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ALGEBROGEOMETRIC AND TOPOLOGICAL METHODS IN CONTROL THEORY

THESIS SUBMITTED
FOR THE AWARD OF THE Ph.D DEGREE
in MATHEMATICAL CONTROL THEORY

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CONTENTS

CHAPTER 5. THE DETERMINANTAL ASSIGNMENT PROBLEM: A Unifying Approach for Static and Dynamic Compensation

CHAPTER 6. THE POLE PLACEMENT MAP, ITS PROPERTIES AND RELATIONSHIP TO SYSTEM INVARIANTS

CHAPTER 7. POLE, ZERO ASSIGNMENT WITH STATIC REAL CONTROLLERS

CHAPTER 8. POLE ASSIGNMENT BY PI AND BDO CONTROLLERS

CHAPTER 9. DECENTRALISED POLE, ZERO ASSIGNMENT BY STATIC CONTROLLERS

CHAPTER 10. GLOBAL ASYMPTOTIC LINEARISATION OF THE POLE PLACEMENT MAP

ABSTRACT

The aim of this thesis is to provide a unifying framework and tools for the study of a number of Control Theory problems of the determinantal type. These problems are known as Frequency Assignment Problems (FRT) and they include the constant, dynamic, pole, zero assignment by centralised as well as decentralised output feedback and the zero assignment problems via squaring down. It has been shown [Kar.1],[Gia.2] that all such problems may be formulated under the unifying framework of the Determinantal Assignment Problem (DAP), and it can be studied using tools from exterior algebra and algebraic geometry. The main objective of this thesis is to develop further the DAP framework, unify it with other algebrogeometric approaches and develop issues related to computation and parametrisation of solutions when such solutions exist.

The natural setup for the study of solutions of the DAP framework has been the intersection theory of projective varieties. This has been extended by developing the topological properties of the pole, zero placement maps and introducing an equivalent formulation for real intersection based on cohomology theory. The properties of this map, with respect to standard system invariants are also established. This approach allows the derivation of new conditions for constant pole, zero assignment with centralised and decentralised controllers, using conditions based on the height of an appropriate cohomology class. Affine algebraic geometry methods are also used for the derivation of partial results for the dynamic case corresponding to PI and OBD controllers.

An entirely new approach for the study of solvability of DAP, as well as computation of solutions is introduced in terms of the notion of global linearisation of the corresponding pole, zero assignment map around a degenerate point. This is based on the special "blow up" property of the pole placement map at degenerate feedbacks and permits the reduction of the overall DAP to a globally linear problem, the solvability of which is defined by the properties of a new local invariant, the "blow up" matrix.

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NOTATIONS AND ABBREVIATIONS

Throughout this thesis, the following notations and abbreviations will be used:

$\mathbb{F}$	denotes a general field
$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$, $\mathbb{Z}_2$	denotes the field of real, complex numbers, rational functions, and integers modulo two respectively.
$\mathbb{R}[s]$	the ring of polynomials over $\mathbb{R}$.
$\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^n(s)$	denotes the n-dimensional vector spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$.
$G_p(\mathbb{F}^n)$	denotes the Grassmannian, that is the set of all p-dimensional subspaces in $\mathbb{F}^n$
h(p,m)	the height of the first Whitney class of $G_p(\mathbb{F}^{p+m})$
$H^*(X;\Lambda)$	the cohomology ring of X with coefficients in Λ
$\mathcal{V}$, V	denotes a vector space and its matrix representation, respectively.
$\wedge^p\mathcal{V}$	denotes the p-th exterior power of the vector space $\mathcal{V}$.
$C_p(V)$	denotes the p-th compound matrix of the matrix V.
col(V)	the vector formed by superimposing the columns of the matrix V
colspan(V)	the vector space spanned by the columns of matrix V
$\mathbb{P}^n(\mathbb{F})$	the n-dimensional projective space for the field $\mathbb{F}$
row(V)	the dual of col(V) ($= (\text{col}(V^T))^T$)
rowspan(V)	the vector space spanned by the rows of matrix V
vec(V)	the vector corresponding to the matrix V (either row(V) or col(V))
#	number
BIBO	Bounded-Input, Bounded-Output
CDAP	Constant Determinantal Assignment Problem
CPA	Constant Pole Assignment
CPPM	Complex Pole Placement Map
DAP	Determinantal pole assignment problem
det	determinant
dim	dimension
GRD, GLD	Greatest right and greatest left divisor respectively
FRT	Frequency Response Techniques

Im	Image
LKer	Left Kernel
LScat	Lusternik-Snirelmann category
MFD	Matrix Fraction Description
MIMO	many inputs-many outputs
mod	modulo
OBD	Observability index bounded dynamics
PI	Proportional plus Integral
PID	Principal ideal domain
QPR	Quadratic Plucker Relations
RQPR	Reduced Quadratic Plucker Relations
SISO	Single input, single output
SSD	State Space Description
vecat	vector bundle category

DECLARATION

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CHAPTER 1. Introduction

Control theory is the backbone to control system design, since it provides the conceptual framework 'concepts and tools' and the algorithms upon which control system design philosophies, strategies and techniques are based. Control theory and design techniques are well developed, especially in the context of linear systems, which are well defined as far as input-output structure and model parameters are concerned. The two main directions of activity have been the areas of synthesis and design. In a synthesis problem, the assumptions, criteria and objectives are well formulated, the solution satisfies exactly all present objectives and criteria and it is obtained in a closed form as the exact solution of the mathematical problem. The main characteristics of a design problem is that the solution satisfies a set of primary objectives, criteria and, in an approximate sense, a set of secondary objectives and criteria. Furthermore, the solution is not always in a closed form and it is usually a byproduct of a methodology involving iteration. Design problems are closer to the spirit of engineering design. However, synthesis problems are essential for two basic reasons: (i) they are instrumental in the development of advanced design techniques and (ii) the solvability condition of exact problems reveals the potential of the system to possess a good approximate solution to a certain design problem. The present thesis is in the area of synthesis problems, although the adopted framework has the potential to evolve into a design methodology. In particular, the main thrust of the work is in the characterisation of system properties which allow the solvability of certain families of problems referred to as 'frequency assignment problems' for linear systems. In this family, we consider problems of pole assignment by constant or dynamic control schemes and zero assignment under squaring down for systems which satisfy the centralised or decentralised assumption. The present study lies within the general area of linear systems. In the study of properties and problems of linear systems, a variety of approaches have been developed. The classification of the different approaches is based on the model which the approach uses, as well as the tools which are deployed. Linear systems have been under study for a long time and from several different view points.

The systematic study of the System and Control Theory had started in the early 1930's with the development of the classical frequency response techniques (Nyquist, Bode, Root locus approaches) for Single Input and Single Output (SISO) systems. Multivariable Systems and control problems started to become increasingly important in aerospace, process design etc. in the late 1930's. This fact, plus the importance of the time-domain analysis and characteristics in aerospace, led to a resurgence of interest, spearheaded by the work of Bellman and Kalman [Kal.1] in the early 1960's, on the state-space description of linear systems. This led, naturally,to more detailed examinations of the structure of linear systems and to questions of redundancy, minimality, controllability, observability etc. and has eventually evolved into an elegant approach for studying both system properties and synthesis problems, the geometric approach of Wonham, Willems etc. [Won.1],[Wil.3],[Bas.1]. Just as the state space approach was maturing Kalman [Kal.2], and Rosenbrock [R.5], had shown how many of the scalar transfer function concepts could extend to the multivariable case. Rosenbrock [R.5] has placed the foundations of the modern transfer function matrix (algebraic) approach, an offspring of which has been the modern Frequency Response Techniques (FRT), which for systems with many inputs and many outputs (MIMO) (Rosenbrock [R.1], MacFarlane [MacF.5], Doyle and Stein [Doy.2] etc.) have provided generalisations of the classical SISO FRT's. Recent developments in the algebraic theory of MIMO systems have led to the formulation of modern algebraic synthesis approaches (Kucera [Kuc.2], Vidyasagar [Vid.2]), a successor of which is the more recent frequency response approach referred to as $H\infty$-optimization [Fra.1],[Glo.1],[Doy.1],[Kwak.1]. The state space and transfer function descriptions are only two extremes of a whole spectrum of possible descriptions of finite-dimensional linear systems. Hybrid approaches such as the matrix pencil approach [MacF.4],[Jaf.1],[Lois.1] and the algebrogeometric approach [Broc.1],[Kar.1], aim at bridging the state space and algebraic cultures. The geometric, algebraic and algebrogeometric approaches have revealed one part of the system structure; that is the one connected to the system invariants. An alternative aspect of the system structure is that characterised by the interconnection graph. The structural, or graph approach [Shie.1],[Rein.2], has been developed to study the properties of state space models with fixed graph and generic values of the numerical parameter and provides an alternative characterisation of system structure. The work here, is in the area of hybrid approaches and in particular, the one referred to as algebrogeometric.

For the study of problems of linear synthesis which are of the determinantal type (such as pole zero assignment, stabilization) a specific school of thought has been developed which is specially suited to tackle such problems. This framework is referred to as algebrogeometric because it relies on tools from algebra and algebraic geometry. The essence of the problems faced in this set-up is that they are of a multilinear nature and the number of design parameters is not necessarily large. Early attempts to study such problems within the classical control theory framework has been based upon the linearisation of the multilinear problem by assuming special structure controllers such as the dyadic [Kai.1], [Won.1]. This approach has been successful in the case of state feedback but not so in the case of output feedback due to the limited number of parameters.

Alternative methodologies such as those based on the diophantine equation [Kuc.2],[Zag.1], which have been promising for state feedback studies, have not been very successful for determinantal output feedback problems. The main difficulty of the determinantal problems in the case of frequency assignment lies in that the problem is equivalent to finding real solutions to sets of nonlinear and linear equations; in the case of stabilization, this is equivalent to determining solutions of nonlinear equations and nonlinear inequalities. The first of the two problems naturally belongs to the intersection theory of complex algebraic varieties, whereas, the latter belongs to the intersection theory of semialgebraic sets [Boc.1]. Additional difficulties arise (and this makes the use of the above areas, not a straightforward off the shelf application) due to the requirements of determining existence of real solutions for both generic and exact formulations of the problem, as well as the need to study specific dynamic structure control schemes (centralised or decentralised) which make the varieties involved not standard, and the compactification issues quite prominent.

It should be noted that the real intersection theory is not well developed and although determining existence of solutions is very important. Also of paramount interest is the development of procedures and methodologies for computing such solutions whenever they exist. The main emphasis in algebraic geometry has been the general study of properties of varieties such as an intersection theory and the emphasis has been on generic properties depending on discrete invariants rather than continuous parameters. Furthermore, issues such as computation of solutions are hardly addressed in either classical [Hod.1] or modern [Har.1] algebraic geometry. The requirements for the development of a unifying algebrogeometric framework are: (1) it can handle different types of compensation (constant, dynamic, centralised, decentralised); (2) provide necessary as well as sufficient conditions for the existence of generic and exact problems; (3) determine a unifying computational framework of solutions whenever they exist and which is general enough (as far as methodologies and tools are concerned) to tackle the tougher issues on stabilization, approximate solutions and robustness.

Within the algebrogeometric framework that has emerged so far we distinguish two distinct directions:

(i) The affine algebraic geometry approach.
(ii) The projective algebraic geometry approach.

The first [Her.1],[Wil.1],[Bro.1], considers the plant and controller as elements of algebraic varieties of an affine space and studies the solvability of pole assignment, (by output feedback), simultaneous stabilisation etc. by using tools from algebraic geometry, like the dominant morphism theorem. Important conditions for generic solvability of control problems have been derived within this framework but a big disadvantage in this approach is that the nongeneric cases are difficult to handle and that no systematic procedure for computing the controllers, whenever they exist, are suggested (the approach is not constructive). Later on, it was recognised [Broc.1], [Byr.2],[Byr.1],[Gho.1] that the use of compact spaces was more appropriate for the consideration of the pole assignment problem. The Grassmann manifold was then considered to be the set parametrising all the constant multivariable controllers of fixed number of inputs and outputs, and this was achieved by introducing the set of controllers at infinity. The Grassmann manifold was found to be convenient for intersection considerations due to the fact that first, it was a natural compactification for the pole assignment problem and secondly, because there was already an intersection theory for this manifold [Sch.1] ie. the Schubert calculus. Although, with this approach, the existing results were reestablished, additional results for real intersection were given and a better insight to the problem was aquired, the method was still nonconstructive and was orientated towards generic solvability results. The Schubert calculus uses implicitly the so called Plucker embedding of the Grassmannian into an appropriate and suitable projective space. This embedding was first recognised and explicitly used for the pole and zero pole placement problems in [Kar.1],[Gia.1],[Gia.2],[Kar.2] and the overall method was called the *Determinantal Assignment Problem (DAP) Approach.* The DAP approach, as well as the Schubert calculus approach (implicitly), may be considered as projective approaches since the Grassmann manifold is viewed as a subvariety of a projective space via the Plucker embedding (a more refined classification of approaches will be presented in chapter 4).

The DAP approach [Kar.1] has been formulated as a unifying approach for all problems of frequency assignment (pole zero) and its basis lies on the fact that determinantal problems are of a multilinear nature and thus may be naturally split into a linear and multilinear problem (decomposability of multivectors). The final solution is thus reduced to the solvability of a set of linear equations (characterising the linear problem) together with quadratics (characterising the multilinear problem of decomposability). The approach heavily relies on exterior algebra and this has implications on the computability of solutions (reconstruction of solutions whenever they exist) and introduces new sets of invariants (of a projective character) which, in turn, characterise the solvability of the problem. The distinct advantages of the DAP approach are: it provides the means for computing the solutions; it can handle both generic and exact solvability investigations; and it introduces new criteria for the characterisation of solvability of different problems. The computation of solutions is reduced to an optimisation problem of a function with quadratic equality constraints [Gia.1],[Mit.1]. The development of such a technique is essential for the method to become a design technique for frequency assignment problems.

The main difficulty of the algebrogeometric approach is faced in the area of defining conditions for the existence of real solutions. In fact, the general framework of algebraic geometry and, in particular, intersection theory has been established for algebraically closed fields. The need for computing real controllers as solutions to the various problems forces us to consider the case of real intersection theory ie. the intersection theory over a field which is not algebraically closed and for which few results are known. It should be pointed out that the basic principle of intersection theory over algebraically closed fields is the ‘conservation of number of roots’ which , in simple terms, means that the degree of the polynomial determines the number of roots over that field (irrespective of what the coefficients are). In the real case however, such property does not hold true, but what is true is the preservation of the number of roots if it is taken modulo 2. The latter observation provides a strong motivation for the use of cohomology ring with coefficients in $\mathbb{Z}_2$ as the formal framework for real intersection theory and which will be used in this thesis.

The present thesis addresses a number of issues related to pole, zero assignment by constant or dynamic, centralised or decentralised controllers. The approach adopted here, is within the framework of the projective or DAP approach, but is more general with regard to the handling of issues and the use of tools, and in many respects this provides a bridge between the dominant tools used in the projective and the affine direction. In particular, it develops tools from algebraic topology such as cohomology rings, vector bundles, characteristic classes, category, for real intersection theory. It also explores the topological properties of the various types of pole, zero maps and links them with classical system invariant theory. Compactification issues are examined and this provides the passage from the affine to the projective and thus, links the two directions. The study of compactification has no unique solution but what really matters is the development of so called 'natural compactifications'. The study of compactification issues is strongly linked to the study of the characterisations of 'infinity' in the context of our problem and the latter is naturally connected to the study of problem degeneracy [Broc.1]. The study of problem degeneracy, together with the DAP formulation of the problem, provides the basis for a global asymptotic linearisation which enhances the computational suitability of DAP and provides new means for derivation of sufficient conditions. It should be pointed out here, that this alternative approach (global asymptotic linearization) differs considerably from the intersection theory approach dominating most parts of the thesis; in fact, this approach is closer in spirit to 'blow up' techniques [Gri.1] on varieties. In this sense, the work in this thesis, on the one hand, provides a bridge between the affine and the projective directions and on the other, further develops the analytic ability to establish solvability conditions as well as computational capabilities of the framework.

In this thesis we will examine both pole and zero assignment problems, concentrating more upon the former and using constant or dynamic compensators having either centralised or decentralised structure. It is also worth noting that although the main core of this thesis is the solution of frequency assignment problems, a great deal of effort was put into appropriately explaining and reformulating certain mathematical tools and methods which, for many, were considered to be abstract or written in a cryptic language. This thesis can essentially be separated into two parts. The first (Ch,2,3,4,5) concentrates upon the description of mathematical tools, the review of methods and results and the formulation of the problems whilst the second part (Ch.6,7,8,9,10) deals with the examination of the several problems addressed in this thesis. More analytically, the contents of the chapters of this thesis are as follows:

In chapter2 we provide some of the control related mathematical tools and notations. Initially, we briefly present several descriptions for linear systems followed by some basic algebraic theory of systems ie. properties of rational and polynomial matrices. Finally, we discuss some of the aspects of the general feedback configuration.

In chapter3 we present all the mathematical tools to be used in this thesis. The purpose of this chapter is to clarify with examples and explain in simple terms, whenever this is possible, all the key mathematical tools for the frequency assignment problems. We begin with the basic concepts of exterior algebra, which is the starting point for the examination of the DAP. We then proceed to the theory of complex and real varieties and then to the intersection theory of complex algebraic varieties, which is essential for the understanding of all intersection theoretic arguments of our thesis. Finally, we present topology and cohomology of manifolds, which is crucial in the understanding of all topological intersection arguments in this thesis.

Chapter4 deals with the review of methods and relevant results for our problem. Firstly, we examine some parametrisation issues of systems and construct a Plucker type embedding for systems. In this way, the set of systems of p-inputs, m-outputs and n-states becomes a smooth quasi projective variety of dimension n(m+p). Although this structure of system was already known, our result provides an explicit characterisation of the variety in terms of dynamic quadratic Plucker relations, and the embedding used couples nicely with the output feedback problem. This parametrisation of systems puts 'genericity' into a correct context and allows us to use the word 'generic system' in the framework of Zarisky topology, which naturally fits with the tools from algebraic geometry. Secondly, we present relevant approaches and methodologies for our problems and we classify them as: (i) state space and algebraic and (ii) geometric. The approaches we will be dealing with are the geometric, which are further classified into infinitesimal, enumerative geometry, topological intersection, combinatorial geometric and projective techniques. Finally, we present all the background results for zero and pole placement problems.

In chapter 5 we define and formulate all the problems to be examined in this thesis. The feedback pole placement problems are mentioned first, in particular using constant, PI and OBD (Observabilty bounded dynamics) controllers. Then we proceed to the zero placement via squaring down and finally we formulate the decentralised versions of the above problems. In this chapter we demonstrate that all our frequency assignment problems can be formulated as Constant Determinantal Assignment Problems (CDAP). Nevertheless, the application of methods of the constant output feedback problem to the dynamic cases is not straightforward since the individual problems have a special structure. Eventually, we bring the chapter to a close by describing, in very general terms, the way CPA problems can be attempted.

In chapter 6 we establish a number of properties of a very important map related to determinantal problems, namely the pole placement map, which is the multivariable analogue of the root locus map. In this chapter we investigate the pole placement map under complex and real output feedback and especially the properties of the image of the pole placement map as well as its asymptotic properties with respect to a high gain output feedback. The first problem addressed in chapter 6, is the derivation of a reasonable measure for the size of the set of polynomials ,which for a system S(A,B,C) of p-inputs, m-outputs and n-states can be assigned. We choose as a measure of the size of this set, the dimension of the image of the real, or the complex pole placement map (PPM). Although the structure of the image of the complex PPM is different than that of the real PPM (and in fact the complex case is nicer than the real), it is shown that both dimensions of the real and complex PPM (which are invariants of the system) are the same. The above dimensions are also shown to be equal to the rank of the differential of the corresponding PPM at a generic feedback K. The rank of this differential at K=0 was shown [Her.1] to be equal to the rank of F_0=[colCB ,colCAB, ... ,colCA^nB], (the 'col' operation on a matrix implies the formation of a composite vector, obtained by superimposing the columns of the matrix) or at an arbitrary K [Rein.1] to be equal to the rank of F_K=[colCB ,colCHB, ... ,colCH^nB], where H=A+BKC. The latter expression is not very convenient for the calculation of the rank at a generic K ; instead we propose an alternative expression of the form $(DT)_K\bar{P}_S$ where T is a function of K only and $\bar{P}_S$ is the reduced Plucker matrix of the system S and which is a complete invariant [Gia.2],[Kar.1].The relationship between the reduced Plucker matrix and the Markov parameters is established; in fact, it is shown that the

Markov parameters may be computed by selecting certain rows of the Plucker matrix. It is shown that the rank of the Plucker matrix provides us with an upper bound for both the dimensions of the image of the complex and real PPM as well as an upper bound for the set $\{\mathrm{rank}F_K\}$. As a result of the above properties , necessary tests for the pole assignability of a system S(A,B,C) are derived.

An integral part of the study of the root locus map (pole placement map) under general real output feedback K, is the study of the location of the closed loop poles when K becomes unbounded, and this is the next topic to be examined in chapter 6. Given that in the expression of the pole placement map the compensator enters in a composite form [I,K], it is essential to have a representation of this form for the compensator when K is unbounded. It has been accepted by many researchers [Broc.1],[Gho.1] that unbounded gains correspond to [A,B] representations where det(A)=0; this fact is rigorously proven here first and is used for the study of asymptotic properties of the pole placement map. Then, we examine the concept of degeneracy of a MIMO system (the systems for which the pole placement map cannot be extended at infinity), for real gains ($\mathbb{R}$-degeneracy) and we show that the problem: "a system to be degenerate" is an intersection problem of algebraic geometry. In the end, we find a sufficient condition for a generic system to be $\mathbb{R}$-degenerate by using basic theory of vector bundles [Osb.1],[Jam.1]; this condition is $\mathrm{vecat}(o(\gamma_m^p)) > n+1$ where the left hand expression denotes the vector category of the orientation bundle of γ_m^p. This condition is not easy to test and a weaker, but testable condition, is derived here - this is of the form $h(p,m) > n$ where h(p,m) is the height of the first Whitney class of $G_p(\mathbb{R}^{p+m})$, for which a computation procedure is also given.

In chapter7 we examine solvability conditions for pole assignment via constant output feedback and zero assignment via constant squaring down. The first problem we need to resolve is the compactification of the set of constant controllers. This compactification has to be done in such a way that certain intersection theoretic requirements are met. We prove that for our purposes the correct compactification for both pole and zero assignment is a Grassmannian. Secondly, we translate our intersection problem into an inequality involving the height of the first Whitney class of the Grassmannian $G_p(\mathbb{R}^{p+m})$, denoted by h(p,m). Specifically, we produce a sufficient condition for the existence of real constant output feedback shifting any set of symmetric poles of a generic system of p-inputs, m-outputs, n-states. This condition is given by the inequality $h(p,m) \geq n$ and its origin can be found in the sufficient condition for the same problem given in [Gia.1] which involves a computation of large numbers of factorials. For a given triple (p,m,n) , the condition $h(p,m) \geq n$ where h(p,m) is the height of the first Whitney class of a Grassmannian, is equivalent to the factorial condition given in [Gia.1];however the present test is much more easy to apply and may be used for the characterization of the (p,m,n) triples for which the problem is solvable generically. A similar nature condition has been given in [Byr.1] for the arbitrary pole assignment of real poles only in terms of $LScat(p,m) \geq n$, where LScat denotes the Ljusternic-Snirelman category of the same Grassmannian. Although $LScat(p,m) \geq h(p,m)$, our result is stronger than the $LScat(p,m) \geq n$ since it is proven for arbitrary sets of poles and not just real poles. It is worth pointing out that LScat(p,m) cannot be easily calculated and in most cases h(p,m) is used as its approximation. It is worth noting that our result includes the condition $m+p-1 \geq n$ given in [Kim.1] and [Dav.1] which is a sufficient condition for the existence of a dyadic feedback controller shifting arbitrarily any set of poles.

As far as the zero placement is concerned, the condition we produce in Ch.7 is again sufficient and involves the height of the Whitney class of the Grassmannian connected with the zero placement problem. It is proven that for a generic system of p-inputs, m-outputs which has Forney dynamical order δ, a sufficient condition for the existence of a real squaring down compensator arbitrarily shifting the zeros is given by the inequality $h(p,m-p) \geq \delta$. This is equivalent to the result given in [Kar.3] which involves a calculation of a large number of factorials. Our new result is compact and can be tested much more easily. Other results were given in [Sab.1] using state space methodology, but the squaring down compensators they construct are dynamic of a rather large degree. It is important to mention that the height h(p,m) has been calculated (for almost all p,m) by Hiller in [Hil.1], and a relatively simple formula for this can be found in [Sto.1] .This formula is given at the end of this chapter and may be used for the testing of the sufficient condition for arbitrary pole zero assignment. Finally, we prove that the height approach for both pole and zero placement is the best that we can achieve, if we consider odd degree intersections in the Grassmannian.

In chapter8, pole assignment via PI and OBD dynamic controllers is studied. First, the problems are transformed into CPAP framework and the corresponding controller spaces (PI and OBD) are compactified as Grassmannians. This allows us to decompose the problem as a linear and a standard multilinear problem and view the solutions as the zeros of certain linear and quadratic equations and, hence, permits us to use all the computational machinery of the constant output feedback problem. Additionally, we derive necessary conditions for the existence of solutions. These are of the form: (i) $2mp \geq n+p$ and the P_{pi} matrix to have full rank (for the PI case) and (ii) $mp+n_1(m+p) \geq n+n_1$ and the P_{obd} matrix to have full rank (for the OBD case) and furthermore, we prove that the Plucker matrices P_{pi} and P_{obd}, have generically full rank. The derivation of sufficient conditions is a more difficult task and the Grassmannian compactification is not appropriate. Sufficient conditions for the generic solvability of solely the complex case are worked out for both PI and OBD cases. In fact, for the PI case it is shown that the conditions a) $2mp \geq n+p$ and b) where there exists <u>one</u> polynomial that can be assigned to a generic system via complex PI controller, are sufficient generic solvability. The conditions for the case of complex dynamics (for the OBD case) become a) $mp+n_1(m+p) \geq n+n_1$ and b) there exist one polynomial that can be assigned, via a complex n_1 degree OBD controller, to a generic system. An alternative formulation for the study of general dynamic compensation problems which may be applied to these specific cases considered here, is presented in chapter10.

Although the design of single input single output (SISO) PI controllers has been well addressed [Mor.1] (as far as the tuning the parameters using various rules is concerned), the potential of the multivariable PI controllers for solving problems such as pole asignment and stabilisation, has received little attention. Previous attempts to address the pole assignment by PI controllers [Ser.1],[Mun.1],[Youn.1] have been based on the reduction of the problem to an equivalent pole assignment by output feedback on an augmented system; within this framework, state space output feedback tools (like dyadic design) have been used to design the PI controller, whenever some of the sufficient conditions are satisfied. In fact, sufficient conditions via state space methods are very weak since the effort in transforming the problem into a linear one, reduces the number of free parameters of the controller significantly. As an example we mention the sufficient condition given in [Mun.2] $2m+p-1 \geq n+p$ which is considerably weaker than the condition given here $2mp \geq n+p$. Regarding the more general dynamic pole placement, the use of state space or algebraic techniques [Bra.1], [Mun.2],[Chen.1] led to a reduction of the number of parameters of the controller or to a simplification of the structure of the problem, and as a result of these methods, it was possible to attain only weak necessary conditions for fixed degree solutions or to construct very high degree controllers. Our result for OBD controllers $mp+n_1(m+p) \geq n+n_1$ is the best possible result one can have for generic dynamic pole placement (as it was claimed in [Wil.1]). Nevertheless, the conditions for PI and OBD pole placement are proved here only for complex and not for real controllers and the approach is non constructive.

Finally in chapter 9 we examine the pole and zero assignment under the decentralisation assumption. The decentralization assumption implies a partially fixed structure of compensators and this results in the emergence of the following two phenomena. Initially, we have the appearance of the concept of fixed modes [Dav.1],[And.1],[Vid.1]] which may arise in the study of pole assignment by decentralised state, or output feedback and may restrict the assignability property. Secondly, the decentralised controllers may be viewed as a subvariety of a Grassmann variety [Kar.4] and thus its topology and intersection theory is not well established. This subvariety is characterised by the set of Quadratic Plucker Relations and a set of fixed zeros defined by the decentralization characteristic of the given problem [Kar.4]. An alternative compactification was recently introduced in [Wang.2], where the decentralised compensator is viewed as an element of product of Grassmannians. In this chapter we extend the algebrogeometric framework for decentralized problems established in [Wang.2] as well as the framework intoduced in [Kar.4], and derive new sufficient conditions for generic pole assignability. Furthermore, the properties of the pole placement map established in Chapter 6 are extended to the decentralised case and this leads to a new test for avoiding the presence of fixed modes using the notion of decentralised Markov parameters.

The results derived within the exterior algebra and algebrogeometric framework concern the exact as well as the generic solvability conditions. In particular, the use of the decentralised Grassmann representative and associated Plucker matrix [Kar.4] has provided criteria for the characterisation of fixed modes, almost fixed modes and necessary conditions for exact assignment based on the rank of the Plucker matrix. Necessary conditions for generic pole assignability by a decentralised controller was shown to be $\Sigma m_i p_i \geq n$ [Lai.1]. Using the product of Grassmannians framework, it has been recently shown that if the order of the product Grassmannians is odd, then $\Sigma m_i p_i \geq n$ is sufficient condition for generic pole assignability by decentralised real output feedback [Wang.2]. An alternative condition was also derived in [Wang.2], where it was shown that $\Sigma m_i p_i > n$ implies generic pole assignability, when either the number of all inputs or the number of all outputs are equal.

The main aim of this chapter is to extend the previous results by considering the case where $\Sigma m_i p_i \geq n$ and assuming that the m_i, p_i are general with no particular relationship between them. Our approach is similar to that developed in [Gia.1] for the centralised case; in fact, we are looking for odd order subvarieties of the product Grassmannian, which intersect the generic linear space. In this setting, this problem can be reduced to finding the height of a particular cohomology class w of the above product variety. The final result is shown to be that a sufficient condition for generic pole assignment by real decentralised output feedback is of the form $h(w) \geq n$ where $h(w)$ is the height of the class w. The computation of $h(w)$ is also considered and this leads to parametrization of m_i, p_i which guarantees solvability of the problem. The present result is a generalization of the sufficient condition $\Sigma m_i p_i \geq n$ and odd order product, given in [Wang.2]. Our result not only covers the case where the order of the product variety is odd, but also when it is even but there exist lower dimension subvarieties of odd order. However, in the case where the m_i's or p_i's are equal, our result is weaker than the second of the results in [Wang.2]; however, our approach provides new criteria for the cases not covered by the equality of m_i or p_i. Some recent results on the properties of the complex and real pole placement map of chapter6 are extended to the case of decentralised feedback. A new expression of the differential of the decentralised pole placement map allows the derivation of links between the decentralised Plucker matrix and the Markov parameters of the system. A new sufficient condition for avoiding fixed modes is established in terms of a special subset defined from Markov parameters and the decentralisation scheme which is refered to as the Decentralised Markov Parameters. To conclude, the same compactification ie. product of Grassmannians is used for the decentralised squaring down zero placement and the sufficient condition for the real solutions is given by $h(w) \geq \delta$ where w is an appropriate cohomology class and δ is the Forney degree of the plant.

In chapter 10, we adopt an alternative approach for the study of static and dynamic pole placement controllers, which deviates from the previous intersection based philosophy. Extensive use of degenerate solutions to pole placement problem is made to define special sequences of compensators which in the limit converge to the degenerate compensator. The essence of this approach is that the corresponding pole placement map is linearized asymptotically and this reduces the overall solvability to a linear problem. Note that the essential part in this framework is the determination of rank of certain matrices referred to as 'blow up matrices', related to the Plucker matrix and the specific selection of the degenerate point. The approach allows the derivation of sufficient conditions for the existence of real solutions, constant or dynamic, and introduces a systematic computational methodology for defining families of solutions parametrised by the selected degenerate point. The special matrix (blow up matrix) characterising the solvability of the reduced linear problem, has rank properties characterising the size or dimension of the 'blow up' at a particular degenerate point. Solvability of the constant or dynamic problem, is achieved for these degenerate points for which the dimension of the 'blow up' (defined by the rank of the blow up matrix) equals the degree of the closed loop polynomial to be assigned. An integral part of this philosophy is the characterisation of families of systems for which, for the generic degenerate point, we have complete blow up of the pole placement map (in this case, the problem is solvable).

This new methodology is clearly sufficient but it is proved to be quite powerful for large families of systems and in fact, the results derived on the one hand, cover all previous results and introduces new important ones as well as a unifying algorithmic computational procedure for workong out families of solutions. In fact, for the case of constant output, the condition $mp>n$ is shown to be sufficient for the generic solvability of the problem with real controllers and it is extended to the case of dynamic pole placement with n_1 order controllers to that $n_1(m+p)+m>n+n_1$ which was conjectured in [Wil.1] but has been an open question so far.

The latter result provides also the means for computing the generic least degree of arbitrary pole assigning compensators and it is shown to be the least n_1 satisfying the above inequality. It is worth noting that the blow up approach does not cover the boundary cases where $mp=n$ or $n_1(m+p)+m=n+n_1$. However, these specific cases have been tackled within the intersection theory framework as it has been shown in [Broc.1]. The final chapter the 'conclusions' contains a summary of what was achieved, a brief critique of the results and what can be further accomplished by our methods.

Although in this thesis we have merely dealt with frequency assignment problems, we have named it 'Algebrogeometric and Topological methods in control theory' not only because the frequency assignment problems are very central in control theory, but also on account of believing that our methods can be extended to other nonlinear problems in control. Finally, we wish that our methods can be a starting point for the introduction of algebrogeometric and topological methods in fields of control theory where conventional methods are unsatisfactory.

CHAPTER 2. Background System Theory

2.1 Introduction

The aim of this chapter is to set the scene for the control engineering part of the theory that will develop in this thesis and to provide some control related mathematical tools and notations. This chapter does not inspire to be a review of background control theoretic results but rather a brief summary with definitions, fundamental concepts and properties. A more detailed exposition of the background topics is given in the listed references.

In particular, we briefly present several descriptions for linear systems in section 2.2. From these, the ones we will be using in this thesis are state space models and matrix fraction descriptions (MFD) of transfer function models. It is worth noting that since the formulation of our problems (see chapter 5) is in terms of composite representations of MFD's our results may be extended into the more general behavioural approach of systems given in [Will. 2]. In section 2.3 we will briefly present some basic algebraic theory of systems based on the study of properties of rational and polynomial matrices and in section 2.4 we demonstrate some of the aspects of the general feedback configuration. Finally, in section 2.5, we briefly introduce the general problems addressed in this thesis. A proper discussion and formulation of these problems, as well as a review of background literature, is given in subsequent chapters.

2.2 Systems Descriptions and invariants

2.2.1 Introduction

Our study assumes linear time invariant systems of the regular state space type ; however, the present approach can also be applied to singular systems [Lew.1], but such cases are not examined here. We briefly review next, the two main families of models :- the internal and external (input-output) models.

2.2.2 Internal Models

The basic two families of internal models for linear systems for linear systems are the state-space and the polynomial models and these are discussed next.

(i) State Space Models

A state space model for a linear time invariant system is described as:

$$S(A,B,C,D): \begin{cases} \dot{\underline{x}}=A\underline{x} + B\underline{u} & (2.1a) \\ \underline{y}=C\underline{x} + D\underline{u} & (2.1b) \end{cases}$$

where $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{nxp}$, $C \in \mathbb{R}^{mxn}$, $D \in \mathbb{R}^{mxp}$, and $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^p$ are real-value vector functions. 'A' is called the *internal dynamics matrix* and its properties stem from the natural dynamic characteristics of the system. The matrices B, C are called *input-, output- matrices* respectively and they express the coupling of input, output variables $\underline{u}$, $\underline{y}$ to the internal variables $\underline{x}$, known as states; thus B, C represent the cummulative effect of selecting actuators (B matrix) and sensors (C matrix) for the system; because of the latter property, we may also refer to B as the *actuator matrix* and to C as the *sensor matrix* [McF.5]. The internal variables of this model are the states $\underline{x}$ and its derivatives $\dot{\underline{x}}$.

A state space model with D constant is called *proper* or *causal*, whereas if D=0, it is called *strictly proper*. If in (2.1b) D is not a constant but a polynomial matrix in $\tau= d(\cdot)/dt$, then the system is called *nonproper* and it admits a state space realization of the singular type ie. $E\dot{\underline{x}}=A\underline{x}+B\underline{u}$, $\underline{y}=C\underline{x}$ where det(E)=0 [Lew.1]. We shall refer to the models of the S(A,B,C,D) type as regular state space models.

The number of states n in the S(A,B,C,D) model is defined as its *order*. It is always assumed that the measurements and actuation variables are independent and thus $\rho(B)=p$, $\rho(C)=m$, as well as that $m \leq n$, $p \leq n$. If $\tau= d(\cdot)/dt$ denotes the derivative operator, the description (2.1) may be expressed as

$$\begin{bmatrix} \tau I\text{-}A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ -\underline{y} \end{bmatrix} \Longleftrightarrow P(\tau)\ \xi(t) = \begin{bmatrix} \underline{0} \\ -\underline{y} \end{bmatrix} \tag{2.2}$$

where $P(\tau)$ is a special type of polynomial matrix, reffered to as the *system matrix pencil* [R.5] [McF.3]. Matrix pencils [Gan.1], appear as simple linear operators of the type sF-G, F, $G \in \mathbb{R}^{qxk}$, where s is an indeterminate frequently representing the Leplace transform variable, and are naturally associated with state-space type problems. The theory of the structure and invariants of state-space models is described by the structural characteristics of appropriate matrix pencils. Matrix pencil theory [Gan.1], is intimately related to the generalised eigenvalue-eigenvector problem [Wilk.,2] and is thus central to state space computations. Systems for which $\rho(P(\lambda))=\min\{n+m, n+1\}$ for almost all values of the complex number λ will be called, *structurally nondegenerate* otherwise, they will be called *structurally degenerate.*

(ii) Polynomial Models

The state space description of a linear system assumes that the system is described in terms of first order differential equations; however, this is not the most general internal description for linear systems. For a number of processes, the most natural description is that defined by the general differential system [R.5],[Cal.1],[Var.3].

$$\Sigma: \begin{cases} A(\tau)\underline{v}(t) = B(\tau)\underline{u}(t) \\ \underline{y}(t) = C(\tau)\underline{v}(t) + D(\tau)\underline{u}(t) \end{cases} \qquad \tau=\frac{d}{dt} \tag{2.3}$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$ are polynomial matrices in τ [Gan.1] of dimension qxq, qxp, mxq respectively, and $\underline{v}(t)$ is a vector valued function with values in $\mathbb{R}^n$ known as *pseudo-state vector* [Cal.1]. The above description is known as *Polynomial Model Description* (PMD) and may also be represented as

$$\begin{bmatrix} A(\tau) & B(\tau) \\ -C(\tau) & -D(\tau) \end{bmatrix} \begin{bmatrix} \underline{v}(t) \\ \underline{u}(t) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ -\underline{y}(t) \end{bmatrix} \Longleftrightarrow T(\tau)\ \underline{x}(t) = \begin{bmatrix} \underline{0} \\ -\underline{y}(t) \end{bmatrix} \tag{2.4}$$

where T(τ) is known as the *Rosenbrock's System matrix* [R.5]. The relationship between PMDs and state models is extensively treated in [R.5],[Pug.1],[Hay.1],[Schum.1].

2.2.3. External models

The two basic families of external (input-output) models are the time-domain (convolution) and frequency domain (transfer function) models and are briefly presented.

(i) Convolution Models

The time domain, input-output description of linear, causal, time invariant systems, which are assumed to be initially relaxed, gives a mathematical description between the input and output vectors, which is expressed as:

$$\underline{y}(t) = \int_0^t G(t-T)\underline{u}(T)dT = \int_0^t G(T)\underline{u}(t\text{-}T)dT \tag{2.5}$$

where t=0 is the initial time and G(t) is an mxp matrix-valued function defined for $t \geq 0$. The integral in (2.5) is known as a convolution integral and the matrix G(t) as an *impulse response matrix*. For a proper state space model S(A,B,C,D) the impulse response matrix is expressed by

$$G(t)= Ce^{At}B + D\delta(t) \tag{2.6}$$

where $\delta(t)$ is the Dirac impulse. The above description is called a *convolution desription.*

(ii) Transfer Function Matrix Models

For systems which are describable by convolution integrals, it is of great advantage to use Laplace Transform, because it will change a convolution integral in the time domain into an algebraic product in the frequaency domain. Thus, by letting $\underline{y}(s)$, $\underline{u}(s)$ be the Laplace Transforms of $\underline{y}(t)$, $\underline{u}(t)$ vector functions then the convolution description (2.5) becomes

$$y(s)= G(s)\, u(s) \tag{2.7}$$

where

$$G(s)= \int_0^\infty G(t)e^{-st}dt \tag{2.8}$$

The matrix G(s) , defined as the Laplace Transform, of the impulse response matrix is called the *system transfer function* and (2.7) defines a *transfer function matrix model.* Whenever a transfer function is used, the system is always assumed to be relaxed at t=0.

A rational matrix $G(s) \in \mathbb{R}^{mxp}[s]$ is said to *proper* if $G(\infty)$ is a finite constant matrix and *strictly proper* if $G(\infty)=0$; otherwise, if some of the elements in the $G(\infty)$ matrix are infinity it will be called *non proper* . The set of proper rational functions is denoted by $\mathbb{R}_{pr}(s)$. For a state space model S(A,B,C,D) the transfer function matrix is given by

$$G(s)=C(sI-A)^{-1}B+D \tag{2.9}$$

which is a transfer function matrix. For every $G(s) \in \mathbb{R}_{prf}(s)^{mxp}$ there always exists a state space model S(A,B,C,D) for which (2.9) holds true; such state space models are called *realisations* of G(s) and are not uniquely defined. A realisation of G(s) with the least possible order is called *minimal realisation* and this order is called the *MacMillan degree* of G(s) [Kai.1].

Certain factorisation, of transfer functions, which provide alternative representations of the system are the polynomial and rational fractional representations [Vid.2],[Var.3]: such representations are crucial in many of the model control synthesis-design approaches, as well as the $\mathcal{H}_\infty$ approach for Control Design, [Glo.1],[Doy.1],[McF.1]. The polynomial fraction representation is briefly presented next.

(iii) Polynomial Matrix Fraction Description

If $\mathbb{R}[s]$ is the set of polynomials (ring) in s variable and with real coefficients then a rational function $g(s) \in \mathbb{R}(s)$ may be expressed as $g(s)=n(s)/d(s)$, where $n(s) \in \mathbb{R}[s]$ is the numerator and $d(s) \in \mathbb{R}[s]$ is the denominator, ie.

$$g(s)=n(s)\ d(s)^{-1} \tag{2.10}$$

Such a representation of g(s) is called a *polynomial fractional description* ($\mathbb{R}[s]-FD$). Such a description is called *coprime* $\mathbb{R}[s]-FD$ if n(s), d(s) have no common zeros. If $G(s) \in \mathbb{R}(s)^{mxp}$, then it may be also represented as

$$G(s)= N_R(s)\ D_R(s)^{-1}= D_L(s)^{-1}\ N_L(s) \tag{2.11}$$

where $N_R(s)$, $N_L(s) \in \mathbb{R}[s]^{mxp}$, $D_R(s) \in \mathbb{R}[s]^{pxp}$, $D_L(s) \in \mathbb{R}[s]^{mxm}$, with $\det(D_L(s))$, $\det(D_R(s)) \neq 0$. $N_R(s)$, $D_R(s)$, $N_L(s)$, $D_L(s)$ are known [Kail.,1] as $\mathbb{R}[s]$-*Right Matrix Fraction Descriptions* ($\mathbb{R}[s]-R-MFD$) and R[s]-*Left-Matrix Fraction Descriptions* ($\mathbb{R}[s]-L-MFD$), respectively. Every transfer function has $\mathbb{R}[s]-R-MFDs$ and $\mathbb{R}[s]-L-MFDs$ and such descriptions are not uniquely defined. If $G(s)=N_R(s)\ D_R(s)^{-1}= D_L(s)^{-1}N_L(s)$, then $\deg\{\det(D_R(s))\}$, $\deg\{\det(D_L(s))\}$ is defined as the *order* of the R-MFD, L-MFD respectively. A R-MFD, or L-MFD is called *irreducible* if $\deg\{\det(D_R(s))\}$, $\deg\{\det(D_L(s))\}$ is minimal amongst all other MFDs. For all irreducible MFDs (left or right), of proper transfer functions we have [Kail.1]:

$$\min\ \{\deg\{\det(D_R(s))\}\} = \min\{\deg\{\det(D_L(s))\}\} = \delta_m(G(s)) \tag{2.12}$$

Irreducible MFD's are not uniquely defined, but they all provide equivalent minimal representations of G(s). The theory of MFD's is quite rich and plays a key role on the development of the modern algebraic approaches for the analysis and synthesis of multivariable control systems.

2.2.4 System Invariants.

System invariants are functions defined on the model, which remain the same under certain types of transformations; thus, they characterise not only a single model but a whole family (equivalence class). Let $\mathcal{M}$ be a family of linear models, E an equivalence relation defined on $\mathcal{M}$, $E(M)$ the equivalence class of $M \in \mathcal{M}$ and let $\mathcal{M}/E$ be quotient set of orbit (set of all equivalence classes). We may define [MacL.1]:

Definition (2.1) Let $\mathcal{M}$ be a family of models, I a set and E an equivalence relation defined on $\mathcal{M}$.

(i) A function $f:\mathcal{M} \to I$ is called an *invariant* of E, when $M_1 E M_2$ implies $f(M_1)=f(M_2)$. Also, f is called a *complete invariant* for E, when $f(M_1)=f(M_2)$ implies $M_1 E M_2$.

(ii) A set of invariants $\{f_i: f_i:\mathcal{M} \to I_i, \quad i=1,2,...,k\}$ is a *complete set* for E on $\mathcal{M}$, if the map f defined by

$$f: M \to \mathop{\times}_{i=1}^{k} I_i: M \to f(M) = \{f_1(M),...,f_k(M)\} \tag{2.13}$$

is a complete invariant for E on X. The complete set of invariants is called *independent* if there is no subset which is also complete.

□

Note that a complete invariant defines a one-to-one correspondence between $E(M)$ equivalence classes and the image of f in I. The notion of independence is essential in the minimal parametrisation of $E(M)$ by invariants. An important issue for system identification and control analysis is that of the canonical form for $E(M)$.

If $f: M \to \mathop{\times}_{i=1}^{k} I_i$ is a complete and independent invariant for E on $\mathcal{M}$, by specialising the invariant f such that its image C is in $\mathcal{M}$ we define a *canonical element* or *canonical form.*

□

Definition (2.2) A set of *canonical forms*, C for E equivalence on $\mathcal{M}$, is a subset of $\mathcal{M}$ such that for every $M \in \mathcal{M}$ there exists a unique $C \in C$ for which MEC.

□

Canonical forms are uniquely defined elements of $\mathcal{M}$, which have the simplest possible structure (least number of parameters) and which describe the invariant in terms of a simple model. Canonical forms are often used as analysis tools and desribe the simplest possible type of model that may be defined under the set of tranformations defining the equivalence relation. The set of canonical forms provides a system of canonical distinct representatives for $\mathcal{M}/E$.

In the case of systems, different types of invariants are introduced for equivalence classes defined by transformation groups. An account of these different types of invariants may be found in [Kai.1],[Kar.7].

2.3 Basic Algebraic theory

In this section, we briefly review some of the basics from the theory of polynomial and rational matrices which are essential for our present studies

2.3.1 Polynomial Matrices: General Properties

Some of the basic properties on polynomial matrices related to invariants and canonical forms under different types of unimodular equivalence are summarised first below. Note that most of these properties, also hold true over any Principal Ideal Domain (PID).

Definition (2.3) A non-singular square polynomial matrix $U(s) \in \mathbb{R}^{q \times q}[s]$ whose determinant is not a function of s is called *unimodular matrix* (i.e. det $U(s) = c \in \mathbb{R}-\{0\}$). □

Note that unimodular matrices represent products of elementary row, column operations on polynomial matrices. In fact, post-multiplication by a unimodular matrix corresponds to products of elementary column operations, while pre-multiplication is equivalent to products of elementary row operations. By elementary operations we can reduce polynomial matrices to several "canonical" forms.

Theorem (2.1) *Column Hermite Form* [Kai, 1]: Any polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$, $\rho\{M(s)\} = t$ with $t \leq \min\{p,q\}$ can be reduced by elementary row operations (i.e. by premultiplication by a unimodular matrix) to a (lower or upper) quasi-triangular form in which

(i) the last p−t rows are identically zero;

(ii) in column j, $1 \leq j \leq t$, the diagonal element is monic and of higher degree that any (non-zero) element above it;

(iii) in column j, $1 \leq j \leq t$, if the diagonal element is unity, then all elements above it are zero; and

(iv) no particular statements can be made about the elements in the last q−t columns and the first t rows □

By interchanging the roles of rows and columns a similar *row-Hermite form* can be obtained.

Remark(2.1) [Marc.2]: The column and row Hermite forms are canonical forms under right or left $\mathbb{R}[s]$ equivalence and thus all elements are uniquely defined.

Theorem (2.2) *Smith Form* [Kai.1]: For any polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$, $\rho\{M(s)\} = t$ with $t \leq \min\{p,q\}$, we can find elementary row and column operations or corresponding unimodular matrices $U(s) \in \mathbb{R}^{p \times p}[s]$, $V(s) \in \mathbb{R}^{q \times q}[s]$, such that

$$U(s)\, M(s)\, V(s) = S(s) \tag{2.14}$$

where

$$S(s) = \left[\begin{array}{ccc|ccc} f_1(s) & & & 0 & & \\ & \ddots & & & \ddots & \\ & & f_t(s) & & & 0 \\ 0 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 0 \end{array}\right] \begin{array}{l} \uparrow \\ t \\ \downarrow \\ \uparrow \\ p-t \\ \downarrow \end{array} \tag{2.15}$$
$$\quad \leftarrow t \rightarrow \qquad \leftarrow q-t \rightarrow$$

and the set $\{f_i(s),\ i = 1, \cdots, t\}$ is uniquely defined modulo $c \in \mathbb{R}$ $(c \neq 0)$ and they satisfy the divisibility conditions

$$f_i(s) \;/\; f_{i+1}(s), \quad i = 1, \cdots, t-1 \tag{2.16}$$

If $D_i(s)$ denote the greatest common divisor of all i^{th}-order minors of M(s), then the set of $f_i(s)$ polynomials is defined by the *Smith Algorithm* i.e.

$$f_i(s) = D_i(s) \;/\; D_{i-1}(s), \quad D_0(s) = 1, \quad i = 1, 2, \cdots, t \tag{2.17}$$

The matrix S(s) is called the *Smith form of* M(s). The $\{D_i(s), \ i = 1, \cdots, t\}$ are called the *determinant divisors of* M(s) and $\{f_i(s), i = 1, \cdots, t\}$ the *invariant polynomials of* M(s).

Definition (2.4) [Kai, 1]: A square polynomial matrix $Q(s) \in \mathbb{R}^{q \times q}[s]$ is said to be a *right divisor (R.D)* of the polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$, with $p \geq q$, if and only if there exists a polynomial matrix $M_1(s) \in \mathbb{R}^{p \times q}[s]$, such that

$$M(s) = M_1(s)\, Q(s) \tag{2.18}$$

Let $Q_G(s)$ be a R.D. of M(s). Then $Q_G(s)$ is said to be a *greatest right divisor (G.R.D)* of M(s) if and only if deg {det $Q_G(s)$} $\geq$ deg {Q(s)} for every R.D. Q(s) of M(s). □

Remark (2.2): Greatest right divisors of polynomial matrices are not unique. They differ only by unimodular (left) factors. □

Definition (2.5) [Kai, 1]: A polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$, $p \geq q$, $\rho\,\{M(s)\} = q$ is said to be *irreducible or least degree* if the following equivalent conditions are satisfied:

(i) all the G.R.D. of M(s) are unimodular matrices;

(ii) the Smith Form of M(s) is $[I_q, 0]^T$;

(iii) the greatest common divisor of all q-order minors of M(s) is 1;

(iv) $\rho\{M(s)\} = q$, for every $s \in C$. □

Definition (2.6) [Kai, 1]: A square polynomial matrix $Q(s) \in \mathbb{R}^{q \times q}[s]$ is said to be a *greatest common right divisor (G.C.R.D)* of the two polynomial matrices $M_1(s) \in \mathbb{R}^{p \times q}[s]$, $M_2(s) \in \mathbb{R}^{m \times q}[s]$ if it satisfies the following properties:

(i) $Q(s)$ is a common right divisor of $M_1(s)$ and $M_2(s)$;

(ii) if $Q'(s) \in \mathbb{R}^{q \times q}[s]$ is any other common right divisor of $M_1(s)$ and $M_2(s)$, then $Q'(s)$ is a right divisor of $Q(s)$, or in other words $\deg\left\{\det\{Q(s)\}\right\} \geq \deg\{\det Q'(s)\}$. □

Remark (2.3): Greatest common divisors of two polynomial matrices are not unique. They differ only by unimodular factors. □

Definition (2.7) [Kai, 1]: Two polynomial matrices $M_1(s) \in \mathbb{R}^{p \times q}[s]$, $M_2(s) \in \mathbb{R}^{m \times q}[s]$ with $\rho\left\{\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}\right\} = q$ are said to be *relatively right prime or right coprime* if and only if one of the following equivalent conditions is satisfied:

(i) all G.C.R.D of $M_1(s)$ and $M_2(s)$ are unimodular matrices;

(ii) the Smith form of $\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ is $\begin{bmatrix} I_q \\ 0 \end{bmatrix}$;

(iii) the greatest common divisor of all q-order minors of $\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}$ is 1;

(iv) $\rho\left\{\begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix}\right\} = q$, for every $s \in \mathbb{C}$. □

Left divisors (L.D.), Greatest Left Divisors (G.L.D.) and Greatest Common Left Divisors (G.C.L.D) can be defined with the obvious changes. For convenience, we shall henceforth only consider the case of right divisors.

Remark (2.4): A right MFD (left MFD) $\{D_R(s), N_R(s)\}$ ($\{D_L(s), N_L(s)\}$) of a transfer function matrix $G(s)$ is called a *right coprime MFD* (a left co-prime MFD), iff the matrices $(D_R(s), N_R(s))$ $(D_L(s), N_L(s))$ are right coprime (left coprime). □

Let $M(s) \in \mathbb{R}^{p \times q}[s]$, $p \geq q$ be a polynomial matrix with $\rho\{M(s)\} = q$ and let us write it in terms of its q column polynomial vectors as $M(s) = [\underline{m}_1(s), \cdots, \underline{m}_q(s)]$ where $\underline{m}_i(s) = [m_{1i}(s), \cdots, m_{pi}(s)]^T$, $i = 1, \cdots, q$. Then we may define [R.5], [Wol.1]:

Definition (2.8) : For a polynomial matrix we define:

(i) The *degree of the polynomial vector* $\underline{m}_i(s)$ is the highest degree occuring among the degrees of its polynomial elements $m_{ji}(s)$, i.e $\in \mathbb{R}^q[s]$

$$\deg \underline{m}_i(s) = \max \{\deg m_{ji}(s) \quad j = 1, \cdots, q\} \tag{2.19}$$

(ii) The *complexity c of* M(s) is the sum of the degrees of its column polynomial vectors, i.e.

$$c = \sum_{i=1}^{q} \deg \{\underline{m}_i(s)\} \tag{2.20}$$

(iii) The *matrix degree d* of M(s) is the highest degree occuring among the degrees of all its q-order minors.

□

Since a q-order minor of M(s) is a sum of products of polynomials one from each column, the maximum degree occuring among all the q-order minors of M(s), i.e. its degree d can not exceed its complexity c, i.e., we have [R.5] [Wol.1] $c \geq d$. Let now $\delta_i = \deg \{\underline{m}_i(s)\}$, $i = 1, \cdots, q$, and write

$$\underline{m}_i(s) = \underline{m}_i^0 + \underline{m}_i^1 s + \cdots + \underline{m}_i^{\delta_i} s^{\delta_i} = \sum_{k=0}^{\delta_i} \underline{m}_i^k s^k, \quad i = 1, \cdots, q \tag{2.21}$$

Then M(s) can be written as

$$M(s) = [\underline{m}_1(s), \cdots, \underline{m}_q(s)] = [\underline{m}_1^{\delta_1}, \cdots, \underline{m}_q^{\delta_q}] \begin{bmatrix} s^{\delta_1} & & 0 \\ & \ddots & \\ 0 & & s^{\delta_q} \end{bmatrix} + M_b Z(s) \tag{2.22}$$

where $M_b \in \mathbb{R}^{p \times c}(c = \sum_{i=1}^{q} \delta_i)$, and

$$Z(s) = \begin{bmatrix} \underline{e}_{\delta_1}(s) & & \\ & \ddots & \\ & & \underline{e}_{\delta_q}(s) \end{bmatrix} \in \mathbb{R}^{c \times q}[s] \quad \text{where} \quad \underline{e}_{\delta_i}(s) = [1, s, \cdots, s^{\delta_i - 1}] \tag{2.23}$$

The matrix $[\underline{m}_1^{\delta_1}, \cdots, \underline{m}_q^{\delta_q}] = M_a \in \mathbb{R}^{p \times q}$ is called the *highest (column) degree coefficient matrix* of M(s).

Definition (2.9) [Kai.1]: A polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$ is said to be *column proper or column reduced* if the matrix M_a has full rank q. □

Proposition (2.1) [Kai.1]: A polynomial matrix $M(s) \in \mathbb{R}^{p \times q}[s]$ is column reduced iff its complexity c is equal to its matrix degree d. □

Proposition (2.2) [Kai.1]: Let $M(s) \in \mathbb{R}^{p \times q}[s]$ be a polynomial matrix which is not column reduced. Then there always exists a unimodular matrix $U(s) \in \mathbb{R}^{q \times q}[s]$, det $\{U(s)\} \in \mathbb{R} - \{0\}$, such that the polynomial matrix $M'(s) = M(s)\, U(s)$ is column reduced. □

2.3.2The algebraic Structure of Rational Vector Spaces [Kar.5]

Let $G(s) \in \mathbb{R}^{m \times p}[s]$, $m \geq p$, $\rho\{G(s)\} = p$ be a matrix. Let us also denote by $\mathcal{V}_G$ the set of all linear combinations of the columns of G(s) with multipliers in $\mathbb{R}(s)$, i.e. if $G(s) = [\underline{g}_1(s), \cdots, \underline{g}_p(s)]$, then $\mathcal{V}_G = \text{span}_{\mathbb{R}(s)}\{\underline{g}_1(s), \cdots, \underline{g}_p(s)\}$. Clearly $\mathcal{V}_G$ is a linear vector space over $\mathbb{R}(s)$ and $\dim\mathcal{V}_G = p$, and it is called the *rational vector space generated by G(s)*.

From any rational basis matrix G(s) of $\mathcal{V}_G$ we can generate a polynomial basis of $\mathcal{V}_G$ by means of a right MFD of G(s), i.e. if $G(s) = N(s)\, D^{-1}(s)$ with $N(s) \in \mathbb{R}^{m \times p}[s]$, $D(s) \in \mathbb{R}^{p \times p}[s]$, det $\{D(s)\} \not\equiv 0$, then clearly the columns of N(s) define a polynomial basis of $\mathscr{V}_G$. More precisely, if $N(s) = [\underline{n}_1(s), \cdots, \underline{n}_p(s)]$ then $\text{span}_{\mathbb{R}(s)}\{\underline{n}_1(s), \cdots, \underline{n}_p(s)\} = \mathscr{V}_G$ and $\text{span}_{\mathbb{R}[s]}\{\underline{n}_1(s), \cdots, \underline{n}_p(s)\} = \mathcal{M}_N$ where $\mathcal{M}_N$ denotes the set of all linear combinations of the columns of N(s) with multipliers in $\mathbb{R}(s)$. The set $\mathcal{M}_N$ is a free $\mathbb{R}[s]$–module [Bir, 1] and is called the *polynomial module generated by* N(s). Some of the important properties of such $\mathbb{R}[s]$ modules are summarised below [Bir.1].

Proposition (2.3): Let $\mathcal{M}_{N_1}$, $\mathcal{M}_{N_2}$ be the polynomial modules generated by the polynomial matrices $N_1(s)$, $N_2(s) \in \mathbb{R}^{m \times p}$, with $\rho\{N_1(s)\} = \rho\{N_2(s)\} = p$. If $N_1(s) = N_2(s)\ Q(s)$, where $Q(s) \in \mathbb{R}^{p \times p}[s]$, det $\{Q(s)\} \not\equiv 0$, then $\mathcal{M}_{N_1} \subseteq \mathcal{M}_{N_2}$. □

Proposition (2.4): Let $N_1(s)$, $N_2(s) \in \mathbb{R}^{m \times p}[s]$ be two polynomial bases of the same polynomial module $\mathcal{M}_N$. Then, there exists a unimodular matrix $Q(s) \in \mathbb{R}^{p \times p}$, det $\{Q(s)\} = c \in \mathbb{R} - \{0\}$ such that $N_1(s) = N_2(s)\ Q(s)$. □

Thus, unimodular matrices represent co-ordinate transformations of a polynomial module.

Proposition (2.5): Let $N(s) \in \mathbb{R}^{m \times p}[s]$ be a basis of the polynomial module $\mathcal{M}_N$. Then the degree of N(s) is an invariant of $\mathcal{M}_N$, or in other words if $N_1(s) \in \mathbb{R}^{m \times p}[s]$ is any other basis of $\mathcal{M}_N$ then $\deg\{N(s)\} = \deg\{N_1(s)\}$. □

Proposition (2.6): Let $N_1(s)$, $N_2(s) \in \mathbb{R}^{m \times p}[s]$, $m \geq p$, $\rho\{N_1(s)\} = p$, $\rho\{N_2(s)\} = p$ and let $d_1 = \deg\{N_1(s)\}$, $d_2 = \{N_2(s)\}$. If $N_1(s) = N_2(s)\ Q(s)$, $Q(s) \in \mathbb{R}^{p \times p}[s]$, deg $\{\det Q(s)\} = q \geq 1$. then

(i) $d_1 = d_2 + q$

(ii) $\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2}$

where $\mathcal{M}_{N_1}$, $\mathcal{M}_{N_2}$ are the polynomial modules generated by the polynomial matrices $N_1(s)$, $N_2(s)$, respectively. □

Clearly, the above conditions represent the extraction of a right divisor Q(s) of the polynomial matrix $N_1(s)$. This observation leads us to the following conclusions: Let $N_1(s) \in \mathbb{R}^{m \times p}[s]$, $m \geq p$, $\rho\{N(s)\} = p$ be a polynomial matrix which can be written in terms of its columns as $N_1(s) = [\underline{n}_1^1(s), \cdots, \underline{n}_p^1(s)]$. Let us assume that $N_1(s)$ is not irreducible and let $\mathcal{V} = \text{span}_{\mathbb{R}(s)}\{\underline{n}_1^1(s), \cdots, \underline{n}_p^1(s)\}$, $\mathcal{M}_{N_1} = \text{span}_{\mathbb{R}[s]}\{\underline{n}_1^1(s), \cdots, \underline{n}_p^1(s)\}$ be the rational vector space $\mathcal{V}$ and the polynomial module $\mathcal{M}_{N_1}$ spanned by its columns. Then, if $Q_i(s)$, $i = 1, 2, \cdots$ are right divisors of $N_1(s)$, i.e.

$$N_1(s) = N_{i+1}(s)\ Q_i(s), \qquad i = 1, 2, \cdots \tag{2.24}$$

and the deg $\{\det Q_i(s)\} = q_i \geq 1$ are such that $q_1 \leq q_2 \leq q_3 \leq \cdots$, then

$$\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2} \subset \mathcal{M}_{N_3} \subset \cdots \tag{2.25}$$

and

$$\deg\{N_1(s)\} \geq \deg\{N_2(s)\} \geq \deg\{N_3(s)\} \geq \cdots \tag{2.26}$$

Moreover, if $Q_G(s)$ is a greatest right divisor of $N_1(s)$ so that $N_1(s) = N(s)\ Q_G(s)$, then

$$\mathcal{M}_{N_1} \subset \mathcal{M}_N \text{ and } \deg\{N_i(s)\} \geq \deg\{N(s)\} \tag{2.27}$$

The polynomial module $\mathcal{M}_N$ is the *maximal* $\mathbb{R}$*(s)-module* of the rational vector space $\mathcal{V}$ and all its polynomial bases are least degree, or irreducible polynomial matrices. In other words, if we consider the set of all polynomial vectors in $\mathcal{V}$ then this set coincides with the module $\mathcal{M}_N$ defined above.

Definition (2.10) [For.1]: A polynomial matrix $N(s) \in \mathbb{R}^{m \times p}[s]$, $m \geq p$ and $\rho\{N(s)\} = l$ is said to be *minimal basis* of the rational vector space $\mathcal{V}$, $\mathcal{V}$ = col sp $\{N(s)\}$, if:

(i) N(s) is least degree

(ii) N(s) is column reduced. □

Remark (2.5): Let $N_1(s) \in \mathbb{R}^{m \times p}[s]$, $m \geq p$, $\rho\{N_1(s)\} = l$. If $N(s)$, $N^*(s) \in \mathbb{R}^{m \times p}[s]$ are two minimal bases of the rational vector space $\mathcal{V}$ spanned be the columns of $N(s)$, the $N(s) = N^*(s)\ Q(s)$, where $Q(s)$ is an $\mathbb{R}[s]$-unimodular matrix □

Theorem (2.3) [For.1]: Let $N(s) = [\underline{n}_1(s), \cdots, \underline{n}_p(s)] \in \mathbb{R}^{m \times p}(s)$, $m \geq p$, $\rho\{N(s)\} = p$ be a minimal basis of a rational vector space $\mathcal{V}_N$ = col $sp_{\mathbb{R}(s)}\{N(s)\}$ and let $\delta_i = \deg \underline{n}_i(s)$, i = 1, ⋯, p. The degrees $\{\delta_i, i = 1, \cdots, p\}$ are invariants of $\mathcal{V}_N$. □

Forney has defined the indices $\{\delta_i, i = 1, \cdots, p\}$ as the *invariant dynamical indices* of $\mathcal{V}_N$, and their sum $\delta = \sum_{i=1}^{p} \delta_i$ as the *invariant dynamical order* of $\mathcal{V}_N$. The set $\{\delta_i, i = 1, \cdots, p\}$ does not define a complete [Bir.1] set of invariants for $\mathcal{V}_N$. A complete invariant is defined by the 'echelon form' minimal basis of $\mathcal{V}_N$ [For, 1].

2.3.3 Further Properties of Rational Matrices

Some further results on the properties and structure of rational matrices related to MFDs and minimality of realizations are summarised here.

Proposition (2.7) [Kai.1]: Let $G(s) \in \mathbb{R}^{m \times p}(s)$, $\rho\{G(s)\} = \min\{m, p\}$ be a rational matrix and let $\{D(s), N(s)\}$ be a right MFD of G(s), i.e. $G(s) = N(s)\, D^{-1}(s)$. Then any realisation of G(s) with order equal to the degree of the determinant of the denominator matrix (i.e. $n = \deg\{\det D(s)\}$) will be minimal (or equivalently, observable and controllable), if and only if the MFD is coprime □

Proposition (2.8) [Kai.1]: Suppose $\{N_i(s)\, D_i^{-1}(s), i = 1, 2\}$ are two coprime MFDs of the rational matrix $G(s) \in \mathbb{R}^{m \times p}$, $\rho\{G(s)\} = \min\{m, p\}$. Then there exists a unimodular matrix $Q(s) \in \mathbb{R}^{m \times p}[s]$, such that $D_1(s) = D_2(s)\, Q(s)$ and $N_1(s) = N_2(s)\, Q(s)$. □

Proposition (2.9) [Kai.1]: If $\{D(s), N(s)\}$ is any MFD of $G(s) \in \mathbb{R}^{m \times p}(s)$ with $\rho\{G(s)\} = \min\{m, p\}$ and $\{\overline{D}(s), \overline{N}(s)\}$ is a coprime MFD of G(s), then there exists a polynomial matrix $R(s) \in \mathbb{R}^{p \times p}[s]$, not necessarily unimodular, such that $D(s) = \overline{D}(s)\, R(s)$ and $N(s) = \overline{N}(s)\, R(s)$. □

Proposition (2.10) [Kai.1]: The determinantal degree of the denominator matrix of any right CMFD of $G(s) \in \mathbb{R}^{m \times p}(s)$ with $\rho\{G(s)\} = \min\{m, p\}$ is equal to the determinantal degree of the denominator matrix of any left CMFD of G(s). □

The most important tool in the study of the properties of rational matrices is the

Smith-McMillan form which is defined next [Kai.1].

Theorem (2.6) *(Smith-MacMillan)*: Let $G(s) \in \mathbb{R}^{m \times p}(s)$, $\rho\{G(s)\} = t \leq \min\{m, p\}$. Then there always exist unimodular matrices $Q_1(s) \in \mathbb{R}^{m \times m}[s]$, $Q_2(s) \in \mathbb{R}^{p \times p}[s]$ such that

$$Q_1(s)\; G(s)\; Q_2(s) = M(s) \tag{2.28}$$

where $M(s) \in \mathbb{R}^{m \times p}(s)$ is defined by

$$M(s) = \left[\begin{array}{ccc|ccc} \frac{\epsilon_1(s)}{\psi_1(s)} & & & & 0 & \\ & \ddots & & & & \ddots \\ & & \frac{\epsilon_t(s)}{\psi_t(s)} & & & & 0 \\ & 0 & & & 0 & \\ & & \ddots & & & \ddots \\ & & 0 & & & & 0 \end{array}\right] \begin{array}{l} \uparrow \\ t \\ \downarrow \\ \uparrow \\ m-t \\ \downarrow \end{array} \tag{2.29}$$

$$\qquad \leftarrow \; t \; \rightarrow \qquad \leftarrow \; l-t \; \rightarrow$$

The pairs of monic polynomials $\{\epsilon_i(s), \psi_i(s)\}$ are co-prime $i = 1, \cdots, t$, uniquely defined and satisfy the division properties: $\psi_{i+1}(s)/\psi_i(s)$, $i = 1, \cdots, t\text{-}1$, $\epsilon_i(s)/\epsilon_{i+1}(s)$, $i = 1, 2, \cdots, t-1$. If $D(s)$ is the monic least common multiple of the denominators of the elements of $G(s)$, then $D(s) = \psi(s)$. □

The sum of the deg $\psi_i(s)$, $i = 1, \cdots, t$ is called the *MacMillan degree* of $G(s)$ and it is equal to the determinantal degree of the denominator matrix of any co-prime MFD of $G(S)$.

Proposition (2.11) [Kai.1]:

(i) The (right or left) numerators of coprime MFDs of $G(s)$ all have the same Smith form.

(ii) The denominators of coprime MFDs of $G(s)$ all have the same non-unity invariant polynomials

□

Proposition (2.12) [Kai.1]: For a rational matrix $G(s) \in \mathbb{R}^{m \times p}(s)$ we have the following properties:

(i) If G(s) is a *strictly proper* (proper) rational transfer function matrix and $G(s) = N(s)\, D^{-1}(s)$, then every column of N(s) has degree strictly less than (less than or equal to) that of the corresponding column of D(s).

(ii) If D(s) is column reduced, then $G(s) = N(s)\, D^{-1}(s)$ is strictly proper (proper) if and only if each column of N(s) has degree less than (less than or equal to) the degree of the corresponding column of D(s) □

2.3.4 Poles and Zeros of Rational Matrices

The Smith-MacMillan form of a rational matrix provides the means for a natural extension of the definition of poles and zeros [R.5], [MacF.4] from the scalar to the matrix case

Definition (2.11): Let $G(s) \in \mathbb{R}^{m \times p}(s)$. Then,

(i) The *zeros* of G(s) are defined as the roots of the numerator polynomials $\{\epsilon_i(s)\}$ of the Smith-MacMillan form.

(ii) The *poles* of G(s) are defined as the roots of the denominator polynomials $\{\psi_i(s)\}$ of the Smith-MacMillan form. □

The polynomials defined by

$$z(s) = \prod_{i=1}^{t} \epsilon_i(s), \qquad p(s) = \prod_{i=1}^{t} \psi_i(s) \tag{2.30}$$

are referred to as the *zero* , *pole polynomial* respectively of G(s). From the results of the previous section we have the following alternative characterisation of poles and zeros.

Proposition (2.13) [Kai.1]: Let $G(s) \in \mathbb{R}^{m \times p}(s)$ and let $G(s) = D_L(s)^{-1} N_L(s) = N_R(s)$ $D_R(s)^{-1}$ be left, right coprime MFDs. Then,

(i) The pole polynomial p(s) of G(s) is given by $p(s) = \det\{D_L(s)\} = c \det\{D_R(s)\}$, $c \in \mathbb{R} \neq 0$.

(ii) The zero polynomial z(s) of G(s) is given by the product of the invariant polynomials of $N_L(s)$, or equivalently $N_R(s)$. □

Consider now a $G(s) \in \mathbb{R}^{m \times p}(s)$, $m \geq p$, $\rho\{G(s)\} = p$ and let {D(s), N(s)} be a right coprime MFD pair. If Z(s) is greatest right divisor of N(s), we may write

$$G(s) = N(s)\ D(s)^{-1} = \bar{N}(s)\ Z(s)\ D(s)^{-1} \tag{2.31}$$

where $\bar{N}(s)$ is a least degree basis matrix for col - $\text{span}_{\mathbb{R}(s)}\{G(s)\}$. Using the above factorisation of G(s) we have $p(s) = \det\{D(s)\}$, $z(s) = \det\{Z(s)\}$ and thus

$$C_p(N(s)) = C_p(\bar{N}(s)\ Z(s)) = C_p(\bar{N}(s)) \in \mathbb{R}^{\binom{m}{p} \times 1}[s] \tag{2.32}$$

where $C_p(.)$ denotes the p-th compound matrix [Mar.3]. Clearly, equ. (2.81) implies:

Remark (2.7): If N(s) is a numerator of a coprime MFD of G(s), then the zero polynomial z(s), is the greatest common divisor of the polynomial entries of $C_p(N(s))$. □

Let us suppose now that G(s) is a rational transfer function matrix with $G(s) \in \mathbb{R}^{m \times p}(s)$, $\rho\{G(s)\} = \min\{m, p\}$ and that $\{D_L(s), N_L(s)\}$, $\{D_R(s), N_R(s)\}$ are left and right MFDs of G(s), respectively, not necessarily coprime; then it can be shown [Kai.1] that realisations of G(s) corresponding to those two MFDs are not minimal; furthermore it can be proved [Kai.1] that:

(i) the (Smith) zeros of $[D_L(s), N_L(s)]$ correspond to uncontrollable modes of the equivalent state space realisation and they are termed *input-decoupling* (i.d.) zeros of the MFD

(ii) the (Smith) zeros of $\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}$ correspond to unobservable modes of the equivalent state space realisation and they are termed the *output-decoupling* (o.d.) zeros of the MFD.

To distinguish the transfer function zeros from the decoupling zeros, we often call the transfer function zeros *transmission zeros.* If $z_{i.d.}(s)$, $z_{o.d.}(s)$ are the i.d. zero, o.d. zero polynomials respectively, then we have:

Proposition (2.14): Let $G(s) \in \mathbb{R}^{m \times p}(s)$ with $\rho\{G(s)\} = \min\{m, p\}$ and let $\{D_L(s), N_L(s)\}$, $\{D_R(s), N_R(s)\}$ be left and right MFDs of G(s), respectively, not necessarily coprime. Then

$$C_m([D_L(s)\ N_L(s)]) = C_m\left(T_L(s)\right) z_{i.d.}(s),\quad C_p\left(\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}\right) = C_p\left(T_R(s)\right) z_{o.d.}(s) \tag{2.33}$$

□

2.4 The general feedback configuration.

2.4.1. General aspects of the general feedback configuration.

Consider the following general control system configuration shown below

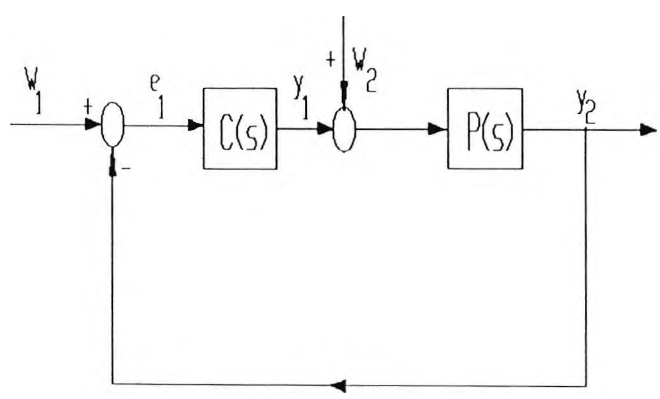


Fig. (2.1): General Control System Configuration

where P(s) represents the mxp transfer matrix of the plant and C(s) the pxm transfer matrix of the controller (in the following, we shall denote P(s), C(s) simply by P, C respectively). The vectors $\underline{w}_1$, $\underline{w}_2$ denote the externally applied inputs, $\underline{e}_1$, $\underline{e}_2$ denote the inputs to the controller, plant and $\underline{y}_1$, $\underline{y}_2$ the vector outputs of the controller, plant

respectively. The transfer matrices P, C are both assumed to be rational and the set $\mathbb{R}_{pr}^{pxm}(s)$ will denote the set of pxm matrices with elements from $\mathbb{R}_{pr}(s)$ (the ring of proper rational functions with no poles inside a prescribed region of the finite complex plane [Var.4]). The configuration of fig(2.1) is quite versatile amd may accommodate several control problems. For instance, in a problem of tracking $\underline{w}_1$ would be a reference signal to be tracked by the plant output $\underline{y}_2$. In a problem of disturbance rejection, or desensitization to noise, $\underline{w}_1$ would be the disturbance/noise vector. Depending on whether $\underline{w}_1$ or $\underline{w}_2$ is the externally applied control signal (as opposed to noise etc.) the configuration can represent either feeedback or cascade control.
The system under study is described by

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix} - \begin{bmatrix} 0 & P \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} , \quad \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \tag{2.34}$$

The system equations can be written as

$$\underline{e} = \underline{w} - FG\underline{e} , \quad \underline{y} = \underline{e} \tag{2.35}$$

where

$$\underline{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}, \ \underline{w} = \begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix}, \ \underline{y} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}, \ F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \ G = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \tag{2.36}$$

It is easy to verify, using the Shur-formula for determinants, that

$$\det(I+FG) = \det(I+PC) = \det(I+CP) = t(s) \in \mathbb{R}(s) \tag{2.37}$$

The system of fig(2.1)1 is said to be *well-formed* [Cal.1] if t(s) is a non zero rational function. This condition is necessary and sufficient to ensure that (2.34) has a unique rational solution for $\underline{e}_1$, $\underline{e}_2$, $\underline{y}_1$, $\underline{y}_1$ corresponding to vectors $\underline{w}_1$, $\underline{w}_2$ of appropriate dimensions.

If the system is well formed then

$$\underline{e} = (I+FG)^{-1}\underline{w} = H(P,C)\ \underline{w} \tag{2.38}$$

$$\underline{y} = G(I+FG)^{-1}\underline{w} = W(P,C)\ \underline{w} \tag{2.39}$$

where

$$\begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix} \xrightarrow{H(P,C)} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \quad , \qquad \text{denoted } H_{\underline{e}|\underline{w}} \tag{2.40}$$

$$\begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix} \xrightarrow{W(P,C)} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \quad , \qquad \text{denoted } H_{\underline{y}|\underline{w}} \tag{2.41}$$

A well formed system allows the existence of various closed loop functions. In the design of feedback systems the 'properness' of these transfer functions is essential, if no signal is to be unduly amplified or otherwise if the smoothness of signals throughout the system is to be preserved. Systems which exhibit this property are said to be 'well-posed', a more formal definition is given below.

Definition (2.12): Let every subsystem of a composite system be described by a rational transfer function. Then the composite system is said to be *well posed* if the transfer function of every subsystem is proper and the closed loop transfer function from any point chosen as an input terminal to every other point along the directed path is well formed and proper.

□

The well posedness property is characterised by the following result:

Theorem (2.4) [Vid.2]: Consider the feedback system of fig 2.1 where P,C are proper rational matrices. The closed loop transfer function $H_{\underline{e}|\underline{w}}$ is proper, iff

$$\det\,(I+C(\infty)\;P(\infty)) = \det\,(I+P(\infty)\;C(\infty)) \in \mathbb{R} \tag{2.42}$$

□

This result implies that if both P,C are proper then condition (2.42) is necessary and sufficient for $(I+PC)^{-1}$, $(I+CP)^{-1}$ to be proper and it follows that all transfer functions associated with the feedback configuration of fig(2.1) will be proper. For systems that are well posed it is possible to obtain several expressions for H(P,C) and W(P,C). Thus for H(P,C) it is readily verified that

$$H(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \tag{2.43}$$

and the following hold true [Vid.2]

$$C(I+PC)^{-1} = (I+CP)^{-1}C \text{ , } P(I+CP)^{-1} = (I+PC)^{-1}P \tag{2.44}$$

$$(I+PC)^{-1} = I - P(I+CP)^{-1}C \text{ , } (I+CP)^{-1} = I - C(I+PC)^{-1}P \tag{2.45}$$

Using the above identities we can obtain the following equivalent expressions for H(P,C).

$$H(P,C) = \begin{bmatrix} I - P(I+CP)^{-1}C & -P(I+CP)^{-1} \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & I - C(I+PC)^{-1}P \end{bmatrix} \tag{2.46}$$

where the first involves only $(I+CP)^{-1}$ and the second $(I+PC)^{-1}$. For the W(P,C) transfer function we have similar expressions, that is

$$W(P,C) = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & P \\ -C & I \end{bmatrix} = \begin{bmatrix} C(I+PC)^{-1} & -CP(I+CP)^{-1} \\ PC(I+PC)^{-1} & P(I+CP)^{-1} \end{bmatrix} \tag{2.47}$$

or alternatively

$$W(P,C) = \begin{bmatrix} C - CP(I+CP)^{-1}C & -CP(I+CP)^{-1} \\ P(I+CP)^{-1}C & P(I+CP)^{-1} \end{bmatrix} = \begin{bmatrix} C(I+PC)^{-1} & -C(I+PC)^{-1}P \\ PC(I+PC)^{-1} & P - PC(I+PC)^{-1}P \end{bmatrix}$$

$$(2.48)$$

An interesting relationship between H(P,C) , W(P,C) denoted in short by H,W is defined below

$$W=F[H-I] \tag{2.49}$$

In fact, (2.49) readily follows from the following arguments taking into account that $F^{-1}=-F$

$$H=(I+FG)^{-1}=\{I+FG-FG\}(I+FG)^{-1}=I-FG(I+FG)^{-1}=I-FW \tag{2.50}$$

Remark (2.8): The transfer function $W\in \mathbb{R}_{pr}^{(m+p)(m+p)}(s)$ iff $H\in \mathbb{R}_{pr}^{(m+p)(m+p)}(s)$

□

Thus in the investigation of stability (external) of the feedback configuration of fig(2.1), the transfer function H(P,C) may be used.

2.4.2 Characteristic Pole Function

The transfer function matrix of the plant and controller may be written as coprime matrix fraction descriptions (MFD's) over the appropriate ring of interest. Since P,C are generally non square we distinguish between left and right MFD's ie.

$$P=A_1^{-1}B_1=B_2A_2^{-1} \tag{2.51}$$

$$C=D_1^{-1}N_1=N_2D_2^{-1} \tag{2.52}$$

By inserting (2.51) and (2.52) into the expression for H(P,L) we have

$$H(P,C) = \begin{bmatrix} A_1 & B_1 \\ -N_1 & D_1 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix}^{-1} \tag{2.53}$$

$$= \begin{bmatrix} D_2 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} D_2 & B_2 \\ -N_2 & A_2 \end{bmatrix}^{-1} \tag{2.54}$$

Proposition (2.15) [Kai 1]: If (A_1, B_1) , (A_2, B_2) , (D_1, N_1) , (D_2, N_2) are coprime pairs then (2.53) defines a left coprime MFD and (2.54) a right coprime MFD of H(P,C).

□

Assume now that both plant and controller transfer functions are represented by coprime MFD's then the expressions for H(P,C), ie. the transfer functions of the feedback configuration are coprime and the characteristic pole function of H(P,C) is given by the determinants of the denominator matrices

$$f \sim \det \begin{bmatrix} A_1 & B_1 \\ -N_1 & D_1 \end{bmatrix} \sim \det \begin{bmatrix} D_2 & B_2 \\ -N_2 & A_2 \end{bmatrix} \tag{2.55}$$

where ' $\sim$ ' denotes equality modulo a non zero real constant.

Proposition (2.16) [Kuc.2]: The characteristic pole function of H(P,C) is given by

$$f \sim \det F_1 \sim \det F_2 \tag{2.56}$$

where

$$F_1 = A_1 D_2 + B_1 N_2 \tag{2.57}$$

$$F_2 = D_1 A_2 + N_1 B_2 \tag{2.58}$$

□

Note the importance of the assumption that both systems within the feedback loop be free of hidden modes. If these assumption were violated, relation (2.55) would not be valid. For systems with hidden modes we have :

Remark (2.9): For systems with hidden modes, equ. (2.56) becomes:

$$f \approx f_0 \, f_p \, f_c \tag{2.59}$$

where f_0 is defined by f Proposition (2.16) and f_p, f_c are the hidden mode pole function of the plant and controller, respectively.

Hidden modes play a key role in characterising the internal stability of a feedback system, in terms of the exterior description.

2.4.3 Internal-External Stability of the Feedback Configuration

The internal stability of the feedback system of the configuration (2.1) is related to its state space description. In the following S_f will denote the state space representation of the composite system of the configuration (2.1) and S_i, i=1,2 will represent the state space models of the controller and plant, respectively. We shall assume that the plant and controller are characterised by the following sets of state space equations

$$S_1: \dot{\underline{x}}_1 = A_1\underline{x}_1 + B_1\underline{e}_1 \quad , \quad \underline{y}_1 = C_1\underline{x}_1 + D_1\underline{e}_1 \tag{2.60a}$$

$$S_2: \dot{\underline{x}}_2 = A_2\underline{x}_2 + B_2\underline{e}_2 \quad , \quad \underline{y}_2 = C_2\underline{x}_2 + D_2\underline{e}_2 \tag{2.60b}$$

The feedback system is assumed to be well posed, so that we have that $\det(I+D_1D_2)= \det(I+D_2D_1) \neq 0$, and the matrices Δ_1, Δ_2 are defined where,

$$\Delta_1=(I+D_1D_2)^{-1}, \qquad \Delta_2=(I+D_2D_1)^{-1} \tag{2.61}$$

The transfer function corresponding to the state space description with input vector $[\underline{w}_1{}^t, \underline{w}_2{}^t]$ and output vector of signals $[\underline{e}_1{}^t, \underline{e}_2{}^t]$ is clearly $H_{e|w}$. Note that

$$\underline{e}_1= \underline{w}_1 - \underline{y}_1, \qquad \underline{e}_2= \underline{w}_2 + \underline{y}_1, \tag{2.62}$$

and thus

$$e_1 = w_1 - C_2 x_2 - D_2(w_2 + C_1 x_1 + D_1 e_1) \tag{2.63}$$

or

$$e_1 = -\Delta_2 D_2 C_1 x_1 - \Delta_2 C_2 x_2 + \Delta_2 w_1 - \Delta_2 D_2 w_2 \tag{2.64}$$

$$e_2 = w_2 - C_1 x_1 + D_1 e_1 = w_2 \tag{2.65}$$

or

$$(I + D_1 D_2) e_2 = C_1 x_1 + D_1 C_2 x_2 + D_1 w_1 + w_2 \tag{2.66}$$

or

$$e_2 = -\Delta_2 C_1 x_1 - \Delta_1 D_1 C_2 x_2 + \Delta_1 D_1 w_1 + \Delta_1 w_2 \tag{2.67}$$

and from the above, we obtain the state-space equations.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\triangleq\, x_f} = \underbrace{\begin{bmatrix} A_1 - B_1\Delta_2 D_2 C_1 & -B_1\Delta_2 C_2 \\ B_2\Delta_1 C_1 & A_2 - B_2\Delta_1 D_1 C_2 \end{bmatrix}}_{\triangleq\, A_f} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\triangleq\, x_f} + \underbrace{\begin{bmatrix} B_1\Delta_2 & -B_1\Delta_2 D_2 \\ B_2\Delta_1 D_1 & B_2\Delta_1 \end{bmatrix}}_{\triangleq\, B_f} \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{\triangleq\, u_f} \tag{2.68}$$

$$\underbrace{\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix}}_{\triangleq\, y_f} = \underbrace{\begin{bmatrix} -\Delta_1 - D_2 C_1 & -\Delta_2 C_2 \\ \Delta_1 C_1 & \Delta_1 D_1 C_2 \end{bmatrix}}_{\triangleq\, C_f} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \Delta_2 & -\Delta_2 D_2 \\ \Delta_1 D_1 & \Delta_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{2.69}$$

$$\triangleq D_f$$

The matrices (A_f, B_f, C_f, D_f) characterize the feedback system completely. The notion of internal and external stability are defined next.

Definition (2.13) [Che.1]: The feedback configuration of Fig(2.1) will be called *internally stable* if the system

$$\dot{\underline{x}}_f = A_f \, \underline{x}_f \tag{2.70}$$

is asymptotically stable. It will be called *bounded-input, bounded-output stable* (BIBO) if the tranfer function $H_{e|w}$ is BIBO stable.

□

To examine the conditions under which the stability of $H_{e|w}$ guarantees internal stability we need the following standard notions.

Definition (2.14) [Won.1]: For a system S(A,B,C,D) we have:
(i) The pair (A,B) is *stabilizable*, if the unstable subspace of $\underline{x} = A\underline{x} + B\underline{u}$ is contained in its controllable subspace.
(ii) The pair (A,C) is *detectable,* if the unreconstructable subspace of $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x}$ is contained in its observable subspace.

□

Remark (2.10) Under the stabilizability and detectability assumptions on a linear system, S(A,B,C,D), the notions of internal and external stability become equivalent. □

With the above standard state space notions, the following results may be stated for the standard feedback configuration.

Proposition (2.17) [Che.1], [Vid.2] : Consider the well-posed feedback system S_f of fig(2.1) with controller S_1 and plant S_2 represented by the quadruples (A_i, B_i, C_i, D_i) i = 1, 2. Then

(i) S_f is controllable, observable iff both S_1 and S_2 are controllable, observable.
(ii) S_f is stabilizable, detectable iff both S_1 and S_2 are stabilizable, detectable.

□

The above leads to the following main result [Vid.2].

Theorem (2.5): Consider the well posed feedback system S_f with the controller S_1 and plant S_2 both stabilizable and detectable. Under these assumptions, S_f is internally stable, iff the transfer function $H_{e|w}$ is BIBO stable. □

Thus under the assumption of well posedness and stabilizability and detectability of plant and controller the closed loop transfer function $H_{e|w}$ defines both internal and external stability. Clearly, since controllability implies stabilizability and observability implies detectability, if S_f is well posed and both plant and controller are free from hidden modes (controllable and observable) then $H_{e|w}$ defines both internal and external stability.

2.5 Conclusions

In this thesis, the emphasis is on the study of pole placement assignment problems for the standard feedback configuration (2.1) which has been discussed in this chapter where the contoller may be constant or dynamic and has centralised or decentralised characteristics. As far as the dynamics are concerned, we are interested in the family of controllers of certain dynamic complexity such as constant PI, and bounded dynamics (bounded controllability observability indices). Our approach, as far as system description goes, will be based on the use of MFD's which have been discussed in detail here. An additional problem, also discussed within the algebraic framework, is the problem of squaring down, which may be considered as part of the effort to form the feedback configuration.

CHAPTER 3. Mathematical Tools

3.1 Introduction

In this chapter we present the main mathematical tools to be used in this thesis. These are : exterior algebra, algebraic geometry and intersection theory, and finally manifolds and cohomology theory. These topics are presented in a rather introductory manner, focusing to those points related to this thesis. In many circumstances (especially in intersection theory) we do not follow the mathematical formalism normally used for the above topics, but try to illustrate the mathematics with specific "down to earth" examples which in turn may be more easily related to our problems. Any additional tools needed subsequently, will be briefly presented in the relative chapters.

3.2 Exterior Algebra and representation of exterior powers of linear maps.

3.2.1 Introduction

This section introduces the main concepts of exterior algebra and representation of exterior maps. These concepts are necessary for the understanding of DAP which has a multilinear skew symmetric nature. In section 3.2.2 we give an abstract mathematical description of the key points and results from exterior algebra and in section 3.2.3 we show how we can represent the exterior powers of maps by certain compound matrices.

3.2.2 Basic notions and results on multilinear algebra [Marc.1, Gre.1]

Let $\mathcal{V}$ and $\mathcal{U}$ be vector spaces over a field $\mathbb{F}$ of characteristic 0. A *p-linear map* from $\mathcal{V}$ to $\mathcal{U}$ is a map ϕ: $\overset{p}{\underset{1}{\times}}\mathcal{V}\rightarrow\mathcal{U}$ which is linear with respect to each argument, i.e.

$$\phi(\underline{x}_1, \cdots, \lambda\underline{x}_i + \mu\underline{y}_1, \cdots, \underline{x}_p) = \lambda\phi(\underline{x}_1, \cdots, \underline{x}_i, \cdots, \underline{x}_p) + \mu(\underline{x}_1, \cdots, \underline{y}_1, \cdots, \underline{x}_p)$$

where λ, $\mu \in \mathbb{F}$ and $\underline{x}_i, \underline{y}_i \in \mathcal{V}$. A p-linear map from $\mathcal{V}$ to $\mathbb{F}$ is called a *p-linear function* in $\mathcal{V}$.

A p-linear map $\phi\colon \overset{p}{\underset{1}{\times}}\mathcal{V} \to \mathcal{U}$ is called *skew-symmetric* if for every permutation σ, $\sigma \in S_p$ (S_p is the group of permutations of p objects)

$$\phi(\underline{x}_{\sigma(1)}, \cdots, \underline{x}_{\sigma(p)}) = \text{sign } \sigma \cdot \phi(\underline{x}_1, \cdots, \underline{x}_p)$$

where sign σ is the sign of the permutation. Every p-linear map ϕ from $\mathcal{V}$ to $\mathcal{U}$ determines a skew symmetric p-linear map ψ which is given by:

$$\psi = \sum_{\sigma} (\text{sign } \sigma)\ \sigma \cdot \phi$$

Example (3.1): Determinants provide an example of multilinearity. For instance, the determinant det(A) of an $n \times n$ matrix, with entries in $\mathbb{F}$ is a function of the columns of A. Let $A = [\underline{a}_1, \cdots, \underline{a}_n] \in \mathbb{F}^{n \times n}$, then the determinant det(A) is a function det: $\overset{n}{\underset{1}{\times}}\mathbb{F}^n \to \mathbb{F}$ for which

$$\det(\underline{a}_1, \cdots, \lambda\underline{a}_i + \mu\underline{a}_j, \cdots, \underline{a}_n) = \lambda\det(\underline{a}_1, \cdots, \underline{a}_i, \cdots, \underline{a}_n) + \mu(\underline{a}_1, \cdots, \underline{a}_j, \cdots, \underline{a}_n)$$

So the determinant is a an n-linear skew symmetric fucntion of $\mathbb{F}^n$.

□

Let $\mathcal{V}$ be an arbitrary vector space and $p \geq 2$ be an integer. Then a vector space $\wedge^p\mathcal{V}$ together with a skew symmetric p-linear map

$$\wedge^p\colon \overset{p}{\underset{1}{\times}}\mathcal{V} \to \wedge^p\mathcal{V}$$

is called a *p-th exterior power of* $\mathcal{V}$ if the following conditions are satisfied:

(i) The vectors $\wedge^p(\underline{x}_1, \cdots, \underline{x}_p)$ generate $\wedge^p\mathcal{V}$.

(ii) If ψ is any skew symmetric p-linear map of $\overset{p}{\underset{1}{\times}}\mathcal{V}$ into an arbitrary vector space $\mathcal{U}$, then there exists a linear map

$$f:\ \wedge^{p}\mathcal{V} \to \mathcal{U} \quad \text{such that} \quad \psi = f \circ \wedge^{p}.$$

It is proved that conditions (i) and (ii) are equivalent to the following condition:

(iii) If ψ is any skew symmetric p-linear map of $\overset{p}{\underset{1}{\times}}\mathcal{V}$ into a vector space $\mathcal{U}$ then there exists a unique linear map f: $\wedge^{p}\mathcal{V} \to \mathcal{U}$ such that $\psi = f \circ \wedge^{p}$

□

The elements of $\wedge^{p}\mathcal{V}$ are called *p-vectors.* A p-vector of the form $\wedge^{p}(\underline{x}_1, \cdots, \underline{x}_p)$ is called *decomposable*, and is denoted by $\underline{x}_1 \wedge \cdots \wedge \underline{x}_p$. Condition (i) states that $\wedge^{p}\mathcal{V}$ is generated by its decomposable elements.

The skew symmetric property of the p-linear map $\wedge^{p}$ implies that for every permutation $\sigma \in S_p$

$$\underline{x}_{\sigma(1)} \wedge \cdots \wedge \underline{x}_{\sigma(p)} = (\text{sign } \sigma) \,.\, \underline{x}_1 \wedge \cdots \wedge \underline{x}_p \tag{3.1}$$

Now suppose that $\{\underline{x}_1, \cdots, \underline{x}_p\}$ are linearly dependent vectors. Then the skew symmetry of $\wedge^{p}$ implies that:

$$\underline{x}_1 \wedge \cdots \wedge \underline{x}_p = 0 \tag{3.2}$$

Conversely, p-vectors which satisfy (3.2) are linearly dependent.

Let us see now how we can construct the p-th exterior power of a finite dimensional vector space. Suppose that $\mathcal{V}$ is a vector space of dimension n over the field $\mathbb{F}$. If $\{\underline{e}_i, i = 1, \cdots, n\}$ is a basis of $\mathcal{V}$, then we consider the following formal products:

$$\underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_p}, \qquad 1 \le i_1 < i_2 < \cdots < i_p \le n$$

where $\wedge$ has the skew symmetry property. There are $\binom{n}{p}$ choices of distinct indices i_1, $\cdots$, i_p from 1 to n, and they can be arranged uniquely in increasing order. The space of all linear combinations of the above products is the required exterior power $\wedge^{p}\mathcal{V}$. Clearly

$$\dim \wedge^p \mathcal{V} = \binom{n}{p}, \qquad p = 0, 1, \cdots, n$$

and $\wedge^p \mathcal{V} = 0$ for $p > n$.

An arbitrary vector space $\wedge^p \mathcal{V}$ is a p-vector and an element of the form $\underline{x}_1 \wedge \cdots \wedge \underline{x}_p$ where $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_p \in \mathcal{V}$ is *decomposable.* Every p-vector $\underline{u}$ of $\wedge^p \mathcal{V}$ can be uniquely represented in the form:

$$\underline{u} = \sum_{<} a_{i_1 i_2 \ldots i_p} \; \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_p} \tag{3.3}$$

where the symbol $<$ indicates that the indices $(i_1, \cdots, i_p)$ are ordered lexicographically ($1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq n$). The coefficients $a_{i_1 i_2 \ldots i_p}$ are called the *co-ordinates of the p-vector* $\underline{u}$ *with respect to the basis* $\{\underline{e}_i, i = 1, \cdots, n\}$ of $\mathcal{V}$.

If we denote by $\wedge \mathcal{V}$ the direct sum $\mathcal{V} \oplus \wedge^1 \mathcal{V} \oplus \wedge^2 \mathcal{V} \oplus \ldots \oplus \wedge^m \mathcal{V}$ where m is the dimension of $\mathcal{V}$, then $(\wedge \mathcal{V}, +, \wedge)$ is an $\mathbb{F}$-graded algebra called the *exterior algebra* of $\mathcal{V}$. The exterior algebra of $\mathcal{V}$ has the property that it is the smallest $\mathbb{F}$-algebra $(\mathcal{A}, +, \cdot)$ which contains $\mathcal{V}$ and satisfies $v \cdot v = 0$ for every $v \in \mathcal{V}$. The next theorem states that a linear map between two linear spaces, can be uniquely extended to a homomorphism of the corresponding exterior algebras.

Theorem (3.1) [Bir. 1]: Let $\mathcal{V}, \mathcal{U}$ be finite dimensional vector spaces over a field $\mathbb{F}$, and let G: $\mathcal{V} \to \mathcal{U}$ be a linear map. Then, there is a unique graded algebra homomorphism $\hat{G}$: $\wedge \mathcal{V} \to \wedge \mathcal{U}$ of the exterior algebras such that $\hat{G}(\underline{x}) = G(\underline{x})$ for any $\underline{x} \in \mathcal{V}$.

□

The map of the above theorem can be constructed as follows: we may define a skew symmetric multilinear map ψ which maps every $(\underline{x}_1, \cdots, \underline{x}_p)$, $\in \overset{p}{\underset{1}{\times}} \mathcal{V}$ to the element $G(\underline{x}_1) \wedge \cdots \wedge G(\underline{x}_p)$ of $\wedge^p \mathcal{U}$. By the property (ii) of the exterior power of a linear space, the map ψ may be lifted to a map $\wedge^p G : \wedge^p \mathcal{V} \to \wedge^p \mathcal{U}$ such that

$$\wedge^p G(\underline{x}_1 \wedge \cdots \wedge \underline{x}_p) = G(\underline{x}_1) \wedge \cdots \wedge G(\underline{x}_p) \tag{3.4}$$

The map $\wedge^p G$ is called the *p-th exterior power of the linear map* G, and the sum of all such powers for $p=0,1,\ldots,\dim\mathcal{V}$, is the map $\hat{G}$ of theorem(3.1). Using this construction we can easily prove:

Corollary (3.1) [Marc.1]: Let F: $\mathcal{V}\to\mathcal{U}$ and G:$\mathcal{U}\to\mathcal{W}$ be linear maps of finite dimensional vector spaces over $\mathbb{F}$. Then

$$\wedge^p(G \circ F) = \wedge^p G \circ \wedge^p F$$

□

3.2.2. Representation of Exterior Powers of Linear Maps [Kar.6]

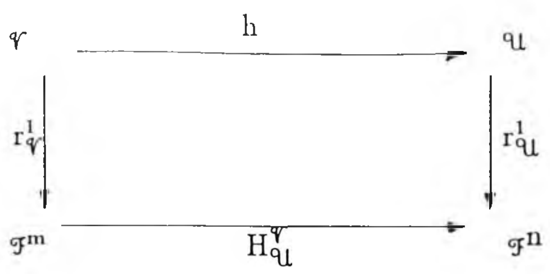
Let $\mathcal{V}$ be an m-dimensional vector space over the field $\mathbb{F}$ and let $\wedge^p\mathcal{V}$, $p \leq m$ be the p-th exterior power of $\mathcal{V}$. If $\{\underline{v}_i,\ i = 1, \cdots, m\}$ is a basis of $\mathcal{V}$, then $\wedge^p\mathcal{V}$ is spanned by the vectors of the basis $\{\underline{v}_{\omega}^{\wedge},\ \omega = (i_1, \cdots, i_p),\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq m,\ \underline{v}_{\omega}^{\wedge} = \underline{v}_{i_1} \wedge \underline{v}_{i_2} \wedge \cdots \wedge \underline{v}_{i_p}\}$. Every vector $\underline{v} \in \wedge^p\mathcal{V}$ may be written as $\underline{v} = \sum_{\omega} a_{\omega}\underline{v}_{\omega}^{\wedge}$. Let $r_{\mathcal{V}}^p$ be the map of $\wedge^p\mathcal{V}$ into $\mathbb{F}^{\binom{m}{p}}$ defined by:

$$r_{\mathcal{V}}^p(\underline{v}) = [\ \cdots, a_{w}, \cdots\ ]$$

Then $r_{\mathcal{V}}^p$ is linear and it is called the *representation map* of $\wedge^p\mathcal{V}$ associated with the basis $\{\underline{v}_i,\ i = 1, \cdots, m\}$. It can be seen that there is such map associated to every basis of $\mathcal{V}$. The image of $\wedge^p\mathcal{V}$ under this map is called the representation of $\wedge^p\mathcal{V}$ relative to the basis $\{\underline{v}_i,\ i = 1, \cdots, m\}$ of $\mathcal{V}$. The following result can be easily verified.

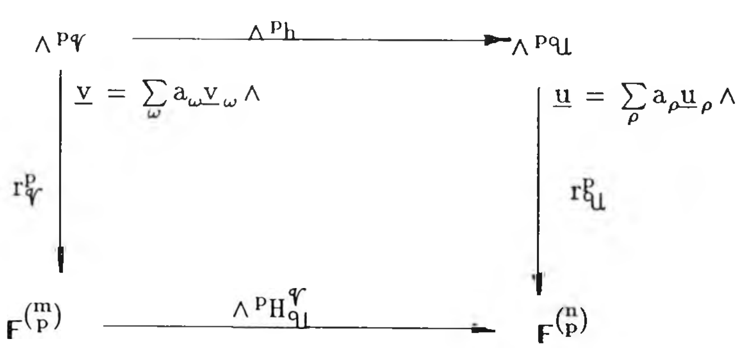
Proposition (3.1) The representations of the p-th exterior power of an m-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$, are linear isomorphisms of $\wedge^p\mathcal{V}$ onto $\mathbb{F}^{\binom{m}{p}}$. □

Let $\mathcal{V}$,$\mathcal{U}$ be two vector spaces over the field $\mathbb{F}$ of dimensions m, n, respectively and let h be a linear map of $\mathcal{V}$ into $\mathcal{U}$. The linear map h can be represented, with respect to the bases $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$ and $B_{\mathcal{U}} = \{\underline{u}_i,\ i = 1, \cdots, n\}$ of $\mathcal{V}$ and $\mathcal{U}$, by a matrix $H_{\mathcal{V}}^{\mathcal{U}}$ which is defined by the following commutative diagram:



where $r^1_{\mathcal{V}}$, $r^1_{\mathcal{U}}$ are the representation maps of $\mathcal{V}$ and $\mathcal{U}$ onto $\mathbb{F}^m$ and $\mathbb{F}^n$ respectively. Because $\mathcal{V}$, $\mathcal{U}$ are isomorphic to $\mathbb{F}^m$, $\mathbb{F}^n$ respectively, $\mathbb{F}^m$, $\mathbb{F}^n$ may be used to represent $\mathcal{V}$, $\mathcal{U}$ and the matrix $H^{\mathcal{V}}_{\mathcal{U}}$ to represent the linear map h.

Let $\wedge^p\mathcal{V}$, $\wedge^p\mathcal{U}$ be the p-th exterior powers of $\mathcal{V}$, $\mathcal{U}$ respectively, where $p \leq \min(m,n)$. Then $h:\mathcal{V}\rightarrow\mathcal{U}$ implies the existence of a linear map $\wedge^p h$: $\wedge^p\mathcal{V}\rightarrow\wedge^p\mathcal{U}$. If we denote by $r^p_{\mathcal{V}}$, $r^p_{\mathcal{U}}$ the representation maps of $\wedge^p\mathcal{V}$, $\wedge^p\mathcal{U}$ with respect to the bases $B_{\mathcal{V}} = \{\underline{v}_i,\ i = 1, \cdots, m\}$ and $B_{\mathcal{U}} = \{\underline{u}_i,\ i = 1, \cdots, n\}$ of $\mathcal{V}$, $\mathcal{U}$ respectively, then applying the representation result for linear maps, we have the following commutative diagram:



and thus the matrix $\wedge^p H^{\mathcal{V}}_{\mathcal{U}}$ is defined by the equation

$$\wedge^p H^{\mathcal{V}}_{\mathcal{U}} \cdot \begin{bmatrix} \vdots \\ a_\omega \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ b_\rho \\ \vdots \end{bmatrix} \tag{3.5}$$

The description of $\wedge^p H^{\mathcal{V}}_{\mathcal{U}}$ in terms of $H^{\mathcal{V}}_{\mathcal{U}}$ will be defined below and this will establish the links between the present subject and the compound matrix theory.

Let $B_{\mathcal{V}} = \{\underline{v}_i, i = 1, \cdots, m\}$, $B_{\mathcal{U}} = \{\underline{u}_i, i = 1, \cdots, n\}$ be bases of $\mathcal{V}$ and $\mathcal{U}$ respectively and let $B^p_{\mathcal{V}} = \{\underline{v}_\omega \wedge = \underline{v}_{i_1} \wedge \cdots \wedge \underline{v}_{i_p}, \omega = (i_1, \cdots, i_p), 1 \leq i_1 < \cdots < i_p \leq m\}$, $B^p_{\mathcal{U}} = \{\underline{u}_\rho \wedge = \underline{u}_{j_1} \wedge \cdots \wedge \underline{u}_{j_p}, \omega = (j_1, \cdots, j_p), 1 \leq j_1 < \cdots < j_p \leq n\}$ be the induced bases of $\wedge^p\mathcal{V}$ and $\wedge^p\mathcal{U}$ respectively. If

$$h(\underline{v}_i) = \sum_{j=1}^{n} c_{ij}\, \underline{u}_j, \quad i = 1, \cdots, m, \quad H^{\mathcal{V}}_{\mathcal{U}} = [c_{ij}] \tag{3.6a}$$

then for all basis vectors $\underline{v}_\omega \in \wedge^p\mathcal{V}$ we have

$$\wedge^p h(\underline{v}_{i_1} \wedge \cdots \wedge \underline{v}_{i_p}) = \Big(\sum_{j=1}^{n} c_{i_1 j}\, \underline{u}_j\Big) \wedge \cdots \wedge \Big(\sum_{j=1}^{n} c_{i_p j}\, \underline{u}_j\Big) \tag{3.6b}$$

However, if we expand the right hand side of (3.6b), using the properties of the exterior product, we have that

$$\wedge^p h(\underline{v}_{i_1} \wedge \cdots \wedge \underline{v}_{i_p}) = \sum H^{j_1, \cdots, j_p}_{i_1, \cdots, i_p}\, \underline{u}_{j_1} \wedge \cdots \wedge \underline{u}_{j_p} \tag{3.7}$$

here the sum is taken for all $\underline{u}_{j_1} \wedge \cdots \wedge \underline{u}_{j_p} \in B^p_{\mathcal{U}}$ and the numbers $H^{j_1, \cdots, j_p}_{i_1, \cdots, i_p}$ are polynomial functions of the c's. A key fact for the representation of exterior powers of maps is that:

$$H^{j_1, \cdots, j_p}_{i_1, \cdots, i_p} = \det\left(\begin{bmatrix} c_{i_1 j_1} & \cdots & c_{i_p j_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 j_p} & \cdots & c_{i_p j_p} \end{bmatrix}\right) \tag{3.8}$$

Clearly, combining equations (3.7) and (3.8), we can deduce that the representation of the linear map $\wedge^p h$: $\wedge^p\mathcal{V} \rightarrow \wedge^p\mathcal{U}$ with respect to the bases $B^p_{\mathcal{V}}$, $B^p_{\mathcal{U}}$ is given by a matrix whose entries are all the pxp minors of $H^{\mathcal{V}}_{\mathcal{U}}$. These types of matrices are called compound matrices and will be examined next.

Lexicographic ordering

(a) $Q_{p,n}$ denotes the set of strictly increasing sequences of p integers ($1 \leq p \leq n$) chosen from $1, \cdots, n$ e.g. $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$. Thus, the number of the sequences belonging to $Q_{p,n}$, is $\binom{n}{p}$. If $\alpha, \beta \in Q_{p,n}$, we say that α precedes β (α<b), if

there exists an integer t ($t \leq t \leq p$) for which $\alpha_1 = \beta_1, \cdots, \alpha_{t-1} = \beta_{t-1}, \alpha_t < \beta_t$, where α_i, β_i denote the elements of α, β respectively; for instance, in the set $Q_{3,8}$, $(3,5,8) < (4,5,6)$. This describes the *lexicographic ordering* of the elements of $Q_{p,n}$. The set of sequences $Q_{p,n}$ from now on will be assumed with its sequences lexicographically ordered and the elements of the ordered set $Q_{p,n}$ will be denoted by $Q_{p,n}(t)$, $t = 1, \cdots, \binom{n}{p}$ or simply by ω.

(b) $Q^{\alpha}_{p,n}$ denotes the subset of $Q_{p,n}$ whose sequences do not contain any of the indices of a given $\alpha \in Q_{p,n}$, such as, for example, $Q^{\alpha}_{2,5} = \{(3,4), (3,5), (4,5)\}$ if $\alpha = (1,2)$. This set has $\binom{n-p}{p}$ elements. The elements of $Q^{\alpha}_{p,n}$ will be denoted by $Q^{\alpha}_{p,n}(t)$ or simply ω_{α}.

(c) If $c_1, \cdots, c_n$ are elements of the field $\mathbb{F}$ and $\omega = (i_1, \cdots, i_p)$ is a sequence in $Q_{p,n}$, $1 \leq p \leq n$, then the product $c_{i_1} \cdots c_{i_p}$ will be denoted by c_{ω}.

(d) Suppose $A = [a_{ij}] \in M_{m,n}(\mathbb{F})$ where $M_{m,n}(\mathbb{F})$ denotes the set of $m \times n$ matrices with elements from the field $\mathbb{F}$; let k,p be positive integers satisfying $1 \leq k \leq m$, $1 \leq p \leq n$ and let $\alpha = (i_1, \cdots, i_k) \in Q_{k,m}$ and $\beta = (j_1, \cdots, j_p) \in Q_{p,n}$. Then $A[\alpha|\beta] \in M_{k,p}(\mathbb{F})$ denotes the submatrix of A which contains the rows $i_1, \cdots, i_k$ and the columns $j_1, \cdots, j_p$. We use the notation $A[\alpha|\beta]$ to designate the submatrix of A wich excludes rows $i_1, \cdots, i_k$ and includes columns $j_1, \cdots, j_p$. The submatrices $A[\alpha|\beta]$ and $A[\alpha|\beta]$ can be similarly defined.

Compound Matrices

Let $A \in \mathbb{F}^{m \times n}$ and $1 \leq p \leq \min(m,n)$, then the *p-th compound matrix* or *p-th adjugate* of A is the $\binom{m}{p} \times \binom{n}{p}$ matrix whose entries are $\det(A[\alpha|\beta])$, $\alpha \in Q_{p,m}$, $\beta \in Q_{p,n}$ arranged lexicographically in α and β. This matrix will be designated by $C_p(A)$. For example, if $A \in \mathbb{F}^{3 \times 3}$and $p = 2$, the $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$ and

$$C_2(A) = \begin{bmatrix} \det\{A[(1,2)|(1,2)]\} & \det\{A((1,2)|(1,3))\} & \det\{A[(1,2)|(2,3)]\} \\ \det\{A[(1,3)|(1,2)]\} & \det\{A[(1,3)|(1,3)]\} & \det\{A[(1,3)|(2,3)]\} \\ \det\{A[(2,3)|(1,2)]\} & \det\{A[2,3)|(1,3)]\} & \det\{A[(2,3)|(2,3)]\} \end{bmatrix}$$

or setting for convenience $\det\{A[\alpha|\beta]\} = a^{\alpha}_{\beta}$ we have

$$C_2(A) = \begin{bmatrix} a^{1,2}_{1,2} & a^{1,3}_{1,3} & a^{1,2}_{2,3} \\ a^{1,3}_{1,2} & a^{1,3}_{1,3} & a^{1,3}_{2,3} \\ a^{2,3}_{1,2} & a^{2,3}_{1,3} & a^{2,3}_{2,3} \end{bmatrix}$$

Properties of Compound Matrices[Marc.3]

(a) If $A \in \mathbb{F}\mathcal{F}^{n \times n}$, $1 \leq p \leq n$ and also A is non-singular

(i) $(C_p(A))^{-1} = C_p(A^{-1})$ (3.9a)

(ii) $C_p(A^*) = (C_p(A))^*$ (3.9b)

where A^* is the conjugate transpose of $A(\mathbb{F} = \mathbb{C})$.

(iii) $C_p(A^T) = (C_p(A))^T$ (3.9c)

where A^T is the transpose of A.

(iv) $C_p(\bar{A}) = \overline{C_p(A)}$ (3.9d)

where $\bar{A}$ is the conjugate of $A(\mathbb{F}=\mathbb{C})$.

(v) $C_p(kA) = k^p C_p(A) \quad \forall\, k \in \mathbb{F}$ (3.9e)

(vi) $C_p(I_n) = I_{\binom{n}{p}}$ (3.9f)

(vii) Sylvester-Franke Theorem

$$\det\{C_p(A)\} = (\det A)^{\binom{n-1}{p-1}} \qquad (3.9g)$$

□

(b) Binet-Cauchy Therorem: If $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times k}$ and $1 \leq p \leq \min(m,n,k)$ then:

$$C_p(AB) = C_p(A)C_p(B) \tag{3.10}$$

□

(c) If $A \in \mathbb{F}^{p \times n}$ and the p rows of A are denoted by $\underline{a}_1^T, \cdots, \underline{a}_p^T$ in succession ($1 \leq p \leq n$), then $C_p(A)$ is an $\binom{n}{p}$-tuple and is called the *Grassmann product* or *skew symmetric product* of the vectors $\{\underline{a}_1^T, \cdots, \underline{a}_p^T\}$ for reasons which will become apparent later on. The usual notation for this $\binom{n}{p}$-tuple of subdeterminants of A is $\underline{a}_1^T \wedge \cdots \wedge \underline{a}_p^T$ and denotes a row vector. The Grassmann product of the columns of a matrix $A \in \mathbb{F}^{n \times p}$ ($1 \leq p \leq n$) may be defined in a similar manner; the product in this case, however, will be an $\binom{n}{p}$-column vector. If $\{\underline{a}_1, \cdots, \underline{a}_p\}$ are the columns of A, in this case, then this $\binom{n}{p}$-tuple of subdeterminants of A will be denoted by $\underline{a}_1 \wedge \cdots \wedge \underline{a}_p$. By the properties of determinants, if $\sigma \in S_p$ (where S_p denotes the totality of permutations of $1, \cdots, p$), then

$$\underline{a}_{\sigma(1)} \wedge \cdots \wedge \underline{a}_{\sigma(p)} = \text{sign } \sigma \; \underline{a}_1 \wedge \cdots \wedge \underline{a}_p \tag{3.11}$$

If $B \in \mathbb{F}^{n \times n}$, $A \in \mathbb{F}^{n \times p}$, then by the Binet-Cauchy theorem it follows that:

$$C_p(B)\underline{a}_1 \wedge \cdots \wedge \underline{a}_p = B\underline{a}_1 \wedge \cdots \wedge B\underline{a}_p$$

Grassmann products suitably deployed may greatly reduce the complexity of the expressions in compound matrices. Thus, let $A \in \mathbb{F}^{m \times n}$ and $1 \leq p \leq \min(m,n)$. The matrix A may be written in terms of its rows or columns respectively as

$$A = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_m^T \end{bmatrix} \qquad \text{or } A = [\underline{a}_1, \cdots, \underline{a}_n]$$

Let $\omega = \{i_1, \cdots, i_p\} \in Q_{p,m}$ and $\phi = \{j_1, \cdots, j_p\} \in Q_{p,n}$ and let us denote by $\underline{a}_\omega^t \wedge$ the Grassmann product $\underline{a}_{i_1}^T \wedge \cdots \wedge \underline{a}_{i_p}^T$ and by $\underline{a}_\phi \wedge$ the Grassmann product $\underline{a}_{j_1} \wedge \cdots \wedge \underline{a}_{j_p}$. The p-th compound matrix of A may then be expressed in either of the following forms:

$$C(A) = \begin{bmatrix} \vdots \\ \underline{a}_{\omega}^{T} \wedge \\ \vdots \end{bmatrix}, \quad \omega \in Q_{p,m}, \quad \text{or} \quad C_p(A) = [\, \cdots, \underline{a}_{\phi} \wedge, \cdots \,]$$

Having defined the compounds of matrices we will now be able to represent exterior powers of linear maps in terms of their representation of the original linear map. Let h be a linear map from $\mathcal{V}$ to $\mathcal{U}$ as in eq.(3.6a) then

$$[h(\underline{v}_1), h(\underline{v}_2), \cdots, h(\underline{v}_m)] = [\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n] \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix}$$

is the matrix representation of h: $\mathcal{V} \to \mathcal{U}$ with respect to the bases $B_{\mathcal{V}}$ and $B_{\mathcal{U}}$ of $\mathcal{V}$, $\mathcal{U}$ respectively. Note that $H_{i_1,\cdots,i_p}^{j_1,\cdots,j_p}$ is the p-th order minor of $H_{\mathcal{U}}^{\mathcal{V}}$ that lies on the $\{j_1, \cdots, j_p\}$ rows and $\{i_1, \cdots, i_p\}$ columns. If we define $\underline{h}_{\omega} \wedge = \underline{h}_{i_1} \wedge \cdots \wedge \underline{h}_{i_p}$, where $\{\underline{h}_{i_1}, \cdots, \underline{h}_{i_p}\}$ are the columns of H that correspond to the indices $(i_1, \cdots, i_p) \in Q_{p,m}$, then eq.(3.7) may be written as:

$$\wedge^p h(\underline{v}_{\omega} \wedge) = [\, \cdots, \underline{u}_{\rho} \wedge, \cdots \,]\, \underline{h}_{\omega} \wedge, \quad \rho \in Q_{p,n}$$

Since the above equality holds for all $\omega \in Q_{p,m}$, it may be written collectively as:

$$[\, \cdots, \wedge^p h(\underline{v}_{\omega} \wedge), \cdots \,] = [\, \cdots, \underline{u}_{\rho} \wedge, \cdots \,]\,[\, \cdots, \underline{h}_{\omega} \wedge, \cdots \,] = B_{\mathcal{U}}^{p} \wedge^p H_{\mathcal{U}}^{\mathcal{V}} = B_{\mathcal{U}}^{p} C_p(H_{\mathcal{U}}^{\mathcal{V}}) \quad (3.12)$$

where $\wedge^p H_{\mathcal{U}}^{\mathcal{V}} = C_p(H_{\mathcal{U}}^{\mathcal{V}})$ is the matrix representation of $\wedge^p h$ with respect to the bases $B_{\mathcal{V}}^{p}$, $B_{\mathcal{U}}^{p}$ and it is defined by the p-th compound matrix of $H_{\mathcal{U}}^{\mathcal{V}}$. These considerations lead to the following result.

Theorem (3.2): Let $\mathcal{V}$, $\mathcal{U}$ be two vector spaces over $\mathbb{F}$, with $\dim\mathcal{V} = m$, $\dim\mathcal{U} = n$ and let h: $\mathcal{V} \to \mathcal{U}$ be a linear map of $\mathcal{V}$ into $\mathcal{U}$. Let $B_{\mathcal{V}} = \{\underline{v}_i, i = 1, \cdots, m\}$, $B_{\mathcal{U}} = \{\underline{u}_j, j = 1, \cdots, n\}$ be bases of $\mathcal{V}$, $\mathcal{U}$ respectively and let $H_{\mathcal{U}}^{\mathcal{V}}$ be the matrix representation of h with respect to the bases $B_{\mathcal{V}}$, $B_{\mathcal{U}}$. If $\wedge^p h$: $\wedge^p\mathcal{V} \to \wedge^p\mathcal{U}$ $1 \le p \le \min(n,m)$, is the p-th exterior power of h, then $\wedge^p h$ may be represented with respect to the induced bases $B_{\mathcal{V}}^{p}$

$= \{\underline{v}_\omega \wedge,\ \omega \in Q_{p,m}\}$, $B^p_{\mathcal{U}} = \{\underline{u}_\rho \wedge,\ \rho \in Q_{p,n}\}$ of $\wedge^p\mathcal{V}$, $\wedge^p\mathcal{U}$ respectively, by the matrix $\wedge^p H^{\mathcal{V}}_{\mathcal{U}} = C_p(H^{\mathcal{V}}_{\mathcal{U}})$, where $C_p(H^{\mathcal{V}}_{\mathcal{U}})$ is the p-th compound matrix of $H^{\mathcal{V}}_{\mathcal{U}}$.

□

3.3 Decomposability of multivectors, and the Grassmann Representative of a vector space[Marc.1]

3.3.1 Introduction.

In this section we will be dealing with two important topics related to exterior algebra which, in turn, arise from our determinantal problems. The first topic concerns decomposability of multivectors which, in simple terms, means that the multivector can be written as an exterior product of vectors or equivalently, as a compound of a matrix. The conditions for decomposability are important for the study of the multilinear part of the determinantal assignment problem as defined in section 5.5. The second topic is on an exterior algebra characterisation of subspaces of a vector space and this is in terms of the exterior product of the basis which is called the Grassmann representative of the subspace. This representative completely characterises the subspace and can be used for the parametrisation of vector subspaces of equal dimension as points in a certain projective space.

3.3.2 Decomposability of multivectors.

As we have seen in section 3.2 an element $\underline{x}$ of $\wedge^m\mathcal{U}$, where $\mathcal{U}$ is an n-dimensional vector space, is determined by $\binom{n}{m}$ scalars, the coefficients of the unique expansion $\underline{x}$ with respect to a basis of $\wedge^m\mathcal{U}$. Not all elements of $\wedge^m\mathcal{U}$ are decomposable ie. they are of the form $\underline{v}_1 \wedge \cdots \wedge \underline{v}_m$ where $\underline{v}_1, \cdots, \underline{v}_m$ are vectors of $\mathcal{U}$. The decomposability of an element must be reflected in its $\binom{n}{m}$ coordinates, which completely characterises that element.

Proposition (3.2): Let $\mathcal{U}$ be a vector space over $\mathbb{F}$ with dim $\mathcal{U} = n$ and let $\underline{z} \neq 0 \in \wedge^m\mathcal{U}$, $m < n$. Then $\underline{z}$ is decomposable if and only if there exists a linearly independent set of vectors $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_m$ in $\mathcal{U}$ such that

$$\underline{u}_i \wedge \underline{z} = 0 \quad (i = 1, \cdots, m)$$

□

If we write the equation of the above proposition in a matrix form then the decomposability condition of proposition(3.2) can be given in terms of the rank of the coefficient matrix of the equation which ,in turn, is in terms of the multivector $\underline{z}$. This matrix is called the *Grassmann matrix* of the multivector $\underline{z}$ and its properties were examined in [Kar.6]. The next proposition gives us an equivalent definition of decomposability in terms of compounds.

Proposition (3.3): Let $\mathcal{U}$ be a vector space over $\mathbb{F}$ with dim $\mathcal{U} = n$ and let $B_{\mathcal{U}}=\{\underline{u}_1,\underline{u}_2,\cdots, \underline{u}_n\}$ be a basis of $\mathcal{U}$. The induced basis of $\wedge^m\mathcal{U}$, $m < n$ is $B_{\mathcal{U}}^m = \{\underline{u}_\omega\wedge, \omega \in Q_{m,n}\}$ and thus any $\underline{z} \in \wedge^m\mathcal{U}$ can be written as:

$$\underline{z} = \sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge \tag{3.13a}$$

This vector is decomposable iff there exists a matrix $A \in \mathbb{F}^{n \times m}$ such that

$$a_\omega = \det(A[\omega|\, 1, 2, \cdots, m]), \quad \omega \in Q_{m,n}, \quad \text{or} \quad C_m(A) = [\cdots, a_\omega, \cdots] \tag{3.13b}$$

□

Proposition (3.4): Let $\mathcal{U}$ be a vector space over $\mathbb{F}$ with dim $\mathcal{U} = n$, then any vector of $\wedge^{n-1}\mathcal{U}$ is decomposable. □

The next theorem gives us the strongest conditions for decomposability and says that a multivector is decomposable iff its coordinates satisfy certain homogeneous quadratic equations.

Theorem (3.3): Let $\underline{c}=(\cdots, c_\omega, \cdots)$, $\omega \in Q_{m,n}$ be an element of $\mathbb{P}^v(\mathbb{F})$ where $v=\binom{n}{m}-1$. Then $\underline{c}$ is decomposable iff it satisfies the quadratic equations:

$$\sum_{k=1}^{m+1} (-1)^{k-1} c_{i_1,\cdots,i_{m-1}} \cdot c_{j_1,\cdots,j_{k-1}j_{k+1},\cdots j_{m+1}} = 0 \tag{3.14}$$

where $1 \le i_1 < i_2 < \cdots < i_{m-1} \le n$ and $1 \le j_1 < j_2 < \cdots < j_{m+1} \le n$. □

The set of the quadratic equations defined by eq.(3.14) is known as the set of *Quadratic Plücker Relations (QPR)* [Hod.1]. A useful parametrization of the Quadratic Plücker Relations (QPR) in terms of a minimal set of algebraically independent quadratics have been obtained in [Gia.2]. Their results rely heavily upon the following theorem.

Theorem (3.4) [Hod.1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathbb{F}^\sigma$, $\sigma=\binom{p}{q}$, be a decomposable vector satisfying the set of QPRs and let $p_{a_1,\cdots,a_q}$ be a non-zero co-ordinate of $\underline{k}$. If we define by:

$$h_{ij} = p_{a_1,\cdots,a_{i-1}j,a_{i+1},\cdots,a_q}, \quad i \in \underline{q}, \ j \in \underline{p} \tag{3.15}$$

then $C_q(H)= \underline{k}$ where $H = [h_{ij}]$ □

Corollary (3.2) [Hod.1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathbb{F}^\sigma$, $\sigma=\binom{p}{q}$, be a decomposable vector and let the first co-ordinate of $\underline{k}$ be non-zero. The H matrix defined by theorem (3.4) has the form

$$H = [p_a I_q, X^T]^T \in \mathbb{F}^{p \times q}, \text{ where } p_a = p_{1,2,\cdots,q} \neq 0 \tag{3.16a}$$

or in a more detailed form, is expressed as in (3.16b), where by H_0 we denote the matrix that corresponds to the first non-zero Plücker co-ordinate

$$H_0 = \begin{bmatrix} p_{1,2,\cdots,q} & 0 & & 0 \\ 0 & p_{1,2,\cdots,q} & & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & p_{1,2,\cdots,q} \\ & & & \\ (-1)^{q-1}p_{2,\cdots,q,q+1} & (-1)^{q-2}p_{1,3,\cdots,q,q+1} & \cdots & p_{1,2,\cdots,q-1,q+1} \\ & \vdots & & \\ (-1)^{q-1}p_{2,\cdots,q,p} & (-1)^{q-2}p_{1,3,\cdots,q,p} & \cdots & p_{1,2,\cdots,q-1,p} \end{bmatrix}$$

(3.16b)

□

Example (3.2): Let $[p_0,p_1,p_2,p_3,p_4,p_5]^T$ be a decomposable vector of the projective space $\mathbf{P}^5(\mathbb{F})$. A basis of the subspace $\mathcal{V}$ whose Plücker co-ordinates of the given point is:

$$H_0 = \begin{bmatrix} p_0 & 0 \\ 0 & p_0 \\ -p_3 & p_1 \\ -p_4 & p_2 \end{bmatrix}$$

under the assumption $p_0 \neq 0$. If we assume that $p_2 \neq 0$ then a basis of the same subspace is given by:

$$H_2 = \begin{bmatrix} p_2 & 0 \\ p_4 & p_0 \\ p_5 & p_1 \\ 0 & p_2 \end{bmatrix}$$

It can be seen that $H_2 = H_0\, Q$, where

$$Q = \begin{bmatrix} \frac{p_2}{p_0} & 0 \\ \frac{p_4}{p_0} & 1 \end{bmatrix}, \quad \det Q \neq 0$$

Note that $C_2(H_0) = [p_0^2, p_0p_1, p_0p_2, p_0p_3, p_0p_4, p_1p_4-p_2p_3]$ which implies that $p_5=(p_1p_4-p_2p_3) / p_0$, or equivalently $p_0p_5-p_1p_4+p_2p_3 = 0$ □

The above example suggests a method for writing down an independent set of QRPs which completely describe all decomposable vectors with having a certain coordinate nonzero; such a set will be referred to as the *Reduced Quadratic Plücker Relations (RQPR)*. For a given nonzero coordinate, the number of RQPR is $\binom{p}{q} - q(p-q)-1$.

Proposition (3.5) [Gia, 1]: Let $\underline{k} = [\cdots, p_\omega, \cdots]^T \in \mathbb{F}^\sigma$, $\sigma=\binom{p}{q}$ be a decomposable vector and let the first co-ordinate of $\underline{k}$ be non-zero. If H_0 is the matrix which is defined by Corollary (3.5), then the equation

$$C_q(H_0) = [\cdots, p_\omega, \cdots]^T \, p_{1,2,\cdots,q}^{q-1} \tag{3.17}$$

defines a set of RQPR with respect to $p_{1,2,\cdots,q}$ co-ordinate. □

A similar procedure can be applied for any non-zero co-ordinates of $\underline{k}$.

Example (3.3): Let $\underline{k} = [p_0, p_1, p_2, \cdots, p_9]$ be a point of the Grassmann variety $\Omega(3,5)$ of the projective space $\mathbb{P}^9(\mathbb{F})$. We can reconstruct H_0 according to corollary (3.4) assuming $p_0 \neq 0$ as follows:

$$H_0 = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \\ p_6 & -p_3 & p_1 \\ p_7 & -p_4 & p_2 \end{bmatrix}$$

The number of RQPR in this case is $\binom{5}{3} - 3(5-3)-1 = 3$. Equation (3.17) gives the following set of equations:

$$p_0^3 = p_0 p_0^2, \quad p_0^2 p_1 = p_1 p_0^2, \quad p_0 p_2 = p_2 p_0^2, \quad p_0^2 p_3 = p_3 p_0^2$$
$$p_0^2 p_4 = p_4 p_0^2, \quad p_0^2 p_6 = p_6 p_0^2, \quad p_0^2 p_7 = p_7 p_0^2 \tag{3.18a}$$

$$p_0 (p_1 p_4 - p_2 p_3) = p_5 p_0^2, \qquad p_0 (p_1 p_7 - p_2 p_6) = p_8 p_0^2$$
$$p_0 (p_3 p_7 - p_4 p_6) = p_9 p_0^2 \tag{3.18b}$$

It is clear that the set (3.18a) is trivial, whereas the set (3.18b) obtained after the cancelation of p_0 is a minimal set. □

3.3.3. Grassmann representative of a vector space

The use of exterior algebra is convenient for the parametrisation of a set of linear subspaces contained in a linear space. The effect of the exterior product is to compress the subspaces into one dimensional linear spaces. In this way, the original subspaces are equal iff the two auxiliary one dimensional spaces are equal. In terms of bases, we have to check whether two vectors are colinear as stated in the next proposition.

Proposition (3.6): Let $\mathcal{U}$ be an n-dimensional vector space over $\mathbb{F}$; $\underline{y}\wedge = \underline{y}_1 \wedge \cdots \wedge \underline{y}_m$, $\underline{z}\wedge = \underline{z}_1 \wedge \cdots \wedge \underline{z}_m$ be two decomposable non-zero elements of $\wedge^m \mathcal{U}$ (m<n). Then the two m-dimensional vector spaces $\mathcal{V}_y = \text{span}\{\underline{y}_1, \cdots, \underline{y}_m\}$ and $\mathcal{V}_z = \text{span}\{\underline{z}_1, \cdots, \underline{z}_m\}$ are equal iff there exists $q \in \mathbb{F}-\{0\}$ such that

$$\underline{y}_1 \wedge \cdots \wedge \underline{y}_m = q\, \underline{z}_1 \wedge \cdots \wedge \underline{z}_m \tag{3.19}$$

□

Definition (3.1): Let $\mathcal{V}$ be an m-dimensional vector space in $\mathbb{F}^n$, then any non-zero decomposable element $\underline{x}_1 \wedge \cdots \wedge \underline{x}_m$, $\underline{x}_i \in \mathcal{V}$, $i \in \underline{m}$ is called a *Grassmann representative* for $\mathcal{V}$ [Mar., 1]. The Grassmann representatives all differ only by non-zero scalar factors so that we shall denote any of them by $\underline{g}(\mathcal{V})$. □

The characterisation of a vector space $\mathcal{V}$ by its Grassmann representative $\underline{g}(\mathcal{V})$ provides the means for the definition of the classical Plücker embedding [Hod.1]. The $\binom{n}{m}$ coordinates of the grassmann representative, $(\ldots\ a_\omega \ldots)$ may be regarded as the homogeneous co-ordinates of a point in the projective space $\mathbf{P}^v(\mathbb{F})$ where $v = \binom{n}{m} - 1$. Then this point depends only on the subspace $\mathcal{V}$ and not on the choice of basis. What we have constructed so far is a well defined projective mapping p: $G_m(\mathbb{F}^n) \to \mathbf{P}^v(\mathbb{F})$ with $p(\mathcal{V}) = \underline{g}(\mathcal{V})$, where $G_m(\mathbb{F}^n)$ is the set of all m-dim linear subspaces of $\mathbb{F}^n$ called the *Grassmannian.* The co-ordinates of the point $\{a_\omega\}$ in $\mathbf{P}^v(\mathbb{F})$ are called the *Plücker co-ordinates* of $\mathcal{V}$ and the mapping p is the *Plücker embedding* of the Grassmannian $G_m(\mathbb{F}^n)$ into the projective space $\mathbf{P}^v(\mathbb{F})$. The Plucker embedding of the Grassmannian is a very important tool for our problems and will be used extensively throughout this thesis. As a matter of fact, the Plucker embedding embeds the Grassmannian into $\mathbf{P}^v(\mathbb{F})$ as a projective variety defined by the QPR's and allows us to examine our problems in the framework of algebraic geometry.

3.4 Compex and real varieties [Mum.1]

3.4.1. Affine and Projective Varieties

An *affine variety* is defined to be the set of all points of $\mathbb{F}^n$ whose cooordinates satisfy the (not necessarily homogeneous) polynomial equations:

$$f_i(x_1,...,x_n) = 0 \qquad i=1, 2,..., t.$$

On the other hand, if $\mathbb{P}^n(\mathbb{F})$ is a projective space over the field $\mathbb{F}$, then the set of all the points of $\mathbb{P}^n(\mathbb{F})$ whose coordinates satisfy the homogenous polynomial equations:

$$f_i(x_0, x_1,...,x_n) = 0 \qquad i=1,2,...,t$$

is called a *projective variety.*

The structures of affine and projective varieties are strongly connected. Indeed every affine variety $\mathfrak{G}$ in $\mathbb{F}^n$ can be compactified to a projective variety $\bar{\mathfrak{G}}$ in $\mathbb{P}^n(\mathbb{F})$ and conversely , as follows: $\mathbb{P}^n(\mathbb{F})$ can be covered by the sets $\mathbb{F}^n_i$ for $0 \leq i \leq n$ such that

$$\mathbb{F}^n_i=\left\{ (x_0, x_1,...,x_n) \in \mathbb{P}^n(\mathbb{F}) \ : \ x_i \neq 0 \right\}$$

Every $\mathbb{F}^n_i$ is isomorphic to $\mathbb{F}^n$ (via division by x_i) and thus are called *affine open subsets* of $\mathbb{P}^n(\mathbb{F})$. For every projective variety , $\mathcal{Y}$, of $\mathbb{P}^n(\mathbb{F})$ the i-th affine open subset of $\mathcal{Y}$, called $\mathcal{Y}_i$, is defined to be equal to $\mathcal{Y} \cap \mathbb{F}^n_i \subset \mathbb{F}^n_i \approx \mathbb{F}^n$ and is an affine variety of $\mathbb{F}^n$. Conversely, every affine variety $\mathfrak{G}$ in $\mathbb{F}^n$ can be considered as the i-th affine open subset of a projective variety $\bar{\mathfrak{G}}$ in $\mathbb{P}^n(\mathbb{F})$.

Example (3.4) For example consider the variety $\mathcal{Y}$: $x^2+xy-y^2=0$ in $\mathbb{P}^1(\mathbb{C})$. Dividing by x^2 and setting $z=y/x$ we get $\mathcal{Y}_0$ to be $1+z-z^2=0$ in $\mathbb{C}$. Dividing by y^2 and setting $z=x/y$ we get $\mathcal{Y}_1$ to be $z^2+z-1=0$ in $\mathbb{C}$. Conversely, the affine variety $z^2+z-1=0$ in $\mathbb{C}$

can be considered as the 1st affine subset of the projective variety $x^2+xy-y^2=0$ in $\mathbb{P}^1(\mathbb{C})$ and this by setting $z=x/y$ and multiplying by y^2 (this process is called homogenisation). □

3.4.2. Basic notions on varieties

A *subvariety* of a variety $\mathcal{V}$ is a subset of $\mathcal{V}$ satisfying an additional set of equations. We may topologize $\mathcal{V}$ defining its closed sets to be all its subvarieties; this topology is called *Zarisky topology.* The open sets of this topology are called *Zarisky open sets.*

A variety $\mathcal{V}$ is said to be ***reducible*** if it can be expressed as a sum of two proper algebraic subvarieties. If $\mathcal{V}$ is not reducible, it is called *irreducible.*

Another basic concept is that of the tangent space of the variety $\mathcal{V} \subset \mathbb{F}^n$ at the point $v \in \mathcal{V}$. Suppose first that $\mathcal{V}$ is affine and let $f_i(x)$ be the equations defining this variety; the tangent space at a is defined to be the linear space

$$\mathrm{Tan}(v)=\left\{(x_1,x_2,\ldots,x_n) \in \mathbb{F}^n \colon (x_1-a_1,x_2-a_2,\ldots,x_n-a_n).J_V(f)=0\right\}$$

where $J_V(f)=\left(\frac{\partial f_i}{\partial x_j}\right)_v$ is the Jacobian evaluated at a. Clearly the dimension of Tan(v) as an $\mathbb{F}$ vector space is equal to $n-\mathrm{rank}(J_V(f))$.This dimension is constant, say d, for all "v" in a nonempty Zarisky open subset of $\mathcal{V}$ which is called the set of smooth points of $\mathcal{V}$ and symbolised by Smooth($\mathcal{V}$). The complementary set is called the set of singular points of $\mathcal{V}$ and symbolised by Sing($\mathcal{V}$); if $v \in \mathrm{Sing}(\mathcal{V})$ then $\dim\mathrm{Tan}(v) \geq d$. If $\mathcal{V}$ is a projective variety in $\mathbb{P}^n(\mathbb{F})$ and $v \in \mathcal{V}$ then we consider the affine open subset of $\mathcal{V}$ to which 'v' belongs and then proceed as in the affine case presented above.

The dimension of a variety $\mathcal{V}$ is the minimum number of independend parameters defining it. For example the variety of $\mathbb{C}^3$ defined by the equation $x+y+z=0$ is two dimensional since two of the three variables x,y,z can be considered free. On the other hand if we consider the variety defined from the equations: (i) $x+y+z=0$ and (ii) $z^2-z=x^2+y^2-x-y+2xy$, is not one dimensional as one could expect but two dimensional again, since the second equation is redundant. However, the concepts of dimension of a variety and the number of minimal equations defining it are not always clearly connected as the following example demonstrates.

Example(3.5) Let $\mathcal{V}$ be the affine variety of all $(x,y,z) \in \mathbb{C}^3$ defined by the following 3 equations:
(i) $y^2 - xz=0$, (ii) $x^3 - yz=0$, (iii) $z^2 - x^2y=0$. It can be easily seen that $\mathcal{V}$ is the set of all triples (t^3,t^4,t^5) where $t \in \mathbb{C}$ and therefore it is one dimensional. Furthermore we have that $z(y^2 - xz)+y(x^3 - yz)+x(z^2 - x^2y)=0$ and thus we might expect that one of the three equations defining $\mathcal{V}$ must be redundant. But this is not the case, since if for instance we ommit (iii) then the two remaining equations define a variety which contains $\mathcal{V}$ as well as the set $(0,0,t)$, $t \in \mathbb{C}$ (similarly we can examine the rest of the cases).

□

An easy way to calculate the dimension, is to locally linearise the set of our equations by using their Jacobian.Then the dimension of the variety is given by $n - \text{rank}(J)$ where the rank of the Jacobian is calculated at a smooth point of the variety and n is the dimension of the underlying space; this calculation can be easily carried out and allows us to give the following definition of the dimension: the *dimension* of an irreducible variety $\mathcal{V}$ can be defined as the vector space dimension of the tangent space of a smooth point of $\mathcal{V}$.

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be two varieties in $\mathbb{P}^n(\mathbb{F})$ given by the equations

$$\text{a) } f_i(x_0, x_1,\dots,x_n) = 0 \qquad i=1,2,\dots,t_1$$
$$\text{b) } g_j(x_0, x_1,\dots,x_n) = 0 \qquad j=1,2,\dots,t_2$$

respectively. Then the points of $\mathbb{P}^n(\mathbb{F})$ which satisfy both sets of equations simultaneously define the *intersection of the varieties* $\mathcal{V}_1$, $\mathcal{V}_2$ denoted by $\mathcal{V}_1 \cap \mathcal{V}_2$. Next we give a condition for two projective varieties to intersect(projective intersection lemma).

Lemma(3.1)

a) If for two projective varieties $\mathcal{X},\mathcal{Y}$ in $\mathbb{P}^k(\mathbb{C})$ we have that

$$\dim(\mathcal{X})+\dim(\mathcal{Y}) \geq k$$

then the variety $\mathcal{X}\cap\mathcal{Y}$ is nonvoid and $\dim(\mathcal{X}\cap\mathcal{Y}) \geq \dim(\mathcal{X})+\dim(\mathcal{Y})-k$ (generically $\dim(\mathcal{X}\cap\mathcal{Y})=\dim(\mathcal{X})+\dim(\mathcal{Y}) - k$).

b) If for two projective varieties $\mathcal{X},\mathcal{Y}$ in $\mathbb{P}^k(\mathbb{C})$ we have that

$$\dim(\mathcal{X})+\dim(\mathcal{Y}) < k,$$

then the variety $\mathcal{X}\cap\mathcal{Y}$ is generically empty. □

For two affine varieties $\mathcal{X},\mathcal{Y}$ in $\mathbb{C}^k$ we can state a lemma similar to the lemma(3.1) (affine intersection lemma).

Lemma (3.2) For two irreducible affine varieties $\mathcal{X},\mathcal{Y}$ in $\mathbb{C}^k$ we have that either

a) $\mathcal{X}\cap\mathcal{Y}=\emptyset$

or

b) $\dim(\mathcal{X}\cap\mathcal{Y}) \geq \dim\mathcal{X}+\dim\mathcal{Y}-k$ □

In the case where $\mathcal{X}$ and $\mathcal{Y}$ are Zarisky open subsets of projective varieties we can use the closures $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ in $\mathbb{P}^k(\mathbb{C})$ and Lemma (3.1) to study their intersections.

The points of $\mathbb{P}^n(\mathbb{F})$ satisfying the equations:

$$f_i(x_0, x_1,\ldots,x_n)\, g_j(x_0, x_1,\ldots,x_n) = 0 \qquad \text{for} \quad i=1, 2,\ldots,t_1 \text{ and } j=1, 2,\ldots,t_2$$

define the *union* $\mathcal{V}_1 \cup \mathcal{V}_2$.

Example (3.6) The Grassmann variety

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. As it has been defined in 3.3.3 the Grassmannian $G_p(\mathbb{F}^{p+r})$ represents the set of all p-dimensional linear subspaces of $\mathbb{F}^{p+r}$. It can be given the structure of a projective variety by the classical Plucker embedding as in 3.3.3 by assigning to every p dimensional subspace, $\mathcal{V} \in G_p(\mathbb{F}^{p+r})$ its Grassmann representative modulo dilations. Thus, the Plucker embedding is a (polynomial) map

$$\mathcal{P}: \; G_p(\mathbb{F}^{p+r}) \rightarrow \mathbb{P}^{\sigma-1}(\mathbb{F})$$

where $\sigma=\binom{p+r}{p}$, such that if V is a px(p+r) representation matrix of $\mathcal{V}$ then $\mathcal{P}(\mathcal{V})=C_p(V)$mod-dilations. By proposition (3.6), $\mathcal{P}$ is one to one and by calculating the differential of this map one can see that $\mathcal{P}$ is an embedding, of the pr-dimensional manifold $G_p(\mathbb{F}^{p+r})$ (see 6.5.3) into $\mathbb{P}^{\sigma-1}(\mathbb{F})$. The image of $\mathcal{P}$ in the projective space is exactly the set defined by the Quadratic Plucker Relations (QPR) as theorem(3.3) states, thus Im($\mathcal{P}$) is a projective variety , named Grassmann variety. The dimension of this variety is pr since it is also a pr-dimensional manifold, but the quadratic Plucker relations cannot be reduced to the expected number σ-1-pr, on the whole Grassmannian. However, this reduction is possible if one of the projective coordinates of the Grassmannian is non zero. In this case, we have the RQPRs as proposition(3.5) states and can construct a basis matrix for the vector space of the decomposable multivector as in theorem(3.4).

□

3.4.3.Morphisms of Complex Algebraic Varieties

Let $\mathcal{X},\mathcal{Y}$ be two affine varieties. Then a *morphism* f: $\mathcal{X}\rightarrow\mathcal{Y}$ is a map that can be described as $f=(f_1,f_2,\ldots,f_n)$ where f_j are polynomial functions. If $\mathcal{X},\mathcal{Y}$ are projective then a morphism f: $\mathcal{X}\rightarrow\mathcal{Y}$ is a map described by polynomial functions, f_j, as above with the additional requirement that these functions must be homogeneous and of the same degree.

Example(3.7) Let (x_1,x_2) be the homogeneous coordinates of $\mathbb{P}^1(\mathbb{C})$, and (x_1,x_2,x_3) be those of $\mathbb{P}^2(\mathbb{C})$. Then the map f: $\mathbb{P}^1(\mathbb{C})\rightarrow\mathbb{P}^2(\mathbb{C})$ such that $f((x_1,x_2))=(x_1^2,x_1x_2,x_2^2)$ is a morphism. It is worth noting that the image of f is the variety $x_2^2-x_1x_3=0$ in $\mathbb{P}^2(\mathbb{C})$; this reflects a very important property of the projective varieties , as is described in the following theorem:

Theorem(3.5) The image of a projective variety through a morphism is always a variety. □

Next we give an example of a morphism whose image is not closed.

Example(3.8) Let (x_1,x_2,x_3) be the affine coordinates of $\mathbb{C}^3$ and (x_1,x_2) be those of $\mathbb{C}^2$. Define $\mathcal{Y}$: $x_1x_3\text{-}x_2=0 \subseteq \mathbb{C}^3$ an irreducible affine variety. Let π: $\mathbb{C}^3 \rightarrow \mathbb{C}^2$ be the projection map $(x_1,x_2,x_3) \rightarrow (x_1,x_2)$. Then $\pi/\mathcal{Y}$: $\mathcal{Y} \rightarrow \mathbb{C}^2$ is a morphism and the image $\pi(\mathcal{Y})$ is neither open nor closed. Indeed it is easy to check that $\pi(\mathcal{Y})$ is the union of the Zariski open subset $\mathbb{C}^2\text{-}\{x_1: x_1=0\}$ and the origin (0,0). □

The above set $\pi(\mathcal{Y})$ is a finite union of locally closed sets $\mathcal{Y}_1 \cup \mathcal{Y}_2 \ldots \cup \mathcal{Y}_n$ or else sets that can be written as $\mathcal{Y}_i = \mathcal{Y}_i' - \mathcal{Y}_i''$ where $\mathcal{Y}_i'$ is a variety and $\mathcal{Y}_i'' \subset \mathcal{Y}_i'$ is a smaller subvariety. Sets that are finite union of locally closed sets are called *constructible sets.*

Theorem(3.6) Let f: $\mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of affine varieties then f maps constructible sets to constructible sets. □

Dominant morphisms. According to what was said previously, if f is a morphism from $\mathcal{X}$ to $\mathcal{Y}$ then $f(\mathcal{X})$ is very close to being a variety (since $f(\mathcal{X})$ is always a constructible set) or in other words $f(\mathcal{X})$ is very close to its Zarisky closure $\overline{f(\mathcal{X})}$. A very interesting case of morphisms is when $\mathcal{X}$ is irreducible and $\overline{f(\mathcal{X})}=\mathcal{Y}$ or else $f(\mathcal{X})$ is Zarisky dense in $\mathcal{Y}$, in this case we say that f is *dominant.* A dominant morphism is very close to be onto ,that is :

Theorem(3.7) Let f: $\mathcal{X} \rightarrow \mathcal{Y}$ be a dominant morphism of varieties then there is a Zarisky open subset of $\mathcal{Y}$, say $\mathcal{U}$, such that $\mathcal{U} \subset f(\mathcal{X})$. □

To see whether a morphism f: $\mathcal{X} \rightarrow \mathcal{Y}$ is dominant it is sufficient to find a point $x \in \mathcal{X}$ where f is locally onto since this fact ensures us that $\overline{f(\mathcal{X})}=\mathcal{Y}$. We may test whether f is onto locally at a point $x \in \mathcal{X}$ by using the differential $(Df)_X$:Tan(x)→Tan(f(x)); if this map is onto then f is locally onto at x. To summarise:

Corollary(3.3) Let $f:\mathfrak{X}\to\mathfrak{Y}$ be a morphism of varieties such that $\exists x\in\mathfrak{X}$: the differential $(Df)_x$ is onto, then f is almost onto (in the sense of Th.(3.1)).

The above corollary can easily be applied, since it only involves the computation of the rank of the linear map $(Df)_x$.

Finite morphisms. This is a case of dominant morphism $f:\mathfrak{X}\to\mathfrak{Y}$ where for every $y\in Y$, $\#f^{-1}(y)$ is finite and constant. Finite morphisms normally arise from projections, for example : Let $\mathfrak{X}\subset\mathbb{P}^n(\mathbb{C})$ be an n-1 dimensional variety given by one homogeneous equation $F(x)=0$, such that $a=(1,0,\dots,0)\notin\mathfrak{X}$. Consider now the projection :

$$p_a: \mathbb{P}^n(\mathbb{C})-a\to\mathbb{P}^{n-1}(\mathbb{C})$$

$p_a((x_0,x_1,\dots,x_n))=((x_1,x_2,\dots,x_n))$. Since $a\notin\mathfrak{X}$ we have that $F(a)\neq 0$ so $F(x)$ can be written as:

$$F(x)=x_0^d+F_1((x_1,x_2,\dots,x_n))x_0^{d-1}+F_2((x_1,x_2,\dots,x_n))x_0^{d-2}+\ .\ .\ .\ +F_n((x_1,x_2,\dots,x_n))$$

We easily see that for every $y=(x_1,x_2,\dots,x_n)\in p_a(\mathfrak{X})$ there exist exactly 'd' inverse images via p_a.

By combining projections of the above form we can define the projection $p_{\mathcal{L}}$ with center $\mathcal{L}$ as follows:

Definition(3.2) Let $\mathcal{L}$ be the Kernel of a linear map $F:\mathbb{C}^{n+1}\to\mathbb{C}^{m+1}$ then F naturally induces the linear map $p_{\mathcal{L}}:\mathbb{P}^n(\mathbb{C})-\mathcal{L}\to\mathbb{P}^m(\mathbb{C})$, this map is called a *central projection* with centre (or base locus) $\mathcal{L}$.

We can state a similar result with the above , that is if $\mathfrak{X}$ is a variety such that $\mathfrak{X}\cap\mathcal{L}=\emptyset$ then $p_{\mathcal{L}}/\mathfrak{X}$ is finite. We can also prove the following theorem:

Theorem(3.8) (Noether Normalization Lemma) [Mum.1] Let $\mathfrak{X}$ be an r-dim subvariety of $\mathbb{P}^n(\mathbb{C})$ then:

a)there exists a linear subspace $\mathcal{L}$ of dimension n-r-1 such that $\mathcal{L} \cap \mathfrak{X} = \emptyset$.

b)For all such $\mathcal{L}$ the projection $p_{\mathcal{L}}$ restricts to a "finite to one" surjective closed map:

$$p_{\mathcal{L}}: \mathfrak{X} \to \mathbb{P}^r(\mathbb{C})$$

□

Definition (3.3) The constant number $\# p_{\mathcal{L}}^{-1}(c)$ of theorem (3.7),where $c \in \mathbb{P}^r(\mathbb{C})$ is called the *degree* of the projective variety $\mathfrak{X}$, symbolised by $deg_{\mathbb{C}}(\mathfrak{X})$.

□

It can be proved that it is equal to the number of points of the intersection of $\mathfrak{X}$ by a generic n-r dimensional linear subspace of $\mathbb{P}^n(\mathbb{C})$[Mum.1]. We can more generally define the degree of a finite mapping f: $\mathfrak{X} \to \mathfrak{Y}$ between two projective varieties , symbolised by $deg_{\mathbb{C}}(f)$, to be the constant number $\# f^{-1}(y)$ and it can be proved that [Mum.1]:

$$deg_{\mathbb{C}}(\mathfrak{X}) = deg_{\mathbb{C}}(f) deg_{\mathbb{C}}(\mathfrak{Y})$$

A more generalised definition of degree can be found in sec.3.6 where the degree of a compact oriented manifold is defined using homology theory. The degree of a smooth complex projective variety can be defined in this context (since it is a compact oriented manifold) and it can be proved that this definition is equivalent to the one above.

3.4.4. Morphisms of Real Algebraic Varieties

Let $\mathfrak{X}, \mathfrak{Y}$ be two real affine varieties. Then a *morphism* f: $\mathfrak{X} \to \mathfrak{Y}$ is a map that can be described as $f=(f_1,f_2,...,f_n)$ where f_j are polynomial functions. If $\mathfrak{X}, \mathfrak{Y}$ are projective, then a morphism f: $\mathfrak{X} \to \mathfrak{Y}$ is a map described by polynomial functions, f_j, as above with the additional requirement that these functions must be homogeneous and of the same degree.

The Image of such a morphism is not an algebraic variety but a semialgebraic set. To illustrate this fact, it is sufficient to see the real variety $y=x^2$; the projection $p:\mathbb{R}^2 \to \mathbb{R}$ on the y-coordinate is a regular function mapping the variety to the set $y \geq 0$. A *semialgebraic set* in $\mathbb{R}^n$ is the solution set of polynomial inequalities and equalities. A map f: $\mathfrak{X} \to \mathfrak{Y}$ is called semialgebraic, iff its graph is a semialgebraic set in $\mathfrak{X} \times \mathfrak{Y}$.

Theorem (3.9) A semialgebraic map f: $\mathfrak{X}\to\mathfrak{Y}$ between semialgebraic sets, maps semialgebraic sets to semialgebraic sets.

□

Dominant morphisms. Let $f:\mathfrak{X}\to\mathfrak{Y}$ be a morphism of real varieties, where $\mathfrak{X}$ and $\mathfrak{Y}$ are irreducible, then f is dominant iff $\overline{f(\mathfrak{X})}=\mathfrak{Y}$. Unlike the complex case the dominance of a map does not imply that $f(\mathfrak{X})$ covers almost the whole of $\mathfrak{Y}$. What actually happens is that $f(\mathfrak{X})$ is an object defined by inequalities and having dimension equal to the dimension of $\mathfrak{Y}$. For example if $f:\mathbb{R}\to\mathbb{R}$ is the function $f(x)=x^2$ then image of f is the set of positive numbers. The dominance of a morphism can still be tested via the rank of its differential at some point $x\in\mathfrak{X}$.

Finite morphisms. Morphisms f: $\mathfrak{X}\to\mathfrak{Y}$ arising from projections are 'finite to one' but the number $\#f^{-1}(y)$ is not conserved as y varies in $\mathfrak{Y}$. For example, consider $\mathfrak{X}\subset\mathbb{P}^2(\mathbb{R})$ to be the set of all (x,y,z) such that $y^2=xz$ and let a=(1,0,0) then $p_a(\mathfrak{X})$ contains all $(y,z)\in P^1(\mathbb{R})$ such that $yz\geq 0$ and thus $p_a^{-1}(y)$ contains either 2 or 0 points. For such morphisms what is conserved is $\#f^{-1}(y)$ mod2. The next theorem is analogous to the Noether Normalisation Lemma but refers to projective varieties.

Theorem(3.10) Let $\mathfrak{X}$ be an r-dim subvariety of $\mathbb{P}^n(\mathbb{R})$ then:

a) there exists a linear subspace $\mathcal{L}$ of dimension n-r-1 such that $\mathcal{L}\cap\mathfrak{X}=\emptyset$.

b) For all such $\mathcal{L}$ the projection $p_{\mathcal{L}}$ restricts to a 'finite to one' (in the sense defined previously) map:

$$p_{\mathcal{L}}: \mathfrak{X}\to\mathbb{P}^r(\mathbb{R})$$

□

The constant number $\#p_{\mathcal{L}}^{-1}(c)$mod2 is called the degree of $\mathfrak{X}$ and is symbolised by $\deg_{\mathbb{R}}(\mathfrak{X})$. There is an obvious relation of the above degree of $\mathfrak{X}$ and the degree of its complexification $\mathfrak{X}_{\mathbb{C}}$. In fact, we have:

$$\deg_{\mathbb{R}}(\mathfrak{X})=\deg_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}})\text{mod}2$$

3.5 Intersection Theory of Complex Algebraic Varieties

3.5.1 Compactification

Many problems considered in this thesis are problems involving the solution of algebraic equations which in the setting of the previous chapter are problems of intersection of varieties. One set of the varieties in this intersection problem is parametrised by another variety, the variety of the systems of p-inputs, m-outputs and n-states. Such a set of intersections, parametrised by a variety, can be viewed as a certain element of an intersection ring of a variety. In this section we present some basic intersection theory for complex varieties which will provide the foundation for the understanding of the specific intersection theory involved in our problems. The more specific intersection theory based on homology of manifolds will be presented later and will help us to derive necessary as well as sufficient conditions for the existence of solutions of our system of equations; it will also help us to gain a better insight of the special nature of the equations and suggest ways of solving them.

The natural field for the intersection theory of varieties is the field of complex numbers $\mathbb{C}$. This field is algebraically closed, which means that every polynomial equation in one complex variable can always be solved. In fact the number of solutions, counted with multiplicities, equals the degree of the polynomial. In the case where we have n (algebraically independent) equations in n unknowns we expect that after succesive elimination of the n-1 unknowns we end up with one equation having one unknown which as we said is always solvable. There are, however, cases where such a system of polynomial equations is not solvable, such as the system xy=1 and xy=-1. In this case we say that the two equations intersect at infinity. Here 'infinity' means that if we projectivise them into $xy=z^2$ and $xy=-z^2$ then they intersect only if z=0 , the infinity space of the projectivisation. What actually happens is that two projective varieties $\mathfrak{X},\mathfrak{Y}\subset\mathbb{P}^n(\mathbb{C})$ always intersect provided $\dim\mathfrak{X}+\dim\mathfrak{Y}\geq n$ (lemma3.1) in this case we call the intersection *proper* if every irreducible component of $\mathfrak{X}\cap\mathfrak{Y}$ has dimension $\dim\mathfrak{X}+\dim\mathfrak{Y}-n$. The advantage of projective varieties (or compact manifolds) as spaces of parametrised intersections, lies on the fact that as the parameters vary, the number of points of the intersection (if finite) does not change. This *conservation of number* may not occur in parametrised intersections on affine varieties (or more generally non compact manifolds), since as the parameters vary, some of the points of the intersection may disappear at infinity. This point is illustrated in the following example.

Example(3.9) Consider the variety $\mathcal{V}_{a,b}$ of all points $(x,y) \in \mathbb{C}^2$ such that

$$x+y=0$$
$$ax+by=1$$

As the pair (a,b) vary, $\mathcal{V}_{a,b}$ contains only one point with the exception when a=b in which case it contains none. On the other hand, if we projectivise $\mathcal{V}_{a,b}$ we get a projective variety $\overline{\mathcal{V}_{a,b}}$ in $\mathbb{P}^2(\mathbb{C})$ containing all points $(x,y,z) \in \mathbb{P}^2(\mathbb{C})$ such that:

$$x+y=0$$
$$ax+by=z$$

The variety $\overline{\mathcal{V}_{a,b}}$ always contains one point, since if a=b there is a common solution of the above, namely $(1,-1,0) \in \mathbb{P}^2(\mathbb{C})$

□

The above discussion leads us to consider projective instead of affine complex varieties for intersection purposes. The new projective variety, coming from the affine one, is called *compactification* and it is constructed by sticking together a negligigle set of points of the affine variety called *points at infinity*. The way we compactify $\mathbb{C}^n$ into a projective variety is not unique, for instance $\mathbb{C}^4$ can be compactified into $\mathbb{P}(\mathbb{C}^4)$, $G_2(\mathbb{C}^6)$ and $\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})$ as shown in the following example:

Example (3.10) Let k_1, k_2, k_3, k_4 be the four independent coordinates of $\mathbb{C}^4$. We can consider the three following embedings of $\mathbb{C}^4$:

$$f_1(k_1, k_2, k_3, k_4) = (1, k_1, k_2, k_3, k_4\ ) \in \mathbb{P}(\mathbb{C}^4)$$

$$f_2(k_1, k_2, k_3, k_4) = (1, k_1) \otimes (1, k_2) \otimes (1, k_3) \otimes (1, k_4) \in \mathbb{P}(\mathbb{C}^{15})$$

where $\otimes$ denotes tensor product, or

$$f_3(k_1, k_2, k_3, k_4) = C_2\left(\begin{bmatrix} 1 & 0 & k_1 & k_2 \\ 0 & 1 & k_3 & k_4 \end{bmatrix}\right) \in \mathbb{P}(\mathbb{C}^5)$$

Consider now the Zarisky closures of every map in the corresponding projective space and call them $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, respectively. The variety $\mathcal{A}_1$ in $P(\mathbb{C}^4)$ consists of all 5-tuples $(\lambda, k_1, k_2, k_3, k_4)$ modulo dilations; the infinity here is represented by the subvariety $\lambda=0$. In the second case, $\mathcal{A}_2$ is $\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})x\mathbb{P}(\mathbb{C})$ which consists of all 16-tuples of the form $(\lambda_1, k_1)\otimes(\lambda_2, k_2)\otimes(\lambda_3, k_3)\otimes(\lambda_4,k_4)$ modulo dilations; in this case the infinity is represented by the subvariety $\lambda_1.\lambda_2.\lambda_3.\lambda_4=0$. In the last case $\mathcal{A}_3$ is the Grassmannian $G_2(\mathbb{C}^6)$ which is given by all the 6-tuples of the form

$$C_2\left(\begin{bmatrix} \lambda_1 & \lambda_2 & k_1 & k_2 \\ \lambda_3 & \lambda_4 & k_3 & k_4 \end{bmatrix}\right)$$

In this case, the infinity is given by the subvariety $\lambda_1.\lambda_4 - \lambda_2.\lambda_3=0$. We can see the difference in the three previous compactifications of $\mathbb{C}^4$ in that in each case we can approach infinity from different directions. For example, if we consider the unbounded sequence $(n, 1, 2n, 3) \in \mathbb{C}^4$ then the limit in the first compactification will be

$$(1, n, 1, 2n, 3)=(1/n, 1, 1/n, 2, 3/n) \to (0, 1, 0, 2, 0) \in P(\mathbb{C}^4)$$

The limit of the same sequence considered in the second compactification can be calculated as follows:

$$(1, n)\otimes(1, 1)\otimes(1,2n)\otimes(1, 3) = (1/n, 1)\otimes(1, 1)\otimes(1/n, 2)\otimes(1, 3) \to (0,1)\otimes(1,1)\otimes(0,2)\otimes(1, 3)$$

Similarly, in the case of the Grassmannian, the limit can be calculated as follows:

$$C_2\left(\begin{bmatrix} 1 & 0 & n & 1 \\ 0 & 1 & 2n & 3 \end{bmatrix}\right) = C_2\left(\begin{bmatrix} 1/n & 0 & 1 & 1/n \\ -2 & 1 & 0 & 1 \end{bmatrix}\right) \to C_2\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}\right)$$

□

The compactification of the affine space depends upon the individual intersection problem we consider and reflects its nature. We normally expect that a good compactification should be smooth and that the variety of solutions at infinity have smaller dimension than the variety of the finite solutions. In such a case we can deduce that whenever the intersection is nonvoid on the compactified space, it must contain a finite point. In the case where we have equal number of equations and unknows, and therefore expect finite number of solutions, the existence of solutions at infinity does not allow us to count correctly the number of finite solutions. These ideas are illustrated in the following example.

Example(3.11) Consider the set of algebraic equations on $\mathbb{C}^2$:

$$c_1 + a_1x_1 + a_2x_2 + a_3x_1x_2 = 0$$

$$c_2 + b_1x_1 + b_2x_2 + b_3x_1x_2 = 0$$

where $a_3, b_3 \neq 0$. The solution set of these equations is either empty or zero dimensional (points) depending upon the coefficients. If we compactify $\mathbb{C}^2$ into $\mathbb{P}^2(\mathbb{C})$, this amounts to homogenising the above equations as

$$\lambda^2c_1 + \lambda a_1x_1 + \lambda a_2x_2 + a_3x_1x_2 = 0$$

$$\lambda^2c_2 + \lambda b_1x_1 + \lambda b_2x_2 + b_3x_1x_2 = 0$$

then the new set of equations has always solutions no matter what the coefficients are. These solutions are given when $\lambda=0$ and $x_1x_2=0$ - that is they are the two points (0,0,1) and (0,1,0) which are both solutions at infinity (since $\lambda=0$). Therefore, the solution set of the equations at infinity is zero dimensional which means that it is not smaller than the finite solution set. This fact has the difficulty that if we consider both finite and infinite solutions of the equations (which as mentioned is a nonvoid set) we cannot conclude by using dimension arguments that this set contains always a finite solution. However, since we have calculated that we always have two solutions at infinity we can calculate the number of finite solutions by substracting the number of infinite solutions from the total number of solutions. The total number of solutions is given by the product of the degrees of the two equations and is therefore equal to 4, and hence the number of finite solutions is $4-2=2$. Although, when the compactification is the projective space the total number of solutions can always be calculated by multiplying the degrees, the computation of number of solutions at infinity is problematic especially when the infinity solutions contain a variety of excess dimension. The problem becomes considerably easier to resolve if we consider a compactification with no solutions at infinity. For instance, in our case, by an introduction of two parameters λ_1, λ_2 , we get:

$$\lambda_1\lambda_2 c_1 + a_1\lambda_2 x_1 + a_2\lambda_1 x_2 + a_3 x_1 x_2 = 0$$

$$\lambda_1\lambda_2 c_2 + b_1\lambda_2 x_1 + b_2\lambda_1 x_2 + b_3 x_1 x_2 = 0$$

and the compactification becomes $\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})$. The solutions at infinity are given when $\lambda_1\lambda_2=0$ and we can easily see that for almost all $\{a_i,b_i\}_{i=0}^{3}$ there are no such solutions. Therefore, in this case, the number of all solutions in $\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})$ is equal to the number of finite solutions. The total number of solutions cannot be calculated as easily as in the case of $\mathbb{P}^2(\mathbb{C})$ where we multiplied the degrees of the equations, but it can be calculated solely via the intersection ring of $\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})$, as follows: the intersection ring of $\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})$ is given by:

$$\mathcal{A}^*\big(\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})\big) = \mathbb{Z}[a]/<a^2=0> \otimes \mathbb{Z}[b]/<b^2=0>$$

where a,b are linear hypersurfaces in $\mathbb{P}^1(\mathbb{C})$. Each one of our equations can be represented by the element $a+b \in \mathcal{A}^*\big(\mathbb{P}(\mathbb{C})\mathrm{x}\mathbb{P}(\mathbb{C})\big)$ and therefore their intersection by $z=(a+b)^2$. By expanding z we get $z=a^2+b^2+2ab$ and because of the relations $a^2=0$, $b^2=0$, which define the ring, z must be equal to 2ab which in turn implies that our equations contain only two common solutions.

□

Therefore, the calculation of a nice compactification for our equations, reduces the problem (as far as counting number of solutions) to the examination of certain elements of the intersection ring of the compactification, which is an algebraic problem. The intersection rings of the spaces to be used in this thesis have either been already calculated or can be calculated using the intersection rings of well known compactifications. The construction of an intersection ring comes originally from the ideas of Schubert on the conservation of number of parametrised intersections as well as his *specialisation principle* which was a controversial issue in his time [Klei.1]. Later these ideas were rigorously formulated and put into correct context in terms of modern algebraic geometry [Ful.1] and algebraic topology [Klei.1].

3.5.2 Intersection ring

The construction of the *intersection ring* of a smooth variety $\mathcal{V} \in \mathbb{P}^n(\mathbb{C})$ is based on the idea of representing every subvariety $\mathcal{W} \subset \mathcal{V}$ by the equivalence class $<\mathcal{W}>$ of an appropriate equivalence relation defined on the set of all formal sums $\sum n_i \mathcal{W}_i$ of irreducible subvarieties of $\mathcal{V}$. The (additive) group of all equivalence classes on $\mathcal{V}$ is denoted by $\mathcal{A}_*\mathcal{V}$. The intersection of varieties now enters as product '.' in $\mathcal{A}_*\mathcal{V}$ as follows: if $<\mathcal{W}_1>$ and $<\mathcal{W}_2>$ are two classes such that the intersection $\mathcal{W}_1 \cap \mathcal{W}_2$ is proper, then $<\mathcal{W}_1> . <\mathcal{W}_2>$ is a linear combination of the irreducible components of $\mathcal{W}_1 \cap \mathcal{W}_2$ with coefficients the intersection multiplicities. The dual of $\mathcal{A}^*\mathcal{V}$ apart from being an additive group, can be endowed the graded ring structure. Every variety $\mathcal{W} \subset \mathcal{V}$ of codimension n corresponds to a class $<\mathcal{W}>$ belonging to $\mathcal{A}^n\mathcal{V}$, the n-th graded component of $\mathcal{A}^*\mathcal{V}$, and the dual of the intersection product, the cup product, is the multiplication of the ring. The ring $\mathcal{A}^*\mathcal{V}$ has also the structure of $\mathbb{Z}$ module which is naturally induced by the additive structure. An important case of intersection ring is when it is finitely generated in which case there is a finite basis $c_{ij}=<\mathcal{V}_j^i>$ for every graded component $\mathcal{A}^j\mathcal{V}$. In this case to determine the multiplication of the ring it is sufficient to determine only how elements of the basis intersect with each other.

In the previous setting, if we want to find the expansion of a k-codimensional variety $\mathcal{W} \subset \mathcal{V}$ in the finitely generated ring $\mathcal{A}^*\mathcal{V}$ with respect to the basis $\{c_{ij}\}$ we consider the intersection $\mathcal{W} \cap \mathcal{V}_k^{i}$ such that it is proper, in this case finite, and equal say to $\delta_i(\mathcal{W})$ then the expansion of $<\mathcal{W}>$ in $\mathcal{A}^k\mathcal{V}$ is :

$$<\mathcal{W}>=\sum_i \delta_i(\mathcal{W})\ c_{ik}$$

If we also have a second variety $\mathcal{G} \subset \mathcal{V}$ such that:

$$<\mathcal{S}> = \sum_i \delta_i(\mathcal{S})\ c_{id}$$

then the class of the intersection $\mathcal{W} \cap \mathcal{S}$ in $\mathcal{A}^*\mathcal{V}$ is equal to:

$$<\mathcal{W} \cap \mathcal{S}> = \left(\sum_i \delta_i(\mathcal{W})\ c_{ik}\right) \cdot \left(\sum_i \delta_i(\mathcal{S})\ c_{id}\right)$$

which is equivalent to:

$$<\mathcal{W} \cap \mathcal{S}> = \sum_{i,j} \delta_i(\mathcal{W})\delta_j(\mathcal{S})\ c_{ik} \cdot c_{jd} \tag{3.20}$$

Thus, to find the class of the intersection of two subvarieties of $\mathcal{V}$ amounts to finding their expansion with respect to a $\mathbb{Z}$-basis in the $\mathcal{A}^*\mathcal{V}$ and then to applying the formula (3.20), provided that we know how the elements of the basis are intersecting each other.

Remark(3.1) It is important to note that the structure of the intersection ring and thus the type of the intersection theory we use, depends essentially on how small is the equivalence class $<\mathcal{W}>$ to which $\mathcal{W}$ belongs. The smaller the equivalence class, the more refined the intersection theory is. A complete account of a refined intersection theory is given in [Ful] where the equivalence relation is the rational equivalence and the (refined) intersection product allows nonproper intersections.

□

The intersection ring of the projective space $\mathbf{P}^n(\mathbf{C})$ [Ful]. Let $\mathcal{L}^k$ be a k-codimensional linear subspace of $\mathbb{P}^n(\mathbb{C})$ then every k-codimensional irreducible variety of degree d is rationally equivalent to $d<\mathcal{L}^k>$. Thus the graded component $\mathcal{A}^k(\mathbb{P}^n(\mathbb{C}))$ is generated by $<\mathcal{L}^k>$ which implies that

$$\mathcal{A}^k(\mathbb{P}^n(\mathbb{C})\ ) \approx \mathbb{Z} \qquad k=0,1,\ldots,n$$

and $\mathcal{A}^k(\mathbb{P}^n(\mathbb{C}))$ is isomorphic to 0 for $k > n$. The intersection product of $\mathcal{A}(\mathbb{P}^n(\mathbb{C}))$ is given by:

$$<\mathcal{L}^p> . <\mathcal{L}^k>=<\mathcal{L}^{p+k}>$$

Therefore the ring $\mathcal{A}(\mathbb{P}^n(\mathbb{C}))$ is isomorphic to $\mathbb{Z}[<\mathcal{L}>]$ with $<\mathcal{L}>^{n+1}=0$. The above description of the intersection ring of the projective space may help us to solve enumerative problems like the problem of determining the number of points the irreducible homogeneous hypersurfaces $\mathcal{V}_i$ i=1,2,...,n intersect in $\mathbb{P}^n(\mathbb{C})$. In this case, we first look at the class of $\mathcal{V}_i$ in $\mathcal{A}(\mathbb{P}^n(\mathbb{C}))$ which can be written as:

$$<\mathcal{V}_i>=d_i<\mathcal{L}^1>$$

where d_i is the degree of $\mathcal{V}_i$. The class of the intersection is equal to the product of the classes

$$<\cap\mathcal{V}_i>= d_1d_2 \dots d_n \left(<\mathcal{L}^1>\right)^n$$

which implies that the number of points is equal to the product of the degrees of the hypersurfaces (Bezout's theorem).

□

The intersection theory which will be used for the purposes of this thesis is called *cohomology theory*. This is a topological intersection theory and will be analytically presented in section 3.6. In such an intersection theory, each algebraic subset of a variety is assigned a cohomology class. Continuously varying the subset, yields another subset with the same cohomology class; in other words, the two subsets are homologically equivalent. If two algebraic subsets are in general position, then their intersection is assigned the (cup) product of their cohomology classes and their union is assigned the sum. This way the cohomology ring $H^*(\mathcal{V};\mathbb{Z})$ of the variety $\mathcal{V}$ becomes a graded ring like $\mathcal{A}^*(\mathcal{V})$ with the difference that the cohomology ring is graded relatively to the real dimension of the subvarieties of $\mathcal{V}$ (this being twice as much as the complex one).

The cohomology ring of a space is related to certain topological properties of its space. Particularly, the torsion free part of $H^i(\mathcal{V};\mathbb{Z})$, $i>0$ measures the number of i-dimensional holes and $H^0(\mathcal{V};\mathbb{Z})$ the number of connected components of $\mathcal{V}$. Connected spaces without holes like $\mathbb{C}^n$ have trivial cohomology rings $H^*(\mathcal{V};\mathbb{Z})=H^0(\mathcal{V};\mathbb{Z})=\mathbb{Z}$ and thus their use cannot produce results. This is an additional reason why we will be examining our intersection problems in the compactified $\mathbb{C}^n$. The new compact space is normally much richer topologically than $\mathbb{C}^n$; this implies that the corresponding cohomology ring is much more complex and interesting for calculations. This new topology is exclusively due to the glueing of the 'points at infinity' of $\mathbb{C}^n$. This process creates certain holes whose number and dimension depends on what we consider as being 'points at infinity' and the way we glued them together.

The cohomology ring of a space can be seen as a subring of the intersection ring $\mathcal{A}^*(\mathfrak{S})$, of cocycles under rational equivalence, via the cocycle map [Ful]:

$$\text{co:}\ \ H^*(\mathfrak{S};\mathbb{Z}) \rightarrow \mathcal{A}^*(\mathfrak{S})$$

In certain cases, such as when $\mathfrak{S}$ is the Projective space or the Grassmannian, the above map is an isomorphism and these are the cases considered in this thesis. Although we can equally use the ring of cocycles under rational equivalence, we prefer to use homology (the cohomology ring), which, as we have mentioned is a topological intersection theory. The main reason for this is that homology theory can be defined for real varieties and thus results may be obtained for real solutions to our problems; this is also compatible with the theory of vector bundles.

In the next section we will analytically be looking at the way the cohomology ring is defined; first, for a general topological space and then for a (oriented) manifold where the Poincare duality theorem leads us to seeing it as an intersection ring.

3.6 TOPOLOGY, MANIFOLDS AND COHOMOLOGY RINGS

3.6.1. Topological spaces.

Let $\mathcal{S}$ be a set. A *topological structure*, or a *topology*, on $\mathcal{S}$ is a collection of subsets of $\mathcal{S}$, called *open* sets, satisfying the axioms

(i) the union of any number of open sets is open

(ii) the intersection of any finite number of open sets is open

(iii) the set $\mathcal{S}$ and the empty set $\emptyset$ are open

A set $\mathcal{S}$ with topology is a called a *topological space.*

A *basis* for a topology is a collection of open sets, called *basic open sets*, with the following properties

(i) $\mathcal{S}$ is the union of basic open sets

(ii) a nonoempty intersection of two basic open sets is a union of basic open sets

A *neighbourhood* of a point p of a topological space is any open set which contains p.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be topological spaces and F a mapping $F: \mathcal{S}_1 \to \mathcal{S}_2$. The mapping F is *continuous* if the inverse image of every open set of $\mathcal{S}_2$ is an open set of $\mathcal{S}_1$. The mapping F is *open* if the image of an open set of $\mathcal{S}_1$ is an open set of $\mathcal{S}_2$. The mapping of F is a *homeomorphism* if it is a bijection and both continuous and open.

If F is an homeomorphism, the inverse mapping F^{-1} is also a homeomorphism.

Two topological spaces $\mathcal{S}_1$, $\mathcal{S}_2$ such that there is a homeomorphism $F: \mathcal{S}_1 \to \mathcal{S}_2$ are said to be *homeomorphic.*

Two continuous maps, α_1,α_2: $\mathcal{S}_1 \to \mathcal{S}_2$ are called homotopic iff there exists a function F such that

$$F : \mathcal{S}_1 \times [0, 1] \to \mathcal{S}_2 , \quad \text{F continuous}$$

and F satisfies

$$F(x, 0) = \alpha_1(x)$$

$$F(x, 1) = \alpha_2(x)$$

A subset $\mathcal{U}$ of a topological space is said to be *closed* if its complement $\bar{\mathcal{U}}$ in $\mathcal{T}$ is open. It is easy to see that the intersection of any number of closed sets is closed, the union of any finite number of closed sets is closed, and both $\mathcal{T}$ and $\emptyset$ are closed.

If $\mathcal{T}_0$ is a subset of a topological space $\mathcal{T}$, there is a unique open set, noted $\mathrm{int}(\mathcal{T}_0)$ and called the *interior* of $\mathcal{T}_0$, which is contained in $\mathcal{T}_0$ and contains any other open set contained in $\mathcal{T}_0$. Likewise, there is a unique closed set, noted $\mathrm{cl}(\mathcal{T}_0)$ and called the *closure* of $\mathcal{T}_0$, which contains $\mathcal{T}_0$. In fact, $\mathrm{cl}(\mathcal{T}_0)$ is the intersection of all closed sets which contain $\mathcal{T}_0$.

A subset of $\mathcal{T}$ is said to be *dense* in $\mathcal{T}$ if its closure coincides with $\mathcal{T}$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topological spaces, then the cartesian product $\mathcal{T}_1 \times \mathcal{T}_2$ can be given a topology taking as a basis the collection of all subsets of the form $\mathcal{U}_1 \times \mathcal{U}_2$, with $\mathcal{U}_1$ a basic open set of $\mathcal{T}_1$ and $\mathcal{U}_2$ a basic open set of $\mathcal{T}_2$. This topology on $\mathcal{T}_1 \times \mathcal{T}_2$ is sometimes called the *product topology.*

If $\mathcal{T}$ is a topological space and $\mathcal{T}_1$ a subset of $\mathcal{T}$, then $\mathcal{T}_1$ can be given a topology taking as open sets the subsets of the form $\mathcal{T}_1 \cap \mathcal{U}$ with $\mathcal{U}$ any open set in $\mathcal{T}$. This topology on $\mathcal{T}_1$ is sometimes called the *subset topology.*

Let $F: \mathcal{T}_1 \to \mathcal{T}_2$ be a continuous mapping of topological spaces, and let $F(\mathcal{T}_1)$ denote the image of F. Clearly, $F(\mathcal{T}_1)$ with the subset topology is a topological space. Since F is continuous, the inverse image of any open set of $F(\mathcal{T}_1)$ is an open set of $\mathcal{T}_1$. However, not all open sets of $\mathcal{T}_1$ are taken onto open sets of $F(\mathcal{T}_1)$. In other words the mapping $F': \mathcal{T}_1 \to F(\mathcal{T}_1)$ defined by $F'(p) = F(p)$ is continuous but not necessarily open. The set $F(\mathcal{T}_1)$ can be given another topology, taking as open sets in $F(\mathcal{T}_1)$ the images of open sets in $\mathcal{T}_1$. It is easily seen that this new topology, sometimes called the *induced topology,* contains the subset topology (ie. any set which is open in the subset topology is open also in the induced topology), and that the mapping of F' is now open. If F is an injection, the $\mathcal{T}_1$ and $F(\mathcal{T}_1)$ endowed with the induced topology are homeomorphic.

A topological space $\mathcal{T}$ is said to satisfy the *Hausdorff separation axiom* (or, briefly, to be an Hausdorff space) if any two different points p_1 and p_2 have disjoint neighbourhoods.

Examples (3.11):

(i) The n-dimensional Euclidean space $\mathbb{R}^n$- we define the basic open sets to be all balls B(a, r) where

$$B(a, r) = \{x \in \mathbb{R}^n : \|x-a\|_2 \leq r\}$$

This topology on $\mathbb{R}^n$ is called the natural or Euclidean topology.

(ii) The n-dimensional projective space $\mathbb{P}^{n-1}(\mathbb{R})$- this is the set of all lines in $\mathbb{R}^n$ and can be given the induced topology of the projection map:

$$p : \mathbb{R}^n - \{0\} \rightarrow \mathbb{P}^{n-1}(\mathbb{R})$$

(iii) The Stiefel space $V_p(\mathbb{R}^{p+r})$ - it is the set of all px(p+r) full rank matrices. This is an open subset of $\mathbb{R}^{p(p+r)}$ and can be given the subset topology.

(iv) The Grassmannian $G_p(\mathbb{R}^{p+r})$ - this is the set of all p-dim linear subspaces $\mathbb{R}^{p+r}$ and can be given the induced topology of the projection :

$$\pi : V_p(\mathbb{R}^{p+r}) \rightarrow G_p(\mathbb{R}^{p+r})$$

(v) Zarisky topology of a variety. This is a non euclidean topology defined on a algebraic variety $\mathcal{V}$ such that all the closed sets are the subvarieties of $\mathcal{V}$(see 3.4.2). This topology contains considerably fewer open sets than the Euclidean topology of $\mathcal{V}$ and is not Hausdorff(hence it cannot be described by a metric). The advantage of this topology is that it is the weakest topology such that any polynomial map $f:\mathcal{V} \rightarrow \mathcal{W}$ is continuous, and thus it is convenient for the study of polynomial maps between algebraic varieties. The open sets of the Zarisky topology are all $\mathcal{U} = \mathcal{V} - \mathcal{V}'$ where $\mathcal{V}'$ is a subvariety of $\mathcal{V}$ and therefore if an open set is not empty, it is the whole of $\mathcal{V}$ apart from a set of measure zero. This shows that if a property holds for every "u" in a nonempty Zarisky open set, then it holds for almost all $u \in \mathcal{V}$ and suggests that the Zarisky topology is appropriate for genericity arguments.

□

3.6.2.Manifolds

The most classical and familiar examples of smooth manifolds are curves and surfaces in the coordinate space $\mathbb{R}^3$. Generalising the classical description of curves and surfaces, we will consider n-dimensional objects in a coordinate space $\mathbb{R}^{\ell}$.

A set $\mathcal{M}$ is an n-dimensional *differentiable manifold* iff:
i) $\mathcal{M}$ is a topological space. ii) $\mathcal{M}$ is provided with a family of pairs $\{(\mathcal{M}_\alpha, \psi_\alpha)\}$. iii) The $\mathcal{M}_\alpha$ are a family of open sets which cover $\mathcal{M}$: $\cup_{\alpha}, \mathcal{M}_\alpha = \mathcal{M}$. The ϕ_α are homeomorphisms from $\mathcal{M}_\alpha$ to an open subset 0_α of $\mathbb{R}^n$, ϕ_α: $\mathcal{M}_\alpha \rightarrow 0_\alpha$. (iv) Given $\mathcal{M}_\alpha$, $\mathcal{M}_\beta$ such that $\mathcal{M}_\alpha \cap \mathcal{M}_\beta \neq \phi$, the map $\phi_\beta \circ \phi_\alpha{}^{-1}$ from the subset $\phi_\alpha(\mathcal{M}_\alpha \cap \mathcal{M}_\beta)$ of $\mathbb{R}^n$ to the subset $\phi_\beta(\mathcal{M}_\alpha \cap \mathcal{M}_\beta)$ of $\mathbb{R}^n$ is infinitely differentiable, (written $\mathbb{C}^\infty$). The family $\{(\mathcal{M}_\alpha,\phi_\alpha)\}$ satisfying (ii), (iii) and (iv) is called an *atlas*. The individual members $(\mathcal{M}_\alpha,\phi_\alpha)$ of the family are called *charts*.

□

Now consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ and a function f: $\mathcal{M} \rightarrow \mathcal{N}$ with atlases $\{(\mathcal{M}_\alpha,\phi_\alpha)\}$, $\{(\mathcal{N}_\alpha,\psi_\alpha)\}$, respectively. Then f is *smooth* at the point x of $\mathcal{M}$ iff for every charts $(\mathcal{M}_\alpha,\phi_\alpha)$, $(\mathcal{N}_\alpha,\psi_\alpha)$ such that $x \in \mathcal{M}_\alpha$ and $f(x) \in \mathcal{N}_\alpha$ we have that the function

$$f' = \psi_\alpha \circ f \circ \phi_\alpha^{-1}$$

is smooth. The function f′ is called the *representation* of f with respect to charts $(\mathcal{M}_\alpha,\phi_\alpha)$, $(\mathcal{N}_\alpha,\psi_\alpha)$. It is important to note that the definition of smoothness does not depend on the choice of local representation of f at x.

The function f: $\mathcal{M} \rightarrow \mathcal{N}$ is *smooth* if it is smooth at x for every $x \in \mathcal{M}$. A function f: $\mathcal{M} \rightarrow \mathcal{N}$ is called a *diffoemorphism* if f is one-to-one-onto, and if both f and the inverse fuction f^{-1}: $\mathcal{N} \rightarrow \mathcal{M}$ are smooth.

The concept of a tangent vector can be defined as follows. Let x be a fixed point of $\mathcal{M}$, and let $(-\epsilon,\epsilon)$ denote the set of real numbers t with $-\epsilon < t < \epsilon$. A *smooth path* through x in $\mathcal{M}$ will mean a smooth function

$$p: (-\epsilon,\epsilon) \to \mathcal{M}$$

defined on some interval $(-\epsilon,\epsilon)$ of real numbers, with $p(0)=x$. The *velocity vector* of such a path is defined to be the vector

$$\left(\frac{dp'}{dt}\right)_{t=0} \in \mathbb{R}^n$$

whose α-th component is $\frac{dp'_\alpha(0)}{dt}$. A vector $v \in \mathbb{R}^n$ is tangent to an n-dim manifold $\mathcal{M}$ at x if v can be expressed as the velocity vector of some smooth path through x in $\mathcal{M}$. The set of all such tangents will be called the *tangent* space of $\mathcal{M}$ are x, and will be denoted by $T_x\mathcal{M}$.

Any map f: $\mathcal{M} \to \mathcal{N}$ which is smooth at x determines a linear map Df_x from the tangent space $T_x\mathcal{M}$ to $T_x\mathcal{N}$ as follows. Given $v \in T_x\mathcal{M}$ express as the velocity vector

$$v = \left(\frac{dp'}{dt}\right)_{t=0}$$

of some path through x in $\mathcal{M}$, and define $Df_x(v)$ to be the velocity vector

$$\left(\frac{d(f \circ p)'}{dt}\right)_{t=0}$$

of the image path $f \circ p$: $(-\epsilon,\epsilon) \to \mathcal{N}$. It is easily seen that this definition does not depend on the choice of p, and that Df_x is a linear mapping. The linear transformation Df_x is called the *derivative* or the *Jacobian* of f at x and we can easily observe that with respect to the local charts $(\mathcal{M}_\alpha,\phi_\alpha)$, $(\mathcal{N}_\alpha,\psi_\alpha)$, Df_x can be given as:

$$Df_x = J_x(f')$$

Examples(3.12)

Hypersurfaces in $\mathbb{R}^m$. Let $\mathcal{U}$ be an open set of $\mathbb{R}^m$ and let $\lambda_1,\ldots,\lambda_{m-n}$ be real-valued smooth functions defined on $\mathcal{U}$. Let $\mathcal{N}$ denote the (closed) subset of $\mathcal{U}$ on which all functions $\lambda_1,\ldots,\lambda_{m-n}$ vanish, ie. let

$$\mathcal{N} = \{ x \in \mathcal{U} : \lambda_i(x) = 0,\ 1 \leq i \leq m\text{-}n \}.$$

Suppose the rank of the Jacobian matrix

$$\begin{bmatrix} \dfrac{\partial \lambda_1}{\partial x_1} & \cdots & \dfrac{\partial \lambda_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \lambda_{m-n}}{\partial x_1} & \cdots & \dfrac{\partial \lambda_{m-n}}{\partial x_m} \end{bmatrix}$$

is m-n at all $x \in \mathcal{N}$. Then $\mathcal{N}$ is a smooth manifold of dimension n.

The proof of this essentially depends on the Implicit Function Theorem, and uses the following arguments. Let $x^0 = (x_1^0, \ldots, x_n^0, x_{n+1}^0, \ldots, x_m^0)$ be a point of $\mathcal{N}$ and assume, without loss of generality, that the matrix

$$\begin{bmatrix} \dfrac{\partial \lambda_1}{\partial x_{n+1}} & \cdots & \dfrac{\partial \lambda_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \lambda_{m-n}}{\partial x_{n+1}} & \cdots & \dfrac{\partial \lambda_{m-n}}{\partial x_m} \end{bmatrix}$$

is nonsingular at x^0. Then, there exist neighbourhoods $\mathcal{A}_0$ of $(x_1^0, \ldots, x_n^0)$ in $\mathbb{R}^n$ and $\mathcal{B}_0$ of $(x_{n+1}^0, \ldots, x_m^0)$ in $\mathbb{R}^{m-n}$ and a smooth mapping $G : \mathcal{A}_0 \to \mathcal{B}_0$ such that

$$\lambda_i(x_1, \ldots, x_n, g_1(x_1, \ldots, x_n), \ldots, g_{m-n}(x_1, \ldots, x_n)) = 0$$

for all $1 \leq i \leq m\text{-}n$. This makes it possible to describe points of $\mathcal{N}$ around x^0 as m-tuples $(x_1, \ldots, x_m)$ such that $x_{n+i} = g_i(x_1, \ldots, x_n)$ for $1 \leq i \leq m\text{-}n$. In this way one is able to construct a coordinate chart around each point x^0 of $\mathcal{N}$ and the coordinate charts thus defined form a smooth atlas.

A manifold of this type is sometimes called a smooth *hypersurface* in $\mathbb{R}^m$. An important example of hypersurface is the *sphere* $\mathbb{S}^{m-1}$, defined by taking n=m-1 and

$$\lambda_1 = x_1^2 + x_2^2 + \dots + x_m^2 - 1.$$

The set of points of $\mathbb{R}^m$ on which $\lambda_1(x)= 0$ consists of all the points on a sphere of radius 1 centered at the origin. Since

$$\left[\frac{\partial \lambda_1}{\partial x_1} \quad \cdots \quad \frac{\partial \lambda_1}{\partial x_m} \right]$$

never vanishes on this set, the required conditions are satisfied and the set is a smooth manifold, of dimension m-1 .

□

3.6.3. Homology and Cohomology theory

To every topological space $\mathfrak{X}$, a ring $H^*(\mathfrak{X};\Lambda)$ can be assigned, called the *cohomology ring* of $\mathfrak{X}$ with coefficients in Λ (Λ is a commutative ring). This is a positively graded ring up to the dimension of $\mathfrak{X}$; that is for a n-dim topological space $\mathfrak{X}$,

$$H^*(\mathfrak{X};\Lambda)= \bigoplus_{i=0}^{n} H^i(\mathfrak{X};\Lambda)$$

where $H^i(\mathfrak{X};\Lambda)$ is an Λ-module called the i-th cohomology module of $\mathfrak{X}$ with coefficients in Λ and the grading is called cup product. The torsion free part of $H^i(\mathfrak{X};\Lambda)$ when $\Lambda=\mathbb{Z}$ measures the number of i-dimensional holes of $\mathfrak{X}$. The cohomology ring with coefficients in $\mathbb{Z}$ is an intersection ring for a closed orientable manifold $\mathcal{M}$, for non orientable manifolds like the real Grassmanian or the real Projective space we use coefficients in $\mathbb{Z}_2$. The following subsections constitute a brief description of homology and cohomology in terms of singular simplexes having the purpose of setting the formalism of this theory rather than trying to make this theory understood by an engineer. Although in the largest part of this section the use of cohomology for the purpose of our thesis is not clear, it is stated, in the concluding part under the title Poincare Duality theorem, that the cohomology and the intersection operations are dual. This fact allows us to view the cohomology ring as an intersection ring and to use it for our problems which are mainly intersection problems.

Homology theory of topological spaces

The *standard* n-*simplex* is the convex set $\Delta^n \subset \mathbb{R}^{n+1}$ consisting of all (n+1)-tuples of real numbers with

$$t_i \geq 0, \quad t_0 + t_1 + \ldots + t_n = 1$$

Any continuous map from Δ^n to a topological space $\mathfrak{X}$ is called a *singular* n-*simplex* in $\mathfrak{X}$. The i-th *face* of a singular n-simplex σ: $\Delta^n \to \mathfrak{X}$ is the singular (n-1)-simplex

$$\sigma \circ \phi_i : \Delta^{n-1} \to \mathfrak{X}$$

where the linear embedding $\phi_i : \Delta^{n-1} \to \Delta^n$ is defined by

$$\phi_i(t_0,\ldots, t_{i-1}, t_{i+1},\ldots, t_n) = (t_0,\ldots, t_{i-1}, 0, t_{i+1},\ldots, t_n)$$

For each $n \geq 0$ the *singular chain group* $C_n(\mathfrak{X};\Lambda)$ with coeficients in a *commutative* ring Λ is the free Λ-module having one generator $[\sigma]$ for each singular n-simplex σ in $\mathfrak{X}$. For n<0, the group $C_n(\mathfrak{X};\Lambda)$ is defined to be zero. The *boundary homomorphism*

$$\partial : C_n(\mathfrak{X};\Lambda) \to C_{n-1}(\mathfrak{X};\Lambda)$$

is defined by

$$\partial\, [\sigma] = [\sigma \circ \phi_0] - [\sigma \circ \phi_1] + -\ldots+ (-1)^n[\sigma \circ \phi_n].$$

The identity $\partial \circ \partial = 0$ is easily verified. Hence, we can define the n-th *singular homology group* $H_n(\mathfrak{X};\Lambda)$ to be the quotient module $Z_n(\mathfrak{X};\Lambda)/B_n(\mathfrak{X};\Lambda)$, where $Z_n(\mathfrak{X};\Lambda)$ is the kernel of $\partial : C_n(\mathfrak{X};\Lambda) \to C_{n-1}(\mathfrak{X};\Lambda)$ and $B_n(\mathfrak{X};\Lambda)$ is the image of $\partial : C_{n+1}(\mathfrak{X};\Lambda) \to C_n(\mathfrak{X};\Lambda)$. Here and elsewhere the word 'group' is used, although a 'left Λ-module' is what we really mean.

The *cochain group* $C^n(\mathfrak{X};\Lambda)$ is defined to be the dual module $Hom_\Lambda(C_n(\mathfrak{X};\Lambda), \Lambda)$

consisting of all Λ-linear maps from $C_n(\mathfrak{X};\Lambda)$ to Λ. The value of a cochain $c \in C^n(\mathfrak{X};\Lambda)$ is defined to be the cochain $\delta c \in C^{n+1}(\mathfrak{X};\Lambda)$ whose value on each (n+1)-chain α is defined by the identity

$$<\delta c, \alpha> + (-1)^n < c, \delta\alpha> = 0$$

Thus we obtain corresponding modules

$$H^n(\mathfrak{X};\Lambda) = (\text{kernel } \delta)/\delta\ C^{n-1}(\mathfrak{X};\Lambda)$$

which are called the *singular cohomology groups* of $\mathfrak{X}$.

The cup product

Given cochains $c \in C^m(\mathfrak{X})$ and $c\prime \in C^n(\mathfrak{X})$, the product $cc\prime = c \cup c\prime \in C^{m+n}(\mathfrak{X})$ is defined as follows. Let $\sigma : \Delta^{m+n} \to \mathfrak{X}$ where

$$\alpha_m(t_0,\dots,t_m) = (t_0,\dots, t_m, 0 ,\dots, 0)$$

Similarly, the *back* n-*face* of σ is the composition $\sigma \circ \beta_n$ where

$$\beta_n(t_m, t_{m+1},\dots, t_{m+n}) = (0,\dots, t_m, t_{m+1},\dots, t_{m+n})$$

Now define $cc' = c \cup c'$ by the identity

$$<cc', [\sigma]> = (-1)^{mn} < c,[\sigma \circ \alpha_m] > <c',[\sigma \circ \beta_n]> \in \Lambda$$

This product operation is bilinear and associative, but is not commutative. The constant cocycle $1 \in C^0$ serves as the identity element. The formula

$$\delta(cc') = (\delta c)c' + (-1)^m c(\delta c')$$

is easily verified. This implies that there is a corresponding product operation $H^m(\mathfrak{X}) \otimes H^n(\mathfrak{X}) \to H^{m+n}(\mathfrak{X})$ of cohomology classes. On the cohomology level the product operation does commute, up to sign. In fact, for $a \in H^m(\mathfrak{X})$, $b \in H^n(\mathfrak{X})$, one has $ba=(-1)^{mn}ab$. In dealing with graded groups, this property is called commutativity and we say that the cohomology ring $H^*(\mathfrak{X})= (H^0(\mathfrak{X}), H^1(\mathfrak{X}), H^2(\mathfrak{X}),....)$ is *commutative as a graded ring.*

Remark(3.2) Now suppose that one is given a pair of spaces $\mathfrak{X} \supset \mathcal{A}$. If the cochain c belongs to the subset $C^m(\mathfrak{X},\mathcal{A}) \subset C^m(\mathfrak{X})$ (that is if $c[\sigma]=0$ for every $\sigma: \Delta^m \to \mathcal{A} \subset \mathfrak{X}$) and if $c' \in C^n(\mathfrak{X})$, then clearly cc' belongs to $C^{m+n}(\mathfrak{X},\mathcal{A})$. This gives rise to a product operation

$$H^m(\mathfrak{X},\mathcal{A}) \otimes H^n(\mathfrak{X}) \to H^{m+n}(\mathfrak{X},\mathcal{A})$$

More generally, consider two subsets $\mathcal{A}$, $\mathfrak{B} \subset \mathfrak{X}$ which satisfy the following. We can similarly define an operation

$$H^m(\mathfrak{X},\mathcal{A}) \otimes H^n(\mathfrak{X},\mathfrak{B}) \to H^{m+n}(\mathfrak{X}, \mathcal{A} \cup \mathfrak{B})$$

□

The cross product and cohomology of the products of topological spaces.

Suppose one is given cohomology classes

$$a \in H^m(\mathfrak{X},\mathcal{A}) \quad , \quad b \in H^n(\mathfrak{Y},\mathfrak{B})$$

where $\mathcal{A}$ is an open subset of $\mathfrak{X}$ and $\mathfrak{B}$ is an open subset of $\mathfrak{Y}$. Using the projection maps

$$p_1 : (\mathfrak{X} \times \mathfrak{Y}, \mathcal{A} \times \mathfrak{Y}) \to (\mathfrak{X},\mathcal{A})$$
$$p_2 : (\mathfrak{X} \times \mathfrak{Y}, \mathfrak{X} \times \mathfrak{B}) \to (\mathfrak{Y},\mathfrak{B})$$

the *cross product* (or *external product*) $a \times b$ is defined to be the cohomology class

$$(p_1^*a) \cup (p_2^*b) \in H^{m+n}(\mathfrak{X} \times \mathfrak{Y}, (\mathcal{A} \times \mathfrak{Y}) \cup (\mathfrak{X} \times \mathfrak{B})$$

Now consider two spaces $\mathscr{X}$ and $\mathscr{Y}$. The cross product operation gives rise to a homomorphism

$$\times : \bigoplus_{i+j=n} H^i(\mathscr{X}) \otimes H^j(\mathscr{Y}) \rightarrow H^n(\mathscr{X} \times \mathscr{Y})$$

Theorem(3.11) Let $\mathscr{X}$ and $\mathscr{Y}$ be topological spaces such that each $H^i\mathscr{X}$ is a torsion free Λ-module and such that $\mathscr{Y}$ has only finitely many cells in each dimension. Then the direct sum $\bigoplus_{i+j=n} H^i(\mathscr{X}) \otimes H^j(\mathscr{Y})$ maps onto $H^n(\mathscr{X} \times \mathscr{Y})$.

Remark(3.3) The above Kunneth decomposition cannot be performed when $\mathscr{X}$ and $\mathscr{Y}$ are real projective planes (using integer coefficients), but if we use as coefficients integers reduced to mod2 then the torsion free requirement holds true and we can apply the above theorem.

HOMOLOGY THEORY OF MANIFOLDS

The Fundamental Homology Class of a Manifold

We will now use the infinite cyclic group $\mathbb{Z}$ as coeficient domain. For each $x \in \mathscr{M}$, recall that

$$H_i(\mathscr{M},\mathscr{M}\text{-}x;\mathbb{Z}) = H_i(\mathbb{R}^n,\mathbb{R}^n\text{-} 0;\mathbb{Z})$$

is infinite cyclic for $i \neq n$.

Definition(3.) A *local orientation* μ_x of $\mathscr{M}$ at x is a choice of one of the two possible generators for $H_n(\mathscr{M},\mathscr{M}\text{-}x;\mathbb{Z})$.

□

Note that such a μ_x determines local orientations μ_y for all points y in a small neighbourhood of x. To be more precise, if $\mathscr{B}$ is a ball about x (in terms of some local coordinate system), then for each $y \in \mathscr{B}$ the isomorphisms

$$H_*(\mathcal{M}, \mathcal{M}\text{-x}) \xrightarrow{\rho_x} H_*(\mathcal{M}, \mathcal{M}\text{-}\mathcal{B}) \xrightarrow{\rho_y} H_*(\mathcal{M}, \mathcal{M}\text{-y})$$

determine a local orientation μ_y.

Definition(3.) An *orientation* for $\mathcal{M}$ is a function which assigns to each $x \in \mathcal{M}$ a local orientation μ_x which ' varies continuously ' with x , in the following sense: for each x there should exist a compact neighbourhood $\mathcal{N}$ and a class $\mu_{\mathcal{N}} \in H_n(\mathcal{M}, \mathcal{M}\text{-}\mathcal{N})$ so that $\rho_y(\mu_{\mathcal{N}}) = \mu_y$ for each $y \in \mathcal{N}$.

□

The pair consisting of manifold and orientation is called an *oriented manifold.*

Theorem(3.12) For any oriented manifold $\mathcal{M}$ and any compact $\mathcal{K} \subset \mathcal{M}$, there is one and only one class $\mu_{\mathcal{K}} \in H_n(\mathcal{M}, \mathcal{M}\text{-}\mathcal{K})$ which satisfies $\rho_x(\mu_{\mathcal{K}}) = \mu_x$ for each $x \in \mathcal{K}$.

In particular, if $\mathcal{M}$ itself is compact, then there is one and only one $\mu_{\mathcal{K}} \in H_n\mathcal{M}$ with the required property. This class $\mu = \mu_{\mathcal{M}}$ is called the *fundamental homology class* of $\mathcal{M}$.

THE CAP PRODUCT OPERATION

For any space $\mathcal{S}$ and any coefficient domain, there is a bilinear pairing operation

$$\cap : C^i(\mathcal{S}) \otimes C_n(\mathcal{S}) \to C_{n-i}(\mathcal{S})$$

which can be characterised as follows. For each cochain $b \in C^i(\mathcal{S})$ and each chain $\xi \in C_n(\mathcal{S})$ the cap product $b \cap \xi$ is the unique element of $C_{n-i}(\mathcal{S})$ such that

$$< a, b \cap \xi > = < ab, \xi > \qquad (3.21)$$

for all $a \in C^{n-i}(\mathcal{S})$. More explicitly, for each generator $[\sigma]$ of $C_n(\mathcal{S})$, the cap product $b \cap [\sigma]$ can be defined as the product of the ring element $(-1)^{i(n-i)} <$ b, [back i-face of $\sigma]$ > with the singular simplex [front (n-1)-face of σ].

Combining the identity (3.21) with the standard properties of cup products, one can derive the following rules:

$$(bc) \cap \xi = b \cap (c \cap \xi)$$
$$1 \cap \xi = \xi$$
$$\partial (b \cap \xi) = (\delta b) \cap \xi + (-1)^{dim\ b}\, b \cap \partial \xi$$

From (4) it follows that there is a corresponding operation

$$H^{\cdot}(\mathcal{M}) \otimes H_n(\mathcal{M}) \rightarrow H_{n-1}(\mathcal{M})$$

which will also be denoted by $\cap$.

In terms of this operation we can now state the duality theorem for compact manifolds, using any coefficient domain.

POINCARE DUALITY THEOREM.

The Poincare duality theorem can be described in two equivalent forms:

i)Duality between homology and cohomology groups

If M is compact, connected and oriented, then $H^{\cdot}(\mathcal{M})$ is isomorphic to $H_{n-i}(\mathcal{M})$ under the correspondance

$$a \mapsto a \cap \mu_m$$

ii)Duality between cohomology groups

If $\mathcal{M}$ is compact, connected and oriented, then $H^{\cdot}(\mathcal{M})$ is isomorphic to $H^{n-1}(\mathcal{M})$ by the bilinear mapping:

$$b : H^i(\mathcal{M}) \otimes H^{n-i}(\mathcal{M}) \to \mathbb{Z}$$

given by :

$$b(\alpha , \beta) = < \alpha\beta , \mu_m >$$

□

Remark(3.4) For a non-orientable manifold the duality theorem is still true, but only if one uses the coefficient domain $\mathbb{Z}/2$.

□

The importance of Poincare duality theorem is that the cup product on cohomology is transformed under this duality isomorphism to the intersection pairing of homology classes. This pairing is a bilinear function

$$\iota : H_i(\mathcal{M}) \otimes H_{n-i}(\mathcal{M}) \to \mathbb{Z}$$

where the number $\iota(\alpha, \beta)$ assigned to (α, β) is called the intersection number of the two homology classes α , β and is defined as follows:

Let $\alpha=[\mathcal{N}]$ and $\beta=[\mathcal{N}']$ be two homology classes that correspond to two compact orientable submanifolds of $\mathcal{M}$, with fixed orientations such that

$$\dim \mathcal{N}=i \text{ and } \dim \mathcal{N}'=n-i$$

Now take a point $p \in \mathcal{N} \cap \mathcal{N}'$ such that

$$T\mathcal{M}_p = T\mathcal{N}_p \oplus T\mathcal{N}'_p$$

In this case we say that the two submanifolds meet in *general position* at p. Now, fixing an orientation for $\mathcal{N}$ means that it makes sense when a basis for each tangent space is "positively" or "negatively" orientated. Let us say that $\mathcal{N}$ and $\mathcal{N}'$ meet at p in a positive way if () is satisfied, and if putting together a positively oriented basis for $T\mathcal{M}_p$. Otherwise (and if they meet in general position) they are said to meet at p in a negative way.

Suppose that $\mathcal{N}$ and $\mathcal{N}'$ meet in general position at each point of intersection. Then

$$i(\alpha,\beta) = \sum_{p \in \mathcal{N} \cap \mathcal{N}'} \text{sign}(p) \tag{3.22}$$

Here, the sign(p) is +1 or -1 according to whether the submanifolds meet in a positive or negative way.

Determining the orientations of the intersections is often an obstacle to determining the intersection number using formula (3.22). Working in the categories of complex analytic instead of real manifold removes this obstacle. The manifold $\mathcal{M}$ has a complex manifold structure if a set of coordinate charts is given, setting up coordinates in $\mathbb{C}^m$, with the transition maps between the charts given by complex analytic functions. A submanifold ι: $\mathcal{N}\to\mathcal{M}$ is said to be complex if the map is complex.

Such a complex structure on manifold $\mathcal{M}$ determines an orientation for the manifold $\mathcal{M}$. In terms of this orientation, two complex submanifolds always meet with positive orientation. Thus, the sum on the right-hand side of (3.22) only involves plus signs. In particular, $i(\alpha,\beta)$ is equal to the number of intersections of the submanifolds $\mathcal{N}$, $\mathcal{N}'$, provided they meet in general position.

Example(3.13)

1. The cohomology ring of the projective space

It is well known [Dold1] that the homology groups of the projective space are equal

$$H_i(\mathbb{P}^n(\mathbb{R});\mathbb{Z}_2)=\mathbb{Z}_2.<\mathcal{L}_i>$$

where $\mathcal{L}_i$ is an i-dimensional linear subspace of $\mathbb{P}^n(\mathbb{R})$. Since $\iota(<\mathcal{L}_i>,<\mathcal{L}_k>)=<\mathcal{L}_{n-i-k}>$ and if $\alpha \in H^1(\mathbb{P}^n(\mathbb{R});\mathbb{Z}_2)$ is the Poincare dual of $<\mathcal{L}_{n-1}>$, then the Poincare duality implies that

$$H^i(\mathbb{P}^n(\mathbb{R});\mathbb{Z}_2)=\mathbb{Z}_2.\alpha^i$$

and that

$$\alpha^i\alpha^j = \alpha^{i+j} \quad \text{and} \quad \alpha^{n+1} = 0$$

Therefore the cohomology ring of $\mathbb{P}^n(\mathbb{R})$ is given by $H^*(\mathbb{P}^n(\mathbb{R});\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ subject to the relation $\alpha^{n+1} = 0$.

2. The cohomology ring of the Grassmannian

$$H^n(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2) = \oplus\ \mathbb{Z}_2\langle a_1,a_2,\ldots a_p\rangle$$

where the sum is taken for all $\langle a_1,a_2,\ldots a_p\rangle$ satisfying :

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_p \leq m \quad \text{and} \quad \sum a_i = n$$

For example,

$$H^1(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2) = \mathbb{Z}_2\langle 0,0,\ldots,1\rangle \qquad (\approx \mathbb{Z}_2)$$

$$H^2(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2) = \mathbb{Z}_2\langle 0,0,\ldots,1,1\rangle \oplus \mathbb{Z}_2\langle 0,0,\ldots,2\rangle \qquad (\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2)$$

$$H^{pm}(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2) = \mathbb{Z}_2\langle p,p,\ldots,p\rangle \qquad (\approx \mathbb{Z}_2)$$

The multiplicative structure of $H^*(G_p(\mathbb{R}^{p+m}), \mathbb{Z}_2)$ is described by the classical formulas of Pieri and Giambelli. Giambelli's formula expresses a general Schubert cocycle as a polynomial in the special Schubert cocycle $\overline{w_j}$ and Pieri's formula explains how a Schubert cocycle is multiplied with a Schubert cocycle. More analytically, let $\overline{w_j} = \langle 0,0,\ldots,0,j\rangle$ where $j=1,2,\ldots,m$ then the following hold true

Pieri's formula:

$$\langle a_1,\ldots, a_p\rangle \,.\, \overline{w_j} = \sum \langle b_1,\ldots,b_p\rangle$$

where

$$a_i \leq b_i \leq a_{i+1} \quad \text{and} \quad \Sigma b_i = j + \Sigma a_i$$

Giambelli's formula:

$$\langle a_1,\ldots, a_p\rangle = \det\left(\overline{w_{a_i+i-j}}\right)$$

where $\overline{w_0}=1$ and $\overline{w_j}=0$ when $j<0$ or $j>m$.

□

3.6.4. Vector bundles

The theory of vector bundles arises from the need to examine vector spaces which are parametrised by certain sets having nice topological structure, like manifolds, varieties etc. For example the set of 2-dimensional vector subspaces $\mathcal{V}$ of $\mathbb{R}^3$:

$$\mathcal{V}=\text{rowspan}\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

where $(a_1)^2+(a_2)^2=1$, is a vector bundle of rank two over the circle. Now let us see some definitions, examples and preliminary results on the theory of vector bundles.

A *real quasi-vector bundle* ψ over a manifold $\mathcal{M}$ (called the base space $\mathcal{B}(\psi)$) is an attachment of a real vector space $\mathcal{E}_m$(fiber) to every point of $\mathcal{M}$ in a continuous way , we also require the vector space $\mathcal{E}_m$ to have the same dimension for every point $m \in \mathcal{M}$ which is called the *rank* of the quasi-vector bundle $(r(\psi))$; the space created this way is called total space of the bundle ψ and is symbolized by $\mathcal{E}(\psi)$. If there is an open covering $\{\mathcal{U}_i\}$ of $\mathcal{M}$ such that $\mathcal{E}(\psi)$ is locally homeomorphic to $\mathcal{U}_i x\mathbb{R}^{r(\psi)}$,then ψ is called a vector bundle over $\mathcal{M}$. The minimum cardinality of such coverings is called the category of the vector bundle ψ, written vecat(ψ).If $\mathcal{E}(\psi) \approx \mathcal{M}x\mathbb{R}^{r(\psi)}$ then the vector bundle ψ is called trivial. Vector bundles of rank one are called *line bundles.* A *section* of a vector bundle over $\mathcal{M}$ is a continuous assignment of a vector $v_m \in \mathcal{E}_m$ for every $m \in \mathcal{M}$. A bundle map , F , between two bundles $(\mathcal{M}_1,\psi_1)$ and $(\mathcal{M}_2,\psi_2)$ of equal rank is a continuous map F: $\mathcal{E}(\psi_1)\rightarrow \mathcal{E}(\psi_2)$ such that $F(\mathcal{M}_1) \subset \mathcal{M}_2$, $F(\mathcal{E}_x) \subset \mathcal{E}_{F(x)}$ and F restricted on $\mathcal{M}_x$ is a linear isomorphism. If we call $\bar{f}$ the restriction of F on $\mathcal{M}_1$ then we say that the bundle map F covers the map $\bar{f}$: $\mathcal{M}_1\rightarrow\mathcal{M}_2$. An *orientation* of a vector bundle ψ over $\mathcal{M}$ is a continuous assignment of orientation to every vector space $\mathcal{E}_m$ $\forall m \in \mathcal{M}$. A bundle which has orientation is called *orientable.* An easy consequence of the above is that a line bundle is orientable iff it has a nonzero section or equivalently iff it is trivial. It is worth noting that to every bundle $(\mathcal{M},\psi)$ of rank n we can assign a line bundle $(\mathcal{M}, \wedge^n\psi)$ such that to every $m \in \mathcal{M}$ we attach the 1-dim vector space $\wedge^n \mathcal{E}_m$ where $\mathcal{E}_m$ is the fiber of ψ at m. This bundle is called the orientation bundle of ψ and symbolised $o(\psi)$. The bundle ψ is orientable iff $o(\psi)$ is orientable that is $o(\psi)$ has a nonzero section or equivalently iff $o(\psi)$ is trivial.

Examples (3.14) (a) The tangent bundle $\tau(\mathcal{M})$ of a n-dimensional manifold $\mathcal{M}$ is a vector bundle of rank n. A tangent vector field of $\mathcal{M}$ is a section of its tangent bundle.
b) The canonical bundle γ_n^1 over $\mathbb{P}^n(\mathbb{R})$ is a rank one vector bundle created by attaching to every point of the projective space the line which defines it. The total space $\mathcal{E}(\gamma_n^1)$, can be regarded as a subset of $\mathbb{P}^n(\mathbb{R})\mathrm{x}\mathbb{R}^{n+1}$ as follows:

$$\mathcal{E}(\gamma_n^1)=\left\{(x,v) \in \mathbb{P}^n(\mathbb{R})\mathrm{x}\mathbb{R}^{n+1}\text{: } v \in x\right\}$$

Note that γ_n^1 is not orientable thus it is not trivial .Therefore $\mathrm{vecat}(\gamma_n^1) > 1$.
c)The canonical bundle γ_m^p on $G_p(\mathbb{R}^{p+m})$ is the vector bundle of rank p for which to every point $x \in G_p(\mathbb{R}^{p+m})$ we attach the p-dim vector space $\mathcal{E}_x$ that defines x. The total space $\mathcal{E}(\gamma_m^p)$, can be regarded as a subset of $G_p(\mathbb{R}^{p+m})\mathrm{x}\mathbb{R}^{m+p}$ as follows:

$$\mathcal{E}(\gamma_m^p)=\left\{(x,v) \in G_p(\mathbb{R}^{p+m})\mathrm{x}\mathbb{R}^{m+p}\text{: } v \in x\right\}$$

The total space of the orientation bundle $o(\gamma_m^p)$ is given by:

$$\mathcal{E}\left(o(\gamma_m^p)\right)=\left\{(x,v) \in G_p(\mathbb{R}^{p+m})\mathrm{x}\mathbb{R}^{\sigma}\text{: } v \in \wedge^p x\right\}$$

The bundle γ_m^p is not orientable implying that $o(\gamma_m^p)$ is nontrivial. Thus $\mathrm{vecat}(o(\gamma_m^p)) > 1$.
d)The tangent bundle $\tau(\mathbb{S}^1)$ is given by attaching to every point (x,y) of $\mathbb{S}^1$ the line perpendicular to the vector (x,y).The correspondence $(x,y) \to (y,-x)$ defines a nonzero section of $\tau(\mathbb{S}^1)$ proving that this bundle is orientable.

□

Let ψ be a vector bundle of rank m $(m \leq n)$ over an n dimensional manifold $\mathcal{M}$, then we can construct some invariants related to the vector bundle ψ, as follows: For every k such that $1 \leq k \leq m$ consider k generic vector fields $v_1, v_2, \ldots, v_k$ of ψ. The points of $\mathcal{M}$ where the exterior product $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ vanishes, i.e, where the vectors are linearly dependent, form a $n-m+k-1$ dimensional submanifold. It can be proved that the submanifolds of $\mathcal{M}$ defined in this way, are topological invariants of the bundle ψ. In fact, no matter how we select the vector fields $v_1, v_2, \ldots, v_k$ of ψ (provided some generic transversality conditions hold true), the above vanishing submanifold belongs to the same homology class mod2. In this way, from the vector bundle ψ, we can get m homology classes. These, in turn, give rise to m cohomology classes mod2 (by Poincare duality) which are the *Stiefel-Whitney classes* of ψ, and are symbolised by: $w_1, w_2, \ldots, w_m$ where $w_i \in H^i(\mathcal{M};\mathbb{Z}_2)$. The class $w=1+w_1+\ldots+w_m \in H^*(\mathcal{M};\mathbb{Z}_2)$ is called the *total Stiefel-Whitney class* of the bundle ψ.

Example(3.15) Consider $\mathcal{M}$ to be the two dimensional sphere ie. all triples $(x,y,z) \in \mathbb{R}^3$ such that $x^2+y^2+z^2=1$ and ψ to be the vector bundle over $\mathcal{M}$ constructed by attaching to every $(x,y,z) \in \mathcal{M}$ the two dimensional vector space of all vectors $(a,b,c) \in \mathbb{R}^3$ such that $ax+by+cz=0$. One can prove that a generic vector field of ψ has exactly two vanishing points (take , for instance, the vector field $(-y,x,0)$) and , therefore, the second Stiefel Whitney class of ψ is zero. Additionaly, the vanishing submanifold of $v_1 \wedge v_2$ of two generic vector fields v_1, v_2 of ψ is a one dimensional submanifold of the two sphere and, therefore, homologous to zero. Thus the second Whitney class of ψ is also zero.

□

The Stiefel Whitney classes are homotopy invariant of the vector bundle and can be defined and constructed in a purely topological framework independent of the vector field description given above. These classes are a measure of how far the bundle is from being non-trivial in the sense that if at least one of these classes is nonzero then the bundle is non-trivial. More importantly, the theory of characteristic classes relates vector bundle theory to cohomology and allows us to transform geometric problems into algebraic ones. The Stiefel Whitney classes satisfy the following four axioms.

1) To every vector bundle ψ of rank n we correspond a sequence of cohomology classes $w_i(\psi) \in H^i(\mathfrak{B};\mathbb{Z}_2) = 0,1,\ldots,n$ called the *Stiefel-Whitney classes* of ψ. The sum of these classes is called the total Whitney class and is denoted by $w(\psi)$.

2) Given a continuous function f: $\mathfrak{X} \to \mathfrak{Y}$ and a bundle ψ over $\mathfrak{Y}$ then $f^*(w_i(\psi))=w_i(\hat{f}(\psi))$

3) Whitney product formula : If ξ and η are real vector bundles over the same $\mathfrak{X}$, with Whitney sum $\xi \oplus \eta$ over $\mathfrak{X}$, then $w(\xi) \cup w(\eta) = w(\xi \oplus \eta) \in H^*(\mathfrak{X})$, for the cup product $w(\xi) \cup w(\eta)$.

4) Normalization : If γ_1^1 is the canonical real line bundle over $\mathbb{P}(\mathbb{R})$, then $w(\gamma_1^1)=1+\alpha$, where α is the generator of $H^*(\mathbb{P}(\mathbb{R}))$.

3.7 Conclusions

In this chapter, we have briefly reviewed the mathematical tools involved in the present approach. Effort was made to simplify the standard mathematical formalism as well as to illustrate advanced concepts and tools with numerous examples and to demonstrate the underlined mathematical ideas in an intuitive manner. Whenever there is a need for further mathematical results, they will be treated in the appropriate chapter.

CHAPTER 4. Reviews of Algebrogeometric Approaches and Results

4.1 Introduction

The main purpose of this chapter is to review of all relevant approaches and results developed so far, for the problems of this thesis from a geometric point of view and to also provide some necessary unifying terminology and background definitions. The approaches examined in this chapter have been developed for the problem of pole-zero assignment by constant, or dynamic compensators of the centralised, or decentralised type. For most of the approaches topological or algebrogeometric intersection theory (see Ch.3) is used and the results produced are rather qualitative and orientated towards a search for **generic solvability** conditions. When we say that a certain property holds for a generic system S of p-inputs, m-outputs and n-states we mean that this property holds for all such systems except, possibly, for some belonging to a 'negligible' set. In our case we define genericity from the algebrogeometric point of view; this is where the set of systems is given the structure of a variety and a 'negligible' set is a subvariety of strictly lower dimension. The parametrisation of the set of linear time invariant systems of p-inputs, m-outputs and n-states as an algebraic manifold is given in Sec. 4.2 and a new projective parametrisation of the same set as a quasiaffine algebraic variety is given in Sec. 4.3. This parametrisation is essential not only for the review in Sec 4.4 and 4.5, but throughout the whole thesis whenever the term 'generic system' is used. Although the majority of the results in this thesis deal with the generic systems case, results for nongeneric systems which deal with the exact problem are also given using some additional system invariants.

In the context of systems described above, the problem of **generic** pole assignment via output feedback has to do with the study of the conditions under which we can assign any set of poles via output feedback to a generic system of p-inputs, m-outputs, n-states, (generic= the whole variety of systems except a subvariety) . Most of the geometric approaches in Sec 4.4 are oriented towards the generic side of Determinantal Assignment Problem (DAP) which is the general problem examined in this thesis and will be analytically exposed in Ch.5. These are classified as Infinitesimal techniques, Schubert calculus techniques, Combinatorial Geometric and Projective techniques. The Projective approach is the one followed in this thesis and is directed towards the generic as well as exact solvability of the problem, but with a more cunstructive flavour, based on the Plucker embedding - a natural embedding for determinantal problems. As we shall see in the following chapters, the naturallity of this embedding will help us not only to find necessary and sufficient conditions for the DAP but also to produce new system invariants and to propose computational methods for the solution of the generic as well as exact problems. Finally, Sec. 4.5 contains all background results for the DA problems to be examined in this thesis and although this is not the purpose of this thesis, we include, for completeness, some of the classical standard non-geometric results.

4.2 The Geometric structure of the family of linear systems

The family of problems examined in this thesis are formulated as problems involving relationships between geometric objects. These objects are parametrised by the set of linear systems and thus it is essential to endow to the above set of systems a 'nice' geometric structure. This structure will first enable us, to give to the parametrisation of our objects a richer characterisation (i.e we will be able to speak about genericity) and second, to study our objects within a suitable "nice" geometrical theory.

The purpose of this section is to parametrise the set of linear time invariant systems p-inputs, m-outputs and n-states, which we denote by $\sum_{m,p}^{n}$. To do so we start by considering a linear time invariant strictly proper system of p-inputs, m-outputs and n-states

$$\begin{aligned} \dot{\underset{\sim}{x}} &= A\underset{\sim}{x}+B\underset{\sim}{u} \\ \underset{\sim}{y} &= C\underset{\sim}{x} \end{aligned} \tag{4.1}$$

The relation between the input $\underset{\sim}{u}$ and the output $\underset{\sim}{y}$ (transfer function) will not change if we transform the states $\underset{\sim}{x}$ via:

$$\underset{\sim}{x}' = T\ \underset{\sim}{x}$$

where T is a constant nxn nonsingular matrix. Then the equations (4.1) change to:

$$\dot{\underline{x}}' = T^{-1}A\,T\underline{x}' + T^{-1}B\underline{u}$$
$$\underline{y} = CT\underline{x}' \tag{4.2}$$

Thus instead of parametrising the set of strictly proper systems as the set of all minimal (controllable and observable) triples (C,A,B) we parametrise them as a set of equivalence classes of an equivalence relation defined on this set of triples. This equivalence relation is defined by:

$$(C,A,B) \approx (C',A',B')$$

iff there exists $T \in GL(\mathbb{R},n)$ such that

$$(C',A',B') = (CT, T^{-1}A\,T, T^{-1}B) \tag{4.3}$$

The question that naturally arises then is what structure we can give to $\sum_{m,p}^{n}$. Following [Haz.1] we first embed the set of all controlable pairs (A,B) into the Stiefel manifold $V_n(\mathbb{R}^{np})$ by the function Φ:

$$\Phi\big((A,B)\big) = (B, AB, \ldots, A^{n-1}B)$$

The orbits of the action '$*$'

$$T*(A,B) = (TA\,T^{-1}, TB) \tag{4.4}$$

of $GL(\mathbb{R},n)$ on $\mathbb{R}^{n^2+np}$, can be mapped into $G_n(\mathbb{R}^{np})$ via $\bar{\Phi}$ given by

$$\bar{\Phi}\big(orb(A,B)\big) = \text{rowspan}(B, AB, \ldots, A^{n-1}B) \tag{4.5}$$

and the following important fact has been proven by Hazewinkel and Kalman in [Haz.1]:

Theorem(4.1) The map $\bar{\Phi}$ is well defined, one to one, and its image is an algebraic manifold of dimension np. Therefore the set $\mathcal{O}$ of all orb(A,B) ($\approx \text{Im}\bar{\Phi}$) becomes an algebraic manifold of dimension np.

□

In this context, the triple ($\text{Im}\Phi$,p,$\mathcal{O}$), where p is the natural projection $\text{Im}\Phi \rightarrow \text{Im}\bar{\Phi}$ is a principal fiber bundle over $\mathcal{O}$ with fiber GL($\mathbb{R}$,n). If we now change the fiber G=GL($\mathbb{R}$,n) of the previous principal bundle with $\mathbb{R}^{mn}$ (which is the set of all matrices C), we get a vector bundle over $\mathcal{O}$ with total space the set $\mathbb{R}^{mn}\underset{G}{x}\text{Im}\Phi$ (where G acts on $\mathbb{R}^{mn}$ by right multiplication). The space $\mathcal{A}=\mathbb{R}^{mn}\underset{G}{x}\text{Im}\Phi$ is an algebraic manifold of dimension np+nm=n(p+m) and it is isomorphic to the set of all equivalence classes of triples (C,A,B), with (A,B) controllable. Subtracting now from every fiber $\mathbb{R}^{mn}$ over orb(A,B) all matrices C such that (C,A) is not observable we get an open and dense subset $\mathcal{B}$ of $\mathcal{A}$ which corresponds to the orbits of all minimal triples. Thus $\mathcal{B}$ is the object parametrising the set of all linear time invariant, strictly proper systems of p-inputs, m-outputs and n-states and it is an n(m+p) dimensional algebraic manifold.

A second non-constructive parametrisation based on invariant theory of algebraic group actions on complex varieties is as follows [Hum.1]: Consider the complex quasi-affine variety of all minimal triples $(C,A,B) \in \mathbb{C}^{mxn}x\mathbb{C}^{nxn}x\mathbb{C}^{nxp}$ then the state coordinate action of GL($\mathbb{C}$,n) (defined by eq(4.3)) is a closed orbit action [Hum.1]. Since GL($\mathbb{C}$,n) is a linear reductive group the orbit space of this action can be endowed with the structure of a complex algebraic variety[Hum.1].

In the next section we will give an alternative parametrisation of $\sum_{m,p}^{n}$ based on the Plucker invariant of the characteristic maximal $\mathbb{R}$[s] module of the system. The new parametrisation is an embedding one and is introduced via the Plucker map. This map compresses a p-dimensonal vector space (or module) to a line and can be used to parametrise a set of vector spaces as a set of points in an appropriate projective space.

4.3. Grassmann and Plucker invariants of rational vector spaces and a new parametrisation of the set of strictly proper systems.

4.3.1 Rational vector spaces, polynomial modules and systems

Rational vector spaces and Polynomial modules arise naturally in control theory in the study of the transfer functions of systems in the frequency domain. The transition from the time domain to the frequency domain is made, as far as the linear time invariant systems are concerned, via the Laplace transform. In this case the derivative operator d/dt is transformed into the indeterminate 's' of the polynomials in $\mathbb{R}[s]$. The transfer function $G(s)=C(sI_n-A)^{-1}B$ which relates the steady state response of the outputs to the inputs, is an mxp rational matrix of McMillan degree n. The notion of the polynomial module comes naturally when we consider matrix fraction descriptions of the transfer function G(s).

Rational strictly proper matrices of McMillan degree n can be written as $G(s)=N(s)D(s)^{-1}$ where $N(s)\in\mathbb{R}[s]^{mxp}$, $D(s)\in\mathbb{R}[s]^{pxp}$ and the column degrees of D(s) are strictly larger than the corresponding column degrees of N(s). The object that naturally characterises the MFDs of a rational function is the composite polynomial matrix:

$$M(s)=\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{4.6}$$

Using the previous MFD, we can readily obtain an alternative MFD for G(s) as:

$$G(s)=\big(N(s)K(s)\big)\big(D(s)K(s)\big)^{-1} \tag{4.7}$$

It is easy to see that the MFD given by eq(4.7) has composite polynomial matrix M(s).K(s). All possible MFDs for the same transfer function G(s) do not correspond to the same system. What actually happens is described in the following important isomorphism theorem[R.5]:

Theorem(4.2) For a given right MFD of a strictly proper rational function, with composite matrix M(s), there correspond one and only one controllable system given by the triple (A,B,C). For the MFD with composite matrix M(s).U(s) where U(s) is unimodular, there corresponds the same controllable system. If additionally M(s) has no zeros (MFD is coprime) then the corresponding system is minimal (controllable and observable).

□

We can now naturally define the following equivalence relation in the set of composite polynomial matrices:

$$M_1(s) \sim M_2(s) \text{ iff } M_1(s)=M_2(s)U(s) \tag{4.8}$$

for some unimodular polynomial matrix U(s). It is apparent that every equivalence class of the above relation corresponds to a unique MFD of G(s) and therefore, to a unique system (which may be nonminimal). A complete invariant of the above equivalence relation is the $\mathbb{R}[s]$ module produced by the $\mathbb{R}[s]$ linear combinations of the columns of M(s). Since all these submodules are included into the rational vector space,

$$\mathcal{R}_{G(s)} = \mathbb{R}(s)\text{-colsp}([I_p, G(s)^T]^T \tag{4.9}$$

we have the 1-1 correspondence:

$$\left\{\text{Systems with transfer function } G(s)\right\} \leftrightarrow \left\{\mathbb{R}[s] \text{ polynomial submodules of the vector space } \mathcal{R}_{G(s)}\right\}$$

The submodule that corresponds to the minimal system is the unique maximal polynomial submodule of $\mathcal{R}_{G(s)}$. Therefore we have the following 1-1 correspondences (which will help us to give a second parametrisation of linear systems)

$$\text{minimal systems} \leftrightarrow \text{rational vector spaces } \mathcal{R}_{G(s)} \leftrightarrow \text{maximal pol. submodule of } \mathcal{R}_{G(s)}$$

This correspondence suggests that we can view the set of minimal systems as a set of rational vector spaces or a set of polynomial modules which, as shall be seen in the next section, can be parametrised as a variety via a Plucker type embedding.

Remark(4.1) The consideration of the maximal module as a representative for a system, also arises naturally from the determinantal equation

$$\det([I_p,K].M(s))=p(s)$$

where M(s) is the composite matrix for a coprime MFD for G(s). This equation appears in the pole placement problem via constant output feedback (see Ch.5). In such cases the equation is invariant under unimodular column transformations of M(s). This fact suggests that what really affects the equation is the module spanned by the columns of M(s) and not the matrix M(s) itself.

□

Next, we will describe how we can achieve alternative representation and invariants for rational vector spaces by using exterior algebra.

4.3.2 Grassmann and Plucker invariants of rational vector spaces.

With the help of the Plucker embedding we can produce a complete basis-free invariant for any rational vector space. The characterisation of a rational vector space can be naturally given in terms of a basis matrix of its unique maximal polynomial module. In this case the discrete invariants for the module are the column degrees of the basis. The new invariants, Grassmann and Plucker, contain only one (obvious) discrete invariant the complexity of the module which is the sum of the column degrees. The column degrees are hidden due to the compression property of the Plucker embedding, and can be recovered by examining certain algebraic relations on the continuous parameters.

The Plucker embedding (see also 3.3.3) is a classical embedding used to parametrise the Grassmannian $G_p(\mathbb{F}^{p+r})$, the set of all p-dimensional $\mathbb{F}$- linear spaces in $\mathbb{F}^{p+r}$, as an algebraic variety over $\mathbb{F}$. This embedding corresponds every p-dimensional linear space $\mathcal{V}$ to the one dimensional linear space $\wedge^p(\mathcal{V})$, the p-th exterior power of $\mathcal{V}$. This correpondence is readily basis free. If we introduce a basis matrix $V \in \mathbb{F}^{(p+r)xp}$ for the vector space $\mathcal{V}$, then the corresponding basis vector for $\wedge^p(\mathcal{V})$ will be the compound

$C_p(V) \in \mathbb{F}^{\sigma x1}$ (which is the exterior product of the column vectors of V) where $\sigma=\binom{p+r}{p}$. Let now a possible change in the basis of $\mathcal{V}$ to be:

$$V'=A.V \tag{4.10}$$

then, (by taking compounds) this corresponds to the following change of basis of $\wedge^p(\mathcal{V})$:

$$C_p(V')=\det(A).C_p(V) \tag{4.11}$$

which is a multiplication of the original basis vector $C_p(V)$ by the scalar $\det(A) \in \mathbb{F}-\{0\}$. Conversely, if $C_p(V)$ and $C_p(V')$ are two basis vectors for $\wedge^p(\mathcal{V})$ and $\wedge^p(\mathcal{V}')$ such that they are colinear, then $\mathcal{V}=\mathcal{V}'$. All the above are summarised in prop.(3.6).

Remark(4.2) Proposition(3.6) states that the map:

$$\mathcal{V} \rightarrow \wedge^p(\mathcal{V}')$$

is one to one.

□

The importance of using exterior products (or compounds) is now obvious. Instead of checking that the columns of two matrices span the same space, we can test whether two vectors ($C_p(V')$ and $C_p(V)$) are colinear, which is a much easier task. It is also apparent that it is more convenient to deal with one dimensional objects than with higher dimensional ones. As a consequence, we can represent any vector space $\mathcal{V}$ with basis matrix V, by its Grassmann representative $g(\mathcal{V}) \in \mathbb{R}^{\sigma}$

$$g(\mathcal{V})=(\ldots,\ v_{\omega}\ ,\ldots)$$

It is worth baring in mind (see Ch.3) that the vector $g(\mathcal{V})$ is the σx1 vector formed by the pxp subdeterminants of V, v_{ω}, lexicographically ordered.

The Grassmann representative is not a canonical representative since it depends on the particular basis matrix V. However, it can easily become canonical by a suitable selection of basis. This will be examined next for the case $\mathbb{F}=\mathbb{R}(s)$(or $\mathbb{C}(s)$) and when $\mathcal{V}=\mathcal{R}_{G(s)}$ given by eq(4.9). If we let M(s) be an $\mathbb{R}(s)$ basis matrix for $\mathcal{R}_{G(s)}$ then we get two types of Grassmann representatives of $\mathcal{R}_{G(s)}$: (i) the rational one, when the vector $\underline{m(s)}=C_p(M(s))$ has rational coordinates and (ii) the polynomial one, if the coordinates of the above representative are polynomials. It is apparent that polynomial Grassmann representatives correspond to polynomial bases M(s) of $\mathcal{R}_{G(s)}$ whose degree varies from n (the McMillan degree of G(s)) and upwards. Thus, a good (starting) choice for a canonical representative $C_p(M(s))$, will be the one where M(s) has the lowest possible degree, ie. n, and this because of two reasons: a)We may expect that it will also contain the least number of independent continuous parameters (a common characteristic of canonical representatives) and b) Such an M(s) is a basis matrix of the maximal module $\mathcal{M}_{max}$ which is unique for every $\mathcal{R}_{G(s)}$(see previous subsection). The problem of canonically selecting a basis matrix M(s) for $\mathcal{M}_{max}$ is not an easy task; it is actually the well known problem of putting M(s) in a canonical echelon form via column transformations. On the other hand, if we look at $\underline{m(s)}=C_p(M(s))$, any unimodular column transformation of M(s) becomes now a multiplication of $\underline{m(s)}$ by a real number. In this way, a possible change of the basis of the maximal module amounts only to the multiplication of the Grassmann representative by a constant number. Thus, the difficult task of putting M(s) into a canonical echelon form via column transformations is transformed into the easy problem of multiplying (or dividing) $C_p(M(s))$ by a suitable nonzero real number.

Definition(4.1)[Kar.1] Let $\mathcal{R}_{G(s)}$ be an $\mathbb{R}(s)$ a rational vector space arising from a strictly proper system with transfer function G(s) and M(s) be one polynomial basis for $\mathcal{M}_{max}$, then if $\underline{m(s)}=(\ldots, m(s)_{\omega},\ldots)^T$ is the Grassmann representative (sec. 3.3.3) of $\mathcal{R}_{G(s)}$ with respect to the basis matrix M(s) , the *canonical polynomial Grassmann representative* of $\mathcal{R}_{G(s)}$ is defined to be :

$$GR(\mathcal{R}_{G(s)})= \frac{1}{p_{ni}} \underline{m(s)} \tag{4.12}$$

where n is the complexity of M(s) (equals to the McMillan degree of G(s)) and p_{ni} is the first nonzero coefficient of s^n in the polynomials appearing in $\underline{m}(s)$.

□

Remark(4.3) The motivation for the above definition is that we want the first polynomial coordinate of $C_p(M(s))$, which is equal to det(D(s)), to be monic, as it is when M(s) is in a canonical echelon form. However, we can divide with alternative numbers thus, obtaining slightly different canonical representatives (see [Kar.1]).

□

Remark(4.4) As a justification of why the canonical Grassmann representative was produced more easily than the (equally canonical) Popov form [Kai.1], we can say that the Grassmann contains only one discrete invariant for the rational vector space, namely its complexity, whereas the Popov form contains the column degrees and the pivot indices. However, these descrete invariants do exist in the grassmann representative, but are hidden and can be recovered by examining certain algebraic relations on the continuous parameters.

□

The canonical polynomial grassmann representatitive of $\mathcal{R}_{G(s)}$ given above is well defined and it is in one to one correspondence with $\mathcal{R}_{G(s)}$ [Kar.1]. With the help of $GR(\mathcal{R}_{G(s)})$ we can define a second complete invariant for $\mathcal{R}_{G(s)}$ which is no more a polynomial invariant but is based on the coefficients.

Definition(4.2) The *canonical Plucker representative* of the rational vector space $\mathcal{R}_{G(s)}$, denoted by P_G, is defined by the following equality:

$$GR(\mathcal{R}_{G(s)})= P_G\,[s^d,\ldots, s^2, s, 1]^T \tag{4.13}$$

□

Next, we will be giving a second parametrisation for the set of linear systems based on the above canonical representatives.

4.3.3 A new parametrisation of the set of strictly proper systems.

We now propose an alternative parametrisation of such systems based on the matrix fraction description of their transfer function and the invariants defined previously. We give to $\sum_{m,p}^{n}$ the stronger structure of irreducible real smooth quasi-affine variety without using (general) theorems of existence of such a structure (invariant theory, schemes , etc.). Our way is totally constructive and rather elementary; we use only a (generalised) Plucker embedding for polynomial modules and some basic theory of MFD's of systems. Since our approach is embedding, we are also able to calculate the closure of the variety of systems $\sum_{m,p}^{n}$.

The set of all modules $\mathcal{M}_{max}$ that correspond to strictly proper systems of p-inputs,m-outputs denoted by $\mathcal{M}_{max}^{p,m,n}$ and is obviously in 1-1 correspondance with the set $\sum_{m,p}^{n}$ (see 4.3.1). For a natural embedding of the set of modules $\mathcal{M}_{max}^{p,m,n}$ to an affine space $\mathbb{R}^{a}$, we will use the Plucker embedding. For a given basis matrix M(s) of a $\mathcal{M}_{max}$, this embedding maps :

$$M(s) \rightarrow C_p(M(s))$$

In this way M(s) is mapped to the σx1 polynomial vector $\underline{m(s)}=\left(\ldots,\ m(s)_{\omega},\ldots\right)$ the grassmann representative with respect to the basis M(s). Any unimodular column transformation of M(s) becomes now a multiplication of $\underline{m(s)}=C_p(M(s))$ by a real number. If we denote $\underline{m}_{\omega}$ as being the vector of the coefficients of $m(s)_{\omega}$ then the Plucker type embedding $\mathcal{P}_d$ which maps M(s) to $\underline{m}=\left(\ldots,\ \underline{m}_{\omega},\ \ldots\right)$ is called the *dynamic Plucker embedding* and maps $\mathcal{M}_{min}^{p,m,n}$ into $P^{n\sigma}(\mathbb{R})$. Polynomial bases M(s) that correspond to the same $\mathbb{R}[s]$ module are mapped via $\mathcal{P}_d$ to the same element of the projective space $P^{n\sigma}(\mathbb{R})$ thus the image of $\mathcal{P}_d$ in $P^{n\sigma}(\mathbb{R})$ is into 1-1 correspondence with $\sum_{m,p}^{n}$. Not all vectors $\underline{m} \in P^{n\sigma}(\mathbb{R})$ can be written as $\mathcal{P}_d(M(s))$ for some matrix M(s) corresponding to a strictly proper system of p-inputs, m-outputs and n-states. The ones that are able to do so, are those satisfying the *Dynamic QPRs* and since deg(det(D(s))=n the first coordinate of $\underline{m}$ must be non-zero. These facts make $\sum_{m,p}^{n}$ a real quasi-affine variety.

Example(4.1) Let M(s) be the composite matrix of a strictly proper systems of 2-inputs, 2-outputs and 2-states. Then the compound of M(s) is given by the vector

$$[s^2 + a_{11}s + a_{12},\quad a_{21}s + a_{22},\quad a_{31}s + a_{32},\quad a_{41}s + a_{42},\quad a_{51}s + a_{52},\quad a_{61}]$$

satisfying only one QPR, namely

$$a_{61}(s^2+a_{11}s+a_{12}) - (a_{41}s+a_{42})(a_{21}s+a_{22}) + (a_{31}s+a_{32})(a_{51}s+a_{52}) = 0$$

Equating the coefficients of s^2, s, 1 to 0 we get three equations which are the Dynamic QPR's describing $\sum_{2,2}^{2}$ in $\mathbb{P}^{11}(\mathbb{C})$.

□

The dimension of this variety can be found by measuring the maximum number of independent parameters appearing in a canonical form for the colspan(M(s)), and this is examined in the following theorem.

Theorem(4.3) The dimension of $\sum_{m,p}^{n}$ is equal to n(m+p)

Proof

For the purposes of this proof we will use the well- known Popov form which is a canonical echelon form for systems. This form contains two types of system invariants: a)discrete invariants (controllability indices and pivot indices), b) continuous invariants (those we want to count). Let $\kappa=\{k_1, k_2,\dots, k_p\}$ be the set of controllability indices and $\pi=\{p_1, p_2,\dots, p_p\}$ the set of pivot indices then:

$U_{\kappa,\pi}=\Big\{\underline{x} \in \mathbb{R}^v$: $\underline{x}$ is the vector having as coordinates all the continuous parameters appearing in the canonical Popov form, of the composite matrix of a minimal MFD of a system in $\sum_{m,p}^{n}$, with discrete invariants $\kappa,\pi\Big\}$

The set $U_{\kappa,\pi}$ is a Zarisky open subset of $\mathbb{R}^v$ since it does not contain the $\underline{x} \in \mathbb{R}^v$ which correspond to a a Popov form of a composite matrix that is not coprime. This set can be embedded by a Plucker type map into the variety $\sum_{m,p}^{n} \subseteq P^{n\sigma}(\mathbb{R})$.

$$\mathcal{P}_{\kappa,\pi}\colon\ U_{\kappa,\pi} \longrightarrow \sum_{m,p}^{n} \subseteq P^{n\sigma}(\mathbb{R})$$

$\mathcal{P}_{\kappa,\pi}$ is a polynomial map and an embedding (1-1 and the differential is injective). The images of all possible $\mathcal{P}_{\kappa,\pi}$ cover the variety $\sum_{m,p}^{n}$ and partition it into a set of disjoint quasi-affine varieties of unequal dimensions. Readily, the dimension of $\sum_{m,p}^{n}$ will be the maximum of the dimensions of all $U_{\kappa,\pi}$ that is the maximum number of parameters contained in the Popov canonical forms for all κ, π.

Let $n=pk+u$ $u<p$, then we can see by inspection that the Popov form with the largest number of parameters is the one with $\kappa=(k,k,..k,k+1,k+1,...,k+1)$ and that this number equals to $n(m+p)$.

□

We can, conversely, construct from a decomposable $\underline{m} \in P^{n\sigma}(\mathbb{R})(m_1 \neq 0)$, a matrix M(s) corresponding to a strictly proper system as follows: follow procedure of corollary(3.2) for $\mathbb{F}=\mathbb{R}(s)$ and get a matrix $L(s)^T=[I_p,G(s)^T]^T$, $G(s) \in \mathbb{R}(s)^{mxp}$. This matrix will be such that:

$$C_p(L(s))= \underline{m}(s)/m(s)_{\omega_1}$$

the matrix G(s) represents the transfer function of strictly proper system of p-inputs , m-outputs and at most n-states whose pole polynomial is equal to $m(s)_{\omega_1}$. Now if $N(s)D(s)^{-1}$ is a coprime MFD for G(s) we get:

$$C_p(M(s)) = C_p(L(s)D(s)) = C_p(L(s))\ \det(D(s)) = C_p(L(s))m(s)_{\omega_1} = \underline{m}(s)$$

therefore, M(s) is mapped to $\underline{m}$ via $\mathcal{P}_d$.

4.4 Approaches and methodologies of control problems

4.4.1 Introduction

As we shall see in Ch.5, all the control problems to be examined in this thesis belong to the same problem family ie. the determinantal assignment problem (DAP). This problem is to solve the following equation with respect to polynomial matrix H(s):

$$\det(H(s).N(s))=p(s) \tag{4.14}$$

where p(s) is a polynomial of an appropriate degree d. The difficulty with this problem is mainly due to its multilinear nature induced by the determinant. An additional complication is due to the fact that we need the solution H(s) of (4.14) to be a polynomial matrix. However, as shall be shown in Ch.5, in all the cases we examine, the dynamics of H(s) can be shifted to N(s), which, in turn, transforms the problem to an equivalent constant DAP. This constant DAP, may be described as follows: Let $M(s) \in \mathbb{R}^{(p+r)xp}[s]$ such that rank(M(s))=p and let $\mathcal{H}$ be a family of full rank px(p+r) matrices having a certain structure. Solving with respect to $H \in \mathcal{H}$ the equation:

$$\det(H.M(s))=p(s) \tag{4.15}$$

where p(s) is an arbitrary polynomial of an appropriate degree d.

We classify all the approaches for DA-problems into two general classes: (i) conventional state- space and algebraic and (ii) geometric techniques, respectively. Since our approach is based on algebraic geometry and topology, we will mainly be reviewing the second class and only presenting the results of the first, without paying much attention to the details.

STATE-SPACE AND ALGEBRAIC TECHNIQUES Here we include all methods, within the bounds of the standard linear systems theory, for the solution of DA problems. These methods which have been devised mainly for the output feedback pole placement problem, are straightforward and algorithmic, and thus very convenient for design purposes. However, such methods were not able to resolve some of the fundamental non-linear features of DAP and hence, their use is restricted.

These techniques can be further classified into dyadic and full rank. The dyadic or rank one design, in the case of pole placement by output feeback, is based on the fact that if the matrix K appearing in $H=[I_p,K]$ is a product of two vectors (rankK=1), then equation (4.14 or 4.15) aquires a simple form, (linear with respect to the unknown K) and can be solved, under certain conditions, with respect to a real K using algorithms described in [Chen.1],[Bra.1]. Although these algorithms have considerable elegance and simplicity, the resulting closed loop systems have poor disturbance rejection properties compared with their full rank counterparts. Furthermore, although the dyadic has been successfully used in the state feedback, its use in the output feedback further reduces the degrees of freedom and thus weakens the solvability conditions. The full rank designs [Mun.2] are more complex, and the main objective is to construct a full rank matrix K solving the output feedback pole placement problem. A detailed account of the nongeneric techniques and results for pole assignment may be found in [Mun.1]; of special interest are those techniques which are based on a canonical description of the system such as those developed in [Var.1]. These techniques are general and may be used also for looking at the Rosenbrock's problem [R.4], which is the assignment of invariant polynomials rather than the determinant. It is worth mentioning that Rosenbrock's problem has been considered only for the state feedback case whereby the output feedback case is still open.

Rosenbrock's problem may be considered as a special case where we try to solve the equation

$$H(s)\,M(s) = P(s) \tag{4.16}$$

where P(s) is a square matrix with a certain structure, M(s) represents the corresponding system representation and H(s) the controller form. The above equation is in the form of a model matching problem; the sovability of (4.16), however, is affected by the selection of P(s). This formulation is not equivalent to the determinant formulation. Every solution of (4.16) implies also a solution of the determinantal equation, but the selection of P(s) is rather arbitrary and cannot be systematically chosen to yield a solution. If we allow H(s) to have any dynamic complexity (degree of H(s)), then Equation(4.16) is linear and can be solved using linear algebra, subject ot some coprimeness conditions (see model matching [Var.2],[Scot.1]).

The study of zero assignment by squaring down has been largely considered within the framework of state space techniques [Kar.2], [Sab.1] with the exception of the work of [Kar.3] which is within the exterior algebra and algebraic geometry techniques considered subsequently. The work in [Kouv.1], [Kar.2], [Kar.3] deal with the constant squaring down whereas that in [Sab.1] also considers the dynamic case. The first two deal with sufficient conditions for assigning part of the set of assignable zeros using techniques for selection of appropriate zero directions [MacF.4]. The work in [Sab.1] is based upon special coodinate descriptions which allow zero assignment if the dynamics of the squaring down compensator are chosen sufficiently high. Such a technique, however, can be applied only in the input squaring down case since in general, dynamic output squaring dowm does not make sense from the engineering point of view (output squaring down of the variables to be controlled and thus any dynamics involved characterize the dynamics of the corresponding sensors). The dynamic squaring down may be formulated as a model matching problem if it is used as input squaring down as this has been discussed [Kar..3]. The only solvability conditions and general approach for computing solutions for the constant squaring down has been given in [Kar.3].

There are certain similarities between the squaring down and the output feedback case. In fact, Aplevitch has demonstrated how the one problem may be transformed into the other [Apl.1]. In general, however, techniques such as dyadic compensators which linearise the problem cannot be used in a straight forward manner since the post compensators or precompensators must have full rank. Full rank linearization may be used for zero assignment but these opportunities have not been properly examined yet.

The pole-zero assignment problems, when the centralisation assumption fails, has been an active area in the last twenty years. We distinguish four main approaches for the study of such problems. The first is based upon the use of standard state space methodology and the most definitive results are those in [Dav.1],[Cor.1] where the existence of both pole assignment and stabilization have been investigated. The particular chracteristics of the decetralised problems is that feedback has a block diagonal structure and thus fewer degrees of freedom. Because of the decentralisation, fixed modes may appear despite the controllability and observability assumption. The work in [Cor.1] deals with the solvability of pole assignment and stabilizability but involves a mixture of both static and dynamic feedback. In fact, their basic philosophy is to determine conditions under which the system can be made controllable and observable from the input and output variables by static feedback applied by the other controllers, then dynamic compensation can be employed at these controllers for pole placement. Solvability conditions are expressed in terms of topological conditions characterising the structure of interconnections (strongly connected assumption). The approach of Davison and Wang has a similar philosophy, although there is no explicit attempt to make all the strongly connected subsystems controllable and observable from a single controller.

An alternative approach based on transfer function MFD tools, has been developed [And.1] for tackling dynamic compensation issues but most of the results deal once more with the characterisation of fixed modes. An attempt to exploit the graph structure of the underline interconnection and define natural system decompositions and decentralisation sheme is described within the overall framework of graph methods [Rein.1]. Some work has recently been done which tries to extend the exterior algebra and Algebraic geometry framework to decentralised pole zero assignment problems [Kar.4]. This work deals mainly with the definition of a framework for the study of these problems as well as the investigation of certain necessary conditions. So far, there are general results expressing necessary and sufficient conditions for generic as well as exact pole, zero assignability which are independent from the underlying system graph.

GEOMETRIC TECHNIQUES. Although (what we regarded to be) 'state-space and algebraic techniques' are simple and constructive, they lack motivation since they concentrate more in constructing solutions than in understanding the nature of the problem. The characteristic of these techniques is that they do not use all possible degrees of freedom of H but rely on special forms which simplify the problem. On the other hand, in what we consider as geometric approaches, the DA problem is examined more thoroughly with the least possible simplifications. We choose the word 'geometric' because the problem is reduced to the study of relations (maps, intersections etc.) of auxiliary geometrical objects like linear spaces, algebraic varieties, manifolds. It is apparent that the nature of these approaches is general, qualitative and more oriented to producing theorems for the existence of solutions rather than algorithms for the computation of solutions. However, it is strongly believed that a better understanding of the nature of the problem may lead to a better computation of the solutions.

The most crucial point in all geometrical techniques is that for a generic system the auxiliary geometrical objects considered, have to be located in a canonical way in the underline space. This is what we call transversality or general position and in most of the cases the measure for this transversality property of the objects is expressed as a rank of a matrix depending upon the parameters of the system.

Next we will examine the five main approaches developed so far for the DA problems.

1) Infinitesimal techniques. One way of looking at the (full) DA problem is to consider the polynomial map:

$$\chi: \mathbb{F}^{\mu} \rightarrow \mathbb{F}^{d}$$

where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, μ are the degrees of freedom of H in eq(4.15) and χ maps H to the coefficient vector of the polynomial p(s) of eq.(4.15). This map was first defined and examined in [Her.1] and [Wil.1] for the output feedback pole placement problem, and the solvability of this problem was reduced to test if this map was onto. The onto properties of a polynomial map can be easily examined using the "dominant morhism theorem" for complex algebraic varieties. According to this theorem, it can be proven that a map is onto if the differential of this map at some point is onto (as a linear map). In the case examined in [Wil.1] (when $\mathbb{F}=\mathbb{C}$), the differential of χ at K=0 was calculated to be:

$$(d\chi)_0 = (col\mathrm{CB},\ col\mathrm{CAB}, \ldots,\ col\mathrm{CA}^{n-1}\mathrm{B})$$

and the generic (in $\sum_{m,p}^{n}$) rank of the mpxn matrix $(d\chi)_0$ was computed to be equal to n (when $mp \geq n$). Using then the previous result along with the donimant morphism theorem (see sec 3.4.3), a necessary and sufficient condition for generic pole assignability was derived.

As far as the case $\mathbb{F}=\mathbb{R}$ is concerned, there is no dominant morphism theorem as in the $\mathbb{F}=\mathbb{C}$ case. In this case if the differential of χ at some point x, is onto, then χ is onto in a neighbourhood around $\chi(x)$. In this context, Reinschke calculated the differential of χ (for the case of pole placement via output feedback) at a point K to be:

$$(d\chi)_K=(col CB,\ col C(A+BKC)B, \ldots,\ col C(A+BKC)^{n-1}B)$$

and produced some results for local pole assignability via real constant output feedback.

2) **Enumerative geometry techniques (Schubert calculus)** . Alternatively we can view the complex DAP as a problem of intersection of hypersurfaces on a Grassmannian as follows: First, since the poles of det(H.M(s)) depend only upon the rowspan(H) and not on H, what we are interested in is the set $\mathcal{H}=\{rowspan(H):H \in \mathcal{H}\}$ and not the set $\mathcal{H}$. In fact, $\mathcal{H}$ is a subset of the manifold $G_p(\mathbb{C}^{p+r})$ which contains all p-dim subspaces of $\mathbb{C}^{p+r}$. Letting $\{s_i\}_d^{i=1}$ be the set of complex conjugate roots of p(s) then by (2.21) we require to find H such that: $\det(H.M(s_i))=0$ for every i=1,...,d.From this, if we set:

$$\mathcal{V}(s_i)=LKerM(s_i)$$

and

$$\sigma(s_i)=\{\mathcal{V} \text{ is a linear subspace of } \mathbb{C}^{p+r}\text{: } \dim\mathcal{V}=p \text{ and } \dim(\mathcal{V}\cap\mathcal{V}(s_i))\geq 1\}$$

then the solution set of gains of the complex DAP can also be given as the intersection:

$$\mathcal{K}_{\mathbb{C}}(p(s))=\bigcap_{i=1}^{d}\sigma(s_i)\cap\mathcal{H}$$

In all cases we examine, $\mathcal{H}$ is a variety or a Zarisky open subset of a variety in $G_p(\mathbb{C}^{p+r})$. Each of the $\sigma(s_i)$ is a Schubert variety of the form $<m-1,m,\ldots,m>_{\mathbb{C}}$ and thus

$\mathcal{K}_C(p(s))$ is a subvariety of $\mathcal{H}$ defined by n linear equations(a more general and constructive account of this fact will be given in Ch.5 Sec.5). When $\mathcal{H}$ is a variety, a necessary and sufficient condition for generic (complex) solvability of the problem can easily be derived by counting dimensions. On the other hand, however, when $\mathcal{H}$ is a Zarisky open subset of a variety (like the constant output feedback pole placement case) additional work has to be done in order to make sure that the solution set does not contain only infinite elements.

As far as the real solutions of the problem are concerned, the above approach allows us to use an intersection theory on the Grassmannian known as Schubert Calculus which is involved in the enumeration of subspaces located in a larger space and pocess certain properties. More analytically:

a) The special case where d is equal to the dimension of $\mathcal{H}$ was considered in [Bro.1]. In this case $\mathcal{K}_C(p(s))$ contains finite number of points whose number can be calculated using Schubert enumerative calculus. If this number is odd then a real solution exist.

b) The more general case where d is arbitrary was examined in [Gia.1]. Here, the intersection of $\mathcal{K}_C(p(s))$ with an appropriate Schubert variety was considered in such a way that it contains a finite number of points. This number can be calculated via Schubert enumerative calculus and if it is odd we then have a real solution.

3) Topological intersection techniques. These techniques are introduced for the examination of the generic solvability of the DAP, especially when we are looking for real solutions. Here the set of solutions $\mathcal{K}_R$ is a submanifold of the compactified parameter space $\mathcal{H}$ and corresponds to a certain element of a topological intersection ring of $\mathcal{H}$, namely the cohomology ring of $\mathcal{H}$ with coefficients in $\mathbb{Z}_2$, symbolised by $H^*(\mathcal{H};\mathbb{Z}_2)$. Then, a sufficient condition for the existence of real solutions is for the above cohomology class to be non-zero. This approach is essentially a mod2-intersection theory, which is normally applied when we consider intersections of smooth real varieties, where a $\mathbb{Z}$-intersection theory (like the Schubert calculus) is not available. It is important to note that the results we derive with this theory can only be sufficient conditions.

Other topological approaches to deal with intersections of real varieties is the use of various topological invariants of spaces or maps which can be derived from our intersection problems. For example we can express the problem of real pole placement of a generic system via constant output feedback, in terms of a topological category (LS category), as follows: As we have seen in the Schubert Calculus techniques, this problem can be written as an intersection of n Schubert hypersurfaces $\sigma_S(s_i)$, $1 \leq i \leq n$ in $G_p(\mathbb{R}^{p+r})$. If this intersection is empty then the complementary set $\bigcup_{i=1}^{n}\left(G_p(\mathbb{R}^{p+r})-\sigma_S(s_i)\right)$ must be the whole of $G_p(\mathbb{R}^{p+r})$. This covering of the Grassmannian by the above n contractible sets is not always possible. The obstruction to this is called Ljusternick-Snirelmann category of $G_p(\mathbb{R}^{p+r})$, LScat(p,r), which is the minimum number of contractible sets which can cover the above Grassmannian. Thus in order that intersection is nonempty we must have LScat(p,r) $\geq$ n. We can also use other topological category numbers according to which topological theory we use; if, for example, vector bundle theory is used, the obstruction here will be the vector category of some bundle.

It is important to note that in these topological intersection approaches, we normally use a compactified parameter space of solutions and not the solution space itself which in many cases is topologically uninteresting. We obtain the new compactified space by introducing the so called 'solutions at infinity', some of which (namely, the degenerate points) are not desirable. Thus, although the topological intersection approach may present nice results we have to be extra carefull in the compactification we use. This has to be natural (in the sense of Ch.3) so that a possible nonvoid intersection on the compactified space contains always a finite solution.

4) Combinatorial Geometric techniques. This approach was first proposed in [Kim.1] by H. Kimura for the output feedback pole placement problem. The main idea is to observe that the solvability of the DAP is equivalent to finding a p dimensional linear subspace $\mathcal{V}$ of $\mathbb{R}^{p+r}$ such that it intersects all the r-dimensional subspaces $\mathcal{V}(s_i)$ i=1,...,n as defined in the Schubert calculus techniques. A straightforward solution for this problem (without using Grassmannians and enumerative calculus) contains an

interesting combination of the geometry of linear subspaces and combinatorics involving dimension counting of certain subspaces. To illustrate this let us try to find n (as a function of r,p) such that we can find a p-dimensional $\mathcal{V}$ intersecting all $\mathcal{V}_i=\mathcal{V}(s_i)$: first, choose a plane $\mathcal{W}$ of dimension r-p+1 intersecting $\mathcal{V}_1,\dots,\mathcal{V}_{r-p+1}$ nontrivially (this can be done by selecting $r-p+1$ linearly independent vectors each from every subspace). Given any two planes $\mathcal{V}_i$, $\mathcal{V}_j$ we can find a vector v such that $\mathcal{W}_1=\text{span}\{\mathcal{W},v\}$ intersects $\mathcal{V}_1,\dots,\mathcal{V}_{r-p+1},\mathcal{V}_i,\mathcal{V}_j$ nontrivially (since $\dim\big((\mathcal{W}_1+\mathcal{V}_j)\cap\mathcal{V}_i\big)=r+p+1-(r+p)=1)$). By adding $p-1$ vectors one gets an r-dimensional plane $\mathcal{W}_{p-1}$ which intersects $r-p+1+2(p-1)=r+p-1$ planes. Thus if we have a number of $n=r+p-1$ (or less) p-dimensional planes in $\mathbb{R}^{p+r}$, we can always find an r-dimensional plane intersecting them. Obviously $\mathcal{W}$ has not been chosen in the best way. By choosing $\mathcal{V}_1,\dots,\mathcal{V}_k$ inductively, until a linear dependence relation appears, one can improve the algorithm considerably. This was done in [Ros.2] where a more refined result of the same problem can be found.

5) Projective techniques. In [Kar.1],[Kar.4] it has been realised that the DAP is of a multilinear skew-symmetric nature and the natural splitting of the multilinear function into a standard multilinear (skew-symmetric) map and a linear map [Gre.1] may be applied. This approach reduces the overall solvability to the problem of determining common solutions of a set of linear and quadratic equations which are defined in the appropriate projective space associated with the standard exterior map of the above splitting.

Classical algebraic geometry in a projective rather than affine space, is used to determine the existence of solutions. The approach relies on exterior algebra to construct the embedding map which is the Plucker embedding. Although the approach has been based on the explicit construction and use of the Plucker embedding, this is not the only type of embedding that may be used for such problems (it will be seen in this thesis that decentralised problems induce other types of embedding such as tensor products of Plucker embeddings).

The use of the projective space as the space of the embedding automatically handles unbounded gain solutions, although additional conditions have to be used if bounded gain controllers are sought. One of the major advantages of this framework is that it introduces new sets of invariants (of projective character) which may be used to chracterise solvability conditions. Both generic and exact solvability conditions may be investigated and a unifying framework for computing solutions wherever they exist. In fact, the solvability may be reduced into an optimisation problem [Mit.1]. Finally, this approach also allows the use of classical intersection tools such as Schubert calculus.

The extensive presentation of this framework will be considered in Chapter 5. One of the aims of this thesis is the extension and enrichment of this framework with tools from the techniques mentioned previously, as well as its full growth and extension to problems such as dynamic and decentralised control.

4.5 Background results on pole zero assignment

POLE ASSIGNMENT BY OUTPUT FEEFBACK

The problem of pole assignment by dynamic compensation has always a solution under the assumptions of controllability and observability and it is given by designing an apropriate observer of the same dynamic order as the plant. Determining reduced order dynamic schemes which allow arbitrary pole placement with order less than n, has been an important research area. Within this area of work we may distinguish the following two types of problems:

(a) What can be achieved with a fixed type controller (constant or PI etc)

(b) Sufficient conditions as well as necessary which allow pole placement with dynamic order less than the full order η.

Regarding the dynamic pole placement, the result was given by Brasch and Pearson [Bra.1] who have shown that sufficient conditions for dynamic pole placement is that the order of dynamics satisfies:-

$$r=\min\{\mu_o-1 , \mu_c-1\}$$

where μ_c and μ_o are the controllability and observability indices, respectively. A different but equally nice result is the one obtained by Kimura [Kim.1] and by Davison and Wang [Dav.1] where it was shown that generic pole assignability holds, provided that

$$r \geq n - m - p + 1$$

Obviously, the former result tends to be sharper if n is large compared to m and p. Kimura has in fact managed to improve this bound [Kim.2] but the results in [Kim.2] required what is known as 'm-decomposability' which is not generic [Will.1]. A necessary condition for generic pole assignability by output feedback was derived by Willems and Hesselink [Wil.1] who showed that

$$r(m+p)+mp \geq n+r$$

In the case where $m+p-1<n$, it has been shown by Munroe and Novin-Hirbod [Mun.2] that there exists a full rank dynamic compensator assigned arbitrary close to a predefined set of closed loop poles if:

$$r \geq \frac{n - (m+p-1)}{\max\{m,p\}}$$

For the case where the controller is of the multivariable proportional plus integral type it has been shown in [Nov.1] that a standard condition for pole assignment by constant feedback [Kim.1], [Dav.1] can be extended to provide a sufficient condition for arbitrary pole placement and this is given as

$$2m+p-1 \geq n+p$$

Regarding the problem of denominator matrix assignment it has been shown [Che.1] that we can assign denominator matrix with column degrees $k+d_i$ via output feedback dynamic compensation of McMillan degree kp if

$$k \geq \mu_0 - 1$$

where μ_0 is the observability index of the open loop plant and i is its controllability (column degrees) indices.

Finally, a number of results regarding denominator assignment, were given by Djaferis [Dja,1.] who have shown that for a generic system of p-inputs, m-outputs and n-states, there exists a dynamic compensator of order q which places a min(n+q,(q+1)m+q+b(p-1)) closed loop poles for any b.

The study of pole assignment by constant output feedback has been given special attention and belongs to the case of pole assignment by fixed structure controllers. Some of the first results for the case of output feedback (or constant precompensation) pole assignment are those of Kimura, [Kim.1], and Davison and Wang [Dav.1] who have shown that k exists, allowing the closed-loop poles to be assigned arbitrarily close to a desired set $\{\gamma_i\}$ if

$$m+p-1 \geq n$$

The above result was proved with combinational geometric techniques [Kim.1] and standard state space techniques [Dav.1]. Using infinitesimal techniques Herman and Martin [Her.1] have shown that a necessary and sufficient condition for generic pole assignment of strictly proper system via complex constant output feedack is that

$$mp \geq n$$

As far as real solutions are concerned, the problem is much more complex and the main difficulty arises from the fact that the field $\mathbb{R}$ is not algebraically closed. Although a special case where mp=n and a number d(m,p) is odd, was proved to be a sufficient condition for generic pole assignment by real output feedback [Bro.1] , the inequality, $mp \geq n$, does not generally imply real solution for the problem, as it has been shown in [Will.1]. The sufficient condition mp=n and d(m,p)=odd has been extended in [Gia.1] to

$$mp \geq n$$

and

$$A(a_1,a_2,\ldots a_p)_C \ \text{ to be odd}$$

where $A(a_1,a_2,...a_p)_C$ is the order of Schubert variety $<a_1,a_2,...a_p>_C$ for $a_1,a_2,...a_p$: $0 \leq a_{12} \leq a_2 \leq ... \leq a_p \leq m$ and $\sum a_i = n$. Using Combinatorial Geometric methods Rosenthal also extented Kimura's sufficient condition to

$$m + \left[\frac{p}{1}\right] + \left[\frac{p}{2}\right] + ... + \left[\frac{p}{k}\right] - 1 \geq n$$

where $k = \left[\frac{m}{p}\right]$ and the bracket denotes the integral part function. Finally using Schubert calculus techniques and topological intersection theory, Byrnes found the strongest sufficient condition for arbitrary pole placement via constant output feedback, namely

$$LScat(p,m) \geq n$$

where LScat(p,m) denotes the Lusternic-Snirelman category of the Grassmannian $G_p(\mathbb{R}^{p+m})$.

ZERO ASSIGNMENT VIA SQUARING DOWN

The study of zero assignment was initiated by the work of Rosenbrock [R.4] [R.5], on the possible zero structure Smith forms that may be assigned to a controllable pair (A,B) by selection of the matrix C of the resulting square system; however, no algorithm for selection of C that assigns the zeros, were given there. The first results on the general squaring down were derived by Kouvaritakis and MacFarlane [Kouv.1] who have suggested methods for assigning parts of the zero structure and some cases the zero structure under squaring down. Nonetheless, they did not give any general solvability conditions for the problem. For the case of designing C, such that the resulting square (A,B,C) triple has a given zero structure, a simple algorithm based on eigenvector assignment techniques were given by Karcanias and Kouvaritakis [Kar.2]. An attempt to generalise Rosenbroch's zero structure assignment for the squaring down case was made in [Var.1], where some necessary conditions were derived. The link of the general squaring down to the pole assignment by output feedback was noted in [Apl.1]. The derivation of necessary and sufficient conditions for the zero assignment under constant squaring dowm was considered by Karcanias and Giannakopoulos [Gia.2],[Kar.3] using the techniques of the projective geometry framework. These are the only existing solvability conditions. It has been shown in [Kar.3] that

$$p(m\text{-}p) \geq \delta+1$$

where δ is the Forney degree of the module colspan[N(s)], N(s) being the numerator of a coprime MFD of the transfer function of the plant, is a necessary and sufficient condition for zero assignment. An additional condition in terms of the rank of the Plucker matrix of N(s) is also a necessary condition for exact solvability. A sufficient condition for generic real solvability was also produced and this is

$$p(m\text{-}p) \geq \delta+1$$

and

$$A(a_1,a_2,...a_p)_C = 1 \bmod 2$$

where $A(a_1,a_2,...a_p)_C$ is the order of Schubert variety $<a_1,a_2,...a_p>_C$ for $a_1,a_2,...a_p$: $0 \leq a_1 \leq a_2 \leq ... \leq a_p \leq m\text{-}p$ and $\sum a_i = \delta+1$.

The derivation of algorithms for squaring down was considered by [Sab.1] using state space methodology. Their approach is based upon the use of a special coordinate framework and aims at assigning the zeros by using dynamic squaring down, rather than addressing the issues related to solvability of the problem with either constant or limited dynamics squaring dowm compensators.

For the case of decentralised control, the zero assignment via decentralised squaring down (applied locally at the subsystem level) has been considered in [Kar.4], [Lai.1] within the projective geometry and exterior algebra framework; their work, however, has been more concerned with the characterisation of fixed and almost fixed zeros, rather than the derivation of solvability conditions.

As far as pole placement via decentralised controllers the condition

$$\Sigma m_i p_i \geq n$$

was proved to necessary and sufficient for complex solutions. A result related to this work was given in [Wang.3] for the case $\Sigma m_i p_i = n$ based on the compactification of the decentralised controllers as a product of Grassmannians. In this case the degree of the product of Grassmannians was calculated and a sufficient condition for arbitrary pole placement was derived using the above degree. In the same paper [Wang.3] it was also shown that $\Sigma m_i p_i \geq n$ implies generic pole assignability, when either the number of inputs or the number of outputs are equal for all channels.

4.6 Conclusions

The emphasis in this chapter has been on the reviewing of the different approaches of the algebrogeometric framework and briefly summarise the existing results related to the problems of pole, zero assignment examined in this thesis. Results derived within the same framework, but dealing with problems not examined here, such as issues related to simultaneous design [Gho.1], have not been examined but may be found in the references. Certain issues related to system parametrisation which are used in the subsequent chapters, have also been developed.

CHAPTER 5. The Determinantal Assignment Problem: A Unifying Approach for Static and Dynamic Compensation

5.1 Introduction.

In this chapter, we will be presenting the mathematical formulation of the problems which will be considered in this thesis. These problems belong to a general group of problems named as 'frequency assignment problems'. These are concerned with the following issues: a) moving the poles of a system using state or output feedback which can be either: (a) constant or dynamic, centralised or decentralised and (b) moving the zeroes of a system using squaring down. In section (5.2) we present the pole placement problems via constant, PI, and Observability index Bounded Dynamics (OBD) controllers (Definition 5.1). In section (5.3) the zero assignment problem via squaring down is discussed, and finally, section (5.4) deals with the decentralised versions of constant output feedaback pole placement and zero assignment via constant squaring down. We will show that all the above problems can be reduced to solving a determinantal equation with respect to a constant matrix. This problem is called the Constant Determinantal Assignment Problem (CDAP) [Kar.1, Kar.2] and it is naturally connected with all the frequency assignment problems via constant or dynamic controllers. In the last section we formulate the DAP (constant and dynamic) and a general approach to this problem will be given, which lies within the framework of the projective methods described in the previous chapter. The present approach is a natural extension of the one introduced by Karcanias and Giannacopoulos in [Kar.1], [Kar.4] and treats both static and dynamic problems in a unifying manner for both centralised and decentralised problems. Throughout the thesis our plant will be a linear time-invariant (strictly) proper system of p-inputs, m-outputs and n-states, described by the following state-space description (SSD)

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x} + B\,\underline{u}(t) \\ \underline{y}(t) &= C\,\underline{x} \end{aligned} \tag{5.1}$$

where $\underline{x} \in \mathbb{R}^n$, $\underline{y} \in \mathbb{R}^m$, $\underline{u} \in \mathbb{R}^p$, are the state, the input, and the output vectors respectively and $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxp}$, $C \in \mathbb{R}^{mxn}$ are real constant matrices, or the equivalent transfer-function matrix description:

$$\underline{y}(s) = G(s)\,\underline{u}(s) \tag{5.2}$$

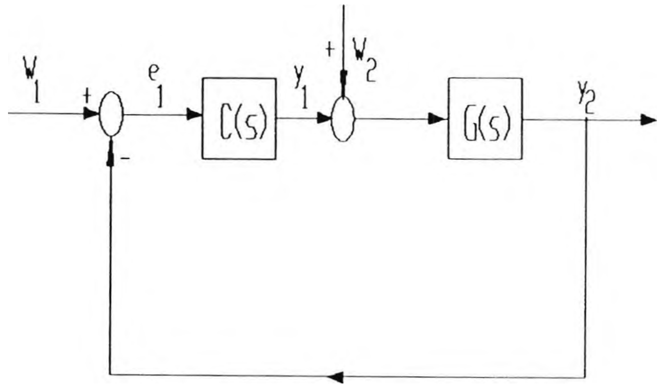
where s is the variable of the Laplace transform and $\underline{y}(s)$ and $\underline{u}(s)$ are the Laplace transforms of the output $\underline{y}(t)$ and input $\underline{u}(t)$ of the control system.

5.2. Pole assignment by precompensation-feedback.

The problem of pole assignment has been extensively studied by many researchers since the 1970s. It is concerned with moving poles of a given time invariant multivariable linear system to a specified set of locations in the s-plane by means of state, or output feedback. The state feedback approach is well established and it has been proven that, provided a system is controllable, all its poles can be arbitrarily assigned by state feedback [Won.,1967]. Since the complete state observation does not hold in most practical situations, it has been desirable to find the condition under which the system is pole assignable with incomplete state observation. In the following section, we will be examining the pole placement problem by using output feedback, or precompensation according to the dimensionality of the plant (relationship between m and p).

5.2.1 The general feedback configuration.(for more details see Ch.2)

Consider the general feedback configuration of plant and controller which are assumed to be both controllable and observable (ie. transfer function describe the systems completely).



If $G(s) \in \mathbb{R}_{pr}(s)^{mxp}$, $C(s) \in \mathbb{R}(s)^{pxm}$, and assume coprime MFD's as

$$G(s)=D_l(s)^{-1}N_l(s)=N_r(s)D_r(s)^{-1} \tag{5.3}$$

$$C(s)=A_l(s)^{-1}B_l(s)=B_r(s)A_r(s)^{-1} \tag{5.4}$$

It is well known [Kuc.1], [Cal.&Des.1] that the finite characteristic polynomial of the closed-loop system is expressed by:

$$f(s)=\det\{F_1(s)\}=\det\{F_2(s)\} \tag{5.5}$$

where,

$$F_1(s)=D_l(s)A_r(s)+N_l(s)B_r(s) \tag{5.6}$$

$$F_2(s)=A_l(s)D_r(s)+B_l(s)N_r(s) \tag{5.7}$$

Thus we can write:

$$f(s) = \det\left\{[D_l(s),N_l(s)]\begin{bmatrix}A_r(s)\\B_r(s)\end{bmatrix}\right\} \tag{5.8}$$

or

$$f(s)= \det\left\{[A_l(s),B_l(s)]\begin{bmatrix}D_r(s)\\N_r(s)\end{bmatrix}\right\} \tag{5.9}$$

i) if $p \leq m$, then $C(s)$ may be interpreted as FEEDBACK COMPENSATOR and we will use the expression of the closed loop polynomial described by (5.9).

ii) if $p \geq m$, then $C(s)$ may be interpreted as PRECOMPENSATOR and we will use the expression of the closed loop polynomial described by (5.8).

As far as case i) is concerned , the problem of pole assignment by dynamic output feedback (DPAP) for a system of p-inputs m-outputs and n-states ($p \leq m$) is defined as follows: **Given a polynomial, p(s) ,solve the equation (5.9) with respect to** $[A_l(s),B_l(s)]$. A similar problem can be defined for case ii) ;this is actually dual to DPAP by taking transposes in (5.8) and interchanging the roles of p and m.Thus it is sufficient to examine only case i) where $p \leq m$.

First, we will consider DPAP for each of the following families of feedback compensators C(s):

i) constant controllers   ii) PI controllers   iii) OBD controllers where the OBD controllers are defined as follows:

Definition(5.1) The Observability index Bounded Dynamics (OBD) controllers are those defined by the property that their McMillan degree is equal to pk, where p is the number of outputs and k is the observability index of the controller.

□

We will also examine the case of decentralised versions of the above problems in a separate section (Sec. 5.4). In the next subsections, we formulate each of the three above cases separately, and as will be seen, all of these problems are reduced to a constant determinantal assignment problem.

5.2.2 Pole assignment via constant controllers

The simplest case of a feedback compensator C(s) is a gain compensator $K \in \mathbb{R}^{p \times m}$. Then, the closed loop system is a well defined strictly proper system with the same McMillan degree as the open loop system. The closed loop polynomial of eq.(5.5), now becomes:

$$f(s)=\det\{D_R(s)+KN_R(s)\} \tag{5.10}$$

or equivalently eq.(5.9) becomes

$$f(s)=\det\left\{[I_p,K]\begin{bmatrix}D_R(s)\\N_R(s)\end{bmatrix}\right\} \tag{5.11}$$

So the problem of pole assignment by constant output feedback is reduced to the following problem:

For a given strictly proper system S of p-inputs, m-outputs and n-states and a given monic polynomial f(s) of degree n, solve the determinantal equation (5.11) with respect to $K \in \mathbf{R}^{p \times m}$.

Remark (5.1) The same problem can be defined for proper systems ($D \neq 0$) with the only difference being that we will have some properness conditions for the resulting closed loop system and f(s) may not be monic.

□

5.2.3 Pole assignment via PI controllers

These controllers contain a constant gain and an integrator and thus they are of the form:

$$C(s)=K_0+\frac{1}{s}K_1$$

where $K_0,K_1 \in \mathbb{R}^{pxm}$. A left MFD for C(s) is defined by

$$C(s)=[sI_p]^{-1}[sK_0+K_1] \tag{5.12}$$

Remark(5.2) The MFD given by eq.(5.12) is coprime at any $s \in \{\infty\} \cup \mathbb{C}-\{0\}$; thus we have to test coprimeness only at s=0. The MFD (5.12) is coprime, iff rank(K_1)=p.

□

If we now apply output feedback PI control to a (minimal) strictly proper plant of n-states, then the resulting closed loop system will be a well defined strictly proper system of n+p states. Substituting (5.12) into (5.9) we have that the pole polynomial f(s) of the closed loop system is given by:

$$f(s)= \det\left\{ [sI_p\ ,\ sK_0+K_1]\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} \right\} \tag{5.13a}$$

which is equivalent to

$$f(s)= \det\left\{ [I_p\ ,\ K_0\ ,K_1]\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix} \right\} \tag{5.13b}$$

Thus the pole assignment problem by PI controllers is reduced to solving the following problem:

For a given strictly proper system S of p-inputs, m-outputs and n-states and a given monic polynomial f(s) of degree n+p, solve the determinantal equation (5.13b) with respect to $[K_0\ ,K_1] \in \mathbf{R}^{px2m}$.

Remark (5.3) If the plant S is proper then the resulting closed loop system may not be proper and poles may appear at $s=\infty$. However under the properness assumption of the closed loop system, and the above statement of the problem is still valid, with the only difference being that f(s) will not necessarily be monic.

□

5.2.4 Pole assignment via OBD controllers

An extension of the families of constant , PI controllers are those for which the McMillan degree is fixed. A special class of such family of controllers is the Observability index Bounded Dynamics (OBD) family which is, defined by def(5.1) and they have the form:

$$C(s)=A_1(s)^{-1}B_1(s)$$

where

$$[\ A_1(s),B_1(s)]=\ T_k s^k+...+T_0 \tag{5.14}$$

$T_k, T_{k-1},..., T_0 \in \mathbb{R}^{px(p+m)}$ and

$$T_k=[I_p,X] \tag{5.15}$$

Remark(5.4) The above MFD is coprime iff rank$[A_1(s),B_1(s)]=p\ \forall s \in \mathbb{C}$ or else iff

$$C_p\Big([\ A_1(s),B_1(s)\ ]\Big)\neq 0\ \forall s\ \ \in \mathbb{C} \tag{5.16}$$

In this case $[A_1(s),B_1(s)]$ corresponds to a minimal system and k is the observability index of the system. In addition, due to (5.15) the McMillan degree of these controllers is equal to pk. If the noncoprimeness assumption does hold, then the Mc Millan degree of the controller will be less than pk.

□

If we now use an output feedback OBD controller to a (minimal) strictly proper plant of n-states, then the resulting closed loop system will be a well defined strictly proper system of n+pk states. Substituting (5.14) to (5.9) we have that the closed loop pole polynomial is given by:

$$f(s)= \det \left\{ (T_k s^k + \ldots + T_0) \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \right\} \Leftrightarrow$$

$$\Leftrightarrow f(s)= \det \left\{ (T_k s^k + \ldots + T_0)\ M(s) \right\} \tag{5.17a}$$

or equivalently

$$f(s)= \det \left\{ [\, T_k,\ T_{k-1}, \ldots,\ T_0] \begin{bmatrix} s^k M(s) \\ s^{k-1} M(s) \\ \vdots \\ M(s) \end{bmatrix} \right\} \tag{5.17b}$$

Thus the problem of pole placement via OBD controllers is equivalent to:

For a given strictly proper system S of p-inputs, m-outputs and n-states and a given monic polynomial f(s) of degree n+pk, solve the determinantal equation (5.17b) with respect to $[T_k, T_{k-1}, \ldots, T_0]$.

Remark (5.5) Similarly with the two previous cases (constant, PI), if the plant is a proper system we can still state the previous problem with the only difference being that we have to assume properness of the closed loop system and that f(s) may not be monic.

□

5.2.5 Pole assignment via general dynamic controllers of fixed McMillan degree.

The general problem of assigning the poles of a system via dynamic output feedback controllers of fixed McMillan degree n_1, can be described as:

For a given strictly proper system S of p-inputs, m-outputs and n-states and a given monic polynomial f(s) of degree $n+n_1$, solve the determinantal equation (5.9) with respect to $[A_1(s),B_1(s)]$.

where $[A_1(s),B_1(s)]$ is the composite matrix of a coprime left MFD of the feedback controller. Readily, this problem is a generalisation of the case of the OBD controllers of previous subsection. Indeed, if n is divisible by p, the number of inputs of our plant, then the family of dynamic controllers becomes of the OBD type and as we have seen, the pole placement problem can be reduced into a *constant* determinantal assignment problem. The more general, case when n is not divisible by p, cannot in general be reduced to a constant DAP since eq(5.14) does not hold true. However if we consider the partial problem of finding dynamic output feedback controllers with fixed observability indices, we can reduce it into a constant DAP; the structure of this general DAP with dynamic general controllers is out of the scope of this thesis.

5.3 Zero assignment by squaring down

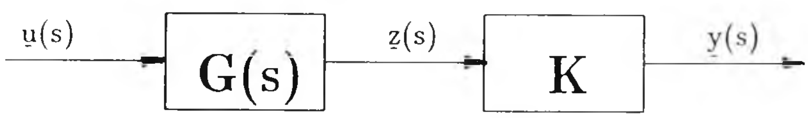
5.3.1 Output squaring down

Frequency response techniques for the design of multivariable systems such as the characteristic locus method [McF.2, Kouv.2], inverse Nyquist array[R.2,R.3,R.1], and generalized Nyquist Root Locus Method [Kouv.3,Maci.1], consider feedback loops between a set of selected measurements and an equal number of control inputs. Thus, common to all these techniques is the assumption that the plant transfer function G(s) is square. For a nonsquare plant whose number of measured output variables is greater than the number of control inputs ($m > p$), the problem of combining all outputs together into a new set of outputs, whose number is equal to the number of control inputs has been called the "squaring down" problem [R.4, Kouv.1, Kar.3, Apl.1,Var.1]. It is evident that the solution of the general squaring down problem has significant consequences on the zero structure of the corresponding loop transmission transfer function matrix and therefore, it vitally affects the final control design process.

We consider a system S whose input-output behaviour is described by the transfer function $G(s) \in \mathbb{R}_{pr}(s)^{mxp}$ where m>p. G(s) may be represented by a right coprime matrix fraction description as:

$$G(s)=N_r(s)D_r(s)^{-1} \tag{5.18}$$

Under the coprimeness assumption, the zeros of the system S are given by the zeros of the numerator $N_r(s)$. If $\underline{z}(s)$ is the Laplace transform of system outputs, then the squaring down problem is defined as the problem of selecting K such that $\underline{y}(s)=K\,\underline{z}(s)$ where $\underline{y}$ is a p-dimensional vector. Then $\underline{y}$ is refered to as the effective output vector.



Squaring down at the plant outputs makes sense as a postcompensation with dynamics representing those of the sensors used [Kar.&Gia.], or constant. In this paper we consider the case where K is free and constant. If we use a squaring down postcompensator $K \in \mathbb{R}^{pxm}$, the squared down pxp transfer function of the composite system $\tilde{S}$ may now be expressed as:

$$\tilde{G}(s)=KG(s)=\left(KN_r(s)\right)D_r(s)^{-1} \tag{5.19}$$

Then the zeros of $\tilde{S}$ are given by the zeros of $KN_r(s)$ and the invariant zero polynomial of $\tilde{S}$ is given by:

$$z_k(s)=\det\left(KN_R(s)\right) \tag{5.20}$$

If now Z(s) is a greatest right divisor of $N_R(s)$ then (5.20) can be rewritten as:

$$z_k(s)=\det\left(K\bar{N}(s)\right)\det(Z(s)) \tag{5.21}$$

where $\bar{N}(s)$ is the least degree polynomial matrix of the rational vector space colsp{G(s)}. It is clear that det(Z(s)) is a fixed divisor of $z_k(s)$ for all K (invariance of existing zeros under squaring down [Kar.3], [Kar.2]). The newly introduced zeros, are the zeros of the polynomial

$$f(s)=\det\left(K\bar{N}(s)\right) \tag{5.22}$$

where f(s) is a polynomial with degree equal to the Forney's dynamical order δ [Kar.1] of the previous rational vector space. The problem of zero assignment by squaring down can be defined as follows:

For a given proper system S of p-inputs, m-outputs and n-states and a given polynomial f(s) of degree δ, solve the determinantal equation (5.22) with respect to the full rank matrix K

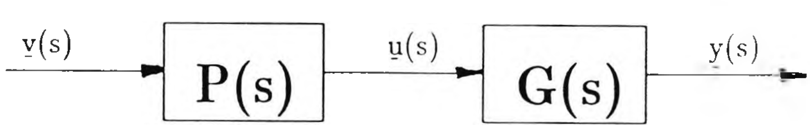
From the previous analysis, we also have:

Remark(5.3). The zeros of G(s) are invariant under constant squaring down and the maximal number of *new* zeros that may be created is equal to the Forney dynamical order δ.

□

5.3.2 Input squaring down

For systems with p>m, the problem of input squaring down may also be defined as follows: Given $G(s)\in\mathbb{R}^{mxp}(s)$ with p>m and $\underline{u}$ inuts, $\underline{y}$ outputs, define a precompensator $P(s)\in\mathbb{R}^{mxp}(s)$ and a set of new inputs $\underline{v}$, such that $\underline{u}(s)=P(s)\,\underline{v}(s)$, as decribed below



such that the squared system $\tilde{S}$ has a transfer fuction matrix $\tilde{G}(s)$ with desirable properties. Note that

$$\tilde{G}(s)= G(s)\ P(s)$$

and in this case P(s) can be any type of dynamic compensator, since any precompensator may be used. Designing P(s), such that $\tilde{G}(s)$ has certain properties, is equivalent to a Model Matching Problem [Scot.1,Kuc.1,Cheng1,Var.2]. Certain aspects of this problem related to zero structure assignment may be discussed within the present framework of DAP and these are summarised below: Let G(s) and P(s) be represented by the following coprime MFD's.

$$G(s)= D_l(s)^{-1}N_l(s)$$

$$P(s)=B_r(s)\ A_r(s)^{-1}$$

Then

$$\tilde{G}(s)= G(s)\ P(s) = D_l(s)^{-1}N_l(s)\ B_r(s)\ A_r(s)^{-1}$$

The zero structure of the resulting system is defined by the matrix

$$Z(s)=N_l(s)\ B_r(s)$$

whereas the pole structure is defined by the $D_l(s)$, $A_r(s)$ denominators. Clearly, the above MFD for $\tilde{G}(s)$, is not necessarily coprime. The designed compensator introduces poles, as those defined by the zeros of $A_r(s)$, as well as contributing in the formation of the zero structure. A zero assignment problem may thus be formed as shown below: Define a pair of polynomial matrices ($B_r(s)$, $A_r(s)$) which are of the appropriate dimension (pxm, mxm) and coprime such that

$$z(s)= \det\ (Z(s)) = \det(N_l(s)B_r(s))$$

is a given polynomial and $B_r(s)A_r(s)^{-1}$ is a proper controller. This may be referred to as *Precompensation Zero Assignment Problem* (PZAP) and various versions of it may be considered according to the nature of $B_r(s)$ matrix. Thus we may have a selection of $B_r(s)$ as a general polynomial matrix ie.

$$B_r(s)= Q_0 + sQ_1 +... + s^{\mu}Q_{\mu}$$

and such a problem is clearly equivalent to the following constant DAP

$$z(s)=\det\left([Q_{\mu}, Q_{\mu-1},\ldots,Q_1]\begin{bmatrix} s^{\mu}P(s) \\ s^{\mu-1}P(s) \\ \vdots \\ P(s) \end{bmatrix}\right)$$

Note that if $B_r(s)$ is selected as above, then $A_r(s)$ has to be selected appropriately for the properness condition to be satisfied .

Remark (5.6) The above version of PZAP aims at producing a system with given zero structure and thus it is more general than a model matching problem, where $\bar{G}(s)$ is given and the existence of an appropriate P(s) is sought. The important question here is 'What is the minimum order of numerator dynamics which is needed to assign arbitrarily the zero structure ?'.

□

Remark (5.7): If the plant transfer function G(s) has zeros, then using similar arguments as in the case of output squaring down, these zeros are fixed and cannot be assigned. In this case the essential problem is the assignment of the newly introduced zeros.

□

5.4 Decentralized pole and zero assignment

5.4.1 Introduction

Decentralised control systems are defined to be large dynamic systems with several automatic controllers each operating on the system with partial information on the states of the controlled system. This decentralisation assumption implies that only local measurements may be used for control actions. Thus the output feedback and squaring down compensators have a block diagonal structure and can be constant or dynamic. For the purposes of this thesis we will consider only *constant* decentralised controllers; however, the cases of PI and OBD controllers of decentralized nature may be treated along similar lines if we assume that the number of channels is κ. In this case, these controllers can be written as:

$$K_{dec} = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & K_\kappa \end{bmatrix} \tag{5.23}$$

where $K_i \in \mathbb{R}^{p_i x m_i}$, $\sum p_i = p$ and $\sum m_i = m$. It is evident that the previously defined problems of pole assignment via constant output feedback and zero assignment via constant squaring down can be restated using decentralised compensation.

5.4.2 Pole placement via constant decentralised controllers.

Consider an κ-channel linear time-invariant strictly proper system of p-inputs, m-outputs and n-states:

$$\begin{aligned} \dot{x} &= Ax + \sum_{i=1}^{\kappa} B_i u_i \\ y_i &= C_i x \end{aligned} \tag{5.24}$$

here x, u_i, y_i, are n, m_i, p_i vectors, respectively, and u_i and y_i are the input and output of the ith channel and let a right coprime MFD of the transfer function G(s) of the above system be:

$$G(s)=N(s)D(s)^{-1} \tag{5.25}$$

If local output feedback laws

$$\underline{u}_i = K_i \underline{y}_i + \underline{u}_i \tag{5.26}$$

for i= 1, 2,...., κ, are applied to each channel, the closed-loop system becomes

$$\begin{aligned} \dot{\underline{x}} &= (A + \sum_{i=1}^{\kappa} B_i K_i C_i) \underline{x} + \sum_{i=1}^{\kappa} B_i \underline{u}_i \\ \underline{y}_i &= C_i \underline{x} \end{aligned} \tag{5.27}$$

and the pole polynomial, f(s), of the closed loop system is:

$$f(s) = \det (sI - A - \sum_{i=1}^{r} B_i K_i C_i) \tag{5.28}$$

or equivalently

$$f(s)=\det\left([I_p , K_{dec}] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) \tag{5.29}$$

where K_{dec} is given by eq.(5.23). The problem, now, of pole assignment by decentralised constant output feedback can be defined as follows:

For a given strictly proper system S of p-inputs, m-outputs and n-states and a given monic polynomial f(s) of degree n, solve the determinantal equation (5.29) with respect to the block-diagonal K_{dec}.

Finally, if we assume that our decentralised controllers have dynamics, then we can define the dynamic decentralised pole placement problem in a similar way. Furthermore, if the controllers are PI or OBD we can transform the problem to a constant one as in equations (5.13) and (5.17).

5.4.3 Zero assignment via decentralised squaring down compensators

Let us consider a proper plant S of p-inputs, m-outputs and n-states such that p<m whose transfer function G(s) is represented by the coprime MFD (5.18). The squaring down of this system amounts to a postcompensation of S by a pxm constant controller as it was explained in Sec.(5.3). If we assume decentralisation in κ channels,

then the zero polynomial of the squared down system is given by:

$$z_{\kappa}(s)=\det\left(K_{dec}N_R(s)\right) \tag{5.30}$$

where K_{dec} is given by eq.(5.23). Dividing now both sides of eq(5.30), by det(Z(s)), the zero polynomial of S, we get

$$f(s)=\det\left(K_{dec}\tilde{N}(s)\right) \tag{5.31}$$

where $\tilde{N}(s)$ is a least degree polynomial numerator matrix of the rational vector space colsp{G(s)} and f(s) is a polynomial with degree equal to the Forney's dynamical order δ of the previous rational vector space [For.1].

The problem of zero assignment by squaring down can be defined as follows:

For a given proper system S of p-inputs, m-outputs and n-states and a given polynomial f(s) of degree δ, solve the determinantal equation (5.31) with respect to the full rank matrix K_{dec}

5.5. The general determinantal assignment problem

5.5.1 Description of the problem

All the problems introduced in the previous sections belong to the same problem family ie. the determinantal assignment problem (DAP). This problem is to solve the following equation with respect to polynomial matrix H(s):

$$\det(H(s).N(s))=f(s) \tag{5.32a}$$

where f(s) is a polynomial of an appropriate degree d. The difficulty for the solution of DAP is mainly due to the multilinear nature of the problem as this is described by its determinantal character. An additional complexity lies in the fact that we want the solution of (5.32a) to be a polynomial matrix. However, in all cases mentioned previously, all dynamics can be shifted from H(s) to N(s), which, in turn, transforms the problem to a constant DAP. This problem may be described as follows:

Let $M(s) \in \mathbf{R}^{(p+r)xp}[s]$ such that rank(M(s))=p and let $\mathcal{H}$ be a family of full rank px(p+r) constant matrices having a certain structure. Solve with respect to $H \in \mathcal{H}$ the equation:

$$\det(H.M(s))=f(s) \tag{5.32b}$$

where f(s) is a real polynomial of an appropriate degree d.

Remark(5.8) The degree of the polynomial f(s) depends firstly upon the degree of M(s) and secondly, upon the structure of H. However in most of our problems the degree of p(s) is equal to the degree of M(s).

□

The constant DAP being similar to the output feedback pole placement problem allows us to apply the same techniques. However the structure of the matrices H and M(s), in the different special versions, may be different from those of the output feedback problem and for this reason we have to apply carefully these techniques to each of our problems and should take into account their individual peculiarities.

The determinantal assignment problem has two main aspects. The first has to do with the solvability conditions for the problem and the second, whenever this problem is solvable, to provide methods for constructing these solutions. In this thesis we will mainly be involved with the first - that is, giving solvability conditions for the DAP. We classify the solvability conditions for the constant DAP into two classes: i) exact solvability conditions and ii) generic solvability conditions.

By the term 'exact solvability conditions' we mean that the matrix M(s) is fixed and we are looking for necessary as well as sufficient conditions -depending on both continuous and discrete parameters of M(s)- such that the constant DAP is solvable for all or almost all polynomials f(s). Since in all the cases we examine, the family of M(s) is parametrised by the set of systems, we will be able to construct new system invariants that arise naturally from our problems. On the other hand for the "generic solvability conditions" the polynomial matrix M(s) is not fixed but generic and we are looking for necessary as well as sufficient conditions -depending only on the generic values of the discrete parameters of M(s)- such that the constant DAP is solvable for all or almost all polynomials f(s).

5.5.2 Brief description of our approach for the DA problems of this thesis.

The essence of our approach is *projective*, that is, we use a natural embedding for determinantal problems to embed the space of the unknowns ,$\mathcal{H}$, of DAP, into an appropriate projective space [Kar.1]. In this way we can see our problem as search for common solutions of some set of linear equations and another set of second order polynomial equations. This also allows us to compactify $\mathcal{H}$ into $\bar{\mathcal{H}}$ and then use *algebraic geometric*, or *topological intersection theory* methods to determine existence of solutions for the above sets of equations. In addition we can utilize this embedding for an alternative application of the *infinitesimal methods* [Her.1]. Particularly, we note that

$$\underline{\mathcal{H}}=\{ \text{ linespan } [C_p(H)] : H \in \mathcal{H}\}$$

can be viewed as a subset of the projective variety $G_p(\mathbb{R}^{p+r})$ in $P\mathbb{R}^{\sigma-1}$, where $\sigma=\binom{p+r}{p}$. By the Binnet-Cauchy theorem [Mar.1] (5.32b) can be written as:

$$C_p(H).C_p(M(s))=p(s) \tag{5.33}$$

where $C_p(H) \in \mathbb{R}^{1x\sigma}$ and $C_p(M(s)) \in \mathbb{R}^{\sigma x1}[s]$ and DAP can be reduced to two problems [Kar.1] :

(i) LINEAR PROBLEM: For a given $p(s) \in \mathbb{R}[s]$ of appropriate degree d and $\underline{h}(s) \in \mathbb{R}^{\sigma x1}[s]$ solve the linear equation,

$$p(s)= \underline{k}.\underline{h}(s) \tag{5.34}$$

with respect to $\underline{k} \in \mathbb{R}^{\sigma-1}_{\omega} \subseteq P(\mathbb{R})^{\sigma-1}$

(ii)MULTILINEAR PROBLEM: Assume that the linear problem has a non void solution set ,say $\mathcal{L}$, then determine whether there exists $\underline{k} \in \mathcal{L}$ which belongs to $\underline{\mathcal{H}}$ that is

k=linespan$[C_p(H)]$ for some $H \in \mathcal{H}$.

□

The importance of this approach is that we look at the problem in the natural embedding space of the Grassmannian, the projective space $P\mathbb{R}^{\sigma-1}$. The corresponding equations are easy to be produced and have comparatively small degree (1 or 2) and thus this method is convenient for calculations, the only disadvantage being that the number of equations and unknowns are rather large. In addition to this, the factoring of the initial problem to a linear and a multilinear problem gives us a better insight into the problem. The multilinearity of the problem does not depend upon the matrix M(s) but only on $\mathcal{H}$. The equations giving $\mathcal{H}$ in $P\mathbb{R}^{\sigma-1}$ are the Quadratic Plucker Relations (QPR) expressing decomposability and some linear equations imposed by the structure of $H \in \mathcal{H}$.

In order to use topological or algebrogeometric intersection theory to examine solvability conditions for the constant DAP, we may compactify $\mathcal{H}$ into $\overline{\mathcal{H}}$ in $G_p(\mathbb{R}^{p+r}) \subseteq P\mathbb{R}^{\sigma-1}$ by taking the topological closure with respect to the Zarisky topology (that is, $\overline{\mathcal{H}}$ is the smallest variety containing $\mathcal{H}$) [Mum.1]. In this setting we can view the set $\mathcal{K}_{\mathbb{R}}(p,M)$, of all $H \in \overline{\mathcal{H}}$ that solve eq(5.32b) as

$$\mathcal{K}_{\mathbb{R}}(p,M)=L\mathbb{R}(p,M) \cap \overline{\mathcal{H}}$$

where $L\mathbb{R}(p,M)$ is the linear subset of $P\mathbb{R}^{\sigma-1}$ whose elements satisfy the linear equation (5.34). To study solvability of DAP in $\mathbb{C}$, we consider the set of complex solutions $\mathcal{K}_{\mathbb{C}}(p,M)$; this set is an intersection of two other complex projective varieties- the linear one and $\overline{\mathcal{H}}$ - and we can, therefore, use lemma(3.1) for complex projective varieties. The examination of the existence of real solutions is a more complicated matter (since there is no theorem like lemma(3.1) for real varieties); this problem can be tackled in the following two ways:

a) First, consider the set of complex solutions $\mathcal{K}_{\mathbb{C}}$ which is a subvariety of the Grassmannian $G_p(\mathbb{C}^{p+r})$. If we consider another subvariety, V, of $G_p(\mathbb{C}^{p+r})$ of complementary dimension to $\mathcal{K}_{\mathbb{C}}$ and meeting $\mathcal{K}_{\mathbb{C}}$ transversely, then the intersection $\mathcal{K}_{\mathbb{C}} \cap V$ is finite. In this case, if # $(\mathcal{K}_{\mathbb{C}} \cap V)$ is odd, we have a real solution, since the points of intersection $\mathcal{K}_{\mathbb{C}} \cap V$ occur in conjugate pairs. Thus, such a method amounts to enumerate certain finite intersections of complex varieties on a Grassmannian and therefore lies in the bounds of complex intersection theory and enumerative calculus on a Grassmannian (Schubert Calculus).

b)Compute the cohomology class of $\mathcal{K}_{\mathbb{R}}(p,M)$ for a generic p and M in $H^*(G_p(\mathbb{R}^{p+r});\mathbb{Z}_2)$ and then examine under which conditions this class is nonzero. These conditions will be sufficient for the existence of real solutions. It is worth noting that in the cases we examine, methods (a) and (b) are equivalent.

Remark(5.9) We may also use topological category arguments for appropriate spaces and maps as it was explained in ch.4. It is worth noting that the Lusternic-Snirelman category can be used only if we have intersection of Schubert hypersurfaces on a Grassmannian, since the complementary of a Schubert hypersurface is homeomorphic to $\mathbb{R}^{r\times p}$ which is contractible. Alternatively we may use vector bundle categories which do not require contractibility.

□

We can also use infinitesimal arguments to study the characteristic map of the DAP problem which is defined to be:

$$\chi: \mathbb{F}^{\mu} \rightarrow \mathbb{F}^{d}$$

where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, μ are the degrees of freedom of H in eq(5.32b) and χ maps H to the coefficient vector of the polynomial p(s) of eq.(5.32b). The decomposition of the problem into a linear and a multilinear one leads to the following factorisation of χ:

$$\chi: \mathbb{F}^{\mu} \overset{\mathcal{P}}{\rightarrow} \mathbb{F}^{\sigma} \overset{L}{\rightarrow} \mathbb{F}^{d}$$

where the left map is the Plucker embedding and the right is a σxd linear map induced in the linear problem and depends only on M(s). In this way the calculation of differential of χ amounts only to the calculation of the differential of of the Plucker embedding (which is the same for all problems) since L is linear. The rank of the differential of χ, which is important for our analysis, has as upper bound the rank of the Plucker matrix and as we shall see, there exist cases where these two numbers are equal. Finally this decomposition of χ may allow us to produce results which are global (independent from the specific point of $\mathbb{F}^{\mu}$ we differentiate) and produce new system invariants relative to the DAP solvability property.

5.6 Conclusions

The present chapter has introduced the framework of the determinantal assignment problem and has described the particular control theory problems which are examined in this thesis. The aim has been to review the existing background results and introduce some unifying formalism and notation. In a sense, this chapter is a prelude of the developments that follow the subsequent chapters.

CHAPTER 6. The Pole Placement Map, Its Properties and Relationship to System Invariants

6.1. Introduction

The aim of this chapter is to establish a number of properties of a very important map related to determinantal problems, the pole placement map. This map is a special case of the χ function as defined in section 4.4 and is the one that maps the set of the unknowns $\mathcal{H}$ of DAP to the set coefficients of the polynomial p(s) under the rule described by equ(5.10). In this chapter we will examine the pole placement map under complex and real output feedback and especially the properties of the image of the pole placement map as well as its asymptotic properties with respect to a high gain output feedback.

One of the important questions connected with the pole placement problem under constant (or dynamic) output feedback, is the derivation of a reasonable measure for the size of the set of polynomials ,which for a system S(A,B,C) of p-inputs, m-outputs and n-states can be assigned. We choose as a measure of the size of this set ,the dimension of the image of the real, or the complex pole placement map (PPM). The geometry of this image, as well as, its dimension (which is a new system invariant) will be the main topic of section 2 of this chapter. Infinitesimal analysis will be used for the calculation of the dimension as well as for the derivation of its relationship with the old system invariants (section 6.3).

The pole placement map is the anologue of the root locus map for the multivariable case and thus to obtain a complete picture of such a map, its extension at infinity (as in the SISO case) is required. An integral part of the asymptotic analysis of the pole placement map is the study of the high gain (multivariable) feedback which is examined in section 6.4. Finally, having rigorously defined the concept of high gain (infinite) feedback controllers, we end this chapter (section 6.5) with the study of systems where the pole placement map does not extend at infinity, the so called *family of degenerate systems.*

6.2 General properties of the pole placement map.

6.2.1 Introduction

Let S(A,B,C) be the state space description of a linear strictly proper system of p inputs, m outputs, n states. Let also $G(s)=N(s)D(s)^{-1}$ be a coprime matrix fraction description of the transfer function of the system. The pole placement problem via constant output feedback (section 5.2) is to examine the solvability of equation (6.1) for a given $(p_n,...,p_1) \in \mathbb{R}^n$, ie.

$$\det\left([I,K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) = \det([I,K]\ M(s)) = s^n+p_n s^{n-1}+...+p_1 \tag{6.1}$$

where M(s) is a column reduced and least degree composite matrix for S, or equivalently, the equation

$$\det(Is-A-BKC)= s^n+p_n s^{n-1}+...+p_1 \tag{6.2}$$

with respect to $K \in \mathbb{R}^{p\times m}$. Of particular interest is to examine the size of this set of n-tuples. This is the same as finding how large the image of the function χ is. The function, χ, from $\mathbb{R}^{pm}$ to $\mathbb{R}^n$, maps every K to $(p_n,...,p_1)$ under the relation (6.1), or the equivalent relation (6.2), and is called the pole placement map (PPM) [Her1,Gho1]. Its extension $\hat{\chi}$, from $\mathbb{C}^{pm}$ to $\mathbb{C}^n$, is called the complex pole placement map (CPPM). It is apparent that the image of the real or complex PPM plays an important role in the characterisation of the pole assignability properties of system via output feedback. The determination of the properties of this image is not an easy task, since this map is nonlinear as the following example shows.

Example(6.1) Consider the strictly proper system S whose transfer function G(s) is expressed as a right coprime MFD as:

$$G(s)=\begin{bmatrix} 0 & s^{-1} \\ -s^{-3} & s^{-2} \end{bmatrix}=\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ s & s^2 \end{bmatrix}^{-1}$$

If we apply to G(s) constant output feedback

$$K=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

then the closed loop pole polynomial is given by:

$$p(s)= s^4+k_{21}s^3+k_{22}s^2-k_{12}s+k_{22}k_{11}- k_{12}k_{21}$$

and so, the pole placement map defined previously is given by:

$$[k_{11},k_{12},k_{21},k_{22}] \to [k_{21}, k_{22},-k_{12}, k_{22}k_{11}- k_{12}k_{21}]$$

□

Next, we will examine the structure of the pole placement map as well as the structure and the size of its image.

6.2.2.The real and complex pole placement maps and their image

The complex (as well as the real) PPM is a regular map, that is its coordinate functions are polynomials (of pm variables). However this map has some special properties imposed by the determinantal nature of the equation (6.1). Following the ideas described in section 5.5 we can decompose $\hat{\chi}$ into two maps:

i) the Plucker embedding, T, from $\mathbb{C}^{pm}$ to $\mathbb{C}^{\sigma}$ where $T(K)=C_p[I,K]$ and $\sigma=\binom{p+m}{p}$.

ii) a linear map $(\bar{P}_S)^r:\mathbb{C}^\sigma \to \mathbb{C}^n$ which is a composition of two linear maps: a) $(P_S)^r:\mathbb{C}^\sigma \to \mathbb{C}^{n+1}$ where P_S is the Plucker matrix of S (see Chapter 5), which is a linear map defined by:

$$C_p\big(M(s)\big)=P_S.[s^n, s^{n-1},...,1]^t \tag{6.3}$$

and (b) the projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$ annihilating the first coordinate (in order to get rid of the 1, the first coefficient of the monic polynomial).

Remark(6.1) Since M(s) is the column reduced composite matrix for the strictly proper system S , we obviously have $P_S=[e_1, \bar{P}_S]$ where e_1 is a σx1 vector having the (1,1) entry 1 and all the rest 0.

□

We can now view $\hat{\chi}(K)$ as a multiplication of the vector T(K) and the matrix $\bar{P}_S$, that is

$$\hat{\chi}(K)=T(K)\ \bar{P}_S \tag{6.4a}$$

we can similarly obtain a decomposition for the real PPM as

$$\chi(K)=T(K)\ \bar{P}_S \tag{6.4b}$$

Remark(6.2) The importance of the descriptions 6.4a and b for the PPM is that the information about the feedback compensator and the system are separated. The map T is a function of K only, incorporates the multilinear properties of the PPM and it is the same for all systems S. On the other hand, the reduced Plucker matrix $\bar{P}_S$ is a complete invariant of the system [Kar.1] and is the one that changes the PPM when S alters.

□

For a rough investigation of the structure of the image of the complex PPM we will use only the fact that it is a regular map between complex algebraic varieties and not the decomposition (6.4). The image of such a map is a constructible set [Hum.1] (theorem(3.)). This means that although $\mathrm{Im}\hat{\chi}$ may fail to be a variety, it is very close to be one - that is, it contains a Zarisky open subset of its closure. Therefore we may use, as a measure of the size of the $\mathrm{Im}\hat{\chi}$, the dimension of its Zarisky Closure (the smallest variety containing $\mathrm{Im}\hat{\chi}$ and denoted by $\mathrm{ZC}(\mathrm{Im}\hat{\chi})$) which is well defined since $\mathrm{ZC}(\mathrm{Im}\hat{\chi})$ is a variety. This dimension is what we define as dimension of $\mathrm{Im}\hat{\chi}$ as the following definition indicates.

Definition (6.1)

$$\dim\mathrm{Im}\hat{\chi} \stackrel{def}{=} \dim \mathrm{ZC}(\mathrm{Im}\hat{\chi}) \qquad \square$$

In our case, we can view the image of CPPM as an affine variety of $\mathbb{C}^n$ without possibly a subset of a subvariety of strictly lower dimension (note that in the real case the image of the PPM does not admit such a nice structure). If we also take into account that a variety of dimension n in $\mathbb{C}^n$ is the whole of this space, then we can easily verify the following:

Theorem (6.1) For a system S(A,B,C) the next two statements are equivalent:

i) $\dim \mathrm{Im}(\hat{\chi})=n$

ii) Almost all complex monic polynomials of degree n can be assigned by a complex output feedback.

□

Given that $\mathbb{R}^n$ is Zarisky dense in $\mathbb{C}^n$ we also have:

Corollary (6.1) For a system S the next two statements are equivalent:

i) $\dim \mathrm{Im}(\hat{\chi})=n$

ii) Almost all real monic polynomials of degree n can be assigned by a complex output feedback.

□

As far as the real PPM χ is concerned, it is once again a regular map in pm (real) variables. The image of such a map usually fails to be a variety; however we know that it is a semialgebraic set, ie. the solution set of inequalities [Boc.1]. To illustrate this fact, it is sufficient to see the real variety $y=x^2$; the projection $p:\mathbb{R}^2 \to \mathbb{R}$ on the y-coordinate is a regular function, mapping the variety to the set $y \geq 0$. The dimension of a semialgebraic set is defined [Boc.1] as the dimension of the smallest real variety (Zarisky closure) containing the set.Thus

Definition (6.2)

$$\dim\mathrm{Im}\chi \overset{def}{=} \dim ZC(\mathrm{Im}\chi)$$

□

Unlike the complex case , the image of a real regular map is far from being a real variety. Thus if $\dim\mathrm{Im}\chi=n$, this would not mean that $\mathrm{Im}\chi$ is almost the whole of $\mathbb{R}^n$, but that it is rather an n-dimensional object defined by inequalities like the n-dimensional unit ball, or the set of all n-tuples having positive coordinates. In the CPPM case we have the easier 'black or white' situation where we have only to examine whether $\dim\mathrm{Im}\hat{\chi}$ is equal to n or not, if we want to determine whether we can assign almost any poles. In the real case, the dimension is not the only parameter determining the size of $\mathrm{Im}\chi$; having found the dimension of $\mathrm{Im}\chi$ we have one of the parameters defining the size, but this is not the only one.

6.3 Relationships of the pole placement map and known system invariants

6.3.1 Introduction

The purpose of this chapter is to establish relationships between the PPM and known system invariants. This will be achieved via infinitesimal analysis of the PPM, which will give us local as well as global results and will be convenient for calculations. The main result of this section is the relation of the image of PPM with the rank of its differential. This differential is strongly related to the Plucker matrix of the system , which is a complete invariant of S [Kar.1]. By calculating the differential of PPM using both expressions (6.1) and (6.2) we can relate the Plucker matrix to the Markov parameters of the system.

6.3.2 The Sard's theorem for varieties and the differential of the pole placement map.

The rank of the differential of a regular map is closely related to the dimension of its image; in fact, the following theorem holds true (this is a version of Sard's theorem for complex algebraic varieties):

Theorem (6.2) [Mum.1] The rank of the differential $(D\Phi)_x$ of a regular map $\Phi:X\to Y$ between complex varieties is equal to the dimension of $\mathrm{Im}\Phi$ for every x in a Zarisky open subset of X .

□

In order to find the dimension of the image of the CPPM, it is necessary and sufficient to calculate the rank of the differential $(D\hat{\chi})_K$. Next, we will be calculating the differential of $\hat{\chi}$ using the matrix fraction description of CPPM (6.1). In order to calculate the differential of $\hat{\chi}$ we will calculate the differential of T, the Plucker embedding,and we will multiply it with the remaining linear map $\bar{P}_S$. The matrix $\bar{P}_S$ is computed by (6.3) and its rows are the coefficients of the pxp polynomial minors of M(s) which are lexicographically ordered.

Remark (6.3) [Kar.1]: The Grassman vector $C_p\big(M(s)\big)$, or equivalently the Plucker matrix $\bar{P}_S$ is a complete invariant of the family of all minimal realisations of G(s).

□

To calculate the differential of T we use an alternative expression of the compound $C_p[I,K]$, that is the exterior product of the rows of [I,K]; so if the rows of [I,K] are $\underline{k}^t_i$ $1 \leq i \leq p$, then $T(K)= \underline{k}^t_1 \wedge \underline{k}^t_2 \dots \wedge \underline{k}^t_p$. We also symbolise the entries of K by k_{ij} $1 \leq i \leq p$ and $1 \leq j \leq m$ and it is now sufficient to calculate the partial derivatives of T for every k_{ij}; this can be done easily using a chain rule for the exterior product and thus the partial derivative of T with respect to k_{ij} is

$$\frac{\partial T}{\partial k_{ij}} = \underline{k}^t_1 \wedge \underline{k}^t_2 \ldots \wedge \underline{k}^t_{i-1} \wedge \underline{e}^t_{p+j} \wedge \underline{k}^t_{i+1} \ldots \wedge \underline{k}^t_p \tag{6.5}$$

where e_{p+j} is a (p+m)x1 vector such that all its entries are zero apart from the p+j-th which is one. Obviously the differential, $(DT)_K$, of the map T at a point K is a $pm\mathrm{x}\binom{p+m}{m}$ matrix having as rows the σx1 vectors described in (6.5); therefore, we have:

Proposition (6.1) The differential of the CPPM is given by:

$$(D\hat{\chi})_K = (DT)_K.\bar{P}_S \tag{6.6}$$

where the rows of $(DT)_K$ are given by the right hand side of (6.5).

□

From the above, it is readily seen that:

Corollary (6.2) For every $K \in \mathbb{C}^{p\mathrm{x}m}$ we have that : $\mathrm{rank}\{\bar{P}_S\} \geq \mathrm{rank}\{(D\hat{\chi})_K\}$

□

The rank of the Plucker matrix characterizes the feedback properties of S in the following way:

Corollary (6.3) $\mathrm{rank}\{\bar{P}_S\} \geq \dim(\mathrm{Im}\hat{\chi})$

□

The above result readily follows from corollary (6.2) and theorem (6.2). Proposition(6.1) and its following corollaries suggest a method for calculating the dimension of the image of the CPPM as shown below:

Remark(6.4). First we calculate the rank of the Plucker matrix $\bar{P}_S$, which constitutes an upper bound for $\dim(\mathrm{Im}\hat{\chi})$. Then, because of eq(6.6), whenever $mp \geq n$, the set

$$W = \left\{K \in \mathbb{C}^{p\mathrm{x}m} : \mathrm{rank}(D\hat{\chi})_K = \mathrm{rank}\bar{P}_S \right\}$$

is a Zarisky open set in $\mathbb{C}^{pxm}$. If now, W is nonvoid, that is if there exists $K_0 \in \mathbb{C}^{pxm}$ such that $\text{rank}(D\hat{\chi})_{K_0} = \text{rank}\bar{P}_S$, then $\dim(\text{Im}\hat{\chi})=\text{rank}\bar{P}_S$.

□

Example(6.2) If we consider the same system as in example (6.1) then we have that:

$$\bar{P}_S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and thus, $\text{rank}(\bar{P}_S)=4$, and the above necessary conditions are satisfied. Since the rank of $\bar{P}_S$ is always greater than or equal to the dimension of the image of the complex pole placement map we have that : $\dim(\text{Im}\hat{\chi}) \leq 4$. As it has been mentioned the pole placement map $\hat{\chi}$ is a map from $\mathbb{C}^4$ to $\mathbb{C}^4$ given by:

$$[k_{11},k_{12},k_{21},k_{22}] \to [k_{21},\ k_{22},-k_{12},\ k_{22}k_{11}-\ k_{12}k_{21}]$$

The differential at the point $K_0=[2,3,3,5]$ is:

$$(D\hat{\chi})_{K_0} = \begin{bmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & -3 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and $\text{rank}((D\hat{\chi})_{K_0})=4$; therefore $\dim(\text{Im}\hat{\chi})=4$

□

The above analysis provides an alternative proof to the following known result [Gia.1]:

Theorem (6.3) For a system S with n-states p-inputs and m-outputs, necessary condition for the assignment of almost all complex (or real) monic polynomials of degree n by complex output feedback is that $mp \geq n$ and $\text{rank}\{\bar{P}_S\}=n$.

Proof

Note that by Theorem(6.): $\text{Im}\hat{\chi}=n$. For a generic K, it follows that $\text{rank}(D\hat{\chi})_K=n$; however the differential is an mpxn matrix so $mp \geq n$. On the other hand, by Corollary (6.2) we get that $\text{rank}\{\bar{P}_S\} \geq \text{Im}\hat{\chi}=n$, which implies that $\text{rank}\{\bar{P}_S\}=n$ since $\bar{P}_S$ is a σxn matrix.

□

As far as determing the dimension of Imχ, the rank of the differential of PPM plays important role again. The version of Sard's theorem for semialgebraic sets may be stated as follows:

Theorem (6.4) [Boc.1] Let Φ:X→Y be a semialgebraic function between semialgebraic sets. Then the set of points $\{\Phi(x): x \in X$ and rank $(D\Phi)_x < \dim Y\}$ is a semialgebraic subset of Y having dimension strictly less than the dimension of Y.

□

In our case the set $\{ x \in \mathbb{R}^{mp}$: rank$(D\chi)_x < \dim(\text{Im}\chi)\}$ is algebraic. Indeed, the equations for the differential are the same as those defining $\hat{\chi}$, the only difference being that $K \in \mathbb{R}^{mp}$,that is:

$$(D\chi)_K=(DT)_K.\bar{P}_S \tag{6.7}$$

By equating to zero the dimImχ-compound of the right hand side of (5.1) we have the required set of equations. The above set is called the set of critical points of χ and is symbolized by Crit(χ); the image of Crit(χ), through χ, is a semialgebraic set and because of the previous theorem, it has dimension strictly smaller than the dimension of ZC(Imχ). Thus Crit(χ) is a proper subvariety of $\mathbb{R}^{mp}$, which implies that rank$(D\chi)_K$ is equal to dimZC(Imχ) for every K in a nonempty Zarisky open set of $\mathbb{R}^{mp}$. Thus, like the complex case, in order to find the dimension of the image of the PPM it is necessary and sufficient to calculate the rank of the differential $(D\chi)_K$ at a generic $K \in \mathbb{R}^{mp}$. The dimensions of the images of the two maps $\hat{\chi}$, χ are invariants of the system S; the rank description of the dimension simplifies the comparison of these two invariants.

Theorem(6.5) $\dim(\text{Im}\chi) = \dim(\text{Im}\hat{\chi})$

Proof

Since the dimension of the image is given by the rank of the matrix $(DT)_K.\bar{P}_S$ at a generic $K \in \mathbb{R}^{pm}$ for χ and at a generic $K \in \mathbb{C}^{pm}$ for $\hat{\chi}$, we readily deduce that

$\dim(\mathrm{Im}\chi) \leq \dim(\mathrm{Im}\hat{\chi})$. On the other hand, since $\mathbb{R}^{pm}$ is Zarisky dense in $\mathbb{C}^{pm}$, the Zarisky closed set of $K \in \mathbb{C}^{pm}$ such that $\mathrm{rank}(DT)_K.\bar{P}_S < \dim(\mathrm{Im}\hat{\chi})$ cannot contain $\mathbb{R}^{pm}$; thus for a generic $K \in \mathbb{R}^{pm}$ we have that $\mathrm{rank}(DT)_K.\bar{P}_S \geq \dim\mathrm{Im}\hat{\chi}$, therefore $\dim\mathrm{Im}\chi \geq \dim\mathrm{Im}\hat{\chi}$ and the result follows.

□

Although the dimension of the images of CPPM and PPM are equal, the actual images are far from having the same structure. As we have said before, $\mathrm{Im}\hat{\chi}$ is a Zarisky open subset of $ZC(\mathrm{Im}\hat{\chi})$, whereas $\mathrm{Im}\chi$ is a semialgebraic subset of $ZC(\mathrm{Im}\chi)$.To illustrate this we consider the following example (this is actually similar to the one given in [Wil.1]).

Example(6.3) Let S be a generic strictly proper system of 2-inputs , 2-outputs and 4-states. According to [Gia.1] the set of all constant 2x2 feedback compensators (considered in the form $(1,k_1,k_2,k_3,k_4,k_5)$ via the embedding T) assigning to S a polynomial $s^4+p_4s^3+p_3s^2+p_2s+p_1$ is given as the solution set of the equations

$$k_5-k_1k_4+k_2k_3=0 \tag{6.8a}$$

$$\underline{p}=\underline{k}.\bar{P}_S \tag{6.8b}$$

where $\underline{k}=(1,k_1,k_2,k_3,k_4,k_5)$, $\underline{p}=(p_4,p_3,p_2,p_1)$ and $\bar{P}_S$ is the 6x4 reduced Plucker matrix of S. As we shall see later, $\bar{P}_S$ has generically a full rank 4x4 submatrix which does not contain the first row.Thus we can solve the set (6.8b) of linear equations with respect to k_1,k_2,k_3,k_4 and these solutions will be of the form:

$$k_i= f_i(k_5,\underline{p}) \quad i=1,2,3,4 \tag{6.9}$$

where f_i are linear functions. Substituting (6.9) to (6.8a) we get

$$F_2(\underline{p})k_5{}^2+ F_1(\underline{p})k_5+F_0(\underline{p})=0 \tag{6.10}$$

where F_i are some appropriate polynomial functions of p of degree ≤ 2. The image of $\hat{\chi}$ will be all $\underline{p} \in \mathbb{C}^4$ such that (6.10) is solvable with respect to $k_5 \in \mathbb{C}$. We can easily see that if $F_2(\underline{p}) \neq 0$ then (6.10) is always solvable, thus $\mathrm{Im}\hat{\chi} \supseteq \{\underline{p} \in \mathbb{C}^4 : F_2(\underline{p}) \neq 0\}$. Similarly, the image of χ will be all $p \in \mathbb{R}^4$ such that (6.10) is solvable with respect to $k_5 \in \mathbb{R}$. Thus $Im\chi = A \cup B \cup C$ where $A=\left\{p \in \mathbb{R}^4 : F_2(p) \neq 0 \text{ and } F_1(p)^2 - 4F_0(p)F_2(p) \geq 0\right\}$, $B=\left\{p \in \mathbb{R}^4 : F_2(p)=0 \text{ and } F_1(p) \neq 0\right\}$ and $C=\left\{p \in \mathbb{R}^4 : F_2(p)=0 , F_1(p)=0 \text{ and } F_0(p)=0\right\}$. Comparing now the two images we can see that on the one hand both are 4-dimensional, but on the other $Im\hat{\chi}$ covers almost the whole of $\mathbb{C}^4$, whereas $Im\chi$ mainly consists of the half-space $p \in \mathbb{R}^4$: $F_1(p)^2 - 4F_0(p)F_2(p) \geq 0$.

□

Remark(6.5) The previous example can also be found in [Ros.2]. There, the above arguments on the solvability of eq(6.10) are explained geometricaly in terms of intersections of graphs in a projective space.

□

6.3.3 Applications.

In the previous sections we had seen that the differential of PPM is $(DT)_K \bar{P}_S$, where T is the Plucker embedding (which is a map indepented of S) and $\bar{P}_S$ is the reduced Plucker matrix, which is a complete invariant of the similarity equivalence orbit of S. On the other hand there is an alternative expression of the same differential but from the state space point of view [Rein.1]; this allows the establishment of the links between the Plucker matrix invariant and the state space description.

Lemma (6.1)[Rein.1] For a fixed K and a system S(A,B,C)we have that

$$(D\chi)_K = \begin{bmatrix} \underline{c}^t{}_1\underline{b}_1 & \underline{c}^t{}_1 H\underline{b}_1 & \cdots & \underline{c}^t{}_1 H^n\underline{b}_1 \\ \vdots & \vdots & & \vdots \\ \underline{c}^t{}_j\underline{b}_i & \underline{c}^t{}_j H\underline{b}_i & \cdots & \underline{c}^t{}_j H^n\underline{b}_1 \\ \vdots & \vdots & & \vdots \\ \underline{c}^t{}_p\underline{b}_m & \underline{c}^t{}_p H\underline{b}_m & \cdots & \underline{c}^t{}_p H^n\underline{b}_m \end{bmatrix} \cdot Q \qquad (6.11)$$

where K and p_i satisfy (2.2) and H=A+BKC

$$
Q = \begin{bmatrix} 1 & p_n & \cdots & p_2 \\ & 1 & \cdots & p_3 \\ & & \ddots & \vdots \\ \mathbf{0} & & & 1 \end{bmatrix}
$$

□

Thus we may write (6.11) as

$$
(D\chi)_K = [\ colCB\ ,colCHB, \ldots ,colCH^nB\ ]\ .\ Q \tag{6.12}
$$

where Q is a full rank nxn matrix and 'col' maps an mxl matrix to the mlx1 matrix formed by superimposing its columns. Now by compairing (6.7) with (6.12) we have the result:

Corollary (6.4) For a given system S , $\forall K \in \mathbb{R}^{pxm}$ and H=A+BKC the following holds true:

$$
(DT)_K.\bar{P}_S = [\ colCB\ ,colCHB, \ldots ,\ colCH^nB\ ]\ .\ Q
$$

and

$$
rank\ [\ colCB\ ,colCHB, \ldots ,\ colCH^nB] \le rank\ \bar{P}_S\ (\le n)
$$

□

Now we are also in a position to calculate the Marcov parameters of the system S using the Plucker matrix. Indeed if we put K=0 in (6.12) we have that $(D\chi)_0$=[colCB ,colCAB, ..., colCA^{n}B].Q , where Q can be constructed using the coefficients of the open loop polynomial of the system (i.e detD(s)). On the other hand if we put K=0 in (6.7) we have that $(D\chi)_0=(DT)_0.\ \bar{P}_S$; in addition $(DT)_0$ can be calculated by (6.5), and it is a pmxσ matrix whose entries are :

$$
R_{ij\ ,\ \omega} = \begin{cases} (-1)^{i-1} & \text{when } \omega=(1,2,\ldots,i-1,i+1,\ldots,p,p+j) \\ 0 & \text{otherwise} \end{cases} \qquad 1 \le i \le p \text{ and } 1 \le j \le m
$$

All the above lead to the following remark describing how we can calculate the Markov parameters using the Plucker matrix.

Remark(6.6) To calculate [colCB ,colCAB, ... ,colCA^{n}B] we select the (1,2,...,i−1,i+1,...,p,p+j) rows of the Plucker matrix $\forall\ 1 \le i \le p,\ 1 \le j \le m$, multiply them by $(-1)^{i-1}$, forming a pmxn matrix and we postmultiply it by Q^{-1}.

□

An alternative way to find the above relation between the Plucker matrix and the Markov parameters is the following:

Remark (6.7) Consider the matrix fraction description of S, $G(s)=N(s)D(s)^{-1}$. By the transposed form of Cramer's rule every entry $g_{ij}(s)$ of G(s) is given by

$$g_{ij}(s)=\frac{\det D^{ij}(s)}{\det D(s)} \tag{6.13}$$

where $D^{ij}(s)$ is the matrix obtained by replacing the jth row of D(s) by the ith row of N(s); that is the coefficient vector of the polynomial $\det D^{ij}(s)$ is the (1,2,...,i-1,i+1,...,p,p+j) row of $\bar{P}_S$ multiplied by $(-1)^{i-1}$. On the other hand

$$G(s)=CBs^{-1}+CABs^{-2}+...+CA^{n-1}Bs^{-n}+... \tag{6.14}$$

Substituting now (6.13) to (6.14) , multiplying by det D(s) and finally equating the coefficients of $1,s,...,s^{n-1}$ we can get the required relation.

□

Corollary (6.5) For a generic S in the variety of strictly proper systems of p-inputs,m-outputs and n-states for which $mp \ge n$,the reduced Plucker matrix $\bar{P}_S$ (having n-rows and σ-columns) contains an mpxn submatrix which has rank equal to n.

□

Indeed the mpxn submatrix formed by the (1,2,...,i-1,i+1,...,p,p+j) rows of $\bar{P}_S$ for $1 \le i \le p$ and $1 \le j \le m$, has the rank equal to the rank of the matrix [colCB,colCAB,...,colCA^{n}B]. The last is proved [Wil.1] to have full rank in a Zarisky open subset of the variety of strictly proper systems of p-inputs,m-outputs and n-states.This also proves that the Plucker matrix has generically rank equal to n and that the image of CPPM (or PPM) has generically dimension equal to n when $mp \ge n$.

6.4 Unbounded gain and composite representation.

6.4.1 Introduction.

An integral part of the study of the root locus map (pole placement map) under general real output feedback K is the study of the location of the closed loop poles when K becomes unbounded. Given that in the expression of the pole placement map the compensator enters in a composite form [I,K], it is essential to have a representation of this form for the compensator when K is unbounded. It has been accepted by many researchers [Broc.1],[Gho.1] that unbounded gains correspond to [A,B] representations where det(A)=0; The rigorous explanation and proof of this fact is the main purpose of this section.

6.4.2.Formulation of the problem and preliminary results

The PPM maps the real gain K to the coefficients of the polynomial of the corresponding closed loop system. This map can serve the role of the MIMO root locus map for finite K. To have a complete root locus map we have to extent PPM when K is unbounded. In the SISO case we know that this extension is always possible and that the actual value of the root locus map will be equal to the zeros of the open loop system. In the MIMO case where the feedback compensator is of the form λK where λ varies and K is constant we have similar results with the SISO case, that is the closed loop poles approach the open loop zeroes as λ tends to infinity. If we allow all the paremeters of K (pm) to vary, then the problem becomes more complicated and one reason for this is that the way we can approach infinity is not unique. To see this let us consider the equation(6.1) which describes the PPM:

$$s^n+p_n s^{n-1}+\ldots+p_1=\det\left\{[I_p,K]\begin{bmatrix}D_R(s)\\N_R(s)\end{bmatrix}\right\}$$

We observe that the roots of the polynomial $\chi(K)$ depend only on rowspan$[I_p,K]$ and not really on K. So for the purposes of our analysis it is better to parametrise the finite gains as rowspan$[I_p,K]$. As we will show in section 3, if we have a sequence of finite gains $\{K_n\}$, which tends to infinity in the usual sense, then the limit corresponds to representations of the form rowspan[A,B] such that A is singular. Thus, instead of having one point representing infinity for the unbounded compensators, we have a set of subspaces having a certain representation, and this explains the complexity of the problem of the extension at infinity of PPM in the MIMO case. In this case, it is not always possible to extent PPM as it will be explained in the following remark.

Remark(6.8) It is important to mention that the gains rowspan[A,B] for which the PPM is not extentable are the ones such that

$$\det\Big([A,B]\ M_R(s)\Big) = 0 \tag{6.15}$$

and this because if we consider the sequence $S_n=[A,B]+1/n[I_p,K]$, then although $S_n \to [A,B]$ for every K, the poles of $\det\big(S_n M_R(s)\big)$ at the limit ($n=\infty$) depend on K (see chapter 10) and thus for this [A,B] we cannot correspond a unique set of poles. In our case, where the system we examine is strictly proper, eq.(6.15) cannot be satisfied by a finite gain (something possible in the case of proper systems).

□

In the following chapter we will deal with the parametrisation of gains (both finite and infinite) as rowspan[A,B] where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$. The parameter space is the Grassmann manifold and the set of infinite gains is a submanifold of codimension 1. We will show how we can examine convergence of sequences on this manifold and finally we will justify the above given composite representation of infinite gains, as linear spaces rowspan[A,B] with det(A)=0. The composite representation of the feedback matrix as an element of an appropriate manifold is considered next.

6.4.3. The Stiefel and the Grassmann manifolds and the composite representation of the family of unbounded feedbacks.

The *Stiefel manifold* $V_p(\mathbb{R}^{p+m})$ is called the set of all full rank px(p+m) matrices having the induced topology of $\mathbb{R}^{p(p+m)}$.The Stiefel manifold is actually an open subset of $\mathbb{R}^{p(p+m)}$. Now the matrix multiplication defines an action of GL(p,$\mathbb{R}$) on $V_p(\mathbb{R}^{p+m})$.The set of all orbits of this action is called the *Grassmannian* ,$G_p(\mathbb{R}^{p+m})$, and it is the set of all p-dimensional linear subspaces of $\mathbb{R}^{p+m}$. We can topologize $G_p(\mathbb{R}^{p+m})$ giving it the topology induced by the projection map:

$$\pi: V_p(\mathbb{R}^{p+m}) \to G_p(\mathbb{R}^{p+m}) \tag{6.16}$$

Remark(6.9) The projection map π maps every full rank matrix $K \in \mathbb{R}^{px(p+m)}$ to the p-dimensional vector subspace of of $\mathbb{R}^{p+m}$ spanned by the rows of K.

□

We can view $G_p(\mathbb{R}^{p+m})$ as a pm-dimensional manifold as follows: For every $1 \le i_1 < i_2 < \dots < i_p \le p+m$, $U_{i_1,i_2,\dots,i_p}$ is the open subset of $G_p(\mathbb{R}^{p+m})$ such that $\pi^{-1}(U_{i_1,i_2,\dots,i_p})$ contains all full rank matrices K having the pxp matrix $K_{i_1,i_2,\dots,i_p}$ formed by the $i_1,i_2,\dots,i_p$ columns of K invertible. Let $A \in U_{i_1,i_2,\dots,i_p}$ then choose an arbitrary $K \in \pi^{-1}(A)$, the matrix $(K_{i_1,i_2,\dots,i_p})^{-1}K$ has its pxp submatrix formed by the $i_1,i_2,\dots,i_p$ columns equal to I_p and we call the remaining pxm submatrix $\phi_{i_1,i_2,\dots,i_p}(A)$. The $(U_{i_1,i_2,\dots,i_p},\phi_{i_1,i_2,\dots,i_p})$ form an atlas for $G_p(\mathbb{R}^{p+m})$ giving it the structure of pm-dimensional manifold. The Grassmann manifold $G_p(\mathbb{R}^{p+m})$ is compact; it is a natural compactification of $\mathbb{R}^{pxm}$ via:

$$T: L \to \pi([I_p, L]) \tag{6.17}$$

where $L \in \mathbb{R}^{pxm}$.This way sequences in $\mathbb{R}^{pxm}$ that did not converge now have a convergent subsequence in $G_p(\mathbb{R}^{p+m})$.

Supposing we want to see where a sequence $\pi(K_n)$,$K_n \in V_p(\mathbb{R}^{p+m})$, converges in $G_p(\mathbb{R}^{p+m})$. First, if K_n converges to K in $V_p(\mathbb{R}^{p+m})$ then $\pi(K_n)$ will converge to $\pi(K)$ in $G_p(\mathbb{R}^{p+m})$, since the Grassmannian is a quotient of the Stieffel manifold. In the case that K_n does not converge in $V_p(\mathbb{R}^{p+m})$, then $\pi(K_n)$ has a convergent subsequence in $G_p(\mathbb{R}^{p+m})$ and this because the Grassmannian is a compact manifold. The limit of this subsequence is located in one of the open patches, $U_{i_1,i_2,\ldots,i_p}$, $1 \le i_1 < i_2 < \ldots < i_p \le p+m$, of the Grassmann manifold, and to find this limit we have to find an appropriate pxp matrix A such that $A.K_n$ converges to a point K in $V_p(\mathbb{R}^{p+m})$ (and this to take advantage of the fact that $\pi(K_n)$ stays invariant under the row transformation $A.K_n$). Then $\pi(K_n)$ will tend to $\pi(K)$ as n tends to infinity.

Example (6.4) Let

$$\pi(K_n) = \pi\left(\begin{bmatrix} 1 & 0 & \frac{1}{n} \\ 0 & \frac{1}{n+1} & 2 \end{bmatrix}\right)$$

be a sequence in $G_2(\mathbb{R}^3)$. Since K_n converges to

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

in $V_2(\mathbb{R}^3)$, $\pi(K_n)$ converges to $\pi(K)$ in $G_2(\mathbb{R}^3)$.

□

Example (6.5) Let

$$\pi(K_n) = \pi\left(\begin{bmatrix} n & 0 & n \\ 0 & n^3 & n \end{bmatrix}\right)$$

be a sequence in $G_2(\mathbb{R}^3)$.According to what we said, to find the limit of this sequence we have to find the limit of:

$$(K_n^I)^{-1}K_n=\begin{bmatrix} n & 0 \\ 0 & n^3 \end{bmatrix}^{-1}\begin{bmatrix} n & 0 & n \\ 0 & n^3 & n \end{bmatrix}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{n^2} \end{bmatrix}$$

This obviously is

$$K=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

in $V_2(\mathbb{R}^3)$; thus our original sequence $\pi(K_n)$ tends to $\pi(K)$ in $G_2(\mathbb{R}^3)$.

□

Next, we will examine how we can represent unbounded gains as certain points of the Grassmannian. Supposing that we have a sequence of gains $L_n \in \mathbb{R}^{p\mathrm{x}m}$ such that $L_n \to \infty$. The corresponding sequence in $G_p(\mathbb{R}^{p+m})$ via the function π (6.16) is $\pi\big([I_p,L_n]\big)$. This sequence must have a convergent subsequence (because of the compactness of the Grassmannian) having limit say $\pi([L_1,L_2])$. The following lemma says that such a limit must have the property that $\det L_1=0$.

Lemma (6.2) If a sequence of the form $\pi\big([I_p,L_n]\big)$, $L_n \to \infty$, converges in $G_p(\mathbb{R}^{p+m})$,then its limit is of the form $\pi([L_1,L_2])$,$\det L_1=0$.

Proof

Here we will examine the case where the pxm entries of the sequence L_n can be given as rational functions of n. Then we set n=1/s and we write the transformed L_n in a left coprime MFD, namely $L(s)=D(s)^{-1}N(s)$; the fact that $L_n \to \infty$ becomes $\det(D(0))=0$. Taking limits we get:

$$\lim_{n\to\infty} \pi\Big([I_p,L_n]\Big) = \lim_{s\to 0} \pi\Big([I_p,D(s)^{-1}N(s)]\Big)$$

Since π is invariant under row transformations we get

$$\lim_{s\to 0} \pi\Big([I_p,D(s)^{-1}N(s)]\Big) = \lim_{s\to 0} \pi\Big([D(s),N(s)]\Big)$$

and finally

$$\lim_{s\to 0} \pi\Big([D(s),N(s)]\Big) = \pi\Big([D(0),N(0)]\Big)$$

which proves the result.

□

Remark(6.10) In the more general case where L_n is arbitrary, we first find, L_n^I, the pxp submatrix of $[I_p,L_n]$ having the greatest determinant; then $\pi\Big([I_p,L_n]\Big)$ is equal to $\pi\Big([\left(L_n^I\right)^{-1},\left(L_n^I\right)^{-1}L_n]\Big)$ and $\left(L_n^I\right)^{-1}$ tends to L_1, $\det L_1=0$. □

The following lemma proves the converse, that is if we have an element of the Grassmannian of the form $\pi([L_1,L_2])$,$\det L_1=0$ then we can construct an unbounded sequence of gains converging to that element.

Lemma (6.3) Let $h=\pi([L_1,L_2])$,$\det L_1=0$ be an element of $G_p(\mathbb{R}^{p+m})$ then there exists a sequence of the form $h_n=\pi\Big([I_p,L_n]\Big)$, $L_n \to \infty$, which converges to h in $G_p(\mathbb{R}^{p+m})$.

<u>Proof</u>

Consider the pencil

$$P(s)=[L_1+sI_p,\ L_2]$$

then since $P(s)$ has not a zero at $s=0$ and L_1+sI_p has a zero at $s=0$, the rational function $G(s)=(L_1+sI_p)^{-1}L_2$ has poles at $s=0$. Clearly as s tends to zero $\pi\Big(P(s)\Big)$ tends to

$\pi([L_1,L_2])$ or equivalently (since π is invariant under row transformations).

$$\lim_{s\to 0}\pi\big(I_p,G(s)\big)=\pi([L_1,L_2])$$

Setting now $L_n=G(1/n)$ we get the required sequence.

□

Taking the above into consideration we are lead to the following definitions

Definition(6.3) We call finite (multivariable) gains (FG) of p-outputs and m-inputs, all the elements of $G_p(\mathbb{R}^{p+m})$ of the form $\pi([K_1,K_2])$ where $\det K_1 \neq 0$.

Definition(6.4) We call infinite gains (IG) of p-outputs and m-inputs, the elements of $G_p(\mathbb{R}^{p+m})$ of the form $h=\pi([K_1,K_2])$ where $\det K_1=0$.

□

Next we will see how we can apply this new representation of the infinite gains to examine the asymptotic properties of multivariable systems.

6.5 Real degeneracy of systems and new conditions.

6.5.1.Introduction.

The classical technique of the root locus analysis of MIMO systems under scalar output feedback is effective in many cases; however the actual problem of location of closed loop poles is much more complex, since the degrees of freedom of the constant output feedback compensator equals to (#inputs)x(#outputs) and the feedback is not always a square matrix. If we try to define the root locus map using not only one, but all possible degrees of freedom of the output feedback compensator (in which case we call it full), then contrary to the SISO case there are MIMO systems for which this is not possible. The property of degeneracy of a system was first introduced in [Broc.1] and characterises the asymptotic behaviour of the closed loop poles of a system with respect to a high gain complex (full) output feedback, in the following way: an (open loop) MIMO system is nondegenerate, iff the root locus map of the corresponding closed loop system under complex (full) output feedback, can be defined for all compensators. If there exists at least one compensator for which the map is not defined, then it is called *degenerate.* A necessary and sufficient condition for a generic system to be (complex) degenerate is $mp > n$ [Broc.1] where p,m,n are the number of inputs, outputs and states of the system. The need to define the root locus map of a closed loop multivariable system when the output feedback compensator is real, leads us to the study of the concept of real degeneracy ($\mathbb{R}$-degeneracy) or $\mathbb{R}$-nondegeneracy ie. finding the conditions under which the Root Locus map is not defined or is defined under a real output feedback and this is studied here. As we will shall see in this chapter, the problem: a 'system to be degenerate' is an intersection problem of algebraic geometry on the Grassmannian which in the complex case can be examined via Schubert Calculus. The real case of the degeneracy problem is much more complicated and basic theory of vector bundles is utilised for the derivation of a sufficient condition for a generic system to be $\mathbb{R}$-degenerate.

6.5.2 The concept of degeneracy.

The concept of degeneracy arises in the examination of the asymptotic behaviour of a closed loop multivariable system with respect to a high gain static output feedback compensation. Obviously, since our system is strictly proper, χ is defined on the set of finite gains. Our problem now is to examine whether χ can be extended continuously to a function $\bar{\chi}$ defined on both finite and infinite gains which assigns to each gain a non zero polynomial, that is:

we aim to extent χ to $\bar{\chi} : G_p(\mathbb{R}^{p+m}) \rightarrow \mathbb{P}^n(\mathbb{R})$.

Definition(6.5). A system S whose PPM can be extented to $\bar{\chi}$ is called $\mathbb{R}$-nondegenerate.

□

Remark(6.11). If we replace $\mathbb{R}$ with $\mathbb{C}$,the new extension problem can be solved using Schubert calculus [Broc.1], and the necessary and sufficient condition for a generic system to be $\mathbb{C}$-nondegenerate is $mp \leq n$.

□

The special structure of PPM allows us to transform this extension problem to an equivalent intersection problem of algebraic varieties. This structure is induced by the skew-symmetric nature of the determinant and had been explained in subsection 6.2.2, the pole placement map can be decomposed into the standard Plucker embedding $\mathcal{P}$ and a linear map P_S. The linear map P_S defines a map:

$$p_B : \mathbb{P}^{\sigma-1}(\mathbb{R}) - B \rightarrow P^n(\mathbb{R})$$

where $p_B(\underline{v}) = \underline{v}\ P_S$ and $B = LKer(P_S)$. The map p_B is called projection with base B. We can define this projection on a set $D \subseteq P^{\sigma-1}(\mathbb{R})$, iff $D \cap B = \emptyset$. Since $\mathcal{P}(FG) \cap B = \emptyset$, the projection p_B is well defined on $\mathcal{P}(FG)$ and the actual images of the elements of this set through the projection are the coefficients of the closed loop polynomials, as we previously mentioned. The problem now involves examining whether we can define this projection on the whole of $G_p(\mathbb{R}^{p+m})$ or else whether $\mathcal{P}\left(G_p(\mathbb{R}^{p+m})\right) \cap B = \emptyset$.Thus

Remark(6.12) The existence of the extension $\bar{\chi}$ of χ is equivalent to the intersection problem $\mathcal{P}\left(G_p(\mathbb{R}^{p+m})\right) \cap B = \emptyset$ or equivalently for every $\underline{v} \in \mathcal{P}\left(G_p(\mathbb{R}^{p+m})\right)$ we must have

$$\underline{v}\ P_S \neq 0$$

□

or equivalently,

Remark(6.13) A system S is $\mathbb{R}$-degenerate iff there exists $\underline{v} \in \mathcal{P}\left(G_p(\mathbb{R}^{p+m})\right)$ such that:

$$v \; P_S = 0 \qquad \square$$

If instead of $\mathbb{R}$ we have $\mathbb{C}$ the above intersection problem, or the equivalent extension one can be easily solved by the use of intersection theory [Broc.1] which cannot be applied here. The method we suggest to examine this problem is topological and will provide us with a necessary condition. Specifically, if we could extend the map χ to $\bar{\chi}$ then the actual domain and range of the second map becomes topologically much richer than the ones of the former and this is possibly one of the facts which obstruct us into extending χ. Under the condition that $\bar{\chi}$ can be covered by a suitable bundle map, this obstruction can be expressed by means of an inequality involving the vector category of certain vector bundle over $G_p(\mathbb{R}^{p+m})$ (see lemma(6.4)). To make the above clearer we will briefly present some basic theory on vector bundles.

6.5.3.A vector bundle approach to the problem of degeneracy.

The theory of vector bundles arises from the need to examine vector spaces which are parametrised by certain sets having nice topological structure, like manifolds, varieties etc (see section 3.6.4). For the present purposes, a certain vector bundle (homotopy) invariant, the category of the vector bundle (see section 3.6.4) will play a key role for the derivation of necessary conditions for real nondegeneracy. The next lemma [Jam.1] gives us a necessary and sufficient condition for the vector category of a bundle to be less than or equal to a number.

Lemma (6.4) Let ψ be a line bundle over M then $\text{vecat}(\psi) \leq n+1$ iff there exists a map $f: M \to \mathbb{P}^n(\mathbb{R})$ covered by a bundle map F from (M,ψ) to $(\mathbb{P}^n(\mathbb{R}), \gamma_n^1)$.

$\square$

This theorem can be related to the nondegeneracy as follows: if a system is nondegenerate, then the real PPM can be extended to a map $\bar{\chi} : G_p(\mathbb{R}^{p+m}) \to \mathbb{P}^n(\mathbb{R})$; this map then can be naturally lifted to a bundle map, and by applying lemma (6.4) we can derive an inequality involving p,m,n. This inequality will be a necessary condition for the existence of nondegenerate systems.

6.5.4.Necessary conditions for real nondegeneracy.

Using the above tools we may now describe the main result:

Theorem (6.5) If there exists an $\mathbb{R}$-nondegenerate system S of p-inputs,m-outputs and n-states then $\text{vecat}(o(\gamma_m^p)) \leq n+1$.

Proof

Let S be an $\mathbb{R}$-nondegenerate system of p-inputs ,m-outputs and n-states.The pole placement map χ of Sec.3 for S can then be extended to a map

$$\bar{\chi} : G_p(\mathbb{R}^{p+m}) \rightarrow \mathbb{P}^n(\mathbb{R})$$

by definition. Note that we consider $G_p(\mathbb{R}^{p+m})$ embeded in $\mathbb{P}^{\sigma-1}(\mathbb{R})$ via the Plucker embeding. We can extend $\bar{\chi}$ to a bundle map X: $E\big(o(\gamma_m^p)\big) \rightarrow E\big(\gamma_n^1\big)$ as follows: let $(x,v) \in E\big(o(\gamma_m^p)\big) \subseteq G_p(\mathbb{R}^{p+m}) x \mathbb{R}^{\sigma}$ then $X(x,v)=(\bar{\chi}(x),vP_S)$. In this way X is linear map between the two fibers attached on x and $\bar{\chi}(x)$ since X acts as the matrix P_S. In addition to this, X restricted to x is a linear isomorphism for every x since if S is $\mathbb{R}$-nondegenerate we have that $\forall\ x \in G_p(\mathbb{R}^{p+m})$, $\wedge^p x \cap \text{Ker}(P_S)=\{0\}$ (see remark (6.13)). The map $\bar{\chi}$ satisfies the requirements of Lemma 6.4 thus for the bundle $o(\gamma_m^p)$ we get that $\text{vecat}(o(\gamma_m^p)) \leq n+1$.

□

The negation of the above result leads to sufficient conditions for $\mathbb{R}$-degeneracy state below:

Corollary (6.6). If $\text{vecat}(o(\gamma_m^p)) > n+1$ then all systems of p-inputs, m-outputs and n-states are $\mathbb{R}$-degenerate.

□

The number vecat($o(\gamma_m^p)$) is a function of p,m and from now on will be symbolised by v(p,m). It is not easy to calculate the numbers v(p,m) for all the values of p and m. We can instead use as lower bounds the nilpotency index of the Adams closure of $[o(\gamma_m^p)]$-1 in the Grothendieck ring $KO(G_p(\mathbb{R}^{p+m}))$, or the Stiefel Whitney index of $o(\gamma_m^p)$ or the Euler-Pontragin index of $o(\gamma_m^p)$. In our case the Stiefel Whitney index of $o(\gamma_m^p)$ is the nilpotency index of the subring of $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ generated by the Stiefel Whitney class $w_1(o(\gamma_m^p))=w_1$, the first Stiefel-Whitney class of the canonical bundle γ_m^p ;thus the Stiefel-Whitney index in this case is equal to h(p,m)+1 where h(p,m) is the height of the first Stiefel-Whitney class of the canonical bundle of the Grassmannian $G_p(\mathbb{R}^{p+m})$. Thus corollary(6.1) implies that:

Corollary (6.7). If $h(p,m) > n$ then all systems of p-inputs, m-outputs and n-states are $\mathbb{R}$-degenerate.

□

The height h(p,m) has been calculated for every p,m in [Hill.1],[Sto.1].The formula for h(p,m) when $1 < p \leq m$ is given as follows:

$$h(p,m)=\begin{cases} 2^{s+1}-2 & \text{if } p=2 \text{ or if } p=3 \text{ and } m+p-1=2^s \\ 2^{s+1}-1 & \text{otherwise} \end{cases}$$

where s is an integer such that $2^s \leq m+p-1 < 2^{s+1}$.

Examples (6.6)

a)Let p=4,m=10 then p+m-1=13 which gives s=3, thus $h(4,10)=2^4-1=15$. Therefore if $n<15$ then all systems of 4-inputs ,10-outputs and $n<15$ states are $\mathbb{R}$-degenerate (Cor(6.7)).

6.6 Conclusions

The first objective in the further development of the DAP framework has been the detailed study of the topological and high gain properties of the pole placement map. This was done by developing results on the dimension of its image and establishing links between the fundamental Plucker matrix invariant and the standard Markov parameters. The issue of real degeneracy was examined using topological tools a framework for examining high gain feedback as finite points in an appropriate compactification has been established. The concepts of degeneracy studied here is further examined in chapter 10 where it is shown to provide one of the key tools for the complete study of the pole placement map.

CHAPTER 7. Pole, Zero Assignment with Static Real Controllers

7.1 Introduction

As we have seen in chapter 5, pole and zero assignment problems can be viewed as intersection problems between sets of polynomial equations defined on the set of feedback or squaring down compensators, which may be noncompact sets. To utilise intersection theory on compact manifolds (homology theory) or intersection theory on projective algebraic varieties, as has been explained in chapters 3 and 5, we need to appropriately compactify the above (possibly) noncompact sets of compensators. This can be done by introducing some special compensators, namely the compensators at infinity, which play the role of sticking together the boundaries of the set of compensators $\mathbb{C}$, thus giving a new set $\bar{\mathbb{C}}$ which is compact. In chapter 6, an interpretation from the engineering point of view of static infinite feedback compensators, associated with a pole placement problem, was made and furthermore, the compactified $\bar{\mathbb{C}}$ obtained by introducing these controllers was seen to be a Grassmannian. In the case of squaring down compensators we do not need to consider compensators at infinity since this set of compensators is already compact.

A compactification of the space of the solutions is not always appropriate for intersection theory considerations. As was explained in chapter(3), the intersection on the compactified set may contain points which are not desirable- the so called degenerate points. A good compactification must have the property that a generic intersection on this set, if non void, contains only a negligible set of degenerate points (ie. its dimension must be less than the dimension of the intersection set). In section 7.2 we will consider this problem and prove that the Grassmannian indeed enjoys this property for both zero and pole placement problems.

After having resolved the above essential property of a good compactification, we then proceed to examining the intersection ring of this set. The set corresponding to the intersection of polynomial equations can be regarded as a certain element of this ring, and should this be nonzero then the intersection will also be nonzero. The conditions for the above to be nonzero depend upon the structure of the intersection ring, or equivalently, on the topology of the compactified space. For both cases of compensators that are examined in this chapter, the correct compactified space is a Grassmannian. Using the properties of its intersection ring we derive new sufficient conditions for the existence of real (and complex) solutions of the polynomial equations, for both the pole assignment and squaring down problem.

7.2 The Grassmannian as a compactification for the zero and pole placement problems.

Consider again the equations defining the output feedback pole placement problem via constant controllers:

$$\det([I,K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=p(s) \tag{7.1}$$

An obvious compactification of the above equations is to introduce one homogeneous parameter λ and thus we get

$$\det([I,\lambda^{-1}K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=p(s) \tag{7.2}$$

Multiplying equation(7.2) by λ^{p} we obtain

$$\det([\lambda I,K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=\lambda^{p}p(s) \tag{7.3}$$

This new set of equations is an extension of the original ones on the projective space $\mathbb{P}^{mp}(\mathbb{F})$, in which case infinity is represented by the hyperplane corresponding to $\lambda=0$. Hence, the equations defining the solutions at infinity are:

$$\det([0,K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=0$$

From this, we can see that no matter what the system is, the set $\mathcal{J}$ of all [0,K] such that K has not full rank is a subset of the set of solutions at infinity. The set $\mathcal{J}$ is a $mp-m-1$ dimensional subvariety of $\mathbb{P}^{mp}(\mathbb{F})$ and is given by all the matrices of the form

$$\left[\begin{array}{c|cccccc} \mathbf{O}_{pxp} & \begin{matrix} 0 & 0 & \dots & \dots & 0 & 0 \\ 1 & x & \dots & \dots & x & x \\ x & x & \dots & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & \dots & x & x \end{matrix} \end{array}\right]$$

Therefore, although $mp \geq n$ implies that the solution set $\mathcal{S}$ of our equations is nonvoid in the projective space(projective intersection theorem,(chapter 3)), this is nothing new since the points $[0,K] \in \mathfrak{I}$, as we already know, are contained in the solution space, no matter what the system is. These points do not have a nice control theoretic interpretation and thus have to be excluded from the solution space. However, as we have seen, $\mathfrak{I}$, has dimension $mp-m-1$, is independent from the system and ,therefore, cannot be easily excluded from the solution set. For a good compactification, we expect that the solutions at infinity can all be approached by finite solutions and, hence, the dimension of the set of solutions at infinity must be less than that of the set of the finite solutions.

We may homogenise equ(7.1) as

$$\det([I,\Lambda_{diag}{}^{-1}K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=p(s) \tag{7.4a}$$

where Λ_{diag} is given by

$$\Lambda_{diag}=\begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \vdots & \\ \mathbf{0} & & & \lambda_p \end{bmatrix}$$

and, therefore, equ(7.4a) can be written as

$$\det([\Lambda_{diag},K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=\lambda_1\ \lambda_2\ \dots\ \lambda_p\ p(s) \tag{7.4b}$$

and in this way, we compactify our gain space $\mathbb{F}^{pm}$ as the product $\underbrace{\mathbb{P}^m(\mathbb{F})x...x\mathbb{P}^m(\mathbb{F})}_{p}$. We can then inspect that matrices of the form

$$\begin{bmatrix} 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 & x & x & \dots & x \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 & x & x & \dots & x \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 & x & x & x & \dots & x \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & x & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & x & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & x & x & x & \dots & x \end{bmatrix}$$

where the first two rows are equal, constitutes solutions of the homogenised set of equations (7.4b) and it is again independent from the system. Therefore for reasons similar to those of the previous compactification, this one is also inappropriate.

Finally, among all possible homogenisation of our equations the one to be considered, which as will be seen does not have the abnormality of a large set of degenerate points, is given as follows:

$$\det([I,\Lambda^{-1}K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=p(s) \tag{7.5}$$

where Λ is a pxp matrix having as entries p^2 homogenisation variables. By multiplying both sides of (7.5) by $\det(\Lambda)$ we get

$$\det([\Lambda,\ K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix})=\det(\Lambda)p(s)$$

This homogenisation corresponds to a compactification of the set of constant feedback controllers into a Grassmannian and as it has been shown in section 5.5, the above equations may be reduced to two problems: a linear and multilinear. In this way, the solution set of compensators is given by the intersection of two varieties- the Grassmann variety $G_p(\mathbb{F}^{p+m})$ and a linear variety of codimension n. The degenerate points of this intersection are defined as follows:

Definition (7.1) Given a system of p-inputs, m-outputs and n-states, a point of the Grassmannian rowspan[A,K] is called a degenerate point of the system if

$$\det([A,K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}) = 0 \tag{7.6}$$

□

Remark (7.1) As far as the zero placement problem via squaring down is concerned, the set of compensators is already compact and is the Grassmannian $G_p(\mathbb{F}^m)$. In this case, the degenerate points are given by all those rowspan(K) such that

$$\det(K\ N(s))=0 \tag{7.7}$$

□

Next we will see one of the fundamental properties of the Grassmannian as far the two determinantal assignment problems which are considered. This property is, that the Grassmannian does not contain a set of degenerate points which is system independent, a property that was not possessed by all the other inappropriate compactifications. To prove that, we will construct a system with no degenerate points, but first, we require the following definitions:

Definition (7.2) [Gri.1] A curve $\underline{f}(s)=(f_1(s),\ f_2(s),\ \ldots,\ f_r(s))^T$ of $\mathbb{F}^r$ is called degenerate, iff there exist a constant 1xr vector $\underline{u}^T$ such that $\underline{u}^T\ \underline{f}(s)=0$. This amounts to the fact that the curve is contained into an r-1 dimensional linear subspace of $\mathbb{F}^r$.

□

Definition (7.3) [Gri.1] Let $\underline{f}(s)$ be a curve of $\mathbb{F}^r$ then the kth associated curve of $\underline{f}(s)$ symbolised by $\underline{f}_k(s)$ is defined to be

$$\underline{f}_k(s) = \underline{f}(s) \wedge \underline{f}'(s) \wedge \ldots \wedge \underline{f}^k(s)$$

□

Next, we will present a theorem that describes a condition of degeneracy of a curve in terms of the rth associated curve.

Lemma (7.1) A curve f(s) of $\mathbb{F}^r$ is degenerate iff $\underline{f}_{r-1}(s)=0$.

Proof:

(⇒) Since $\underline{u}^T\underline{f}(s)=0$ we must also have that $\underline{u}\ \underline{f}^k(s)=0$ for all $k \geq 0$. Therefore, $\underline{u}$ belongs to the left Kernel of the rxr matrix $F(s)=[\underline{f}(s),\ \underline{f}'(s),\ldots,\ \underline{f}^{r-1}(s)]$. This means that

$$\underline{f}_{r-1}(s)= \det(F(s))=0$$

($\Leftarrow$) consider the maximum $k \leq r\text{-}1$ such that $\underline{f}_{k-1}(s) \neq 0$ and $\underline{f}_k(s)=0$. Then, there exist rational functions $\lambda_1(s)$, $\lambda_2(s)$,..., $\lambda_k(s)$ such that:

$$\underline{f}^k(s)=\lambda_1(s)\ \underline{f}(s) + \lambda_2(s)\ \underline{f}'(s) + \ldots + \lambda_k(s)\ \underline{f}^{k-1}(s)$$

Consider now the derivative of the k-1 associated curve of $\underline{f}(s)$

$$\left(\underline{f}_{k-1}(s)\right)' = \left(\underline{f}(s) \wedge \underline{f}'(s) \wedge \ldots \wedge \underline{f}^{k-1}(s)\right)' = \underline{f}(s) \wedge \underline{f}'(s) \wedge \ldots \wedge \underline{f}^{k-2}(s) \wedge \underline{f}^{k}(s) = \lambda_k(s)\ \ \underline{f}_{k-1}(s)$$

This means that $\underline{f}_{k-1}(s)$ is constant (modulo multiplication by polynomials) and therefore the k dimensional subspace of $\mathbb{F}^r$, $\mathcal{V}=$ colspan$[\underline{f}(s),\ \underline{f}'(s),\ldots,\ \underline{f}^{k-1}(s)]$ is independent of s. Hence, we can choose a constant 1xr vector $\underline{u}$ which annihilates $\mathcal{V}$ from the left. Obviously, for this $\underline{u}$ we have that $\underline{u}\ \underline{f}(s)=0$ which proves that f is degenerate.

□

Consider now $\underline{f}(s)$ to be a curve of degree n ($n \geq p$) that is:

$$\underline{f}(s)=A\ \underline{e}_n(s) \tag{7.8}$$

where $\underline{e}_n(s)=[1,s,s^2,\ldots,\ s^n]^T$ and A is an pxn constant matrix. From the definition of degeneracy, the curve $\underline{f}(s)$ is nondegenerate iff A has full rank. On the other hand from theorem(7.1) f(s) is nondegenerate iff $\underline{f}_{r-1}(s)$ is not identically zero. Therefore

Lemma (7.2) Let $\underline{f}(s)$ be a curve of degree n, given by equation(7.8), then $\underline{f}_{r-1}(s)$ is nonzero iff A has full rank.

□

And now with the help of the above lemma, we can present an example of a strictly proper system of p-inputs, m-outputs and mp-states which does not contain any degenerate points.

Example (7.1) Consider a system whose composite denominator, numerator matrix is the matrix $M(s)=[\underline{e}_n(s),\ \underline{e}_n(s)',\ldots,\ \underline{e}_n(s)^{(p-1)}]$ where $n=m+p-1$. This represents a strictly proper system of p-inputs, m-outputs and mp-states. If this system contained degenerate points, then there would exist a composite (full rank) feedback matrix [A,K] such that det([A,K]M(s))=0, which in turn is equivalent to the fact that the n-1th associate curve of $\underline{e}_n(s)$ is identically zero. By lemma (7.1) [A,K] has not full rank, which is a contradiction. Therefore, the above constructed system does not contain any degenerate points.

□

As far as zero assignment is concerned we can construct the following example.

Example (7.2) Consider a system whose numerator matrix is given by $N(s)=[\underline{e}_n(s), \underline{e}_n(s)', \ldots, \underline{e}_n(s)^{(p-1)}]$ where $n=m-1$. Following similar arguments with the ones of the previous example we can conclude that $\det(KN(s)) \neq 0$ for all rowspan(K) in $G_p(\mathbb{F}^m)$.

□

If n<mp (or δ<p(m-p) for the zero assignment problem) then we cannot find a system with no degenerate points, since the dimension of the set of degenerate points is at least mp-n-1 (p(m-p)-δ-1 for the zero assignment case). What we will be proving, however, is that this dimension is generically mp-n-1. But first, we will look at what is happening when we restrict the degeneracy problem to a Schubert subvariety Grassmannian which is n dimensional. In this case, we expect that the set of degenerate points on this subvariety must be n-n-1=-1 dimensional, that is empty.

Proposition (7.1) There exists a Schubert variety of the type $<a_1,a_2,\ldots a_p>_C$ such that $\sum a_i = n \leq mp$, and a system of p-inputs, m-outputs, n-states whose set of degenerate points do not intersect this Schubert variety.

Proof

Let $n_i = a_i + i$ for $i=1,2,..,p$ and $k_i = n_p - n_{p-i}$ for $i=i,2,\ldots,p-1$. Then consider the system whose composite denominator, numerator matrix M(s) is given by

$$M(s)=\left(\underline{e}(s), \underline{e}(s)^{(k_1)}, \underline{e}(s)^{(k_2)}, \ldots\ldots, \underline{e}(s)^{(k_{p-1})}\right) \tag{7.9}$$

where $\underline{e}(s)=[s^{n_p-1}, s^{n_p-2}, \ldots, s^2, s, 1, 0, 0, \ldots, 0]^T \in \mathbb{R}[s]^{(m+p)\times 1}$. Now, let V_i (for i=1,2,...p) be the n_i dimensional subspace of $\mathbb{C}^{1\times(m+p)}$ which contains all vectors whose all $m+p-n_i$ rightmost coordinates are zero. Then, the flag $V_1 \subset V_2 \subset \ldots \subset V_p$ defines a

Scubert variety of the type $<a_1,a_2,...a_p>_{\mathbb{C}}$ which contains all p-dimensional subspaces of $\mathbb{C}^{m+p}$ whose right Hermite forms $\mathcal{H}(j_1,j_2,...,j_p)$ have the property that $j_i \leq n_i$ for i=1,2,...,p. It is now easy to see that if rowspan[A,K] has a Hermite form possessing the latter property, then $\det([A,K]\ M(s)) \neq 0$ which proves our lemma.

□

Proposition (7.2) There exists a Schubert variety of type $<a_1,a_2,...a_p>_{\mathbb{C}}$ such that $\sum a_i = \delta \leq (m-p)p$ and a numerator matrix of dynamic degree δ, whose degenerate points do not intersect this Schubert variety.

Proof

Let $n_i = a_i + i$ for $i=1,2,..,p$ and $k_i = n_p - n_{p-i}$ for i=i,2,...,p-1. Then consider the system whose numerator matrix N(s) is given by

$$N(s)=\left(\underset{\sim}{e}(s),\ \underset{\sim}{e}(s)^{(k_1)},\ \underset{\sim}{e}(s)^{(k_2)},\ ......,\ \underset{\sim}{e}(s)^{(k_{p-1})}\right) \tag{7.10}$$

where $\underset{\sim}{e}(s)=[s^{n_p-1}, s^{n_p-2},...., s^2, s, 1, 0, 0,....,0]^T \in \mathbb{R}[s]^{m\times 1}$. Now, let V_i (for i=1,2,...p) be the n_i dimensional subspace of $\mathbb{C}^{1\times m}$ which contains all vectors whose all $m-n_i$ rightmost coordinates are zero. Then, following the same lines with the previous lemma we get that $\det(KN(s)) \neq 0$ for all rowspan(K) in the Schubert variety defined by the flag $V_1 \subset V_2 \subset ... \subset V_p$.

□

Theorem (7.1) The set of all systems of p-inputs, m-outputs and n-states that do not intersect the Schubert variety, $\mathcal{V}$, of type $<a_1,a_2,...a_p>_{\mathbb{C}}$, $\sum a_i = n \leq mp$, defined by the flag $V_1 \subset V_2 \subset ... \subset V_p$ of proposition(7.1) is a nonvoid Zarisky open subset and this is, thereforea a generic system does not have degenerate points on $\mathcal{V}$.

Proof

Consider the set $\mathfrak{D}$ of all pairs $(x, S) \in \mathcal{V} \times \Sigma_{m,p}^{n}$, such that x is a degenerate point of S. Then $\mathfrak{D}$ is a variety defined by equation (7.6). Therefore, the image of the projection on the second coordinate, $P_2(\mathfrak{D})$, is a subvariety of $\Sigma_{m,p}^{n}$. The set $P_2(\mathfrak{D})$ contains all systems S that do have at least one degenerate point on $\mathbb{S}$, and is a proper subset of $\Sigma_{m,p}^{n}$ because of lemma(7.2). Hence, the set of systems that do not have any degenerate points on $\mathbb{S}$, that is the set $\Sigma_{m,p}^{n} - P_2(\mathfrak{D})$, is a nonvoid Zarisky open subset of $\Sigma_{m,p}^{n}$. This proves the fact that a generic system of $\Sigma_{m,p}^{n}$ does not have a degenerate point on $\mathbb{S}$.

□

Following the same arguments, we can prove a similar theorem for the zero placement case.

Theorem (7.2) The set of all mxp numerator matrices of degree δ, that do not intersect the Schubert variety, $\mathbb{S}$, of type $<a_1,a_2,...a_p>_{\mathbb{C}}$, $\sum a_i = \delta \leq (m\text{-}p)p$, defined by the flag $V_1 \subset V_2 \subset ... \subset V_p$ of the proposition(7.2) is a nonvoid Zarisky open subset. Therefore a generic numerator does not have degenerate points on $\mathbb{S}$.

□

Using the above results, we will show that for the Grassmannian compactification, the set of the degenerate points of the pole placement map of a system S of p-inputs, m-outputs and n-states has the correct dimension mp-n-1 (the dimension of the Grassmann variety minus the number of linear hypersurfaces that define the equations of degenerate points)

Corollary(7.1) The variety $\mathfrak{D}(S) \subset G_p(\mathbb{C}^{p+m})$ of all degenerate points of a generic system S of p-inputs, m-outputs , n-states has dimension mp-n-1.

Proof

Since $\mathfrak{D}(S)$ is defined by n+1 equations, its dimension has to be greater than or equal to mp-n-1. If it was strictly greater, then consider the Schubert variety, $\mathcal{V}$ of type $<a_1,a_2,...a_p>_{\mathbb{C}}$ of proposition(7.1). Then by the projective intersection theorem(Ch3), we get

$$\dim(\mathfrak{D}(S) \cap \mathcal{V}) \geq \dim \mathfrak{D}(S) + \dim(\mathcal{V}) - mp = (mp - n - 1 + k) + n - mp = k - 1 \geq 0$$

which means that $\mathfrak{D}(S)$ intersects $\mathcal{V}$ contradicting theorem (7.2). Therefore, $\dim(\mathfrak{D}(S))$=mp-n-1.

□

Similarly, for the zero assignmnent problem

Corollary (7.2) The variety $\mathfrak{D}(N) \subset G_p(\mathbb{C}^m)$ of all degenerate points of a numerator matrix of a generic system S of p-inputs, m-outputs , n-states has dimension $(m-p)p-\delta-1$.

□

Example(7.3) Consider the following 4x2 numerator matrix of degree 2:

$$N(s)=\begin{bmatrix} s & 2 \\ 0 & s \\ 2s & 1 \\ 1 & 0 \end{bmatrix}$$

From the previous corollary we expect $\mathfrak{D}(N)$ to have dimension 2x2-2-1=1. Indeed, the set of all degenerate points for N(s) is given by the set of all solutions of $\det(KN(s))=0$ where rowspan(K) belongs to the Grassmannian $G_2(\mathbb{C}^4)$. These equations are:

(i) the only one QPR for $G_2(\mathbb{C}^4)$ in $P\mathbb{C}^5$

$$k_1k_6-k_2k_5+k_3k_4=0$$

(ii) the linear equations

$$k_1-2k_4=0$$
$$3k_2-k_5=0$$
$$-2k_3+k_6=0$$

Solving the linear equations ,with respect to k_1,k_5,k_6 , and substituting to the quadratic we get that $\mathfrak{D}(N)$ is the set of all $(2k_4,k_2,k_3,k_4,3k_2,2k_3)\in P\mathbb{C}^5$ such that $5k_3k_4-3k_2^2=0$. This is a one dimensional subvariety of $\mathbb{P}^5(\mathbb{C})$.

□

7.3 The general Philosophy behind the search of real solutions.

In sec. 5.5, it has been seen that CPAP may be reduced to two problems; a linear and a multilinear. This way, the solution set (of compensators) is given by the intersection of two real varieties; A variety which corresponds to the solution set of the multilinear problem and $L\mathbb{R}$,the linear variety which is the generic solution set of the linear problem (both varieties are considered as subvarieties of $\mathbb{P}^{\sigma-1}(\mathbb{R})$. For complex varieties there is intersection theory , however there is no intersection theory for real varieties due to the fact that $\mathbb{R}$ is not algebraically closed. The method we use to examine whether there exist real solutions of a real polynomial equation ,has been suggested in [Kar.1] for the above intersection problem.That is,we consider the complex solutions of the real polynomial equation and if their number is odd then one of them must be real (since the solutions occur in conjugate pairs).

In our case, the complex solutions of the multilinear problem are given by the set $G_p(\mathbb{C}^{p+m})$ and the solution set of the linear problem is L$\mathbb{C}$; the subvariety of $\mathbb{P}^{\sigma-1}(\mathbb{C})$ given by the same set of equations as L$\mathbb{R}$. The two varieties,$G_p(\mathbb{C}^{p+m})$ and L$\mathbb{C}$ have dimensions mp and σ-1-n,respectively. The set of complex solutions of our problem is $G_p(\mathbb{C}^{p+m}) \cap L\mathbb{C}$ (it is nonvoid iff $mp \geq n$). This set is finite, iff the two varieties have complementary dimensions.If this is not the case, then we replace $G_p(\mathbb{C}^{p+m})$ with a Schubert subvariety $<a_1,a_2,...a_p>_C$ satisfying the above dimension requirement (that is $\sum a_i=n$). Finally, if the set $<a_1,a_2,...a_p>_C \cap L\mathbb{C}$ has odd parity then we have a real solution. In fact this parity is $A(a_1,a_2,...a_p)_C$,the order of $<a_1,a_2,...a_p>_C$.All the above arguments lead to the theorem stated below:

Theorem(7.3) [Kar.1] If there exist $a_1,a_2,...a_p:0 \leq a_1 \leq a_2 \leq ... \leq a_p \leq m$, $\sum a_i=n \leq mp$ and $A(a_1,a_2,...a_p)_C$ is odd then, to a generic system of p-inputs, m-outputs, n-states we can assign almost any set of complex conjugate poles by real output feedback.

Proof

For every $a_1,a_2,...a_p:0 \leq a_1 \leq a_2 \leq ... \leq a_p \leq m$, $\sum a_i =n \leq mp$ consider a Schubert variety $\mathcal{V}$ of the type $<a_1,a_2,...a_p>_C$ that do not intersect the base locus (the set of degenerate points) of a generic system of p-inputs, m-outputs and n-states (this can always be done as indicated by theorem (7.2)). Then consider the map :

$$\chi_{\mathcal{V}} : \mathcal{V} \longrightarrow P\mathbb{C}^n$$

defined by

$$\chi_{\mathcal{V}}([c,\underline{k}])=[c,\underline{k}]\ P_S$$

for every multivector $[c,\underline{k}] \in \mathcal{V} \subseteq G_p(\mathbb{C}^{p+m})$ and where P_S is a full rank plucker matrix of a generic system. One can easily see that $\chi_{\mathcal{V}}$ is the restriction of the complex pole placement map on the Schubert variety $\mathcal{V}$. By theorem(3.8) $\chi_{\mathcal{V}}$ is onto, which proves the well known result that, for a generic system, any complex pole polynomial can be assigned via complex output feedback when $mp \geq n$. Additionally, by theorem(3.8), $\chi_{\mathcal{V}}$ is a finite to one map which means that the parity $\#\chi_{\mathcal{V}}^{-1}(\underline{p})$ is equal to the degree $\deg(\mathcal{V})=A(a_1,a_2,...a_p)_{\mathbb{C}}$ for all $\underline{p} \in \mathbb{P}^n(\mathbb{C})$.

Consider now the real element $[1, \underline{p}^T]^T=[1, p_n,...,p_1]^T \in \mathbb{P}^n(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{C})$ where $[1, \underline{p}^T]^T$ is the coefficient vector of real pole polynomial p(s). Then according to the previous analysis

$$\#\chi_{\mathcal{V}}^{-1}\left([1, \underline{p}^T]^T\right)=A(a_1,a_2,...a_p)_{\mathbb{C}}$$

The inverse image $\chi_{\mathcal{V}}^{-1}\left([1, \underline{p}^T]^T\right)$ contains all feedback controllers in $\mathcal{V}$ that assign the pole polynomial p(s), or equivalently it is the solution of

$$[1,\underline{k}^T]^T\, P_S = [1, \underline{p}^T]^T \in \mathbb{P}\mathbb{R}^n$$

with respect to $[1,\underline{k}^T]^T \in \mathcal{V}$. Since the above equation has real coefficients, the controllers in $\chi_{\mathcal{V}}^{-1}\left([1, \underline{p}]\right)$ occur in conjugate pairs. Therefore if $A(a_1,a_2,...a_p)_{\mathbb{C}}$ is odd then we have a real element in $\chi_{\mathcal{V}}^{-1}\left([1, \underline{p}]\right)$ which in turn is a real controller that assigns p(s).

□

Similarly, for the zero assignment case, we have the following theorem

Theorem (7.4)[Kar.1] If there exist $a_1,a_2,...a_p{:}0 \leq a_1 \leq a_2 \leq \cdots \leq a_p \leq m\text{-}p$, $\sum a_i = \delta \leq (m\text{-}p)p$ and $A(a_1,a_2,...a_p)_{\mathbb{C}}$ is odd then, to a generic system of p-inputs, m-outputs, n-states we can assign almost any set of complex conjugate zeros by real squaring down.

Proof

For every $a_1,a_2,...a_p{:}0 \leq a_1 \leq a_2 \leq \cdots \leq a_p \leq \delta$, $\sum a_i = \delta \leq (m\text{-}p)p$ consider a Schubert variety $\mathcal{V}$ of the type $<a_1,a_2,...a_p>_{\mathbb{C}}$ that do not intersect the base locus (the set of degenerate points) of the numerator of a generic system of p-inputs, m-outputs and n-states (this can always be done as indicated by theorem (7.2)). Then the proof follows along the same lines as that of theorem (7.3).

□

The requirement that the Schubert variety of type $<a_1,a_2,...a_p>_C$ does not intersect the set of degenerate points of the system is essential. If there was a degenerate point on the Schubert variety, then the number of compensators on this variety assigning certain pole polynomial is not $A(a_1,a_2,...a_p)_C$, the degree of the Scubert variety, but something less than this. This is illustrated in the following examples.

Example (7.4) Consider the variety $\mathcal{V}$: $xy^2=z^3$ of the projective space $\mathbb{P}^2(\mathbb{C})$. This is a one dimensional irreducible variety which has degree three. Next consider the following two cases of linear projections P

(i) $P(x,y,z)=(x,y) \in \mathbb{P}^1(\mathbb{C})$

The degenerate points of P are given by all $(x,y,z) \in \mathbb{P}^2(\mathbb{C})$ such that $P(x,y,z)=(0,0)$ and , therefore, we have only one degenerate point namely, the point (0,0,1) which does not belong to $\mathcal{V}$. Consider now the linear subspace of $\mathbb{P}^2(\mathbb{C})$, $\mathcal{L}=P^{-1}(a,b)$ where (a,b) is fixed. The intersection $\mathcal{L} \cap \mathcal{V}$ contains all the points $(a,b,z) \in \mathbb{P}^2(\mathbb{C})$ such that $ab^2=z^3$ and therefore, we have as many solutions as the cubic roots of ab^2, ie. three. In this case, the number of points in $\mathcal{L} \cap \mathcal{V}$ is equal to the degree of $\mathcal{V}$ and this is because we had no degenerate points on $\mathcal{V}$.

(ii) $P(x,y,z)=(x,z) \in \mathbb{P}^1(\mathbb{C})$

Here, we have one degenerate point, the point (0,1,0) which , in fact , belongs to $\mathcal{V}$. The linear space $\mathcal{L}=P^{-1}(a,b)$ now intersects $\mathcal{V}$ at all points $(a,y,b) \in \mathbb{P}^2(\mathbb{C})$ such that $ay^2=b$ for fixed a and b, ie. $\mathcal{L} \cap \mathcal{V}$ contains two points. In this case the intersection of the variety $\mathcal{V}$ with the linear subspace of complementary dimension, $\mathcal{L}$, contains less points than the degree of $\mathcal{V}$.

□

Example (7.5) Consider the Schubert subvariety $\mathbb{S}$ of $G_2(\mathbb{C}^4)$ of type <0,2> which contains all elements of the form

$$\text{rowspan}\begin{bmatrix} x & 0 & 0 & 0 \\ x & x & x & x \end{bmatrix}$$

Then if we embed $\mathbb{S}$ into $\mathbb{P}^5(\mathbb{C})$ via the Plucker embedding, we can view $\mathbb{S}$ as the set of all $(x_1,x_2, x_3,x_4,x_5,x_6) \in \mathbb{P}^5(\mathbb{C})$ such that $x_4= x_5= x_6=0$. Therefore, $\mathbb{S}$ is a linear two dimensional variety with degree one (since it is linear). Now, according to theorem(7.3), the set of all gains in $\mathbb{S}$ assigning a generic pole polynomial to a system must be equal to the degree of $\mathbb{S}$, ie.1, provided the system does not have a degenerate point on $\mathbb{S}$. Indeed, consider the system

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s & 1 \\ 0 & s \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which does not have any degenerate points on $\mathbb{S}$. Then, for every monic polynomial $p(s)=s^2+\alpha s+\beta$, we have only one element of $\mathbb{S}$, namely,

$$\text{rowspan}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & -\beta \end{bmatrix}$$

which assigns p(s) to the system. On the other hand, if we consider the system

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then, this has a degenerate point in $\mathbb{S}$, namely,

$$\text{rowspan}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We can easily see that a generic polynomial cannot be assigned to this system via a feedback controller belonging to $\mathbb{S}$. This is one of the nongeneric cases of theorem(7.3) where the number of compensators assigning a polynomial is not equal to the degree of the Schubert variety and this is because of the existence of a degenerate point on this variety.

□

Thus, according to theorems (7.3),(7.4) in order to decide whether there exist a solution to our frequency assignment problems, we have to check if one of the numbers $A(a_1,a_2,...a_p)_C$ is odd under the conditions:

$$0 \le a_1 \le a_2 \le ... \le a_p \le m \quad \text{and} \quad \sum a_i = \delta \tag{7.11}$$

for the pole placement problem, or the conditions

$$0 \le a_1 \le a_2 \le ... \le a_p \le m\text{-}p \quad \text{and} \quad \sum a_i = n\text{-}p \tag{7.12}$$

for the zero placement problem.

A formula for these numbers can be found in [Hod.1].This involves a large number of factorials and has to be computed for every p-tuple satisfying (7.11).Due to this fact for relatively large p,m,n theorem (7.3) is rather difficult to be used.

In the following sections we attempt to simplify theorems (7.3) and (7.4) by providing an equivalent testable form. Note that the numbers $A(a_1,a_2,...a_p)_C$ are odd iff the numbers $A(a_1,a_2,...a_p)_C \bmod 2$ do not vanish. All these numbers under the conditions (7.11) appear in a single formula occuring in the $\mathbb{Z}_2$-cohomology ring of $G_p(\mathbb{R}^{p+m})$. The structure of this ring is described next.

7.4. The cohomology of complex and real Grassmann varieties.

The intersection theory on the complex Grassmann variety $G_p(\mathbb{C}^{p+m})$, known as Schubert calculus [Klei.1], can be regarded by the means of algebraic topology as the cohomology ring of $G_p(\mathbb{C}^{p+m})$ with coefficients in $\mathbb{Z}$, symbolised by $H^*(G_p(\mathbb{C}^{p+m});\mathbb{Z})$.This ring is a positively graded ring up to 2pm, the real dimension of $G_p(\mathbb{C}^{p+m})$.The i-th graded component of $H^*(G_p(\mathbb{C}^{p+m});\mathbb{Z})$ is a $\mathbb{Z}$-module and is symbolised by $H^i(G_p(\mathbb{C}^{p+m});\mathbb{Z})$.Every subvariety ,X ,of $G_p(\mathbb{C}^{p+m})$ of codimension n appear as an element $x \in H^{2n}(G_p(\mathbb{C}^{p+m});\mathbb{Z})$ and the interesection between varieties can be seen as the multiplication (cup product) of the ring. Particularly the Schubert varieties $<a_1,a_2,...a_p>_C$ correspond to elements $\{m\text{-}a_p,m\text{-}a_{p\text{-}1},...,m\text{-}a_1\}_C$ called Schubert cocycles. Every $H^i(G_p(\mathbb{C}^{p+m});\mathbb{Z})$ where i=even is freely generated by its

Schubert cocycles(whereas for i=odd the i-th cohomology module vanishes) in the following way[Klei.1]:

Lemma (7.3) (basis theorem)Let X be a subvariety of $G_p(\mathbb{C}^{p+m})$ of codimension n and x its corresponding cohomology class in $H^{2n}(G_p(\mathbb{C}^{p+m});\mathbb{Z})$, then x can be written as:

$$x=\sum\delta(x;a_1,a_2,...a_p)_C\{a_1,a_2,...a_p\}_C \tag{7.13}$$

where the sum is taken for all $a_1,a_2,...a_p$ satisfying (7.11) and $\delta(x;a_1,a_2,...a_p)_C$ is the number of points with multiplicity in the intersection $X\cap<a_1,a_2,...a_p>_C$.

□

Lemma (7.3) suggests that we may find a specific element of the cohomology ring whose expansion (7.13) into the Schubert basis has as coefficients all the orders of the theorem (7.3). Indeed this element is c_1^n the n-th power of the first Chern class, $c_1=\{0,0,...,1\}_C$, of the complex Grassmannian.

Theorem (7.5) The n-th power of the element $\{0,0,...,1\}_C$ can be written as

$$\{0,0,...,1\}_C^n=\sum A(a_1,a_2,...a_p)_C\{a_1,a_2,...a_p\}_C \tag{7.14}$$

where the sum is taken for all $a_1,a_2,...a_p$ satisfying (7.11).

Proof

By Lemma (7.3) the coefficient of $\{a_1,a_2,...a_p\}_C$ in the expansion (7.14) of $c_1^n=\{0,0,...,1\}_C^n$ is given by the number of points of the intersection $X\cap<a_1,a_2,...a_p>_C$ where X is the variety corresponding to the cohomology class c_1^n. But the variety corresponding to c_1 is $<m-1,m,...,m>_C$ and so $\bigcap_{i=1}^{n}<m-1,m,...,m>_C$ is the one that corresponds to c_1^n. Therefore the required coefficient is the order $A(a_1,a_2,...a_p)_C$.

□

To apply theorem (7.3) we would prefer to have a formula similar to (7.14) but with coefficients reduced mod2. Such a formula appears in the cohomology ring of the real Grassmannian $G_p(\mathbb{R}^{p+m})$ with coefficients in $\mathbb{Z}_2$, symbolised by $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$, [Cher.1,Hill.1].This ring provides us with a mod2-intersection theory on $G_p(\mathbb{R}^{p+m})$, it is a positively graded ring up to dimension mp and the subvarieties of codimension n correspond to elements of the n-th graded component $H^n(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ -which is actually a $\mathbb{Z}_2$ vector space. Particularly the Schubert varieties $<a_1,a_2,...a_p>_{\mathbf{R}}$ correspond to elements $\{m-a_p,m-a_{p-1},...,m-a_1\}_{\mathbf{R}}$ called Schubert cocycles , these cocycles form a $\mathbb{Z}_2$-basis for the graded component they belong. All of these can be summarised in the following equations:

$$H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)=\bigoplus_{i=0}^{mp} H^i(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2) \tag{7.15}$$

$$H^n(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)=\oplus\,\mathbb{Z}_2\{a_1,a_2,...a_p\}_{\mathbf{R}} \tag{7.16}$$

where $a_1,a_2,...a_p$ satisfy (7.11).

Remark(7.2) An analytic description of the multiplication (cup product) of the ring $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ can be found in [Cher.1, Hill.1, Ber.1].

□

It is important to note that the multiplication rule in $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ is the same as the one of $H^*(G_p(\mathbb{C}^{p+m});\mathbb{Z})$ the only difference being that the numbers appearing are reduced mod2 (see for example [Ber.1], remark, page 132). This and theorem(7.6) lead us to the conclusion that the expansion of w_1^n where $w_1=\{0,...,0,1\}_{\mathbf{R}}$ is

$$w_1^n=\sum A(a_1,a_2,...a_p)_{\mathbf{R}}\{a_1,a_2,...a_p\}_{\mathbf{R}} \tag{7.17}$$

where the sum is taken for all Schubert symbols satisfying (7.11) and

$$A(a_1,a_2,...a_p)_{\mathbf{R}}=A(a_1,a_2,...a_p)_{\mathbf{C}}\text{mod}2 \tag{7.18}$$

The Schubert cocycle w_1 is the generator of $H^1(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ and is called the first Whitney class of the Grassmann variety $G_p(\mathbb{R}^{p+m})$.The height of this class will play a crucial role in the solution of the problem.

Definition (7.1) .The number h(m,p) is defined to be the maximum exponent "a" such that w_1^a is non zero in $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ and is called the height of the first Whitney class of the Grassmannian $G_p(\mathbb{R}^{p+m})$.

□

Remark (7.3) .Note that the ring $H^*(X;R)$ has zero-divisors and thus the above definition makes sense.

7.5 New sufficient conditions for generic pole, zero assignment

The above results provide alternative means for testing equation (7.17) incorporates all the conditions described in theorem (7.3) and leads to the following main result

Theorem (7.6) The following two statements are equivalent:

(a) $\exists\, a_1,a_2,\ldots a_p$: $0 \le a_1 \le a_2 \le \ldots \le a_p \le m$, $\sum a_i = n$ and $A(a_1,a_2,\ldots a_p)_C$=odd

(b) $h(p,m) \ge n$

Proof:

Considering (7.17) and the fact that all cocycles $\{a_1,a_2,\ldots a_p\}_R$ such that $\sum a_i = n$ are linearly independent in $H^n(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$ we have:

w_1^n is non zero iff $\exists\, \{a_1,a_2,\ldots a_p\}_R$ such that $\sum a_i = n$ and $A(a_1,a_2,\ldots a_p)_R = 1$

The last statement due to (7.18) is equivalent to

w_1^n is non zero iff $\exists \{a_1,a_2,\ldots a_p\}_C$ such that $A(a_1,a_2,\ldots a_p)_C$=odd and $\sum a_i = n$

On the other hand,w_1^n is non zero iff $h(p,m) \ge n$. Hence,the theorem is proved.

□

Similarly for the zero placement problem:

Remark (7.4) The following two statements are equivalent:

(a) $\exists\, a_1,a_2,\ldots a_p$: $0 \le a_1 \le a_2 \le \ldots \le a_p \le m\text{-}p$, $\sum a_i = \delta$ and $A(a_1,a_2,\ldots a_p)_C$=odd

(b) $h(p,m\text{-}p) \ge \delta$

□

Corollary (7.3) A sufficient condition for the existence of real solution of the output feedback pole placement problem for a generic system with p-inputs,m-outputs and n-states is $h(p,m) \geq n$.

□

The above can be readily deduced by combining theorems (7.3) and(7.6). For the zero assignment, we have the following result.

Corollary (7.4) A sufficient condition for the existence of real solution of the squaring down zero placement problem for a generic system of p-inputs, m-outputs and δ Forney dynamical order is $h(p,m\text{-}p) \geq \delta$.

□

The above corollary follows easily by a combination of theorem (7.4) and remark(7.4). Furthermore the height result is the best possible result, as far as odd intersections are concerned. This is established by the following result indicating the limits of the present approach.

Theorem(7.7) The following two statements are equivalent:

(a) $\exists$ an n dimensional subvariety $\mathfrak{B}$ of $G_p(\mathbb{C}^{p+m})$ such that $\deg(\mathfrak{B})$=odd

(b) $h(p,m) \geq n$

Proof

(b)$\Rightarrow$(a) Obvious from theorem(7.6)

(a)$\Rightarrow$(b) By Lemma(7.3) the cohomology x class of $\mathfrak{B}$ is an element of $H^{2(mp-n)}(G_p(\mathbb{C}^{p+m};\mathbb{Z})$ and can be expanded as:

$$x=\sum \delta(x;\ a_1,a_2,\ldots a_p)_{\mathbb{C}}\{m\text{-}a_p,m\text{-}a_{p-1},\ldots,m\text{-}a_1\}_{\mathbb{C}} \tag{7.19}$$

where the sum is taken for all $a_1,a_2,\ldots a_p$: $0 \leq a_1 \leq a_2 \leq \ldots \leq a_p \leq m$, $\sum a_i = n$, and

$$\delta(x;\ a_1,a_2,\ldots a_p)_{\mathbb{C}} = \#(\mathfrak{B} \cap \langle m\text{-}a_p,m\text{-}a_{p-1},\ldots,m\text{-}a_1 \rangle_{\mathbb{C}})$$

multiplying now (7.14) with (7.19) we get

$$\deg(\mathcal{G})=\sum \delta(x;\, a_1,a_2,\ldots a_p)_C A(a_1,a_2,\ldots,a_p)_C$$

Therefore if $\deg(\mathcal{G})$ is odd, then at least one of the numbers $A(a_1,a_2,\ldots,a_p)_C$ must be odd, and hence by theorem(7.7) $h(p,m)$ must be $\geq n$.

□

The set up based on the cohomology ring allows the unification of the study of CPAP for both complex and real cases. In fact:

Remark(7.5) In both complex and real cases the intersections $G_p(\mathbb{C}^{p+m}) \cap L\mathbb{C}$ and $G_p(\mathbb{R}^{p+m}) \cap L\mathbb{R}$ may be represented as certain elements of the corresponding cohomology rings; the existence of intersection is then reduced to whether these elements are nonzero.In particular:

1)The set of all complex gains (infinite and finite) generically solving our problem is $G_p(\mathbb{C}^{p+m}) \cap L\mathbb{C}$ and is represented in $H^*(G_p(\mathbb{C}^{p+m});\mathbb{Z})$ by $\{0,\ldots,1\}^n_C$.For this element to be nonzero n has to be less than or equal to height of the class $\{0,0,\ldots1\}_C$ which is mp [Ber.1]. So,we have generically a complex solution iff $n \leq mp$. This provides an alternative proof of the well known result [Broc.1, Her.1, Gia.1]. Similarly,the corresponding result [Kar.3] for zero assignment is established.

2)The set of all real gains (both infinite and finite) generically solving the problem is $G_p(\mathbb{R}^{p+m}) \cap L\mathbb{R}$.This intersection (considered reduced mod2 since there is not a $\mathbb{Z}$-intersection theory as in the complex case) is represented by w_1^n in $H^*(G_p(\mathbb{R}^{p+m});\mathbb{Z}_2)$.The height of the class w_1 is $h(m,p)$.So we have generically a real solution if $n \leq h(p,m)$.It is worth pointing out that in the real case the conditions based on the height are only sufficient and not necessary and this because we use mod2-reduced results.

□

It has been shown that the height $h(p,m) \geq n$ is a useful criterion for pole assignability and the computation of $h(p,m)$ is examined in this section. One of the basic results on the height is the following:

Lemma (7.4) [Sto.1] If $1 < p \leq m$ and 's' is such that $2^s \leq m+p-1 < 2^{s+1}$ then the height of the first Whitney class of $G_p(\mathbb{R}^{p+m})$ is :

$$h(p,m)=\begin{cases} 2^{s+1}-2 & \text{if } p=2 \text{ or if } p=3 \text{ and } m+p=2^s+1 \\ 2^{s+1}-1 & \text{otherwise} \end{cases}$$

□

Remark (7.6) We can always find such an 's' ; it is actually the integral part of $\log_2(m+p-1)$.For example for p=3 , m=8 we have that m+p-1=10 so s=3 , thus h(p,m)=16−1=15 and the sufficient condition becomes $n \leq 15$.

Using lemma (7.4) we can easily see that $m+p-1 \leq h(p,m)$ holds true (apart from the case where p=2 and $m=2^{s+1}-2$); combining this with the main result of [Ber.1] we derive the following lemma which helps us to establish links between the various criteria.

Lemma (7.5) For the h(p,m) we have that:

$$m+p-1 \leq h(p,m) \leq mp$$

The right equality holds true iff min(m,p)=1 or min(m,p)=2 and $\max(m,p)=2^r-1$.

□

Remark (7.7) Kimuras result [Kim.1] is an immediate cosequence of Lemma (7.5) and Corollary (7.3).

□

Remark (7.8) Note that since $LScat(p,m) \geq h(p,m)$ it may seem that a criterion based on LScat [Byr.1] is stronger than the height .However the LScat criterion [Byr.1] has been proved only for real poles whereas the one based on the height is for any symmetric set of poles.Thus as far as the generic pole assignment of generic sets of poles is concerned the height criterion is stronger.

Remark (7.9) So far, we have treated only strictly proper systems (as far as pole placement is concerned). The case of proper systems can be dealt with similarly, the only difference being that we may get degenerate points which are finite. However, all theorems of this chapter can be formulated and stated almost word by word and with only minor modifications for the case of proper systems and the conditions will be the same.

□

7.6 Conclusion

The study of static output feedback and squaring down problems have been the main objects of this chapter. The work here provides an extension of the previous results in [Kar.1] [Kar.3] by establishing the alternative topological aspects based on the use of cohomology rings for the derivation of the sufficient conditions for existence of real solutions. The advantage of this new approach is that it provides a framework for testing the solvability conditions by computations of height rather than the ad hoc methodology based on the factorial formulas [Kar.1]. The present topological framework is based on the existence of an odd intersection on an appropriate compactification and as such, it has certain restrictions. An alternative methodology reffered to as global linearisation will be presented in chapter 10.

CHAPTER 8. Pole Assignment by PI and BDO Controllers

8.1 Introduction.

As it has been seen in the previous chapter, if the problem of arbitrary pole assignment by constant output feedback, or constant precompensation is solvable then we must have that $mp \geq n$. If this is not the case then to get arbitrary pole assignability we need to use a more complex family of controllers. The family of Proportional plus Integral (PI) controllers is the next more complex family (as far as dynamic complexity is concerned), within the general family of proper controllers, since they are defined in terms of two matrices and their poles are fixed. The family of PI controllers is widely used, especially in the area of process control, due to its inherent characteristics for steady state tracking and disturbance rejection. Although the design of single input single output (SISO) PI controllers has been well addressed [Mor.1] as far as tuning the parameters using various rules, the potential of the multivariable PI controllers for solving problems such as pole asignment and stabilisation, has received little attention with the exception of some state space based results which try to transfer known results from the constant pole assignment to the PI case [Ser.1],[Nov.1]. Here, one of the purposes of this chapter is to address the pole assignment by PI controllers within the DAP framework as it was presented in Chapter5. Secondly, we will examine the pole placement problem for a more general family of controllers the Observability index Bounded Dynamics controllers (OBD) once more within the DAP framework.

Although as we saw in chapter 5 both cases (PI and OBD) can be reduced to a constant DAP, that is a problem similar to the constant output feedback problem, the results of Chapter 7 concerning solvability, cannot be directly applied. This happens mainly because the corresponding linear subproblems of the constant DAPs, arising from the PI or OBD pole assignment, have a special structure, different from the one of the output feedback pole placement problem. The determination of solvability conditions is closely related to the calculation of the rank of the Plucker matrices of the linear subproblems as well as with the study of intersections on a appropriately compactified controller space. The compactification we present here, for both the PI and OBD case, is the Grassmann variety. This parametrisation arises from the reduction of the problem to a constant DAP, and therefore the infinite PI or OBD controllers obtain a similar meaning with those of the constant case. However, as we will see, the Grassmannian is not the best compactification for the PI and OBD cases and the sufficient conditions for the generic solvability of the problem will be derived via affine methods.

8.2 The parametrization of PI and OBD controllers as Grassmannians and decomposition of the problem.

8.2.1 Introduction

The parametrisation of PI and OBD feedback controllers is essential for the examination of pole assignment problem. A parametrisation provides us with a description of the controllers in a form convenient for topological, algebrogeometrical or other (depending upon the nature of the parametrisation) considerations. Since the pole placement problem is expressed as an intersection problem of varieties (see chapters 4,5), we may compactify the set of controllers as a projective variety in order to use the rich algebraic and topological intersection theory for such varieties. As it was explained in section 3.5, a good compactification must naturally arise from the specific problem and is done by introducing some elements representing "infinity" (see section 6.4). As we will see, both PI and OBD controllers can be naturally parametrised (and compactified at the same time) as Grassmann varieties embedded in an appropriate projective space via the Plucker embedding. In this way, the set of feedback controllers (finite and infinite), which assign a certain closed loop polynomial, can be taken as an intersection of the Grassmann variety with a linear one, in the above projective space.

8.2.2 Parametrisation of the family of PI controllers

A first look at the structure of PI controllers gives the impression that a first rough parametrisation of them is provided by the affine space $\mathbb{R}^{p\times 2m}$, since the free parameters are K_0 ,K_1. However the special structure of (5.13a) can give us a better insight for this parametrisation. In fact, equation (5.13a) may be equivalently expressed as

$$f(s)=\det\left\{[I_p\ ,\ K_0\ ,K_1]\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix}\right\} \tag{8.1}$$

Note that a row transformation of $[I_p,K_0,K_1]$ yields the same closed loop polynomial modulo multiplication by constants. Thus, it is the rowspan of $[I_p,K_0,K_1]$ and not the matrix $[I_p,K_0,K_1]$ which defines the closed loop poles. This motivates us to view the PI controllers of p-inputs m-outputs, as a subset of the Grassmann manifold $G_p(\mathbb{R}^{p+2m})$. The elements of this manifold are the p dimensional vector spaces defined by rowspan[A,B,C], where $A \in \mathbb{R}^{pxp}$ and $B,C \in \mathbb{R}^{pxm}$ and correspond to generalised PI controllers of the form [sA,sB+C] .We classify the elements of $G_p(\mathbb{F}^{p+2m})$ where $\mathbb{F}$ is $\mathbb{R}$, or $\mathbb{C}$ into two complementary classes:

Definition(8.1) We may define the following two classes of PI controllers:

8.1(a) **Regular PI controllers (RPI)**: These are PI controllers of p-inputs ,m-outputs and are defined as elements of $G_p(\mathbb{F}^{p+2m})$ of the form rowspan[A,B,C] such that rank(A)=p. These correspond to controllers $C(s)=[sI_p]^{-1}[sK_0+K_1]$, the whole family is denoted by $\mathcal{C}^r_{pi}$, and may be classified to:

(i) **Full dynamics RPI controllers (FRPI)** defined by the additional condition that rank(C)=p and corresponding to $C(s)=[sI_p]^{-1}[sK_0+K_1]$ with $\text{rank}(K_1)=p$.

(ii) **Reduced dynamic RPI controllers (RRPI)**, when $\text{rank}(K_1) < p$.

8.1(b) **Infinite PI controllers (IPI)**: which is the complement of PI in $G_p(\mathbb{F}^{p+2m})$ and are also the elements of $G_p(\mathbb{F}^{p+2m})$ of the form rowspan[A,B,C], such that $\text{rank}(A) < p$. These controllers are refered to as infinite gain PI controllers for reasons discussed in Ch.6, and may be further classified along similar lines to RPI. This family of controllers is denoted by $\mathcal{C}^{in}_{pi}$.

□

Remark(8.1) (see Ch.6) In order to justify briefly the definition of infinite PI controllers, we consider a sequence $\mathcal{A}_n$=rowspan$[I_p,K_n]$ in $G_p(\mathbb{F}^{p+2m})$ such that $K_n \to \infty$ in the usual sense. Then, since the Grassmannian is compact, $\mathcal{A}_n$ must converge to some

$\mathcal{A}$=rowspan[A,B] in $G_p(\mathbb{F}^{p+2m})$. To find this limit, it is sufficient to find an $A_n \in \mathbb{F}^{pxp}$ such that $A_n[I_p,K_n]=[A_n,A_nK_n]$ converges to a finite $[A,B] \in \mathbb{F}^{px(p+2m)}$ in the usual sense; finally, to ensure that A_nK_n converges to a finite matrix B, A_n must converge to a singular matrix A.

□

The above Grassmannian $G_p(\mathbb{F}^{p+r})$ which represents the family of PI controllers may be embedded in the projective space and this is done by the classical Plucker embedding $\mathcal{P}$, (see Ch.3)

$$\mathcal{P}: \; G_p(\mathbb{F}^{p+r}) \to P(\wedge^p \mathbb{F}^{p+r}) = P(\mathbb{F})^{\sigma-1} \quad \text{where} \quad \sigma=\binom{p+r}{p} \tag{8.2}$$

as follows: Let $\mathcal{A} \in G_p(\mathbb{F}^{p+r})$, where as it has been explained before, $\mathcal{A}$ is the p-dimensional subspace of $\mathbb{F}^{p+r}$ spanned by the set of rows $\underline{k}_1^t$, $\underline{k}_2^t$, ... , $\underline{k}_p^t$,of the matrix K. Then,

$$\mathcal{P}(\mathcal{A}) \stackrel{def}{=} \underline{k}_1^t \wedge \underline{k}_2^t \wedge \ldots \wedge \underline{k}_p^t \tag{8.3}$$

or

$$\mathcal{P}(\mathcal{A}) \stackrel{def}{=} C_p(K) \tag{8.4}$$

where $C_p()$ denotes the p-th compound of a matrix (see Ch.3). Specifically, C_p maps K to (. . ., $\det K_\omega$,. . .) where K_ω is the ω-th pxp minor of K and the multiindices ω are lexicographically ordered. The image of $\mathcal{P}$ in the projective space is cut by certain homogeneous polynomial equations called quadratic Plucker relations QPR [Hod.1] which, in turn, can be reduced to σ-1-pr equations - the so called Reduced QPR (RQPR) [Kar.1],[Gia.2] (see also Ch.3).

This way, the set of PI controllers is endowed the structure of an algebraic variety through its image via the Plucker embedding:

$$\mathcal{P}: \; G_p(\mathbb{R}^{p+2m}) \to P\mathbb{R}^{\sigma_1-1} \tag{8.5}$$

where $\sigma_1=\binom{p+2m}{p}$ .The previous classification of PI controllers becomes:

Remark (8.2) : In the projective space, the parametrisation of PI controller families becomes:

a) Regular PI controllers (RPI): Is the subset of $P\mathbb{R}^{\sigma_1 - 1}$ containing all $1x\sigma_1$ vectors $[...p_\omega...]$, which for $\omega_0=(1,2,3...,p)$ satisfies the conditions:

i)The RQPRs ii) $p_{\omega_0} \neq 0$

b) Irregular PI controllers (IPI): Is the subset of $P\mathbb{R}^{\sigma_1 - 1}$ containing all $1x\sigma_1$ vectors $[...p_\omega...]$ which for $\omega_0=(1,2,3...,p)$ satisfies the conditions:

i)The QPRs ii) $p_{\omega_0}=0$

□

Remark(8.3). The equation $p_{\omega_0}=0$ expresses the fact that the PI controller rowspan[A,B,C] is such that rankA < p. This way the set of IPI is a subvariety of $G_p(\mathbb{R}^{p+2m})$ of dimension 2mp-1. The complementary set RPI is 'almost' the whole of $G_p(\mathbb{R}^{p+2m})$ in the sense that it does not contain solely a set of zero measure (the set of IPI).

□

8.2.3 Parametrisation of the family of OBD controllers

The pole placement equation via OBD controllers, as it was explained in chapter 5, can be written in the following determinantal form:

$$f(s)= \det\left\{ [\, T_k, T_{k-1},..., T_0] \begin{bmatrix} s^k M(s) \\ s^{k-1} M(s) \\ \vdots \\ M(s) \end{bmatrix} \right\} \tag{8.6}$$

We can again follow the same arguments for the PI case and parametrise the generalised BDO as the Grassmannian $G_p(\mathbb{R}^{(k+1)(p+m)})$,having as elements the p-dim subspaces of $\mathbb{R}^{(k+1)(p+m)}$,rowspan$[T_k, T_{k-1},..., T_0]$. We may partition this Grassmannian, into two complementary sets as shown below:

Definition(8.2): We may define the following two classes of OBD controllers:

a) Regular or finite OBD controllers (ROBD). These are the elements defined by rowspan$[T_k, T_{k-1},..., T_0]$, such that the first pxp submatrix of T_k has full rank. These correspond either to OBD controllers of McMillan degree pk (if $C_p[T_k s^k+...+T_0] \neq 0$ $\forall$s) which are called full order McMillan degree OBD controllers, or to OBD with McMillan degree less than pk which are called reduced order OBD controllers.

b) Irregular or infinite OBD (IOBD) controllers: These are the elements defined by rowspan$[T_k, T_{k-1},..., T_0]$, such that the pxp submatrix formed by the first p columns of T_k (from the left) is singular. These corresponds to limits of regular OBD controllers, rowspan$[T_k, T_{k-1},..., T_0]$, on the Grassmannian, when $[T_k, T_{k-1},..., T_0]]$ tends to infinity in the usual sense.

□

By using the Plucker embedding

$$T: \ G_p(\mathbb{R}^{(k+1)(p+m)}) \rightarrow P\mathbb{R}^{\sigma_3 - 1} \tag{8.7}$$

where $\sigma_2=\binom{(k+1)(p+m)}{p}$,which maps the rowspan$[T_k, T_{k-1},..., T_0]$ to $C_p[T_k, T_{k-1},..., T_0]$. The previous classification of the OBD controllers becomes:

Remark (8.4):In the projective space, the parametrisation of OBD controller families becomes:

a) Regular OBD controllers(ROBD): Is the subset of $P\mathbb{R}^{\sigma_2 - 1}$ containing all 1xσ_2 vectors $[...p_\omega...]$ which for ω_0=(1,2,3...,p) satisfies the conditions:

i)The RQPRs ii) $p_{\omega_0} \neq 0$

b) Irregular OBD controllers(IOBD): Is the subset of $P\mathbb{R}^{\sigma_2 - 1}$ containing all 1xσ_2 vectors $[...p_\omega...]$ which for ω_0=(1,2,3...,p) satisfies the conditions:

i)The RQPRs ii) $p_{\omega_0}=0$ □

To summarise, we have parametrised the families of regular PI and OBD feedback controllers as certain nonvoid Zarisky open subsets of the Grassmann varieties $G_p(\mathbb{R}^{p+r}) \subseteq P\mathbb{R}^{\sigma-1}$, where $\sigma=\binom{p+r}{p}$. That is they are the whole of the Grassmann variety apart from a subvariety of strictly lower dimension corresponding to the set of 'infinite' controllers. Note that the above Zarisky open sets are contained in the affine open set $\mathbb{R}^{\sigma-1}_{\omega_0} \subseteq P\mathbb{R}^{\sigma-1}$ ω_0=(1,2,3,...,p), which corresponds to the requirement that in the parametrisation considered, the pxp submatrix formed by the first p columns (considered from the left) must be I_p in both cases.

Remark (8.5) We can use the same arguments to parametrise the corresponding families of complex controllers. In this case, the real Grassmann variety $G_p(\mathbb{R}^{p+r})$, will become $G_p(\mathbb{C}^{p+r})$, but everything else will be the same.

□

8.2.4 Decomposition of the problem

The set of PI controllers is viewed as a certain Grassmannian and this provides the necessary tools to study pole assignment by PI controllers. In this section it is shown that this form of the Determinantal Assignment Problem (DAP) is a real intersection problem in a projective space. Following what was mentioned in Ch.5, given that the closed loop pole polynomial is expressed by (8.1), then using the Binnet-Cauchy theorem, f(s) may be expressed as:

$$f(s)= C_p\{H\}C_p\{M_1(s)\} \tag{8.8}$$

where

$$H=[I_p\ ,\ K_0\ ,K_1] \text{ and } M_1(s)= \begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix} \tag{8.9}$$

The above problem is formulated as a *Constant Determinantal Assignment Problem* (CDAP) and in this specific form it will be referred to as *PI-DAP*; this problem can be reduced to the following two problems, one linear and one standard multilinear as described below:

(i) LINEAR PROBLEM: For a given $p(s) \in \mathbb{R}[s]$ of degree n+p and $\underline{h}(s)= C_p\{M_1(s)\} \in \mathbb{R}^{\sigma \times 1}[s]$ solve the equation,

$$p(s)= [1\ ,\ \underline{k}^T]^T.\ \underline{h}(s) \tag{8.10}$$

with respect to $[1\ ,\ \underline{k}^T]^T \in \mathbb{R}^{\sigma-1}_{\omega} \subseteq P(\mathbb{R})^{\sigma-1}$. Equivalently, if we write

$$\underline{h}(s)=C_p(M_1(s))= P_{pi}\ [s^{p+n},\ s^{p+n-1},\ \dots\ ,\ 1]^t \tag{8.11}$$

then (8.10) is reduced to the linear system

$$[1\ ,\ \underline{p}^T] = [1\ ,\ \underline{k}^T]\ P_{pi} \tag{8.12}$$

which, in turn, defines a linear variety $\mathcal{L}$ in the projective space. The matrix P_{pi} is a σx(n+p+1) matrix, and will be reffered to as the *PI Plucker Matrix* of the system; its properties play a crucial role in the solvability of this specific form of DAP.

(ii)MULTILINEAR PROBLEM: Assume that the linear problem has a non void solution set ,say $\mathcal{L}$. Determine whether there exists $[1\ ,\ \underline{k}^T] \in \mathcal{L}$ which belongs to $G_p(\mathbb{R}^{p+2m})$, that is $[1\ ,\ \underline{k}^T]=C_p(H)$ for some $H=[I_p,K_0,K_1]$. The special structure of $M_1(s)$ implies that the PI Plucker matrix must be of the form:

$$P_{pi}=\begin{bmatrix} 1 & | & \underline{a} \\ — & + & — \\ \underline{0}^T & | & \hat{P}_{pi} \end{bmatrix} \tag{8.13}$$

which in turn implies that the equations for the linear variety defined by (8.12) can be rewritten as:

$$\underline{0}=[1\ ,\ \underline{k}^T]\begin{bmatrix} \underline{a}-\underline{p} \\ ------ \\ \hat{P}_{pi} \end{bmatrix} \tag{8.14}$$

Similarly, for the case of OBD controllers we have a linear and multilinear decomposition and the *OBD Plucker matrix* P_{obd} is defined by:

$$P_{obd}\,[s^{n_1+n}, s^{n_1+n-1}, \dots, 1]^t = C_p\left(\begin{bmatrix} s^k M(s) \\ s^{k-1}M(s) \\ \vdots \\ M(s) \end{bmatrix}\right)$$

As far as the equations of the multilinear problem, these are defined as those expressing the decomposability of the multivectors $[1\ ,\ \underline{k}^T]$, that is the RQPRs. In other words, they are the set of RQPRs [Gia.2] for $\mathbb{R}^{\sigma_1 - 1}_{\omega} \subseteq P\mathbb{R}^{\sigma_1 - 1}$,ω=(1,2,3,...,p) and they have been defined in chapter3. To illustrate the procedure so far we give the following example.

Example (8.1) Consider the system S which is decscribed by the transfer function matrix:

$$G(s)=\begin{bmatrix} \frac{-1+s}{s^2-2} & \frac{-2+s}{s^2-2} \\ \frac{1+2s}{s^2-2} & \frac{4+s}{s^2-2} \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} s & 2+s \\ 1 & s+1 \end{bmatrix}^{-1}$$

We want to find an output feedback (or precompensating) PI controller which assigns to the closed loop system, the pole polynomial $s^4-10s^2-11s-3$. In accordance with the procedure we described previously we have:

$$M_1(s)=\begin{bmatrix} s^2 & 2s+s^2 \\ s & s+s^2 \\ s & 2s \\ 2s & s \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \rightarrow \quad C_p\{M_1(s)\}=\begin{bmatrix} s^4-2s^2 \\ s^3-2s^2 \\ -s^3-4s^2 \\ s^2-2s \\ -s^2-4s \\ -s^3+s^2 \\ -2s^3-s^2 \\ -s^2+s \\ -2s^2-s \\ -3s^2 \\ 0 \\ -3s \\ 3s \\ 0 \\ -3 \end{bmatrix}$$

Therefore the linear equations (8.12) for our problem are given by:

$$[1,0,-10,-11,-3] = [1,k_2,k_3,\dots,k_{15}] \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The RQPR for $\mathbb{R}^{14}_{\omega} \subseteq \mathbb{PR}^{14}$, ω=(1,2) (that is when k_1=1) are:

$$k_{10}=k_2k_7-k_3k_6 \qquad k_{11}=k_2k_8-k_4k_6 \qquad k_{12}=k_2k_9-k_5k_6$$

$$k_{13}=k_3k_8-k_4k_7 \qquad k_{14}=k_3k_9-k_5k_7 \qquad k_{15}=k_4k_9-k_5k_8$$

Thus all the possible solutions (PI controllers) to our problem are embedded in $\mathbb{R}^{14}_{\omega} \subseteq \mathbb{PR}^{14}$ via the Plucker embedding and they are the solutions of the sets of linear equations and the RQPR described above. A solution for the above set of equations is:

$$\underline{k}=[1,0,1,0,1,-1,0,-1,0,1,0,1,-1,0,1]$$

This, according to Lemma 3.1, decomposes to:

$$[I_2,K_0,K_1]=\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

which in turn gives the desired solution:

$$K_0+\frac{1}{s}K_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\frac{1}{s}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

The above example demonstrate the nature of the equations we have to solve. The solvability of these equations is examined in the following section.

8.3 Plucker matrices and their properties.

8.3.1 Introduction

The Plucker matrices of our problems play a key role in the study of the corresponding DA problems since they define the linear subproblem which embodies all the information of the open loop plant as well as some of the information of the generic dynamic structure of the feedback controller. As it has been previously seen, the set of finite and infinite feedback controllers which assign a given closed loop polynomial to a plant S, is given by the intersection of a Grassmann variety with a linear set given by the null space of the corresponding Plucker matrix. Thus the determination of the rank of the Plucker matrices P_{pi} and P_{obd} is essential for the examination of the above intersection and therefore the derivation of solvability conditions for the PI and OBD pole placement problems.

8.3.2 The PI Plucker matrix.

One of the most important properties of the PI-Plucker matrices is that describing the generic rank of the PI-Plucker matrix and this is defined by the following result:

Proposition(8.1) If $2mp \geq n$, then the rank of the PI-Plucker matrix for a generic system of p-inputs, m-outputs and n-states is equal to $n+p+1$.

Proof

Let $\sum_{m,p}^{n}$ be the algebraic variety of strictly proper systems of p-inputs, m-outputs and n-states. Then the subset of all systems such that their PI-Plucker matrix ,

P_{pi} ,has rank less than n+p+1 is determined by the zeros of all (n+p+1)x(n+p+1) determinants of submatrices of P_{pi} and thus, it is a subvariety of $\sum^{n}_{m,p}$. Subsequently, the set of all systems with p-inputs, m-outputs and n-states, which have full rank PI-Plucker matrix, say $\mathcal{F}_{pi}$,is a Zarisky open subset of $\sum^{n}_{m,p}$. To prove that the property of a system to have a full rank PI-Plucker matrix is generic, is equivalent to proving that $\mathcal{F}_{pi}$ is nonvoid. Therefore we have to prove that for every p,m,n such that $2mp \geq n$ there exists a system of p-inputs, m-outputs and n-states whose PI-Plucker matrix has full rank. This can be looked at by considering the following two cases:

i) Let n+p=kp

Consider a system of p-inputs, m-outputs and n-states whose composite matrix

$M(s)=\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ is of the form:

$$M(s)=\begin{bmatrix}
s^k & & & & & \\
s^{k-2} & s^k & & & \mathbf{0} & \\
s^{k-4} & s^{k-2} & s^k & & & \\
: & s^{k-4} & s^{k-2} & : & & \\
: & : & : & : & : & \\
: & : & : & : & : & s^k \\
: & : & : & : & : & s^{k-2} \\
: & : & : & : & : & s^{k-4} \\
s^4 & : & : & : & : & : \\
s^k & s^4 & : & : & : & : \\
1 & s^2 & s^4 & : & : & : \\
 & 1 & s^2 & : & : & : \\
 & & 1 & : & : & : \\
 & \mathbf{0} & & : & : & s^4 \\
 & & & & : & s^2 \\
 & & & & & 1
\end{bmatrix}$$

The matrix M(s) can be formed if $2mp \geq n$. The corresponding PI composite matrix

$M_1(s)=\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix}$ is given by

$$
M_1(s)=\begin{bmatrix}
s^{k+1} & & & & & \\
s^{k-1} & s^{k+1} & & & \mathbf{0} & \\
s^{k-1} & s^{k-1} & s^{k+1} & & & \\
\vdots & \vdots & \vdots & \vdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & s^{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & s^{k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & s^{k-3} \\
s^5 & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^3 & \vdots & \vdots & \vdots & \vdots & \vdots \\
s & s^3 & \vdots & \vdots & \vdots & \vdots \\
 & s & s^3 & \vdots & \vdots & \vdots \\
 & & s & \vdots & \vdots & \vdots \\
 & \mathbf{0} & & \vdots & \vdots & s^3 \\
 & & & & & s \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^4 & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^2 & s^4 & \vdots & \vdots & \vdots & \vdots \\
1 & s^2 & \vdots & \vdots & \vdots & \vdots \\
 & 1 & s^2 & \vdots & \vdots & \vdots \\
 & & 1 & \vdots & \vdots & \vdots \\
 & & & \vdots & \vdots & \vdots \\
 & \mathbf{0} & & & \vdots & s^2 \\
 & & & & & 1
\end{bmatrix}
$$

By the definition of the PI-Plucker matrix, to calculate the P_{pi} we have to calculate all the pxp minors of $M_1(s)$. These determinants are polynomials in 's' whose degree varies from 0 to n+p and the PI-Plucker matrix is the coefficient matrix of these polynomials. Our aim is to select n+p+1 of these polynomials whose coefficient matrix has full rank.

Let h: $0 \leq h \leq n+p$ and $h=(k+1)j+u$ where $0 \leq u < k+1$ then:

a) If u is odd, consider the pxp submatrix of $M_1(s)$ formed by its first j rows ie.

$$
A_h=\begin{bmatrix}
s^{k+1} & & & & & & & & \\
x & \vdots & & & & & \mathbf{0} & & \\
x & x & \vdots & & & & & & \\
x & x & x & s^{k+1} & & & & & \\
x & x & x & x & s^{u} & x & x & x & x \\
 & & & & & 1 & x & x & x \\
 & & & & & & \vdots & x & x \\
 & \mathbf{0} & & & & & & \vdots & x \\
 & & & & & & & & 1
\end{bmatrix}
$$

then $\det(A_h)=s^h$

b) If u is even then consider the pxp submatrix of $M_1(s)$ formed by its first j columns

$$B_h=\begin{bmatrix} s^{k+1} & & & & & & & & & \\ x & \vdots & & & & & & \mathbf{0} & & \\ x & x & \vdots & & & & & & & \\ x & x & x & s^{k+1} & & & & & & \\ x & x & x & x & s^{u-1} & x & x & x & x & x \\ & & & & & s & x & x & x & x \\ & & & & & & 1 & x & x & x \\ & & & & & & & 1 & x & x \\ & \mathbf{0} & & & & & & & \vdots & x \\ & & & & & & & & & 1 \end{bmatrix}$$

$\det(B_h)=s^h$. Thus the above selection of determinants will give us the set $\{1, s, s^2,\ldots, s^{n+p}\}$ whose coefficient matrix is the identity matrix I_{n+p+1}. This imply that the above PI-Plucker matrix has full rank and our proposition is proved.

ii) $n=kp+\omega$ where $0<\omega<p$

In this case, we can modify the proof given above along the following line. Since $\omega<p$ we can select the first ω columns of M(s) and can increase the degrees from k to k+1; the remaining columns are left with the same degree ie. k. The rest of the proof then follows, as indicated in part (i), with the obvious changes. □

The above result establishes the important property that if $2mp \geq n$, then the PI-Plucker matrix has full rank for a generic system. This allows the computation of the dimension of the generic linear variety that enters in the study of solvability conditions of the problem.

8.3.3 The OBD Plucker matrix.

The generic rank of the OBD Plucker matrix is once more an important property and it is defined by the following result:

Proposition(8.2) If $n \leq n_1 m + mp$ where $n_1=kp$ then the rank of the OBD Plucker matrix for a generic system of p-inputs, m-outputs and n-states, is equal to $n+n_1+1$.

Proof

Using similar arguments to those used in the proof of Proposition(8.1) , to prove the above result we have to find a system whose OBD Plucker matrix has full rank. Let

$n=n_1m + mp = mp(k+1)$ and consider the system defined by:

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = M(s) = \begin{bmatrix} s^{m(k+1)} & & & & \\ 1 & s^{m(k+1)} & & & \mathbf{0} \\ & 1 & \vdots & & \\ & & \vdots & & \\ \mathbf{0} & & & 1 & s^{m(k+1)} \\ & & & & s^{(m-1)(k+1)} \\ & & & & s^{(m-2)(k+1)} \\ & & & & \vdots \\ & & & & s^{k+1} \\ & & & & 1 \end{bmatrix}$$

One can easily see that the OBD Plucker matrix of this system has full rank. The case $n < n_1m + mp$ can be treated similarly with only minor modifications.

□

8.4 Necessary conditions for Pole assignment.

To derive necessary conditions for our problem we will approach it in a similar manner to that deployed in [Wil.1]. First, we consider the pole placement map defined by:

$$\chi : \mathbb{C}^r_{pi} \rightarrow \mathbb{R}^{n+p} \tag{8.15}$$

where $\mathbb{C}^r_{pi}$ is the set of regular PI feedbac

coefficients of s^i: $0 \leq i \leq n+p-1$ in the polynomial $f(s)$ given by (8.1). Then the problem of pole placement becomes one of examining whether χ is onto. The map χ is a differentiable map between two manifolds. If the dimension of RC is less than n+p, then $Im(\chi)$ is a set of measure zero in $\mathbb{R}^{n+p}$ and χ cannot be onto. Therefore, a necessary condition for the map χ to be onto is:

$$\dim\mathbb{C}^{r}_{pi} \geq n+p$$

or equivalently

$$2mp \geq n+p \tag{8.16}$$

since $\dim\mathbb{C}^{r}_{pi}$ is equal to 2mp, the number of free parameters in the matrix $[I_p,K_0,K_1]$. We can additionally impose a second necessary condition which is specially useful for the nongeneric S. The decomposition of the problem into a linear and a multilinear one leads to the following factorisation of χ:

$$\chi:\mathbb{C}^{r}_{pi} \xrightarrow{\mathcal{P}} \mathbb{R}^{\sigma} \xrightarrow{L} \mathbb{R}^{n+p}$$

where the left map is the Plucker embedding and the right is a σx(n+p) linear map whose n+p columns are the right n+p columns of P_{pi}. So it is apparent that, if the linear map P_{pi} has not full rank, then it is impossible for χ to be onto. These arguments lead to the following result:

Theorem (8.1) A necessary condition for the existence of a real (and complex) PI controller assigning every real monic polynomial of degree p+n to a strictly proper system S of p-inputs, m-outputs and n-states is that $2pm \geq p+n$ and $\text{rank}(P_{pi})=n+p+1$.

□

It is apparent that in the single input-many output case the problem becomes linear and the above condition becomes necessary and sufficient.

Corollary (8.1) A necessary and sufficient condition for the existence of a real (and complex) PI assigning every real monic polynomial of degree 1+n to a strictly proper system S of 1-input, m-outputs and n-states is $2m \geq 1+n$ and $\text{rank}(P_{pi})=n+2$.

□

Similarly for the OBD feedback controllers case:

Theorem (8.2) A necessary condition for the existence of a real (and complex) OBD controller, of dynamic degree $n_1=kp$, assigning every real monic polynomial of degree n_1+n to a strictly proper system S of p-inputs, m-outputs and n-states is that $pm+n_1(m+p) \geq n_1+n$ and $\text{rank}(P_{obd})=n+n_1+1$.

□

8.5 Sufficient conditions for PI solutions.

For a given monic polynomial p(s) of degree n+p, the set of all PI controllers that assign it as a closed loop pole polynomial, is given as a set of zeros of n+p polynomial equations in 2mp unknowns with coefficients parametrised by $\sum_{m,p}^{n}(\mathbb{C})$, the set of strictly proper systems of p-inputs, m-outputs and n-states. A basic theorem on complex varieties says that, all sets of n+p equations in 2mp unknowns, except from some nongeneric ones, have a solution, provided $2mp \geq n+p$. Unfortunately, this cannot be applied directly to our case since our equations have a special structure and furthermore the space of coefficients is not free but is parametrised by $\sum_{m,p}^{n}(\mathbb{C})$ and therefore they may lie in the nongeneric set of equations of the previous theorem.

One way to get around this difficulty is to consider the above n+p equations defined on the set of compactified PI controllers which according to our previous construction is the Grassmannian $G_p(\mathbb{C}^{p+2m})$. The set of the n+p equations, now, has always solution provided $2mp \geq n+p$, according to the projective intersection theorem (Chapter 3). However, we can easily see that all $\text{rowspan}[A,K_0,K_1] \in G_p(\mathbb{C}^{p+2m})$ such that:

$$\text{rank}[sA,\ sK_0+K_1] < p \tag{8.17}$$

satisfy the n+p equations on the Grassmannian, no matter what the dimensions or the system are. In fact the above points are degenerate points of the intersection, that is they satisfy:

$$\det\left([A,\ K_0,K_1]\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix}\right)=0 \tag{8.18}$$

The system theoretic interpretation of this type of controllers is that, they are controllers for which the the feedback loop is not well posed, or such that the closed loop system is degenerate. Hence the existence of solutions produced by the projective intersection theorem on the Grassmannian does not help us at all, since the controllers in the solution may all as well be degenerate. This suggests that the Grassmannian compactification of PI controllers is not adequate for our intersection problem and probably we may have to construct a better compactification if we are to tackle this problem.

Instead of producing a new compactification, the approach we will adopt here tackles the problem using techniques that can be applied to non projective, that is noncompact, complex algebraic varieties. Let $\mathcal{U}$ be the set of all PI controllers $K_0+\frac{1}{s}K_1$ such that the first pxp submatrix of K_1, denoted by $\tilde{K}_1$ has full rank. This is a Zarisky open subset of $\mathbb{C}^{2mp}$ which contains all $[K_0, K_1]$ such that $\det(\tilde{K}_1) \neq 0$. For a given polynomial p(s), consider now, the subvariety $\mathfrak{S}(p(s))$ of $\mathcal{U}\mathrm{x}\sum^n_{m,p}(\mathbb{C})$, defined as follows:

$$\mathfrak{S}=\left\{([K_0, K_1]\ , S) \in \mathcal{U}\mathrm{x}\textstyle\sum^n_{m,p}(\mathbb{C}), \text{ such that } \det\left\{[I_p\ , K_0\ ,K_1]\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix}\right\}=p(s)\right\} \quad (8.19)$$

where $\sum^n_{m,p}(\mathbb{C})$ is the algebraic variety of strictly proper systems of p-inputs, m-outputs and n-states defined in Chapter 4. The set $\mathfrak{S}$ contains as fibers all possible PI controllers in $\mathcal{U}$ that assign the closed loop polynomial p(s) for all S in $\sum^n_{m,p}(\mathbb{C})$; consequently it is the most appropriate set for the examination of the pole assignability properties of a generic S. Thus we have to calculate the dimension of a fiber of $\mathfrak{S}$ under the projection on the second coordinate, for a generic S in $\sum^n_{m,p}(\mathbb{F})$. First we need to calculate the dimension of $\mathfrak{S}$, and to do so we will prove the following Lemma.

Lemma(8.1) The map

$$f: \textstyle\sum^n_{m,p} \rightarrow \mathbb{C}^{n+p} \quad (8.20)$$

Such that $f(S)$, $S \in \sum_{m,p}^{n}$, is the coefficient vector of the powers s^{n+p-1}, s^{n+p-2}, ... , s, 1 of the polynomial $\det(sD(s)+\tilde{N}(s))$, where $\tilde{N}(s)$ is the top pxp submatrix of the numerator $N(s)$ of S, is a dominant morphism.

Proof

Indeed, every monic polynomial $p(s)$ of degree $n+p$ can be factorised as

$$p(s)=p_p(s)\, p_{p-1}(s) \dots\, p_1(s)$$

where $p_i(s)$ are monic polynomials with degree greater than or equal to one. If we let

$$p_i(s)= sa_i(s)+b_i \qquad i=1,2,\dots,p$$

Then for every $p(s)$ such that $p(0) \neq 0$ the system defined by:

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} a_p(s) & 0 & \dots & 0 \\ 0 & a_{p-1}(s) & & \vdots \\ \vdots & & \vdots & 0 \\ 0 & \dots & 0 & a_1(s) \\ b_p & 0 & \dots & 0 \\ 0 & b_{p-1} & 0 & 0 \\ \vdots & & \vdots & 0 \\ \vdots & \dots & 0 & b_1 \\ \vdots & & & 0 \\ 0 & \dots & \dots & 0 \end{bmatrix}$$

is a well defined strictly proper system with n-states and satisfies the following condition:

$$\det[sD(s)+\tilde{N}(s)]= \prod_i (sa_i(s)+b_i)=\prod_i p_i(s)=p(s)$$

which proves the fact that the Image of f includes all $\underline{c} \in \mathbb{C}^{n+p}$ such that $c_0 \neq 0$. Hence f is dominant.

□

The variety $\mathfrak{S}$ is given by n+p equations on $\mathcal{U} x \sum_{m,p}^{n}(\mathbb{C})$, therefore $\mathfrak{S}$ is either empty or has dimension greater than or equal to $\dim(\mathcal{U})+\dim\left(\sum_{m,p}^{n}(\mathbb{C})\right)$ $-(n+p)=2mp+n(m+p)-n-p$. In fact as the next theorem states the dimension of $\mathfrak{S}$ is exactly equal to $2mp+n(m+p)-n-p$.

Theorem(8.3) The dimension of the variety $\mathfrak{S}$ for a generic closed loop pole polynomial p(s) is given by:

$$\dim(\mathfrak{S})=n(m+p)+2mp-n-p \tag{8.21}$$

Proof

Consider the natural projection of $\mathfrak{S}$ on $\mathcal{U}$

$$p_1: \mathfrak{S} \rightarrow \mathcal{U}$$

Then the following equality holds true:

$$\det\left\{[I_p , K_0 , K_1]\begin{bmatrix} sD_R(s) \\ sN_R(s) \\ N_R(s) \end{bmatrix}\right\}=\det\left\{ [I_p, 0_{pxm} , I_p , 0_{px(m-p)}]\begin{bmatrix} sD_R(s)+sK_0N_R(s) \\ sA(K_1)N_R(s) \\ A(K_1)N_R(s) \end{bmatrix}\right\}$$

where

$$A(K_1)=\begin{bmatrix} K_1 \\ \hline 0_{(m-p)xp} \mid I_{m-p} \end{bmatrix}$$

which implies that

$$p_1^{-1}([K_0, K_1]) \approx p_1^{-1} [I_p, 0_{pxm} , I_p , 0_{px(m-p)}]$$

since if a system S, having composite M(s), is in $p_1^{-1}([K_0, K_1])$ then the system S′ with composite $B(K_0, K_1)M(s)$ is in $p_1^{-1} [I_p, 0_{pxm} , I_p , 0_{px(m-p)}]$, where

$$B(K_0, K_1)=\begin{bmatrix} I_p & K_0 \\ 0 & A(K_1) \end{bmatrix}$$

Therefore the above equality of fibers implies that the dimension of all these fibers is equal to the dimension of $p_1^{-1} [I_p, 0_{pxm} , I_p , 0_{px(m-p)}]$ which in turn is equal to n(m+p) − (n+p) because of the previous lemma. Therefore, the dimension of $\mathcal{G}$ is given by:

$$\dim(\mathcal{G}) = n(m+p) - (n+p) + \dim(\mathcal{U})$$

or equivalently,

$$\dim(\mathcal{G}) = n(m+p)-(n+p)+2mp$$

□

To calculate the dimension of the generic fiber of the projection of $\mathcal{G}$ on the $\sum_{m,p}^{n}(\mathbb{C})$ we need the following lemma.

Lemma(8.2) Let $\mathcal{G}_1$, $\mathcal{G}_2$ be two complex varieties such that $\dim(\mathcal{G}_1) \geq \dim(\mathcal{G}_2)$ and

$$f_a\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$$

be a family of maps where 'a' belongs to a complex variety $\mathcal{V}$. Then the set of all 'a' such that f_a is dominant is a Zarisky open subset of $\mathcal{V}$.

Proof

We know from the dominant morphism theorem (see Chapter 3) that f_a is dominant, iff there exists $x \in \mathcal{G}_1$ such that the differential $(Df_a)_x$ has full rank, or else it has rank equal to $\dim(\mathcal{G}_2)$. Therefore, consider the set

$$\mathcal{Z}=\{\ (x,a) \in \mathcal{G}_1 x\mathcal{V}\colon \operatorname{rank}(Df_a)_x=\dim(\mathcal{G}_2)\}$$

This set is a Zarisky open subset of $\mathcal{G}_1 x\mathcal{V}$ since it is required that at least one of the maximal minors of $(Df_a)_x$ to be nonzero. The image of the projection

$$\mathcal{Z} \longrightarrow \mathcal{V}$$

on the second coordinate contains all $a \in \mathcal{V}$ such that f_a is dominant. This set is a Zarisky open subset of $\mathcal{V}$ since it is the image of the projection of another Zarisky open set.

□

Corollary(8.2) Let

$$p_{\underline{c}}: \mathfrak{G} \rightarrow \sum{}^{n}_{m,p}(\mathbb{C}) \tag{8.22}$$

be the family of projections parametrised by the set $\mathbb{C}^{n+p}$ of coefficients of closed loop polynomials. Then, if $\dim(\mathfrak{G}) \geq n(m+p)$ and there exists one $c_0 \in \mathbb{C}^{n+p}$ such that p_{c_0} is dominant, $p_{\underline{c}}$ is dominant for all $\underline{c}$ in a nonvoid Zarisky open subset of $\mathbb{C}^{n+p}$.

Proof

From the previous Lemma the set of all $\underline{c} \in \mathbb{C}^{n+p}$ such that $p_{\underline{c}}$ is dominant is Zarisky open, and furthermore is nonvoid because of the existence of c_0.

□

Theorem(8.4) Let us assume that:

(i) $2mp \geq n+p$

(ii) There exists one polynomial of degree n+p that can be assigned to a generic system $S \in \sum{}^{n}_{m,p}$ by a PI controller in $\mathfrak{U}$.

Then, for a generic complex polynomial p(s) of degree n+p and a generic system $S \in \sum{}^{n}_{m,p}(\mathbb{C})$ the set of all PI controllers in $\mathfrak{U}$ assigning p(s) to S is nonempty and has dimension $2mp-(n+p)$.

Proof

Because of the assumption (i) and theorem(8.3), we get that $\dim(\mathfrak{G}) \geq n(m+p)$. Combining the latter with the assumption (ii) and corollary(8.2), we can conclude that the projection :

$$p_{\underline{c}}: \mathfrak{G} \rightarrow \sum{}^{n}_{m,p}(\mathbb{C})$$

is dominant for all $\underline{c}$ in a nonvoid Zarisky open set of $\mathbb{C}^{n+p}$. Therefore, for a generic $S \in \sum{}^{n}_{m,p}(\mathbb{C})$ the fiber has dimension $\dim(\mathfrak{G}) - \dim(\sum{}^{n}_{m,p}(\mathbb{C}))=2mp-(n+p)$ and this proves the theorem.

□

Corrolary(8.3) Assuming (i) and (ii) of theorem(8.4), then for a generic real polynomial p(s) of degree n+p and a generic system $S \in \sum{}^{n}_{m}$

Proof

It is an easy consequence of theorem(8.4) and the fact that the sets of real polynomials and real systems are Zarisky dense in the sets of complex polynomials and complex systems respectively.

□

To apply theorem(8.4) or corrolary(8.3) we need to check conditions (i) and especially (ii). Condition ii although seems simple, is in fact as difficult to be checked as the original problem of assignability of an arbitrary polynomial. Therefore theorem(8.4) has mainly theretical than practical interest. To answer the question of arbitrary pole assignability we have again to resort to the calculation of the differential of the PI pole placement map as the following example indicates:

Example(8.2)

Consider the system of example(8.1). The the closed loop pole polynomial via a PI feedback compensator $K_0+\frac{1}{s}K_1$ is given by:

$$p(s)=s^4+p_3(K)s^3-p_2(K)s^2-p_1(K)s+p_0(K)$$

where $K=[K_0, K_1]=(k_{ij})$ and

$$p_3(K)= k_{11}+ 2k_{12}+ k_{21}- k_{22}$$

$$p_2(K)= 3k_{11}k_{22}- 3k_{12}k_{21}+ k_{11}-k_{12}-k_{13}-2k_{14}+ 2k_{21}+ 4k_{22}- k_{23}+ k_{24}+ 2$$

$$p_1(K)= 3k_{11}k_{24}-3k_{12}k_{23}+ 3k_{13}k_{22} -3k_{14}k_{22} + k_{13} + 2k_{23} + 4k_{24}$$

$$p_0(K)= 3k_{14}k_{23}-3k_{13}k_{24}$$

The PI pole placement map is given by: $\chi(K)= (p_3(K), -p_2(K), -p_1(K), p_0(K))$ and its differential can be easily calculated to be

$$(D\chi)_K=\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & 0 & 0 \\ -3k_{22}-1 & 3k_{21}+1 & 1 & 2 & 3k_{12}-2 & -3k_{11}-4 & 1 & -1 \\ -3k_{24} & 3k_{23} & -3k_{22}-1 & 3k_{21}+1 & 3k_{14} & -3k_{13} & k_{12}-2 & -3k_{11}-4 \\ 0 & 0 & -3k_{24} & 3k_{23} & 0 & 0 & 3k_{14} & -3k_{13} \end{bmatrix}$$

for $K=[I_2,I_2]$ the differential becomes equal to:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & 0 & 0 \\ -4 & 1 & 1 & 2 & -2 & -7 & 1 & -1 \\ -3 & 0 & -4 & 1 & 0 & -3 & -2 & -7 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

which has full rank. Therefore by the dominant morphism theorem (see Chapter3) χ is dominant, and thus almost every closed loop pole polynomial may be assigned by an appropriate PI feedback compensator.

□

The conditions established so far deal with the existence of complex solutions. The search for real solutions centers around certain tests related to the existence of real intersections; this, however, is not considered here.

8.6 Sufficient conditions for OBD solutions.

In this section we try to extend the results of the previous section to the case of OBD compensators. Let $n \geq n_1$ and $n_1=kp$, we may first state the following result:

Lemma(8.3) The map

$$f:\Sigma_{m,p}^{n} \rightarrow \mathbb{C}^{n+n_1}$$

such that $f(S)$, $S \in \Sigma_{m,p}^{n}$, is the coefficient vector of the powers $s^{n+n_1-1}, \ldots, s, 1$ of the polynomial $\det(s^{n_1}D(s)+\tilde{N}(s))$, is dominant, ie. onto for almost all $\mathbb{C}^{n+n_1}$.

Proof

Consider all $[c_{n+n_1-1}, c_{n+n_1-2}, \ldots, c_0] \in \mathbb{C}^{n+n_1}$ such that $c_0 \neq 0$ corresponds to all monic complex polynomials such that $p(s) \neq 0$. Then such a $p(s)$ can be written as

$$p(s)=p_p(s)\, p_{p-1}(s) \ldots p_1(s)$$

where $\deg(p_i(s)) \geq k$. Then we may write

$$p_i(s)=s^k a_i(s)= b_i(s)$$

and consider the system such that

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} a_p(s) & 0 & \dots & 0 \\ 0 & a_{p-1}(s) & & : \\ : & & : & 0 \\ 0 & \dots & 0 & a_1(s) \\ b_p(s) & 0 & \dots & 0 \\ 0 & b_{p-1}(s) & 0 & 0 \\ : & & : & 0 \\ : & \dots & 0 & b_1(s) \\ : & & & 0 \\ 0 & \dots & \dots & 0 \end{bmatrix}$$

This is a well defined strictly proper system of n states which satisfies

$$\det(s^{n_1}D(s)+\tilde{N}(s))=p(s)$$

□

For a given p(s) let us define the set

$$\mathfrak{X}_{obd} = \left\{(S_1,S) \in \text{ OBD } \times \Sigma^n_{m,p} \text{ satisfying equ. (8.6)}\right\}$$

Then the dimension of $\mathfrak{X}_{obd}$ is given by the following theorem:

Theorem (8.5) For a generic polynomial p(s), the dimension of the variety $\mathfrak{X}_{obd}$ is equal to:

$$n(m+p)+ n_1(p+m) +pm-(n+n_1)$$

Proof

Consider the natural projection of $\mathfrak{X}_{obd}$ on the OBD set

$$p_1 : \mathfrak{S}_{obd} \to OBD$$

Then, the fibre $p_1^{-1}(S_1)$ of a generic S_1 is equal to the fibre of the map

$$f_{S_1} : \Sigma_{m,p}^{n} \to \mathbb{C}^{n+n_1}$$

where f_{S_1} is the map assigning every open loop plant $S \in \Sigma_{m,p}^{n}$ to the coefficient vector of the monic pole polynomial after closing the loop with the feedback OBD compensator S_1. Because of lemma(8.2) and lemma(8.3), f_{S_1} must be onto and ,therefore, a generic fibre of f_{S_1} has dimension equal to $\dim(\Sigma_{m,p}^{n}) - \dim(\mathbb{C}^{n+n_1}) = n(m+p) - (n+n_1)$. Hence, the dimension of $p_1^{-1}(S_1)$ is equal to $n(m+p) - (n+n_1)$ and, therefore, the dimension of $\mathfrak{S}_{obd}$ is given by

$$\dim(\mathfrak{S}_{obd}) = \dim(p_1^{-1}(S_1)) + \dim(OBD) = n(m+p) - (n+n_1) + n_1(p+m) + pm$$

□

Theorem(8.6) Let us assume that:

(i) $(n+n_1) \le n_1(p+m)+pm$

(ii) There exists one polynomial of degree $n+n_1$ that can be assigned to a generic system $S \in \Sigma_{m,p}^{n}$ by an OBD controller.

Then, for a generic complex polynomial $p(s)$ of degree $n+n_1$ and the generic system $S \in \Sigma_{m,p}^{n}$ the set of all OBD controllers assigning $p(s)$ to S is nonempty and has dimension $(n_1(p+m)+pm) - (n+n_1)$.

Proof

Consider the projections

$$p_{\underline{c}} : \mathfrak{S}_{obd} \to \Sigma_{m,p}^{n}(\mathbb{C})$$

parametrised by $\underline{c}$, the coefficient vector of the closed loop polynomial. Then because of the assumption (ii) one of those projections is dominant and hence almost all of them must be dominant by lemma(8.2). Therefore, for a generic $S \in \Sigma_{m,p}^{n}(\mathbb{C})$ the fiber has dimension $\dim(\mathfrak{S}_{obd}) - \dim(\Sigma_{m,p}^{n}(\mathbb{C})) = n(m+p) - (n+n_1) + n_1(p+m) + pm - (n+p)$ and this proves the theorem.

□

Corrolary(8.4) Assuming (i) and (ii) of theorem(8.6), then for a generic real polynomial p(s) of degree $n+n_1$ and a generic system $S \in \Sigma_{m,p}^{n}(\mathbb{R})$ the set of all OBD (complex) controllers assigning p(s) to S is nonempty and has dimension $n_1(p+m)+pm-(n+n_1)$.

Proof

It is an easy consequence of theorem(8.6) and the fact that the sets of real polynomials and real systems are Zarisky dense in the sets of complex polynomials and complex systems respectively.

□

8.8 Conclusions.

The study of dynamic DAP in the special case of dynamic controllers such as the PI and OBD has been considered here. the main difficulty in extending the DAP framework to the dynamic case has been the associated compactification issues. By reducing the dynamic problem to a modified constant DAP, a compactification as a Grassmannian is obtained for both cases and necessary conditions for the solvability have been derived. By using affine methods, we derive sufficient conditions for the existence of complex controllers for both PI and OBD cases. Deriving sufficient conditions for the existence of real controllers for these two families is still an open issue. The main obstacle is presented not only by the compactification issues but also by the intersection properties on the associated compactified variety (assuming that this can be constructed). In fact, such a variety is non standard and its intersection ring is not well studied. The alternative framework based on global linearization discussed in chapter 10 seem to be more suitable.

CHAPTER 9. Decentralised Pole, Zero Assignment by Static Controller

9.1. Introduction

As we have seen in chapter7, the centralization assumption enables us to view the compensator as an element of a certain Grassmann variety and thus use its properties and the well established topology in the study of intersections with the linear varieties associated with the corresponding problems. The decentralization assumption implies a partially fixed structure of compensators and this results in the emergence of the following two phenomena. First, we have the appearence of the concept of fixed modes [Cor.1],[And.1],[Kar.4] which may arise in the study of pole assignment by decentralised state, or output feedback and may restrict the assignability property. Secondly, the decentralised controllers may be viewed as a subvariety of a Grassmann variety [Kar.4] and thus its topology and intersection theory is not well established. This subvariety is characterised by the set of Quadratic Plucker Relations and a set of fixed zeroes defined by the decentralization characteristic of the given problem [Kar.4]. An alternative compactification was recently introduced in [Wang.2], where the decentralised compensator is viewed as an element of product of Grassmannians. In this chapter we extend the algebrogeometric framework for decentralized problems established in [Kar.4] as well as the framework intoduced in [Wang.2], and derive new sufficient conditions for generic pole assignability. Furthermore, the properties of the pole placement map established in Chapter6 are extended to the decentralised case and this leads to a new test for avoiding the presence of fixed modes using the notion of decentralised Markov parameters.

9.2. Problem formulation

As it was explained in Ch.5 the pole placement problems via decentralised controllers can be reduced to the solution of the equation:

$$p(s) = \det\left(sI - A - \sum_{i=1}^{\kappa} B_i K_i C_i \right) \tag{9.1a}$$

or

$$p(s)=\det\left([\ I_p\ ,\ K_{dec}]\begin{bmatrix}D(s)\\N(s)\end{bmatrix}\right) \tag{9.1b}$$

with respect to K_{dec}, where

$$K_{dec}=\begin{bmatrix}K_1 & 0 & \dots & 0\\ 0 & K_2 & 0 & :\\ : & 0 & & 0\\ 0 & \dots & 0 & K_k\end{bmatrix} \tag{9.2}$$

$K_i \in \mathbb{R}^{p_i x m_i}$, $\sum p_i=p$ and $\sum m_i=m$. Similarly, the zero assignment problem may be reduced to the solvability of the equation

$$f(s) = \det\ (\ K_{dec}\bar{N}(s)) \tag{9.3}$$

with respect to K_{dec} given by (9.2).

The polynomial equations (9.1b),(9.3), which we are interested in solving, can be reduced to n algebraic equations by equating the corresponding coefficients of the left and right hand side polynomials. These equations have to be solved with respect to K_{dec} which, in turn, contains $\Sigma m_i p_i$ unknowns. Thus we now have to solve a system of n algebraic equations in $\mathbb{R}^{\Sigma m_i p_i}$ or else, find the intersection of n algebraic hypersurfaces in the same affine space.

To study such intersections we normally need to compactify the parameter space of the unknowns and then use the intersection theory available for compact manifolds. The new compact parameter space contains the initial parameter space, which covers almost the whole of it, and a negligible set of points at "infinity". In this way, the extended solution set of our system of equations may contain points at "infinity" some of which are not desirable (see Sec.3.5.1 and Sec.6.5), but provided the compactification is natural enough, we can deduce that, generically, if the solution set of the system of equation is nonvoid then it must contain a finite point (see Sec.3.5.1, and [Byr.1]).

The way in which we extend the parameter space of the unknowns into a compact set depends mainly upon the nature of the equations and the way we define infinity. Here we will present two compactifications for the set of decentralised feedback controllers ($\approx \mathbb{R}^{\Sigma m_i p_i}$) and our approach will be projective (here will not need to compactify the set of decentralised squaring down compensators) . The method used will involve embedding $\mathbb{R}^{\Sigma m_i p_i}$ into $\mathbb{R}^{\sigma\text{-}1}$ in a way the nature of the problem suggests. Subsequently, we will consider Zarisky closure of the image of this embedding and thus get a projective variety in $P\mathbb{R}^{\sigma\text{-}1}$; in this way, the extended parameter space obtains the stronger structure of the projective variety, rather than the structure of the compact manifold (this makes no difference in our subsequent analysis since the intersection theory we will be using will be cohomology with coefficients in $\mathbb{Z}_2$). The special structure of the problem arises from the following two characteristics of the expression of the closed loop pole polynomial of eq.(9.1b):

(i) The existence of the determinant. This induces a multilinear skew-symmetric nature to the problem and characterises all determinantal assignment problems (pole assignmnet via output/state feedback[Gia.1], zero assignment via squaring down[Kar.3]).

(ii)The block diagonal structure of the output feedback matrix, which is due to the decentralisation assumption.

The two equivalent compactifications which will be presented in the following two sections can be described briefly as:

First compactification: As a subvariety of the Grassmannian $G_p(\mathbf{R}^{p+m})$ embedded in $\mathbf{PR}^{\sigma\text{-}1}$, $\sigma=\binom{p+m}{p}$ via the Plucker embedding. This was first introduced in [Kar.4] as decentralised Grassmann variety.

Second compactification: As a product of Grassman varieties embedded in $\mathbf{PR}^{\sigma_1\text{-}1}$ via a combination of the Plucker and Segre embeddings, where $\sigma_1=\prod_{i=1}^{\kappa}(n_i+1)$ and $n_i=\binom{p_i+m_i}{p_i}$. This was first introduced in [Wang.3].

9.3. Decentralised Grassmann variety and invariants

The multilinear skew-symmetric nature of the problem, leads us to breaking this problem into two : one which is pure multilinear and a second, which is linear and suggests the following compactification of $\mathcal{F}_{dec}$ [Kar.4]. Using the Binet-Cauchy theorem, we derive

$$\det\left([\, I_p \,,\, K_{dec}] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) = C_p([\, I_p \,,\, K_{dec}])C_p\left(\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) \tag{9.4}$$

where $C_p(A)$ is the p-th compound matrix of A which, in other words, is the p-th exterior or skew product power ($\wedge^p$) naturally used for the factorisation of multilinear skew-symmetric functions. If we define $[1,\ \underline{k}^T] \in \mathbb{R}^{1x\sigma}$, $\sigma=\binom{p+m}{p}$, to be a vector such that

$$[1,\ \underline{k}^T]=C_p([\, I_p \,,\, K_{dec}]), \tag{9.5}$$

P_S to be a $\sigma x(n+1)$ Plucker matrix coefficient of the polynomial Grassmann representative [Kar.1]

$$\underline{g}(s)= C_p\left(\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) = P_S\, [\, s^n,\ s^{n-1},\ldots,\ 1]^T \tag{9.6}$$

and $[1,\ \underline{p}^T] \in \mathbb{R}^{1x(n+1)}$ to be a vector such that

$$p(s)=[1,\ \underline{p}^T]\ [\, s^n,\ s^{n-1},\ldots,\ 1]^T$$

then the solution of eq.(9.1) can be reduced to:

(i) LINEAR PROBLEM: For a given $[1,\ \underline{p}^T] \in \mathbb{R}^{1x(n+1)}$, solve the linear equation,

$$[1,\ \underline{p}^T]=[1,\ \underline{k}^T].P_S \tag{9.7}$$

with respect to $[1,\ \underline{k}^T] \in \mathbb{R}^{1x\sigma}$.

(ii) MULTILINEAR PROBLEM: Assume that the linear problem has a non void solution set, say $\mathcal{K}$,then determine whether there exists $[1,\ \underline{k}^T] \in \mathcal{K}$ such that $[1,\ \underline{k}^T]=C_p([\, I_p \,,\, K_{dec}])$ for some $[I_p \,,\, K_{dec}]$; the vectors $[1,\ \underline{k}^T]$, having the above properties, have coordinates which satisfy the Quadratic Plucker Relations (see Sec.3.3.2). The block diagonal structure of $[I_p, K_{dec}]$ implies that the multivector $[1, \underline{k}^T]$ has fixed zeroes at certain locations. These locations define the decentralisation characteristic which was introduced and calculated in [Kar.4]. If $\mathbb{D}$ is the decentralization characteristic (ie. the sequences $\omega \in Q_p{}^{m+p}$ for which the corresponding coordinates in (9.5) are identically zero and $\mathbb{D}^C$ is the complementary set with respect to $Q_p{}^{m+p}$, then the subvector defined from $\underline{g}(s)$ by dropping the $\mathbb{D}$ coordinates is denoted by

$$\underline{g}_{dec}(s)=P_{dec}\,[s^n, s^{n-1},...,1]^T$$

and it is refered to as the *decentralised Grassmann representative*; the matrix coefficient P_{dec} is the *decentralized Plucker matrix* [Kar.4].

Remark (9.1) [17]: The fixed modes of the problem are defined as the zeros of the greatest common devisor of the entries of $\underline{g}_C(s)$. Necessary conditions for arbitrary pole assignability is that $\underline{g}_C(s)$ is coprime and rank of P_C is equal to n+1.

□

In the following we shall assume that we have no fixed modes for the generic $\mathcal{S}(A,B,C)$ system.

The above analysis suggests to embed $\mathcal{F}_{dec}=\mathbb{R}^{\Sigma m_i p_i}$ in $\mathbb{R}^\sigma$ via $\phi= \wedge^p \circ \delta$ as follows:

$$K_{dec} \xrightarrow{\delta} [\, I_p\, ,\, K_{dec}] \xrightarrow{\wedge^p} [1, \underline{k}^T]= C_p([\, I_p\, ,\, K_{dec}]) \tag{9.8}$$

Now if we consider the Zarisky closure of $\phi(\mathcal{F}_{dec})$ in $P\mathbb{R}^{\sigma-1}$, we obtain a characterisation of the Decentralised Grassmann varieties as shown below:

$$\left(\mathcal{F}_{dec} \approx \right) \overline{\phi(\mathcal{F}_{dec})}=\Big\{ [\lambda, \underline{k}^T] \in P\mathbb{R}^{\sigma-1}\text{: has zero coordinates at the decentralisation characteristics and satisfies the QPR (Quadratic Plucker Relations)}\Big\}$$

Remark (9.2) $\overline{\phi(\mathcal{F}_{dec})}$ can be considered as a subvariety of the Grassmann variety

$G_p(\mathbb{R}^{p+m})$ of p-dimensional subspaces of $\mathbb{R}^{p+m}$. The dimension of this subvariety is equal to $\Sigma m_i p_i$ and the equations defining it are given by the QPRs and the zero coordinates corresponding to the decentralisation characteristic.

□

If we write the Plucker matrix P_S as:

$$P_S = \begin{bmatrix} 1 & | & \underline{a}^T \\ — & + & — \\ \underline{0} & | & \hat{P}_S \end{bmatrix} \tag{9.9}$$

The linear equations (9.7) can then be homogenised as:

$$\underline{0} = [\lambda \ , \ \underline{k}^T] \begin{bmatrix} \underline{a}^T - \underline{p}^T \\ \text{------} \\ \hat{P}_S \end{bmatrix} \tag{9.10}$$

These n linear equations in $P\mathbb{R}^{\sigma-1}$ define a projective variety of dimension greater than or equal to σ-1-n symbolised by $L\mathbb{R}(S,p(s))$. The intersection $L\mathbb{R}(S,p(s)) \cap \overline{\mathcal{F}_{dec}}$ contains all decentralised compensators (finite and infinite) which assign the polynomial p(s) to the system S via output feedback. The desired controllers are those with $\lambda \neq 0$ (finite controllers) and correspond to composite representations $[I_p, K_{dec}]$.

Example(9.1) Consider the decentralised scheme with $m_1 = p_1 = 2$, $m_2 = p_2 = 2$. The composite feedback matrix $F_{dec} = [I_4, K_{dec}]$ may be expressed as

$$[\ I_4 \ , K_{dec}] = \begin{bmatrix} 1 & 0 & 0 & 0 & k_{11} & k_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 & k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & 0 & 1 & 0 & 0 & k_{43} & k_{44} \end{bmatrix}$$

and the closed loop pole polynomial may be expressed as

$$p(s) = \det\left([I_p, \ K_{dec}] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right)$$

Let Q be the coordinate transformation defined by

$$Q=\begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & I_2 \end{bmatrix}$$

Clearly, QQ=I and thus we may write

$$p(s)= \det\left([I_p, K_{dec}]\ Q\cdot Q\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\right) = \det(\tilde{F}_{dec}\ \tilde{T}(s))$$

where

$$\tilde{T}(s)= Q\begin{bmatrix} D(s) \\ N(s)\end{bmatrix},\quad \tilde{F}_{dec} = [I_4, K_{dec}]\ Q$$

We may notice that

$$\tilde{F}_{dec} = \begin{bmatrix} 1 & 0 & k_{11} & k_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & k_{21} & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & k_{33} & k_{34} \\ 0 & 0 & 0 & 0 & 0 & 1 & k_{43} & k_{44} \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix}$$

and thus for $C_4(\tilde{F}_{dec})$ we are able to make the following two observations:

(i) The nonidentically zero coordinates in $C_4(\tilde{F}_{dec})$ correspond to the sequences $\omega=(i_1, i_2, i_3, i_4) \in Q_4^8$ where (i_1, i_2) take values in [1,2,3,4] and (i_3, i_4) take values in [5,6,7,8]. These sequences in Q_4^8 are reffered to as *non degenerate* sequences. All other remaining sequences in Q_4^8 have the property that the corresponding minors are identically zero and are reffered to as *degenerate* sequences of Q_4^8.

(ii) The degenerate sequences are those for which either fewer than two, or more than two of the integers are taken from the [1,2,3,4] or [5,6,7,8] intervals. For instance $a_{1,2,3,6}=0$ and $a_{3,6,7,8}=0$.

(iii) The coordinates in $C_4(\tilde{F}_{dec})$ which correspond to the nondegenerate sequences $\omega=(i_1, i_2, i_3, i_4)$ may be expressed as

$$a_\omega = a_{1,2,3,4} = c^{1}_{i_1 i_2} \cdot c^{2}_{i_3 i_4}$$

where $c^{1}_{i_1 i_2} \cdot c^{2}_{i_3 i_4}$ are the coordinates in $C_2(\tilde{F}_2)$ (note that the columns of $\tilde{F}_1$ are numbered as [1,2,3,4] and those of $\tilde{F}_2$ as [5,6,7,8]). The above property clearly suggests that if

$$\tilde{F}_1 = \begin{bmatrix} \underline{f}^t_{11} \\ \underline{f}^t_{12} \end{bmatrix}, \tilde{F}_1 = \begin{bmatrix} \underline{f}^t_{21} \\ \underline{f}^t_{22} \end{bmatrix}$$

then the nonzero coordinates in $C_4(\tilde{F}_{dec})$ are given as the tensor product of the exterior product of $\tilde{F}_1$, $\tilde{F}_2$ ie.

$$[C_4(\tilde{F}_{dec})]_{\neq 0} = (\underline{f}^t_{11} \wedge \underline{f}^t_{12}) \otimes (\underline{f}^t_{21} \wedge \underline{f}^t_{22})$$

The two alternative compactifications adopted here reflect the above observations, ie.

(a) We may consider $C_4(\tilde{F}_{dec})$ as a vector in $\mathbb{P}(\mathbb{R}^{\binom{8}{4}})$ with certain fixed zeros coordinates, which also belongs to the corresponding Grassmann variety. The set of all such (decomposable) vectors of $\mathbb{P}(\mathbb{R}^{\binom{8}{4}})$, is the decentralised Grassmann variety described in this chapter.

(b) We can consider only the nondegenerate subset of coordinates of $C_4(\tilde{F}_{dec})$ which as a vector, is the tensor product of the exterior products of the rows of $\tilde{F}_1$ and $\tilde{F}_2$. All these vectors which can be written as a tensor product of (two) decomposable vectors constitute a product of (two) Grassmannians [Wang.3]. This set will be considered next.

□

9.4. The Decentralised problem and the product compactification formulation

The entries of $[\lambda, k^T]$ corresponding to the degenerate sequences, do not play a role in the decentralised pole placement and can be omitted. The remaining vector $[\lambda, k^T{}_{dec}]$ lies in a projective space $P\mathbb{R}^{\sigma_1 - 1}$ of lower dimension and can be taken as products of the Plucker coordinates of the individual blocks as was shown in the previous example. This suggest a second compactification for $\mathcal{F}_{dec}$ as a product of Grassmannians embedded in an appropriate projective space by a combination of the Plucker and Segre embedding (tensor product) and was first presented in [Wang.3].

The set $\mathcal{F}_{dec} \approx \mathbb{R}^{\Sigma m_i p_i}$ of decentralised controllers can be written as the following product:

$$\mathcal{F}_{dec} = \mathop{\times}_{i=1}^{\kappa} \mathcal{F}_i \tag{9.11}$$

where $\mathcal{F}_i$ is the set of local controllers and it is isomorphic to $\mathbb{R}^{m_i p_i}$ for every $i=1, 2,\dots,\kappa$. Each $\mathcal{F}_i$ can be embedded into $\mathbb{R}^{n_i}$ where $n_i = \binom{p_i+m_i}{p_i}$, via the embedding ϕ_i defined below:

$$K_i \xrightarrow{\delta} [\, I_p \,,\, K_i] \xrightarrow{\wedge^p} C_p([\, I_p \,,\, K_i]) \tag{9.12}$$

The closure of the image $\phi_i(\mathcal{F}_i)$ is the Grassmann variety $G_{p_i}(\mathbb{R}^{p_i+m_i})$ in $P\mathbb{R}^{n_i}$ and thus, $\overline{\mathcal{F}_{dec}}$ is considered to be the product:

$$\overline{\mathcal{F}_{dec}} = G_{p_1}(\mathbb{R}^{p_1+m_1}) \times G_{p_2}(\mathbb{R}^{p_2+m_2}) \times \dots \times G_{p_k}(\mathbb{R}^{p_k+m_k}) \tag{9.13}$$

in $P\mathbb{R}^{n_1} \times \dots \times P\mathbb{R}^{n_\kappa}$. We can further embed the above product into $P\mathbb{R}^{\sigma_1 - 1}$, where $\sigma_1 = \prod_{i=1}^{\kappa} (n_i+1)$ via the Segre embedding Seg [Sha.1]:

$$\mathrm{Seg}\,(x_1,\dots, x_\kappa) = x_1 \otimes x_2 \otimes \dots \otimes x_\kappa \tag{9.14}$$

The image of $G_{p_1}(\mathbb{R}^{p_1+m_1}) \times G_{p_2}(\mathbb{R}^{p_2+m_2}) \times \dots \times G_{p_k}(\mathbb{R}^{p_k+m_k})$ in $P\mathbb{R}^{\sigma_1 - 1}$ via the Segre embedding is a projective variety which is the required structure for the set $\overline{\mathcal{F}_{dec}}$.

One can easily see that the equations for the pole placing decentralised controllers in $P\mathbb{R}^{\sigma_1 -1}$ are:

(i) The equations defining the above product of Grassmannians in $P\mathbb{R}^{\sigma_1 -1}$, $\overline{\mathcal{F}_{dec}}$.

(ii) The n linear equations:

$$[\lambda \, , \, \underline{k}^T{}_{dec}] \, \hat{P}_{dec} = 0 \tag{9.15}$$

where $\hat{P}_{dec}$ can be taken if we omit the rows, of the matrix of eq.(9.10), corresponding to the decentralisation characteristic [Kar.4] $\mathbb{D}$.

Remark(9.3) It is worth rewriting the n linear equations in a pole placement formulation rather than in the polynomial coefficient formulation which was given previously in terms of the decentralised Plucker matrices. In fact, let $s_1, s_2, \ldots, s_n$ be the set of roots of the polynomial p(s) of equation(9.1). Then equation(9.1) is equivalent to

$$\det\left([I, K_{dec}] \begin{bmatrix} D(s_i) \\ N(s_i) \end{bmatrix}\right) = 0 \qquad \forall \, i=1,2,..,n \tag{9.16}$$

If $\underline{\ell}_i = C_p\left(\begin{bmatrix} D(s_i) \\ N(s_i) \end{bmatrix}\right) \in \mathbb{C}^{\sigma x1}$ and $L=[\underline{\ell}_1, \underline{\ell}_2, \ldots, \underline{\ell}_n] \in \mathbb{C}^{\sigma xn}$ then the linear equations(9.15) can be

rewritten as $[\lambda, \, \underline{k}^T] \, L=0$ and if we keep the rows of L that correspond to decentralisation characteristics and omit the others we get

$$[\lambda, \, \underline{k}^T{}_{dec}] \, \hat{L}=0. \tag{9.17}$$

□

Remark (9.4) Note that the equations defining $\overline{\mathcal{F}_{dec}}$ in $P\mathbb{R}^{\sigma_1 -1}$ are given by the QPRs of $P\mathbb{R}^{\sigma_1 -1}$ with the Plucker coordinates corresponding to the decentralisation characteristic being set to zero.

□

The product compactification defined as above is not different from the one defined in the previous chapter. In fact the two compactifications are isomorphic as projective varieties as it is shown by the following result:

Theorem(9.1) The product of Grassmannians $G_{p_1}(\mathbb{R}^{p_1+m_1})$ x $G_{p_2}(\mathbb{R}^{p_2+m_2})$ x ... x $G_{p_k}(\mathbb{R}^{p_k+m_k})$ considered as subvariety of $P\mathbb{R}^{\sigma_1-1}$ is isomorhic to $\overline{\phi(\mathcal{F}_{dec})}$ $\subset G_p(\mathbb{R}^{p+m}) \subset P\mathbb{R}^{\sigma-1}$.

Proof

We first construct a map

$$f: \quad G_{p_1}(\mathbb{R}^{p_1+m_1}) \times G_{p_2}(\mathbb{R}^{p_2+m_2}) \times \dots \times G_{p_k}(\mathbb{R}^{p_k+m_k}) \ \left(\subset P\mathbb{R}^{\sigma_1-1}\right) \longrightarrow P\mathbb{R}^{\sigma-1}$$

by considering an $\underline{x} \in G_{p_1}(\mathbb{R}^{p_1+m_1})$ x $G_{p_2}(\mathbb{R}^{p_2+m_2})$ x ... x $G_{p_k}(\mathbb{R}^{p_k+m_k})$ $\subset P\mathbb{R}^{\sigma_1-1}$ which, in turn, can be written as

$$\underline{x} = \underline{x}_1 \otimes \underline{x}_2 \otimes \dots \otimes \underline{x}_\kappa$$

where $\underline{x}_i$ are decomposable vectors of $P\mathbb{R}^{n_i}$ for i=1,2,..,κ respectively. A new vector, composed of the vector $\underline{x}$ and some extra zero coordinates, is constructed. These new coordinates correspond to the set of degenerate sequences $\mathbb{D}^c$ which is the complementary set of the set of decentralisation characteristics $\mathbb{D}$. This new vector is the vector $f(\underline{x})$ and belongs to $P\mathbb{R}^{\sigma-1}$ by construction.

The map f is obviously one to one and $f(\underline{x})$ is a decomposable vector of $P\mathbb{R}^{\sigma-1}$. In this way, f injects the above product of Grassmannians into the larger Grassmannian $G_p(\mathbb{R}^{p+m})$ and consequently this product can be identified by Im(f) contained in the Grassmannian $G_p(\mathbb{R}^{p+m})$. Readily, the image of f contains $\phi(\mathcal{F}_{dec})$ and thus its closure $\overline{\phi(\mathcal{F}_{dec})}$, and since $\dim(\mathrm{Im}(f))=\dim(\phi(\mathcal{F}_{dec}))=\Sigma m_i p_i$ and Im(f) is irreducible, we can imply that $\mathrm{Im}(f)=\overline{\phi(\mathcal{F}_{dec})}$.

□

The previously established equivalence of the two compactifications allows us to use either of them for the study of the decentralised compensation. In the next chapter we will examine the problem of finding sufficient conditions for the solvability of the arbitrary pole placement problem using decentralised output feedback. The approach will be topological and the compactification as product of Grassmannians will be used. The reason behind this, is that we can easily compute the cohomology ring of the product in terms of the cohomology rings of the particular Grassmannians via the Kunneth decomposition [Dold.1] , and thus we can relate the various topological invariants of the product with some well established invariants of the Grassmannians.

9.5.Sufficient conditions for the existence of solutions

To derive sufficient conditions for the existence of solutions we use the product compactification of the decentralised controllers, that is to say

$$\overline{\mathcal{F}_{dec}} = G_{p_1}(\mathbb{R}^{p_1+m_1}) \times G_{p_2}(\mathbb{R}^{p_2+m_2}) \times \dots \times G_{p_k}(\mathbb{R}^{p_k+m_k}) \tag{9.18}$$

As we mentioned previously the linear problem defines the n-linear equations (9.15) on $\overline{\mathcal{F}_{dec}}$. Each of these linear equations defines a linear hypersurface on $\overline{\mathcal{F}_{dec}}$ which, in turn, defines an element c in $H^2(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$ (the first cohomology group of the cohomology ring $H^*(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$) and an element w in $H^1(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$ (the first cohomology group of the cohomology ring $H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$). All n-equations will define the intersection of n-linear hypersurfaces on $\overline{\mathcal{F}_{dec}}$ and the corresponding cohomology elements for this intersection will be c^n in $H^{2n}(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$ and w^n in $H^n(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$. Following [Lev.1] (see also Ch.3 and 7) we can express c^n as a $\mathbb{Z}$ linear combination of the elements of the basis of $H^{2n}(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$. These coefficients are the orders of the subvarieties of $\overline{\mathbb{C}\mathcal{F}_{dec}}$ represented by the basis and if one of them is odd, then we can find a real solution since the solutions occur in conjugate pairs.

Remark (9.5): The set of points corresponding to the intersection of the linear variety with the odd order subvariety may contain a subset of degenerate points (as in chapter 7); these are the points rowspan$[A,K_{dec}] \in \overline{\mathbb{C}\mathcal{F}_{dec}}$ such that:

$$\det\left([\, A,\ K_{dec}] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right)=0 \quad \forall s$$

however, as it has been shown in [Wang.3], when the dimensions of linear and the odd order family of subvarieties representing the basis are complementary, then we can always select an element of the family to intersect the linear subvariety without degenerate points.

□

The coefficients of the expansion of c^n appear reduced mod2 in the expansion of w^n in $H^n(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$. Thus, as far as examining whether there exist an odd coefficient of the expansion of c^n, is equivalent to examining whether w^n is non-zero. The condition for this will be of the form $n \le h(w)$ where $h(w)$ is the maximum integer k such that w^k is non-zero in $H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$ and is called the height of the cohomology class w. Given that $h(w)$ is entirely defined by the partitioning of the inputs and outputs into (m_i,p_i), $1 \le i \le \kappa$ sets, $h(w)$ will be reffered to in the following as the *decentralisation height.* The inequality $n \le h(w)$ gives us a sufficient condition for the existence of a real decentralised static output feedback arbitrarily placing the poles of a generic system and it is stated below:

Proposition (9.1) For a given set (m_i, p_i) $i \in \underline{k}$, a sufficient condition for decentralised pole assignability is that the number of states must be less or equal to the maximum k such that w^k is non-zero in $H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$, or equivalently $h \le h(w)$.

□

To calculate both c^n and w^n it is convenient to use the Kunneth decomposition [Dold.1] of $H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$ and $H^*(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$. This tells us that

$$H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2) = H^*(\ G_{p_1}(\mathbb{R}^{p_1+m_1});\ \mathbb{Z}_2) \otimes \ldots \otimes H^*(G_{p_k}(\mathbb{R}^{p_k+m_k});\mathbb{Z}_2) \quad (9.19a)$$

and

$$H^*(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z}) = H^*(\ G_{p_1}(\mathbb{C}^{p_1+m_1});\ \mathbb{Z}) \otimes \ldots \otimes H^*(G_{p_k}(\mathbb{C}^{p_k+m_k});\mathbb{Z}) \quad (9.19b)$$

which in turn gives

$$H^1(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2) \approx H^1(\ G_{p_1}(\mathbb{R}^{p_1+m_1});\ \mathbb{Z}_2) \otimes \ldots \otimes H^1(G_{p_k}(\mathbb{R}^{p_k+m_k});\mathbb{Z}_2) \quad (9.20a)$$

and

$$H^1(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z}) \approx H^1(\ G_{p_1}(\mathbb{C}^{p_1+m_1});\ \mathbb{Z}) \otimes \ldots \otimes H^1(G_{p_k}(\mathbb{C}^{p_k+m_k});\mathbb{Z}) \quad (9.20b)$$

Each $H^1(\ G_{p_i}(\mathbb{R}^{p_i+m_i});\ \mathbb{Z}_2)$ is a $\mathbb{Z}_2$ vector space produced by w_i the first Whitney class of the canonical bundle [Gri.1, Sto.1] of this Grassmannian $G_{p_i}(\mathbb{R}^{p_i+m_i})$. This means that $H^1(G_{p_i}(\mathbb{R}^{p_i+m_i}); \mathbb{Z}_2)$ is the additive group $\{0, w_i\}$ such that $w_i+w_i=0$. Since w belongs to $H^1(G_{p_1}(\mathbb{R}^{p_1+m_1}); \mathbb{Z}_2) \otimes \ldots \otimes H^1(G_{p_k}(\mathbb{R}^{p_k+m_k});\mathbb{Z}_2)$, it can be decomposed into:

$$w=\sum_{i=1}^{k} \epsilon_i w_i \tag{9.21}$$

where ϵ_i are in $\mathbb{Z}_2$ (are either 0 or 1). Similarly, c can be decomposed in $H^2(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z})$ as:

$$c=\sum_{i=1}^{k} a_i c_i \tag{9.22}$$

where a_i are in $\mathbb{Z}$ and c_i are the first Chern classes [Gri.1] of the Grassmannians $G_{p_i}(\mathbb{R}^{p_i+m_i})$ and w_i is the corresponding Whitney class. The following theorem indicates that the values of all ϵ_i's and a_i's are 1.

Theorem (9.2) The cohomology class $c \in H^2(\overline{\mathbb{C}\mathcal{F}_{dec}};\mathbb{Z}_2)$ (complex case) which corresponds to linear hypersurface X of $\overline{\mathcal{F}_{dec}}$ is equal to:

$$c=\sum_{i=1}^{k} c_i \tag{9.23a}$$

Similarly, the w cohomology class $H^1(\mathcal{F}_{dec};\mathbb{Z})$ may be expressed as:

$$w=\sum_{i=1}^{k} w_i \tag{9.23b}$$

Proof

To find the cohomology class that corresponds to the continuous family of hypersurfaces on $\overline{\mathbb{C}\mathcal{F}_{dec}}$ which are parametrised by the set of strictly proper systems of p-inputs, m-outputs and n-states it is sufficient to find the cohomology class corresponding to a special representative of this family [Klei.1] . This representative arises from a system of the form:

$$\begin{bmatrix} D(s)' \\ N(s)' \end{bmatrix} = \begin{bmatrix} D_1(s) & 0 & \dots & 0 \\ 0 & D_2(s) & & : \\ : & & : & 0 \\ 0 & \dots & 0 & D_k(s) \\ N_1(s) & 0 & \dots & 0 \\ 0 & N_2(s) & & : \\ : & & : & 0 \\ 0 & \dots & 0 & N_k(s) \end{bmatrix} \tag{9.24}$$

we can easily see that the closed loop pole polynomial p(s) is given by:

$$p(s) = \det\left\{[I_p, K_{dec}] \begin{bmatrix} D(s)' \\ N(s)' \end{bmatrix}\right\} = \prod_{i=1}^{k} \left\{\det\left\{[I_{p_i}, K_i] \begin{bmatrix} D_i(s) \\ N_i(s) \end{bmatrix}\right\}\right\} \tag{9.25}$$

A linear hypersurface on $\overline{\mathbb{C}\mathcal{F}_{dec}}$ corresponds to the assignment of 1 pole say at s_0 (see Remark(9.1)) and this equation becomes:

$$\prod_{i=1}^{k} \left\{\det\left\{[I_{p_i}, K_i] \begin{bmatrix} D_i(s_0) \\ N_i(s_0) \end{bmatrix}\right\}\right\} = 0 \tag{9.26}$$

Equivalently, this corresponds to a union of the linear hypersurfaces

$$\det\left\{[I_{p_i}, K_i] \begin{bmatrix} D_i(s_0) \\ N_i(s_0) \end{bmatrix}\right\} = 0 \tag{9.27}$$

for i=1,2,...,k on every $G_{p_i}(\mathbb{C}^{p_i+m_i})$ respectively. It is known [Lev.1] that each of these hypersurfaces corresponds to c_1 in $G_{p_i}(\mathbb{C}^{p_i+m_i})$, the first Chern class of the Grassmannian. Finally, the union of these hypersurfaces correspond to the sum of the above classes.

□

Remark (9.6): The above result indicates that the original decentralized problem concerning the placement of one pole is topologically equivalent to a determinantal problem defined on a subsystem. □

For the subsequent analysis, we need some further definitions.

Definition(9.1) A binary partition t , of the number n of length k is a sequence of *non-negative integers* t(1), t(2),...,t(κ), such that:

$$n=t(1)+t(2)+...+t(k) \tag{9.28}$$

and for every j, there is at most one 1 in all jth digits of the binary representations of t(1),t(2),...,t(κ).

□

Example (9.2) Consider the number n=10 and k=2. Then n is represented in a binary way as (1010) and its binary partitions are

$$(1010)=(1000)+(0010), \qquad (1010)=(1010)+(0000)$$

Note that although

$$(1010)=(0110)+(0100)$$

the above partition is not binary since the 1's appear in the same location.

□

With the above definition in mind we may state the following result:

Theorem(9.3) The height of $w=\sum_{i=1}^{k} w_i$, (denoted by h(w)), in $H^*(\overline{\mathcal{F}_{dec}};\mathbb{Z}_2)$ is equal to the maximum integer n for which there exists a binary partition t of n of length κ such that:

$$t(i) \leq h(p_i,m_i) \qquad i=1,2,....\kappa \tag{9.29}$$

Proof

Let us write the exponent n of w in the binary form:

$$n=2^{h_r}+2^{h_{r-1}}+\ldots+2^{h_2}+2^{h_1}$$

where $h_r > h_{r-1} > \ldots > h_2 > h_1 \geq 0$. For every h_i we have that

$$w^{2^{h_i}}=\Big(\sum_{i=1}^{k} w_i\Big)^{2^{h_i}}=\Big(\sum_{i=1}^{k} w_i^{2^{h_i}}\Big)$$

since the coefficients in the above expansion are taken reduced mod2 . Therefore,

$$w^n=\prod_{j=1}^{r}\Big(\sum_{i=1}^{k} w_i^{2^{h_j}}\Big)$$

Thus it can be readily shown that

$$w^n=\sum_{t} w_1^{t(1)}w_2^{t(2)}\cdots w_{\kappa-1}^{t(\kappa-1)}w_{\kappa}^{t(\kappa)} \tag{9.30}$$

where the sum is taken for all binary partitions of n, t, of length κ. Now it is apparent that w^n is nonzero, iff there exist at least a nonzero summand in the expansion (9.30). This happens, iff the exponents of w_i in this summand are less than the corresponding heights $h(p_i,m_i)$ and this proves the result.

□

An alternative proof of the above result is to consider the binomial expansion. It should be pointed out that terms in this expansion disappear for two reasons: (i) the coefficient is an even number and (ii) at least one of the exponents of w_i^k in the expansion is more than the corresponding height. If we elaborate on these two cases we derive the conditions of the above theorem. The above result reduces the search for sufficient conditions into a problem testing whether a certain set of binary partitions satisfy the height conditions (9.29). Note that the individual heights $h(p_i,m_i)$ may be computed as in [Lev.1]. This provides a methodology for a systematic search for sufficient conditions and it is further examined below. Using the above result and Proposition (9.1) we may now state:

Theorem (9.4): A sufficient condition for arbitrary pole placement by a real static decentralised output feedback for a strictly proper system with n states and κ (m_i,p_i) channels, is that there exists a κ lenght binary partition of n, say $\{t(1),t(2),...,t(\kappa)\}$ such that

$$t(i) \leq h(p_i,m_i) \qquad \text{for every } i=1,...,\kappa \tag{9.31}$$

Proof: The proof follows by using theorem (9.2) and proposition(9.1).

□

Example(9.3) Let $p_1=2$, $m_1=4$, $p_2=1$, $m_2=3$ and $n=9=(1010)$ then $h(p_1,m_1)=6$ and $h(p_2,m_2)=3$ The two binary partitions of n of length 2 are

$$n=(1000)+(0010)=8+1$$
$$n=(1010)+(0000)=9+0$$

and since 9 and 8>6 and 3, the test cannot be applied. On the other hand if $n=7=(111)$ we have the partition

$$n=(101)+(010)=5+2$$

which satisfies the inequalities $5<h(p_1,m_1)=6$ and $2<h(p_2,m_2)=3$ and therefore in this case the generic pole assignability property holds true.

□

Corollary(9.1) The height for the class w takes its maximum value $\Sigma m_i p_i$, iff the following conditions hold true:

i) The particular heights for subsystems take their maximum value ie.

$$h(p_i,m_i)= p_i m_i \qquad \text{for all } i \tag{9.32}$$

ii) The binary representations t(i) of $p_i m_i$ constitute a binary partition for $\Sigma m_i p_i$.

Proof:

Again this is an obvious application of theorem(9.2).

□

Corollary(9.2) If $\sum m_i p_i \geq n$ and the conditions (i) and (ii) of corollary(9.1) hold true then, we have arbitrary pole placement via real decentralised static output feedback for a generic strictly proper system of p-inputs, m-outputs and n-states. □

Corollary (9.2) is equivalent to the result in [Wang.3] which gives a sufficient condition for existence of solution when the degree of the product Grassmannian is odd. The oddness of the degree of the product of Grassmannians can occur only under the very strong assumptions (i) and (ii) of corollary(9.1) or the equivalent ones of [Wang.3]. Thus the sufficient condition of Corollary (9.2) has only restricted use. In the very likely case that the above degree is even, we may have lower dimensional subvarieties which have odd degree. In such a case, theorem(9.3) can be applied and a sufficient condition can therefore be derived. The following example demonstrates the above argument.

Example(9.4) Let $p_1=2$, $m_1=4$, $p_2=1$, $m_2=3$, then we may compute the heights and have $h(p_1,m_1)=6$ and $h(p_2,m_2)=3$. In this case, since $p_1.m_1=2.4=8> h(p_1,m_1)=6$, the degree of the product of Grassmannians must be even. However, we can apply theorem(9.3) and derive that if n=7 then we have generic pole assignability(see example(9.3)).

□

Theorem (9.3) provides two different routes to search for sufficient conditions. The first is to work through all partitions and use the standard expressions for height for each of the subsystems, referred to as the "partitioning approach", and the second is to work out sufficient conditions which are weaker, but are independent from the individual partitioning. We demonstrate the partitioning approach in terms of the following example:

Example (9.5)

$$k=2,\ p_1=2,\ p_2=3,\ m_1=2,\ m_2=3$$

$$n=7 \rightarrow 2^2+2+1=(111) \text{ in binary form}$$

All possible binary partitions are as follows,

$$\{(2^2)\ ,\ (2^1+2^0)\} \quad \{(2^2+2^1)\ ,\ (2^0)\} \quad \{(2^2+2^0)\ ,\ (2^1)\} \quad \{(2^2+\ 2^1+2^0)\ ,\ (0)\}$$

and their permutations.

The *heights* for the given decetralisation are:

(1) $m_1=2,\ p_1=2$
$2^1 \leq 2+2-1 < 2^2$
$h(2,2)= 2^2-2=2$

(2) $m_2=3,\ p_2=3$
$2^2 \leq 3+3-1=7 < 2^3$
$h(3,3)= 2^3-1=7$

Note that for the above problem the results in [Wang.1] cannot be applied since:
(i) We do not have equal numbers in inputs or outputs.
(ii) The product Grassmannian has not odd order. This can be readily seen from the fact that the heights in this case h(2,2)=2, h(3,3)=7 do not take the maximum values which are 4 in the first case and 9 in the second.
According to our result we can have generic pole assignability if one of the above partitions is bounded by the heights. Indeed, for the partitioning (2^2+2^1), (2^0) which give 6 and 1 we have

$$6<7 \quad \text{and} \quad 1<2$$

and thus pole assignability is possible.

□

The second use of the theorem is considered next and we will try to calculate the height of w in terms of the heights $h(p_i,m_i)$. The heights $h(p_i,m_i)$ were defined in [Sto.1] and are given as shown below:

Lemma (9.1) If $1 < p \leq m$ and 'ν' is such that $2^\nu \leq m+p-1 < 2^{\nu+1}$, then the height of the first Whitney class of $G_p(\mathbb{R}^{p+m})$ is give as

$$h(p,m)=\begin{cases} 2^{\nu+1}-2 & \text{if } p=2 \text{ or if } p=3 \text{ and } m+p=2^\nu+1 \\ 2^{\nu+1}-1 & \text{otherwise} \end{cases} \tag{9.33}$$

□

Remark(9.7). In the case where p=1 then $G_p(\mathbb{R}^{p+m}) \approx P\mathbb{R}^m$ and the height h(1,m) equals to m. The maximum possible value for the height h(p,m) is pm and is obtained iff either p=1 or p=2 and $m=2^\nu-1$.

□

Given that the solvability conditions are based on the decentralisation heights, its computation is considered next.

Lemma (9.2): Let

$$\max_i(h(p_i,m_i)) = h(p_a,m_a) \tag{9.34}$$

and let 'ν' be the unique integer such that $2^\nu \le h(p_a,m_a) < 2^{\nu+1}$. Then the decentralisation height satisfies the condition

$$h(w) \le 2^{\nu+1}-1 \tag{9.35}$$

Proof

For a given number d the length of its binary representation is given by the number $\nu+1$ such that $2^\nu \le d < 2^{\nu+1}$. Thus the number $\nu+1$ in the statement of our theorem is the length of the binary representation of the largest of all heights $h(p_a,m_a)$. For a given binary partition t of n such that

$$t(i) \le h(p_i,m_i) \qquad i=1,2,\dots\kappa$$

it is evident that as we sum up the t(i)'s, in binary form, there are no numbers carried over (by the definition of binary partition). Thus the sum of t(i)'s must have the same length as that of max(t(i)). Therefore, the largest possible sum of t(i)'s we can have is the largest number having the same binary length as that of $h(p_a,m_a)$. This number is obviously $2^{\nu+1}-1$.

□

The above result gives an upper bound for the height independent of the partition. In the frame of the present sufficient conditions, Lemma (9.2) provides an estimation for the number of states for which we may find a solution using Corollary(9.1). In this sence, the last result provides an indication for the limitations of our framework. A sufficient condition, which is independent of the partition search and thus simpler, is described below and its proof readily follows from the previous analysis.

Lemma (9.3): Let $\kappa \ge 2$ and assume that the maximum height is $h(p_{a_1},m_{a_1})$ such that $p_{a_1} \ge 2$. If ν_1 is the unique integer such that

$$2^{\nu_1} \le h(p_{a_1},m_{a_1}) < 2^{\nu_1+1}$$

then

$$h(w) = 2^{\nu_1+1} - 1$$

Proof:

Since $\kappa \ge 2$ then we can order the heights as:

$$h(p_{a_1},m_{a_1}) \ge h(p_{a_2},m_{a_2}) \ge \ldots \ge h(p_{a_k},m_{a_k})$$

where $h(p_{a_2},m_{a_2}) \ge 1$ and (a_i) denotes the permutation of original set. Consider now the following partition t (of length κ) of $2^{\nu_1+1} - 1$:

$$t(1) = 2^{\nu_1+1} - 2$$
$$t(2) = 1$$

and

$$t(i) = 0 \qquad i=3,\ldots,\kappa$$

Furthermore, having that $p_{a_1} \ge 2$, then either $h(p_{a_1},m_{a_1}) = 2^{\nu_1+1}-1$ or $2^{\nu_1+1}-2$ (see Lemma(9.1)) and thus we have

$$t(1) \le h(p_{a_1},m_{a_1})$$

We additionally have

$$t(i) \le h(p_i,m_i) \qquad i=2,3,\ldots,\kappa$$

Therefore the binary partition t of $2^{\nu_1+1} - 1$ satisfies the required height inequalities of theorem(9.3) and hence we have that $h(w) \ge 2^{\nu_1+1} - 1$. If we combine it with the inverse inequality of Lemma(9.2) we get the required equality $h(w) = 2^{\nu_1+1} - 1$.

□

Corollary (9.3): Under the assumption of the Lemma (9.3) for (p_i,m_i) i=1,2,...,k set , a sufficient condition for pole assignment by real decentralised static, output feedback is that

$$n \le 2^{\nu_1+1} - 1 \tag{9.36}$$

□

An example illustrating the above result is given below

Example (9.6): If (p_1=2, m_1=4) , (p_2=1, m_2=3) then $h(p_1,m_1)=6$ and $h(p_2,m_2)=3$. The integer ν for which

$$2^{\nu} \le h(p_1,m_1)=6<2^{\nu+1}$$

is ν=2. From Lemma (9.3) $h(w)=2^{2+1}-1=7$. Thus the above Corollary indicates that $n \le 7$ is the sufficient condition for pole assignability. □

9.6. The pole placement map under the decentralisation assumption.

The pole placement map (see Ch.6) is defined to be the function that maps every feedback controller K to the closed loop poles or to the coefficients of the closed loop polynomial. In our case K is block diagonal and the above map can be defined as :

$$\chi^{d} : \mathbb{F}^{\Sigma m_i p_i} \longrightarrow \mathbb{F}^{n}$$

such that

$$\chi^{d}(K_1,K_2,\ldots,K_\kappa)=(p_n,\ p_{n-1},\ldots,\ p_1) \tag{9.37}$$

where $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$ and $(p_n,p_{n-1},\ldots,p_1)$ is the coefficient vector of p(s) of the equation (9.1a) or the equivalent (9.1b). The pole placement map carries all the information as far as the pole placement properties of a given system and thus its examination is very important. The above defined pole placement map for the decentralised case can be considered as a restriction of the general pole placement map of the output feedback case. In fact if χ^c is the centralised pole placement map:

$$\chi^{c} : \mathbb{F}^{mp} \longrightarrow \mathbb{F}^{n}$$

then χ^d can be factored as

$$\chi^{d}: \mathbb{F}^{\Sigma m_i p_i} \xrightarrow{\ E\ } \mathbb{F}^{m \times p} \xrightarrow{\ \chi^c\ } \mathbb{F}^{n} \tag{9.38}$$

where

$$E(\text{row}(K_1), \text{row}(K_2), ..., \text{row}(K_\kappa)) = \begin{bmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & 0 & : \\ : & 0 & & 0 \\ 0 & \dots & 0 & K_k \end{bmatrix} \tag{9.39}$$

As in Ch.6 the rank of the differential of χ^d at a generic point of its domain, is equal to the dimension of the image of χ^d, which in turn constitutes a measure for the size of assignable closed loop pole polynomials of a given system. To calculate the differential of χ^d we will use its decomposition (9.38) as $\chi^c \circ E$. This decomposition implies the following equation:

$$D(\chi^d)_K = D(\chi^c)_{E(K)} \circ D(E)_K \tag{9.40}$$

For now on on we will continue our study by using matrix representation of the previous three differentials. For these we need to specify the basis for the tangent spaces $T(\mathbb{F}^{mp})_K$ and $T(\mathbb{F}^{\Sigma m_i p_i})_K$ with respect to which the differential will represented. To this end we need the following definitions:

Definition(9.2): We may define the following sets of integers:

i) $\Omega = \{ (i,j) : 1 \le i \le p \text{ and } 1 \le j \le m\}$

ii) $\Omega_d = [1,p_1]x[1,m_1] \cup [p_1+1,p_1+p_2]x[m_1+1,m_1+m_2] \cup \dots \cup [\sum_{i=1}^{\kappa-1} p_i+1, p]x[\sum_{i=1}^{\kappa-1} m_i+1,m] \subseteq \Omega$

□

The two sets of indices Ω and Ω_d specify the lower indices of the entries k_{ij} of the centralised and decentralised feedback matrix respectively. For the present purposes, we consider as basis for $T(\mathbb{F}^{mp})_K$ the set of all $(\frac{\partial}{\partial k_\alpha})$, $\alpha \in \Omega$ and for $T(\mathbb{F}^{\Sigma m_i p_i})_K$ the set of all $(\frac{\partial}{\partial k_\beta})$, $\beta \in \Omega_d$, where all the indices are lexicographically ordered. Using these bases we have a representation of the above differentials. The differential $D(E)_K$ has a very simple structure; In fact, since E is a linear map, its differential must also be a linear map independent of K and thus this differential can be described as shown below:

Lemma(9.4) Let K= [row(K_1), row(K_2),..., row(K_κ)] and E(K) is given by eq(9.39), then a representation of differential $D(E)_K$ with respect to the previously defined basis of the tangent spaces, is given by a matrix R(E) $\in \mathbb{F}^{mp \times \Sigma m_i p_i}$ such that:

$\forall\, \alpha \in \Omega,\ \beta \in \Omega$, then $$R(E)_{\alpha\beta}=\begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{otherwise} \end{cases}$$

Proof

Obvious from the definition of E.

□

On the other hand the differential $D(\chi^c)_K$ has been already calculated [Rein.1] and its expression is given by the following lemma.

Lemma (9.5) For a fixed centralised feedback gain K and a system S(A,B,C) we have that a matrix representation of $(D\chi^c)_K$, denoted by $R(\chi^c)_K$, with respect to the basis $(\frac{\partial}{\partial k_\alpha})$ of $T(\mathbb{F}^{mp})_K$ $(\alpha \in \Omega)$, is a n x mp matrix given by:

$$R(\chi^c)_K= Q^T[\ \mathrm{col}CB\ ,\mathrm{col}CHB,\ \ldots\ ,\mathrm{col}CH^{n-1}B\ ]^T \tag{9.41}$$

where 'col' maps an mxp matrix to the mpx1 matrix formed by superimposing its columns, H=A+BKC and Q is given by:

$$Q=\begin{bmatrix} 1 & p_n & \cdots & p_2 \\ 0 & 1 & & p_3 \\ \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

where the p_i's are the coefficients of the closed loop pole polynomial

□

Combining now the two previous results we can now derive a matrix representation for the differential of the decentralised pole placement map χ^d.

Theorem(9.5) For a given decentralised feedback gain K and a system S(A,B,C) we have that a matrix representation of the differential of the decentralised pole placement map $(D\chi^d)_K$, denoted by $R(\chi^d)_K$ is given by

$$R(\chi^d)_K = R(\chi^c)_{E(K)} \cdot R(E) \tag{9.42}$$

Proof

Obvious from eq(9.40)

□

Note that $R(\chi^d)_K$ is obtained from $R(\chi^c)_{E(K)}$ by keeping only those rows of $R(\chi^c)_{E(K)}$ which correspond to the Ω_d set of indices only.

Using Theorem(9.5) we can define the decentralised Markov Parameters. The Markov parameters of a system can be taken by the differential of χ^C at K=0 as Lemma(9.4) indicates. Therefore we can similarly, define as decentralised Markov parameters, by using the differential of the the decentralised Pole Placement Map at $K_{dec}=0$. In fact for $K_{dec}=0$ we have that

$$R(\chi^d)_0 = Q^T[\ \mathrm{col}\hat{CB}\ ,\mathrm{col}\hat{CAB},\ ...\ ,\mathrm{col}\hat{CA^{n-1}B}\ ]^T \tag{9.43}$$

where $\mathrm{col}\hat{CA^iB}$ denotes the reduced column obtained from $\mathrm{col}CA^iB$ after eliminating all the entries that do not correspond to the set of indices Ω_d (this is the result of multiplication by the R(E) matrix). The matrix

$$M_d=[\ \mathrm{col}\hat{CB}\ ,\mathrm{col}\hat{CAB},\ ...\ ,\mathrm{col}\hat{CA^{n-1}B}\ ]=[\ \hat{m}_1\ ;\ \hat{m}_2;\ ...;\ \hat{m}_n]$$

will be referred to as *decentralised Markov matrix* and its importance for our study is considered next.

Proposition(9.2) The matrix M_dQ is a full row submatrix of the decentralised Plucker matrix.

Proof

Since

$$\chi^{d}(K_{dec})=C_p\big([I_p, K_{dec}]\big)\, P_S$$

by differentiating with respect to K_{dec} at 0 we get that $R(\chi^d)_0$ is a matrix A^T such that A is a full row submatrix of the decentralised Plucker matrix. Having also that $R(\chi^d)_0= (M_dQ)^T$, we easily get the required result.

□

Taking into account that the rank properties of the decentralised Plucker matrix [Kar.4] determine the presence or not of fixed modes we have:

Corollary(9.4) If $\text{rank}(M_d)=n$ the system has no fixed modes.

This provides a simple sufficient condition for avoiding fixed modes based on the decentralised Markov matrix. A simple method for calculating M_d is described below.

Remark(9.8) Let $H_i=CA^iB$ for i=0,1,2,...,n-1 , be the Markov parameters of the S(A,B,C) system, and let K be the general form of the decentralised output feedback. For each $H_i=CA^iB$ we zero all entries which correspond to the fixed zeroes of $K_{dec}{}^T$ and let $\hat{H}_i$ be the resulting matrix. By taking $\text{col}\hat{H}_i$ and eliminating the fixed zeroes (matrix supression) we obtain the $\hat{m}_i$ vectors in the Markov matrix. The $\hat{H}_i$ i=1,2,...,n-1 are referred to as *decentralised Markov parameters.*

□

9.7 Conclusions

The previously established framework of sufficient conditions based on the odd order of the corresponding subvarieties and the height [Lev.1] for the centralised case has been extended to the decentralised case. The new sufficient conditions provide tests for solvability of the decentralised pole assignment by output feedback problems for cases not covered by the equality of input or output channel assumption [Wang]. A systematic procedure based on partitioning of the states may be based on theorem (9.2) and this provides stronger sufficient conditions than that of Corollary (9.2). Some further properties of the decentralised pole placement map have been derived and this has led to some new sufficient condition for avoiding fixed modes.

CHAPTER 10. Global Asymptotic Linearisation of the Pole Placement Map

10.1. Introduction.

In this chapter we present a new method for constructing pole placing real constant and dynamic output feedback compensators for proper plants. This method is based on a linearisation of the pole placement map by considering special sequences of feedback compensators which, in the limit, converge to a so called degenerate compensator; the advantage of this approach is that it asymptotically reduces the overall pole placement map to a linear one and thus reduces the the overall solvability of the problem to a linear set of equations. This type of global asymptotic linearization defined around singular solutions is different from the standard methodologies for linearizing the multilinear nature of the map (dyadic feedback, full rank linearization [Gia. 2]). The solutions worked out within this framework are given in a closed form and it is of the type of a one parameter family of multivariable compensators. The essence of this new methodology is that we may rely on large gains in the case of strictly proper systems, but not necessarily large gains in the proper case and assignment is achieved not in an exact sense, but in small neighbourhoods (as small as we want) around a given set of poles. Although the approach is sufficient, it is proved that the methodology can work for a generic proper system of p-inputs, m-outputs , n-states satisfying the condition $mp>n$ for the constant feedback case and the condition $n_1(m+p)+mp > n+n_1$ for the case of dynamic feedback compensators of order n_1.

The condition $mp>n$ has been known as a necessary condition [Wil.1], [Broc.1] for the existence of a real feedback and it has also been shown recently [Wang 2] to be a generic sufficient condition for real output feedback. The present approach not only provides an alternative simpler proof of this important result but also a simpler algorithm for computing families of real feedbacks. As far as construction of constant feedback is concerned, the previous known conditions for which output feedback may be constructed, has been of the type $m+p-1 \geq n$; clearly, the present approach extends the family of systems for which constructive tools are available. Although the original methodology is for the constant feedback case the approach is shown to extend to the case of dynamic global linearising controllers. It is proved that families of such dynamic controllers, based on dynamic degenerate point exist and that the dynamic pole assignment using these controllers can be solved for a generic plant if $n_1(m+p)+mp > n+n_1$. Furthermore the least n_1 that satisfies this condition gives us the least degree family of controllers for which the arbitrary pole assignment is solved on a system.

The overall methodology behind the present global linearisation for both static and dynamic cases is that degenerate feedbacks may be approached asymptotically by regular controllers for which the restricted pole placement map (in the sense that one parameter families of feedback compensators are considered) is linear. The solvability of the linearised version of the problems is well known and can be described in terms of elementary linear algebra or standard theory of polynomial diofantine equations. The essential step though, is to select appropriate degenerate points for which the solvability of the above linear problem is satisfied. It should be stressed that the approach is sufficient , but quit general, since the conditions $mp>n$ (constant) and $n_1(m+p)+mp > n+n_1$ (dynamic) are shown to guarantee the success of the method for a generic system. Another important aspect of the methodology is that pole assignment is not considered in an exact sense but rather in an approximate sense; that is, the designed controllers assign polynomials with poles in arbitrarily small regions around the given set.

The present approach differs from all other approaches which have been considered for similar problems so far, since it does not belong to the family of standard linearising controllers (dyadic, full rank linearisation) and does not belong to the class of intersection theoretic tools which have been considsidered in the previous chapters. Key features of the methodology is that on one hand it provides tools for establishing existence results and on the other leads to a novel computational framework. Within this framework degenerate points for which the problem is solvable parametrise whole family of solutions and not just a single one. Such solutions may be worked out in a closed parametric form.

10.2. System degeneracy and feedback.

Consider the standard feedback configuration shown below:

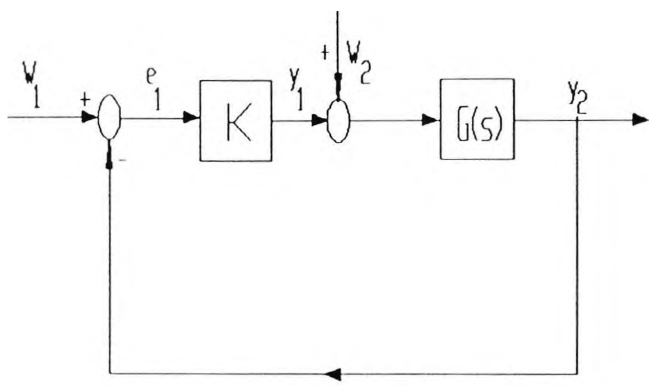


fig. 10.1

It is well known that the closed loop pole polynomial p(s) of the feedback system of fig.10.1 is given by:

$$p(s)=\det\left\{[I,\ K]\begin{bmatrix}D(s)\\N(s)\end{bmatrix}\right\}$$

where $N(s)D(s)^{-1}$ is a coprime MFD of the transfer function of the open loop system. A degenerate point for the feedback configuration is a gain where the configuration has a singularity, in the sense that the feedback system is not well posed. If we consider the generalised gain of the form rowspan$[A,B]\in G_p(\mathbb{R}^{p+m})$ then the feedback gain is degenerate if:

$$\det\left\{[A,B]\begin{bmatrix}D(s)\\N(s)\end{bmatrix}\right\}=0 \quad \forall s \qquad (10.1)$$

Remark(10.1) Degenerate points do not exist in SISO feedback systems since the generalised closed loop polynomial $ad(s)+kn(s)$, $(a,k)\in\mathbb{P}^1(\mathbb{R})$ can be identically zero, iff $a=k=0$. □

The determination of a (particular) degenerate gain is the first step in our method for constructing pole placing feedback compensators. A first (sufficient) approach for this construction can be found in [Wang 2] where the degenerate gain was constructed via the coefficient matrix of the column $\underline{m}_p(s)$ of the composite matrix $M(s)=[D(s)^T,N(s)^T]^T$, that corresponds to the smallest controllability index c_p. According to our formulation this amounts to selecting p linearly independent vectors from the left kernel of the $(m+p)\times(c_p+1)$ coefficient matrix of $\underline{m}_p(s)$. Although this can be done if $mp>n$, it is not always adequate for the purposes of our method, since we need a degenerate gain with certain properties. A more general result which gives all possible degenerate gains for a given system is described next.

Proposition(10.1) For the standard feedback configuration and for a plant described by the right coprime pair (N(s), D(s)), or the composite matrix $M(s)=[D(s)^T,N(s)^T]^T$, a p-dimensional vector space $\mathcal{V}$=rowspan[A,B] corresponds to a degenerate gain, iff either of the following equivalent conditions holds true:
(i) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathbb{R}[s]$-colspan(M(s)) such that $[A,B]\ \underline{m}(s)=0\ \forall s$. (ii) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathbb{R}[s]$-colspan(M(s)) with coefficient matrix $C_{m(s)}$ such that rank$C_{m(s)} \leq m$.
Proof
Since $\det([A,B]M(s))=0\ \forall s$, the pxp polynomial matrix [A,B]M(s) has nontrivial right kernel. Thus there exists a px1 polynomial vector $\underline{v}(s)$ such that $[A,B]M(s)\ \underline{v}(s)=0\ \forall s$. This is equivalent to the fact that [A,B] is basis matrix for a p-dimensional vector subspace of the left kernel of the coefficient matrix of $\underline{m}(s)=M(s)\ \underline{v}(s) \in \mathbb{R}[s]$-colspan(M(s)). For a given $\underline{m}(s) \in \mathbb{R}[s]$-colspan(M(s)), its coefficient matrix is an (m+p)x(d+1) constant matrix $C_{m(s)}$ where d is the degree of $\underline{m}(s)$. If r is the rank of $C_{m(s)}$, then there exists a p-dimensional subspace of the left kernel of $C_{m(s)}$, iff $m+p-r \geq p$, or equivalently, iff $m \geq r$.

□

Proposition(10.1) provides the means for calculating all degenerate points of a given system. First we construct a coprime (right) MFD for the transfer function and then consider an $\mathbb{R}[s]$ linear combination $\underline{m}(s)$ of the columns of the composite M(s). For this $\underline{m}(s)$, if the rank of the coeficient matrix is less than or equal to m (number of outputs), then we construct a basis for its left kernel; Subsequently, we select all possible p-tuples of linearly independent vectors and thus we get all possible degenerate points with respect to this $\underline{m}(s)$. The determination of all degenerate points may be achieved if we repeat the above procedure for all $\underline{m}(s) \in \mathbb{R}[s]$-colspan(M(s)).

Example(10.1) Consider the proper system of 2-inputs, 3-outputs and 5-states with transfer function G(s) given by

$$G(s)=\begin{bmatrix} -s^{-3} & 1 \\ -s^{-4} & s^{-1} \\ s^{-5} & s^{-2} \end{bmatrix} = \begin{bmatrix} 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^3 & 0 \\ 1 & s^2 \end{bmatrix}^{-1}$$

and its composite matrix M(s) is therefore given by,

$$M(s)=\begin{bmatrix} s^3 & 0 \\ 1 & s^2 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix} = [\underline{m}_1(s), \underline{m}_2(s)]$$

Then consider the following two cases:

a) Let $\underline{m}(s)= 0\underline{m}_1(s)+1\ \underline{m}_2(s) = \underline{m}_2(s) =[0,s^2,s^2,s,1]^T$, with coefficient matrix ,

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The left kernel of C is a two-dimensional vector space spanned by the rows of the following matrix:

$$[A,B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

and thus rowspan[A,B] defines a degenerate point for the above system.

b) If we now let $\underline{m}(s)=1\underline{m}_1(s)+0\underline{m}_2(s) = \underline{m}_1(s)=[s^3,1,0,0,0]^T$, the new coefficient matrix is

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and its left kernel $\mathcal{V}$ consists of all vectors of the form [0,0,x,y,z]. All 2-dimensional subspaces of $\mathcal{V}$ will therefore give us all degenerate gains produced from $\underline{m}_1(s)$, which in fact are all rowspan$[0_{2x2}$, K] where $K \in \mathbb{R}^{2x3}$ and rank(K)=2.

Furthermore we can prove that all the other degenerate gains produced by linear combinations $\underline{m}(s)= p_1(s)\ \underline{m}_1(s)+p_2(s)\ \underline{m}_2(s)$ belong either to case a) or the case b), which proves that the above cases give us all possible degenerate gains. □

10.3 Output feedback compensators converging to degenerate solutions.

The examination of asymptotic properties of the pole placement map can be achieved via the compactification of the gain space to a Grassmannian. In that case the gains K can be considered in the composite form [I,K] or more generally as ***rowspan***[A,K], where [A,K] has full row rank (see also Sec. 6.5.3). If A has full rank then ***rowspan***[A,K] is called a *finite gain* and the corresponding feedback compensator is $A^{-1}K$. One of the most important advantages of this interpretation is that allows us to view unbounded gains as a certain ***rowspan***[A,B]. Indeed, if we consider an one parameter family of gains K_ϵ such that $\|K_\epsilon\|\to\infty$ as $\epsilon\to0$, then according to the composite notation, this corresponds to an one parameter family of generalised gains $\hat{K}_\epsilon$ of the form $\hat{K}_\epsilon$=***rowspan***$[I,K_\epsilon]$. If K_ϵ is a rational matrix in ϵ, then it has a coprime MFD of the form $K_\epsilon=A(\epsilon)^{-1}B(\epsilon)$ such that A(0) is singular and therefore we have:

$$\lim_{\epsilon\to0}\hat{K}_\epsilon=\lim_{\epsilon\to0}\textbf{\textit{rowspan}}[I,A(\epsilon)^{-1}B(\epsilon)]=\lim_{\epsilon\to0}\textbf{\textit{rowspan}}[A(\epsilon),B(\epsilon)]=\textbf{\textit{rowspan}}[A(0),B(0)]$$

This shows that in the composite notation, unbounded gains correspond to **rowspan**[A,B] such that A is singular. In this setting we can classify the degenerate gains ***rowspan***[A,B] constructed as shown in the previous section, into finite and infinite ones, depending upon whether A is nonsingular or not. Every degenerate gain ***rowspan***[A,B] can be approached via:

$$\hat{K}_\epsilon= \textbf{\textit{rowspan}}\ ([A,B]+\epsilon[A',B'])$$

as $\epsilon\to0$, where $[A',B']$ is a px(m+p) scalar matrix, and can be seen as the direction via which we approach the degenerate gain ; If A' is chosen such that $A+\epsilon A'$ is not identically zero then:

$$\hat{K}_\epsilon= \textbf{\textit{rowspan}}\ ([\ I_p\ ,\ (A+\epsilon A')^{-1}(B+\epsilon B')])$$

and therefore $\hat{K}_\epsilon$ is an one parameter family of finite gains corresponding to an one parameter feedback compensators of the type:

$$K_\epsilon=(A+\epsilon A')^{-1}(B+\epsilon B')$$

Example(10.2) Let

$$\textbf{\textit{rowspan}}[A,\ B] = \textbf{\textit{rowspan}}\begin{bmatrix} 1 & 3 & 7 & 4 & 1 \\ 0 & 0 & -2 & 1 & -5 \end{bmatrix}$$

be an infinite degenerate point for a strictly proper system of 2-inputs and 3-outputs. Then consider the direction

$$[A',\ B']_1 = \begin{bmatrix} 1 & 0 & 1 & 7 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{bmatrix}$$

an one-parameter family of generalised gains approaching ***rowspan***[A,B], as $\epsilon \to 0$, in the direction $[A',\ B']_1$ is given by:

$$\hat{K}_\epsilon = \textbf{\textit{rowspan}}\ ([A,B]+\epsilon[A',B']) = \textbf{\textit{rowspan}}\begin{bmatrix} 1 & 0 & 1+6/\epsilon & 7-3/\epsilon & -\dfrac{11\epsilon+15}{\epsilon(\epsilon+1)} \\ 0 & 1 & -2/\epsilon & 1/\epsilon & 4+5/\epsilon \end{bmatrix}$$

which gives us an unbounded (as $\epsilon \to 0$) one parameter family of feedback compensators:

$$K_\epsilon = \begin{bmatrix} 1+6/\epsilon & 7-3/\epsilon & -\dfrac{11\epsilon+15}{\epsilon(\epsilon+1)} \\ -2/\epsilon & 1/\epsilon & 4+5/\epsilon \end{bmatrix}$$

By changing now the direction we can find a different unbounded family of gains converging to the same degenerate gain. For example if we let

$$[A',B']_2 = \begin{bmatrix} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 6 & 1 \end{bmatrix}$$

be a new direction and follow the previous procedure, we can find a new one parameter family of feedback compensators

$$L_\epsilon = \begin{bmatrix} \dfrac{2(\epsilon^2+2\epsilon+3)}{\epsilon(1+\epsilon)} & \dfrac{3\epsilon^2-14\epsilon-3}{\epsilon(1+\epsilon)} & \dfrac{5\epsilon^2-2\epsilon-15}{\epsilon(1+\epsilon)} \\ 1-2/\epsilon & 6+1/\epsilon & 1+5/\epsilon \end{bmatrix}$$

which converge to the same degenerate point (as $\epsilon \to 0$).

□

Approaching a degenerate point via different directions is very crucial for our method and as we will see in the subsequent sections the pole placement problem becomes linear with respect to these directions.

10.4. Asymptotic properties of the pole placement map around degenerate points.

The degenerate points are points of the Grassmannian where the pole placement map cannot be continuously extended. In fact, degenerate points possess a very important singularity, they scatter sequences of gains approaching them. As the following example shows we may have two sequences of gains converging to a degenerate point, as $\epsilon \to 0$, and yet the corresponding sequences of closed loop poles to converge into two different limits.

Example(10.3) Consider the system and its degenerate gain of example(10.2). The two one dimensional fimilies of gains

$$\hat{K}_\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+\epsilon & 0 & 0 \end{bmatrix}$$

$$\hat{L}_\epsilon = \begin{bmatrix} 1 & 0 & \epsilon & 0 & 0 \\ 0 & 1 & -1+\epsilon & 0 & 0 \end{bmatrix}$$

both converge to the degenerate gain [A,B] as $\epsilon \to 0$. On the other hand, if we calculate the closed loop pole polynomials corresponding to these two families we get that

$$\det(\hat{K}_\epsilon M(s)) = \epsilon\, s^5$$

$$\det(\hat{L}_\epsilon M(s)) = \epsilon\, s^5 \; - \epsilon\, s^2$$

Therefore as we approach the degenerate point [A,B] by the first family of gains, the closed loop poles are all zero. However, as we approach it by the second family of gains, two of the closed loop poles approach one and the rest of them zero. This indicates the spacial singularity of the pole placement map at a degenerate point.

□

Next we will examine carefully this scattering of sequences in connection with the pole placement map as we approach a degenerate point. The remarkable fact is that this happens in a linear way and thus it allows linearization of the pole placement problem via constant output feedback.

10.5. Global linearisation of the output feedback problem and computation of solutions.

Consider the composite gain sequences of the form

$$S_\epsilon = [A,\ B] + \epsilon[A',\ B'] \quad , \quad \det(A+\epsilon A') \neq 0 \tag{10.2}$$

These sequences converge to [A, B] as ϵ tends to zero. We consider the DAP for the finite gain sequences S_ϵ, which converges to a degenerate point [A, B] . Thus, we have:

$$\det\left\{ [A,\ B] + \epsilon[A',\ B'] \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right\} = p_\epsilon(s) \tag{10.3}$$

Since the roots of a polynomial do not change if we multiply (or divide) it by a number, it is more appropriate to consider the coefficient vector of the polynomial modulo dilations(multiplication by scalar), or alternatively to make the polynomial monic. In this way, to examine the convergence of the roots of $p_\epsilon(s)$ as $\epsilon \to 0$, we will regard the coefficient vector $\underline{p}_\epsilon \in \mathbb{R}^{n+1}$ of $p_\epsilon(s)$ modulo dilations(multiplications by scalar), as a sequence $<\overline{p_\epsilon}> \in \mathbb{P}(\mathbb{R})^n$ (the set of lines in $\mathbb{R}^{n+1}$ passing through the origin); being a sequence in the compact space $\mathbb{P}(\mathbb{R})^n$, $<\underline{p}_\epsilon>$, implies that it should have at least one cluster point as $\epsilon \to 0$. In fact, the next theorem states that this cluster point is unique, as $\epsilon \to 0$, and the relation between this point and the direction $[A', B']$ is linear.

Theorem(10.1) Let ***rowspan***[A,B] be a degenerate gain and S_ϵ be a sequence of finite gains converging to it. Then the corresponding sequence of closed loop polynomial vectors $<\underline{p}_\epsilon>$ (modulo dilations) converges as $\epsilon \to 0$ to a vector $<\underline{p}> \in \mathbb{P}(\mathbb{R})^n$; furthermore the function T which maps the direction $[A', B']$ to $<\underline{p}>$ is linear.

Proof

The determinant appearing in (10.3) is a polynomial in ϵ and can be computed (by the Binet Cauchy theorem) as:

$$p_\epsilon(s) = C_p\left([A,B] + \epsilon[A', B']\right) \cdot C_p\left(\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) \tag{10.4}$$

Subsequently, we set

$$C_p\left(\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) = P_S \, \underline{e}_n(s) \tag{10.5}$$

where $\underline{e}_n(s) = [\, s^n, s^{n-1}, \dots, s, 1]^T$, and expand $C_p([A,B] + \epsilon[A', B'])$ as follows:

$$C_p([A,B] + \epsilon[A', B']) = C_p\left(\begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \vdots \\ \underline{a}_p^T \end{bmatrix} + \epsilon \begin{bmatrix} \underline{b}_1^T \\ \vdots \\ \vdots \\ \underline{b}_p^T \end{bmatrix}\right) = \left(\underline{a}_1^T + \epsilon\, \underline{b}_1^T\right) \wedge \dots \wedge \left(\underline{a}_1^T + \epsilon\, \underline{b}_1^T\right) \tag{10.6}$$

where $\wedge$ denotes the exterior product. Expanding now the right hand side of () using the distributive property of the exterior product we get:

$$C_p\,([A,B]+\epsilon[A',\,B']\,) = \Delta + \epsilon(\sum_{i=1}^{p}\Delta_i) + \epsilon^2(\sum_{i<j}\Delta_{ij}) + \epsilon^3(\sum\Delta_{ijk}) + \ldots + \epsilon^p\,\Delta_{12..p} \qquad (10.7)$$

where

$$\Delta = \underline{a}_1^T \wedge \ldots \wedge \underline{a}_p^T$$

$$\Delta_i = \underline{a}_1^T \wedge \ldots \wedge \underline{a}_{i-1}^T \wedge \underline{b}_i^T \wedge \underline{a}_i^T \wedge \ldots \wedge \underline{a}_p^T \qquad (10.8)$$

and more generally $\Delta_{ij..k}$ is the exterior product (or compound matrix) taken from $\underline{a}_1^T \wedge \ldots \wedge \underline{a}_p^T$ if we substitute the vectors $\underline{a}_i^T$, $\underline{a}_j^T$,..., $\underline{a}_k^T$ with $\underline{b}_i^T$, $\underline{b}_j^T$,..., $\underline{b}_k^T$ respectively. Combining now (10.4) and (10.7) we get:

$$p_\epsilon(s) = [\Delta\, P_S + \epsilon(\sum_{i=1}^{p}\Delta_i)\, P_S + \epsilon^2(\sum_{i<j}\Delta_{ij})\, P_S + \epsilon^3(\sum\Delta_{ijk})\, P_S + \ldots + \epsilon^p\,\Delta_{12..p}P_S)]\,\underline{e}_n(s) \qquad (10.9)$$

since ***rowspan***$[A,B]$ is a degenerate gain we have that $\det\left\{[A,B]\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\right\}=0$, $\forall s$ which implies (Binet Cauchy):

$$C_p\,([A,B])\,.\,C_p\left(\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\right)=0 \quad \forall s$$

or equivelently:

$$\Delta\, P_S=0$$

Therefore the coefficient vector $\underline{p}_\epsilon$ of $p_\epsilon(s)$ is given by:

$$\underline{p}_\epsilon = \epsilon(\sum_{i=1}^{p}\Delta_i)\, P_S + \epsilon^2(\sum_{i<j}\Delta_{ij})\, P_S + \epsilon^3(\sum\Delta_{ijk})\, P_S + \ldots + \epsilon^p\,\Delta_{12..p}\, P_S \qquad (10.10)$$

If we consider now $\underline{p}_\epsilon$ modulo dilations (multiplication by constant), we get the vector $<\underline{p}_\epsilon> \in \mathbb{P}(\mathbb{R})^n$. Then because of the invariance of $<\underline{p}_\epsilon>$ with respect to scalar multiplication we get:

$$<\underline{p}_\epsilon>=<\tfrac{1}{\epsilon}\,\underline{p}_\epsilon>=<\Big(\sum_{i=1}^{p}\Delta_i\Big)\,P_S+\epsilon\Big(\sum_{i<j}\Delta_{ij}\Big)\,P_S+\epsilon^2\Big(\sum\Delta_{ijk}\Big)\,P_S+\ldots+\epsilon^{p-1}\,\Delta_{12..p}\,P_S>\in\mathbb{P}(\mathbb{R})^n$$

letting now $\epsilon\to0$ we get that $<\underline{p}_\epsilon>$ converges to a unique point namely to:

$$<\underline{p}>=<\Big(\sum_{i=1}^{p}\Delta_i\Big)\,P_S> \tag{10.11}$$

Furthermore, since the exterior product of vectors is linear with respect to each of the vectors, the function $\left(\sum_{i=1}^{p}\Delta_i\right)P_S$ is linear with respect to $\underline{b}_i^T$, the rows of the direction matrix $[A',B']$.

□

It is apparent, from the previous theorem, that the closed loop poles for the sequence of finite gains $S_\epsilon=$***rowspan***$\big([A,B]+\epsilon[A',B']\big)$ tend to the roots of the polynomial:

$$\boxed{p(s)=\Big(\sum_{i=1}^{p}\Delta_i\Big)\,P_S\,\underline{e}_n(s)} \tag{10.12}$$

as $\epsilon\to0$. The polynomial given by () will be called *prime polynomial* with respect to the degenerate point ***rowspan***$[A,B]$ and the direction $[A',B']$; as the previous theorem states the coefficient vector $\overline{p}$ of this polynomial depends linearly on the parameters of the direction $[A',B']$. As a matter of fact the prime polynomial can be written as a linear combination $\sum(b_{ij}\,p_{ij}(s))$ where b_{ij} are all the elements of the matrix $[A',B']$ and $p_{ij}(s)$ are some polynomials depending on the particular system and the selected degenerate point.

Theorem(10.2) Let $\mathfrak{D}=$***rowspan***$[A,B]$ be a degenerate point of a system defined by the coprime composite representation $M(s)=[D(s)^T,N(s)^T]^T$, then the prime polynomial of this system with respect to $\mathfrak{D}$ and the direction $[A',B']=(b_{ij})$ $1\le i\le p$, $1\le j\le p+m$, can be written as:

$$p(s)=\sum(b_{ij}\,p_{ij}(s)) \tag{10.13}$$

where $p_{ij}(s)$ is the determinant of the pxp polynomial matrix $D_{ij}(s)$ having the same rows with the matrix $AD(s)+BN(s)$ apart from the i-th, which is replaced by the j-th row of M(s).

Proof

From eq.(10.12) we have that:

$$p(s) = \Big(\sum_{i=1}^{p}\Delta_i\Big)\, P_S\, \underline{e}_n(s) = \sum_{i=1}^{p}\Big(\Delta_i\ C_p(M(s))\Big) \tag{10.14}$$

developing the exterior product Δ_i with respect to the i-th row we get:

$$\Delta_i = \sum_{j=1}^{m+p} b_{ij}\ \underline{a}_1^T \wedge \ldots \wedge \underline{a}_{i-1}^T \wedge \underline{e}_j^T \wedge \underline{a}_i^T \wedge \ldots \wedge \underline{a}_p^T \tag{10.15}$$

where $\underline{e}_j^T$ is a 1x(m+p) vector having the j-th entry one and all the others zero. Substituting now (10.15) into (10.14) we get:

$$p(s) = \sum_{i,j} b_{ij}\Big(\underline{a}_1^T \wedge \ldots \wedge \underline{a}_{i-1}^T \wedge \underline{e}_j^T \wedge \underline{a}_i^T \wedge \ldots \wedge \underline{a}_p^T\ C_p(M(s))\Big) \tag{10.16}$$

if we let $[A,B]_{ij}$ to be the px(m+p) matrix having the same rows with the matrix [A,B] apart from the i-th which is replaced by $\underline{e}_j^T$, then (10.16) can be rewritten as:

$$p(s) = \sum_{i,j} b_{ij}\Big(C_p([A,B]_{ij})\ C_p(M(s))\Big)$$

By the Binet Cauchy theorem this is equivalent to:

$$p(s) = \sum_{i,j} b_{ij}\ \det\Big([A,B]_{ij}\ M(s)\Big)$$

If now set $D_{ij}(s)=[A,B]_{ij}\ M(s)$ then:

$$p(s) = \sum_{i,j} b_{ij}\ \det(\ D(s)_{ij}\ )$$

and the theorem is proved.

□

Example(10.4) Consider the system defined by:

$$M(s)=\begin{bmatrix} s^3 & 0 \\ 1 & s^2 \\ s^2+1 & s+1 \\ s+3 & s \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

A degenerate point for this system is defined by $\mathfrak{D}$=***rowspan***[A, B] where:

$$[A,B]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

and this since:

$$\det(AD(s)+BN(s))= \det\begin{bmatrix} s^3 & 0 \\ s^2-2s-3 & 0 \end{bmatrix}=0$$

Let us now approach the degenerate point $\mathfrak{D}$ via the direction:

$$(b_{ij}) = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \end{bmatrix}$$

for this direction we have $b_{12}=2$, $b_{22}=1$, $b_{24}=-3$ and all the rest of the b's are zero. Due to the previous theorem, the prime polynomial for this direction is given by

$$p(s)=\sum(b_{ij}\, p_{ij}(s))=2p_{12}(s)+ p_{22}(s)-3p_{24}(s)$$

where

$$p_{12}(s)= \det\begin{bmatrix} 1 & s^2 \\ s^2-2s-3 & 0 \end{bmatrix} = -s^4+2s^3+3s^2,\quad p_{22}(s)= \det\begin{bmatrix} s^3 & 0 \\ 1 & s^2 \end{bmatrix} = s^5,\quad p_{24}(s)=\det\begin{bmatrix} s^3 & 0 \\ s+1 & s \end{bmatrix}=s^4$$

and therefore

$$p(s)=2(-s^4+2s^3+3s^2)+s^5-3s^4=s^5-5s^4+4s^3+6s^2$$

□

From all the above, it is now evident that the relation of the directional parameters b_{ij} and the pole polynomial at the limit (as $\epsilon\to 0$) is linear, and the matrix representing this linear relationship is the coefficient matrix of the polynomial vector [$p_{11}(s)$, $p_{12}(s)$,...,

$p_{ij}(s),\ldots, p_{p\ m+p}(s)]$.

Remark(10.2) For a given (feedback) degenerate point $\mathfrak{D}$=***rowspan***[A,B] of a system, the linear function that maps the direction $[A', B']=(b_{ij})$ to the coefficient vector $\underline{p}$ of the corresponding prime polynomial p(s), has a matrix representation denoted by $L_{\mathfrak{D}}$, which is the p(m+p)x(n+1) coefficient matrix of the polynomial vector [$p_{11}(s), p_{12}(s),\ldots, p_{ij}(s),\ldots, p_{p\ m+p}(s)]$. In this setting $\underline{p}$ can be written as:

$$\underline{p} = vec(b_{ij})\ L_{\mathfrak{D}} \tag{10.17}$$

□

For a given degenerate point $\mathfrak{D}$, the arbitrary prime polynomial assignability by sequences of feedback compensators converging to $\mathfrak{D}$, depends readily on $L_{\mathfrak{D}}$. In fact,

Corollary(10.1) Let $\mathfrak{D}$ be a degenerate point for a system. Then an arbitrary prime polynomial can be assigned via a seqence of feedback compensators converging to $\mathfrak{D}$, iff $rank(L_{\mathfrak{D}})=n+1$. In that case the appropriate direction can be found by solving eq(10.17) with respect to $vec(b_{ij})$.

□

This suggest the following procedure for the construction of (approximate) pole placing compensators:

a) Construct a degenerate point $\mathfrak{D}$=***rowspan***[A, B], as described in section(10.2).

b)Calculate the matrix $L_{\mathfrak{D}}$

c)If $rank(L_{\mathfrak{D}})=n+1$, then solve the linear equation (10.17) with the direction $(b_{ij})=[A',B']$, else go to step a).

d)The one parameter family $K_\epsilon=(A+\epsilon A')^{-1}(B+\epsilon B')$ of pxm matrices, is the family of real constant feedback compensators placing the poles of the system at the given set, as $\epsilon \to 0$.

e)Select a small enough ϵ (in K_ϵ), to approach the given closed loop pole polynomial as close as you like.

Remark(10.3) Although we have not described how we can select a degenerate gain $\mathfrak{D}$ such that $L_{\mathfrak{D}}$ has rank equal to n+1, as we will see in the next section, if $mp > n$ then for a generic system, $L_{\mathfrak{D}}$ satisfies this requirment. This means that for almost all systems such that mp>n the above procedure can be carried out without even needing to repeat steps a,b,c for a second time. In fact the set of systems for which the above cannnot be applied is a "negligible" set.

□

Example(10.5) Consider the same system and degenerate point $\mathfrak{D}$ of example(10.4). Then the polynomials $p_{ij}(s)$ are given by:

$p_{11}(s)=0$, $p_{12}(s)=-s^4+2s^3+3s^2$, $p_{13}(s)=-s^3+s^2+5s+3$, $p_{14}(s)=-s^3+2s^2+3s$,

$p_{15}(s)=-s^2+2s+3$, $p_{21}(s)=0$, $p_{22}(s)=s^5$, $p_{23}(s)=s^4+s^3$, $p_{24}(s)=s^4$, $p_{25}(s)=s^3$.

Therefore,

$$L_{\mathfrak{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 5 & 3 \\ 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We can easily see that $\text{rank}(L_{\mathfrak{D}})=6$; a fact that means that every prime polynomial can be assigned via a certain sequence of feedback compensators converging to $\mathfrak{D}$. Suppose now that we want to assign the stable polynomial $p(s)=s^5+5s^4+10s^3+\frac{117}{9}s^2+\frac{51}{9}s+1$. To do so we have to solve the linear equation

$$\text{vec}(b_{ij})\ L_{\mathfrak{D}}=[1,\ 5,\ 10,\ \tfrac{117}{9},\ \tfrac{51}{9},1]$$

with respect to $\text{vec}(b_{ij})=[b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{21}, b_{22}, b_{23}, b_{24}, b_{25}]$. One of the solutions of this equation is:

$$\text{vec}(b_{ij})=[0,\ 10/3,\ 0,\ 5/3,\ 1/3,\ 0,\ 1/3,\ 0,\ 25/3,\ 15/3]$$

which gives the direction

$$(b_{ij})=[A',B'] = \frac{1}{3}\begin{bmatrix} 0 & 10 & 0 & 5 & 1 \\ 0 & 3 & 0 & 25 & 15 \end{bmatrix}$$

As in section(10.3) the sequence of finite gains that assigns (as $\epsilon \to 0$) the polynomial p(s) is

$$[A,B]+\epsilon[A',B']=\begin{bmatrix} 1 & 10\epsilon/3 & 0 & 5\epsilon/3 & \epsilon/3 \\ 0 & \epsilon & 1 & (25\epsilon-3)/3 & 5\epsilon-1 \end{bmatrix}$$

therefore the required sequence (or one parameter family) of 2x3 feedback compensators is given by:

$$K_\epsilon=(A+\epsilon A')^{-1}(B+\epsilon B')=\begin{bmatrix} -10/3 & 5(6-47\epsilon)/9 & (10-49\epsilon)/3 \\ 1/\epsilon & (25\epsilon-3)/3\epsilon & (5\epsilon-1)/\epsilon \end{bmatrix}$$

Indeed, one can easily verify that

$$\det([I,K_\epsilon]\ M(s))= s^5+5s^4+(10-191\tfrac{\epsilon}{9})s^3+ \frac{117-902\epsilon}{9}\ s^2+ \frac{51+235\epsilon}{9}\ s+1+33\epsilon$$

which tends to p(s) as $\epsilon \to 0$. Selecting, for instance, $\epsilon=0.01$ we get

$$K_{0.01}=\begin{bmatrix} -3.333 & 3.07 & 3.17 \\ 100 & -91.667 & -95 \end{bmatrix}$$

with closed loop polynomial $p_{0.01}(s)=s^5+5s^4+9.789\ s^3+ \frac{107.98}{9}\ s^2+ \frac{53.35}{9}\ s+1.33$. If we want to approach p(s) more accurately we simply select a smaller ϵ.

□

Next we will show that the pole assignment conditions described in Corollary() are not vague and rarely satisfied. In fact, if mp>n, for a generic strictly proper system, there exists a degenerate point with the property that $L_{\mathcal{D}}$ has full rank. This will ensure us that for an arbitrary system of p-inputs m-outputs and n-states such that mp>n our method is almost always succesfull.

10.6 Conditions for generic pole assignability

In this section, we will establish the existence of a broad family of systems for which generically the above described method can be successfully applied for arbitrary pole assignment. It should be pointed out that the essence of genericity involved in the present framework means that the family of systems for which the methods fail is a proper subvariety of the systems variety and thus can be considered negligible. The method for establishing a properness of the subvariety of systems for which the property fails, is by establishing one example of system for which the method works.

Theorem (10.3) When mp>n, then for a generic system of p-inputs, m-outputs and n-states, there exists a degenerate point $\mathfrak{D}=$***rowspan***[A,B] such that $L_{\mathfrak{D}}$ has full rank.

Proof:

For a generic system of p-inputs,m-outputs and n-states it is known that the controllability indices are:

$$\underbrace{d+1,d+1,\dots d+1}_{r},\ \underbrace{d,d,\dots,d}_{p-r}$$

where d,r are defined by the Euclidean division of integers: n=pd+r where $r<p$. Consider the following coprime matrix:

$$M(s)=\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} s^{d+1} & & & & & & \\ 1 & s^{d+1} & & & & & \\ & 1 & s^{d+1} & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & s^{d} & \\ & & & & & 1 & s^{d} \\ & & & & & & s^{d-1} \\ & & & & & & s^{d-2} \\ & & & & & & s^{d-3} \\ & & & & & & \vdots \\ & & & & & & 1 \\ & & & & & 0 & \vdots \\ & & & & 0 & 1 & 0 \end{bmatrix}$$

This matrix can be formed since m>d (it is an easy implication of mp>n). The above system has the following degenerate point (with respect to the rightmost column):

$$[A,\ B]=\begin{bmatrix} 1 & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & \\ & & & \vdots & & & & & & & & & & \\ & & & & \vdots & & & & & & & & & \\ & & & & & 1 & & & & & & & & \\ 0 & 0 & 0 & .. & .. & .. & 0 & 0 & 0 & .. & .. & 0 & 1 \end{bmatrix}$$

in fact,

$$AD(s) + BN(s)=\begin{bmatrix} s^{d+1} & 0 & .. & & & .. & 0 \\ 1 & s^{d+1} & & & & & 0 \\ 0 & 1 & s^{d+1} & & & & \vdots \\ \vdots & & 1 & & & & \\ & & & .. & & & \vdots \\ \vdots & & & & .. & s^{d} & 0 \\ 0 & .. & & & 0 & 1 & 0 \end{bmatrix}$$

To find $L_{\mathfrak{D}}$ we need to find all $p_{ij}(s)=\det(D_{ij}(s))$ where $D_{ij}(s)$ by p-1 rows of AD(s) + BN(s) and one row of M(s) (see theorem(10.2)). Since the rightmost column of AD(s) + BN(s) is 0, the determinant of $D_{ij}(s)$ can be written as a product of the minor formed by the (1,2,...,i,i+1,...,p) rows and (1,2,...,p-1) columns of AD(s) + BN(s) and the polynomial element $m_{pj}(s)$ of M(s). Consequently we need to calculate all the principal minors of the px(p-1) matrix formed by the (1,2,...,i,i+1,...,p) rows of AD(s) + BN(s) and multiply them with each of the elements of $[s^d,s^{d-1},...,s^2,s,1]^T$. One can readily see that these minors are the polynomials 1, s^{d+1}, $s^{2(d+1)}$,..., $s^{r(d+1)}$, $s^{(r+1)d+r}$, $s^{(r+2)d+r}$,...,$s^{(p-1)d+r}$. If we multiply them with all the polynomials $s^d,s^{d-1},...,s^2,s,1$ we get the polynomials 1, s, s^2,..., s^n. This proves that the coefficient matrix $L_{\mathfrak{D}}$ of the polynomials $p_{ij}(s)$ has rank equal to n+1, or in other words that it has full rank.

□

Corollary(10.2) If mp>n for a generic strictly proper system of p-inputs, m-outputs and n-states every closed loop polynomial can be (approximately) assigned by sequences of feedback controlllers converging to a degenerate gain.

□

A consequence of the above is the following generic pole assignment theorem:

Theorem(10.4) If mp>n, then for a generic strictly proper system of p-inputs, m-outputs and n-states a generic closed loop polynomial can be (exactly) assigned by a real constant output feedback controller.

Proof

It is sufficient to prove that the image of the (real) pole placement map contains a nonvoid Zarisky open set. Indeed, since the map is regular, this image is a semialgebraic subset of $\mathbb{R}^n$(a set defined by inequalities). Then we have the following two possibilities:

a)Either this image contains a nonvoid Zarisky open set, or

b)There is an n-dimesional semialgebaric subset of $\mathbb{R}^n$ which is not contained in this image.

Because of corollary(10.2), for a generic system, all points of $\mathbb{R}^n$ can be approximated by points of the image of the pole placement map of this system; therefore the case b) is impossible for a generic system, and this proves the result. □

10.7. Globally linearising dynamic feedback controllers: The case of the dynamic pole placement map

Next we will examine the case of using dynamic feedback compensators (on a plant of p-inputs, m-outputs and n-states) for pole placement purposes when the use of constant controllers is not effective (for example when mp<n). In this case we will be interested to find a family of controllers with the smallest possible McMillan degree n_1 such that arbitrary pole assignability is possible. In this setting it is natural to parametrise the controllers according to their number of inputs, outputs and states. The family of all proper controllers of m-inputs, p-outputs and n_1-states is denoted by $P\Sigma_{p,m}^{n_1}$ and consists of all $\mathbb{R}[s]$-***rowspan***$[D_1(s),N_1(s)]$ where $D_1(s),N_1(s)$ is a left coprime MFD pair of a transfer function of a proper system of m-inputs, p-outputs and n_1-states. In this case we want to solve:

$$p(s)=\det\left\{[D_1(s),N_1(s)]\begin{bmatrix}D(s)\\N(s)\end{bmatrix}\right\} \tag{10.18}$$

with respect to $[D_1(s),N_1(s)]$, given that p(s) is an arbitrary polynomial of degree $n+n_1$, and that the transfer function of the plant has a coprime right MFD $N(s)D(s)^{-1}$. As in the constant, PI and OBD cases the dynamic pole placement problem induces a pole placement map

$$F: P\Sigma_{p,m}^{n_1} \longrightarrow \mathbb{R}^{n+n_1} \tag{10.19}$$

where F($[D_1(s),N_1(s)]$) is the coefficient of the closed loop polynomial given by (). The arbitrary pole placement question is translated to examining whether F is onto. Since the dimension of $P\Sigma_{p,m}^{n_1}$ is $n_1(m+p)+mp$ (by counting the free parameters of a generic Popov form as in 4.3.3), a neccessary condition for arbitrary pole assignability will be

$$n_1(m+p)+mp \geq n+n_1 \tag{10.20}$$

(see also [Wil.1]). In [Wil.1], it was conjectured that this also is a sufficient condition. If this is the case by selecting the minimum n_1 that (10.20) is satisfied, we will have the minimum order family of controllers that arbitrarily assign the poles of a system of p-inputs, m-outputs anf n-states. Using the global linearisation methods applied in the constant case, we will prove that $n_1(m+p)+mp>n+n_1$ is a sufficient condition for F to be onto. Furthermore, for every pole polynomial we will construct a one parameter family of compensators in $P\Sigma_{p,m}^{n_1}$ that assign this polynomial to a given plant via feedback.

To apply the global linearisation method (as applied to the constant controllers case) we need to appropriately define the concept of degenerate controllers for the given feedback configuration. These controllers are not necessarily proper systems but they must belong to the closure of $P\Sigma_{p,m}^{n_1}$. Like the case of constant controllers the set $P\Sigma_{p,m}^{n_1}$ is not compact and can be compactified into $\overline{P\Sigma_{p,m}^{n_1}}$ by introducing a set of generalised controllers. In fact the compact set $\overline{P\Sigma_{p,m}^{n_1}}$ contains all p-dimensional $\mathbb{R}[s]$-***rowspan***$[D_1(s),N_1(s)]$ whose polynomial degree is $\leq n_1$. In this setting we have the following definition:

Definition(10.1) A generalised dynamic controller ***rowspan***$[D_1(s),N_1(s)] \in \overline{P\Sigma_{p,m}^{n_1}}$ is a degenerate controller iff:

$$\det\left\{[D_1(s),N_1(s)]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right\}=0 \quad \forall s \tag{10.21}$$

□

Like the constant case we have the following proposition for the calculation of degenerate controllers:

Proposition(10.2) For the standard feedback configuration and for a plant described by the right coprime pair N(s), D(s) or the composite matrix $M(s)=[D(s)^T,N(s)^T]^T$, a p-dimensional R[s] module $\mathcal{V}=$ ***rowspan***$[D_1(s),N_1(s)] \in \overline{P\Sigma^{n_1}_{p,m}}$ corresponds to a dynamic degenerate compensator iff either of the following equivalent conditions hold true:

(i) there exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s)\in\mathbb{R}[s]$-colspan$(M(s))$ such that $[D_1(s),N_1(s)]\underline{m}(s)=0\ \forall s$.

(ii) there exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s)\in\mathbb{R}[s]$-colspan$(M(s))$, whose left kernel contains p $\mathbb{R}[s]$-linearly independent polynomial vectors $\underline{w}_1(s)^T,\ldots,\underline{w}_p(s)^T$ such that the the px(m+p) polynomial matrix $[D_1(s),N_1(s)]=[\underline{w}_1(s),\ \ldots\ ,\underline{w}_p(s)]^T$ has polynomial degree $\le n_1$.

Proof

The proof is similar with the one of the proposition(10.1).

□

From the above proposition it is evident that to finding a degenerate point we need to solve diofantine equations of the type:

$$\underline{w}(s)^T\,\underline{m}(s) = 0$$

with respect to the polynomial vector $\underline{w}(s)^T$. This equation is linear with respect to the coefficient vector of $\underline{w}(s)^T$ and can be written in a scalar matrix equation form in terms of an appropriate Toeplitz matrix.

Proposition (10.3) Consider the equation

$$\underline{x}(s)^T\,\underline{a}(s) = 0 \tag{10.22}$$

where $\underline{x}(s)^T\in\mathbb{R}^{1\times\ell}[s]$ and $\underline{a}(s)\in\mathbb{R}^{\ell\times 1}[s]$. If we write $\underline{x}(s)^T= s^{d_1}\underline{x}_{d_1}{}^T+\ldots+\underline{x}_0{}^T$, $a(s)= s^d\,\underline{a}_d+\ldots+\underline{a}_0$ then eq.() can be rewritten as:

$$\text{vec}(\,\underline{x}\,)\,\mathcal{T}_{d_1}(\,\underline{a}\,) = \underline{0} \tag{10.23}$$

where

$$\text{vec}(\underline{x}) = [\, \underline{x}_{d_1}^{T}, \dots, \underline{x}_0^{T} \,] \in \mathbb{R}^{1 \times \ell(d_1+1)}$$

and

$$\mathcal{T}_{d_1}(\underline{a}) = \begin{bmatrix} \underline{a}_d & \underline{a}_{d-1} & \cdots & \underline{a}_0 & 0 & \cdots & 0 \\ 0 & \underline{a}_d & \underline{a}_{d-1} & \cdots & \underline{a}_0 & \cdots & 0 \\ \vdots & 0 & \underline{a}_d & \cdots & & \underline{a}_0 & \\ \vdots & \cdots & \cdots & \cdots & & & \vdots \\ 0 & & & \underline{a}_d & \underline{a}_{d-1} & \cdots & \underline{a}_0 \end{bmatrix} \in \mathbb{R}^{\ell(d_1+1) \times (d+d_1+1)}$$

□

It is apparent that every polynomial vector x(s) of degree d_1 or less, that satisfies eq(10.22) also satisfies the Toeplitz equation (10.23), and conversely. Although the two equations are equivalent, the Toeplitz equation has the advantage that allows us to find all the solutions of the equation $x(s)^T a(s)=0$ of degree d_1 or less, and this by considering the (scalar) Left Kernel of the Toeplitz matrix $\mathcal{T}_{d_1}(\, a \,)$. Therefore

Remark (10.4) By selecting a basis for the left Kernel of $\mathcal{T}_{d_1}(\, a \,)$ of generic polynomial vector a(s), we automatically select a basis for the solution set of $x(s)^T\ a(s)=0$ of polynomial vectors $x(s)^T$ of degree d_1 or less. □

Corollary (10.3) For a generic polynomial vector $m(s) \in \mathbb{R}[s]^{(m+p)x1}$ and whose degree is d, to have p independent polynomial vectors of degree d_1 or less in its left Kernel, we must have

$$d_1(m+p) + m > d+d_1$$

Proof

Since, by Proposition (10.3), the Toeplitz matrix $\mathcal{T}_{d_1}(\,m\,)$ has $(m+p)(d_1+1)$ rows and $(d+d_1+1)$ columns, the generic dimension of its left Kernel is $(\#rows)-(\#columns)=(m+p)(d_1+1)-(d+d_1+1)$. For this to be greater than or equal to p, we must have $(m+p)(d_1+1)-(d+d_1+1) \geq p$ or equivalently, $d_1(m+p)+m \geq d+d_1+1$ QED.

□

A consequence of the above is the following theorem, which guarantees the existence of degenerate points when $n_1(m+p)+mp > n+n_1$.

Theorem (10.5) A generic plant of p-inputs, m-outputs, n-states subject to dynamic feedback by compensators from the family $\overline{P\Sigma^{n_1}_{p,m}}$ for which $n_1(m+p)+mp > n+n_1$, has always a degenerate point.

Proof

As previously, the controllability indices of a generic system of p-inputs,m-outputs and n-states are:

$$\underbrace{d+1,d+1,\ldots,d+1}_{r},\ \underbrace{d,d,\ldots,d}_{p-r}$$

where d,r are given by the Euclidean division of integers $n=pd+r$, $0 \leq r<p$. Let d_1 be the smallest integer such that $d_1(m+p)+m > d+d_1$ is satisfied. Then, pd_1 is the smallest integer κ such that $\kappa(m+p)+mp > pd+\kappa$, or equivalently, $\kappa(m+p-1)+mp > n-r$ is satisfied; however, this inequality is also satisfied for $\kappa=n_1$ by our assumption. Since pd_1 is the smallest integer having this property, we get $pd_1 \leq n_1$.

Let now $\underline{m}(s)$ be the column of the composite representation M(s) of the plant corresponding to the smallest controllability index which, as we have said, is d. According to Proposition (10.2), a degenerate controller for this system can be obtained by taking p linearly independent polynomial vectors of appropriate degree in the left Kernel of m(s). By the genericity assumption of the plant, we can assume that m(s) is a generic polynomial vector of degree d and therefore, using Corollary(10.3), these p vectors exist iff $d_1(m+p)+m > d+d_1$ (assuming that their degree is d_1 or less). The total degree $\hat{n}_1$ of the matrix M′(s) formed by these p polynomial vectors is obviously less than or equal to pd_1 which, in turn, is less than or equal to n_1, as we previously proved.

Therefore, the polynomial matrix M′(s) constructed this way, satisfies det(N(s)M(s))=0 and its degree $\hat{n}_1$ is less than or equal to n_1 QED.

□

Having seen the way we construct dynamic degenerate points we have to proceed to the crucial part of examining the asymptotic properties of the pole placement map close to a degenerate point. Let ***rowspan***(M′(s)) be a degenerate point for a system with composite M(s). We will approach this degenerate point with sequences of the form $M'(s)+\epsilon B(s)$ as ϵ tends to zero. The matrix B(s) is the direction as in the constant case and we require that B(s) is such that the sequence $M'(s)+\epsilon B(s)$ corresponds to composite representation of proper systems of degree n_1 for almost all ϵ in a neighbourhood of zero. By examining the determinant $\det\big((M'(s)+\epsilon B(s))\ M(s)\big)$ and letting ϵ tend to zero we can prove along similar lines to the constant case the following linearisation theorem:

Theorem(10.6) Let ***rowspan***[M′(s)] be a degenerate gain and $S_\epsilon = M'(s)+\epsilon B(s)$ be a sequence of finite gains converging to it. Then the corresponding sequence of closed loop polynomial vectors $<\underline{p}_\epsilon>$ (modulo dilations) converges as $\epsilon \to 0$ to a vector $<\underline{p}> \in \mathbb{P}(\mathbb{R})^{n+n_1}$. Furthermore the function T which maps the direction B(s) to $<\underline{p}>$ is linear.

Proof

The proof is exactly the same as that of Theorem(10.1) for the constant controllers.

□

Additionally, we have a similar formula for the prime polynomial for the dynamic case

Theorem(10.7) Let $\mathcal{D}$=***rowspan***[M′(s)] $\in \overline{P\Sigma^{n_1}_{p,m}}$ be a degenerate point of a system given by the coprime composite $M(s)=[D(s)^T,N(s)^T]^T$, then the prime polynomial of this system with respect to $\mathcal{D}$ and the direction $B(s) = (b_{ij}(s))$ $1 \leq i \leq p$, $1 \leq j \leq p+m$, can be written as:

$$p(s)=\sum(b_{ij}(s)\ p_{ij}(s)) \tag{10.24}$$

where $p_{ij}(s)$ is the determinant of the pxp polynomial matrix $D_{ij}(s)$ having the same rows with the matrix $\left(M'(s)M(s)\right)$ apart from the i-th which is replaced by the j-th row of M(s).

Proof

The same as that of Theorem(10.2).

□

Therefore, the assignment of prime polynomial is reduced to the solvability of a linear polynomial equation. The requirement that the sequence $M'(s)+\epsilon B(s)$ has to belong to $P\Sigma_{p,m}^{n_1}$ for almost all ϵ imposes certain restrictions on the column degrees of B(s). As a matter of fact, is $n_1= pf+g$ with $g<p$ then we can consider that the column degrees of B(s) all less than f+1. This allows as to write eq(10.24) in a Toeplitz form and then apply elementary linear algebra.

Therefore, we have the following procedure for the construction of (approximate) pole placing dynamic compensators:

a) Select an n_1 such that $n_1(m+p)+mp > n+n_1$.

b) Construct a degenerate point $\mathfrak{D}=$***rowspan*** $M'(s) \in \overline{P\Sigma_{p,m}^{n_1}}$, as described in Propositions (10.2), (10.3).

c) Find upper bounds for the row degrees of the dynamic direction $B(s)=(b_{ij}(s))$ such that $M'(s)+\epsilon B(s)$ to belong to $P\Sigma_{p,m}^{n_1}$.

d)For this B(s) solve the diofantine equation (10.24), or its corresponding Toeplitz equation, if the diofantine equation is not solvable, go to step b).

e)If B(s) is a solution of the diofantine equation then, the one parameter family $M'(s)+\epsilon B(s)$ corresponds to the family of feedback compensators placing the poles of the system at the given set, as $\epsilon\to 0$.

f)Select a small enough ϵ (in $M'(s)+\epsilon B(s)$), to approach the given closed loop pole polynomial as close as you like.

The next example shows how we can utilise all the above to find the smallest order family of compensators that arbitrarily places the poles of a system, and construct the solution.

Example (10.6)

Consider the system of 2-inputs 2-outputs and 8-states whose composite matrix representation of one MFD of its transfer function is given by:

$$M(s) = \begin{bmatrix} s^4 & 0 \\ s^3 & s^4 \\ s & 1 \\ 1 & 0 \end{bmatrix}$$

since 2x2<8 the system does not have the arbitrary pole assignability property via consatant output feedback. The least degree family of controllers that makes this system having this property is the least number n_l such that $4n_l+4 \geq 8+n_l$ that is $n_l=2$. So using this family of controllers we want to assign an arbitrary pole polynomial of the form

$$p(s)=s^{10}+a_9s^9+ a_8s^8+ a_7s^7+ a_6s^6+ a_5s^5+ a_4s^4+ a_3s^3+ a_2s^2+ a_1s + a_0 \tag{10.25}$$

To do so we need to consider a degenerate point in $\overline{P\Sigma_{2,2}^2}$ and approach it via an appropriate direction B(s):

$$B(s)=\begin{bmatrix} b_{11}(s) & b_{12}(s) & b_{13}(s) & b_{14}(s) \\ b_{21}(s) & b_{22}(s) & b_{23}(s) & b_{24}(s) \end{bmatrix}$$

which has to be calculated. To do so consider the degenerate point:

$$M'(s)=\begin{bmatrix} 1 & -s & 0 & 0 \\ 0 & 0 & 1 & -s \end{bmatrix}$$

for this we have:

$$M'(s)\ M(s) = \begin{bmatrix} 0 & -s^5 \\ 0 & 1 \end{bmatrix}$$

Then, the polynomials $p_{ij}(s)$ of the diofantine equation (10.24) are:

$p_{11}(s)=s^4$ $\quad p_{12}(s)=s^3$ $\quad p_{13}(s)=s$ $\quad p_{14}(s)=1$

$p_{21}(s)=s^9$ $\quad p_{22}(s)=s^8$ $\quad p_{23}(s)=s^6$ $\quad p_{24}(s)=s^5$

To solve the approximate pole assignment problem as theorem(10.7) states we need to solve eq(10.24) when p(s) is the monic polynomial of degree 10 given by (10.25), with respect to $B(s)=(b_{ij}(s))$ .It is obvious that in order for the $M'(s)+\epsilon B(s)$ to be -composite denominator, numerator matrix of- a second order system, the rows of B(s) must have degree equal to one. We can now see that p(s) can always be written as:

$$p(s)= s^9(s+a_9) + a_8s^8 + s^6(a_7s+a_6) + a_5s^5 +s^3(a_4s+a_3) + s(a_2s +a_1) + a_0$$

and thus the required direction B(s) is:

$$B(s)=\begin{bmatrix} 0 & a_4s+a_3 & a_2s+a_1 & a_0 \\ s+a_9 & a_8 & a_7s+a_6 & a_5 \end{bmatrix}$$

For instance, if we want to assign the pole polynomial $s^{10}+ s^9+ s^7+ 2s^4+ s^3+ s^2+ s + 1$ then the required direction is:

$$B(s) = \begin{bmatrix} 0 & 2s+1 & s+1 & 1 \\ s+1 & 0 & s & 0 \end{bmatrix}$$

the one parameter family of dynamic feedback compensators that assigns the above polynomial (as $\epsilon\to 0$) is $M'(s)+\epsilon B(s)$. In fact the closed pole loop polynomial for this family is

$$\det\Big((M'(s)+\epsilon B(s))\ M(s)\Big)= \epsilon\ p_\epsilon(s)$$

where $p_\epsilon(s)$ is given by:

$$p_\epsilon(s)= s^{10}(1-2\epsilon)+ s^9(1-3\epsilon) - \epsilon s^8+ s^7(1-2\epsilon) - 2\epsilon s^6+ 2s^4+s^3+ s^2+s(\epsilon+1)+1$$

and we can easily see that $p_\epsilon(s)$ tends to $s^{10}+s^9+s^7+2s^4+s^3+s^2+s+1$ as $\epsilon\to 0$.

□

Next we will prove that the above method can be carried out for almost all systems. Similarly to the case of constant controllers we need only to construct a system of p-inputs, m-outputs and n-states (for every p,m,n) such that our method works.

Theorem(10.8) For every p,m,n there exists a plant of p-inputs, m-outputs and n-states possesing a degenerate point in the family $\overline{P\Sigma_{p,m}^{n_1}}$, where n_1 is such that $n_1(m+p)+mp > n+n_1$. Furthermore this degenerate point has the property that, we can approach all monic closed loop pole polynomials of degree $n+n_1$, via sequences of degree n_1 controllers approaching this degenerate point.

Proof

Like theorem(10.3), the controllability indices of a generic system of p-inputs,m-outputs and n-states are:

$$\underbrace{d+1,d+1,\ldots d+1}_{r},\ \underbrace{d,d,\ldots,d}_{p-r}$$

where d,r are given by the Euclidean division of integers: n=pd+r where $0\leq r<p$. Let d_1 be the smallest positive integer such that $d_1(m+p)+m > d+d_1$. Then we can form the following p-1 (m+p)x1 polynomial vectors:

$$\underline{u}_1(s)=(\underbrace{1,\ s^{d+1}-1,\ s^{d-d_1+1},\ s^{d-2d_1+1},\ s^{d-3d_1+1},\ \ldots,\ s^{d-(p-1)d_1+1}}_{p+1},\ 0,\ 0,\ \ldots,\ 0)^T$$

$$\underline{u}_2(s)=(\underbrace{s^{d_1},\ 1,\ s^{d+1}-1,\ s^{d-d_1+1},\ s^{d-2d_1+1},\ s^{d-3d_1+1},\ \ldots,\ s^{d-(p-2)d_1+1}}_{p+1},\ 0,\ 0,\ \ldots,\ 0)^T$$

⋮

$$\underline{u}_r(s)=(\underbrace{s^{(r-1)d_1},\ s^{(r-2)d_1},\ldots,1,\ s^{d+1}-1,\ s^{d-d_1+1},\ \ldots,\ s^{d-(p-r)d_1+1}}_{p+1},\ 0,\ 0,\ \ldots,\ 0)^T$$

$$\underline{u}_{r+1}(s)=\Big(\underbrace{s^{rd_1}, s^{(r-1)d_1},\dots,1, s^{d}-1, s^{d-d_1},\dots, s^{d-(p-r-1)d_1}}_{p+1}, 0, 0, \dots, 0\Big)^T$$

$\vdots$

$\vdots$

$$\underline{u}_{p-1}(s)=\Big(\underbrace{s^{(p-2)d_1},\dots, s^{2d_1}, s^{d_1}, 1, s^{d}-1, s^{d-d_1}}_{p+1}, 0, 0, \dots, 0\Big)^T$$

Furthermore d_1 being the smallest positive integer such that $d_1(m+p) + m > d+d_1$ is satisfied, d can be written as

$$d=d_1(m+p-1) + r_1 \qquad -p \leq r_1<m$$

and we can select the last polynomial vector depending upon whether r_1 is positive or negative. Specifically, if $r_1 \geq 0$ we have

$$\underline{u}_p(s)= \Big(s^d, s^{d-d_1}, s^{d-2d_1},\dots, s^{(r_1+1)d_1+r_1}, \underbrace{s^{r_1d_1+r_1},\dots, s^{2d_1+2}, s^{d_1+1}, 1}_{r_1+1}\Big)^T$$

otherwise if $r_1< 0$ we can form

$$\underline{u}_p(s)= \Big(s^d, s^{d-d_1}, s^{d-2d_1},\dots, s^{(r_1+1)d_1-r_1}, \underbrace{s^{r_1d_1-r_1},\dots, s^{2d_1-2}, s^{d_1-1}, 1}_{-r_1+1}\Big)^T$$

We consider now a plant of p-inputs, m-outputs and n-states, such that its composite (denominator, numerator) matrix is given by:

$$M(s)=\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = [\,\underline{u}_1(s), \underline{u}_2(s),\dots, \underline{u}_p]$$

Then we can see that a degenerate point for M(s) is given by

$$M'(s)=\begin{bmatrix} -1 & s^{d_1} & 0 & \cdots & & 0 \\ 0 & -1 & s^{d_1} & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & -1 & s^{d_1} & 0 \end{bmatrix} \in \mathbb{R}[s]^{p\times(m+p)}$$

indeed,

$$M'(s)\,M(s)=\begin{bmatrix} s^{d+d_1+1}-s^{d_1}-1 & 0 & & \cdots & & & 0 \\ 1 & \vdots & & & & & \\ 0 & \vdots & \vdots & & & & \vdots \\ & & \vdots & s^{d+d_1+1}-s^{d_1}-1 & & & \\ \vdots & & & 1 & s^{d+d_1}-s^{d_1}-1 & & \vdots \\ & & & & 1 & \vdots & \\ & & & & & \vdots \quad s^{d+d_1}-s^{d_1}-1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{bmatrix}$$

and additionally $M'(s)$ has degree pd_1 ($<n_1$) and therefore belongs to $\overline{P\Sigma^{n_1}_{p,m}}$. Since the p-th column of $M'(s)M(s)$ is zero, the polynomials $p_{ij}(s)$ of the expansion (10.24) of the prime polynomial $p(s)$ are given (up to sign) by:

$$p_{ij}(s)=\alpha_i(s)\,\beta_j(s)$$

where $\alpha_i(s)$ is the determinant of the matrix formed by the 1,2,3,...,p-1 columns and the $1,2,\ldots i-1,i+1,\ldots p$ rows of $M'(s)M(s)$ and $\beta_j(s)$ is the j-th element of the vector $\underline{u}_p(s)$. The polynomials $\alpha_i(s)$ can be easily calculated due to the bidiagonal structure of $M'(s)M(s)$. Specifically,

$$\alpha_i(s)=\left(s^{d+d_1+1}-s^{d_1}-1\right)^{i-1} \quad \text{when} \quad 1\le i\le r+1$$

and

$$\alpha_i(s)=\left(s^{d+d_1+1}-s^{d_1}-1\right)^{r}\left(s^{d+d_1}-s^{d_1}-1\right)^{i-r-1} \quad \text{when } r+2\le i\le p$$

From this, we can see that the degrees of $p_{ij}(s)$ are distinct, and vary from $n+(p-1)d_1$ (the degree of $p_{p1}(s)$) to 1 (the degree of $p_{1(p+m)}(s)$). If we reorder the $p_{ij}(s)$ with respect to their degrees, we can see that the difference of degrees of two consecutive ones is not more than d_1+1. These facts can be used to prove that the diofantine equation

$$p(s)=\sum (b_{ij}(s)\ p_{ij}(s))$$

has an appropriate solution $b_{ij}(s)$, when $p(s)$ is a generic polynomial of degree $n+n_1$. Indeed, consider first the polynomial Euclidean division:

$$p(s)=\pi(s)\ p_{ij}(s) + r(s) \qquad deg(r(s)) < deg(p_{ij}(s))$$

where $p_{ij}(s)$ is the largest degree polynomial (namely $p_{p1}(s)$). Next divide $r(s)$ by the $p_{ij}(s)$ of the next degree and continue this way until the $p_{ij}(s)$ of the least degree, that is one. This way for every $p_{ij}(s)$ we get a corresponding quotient $b_{ij}(s)$, and the construction is done in such a way that eq(10.24) is satisfied. Furthermore apart from $b_{p1}(s)$ whose degree is n_1-$(p-1)d_1$ all the others have degree less than or equal to d_1. This proves that the $b_{ij}(s)$ constructed this way, is a solution of the diofantine equation (10.24) and that the sequence $M'(s)+\epsilon B(s)$ when $\epsilon \neq 0$, corresponds to systems of degree n_1. By theorem (10.7) the sequence of $M'(s)+\epsilon B(s)$ is the sequence of feedback compensators of degree n_1 which can assign to the plant closed loop poles arbitrarily close to the roots of the generic polynomial $p(s)$. □

With the help of the theorem(10.8) we can prove the following generic pole assignment result

Theorem(10.9) If $n_1(m+p)+mp > n+n_1$, then for a generic strictly proper system of p-inputs, m-outputs and n-states, a generic closed loop polynomial can be (exactly) assigned by a dynamic output feedback controller of degree n_1.

Proof

The dynamic pole placement map F of (10.19) is a regular map, therefore with similar arguments to the ones of theorem(10.4) and with the help of theorem(10.19) we can prove that the approximate pole assignability implies exact pole assignability.

□

10.8 Conclusions

Due to a remarkable property of the degenerate points of a feedback configuration, we derived a new method of constructing constant and low degree pole placing compensators. Although the approach is asymptotic, it has important consequences for the exact problem. In fact with this method it was proved that $mp>n$ is a sufficient condition for generic pole placement via constant controllers and that $n_1(m+p)+mp > n+n_1$ is sufficient for the generic pole placement problem via dynamic controllers of degree n_1.

CHAPTER 11. Conclusions

CONCLUSIONS

The main objective of this thesis was to develop and enhance with new concepts and tools, the overall framework of the Determinantal Assignment Problem (DAP). Previous work [Gia.2] was focused on the constant DAP and any work as far as dynamics is concerned was at a rather preliminary stage of development. Work on the decentralised DAP was initiated in [La.1], but the main effort there concentrated upon the exterior algebra based implications rather than solvability conditions of the relevant problems. The starting points of this thesis were: (i) to enrich the algebrogeometric framework in [Gia.2] by formulating and translating many of the intersection results in a cohomology theoretic framework, (ii) to develop the approach to decentralised problems (like those defined in[La.1]), and (iii) to extend the approach to dynamic problems of either a specific controller type (such as PI, Observability index bounded dynamics) or fixed McMillan degree type and provide a unifying systematic framework for computation and parametrisation of families of solutions, whenever such solutions exist. The need to develop the topological dimension of intersection theory of DAP, was motivated by the difficulties faced in the testing of sufficient solvability conditions based on the calculation of degrees of varieties, as well as the problems emerging within the existing framework in extending the odd intersection test to decentralised and dynamic problems. Although, the link of system invariants to solvability conditions has been established in [Kar.1] and [Gia.2], the topological properties (local and global) of the pole, zero placement maps have not been looked up properly (with the exception the work in [Her.1] and [Rein.1]). A great deal of effort was spent on the derivation of solvability conditions for the various versions of DAP, however, little work has been done on the computation of solutions of DAP whenever the solvability conditions are satisfied. DAP has been formulated as an optimisation problem [Gia.2], [Mit.1], but this approach is still in its early stages of development as a systematic method. Furthermore, this optimisation framework allows computation of one solution at a time and does not provide parametrisation. The lack of a systematic computation framework has been evident from the early beginning of this work. The observation, that for the solvability of DAP, the number of controller free parameters must be greater than or equal to the number of constraints, suggests that the solvability of DAP has the complexity of a variety intersection problem only in the boundary case when the number of free parameters is equal to the number of constraints. In the case where the degrees of freedom are greater than the constraints, it was noted that there is a need for alternative methods which explore efficiently the additional degrees of freedom, rather than adopting the sufficient approach that tries to reduce these problems to intersection type, by fixing a certain number of free parameters.

The work in the thesis has three natural sections: the development of the geometric properties of the DAP map, the derivation of new solvability conditions for families of problems stated above using cohomology theory, and the introduction of an entirely new framework for studying DAP in the case where the number of free parameters are greater than the number of constraints. The mathematical tools needed for the above problems come from the areas of algebraic geometry and topology and our objective has been to use these advanced tools without confusing the underlying system theory issues. Thus, a lot of effort was put into explaining the system significance of the tools, and into illustrating their importance by numerous examples and by avoiding unnecessary abstraction and mathematical formalism, which does not seem to be appropriate for the study of the problems under consideration. Although the majority of mathematical tools used here have been standard, to apply them to the special problems it was necessary to adjust them to the specific context and to further develop certain aspects of them. Some of these new developments, and in particular, issues related to compactification of dynamical systems and the work on global linearisation, have a mathematical interest of their own, independent of system theory. In fact, the compactification of dynamical systems and its topology has relevance to issues in Young-Mills theory, whereas the global linearisation may have implications in the study of real algebraic regular maps. However, the examination of such issues has been outside the scope of the thesis and it is an area of future work.

The work in chapter 3, aimed at reviewing and popularising important mathematical tools without the unecessary formalism found in most of the textbooks and has been included to help the reader, with a systems and control background, deal with these issues. The material covered in Ch.3 may be further developed by exploring its significance to further issues in system theory, but due to the emphasis on DAP, this was not done. Chapter 4 has served as a review of the algebrogeometric approaches of system and control theory and its content was restricted to the areas related to DAP. In fact, results of the algebrogeometric framework which do not belong to the DAP range of problems were not considered. In the same Chapter, issues related to the parametrisation of systems which are essential for the algebraic geometry formulation of DAP, were examined. This latter work presents a simple constructive way of structuring the families of systems as real varieties embedded in a projective space via a Plucker type embedding. The properties of this variety and its closure in the projective space, as well as the relations of its topological properties to system invariants, is still under investigation. Chapter 5 provides a detailed formulation of the various DAPs and presents its basic analysis based on exterior algebra. This chapter serves as a prelude to the issues dealt subsequently.

The main part of the work starts with chapter 6, which provides a detailed study of the geometry of the DAP map and feedback related issues. In particular, the properties of the image of the map, such as dimension and structure, and their relations to standard system invariants were examined. This work needs further development in the area of specifying the more detailed semialgebraic structure properties of this map (especially when arbitrary pole assignment is not possible) – this has important implications in the study of the stabilisation problem (instead of arbitrary pole assignment, we require the map to be Hurwitz). The representation of infinite gains as finite points on a certain compactification, has been an integral part of the study of the pole placement map and the present work provides a rigorous treatment, of otherwise intuitive ideas previously used in system theory (treatment of high gain feedback). Further work is in the area of extending the framework to dynamic infinite gains, as well as linking the present approach to duality theory on dynamical systems (define equivalent system problems with finite gains) [Kalo.1]. The notion of system degeneracy, has been systematically examined and relations with the theory of vector bundles have been established. As far as system degeneracy is concerned, the work in chapter 6 has been of a rather general character, and a more detailed study related to global linearisation was made in Chapter 10.

The study of of pole placement by output feedback and zero placement by squaring down, using the tools of the cohomology theory was made in chapter 7 for the case of centralised static compensators. It has been shown rigorously that for such problems the grassmannian provides the correct compactification framework. The overall philosophy for the odd intersection theory [Gia.1] on specially selected subvarieties of the Grassmannian, and in particular Schubert varieties, has been formulated and established in a rigorous way and the overall intersection problem has been translated into an equivalent formulation using cohomology rings. The advantage of the cohomology ring approach is that the odd property of a large number of Schubert varieties can be checked using only the height of the first Whitney class, which has been already calculated using alternative means [Sto.1]; this new test was shown to be the best that can be achieved when the odd intersection philosophy is used. This approach allows the characterisation of systems for which the odd order sufficient condition generically holds, and this is an advantage in comparison to the previous test [Gia.1] based on factorials. The extension of the cohomology ring approach to dynamic DAP problems is still an open issue; work in this area invilves the construction of a "nice" compactification and the more difficult issue of calculating its cohomology ring in a form suitable for the calculation of certain heights. An important issue of this framework, which is related to the computation of solutions is the construction of the representation of the Schubert varieties for which the odd order based test is satisfied. This may allow parametrisation of families of feedbacks for which the DAP is solvable and the solutions explicitly computable.

Chapter 8 has been concern with the study of DAP to two specific families of dynamic pole assignment, ie PI and OBD families of controllers. An attempt to reduce this problem to an equivalent constant DAP was made, but the corresponding compactification as a Grassmannian presented problems. As a result tools used in the previous Chapter were unable to be applied to these cases. Instead, an alternative approach using noncompact varieties, were used for the derivation of necessary as well as sufficient conditions for the existence of complex solutions to these problems. The study of the existence of real solutions and their computation still remains an open question. The search for an appropriate compactification of the variety of PI and OBD controllers is the key issue for these problems, if we aim at using the framework and tools deployed in ch.7. An alternative method for these problems is that based on the results of Chapter 10.

The study of decentralised pole, zero assignment by static controllers has been the subject of Chapter 9. The approach to the study of decentralised DAP suggested in [Kar.4] and [La.1] and based upon decentralised Grassmann variety, was used together with the equivalent (in terms of an isomorphism) formulation of the product of Grassmannians [Wang.3]. The cohomology ring framework based on height was extended to this case and new sufficient conditions for the generic existence of real solutions for the constant output feedback and squaring down decentralised problems were derived. These conditions were in terms of a height of an appropriately constructed class and this height has been computed in terms of the formuli given for the heights of the Grassmannians involved in the product. The results of Chapter6 on the general properties of pole placement map, were extended to the decentralised case and this provided a better understanding of the system theory content of the problem since properties of this map to system invariants, (defined as decentralised Markov parameters), were established. The results in this chapter may be used for the study of suitability of alternative decentralisation schemes, based on the decentralised height criterion and decentralised Markov parameters. The computation of solutions is still an open question within this framework, if we want to avoid the optimisation approach [La.1]. The representation of the odd order Schubert varieties will provide useful computational tools and the alternative framework discussed in Chapter 10 may also introduce some alternative way of thinking for this problem. It should be stressed that it is still an open question, the relationship between the decentralised invariants of the decentralised DAP framework and the special graph based properties of the system that permit the solvability of the decentralised pole assignment problems.

The work in chapter 10 introduces a new framework for the solvability as well as computation of solutions of the constant and dynamic DAP which is different from the intersection dominated philosophy adopted through out the previous chapters. As such, this framework avoids all difficult compactification issues (although some compactification is assumed for defining degenerate points) and apart from new strong solvability conditions provides the means for the parametrisation of families of such solutions. Implicit in this methodology is the detailed study of degenerate feedbacks (static and dynamic case) which characterise the special points where the pole placement map blows up. This property is shown to be behind the global linearisation of the pole placement map and introduces new invariants characterising completely the solvability of the problem ie. the "blow up" matrix. Searching through the various degenerate points and calculating the rank of the blow up matrix allows the assessment of solvability or not of the problem. The characterisation of families of systems ($mp>n$ for static case, and $n_1(m+p)+mp>n+n_1$ for the dynamic case) for which the maximal generic dimension of the blow up is equal to the degree of the assignable polynomials gives the strongest sufficient condition so far on static and dynamic feedback; this also confirms the previous conjecture of Willems [Wil.1] which has been an open problem for the past fifteen years and suggests the way for computing the least required dynamic order for generic pole assignment. A distinct advantage of the global linearisation framework is that it permits the calculation of feedbacks as solutions of a simple linear set of equations and also allows the parametrisation of all such solutions based on the system Plucker matrix and the blow up term associated with the degenerate point. The numerical aspects (ie sensitivity, robustness) of this scheme is to be examined and this is one important area for further work. The extension of this approach to decentralised problems is not a trivial application since such extension heavily relies on the fine algebraic structure aspects of rational vector spaces and modules involved in the definition of the appropriate degenerate points.

The range of problems examined in this thesis by no means exausts those for which the algebrogeometric framework and tools developed in this thesis may be applied. Problems such as simultaneous design, as this is defined by problems of simultaneous pole, zero assignment may be discussed using the same tools. Applying these tools to system parametrisation issues, establishing links between Plucker invariants (integral part of the present framework) and standard invariants as the recent work in [Kar.8] on relations with Kronecker invariants is essential in extending the importance of this approach not only to the study of control problems but also in the better understanding of underlying system theory issues. A rather important limitation of the present overall approach is that it is based on the assignment of coefficients (or roots) of polynomials. Extension of this framework to the stabilisation problems is very important, but not a straight forward exercise. Although the basis exterior algebra initial analysis is still valid, the theory of semialgebraic sets becomes now central rather than algebraic geometry. This area is an open challenge.

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