we break our steps towards Painlevé integrability into two stages

1. **Pass the Painlevé test.** Or in other words, satisfy the *necessary criteria* of finding as many positive integer resonances as the order of the system[1]

2. **Possess the Painlevé property.** As detailed above, this is *sufficient criteria* for the system to be integrable

The machinery developed in chapter 3 provides a straight forward way of flagging potential new integrable Toda field theory candidates that accomplish the conditions stated in stage 1, the majority of this chapter will focus on providing a stronger argument for the stage 2, for any new integrable candidates we find from 1.

[1] As previously discussed, for all Toda field theories based on algebras with a Cartan matrix K, the order of the system is equal to the rank of K.

4.1.1 Towards the Painlevé Property of Toda Field Theories

Our approach for achieving step 2 starts with the model or set of models we have found to pass the Painlevé test. Passing this test will have given us enough positive integer resonances, which we denote as N due to it identically matching the N real dimensions of the analytic manifold. We also have that $N = r$ where r is the rank of the underlying algebra, the associated Cartan matrix, and hence the order of the Toda field theory's PDE we are interested in. In general, for Toda systems, to pass the test we require that there are N positive integer resonances. It is also possible to achieve the Painlevé property with negative integer resonances [97], and the so called weak Painlevé property can be satisfied with non-integer resonances [96]. However, our analysis will stay consistent with previous chapters and only focus on models analogous to Toda field theories based on finite and affine Lie algebras, so we will still only be dealing with enough positive integer resonances to match their rank.

At first sight, one may expect to find $2N - 1 = 2r - 1$ resonances in an integrable Toda system corresponding to the $2r - 1$ arbitrary functions occurring along the analytic manifold, ϕ coming from a meromorphic function, $f = f(z_1, \cdots, z_{2N})$, $\{z_1, \cdots, z_{2N}\} \in \mathbb{C}$, with no moveable critical singularities occurring on ϕ. This agrees with the notion of the system having Liouville integrability of N real degrees of freedom with N analytic single valued global integrals of motion in involution. However, as noted in Flaschka in [98], we expect to see the behaviour of $N = r$ resonances for the *lowest balance* Painlevé expansion, and moreover for Toda systems the lowest balance is unique. To understand this notion more concretely, we define what is meant by a balance and how this fits in with resonances and the Painlevé property of some Toda field theories.

In general, a balance is a class of Painlevé expansions with the most negative power in the series fixed, meaning that the dominant singular behaviour is most dependent at this point. At certain powers of the expansion arbitrary parameters can be introduced, the values of the powers in which this is possible are, as we know, identified as the resonances. Painlevé expansion solutions that contain $2N - 1$ resonances and arbitrary functions are known as *principle balances*, whereas as mentioned above, those containing N resonances and arbitrary functions are the lowest balances. Balances with amounts of resonances between the principle and lowest balance values are also clearly possible, and more details regarding these are discussed by Flaschka in [98] and references within.

The lowest balance is unique for a Toda PDE solution as it contains information directly inherited from the $2N$ free parameters of the system. This was argued using the technology of affine varieties and Schubert cells in [98], which we will spare all the details of for our purposes here. Heuristically however, their argument may be understood as the $2N$ complex constants of motion in involution for the integrable Toda system being viewed as an affine variety. This may be done as we can find a function which takes these $2N$ constants to zero[2], and hence this level set is also an affine variety in the affine space

[2] For our examples in this chapter and chapter 3, this function can be seen as $\det[T^{(k)}] = 0$. The

of solutions. This affine variety can be compactified by the addition of ideal points at infinity, achieved through the addition of varieties of real dimension $N-1$, each of which is parameterized by $N-1$ of the $2N$ free parameters of the principle balance. In other words, one can consider N of the $2N$ parameters to be used up to specify the values of the N constants of motion of the Toda system as both currents of their W-algebras [99] and exponents of the Lie algebra, as we did in the discussion within section 3.6. Moreover, for all the models we examine in this chapter, we show explicitly that only the lowest balance is possible in terms of the maximum possible number of positive integer resonances that may be discovered.

For the Toda field theories considered in this chapter, we will thus use the lowest balance to construct a meromorphic solution on the singular non-characteristic manifold, ϕ, and show that it can possess the Painlevé property if it satisfies the definition in the previous subsection. To do so, we will need to take each model that passes the Painlevé test and solve the recursion equation, showing that we can find $N=r$ free functions in the general solution to the Painlevé expansion, corresponding to values found at the resonances values hence showing that the solution is meromorphic on ϕ. We also expect to uncover one final arbitrary functions, bringing us to a total of $r+1$, which will always be related to the arbitrary expansion parameter.

For the $r=2$ cases that we focus on, this means that we will uncover three arbitrary functions in their general solutions. We will show that these solutions only depend on these three arbitrary functions and the expansion parameter, meaning that any singularities that occur in each expansion solution must be moveable due to the dependence on the arbitrary functions, or equivalently that the solutions are meromorphic on the singularity manifold ϕ. Hence, providing formal evidence for the Painlevé property and the Painlevé integrability of the given model.

It is important to stress the formal nature of the evidence for the Painlevé property and that at this point of analysis we have no information about the series convergence of the solution. As noted for example in [90], there is the possibility of sequences of singularity manifolds for single valued solutions which combine into a more complex singularity that we cannot Painlevé expand. To rule out this possibility and to absolutely conclude the system is integrable we must estimate the radius of convergence of the series. However, there are, to the authors knowledge, no known examples of such models that pass the Painlevé test, and are subsequently show to have Painlevé property via WTC methods or otherwise, but have divergent series. We do not conduct this additional analysis here, and only take note of this possibility.

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remaining r resonances may be found at negative values, as we show in section 3.6.4 for the perturbed n-extended Lorentzian Toda theories.

4.3 Rank-2 Models that pass the Painlevé Test

Recall from section 3.6.4 that the Painlevé test could not be passed for the class of n-extended Lorentzian Kac-Moody algebras that had been the focus of previous chapters. In this section we search for new models that do pass the Painlevé test by determining conditions on the Cartan matrix, K that allow for the test to be passed. We find that some of these models must be described with a Lorentzian metric in their root space. For some specific examples that we find to pass the Painlevé test, we prove they possess the Painlevé property in later sections.

We restrict our introductory analysis to rank-2 Cartan matrices, K, as to simplify calculations, although we make note of some higher rank solutions that follow the uncovered patterns that also pass the tests of this section. We also note that our analysis is not complete in the sense of proving that we have uncovered every general rank-2 Cartan that passes the Painlevé test, as our main motivation here is to find the most reasonable candidates to test their Painlevé integrability, so we stop short of exhausting all possibilities.

Our results are split into roughly two categories of solutions that we find to pass the Painlevé test. We only will perform further Painlevé analysis on one of these categories of solutions due to physical reasons presented in the next section. Working with the definitions given in chapter 3 for the Painlevé test, we start our analysis with

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad D = \frac{1}{ad-bc}\begin{pmatrix} d-b & 0 \\ 0 & a-c \end{pmatrix}, \tag{4.2}$$

where $\{a, b, c, d\} \in \mathbb{R}$. Hence, the Painlevé matrix, $P = 2DK$, takes the form

$$P = \frac{1}{ad-bc}\begin{pmatrix} 2a\,(d-b) & 2b\,(d-b) \\ 2c\,(a-c) & 2d\,(a-c) \end{pmatrix}. \tag{4.3}$$

The associated eigenvalues of P are calculated to be $\lambda = \{2, -\frac{2(a-c)(b-d)}{ad-bc}\}$, where the first value is always fixed at 2, corresponding to the conserved quantity of the spin-2 current of the energy-momentum tensor from the Toda field theory. Again, as discussed in section 3.6, it is well understood that the resonances, n are related to λ through $\lambda = n(n-1)$, so we seek to solve

$$-\frac{2(a-c)(b-d)}{ad-bc} = n(n-1), \quad n \in \mathbb{Z}^{+}. \tag{4.4}$$

Meaning that we would have two positive integer resonances for the Toda field theory associated with K, thus passing the Painlevé test.

Solving equation (4.4) gives us two main categories of solutions, with the first that we present below being a superset of the second. The first category has solutions of K that take a form in which we have three free parameters and one fixed, giving the following four possibilities for K that we split into two groups depending on the diagonal or off-diagonal position of the dependant value for ease of reference,

$$K = \begin{pmatrix} \frac{c(b((n-1)n+2)-2d)}{2b+d(n-2)(n+1)} & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & \frac{b(c((n-1)n+2)-2a)}{a(n-2)(n+1)+2c} \end{pmatrix} \tag{4.5}$$

$$K = \begin{pmatrix} a & b \\ \frac{a(2b+d(n-2)(n+1))}{b((n-1)n+2)-2d} & d \end{pmatrix}, \begin{pmatrix} a & \frac{d\left(a\left(-n^2+n+2\right)-2c\right)}{2a+c(-n^2+n-2)} \\ c & d \end{pmatrix} \tag{4.6}$$

Subbing the above equations (4.5) and (4.6) back into equation (4.3) results in the eigenvalues of P being $\lambda = \{2, n(n-1)\}$ as expected. It is not hard to see that we can recover results of the finite Lie algebras that are known to pass the Painlevé test from the forms of K in equations (4.5) and (4.6). For example, taking the second solution of K from equation (4.6) with $a = d = 2$ and $c = -1$ results in $K = \left(\begin{smallmatrix} 2 & \frac{36}{n(n-1)+6}-4 \\ -1 & 2 \end{smallmatrix}\right)$, reproducing

$$K_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad K_{C_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad K_{G_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{4.7}$$

Which matches up precisely for the resonance of $n = 2, 3, 6$ values expected in the A_2, C_2 and G_2 Lie algebras respectively. A similar calculation on our other solution in equation (4.6) would clearly result in the K for D_2, hence reproducing all the finite Lie algebras at rank-2. Equation (4.5) can be seen as the K solutions of taking an non-normalized root space, i.e. $\alpha_i \cdot \alpha_j = K_{ij}$ in opposed to the $\frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} = K_{ij}$ used for simple Lie algebras, where α_i for $i \in \{1, \cdots, r\}$ are the simple roots of the Lie algebra as before.

Continuing to examine the solutions (4.5) and (4.6), we immediately ask what other values of n could be reasonable in terms of other potentially unfound algebras that will pass the Painlevé test. For the series of known finite algebras in (4.7), we see that no other values of n are valid for $\frac{36}{n(n-1)+6} - 4$ to be an integer. It may be the case that consistent combinations that give integer values of $\{a, b, c, d\}$ and positive integer n could be found for in (4.5) and (4.6), at this stage we could perform an analysis similar to that in section 3.6.4 to get further restrictions on their form according to the definiteness of K. However, we will restrict our analysis to a subcategory of equations (4.5) and (4.6), with the idea in mind that the novel form of the solutions in this subcategory will likely present distinct physical behaviour in comparison to the finite and affine Toda field theories.

The second category of solutions gives four possibilities with two free variables, which we again make the distinction between zeros that occur in diagonals and off-diagonals

$$K = \begin{pmatrix} 0 & b \\ c & \frac{1}{2}b(n-1)n + b \end{pmatrix}, \begin{pmatrix} \frac{1}{2}c(n-1)n + c & b \\ c & 0 \end{pmatrix}, \tag{4.8}$$

$$K = \begin{pmatrix} a & -\frac{1}{2}d\left(n^2-n-2\right) \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & 0 \\ -\frac{1}{2}a\left(n^2-n-2\right) & d \end{pmatrix}. \tag{4.9}$$

This second category immediately stands out from any Cartan matrices we have previously encountered through the presence of a zero value in their rows and columns. When zero values occur in the diagonals of K we refer to these as *null roots*, due to our interpretation that the roots have the property that $\alpha_i \cdot \alpha_i = 0$. For zero values in off-diagonals, we view this as non-commutativity in root space and clearly have that $\alpha_i \cdot \alpha_j \neq \alpha_j \cdot \alpha_i$ with $\alpha_i \cdot \alpha_j = 0$ for some α_i.

Due to the presence of zeros in their K's, the solutions (4.8), (4.9) will likely have distinct physical behaviour with respect to their super category (4.5) and (4.6), which reproduce the known finite and associated affine Toda field theory results [84]. Narrowing down our search to the models of equations (4.8), (4.9), in the next subsection we will use the methods developed in chapter 3 to examine the physical properties of Toda field theories associated to these Cartan matrices. In examining the physics of (4.8), (4.9) we hope to rule out any trivial behaviour before conducting in-depth Painlevé analysis that may occur in some of these models, which we cannot identify at the level of analysis of K alone.

4.4 Physical Behaviour of the Integrable Candidates

Writing down the Toda field theory Lagrangian, as we did in equation (3.6) of chapter 3, as

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{g}{\beta^2}\sum_{i=1}^{r} e^{\beta \alpha_i \cdot \varphi} \tag{4.10}$$

where we denote φ as the fields to distinguish in our notation the singular manifold, ϕ of equation (4.1), and all other quantities are as before. Following standard practice and using $\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^a)} = \frac{\partial \mathscr{L}}{\partial \varphi^a}$ with $a \in \{1, \cdots, r\}$, the Euler-Lagrange equations result from equation (4.10), and can be written as

$$\begin{aligned} \partial_\mu \partial^\mu \varphi + \frac{g}{\beta}\sum_{i=1}^{r} \alpha_i^a e^{\beta \alpha_i \cdot \varphi} &= 0 \\ \partial_\mu \partial^\mu \alpha_j \cdot \varphi + \frac{g}{\beta}\sum_{i=1}^{r} \alpha_j \cdot \alpha_i^a e^{\beta \alpha_i \cdot \varphi} &= 0, \end{aligned} \tag{4.11}$$

making the identity $\Phi := \alpha_i \cdot \varphi - \frac{1}{\beta}\ln \chi_i$, where χ_i is to be found, substituting this in to equation (4.11) we have that

$$\begin{aligned} \partial_\mu \partial^\mu \Phi_j + \frac{g}{\beta}\sum_{i=1}^{r} \chi_i (\alpha_j \cdot \alpha_i^a) e^{\beta \alpha_i \cdot \Phi} &= 0 \\ \partial_\mu \partial^\mu \Phi_j + \frac{g}{\beta}\sum_{i=1}^{r} K_{ji} e^{\beta \alpha_i \cdot \Phi} &= 0, \end{aligned} \tag{4.12}$$

for $\chi_i = \frac{2}{\alpha_i^2}$ where we identify K_{ji} as the Cartan matrix. Concentrating on rank-2 models with K of the form in equation (4.2) we find that the equations of motion from above take the form

$$\begin{aligned}\partial_\mu\partial^\mu\Phi_1 + \frac{g}{\beta}(ae^{\beta\Phi_1} + be^{\beta\Phi_2}) = 0\\ \partial_\mu\partial^\mu\Phi_2 + \frac{g}{\beta}(ce^{\beta\Phi_1} + de^{\beta\Phi_2}) = 0.\end{aligned} \tag{4.13}$$

Hence, we see that taking $c \to 0$ or $b \to 0$ corresponding to equations with zeros in off-diagonal positions as in equation (4.9) means that equation (4.13) partially decouples into unconnected equations of motion. Whereas taking $a \to 0$ or $d \to 0$ we do not see this partial decoupling. The latter behaviour we term as belonging to the *null-root* models, whereas the former we refer to as *partially decoupled* models.

The behaviour of these partially decoupling models seems to be more trivial than that of the null-root models. For example, take $c = 0, d = 2$ (or $b = 0, a = 2$), then we have decomposition into an A_1 Toda field theory equation of motion for the first (resp. second) equation of (4.13), with the equation of motion for the other field being dependent on both fields. Studying the behaviour of these partially decoupled models could be interesting in its own right, for example as a nonsymmetric perturbation of $(A_1)_1$ with Φ_1 by another field Φ_2 from $(A_1)_2$, such analysis is similar but not identical to that conducted in [100], in which they examine an A_1 Toda system, perturbing it with another field such that the resulting model has behaviour halfway between the Louisville equation and the sinh-Gordon equation. Here however, we make the decision to not study these models further in this chapter's discussion, and we focus our Painlevé analysis on the non-decoupling null-root models we have uncovered in above calculations.

4.5 Painlevé Integrability and the Painlevé Property

As previously discussed, it is well known that Toda theories based on finite and affine Lie-algebras are integrable, in the classical sense [91][43]. They are also integrable as quantum field theories, and it is even possible to find exact S-matrices of these models [84][52]. As exemplified in the provided references above, often the integrability of these systems is shown through finding Lax pairs or the zero-curvature representation of the system. For this subsection we focus on the Painlevé integrability of A_2, for which the zero-curvature representation and Lax pairs have been found [101], and therefore A_2's integrability established.[3]

Even though the integrability of A_2 has been established, the examination of section (4.5.1) is useful for several reasons. Firstly, as far as we could find in the literature, the Painlevé analysis on the finite and affine Toda theories has not been previously under-

[3]As mentioned, and shown with the references of this paragraph, the Lax pairs and zero-curvature representations have been found for all affine and finite Toda field theories based on ADE algebras. We stress this for A_2 as this is the example model that we will also show Painlevé integrability for.

taken, which is understandable as their integrability has been already established by the mean discussed above. Hence, it will be beneficial to see a concrete working example of Painlevé integrability. Secondly, A_2 will provide us with a good idea of what the Painlevé integrability should look like when we examine the rank-2 null-root models in section 4.5.2.

4.5.1 Painlevé Property of A_2

For the A_2 Toda theory we start with

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} -\sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \tag{4.14}$$

where K is Cartan matrix and M is a 2-dimensional matrix representation of the simple roots, $\{\alpha_1, \alpha_2\}$, such that $M \cdot M^T = K$, and all dot products will be taken with a Euclidean inner product for A_2 Subbing these values, along with $D_1, D_2 = 1$ for A_2, into the equations from section 3.6 for the recurrence relation, $T^{(k)} \cdot X^{(k)} = Y^{(k)}$, derived from the Painlevé equation (3.59), we have

$$T^{(k)} = \begin{pmatrix} k\dot{\phi} & 0 & -\sqrt{2}n_p n_q \dot{\phi}\acute{\phi} & 0 \\ 0 & k\dot{\phi} & \frac{1}{\sqrt{2}} n_p n_q \dot{\phi}\acute{\phi} & \sqrt{\frac{3}{2}} n_p n_q \dot{\phi}\acute{\phi} \\ -\sqrt{2} & \frac{1}{\sqrt{2}} & (k - n_p)\dot{\phi} & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & (k - n_p)\dot{\phi} \end{pmatrix} \tag{4.15}$$

where n_p, n_q are the balances that will be discussed shortly, and $\phi := \phi(u, v)$ is the field in light-cone coordinates (u, v) that we aim to show the solutions of the recurrence relations are meromorphic, i.e. possessing moveable no critical singularities in the limit $\phi(u, v) \to 0$. The other quantities in the recurrence relation take the form

$$Y^{(k)} = \begin{pmatrix} -\sum_{i=1}^{k-1} \sqrt{2} b_1^{(i)} a_1^{(k-i)} \\ \sum_{i=1}^{k-1} \left(\frac{b_1^{(i)}}{\sqrt{2}} + \sqrt{\frac{3}{2}} b_2^{(i)} \right) a_2^{(k-i)} \\ 0 \\ 0 \end{pmatrix}, \qquad X^{(k)} = \begin{pmatrix} a_1^{(k)} \\ a_2^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \end{pmatrix} \tag{4.16}$$

where the a_i and b_i are the expansion coefficients from the Painlevé expansion that we expect at least 2 to be arbitrary if this A_2 Toda field theory is to possess the Painlevé property. We find that the determinate of $T^{(k)}$ for A_2 takes the form

$$\det T^{(k)} = \left(k^2 - n_p \left(k + n_q\right)\right) \left(k^2 - n_p \left(k + 3n_q\right)\right) \dot{\phi}^2 \acute{\phi}^2 \tag{4.17}$$

At this point, to continue with our analysis we would like to know what balances, $n_p, n_q > 0$ are admissible for valid resonances of the A_2 Toda theory. We find that if we require $\det T^{(2)} = \det T^{(3)} = 0$, the only possible integer solutions for the balances are

$n_p = 1, n_q = 2$, resulting in

$$\det T^{(k)} = (k-3)(k-2)(k+1)(k+2)\dot{\phi}^2\acute{\phi}^2, \tag{4.18}$$

where the resonances occur identically at the values of k that satisfy this equation. Evidently $\det T^{(k)} = 0$ for some $k < 0$, meaning that, for the reasons that we discussed in section 4.1.1, this is the lowest balance solution that we expected to find.[4]

It is also no surprise that this solution clearly shows that A_2 passes the Painlevé test, containing enough positive integer resonances, $k = \{2, 3\}$, to match its rank. To show that A_2 also possesses the Painlevé property, we now aim to solve the recurrence relations and expect to find three free parameters in doing so. As is standard practice in the WTC Painlevé method [89], we employ the Kruskal simplification on our field, ϕ, so that $\phi(u, v) = u - \xi(v)$, where $\xi(v)$ is an arbitrary choice of function at this stage. From here on, we shall denote $\xi := \xi(v)$ and its derivative as ξ', for notation's sake.

To solve the Painlevé expansions equations (3.59), for A_2 we seek to solve

$$\begin{pmatrix} a_1^{(k)} \\ a_2^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{(k-1)\left(\left(k^2-k-4\right)\sum_i^{k-1}\sqrt{2}b_1^{(i)}a_1^{(k-i)}+2\sum_i^{k-1}\left(\frac{b_1^{(i)}}{\sqrt{2}}+\sqrt{\frac{3}{2}}b_2^{(i)}\right)a_2^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \\ \frac{(k-1)\left(\left(-k^2+k+4\right)\sum_i^{k-1}\left(\frac{b_1^{(i)}}{\sqrt{2}}+\sqrt{\frac{3}{2}}b_2^{(i)}\right)a_2^{(k-i)}+2\sum_i^{k-1}-\sqrt{2}b_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \\ \frac{\left(2k^2-2k-6\right)\sum_i^{k-1}\sqrt{2}b_1^{(i)}a_1^{(k-i)}+k(k-1)\sum_i^{k-1}\left(\frac{b_1^{(i)}}{\sqrt{2}}+\sqrt{\frac{3}{2}}b_2^{(i)}\right)a_2^{(k-i)}}{\sqrt{2}(k-3)(k-2)(k+1)(k+2)\xi'} \\ -\frac{\sqrt{\frac{3}{2}}\left(\left(-k^2+k+4\right)\sum_i^{k-1}\left(\frac{b_1^{(i)}}{\sqrt{2}}+\sqrt{\frac{3}{2}}b_2^{(i)}\right)a_2^{(k-i)}-2\sum_i^{k-1}\sqrt{2}b_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \end{pmatrix} \tag{4.19}$$

to find the expansion parameters at each level k. Or at levels in which the RHS of equation (4.19) we must show that any potential arbitrary values of the expansion parameters found can be substituted into all levels of the recursion relations $T^{(k)} \cdot X^{(k)} = Y^{(k)}$ consistently. This is show systematically through each level in the proceeding calculations.

Level $k = 0$

Using equation (3.63) with the derivative of the field from Kruskal simplification, ξ', and appropriate constants for A_2, $n = -1$ and $r = 2$, we find that

$$a_i^{(0)} = 2\xi' D_i, \quad b_\alpha^{(0)} = 2\sum_{i=1}^{2}(M^T)_{\alpha i}\xi' D_i, \tag{4.20}$$

[4]Further basic analysis on the roots of equation (4.17) shows that only the lowest balance solutions will be possible as negative values of k are unavoidable. Explaining why we uncover the $k = -1, -2$ resonances in this example, and also similar negative resonances in further examples, which we shall examine shortly.

resulting in

$$\begin{pmatrix} a_1^{(0)} \\ a_2^{(0)} \\ b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \begin{pmatrix} \xi' \\ 2\xi' \\ -\sqrt{2}\,\xi' \\ \sqrt{6}\,\xi' \end{pmatrix}, \tag{4.21}$$

where ξ is our first discovered arbitrary function, we will need to check that ξ stays arbitrary at all higher levels too.

Level $k = 1$

At this level $T^{(1)}$ is also invertible, hence we can easily find the RHS of equation (4.19) through substitution of the appropriate quantities resulting in

$$\begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ b_1^{(1)} \\ b_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.22}$$

Level $k = 2$

$T^{(2)}$ is singular, corresponding to the resonance related to the spin-2 energy-momentum tensor of A_2. Comparing both sides of $T^{(2)} \cdot X^{(2)} = Y^{(2)}$ gives

$$\begin{pmatrix} -2\left(a_1^{(2)} - \sqrt{2}b_1^{(2)}\right)\xi' \\ -\left(\left(2a_2^{(2)} + \sqrt{2}b_1^{(2)} + \sqrt{6}b_2^{(2)}\right)\xi'\right) \\ -\sqrt{2}a_1^{(2)} + \frac{a_2^{(2)}}{\sqrt{2}} + b_1^{(2)} \\ \sqrt{\frac{3}{2}}a_2^{(2)} + b_2^{(2)} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}a_1^{(1)}b_1^{(1)} \\ \frac{a_2^{(1)}\left(b_1^{(1)}+\sqrt{3}b_2^{(1)}\right)}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}. \tag{4.23}$$

Substituting in the values in from level-1 and solving results in

$$\begin{pmatrix} a_2^{(2)} \\ b_1^{(2)} \\ b_2^{(2)} \end{pmatrix} = \begin{pmatrix} a_1^{(2)} \\ \frac{1}{\sqrt{2}}a_1^{(2)} \\ -\sqrt{\frac{3}{2}}a_1^{(2)} \end{pmatrix}, \tag{4.24}$$

where $a_1^{(2)}$ is the second arbitrary function that we will see stays arbitrary at all higher levels.

Level $k = 3$

$T^{(3)}$ is also singular, so we proceed analogously to the above $k = 2$ case and find that $T^{(3)} \cdot X^{(3)} = Y^{(3)}$ yields

$$\begin{pmatrix} \left(2\sqrt{2}b_1^{(3)} - 3a_1^{(3)}\right)\xi' \\ -\left(\left(3a_2^{(3)} + \sqrt{2}b_1^{(3)} + \sqrt{6}b_2^{(3)}\right)\xi'\right) \\ -\sqrt{2}a_1^{(3)} + \frac{a_2^{(3)}}{\sqrt{2}} + 2b_1^{(3)} \\ \sqrt{\frac{3}{2}}a_2^{(3)} + 2b_2^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.25}$$

after subbing in the values from levels $k = 1, 2$ into the RHS. Solving for the expansion coefficients gives

$$\begin{pmatrix} a_2^{(3)} \\ b_1^{(3)} \\ b_2^{(3)} \end{pmatrix} = \begin{pmatrix} -a_1^{(3)} \\ \frac{3}{2\sqrt{2}}a_1^{(3)} \\ \frac{1}{2}\sqrt{\frac{3}{2}}a_1^{(3)} \end{pmatrix}, \tag{4.26}$$

where $a_1^{(3)}$ is arbitrary. Hence, we have uncovered three arbitrary functions, and as $T^{(k)}$ is not singular at any higher levels no more will be introduced.

General Painlevé Expansion Solutions

To proceed we will examine the form of equation (4.19) to find solutions to the Painlevé expansions, and show that these solutions are meromorphic on the singular analytic manifold we formed these expansions around. From equation (4.19) we can see that the highest power of the expansion parameter on the RHS corresponds to one less than that on the LHS. However, as $a_i^{(1)} = b_i^{(1)} = 0$ for $i = \{1, 2\}$ we can identify that the highest terms that occur will be two powers less on the RHS to the LHS. Taking each of the values from the $k = 0, 1, 2, 3$ level solutions found above and subbing these first into (4.19) and then the Painlevé expansion gives

$$\begin{aligned} P_1 = &-\sqrt{2}\xi'\phi^{-1} + \frac{3a_1^{(3)}\phi}{2\sqrt{2}} + \frac{(a_1^{(2)})^2\phi^2}{10\sqrt{2}\xi'} + \frac{15a_1^{(2)}a_1^{(3)}\phi^3}{28\sqrt{2}\xi'} \\ &+ \frac{15a_1^{(2)}a_1^{(3)}\phi^4}{28\sqrt{2}\xi'} + \frac{\phi^5\left(15(a_1^{(3)})^2\xi' + 4(a_1^{(2)})^3\right)}{280\sqrt{2}\xi'^2} + \frac{3(a_1^{(2)})^2a_1^{(3)}\phi^6}{20\sqrt{2}\xi'^2} \\ &+ \frac{\phi^7\left(50(a_1^{(3)})^2a_1^{(2)}\xi' + 3(a_1^{(2)})^4\right)}{1400\sqrt{2}\xi'^3} + \frac{3a_1^{(3)}\phi^8\left(17(a_1^{(3)})^2\xi' + 30(a_1^{(2)})^3\right)}{2464\sqrt{2}\xi'^3} \\ &+ \frac{\phi^9\left(1285(a_1^{(3)})^2(a_1^{(2)})^2\xi' + 28(a_1^{(2)})^5\right)}{86240\sqrt{2}\xi'^4} + \frac{3a_1^{(3)}\phi^{10}\left(3275(a_1^{(3)})^2a_1^{(2)}\xi' + 1382(a_1^{(2)})^4\right)}{509600\sqrt{2}(\xi')^4} \\ &+ \ldots \end{aligned} \tag{4.27}$$

Where we have taken terms up to ϕ^{10} but could generate terms up to arbitrary order. P_2 takes the form

$$
\begin{aligned}
P_2 =& \frac{\sqrt{6}\xi'}{\phi} - \sqrt{\frac{3}{2}}a_1^{(2)}\phi + \frac{1}{2}\sqrt{\frac{3}{2}}a_1^{(3)}\phi^2 - \frac{\sqrt{\frac{3}{2}}(a_1^{(2)})^2\phi^3}{10\xi'} + \frac{5\sqrt{\frac{3}{2}}a_1^{(2)}a_1^{(3)}\phi^4}{28\xi'} \\
& - \frac{\sqrt{\frac{3}{2}}\phi^5\left(15(a_1^{(3)})^2\xi' + 4(a_1^{(2)})^3\right)}{280\xi'^2} + \frac{\sqrt{\frac{3}{2}}(a_1^{(2)})^2 a_1^{(3)}\phi^6}{20\xi'^2} - \frac{\sqrt{\frac{3}{2}}\phi^7\left(50(a_1^{(3)})^2 a_1^{(2)}\xi' + 3(a_1^{(2)})^4\right)}{1400\xi'^3} \\
& + \frac{\sqrt{\frac{3}{2}}a_1^{(3)}\phi^8\left(17(a_1^{(3)})^2\xi' + 30(a_1^{(2)})^3\right)}{2464\xi'^3} - \frac{\sqrt{\frac{3}{2}}\phi^9\left(1285(a_1^{(3)})^2(a_1^{(2)})^2\xi' + 28(a_1^{(2)})^5\right)}{86240\xi'^4} \\
& + \frac{\sqrt{\frac{3}{2}}a_1^{(3)}\phi^{10}\left(3275(a_1^{(3)})^2 a_1^{(2)}\xi' + 1382(a_1^{(2)})^4\right)}{509600\xi'^4} + \ldots
\end{aligned}
\tag{4.28}
$$

For the Q_i's we have that

$$
\begin{aligned}
Q_1 =& \frac{2\xi'}{\phi^2} - a_1^{(2)} + a_1^{(3)}\phi + \frac{3(a_1^{(2)})^2\phi^2}{10\xi'} + \frac{5a_{12}a_1^{(3)}\phi^3}{7\xi'} + \frac{\phi^4\left(15(a_1^{(3)})^2\xi' + 4(a_1^{(2)})^3\right)}{56\xi'^2} \\
& + \frac{3(a_1^{(2)})^2 a_1^{(3)}\phi^5}{10\xi'^2} + \frac{\phi^6\left(50(a_1^{(3)})^2 a_{12}\xi' + 3(a_1^{(2)})^4\right)}{200\xi'^3} + \frac{a_1^{(3)}\phi^7\left(17(a_1^{(3)})^2\xi' + 30(a_1^{(2)})^3\right)}{308\xi'^3} \\
& + \frac{9\phi^8\left(1285(a_1^{(3)})^2(a_1^{(2)})^2\xi' + 28(a_1^{(2)})^5\right)}{86240\xi'^4} + \frac{a_1^{(3)}\phi^9\left(3275(a_1^{(3)})^2 a_1^{(2)}\xi' + 1382(a_1^{(2)})^4\right)}{50960\xi'^4} \\
& + \ldots
\end{aligned}
\tag{4.29}
$$

and

$$
\begin{aligned}
Q_2 =& \frac{2\xi'}{\phi^2} + a_1^{(2)} - a_1^{(3)}\phi + \frac{3(a_1^{(2)})^2\phi^2}{10\xi'} - \frac{5a_{12}a_1^{(3)}\phi^3}{7\xi'} + \frac{\phi^4\left(15(a_1^{(3)})^2\xi' + 4(a_1^{(2)})^3\right)}{56\xi'^2} \\
& - \frac{3(a_1^{(2)})^2 a_1^{(3)}\phi^5}{10\xi'^2} + \frac{\phi^6\left(50(a_1^{(3)})^2 a_{12}\xi' + 3(a_1^{(2)})^4\right)}{200\xi'^3} - \frac{a_1^{(3)}\phi^7\left(17(a_1^{(3)})^2\xi' + 30(a_1^{(2)})^3\right)}{308\xi'^3} \\
& + \frac{9\phi^8\left(1285(a_1^{(3)})^2(a_1^{(2)})^2\xi' + 28(a_1^{(2)})^5\right)}{86240\xi'^4} - \frac{a_1^{(3)}\phi^9\left(3275(a_1^{(3)})^2 a_1^{(2)}\xi' + 1382(a_1^{(2)})^4\right)}{50960\xi'^4} \\
& + \ldots
\end{aligned}
\tag{4.30}
$$

up to terms including terms of the order ϕ^9, where like the P_i terms we can generate terms of Q_i up to arbitrary order in ϕ. From the form of these four solutions, we can show that up to any arbitrary order the expansions may be written solely in terms of the three arbitrary functions $a_1^{(2)}, a_1^{(3)}$ and ξ, allowing us to conclude that these expansions give a meromorphic solution fitting with the definition of the Painlevé property, giving strong indication that the system is Painlevé integrable as expected.

4.5.2 Painlevé Property of spin-3, Rank-2 Null-Root Models

Now we have seen the Painlevé property argued for an example that we already know to be integrable, we move on to examining the null-root solutions that we found to pass the Painlevé test. We follow an analogous process to the previous subsection, with the only difference being we will use a Lorentzian inner product throughout and the Euclidean inner product used for A_2. Hence, we argue for the Painlevé property for systems in the form of equation (4.8), in which we set $b, c \to -dc$ for clarity in calculation and $n \to 3$, so that we have spin-3 null-root model solutions as desired[5], meaning we have

$$K = \begin{pmatrix} 0 & -dc \\ -dc & -4dc \end{pmatrix}, \quad M = \begin{pmatrix} 0 & d \\ c & 2d \end{pmatrix}, \tag{4.31}$$

where again $M \cdot M^T = K$, with the rank-2 Lorentzian inner product such that

$$w \cdot z = -w_1 z_2 - w_2 z_1, \tag{4.32}$$

for the vectors $w = (w_1, w_2)$ and $z = (z_1, z_2)$. Proceeding as we did with A_2, we find that

$$T^{(k)} = \begin{pmatrix} k\dot{\phi} & 0 & 0 & \frac{6\dot{\phi}\acute{\phi}}{c} \\ 0 & k\dot{\phi} & -\frac{2\dot{\phi}\acute{\phi}}{d} & -\frac{4\dot{\phi}}{c} \\ -d & -2d & (k-1)\acute{\phi} & 0 \\ 0 & -c & 0 & (k-1)\acute{\phi} \end{pmatrix}, \qquad X^{(k)} = \begin{pmatrix} a_1^{(k)} \\ a_2^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \end{pmatrix}, \tag{4.33}$$

where we have transformed M^T as to absorb the Lorentzian inner product so that we can use regular matrix multiplication with the Euclidean inner product, doing so we calculate the determinant of $T^{(k)}$ to be

$$\det T^{(k)} = (k-3)(k-2)(k+1)(k+2)\dot{\phi}^2\acute{\phi}^2, \tag{4.34}$$

we have again $k = \{2, 3\}$ as the resonances we expected to find, so we can confirm by this explicit calculation that this rank-2 null root model passes the Painlevé test. $Y^{(k)}$ takes the form

[5] We set $n = 3$ as we choose to study resonances $\{2, 3\}$, but clearly this method can be applied to higher spin solutions of these null-root models by altering the value of n to the resonance and hence spin of our choosing.

$$
Y^{(k)} = \begin{pmatrix} -\sum_{i=1}^{k-1} db_1(i)a_1(k-i) \\ -\sum_{i=1}^{k-1} a_2(k-i)\left(2db_1(i)+cb_2(i)\right) \\ 0 \\ 0 \end{pmatrix}, \tag{4.35}
$$

where d, c remain as arbitrary real constants throughout.

Next, we again employ the Kruskal's simplification on, ϕ, so that $\phi(u,v) = u - \xi(v)$, with ξ an arbitrary function. For $k \geq 1$ the expansion coefficients for the Painlevé expansion solutions are found to be

$$
\begin{pmatrix} a_1^{(k)} \\ a_2^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \end{pmatrix} = \begin{pmatrix} -\frac{(k-1)\left(6\sum_i^{k-1} a_2^{(k-i)}\left(2db_1^{(i)}+cb_2^{(i)}\right)+\left(-k^2+k+8\right)\sum_i^{k-1} +db_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \\ \frac{(k-1)\left((k-1)k\sum_i^{k-1} a_2^{(k-i)}\left(2db_1^{(i)}+cb_2^{(i)}\right)+2\sum_i^{k-1} db_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \\ -\frac{d\left(2\left(-k^2+k+3\right)\sum_i^{k-1} a_2^{(k-i)}\left(2db_1^{(i)}+cb_2^{(i)}\right)+\left(-k^2+k+4\right)\sum_i^{k-1} db_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \\ \frac{c\left((k-1)k\sum_i^{k-1} a_2^{(k-i)}\left(2db_1^{(i)}+cb_2^{(i)}\right)+2\sum_i^{k-1} db_1^{(i)}a_1^{(k-i)}\right)}{(k-3)(k-2)(k+1)(k+2)\xi'} \end{pmatrix}, \tag{4.36}
$$

so that we may now proceed to solve at each level and check if we may find the two remaining arbitrary parameters that we would need to form a meromorphic solution to the Painlevé expansion.

Level $k = 0$

For $k = 0$ we use equations (4.20) with $D_1 = \frac{3}{cd}$, $D_2 = -\frac{1}{cd}$, and find that

$$
\begin{pmatrix} a_1^{(0)} \\ a_2^{(0)} \\ b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \begin{pmatrix} \frac{6\xi'}{dc} \\ -\frac{2\xi'}{dc} \\ -\frac{2\xi'}{d} \\ \frac{2\xi'}{c} \end{pmatrix}, \tag{4.37}
$$

where ξ remains an arbitrary parameter.

Level $k = 1$

$T^{(1)}$ is invertible, hence solving (4.36) for $k = 1$ results in

$$
\begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ b_1^{(1)} \\ b_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.38}
$$

Level $k = 2$

As $k = 2$ corresponds to the conserved spin related to the energy-momentum tensor, we will again find that $T^{(2)}$ is singular. As before, comparing both sides of $T^{(2)} \cdot X^{(2)} = Y^{(2)}$ yields

$$\begin{pmatrix} -\frac{2\left(a_1^{(2)}c+3b_2^{(2)}\right)\xi'}{c} \\ \frac{2\xi'\left(a_2^{(2)}(-d)c+2b_2^{(2)}d+b_1^{(2)}c\right)}{dc} \\ b_1^{(2)} - \left(a_1^{(2)} + 2a_2^{(2)}\right) d \\ b_2^{(2)} - a_2^{(2)} c \end{pmatrix} = \begin{pmatrix} a_1^{(1)} b_1^{(1)} (-d) \\ a_2^{(1)} \left(-2b_1^{(1)} d - b_2^{(1)} c\right) \\ 0 \\ 0 \end{pmatrix}. \tag{4.39}$$

Substituting in the values in from level-1 and solving results in

$$\begin{pmatrix} a_2^{(2)} \\ b_1^{(2)} \\ b_2^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{3b_2^{(2)}}{c} \\ \frac{b_2^{(2)}}{c} \\ -\frac{b_2^{(2)}d}{c} \end{pmatrix}, \tag{4.40}$$

where for this model we find $b_2^{(2)}$ as a second arbitrary function.

Level $k = 3$

$T^{(3)}$ is again singular, and we find that after substituting in the above results from $k = 0, 1, 2$, $T^{(3)} \cdot X^{(3)} = Y^{(3)}$ gives

$$\begin{pmatrix} -\frac{3\left(a_1^{(3)}c+2b_2^{(3)}\right)\xi'}{c} \\ \frac{\xi'\left(-3a_2^{(3)}dc+4b_2^{(3)}d+2b_1^{(3)}c\right)}{dc} \\ 2b_1^{(3)} - \left(a_1^{(3)} + 2a_2^{(3)}\right) d \\ 2b_2^{(3)} - a_2^{(3)} c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.41}$$

after subbing in the values from levels $k = 1, 2$ into the RHS. Solving for the expansion coefficients gives

$$\begin{pmatrix} a_1^{(3)} \\ b_1^{(3)} \\ b_2^{(3)} \end{pmatrix} = \begin{pmatrix} -a_2^{(3)} \\ \frac{1}{2} a_2^{(3)} d \\ \frac{1}{2} a_2^{(3)} c \end{pmatrix}, \tag{4.42}$$

where $a_2^{(3)}$ is now arbitrary. Hence, we have uncovered three arbitrary functions and as we can see from the determinant of $T^{(k)}$, it is not singular at any higher levels, so no more arbitrary functions will be introduced.

General Painlevé Expansion Solutions

Substituting the relations found for levels $k = 0, 1, 2, 3$ into equations (4.36) allows us to write down each term of the Painlevé expansion solutions power by power. We can do this up to arbitrarily high powers of ϕ, and present here up to ϕ^6:

$$\begin{aligned}P_1 =& -\frac{2\xi'}{d}\phi^{-1} - \frac{b_2^{(2)} d\phi}{c} + \frac{1}{2}a_2^{(3)} d\phi^2 + \frac{b_2^{(2)2} d\phi^3 \left(3c^2 - 2d^2\right)}{10c^2\xi'} + \frac{a_2^{(3)} b_2^{(2)} d\phi^4 \left(17c^2 - 14d^2\right)}{84c\xi'} \\ &+ \frac{d\phi^5 \left(5a_2^{(3)2} c^3 \left(41d^2 + 27c^2\right)\xi' + 12b_2^{(2)3}\left(25d^4 - 47d^2c^2 + 18c^4\right)\right)}{3360c^3\xi'^2} \\ &- \frac{a_2^{(3)} b_2^{(2)2} d\phi^6 \left(91d^4 + 262d^2c^2 - 299c^4\right)}{5040c^2\xi'^2} + ...\end{aligned} \tag{4.43}$$

Where we have taken terms up to ϕ^{10} but could generate terms up to arbitrary order. P_2 takes the form

$$\begin{aligned}P_2 =& \frac{2\xi'}{c}\phi^{-1} + b_2^{(2)}\phi + \frac{1}{2}a_2^{(3)} c\phi^2 + \frac{b_2^{(2)2}\phi^3\left(2c^2 - 3d^2\right)}{10c\xi'} + \frac{a_2^{(3)} b_2^{(2)}\phi^4\left(10c^2 - 7d^2\right)}{84\xi'} \\ &+ \frac{\phi^5\left(5a_2^{(3)2}c^3\left(29d^2 + 15c^2\right)\xi' + 12b_2^{(2)3}\left(17d^4 - 23d^2c^2 + 10c^4\right)\right)}{3360c^2\xi'^2} \\ &+ \frac{a_2^{(3)} b_2^{(2)2}\phi^6\left(-133d^4 - 82d^2c^2 + 161c^4\right)}{5040c\xi'^2} + ...\end{aligned} \tag{4.44}$$

For the Q_i's we present up to ϕ^5:

$$\begin{aligned}Q_1 =& \frac{6\xi'}{dc}\phi^{-2} + \frac{3b_2^{(2)}}{c} - a_2^{(3)}\phi - \frac{3b_2^{(2)2}\phi^2\left(c^2 - 4d^2\right)}{10c^2\xi'} - \frac{a_2^{(3)} b_2^{(2)} c\phi^3}{7\xi'} \\ &- \frac{\phi^4\left(5a_2^{(3)2}c^3\left(17d^2 + 3c^2\right)\xi' + 12b_2^{(2)3}\left(9d^4 + d^2c^2 + 2c^4\right)\right)}{672c^3\xi'^2} \\ &- \frac{a_2^{(3)} b_2^{(2)2}\phi^5\left(-175d^4 + 98d^2c^2 + 23c^4\right)}{840c^2\xi'^2} + ...\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}Q_2 =& -\frac{2\xi'}{dc\phi^2} + \frac{b_2^{(2)}}{c} + a_2^{(3)}\phi + \frac{3b_2^{(2)2}\phi^2\left(2c^2 - 3d^2\right)}{10c^2\xi'} + \frac{a_2^{(3)} b_2^{(2)}\phi^3\left(10c^2 - 7d^2\right)}{21c\xi'} \\ &+ \frac{\phi^4\left(5a_2^{(3)2}c^3\left(29d^2 + 15c^2\right)\xi' + 12b_2^{(2)3}\left(17d^4 - 23d^2c^2 + 10c^4\right)\right)}{672c^3\xi'^2} \\ &+ \frac{a_2^{(3)} b_2^{(2)2}\phi^5\left(-133d^4 - 82d^2c^2 + 161c^4\right)}{840c^2\xi'^2} + ...\end{aligned} \tag{4.46}$$

We can clearly see that to arbitrary order of expansion we may write our solutions in terms of $a_2^{(3)}, b_2^{(2)}$ and ξ, giving us meromorphic solutions, and providing strong evidence for the Painlevé property of this rank-2 null-root model. Hence, the class of rank-2 null-root models with Cartan matrices of the form (4.31) have strong evidence that they are integrable.

4.6 Summary

In this chapter, we examined a new class of rank-2 integrable Toda field theories based on Lorentzian root lattices, and showed their integrability through a Painlevé analysis. We started by discussing the limitations of the Painlevé test, emphasising that it only gives necessary but not sufficient conditions for integrability of a given model, and therefore is effective for its usage in chapter 3 for ruling out integrability, whereas for proving integrability more analysis must be conducted. Sufficient conditions are, however, given through possession of the Painlevé property, which we proved first for a known integrable model - the finite A_2 Toda theory - and then for the new class of rank-2 integrable Toda field theories.

Starting from the Painlevé test, we highlighted possible rank-2 Toda field theories, which satisfy the necessary condition of possessing enough positive integer resonances to pass the test. From this analysis, we recovered the Cartan matrices for all the expected finite Toda theories, but also uncovered several categories of theories that previously had not been examined. Of these categories, the *null root* models were non-trivial and did not decouple into any known Toda systems, so these were singled out as candidates to conduct further Painlevé analysis on.

As a proof of concept, we first showed that the integrable A_2 Toda theory possessed the Painlevé property, and continued to show that all the rank-2 null root Toda field theories based on Lorentzian root lattices possess the Painlevé property. This was shown through examination of the Painlevé equations at various levels of their recursion - in doing so, we showed that there existed enough arbitrary functions within the Painlevé equations, so that all solutions to an arbitrary high order of recursion depend only on these arbitrary functions. This meant that any singularities in the solutions for the A_2 Painlevé equations must not be movable, hence illustrating that they possess the Painlevé property by definition.

The null root models possessing the Painlevé property gives very strong indication of their integrability. We noted that there is the possibility of the Painlevé solutions diverging, so that sequences of singular manifolds, for which we have free solutions on, combining into a more complex singularity that we can no longer solve through the Painlevé expansion methods. In adjacent work [102], the authors eliminated the possibility of this divergence through directly showing the convergence of the solutions. However, as these authors noted, there are no known examples of this behaviour, and even contriving one appears to be counterintuitive - so at this stage we still conclude that the null root models are new integrable theories, and leave such additional analysis to future exploration as it will provide interest in its own right.

There is nothing special about the spin-3 or rank-2 nature of the null root models studied in this chapter. In fact, a simple calculation on the Painlevé matrix of higher rank models that also possess a null root in certain positions also pass the Painlevé test, and very likely will also possess the Painlevé property, and this is also true for higher spin models at rank-2 or above. Our choice of rank-2 was only illustrative, as the number of Painlevé expansion equations to solve, and constants within each, both increase with the rank, meaning such calculations are harder to present without more computational automation with this current method. However, as is the case with the finite and affine series of Lie algebras, well-defined patterns in the Cartan matrices of the null root models will lead to more classifications of potential integrable Toda field theories based on Lorentzian root lattices, but also new Kac-Moody like algebraic structure which can be studied independently of their field theory properties.

Chapter 5

Conclusion

5.1 Overview

Throughout this thesis, we have examined the integrability and other physical features of field theories that are built from Lorentzian Kac-Moody algebras. To do so, we have developed a framework to study a new class of Lorentzian Kac-Moody algebras, which we named the n-extended Lorentzian Kac-Moody algebras, $\mathfrak{g}_{-n}$. Utilizing aspects of the framework used to form $\mathfrak{g}_{-n}$, we built new classes of Lorentzian Toda field theories based on perturbed $\mathfrak{g}_{-n}$ structures, analysed a number of these theories physical features and, for some models, determined their non-integrability through the Painlevé test. Finally, we conducted more Painlevé analysis on rank-2 Toda field theories, and found that there were more options than those already known that passed the Painlevé test and therefore were likely to be integrable. To illustrate this, for one Lorentzian rank-2 spin-3 Toda field theory, we proved through the possession of the Painlevé property that this theory is integrable - demonstrating that the other theories in this new Lorentzian class of Toda field theories are likely to also be integrable.

The journey throughout this thesis began with an overview of the status and importance of field theories within physics, especially highlighting the historical successes of integrability and its uses in exactly solving the equations of motions for a given theory. We also highlighted the importance of Lorentzian Kac-Moody algebras in describing the symmetry groups of string theory, whereby the hyperbolic and Lorentzian Kac-Moody algebras E_{10} and E_{11}, respectively, play an essential role in string theories unification through M-theory.

Continuing from this motivation, we developed a framework to define the new class of Lorentzian Kac-Moody algebra, $\mathfrak{g}_{-n}$ mentioned above, through building from rank-r finite, $\mathfrak{g}_f$, and affine, $\mathfrak{g}_a$, Kac-Moody algebras and extending their simple roots, α_i for $i \in \{1, \cdots, r\}$ on top of the affine root α_0. In this way, we constructed Dynkin diagrams for $\mathfrak{g}_{-n}$ and corresponding root and weight lattices for these n-extended simple roots, $\alpha_i^{(n)}$ for $i \in \{-n, \cdots, 0, 1, \cdots, r\}$, with fundamental weights $\lambda_i^{(n)}$ - both of which we gave closed formulas for all orders of n, from any starting root or weight basis of $\mathfrak{g}_f$. We identified the set of constants $D_i^{(n)}$, which occurred naturally from summing over indices of the inverse Cartan matrix, but also occurred within the context of 3-dimensional principal subalgebras of $\mathfrak{g}_{-n}$. The $D_i^{(n)}$ constants led us to discover that certain $\mathfrak{g}_{-n}$ decompose into sub-$\mathfrak{g}_{-n}$ and $\mathfrak{g}_f$ algebras, through a novel and very natural decomposition procedure, and we gave an exhaustive list of all of these possible decompositions.

In chapter 3 we introduced various flavours of Toda field theories, and developed a method, based on the extension method from the previous chapter, to extend Toda field theories based on $\mathfrak{g}_f$ and $\mathfrak{g}_a$. This meant that the resulting extended theories had fields, ϕ_i, based on the Lorentzian root system $\mathfrak{g}_{-n}$, and perturbations around its corresponding simple roots, $\alpha_i^{(n)}$, again with $i \in \{-n, \cdots, 0, 1, \cdots, r\}$. We named these new models *Lorentzian Toda field theories*, and observed that the behaviour of these theories alternated between conformal and massive, on addition of each simple root in the extension procedure. Through use of the Painlevé test, we showed that these theories, conformal or massive, were not integrable. Focusing on the massive theories, we calculated mass ratios for several examples, observing that mass ratios for the finite parts of the extended theory can be maintained, a feature that we attributed to the integrability of the finite theory that we perturbed during the n-extension procedure.

Chapter 4 focused on using the Painlevé test and Painlevé property to find new integrable Toda field theories with simple roots dictating field content based on Lorentzian lattices. Focusing on rank-2 models, the Painlevé test was initially used to discover all categories of rank-2 Cartan matrices that had enough positive integer *resonances* to pass the test. From this, we recovered all the expected known integrable Toda field theories based on $\mathfrak{g}_f$, but also found some non-trivial unknown solutions that contained a simple root of 0 length. We named these models, *null root* theories, and they could only be understood naturally through their simple roots existing on a non-Euclidean lattice, which we chose to be Lorentzian for reasons discussed in chapters 2 and 3. Taking one rank-2 spin-3 null root model as an example, we showed that it not only passed the Painlevé test, but also possessed the Painlevé property, meaning this Lorentzian Toda field theory is an integrable model. We concluded by describing how this example could be generalised, and that other null root models of higher rank and spin are also very likely to be integrable in the sense of passing the Painlevé test and possessing the Painlevé property.

5.2 Outlook

There are a number of questions that remain unanswered in light of the results uncovered over the course of this thesis. A tough and pure mathematical question to answer regarding the n-extended Lorentzian Kac-Moody algebras, would be to explain why the decomposition occur on certain $\mathfrak{g}_{-n}$ but not others. Kac has some insight into this reason [103], in that the centre of certain constructions of Kac-Moody algebras is generated by the Casimir operator, potentially explaining why decomposition occurs when a $D_i = 0$.

However, to prove this in terms of theorem 2 in [103], we would need greater understanding of the Weyl group of the given $\mathfrak{g}_{-n}$, amongst other tools that still have the potential to be developed for Lorentzian Kac-Moody algebras in a more general setting.

From a more physical perspective, in the Lorentzian Toda field theories we did not analyse the features of the conformal Lorentzian models, which could already be studied further using known methods. For the conformal and massive Lorentzian Toda field theories, in a similar way to the finite and affine theories, an algebraically independent formulation could be developed to complement the simple root and fundamental weight construction explored in this thesis. Furthermore, standard calculations such as mass renormalizations for Lorentzian Toda theories, or study of the flows between models could be conducted. It would also be interesting to develop other field theories that use $\mathfrak{g}_{-n}$ algebras, such as Calogero-Moser-Sutherland models that also use the roots of a Kac-Moody algebra, but this issue is also non-trivial as it would again require the Lorentzian Weyl group which is currently only known in a few special examples, such as for AE_3, the extension of the modular group $\mathrm{PSL}(2, \mathbb{Z})$, and other similar modular group correspondences.

Regarding the integrable Lorentzian Toda models we discovered, it was already mentioned that convergence analysis could be conducted to further convince ourselves of their integrability - however, other methods such as Lax pairs, or use of the zero curvature condition may prove equally insightful, as they could potentially help in finding the status of integrability for higher rank and spin Lorentzian null root Toda field theories. Investigation into the W-algebras and associated W-currents of Lorentzian Toda field theories could also provide important insights into both the mathematics and physics. Of course, the other standard calculations mentioned above could also be carried out for the null root models as they were, or were proposed, for the other Lorentzian Toda field theories considered.

In conclusion, this thesis has substantially rooted a framework to study extended Lorentzian algebras in their own right, and in relations to integrable quantum field theories. There are many avenues for further study, and we hope the mathematical and physics communities will build upon these frameworks with us, to ultimately better understand the most fundamental questions that our joint fields aim to elucidate.

Appendix A

The Perron-Frobenius Correspondence

This appendix aims to answer some of the questions raised in chapter 2 regarding the following two points:

1. Why the representation of the simple roots, α_i, for a rank-r Kac-Moody algebra, in matrix form, M, must be a square matrix to form a coherent Toda field theory mass matrix, $(M^2)^{ab}$, and how this relates to the so-called Perron-Frobenius eigenvector.

2. Under what conditions the values of the Perron-Frobenius eigenvector can give the correct values of $(M^2)^{ab}$ for Lorentzian or Hyperbolic Toda field theories, in which non-square representations of the simple roots matrix, M, is used, and where the Kac labels, n_i, are less well-defined.

A.1 The Toda Mass Matrix

Starting with theorem (5.2) from [104], which follows from [61], then we may write the equation associated to this theorem as

$$\tilde{M} = \langle \alpha_i, e \rangle \langle \alpha_i, e^* \rangle. \tag{A.1}$$

This may be rewritten in our matrix representation of roots, $M_{i\alpha}$, from equation (3.39), as

$$\tilde{M} = \langle M_{i\alpha}, e_{\alpha\beta} \rangle (\langle M_{j\beta}, e^*_{\beta\gamma} \rangle)^T = M_{i\alpha} e_{\alpha\beta} e^*_{\beta\gamma} (M^T)_{\gamma j}, \tag{A.2}$$

where the indices $i, j \in \{1, \cdots, r\}$ and $\alpha, \beta, \gamma \in \{1, \cdots, N\}$, for a rank-$r$ algebra with an N-dimensional representation of the simple roots.

The step operators, e, in the Cartan-Weyl representation are also Weyl vectors and belong to the Cartan subalgebra in apposition [104], i.e. $e \in \mathfrak{h}'$, may be chosen most simply as[1]

$$e_{\alpha\beta}e^*_{\beta\gamma} = \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_{r+1} \end{pmatrix}, \tag{A.3}$$

with $n_{r+1} = 1$, and the rest are the usual Kac labels for the given $\mathfrak{g}$. Thus, for this initial section of discussion, we may only choose $\alpha, \beta, \gamma \in \{1, \cdots, r+1\}$, and hence we are limited to finding a square representation of $M_{i\alpha}$ meaning that if we put in the correct values for the affine root in the $(r+1)^{\text{th}}$ position of each simple root in an r-dimensional representation we reproduce the exact same formula for the mass matrix for the affine Toda field theory case [84].

In other words, this is the exact reason why the affine Toda mass matrix commonly presented, such as we give in equation (3.44) or as in [84], utilizes a r-dimensional representation of the roots to reproduce the results as explained in [61], and also links, through algebraically reasoning, the various representations of the Lagrangian written in the literature. Explaining why the $\mathfrak{g}_a$ Toda theories based on $\mathfrak{g}_f$ give mass spectrums with the same ratios of those values found in the Perron-Frobenius eigenvector of the Cartan matrix associated to the given $\mathfrak{g}_f$.

Now, to extend this idea beyond $\mathfrak{g}_f$ and $\mathfrak{g}_a$, if we reduce the limitation of taking the most canonical representation of ee^* as in equation (A.3), and instead take an N-dimensional representation of the simple roots, we may still replicate the mass matrix $(M^2)^{ab}$ we get for cases. We can test this against the result that such a mass matrix $\tilde{M} = (M^2)^{ab}$ reproduces the Perron-Frobenius eigenvector values as its eigenvalues, and show this exactly for certain cases through direct calculation. Therefore, the remaining discussion in A.2 does not assume an r-dimensional representation of the simple roots, and hence, does not assume a square representation of M.

A.2 The Correspondence

Taking Λ as a matrix with its rows composed of fundamental weights, λ_i of the Lie algebra, we may act on $(M^2)^{ab}$ as

$$\Lambda(M^2)^{ab}M^T, \tag{A.4}$$

and we find that the eigenvalues, denoted by the function γ, are

$$\gamma[\Lambda(M^2)^{ab}M^T] = \gamma[n_{aa}(K_{ij} + X_{0i})_{ab}], \tag{A.5}$$

where n_{aa} is a diagonal matrix composed of the Kac labels, n_i, and X_{0i} is a column matrix of zeros except for one column of ones in the column associated to the node of the finite

[1] See [104] for definitions of e on page 13.

Dynkin diagram to which the highest root attaches, forming the affine Dynkin diagram and associated Cartan matrix, $\tilde{K}_{ij}$, where now $i, j \in \{1, \cdots, r+1\}$.

It follows from the fact that $\tilde{K}_{ij}$ is the affine version of the matrix $[K_{ij} + X_{0i}]_{ab}$. Hence, we may write the simplified relation

$$\gamma[\Lambda(M^2)^{ab}M^T] = \gamma[\tilde{n}_{aa}(\tilde{K})_{ab}] - \{0\}, \tag{A.6}$$

where the $\{0\}$ represents the 0 eigenvalue, which we get as a result of $\tilde{K}$ having a zero determinant. This is immediately generalizable to Lorentzian Toda field theories, whereby we take $\tilde{K}_{ab}$ to be the Cartan matrix associated to the massive extended theory, such as that of the $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2}}$ theory derived in section 3.5.2, or any $\mathscr{L}_{\mathring{\mathfrak{g}}_{-2n}}$ theory through the same reasoning.

It is important to note that the correspondence (A.6) holds solidly for the finite cases only, in which the Kac labels are well-defined, along with their step operators e. We can however find specific representations analogies of step operators and Kac labels for hyperbolic and Lorentzian algebras, but the assumption must be made that we are expecting to reproduce the Perron-Frobenius eigenvector values in $(M^2)^{ab}$, otherwise we do not have anything to measure our theory against and values of Kac labels and all representations of step operators become arbitrary. For example, some of this ambiguity comes from not knowing what a *true* $\tilde{K}_{ab}$ version of such an algebra would look like, since the n-extension procedure is only one of many possible ways of extending, and thus forcing $\det \tilde{K} = 0$, the Lorentzian or hyperbolic Kac-Moody algebra.

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