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# PhD Thesis on Liquidity of Bond Markets



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A thesis submitted for the degree of

*Doctor of Philosophy*

July 2, 2013

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## Acknowledgements

First and foremost, I want to thank my supervisor, Professor Stewart Hodges. It has been a great honour to be his last PhD student. Professor Hodges has been providing a patient and generous guidance in economics and modelling along the whole period of the thesis. I appreciate all his contributions of time and ideas to make my PhD experience productive and stimulating. Without his support this PhD thesis would have not been written.

The presentations at the 5th Financial Risks International Forum in Paris on ‘Systemic Risk’ and the 5th EMG-ESRC Workshop on the Microstructure of Financial Markets at Cass have been very motivating. I am very grateful to the discussants Dr Sophie Moinas and Professor Giovanni Urga for their helpful comments.

I would like to thank the co-director of the PhD programm at Cass, Professor Ian Marsh. I appreciate his time and effort to look for people who do proof reading for PhD thesis.

My time at Cass was made enjoyable in large part due to the many friends. I am very grateful for time spent with Dr Peter Friis, for his proof reading of the previous drafts of this thesis. My time at Cass was also enriched by Chen Yifan, Liu Wei, Yan Cheng, Zhao Bo, Zhao Hongbiao and our table tennis and badminton competitions.

I am also grateful to the Support Staff: Pratt Malla, Abdul Momin, and many others for their help.

Lastly, I would like to thank my family for all their love and encouragement. For my parents who raised me and supported me in all my pursuits. The faithful support, encouragement, and patience of Ning Jiasheng during the period of this PhD is so appreciated. Thank you.

I would like to dedicate this thesis to my loving family.  
This thesis is also dedicated to Professor Stewart Hodges on the  
occasion of his retirement.

# Abstract

This thesis consists of empirical and theoretical studies on the liquidity of bond markets.

In the first study, we present an extended model for the estimation of the effective bid-ask spread that improves the existing models and offers a new direction of generalisation. The quoted bid-ask spread represents the prices available at a given time for transactions only up to some relatively small trade size. Trades can be executed inside or outside the quoted bid-ask spread. Thus, we extend Roll's model to include multiple spreads of different sizes and their associated probabilities. The extended model is estimated via a Bayesian approach, and the fit of the model to a time series of a year of corporate bond transaction data is assessed by a Bayesian model selection method. Results show that our extended model fits the data better.

Our second study examines the relationships between different liquidity proxies and the non-default corporate yield spread as well as the effective bid-ask spread. We first separate the non-default component of bond spreads from the default one by using the information contained in credit default swaps. We then apply our state-space extension of the Roll model to disentangle the unobservable non-default yield spread from the effective bid-ask spread. The empirical results show that the non-default yield spread has a nonlinear relationship with time to maturity and a positive correlation with the bid-ask spread as well as with the default risk, and therefore may reflect the future expected liquidity. We find that the effective bid-ask spread is related to bond characteristics associated with illiquidity (e.g. time-to-maturity and issue amount) and trading activity measures (e.g. daily turnover, and daily average trade size), indicating that transactions costs are more likely to be associated with the current level of liquidity rather than the future expected liquidity. We also find that the non-default component accounted for a bigger proportion of the yield spread before the financial crisis 2007 - 2009, whereas during the crisis credit risk played a more influential role in determining

the yield spread. Common factors such as the underlying volatility and CDS spread explain more of the variation in the non-default yield spread and the bid-ask spread than idiosyncratic factors such as time-to-maturity, issue size, and trading activity proxies do.

The third study presents an equilibrium model in which the heterogeneity of liquidity among bonds is determined endogenously. In particular we show that bonds differ in their liquidity despite having identical cash flow, riskiness and issue amount. Under certain conditions, we show that investors have strong preference for concentrating trading on a small number of bonds. We conjecture that the identity of the ones which are traded may result from a ‘Sunspot’ equilibrium where it is optimal for traders to randomly label a subset of the bonds as the ‘liquid’ ones and concentrating trading on them. We also show that changing the model assumptions leads to different equilibrium configurations where trading is spread over the bonds. In addition, by utilising the concepts of stochastic dominance, utility indifference pricing, and some specific assumptions on asset value and order arrival rate, the equilibrium prices and bid-ask spreads can be quantified.

# Chapter 1

## Introduction

The financial crisis of 2007-2009 has been the most serious financial crisis since the Great Depression. The immediate cause of the crisis was the bursting of the American housing bubble which peaked in 2005-2006 approximately. As part of the housing and credit booms, financial innovation facilitated the development of complex financial products designed to achieve particular client objectives, such as offsetting a particular risk exposure or to assist with obtaining financing.

Banks and non-bank financial institutions use off-balance-sheet entities to fund investment strategies. The strategy of investing in long-term structured asset backed securities (ABS), such as Mortgage Backed Securities (MBS), Collateralised Debt Obligations (CDOs) and Collateralised Loan Obligations (CLOs), and issuing short-maturity papers in the form of asset-backed commercial papers (ABCPs) exposes them to funding liquidity risk.<sup>1</sup> This means that difficulties with refinancing in credit markets could force them to liquidate their long-term assets. Furthermore, since market liquidity may decrease when you need to sell, investors also face market liquidity risk.<sup>2</sup> For instance, during the crisis a number

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<sup>1</sup>An investor has good funding liquidity if it has enough available funding from its own capital or from (collateralised) loans. Funding liquidity risk is the risk that a trader cannot fund his position and is forced to unwind.

<sup>2</sup>A security has good market liquidity if it is easy to trade, that is, has a low bid-ask spread, small price impact, and high resilience. Market liquidity risk is the risk that the market liquidity worsens when you need to trade.

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of markets were virtually shut down (no bids), such as those for certain asset-backed securities and convertible bonds. The shortage of market liquidity during the crisis is widely regarded as one of the causes of the dramatic drop in asset prices.<sup>1</sup>

Nevertheless, it was still not clear which factor (credit risk or market liquidity risk) was the main force driving up the yield spreads for defaultable securities, especially when both credit and market liquidity risks increased at the same time. In particular, corporate bond yield spreads above Treasury bond yields widened dramatically during the crisis. The yield spreads became much larger than can be explained by expected losses arising from default. This leads to more fundamental questions: can the non-default yield spreads be explained by market liquidity (or illiquidity), and how does market liquidity affect asset prices? This thesis will try to answer the above questions and address related issues on the liquidity of bond markets.<sup>2</sup>

The reasons why we choose to focus on the U.S. corporate bond market are the following: the first reason is the importance of the U.S. corporate bond market. Corporate bonds form one of the largest asset classes in the financial markets. According to the Securities Industry and Financial Markets Association (SIFMA), as of Q2 2011, the U.S. corporate bond market size was \$7.7 trillion with 24% of the total U.S. bond market size;<sup>3</sup> secondly, corporate bonds are clearly subject to credit risk, as corporations sometimes default and the extra yield spread is normally regarded as the compensation for credit risk. Recently, some papers find that yield spreads for corporate bonds are too high to be explained by credit risk alone and suggest that the unexplained portion of corporate yield spreads could be due to liquidity risk.<sup>4</sup> However, more studies are needed to explore this topic. Finally, the most interesting markets for us to study liquidity in are

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<sup>1</sup>The two forms of liquidity are linked and can reinforce each other in liquidity spirals. See Brunnermeier (2009) and Brunnermeier & Pedersen (2008) on the liquidity spiral.

<sup>2</sup>Since market liquidity and market liquidity risk are the focuses of our studies, for convenience, hereafter we will refer to market liquidity as liquidity, and market liquidity risk as liquidity risk. Otherwise, funding liquidity and funding liquidity risk will be explicitly expressed.

<sup>3</sup>The total outstanding issuance of the US corporate bond market was around \$5 trillion, according to the Global Financial Stability Report of the international Monetary Fund (IMF), October 2008.

<sup>4</sup>See, for example, Huang & Stoll (1997), Collin-Dufresne *et al.* (2001), Longstaff *et al.* (2005), Bao *et al.* (2011) and Dick-Nielsen *et al.* (2012).



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those where liquidity is a real problem. In fact, the majority of corporate bonds issued by private and public corporations are traded over-the-counter (OTC). This feature makes corporate bonds more likely to be subject to liquidity risk compared with exchange traded securities such as equity.

Having chosen the type of market and asset that we want to study, a natural question arises as to what portion of corporate yield spreads is attributable to liquidity risk, on top of credit risk. To answer the question, we review previous literature which evaluate the implication of illiquidity on corporate yield spreads. We find that the literature is divided into two main strands. One strand focuses on using the theoretical models to obtain the component that is purely due to credit risk and attribute the portion that is left over to illiquidity. Other studies regress yield spreads on either direct or indirect measures of illiquidity to assess how much of the yield spreads can be explained by liquidity risk after controlling for other factors such as credit risk or tax etc.

However, both methods have pros and cons. We decide to adopt a kind of hybrid approach which combines the above two methods. Hence, we need data from three different and independent sources in order to decompose the yield spreads. First, as we are interested in decomposing the corporate yield spreads, the corporate bond prices are needed. Second, we need some independent data which is, ideally, only associated with pure risk of default. The best example would be credit default swaps which are essentially insurance contracts insuring against losses due to credit events on the corporation issuing the bonds. The final ingredient is the term structure of the risk free rate. One option would be the yields of Treasury bonds which can be downloaded from the Federal Reserve website.

Having analysed the corporate bond and credit default swaps data obtained, we can make the following observations. First, some of the credit risk models failed to either predict credit spreads or generate flexible default intensity term structures. Second, some empirical market microstructure models that have been used previously by other papers to estimate the effective bid-ask spread suffered from misspecification problems when applied to our data set. Third, we find that empirically there exist heterogeneous levels of liquidity and liquidity premia among bonds issued by a single company. This brings us to the forth observation

that very few existing theoretical studies have success in explaining why otherwise identical bonds differ significantly in their liquidity.

## 1.1 Summary of the Dissertation

How do we measure illiquidity? What portion of yield spreads is attributable to liquidity risk? How does liquidity affect bond prices? Our understanding of these fundamental questions still remains limited. This thesis tries to tackle these questions by addressing three closely related issues on the liquidity of bond markets: the estimation of the effective bid-ask spread, the impact of illiquidity on the corporate yield spreads, and the equilibrium bond prices in the presence of illiquidity.

In our first study we present an extended model for the estimation of the effective bid-ask spread. The bid-ask spread is normally regarded as the transaction costs which consist of, firstly, the order processing costs which are associated with the costs incurred when handling a transaction, secondly, the inventory holding costs which are charged by the market maker as compensation for the risk that his inventory value may change adversely when supplying immediacy, and finally, the adverse selection costs which are associated with asymmetric information. Despite of its different components and different interpretations, the bid-ask spread is widely used as a measure of the level of market illiquidity and plays a very important role in asset pricing and market microstructure theories. The motivation for the first study is that a parsimonious spread estimation model with as few assumptions as possible is needed to fit our data set as better as possible. The quoted bid-ask spread represents the prices available at a given time for transactions of some relatively small amounts. Furthermore, as we know, trades can be executed either inside or outside the quoted spread, depending on the type of the trader and the size of the trade. In the original Roll model, it is not possible to distinguish multiple spreads.

Therefore, in our first study we extend the original Roll model to allow trades to be executed at different spreads, by adding an extra parameter  $\lambda$ , the so-called

‘spread multiplier’, which is constructed to separate spreads of different magnitudes. In other words, we generalise Roll’s spread estimator (a scalar) to a vector of spreads with associated probabilities. The parameters in the extended model are estimated by using a Bayesian estimation method proposed by [Hasbrouck \(2004\)](#), and the value of  $\lambda$  is determined via a Bayesian model selection method, suggested by [Chib \(1995\)](#). In the empirical application, we show that the extended model fits the transaction data better.

In our second study we try to understand empirically how liquidity and credit risk affects corporate bond prices, and, in particular, why bonds issued by a single company exhibit heterogenous levels of liquidity and liquidity premia. Firstly, we intend to understand how much of yield spreads is due to default risk and what proportion is associated with the risk of illiquidity. Secondly, we try to find out which factors and how these factors determine the level of illiquidity as well as the liquidity premium. Answering these questions requires first separating credit risk from liquidity risk, and then distinguishing the permanent component of the non-default price residuals, namely the liquidity premium, from the transitory component of the non-default price residuals, e.g. the component arises from illiquidity.

Therefore, we first calculate the non-default price residuals by applying a non-parametric reduced-form credit risk model to price credit default swaps and corporate bonds simultaneously. We then apply a generalised version of the Roll model to the non-default price residuals to separate the liquidity premium from the ‘illiquidity component’.

We find that for the bonds in our sample credit risk played a more important role in determining the yield spreads during the crisis than it did before the crisis. By using the panel data during 2006 - 2010, we examine the relations of both the permanent and transitory components of the non-default price residuals with a group of bond characteristics, some of which are considered to be associated with illiquidity. We find that the transitory component of the non-default price residual is related to both direct and indirect liquidity measures, indicating that the ‘illiquidity component’ may reflect the current level of market illiquidity. The empirical results also show that the permanent component or the non-default yield spread is positively correlated with time-to-maturity and the

bid-ask spread. This implies that the liquidity premium may reflect the future expected illiquidity. Both the liquidity premium and the ‘illiquidity component’ show increasing and concave term structures, and both are positively correlated with default risk. Common factors, such as volatility and CDS spread, account for more of the variation in the liquidity premium and the ‘illiquidity component’ than idiosyncrasy factors, e.g. issue size, time-to-maturity and trading activity measures.

The third study tries to understand the liquidity of bond markets in a theoretical framework. Such a model may help us understand how liquidity affects bond prices through the behaviors and interactions of market participants. Existing theoretical studies usually impose *ex ante* assumptions on bond liquidity. We propose an equilibrium model in which the identity of which bonds are liquid, and the size of their spreads and liquidity discounts, are determined endogenously. The equilibrium is essentially determined by traders optimally choosing their trading strategies and taking into account actions by themselves and others. We show that heterogeneous levels of liquidity can arise even when bonds have identical cash flows, riskiness, and issue amounts. Long-term investors have strong preference for trading concentration, whereas liquidity constrained traders are forced to spread trading across bonds. We show that the identity of which bonds are tradable may result from a ‘Sunspot’ equilibrium. The equilibrium bond prices and bid-ask spreads can be quantified under some specific assumptions on asset value and order arrival rate, by using the concepts of stochastic dominance and utility indifference pricing.

## 1.2 Plan of the Dissertation

Chapter 2 reviews the most recent empirical and theoretical literature about the liquidity of bond markets. For the purpose of this thesis, we classify the literature into three broad categories: the first strand includes the literature about the methods and techniques used to decompose corporate yield spreads; the second strand reviews the literature about the models and useful tools to extract the effective bid-ask spreads from real transaction data; the third strand

particularly focuses on the theoretical models which study liquidity in context of asset pricing. Before discussing these literature, the concept of liquidity is briefly introduced, followed by a description of the microstructure of the U.S. corporate bond market. Chapter 3 sets out an extended model used for estimating the effective bid-ask spread as well as the underlying return volatility. Our extended model accounts for multiple spreads with their associated probabilities. Empirical results based on real transaction data show that the extended model fits better than Roll's model. Chapter 4 begins the empirical investigation of how liquidity and credit risk affects corporate yield spreads, and especially why bonds issued by a single company exhibit heterogenous levels of liquidity and liquidity premia. This chapter confirms that credit and liquidity risks played important roles in determining the yield spreads both before and during the financial crisis, and the differentials and variations in the level of liquidity and liquidity premium are associated with both common and idiosyncratic factors which are considered to be linked to bond illiquidity. Chapter 5 introduces an equilibrium model to explain why bonds differ in liquidity, and especially why some bonds are more liquid and more expensive than other (otherwise identical) bonds. Chapter 6 outlines possible future extensions of our current research. Chapter 7 draws together the conclusions for the theory, findings and their implications.

# Chapter 2

## Literature Review

This chapter reviews the empirical and theoretical studies on asset liquidity, and in particular the liquidity in bond markets. Accordingly, the chapter touches only briefly on recently developed theories of asset pricing, financial econometrics and market microstructure, and concentrates on the studies on bond markets.

Section 2.1 begins with a brief introduction of the concept of liquidity as a basis for reviewing the literature on liquidity and its developments.

Section 2.3 considers mainly the methodologies and results of empirical studies on decomposing the corporate yield spread. In this section we discuss three methods to decompose the corporate yield spread, namely using structural credit risk models, using reduced-form credit risk models, and running regressions with control variables. The last method normally involves panel data analysis which recently has become a widely used tool to analyze cross-sectional time series data (e.g. time series of yield spreads of several bonds).

Section 2.4 covers on to the literature on empirical market microstructure models used to estimate the effective bid-ask spread. The literature includes two main types of models, namely serial covariance spread estimation models and order flow spread estimation models. This section also introduces some techniques that are applied to estimate these models, concentrating on the methods developed recently, particularly those from Bayesian econometrics.

Section 2.5 reviews the recent theoretical literature which addresses address

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the issues related to illiquidity. In this section we first introduce the models that focus on the inventory costs component of the bid-ask spread followed by the models dealing with asymmetric information. Then we move on to review the models describe liquidity differentials among multiple assets. Finally models in which the market functions as a limit order book are also reviewed.

### 2.1 The Concept of Liquidity

In this section we will describe the potential sources of illiquidity in terms of market liquidity, which is related to our topics (ignoring funding liquidity), followed by the concept of market liquidity.

Generally speaking, liquidity is the ease of trading a security. One source of illiquidity is exogenous transaction costs such as brokerage fees, order-processing costs, or transaction taxes. Every time a security is traded, the buyer and/or seller incurs a transaction cost; in addition, the buyer anticipates further costs upon a future sale, and so on, throughout the life of the security. [Amihud & Mendelson \(1986\)](#) argue that transaction costs result in liquidity premia in equilibrium, reflecting the differing expected returns for investors with different trading horizons who have to defray their transaction costs. There is an implicit clientele effect due to which securities which are more illiquid, and which are cheaper as a result, are held in equilibrium by investors with longer holding periods.

Another source of illiquidity is inventory holding risk. Agents are not present in the market at all times, which means that if an agent needs to sell a security quickly, then the neutral buyers may not be immediately available. As a result, the seller may sell to a market maker who buys in anticipation of being able to lay off the position later. The market maker, being exposed to the risk of price changes while he holds the asset in inventory, must be compensated for this risk.

Illiquidity can also arise from asymmetric information. The seller and/or buyer may worry that his counterparty has private information about the fundamentals or order flow of the security. These costs of illiquidity should reflect the risk of trading against traders who possess private information. In addition, because liquidity varies over time, risk-averse investors may require a compensation for being exposed to liquidity risk.

According to [Kyle \(1985\)](#), market liquidity can be summarized in three dimensions, namely, tightness, depth, and resilience. Tightness shows the difference between trade price and actual price, and is usually measured as the bid-ask spread. Depth shows the volume which can be traded at the current price level, and resilience is defined as the speed of convergence from the price level which has been brought by random price changes.



## 2.2 The U.S. Corporate Bond Market

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The bid-ask spread is the measure which has been most widely used in recent studies on market liquidity. The bid-ask spread corresponds to tightness and provides information about how much the cost will be in the case of an immediate transaction. However, the bid-ask spread only expresses the transaction cost for those who wish to execute a marginal trade in the market, and does not provide information about how many units will be absorbed or about the extent to which a price will move after limit orders at the best quoted price have been digested.

Market depth is a dynamic indicator of market liquidity, providing information about the ability of the market to absorb trades as changes in price, which take place upon trade execution. Another dynamic indicator measuring market liquidity is market resilience. This indicator provides information about how the market automatically returns its original state after a certain shock has been added to the market.

Data about the microstructure, such as order flow or volume is required to measure market depth and resilience. However, these types of data are not normally available for securities trading over-the-counter, such as corporate bonds. Empirical models have been developed to estimate the effective bid-ask spreads if the information about order flows or volume is not available. We will review the literature on the estimation of the effective bid-ask spread later in this chapter.

## 2.2 The U.S. Corporate Bond Market

Corporate bonds are a principal source of external financing for U.S. firms. For decades, most U.S. corporate bonds primarily traded in an OTC dealer market.<sup>1</sup> Broker-dealers execute the majority of customer transactions in a principal capacity, and trade among themselves in the inter-dealer market to obtain securities desired by customers or to manage their inventories.

**Biais & Green (2007)** give a good description of the OTC market. The OTC market is made by dealers within and between their offices at prices established by individual negotiation, that is, through bid and ask prices. A dealer creates

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<sup>1</sup>The majority of trades of Municipal bonds, State bonds, Treasury bonds, and Utility bonds are also conducted over the counter.

## 2.2 The U.S. Corporate Bond Market

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and maintains a market for any issue of bonds by announcing openly to the other dealer and broker houses that he stands ready both to buy and sell that security at the bid price and the ask price that he quotes to those who inquire. The OTC market dealers include investment banking houses, OTC houses, stock exchange firms which operate OTC trading departments, and dealer banks. A house that makes a market in an issue usually maintains a position in the security by trading (buying and selling) against its position in the issue. It buys and sells for its own account and risk as principal. The price charged by the OTC dealer will be a net price. The equivalent of a commission is already included in the price. The U.S. corporate bond market is such an OTC dealer market, where non-binding indications of interest are distributed to preferred clients, with trading conducted primarily through telephone and e-mail negotiations.

During the early decades of the twentieth century, corporate bonds were predominantly traded on the New York Stock Exchange's transparent limit order market. However, corporate bond trading largely migrated away from the New York Stock Exchange to an OTC dealer market during the 1940s. [Biais & Green \(2007\)](#) find that this migration was coincident with the growth in bond trading on the part of institutional investors (like pension funds, insurance companies, mutual funds, and endowments), who fare better than individuals in the OTC market. The dealer market for corporate bonds is dominated by large institutional investors. OTC corporate bond trades tend to be large and infrequent. [Bessembinder & Maxwell \(2008\)](#) document an average trade size of \$ 2.7 million for institutional trades in the OTC dealer market. [Edwards & Piwowar \(2007\)](#) report that individual bond issues did not trade 48% of days in their sample.

Corporate bonds are a preferred investment for insurance companies and pension funds, whose long-horizon obligations can be matched reasonably well to the relatively predictable, long-term stream of coupon interest payments from bonds. As a result, most or all of a bond issue is often absorbed into stable 'buy-and-hold' portfolios soon after issue.

Dealer quotations in corporate bonds are not disseminated broadly or continuously. Quotations are generally available only to institutional traders, mainly in response to phone requests. In addition to telephone quotations, some institutional investors have access to 'indicative' quotations through electronic mes-

## 2.2 The U.S. Corporate Bond Market

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saging systems provided by vendors such as Bloomberg. However, these price quotations mainly serve as an indication of the desire to trade, not a firm obligation on price and quantity. Prior to the introduction of Transaction Reporting and Compliance Engine (TRACE), transaction prices were not reported except to the parties involved in a trade.

On January 31, 2001, the Securities and Exchange Commission initiated post-trade transparency in the corporate bond market when it approved rules requiring the National Association of Security Dealers to compile data on all OTC transactions in publicly issued corporate bonds. For each trade, the dealer is required to identify the bond and to report the date and time of execution, trade size, trade price, yield, and whether the dealer bought or sold in the transaction.<sup>1</sup> Not all of the reported information is disseminated to the public: investors receive bond identification, the date and time of execution, and the price and yield for bonds specified as TRACE-eligible. Trade size is provided for investment-grade bonds if the face value transacted is \$ 5 million or less, and for non-investment-grade bonds if the face value transacted is \$ 1 million or less (otherwise, an indicator variable denotes a trade exceeding the maximum reported size). As argued by Bessembinder & Maxwell (2008), investors have benefited from the increased transparency through the introduction of the TRACE system. The availability of this transaction-level data also enables us to study the implications of liquidity on the U.S. corporate bonds.

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<sup>1</sup>Trade direction is available since 3 November 2008.

## 2.3 Decomposition of Corporate Yield Spreads

Having briefly covered market liquidity, we proceed to review the literature on the liquidity of corporate bonds. One of the motivations for studying the liquidity component in corporate bond yield spreads has been the ‘credit risk puzzle’. That is, there is a significant non-default component of corporate bond spreads which cannot be explained by empirical default risk measures or traditional credit risk models. The first strand of literature focuses on decomposing yield spreads by taking advantage of the advent of credit sensitive securities and credit risk models. The second strand relies on measures of illiquidity and modern econometrics models.

### 2.3.1 Decomposition by Structural Models

In this section we will review the literature on the mainstream of structural models before looking at how structural models are used to decompose yield spreads or, perhaps more precisely, how structural models fail to predict credit spreads.

The central distinguishing point of structural models from reduced-form models is the view of debt, equity, and other claims issued by a firm as contingent claims on the firm’s asset value. **Black & Scholes (1973)** first used the idea that the bondholders own the company’s assets, but they give options to the stockholders to buy the assets back. **Merton (1974)** expands this idea to model credit risk. Under absolute priority rules, equity shareholders are residual claimants on the assets of the firm, since bondholders are paid first in event of default. Equity shareholders, in effect, hold a call option on the assets of the firm, with a strike price equal to the debt owed to the bondholders. Similarly, the value of the debt owed by the firm is equivalent to a default-free bond plus a short position on a put option on the assets of the firm.

We will now briefly review the literature on structural models.

#### Default-at-maturity Model

Given a filtered probability space  $\{(\Omega, \mathcal{G}, P), (\mathcal{G}_t : t \in [0, \bar{T}])\}$ , a firm borrows funds in the form of a zero-coupon bond promising to pay a dollar (the face

## 2.3 Decomposition of Corporate Yield Spreads

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value) at its maturity  $T \in [0, \bar{T}]$ . Its price at time  $t \leq T$  is denoted by  $v(t, T)$ . Assume this is the only liability of the firm. Default free zero-coupon bonds of all maturities are also traded, with the default free spot rate of interest denoted by  $r_t$ . Markets for the firm's bond and the default free bonds are assumed to be arbitrage free, hence, there exists an equivalent probability measure  $Q$  such that all discounted bond prices are martingales with respect to the information set  $\{\mathcal{G}_t : t \in [0, \bar{T}]\}$ . The discount factor is  $e^{-\int_0^t r_s ds}$ . Markets need not be complete, so that the probability  $Q$  may not be unique.

Let the firm's asset value be denoted by  $A_t$ . Then,  $\mathcal{F}_t = \sigma(A_s : s \leq t) \subset \mathcal{G}_t$ . Let the firm's asset value follow a diffusion process that remains non-negative:

$$dA_t = A_t \alpha(t, A_t) dt + A_t \sigma(t, A_t) dW_t \quad (2.1)$$

where  $\alpha_t, \sigma_t$  are well defined, and  $W_t$  is a standard Brownian motion. Given a simple debt structure of the firm, a single zero-coupon bond with maturity  $T$  and face value  $K$ , default can only happen at time  $T$ . Assume there is no liquidation costs and renegotiation. Absolute priority holds in the event of default. Furthermore, default happens only if  $A_T \leq K$ . Thus, the probability of default for this firm at time  $T$  is given by  $P(A_T \leq K)$ . The time 0 value of the firm's debt is

$$v(0, T) = E^Q \left( [\min(A_T, K)] e^{-\int_0^T r_s ds} \right) \quad (2.2)$$

Assuming that interest rates  $r_t$  are constant, and that the diffusion coefficient  $\sigma(t, A_t)$  is constant, this expression can be evaluated in closed form. The formula is

$$v(0, T) = e^{-rT} K N(d_2) + A_0 N(-d_1) \quad (2.3)$$

where  $N(\cdot)$  is the cumulative standard normal distribution function,  $d_1 = [\ln(A_0/K) + (r + \sigma^2/2)T] / \sigma\sqrt{T}$ , and  $d_2 = d_1 - \sigma\sqrt{T}$ . The credit spread  $s(0, T)$  is given by

$$s(0, T) = -\frac{1}{T} \ln \frac{v(0, T)}{e^{-rT} K} \quad (2.4)$$

This is the original risky debt model presented in [Merton \(1974\)](#), where the

## 2.3 Decomposition of Corporate Yield Spreads

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firm's equity is viewed as a European call option on the firm's assets with maturity  $T$  and a strike price equal to the face value of the debt. The time  $T$  value of the firm's equity is:  $A_T - \min(A_T, K) = \max(A_T - K, 0)$ .

Early empirical tests were not encouraging. [Jones \*et al.\* \(1984\)](#) find that predicted prices are, on average, 4.5% higher than prices observed in the market. The largest differentials are observed for speculative-grade bonds. [Ogden \(1987\)](#) also finds that the Merton model underpredicts by 104 basis points on average. The combination of restrictive theoretical assumptions and empirical shortcomings gave rise to an enormous theoretical literature generalizing the original model.

### First Passage Models

[Black & Cox \(1976\)](#) generalize the model to allow default prior to time  $T$  if the asset's value hits some prespecified default barrier,  $L_t$ . The economic interpretation is that the default barrier represents some debt covenant violation. In this formulation, the barrier itself could be a stochastic process. Then, the information set becomes  $\mathcal{F}_t = \sigma(A_s, L_s : s \leq t)$ . Assume that in the event of default, the debt holders receive the value of the barrier at time  $T$ . In this generalization, the default time becomes a random variable and it corresponds to the first hitting time of the barrier:

$$\tau = \inf\{t > 0 : A_t \leq L_t\}. \quad (2.5)$$

Here, the default time is a predictable stopping time. Intuitively, a predictable stopping time is “known” to occur “just before” it happens, since it is “announced” by an increasing sequence of stopping times. It is not a “true surprise” to the modeler, since it can be anticipated with almost certainty by watching the path of the asset's value process.

Given the default time in expression (2.5), the value of the firm's debt is given by

$$v(0, T) = E^Q \left( [1_{\{\tau \leq T\}} L_\tau + 1_{\{\tau > T\}} K] e^{-\int_0^T r_s ds} \right) \quad (2.6)$$

If the interest rate  $r_t$  is constant, the barrier is a constant  $L$ , and the asset's

## 2.3 Decomposition of Corporate Yield Spreads

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volatility  $\sigma_t$  is constant; then, expression (2.6) has closed form solution as

$$v(0, T) = Le^{-rT}Q(\tau \leq T) + Ke^{-rT}[1 - Q(\tau \leq T)], \quad (2.7)$$

where

$$Q(\tau \leq T) = N(h_1(T)) + \frac{A_0}{K} e^{(1-2r/\sigma^2)T} N(h_2(T)), \quad h_1(T) = [\ln L - \ln A_0 - (r - \sigma^2)T] / \sigma \sqrt{T}, \quad (2.8)$$

and

$$h_2(T) = [2\ln K - \ln L - \ln A_0 + (r - \sigma^2)T] / \sigma \sqrt{T}. \quad (2.9)$$

### Moody's KMV Model

KMV revived the practical applicability of structural models by implementing a modified structural model called the Vasicek-Kealhofer (VK) model (Crosbie & Bohn (2003), Kealhofer (2003a) and Kealhofer (2003b)). MKMV uses the option-pricing equations derived in the VK framework to derive the market value of a firm's assets and the associated asset volatility. The default barrier at different points in time in the future is determined empirically. MKMV combines market asset value, asset volatility, and the default point term-structure to calculate a Distance-to-default (DD) term structure

$$DD_T = \frac{\log[\frac{A}{X_T}] + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad (2.10)$$

where  $A$  is the firm's asset value,  $\mu$  is the drift of the asset return,  $\sigma$  is the volatility of the asset returns, and  $X_T$  is the default barrier. This term structure is translated to a default probability using an empirical mapping between DD and historical default data.

### Remarks

Direct tests of Merton-style models find that the models seriously underpredict the level of long-term corporate bond spreads.

## 2.3 Decomposition of Corporate Yield Spreads

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Huang & Stoll (1997) calibrate several structural risky bond pricing models to historical data on default rates and loss given default. They find that for high-grade debt, only a small fraction of the total spread can be explained by credit risk. For lower quality debt a large part of the spread can be attributed to default risk.

By implementing a structural model, Elton *et al.* (2001) show that expected default accounts for a small fraction of the premium in credit spreads. Tax effects explain a substantial portion of the difference. The remaining spreads are related to risk premium.

Eom *et al.* (2004) find that some extensions of the Merton model (such as Leland & Toft (1996) and Collin-Dufresne & Goldstein (2001)) overpredict spreads for poorly capitalized firms, but continue to underpredict spreads for large, well-capitalized firms.

Using a set of structural models, Ericsson *et al.* (2005) evaluate the price of default protection for a sample of US corporations. In the residuals for bonds, they find strong evidence for non-default components, in particular an illiquidity premium. CDS residuals reveal no such dependence. This finding supports that CDS spreads do not contain liquidity premium as argued by Longstaff *et al.* (2005).

Ericsson & Renault (2006) develop a structural bond valuation model to simultaneously capture liquidity and credit risk. They assume that the probability of liquidity shocks has a time-varying intensity which follows a square-root process. Simultaneously, given a liquidity shock the price offered by any particular trader is assumed to be a random fraction, which is uniformly distributed, of the perfectly liquid price. The bondholder obtains a Poisson quantity of offers and retains the best one. As for other structural models, due to the complex structure, they are unable to fully calibrate their model. Nevertheless, empirically they regress bond yield spreads on two sets of variables, one that controls for credit risk, and one that proxies for liquidity risk. They find decreasing and convex term structures of liquidity spreads and a positive correlation between the illiquidity and default components of yield spreads.

One potential explanation for why Merton-style models tend to underpredict yield spreads is that these models omit a liquidity component. These models are



## 2.3 Decomposition of Corporate Yield Spreads

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not suitable for our purposes, as they fail to predict credit spreads. Furthermore, the model developed by [Ericsson & Renault \(2006\)](#) is too complicated to be fully calibrated. Therefore, now we will turn our attention to the reduced form models.

### 2.3.2 Decomposition by Reduced Form Models

The other major thread of credit risk modelling research focuses on reduced form models of default. This section reviews the literature on reduced form models and how some of these models are employed to decompose the yield spreads.

The reduced form approach assumes a firm's default time is inaccessible or unpredictable, and driven by a default intensity that is a function of latent state variables. [Jarrow & Turnbull \(1995\)](#), [Duffie & Singleton \(1999\)](#), [Hull & White \(2000\)](#), [Jarrow \*et al.\* \(1997\)](#) and [Duffie & Lando \(2001\)](#) present detailed explanations of several well-known reduced form modelling approaches.

#### [Jarrow & Turnbull \(1995\)](#) Model

The key feature for reduced form models is that the modeler observes the filtration generated by the default time  $\tau$  and a vector of state variables  $X_t$ , where the default time is a stopping time generated by a Cox process  $N_t = 1_{\tau \leq t}$  with an intensity process  $\lambda_t$  depending on the vector of state variables  $X_t$ . A Cox process is a point process which is conditional on the information set generated by the state variables over the entire time interval. The conditional process is Poisson with intensity  $\lambda_t(X_t)$ . In reduced form models, the processes are normally specified under the martingale measure  $Q$ . In this formulation, the stopping time is totally inaccessible. Intuitively, a totally inaccessible stopping time is not predictable so that it is a “true surprise” to the modeler. The payoff to the firm's debt in the event of default is called the recovery rate. This is usually given by a stochastic process  $\delta_\tau$ , also assumed to be part of the information set available to the modeler. For convenience, we assume the recovery rate  $\delta_\tau$  is paid at time  $T$ . The probability

## 2.3 Decomposition of Corporate Yield Spreads

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of default prior to time  $T$  is given by

$$\begin{aligned} Q(\tau \leq T) &= E^Q(N(T) = 1 | \sigma(X_s : s \leq T)) \\ &= E^Q(1 - e^{-\int_0^T \lambda_s ds}). \end{aligned} \quad (2.11)$$

The value of the firm's debt is given by

$$v(0, T) = E^Q([1_{\tau \leq T} \delta_\tau + 1_{\tau > T} 1] e^{-\int_0^T r_s ds}). \quad (2.12)$$

Note the recovery rate process is prespecified by a knowledge of the liability structure in the structural approach, while in reduced form models it is exogenously supplied.

If the recovery rate and intensity processes are constants  $(\delta, \lambda)$  and the recovery is paid at time  $T$  in terms of a fraction of the principal (this assumption is named as Recovery of the Face Value(RFV)), then this expression can be evaluated explicitly, generating the model in [Jarrow & Turnbull \(1995\)](#) where the debt's value is given by

$$v(0, T) = p(0, T)(\delta + (1 - \delta)e^{-\lambda T}) \quad (2.13)$$

where  $p(0, T) = e^{-\int_0^T r_s ds}$ .

### **Duffie & Singleton (1999) Model**

In the [Jarrow & Turnbull \(1995\)](#) model, the following assumptions are strong and counterfactual: the recovery rate  $\delta$  is constant, and default is independent of market condition. [Duffie & Singleton \(1999\)](#) relax these assumption at the cost of analytical tractability.

If the asset has not defaulted by time  $t$ , its market value  $V_t$  would be the present value of receiving  $\varphi_{t+1}$  in the event of default between  $t$  and  $t + 1$  plus the present value of receiving  $V_{t+1}$  in the event of no default, meaning that

$$V_t = h_t e^{-r_t} E^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E^Q(V_{t+1}), \quad (2.14)$$

### 2.3 Decomposition of Corporate Yield Spreads

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where  $\varphi_s$  is the recovery in the event of default at  $s$ ,  $h_s$  is the conditional probability at time  $s$  under a risk-neutral probability measure  $Q$  of default between  $s$  and  $s + 1$  given the information available at time  $s$  in the event of no default by  $s$ , and  $r_s$  is the default-free short rate. Assume that the recovery is paid in terms of the market value (this assumption is named as Recovery of Market Value (RMV)) prior to default,

$$E_s^Q(\varphi_{s+1}) = (1 - L_s)E_s^Q(V_{s+1}), \quad (2.15)$$

where  $L_s$  is the expected fractional loss in market value if default were to occur at time  $t$ , conditional on the information available up to time  $t$ . Then the defaultable bond price is given by

$$\begin{aligned} v(t, T) &= e^{-r_t}((h_t\Delta[1 - L_t] + (1 - h_t\Delta))E^Q(v(t + \Delta, T))) \\ &= e^{-r_t}(1 - h_t\Delta L_t)E^Q(v(t + \Delta, T)). \end{aligned} \quad (2.16)$$

where  $h_s\Delta$  is the conditional probability at time  $s$  of default within  $(s, s + \Delta)$  under  $Q$  given no default by time  $s$ . For small  $\Delta$ ,

$$v(t, T) = e^{-r_t + h_t L_t} E^Q(V_{t+1}). \quad (2.17)$$

Therefore, in continuous time,

$$v(t, T) = E_t^Q(e^{-\int_t^T r_s + h_s L_s ds}). \quad (2.18)$$

In this formulation, one can model  $R = r + hL$  directly, and, therefore, easily apply risk-free term structure models. It is possible to allow correlation among  $r, h$  and  $L$  and identify each contribution separately. Importantly, one can also add liquidity effect to  $R$ , i.e.  $R = r + hL + l$ , where  $l$  can be viewed as the fractional carrying cost of the default instrument.

## 2.3 Decomposition of Corporate Yield Spreads

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### Hull & White (2000) Model

Instead of using a hazard rate for the default probability, the Hull & White (2000) model incorporates a default density concept, which is the unconditional cumulative default probability within one period no matter what happens in other periods. By assuming an expected recovery rate, the model generates default densities recursively based on a set of zero-coupon corporate bond prices and a set of zero-coupon treasury bond prices. Then the default density term structure is used to calculate the premium of a credit default swap contract. The two sets of zero-coupon bond prices can be bootstrapped from corporate coupon bond prices and treasury coupon bond prices.

### Ratings Transition Model

Jarrow *et al.* (1997) extends Jarrow & Turnbull (1995) and employs a discrete time, time-homogeneous finite state space per period Markov Chain  $Q$  to model  $Pr_t(\tau^* > T)$  as

$$Q(t, t+1) = \begin{pmatrix} q_{1,1}(t, t+1) & q_{1,2}(t, t+1) & \cdots & q_{1,k}(t, t+1) \\ q_{2,1}(t, t+1) & q_{2,2}(t, t+1) & \cdots & q_{2,k}(t, t+1) \\ \cdots & \cdots & \cdots & \cdots \\ q_{k-1,1}(t, t+1) & q_{k-1,2}(t, t+1) & \cdots & q_{k-1,k}(t, t+1) \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad (2.19)$$

and

$$Q(t, T) = Q(t, t+1)Q(t+1, t+2) \cdots Q(T-1, T). \quad (2.20)$$

If  $q_{ik}$  is  $ik$ th element of  $Q(t, T)$ , then  $Pr_t(\tau^* > T) = 1 - q_{ik}$ ;  $Q(\cdot, \cdot)$  is the risk-neutral probability.

### Incomplete Information Model

Duffie & Lando (2001) consider a model in which the default time is fixed by the firm's managers so as to maximize the value of equity. Investors cannot observe the assets directly, and receive only periodic and imperfect accounting reports. Assuming a given Markov process,  $A = (A_t)_{t \geq 0}$ , where  $A_t$  represents the

## 2.3 Decomposition of Corporate Yield Spreads

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firms value at time  $t$ , [Duffie & Lando \(2001\)](#) “obscure” the process  $A$  so that it can be observed at only discrete time intervals, and add independent noise, and which is observed at times  $t_i$  for  $i = 1, \dots, \infty$ . The authors derive the distribution of the firm’s asset value conditional on investors’ information, and, from this distribution, the intensity of default in terms of the conditional asset distribution and the default threshold.

### Remarks

Many practitioners in the credit trading industry have tended to gravitate toward the reduced form modelling approach given its mathematical tractability. [Jarrow & Protter \(2004\)](#) argue further that, if one is using the model for risk management purposes - pricing and hedging - then the reduced form perspective is the correct one to take. Prices are determined by the market, and the market reaches equilibrium based on the information that it has available to make its decisions. In marking-to-market, or judging market risk, reduced form models are the preferred modelling methodology.

[Driessen \(2005\)](#) provides evidence for a liquidity component in corporate bond spreads using the [Duffie & Singleton \(1999\)](#) reduced-form pricing approach. They model the default intensity as a function of several common factors and one firm-specific factor as well as two terms that allow for correlation with default-free rates. The liquidity component is modelled by a standard square-root diffusion process. The model is calibrated using only corporate bonds.

[Liu \*et al.\* \(2006\)](#) use a five-factor affine term structure model to jointly model the Treasury, Repo, and swap term structures and show that the swap spread is driven by a persistent liquidity process and a rapidly mean-reverting default intensity process. The state variables follow Gaussian processes with a general correlation structure.

Another very important and related paper in this strand of the literature is by [Longstaff \*et al.\* \(2005\)](#), who fit a simple reduced-form model to both credit default swaps and corporate bonds, and find evidence of a significant non-default component in the yield spread which can be related to the liquidity of a bond.

## 2.3 Decomposition of Corporate Yield Spreads

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The ability of reduced form models to price a variety of fixed income and credit risk products makes them very appealing for our purposes. Moreover, with the rapid growth of the credit derivatives market, credit default swaps provide a ideal way to directly measure default risk.

A Credit Default Swap (CDS) is a bilateral financial contract in which one counterparty (the Protection Buyer) pays a periodic fee, typically expressed in basis points per annum, paid on the notional amount, in return for a Contingent Payment by the Protection Seller following a Credit Event with respect to a Reference Entity. A standard definition of Credit Event<sup>1</sup> includes one or more of the following: bankruptcy, filing for protection, failure to pay, obligation default, obligation acceleration and repudiation/moratorium.

It is possible, and increasingly easier, to terminate or unwind a credit default swap before its maturity. In order to exit a credit position in corporate bonds all the investors have to do is to sell the asset at the market. By contrast, a CDS position involves negotiating the terms of the unwinding with the original counterparty (termination), or getting their consent to have a third party step in the trade in their place (novation or assignment). On a termination, the original counterparties on the CDS agree to tear up the contract at a MTM price, payable from Party A to Party B or vice versa depending on which side is in-the-money at the time. The novation/assignment is more complex in the sense that the original counterparty (eg. Party B) is required to consent that Party C steps into the trade to replace Party A. There will be a cash flow between Parties A and C as in the CDS termination case, and party A will then be out of the picture. The trade will remain existing, only between Parties B and C from that point on.

There are important differences between corporate bond and CDS worth mentioning. A bond investor's ability to take a certain amount of a specific credit risk is limited by the possibility of finding such corporate bonds in the marketplace and the availability of funding for the purchase. Illiquidity can arise from limited arbitrage in the corporate bond market, as sometimes it can be difficult and costly to short sell corporate bonds. A CDS investor buying or selling protection, only needs a fraction of the principal (initial collateral margins or upfront fees), and the mark-to-market effect of interest rate fluctuations to a position is usually

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<sup>1</sup>Restructuring has been excluded from the CDS contract by ISDA since April 8th, 2009.

## 2.3 Decomposition of Corporate Yield Spreads

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negligible. In addition, the availability of CDS is not necessarily linked to the aggregate amount of reference bonds outstanding, nor their maturities. With the introduction of the new standard CDS contract and conventions, because of the generic nature of the cash flows, credit default swaps cannot be demanded particularly in the way Treasury securities or popular equities may. Furthermore, it is important to notice that credit default swaps are essentially insurance contracts. Many investors who buy default protection may intend to hold the position for a fixed time period.

Therefore, credit default swaps are much less sensitive to market liquidity, compared to bonds. [Ericsson \*et al.\* \(2005\)](#) evaluate the price of default protection for a sample of US corporations. In bond residuals, they find strong evidence for non-default components, in particular an illiquidity premium. CDS residuals reveal no such dependence. This finding supports the assumption in [Longstaff \*et al.\* \(2005\)](#) that CDS spreads do not contain liquidity premium. [Blanco \*et al.\* \(2005\)](#) provide evidence that changes in the credit quality of the underlying name are likely to be reflected more quickly in the default swap spread than in the bond yield spread. This may be due to a strongly mean-reverting, non-default component in bond spreads that obscure the impact of changes in credit quality. [Ericsson \*et al.\* \(2009\)](#) investigate the linear relationship between theoretical determinants of default risk and default swap spreads. They find that there is limited evidence for a common factor in residuals, indicating that the liquidity premium in credit default swap spread is negligible.

As shown in [Duffie \(1999\)](#), under the no-arbitrage condition the credit default swap premium equals the spread between the par defaultable floating rate note and the par default-free floating rate note, the credit default swap premium is a biased measure of the default component in the yield spread of fixed-coupon corporate bonds. The adjustment from floating-rate spread to fixed-rate spread can be made explicitly by applying a reduced-form model.

Now, our proposed approach to separate liquidity risk from credit risk is clear. That is, we extract default intensities by fitting a reduced-form model to CDS spreads, then use the estimated default intensities to calculate the fair value of a corporate bond, and finally obtain the non-default price residual by taking the

difference between the fair value and market price. We will discuss in details how to obtain the non-default price residual or non-default yield spread in Chapter 3.

### 2.3.3 Decomposition by regressions

A common feature of empirical research in liquidity is that it generally uses transaction data such as trading volume, number of trades or the bid-ask spread to construct measures of illiquidity. Empirical papers which examine liquidity in bond or equity markets use both direct measures (based on transaction data), and indirect measures (based on bond characteristics and/or last prices). However, transaction data for illiquid securities are often rather sparse. Researchers have to resort to indirect proxies. The most popular approach is to regress the yield spreads on a range of proxies for credit risk, liquidity risk and other effects. Meanwhile, panel regression analysis is often applied to take advantage of huge data sets which contains cross-sectional time series data (e.g. time series of prices of different bonds), as some measures are homogenous across individual bonds but vary over time and others are invariant through time but different cross-sectionally.

### Measures of Illiquidity

We will now review some illiquidity measures frequently used in the empirical liquidity literature. Our focus will be mainly on the recent literature.

Direct illiquidity measures include price impact, transaction cost, trading frequencies and trading volume. Microstructure theory suggests that the transaction cost combined with the price impact of trade is a good measure of an asset's liquidity.

Amihud (2002) examines the effect of illiquidity on the cross-section of stock returns using an illiquidity measure that is related to Kyle (1985) price impact coefficient  $\lambda$ . The measure is called  $ILLIQ = |R|/(P * VOL)$ , where  $R$  is daily return,  $P$  is the closing daily price and  $VOL$  is the number of shares traded during the day.  $ILLIQ$  reflects the relative price change that is induced by a given dollar volume. The Amihud illiquidity measure has been used extensively



## 2.3 Decomposition of Corporate Yield Spreads

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in the literature on equity. When applied to bond markets, this measure can be defined as

$$ILLIQ = \frac{|p_t - p_{t-1}|/p_{t-1}}{Q_t} \quad (2.21)$$

where  $p_t$  is the bond price and  $Q_t$  is the dollar volume of a trade. Larger values of the Amihud measure suggest more illiquid bonds, as a given trade size moves prices more.

Transaction costs cause negative serial dependence in successive observed market price changes. The bond price bounces back and forth within the bid-ask bounce, and higher bid-ask bounces lead to higher negative covariance between adjacent price changes. Assuming market efficiency, the effective bid-ask spread can be measured by  $S_t = 2\sqrt{-cov(\Delta P_i, \Delta P_{i-1})}$  where “cov” is the first-order serial covariance of price changes,  $t$  is the time period for which the measure is calculated and  $\Delta$  represents the adjacent price difference. This is the famous Roll measure [Roll \(1984\)](#). Two assumptions are needed: The asset is traded in an informationally efficient market, and the probability distribution of observed price changes is stationary. Roll’s model generates undefined spread estimates almost half of the time when applied to daily transaction data on equities [[Harris \(1990\)](#)]. It may also suffer from misspecification problem when applied to markets with richer structures. However, despite of its simplicity, it is still very useful as it provides a method to estimate the bid-ask spread based on only transaction prices. Later in section ?? we will discuss Roll’s model and its extensions in detail.

[Mahanti et al. \(2009\)](#) investigate the interaction between market liquidity and the price of credit risk by relating the liquidity of corporate bonds to the basis between the credit default swap price of the issuer and the par-equivalent CDS spread of the bond. The liquidity of a bond is measured by the so-called latent liquidity which is defined in [Mahanti et al. \(2007\)](#) as the weighted average turnover of the investors holding the bond, where the weights are their fractional holdings of the bond. They find that their measure has explanatory power for the liquidity component of the CDS-bond basis.

[Chen et al. \(2007\)](#) investigate bond-specific liquidity effects on the yield spread using three separate liquidity measures, namely bid-ask spread, the liquidity

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proxy of zero returns and a model-based transaction cost estimator developed by [Lesmond \*et al.\* \(1999\)](#). In the presence of transaction costs, investors will trade less frequently. [Bekaert \*et al.\* \(2007\)](#) suggest that the magnitude of the percentage of zero returns is a reasonable measure of illiquidity. [Lesmond \*et al.\* \(1999\)](#) show that their estimator measures the underlying liquidity costs better than the percentage of zero returns. They find that liquidity is priced in both the levels and the changes of the yield spread. In our study we are also interested in the cross-sectional differentials of both the level of liquidity and the liquidity premium across individual bonds. However, unlike their study, we will be able to make use of the information from the credit default swap market to control credit risk, and examine the interaction between credit and liquidity risk.

[Bao \*et al.\* \(2011\)](#) examine the connection between their Roll measure and bond trading activity measures (e.g. trade size, number of trades and turnover). They find that bonds which have smaller trade sizes are more illiquid. Bond trading activity measures may be able to capture the liquidity variations at the individual bond level.

Indirect illiquidity measures include bond characteristics such as issue size or outstanding amount, time-to-issue or time-to-maturity, and price volatility.

The earliest example of this kind of research on corporate bonds, that we know of, is by [Fisher \(1959\)](#), who uses the amount outstanding of a bond as a measure of liquidity and the earnings variability as a measure of the credit risk of the firm, and finds that yield spreads on bonds with low issue sizes are higher. As we discussed earlier, one of the sources of illiquidity is related to searching costs. This searching friction is particularly relevant in over-the-counter markets such as the corporate bond market. The amount outstanding measures the overall availability of a bond, and therefore, reflects the liquidity of a bond.

Time-Since-Issue and Time-To-Maturity are popular proxies for bond liquidity. Examining the corporate bond market, [Schultz \(2001\)](#) finds that newly issued bonds trade more than old bonds. More recently, [Edwards & Piwowar \(2007\)](#) study secondary transaction costs in the corporate bond markets. They find that transaction costs decrease significantly with trade size, and highly rated bonds, recently issued bonds, and bonds close to maturity have lower transaction costs than do other bonds.

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Alexander *et al.* (2000) find that bond issues with higher volume tend to be larger and more recently issued. Bao *et al.* (2011) establish a robust connection between their illiquidity measure and liquidity-related bond characteristics. Similarly, they find that the illiquidity measure is higher for older and smaller bonds, and bonds with smaller average trade sizes and higher idiosyncratic return volatility. These papers not only find strong evidence of a seasoning effect, but also they both suggest that issue amount could be an indicator of bond liquidity. Intuitively, bonds with smaller issue amount tend to be held by long-term investors more easily, reducing the tradable amount and thus their liquidity.

Moreover, in the microstructure literature, market makers' inventory holding costs are often related to price volatility. Alexander *et al.* (2000) use the average of absolute price returns to approximate the underlying volatility. However, they find a negative relation between volatility and yields. Gwilym *et al.* (2002) conduct an examination of the determinants of the bid-ask spread and confirm that credit rating, issue size, and price volatility affect liquidity.

Houweling & Vorst (2005) use liquidity-sorted portfolios in the European market to test whether liquidity risk is priced by using nine proxies for liquidity including issued amount, listing, currency, on-the-run or not, age, whether there are missing prices, yield volatility, the number of quote contributors, and yield dispersion.

Dick-Nielsen *et al.* (2012) analyze liquidity components of corporate bond spreads by using TRACE data. They find that illiquidity premia increased dramatically during the crisis. As no liquidity measure is perfect and any measure is able to capture some information about liquidity, they combine four liquidity proxies, namely the price impact of trades (the Amihud measure), transaction costs (Unique roundtrip cost) and the variability of these two variables, to capture most of the liquidity-related variation of spreads.

In addition to the above research on the corporate bond market, by Covitz & Downing (2007) find that liquidity plays a role in the determination of spreads of commercial papers (CP), but credit quality is a more important determinant of spreads. Their regression analysis is based on three liquidity measures (that is trade volume, dollar volume and maturity) and three credit risk proxies (that is Expected Default Frequency (EDF), credit rating and equity volatility). Li

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*et al.* (2009) estimate a market-wide liquidity factor based on transaction data and document a strong positive relation between expected Treasury returns and liquidity. Their findings suggest that term structure modelling should consider liquidity effects in order to capture yield curve dynamics.

Some researchers are interested in cross-market determinants of liquidity. *Chordia et al.* (2005) explores cross-market liquidity dynamics by estimating a vector autoregressive model for liquidity. They find that shocks to stock and bond market liquidity and volatility are significantly correlated, implying that common factors drive liquidity and volatility in these markets. Using time-series regressions, controlling for market risk by using the stock market index return and the change in implied equity index volatility, *De Jong & Driessen* (2007) provide evidence that corporate bond returns are positively related to changes in the equity and bond market liquidity measures.

The recent liquidity crisis in the interbank lending market warns us that great attention has to be paid to the healthy functioning of this market, which is directly related to the monetary policy of governments. *Eisenschmidt & Taping* (2009) propose a theoretical model for the funding liquidity risk of lenders in unsecured term money markets and presents evidence that the unsecured interbank money market rates reached levels that cannot be explained alone by higher credit risk. *Michaud & Upper* (2008) and Taylor and Williams (2008) *Taylor & Williams* (2009) try to decompose money market rates into credit risk, liquidity and other components by applying regression techniques. Treated as independent variables, the credit component is measured by CDS spreads and liquidity is measured by dummy variables or treated as a residual. They demonstrate a significant impact of liquidity on money market rates.

### Remarks

As we can see, each liquidity measure has its strengths and weaknesses. Moreover, some measures are able to explain cross-sectional differences whereas other measures capture common time series variations. We will use both direct and indirect measures in our panel regression analysis, as one of problems is how much of the variations in the level of liquidity and liquidity premium are common and

how much are idiosyncratic. Panel regression techniques will be reviewed in the next section.

### 2.3.4 Panel Data Analysis

Our data consist of a battery of corporate bonds and credit default spreads. These data enable us to observe not only time series but also cross-sectional variations in the level of liquidity as well as the liquidity premium. Recent developments in econometric technique and statistical software offer us some great tools to analyze panel data. In this section we will review some important issues in panel data analysis. In particular, we will discuss two special cases, namely fixed-effects and random-effects models<sup>1</sup>, followed by diagnostic tests.

#### Fixed-Effects and Random-Effects

Consider fitting models of the form

$$y_{it} = \alpha + x_{it}\beta + \nu_i + \epsilon_{it}. \quad (2.22)$$

In this model,  $\nu_i + \epsilon_{it}$  is the residual, which we have little interest in; we want estimates of  $\beta$  or perhaps  $\alpha$ .  $\nu_i$  is the unit-specific residual; it differs between units, but for any particular unit, its value is constant.  $\epsilon_{it}$  is the “usual” residual with the usual properties (mean 0, uncorrelated with itself, uncorrelated with independent variables, uncorrelated with unit-specific residual  $\nu_i$ , and homoscedastic).

If equation 2.22 holds, we must have that

$$\hat{y}_i = \alpha + \hat{x}_i\beta + \nu_i + \hat{\epsilon}_i, \quad (2.23)$$

where  $\hat{y}_i = \sum_t y_{it}/T_i$ ,  $\hat{x}_i = \sum_t x_{it}/T_i$ , and  $\hat{\epsilon}_i = \sum_t \epsilon_{it}/T_i$ . Subtracting equation

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<sup>1</sup>Both Baltagi (2008) and Wooldridge (2009) provide good overviews of fixed-effects and random-effects models. Allison (2009) provides perspective on using fixed-effects and random-effects models.

## 2.3 Decomposition of Corporate Yield Spreads

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2.23 from equation 2.22, we obtain

$$(y_{it} - \hat{y}_i) = (x_{it} - \hat{x}_i)\beta + (\epsilon_{it} - \hat{\epsilon}_i). \quad (2.24)$$

Equation 2.24 provides the fixed-effects (FE) estimator.

Fixed-effects (FE) should be applied if one is only interested in analyzing the impact of variables that vary over time. FE explore the relationship between predictor and outcome variables within an entity (country, person, company, etc.). Each entity has its own individual characteristics which may or may not influence the predictor variables. When using FE it is assumed that something within the entity may impact or bias the predictor or outcome variables and we need to control for this. This is the rationale behind the assumption of the correlation between the entity's error term and predictor variables. FE remove the effect of those time-invariant characteristics from the predictor variables so we can assess the predictors' net effect.

Another important assumption of the FE model is that those time-invariant characteristics are unique to the individual and should not be correlated with other individual characteristics. Each entity is different, therefore, the entity's error term and the constant (which captures individual characteristics) should not be correlated with other entities' error terms and the constants. If the error terms are correlated, then FE is not suitable any more, since inferences may not be correct and you need to model that relationship (probably using random-effects). In other words, the FE model controls for all time-invariant differences between the individuals, so the estimated coefficients of the FE models cannot be biased because of omitted time-invariant characteristics.

One side effect of the features of FE models is that they cannot be used to investigate time-invariant causes of the dependent variables. Technically, time-invariant characteristics of the individuals are perfectly collinear with the entity dummies. FE models are designed to study the causes of changes within an entity. A time-invariant characteristics cannot cause such a change, because it is constant for each entity. In this case, one may resort to the random-effects models.

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The random-effects (RE) estimator turns out to be equivalent to estimation of

$$(y_{it} - \theta \hat{y}_i) = (1 - \theta)\alpha + (x_{it} - \theta \hat{x}_i)\beta + \{(1 - \theta)\nu_i + (\epsilon_{it} - \theta \hat{\epsilon}_i)\}, \quad (2.25)$$

where  $\theta$  is a function of  $\sigma_\nu^2$  and  $\sigma_\epsilon^2$ . If  $\sigma_\nu^2 = 0$ , meaning that  $\nu_i$  is always 0, then  $\theta = 0$  and equation 2.22 can be estimated by OLS directly. Otherwise, if  $\sigma_\epsilon^2 = 0$ , meaning that  $\epsilon_{it}$  is 0, then  $\theta = 1$  and the FE estimator returns all the information available.

The rationale behind random effects (RE) model is that, unlike the FE model, the variation across entities is assumed to be random and uncorrelated with the predictor and with independent variables included in the model. An advantage of RE is that it is possible to include time invariant variables. In the FE model these variables are absorbed by the intercept. RE assume that the entity's error term is not correlated with the predictors which allows for time-invariant variables to play a role as explanatory variables. In RE one needs to specify those individual characteristics which may or may not influence the predictor variables. The problem with this model is that some variables may not be available, therefore, leading to omitted variable bias in the model.

### Diagnostic Tests

According to Baltagi (2008), cross-sectional dependence is a problem in macro panels with long time series. This is not much of a problem in micro panels. The null hypothesis in the Breusch-Pagan LM test of independence is that residuals across entities are not correlated. Alternatively, Pasaran's CD test can be used to test whether or not the residuals are correlated entities. The null hypothesis is that residuals are not correlated. If cross-sectional dependence is present, it can be corrected by using Driscoll and Kraay standard errors (See Driscoll & Kraay (1998)).

A modified Wald test and a likelihood ratio test for groupwise heteroscedasticity are developed (See Baum (2006)). The null hypothesis is homoscedasticity or constant variance. If the tests show that there is heteroscedasticity, then it

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can be corrected by using robust standard errors or Huber-White standard errors (See [Green \(2008\)](#) and [White \(1980\)](#)).

The disturbances may be serially correlated within the panel. Serial correlation causes the standard errors of the coefficients to be smaller than they actually are and higher R-squared values. A Lagrange-Multiplier test for serial correlation is derived by [Wooldridge \(2002\)](#). The null hypothesis is no serial correlation. When first-order autocorrelation is present, then it can be corrected by using Prais-Winsten estimation (See [Wooldridge \(2009\)](#)).

Additionally, to decide between fixed or random-effects one can run a Hausman test where the null hypothesis is that the preferred model is random-effects against fixed-effects (See [Green \(2008\)](#)).



## 2.4 Estimation of the Effective Bid-ask Spread

In the market microstructure literature on estimating and decomposing the bid-ask spread, there are two classes of models: the serial covariance spread estimation model, and the order flow spread estimation model. We found existing models suffered from misspecification problems when applied to our data. Therefore, we also review some techniques which may help when we develop our models.

### 2.4.1 Serial Covariance Spread Estimation Model

In the serial covariance spread estimation model, the spread measures are derived from the serial covariance properties of transaction price changes.

#### Roll (1984) Model

The first such model was developed by Roll (1984). Transaction costs are inferred from serial covariance of daily equity returns. In an efficient market the price dynamics may be stated as

$$m_t = m_{t-1} + \epsilon_t, \quad (2.26)$$

$$p_t = m_t + sq_t, \quad (2.27)$$

where  $m_t$  is the unobservable efficient price,  $p_t$  is the transaction price observed at time  $t$ ,  $s$  is one-half the bid-ask spread, and the  $q_t$  are trade direction indicators, which take the values  $+1$  for buy orders or  $-1$  for sell orders with equal probability.

The changes in transaction prices between two successive trades is

$$p_t - p_{t-1} = m_t + sq_t - (m_{t-1} + sq_{t-1}), \quad (2.28)$$

$$\Delta p_t = s\Delta q_t + \epsilon_t. \quad (2.29)$$

## 2.4 Estimation of the Effective Bid-ask Spread

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Thus,

$$\begin{aligned} Cov(\Delta p_t, \Delta p_{t-1}) &= E[\Delta q_t \Delta q_{t-1} s^2] + E[\Delta q_t \epsilon_{t-1} s] \\ &\quad + E[\Delta q_{t-1} \epsilon_t s] + E[\epsilon_t \epsilon_{t-1}], \end{aligned} \quad (2.30)$$

where  $Cov(\Delta p_t, \Delta p_{t-1})$  is the first-order autocovariance of the price changes.

In deriving the Method of Moments estimator Roll (1984) makes several assumptions:

1. Successive transaction types are independent. Thus,

$$E[q_t q_{t-1}] = 0.$$

2. The half-spread  $s$  is constant.

3. Order flows do not contain information about future fundamental price changes. Thus,

$$E[\Delta q_{t-1} \epsilon_t] = 0.$$

4. Changes in fundamental value cannot predict order flows:

$$E[\Delta q_t \epsilon_{t-1}] = 0.$$

5. The innovations in the fundamental price process reflect public information and are assumed to be independent. Therefore,

$$E[\epsilon_t \epsilon_{t-1}] = 0.$$

Then,

$$Cov(\Delta p_t, \Delta p_{t-1}) = -s^2. \quad (2.31)$$

This gives Roll's Method of Moments spread estimator

$$\hat{s} = \sqrt{-Cov(\Delta p_t, \Delta p_{t-1})}. \quad (2.32)$$

## 2.4 Estimation of the Effective Bid-ask Spread

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If the innovations in the efficient price process are assumed to follow a normal distribution with mean 0 and constant variance  $\sigma_\epsilon^2$ , the Method of Moments variance estimator is

$$\hat{\sigma}_\epsilon^2 = Var(\Delta p_t) - s^2 Var(\Delta q_t) = Var(\Delta p_t) - 2s^2. \quad (2.33)$$

An advantage of Roll's model is that it relies exclusively on transaction price data.

### **Choi *et al.* (1988)) Model**

Choi *et al.* (1988) modify Roll's estimator and introduce a serial correlation assumption regarding transaction type. An implicit hypothesis in Roll's model is that orders are not serially correlated, meaning that the probability of a buy order is equal to that of a sell order, and is independent of the previous orders. However, when large orders are broken up, there could be serial dependence transaction type.

The probability of continuation is equal to

$$Pr(\Delta q_{t+1} = \Delta q_t) = (1 - \gamma), \quad (2.34)$$

and the covariance between the transaction price variations results in

$$Cov(\Delta p_t, \Delta p_{t-1}) = -s^2 \gamma^2. \quad (2.35)$$

whereas the spread estimator becomes

$$\hat{s} = \frac{1}{\gamma} \sqrt{-Cov(\Delta p_t, \Delta p_{t-1})}. \quad (2.36)$$

### **Stoll (1989) Model**

The probability of an order flow reversal or continuation may be different from  $\frac{1}{2}$  when market makers adjust bid-ask spreads to equilibrate inventory. Stoll

## 2.4 Estimation of the Effective Bid-ask Spread

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(1989) models the relation between the bid-ask spread and the serial covariance of transaction price changes as a function of the probability of a price reversal and the magnitude of a price reversal in order to decompose the bid-ask spread into three components: adverse information costs, order processing costs, and inventory holding costs. The model consists of two equations:

$$Cov(\Delta p_t, \Delta p_{t-1}) = s^2[\delta^2(1 - 2\pi) - \pi^2(1 - 2\delta)], \quad (2.37)$$

$$Cov(\Delta M_t, \Delta M_{t-1}) = \delta^2 s^2(1 - 2\pi), \quad (2.38)$$

where  $M_t$  is mid-quote which is calculated as the midpoint of the bid-ask quotes that prevail just before a transaction,  $\delta = \frac{\Delta p_t | p_{t-1}=A_{t-1}, p_t=A_t}{s} = \frac{-\Delta p_t | p_{t-1}=B_{t-1}, p_t=B_t}{s}$  is the magnitude of a price continuation as a percentage of the bid-ask spread. He shows that the compensation earned by a market maker for order processing and inventory costs is  $2(\pi - \delta)s$ . The remainder of the spread  $[1 - 2(\pi - \delta)]s$  reflects the adverse selection costs.

### George *et al.* (1991) Model

George *et al.* (1991) use daily data and consider changing expectations in their model. As in Roll's model, they also assume that the spread is independent of trade size and the probabilities of trades at the bid and ask are equal. Their model for transaction prices can be written as

$$p_t = m_t + \pi s q_t, \quad (2.39)$$

$$m_t = E_t + m_{t-1} + (1 - \pi) s q_t + \nu_t, \quad (2.40)$$

where the “true” expected return of a security,  $E_t$ , is varying over time.

### Remarks

[Harris \(1990\)](#) shows that Roll's spread estimator generates undefined spread estimates almost half of the time when applied to daily transaction data on equity. In particular, if there exist multiple spreads in the market as our data seems to suggest, the model suffers misspecification problem. Moreover, serial correlation spread estimation models rely on a few moments such as variance and covariance. In a richer or more realistic model, it is often difficult to find enough moments without even mentioning how to compute them.

Recently, [Hasbrouck \(2004\)](#) suggests a Bayesian Gibbs approach to estimate Roll's model and applies it to commodity futures transaction data. In Roll's model there are several latent variables such as fundamental prices of a security and trade direction indicators which are not artificial quantities, but are economically meaningful components of a structural model. The Bayesian approach facilitated by the Gibbs sampler (See [Casella & George \(1992\)](#)) provides a very flexible framework to estimate these latent variables and gives possibility to extend Roll's model. Later we will review the literature on Bayesian model estimation and comparison.

### 2.4.2 Order Flow Spread Estimation Model

In another class of models, the bid-ask spread is estimated via order flow regression models.

#### [Glosten & Harris \(1988\)](#) Model

[Glosten & Harris \(1988\)](#) applied this idea to extract the adverse selection spread component by developing an order flow transaction costs estimation model.

With asymmetric information effects, the order flow has an impact on the equilibrium price, as follows:

$$m_t = m_{t-1} + s(1 - \pi)q_t + \epsilon_t, \quad (2.41)$$

## 2.4 Estimation of the Effective Bid-ask Spread

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where  $\pi$  is the parameter indicating order processing costs as a fraction of total costs, and  $(1 - \pi)$  the fraction of adverse selection costs. This implies that the equilibrium price includes the public information as well as the private information revealed by the order flow  $q_t$ .

If the ‘true’ price reflects adverse selection costs, then the transaction price  $p_t$  reflects order processing costs:

$$p_t = m_t + sq_t. \quad (2.42)$$

They allow the order processing cost and adverse selection cost to be function of trade size. Assume the spread consists of an order processing cost component,  $c_t$ , and a price impact component,  $z_t$ :

$$s_t = c_t + z_t. \quad (2.43)$$

Both spread components are linear in trade size  $v_t$ :

$$c_t = c_0 + c_1v_t, \quad (2.44)$$

$$z_t = z_0 + z_1v_t. \quad (2.45)$$

The model is estimated by Maximum Likelihood and intraday data on prices and trading volume (or net order flow) are needed.

### **Madhavan *et al.* (1997) Model**

Madhavan *et al.* (1997) develop an order flow regression model based on Glosten & Harris (1988) to study intra-day patterns in bid-ask spreads, price volatility, the serial correlations of transaction returns and quote revisions. Their model can be written as

$$\Delta p_t = (\phi + \theta)q_t - (\phi + \rho\theta)q_{t-1} + e_t \quad (2.46)$$

## 2.4 Estimation of the Effective Bid-ask Spread

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where  $\theta$  is the adverse selection component,  $\phi$  is the order processing and inventory component.  $\theta$  measures the degree of information asymmetry. Higher values of  $\theta$  indicate larger revisions for a given innovation in order flow.  $\phi$  represents market makers' cost per share for supplying liquidity. Thus,  $\phi$  captures the transitory effect of order flow on prices.  $\rho$  is the first-order autocorrelation of order flow. When order flow is serially uncorrelated,  $\rho$  is equal to 0. Their model also assumes a fixed order size.

### Huang & Stoll (1997) Model

Huang & Stoll (1997) develop a model to decompose the components of the bid-ask spread. In particular, the spread components differ according to trade size. Let us look at their model in details.

The unobservable fundamental value  $m_t$  follows:

$$m_t = m_{t-1} + \alpha s q_{t-1} + \varepsilon_t, \quad (2.47)$$

where  $\alpha$  is the percentage of the half-spread contributable to asymmetric information. Assuming that past trades are of a normal size of one, the quote midpoint is related to the fundamental value according to

$$M_t = m_t + \beta s \sum_{i=1}^{t-1} q_i, \quad (2.48)$$

where  $\beta$  is the proportion of the half-spread attributable to inventory holding costs, and  $\sum_{i=1}^{t-1} q_i$  is the cumulated inventory from the market open until time  $t - 1$  with  $q_1$  as the initial inventory for the day. The transaction price follows

$$p_t = M_t + s q_t + \eta_t, \quad (2.49)$$

where  $\eta_t$  captures the deviation of the observed half-spread from the constant half-spread as well as rounding errors associated with price discreteness.

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Combining the above equations, the model can be written as the form of observed price changes:

$$\Delta p_t = s(q_t - q_{t-1}) + \lambda s q_{t-1} + e_t, \quad (2.50)$$

where  $\lambda = \alpha + \beta$  and  $e_t = \varepsilon_t + \Delta \eta_t$ . The model provides estimates of the traded spread,  $s$ , and the total adjustment of quotes to trades,  $\lambda s$ .

In particular, they also generalize the model to allow coefficient estimates depending on three trader size categories. Then, the model becomes

$$\Delta p_t = s^s q_t^s + (\lambda^s - 1) s^s q_{t-1}^s + s^m q_t^m + (\lambda^m - 1) s^m q_{t-1}^m + s^l q_t^l + (\lambda^l - 1) s^l q_{t-1}^l + e_t, \quad (2.51)$$

where the half-spread  $s$  and  $\lambda$  differ for small, medium, and large trade sizes.

### Ball & Chordia (2001) Model

Ball & Chordia (2001) generalize the Huang & Stoll (1997) model by explicitly allowing for rounding onto the tick grid. More importantly, they allow the bid-ask spread to vary from transaction to transaction and to continuously depend on prior trade size and market depth. Similar to our approach in the first study, the chosen model is determined by the marginal likelihood computed from the Gibbs sampling output as suggested by Chib (1995).

The transaction price is modeled as follows:

$$p_t = [m_t + (1 - \lambda) s_t q_t]_{Round}, \quad (2.52)$$

where  $\lambda$  is the adverse selection component of the spread and the notation  $[\cdot]_{Round}$  represents rounding onto the tick grid. The unobservable fundamental value is assumed to evolve as follows:

$$m_t = m_{t-1} + \lambda s_t q_t + u_t. \quad (2.53)$$

Here they allow the adverse selection to affect the fundamental value.



## 2.4 Estimation of the Effective Bid-ask Spread

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They argue that the bid-ask spread may vary according to a micro information flows and responding to large information-related trades. Therefore, they model the bid-ask spread as follows:

$$\ln(s_t) = \alpha + \beta \ln(s_{t-1}) + \delta \ln(V_{t-1}/D_{t-1}) + \tau TIME_{t-1} + d_1 BEG + d_2 END + e_t. \quad (2.54)$$

where  $V_{t-1}$  is the volume at the previous trade,  $D_{t-1}$  is the corresponding depth of trade for which the then prevailing bid-ask quote held,  $TIME_{t-1}$  is the time between the last trade and the one before it, and  $BEG(END)$  are dummy variables to represent the first (last) hour of the trading day. The model allows the spread to depend on the volume and depth of previous transaction. The dummy variables capture the intra-day seasonality.

They have observations on the transaction price, the trade direction, the size and depth of the previous trade, and the time between trades as well as whether the trade occurred at the beginning or end of the trading day. However, the latent variables the fundamental value and the effective bid-ask spread are unknown. The model is estimated by using Bayesian estimation method which provides a relaxable framework to estimate the model parameters and the latent variables given the observed data. The development in Bayesian econometric provides an opportunity and reflexivity to estimate such complicated models as [Ball & Chordia \(2001\)](#) model and our model in Chapter 3. With such a powerful technique in hand, we will be able to develop a model whose structure is rich enough to fit the market observations better.

Within a Bayesian framework we can also choose the best model from various alternative specifications. For instance, [Ball & Chordia \(2001\)](#) consider four alternative model specifications:

- M1:  $d_1 = d_2 = \tau = 0$ ,
- M2:  $d_1 = d_2 = 0$ ,
- M3:  $\tau = 0$ ,
- M4: No restriction on the parameters.

## 2.4 Estimation of the Effective Bid-ask Spread

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The first model assumes the spread is a first order autoregressive process and also depends on the transaction size  $V$  and the depth  $D$ . The intuition is that the greater the prior volume per unit depth, the wider the spread will become. The second and third models capture the time-between-trades and intro-day effects on the spread. The best model is chosen by comparing the marginal likelihood computed using the Gibbs outputs from the estimation stage. Standard Markov Chain Monte Carlo methods such as Gibbs sampling and Metropolis-Hastings sampling will be reviewed later on.

### Remarks

As explained by Hasbrouck (2004), in the serial correlation spread estimation models, important latent variables such as bids, asks, efficient prices and trade directions, which are economically meaningful information, are ignored. Modern Bayesian econometric techniques boost the development of the order flow spread estimation model. Equipping Bayesian econometrics, order flow spread estimation models are able to capture a variety of market structures, e.g. the underlying value process may depend on volume or depth, the bid-ask spread may take multiple values or vary continuously over time. Under Bayesian framework, one may also infer the trade directions from the transaction data, although there exist some simple rules to help us classify trades. However, Bayesian approach often does not have close-form solution, and is not very attractive analytically. With large data set or more parameters, the numerical estimation procedure may take several hours or more to finally converge. Nevertheless, given many advantages that the Bayesian approach has given, our extended models follow this approach.

In the following we will first review the literature on trade classification and discuss their advantages and problems before introducing some Bayesian econometric techniques recently developed, including Markov Chain Monte Carlo methods, Bayesian model selection, State space model and Kalman filter.

### 2.4.3 Trade Classification

The data series of some products do not contain information about buy/sell identifier for trades. However, trade classification is useful in the estimation of the net order flow, the probability of informed trading, and the effective bid-ask spread.

When bid-ask quotes are available, a natural method is to compare the trade price with the quotes prevailing at the time of the trade. A trade is classified as a buy trade if the transaction price is above the midpoint of the bid-ask quotes. Trades below the midpoint will be classified as sells. The limitation of this algorithm is that transactions executed at the midpoint cannot be classified by the quote rule.

When bid-ask quotes are not available, a technique known as a ‘tick test’ is commonly used. Each trade is classified by the test into four categories: an uptick, a downtick, a zero uptick, and finally a zero downtick. A trade will be classified as an uptick (downtick) if the price is higher (lower) than the price prior to the trade. When the price is the same as the previous price, if the last price change was an uptick (downtick), then the trade is a zero-uptick (zero downtick). A trade will be classified as a buy if it is an uptick or a zero uptick; otherwise it will be classified as a sell. Similarly, a reverse ‘tick test’ uses the next trade price to classify the current trade. If the next trade occurs on an uptick or zero uptick (a downtick or zero downtick), the current trade will be classified as a sell (buy).

The third rule is a hybrid method, suggested by [Lee & Ready \(1991\)](#). [Lee & Ready \(1991\)](#) method combines the quote rule and the tick rule by first classify all trades that do not occur at the midpoint using the quote rule, and then classify midpoint trades by the ‘tick test’. Using intra-day quote and trade prices on equity, [Finucane \(2000\)](#) finds that using either [Lee & Ready \(1991\)](#) algorithm or the ‘tick test’ results in significantly biased estimates of effective bid-ask spreads and signed volume, however, the ‘tick test’ performs better than [Lee & Ready \(1991\)](#) algorithm.

Using a Nasdaq proprietary data set, [Ellis \*et al.\* \(2000\)](#) find that the quote rule, the tick rule, and the [Lee & Ready \(1991\)](#) rule correctly classify 76.4%, 77.66%, and 81.06% of the trades, respectively. But all rules do not enjoy much

success in classifying trades executed inside the quotes. Therefore, [Ellis \*et al.\* \(2000\)](#) propose a new classification rule: all trades executed at the ask (bid) quote are classified as buys (sells). All other trades are classified by the ‘tick test’. They show the new algorithm outperforms other classification rules.

### 2.4.4 Markov Chain Monte Carlo Method

Markov chain Monte Carlo (MCMC) methods are a class of algorithms for sampling from probability distributions based on constructing a Markov chain that has the desired distribution as its equilibrium distribution. The state of the chain after a large number of steps is then used as a sample of the desired distribution. The quality of the sample improves as a function of the number of steps. The first such method is known as the Metropolis-Hastings (MH) algorithm (See [Chib & Greenberg \(1995\)](#)). Another MCMC method is the Gibbs Sampling algorithm (See [Casella & George \(1992\)](#)). Now we will briefly review these two algorithms.

#### Metropolis-Hastings

The general idea of the Metropolis-Hastings algorithm ([Hastings \(1970\)](#)) is to generate a series of samples that are linked in a Markov chain. The MH algorithm is an extension of the original Metropolis algorithm ([Metropolis \*et al.\* \(1953\)](#)).

The candidate-generating density is denoted  $q(x, y)$ , where  $\int q(x, y)dy = 1$ . This density can be interpreted as that when a process is at the point  $x$ , the density generates a value of  $y$  from  $q(x, y)$ . If  $q(x, y)$  does not satisfy the reversibility condition, for some  $x, y$ ,

$$\pi(x)q(x, y) > \pi(y)q(y, x), \quad (2.55)$$

where  $\pi(\cdot)$  is the density from which we would like to generate sample but the transition kernel is unknown. A possible way is to introduce a probability  $\alpha(x, y) < 1$  such that the reversibility condition is satisfied. Then,

$$\pi(x)q(x, y)\alpha(x, y) = \pi(y)q(y, x). \quad (2.56)$$

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Thus,

$$\alpha(x, y) = \pi(y)q(y, x)/\pi(x)q(x, y). \quad (2.57)$$

Therefore the transitions from  $x$  to  $y$  ( $y \neq x$ ) are made according to  $p_{MH}(x, y) = q(x, y)\alpha(x, y)$ . In order for  $p_{MH}(x, y)$  to be reversible, the probability of move must be set to

$$\alpha(x, y) = \min[\pi(y)q(y, x)/\pi(x)q(x, y), 1], \quad (2.58)$$

$$\alpha(x, y) = \min[\pi(y)q(y, x)/\pi(x)q(x, y), 1], \text{ if } \pi(x)q(x, y) > 0, \quad (2.59)$$

$$= 1, \text{ otherwise.} \quad (2.60)$$

In summary, the MH algorithm starts with an arbitrary value  $x^{(0)}$ :

1. Repeat for  $j = 1, 2, \dots, N$ .
2. Generate  $y$  from  $q(x^{(j)}, \cdot)$  and  $u$  from  $u(0, 1)$ .
3. If  $u \leq \alpha(x^{(j)}, y)$ , set  $x^{(j+1)} = y$ .
4. Else set  $x^{(j+1)} = x^{(j)}$ .
5. Return  $\{x^{(1)}, x^{(2)}, \dots, x^{(N)}\}$ .

### Gibbs Sampling

The Gibbs sampling is a technique for generating random variables from a marginal distribution indirectly, without having to calculate the joint density. As a Markov chain Monte Carlo method (Chib & Greenberg (1996)), the Gibbs sampler generates sample values from the distribution of each variable in turn, conditioned on the current values of the other variables.

Suppose we are given a joint density  $f(x, y_1, \dots, y_p)$ , and are interested in obtaining characteristics of the marginal density

$$f(x) = \int \dots \int f(x, y_1, \dots, y_p) dy_1 \dots dy_p, \quad (2.61)$$

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such as the mean or variance. The most natural and straightforward way would be to calculate  $f(x)$  and use it to obtain the desired characteristic. However, there are many situations where such an integration in equation (2.61) are not available in closed form or difficult to compute numerically. In such cases the Gibbs sampler provides an alternative method for obtaining the marginal density.

Instead of computing  $f(x)$  directly, the Gibbs sampler allows us effectively to generate a sample  $X_1, \dots, X_m \sim f(x)$  without requiring  $f(x)$ . By simulating a large enough sample, the mean, variance, or any other characteristic of  $f(x)$  can be calculated to the desired degree of accuracy.

Let us consider a bivariate random variable  $(x, y)$ , and suppose we would like to compute marginal densities,  $p(x)$  and  $p(y)$ . The idea of Gibbs sampler is that it would be easier to consider in terms of a sequence of conditional distributions,  $p(x|y)$  and  $p(y|x)$ , than it is to consider in terms of the marginal densities by integrating the joint density  $p(x, y)$  over  $x$  or  $y$ .

The sampler starts with some initial values  $x^{(0)}$  for  $x$ . Then we can obtain  $y^{(0)}$  by generating a random variable from the conditional distribution  $p(y|x = x^{(0)})$ . The sampler uses  $y^{(0)}$  to generate a new value  $x^{(1)}$ , drawing from the conditional distribution  $p(x|y = y^{(0)})$ . The Gibbs sampler continues as follows:

$$y^{(i)} \sim p(y|x = x^{(i-1)}), \quad (2.62)$$

$$x^{(i)} \sim p(x|y = y^{(i)}). \quad (2.63)$$

This process is repeated for  $k$  times, generating a sequence of length  $k$ , where a subset of pairs  $(x^{(j)}, y^{(j)})$  for  $1 \leq m \leq j < k$  are taken as the draws from the full joint distribution. The Gibbs sequence converges to a stationary distribution which is the target distribution.

### 2.4.5 Bayesian Model Comparison

Bayes factors have been considered as alternatives to classical  $P$ -values for testing hypotheses and for quantifying the degree to which observable data support or

conflict with a hypotheses. Bayesian model comparison can be based on Bayes factors (Kass & Raftery (1995)). An alternative option is the Deviance Information Criterion (DIC) developed by Spiegelhalter *et al.* (2002).

### Bayes Factors

We begin with data  $D$ , assumed to be have arisen under one of the two hypotheses  $H_1$  and  $H_2$  according to a probability density  $Pr(D|H_1)$  or  $Pr(D|H_2)$ . Given a prior probabilities  $Pr(H_1)$  and  $Pr(H_2) = 1 - Pr(H_1)$ , the data produce a posterior probabilities  $Pr(H_1|D)$  and  $Pr(H_2|D) = 1 - Pr(H_1|D)$ . Because any prior opinion gets transformed to a posterior opinion through consideration of the data, the transformation itself represents the evidence provided by the data. In fact, the same transformation is used to obtain the posterior probability, regardless of the prior probability. Once we convert to the odds scale (odds = probability/(1-probability)), the transformation takes a simple form, From Bayes's theorem, we obtain

$$Pr(H_k|D) = \frac{Pr(D|H_k)Pr(H_k)}{Pr(D|H_1)Pr(H_1) + Pr(D|H_2)Pr(H_2)} (k = 1, 2), \quad (2.64)$$

so that

$$\frac{Pr(H_1|D)}{Pr(H_2|D)} = \frac{Pr(D|H_1)}{Pr(D|H_2)} \frac{Pr(H_1)}{Pr(H_2)}, \quad (2.65)$$

and the transformation is simply multiplication by

$$B_{12} = \frac{Pr(D|H_1)}{Pr(D|H_2)}, \quad (2.66)$$

which is the Bayes factor. Alternatively stated, posterior odds = Bayes factor  $\times$  prior odds, and the bayes factor is the ratio of the posterior odds of  $H_1$  to its prior odds, regardless of the value of the prior odds.

Bayes factor allows multiple model comparison. It is consistent and easy to interpret. However, the use of improper priors may lead to indeterminate answers. Chib (1995) develops a method to evaluate Bayes factors numerically.

### Deviance Information Criteria (DIC)

Spiegelhalter *et al.* (2002) develop the DIC as a model choice criterion. Based on the posterior distribution of  $D(\theta)$ , which is defined as

$$D(\theta) = -2\log f(y|\theta) + 2\log f(y), \quad (2.67)$$

it consists of two components: a term that measures goodness-of-fit and a penalty term for increasing model complexity.

The measure of fit consists of the posterior expectation of the deviance

$$E_{\theta|y}[D] = \hat{D}. \quad (2.68)$$

The second part measures the complexity of the model by the effective number of parameters,  $p_D$ , defined as the difference between posterior mean of the deviance and the deviance evaluated at the posterior mean of the parameters:

$$p_D = E_{\theta|y}[D] - D(E_{\theta|y}[D]) = \hat{D} - D(\hat{\theta}). \quad (2.69)$$

The DIC is defined as the sum of both components

$$DIC = \hat{D} + p_D = D(\hat{\theta}) + 2p_D. \quad (2.70)$$

DIC is easily compatible with MCMC method and works with improper priors. However, DIC lacks a formal derivation and it is intrinsically a large sample measure based on point estimation and not a Bayesian measure.

### 2.4.6 State Space Model and Kalman Filter

State space models (See Chapter 50 written by James D. Hamilton in Engle & Mcfadden (1986) and Kim (1999)), which typically deal with dynamic time series models that involve latent variables, have had a wide range of applications in economics and finance. A common tool used to deal with the standard state space



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model is the Kalman filter, a recursive procedure for computing the estimator of the unobservable variable or the state vector at time  $t$ , based on available information at time  $t$ . When the shocks to the model and the initial unobservable variables are normally distributed, the Kalman filter also enables the likelihood function to be calculated via the prediction error decomposition.

The state space model can be written as

$$y_t = x_t \beta_t + \epsilon_t, t = 1, 2, \dots, T, \quad (2.71)$$

$$\beta_t = \mu + F \beta_{t-1} + \nu_t, \quad (2.72)$$

$$\epsilon_t \sim i.i.d.N(0, R), \quad (2.73)$$

$$\nu_t \sim i.i.d.N(0, Q), \quad (2.74)$$

where  $y_t$  is  $1 \times 1$ ;  $x_t$  is  $1 \times k$ ;  $\beta_t$  is  $k \times 1$ ;  $F$  is  $k \times k$ ;  $R$  is  $k \times k$ ;  $Q$  is  $k \times k$  and  $\epsilon_t$  and  $\nu_t$  are independent. If  $\mu = 0$  and  $F = I_k$ , each regression coefficient in  $\beta_t$  follows a random walk.

The Kalman filter is a recursive procedure for computing the optimal estimate of the unobserved state vector  $\beta_t, t = 1, 2, 3, \dots, T$  based on the appropriate information set, assuming that  $\mu, F, R$  and  $Q$  are known. It provides a minimum mean squared error estimate of  $\beta_t$  given the appropriate information set.

The Kalman filter consists of the following two steps:

1. Prediction: At the beginning of time  $t$ , we may want to form an optimal predictor of  $y_t$ , based on all the available information up to time  $t-1$ :  $y_{t|t-1}$ . To do this, we need to calculate  $\beta_{t|t-1}$ .
2. Updating: Once  $y_t$  is realized at the end of time  $t$ , the prediction error can be calculated:  $\eta_{t|t-1} = y_t - y_{t|t-1}$ . This prediction error contains new information about  $\beta_t$  beyond that contained in  $\beta_{t|t-1}$ .

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Thus, after observing  $y_t$ , a more accurate information can be made of  $\beta_t$ .  $\beta_{t|t}$ , an inference of  $\beta_t$  based on information up to time  $t$ , may be of the following form:  $\beta_{t|t} = \beta_{t|t-1} + K_t \eta_{t|t-1}$ , where  $K_t$  is the weight assigned to new information about  $\beta_t$  contained in the prediction error.

The basic filter is described by the following six equations:

1. Prediction:

$$\beta_{t|t-1} = \mu + F\beta_{t-1|t-1}, \quad (2.75)$$

$$P_{t|t-1} = FP_{t-1|t-1}F' + Q, \quad (2.76)$$

$$\eta_{t|t-1} = y_t - y_{t|t-1} = y_t - x_t\beta_{t|t-1}, \quad (2.77)$$

$$f_{t|t-1} = x_tP_{t|t-1}x_t' + R, \quad (2.78)$$

2. Updating:

$$\beta_{t|t} = \beta_{t|t-1} + K_t\eta_{t|t-1}, \quad (2.79)$$

$$P_{t|t} = P_{t|t-1} - K_tx_tP_{t|t-1}, \quad (2.80)$$

where  $K_t = P_{t|t-1}x_t'f_{t|t-1}^{-1}$  is the Kalman gain, which determines the weight assigned to new information about  $\beta_t$  contained in the prediction error.

The discussion above assumes that the model's parameters are known. However, some of these parameters are usually unknown. In this case, one needs to estimate the parameters first; then the estimate of  $\beta_t, t = 1, 2, \dots, T$  is conditioned on these estimated parameters.

For given parameters of the model, the Kalman filter provides us with prediction error ( $\eta_{t|t-1}$ ) and its variance ( $f_{t|t-1}$ ). In addition, if  $\beta_0$  and  $\{\epsilon_t, \nu_t\}_{t=1}^T$  are Gaussian, the distribution of  $y_t$  conditioned on  $\psi_{t-1}$  the information available up to time  $t - 1$  is also Gaussian:

$$y_t|\psi_{t-1} \sim N(y_{t|t-1}, f_{t|t-1}), \quad (2.81)$$

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and the sample log likelihood function is represented by

$$\ln L = -\frac{1}{2} \sum_{t=1}^T \ln(2\pi f_{t|t-1}) - \frac{1}{2} \sum_{t=1}^T \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1}, \quad (2.82)$$

which can be maximized with respect to unknown parameters of the model. For non-stationary  $\beta_t$ , the log likelihood function is evaluated from observation  $\tau + 1$  ( $\tau \gg 1$ ), where  $\tau$  is large enough.

## 2.5 Microstructure Models

### 2.5.1 Market Making

In this section we will discuss the role of the market maker in determining the bid-ask prices or perhaps the bid-ask spreads.

In an efficient market, the price of a security should reflect its fundamental value. The asynchronous arrival of buyers and sellers, who demand quick transactions, limits liquidity in capital markets. The demand for immediacy creates a role for financial intermediary. A market maker is a simple type of financial intermediary. He stands ready to trade on either side the market for his own account when an order arrives. He may buy primary issues and issue a liability to finance an inventory <sup>1</sup>. A market maker may buy and sell the identical securities.

Therefore, transactions essentially can be summarized as a combination of a primary transaction for the underlying security and a secondary transaction for immediacy. The gap between the fundamental value and transaction prices represents a cost of transacting. The price of immediacy is determined by two factors: the costs of market making and the extent of competition among market makers.

By assuming perfect competition in market making explicitly or implicitly, the price of immediacy is determined as the marginal cost of supplying immediacy.

**Garman (1976)** presents a stochastic model of the dealership market which is dominated by a centralized market maker, who possesses a monopoly on all trading. The possible temporal discrepancy between market buy and sell orders, and the obligation to maintain continuous trading, induce the market maker to carry stock inventories. He suggests that the specialists must pursue a policy of relating their prices to their inventories in order to avoid failure.

**Stoll (1978b)** considers an expected utility maximizing dealer whose quoted prices are a function of the cost of taking a position which deviates from his desired position and derives the implication of his inventory policy on the bid-ask spread and on the structure of the dealership market. The holding cost depends on the dollar size of the transaction, the variance of return of the stock being traded,

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<sup>1</sup>The latter issue is normally omitted by theoretical papers.

the size of the initial holdings of all stocks in the dealer's trading account, the covariance between the return on the stock being traded and the return on the trading account, the wealth of the dealer, and his attitude toward risk.

Amihud & Mendelson (1980) extend Garman's model to allow the prices set by the market maker to depend on his stock inventory position. They show that the prices are monotone decreasing functions of the market maker's stock, and that there exists a desired inventory position.

Market makers bear the uncertainty about the return on their inventories as well as the uncertainty about when future transactions will occur. Their attitude toward risk should not be ignored. O'Hara & Oldfield (1986) examine the pricing policy of a risk-averse market maker. They show that as a result of order flow and inventory value uncertainty, a risk-averse market maker may set a smaller spread than a risk-neutral specialist, and that the market maker's inventory affects both the placement and the size of the spread.

Recently, Shen & Starr (2002) show that the cost of immediacy increases with increasing absolute value of market makers' inventories, volatility of security price and order flow. Comerton-Forde *et al.* (2010) show that market makers' financing constraints affect liquidity.

It is also suggested that the market maker in supplying immediacy is facing two kinds of traders, namely liquidity traders, who do not possess any superior information, and information-motivated traders, who are better informed with information that is not yet reflected in the price. In practice, market makers may not only have to be faced with informed traders but also have to manage their inventories. Copelan & Galai (1983) model the dealer's bid-ask spread as a tradeoff between expected losses to informed traders and expected gains from uninformed traders. They characterize the cost of supplying quotes, as writing a put and a call options to an informed trader.

In the information-based models, market makers are assumed to be risk neutral. Glosten & Milgrom (1985) model a risk-neutral competitive specialist who faces on transactions costs, that is, a specialist whose expected profit from each transaction is zero. It is assumed that traders are asymmetrically informed. The spread is the premium that the specialist demands for trading with traders with superior information. Kyle (1985) models the strategic interaction between an

informed trader and a group of market makers who take the informed trader's strategy into account. He shows how information is incorporated into the equilibrium prices. [Easley & O'Hara \(1987\)](#) show that there is a relationship between price and trade size. This relationship arises because the larger the trade size, the more likely it is that the market maker is trading with an informed trader. Therefore, the market maker's optimal pricing strategy depends on quantity.

Trading costs will be determined by competition among market makers, and lower trading costs may induce more trading activity. [Ho & Stoll \(1981\)](#) and [Copelan & Galai \(1983\)](#) analyze that the behavior of competing dealers and the problem of determining the equilibrium market bid-ask spread. They show that the market spread is determined by the second best dealers, while others solve the pricing problem of a monopolistic dealer. [Goldstein & Nelling \(1999\)](#) study the relations between the number of markets, trading activity, and price improvement in Nasdaq stocks, and show that trading frequency may be more important than trade size in determining the number of market makers.

### 2.5.2 Inventory-based Models

A fundamental source of illiquidity is the fragmentation of investors and markets due to the fact that not all investors are present in the same market all of the time. The gap between sellers and buyers is bridged by market makers who provide immediacy through their continuous presence in the market and thus enable continuous trading by any trader who wishes so. In particular, the market maker can buy from the seller and later sell to the buyer. However, the market maker faces a risk that the fundamental price may change adversely against him, and must be compensated for this risk.

#### [Garman \(1976\)](#) Model

[Garman \(1976\)](#), which is perhaps the earliest study of market microstructure, modeled the relation between dealer quotes and inventory levels based on the intuition above.

A single monopolistic market maker who receives the orders, sets the prices, and clears the trades. His objective is to maximize the expected profit per unit of time, subject to the avoidance of bankruptcy which happens whenever he runs out of either inventory or cash. The market maker with an infinite horizon can only decide at which price he fills the orders wishing to buy the asset (the ask price  $p_a$ ) and at which price he fills the orders wishing to sell the asset (the bid price  $p_b$ ), at the beginning of time.

The uncertainty or the risk arises from the arrival of the buy and sell orders, which are assumed to follow independent Poisson processes, with stationary arrival rate functions  $\lambda_a$  and  $\lambda_b$ . At time 0 the market maker holds  $I_c(0)$  units of cash and  $I_s(0)$  units of the asset. Let  $N_a(t)$  ( $N_b(t)$ ) be the cumulative number of shares that have been sold to (bought from) traders. His inventories at time  $t$  are therefore computed as

$$I_c(t) = I_c(0) + p_a N_a(t) - p_b N_b(t), \quad (2.83)$$

$$I_s(t) = I_s(0) + N_b(t) - N_a(t). \quad (2.84)$$

In Garman's model, we can get the probability of failure:

$$\lim_{t \rightarrow \infty} = \left( \frac{\lambda_b p_b}{\lambda_a p_a} \right)^{I_c(0)/\hat{p}} \quad \text{if } \lambda_a p_a > \lambda_b p_b, \quad (2.85)$$

$$= 1 \text{ otherwise}, \quad (2.86)$$

where  $\hat{p}$  is defined to be the average of the bid and ask prices.

To avoid his failure with probability one, the market maker has to set:

$$\lambda_a p_a > \lambda_b p_b, \quad (2.87)$$

$$\lambda_b > \lambda_a, \quad (2.88)$$

which requires that:

$$p_a \gg p_b. \quad (2.89)$$

It follows that market makers must actively adjust not only the prices but the spreads in relation to inventory.

### Stoll (1978b) Model

Stoll (1978b)

 considers a two-period model in which a dealer quotes bid and ask prices. The dealer is risk averse and demands compensation whenever he acts as liquidity supplier. When he moves away from his efficient portfolio frontier, he sets prices such that the expected utility of his terminal wealth when the portfolio is on the efficient frontier,  $W^{EF}$ , is equal to the expected utility of his terminal wealth,  $W^{dealer}$ , when he takes the other side of a order  $x_i$ . This allows the dealer to be compensated for the liquidity he supplies to the market. CARA utility function is assumed with a coefficient of risk aversion equal to  $A$ .

The other relevant assumptions are the following: the dealer chooses an initial portfolio of risky assets equal to:

$$\sum_{h=1}^N p_h q_h^* = W_0 \quad (2.90)$$

where  $p_h$  is the price of the asset  $h$  and  $q_h^*$  is the optimal choice of the asset  $h$ 's holding at time  $t = 0$ ; the terminal wealth of the initial portfolio when no trade occurs between  $t = 0$  and  $t = 1$  is thus equal to:

$$\tilde{W}^{EF} = \sum_{h=1}^N (\tilde{V}_h - p_h) q_h^* = R^* W_0, \quad (2.91)$$

where  $\tilde{V}_h \sim N(\bar{V}, \sigma_h^2)$  is the price of the asset  $h$  at  $t = 1$ , and  $R^*$  is the return on the risky asset portfolio that lies on the dealer's efficient frontier. The terminal wealth of the new portfolio when a trade equal to  $q_i - q_i^* = -x_i$  occurs at time  $t = 0$  is:

$$\tilde{W}^{dealer} = \sum_{h=1}^N (\tilde{V}_h - p_h) q_h^* + (\tilde{V}_i - p_i)(-x_i), \quad (2.92)$$

where  $p_i$  is equal to the ask price  $p_i^A$ , when the incoming trader submits a buy



order of size  $x_i > 0$  and the dealer sells  $-x_i = q_i - q_i^* < 0$ ; conversely,  $p_i$  is equal to the bid price  $p_i^B$ , when the trader sell  $x_i < 0$  and the dealer buys  $-x_i = q_i - q_i^* > 0$  from this trader.

The dealer quotes competitive prices and he sets the price so that the expected utility from the terminal wealth of the initial portfolio is equal to the expected utility from the terminal wealth from the new portfolio, which includes  $x_i$ :

$$E[U(\tilde{W}^{EF})] = E[U(\tilde{W}^{dealer})]. \quad (2.93)$$

Then, the bid ( $p_i^B$ ) and ask ( $p_i^A$ ) prices for the asset  $i$  can be derived as:

$$p_i^A = \bar{V}_i - AW_0\sigma_{*,i} + \frac{A}{2}\sigma_i^2|x_i|, \text{ with } x_i > 0, \quad (2.94)$$

and

$$p_i^B = \bar{V}_i - AW_0\sigma_{*,i} - \frac{A}{2}\sigma_i^2|x_i|, \text{ with } x_i < 0 \quad (2.95)$$

where  $\sigma_{*,i}$  is the covariance of the asset  $i$  with the portfolio of risky asset lying on the efficient frontier and  $\sigma_i^2$  is the variance if the  $i$ th asset. Therefore, the bid-ask spread is  $S_i = p_i^A - p_i^B = A\sigma_i^2|x_i|$ . Since inventory costs increase with the size of the transaction, the equilibrium spread is increasing in the trade size.

### Amihud & Mendelson (1986) Model

Amihud & Mendelson (1986) study the effect of having different types of investors with different expected holding periods. Suppose that an agent of type  $i$ ,  $i = 1, \dots, M$  receives a liquidity shock with probability  $\mu_i$  that forces him to sell and leave the market. Assume that type 1 agent has the highest risk of a liquidity shock, type 2 has the second highest, and so on,  $\mu_1 > \mu_2 > \dots > \mu_M$ , and there are  $j$  securities within which security 1 is most liquid, security 2 is second most liquid, and so on,  $\frac{c_1}{d_1} < \frac{c_2}{d_2} < \dots < \frac{c_N}{d_N}$ , where  $S_j$  is transaction cost per share and  $d_j$  is dividend.

They define the expected spread-adjusted return of asset  $j$  to investor of type  $i$  as the difference between the gross market return on asset  $j$  and its expected

liquidation cost per unit time:

$$r_{ij} = d_j/V_j - \mu_i S_j, \quad (2.96)$$

where  $V_j$  is the ask price, and thus the bid price is  $V_j(1 - S_j)$ .

With borrowing constraints and limited wealth, the optimal trading strategy of agent  $i$  is to invest all his wealth in securities with the highest spread-adjusted return,

$$r_i^* = \max_j r_{ij}, \quad (2.97)$$

with  $r_1^* \leq r_2^* \leq \dots \leq r_M^*$ .

They show that in equilibrium the agents with the shortest holding period hold the riskless security and the illiquid securities with the lowest trading cost. Agents with the next shortest holding period hold a portfolio of securities with the next lowest trading costs and so on.

The gross return required by investor  $i$  on asset  $j$  is given by  $r_i^* + \mu_i S_j$ , which is effectively the sum of the required spread-adjusted return  $r_i^*$  and the expected liquidation cost  $\mu_i S_j$ . The equilibrium market observed return on asset  $j$  is determined by

$$\max_i \{d_j/(r_i^* + \mu_i S_j)\}. \quad (2.98)$$

This implies that the equilibrium value of asset  $j$ , is equal to the present value of its perpetual cash flow, discounted at the market observed return  $r_i^* + \mu_i S_j$ . They show that in equilibrium the gross return is an increasing and concave function of the spread.

### Remarks

**Garman (1976)** explains the market makers' control problem as to avoid bankruptcy which is caused by the non-synchronous arrivals of buyers and sellers. However, he assumes that the market maker can only set the quotes at the beginning of the trading period. **Stoll (1978b)** generalizes Garman's model to allow the market maker to act as a liquidity provider who absorbs temporary order imbalances. The market maker often holds a suboptimal portfolio position, and thus

requires compensation in the form of a bid-ask spread. Using this idea, [Amihud & Mendelson \(1986\)](#) analyze a model in which investors with different expected holding periods trade assets with different relative spreads. The size of the spread is exogenously determined.

### 2.5.3 Information-based Models

Investors with private information choose to trade in order to maximize his profits, and market makers strategically protect themselves against informed traders. The information-based model studies the strategic interaction among risk-neutral market makers, informed traders and liquidity providers.

#### [Glosten & Milgrom \(1985\)](#) Model

[Glosten & Milgrom \(1985\)](#) assume an economy where market makers quote bid-ask prices and trade one single security with two types of agents, informed traders and uninformed traders. The informed traders receive a perfectly informative signal about the security's value prior to trading. All traders are risk neutral and arrive at the market sequentially and choose to buy or sell. The size of the order is equal to one unit, and agents may only trade once. Nature chooses the future value of the security,  $\tilde{V}$ , which can be high at  $\hat{V}$  with probability  $\theta$  and low at  $\underline{V}$  with probability  $1 - \theta$ . The market makers face an informed trader with probability  $\alpha$  and an uninformed trader with probability  $1 - \alpha$ . The informed trader is supposed to choose the trade that maximizes his profit, while uninformed traders will buy and sell with probability  $\frac{1}{2}$ . Trading occurs as a sequence of bilateral trading opportunities.

At time  $t$ , market makers compete on the price of the security, realizing an expected profit,  $\Pi_t$ , equal to zero. There is a competition among dealers, driving expected profits to zero. Market makers will end up setting a price that is equal to their conditional estimate of the future of the asset,  $E(\tilde{V}|\Phi_t)$ , where  $\Phi_t$  denotes the information on the direction of trade. If the incoming order is an buy order ( $B_t$ ), then dealers will offer an ask price,  $a_t$ ; conversely, if the incoming trade is a sell order ( $S_t$ ), then they will offer a bid price,  $b_t$ . This conditional expectation

given an incoming buy order can be interpreted as the opportunity costs that dealers face when selling the asset:

$$E(\Pi_t|B_t) = E[(a_t - \tilde{V})|B_t] = 0. \quad (2.99)$$

Thus,

$$a_t = E(\tilde{V}|B_t), \quad (2.100)$$

and

$$b_t = E(\tilde{V}|S_t). \quad (2.101)$$

The dealers will use the information to update their estimate of the future value of the asset, the process being completed when all the private information is incorporated into prices. This learning process affects only dealers' beliefs, whereas the uninformed traders do not learn anything from the market prices.

The price quoted equals the expected value of the asset conditional on the information on the possible direction of the next trade. Therefore, the bid and ask prices are:

$$a_t = E[\tilde{V}|B_t] = \underline{V}Pr\{\tilde{V} = \underline{V}|B_t\} + \overline{V}Pr\{\tilde{V} = \overline{V}|B_t\}, \quad (2.102)$$

and

$$b_t = E[\tilde{V}|S_t] = \underline{V}Pr\{\tilde{V} = \underline{V}|S_t\} + \overline{V}Pr\{\tilde{V} = \overline{V}|S_t\}. \quad (2.103)$$

By setting  $\theta = 1/2$ , we can get:

$$a_t = E[\tilde{V}|B_t] = \frac{1}{2}\underline{V}(1 - \alpha) + \frac{1}{2}\overline{V}(1 + \alpha), \quad (2.104)$$

and

$$b_t = E[\tilde{V}|S_t] = \frac{1}{2}\underline{V}(1 + \alpha) + \frac{1}{2}\overline{V}(1 - \alpha). \quad (2.105)$$

Therefore, the spread is equal to:

$$S_t = a_t - b_t = \alpha(\overline{V} - \underline{V}). \quad (2.106)$$

Notice the spread arises from the probability of informed trading,  $\alpha$ , and it is due to adverse selection costs. Furthermore, the variance of the risky asset is equal to:

$$Var\tilde{V} = \theta(\bar{V} - E(\tilde{V}))^2 + (1 - \theta)(\underline{V} - E(\tilde{V}))^2 = \theta(1 - \theta)(\bar{V} - \underline{V})^2. \quad (2.107)$$

It follows that the spread is an increasing function of the variance. This is because that the greater the variance is, the greater the informed traders' information advantage is, and thus the greater the market makers' potential losses will be.

### **Kyle (1985) Model**

In a one-period, one asset economy a battery of risk-neutral market-makers faces one monopolistic risk-neutral informed trader and a group of liquidity traders. At time  $t = 0$  the informed trader receives a perfect signal  $\delta$  on the future value of the risky asset which is equal to:

$$\tilde{V} = \hat{V} + \tilde{\delta}, \text{ with } \tilde{\delta} \sim N(0, \sigma_V^0), \quad (2.108)$$

and submits the net demand  $x(V)$ . Thus the informed trader knows the future value of the asset in advance. Noise traders are uninformed and submit a random net demand equal to  $\tilde{z} \sim N(0, \sigma_z^2)$  at time  $t = 1$ . Their reasons for trading may be a sudden need for consumption or idiosyncratic shocks to wealth, or needs related to the cycle. It follows that at time  $t = 1$ , market makers receive a net order flow equal to  $\tilde{w} = x(\tilde{V}) + \tilde{z}$ , which they can only observe in the aggregate. Holding this information, they compete on prices, offering a price function:

$$p(\tilde{w}) = E(\tilde{V}|\tilde{w}). \quad (2.109)$$

Competition drives market makers' expected profits to zero:

$$E(\pi_M|w) = E((p(\tilde{w}) - \tilde{V})\tilde{w}|\tilde{w}) = 0. \quad (2.110)$$

Notice that the market makers' pricing function depends on the informed agent's demand function,  $x(V)$ . If there were no informed traders, the aggregate order flow observed by market makers would be equal to the noise traders' net demand,  $\tilde{w} = \tilde{z}$ . The pricing function that results would reduce to  $p(\tilde{w}) = E(\tilde{V}|z) = \hat{V}$  and the market would be infinitely deep.

The informed traders' strategy depends on the market makers' pricing function. Actually, the informed trader's problem is choosing the demand function,  $x(V)$ , that maximizes his expected end-of-period profits:

$$\tilde{\pi} = E[x(\tilde{V} - p(\tilde{w}))|\tilde{V}]. \quad (2.111)$$

An equilibrium can be obtained by considering that market makers maximize their expected profits given their rational Bayesian interpretation of the information content of the aggregate order flow, and that the informed trader set his own demand function to maximize expected profits given his rational expectations of the impact of his order,  $x(V)$ , on the market price.

He proves that there is a linear equilibrium for this strategic game, such that the market makers' pricing rule is:

$$p^*\tilde{w} = E(\tilde{V}|w = x^*(V) + \tilde{z}) = \hat{V} + \lambda^*\tilde{w}, \quad (2.112)$$

and the informed trader's trading strategy is:

$$x^*(V) = \beta^*(\delta) = \beta^*(V - \hat{V}), \quad (2.113)$$

where

$$\lambda^* = \frac{1}{2} \frac{\sigma_V}{\sigma_z}, \quad (2.114)$$

and

$$\beta^* = \frac{\sigma_z}{\sigma_V}. \quad (2.115)$$

The market liquidity,  $1/\lambda$ , is equal to the inverse of the price impact of an uninformed order and is a proxy for market depth. In equilibrium, the illiquidity

parameter  $\lambda$  increases in  $\sigma_V$  and decreases in  $\sigma_z$ . Intuitively,  $\sigma_V$  measures the informed trader's information advantage, which is positively related to the adverse selection costs. Therefore, the greater  $\sigma_V$  is, the less deep the market becomes. An increase in the volume of uninformed trading,  $\sigma_z$ , increases liquidity. Thus, the greater  $\sigma_z$ , the more liquid the market becomes.

### Remarks

Kyle (1985) shows that all available information will be eventually incorporated in market prices. In his model the market maker does not act strategically, but merely processes market orders, and sets market clearing prices. Glosten & Milgrom (1985) also capture the costs of asymmetric information. In their model, the trade size is assumed to be for one round lot and only the direction of trades reveals information to the market makers. For our application, asymmetric information may not be very relevant as traders in bond markets could manage to evaluate the fair value of bonds based on other publicly traded assets, such as credit default swaps and government bonds. Thus, we decide to directly address illiquidity issues by only considering the inventory holding risk in our model.

In addition, some papers consider the strategic behaviors of traders in limit order book models which will be reviewed in the next section.

### 2.5.4 Models of the Limit Order Book

The limit order book models describe price formation in an order-driven market, where agents trade via a limit order book. In an order-driven market there are no designated market makers who provide liquidity by setting bid and ask prices as in a quote-driven market. Liquidity is supplied by anonymous traders who place orders in the limit order book, and wait until the orders get executed.

#### Foucault *et al.* (2005) Model

Foucault *et al.* (2005) develop a model for a limit order market where there are only strategic liquidity traders and the choice between limit and market orders

depends only on their degree of impatience. Traders value execution speed differently as some of them want to trade as soon as possible. In other words, traders who demand immediacy submit market orders, and traders who supply liquidity submit limit orders.

They assume that a risky security is traded in a continuous double auction market organized as a limit order book. The market participants are strategic and risk-neutral liquidity traders, arrive sequentially and differ in impatience defined as the cost of a delay in order execution. Each trader minimizes his total execution costs by choosing a market or a limit order, conditional on the state of the book. There are two types of traders, type  $P$  traders who are relatively patient and incur a waiting cost of  $\sigma_P$  per unit of time, and type  $I$  traders who are relatively impatient, as they incur a bigger waiting cost of  $\sigma_I$ . The proportion of patient traders is  $\theta_P$  ( $1 > \theta_P > 0$ ). The proportions remain constant over time, and the arrival rate is independent of the type distribution. Prices and spreads are expressed as multiples of the tick size  $\Delta$ . Traders arrive at the market according to a Poisson process with parameter  $\lambda > 0$ .

The optimal order placement strategy of trader  $i$  when the spread is  $s$  is to maximize the price improvement function:

$$\max_{j \in \{0, \dots, s-1\}} \pi_i(j) \equiv j\Delta - \sigma_i T(j), \quad (2.116)$$

where  $T(j)$  is the expected waiting time associated with each  $j$ -order's execution.

They show that in equilibrium patient traders tend to submit limit orders and impatient traders prefer market orders. Traders tend to submit more market orders when either the proportion of patient traders,  $\theta_P$ , or their waiting cost costs,  $\sigma_P$ , increases. They will submit less market orders if the order arrival rate,  $\lambda$ , decreases. They also define a measure of market resilience, which is equal to  $\theta_P^{q-1}$ . This can be interpreted as the probability that the spread reverts back to the competitive level from the current spread after  $q - 1$  consecutive patient traders submitting their orders.



### Rosu (2009) Model

Rosu (2009) develops a dynamic limit order book model where traders are allowed to modify and cancel their limit orders. He considers a market for an asset which pays no dividends. Prices can take any value in a constant range between  $A$  and  $B$  ( $A > B$ ). There are only two types of trades, namely market orders and limit orders. Limit orders can be canceled for no cost at any time.

Traders arrive randomly in the market, following a exogenously determined arrival process. Similar to Foucault *et al.* (2005), once traders arrive, they will choose strategically between market and limit orders. The risk neutral traders are either buyers and sellers. Given the random execution time,  $\tau$ , and the price obtained at  $\tau$ ,  $P_\tau$ , the expected utility of a seller (buyer) with patience coefficient  $\tilde{r}$  is  $f_t = E_t\{P_\tau - \tilde{r}(\tau - t)\}$  ( $g_t = E_t\{P_\tau + \tilde{r}(\tau - t)\}$ ). The patience coefficient takes two values  $r$  for patient trader, and  $r'$  for impatient trader ( $r < r'$ ). The types of equilibrium used in his model are subgame perfect equilibrium and Markov perfect equilibrium. The model implies that higher trading activity and higher competition among liquidity providers lead to smaller spreads and lower price impact.

### 2.5.5 Transaction Costs and Liquidity Differentials

#### Vayanos & Vila (1999) Model

Vayanos & Vila (1999) study an over-lapping-generation model with a risky asset with proportional trading costs and a liquid riskless asset in fixed supply, thus endogenizing the riskless interest rate. They assume agents can invest in two financial assets. Both assets pay dividends at a constant rate  $D$ . The first asset is liquid and does not carry transaction costs, while the second asset is illiquid. The supply of the liquid asset is  $1 - k$  ( $0 < k < 1$ ), its price is  $p$ , and its rate of return is  $r = D/p$ . Similarly, the price of the illiquid asset is  $P$  and its rate of return is  $R = D/P$ . The illiquid asset carries transaction costs that are proportional to the value of traded, i.e. the costs of buying or selling  $x$  shares of the illiquid asset are  $\epsilon x P$ , with  $\epsilon \geq 0$ . An agent of age  $t$  holds  $x_t$  and  $X_t$  shares of the liquid and illiquid assets, respectively. His “liquid wealth” is  $a_t = px_t$ , his

“illiquid wealth” is  $A_t = PX_t$ , and his total wealth is  $w_t = a_t + A_t$ . His dollar investment in the liquid asset is  $i_t = p(dx_t/dt)$ , and his dollar investment in the illiquid asset is  $I_t = P(dX_t/dt)$ .

In equilibrium, they show that the prices  $p$  and  $P$  can be written as

$$p - P = \sum_{l=0}^{\infty} P\epsilon(1 + e^{-r\Delta})e^{-rl\Delta}, \quad (2.117)$$

where  $r$  is the risk-free rate and  $\Delta$  is the minimum holding period of the illiquid asset. This implies that the price has to fall by the present value of the transaction costs that a sequence of marginal investors incur.

The liquidity premium can be expressed as follow:

$$\mu = r \frac{1 + E^{-R\Delta}}{1 - E^{-R\Delta}} \epsilon. \quad (2.118)$$

Thus, the equilibrium liquidity premium which is the difference between the rates of return on the two assets is a function of  $\epsilon$  and the equilibrium  $\Delta$ . For the liquid asset, they show that the rate of return on the liquid asset  $r$  decreases in  $\epsilon$ . For the illiquid asset, they also show that transaction costs decrease the price of the illiquid asset relative to the price of the liquid asset. but increase the price of the liquid asset. If the risky asset has a higher trading cost then the risk-free asset becomes a more attractive alternative. Therefore, the equilibrium price of a risk-free asset is increasing in the trading cost of the risky asset.

### **Huang (2003) Model**

**Huang (2003)** assumes that two consol bonds that are identical except that one is liquid and the other is illiquid, i.e., it incurs proportional transaction costs. Investors are risk averse and have a constant income stream. Each investor is hit by a negative liquidity shock with a Poisson arrival rate and when this happens, the investor must liquidate his securities and exit. He shows that in equilibrium the illiquid asset security whose net return becomes stochastic will have a premium

over the liquid security which exceeds the magnitude of expected transaction costs, reflecting the liquidity-induced risk premium.

At any time, each agent faces a constant probability per unit of time,  $\lambda$ , of experiencing a liquidity shock which, upon arrival, will force him to liquidate his securities and exit the economy. The arrivals of the liquidity shocks for all existing agents are independent Poisson times. There are two financial securities. Both are riskless consol bonds, that are perpetuities that pay a constant flow of consumption dividends at a rate of  $d$  per unit of time. One consol is liquid, with a price of  $p$  per share, and no transactions costs are required for trading it. The other is illiquid, with a price of  $P$  per share, and agents who buy or sell  $x$  shares of it pay a proportional transactions cost totalling  $\varepsilon x P$ . The fractional supply of the liquid consol is  $k \in (0, 1)$ . The dividend flow rate for the liquid and illiquid consols are  $r = d/p$  and  $R = d/P$ .

$\beta$  is the constant time preference rate and  $\gamma$  is the relative risk-aversion of CRRA utility.

They show that in equilibrium the agents holds the liquid consol exclusively if the liquidity premium is lower than  $R^* - r$ , where

$$R^* = r + 2\varepsilon(\beta + \lambda + \gamma r - r/2), \quad (2.119)$$

and holds the illiquid consol exclusively if the liquidity premium is higher than  $R^{**} - r$ , where

$$R^{**} = r + \frac{2\varepsilon(\beta + \lambda + \frac{\gamma r}{1+\varepsilon} - r/2)}{1 - \frac{2\varepsilon}{1+\varepsilon}\gamma}. \quad (2.120)$$

If the liquidity premium falls between  $(R^* - r, R^{**} - r)$ , the agent holds constant non-zero proportions of each consols. The illiquid consol proportion,  $1 - \theta(R)$ , increases with the return of the illiquid consol.

### Remarks

For investors with relatively short holding period, transaction costs are important, and a liquid asset is preferred regardless of its lower yield. For investors making long-term investment, illiquidity is not a concern, and the cheap and illiquid

asset is the better investment. When buying assets, short-term traders will take into account future transaction costs and therefore, require big price discount. Investors do care about the illiquidity of the assets and indeed there is a well-defined clientele for each asset.

However, we believe the liquidity differentials among assets, the heterogeneity of transaction costs of liquidity premia, and ultimately the transaction prices should be a result of an equilibrium where market participants optimally choose their trading strategies. Therefore, assuming exogenous transaction costs or a priori the liquidity condition of assets as in previous literature (e.g. [Vayanos & Vila \(1999\)](#) and [Huang \(2003\)](#)) will not be appropriate.

Therefore, it is necessary to develop an equilibrium model where the heterogeneous levels of liquidity in bonds are generated endogenously. This will be the main research question in Chapter 5.

### 2.5.6 General Equilibrium Model

Consider an economy with  $I$  agents  $i \in \mathcal{I} = \{1, \dots, I\}$  and  $L$  commodities  $l \in \mathcal{L} = \{1, \dots, L\}$ . A bundle of commodities is a vector  $x \in \mathbf{R}_+^L$ . Each agent  $i$  has an endowment  $e^i \in \mathbf{R}_+^L$  and a utility function  $u^i : \mathbf{R}_+^L \rightarrow \mathbf{R}$ . These endowments and utilities are the primitives of the exchange economy, so let  $\varepsilon = ((u^i, e^i)_{i \in \mathcal{I}})$ .

Agents are assumed to take as given the market prices for the goods. The vector of market prices is  $p \in \mathbf{R}_+^L$ ; all prices are nonnegative. Each agent chooses consumption to maximize her utility given her budget constraint. Therefore, agent  $i$  solves:

$$\max_{x \in \mathbf{R}_+^L} u^i(x) \quad s.t. \quad p \cdot x \leq p \cdot e^i. \quad (2.121)$$

The budget set is written as

$$\mathcal{B}^i(p) = \{x : p \cdot x \leq p \cdot e^i\}. \quad (2.122)$$

A Walrasian equilibrium is a vector of prices, and a consumption bundle of each agent, such that

1. every agent's consumption maximizes her utility given prices, and

2. markets clear: the total demand for each commodity just equals the aggregate endowment.

**Definition 1.** A Walrasian equilibrium for the economy  $\varepsilon$  is a vector  $(p, (x^i)_{i \in \mathcal{I}})$  such that:

1. Agents are maximizing their utilities: for all  $i \in \mathcal{I}$ ,

$$x^i \in \arg \max_{x \in \mathcal{B}^i(p)} u^i(x)$$

2. Markets clear: for all  $l \in \mathcal{L}$ ,

$$\sum_{i \in \mathcal{I}} x_l^i = \sum_{i \in \mathcal{I}} e_l^i.$$

Pareto optimality considers the set of feasible allocations and identifies those allocations at which no consumer could be made better off without another being worse off.

**Definition 2.** An allocation  $(x^i)_{i \in \mathcal{I}} \in \mathbf{R}_+^{I \cdot L}$  is feasible if for all  $l \in \mathcal{L} : \sum_{i \in \mathcal{I}} x_l^i \leq \sum_{i \in \mathcal{I}} e_l^i$ .

**Definition 3.** Given an economy  $\varepsilon$ , a feasible allocation  $x$  is Pareto optimal (or Pareto efficient) if there is no other feasible allocation  $\hat{x}$  such that  $u^i(\hat{x}^i) \geq u^i(x^i)$  for all  $i \in \mathcal{I}$  with strict inequality for some  $i \in \mathcal{I}$ .

We need the following assumptions about consumers' preferences and endowments:

1. For all agents  $i \in \mathcal{I}$ ,  $u^i$  is continuous.
2. For all agents  $i \in \mathcal{I}$ ,  $u^i$  is increasing.
3. For all agents  $i \in \mathcal{I}$ ,  $u^i$  is concave.

4. For all agents  $i \in \mathcal{I}$ ,  $e^i \gg 0$ .

Given the prices and endowments, consider the optimization problem facing consumer  $i$ :

$$\max_{\tilde{x}^i} u^i(\tilde{x}^i) \quad s.t. \quad p \cdot \tilde{x}^i \leq p \cdot e^i. \quad (2.123)$$

One can use the Kuhn-Tucker conditions to characterize the optimum. Letting  $\nu^1, \dots, \nu^I$  denote the Lagrange multipliers on the budget constraints of agents  $1, \dots, I$ , the Kuhn-Tucker conditions state that a necessary and sufficient condition for  $(x^1, \dots, x^I, \nu^1, \dots, \nu^I)$  to solve the  $I$  utility maximization problems given prices  $p$  is that for all  $i, l$ :

$$\begin{aligned} \frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l &\leq 0, \\ x_l^i &\geq 0, \\ \left( \frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l \right) \cdot x_l^i &= 0, \end{aligned}$$

and each of the resource constraints bind.

### Remarks

There are three papers which are closely related to our third study. [Economides & Siow \(1988\)](#) using search-based model show that agents prefer to trade in a spot market with high rather than low liquidity. Liquidity in a market can only be increased by increasing the number of traders at that market. [Vayanos & Wang \(2007\)](#) develop a search-based model in which investors of different horizons can invest in two assets with identical payoffs. They show that there exists such an equilibrium where all short-term investors search for the same asset. This equilibrium dominates the one where all investor types split equally across assets. [Weill \(2008\)](#) shows that although the search technology is the same for all assets, heterogeneous trading costs arise endogenously. In equilibrium, an asset return is negatively related to its number of tradable shares, its turnover, its trading volume, and it is positively related to its bid-ask spread. [Vayanos & Weill \(2008\)](#) use search-based model to explain the on-the-run phenomenon. They show that short-sellers can endogenously concentrate in one asset because

## 2.5 Microstructure Models

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of search externalities and the constraint that they must deliver the borrowed asset. These models are all based on the notion that asset trading involves search. However, bonds are normally traded at the over-the-counter markets where trades are executed through the market makers. Moreover, in their models the investors are assumed to be risk averse.

# Chapter 3

## An Extended Model of Estimating Effective Bid-ask Spread

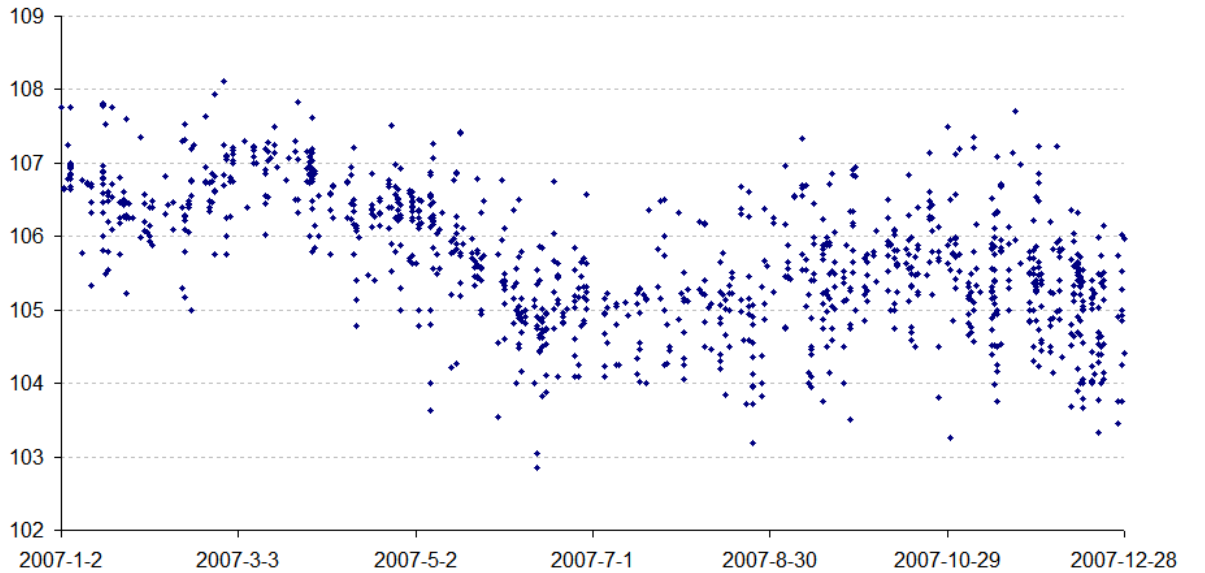
### 3.1 Introduction

In a simple security market, potential traders purchase or sell securities at the ask or the bid prices posted by the specialists. The difference between the bid and the ask is often referred to as the bid-ask spread, which is one of the most widely used measures of illiquidity. The quoted bid-ask spread represents the prices available at a given time for transactions only up to some relatively small amount.

Trades sometimes may execute either inside or outside the quoted spread. A block trade negotiated (typically by institutional investors) with a credible guarantee that it is not information-motivated could receive a better price. [Schultz \(2001\)](#) finds that trading costs are lower for larger trades and small institutions pay more to trade than large institutions (all else being equal) in the over-the-counter corporate bond market. [Feldhutter \(2012\)](#) develops a search-based model



where an asset can simultaneously trade at different prices as large traders negotiate tighter bid-ask spreads due to their stronger outside options. In contrast, given that they wish to trade, informed traders prefer to trade larger amounts at any given price. In information-based models, large traders buy or sell at worse prices than small traders, because large traders are likely to have private information. [Easley & O'Hara \(1987\)](#) argue that uncertainty about whether any individual trader is informed causes prices to worsen for a block trade. Consequently, statistical spread estimation models which do not take multiple spreads into account will produce biased estimates of not only the spreads but also the underlying variance. Figure 3.1 seems to support the assertion of multiple spreads (i.e. most of trades executed at a smaller spread whereas a few trades were being made at a larger spread.). Therefore, in this paper we extend an existing model ([Roll \(1984\)](#)) to incorporate multiple spreads as well as their associated probabilities, and propose a Bayesian approach, which requires no data other than transaction prices, to estimate the extended model.



**Figure 3.1:** Time series plot of the intra-day transaction prices of a corporate bond (CUSIP:172967AZ4) during 02/01/2007 - 28/12/2007

In the market microstructure literature, there are two classes of statistical

models of estimating the effective bid-ask spread: the covariance model, and the trade indicator model.

In the covariance models the spread measures are derived from the serial covariance properties of transaction price changes. The first such model was developed by Roll (1984). By assuming an informationally efficient market and a stationary probability distribution of observed price changes, the effective bid-ask spread is inferred from the serial covariance of daily and weekly equity returns. Choi *et al.* (1988) modify Roll's estimator and introduce a serial correlation assumption regarding transaction type. Stoll (1989) models the relation between the quoted spread and the covariance estimate of the spread as a function of the probability of a price reversal and the magnitude of a price reversal. The empirical results imply that the average realized spread is less than the quoted spread. George *et al.* (1991) show that positively autocorrelated time-varying expected returns lead to downward biases in estimators of both the spread and its components. They introduce an alternative approach based on the serial covariance of the difference between transaction returns and returns calculated using bid prices to adjust transaction returns for time-varying expected returns.

In the trade indicator models, the bid-ask spread is estimated as the coefficient of signed order flow in a price change regression. Glosten & Harris (1988) applied this idea to separate the adverse selection spread component from the one due to inventory costs, clearing fees, and/or monopoly profits. Their model is estimated by integrating unobservable variables (underlying values and trade directions) out of the conditional likelihood function and maximizing the unconditional likelihood function. Madhavan & Smidt (1991) develop a similar model by explicitly modelling the learning process of the market maker. A reduced form model and a measure of information asymmetry are estimated using Box-Jenkins methods. Madhavan *et al.* (1997) develop and estimate a structural model of price formation that decomposes intraday volatility into components attributable to public information shocks and trading frictions. Both Madhavan & Smidt (1991) and Madhavan *et al.* (1997) adopt trade classification procedure to determine the trade directions. Huang & Stoll (1997) also construct a basic trade indicator model which is estimated by GMM. They are able to observe indicator variables and classify them for each size category. Ball & Chordia (2001)

model the observed price as a discrete version of the sum of a permanent informational component and the transient components arising out of the trading mechanism. Since explicitly allowing for rounding onto the tick grid destroys the Gaussian structure, they adopt Monte Carlo Markov Chain (MCMC) method to estimate the models and to decide the best model specification. Recently, [Hasbrouck \(2004\)](#) suggested a Bayesian Gibbs approach by treating trade direction as a latent variable and deriving a probability density for the sign of the trade, and applied the approach to commodity futures transaction data.<sup>1</sup>

Our model belongs to the second category. Our purpose is to develop a parsimonious but realistic model which does not have too many assumptions and requires as few data to estimate as possible. Compared to [Choi \*et al.\* \(1988\)](#), [Stoll \(1989\)](#), and [George \*et al.\* \(1991\)](#), our reduced-form model does not associate price changes to transaction types, and thereby does not require data about quotes and trade directions. Unlike [Glosten & Harris \(1988\)](#), [Madhavan & Smidt \(1991\)](#), [Madhavan \*et al.\* \(1997\)](#), [Huang & Stoll \(1997\)](#), and [Ball & Chordia \(2001\)](#), we do not intend to decompose the bid-ask spread into inventory cost and adverse selection components. Our model focuses on estimating the effective bid-ask spreads as well as the underlying variance. In the original Roll model, it is not possible to distinguish spreads of different magnitudes. The Roll measure is essentially a weighted average of those spreads. We extend Roll’s model by adding an extra parameter  $\lambda$ , the so-called ‘spread multiplier’, to separate different spreads. In other words, we generalize Roll’s spread estimator (a scalar) to a vector of spreads with associated probabilities. Different from [Huang & Stoll \(1997\)](#) where they allow the components of the spread to explicitly depend on the trade size, our extended model does not emphasize this link between the spread and the trade size.

Our estimation procedure is based on a Bayesian Gibbs estimation method proposed by [Hasbrouck \(2004\)](#). The trader direction is treated as a latent variable which can be inferred from the transaction prices using a Bayesian trade

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<sup>1</sup>Both [Glosten & Harris \(1988\)](#) and [Madhavan & Smidt \(1991\)](#) find strong evidence of information asymmetry, whereas [Huang & Stoll \(1997\)](#) find a large order processing component and smaller adverse selection and inventory components. [Ball & Chordia \(2001\)](#) focus on discreteness and find that for large stocks, most of the quoted spread is attributable to the rounding of prices and the adverse selection component is small.

classification method. Therefore, different from [Madhavan & Smidt \(1991\)](#) and [Madhavan \*et al.\* \(1997\)](#), our model does not depend on a tick test to identify trade directions. This estimation approach is very intuitive, and to some extent similar to [Huang & Stoll \(1997\)](#) as we essentially integrate out the latent variables numerically. We treat models with different values of  $\lambda$  as competitive models. The value of  $\lambda$  is determined via a Bayesian model selection method, proposed by [Chib \(1995\)](#).<sup>1</sup>

The simulation analysis shows that the model performs well when the underlying volatility is low, and the difference between the inner and outer (half) spreads is big compared to the underlying volatility. The empirical estimation shows that our extended model fits the transaction data better than Roll's model (estimated using Method of Moments and Bayesian approaches).

We organize this chapter as follows. Section 2 presents the model specification of the original Roll model, followed by the extended model. In Section 3, we introduce the Bayesian model estimation and selection methodology, and the details of the computational procedures. Simulation analysis is carried out in Section 4 to assess the performance of the Bayesian estimators. An example of an application to the actual transaction data is given in Section 5. Section 6 contains our conclusions.

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<sup>1</sup>Bayes factors and various model selection criteria have been applied to compare financial models e.g. [Osiewalski & Pipien \(2004\)](#) and [Deschamps \(2011\)](#). In [Ball & Chordia \(2001\)](#), the chosen model is also determined by the Bayesian model selection method proposed by [Chib \(1995\)](#).

## 3.2 The Model

This section first gives a brief introduction to the basic Roll model. Based on which, we present our extended model and discuss the key parameters.

### 3.2.1 The Roll Model

It has long been recognized that if trades fluctuate between bid and ask prices, then observed price changes will be negatively autocorrelated. Roll (1984) uses this property of transaction prices to derive an estimator of the bid-ask spread. One advantage of his model is that it is based only on published transaction prices.

In an efficient market the price dynamics may be stated as

$$m_t = m_{t-1} + \epsilon_t, \quad (3.1)$$

$$p_t = m_t + sq_t, \quad (3.2)$$

where  $m_t$  is the unobservable efficient price,  $p_t$  is the transaction price observed at time  $t$ ,  $s$  is one-half the bid-ask spread, and the  $q_t$  are trade direction indicators, which take the values  $+1$  for buy orders or  $-1$  for sell orders with equal probability.

The changes in transaction prices between two successive trades is

$$p_t - p_{t-1} = m_t + sq_t - (m_{t-1} + sq_{t-1}), \quad (3.3)$$

$$\Delta p_t = s\Delta q_t + \epsilon_t. \quad (3.4)$$

Thus,

$$\begin{aligned} Cov(\Delta p_t, \Delta p_{t-1}) &= E[\Delta q_t \Delta q_{t-1} s^2] + E[\Delta q_t \epsilon_{t-1} s] \\ &\quad + E[\Delta q_{t-1} \epsilon_t s] + E[\epsilon_t \epsilon_{t-1}], \end{aligned} \quad (3.5)$$

where  $Cov(\Delta p_t, \Delta p_{t-1})$  is the first-order autocovariance of the price changes.

In deriving the Method of Moments estimator Roll (1984) makes several assumptions:

1. Successive transaction types are independent. Thus,

$$E[q_t q_{t-1}] = 0.$$

2. The half-spread  $s$  is constant.

3. Order flows do not contain information about future fundamental price changes. Thus,

$$E[\Delta q_{t-1} \epsilon_t] = 0.$$

4. Changes in fundamental value cannot predict order flows:

$$E[\Delta q_t \epsilon_{t-1}] = 0.$$

5. The innovations in the fundamental price process reflect public information and are assumed to be independent. Therefore,

$$E[\epsilon_t \epsilon_{t-1}] = 0.$$

**Table 3.1:** Joint probabilities of consecutive price changes

		$\Delta q_{t-1}$		
		-2	0	+2
$\Delta q_t$	-2	0	$\frac{1}{8}$	$\frac{1}{8}$
	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
	+2	$\frac{1}{8}$	$\frac{1}{8}$	0

Then, from the joint probabilities in Table 3.1, Equation 3.5, and the assumptions, we obtain

$$Cov(\Delta p_t, \Delta p_{t-1}) = -s^2. \quad (3.6)$$

This gives Roll's Method of Moments spread estimator

$$\hat{s} = \sqrt{-Cov(\Delta p_t, \Delta p_{t-1})}. \quad (3.7)$$

If the innovations in the efficient price process are assumed to follow a normal distribution with mean 0 and constant variance  $\sigma_\epsilon^2$ , the Method of Moments variance estimator is

$$\hat{\sigma}_\epsilon^2 = Var(\Delta p_t) - s^2 Var(\Delta q_t) = Var(\Delta p_t) - 2s^2. \quad (3.8)$$

However, Roll's method of moments spread estimator has performed poorly, generating undefined spread estimates almost half of the time when applied to daily transaction data on equities [Harris (1990)]. In particular, Roll's model is not able to identify the multiple spreads with different magnitudes. The estimator is only a scalar which is, essentially, a weighted average of those spreads. However, several refinements of the Roll's model are possible. Our extended model, which will be introduced in the next section, provides a richer structure, and does not produce undefined spread estimates.

### 3.2.2 The Extended Model

The fact that trades are sometimes executed either inside or outside the posted bid and ask indicates that sometimes the posted spread may not represent the transaction cost that investors can expect. This motivates our extended Roll model aiming at separating spreads with two different magnitudes.

In our model the trade direction indicators  $q_t$  are generalised as follow:

$$q_t \in \{-\lambda, -1, +1, +\lambda\}, \quad (3.9)$$

where  $\lambda$  is the so-called 'spread multiplier' which is used to distinguish different spreads. Thus, for expositional ease, we define  $\lambda \in [1, +\infty)$ <sup>1</sup>.  $\lambda \neq 1$  implies

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<sup>1</sup>Equivalently, we can define  $\lambda \in (0, 1]$ . For computational ease, we choose to use  $\lambda \in (0, 1]$  later in the estimation.

that there exist two different sizes of spreads.  $-\lambda$  and  $+\lambda$  are indicators of the ‘abnormal’ spreads that are outside ‘normal’ bid-ask spreads. Roll’s model is a special case of our extended model corresponding to  $\lambda = 1$ . For estimation purposes we also assume here that  $\epsilon \sim i.i.d.N(0, \sigma_\epsilon^2)$ . We shall not hesitate to drop the subscript of  $\sigma_\epsilon^2$ , writing  $\sigma^2$  wherever convenient.

In summary, the half-spread,  $s$ , and the variance of the efficient price changes,  $\sigma_\epsilon^2$ , are unknown parameters from the regression specification

$$\Delta p_t = s\Delta q_t + \epsilon_t \quad (3.10)$$

$$q_t \in [-\lambda, -1, +1, +\lambda] \quad (3.11)$$

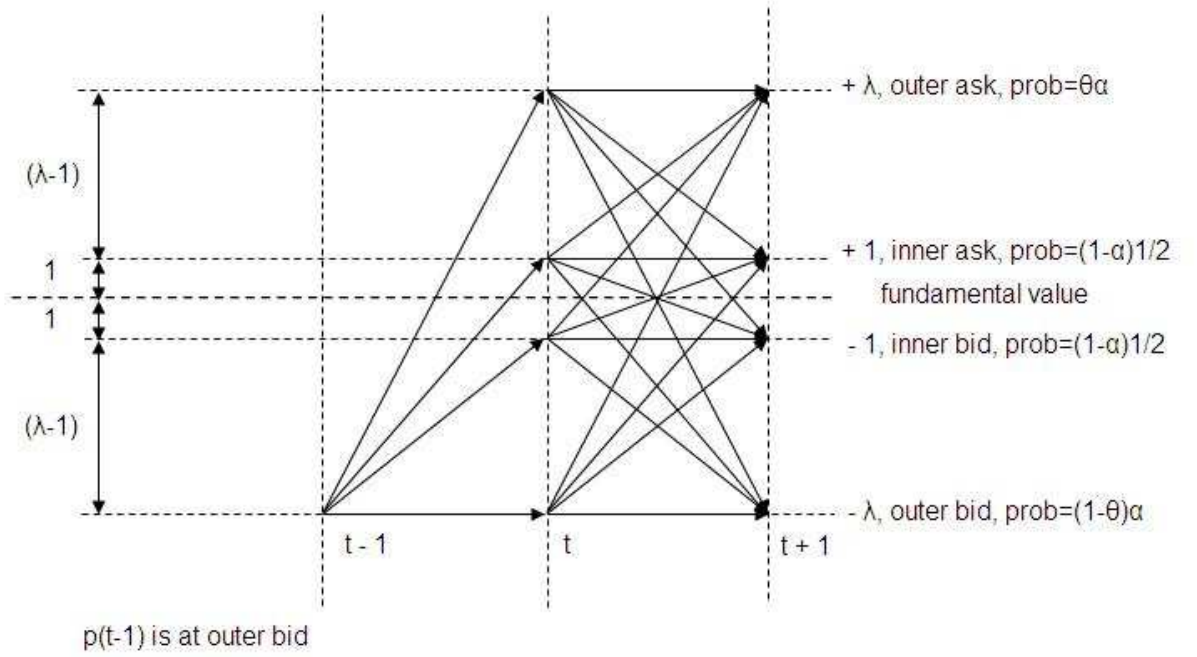
$$\epsilon \sim i.i.d.N(0, \sigma_\epsilon^2), \quad (3.12)$$

where  $q_t$  and  $\lambda$  are latent variables.

A slightly simplified graphical illustration of the extended model is shown in Figure 3.2. At any point in time, there exist four possible values for the security price. They are: inner bid, inner ask, outer bid and outer ask, to which the trade direction indicator  $q_t$  assign corresponding values, e.g.  $-1, +1, -\lambda$  and  $+\lambda$ , respectively. For instance, in Figure 3.2, the security price at time  $t - 1$  is at the quote setter’s outer bid. Assume that no new information arrives in the market about the security, there are 16 possible paths of observed price between successive time periods. For convenient demonstration and comparison, let  $\alpha$  control the probability of a trade occurring at the outer bid or ask, and  $\theta$  determine whether the model has a symmetric structure. Then, the probabilities of  $q_{t+1}$  being  $-1, +1, -\lambda$  and  $+\lambda$  are  $(1 - \alpha)/2$ ,  $(1 - \alpha)/2$ ,  $(1 - \theta)\alpha$ , and  $\theta\alpha$ , respectively. When  $\theta = 0.5$ , the model has a symmetric structure (i.e. the outer bid and the outer ask are equally likely to happen.). When  $\theta > 0.5$ , that implies that trades are more likely to execute at the outer ask than at the outer bid.

It is interesting to look at the only two non-zero second moments, namely autocovariance and variance, under our extended model specification. For computational ease, in the rest of this paper, we let  $\lambda \in (0, 1]$ .





**Figure 3.2:** Price paths.

This figure shows the possible paths of observable market price between successive time periods, given that the price at time  $t - 1$  is at outer bid, and given that no new information arrives in the market.  $\lambda$  is the ‘spread multiplier’ which is used to distinguish different spreads.  $\lambda \neq 1$  implies that there exist two different sizes of spreads. At any point in time, there exist four positions for the security price to be located. They are: inner bid, inner ask, outer bid and outer ask, for which the trade direction indicator  $q_t$  assign corresponding values, e.g.  $-1, +1, -\lambda$  and  $+\lambda$ , respectively.  $\alpha$  controls the possibility of the occurrence of outer bid and ask.  $\theta$  determines whether the model has a symmetric structure for outer bid and ask. Given the position of the price at time  $t - 1$ , there are sixteen possible price paths in total between  $t - 1$  and  $t + 1$ .

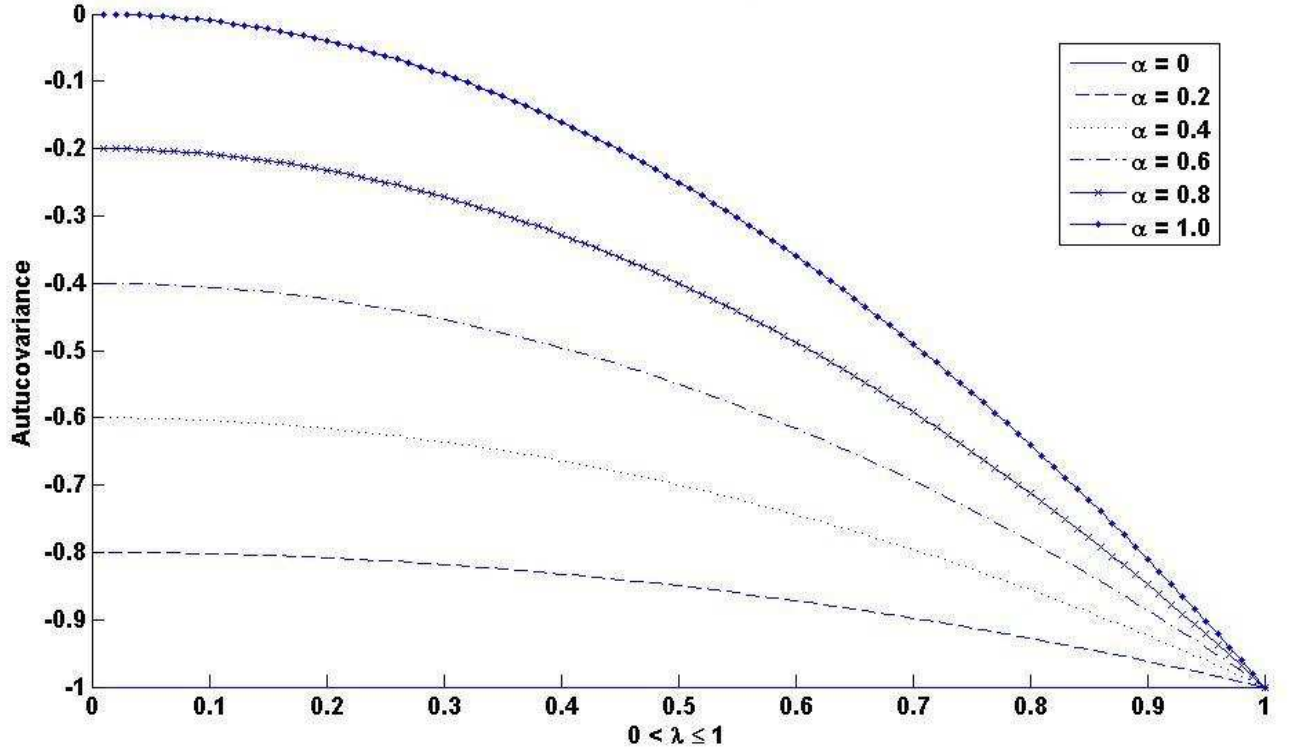
Using the probabilities of different prices as just described, we have,

$$\begin{aligned} Cov(\Delta p_t, \Delta p_{t-1}) = & -s^2(1-\alpha)^3 - s^2\alpha^2(1-\alpha)[\lambda^2(1+4\theta-4\theta^2)+1] \\ & -s^2(1-\alpha)^2\alpha(\lambda^2+2) - 4s^2\alpha^3\lambda^2\theta(1-\theta), \end{aligned} \quad (3.13)$$

and

$$Var(\Delta p_t) = \sigma_\epsilon^2 + s^2[8\lambda^2(\theta-\theta^2)\alpha^2 + 2(\lambda^2+1)\alpha^2(1-\alpha) + 2(\lambda^2+2)(1-\alpha)^2\alpha + 2(1-\alpha)^3]. \quad (3.14)$$

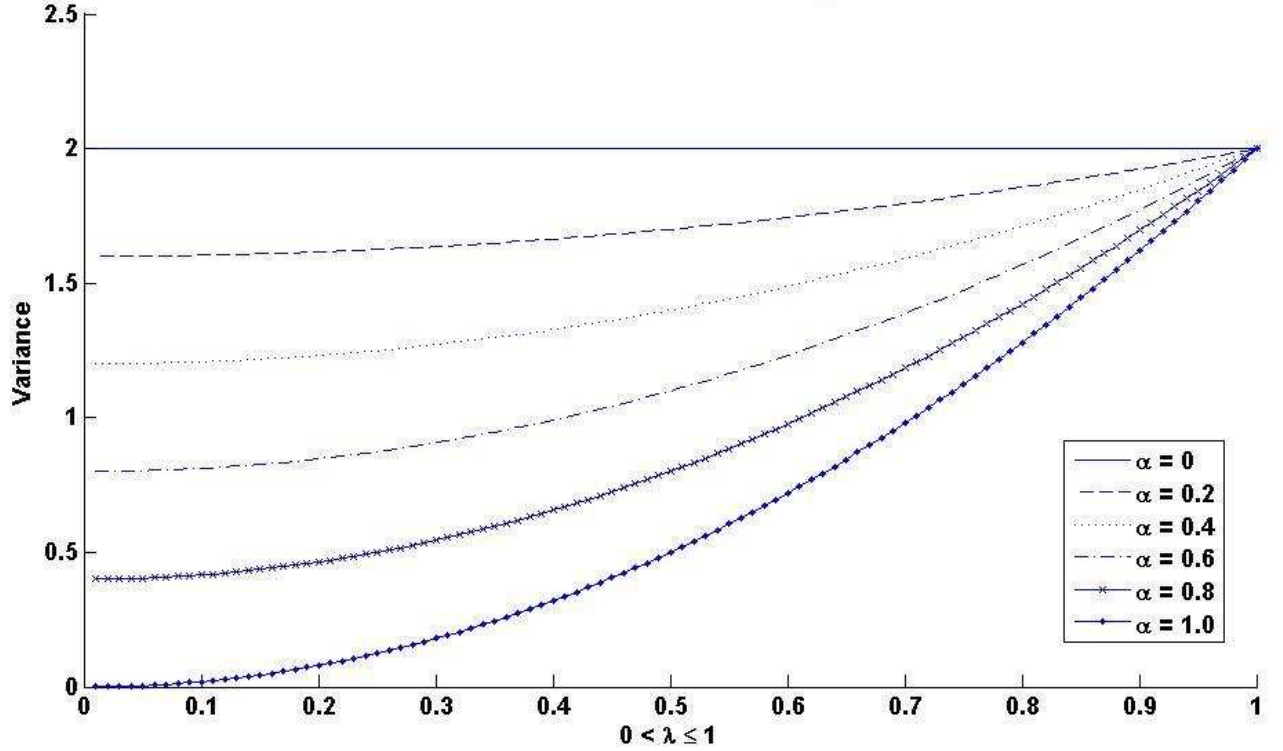
Assuming that  $s = 1$ ,  $\theta = 0.5$ , and  $\lambda \neq 1$ , Roll's spread estimator will be biased as shown in Figure 3.3. The bias increases while  $\alpha$  increases. That is, the bias increases as the probability of existence of the outer spread increases. The bias also increases while  $\lambda$  decreases, as Roll's spread estimator is a weighted average of the inner and outer spreads. Roll's variance estimator is also biased as shown in Figure 3.4. The bias increases as  $\alpha$  increases. This is because that the estimator mistakenly treats some of the transitory effect due to big spreads as the permanent effect. The bias again increases while  $\lambda$  decreases. This is due to the fact that the outer spread increases while  $\lambda$  decreases. That is, if our extended model is the true data generating process, Roll's measure will be biased.



**Figure 3.3:** Plot of autocovariance I

This figure provides the autocovariances calculated under the extended model.  $s$  is the half-spread.  $\lambda$  is the ‘spread multiplier’ which is used to distinguish different spreads.  $\lambda \neq 1$  means that there exist two different sizes of spreads.  $\alpha$  controls the possibility of the occurrence of outer bid and ask.  $\theta$  determines whether the model has a symmetric structure for outer bid and ask. Assume  $s = 1$  and  $\theta = 0.5$ . Different lines in this figure represent autocovariances as a function of  $\lambda$  for a few values of  $\alpha$ . Autocovariance under the extended model is computed as

$$Cov(\Delta p_t, \Delta p_{t-1}) = -s^2(1 - \alpha)^3 - s^2\alpha^2(1 - \alpha)[\lambda^2(1 + 4\theta - 4\theta^2) + 1] - s^2(1 - \alpha)^2\alpha(\lambda^2 + 2) - 4s^2\alpha^3\lambda^2\theta(1 - \theta).$$



**Figure 3.4:** Plot of variance I

This figure provides the variances calculated under the extended model.  $s$  is the half-spread.  $\sigma_\epsilon$  is the variance of the efficient price changes.  $\lambda$  is the ‘spread multiplier’ which is used to distinguish different spreads.  $\lambda \neq 1$  means that there exist two different sizes of spreads.  $\alpha$  controls the possibility of the occurrence of outer bid and ask.  $\theta$  determines whether the model has a symmetric structure for outer bid and ask. Assume  $s = 1$ ,  $\sigma_\epsilon = 0$  and  $\theta = 0.5$ . Different lines in this figure represent variances as a function of  $\lambda$  for a few values of  $\alpha$ . Variance under the extended model is computed as

$$Var(\Delta p_t) = \sigma_\epsilon^2 + s^2[8\lambda^2(\theta - \theta^2)\alpha^2 + 2(\lambda^2 + 1)\alpha^2(1 - \alpha) + 2(\lambda^2 + 2)(1 - \alpha)^2\alpha + 2(1 - \alpha)^3].$$

### 3.3 The Model Estimation and Selection

The latent variables in the extended model are the trade direction indicators  $q_t$  and the spread multiplier  $\lambda$ . Since integrating out the latent variables is cumbersome, there is no attractive closed-form solution for the likelihood function. Moreover, it is difficult to compute the higher moments, using GMM estimation, which are needed in order to estimate the extended model. Therefore, our extended model is estimated via a Bayesian approach proposed by Hasbrouck (2004). More precisely, the model parameters, the half-spread  $s$  and the underlying variance  $\sigma_\epsilon^2$ , are estimated using a Markov chain Monte Carlo (MCMC) method, the Gibbs sampler. Then, we treat models with different specifications of  $\lambda$  as competitive models. The best model is determined via a Bayesian model selection method which is introduced by Chib (1995). The advantage of this method is that it is also based on the Gibbs outputs.<sup>1</sup> The most appealing motivation for the use of the Bayesian approach lies in the fact that the role of the latent variables in the estimation can be intuitively understood. In the following, we describe the parameter estimation procedure in detail, followed by the model selection method.

#### 3.3.1 The Bayesian Estimation Procedure

The extended model has two parameters  $s$ ,  $\sigma_\epsilon$ , and  $T$  latent variables  $q = (q_1, q_2, \dots, q_T)$  given the value of  $\lambda$ . We estimate the model parameters via a Bayesian approach, using the Gibbs sampler. That is, given a known sequence of observations  $p = (p_1, p_2, \dots, p_T)$ , we estimate  $s$  and  $\sigma_\epsilon$  under mean squared error risk rule by taking the mean of the posterior distributions of  $s$  and  $\sigma_\epsilon$  for fixed  $\lambda$ .<sup>2</sup>

Gibbs sampling is a technique for generating random variables from a marginal distribution indirectly, without having to calculate the joint density. As a Markov

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<sup>1</sup>This approach shares some similarities with the Expectation-Maximization (EM) algorithm which is also an iterative method for finding maximum likelihood estimates in statistical models, which the model has some latent variables.

<sup>2</sup>Using the MSE as risk, the Bayes estimate of the unknown parameter is simply the mean of the posterior distribution.

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chain Monte Carlo method, the Gibbs sampler generates sample values from the distribution of each variable in turn, conditional on the current values of the other variables.

Gibbs sampling is, essentially, an iterative procedure. An iteration is generally termed a “sweep”. Initially, i.e.,  $j = 0$ , for fixed  $\lambda$  the parameters and the latent variables are set to any values, where tick test results can be used as initial values for  $q$ , and the Method of Moments estimates for  $s$  and  $\sigma_\epsilon$ . Denote these initial values  $(s^{(0)}, \sigma_\epsilon^{(0)}, q^{(0)})$ , where  $q^{(0)} = (q_1^{(0)}, q_2^{(0)}, \dots, q_T^{(0)})$ .

The steps in the first sweep ( $j = 1$ ) given  $\lambda$  and  $p$  are:

1. Draw  $s^{(1)}$  from  $f_s(s|\sigma_\epsilon^{(0)}, q^{(0)}, p, \lambda)$ ,
2. Draw  $\sigma_\epsilon^{(1)}$  from  $f_{\sigma_\epsilon}(\sigma_\epsilon|s^{(1)}, q^{(0)}, p, \lambda)$ ,
3. Draw  $q^{(1)}$  from  $f_q(q|s^{(1)}, \sigma_\epsilon^{(1)}, p, \lambda)$ ,

where  $f(\cdot|\cdot)$  is the complete conditional density.

Similarly, draws for  $q_t^{(1)}$  for  $t = 1, \dots, T$  are made sequentially using the Gibbs sampler. The steps in the first sweep ( $j = 1$ ) are then:

1. Draw  $q_1^{(1)}$  from  $f_q(q_1|s^{(1)}, \sigma_\epsilon^{(1)}, p, \lambda, q_2^{(0)}, q_3^{(0)}, \dots, q_T^{(0)})$ ,
2. Draw  $q_2^{(1)}$  from  $f_q(q_2|s^{(1)}, \sigma_\epsilon^{(1)}, p, \lambda, q_1^{(1)}, q_3^{(0)}, \dots, q_T^{(0)})$ ,
- $\vdots$
- $\dots$
- T. Draw  $q_T^{(1)}$  from  $f_q(q_T|s^{(1)}, \sigma_\epsilon^{(1)}, p, \lambda, q_1^{(1)}, q_2^{(1)}, \dots, q_{T-1}^{(1)})$ .

Note that, all parameters and latent variables except for the component being drawn are taken as given.

The next iteration starts with a draw of  $s^{(2)}$  conditional on  $\sigma_\epsilon^{(1)}, q^{(1)}, p$  and  $\lambda$ . Repeating this  $n$  times, we generate a sequence of draws  $(s^{(j)}, \sigma_\epsilon^{(j)}, q^{(j)})$  for  $j = 1, \dots, n$ , where  $q^{(j)} = (q_1^{(j)}, q_2^{(j)}, \dots, q_T^{(j)})$ . The Gibbs principle ensures that the limiting distribution of the  $n$ th draw (as  $n \rightarrow \infty$ ) is  $F(s, \sigma_\epsilon, q|p, \lambda)$ . The limiting draw for any parameter is distributed in accordance with the corresponding marginal posterior, i.e., the limiting density of  $s^{(n)}$  is  $f_s(s|p, \lambda)$ . Given some continuous function of the model parameters,  $h(s, \sigma_\epsilon, q)$  and a set of parameter

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draws,  $(s^{(j)}, \sigma_\epsilon^{(j)}, q^{(j)} : j = 1, \dots, n)$ , the corresponding sequence  $(h(s^{(j)}, \sigma_\epsilon^{(j)}, q^{(j)} : j = 1, \dots, n))$  generally has a limiting distribution for  $h(s, \sigma_\epsilon, q)$ .

The consistent estimates of population parameters  $\theta = (s, \sigma_\epsilon)$  are given by the posterior mean  $E_{\theta|p, \lambda}(\theta)$ . If one is interested in some statistic  $g(\theta)$ , then  $E_{\theta|p, \lambda}g(\theta)$  is a consistent estimate of  $g(\theta)$ .

#### 3.3.2 The Bayesian Model Selection

Suppose we have  $K$  models  $M_k$  ( $k = 1, 2, \dots, K$ ), given the model-specific parameter vector  $\theta_k$ . Our prior information on these models can be used to assign each of them a prior probability  $\pi(\theta_k|M_k)$ . A data set  $y = (y_1, y_2, \dots, y_n)$  is used to update these prior probabilities. The marginal likelihood  $m(y|M_k)$  of  $y$  under model  $M_k$  is defined by

$$m(y|M_k) = \int f(y|\theta_k, M_k)\pi(\theta_k|M_k)d\theta_k, \quad (3.15)$$

where  $f(y|\theta_k, M_k)$  is the probability density of  $y$  given the value of  $\theta_k$ , or the likelihood function of  $\theta_k$ .

The Bayes factor is defined as the log of the ratio of the marginal likelihoods for any two competing models  $k$  and  $l$ :

$$B_{kl} = \ln \left( \frac{m(y|M_k)}{m(y|M_l)} \right). \quad (3.16)$$

The preferred model is the one with the highest marginal likelihood. Therefore, evaluating the marginal likelihood is the key to Bayesian model selection.

Using the Basic Marginal Likelihood Identity (BMI), the marginal likelihood can be written as

$$m(y) = \frac{f(y|\theta)\pi(\theta)}{\pi(\theta|y)}, \quad (3.17)$$

where the numerator is the product of the likelihood and the prior, and the denominator is the posterior density. Conveniently, it can be expressed in a

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logarithmic scale as

$$\ln m(y) = \ln f(y|\theta) - \ln \frac{\pi(\theta|y)}{\pi(\theta)}, \quad (3.18)$$

where the first term measures how well the model fits the data given the most probable parameter values, and the second term penalizes the model according to its complexity. Hence, the marginal likelihood automatically incorporates a tradeoff between model fit and its complexity.

The marginal likelihood can be evaluated from the Gibbs output, generated during the simulation stage, as suggested by [Chib \(1995\)](#).

Let  $z$  denote latent data, and suppose that for a given set of vector blocks  $\theta = (\theta_1, \theta_2, \dots, \theta_B)$ , the Gibbs sampling algorithm is applied to the set of  $(B + 1)$  complete conditional densities,  $\{\pi(\theta_r|y, \theta_s (s \neq r), z)\}_{r=1}^B$  and  $\pi(z|y, \theta)$ . The goal is to compute the marginal likelihood  $m(y)$  from the output  $\{\theta^{(j)}, z^{(j)}\}_{j=1}^n$  obtained from these full conditional densities. In some situations, the density  $f(y|\theta) = \int f(y, z|\theta) dz$  is not available in closed form, however, the likelihood  $f(y|\theta, z)$  is. The marginal likelihood  $m(y)$  in this situation has the form

$$m(y) = \frac{f(y|\theta, z)\pi(\theta, z)}{\pi(\theta, z|y)}, \quad (3.19)$$

where the likelihood  $f(y|\theta, z)$ , the prior  $\pi(\theta, z)$ , and the multivariate posterior density  $\pi(\theta, z|y)$  can be evaluated at the selected high density point  $(\theta^*, z^*)$  (e.g. the mode of the posterior density derived from the Gibbs sampler). When estimating  $\pi(\theta, z|y)$ ,  $z$  can be treated as an additional block, e.g.  $z \equiv \theta_{B+1}$ . Evaluating at a particular point  $\theta^*$ , we have

$$\ln m(y) = \ln f(y|\theta^*) + \ln \pi(\theta^*) - \ln \pi(\theta^*|y). \quad (3.20)$$

The posterior density at the selected high density point is given by

$$\pi(\theta^*|y) = \pi(\theta_1^*|y) \times \pi(\theta_2^*|y, \theta_1^*) \times \dots \times \pi(\theta_{B+1}^*|y, \theta_1^*, \dots, \theta_B^*), \quad (3.21)$$



### 3.3 The Model Estimation and Selection

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where the first term on the right hand side is the marginal ordinate, which can be estimated from the draws of the initial Gibbs sampling, e.g.  $\hat{\pi}(\theta_1^*|y) = n^{-1} \sum_{j=1}^n \pi(\theta_1^*|y, \theta_s^{(j)} (1 < s \leq B+1))$ , and the rest of the terms are the reduced conditional ordinates with the typical form of  $\pi(\theta_r^*|y, \theta_1^*, \theta_2^*, \dots, \theta_{r-1}^*)$ , which are given by

$$\int \pi(\theta_r^*|y, \theta_1^*, \theta_2^*, \dots, \theta_{r-1}^*, \theta_{r+1}, \dots, \theta_B) d\pi(\theta_{r+1}, \dots, \theta_B|y, \theta_1^*, \theta_2^*, \dots, \theta_{r-1}^*). \quad (3.22)$$

To estimate the above integral, continue the sampling with the complete conditional densities of  $\{\theta_r, \theta_{r+1}, \dots, \theta_{B+1}\}$ , where in each of these full conditional densities,  $\theta_s$  is set equal to  $\theta_s^*$ , ( $s \leq r-1$ ). If the draws from the reduced complete conditional Gibbs sampling are denoted by  $\{\theta_r^{(g)}, \theta_{r+1}^{(g)}, \dots, \theta_{B+1}^{(g)}\}$ , then an estimate of (3.22) is

$$\hat{\pi}(\theta_r^*|y, \theta_s^* (s < r)) = n^{-1} \sum_{g=1}^n \pi(\theta_r^*|y, \theta_1^*, \theta_2^*, \dots, \theta_{r-1}^*, \theta_l^{(g)} (l > r)). \quad (3.23)$$

Therefore, an estimate of the posterior density is

$$\hat{\pi}(\theta_1^*|y) \prod_{r=2}^B \hat{\pi}(\theta_r^*|y, \theta_s^* (s < r)) \pi(z^*|y, \theta_s^* (s < B+1)). \quad (3.24)$$

The log of the marginal likelihood is

$$\begin{aligned} \ln \hat{m}(y) &= \ln f(y|\theta^*, z^*) + \ln \pi(\theta^*, z^*) - \ln \hat{\pi}(\theta_1^*|y) \\ &\quad - \sum_{r=2}^B \ln \hat{\pi}(\theta_r^*|y, \theta_s^* (s < r)) - \ln \pi(z^*|y, \theta_s^* (s < B+1)). \end{aligned} \quad (3.25)$$

In our situation for  $\theta = (s, \sigma_\epsilon, q)$ , where we suppress  $\lambda$  for convenience, Gibbs sampling is applied to the complete conditional densities

$$\pi(\theta_1|y, \theta_2, z) = f_s(s|p, \sigma_\epsilon, q), \quad (3.26)$$

### 3.3 The Model Estimation and Selection

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$$\pi(\theta_2|y, \theta_1, z) = f_{\sigma_\epsilon}(\sigma_\epsilon|p, s, q), \quad (3.27)$$

$$\pi(z|y, \theta) = f_q(q|p, s, \sigma_\epsilon). \quad (3.28)$$

The goal is to estimate the posterior ordinate  $f(s^*, \sigma_\epsilon^*, q^*|p)$ , which is now expressed as  $f_s(s^*|p)f_{\sigma_\epsilon^*}(\sigma_\epsilon^*|p, s^*)f_q(q^*|p, s^*, \sigma_\epsilon^*)$ . The first term,  $f_s(s^*|p)$ , can be estimated by taking the average of the full conditional density with the posterior draws of  $(\sigma_\epsilon, q)$ , leading to the estimate

$$\hat{f}_s(s^*|p) = n^{-1} \sum_{j=1}^n f_s(s^*|p, \sigma_\epsilon^{(j)}, q^{(j)}). \quad (3.29)$$

Continue sampling for an additional  $n$  iterations with the complete conditional densities  $f_{\sigma_\epsilon}(\sigma_\epsilon|p, s, q)$  and  $f_q(q|p, s, \sigma_\epsilon)$ , where in each of these densities,  $s$  is set equal to  $s^*$ . It can be shown that the draws  $\{q^{(g)}\}$  from this round of iterations follow the density  $f_q(q|p, s^*)$ , as required. Therefore,

$$\hat{f}_{\sigma_\epsilon^*}(\sigma_\epsilon^*|p, s^*) = n^{-1} \sum_{g=1}^n f_{\sigma_\epsilon^*}(\sigma_\epsilon^*|p, s^*, q^{(g)}). \quad (3.30)$$

Substituting the density estimates into (3.25) yields the following estimate of the marginal likelihood:

$$\ln \hat{m}(p) = \ln f(p|s^*, \sigma_\epsilon^*, q^*) + \ln f(s^*, \sigma_\epsilon^*, q^*) - \ln \hat{f}_s(s^*|p) - \ln \hat{f}_{\sigma_\epsilon^*}(\sigma_\epsilon^*|p, s^*) - \ln f_q(q^*|p, s^*, \sigma_\epsilon^*), \quad (3.31)$$

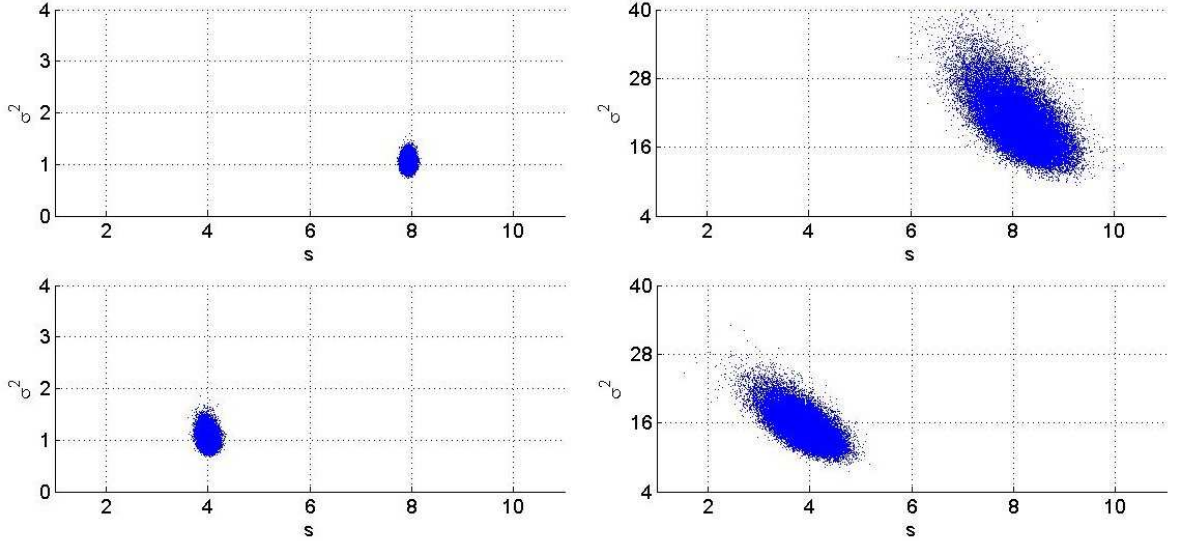
where  $f(s^*, \sigma_\epsilon^*, q^*) = f_s(s^*)f_{\sigma_\epsilon^*}(\sigma_\epsilon^*)f_q(q^*)$  which is the product of the prior densities of  $s$ ,  $\sigma_\epsilon$  and  $q$  evaluated at  $s^*$ ,  $\sigma_\epsilon^*$  and  $q^*$ .

The performance of the Gibbs sampler and an example of the application of the Bayesian model estimation and selection techniques are illustrated in the next two sections.

### 3.4 Simulation Analysis

The performance of the Bayesian estimators may be illustrated by considering simulated samples under three different specifications. The specifications correspond to typical situations in the marketplace. Assume trades can execute at two bid-ask spreads, e.g. the inner spread and outer spread.

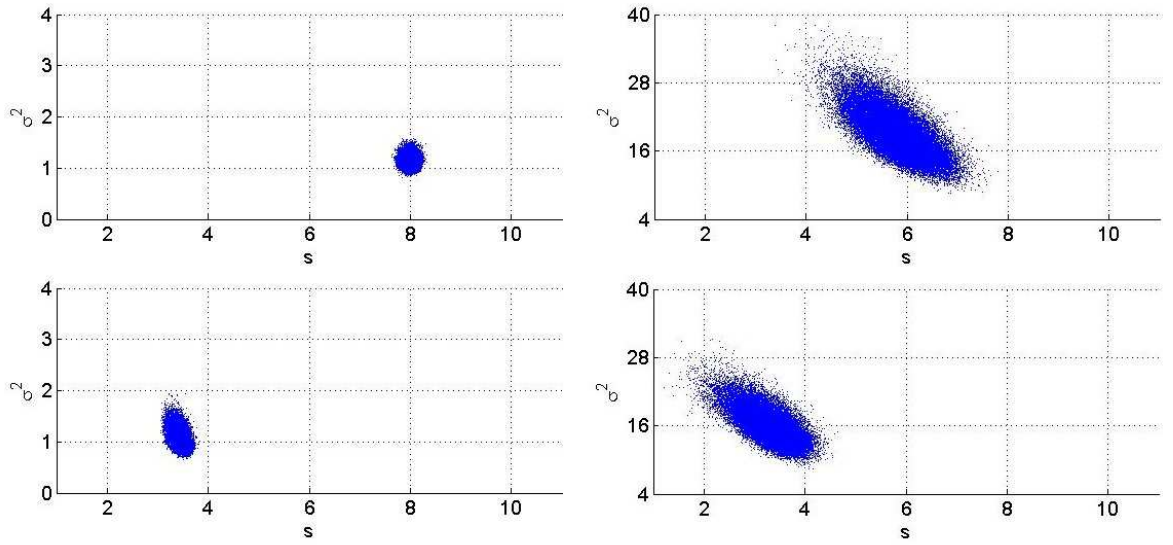
First scenario, trades execute at inner spread and outer spread with equal probabilities. That means  $p(q = \pm\lambda) = p(q = \pm 1)$ , that is,  $p(q = \lambda) = p(q = -\lambda) = p(q = 1) = p(q = -1) = 0.25$  in Figure 3.5, where  $p(\cdot)$  is the probability of the trade direction indicator  $q$ .



**Figure 3.5:** Joint posteriors of half spread  $s$  and variance  $\sigma^2$  I

This figure provides the joint posteriors of half spread  $s$  and variance  $\sigma^2$ , given  $\lambda$  for four simulated series of length 300 with probabilities  $p(q = +\lambda) = 0.25$ ,  $p(q = +1) = 0.25$ ,  $p(q = -1) = 0.25$ ,  $p(q = -\lambda) = 0.25$ : Top-left panel:  $s = 8$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.25$ ; Top-right panel:  $s = 8$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.25$ ; Bottom-left panel:  $s = 4$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.5$ ; Bottom-right panel:  $s = 4$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.5$ .

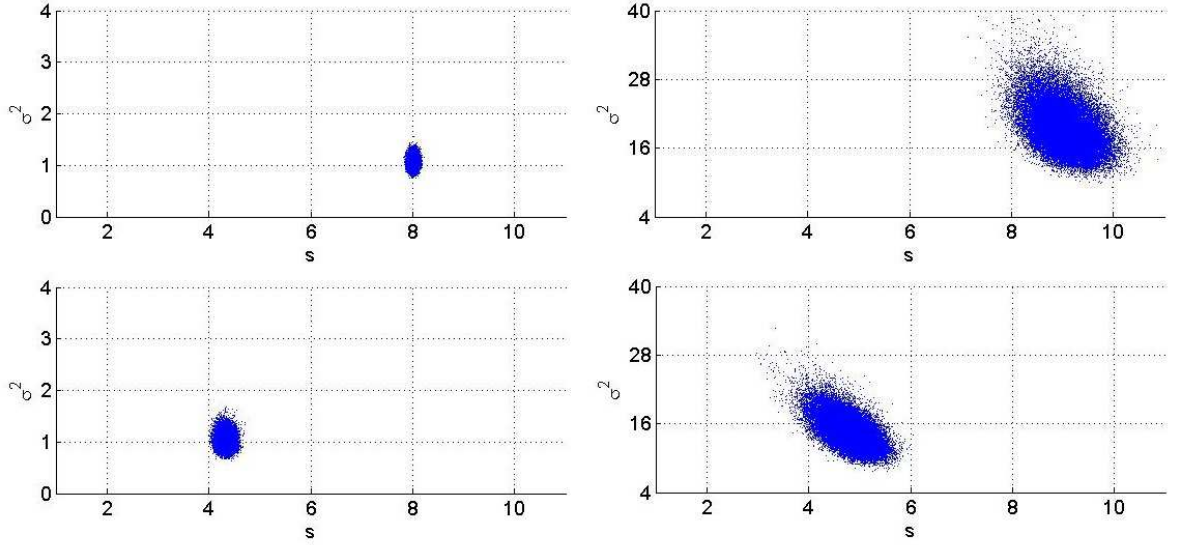
Second scenario, most of trades are executed at the outer spread, while a few orders execute at the inner spread. In other words,  $p(q = \pm\lambda) > p(q = \pm 1)$ , say,  $p(q = \lambda) = p(q = -\lambda) = 0.40$  and  $p(q = 1) = p(q = -1) = 0.10$  in Figure 3.6.



**Figure 3.6:** Joint posteriors of half spread  $s$  and variance  $\sigma^2$  II

This figure provides the joint posteriors of half spread  $s$  and variance  $\sigma^2$ , given  $\lambda$  for four simulated series of length 300 with probabilities  $p(q = +\lambda) = 0.40$ ,  $p(q = +1) = 0.10$ ,  $p(q = -1) = 0.10$ ,  $p(q = -\lambda) = 0.40$ : Top-left panel:  $s = 8$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.25$ ; Top-right panel:  $s = 8$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.25$ ; Bottom-left panel:  $s = 4$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.5$ ; Bottom-right panel:  $s = 4$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.5$ .

Third scenario, most orders are traded at the normal spread. The market maker charges a few orders at a larger spread. Therefore, in this case we have  $p(q = \pm\lambda) < p(q = \pm 1)$ , say,  $p(q = \lambda) = p(q = -\lambda) = 0.10$  and  $p(q = 1) = p(q = -1) = 0.40$  in Figure 3.7.



**Figure 3.7:** Joint posteriors of half spread  $s$  and variance  $\sigma^2$  III

This figure provides the joint posteriors of half spread  $s$  and variance  $\sigma^2$ , given  $\lambda$  for four simulated series of length 300 with probabilities  $p(q = +\lambda) = 0.10$ ,  $p(q = +1) = 0.40$ ,  $p(q = -1) = 0.40$ ,  $p(q = -\lambda) = 0.10$ : Top-left panel:  $s = 8$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.25$ ; Top-right panel:  $s = 8$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.25$ ; Bottom-left panel:  $s = 4$ ,  $\sigma^2 = 1$ ,  $\lambda = 0.5$ ; Bottom-right panel:  $s = 4$ ,  $\sigma^2 = 16$ ,  $\lambda = 0.5$ .

Four sample paths with different values of  $s$ ,  $\sigma^2$  and  $\lambda$  are simulated for each case. All simulated samples are of length 300. For each path the Gibbs sampler is run for 50,000 iterations, with first 20% discarded. The joint posteriors of  $s$  and  $\sigma^2$  given the value of  $\lambda$  for each sample path are characterised by the remaining 80% draws, and displayed in four sub-panels in Figure 3.5, 3.6 and 3.7, respectively.

There are a few similarities amongst these three cases. Firstly, in the left-hand side panels, the joint posteriors are neatly packed together, whereas in the right-hand side panels, the joint posteriors are more scattered. Secondly, the joint posteriors are more negatively sloped in the right-hand side panels than they

are in the left-hand side panels. Finally, the joint posteriors in the bottom-left panels are slightly more dispersed than they are in the top-left panels, whereas the opposite is true for the right-hand side panels.

In Figure 3.5, the joint posteriors are generally centered around the true values. However, in Figure 3.6 the posterior for  $s$  is biased downwards, whereas in Figure 3.7 it is biased upwards, except in the top-left panels.

The Bayesian estimation approach needs to balance the composition of the price changes between the transient component and the permanent component in the security value as well as balance the composition of the transient component itself, which is the combination of small and large spread changes. Hence, the concentration of the joint posterior generally indicates the certainty that the Bayesian trade classification procedure can assign a direction to a particular trade with. Intuitively, when the outer spread  $s$  is relatively larger than the volatility of the efficient price changes  $\sigma$  and the inner spread  $\lambda s$ , the bayesian procedure will be more accurate about whether a spike is caused by a large spread accompanied by a small efficient price increment or a small spread accompanied by a large efficient price increment. The misclassification in the scenario where the inner and outer spreads are similar results in the spreading out and the negative sloping of the joint posterior. Furthermore, when  $\lambda$  is close to 1, it is more difficult to distinguish the outer bids and asks from inner bids and asks due to the existence of non-zero variance.

When the data is not sufficient to determine a direction for a trade, the prior information will play a more important role. In the asymmetric cases (Figure 3.6 and 3.7), when the volatility is sizable compared to the inner and outer spreads, the procedure may rely slightly more on the prior. As a uniform prior is applied, the procedure underestimates the spreads in Figure 3.6 and overestimates the spreads in Figure 3.7.

## 3.5 Empirical Application

The extended model is applied to a time series of intra-day bond prices, as the data in Figure 3.1 suggests that the extended model might be an appropriate underlying model. Table 3.2 summarizes the basic information about the bond and the descriptive statistics of the sample.

The Gibbs sampler is run for 10,000 iterations, with first 20% discarded. Our results for the data are presented in Table 3.3. The Method of Moments estimate for the spread  $s$  is about three standard deviations bigger than the Bayesian estimate under Roll's model. The Bayesian estimate of the spread  $s$  under Roll's model and the Method of Moments estimate are in between the estimated inner and outer spreads (the grand average of the estimated inner and outer spreads is 0.4762), suggesting that Roll's spread measures estimated using both the classic and Bayesian approaches are essentially some weighted averages of the two spreads. For the variance  $\sigma^2$ , the Method of Moments estimate is five standard deviations smaller than the Bayesian estimate under Roll's model, but sixteen standard deviations bigger than the Bayesian estimate under the extended model. The estimated log marginal likelihood is equal to -1182.8434 for Roll's model. By contrast with the extended model, the maximum of the estimated log marginal likelihood is equal to -326.5221 at  $\lambda = 0.13$ . Therefore, the extended model fits the data better than the original Roll's model.

Additionally, the posterior median  $\hat{m}_t$  as the estimate of the efficient price  $m_t$  together with the estimated inner and outer bid-ask bounces, defined as  $\hat{m}_t \pm s$  and  $\hat{m}_t \pm s\lambda$  respectively, are also shown in Figure 3.8. It is clear from the figure that trades executed with different transaction costs are distributed between the corresponding estimated bid-ask spreads. Furthermore, the estimates of  $\alpha$  and  $\theta$  are 34% and 53%, respectively. The estimate of  $\alpha$  indicates that 66% of the trades happened at the inner spread and the estimate of  $\theta$  supports a symmetric structure (i.e. the outer bid and the outer ask are equally likely to happen.). The estimated efficient price changes  $\Delta\hat{m}$  under the extended model exhibits less serial correlation than they do under Roll's model, as shown in Table 3.4. This implies that the permanent component of the transaction prices, which reflect

### 3.5 Empirical Application

the degree of illiquidity and lead to negatively serially correlated price changes, is better estimated under the extended model.

**Table 3.2:** Description and Summary Statistics

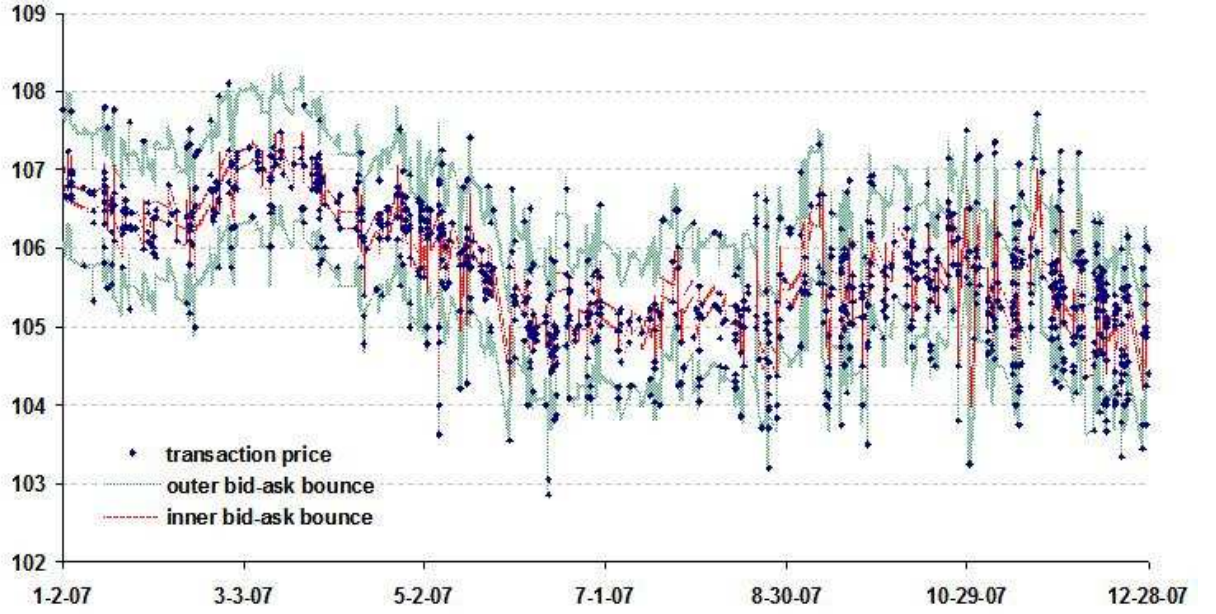
CUSIP ID	172967AZ4
Company name	Citigroup Inc.
Data source	TRACE
Data beginning	02/01/2007
Data ending	28/12/2007
# trading days in samples	197
Total # of trades	1339
Price units	US Dollars
Average price	105.72
Standard deviation	1.00
Maximum price	108.106
Minimum price	102.85
Average daily trades	6.80
Average time between trades	06:27:35

**Table 3.3:** Summary of estimates for bond 172967AZ4.

Parameter	MM, Roll	Post.mean, Roll	Post.mean, Extended
$s$	0.4062	0.3302(0.0225)	0.8428(0.0227)
$\sigma^2$	0.1982	0.3231(0.0235)	0.0898(0.0066)
$\lambda$		1	0.13
$\lambda s$		0.3302	0.1096
Log marginal likelihood		-1182.8434(0.0286)	-326.5221(0.0333)



### 3.5 Empirical Application



**Figure 3.8:** Time series plot of transaction prices with estimated bid-ask bounces

**Table 3.4:** Serial correlation of  $\Delta\hat{m}$ .

Lag	Roll	Extended
1	-0.1646	-0.0623
2	-0.1186	-0.0331
3	-0.1323	-0.0893
4	-0.0786	-0.0814
5	0.0132	-0.0689
6	0.0295	0.0059
7	-0.0153	-0.013
8	0.0074	-0.0424
9	0.0046	-0.0311
10	-0.0056	0.0269

## 3.6 Conclusions

In this paper we extend the Roll model to allow trades to be executed at two different spreads with associated probabilities. The parameters of the extended model are determined via a Bayesian approach, based on only transaction data. A numerical example based on a set of corporate bond transaction data shows that the extended model fits the data better. The estimated efficient price changes under the extended model exhibits less serial correlation than those under Roll's model. The new model together with the model estimation and selection procedures offers an alternative way to estimate the effective bid-ask spread and the underlying variance.

## Chapter 4

# Non-default Yield Spread and Illiquidity of Corporate Bonds

### 4.1 Introduction

During the recent financial crisis, corporate yield spreads widened dramatically. Spreads were much larger than can be explained by expected losses arising from default.<sup>1</sup> The objective of this study is to understand that whether time series and cross-sectional variations of non-default yield spreads can be explained by liquidity, and in particular, why bonds issued by a single company exhibit heterogeneous levels of liquidity and liquidity premia.

Recognizing that a reasonable method to accurately extract the non-default component of yield spreads is essential to our task, we follow Longstaff *et al.* (2005) to compute the fundamental value of a corporate bond, as if it is perfectly liquid, by using the information from credit default swaps. To this point, the difference between our method and the one by Longstaff *et al.* (2005) is that the latter uses a reduced-form approach which includes both default and liquidity yield processes to price corporate bonds and credit default swaps simultaneously, while we adopt a non-parametric approach to extract default intensities from

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<sup>1</sup>See, for example, Bao *et al.* (2011) and Dick-Nielsen *et al.* (2012).

credit default swap spreads of the same name as the bond, and then utilize the default intensities to obtain the fundamental value of the bond which is the price when the market is perfectly liquid. Our approach is capable to generate flexible default term structures and thus avoids the misspecification problem of reduced-form models.

Bid-ask spreads have been widely used as a measure of illiquidity in the literature.<sup>1</sup> We are interested in both the level of illiquidity (the transitory component) of the bonds and the liquidity premium (the permanent component) embedded in bond yields.<sup>2</sup> Order flow spread models are capable of estimating both the two components. However, Roll's model is clearly misspecified when applied to our transaction data which have a non-constant transitory component, and the other models with a richer structure require more detailed data (e.g. quotes or trade directions) which are difficult to obtain for corporate bonds as they are traded in OTC markets.

Therefore, our next task is to develop a parsimonious model which has a reasonably rich structure and does not require extra data to estimate. We developed two competing models based on the original Roll model. The first one is the model developed in the previous chapter which allows two different magnitudes of bid-ask spreads with associated probabilities. The model is estimated via a Bayesian approach. The second one is a Kalman Filter model which allows the transitory component to vary over time and take continuous values. Computationally, the second model is preferred since it works more efficiently when applied to a relatively large data set.

A second difference between this study and previous studies on bond illiquidity is that the latter examine raw transactions data, while we apply our extended Roll model to the non-default price residuals which are calculated by subtracting the corporate bond market prices from the theoretical fundamental values implied from the credit default swap spreads. Therefore, our approach essentially decomposes the non-default price residuals into a transitory component which arises from the illiquidity of the market and a permanent component which reflects

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<sup>1</sup>See, for example, Bessembinder & Maxwell (2008), Edwards & Piwowar (2007) and Goldstein *et al.* (2007).

<sup>2</sup>The liquidity premium in bond yields does not violate market efficiency, but simply reflects the rational response by the investors when taking into account the illiquidity of the asset.

public information including information about future bond liquidity. Our measure of illiquidity is derived from the transitory component, which is supposed to be orthogonal to the fundamental value. The permanent component of the non-default price residuals can be expressed in terms of yield as non-default yield spreads. Longstaff *et al.* (2005) find that non-default yield spreads are strongly related to bond-specific illiquidity. Ericsson & Renault (2006) find evidence of a positive correlation between illiquidity and default components of yield spreads. However, they did not distinguish non-default yield spreads from transitory effects. Chen *et al.* (2007) find that more illiquid bonds earn higher yield spreads. As they use over 4000 corporate bonds to cover the universe of corporate bond market, they could not make use of the information in credit default swap market.

This leads to the third and final difference between our study and other papers related to the liquidity of corporate bond markets. The latter concentrate on either developing illiquidity measures and associating them to bond characteristics which are known to be related to bond illiquidity or trying to explain the unexplained yield spread or return as liquidity premia. In contrast, our study focuses on understanding time series and cross-sectional variations in both illiquidity and liquidity premia of bonds issued by a single corporation. De Jong & Driessen (2007) show that liquidity risk is a pricing factor for the expected returns on corporate bonds, and the credit spread puzzle can be explained by the associated liquidity premia. Bao *et al.* (2011) uses their bond-level illiquidity measure to explain individual bond yield spreads. Their results suggest that for high-rated corporate bonds, the sudden increase in illiquidity during the crisis was the dominating factor in driving up yield spreads. By developing a new illiquidity measure, Dick-Nielsen *et al.* (2012) analyze the liquidity components of corporate bond spreads during 2005-2009. They find that the spread contribution from illiquidity increased dramatically with the onset of the subprime crisis. The increase was slow and persistent for high-rated bonds. Our analysis shows that before the crisis (before July 2007) and during the crisis (from July 2007 to June 2009), the non-default spread accounted for about 80% and 48% of the yield spreads on average, respectively. The proportion dropped to around 23% after the crisis (after June 2009) as a result of the emergency liquidity supply from the U.S. Treasury.

Our study also sheds light on how the common and idiosyncratic factors affect bond illiquidity and non-default yield spreads. By analyzing the variations in both our illiquidity measure and the estimated non-default yield spread during 2006 - 2010, our results show that the illiquidity measure has a nonlinear positive relationship with time-to-maturities, and is negatively correlated with issue sizes and the trading activity measures. Both fundamental volatilities and CDS spreads are positively correlated with the illiquidity measure. The common factors such as fundamental volatilities and CDS spreads can explain 17.57% of the variations in the illiquidity measure. Adding the idiosyncratic factors such as time-to-maturities, issue sizes, and the trading activity measures improves the  $R^2$  by roughly about 3%.

The regression results of non-default yield spreads show that non-default yield spreads are positively correlated with our illiquidity measure. Adding the illiquidity measure in the regression increases the  $R^2$  by 5.85%. Thus, non-default yield spreads are indeed related to bond illiquidity. Our results support a positive correlation between the credit risk and liquidity spreads, and an increasing and concave term structure of liquidity spreads. Finally, 26.85% of the variations in liquidity spreads can be explained by the common factors, such as CDS spreads and fundamental volatilities. The idiosyncratic factors apart from the illiquidity measure contribute to the  $R^2$  by only 0.29%.

The remainder of this chapter is organized as follows. Section II summarizes the data, and Section III describes the methods to estimate the illiquidity measure and the non-default yield spreads. Section IV conducts the regression analysis for the illiquidity measure and the non-default yield spreads. Section V concludes.

## 4.2 Data

In this section, we describe our main data sources that are used to decompose corporate yield spreads, and construct various credit and illiquidity proxies.

Given that we are interested in understanding that how bonds issued by a single company exhibit heterogeneous levels of liquidity and liquidity premia, we will focus on a single but representative corporation (i.e. Citigroup Inc.) which has a relatively large amount of corporate debts outstanding during the sample period.

The two main types of Citigroup data used in this study are mid-quotes of CDS spreads and transaction data of corporate bonds. Since January 2001 FINRA members have been required to report their secondary OTC corporate bond transactions through TRACE.<sup>123</sup> TRACE contains all reported OTC trades in corporate bonds since 2002. By the end of 2004 about 99% of all trades representing about 95% of the dollar value traded were disseminated within 15 minutes. In practice 80% of all transactions are reported within 5 minutes. For each trade the NASD member is required to report: the bond identification, the date and the time of execution, the trade price and the yield, the trade size and the buy or sell side indication. Trade side indications since 3 November 2008 were made available. Our selective corporate bond transaction data set consists of 17 fixed-rate senior unsecured (A rated) dollar-denominated debt obligations of Citigroup during the period from February 7, 2006 to September 30, 2010. Table 4.1 gives the information of the bonds in our sample.

The daily CDS spreads are obtained from Credit Market Analysis (CMA) which sources its CDS data from a robust consortium which consists of around 40 members from the buy-side community (hedge funds, asset managers, and major investment banks) who are active participants in the CDS market.<sup>4</sup> Each of these members contributes their CDS prices to a CMA database which they receive in Bloomberg formatted messages (as well as forms) from their sell-side

<sup>1</sup>The Financial Industry Regulatory Authority formerly named National Association of Security Dealers (NASD)

<sup>2</sup>Trade Reporting and Compliance Engine

<sup>3</sup>For a detailed description of the TRACE system, see Zhou (2005).

<sup>4</sup>CMA has been found to be one of the more reliable CDS data sources by Mayordomo & Schwartz (2010)

Table 4.1: Bond Information

Bond	Coupon	Maturity Date	1st Settle Date	1st Coupon Date	Issuance(MM)	Rating	Maturity	CUSIP
bond01	6.5	2011-1-18	2001-1-16	2001-7-18	2500	A	10	172967BC4
bond02	5.3	2016-1-7	2005-12-8	2007-7-7	1000	A	10	172967DE8
bond03	6	2012-2-21	2002-2-21	2002-8-21	1500	A	10	172967BJ9
bond04	5.125	2014-5-5	2004-5-5	2004-11-5	1750	A	10	172967CK5
bond05	4.625	2010-8-3	2005-8-3	2006-2-3	1250	A	5	172967DA6
bond06	4.125	2010-2-22	2005-2-22	2005-8-22	2000	A	5	172967CU3
bond07	5.125	2011-2-14	2006-2-14	2006-8-14	2000	A	5	172967DH1
bond08	5.1	2011-9-29	2006-9-29	2007-3-29	1100	A	5	172967DU2
bond09	5.25	2012-2-27	2007-2-27	2007-8-27	1300	A	5	172967DZ1
bond10	6	2017-8-15	2007-8-15	2008-2-15	2000	A	10	172967EH0
bond11	5.5	2012-8-27	2007-8-27	2008-2-27	1000	A	5	172967EJ6
bond12	5.3	2012-10-17	2007-10-17	2008-4-17	3000	A	5	172967EL1
bond13	6.125	2017-11-21	2007-11-21	2008-5-21	4000	A	10	172967EM9
bond14	5.5	2013-4-11	2008-4-11	2008-10-11	4750	A	5	172967EQ0
bond15	6.125	2018-5-15	2008-5-12	2008-11-15	3000	A	10	172967ES6
bond16	6.5	2013-8-19	2008-8-19	2009-2-19	3000	A	5	172967EU1
bond17	8.5	2019-5-22	2009-5-22	2009-11-22	3000	A	10	172967EV9



dealers. Because CMA receives prices from the buy-side community, who are constantly receiving quotes from their dealers, these prices are very likely to be tradable or even executable prices.<sup>1</sup> These CDS spreads are mid-market price quotes for contracts covering from 6-month to 10-year maturities. This enables us to bootstrap hazard rates at any maturity within the range to match the maturities of all the bonds issued by the company. Considering both the CDS and corporate bond data we have, our analysis is conducted at a daily frequency.<sup>2</sup>

Finally, we choose Treasury rates as the risk-free rate, as it is widely used by empirical studies in finance. The entire term structure of Treasury rates is available from Federal Reserve H.15 Release<sup>3</sup>.

There are a few essential differences between market characterizations of corporate bonds and CDSs, which are important for our applications.

Firstly, and most importantly, credit default swaps are contracts, not securities. Securities are in fixed supply, whereas the notional amount of credit default swaps can be arbitrarily large. It may take time and effort to find a particular corporate bond, whereas new credit default swaps of a underlying entity can be created whenever they are needed. Moreover, the availability of CDS contracts is not necessarily linked to the aggregate amount of reference bonds outstanding, nor their maturities. In other words, CDS contracts are much less susceptible to the market friction than the underlying corporate bonds.

Secondly, it is important to notice that credit default swaps are essentially insurance contracts. Many investors who buy default protection may intend to hold the position for a fixed time period, and may not generally plan to unwind their position early. Even if an investor wants to unwind his current position, it may be less costly to simply enter into a new swap in the opposite direction

<sup>1</sup>Mayordomo & Rodriguez-Moreno (2011) argue that the nature of the CMA data suppose an advantage for the use of the bid-ask spread as a measure of liquidity because of the use of information from the buy-sell sides.

<sup>2</sup>Bao *et al.* (2011) suggest that at daily frequency illiquidity effect is stronger.

<sup>3</sup>Yields are interpolated by the Treasury from the daily yield curve. This curve, which relates the yield on a security to its time to maturity is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. These market yields are calculated from composites of quotations obtained by the Federal Reserve Bank of New York. The yield values are read from the yield curve at fixed maturities, currently 1, 3 and 6 months and 1, 2, 3, 5, 7, 10, 20, and 30 years.

than to try to short his current position. Therefore, the liquidity of his current position is less relevant since he is able to effectively replicate the short position through other contracts.

Thirdly, a CDS investor buying or selling protection, only needs a fraction of the principal (initial collateral margins or upfront fees), and the mark-to-market effect of interest rate fluctuations to a position is usually negligible. It is generally as easy to sell default protection as it is to buy default protection in CDS markets. In contrast, sometimes it can be difficult and costly to short sell corporate bonds due to limited arbitrage in the corporate bond market.

Finally, with the introduction of the new standard CDS contract and conventions, and because of the generic nature of the cash flows, credit default swaps cannot be demanded particularly in the way as Treasury securities or popular equities may.

In the previous literature, empirical evidence is mixed. [Ericsson \*et al.\* \(2005\)](#) use a set of credit risk models to evaluate the price of credit default swaps as well as corporate bonds. They find that their models tend to systematically underestimate bond spreads and, more importantly, that this is not the case for CDS premia. They also find that bond spread residuals are largely unrelated to default proxies but correlated with non-default proxies (such as illiquidity proxies) and CDS premium residuals are uncorrelated with both. In other words, the evidence supports the presence of illiquidity in bond residuals but not in CDS residuals. [Blanco \*et al.\* \(2005\)](#) also provide evidence that changes in the credit quality of the underlying name are likely to be reflected more quickly in the default swap spreads than in the bond yield spreads. This may be due to a non-default component in bond spreads that obscure the impact of changes in credit quality. [Ericsson \*et al.\* \(2009\)](#) investigate the linear relationship between theoretical determinants of default risk and default swap spreads. The principle component analysis shows that there is limited evidence for a common factor in the regression residuals. That implies that a liquidity-related component in credit default swap spread may be negligible.

By contrast, [Bongaerts \*et al.\* \(2008\)](#) study the effect of liquidity on prices of CDS from 2000 to 2006. They provide evidence of an economically and statistically significant expected liquidity premium attributable to the protection seller.

Using data from 2002 to 2005, [Lin \*et al.\* \(2009\)](#) find both CDS and corporate bond spreads contain significant liquidity components. Since there is no consensus in the literature on whether CDS spreads contain a liquidity premium and it is hard for us to check if CDS spreads contain a liquidity premium as that requires a long history of default data, this study follows [Longstaff \*et al.\* \(2005\)](#) in assuming that CDS spreads do not contain a liquidity premium.

## 4.3 The Methodology

In this section, first, we discuss the valuation of bonds and credit default swaps in our non-parametric framework. Second, we elaborate on the assumption of a constant recovery rate as well as the tenor effects in term structures post the crisis. Third, descriptive statistics of the estimated non-default price residuals are provided. Fourth, a detailed instruction of computing non-default price residuals of a corporate bond is given. Fifthly, we introduce the extended Roll model to estimate our liquidity measure as well as the non-default yield spread. Finally, summary statistics of the non-default yield spread and the liquidity measure are reported.

### 4.3.1 Valuing Bonds and Credit Default Swaps

Following Longstaff *et al.* (2005), we treat CDSs as a benchmark, that is, we explain credit default swap spreads by only default risk (in terms of default probability) and apply the default probabilities backed out from the credit default swap spreads of the same underlying entity to obtain the ‘fundamental value’ of a corresponding corporate bond.<sup>1</sup>

We value credit default swaps and corporate bonds within the reduced-form framework of Duffie & Singleton (1999). But, different from Longstaff *et al.* (2005), we apply a non-parametric valuation approach in order to generate more flexible shapes of default intensity term structures.

Let  $D(t, T)$  denote the riskless discount factors at time  $t$  for maturity  $T$  and assume them to be independent of the default time  $\tau$ . The probability of the name survives up to time  $T$ , observed at time 0, can be expressed as

$$Surv(T) := Prob(\tau > T) \tag{4.1}$$

The conditions for a valid survival probability function are  $Surv(0) = 1$ ,  $Surv(t) > 0$  and  $Surv(\cdot)$  needs to be decreasing. By following the market,  $Surv(t)$  is defined

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<sup>1</sup>The ‘fundamental value’ equals the value of a hypothetical liquid defaultable bond with the same maturity date and coupon rate.

as

$$\exp(-H(t)) = \exp\left(-\int_0^t h(s)ds\right), \quad (4.2)$$

where the hazard rate  $h$  is assumed to be piecewise constant<sup>1</sup> in time, eg.  $h(t) = h_i$  for  $t$  in  $[T_{i-1}, T_i]$ . Let  $\Delta_i$  denote the accrual period from  $T_{i-1}$  to  $T_i$ . The value of the premium of a CDS at time 0 can be decomposed into the value of an annuity paying one basis point and its accrual fee paid upon default, scaled by the spot running spread:

$$\begin{aligned} & PremLeg_N(S_N; D(0, \cdot), Surv(\cdot)) \\ &= S_N \cdot (Annu_N(D(0, \cdot), Surv(\cdot)) + Accr_N(D(0, \cdot), Surv(\cdot))), \end{aligned} \quad (4.3)$$

where

$$Annu_N(D(0, \cdot), Surv(\cdot)) = \sum_{i=1}^N D(0, T_i) \Delta_i \exp(-H(T_i)) \quad (4.4)$$

and

$$Accr_N(D(0, \cdot), Surv(\cdot)) = \sum_{i=1}^N D(0, T_i) \frac{\Delta_i}{2} (\exp(-H(T_{i-1})) - \exp(-H(T_i))). \quad (4.5)$$

Note  $\frac{\Delta_i}{2}$  is the average accrual period from  $T_{i-1}$  to  $T_i$  and  $\exp(-H(T_{i-1})) - \exp(-H(T_i))$  is the probability of a credit event occurring during period  $T_{i-1}$  to  $T_i$ .

By assuming recovery of the face value in the event of default, the valuation of the protection leg can be expressed similarly as

$$\begin{aligned} & ProtLeg_N(R; D(0, \cdot), Surv(\cdot)) \\ &= -(1 - R) \int_0^{T_N} D(0, s) dSurv(s) \\ &\approx (1 - R) \sum_{j=1}^M D(0, t_j) (\exp(-H(t_{j-1})) - \exp(-H(t_j))), \end{aligned} \quad (4.6)$$

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<sup>1</sup>We also tried piecewise linear hazard rate term structure. The results generally remain the same as we have data of CDS spreads at all maturities (1,2,3, up to 10 years).

where the Stieltjes integral is replaced by Riemann sums taken on a low enough discretization time steps  $t_j$ ,  $j = 1, \dots, M$ , given  $t_0 = 0$  and  $t_M = T_N$ .

Equating the values of the two legs and solving for the running spread gives

$$S_N = \frac{(1 - R) \sum_{j=1}^M D(0, t_j) (\exp(-H(t_{j-1})) - \exp(-H(t_j)))}{\sum_{i=1}^N D(0, T_i) \Delta_i [\exp(-H(T_i)) + \frac{1}{2} (\exp(-H(T_{i-1})) - \exp(-H(T_i)))]}. \quad (4.7)$$

Under the same framework, the 'fair value' of a defaultable bond with coupon  $c$ , face value  $F$  and maturity  $N$  can be expressed as

$$\begin{aligned} CP_N = & c \cdot \sum_{i=1}^N D(0, T_i) \Delta_i \exp(-H(T_i)) \\ & + F \cdot D(0, T_N) \exp(-H(T_N)) \\ & + R \sum_{j=1}^M D(0, t_j) (\exp(-H(t_{j-1})) - \exp(-H(t_j))), \end{aligned} \quad (4.8)$$

where  $R$  is the recovery rate which is kept consistent to the recovery assumption in the CDS pricing formulas with corresponding seniority and maturity.

#### 4.3.2 Recovery Rate Assumption

The assumption of a constant recovery rate is often adopted for modelling and estimation purposes in many credit literature. However, very few of them check whether the assumption of recovery rates has any impact on the valuation. Recovery rates generally can be treated in two ways. The first method is to consider it as another parameter, and to estimate it from the data along with other parameters. The second one is to a priori fix a value.

Now let us have a look at the relationship between recovery rates and default probabilities in a simply case: the valuation of a risky zero-coupon bond in one-period. Assuming the riskless rate is zero, consider the price of a defaultable zero-coupon bond

$$\begin{aligned} ZCP &= Prob \cdot R + (1 - Prob) \\ &= 1 - Prob \cdot (1 - R) \end{aligned} \quad (4.9)$$

Then the valuation equation for CDS spreads is given by

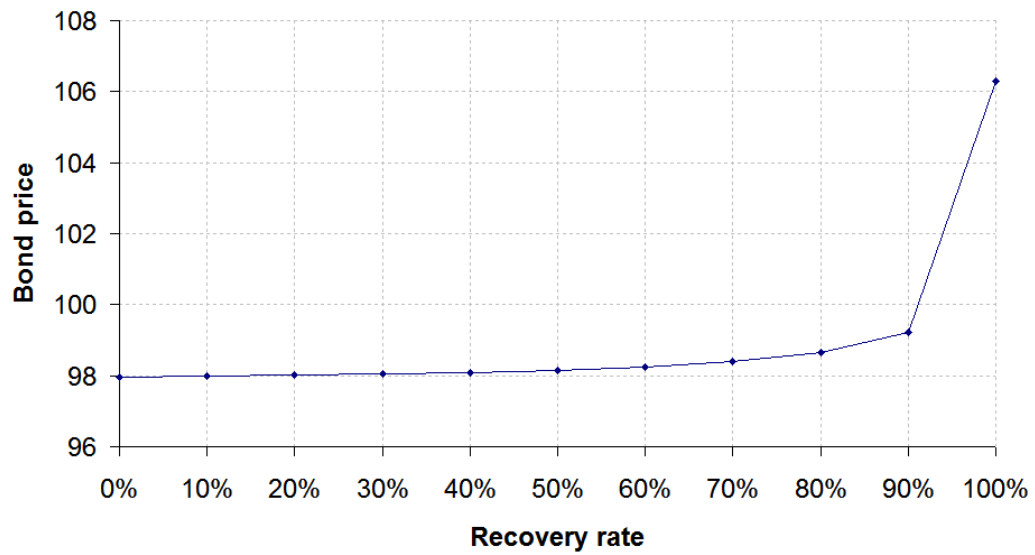
$$S \cdot (1 - Prob) = (1 - R) \cdot Prob. \quad (4.10)$$

We find that the credit spread or CDS spread  $S$  is given by

$$\begin{aligned} S &= -(1 - R) \left(1 - \frac{1}{1 - Prob}\right) \\ &\approx (1 - R) \cdot Prob. \end{aligned} \quad (4.11)$$

Note that both bond prices and CDS spreads are actually expressed by the product of the expected loss  $(1 - R)$  and the default probability  $Prob$ . It turns out that it is hard to identify both the recovery rate and the default probability from only one data source, either bond prices or CDS spreads. This may pose a problem for some applications. For example, if market participants think the recovery rate was 50%, but 40% was put in the formulas for calculating the upfront fee for CDSs, then the hazard rate or the default probability must move down to balance the larger loss incurred in the event of default. The difference between the expected recovery rate and the assumed constant recovery rate will result in a cheaper upfront fee, and therefore affect the conversion.

Fortunately, for our purpose of obtaining the fundamental value of a corporate bond using CDS spreads, this does not cause any problem. Figure 4.1 shows the price of a generic bond with 5-year maturity for varying recovery rates (and thus varying default probabilities). The bond price increases very slowly when the recovery rate is below 70%. As long as the recovery rate is kept between 10% and 70%, the differences between estimated bond yields are essentially within 10 bps. Therefore, the defaultable bond price is relatively insensitive to the assumed recovery rate. Therefore, following the market convention, the recovery rate for senior unsecured corporate bonds is assumed to be 40%, as well as for senior CDS contracts.



**Figure 4.1:** Sensitivity of Bond Prices to the Recovery Rate

A reduced form model with piecewise constant hazard rate term structure is used to price corporate bonds under the assumption of constant recovery rate.



### 4.3.3 Tenor Effects in the Term Structure

Since the beginning of the financial crisis in August 2007, the money markets have exhibited an unprecedented behaviour. Interest rates, like for instance swap rates with the same tenor but based on different floating-leg frequencies, that previously had been following each other closely for a long time, have started to keep a significant distance apart. This divergence in values does not create arbitrage opportunities when credit or liquidity issues are taken into account, which means that the market is not pricing with a unique term structure any more. The assumption of a unique discounting curve has been forsaken by practitioners, who seem to agree on an empirical approach based on the construction of as many curves as possible rate tenors. For each given contract, they select a specific discount curve, which they use to calculate the net present value (NPV) of the contract's future payments, consistently with the contract's features such as tenors or counterparties in question.<sup>1</sup>

All corporate bonds in our sample pay semi-annual coupons. Therefore, tenor effects will not be a concern for our applications, as long as we keep the same practice of building discounting curves for all bonds. To construct a riskless discount curve, we download the Constant Maturity Treasury rates of maturities 0.5-year, 1-year, 2-year, 3-year, 5-year, 7-year and 10-year from Federal Reserve website. We first interpolate these par rates at semiannual intervals by using cubic spline. Then spot rates are obtained by bootstrapping these par rates at semiannual intervals. A whole discount curve is provided by linearly interpolating the corresponding forward curve.

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<sup>1</sup>The valuation of interest rate derivatives under different curves, for generating future rates and for discounting has received a lot of attention in the financial literature recently. Previous works that started to deal with these issues, mainly concerning the valuation of cross currency swaps, are [Boenkost & Schmidt \(2005\)](#), [Kijima \*et al.\* \(2008\)](#) and [Henrard \(2007\)](#). [Bianchetti \(2010\)](#), was the first to apply the methodology to the single currency case, while [Chibane & Sheldon \(2009\)](#) propose methods to extend yield curve bootstrapping to a multi-curve setting. In terms of new pricing models, [Kijima \*et al.\* \(2008\)](#), [Mercurio \(2010a\)](#) and [Mercurio \(2010b\)](#) apply the method using a stochastic volatility LMM, in order to price calets and swaptions, while [Moreni & Pallavicini \(2010\)](#), apply the new framework under the HJM model.

#### 4.3.4 Computing Non-default Price Residuals of a Corporate Bond

To compute the non-default price residuals of a corporate bond, we use the following procedure. Firstly, a riskless discount function  $D(0, \cdot)$  is needed for each trading date. The constant maturity 6-month, 1-year, up to 10-year Treasury rates are downloaded from the Federal Reserve website. Then these par rates are interpolated at semiannual intervals by using a cubic spline algorithm. A discount curve at semiannual intervals can be obtained by bootstrapping the interpolated par rates. A linear interpolation of the corresponding forward rates is applied to provide the value of  $D(0, \cdot)$  at other maturities.

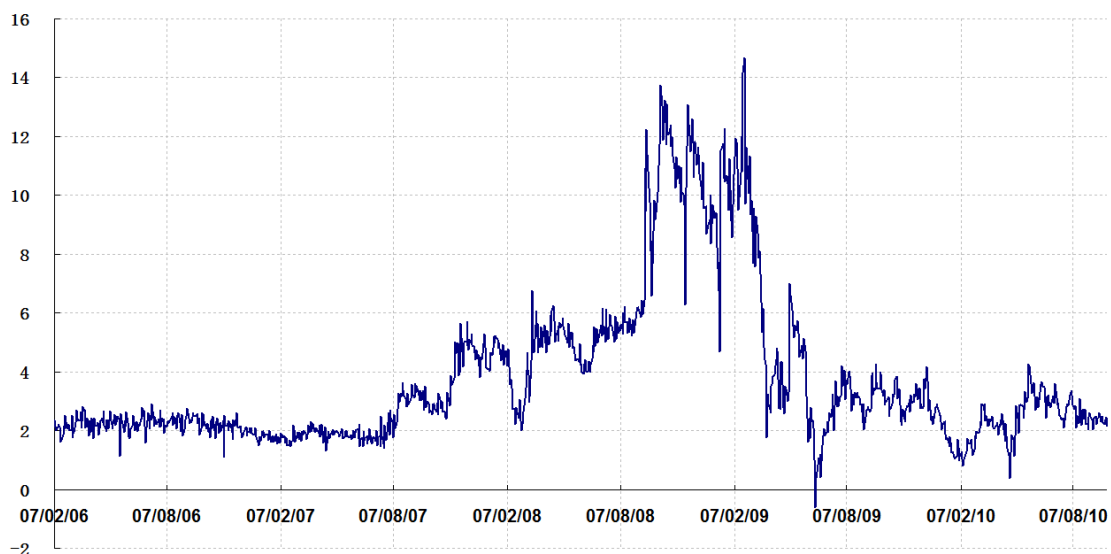
Secondly, given the discount curve and the mid-quotes of the CDS contracts covering from 6-month to 10-year maturities, on each trading day a term structure of hazard rates can be bootstrapped by using Equation 4.7 under the assumption of a piecewise constant hazard rate term structure. Then, a survival probability function  $Surv(\cdot)$  is obtained by using Equation 4.2.

Thirdly, given the discount curve  $D(0, \cdot)$  and the survival probability function  $Surv(\cdot)$ , a ‘fundamental price’ of the corresponding corporate bond can be computed by using Equation 4.8. In other words, the ‘fundamental price’ of a corporate bond is equal to the value of a hypothetical liquid defaultable bond with the same maturity date and coupon rate.

Finally, subtracting the market-based transaction prices from the model-based ‘fundamental prices’ gives the non-default price residuals of a corporate bond. This procedure is similar to the one implemented in Longstaff *et al.* (2005). However, our approach does not depend on a parametric credit risk model, and is able to generate very flexible shapes of hazard rate term structures. It is important to notice that there is no regression involved in the procedure of computing the non-default price residuals.

### 4.3.5 Descriptive Statistics for Non-default Price Residuals

Figure 4.2 plots the time series of the cross-sectional average of the non-default price residuals. Historically, the average non-default price residual has been around \$2 (for the period from 02/2006 to 08/2007). However, in just over a month, it rose to \$3.549 on 14/09/2007, when the Bank of England announced emergency funding to rescue the troubled Northern Rock, one of the UK's largest mortgage lenders. The non-default price residual reached its all time high at \$5.691 on 04/12/2007. Around the same time, large investment banks such as UBS and Lehman Brothers announced huge write downs. On 18/03/2008 the collapse of Bear Sterns led to a price drop of \$6.760. In the latest illiquidity wave following the failure of Lehman Brothers, the non-default price residual was \$13.793 (as of 10/10/2008).



**Figure 4.2:** Time-series Plot of the Average non-default Price Residuals

The plot shows the time series of the non-default price residuals averaged over bonds.

On 23/02/2009, Citigroup announced that the United States government

would take a 36% equity stake in the company by converting \$25 billion in emergency aid into common shares with a US Treasury credit line of \$45 billion to prevent the bankruptcy of the largest bank in the world at the time. The government would also guarantee losses on more than \$300 billion troubled assets and inject \$20 billion immediately into the company. The announcement pushed the price non-default residual to the highest point at \$14.663. The price drops may be triggered by events which may affect the liquidity of the market.

Table 4.2 furnishes descriptive statistics of the non-default price residuals of the individual bonds in our sample. We observe clear variations in cross sectional as well as time series dimensions. For the whole sample period, the time series averages of the non-default price residuals are in the range of \$1.198 to \$8.168. Historically, the difference between the smallest and the biggest averages of the non-default price residuals in Period I was \$2.536. This figure increased to \$5.412 in Period II reaching a historical record high of \$10.241 in Period III before decreasing to \$7.332 post the crisis. This may imply that, not only there exist heterogenous levels of illiquidity among those bond, but also the extant of the divergence is varying over time. It would be interesting to look at what cause these variations across the bonds as well as over time.

### 4.3.6 A Generalized Roll Model

As non-default price residuals are calculated by subtracting transaction prices from model-based fundamental values, two properties can be derived from non-default price residuals. First, illiquidity gives rise to transitory components in transaction prices, which are orthogonal to fundamental values. That means that the transitory components can be estimated from the non-default price residuals instead. Second, there might be non-fundamental risk premia embedded in bond yield spreads, such as liquidity premia which are the return compensation for illiquidity.

Therefore, let us assume that the non-default price residual  $p_t$  consists of two components:

$$p_t = m_t + \theta_t, \tag{4.12}$$

**Table 4.2:** Descriptive Statistics for Non-default Price Residuals.

This table reports descriptive statistics of the non-default price residuals of our sample bonds. The non-default price residuals are computed as the difference between model-based ‘fair values’ and market-based transaction prices. For each bond, we calculate the time series mean, median and standard deviation for the whole period and sub-periods. The whole sample period (February 7, 2006 to September 30, 2010) is divided into four sub-sample periods. Period I is spanning from February 7, 2006 to July 31, 2007; Period II is from August 1, 2007 to July 31, 2008; Period III is from August 1, 2008 to July 31, 2009; Period IV is from August 3, 2009 to September 30, 2010.

Bond	Period I			Period II			Period III			Period IV			Whole Period		
	mean	med	std	mean	med	std	mean	med	std	mean	med	std	mean	med	std
bond01	1.921	1.917	0.623	3.524	3.524	1.007	4.321	4.226	3.751	0.267	0.193	0.541	2.362	1.960	2.361
bond02	3.613	3.846	0.847	6.301	6.306	2.083	11.743	11.369	5.381	3.837	3.842	1.672	5.943	4.316	4.222
bond03	2.174	2.200	0.517	4.046	4.126	1.266	6.626	6.314	4.601	1.086	1.034	0.843	3.251	2.369	3.026
bond04	3.096	3.196	0.743	5.029	5.048	1.795	10.537	10.176	6.079	3.185	3.183	1.139	5.084	3.610	4.159
bond05	1.161	1.255	0.661	1.942	1.883	0.740	2.570	2.502	3.375				1.452	1.310	1.927
bond06	1.076	1.103	0.481	1.773	1.625	0.863	1.502	1.636	2.675				1.198	1.215	1.545
bond07	1.526	1.623	0.458	3.031	3.024	0.988	4.269	3.706	4.031	0.327	0.284	0.593	2.149	1.661	2.437
bond08				3.152	3.055	1.090	5.003	4.257	4.375	0.373	0.345	0.672	2.474	1.749	2.897
bond09				3.242	3.195	1.066	6.194	5.608	4.482	0.916	0.873	0.827	3.092	2.026	3.255
bond10				7.185	7.212	1.667	11.101	10.881	4.336	5.020	4.707	1.750	7.565	6.720	3.761
bond11							6.582	6.352	4.407	1.258	1.202	0.955	3.792	2.842	3.491
bond12							7.263	7.462	4.293	1.556	1.548	0.904	4.148	3.057	3.583
bond13							11.228	10.719	4.687	5.714	5.782	1.756	8.168	7.446	3.878
bond14							7.159	6.532	3.834	2.484	2.457	1.080	4.681	3.585	3.354
bond15							10.714	10.749	4.009	5.751	5.762	1.883	8.142	7.239	3.748
bond16							7.493	7.085	4.274	2.849	2.758	1.128	4.949	3.473	3.780
bond17										7.600	7.848	2.124	7.761	7.919	2.228

The first component  $m_t$  represents the permanent price effect, reflecting the unanticipated information received by market participants, including the information about the future illiquidity of the bonds. The permanent component is assumed to follow a random walk process:

$$m_t = m_{t-1} + \epsilon_t, \quad (4.13)$$

where  $\epsilon_t$  describes changes in prices due to new information.

The second component  $\theta_t$  comes from costs of illiquidity, which is transitory and uncorrelated with the fundamental value. The absolute value of the transitory component  $\theta_t$  is essentially our illiquidity measure, which characterizes the degree of current illiquidity in the market.

Now, let us assume that the transitory component  $\theta_t$  is equal to a serially uncorrelated disturbance,  $\nu_t$ , with mean zero and variance  $H_t$ , and the innovation term of the permanent component  $\epsilon_t$  has mean zero and variance  $Q_t$ . However, in our application both  $\nu_t$  and  $\epsilon_t$  are i.i.d. normally distributed, and both  $H$  and  $Q$  are independent of time.

The estimation of the state space model is carried out via a Kalman filter approach by numerically maximizing the relevant log-likelihood function. Furthermore, we assume  $\nu_t \sim N(0, H)$  with  $H = \exp(c_1)$  and  $\epsilon_t \sim N(0, Q)$  with  $Q = \exp(c_2)$ . The underlying volatility is computed as  $\sqrt{Q}$ . The model is estimated respectively for each bond and each sub-period.

The estimates of the model parameters  $c_1$  and  $c_2$  are reported in Table 4.3. The results show that the cross-sectional average of the underlying volatility was 0.047 prior to the crisis.<sup>1</sup> It jumped to 0.266 in Period II and then reached the highest level at 0.768 in Period III, before coming back to 0.247 post the crisis period. The obvious structural breaks in the data force us to estimate the model period by period.

We can also observe some degree of contemporaneous heteroscedasticity across different bonds. Figure 4.3 plots the underlying volatilities of the 5-year and 10-year bonds, respectively, against their corresponding maturity dates in four

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<sup>1</sup>It is important to notice that these volatilities are those of the unobservable non-default price discounts, as the analysis is focusing on non-default bond residuals.

**Table 4.3:** Estimates of  $c_1$  and  $c_2$ , state space model.

This table reports the Kalman filter estimates of the coefficients of the state-space model. The model is estimated separately for each bond and the indicated sub-period. Underlying volatility is computed as  $\sqrt{Q}$ . Avg vol stands for the volatility averaged over the bonds for the indicated sub-period. The standard errors are reported in brackets. The superscript \* denotes insignificance at the 5% level; the superscript \*\* denotes insignificance at the 10% level. The whole sample period (February 7, 2006 to September 30, 2010) is divided into four sub-periods. Period I is spanning from February 7, 2006 to July 31, 2007; Period II is from August 1, 2007 to July 31, 2008; Period III is from August 1, 2008 to July 31, 2009; Period IV is from August 3, 2009 to September 30, 2010.

$$p_t = m_t + \nu_t, \nu_t \sim N(0, H), H = \exp(c_1),$$

$$m_t = m_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q), Q = \exp(c_2).$$

Bond	Period I		Period II		Period III		Period IV	
	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_1$	$\hat{c}_2$
bond01	-1.448(0.031)	-7.598(0.886)	-0.769(0.092)	-3.105(0.281)	0.593(0.097)	-0.147(0.175)**	-1.657(0.060)	-5.725(0.378)
bond02	-0.537(0.074)	-4.069(0.314)	-0.605(0.101)	-1.354(0.203)	2.311(0.071)	-0.496(0.360)**	0.445(0.082)	-2.815(0.415)
bond03	-1.615(0.047)	-6.863(0.609)	-0.656(0.095)	-2.818(0.260)	1.686(0.067)	-0.951(0.305)	-1.341(0.088)	-3.173(0.254)
bond04	-0.827(0.059)	-5.319(0.467)	-0.229(0.104)	-1.811(0.287)	2.299(0.050)	-0.169(0.207)**	-0.549(0.103)	-2.438(0.285)
bond05	-0.985(0.029)	-7.024(1.072)	-1.314(0.104)	-3.717(0.329)	1.030(0.042)	-1.037(0.185)		
bond06	-1.581(0.026)	-7.819(0.982)	-0.555(0.040)	-4.217(0.570)	0.112(0.074)**	-1.090(0.162)		
bond07	-1.662(0.048)	-7.771(0.728)	-0.811(0.065)	-3.185(0.295)	1.229(0.058)	-0.544(0.218)	-1.376(0.073)	-5.838(0.510)
bond08			-0.886(0.103)	-3.027(0.278)	1.406(0.061)	-0.829(0.288)	-1.609(0.079)	-4.175(0.322)
bond09			-0.751(0.074)	-3.211(0.274)	0.884(0.071)	0.105(0.171)**	-1.271(0.093)	-3.223(0.222)
bond10			-0.175(0.098)	-1.872(0.258)*	1.621(0.083)	-0.261(0.281)**	0.191(0.091)	-2.709(0.091)
bond11					1.069(0.086)	-0.893(0.234)	-0.840(0.077)	-3.180(0.293)
bond12					0.889(0.085)	-0.759(0.242)	-1.072(0.078)	-3.458(0.291)
bond13					1.916(0.038)	-0.380(0.176)	-0.422(0.120)	-1.279(0.217)
bond14					0.361(0.091)	-0.735(0.224)	-1.154(0.098)	-2.956(0.257)
bond15					0.454(0.150)	-0.035(0.231)**	0.065(0.098)*	-1.853(0.267)
bond16					0.662(0.096)	-0.747(0.245)	-0.572(0.066)	-3.057(0.320)
bond17							-0.759(0.125)	-1.336(0.216)
Avg vol		0.047		0.266		0.768		0.247

sub-periods, separately. As we can see, the volatility curves are generally linear and positively sloped, indicating that uncertainty about future liquidity increases in the time-to-maturity. In Period III and IV, there are a few outliers, which are the bonds with small or medium issue sizes, indicating that issue sizes might be the possible omitted variable.

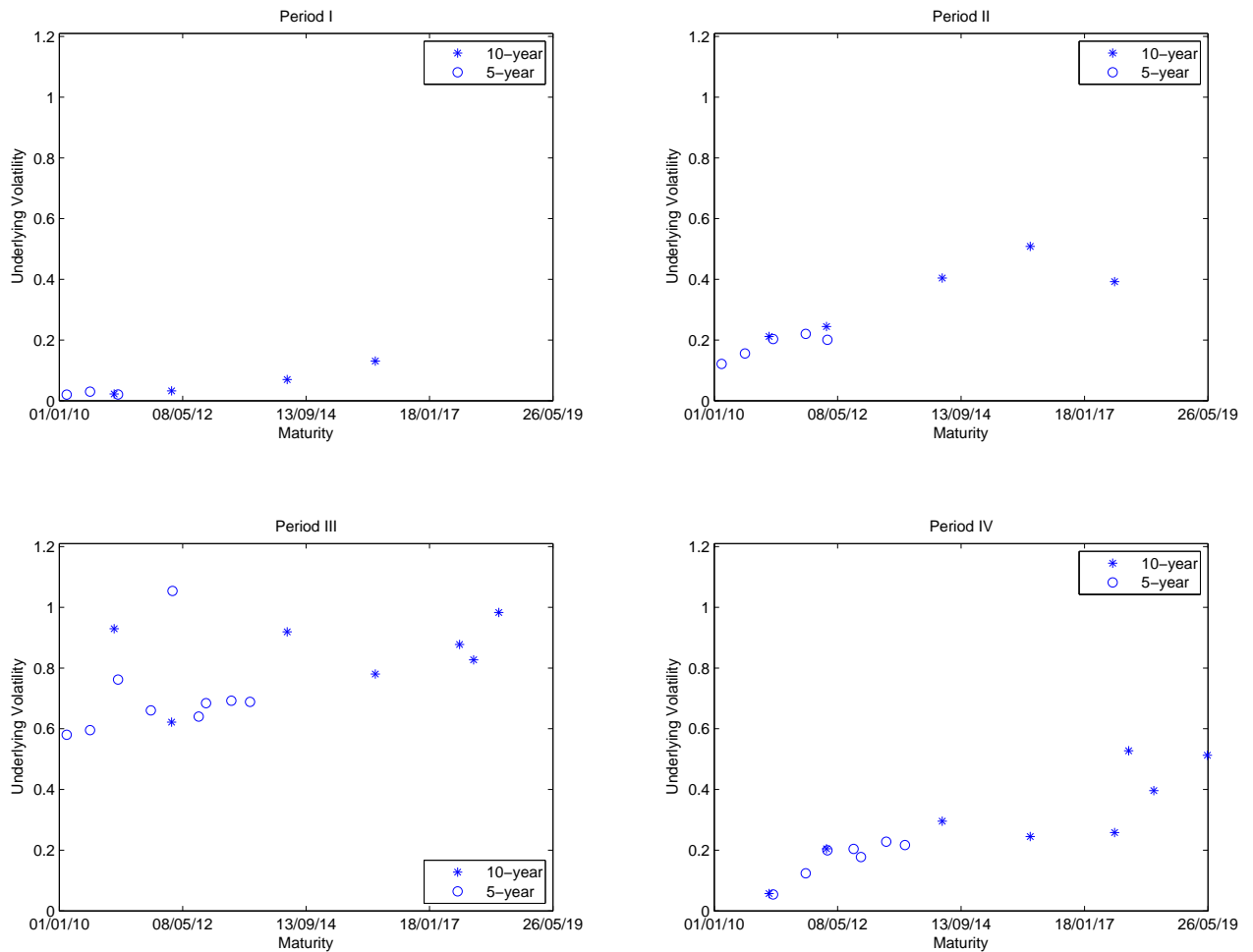
In addition, we check whether the estimated unobservable non-default price residuals  $m_t$  are integrated of order one as the model assumes. In order to determine the order of integration of  $m_t$  for each bond in each sub-period, we use Augmented Dickey-Fuller (ADF) tests with Mackinnon's critical values. Table 4.4 reports the results (t-statistics) of the tests of the null hypothesis of a unit root in  $m_t$ . If the t-statistics differ significantly from zero, the null hypothesis of existence of a unit root is reject and stationarity is concluded. We test for unit roots in  $m_t$  of each bond in each sub-period in terms of levels and first differences. The ADF tests indicate that for almost all the series of  $m_t$  the null hypothesis of existence of a unit root cannot be rejected.

### 4.3.7 Preliminary Analysis on Unobservable Non-default Yield Spread and Illiquidity Measure

Now, it is time to turn to the elements which we are really interested in, namely, the unobservable non-default price discount  $m_t$  (or in terms of yield) and the illiquidity measure  $s_t$ . Table 4.5 reports the means and the standard deviations of the estimated unobservable non-default price discounts  $m_t$  for each bond in the indicated sub-periods. The cross-sectional averages of the means of  $m_t$  rose from a level of about \$2 in Period I to about \$4 in Period II. In Period III, the cross-sectional average increased dramatically to more than \$7. Similar to the volatility, post the crisis the unobservable non-default price discounts came back to about the same level prior to the crisis. The results suggest that there exist common processes that are driving the non-default components across the bonds. Cross-sectionally, the time series averages of  $m_t$  in Period I ranged from \$3.605 to \$1.077, whereas in Period II, III, and IV they ranged from \$7.018 to \$1.761, from \$11.7657 to \$1.505, and \$7.605 to \$0.276. In particular, the difference between



### 4.3 The Methodology



**Figure 4.3:** Cross-sectional Plots of Underlying Volatility against Maturity Date

This figure plots the average underlying volatilities of the 5-year and 10-year bonds against their maturity dates, respectively. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond. Issue size might be the possible omitted variable.

**Table 4.4:** Augmented Dickey-Fuller Tests for Unit Roots in Filtered Unobservable Non-default Price Discount  $m_t$ .

This table reports the results (t-statistics) of tests of the null hypothesis of a unit root in the filtered unobservable non-default price discount  $m_t$ . The results are presented separately for each bond and the indicated sub-period.  $D(\cdot)$  represents the first difference operation. 1% Level represents the critical value at 1% significance level. The whole sample period (February 7, 2006 to September 30, 2010) is divided into four sub-periods. Period I is spanning from February 7, 2006 to July 31, 2007; Period II is from August 1, 2007 to July 31, 2008; Period III is from August 1, 2008 to July 31, 2009; Period IV is from August 3, 2009 to September 30, 2010.

$$p_t = m_t + \nu_t, \nu_t \sim N(0, H), H = \exp(c_1),$$

$$m_t = m_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q), Q = \exp(c_2).$$

Bond	Period I		Period II		Period III		Period IV	
	$m_t$	$D(m_t)$	$m_t$	$D(m_t)$	$m_t$	$D(m_t)$	$m_t$	$D(m_t)$
bond01	-2.403	-18.700	-2.465	-15.485	-3.084	-15.349	-4.313	-19.823
bond02	-3.448	-18.573	-2.588	-15.420	-1.788	-16.225	-4.178	-17.540
bond03	-1.510	-19.727	-2.220	-15.510	-1.803	-9.922	-3.104	-17.189
bond04	-2.184	-20.751	-2.601	-15.155	-2.053	-14.932	-3.117	-15.601
bond05	-1.651	-18.630	-3.583	-14.984	-2.395	-15.538		
bond06	-2.632	-20.037	-2.334	-16.016	-2.772	-16.776		
bond07	-3.304	-21.415	-2.288	-15.334	-2.578	-14.190	-4.191	-21.246
bond08			-2.807	-15.048	-1.926	-9.460	-2.546	-17.722
bond09			-2.510	-14.484	-2.545	-14.476	-2.911	-17.483
bond10			-2.850	-15.011	-2.184	-16.081	-2.363	-16.123
bond11					-2.332	-14.710	-2.554	-18.530
bond12					-2.141	-14.964	-3.063	-17.668
bond13					-2.289	-14.784	-3.625	-15.827
bond14					-2.548	-15.570	-2.257	-16.185
bond15					-2.497	-15.146	-3.095	-16.643
bond16					-2.169	-15.585	-2.440	-17.191
bond17							-2.650	-16.112
1% Level		-3.448		-3.456		-3.457		-3.453

the lowest and the highest averages was \$10.26 in Period III, compared to \$2.528 in Period I. The results clearly indicate a significant level of heterogeneity in  $m_t$ .

Figure 4.4 plots the time series averages of the unobservable non-default price discounts of the 5-year and 10-year bonds respectively against their corresponding maturity dates in four sub-periods. As expected, the averages increase in time-to-maturities. Therefore, it is necessary to convert price discounts (in terms of price) into yield spreads and check whether maturity effects still remain in yield spreads. Figure 4.5 plots the time series averages of the non-default yield spreads of the 5-year and 10-year bonds respectively against their corresponding maturity dates in four sub-periods. The term structures of non-default yield spreads show a variety of shapes. In Period I, II, and IV, the term structures, which may measure future cumulative liquidity costs, were slightly upward sloping.

The most interesting finding, by far, is that in Period III the non-default yield spreads for both the short-term and long-term bonds clearly had humped-shape term structures. The term structures peaked half way around two and half years and five years for the short-term and long-term bonds respectively. This result is seemingly opposite to the findings in previous studies. For instance, [Ericsson & Renault \(2006\)](#) find a decreasing and convex term structure of liquidity spreads, and [Dick-Nielsen et al. \(2012\)](#) find that the liquidity premium increases as the maturity increases. The seasoning effect that we find here is very likely to be related to illiquidity. It may be explained by the argument that, it would be very unlikely that traders can buy their desired bonds just on the day of issuance, as it takes time and human capital for an investor to find a suitable investment and then act on it. Similarly, when it is close to maturity, investors may choose to cash out prior to maturity, as planned, due to imperfect matching of maturities. This effect might be amplified during the crisis period (e.g. Period III) due to fight-to-liquidity, as [Beber et al. \(2009\)](#) find that in the times of market distress investors chase liquidity, although they examine government bond markets.

Table 4.6 reports the means and standard deviations of the illiquidity measure  $s_t$  for each bond in the indicated sub-period. The cross-sectional averages of the means of  $s_t$  increased from \$0.321 in Period I to \$0.5 in Period II, and then jumped to \$1.052 in Period III before falling back to \$0.448 in Period IV. This implies that the liquidity of the bonds got worse and worse as the crisis unfolded,

**Table 4.5:** Summary Statistics of Filtered Unobservable Non-default Price Discount  $m_t$ .

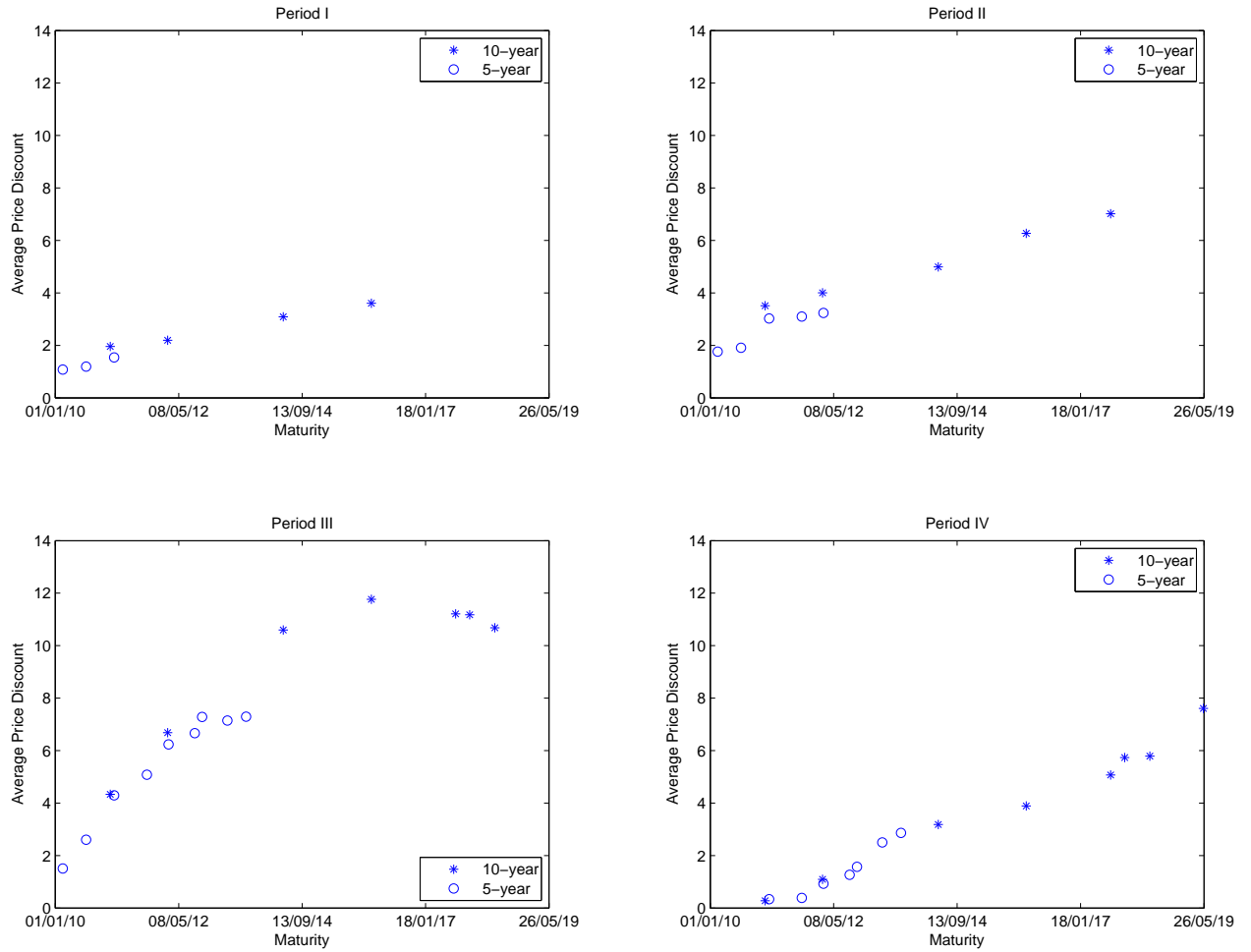
This table reports the means and standard deviations of the filtered unobservable non-default price discount  $m_t$ . The model is estimated separately for each bond and the indicated sub-period. Average stands for cross-sectional average. The whole sample period (February 7, 2006 to September 30, 2010) is divided into four sub-periods. Period I is spanning from February 7, 2006 to July 31, 2007; Period II is from August 1, 2007 to July 31, 2008; Period III is from August 1, 2008 to July 31, 2009; Period IV is from August 3, 2009 to September 30, 2010.

$$p_t = m_t + \nu_t, \nu_t \sim N(0, H), H = \exp(c_1),$$

$$m_t = m_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q), Q = \exp(c_2).$$

Bond	Period I		Period II		Period III		Period IV	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
bond01	1.961	0.198	3.513	0.754	4.331	3.485	0.276	0.319
bond02	3.605	0.359	6.272	1.573	11.765	4.286	3.892	1.204
bond03	2.195	0.258	4.006	1.055	6.681	3.922	1.095	0.664
bond04	3.090	0.340	4.992	1.405	10.594	5.129	3.184	0.849
bond05	1.194	0.257	1.910	0.562	2.601	2.902		
bond06	1.077	0.164	1.761	0.422	1.505	2.456		
bond07	1.541	0.140	3.027	0.732	4.294	3.558	0.338	0.315
bond08			3.100	0.916	5.079	3.833	0.384	0.499
bond09			3.235	0.816	6.234	4.190	0.925	0.633
bond10			7.018	1.682	11.208	3.699	5.072	1.424
bond11					6.662	4.008	1.267	0.693
bond12					7.280	3.975	1.570	0.691
bond13					11.173	3.952	5.732	1.581
bond14					7.146	3.642	2.499	0.916
bond15					10.682	3.831	5.788	1.618
bond16					7.290	4.153	2.865	0.834
bond17							7.605	2.028
Average	2.095	0.245	3.883	0.992	7.158	3.814	2.833	0.951

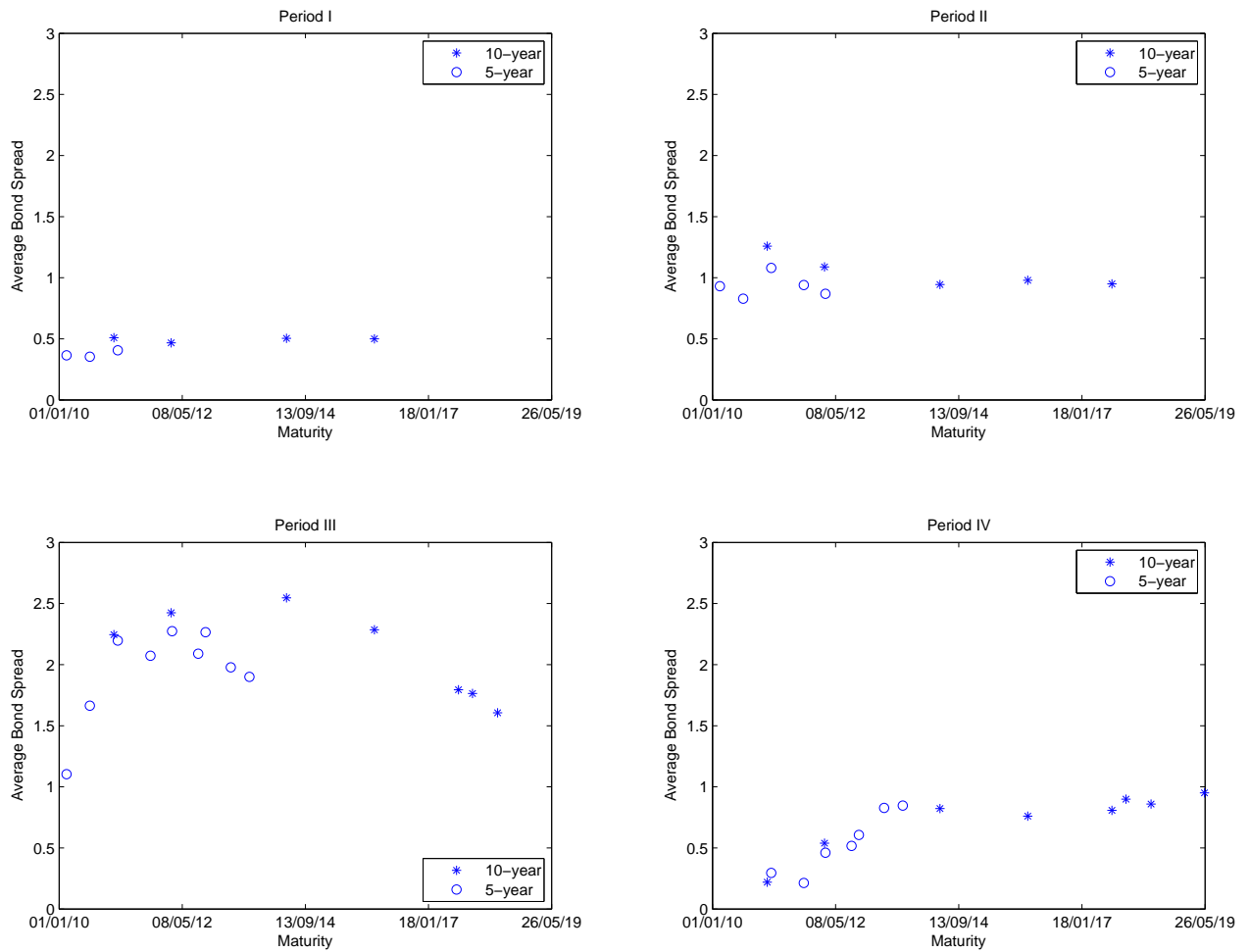
### 4.3 The Methodology



**Figure 4.4:** Cross-sectional Plots of Time Series Averaged Unobservable Non-default Price Discount against Maturity Date

This figure plots time series averaged filtered unobservable non-default price discounts of 5-year and 10-year bonds against their maturity dates. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond.

### 4.3 The Methodology



**Figure 4.5:** Cross-sectional Plots of Time Series Averaged Unobservable Non-default Yield Spread against Maturity Date

This figure plots time series averaged unobservable non-default yield spreads of 5-year and 10-year bonds respectively against their maturity dates. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond. Issue size might be the possible omitted variable.

and recovered post the crisis. The liquidity conditions of the individual bonds were quite different, as the standard deviations of the means of  $s_t$  in Period III was 0.444, compared to 0.178 in Period IV.

Figure 4.6 plots the time series averages of  $s_t$  of the 5-year and 10-year bonds respectively against their corresponding maturity dates for each bond and the indicated sub-period. The illiquidity measure exhibited obvious seasoning effects in all sub-periods except Period I where there were fewer observations. In particular, in Period III  $s_t$  was around four times as big for the most illiquid bond compared to the most liquid bond. Our finding that seasoned bonds are more illiquid than newly issued bonds and bonds close to maturity is consistent with Schultz (2001) and Edwards & Piwowar (2007). Due to a flight-to-liquidity effect, the plots in Period III had more curvature than those in other sub-periods, as investors rebalance their portfolios towards more liquid securities. The volume data in Figure 4.7 support that trading is more active just after the issuance as well as before maturity.

Finally, as we want to explain non-default yield spreads as liquidity premia, it is important to check their relation with the illiquidity measure. Figure 4.8 plots the time series averages of the unobservable non-default yield spreads of the 5-year and 10-year bonds respectively against the time series averages of the illiquidity measures for each bond and the indicated sub-period. The plots show weak signs of a positive relationship between the non-default yield spread and the illiquidity measure. However, this may result from omitting relevant variables, such as issue sizes or time-to-maturities. We will further examine this relationship in the next section.

We are also interested in the time series behaviour of both the default and non-default components. Figure 4.9 gives a time series plot of both the daily default component and the cross-sectionally averaged unobservable non-default component in yield spreads. Before 26 July 2007, the non-default spread was always above the default spread, and accounted for about 80% of the overall spread. During the crisis (July 2007 - June 2009), the non-default component accounted for about 48% of the overall spread on average. Its proportion dropped to around 23% after the crisis (after June 2009). Between July 2007 and August 2008, there were four big jumps in both the default and non-default components,

**Table 4.6:** Summary Statistics of Estimated Half Bid-ask Spread  $s_t$ .

This table reports the means and standard deviations of the estimated half bid-ask spread. The half bid-ask spread  $s_t$  is computed as the absolute value of  $\theta_t$ . The model is estimated separately for each bond and the indicated sub-period. Average stands for cross-sectional average. The whole sample period (February 7, 2006 to September 30, 2010) is divided into four sub-periods. Period I is spanning from February 7, 2006 to July 31, 2007; Period II is from August 1, 2007 to July 31, 2008; Period III is from August 1, 2008 to July 31, 2009; Period IV is from August 3, 2009 to September 30, 2010.

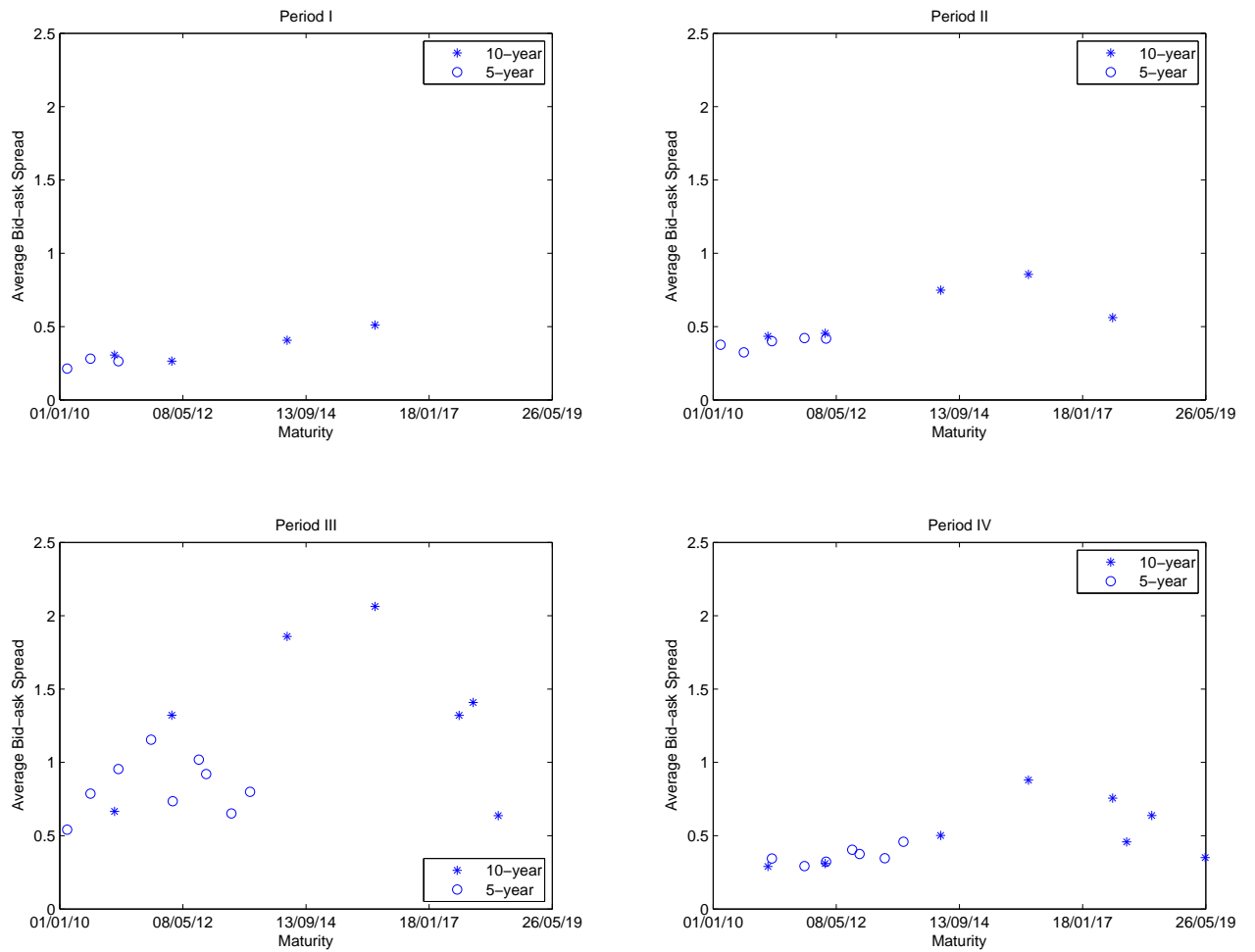
$$p_t = m_t + \nu_t, \nu_t \sim N(0, H), H = \exp(c_1),$$

$$m_t = m_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q), Q = \exp(c_2).$$

Bond	Period I			Period II			Period III			Period IV		
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
bond01	0.306	0.363	0.434	0.390	0.666	0.693	0.290	0.285				
bond02	0.510	0.481	0.856	0.729	2.063	1.913	0.879	0.709				
bond03	0.265	0.338	0.454	0.405	1.322	1.554	0.310	0.281				
bond04	0.407	0.477	0.749	0.567	1.858	2.009	0.501	0.375				
bond05	0.281	0.525	0.324	0.303	0.787	1.165						
bond06	0.214	0.386	0.376	0.591	0.541	0.601						
bond07	0.263	0.332	0.401	0.409	0.955	1.172	0.344	0.324				
bond08			0.421	0.338	1.155	1.276	0.293	0.258				
bond09			0.419	0.417	0.735	0.843	0.322	0.296				
bond10			0.562	0.482	1.321	1.290	0.756	0.624				
bond11					1.018	0.979	0.404	0.391				
bond12					0.920	0.857	0.376	0.328				
bond13					1.408	1.732	0.458	0.369				
bond14					0.652	0.625	0.346	0.301				
bond15					0.637	0.575	0.638	0.569				
bond16					0.799	0.775	0.458	0.461				
bond17							0.350	0.319				
Average	0.321	0.414	0.500	0.463	1.052	1.129	0.448	0.393				



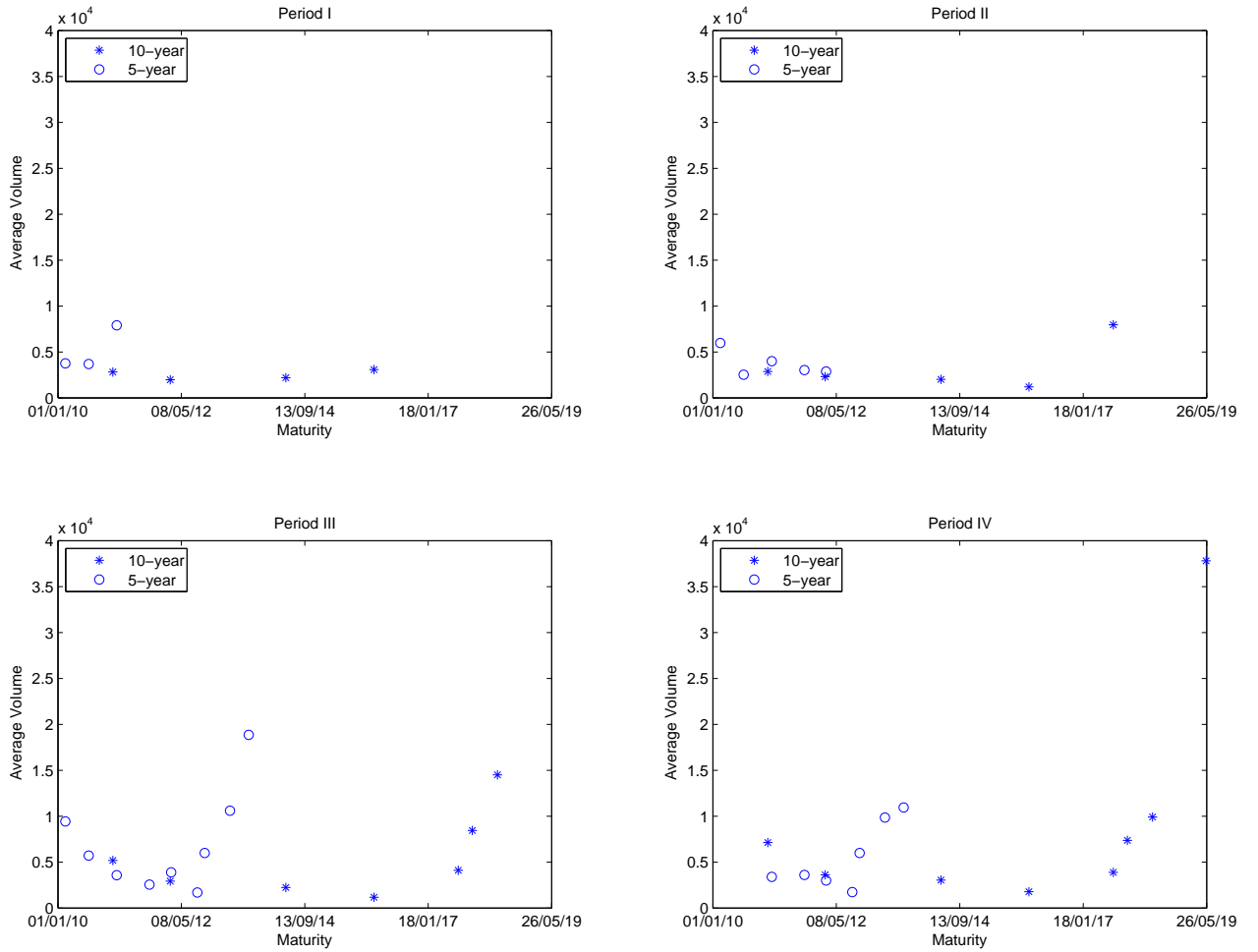
### 4.3 The Methodology



**Figure 4.6:** Cross-sectional Plots of Time Series Averaged Half Bid-ask Spread against Maturity Date

This figure plots time series averaged half bid-ask spreads of 5-year and 10-year bonds respectively against their maturity dates. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond. Issue size might be the possible omitted variables.

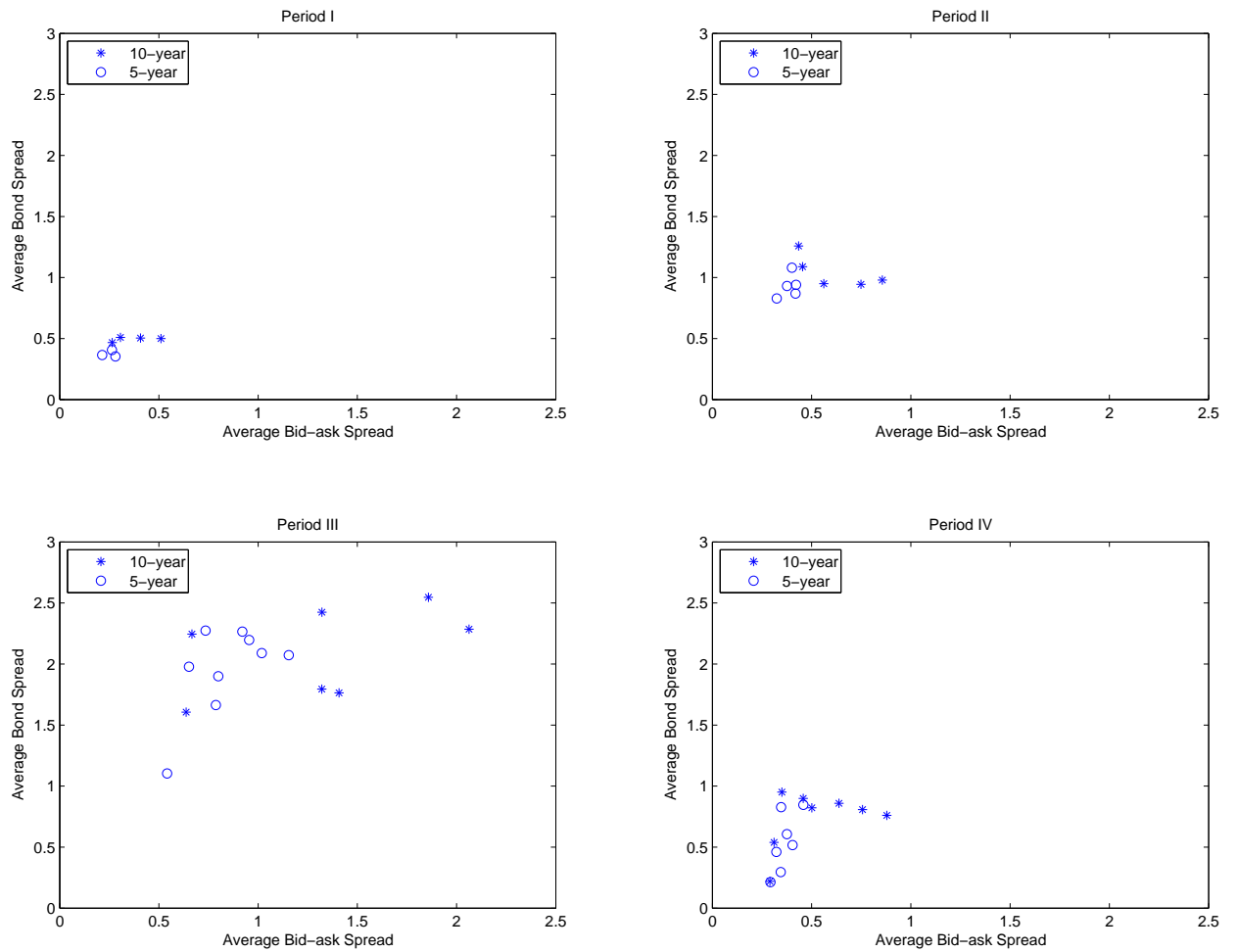
### 4.3 The Methodology



**Figure 4.7:** Cross-sectional Plots of Time Series Averaged Trading Volume against Maturity Date

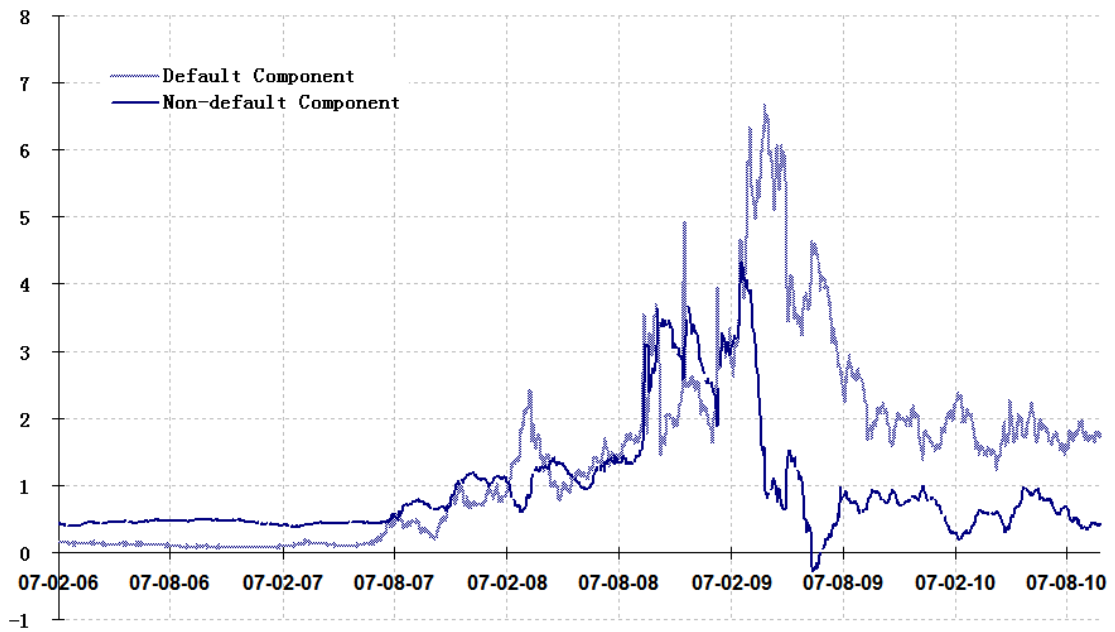
This figure plots time series averaged trading volume of 5-year and 10-year bonds respectively against their maturity dates. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond.

### 4.3 The Methodology



**Figure 4.8:** Cross-sectional Plots of Time Series Averaged Unobservable Non-default Yield Spread against Time Series Averaged Half Bid-ask Spread

This figure plots time series averaged unobservable non-default yield spreads of 5-year and 10-year bonds respectively against time series averaged half bid-ask spreads. \* 10-year denotes 10-year bond; o 5-year denotes 5-year bond.



**Figure 4.9:** Time Series Plot of Default Component and Cross-sectional Averaged Unobservable Non-default Component in Yield Spreads

This figure gives a time series plot of both the daily default component and the cross-sectional averaged unobservable non-default component in yield spreads.

and each jump in the default component occurred ahead of each jump in the non-default component. Between September 2008 and February 2009, both the default and non-default components had sharp increases four times. By contrast, they jumped almost at the same time. What is most surprising to us is that in March 2009 the non-default component dropped sharply well before the default component reached its historical highest point.<sup>1</sup> After 2 March 2009 the default component stayed well above the non-default component until the end of our sample period. The non-default component fell below zero for a while during June and July 2009.

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<sup>1</sup>On February 23 2009, Citigroup announced that the United States government would take a 36% equity stake in the company by converting \$25 billion in emergency aid into common shares with a US treasury credit line of \$45 billion.

## 4.4 Regression Results

Our sample consists of a cross-section of bonds, which allows us to examine the connections amongst non-default yield spreads, our illiquidity measure and various liquidity proxies. In particular, this enables us to answer the questions that why bonds issued by a single company exhibit heterogenous levels of liquidity and liquidity premia, how much of the time series variations are in common, and how much are idiosyncratic. Therefore, we combine estimates over the whole sample period to construct daily non-default yield spreads and the illiquidity measure for each bond, and perform panel regressions on various liquidity proxies. In doing this, it would be better to give some intuitive interpretation of the liquidity proxies, their expected relationships with non-default yield spreads and the illiquidity measure, and theoretical or empirical evidence.

### 4.4.1 Liquidity Proxies

Empirical papers, that examine liquidity in bond or equity markets, use both direct measures (based on transaction data), and indirect measures (based on bond characteristics and/or last prices). In our applications we will use both direct and indirect measures, as they may provide different information about bond liquidity.

The first proxy is the time-since-issue or time-to-maturity of a bond. The intuition for these proxies is that traders buy bonds after the offering and sell them shortly thereafter, as they expect that the bonds will get illiquid; while bonds are close to maturity, some traders will start liquidating their holdings as a result of imperfect match of maturities. Therefore, newly issued bonds and bonds close to maturity may be more liquid than seasoned bonds. The difference in the time-since-issue or time-to-maturity across bonds may explain the cross-sectional difference in illiquidity and liquidity premia. Examining the corporate bond market, [Schultz \(2001\)](#) find that newly issued bonds trade more than old bonds. More recently, [Edwards & Piwowar \(2007\)](#) find, using transaction cost estimates, that highly rated bonds, recently issued bonds, and bonds close to maturity have lower transaction costs than do other bonds.

The second proxy is the issue amount of a bond. Intuitively, bonds with smaller issue amount tend to get locked away by long-term investors more easily, reducing the tradable amounts and thus their liquidity. Thus, the larger an issue, the more liquid an bond is. [Alexander \*et al.\* \(2000\)](#) find that bond issues with higher volume tend to be larger and more recently issued. [Bao \*et al.\* \(2011\)](#) establish a robust connection between their illiquidity measure and liquidity-related bond characteristics. Similarly, they find that the illiquidity measure is higher for older and smaller bonds, and bonds with smaller average trade sizes and higher idiosyncratic return volatilities. Not only these papers found strong evidences of a seasoning effect, but also they both suggested that issue amounts could be an indicator of bond liquidity.

Our third proxy is related to inventory holding costs. In microstructure literature, market makers' inventory holding costs are often related to price volatilities. Following [Alexander \*et al.\* \(2000\)](#), we use the average of absolute price returns to approximate the volatility of fundamental prices.

The final three proxies are based on transaction level data and try to measure trading frequencies. They are the daily turnover, the daily average trade size, and the number of trades in a day of a bond. Obviously, all else equal, bonds which trade more frequently will be more liquid. However, as they are constructed from transaction-level data such as trading volumes, they do not tend to be forward-looking, but rather reflect the current condition of market liquidity.

In addition to those measures of illiquidity, it would be interesting to look at whether CDS spreads or the default risk will affect bond liquidity. There are good reasons why they should be correlated. For example, when default probabilities are high, it is plausible to believe that liquidity will be low, as there are fewer buyers in the market willing to hold bonds with high default risk. Empirically, [Ericsson & Renault \(2006\)](#) find evidence of a positive correlation between the illiquidity and default components of bond yield spreads. [Bao \*et al.\* \(2011\)](#) also find that bonds with higher CDS spreads are less liquid. However, it is important to notice that CDS spreads are merely capturing the fundamental default risk, rather than a measure of bond illiquidity.

Table 4.7 gives details on the definition of each liquidity proxy. It also shows the expected relationship between each measure and the dependent variables,

## 4.4 Regression Results

respectively. We also expect some of these relationships to possess some form of nonlinearity.

**Table 4.7:** Illiquidity Proxies and Their Expected Relationships with Non-default Yield Spreads and the Illiquidity Measure.

This figure shows the illiquidity proxies and control variables we will use in the regression analysis and their expected relationships with the dependent variables.

Liquidity measure	Definition	Relationship	
		Non-default Yield Spread	Illiquidity Measure
Time-Since-Issue	Time between issue date and transaction date in years	positive	positive
Time-To-Maturity	Time between maturity date and transaction date in years	positive	positive
Issue Amount	Total notional in millions of dollars	negative	negative
Turnover	The bond's daily trading volume as a percentage of its issue amount	negative	negative
Trade Size	Daily average trade size of the bond in thousands of dollars of face value	negative	negative
Number of Trades	Total number of trades of the bond in a day	negative	negative
Volatility	Absolute value of daily log returns	positive	positive
CDS Spread	The spread on the five-year on-the-run CDS in percentage	positive	positive

Table 4.8 shows the correlations among the dependent and independent variables. As shown, the fundamental volatility and the CDS spread are highly and positively correlated with the illiquidity measure and the non-default yield spread. In particular, there is a high correlation of 41% between the illiquidity measure and the non-default yield spread, indicating that the non-default yield spread could be related to liquidity risk. As expected, the negative exponential of the time-to-maturity is negatively correlated with the illiquidity measure and



the non-default yield spread. The illiquidity measure is generally negatively correlated with the issue size and the trading activity measures except the number of trades.

### 4.4.2 Illiquidity Measure

Table 5.2 reports the panel regression results with the daily illiquidity measure  $s_t$  as the dependent variable.  $t$ -statistics are reported in squared brackets, using the standard errors computed using White robust method. To avoid collinearity, we keep the time-to-maturity and exclude the time-since-issue from the panel regressions.

The regression results show that the estimated coefficients of all proxies have the signs expected. We find a positive relationship between our illiquidity measure and the time-to-maturity. This implies that there may be maturity clienteles for corporate bonds. That is, bonds with longer time-to-maturities are less liquid. The positive relationship between the illiquidity measure and the time-to-maturity is consistent with Bao *et al.* (2011) where they find that their illiquidity measure is higher for bonds with longer time-to-maturities. Moreover, we find that the illiquidity measure is a concave function of the time-to-maturity, indicating that illiquidity increases nonlinearly in time-to-maturities. The results are generally robust regardless of which control variables are used in these regressions, with  $t$ -statistics around -4.19 on average.

Issue size loosely speaking measure the availability of a bond in the market, and thus indirectly measures the illiquidity of a bond, as smaller issues may be more likely subject to buy-and-hold investors, and thus may be less liquid. We find that our illiquidity measure is higher for bonds with smaller issue amounts. Therefore, the liquidity of bonds issued by the same borrower does increase in their issue sizes.

The availability of the transaction-level data enables us to examine the connection between our illiquidity measure and bond trading frequencies. All three trading activity measures are statistically significant at the 95 percent confidence level. We find that our illiquidity measure is higher bonds with smaller trade sizes and lower turnovers. Unlike Bao *et al.* (2011), we find that bonds with a

**Table 4.8:** Correlation Matrix for Dependent and Independent Variables.

This table shows correlations among the dependent and independent variables. *Nondef* is the daily estimated unobservable non-default yield spread  $m_t$  in percent. *Illiq* is the daily estimated illiquidity  $s_t$  in dollars. *Mat* is the time-to-maturity. *Issue* is the issue amount in millions of dollars. *Turnover* is the daily trading volume as a percentage of the issue amount. *Size* is the daily average trade size in thousands of dollars of the face value. *#Trds* is the total number of trades in a day. *Volatility* is the absolute value of daily log returns on fundamental values. *CDS* is the spread on the 5-year on-the-run CDS in percentage. The whole sample period is from February 7, 2006 to September 30, 2010.

	<i>Nondef</i>	<i>Illiq</i>	<i>Exp(-Mat)</i>	<i>Log(Issue)</i>	<i>Turnover</i>	<i>Log(Size)</i>	<i>Log(#Trds)</i>	<i>Volatility</i>	<i>CDS</i>
<i>Nondef</i>	1								
<i>Illiq</i>	0.41	1							
<i>Exp(-Mat)</i>	-0.20	-0.12	1						
<i>Log(Issue)</i>	0.06	-0.05	-0.04	1					
<i>Turnover</i>	-0.03	-0.04	0.04	0.09	1				
<i>Log(Size)</i>	-0.03	-0.09	-0.01	0.28	0.62	1			
<i>Log(#Trds)</i>	0.04	0.02	0.14	0.34	0.32	0.07	1		
<i>Volatility</i>	0.27	0.36	-0.16	0.09	0.07	0.06	0.04	1	
<i>CDS</i>	0.35	0.33	0.18	0.15	0.06	0.01	0.30	0.32	1

## 4.4 Regression Results

**Table 4.9:** Regression of the Illiquidity Measure on Illiquidity and Credit Proxies.

This table reports the panel regressions with the daily illiquidity measure  $s_t$  as the dependent variable.  $t$ -statistics are reported in square brackets using White robust method. The daily illiquidity measure is in dollars.  $Mat$  is the time-to-maturity in years.  $Issue$  is the issue amount in millions of dollars.  $Turnover$  is the daily trading volume as a percentage of the issue amount.  $Size$  is the daily average trade size in thousands of dollars of the face value.  $\#Trds$  is the total number of trades in a day.  $Nondef$  is the daily estimated unobservable non-default yield spread in percentage.  $Volatility$  is the absolute value of daily log returns on the fundamental price in percentage.  $CDS$  is the spread on the 5-year on-the-run CDS in percentage. The whole sample period is from February 7, 2006 to September 30, 2010.

	Coefficients on equations						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
<i>Cons</i>	0.18** [3.20]	1.94** [2.46]	2.03** [3.27]	1.96** [3.19]	1.92** [3.25]	1.92** [2.98]	1.68** [2.65]
<i>Exp(-Mat)</i>		-1.33** [-4.43]	-1.01** [-4.09]	-0.97** [-4.08]	-0.99** [-3.87]	-0.98** [-4.46]	-0.32** [-2.14]
<i>Log(Issue)</i>		-0.23** [-2.15]	-0.25** [-2.97]	-0.23** [-2.87]	-0.21** [-2.72]	-0.22** [-2.53]	-0.22** [-2.57]
<i>Turnover</i>				-7.24** [-6.24]			
<i>Log(Size)</i>					-0.03** [-4.90]		
<i>Log(#Trds)</i>						-0.04** [-2.29]	
<i>Nondef</i>							0.23** [6.12]
<i>Volatility</i>	0.36** [6.53]		0.34** [6.56]	0.34** [6.70]	0.34** [6.74]	0.33** [6.66]	0.28** [6.13]
<i>CDS</i>	0.17** [5.56]	0.24** [7.49]	0.19** [6.53]	0.19** [6.62]	0.19** [6.56]	0.20** [6.57]	0.12** [6.23]
Obs	10206	13186	10196	10183	10184	10195	10196
$R^2$	17.57%	15.12%	20.50%	21.02%	21.24%	20.52%	26.43%

larger number of trades have smaller bid-ask spreads and are more liquid. Alternatively stated, more trades do lead to more liquidity. However, despite they are statistically significant, their contributions to  $R^2$  appear to be modest. Adding the turnover, the trade size and the number of trades only increases  $R^2$  by 0.52%, 0.74% and 0.02%, respectively.

In addition, we find that, after controlling for the time-to-maturity, the issue size, the fundamental volatility and the other variables, our illiquidity measure is positively correlated with the CDS spread. That is, the more likely the company will default, the less liquid its bonds become. The connection in terms of magnitude and statistical significance is very strong as the  $t$ -statistics are about 6.57 on average. If the CDS spread increases by 1 percent, our illiquidity measure would increase roughly about 19 percent which is sizeable given that the average illiquidity measure before the crisis was only at 32.1 percent. However, the result does not mean that CDS spreads contain any information on bond liquidity. This is merely because that during distressed time there are fewer investors willing to hold bonds with high default risk. Our result also confirms the finding in [Ericsson & Renault \(2006\)](#) and [Bao \*et al.\* \(2011\)](#) that there is a close relationship between credit and liquidity risks.

Interestingly, the absolute returns of the fundamental price as a proxy for price uncertainty is found to be positively correlated with the illiquidity measure. It is well-known that market makers' inventory costs are higher if price uncertainty is higher. An important source of uncertainty is related to the predictability of future price movements. Thus, a higher price uncertainty leads to larger bid-ask spreads, and thus lower liquidity. In addition, the non-default yield spread also has some predictability for the illiquidity measure. Including the non-default yield spread in the regression increases the  $R^2$  by 5.93%.

Finally, among the independent variables, some of them are common across the bonds, such as the fundamental volatility and the CDS spread. Others are viewed as the idiosyncratic factors, for instance, the time-to-maturity, the issue size and the trading activity measures. The fundamental volatility and the CDS spread all together can explain 17.57% of the variations in the illiquidity measure, in which 5.38% is attributable to the fundamental volatility. Adding the

idiosyncratic factors such as the time-to-maturity, the issue size and the non-default yield spread increase the  $R^2$  by 8.86%. The contribution of the trading activity measures to the  $R^2$  appears to be modest (less than 1%). It is important to caution that the variations which are unexplained by our selected factors may include the errors which are introduced in the process of separating the default and non-default price components as well as estimating the illiquidity measure and the non-default yield spread.

### 4.4.3 Non-default Yield Spread

Now we focus on the variations in the non-default component of the yield spreads. Once again we regress the non-default yield spreads of the individual bonds on a number of liquidity proxies and control variables. Table 4.10 reports the regression results.

As many studies explain non-default yield spreads as a liquidity premium which is the compensation required by investors for the transaction costs incurred when trading the assets, it is important to check whether there is a strong relationship between the non-default yield spread and the illiquidity measure. Table 4.10 shows that the non-default yield spread is significantly and positively related to the illiquidity measure. Thus, the size of the non-default yield spread increases, while the liquidity of a bond decreases. This is consistent with the hypothesis that illiquid bonds have larger liquidity premia embedded within their yield spreads than liquid bonds. In particular, the regression coefficient on the illiquidity measure is 0.06 with a  $t$ -statistic of 17.62. This means that for two bonds issued by the company a difference of \$1.0 in the illiquidity measure leads to a difference of 6 bps in the yield spread. This is obviously much smaller than the differences found in Longstaff *et al.* (2005) (a difference of 36.9% for AAA/AA-rated firms) and Chen *et al.* (2007) (42% for Investment Grade bonds). This is possibly because our analysis is limited to a set of single A-rated bonds, issued by a specific company, and our sample (from February 2006 to September 2010) is more recent than the one used in Longstaff *et al.* (2005) (from March 2001 to October 2002) and Chen *et al.* (2007) (from 1995 to 2003).

## 4.4 Regression Results

**Table 4.10:** Regression of Non-default Yield Spread on Illiquidity and Credit Measures.

This table reports the panel regressions with the daily estimated non-default yield spread  $m_t$  as the dependent variable.  $t$ -statistics are reported in square brackets using White robust method. Non-default yield spreads are in percentage.  $Mat$  is the time-to-maturity in years.  $Issue$  is the issue amount in millions of dollars.  $Turnover$  is the daily trading volume as a percentage of the issue amount.  $Size$  is the daily average trade size of in thousands of dollars of the face value.  $\#Trds$  is the total number of trades in a day.  $Illiq$  is the daily illiquidity measure  $s_t$  in dollars.  $Volatility$  is the absolute value of daily log returns on the fundamental prices in percentage.  $CDS$  is the spread on the 5-year on-the-run CDS in percentage. The whole sample period is from February 7, 2006 to September 30, 2010.

	Coefficients on equations							
	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
<i>Cons</i>	0.73** [37.72]	0.29 [1.53]	0.53** [2.07]	0.66** [3.18]	0.70** [3.41]	0.69** [3.30]	0.68** [3.26]	0.66** [3.74]
<i>Exp(-Mat)</i>		-1.81** [-14.61]	-2.04** [-11.21]	-2.26** [-15.95]	-2.28** [-16.49]	-2.27** [-16.20]	-2.27** [-16.05]	-2.31** [-20.00]
<i>Log(Issue)</i>		0.12** [4.85]	0.07** [2.14]	0.03 [1.12]	0.03 [0.92]	0.03 [1.03]	0.03 [1.00]	0.02 [0.67]
<i>Turnover</i>					0.39 [0.68]			
<i>Log(Size)</i>						-0.00 [-0.65]		
<i>Log(#Trds)</i>							0.00 [1.10]	
<i>Illiq</i>								0.06** [17.62]
<i>Volatility</i>	0.01* [1.79]	0.03** [7.40]		-0.00 [-0.94]	-0.00 [-0.82]	-0.00 [-0.85]	-0.00 [-0.90]	-0.00 [-1.56]
<i>CDS</i>	0.19** [23.25]		0.09** [11.10]	0.18** [21.46]	0.19** [22.56]	0.18** [21.96]	0.18** [21.58]	0.23** [29.47]
Obs	10224	10214	13232	10214	10199	10200	10211	10196
$R^2$	26.85%	24.79%	18.22%	27.14%	27.90%	27.54%	27.29%	32.99%

Turning to term structures of liquidity premia, we find that an increasing and concave term structure of liquidity premia. It is consistent with the interpretation that long-term bonds have higher yields than short-term bonds. The upward-sloping term structure is also consistent with the result by [Dick-Nielsen \*et al.\* \(2012\)](#) who find that the liquidity component increases as the maturity increases and contrasts the work of [Ericsson & Renault \(2006\)](#) who show that the liquidity premium due to selling pressure decreases with time-to-maturities.

Adding the CDS spread as a control variable for the fundamental risk of a bond, we find that the CDS spread is significantly positively related to the liquidity premium. In other words, the liquidity premium is large when the default probability is high. The coefficients of the CDS spread in the regressions are on average 0.18 with  $t$ -statistics of about 20. However, controlling for the CDS spread dramatically reduces the economic and statistical significance of the issue size and the fundamental volatility in the regressions. This is because that CDS spreads might measure some similar fundamental risk that the issue size and the fundamental volatility have contained. In particular, the insignificance of the issue size is consistent with [Crabbe & Truner \(1995\)](#) who also find that for bonds issued by the same corporation there is no relation between issue sizes and bond yields, indicating that large and small debts issued by the same borrower are close substitutes.

Unlike [Longstaff \*et al.\* \(2005\)](#), we are able to obtain transaction-level data for trading activities. However, all our three trading frequency proxies, the turnover, the trade size and the number of trades, turn out to be insignificant for the liquidity premium. This may be because liquidity premia capture future expected liquidity, and these trading activity measures based on transaction-level data may just reflect the current liquidity condition in the market.

In addition, the common factors, namely the fundamental volatility and the CDS spread, explain 26.85% of the variations in the non-default yield spreads. Adding the illiquidity in the regression improves the  $R^2$  by 5.85%. The time-to-maturity and the issue size together contribute to the  $R^2$  by only 0.29%.

Finally, Figure [4.10](#) plots the predicted non-default yield spread against the predicted illiquidity measure by the regression models. The predicted values are generated based on regression (3) and (11), using two common factors (the

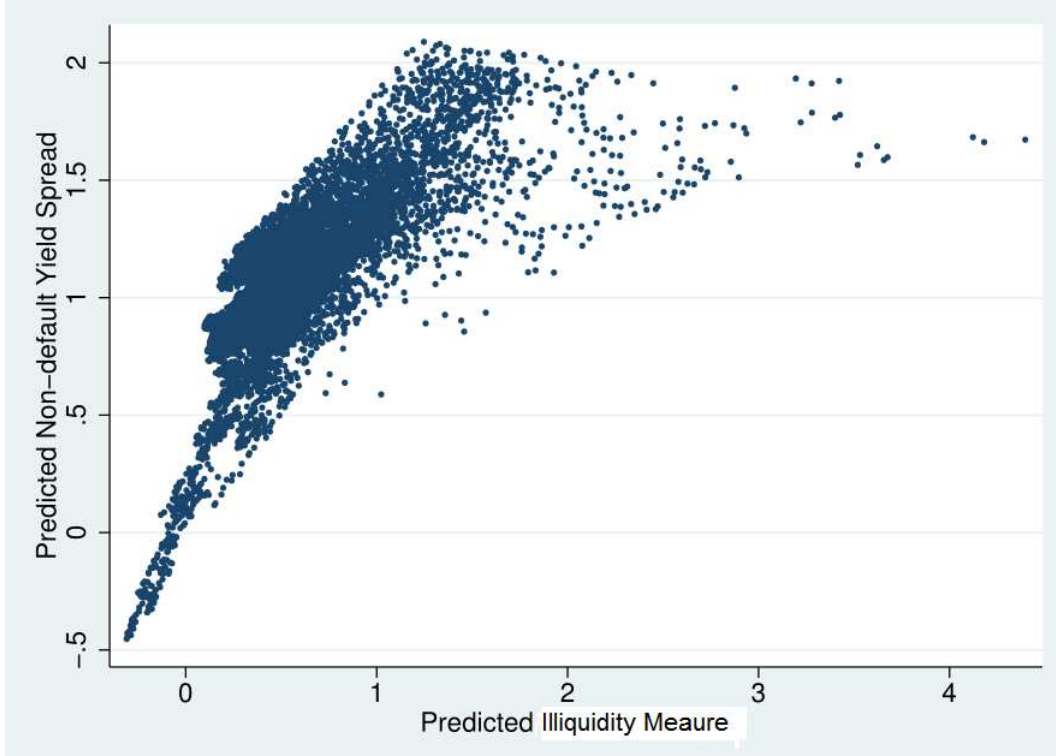
fundamental volatility and the CDS spread) and two idiosyncratic factors (the time-to-maturity and the issue size). As we can see from Figure 4.10, it is obvious that there is a positive relationship between the non-default yield spread and the illiquidity measure. However, there are also a few observations biased towards the right, suggesting that an increase in the illiquidity measure leads to a smaller increase in the yield spread than the increase suggested by most of the data. Consequently, we want to know which factors contribute to the positive relationship and which factors cause a few observations skew to the right. Figure 4.11 shows the contribution of four major factors, namely the time-to-maturity, the issue size, the fundamental volatility and the CDS spread, to the relationship between the non-default yield spread and the illiquidity measure. The positive relationship is generally determined by the time-to-maturity and the CDS spread. As can be seen from Figure 4.11, the reason of a few observation biased to the right in Figure 4.10 is because that the fundamental volatility has a big impact on the illiquidity measure but has insignificant impact on the non-default yield spread. Nevertheless, high volatilities and illiquidity measures are still rarely observed. The issue size is insignificant for the non-default yield spread but has relatively small impact on the illiquidity measure. Both the fundamental volatility and the issue size may contribute to the horizontal spread of the plots in Figure 4.10.

### 4.4.4 Coefficient Stability

Finally, we check the stability of the coefficients across the sub-periods. Our approach is that, by using dummy variables for each sub-period, each independent variable is broken into four variables. Each coefficient of those variables represents the relationship between the independent and dependent variables within each sub-period. We examine one variable at a time and keep all other variables the same as they were in the regressions using the overall sample.

Table A.1 shows the regression results using sub-sample dummy variables for the variations in the illiquidity measure. The coefficients are generally stable and have the expected signs. However, the coefficients of the time-to-maturity slightly decreases throughout the sub-sample periods. The coefficients of the fundamental volatility and the CDS spread in the pre-crisis period have a negative sign. The





**Figure 4.10:** Scatter Plots of Predicted Non-default Yield Spread against Predicted Illiquidity Measure

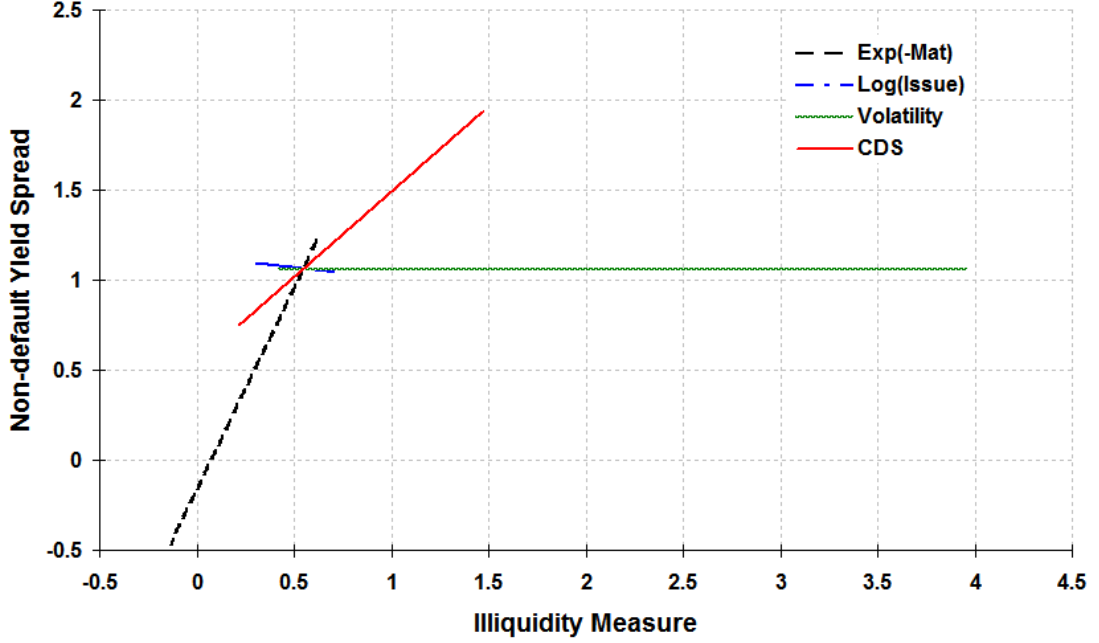
This figure plots the predicted non-default yield spread against the predicted illiquidity measure. The predicted values are generated based on the two regressions:

$$Nondef = \alpha + \beta_1 Exp(-Mat) + \beta_2 Log(Issue) + \beta_3 Volatility + \beta_4 CDS + \epsilon$$

and

$$Illiq = \alpha' + \beta_1' Exp(-Mat) + \beta_2' Log(Issue) + \beta_3' Volatility + \beta_4' CDS + \epsilon'$$

where  $Mat$  is the time-to-maturity in years,  $Issue$  is the issue amount in millions of dollars,  $Volatility$  is the absolute value of daily log returns on the fundamental prices in percentage,  $CDS$  is the spread on the 5-year on-the-run CDS in percentage,  $Illiq$  is the daily illiquidity measure  $s_t$  in dollars and  $Nondef$  is the daily estimated unobservable non-default yield spread  $m_t$  in percentage. The figure suggests a positive relation between the non-default yield spread and the illiquidity measure.



**Figure 4.11:** Contribution of Major Factors to the Relation between the Non-default Yield Spread and the Illiquidity Measure

This figure shows the contribution of the four major factors, namely the time-to-maturity, the issue size, the fundamental volatility and the CDS spread, to the relationship between the non-default yield spread and the illiquidity measure. The figure is constructed based on the coefficients estimated from the two regressions:

$$Nondef = \alpha + \beta_1 Exp(-Mat) + \beta_2 Log(Issue) + \beta_3 Volatility + \beta_4 CDS + \epsilon$$

and

$$Illiq = \alpha' + \beta_1' Exp(-Mat) + \beta_2' Log(Issue) + \beta_3' Volatility + \beta_4' CDS + \epsilon',$$

where  $Mat$  is the time-to-maturity in years,  $Issue$  is the issue amount in millions of dollars,  $Volatility$  is the absolute value of daily log returns on the fundamental prices in percentage,  $CDS$  is the spread on the 5-year on-the-run CDS in percentage,  $Illiq$  is the daily illiquidity measure  $s_t$  in dollars and  $Nondef$  is the daily estimated unobservable non-default yield spread  $m_t$  in percentage. The figure suggests that the positive relationship between the non-default yield spread and the illiquidity measure is mainly due to the time-to-maturity and the CDS spread.

coefficient of the turnover in Period III has the largest magnitude. Both the trade size and the number of trades are insignificant in Period III.

In Table [A.2](#) the coefficients in the non-default yield spread regression show a sign of slight instability. The coefficient of time-to-maturity in Period I is dramatically smaller than those in other periods. The issue size becomes insignificant in Period II and IV and has a negative coefficient in Period I. Coefficients of both the fundamental volatility and the CDS spread have positive signs only in Period III. Coefficients of the illiquidity measure in Period I and IV have negative signs.

We expect some variations in the coefficients across different sub-periods. However, the overall sample is more suitable for our purpose as we generally try to associate the variations in the illiquidity measure and the non-default yield spread with various liquidity proxies.

## 4.5 Summary and Conclusion

In this chapter, we use both credit default swap and corporate bond data of a single firm during 2006 - 2010 to study the relationships between bond liquidity and the non-default component of corporate bond yield spreads as well as our illiquidity measure.

We first separate the non-default component of bond spreads from the default one by using the information in credit default swaps. We then apply our extended Roll model to non-default bond price residuals. The Kalman filter model provides an alternative way to estimate the unobservable non-default yield spread as well as our illiquidity measure.

Our results show that for the bonds in the sample, the non-default component accounted for about 80% of the yield spreads before the financial crisis whereas during and after the crisis default risk played a more important role, and accounted for around 52% and 77% of the yield spreads, respectively. These results indicate that there is a significant non-default component in corporate bond yield spreads. Moreover, this non-default component varies over time as well as across bonds.

We find that our measure of illiquidity is related to several bond characteristics. In particular, the illiquidity measure has a nonlinear and positive relationship with the time-to-maturity. The level of illiquidity decreases while the issue size and the trading frequency increase. In addition, we find that both the fundamental volatility and the CDS spread are positively correlated with the illiquidity measure. The common factors such as the fundamental volatility and the CDS spread can explain 17.57% of the variations in the illiquidity measure whereas the idiosyncratic factors such as the time-to-maturity, the issue size and the non-default yield spread account for 8.86% of the variations.

Empirical results also show that the non-default yield spread is significantly positively correlated with our illiquidity measure. That is, the non-default yield spread is indeed associated with bond illiquidity and increases with the level of illiquidity. Our results support a strong and positive correlation between default risk and liquidity premia. We find an increasing and concave term structure of the liquidity premium. In addition, the trading activity measures are insignificant

for the liquidity premium, suggesting that the liquidity premium may reflect future expected illiquidity of a bond. Finally, 26.85% of the variations in the liquidity premium is explained by the common factors, such as the CDS spread and the fundamental volatility. The illiquidity measure is the most influential idiosyncratic factor of the liquidity premium. Including the illiquidity measure increases the  $R^2$  by 5.85%.

Our main analysis is based on the assumption that CDS spreads do not contain a liquidity premium. However, if they do then it will create a common factor (across bonds of similar maturity) in the residuals. Reduced but still significant liquidity effects would then be estimated. It is also important to caution that our results may not generalize to other bonds or other firms. There may be some doubt about our results for the broader market of corporate bonds. In our future research, we may extend our data set to include a larger number of corporate bonds. Still, for the bonds that we analyzed, these results do suggest that there is a significant non-default component in the corporate bond yield spread which is associated with the illiquidity of a bond, and the illiquidity and liquidity premia vary over time and across bonds deriving by both common and idiosyncratic factors.

# Chapter 5

## An Equilibrium Model of Liquidity in Bond Markets

### 5.1 Introduction

The recent financial crisis has revealed a strong need to gain a better understanding of the influence of illiquidity on asset pricing. In particular, the illiquidity of bond markets has captured the interest of researchers, practitioners, and policy makers. Many empirical studies have found that very similar assets can exhibit significantly different levels of liquidity.<sup>1</sup> Therefore, fundamental asset pricing questions have to be asked: why similar assets differ significantly in liquidity, and in particular, why some bonds are more liquid and more expensive (in terms of price) than other (otherwise identical) bonds.

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<sup>1</sup>Amihud & Mendelson (1991) compare the yields on short-term U.S. Treasury notes and bills with the same maturities, and find that their liquidity is different. Krishnamurthy (2002) documents that on-the-run Treasury bonds are significantly more liquid than off-the-run bonds maturing on nearby dates. In fact, one of long term capital management's (LTCM) main strategies was to exploit the price differences between on-the-run and off-the-run Treasury bonds. Krishnamurthy & Vissing-Jorgensen (2012) examine the yield differences between 6-month Federal Deposit Insurance Corporation (FDIC) insured certificates of deposit (CDs) and 6-month Treasury bills. They suggest that, given FDIC insurance, the yield differences can only be due to liquidity.

To answer the questions, a “partial equilibrium” analysis that imposes ex ante assumptions on the liquidity of assets will be inappropriate. Some models previously developed assume exogenously given transaction costs, the liquidity differentials are explained by exogenously pre-specified differences in transaction costs.<sup>1</sup> In contrast, in this paper we characterize an equilibrium model in which the heterogeneity of liquidity among bonds is determined endogenously, and thereby, we focus on understanding how liquidity externalities arise in an equilibrium framework.

Different from search-based models, our model assumes that a number of (otherwise identical) bonds are exchanged amongst the market participants: a set of risk neutral investors, with exogenously determined cash flow constraints and having in mind their desire and ability to liquidate a proportion of their bonds in a future date prior to maturity as they may experience liquidity shocks or face surprise investment opportunities, purchase the bonds by maximizing their post-investment residual surplus; another set of risk neutral traders, who are willing to buy the bonds unwound by the former investors, come to the marketplace randomly; a single risk averse competitive market maker acts as the whole financial intermediaries by bridging the gaps between buyers and sellers and requires a premium to cover expected inventory-holding costs when supplying immediacy.<sup>23</sup>

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<sup>1</sup>See, for example, Amihud & Mendelson (1986), Huang (2003), Vayanos & Vila (1999) and Vayanos (1998).

<sup>2</sup>Economides & Siow (1988) using search-based model show that agents prefer to trade in a spot market with high rather than low liquidity. Vayanos & Wang (2007) develop a search-based model in which investors of different horizons can invest in two assets with identical payoffs. They show that there exist such an equilibrium where all short-term investors search for the same asset. Weill (2008) show that although the search technology is the same for all assets, heterogeneous trading costs arise endogenously. Vayanos & Weill (2008) use search-based model to explain the on-the-run phenomenon.

<sup>3</sup>Another strand of market microstructure literature (e.g. Grossman (1980), Kyle (1985), Brennan & Subrahmanyam (1996), and Easley *et al.* (1996).) suggests that asymmetric information creates significant illiquidity costs. Our study focuses on inventory-holding risk faced by dealers in bond markets for the following reason. Main investors in corporate bond markets are institutional investors such as pension funds and insurance companies whose long term obligations can be matched reasonably well to the relatively predictable, long term stream of coupon interest payments generated by bonds. As a result, most or all of a bond issue is often absorbed into stable ‘buy-and-hold’ portfolios soon after issue. When pension funds or insurance companies liquidate, they liquidate for cash, rather than short term trades motivated by asymmetric information.

By utilizing the concept of stochastic dominance, we associate the inter-arrival time between offsetting orders with the market maker's inventory holding costs. Furthermore, by capitalizing the argument of utility indifference pricing in addition to some specific assumptions on the asset value, the arrival of orders, and the market maker's utility function, we show that the spread charged by the market maker can be quantified. In particular, the spread is positively correlated with the fundamental volatility, and negatively correlated with the order arrival rate. To our knowledge, this is the first time that stochastic dominance appears in the microstructure literature. Under the risk-neutral pricing framework, the initial bond price depends the expected value of future liquidity, as the marginal investors are willing to pay a premium on bonds that help them in the event of early liquidation.

In our equilibrium model the identity of which bonds are liquid, and the size of their spreads and liquidity discounts, are determined endogenously. The equilibrium mostly depends on the initial choices of the liquidity constrained investors on the offering date, who optimally choose their trading strategies by taking into account the actions by themselves and others.<sup>1</sup> In particular, under the liquidity constraints, we show that bonds can differ in liquidity despite having identical cash flows, riskiness, issue amounts and maturities. The trading concentration is consistent with [Economides & Siow \(1988\)](#), where investors have strong preference for concentrating trading on a limited number of bonds. The liquidity externality is also similar to [Krishnamurthy \(2002\)](#), where asset prices are driven by investors' preference for liquid assets, and [Vayanos & Wang \(2007\)](#), where the clientele equilibrium dominates the one where all investors split trading across assets.<sup>2</sup>

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<sup>1</sup>It is different from [Eisfeldt \(2004\)](#) where the level of liquidity is endogenously determined as a function of productivity.

<sup>2</sup>Our result is also similar to some of the results in the microstructure literature regarding equity markets. [Admati & Pfleiderer \(1988\)](#) developed a model to explain that trading tends to be concentrated at particular times of the day. Intuitively, if traders have discretion over the timing of their trades, they all will tend to trade at times when they expect the others to be trading as well, since that is the time at which liquidity is highest. [Pagano \(1989\)](#) showed that since each trader assesses the absorptive capacity of the market on the basis of his conjectures about the behavior of the others, there may exist an equilibrium where trade concentrates on one of the markets.



When the quantity of bonds which do trade implies that there are some tradable and non-tradable bonds, the identity of the ones which are traded may result from a ‘Sunspot’ equilibrium where it is optimal for traders to randomly label a subset of the bonds as the liquid ones and concentrate trading on them.<sup>1</sup> This self-fulfilling prophecy is analogous to the on-the-run phenomenon where expecting lower liquidity will themselves reduce liquidity.

The difference between our paper and the previous studies is that we relate liquidity demands to liquidity conditions. Our model predicts that liquidity conditions of investors may affect their liquidity demand, and thereby, prices and spreads. The increase of liquid asset holdings can be the result of forwarding looking investors shifting their preferences towards liquid assets, ahead of periods when liquidity is in shortage. Alternatively, the investors with loose liquidity constraints holding more illiquid bonds can be interpreted as long-term investors, whereas the agents with tough liquidity constraints holding more liquid bonds can be interpreted as short-term investors. This clientele effect is consistent with [Vayanos & Vila \(1999\)](#) where they show that agents with short investment horizon buy the liquid asset and agents with long investment horizon hold the illiquid asset.

Our study contributes the existing literature in several ways. First, in previous theoretical studies of asset pricing, ex ante assumptions on asset liquidity are normally made.<sup>2</sup> Our study bridges the gap by characterizing an equilib-

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<sup>1</sup>Market uncertainty can also be driven by extrinsic uncertainty, which includes such variables as market psychology, self-fulfilling prophecies and ‘animal spirits’ ([Keynes \(1936\)](#)), collectively known as ‘sunspots’. The concept of ‘sunspot’ equilibrium was introduced by [Cass & Shell \(1983\)](#). [Duffy \(2005\)](#) interpret Sunspot equilibrium as a means of coordinating the expectations and plans of market participants. For applications of ‘Sunspots’ equilibrium in finance, see [Diamond \(1983\)](#) and [Peck & Shell \(2003\)](#).

<sup>2</sup>[Acharya & Pedersen \(2005\)](#) derive an equilibrium model under exogenous stochastic transaction costs. [Amihud & Mendelson \(1986\)](#) analyze a model in which investors with different expected holding periods trade assets with different relative spreads, which are pre-determined. To jointly explain the equity premium and the low risk-free rate, [Aiyagari & Gertler \(1991\)](#) introduce two kinds of securities in their model. One type of security is freely exchanged, while the other type is costly to trade. By assuming that one asset is liquid while the other asset carries proportional transaction costs, [Vayanos & Vila \(1999\)](#) show that agents buy the liquid asset for short-term investment and the illiquid asset for long-term investment. In other words, there is a well-defined clientele for each asset. More recently, [Huang \(2003\)](#) shows that in equilibrium, by comparing the illiquid bond with the (otherwise identical) liquid bond, the illiquid bond should generate a higher expected return to compensate its holders.

rium model in which the heterogeneity of liquidity among bonds is determined endogenously. Second, our model, which is not search-based, offers an alternative theory that explains why similar assets exhibit heterogeneous levels of liquidity. Third, by utilizing the concepts of stochastic dominance, utility indifference pricing plus some specific assumptions on the asset value, the arrival of orders, the market maker's utility function and agents' risk neutrality, the equilibrium bond prices and spreads can be quantified.<sup>1</sup> Fourth, we show that the identity of which bonds do trade can be a result of a 'Sunspot' equilibrium. Fifth, our model shows that forward looking investors' liquidity preference is affected by their liquidity conditions.

Among microstructure studies on liquidity, [Foucault \*et al.\* \(2005\)](#) develop a dynamic model of a limit order market in which agents are strategic and take liquidity into account when trading. Their model predicts a negative relationship between the order arrival rate and the market resiliency. [Rosu \(2009\)](#) models an order-driven market where strategic traders tradeoff execution prices and waiting costs. The model predicts that higher trading activity and higher trading competition cause smaller spreads and lower price impact. Our model shares some features of these models, moreover, complements this strand of microstructure literature.

The rest of the chapter is structured as follows. Section 2 gives the introduction and discussion of the general model, followed by a specific example. In Section 3, we characterize the equilibrium. Then, we analyze the quantitative implications of the equilibrium in Section 4. Section 5 concludes.

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<sup>1</sup>[Hodges & Neuberger \(1989\)](#) were the first to introduce the concept of utility indifference pricing in the context of option pricing under transaction costs in Black-Scholes model.

## 5.2 The General Model

In this section, we introduce the general model, and provide the motivation, the discussion of the model, and especially the behaviors of different market participants.

### 5.2.1 Financial Market

The economy has three dates, 0, 1 and 2. The financial market consists of a single consumption good, which is used as the numeraire, and  $N$  bonds, indexed by  $j \in \mathcal{N} = \{1, 2, \dots, N\}$ .<sup>1</sup> We assume:

- (A) All the bonds issued on date 0 and maturing on date 2 has a face value of 1 dollar.<sup>2</sup>
- (B) The total supply of the bonds is  $K$ , with the supply of bond  $j$  in terms of dollar value, denoted  $k_j$  (that is,  $\sum_{j=1}^N k_j = K$ ,  $0 < k_j$ ).
- (C) All the bonds share the same level of default risk (i.e. perfectly correlated defaults).<sup>3</sup>

One unique feature of our setup is that in contrast to previous literature we do not impose ex ante assumptions on bond liquidity.<sup>4</sup> The heterogenous levels of liquidity can be determined endogenously by interactions among traders.

### 5.2.2 Agents

We consider a dealership market where there are three types of market participants trading the bonds, namely Type 1 trader, Type 2 trader and the market maker. The following assumptions are made:

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<sup>1</sup>Equivalently, one dollar one consumption good.

<sup>2</sup>The analysis ignores coupons. It applies with coupons if all bonds pay the same coupon rate.

<sup>3</sup>Imperfectly correlated defaults are likely to change our results. However, perfectly correlated defaults are suitable for our purpose, as we try to explain that why otherwise identical bonds (e.g. bonds issued by a single company) differ in their liquidity.

<sup>4</sup>See, for example, Amihud & Mendelson (1986), Huang (2003), Vayanos & Vila (1999) and Vayanos (1998).

- (D) A continuum of risk-neutral traders exist in the economy.<sup>1</sup> Depending on when they trade, the traders fall into two groups, namely Type 1 and Type 2.
- (E) There are  $M$  Type 1 traders subject to exogenous cash flow constraints, indexed by  $i \in \mathcal{M} = \{1, 2, \dots, M\}$ , purchasing bonds on the offering date 0. In addition, they are facing the risk of reselling a proportion of their asset holdings on the interim date 1.
- (F) Since date 1 Type 2 traders arrive at the market randomly to buy the bonds.
- (G) All trades are made through a single centralized risk-averse market maker, who is characterized as a monopolistic competitor, subject to a zero-profit condition. No direct exchanges between the traders are permitted.

One interpretation of Assumption (E) is that Type 1 traders are institutional investors such as pension funds and insurance companies who buy bonds to generate cash flows, which match their long-term liabilities. Meanwhile, subject to a limited budget, they must anticipate possibilities and costs of reselling a proportion of their holdings back to the market at a later point in time. Each Type 1 trader must make her investment decision on date 0 subject to the risk of liquidating some of her holdings (only) on date 1.<sup>2</sup>

Vayanos & Vila (1999) show that agents purchase the liquid asset for short-term investment and the illiquid asset for long-term investment. Amihud & Mendelson (1986) also show that there is a clientele effect whereby investors with longer holding periods select illiquid assets. That is, for a short holding period, transaction costs are important, and a liquid asset is preferred regardless of its lower yield; for a long holding period, illiquidity is not a concern, and thus a cheap and illiquid asset is a better choice. In other words, illiquid long-term investments are more productive but less flexible than liquid short term ones. Although in some theoretical studies agents with pre-specified investment horizons hold uniform assets (either short-term liquid or long-term illiquid assets),

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<sup>1</sup>In search-based models, agents are assumed to be risk-averse. We believe that traders are likely to be less risk averse than the market maker, as they are relatively tolerant to risk.

<sup>2</sup>When Type 1 traders liquidate, they liquidate for cash. Therefore, we do not consider the case where Type 1 traders are allowed to buy bonds on date 1.

in practice investors optimize their asset mix across the spectrum of high-yield illiquid bonds to low-yield liquid ones. In our model investment horizons are determined implicitly and endogenously in terms of the ratio of liquid holdings to illiquid holdings, rather than are ex ante assumed.

Type 2 traders in Assumption (F) can be regarded as buyers in the secondary market who potentially can buy any bond in the market. Their random arrivals are due to the fact that they are liquidity traders who trade on noise. Included in this category are large traders, such as some financial institutions, whose trades reflect the liquidity needs of their clients or who trade for portfolio-balancing reasons.

The centralized market maker in Assumption (G) represents all the market makers in the over-the-counter (OTC) markets such as trading desks in major investment banks and hedge funds.<sup>1</sup> In the OTC markets, investors normally negotiate trades with the market makers, rather than directly search for counterparties themselves. In the absence of information asymmetry, the market maker is willing to take orders from traders, therefore acting as a liquidity provider who accommodates temporary order imbalances.<sup>2</sup> Obviously, the market maker needs to manage the inventory-holding risk of his position, as at the time of trade he does not know when an offsetting transaction will occur, and during the course of waiting for the offsetting order the value of his inventory fluctuates. If the market does not provide a viable hedging vehicle or the hedging cost exceeds the revenue of market making, the risk-averse market maker must be compensated for bearing the inventory-holding risk. This compensation is usually generated in the form of a bid-ask spread, which is charged by the market maker for each trade. In other words, the existence of the market maker merely reduces the market illiquidity (or the search cost in [Vayanos & Wang \(2007\)](#) and [Weill \(2008\)](#)) and express it in the form of a bid-ask spread.

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<sup>1</sup>These market makers may not be present in the market all the time, like the traditional specialists in equity markets who constantly maintain a list of quotations for bid and ask orders. They might be simply the kind of investors who provide liquidity whenever other investors want to buy or sell.

<sup>2</sup>We believe that the information in bond markets is relatively symmetric as interest rates and credit default swap spreads are publicly observable.

The zero profit condition in Assumption (G) implies that the bid-ask spread required by the market maker compensates exactly the expected marginal cost of supplying liquidity, which in our case is the inventory-holding costs.<sup>1</sup> The perfect competition drives down the spread until it is equal to the marginal cost, as market makers who quote a spread which is higher than the marginal cost will not receive any order at all. In the perfectly competitive market, market makers need to rebalance their inventories as quickly as possible so as to provide the competitive spreads. Therefore, the market makers' business depends on how quickly they can rebalance their inventories.

### 5.2.3 The Market Makers' Behavior

In this section we characterize the behavior of the market maker, and then show how the spread charged the market maker is quantified.

We take as given an arbitrage-free setting where each bond has an efficient price  $P$ , which contains all relevant information including publicly available information as well as the information about the future liquidity of the asset.<sup>2</sup> We assume that

- (H) The discounted price process  $P(s)$  (over a short time interval) is a martingale under the 'risk neutral' measure  $\mathbb{Q}$  with respect to some suitable numeraire (e.g. bank account), where  $s$  is the calendar time. In particular, denote the bond price on the offering date (on date 0)  $P^0$ .
- (I) A non-negative stochastic process  $\tilde{t}(s)$ , which is assumed to be independent of  $P(s)$ , represents the random waiting time that the market maker at time  $s$  must wait to see the arrival of the offsetting order.

Under the above two assumptions, over the random waiting time  $\tilde{t}$  which is considered to be much shorter than the life of the bond, the discounted price process  $P(s)$  is a martingale. Apparently, the shorter the waiting time is, the

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<sup>1</sup>Another source of the costs of providing market making services is the order processing costs: order taking, execution, computer, informational service and labor costs. For simplicity the fixed costs are ignored here.

<sup>2</sup>Others also refer to  $P$  as the true price or fair price.

more quickly the market makers can rebalance their inventories, and thus the smaller risks she is facing.

Suppose at time  $s$  the market maker acquires one unit of the asset from a customer. The change of her inventory value or her wealth  $\tilde{w}(s)$  until she unloads the inventory to a new customer is given by

$$\tilde{w}(s) = P(s + \tilde{t}(s)) - P(s), \quad (5.1)$$

where  $s + \tilde{t}(s)$  is the random time at which the position gets unwound.

In Equation (5.1) the change of the inventory value can be positive or negative as the price fluctuates over time. Moreover, it is affected by the random waiting time, because the longer she waits, the more likely the inventory value will go adversely against her. Therefore, apart from the underlying price volatility, the inventory-holding risk borne by the market maker arises from the random waiting time of rebalancing her inventory.

Now we will show that the market maker prefers to make the market for bonds with shorter expected waiting time. Stochastic dominance provides a rather general framework in which to model the market maker's decisions. It will also be sufficient for our purposes.

Given alternative distributions of the random waiting times  $\tilde{t}_A$  and  $\tilde{t}_B$  where  $\tilde{t}_A$  is first-order stochastically dominated by  $\tilde{t}_B$  (i.e.,  $\tilde{t}_A$  provides shorter expected waiting time than  $\tilde{t}_B$  does),  $\tilde{t}_A$  will be preferred by a risk-averse market maker as generating smaller inventory-holding risks through fewer mean-preserving spreads.

This is formalized in Lemma 1:

**Lemma 1.**  *$\tilde{t}_B$  first-order stochastically dominates  $\tilde{t}_A \Rightarrow \tilde{w}_A$  is second-order stochastically dominant over  $\tilde{w}_B$ .*

*Proof.*  $\tilde{t}_A$  has the distribution function  $G_A$  and  $\tilde{t}_B$  the distribution function  $G_B$ , as shown in Figure 5.1 graphically. Denote  $f(w)$  and  $F(w)$  the density and distribution functions of the market maker's inventory value changes. The value dis-

tribution is generally determined by the random variable  $\tilde{t}$ . More precisely, there exists a non-negative real-valued function  $h(w|t)$  such that the density function of  $w_i$ ,  $f_i(w)$ , is defined with respect to  $G_i(t)$  as

$$f_i(w) = \int h(w|t) dG_i(t), \quad i = A, B. \quad (5.2)$$

For all  $t$  in (5.2), there exists a mean-preserving spreads  $\epsilon$  corresponding to the shifting time from  $v$  to  $u = G_B^{-1}(G_A(v))$  (as shown by the dashed arrow sign in Figure 5.1) such that  $\tilde{w}_B \stackrel{d}{=} \tilde{w}_A + \epsilon$ .

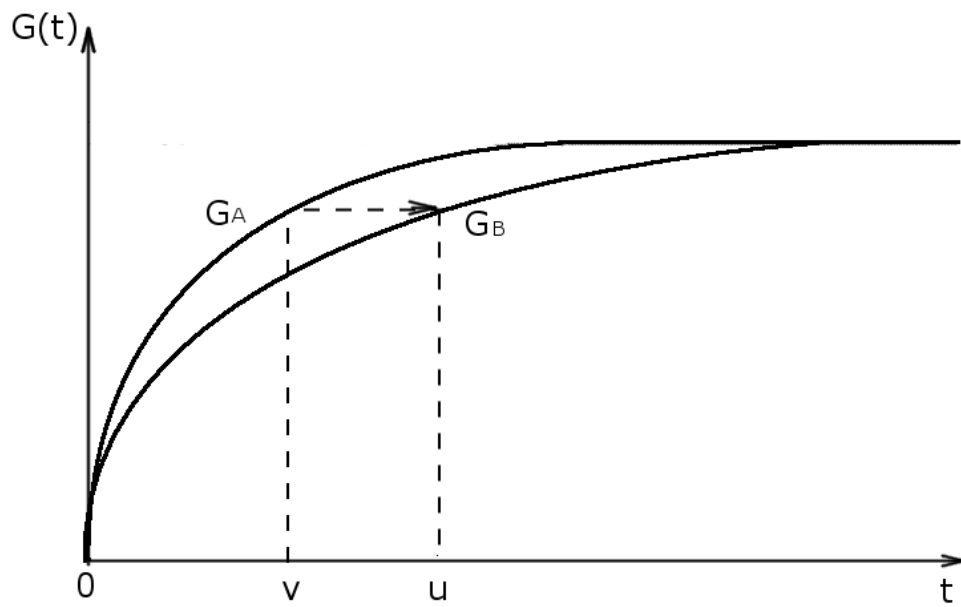
Therefore, by the Rothschild and Stiglitz Theorem,  $\tilde{w}_A$  is second-order stochastically dominant over  $\tilde{w}_B$ . □

As shown by Figure 5.1, changing the distribution from  $G_A$  to  $G_B$  means that  $\tilde{w}_B$  will have more weight in its tails than  $\tilde{w}_A$ . Alternatively stated, shortening the expected waiting time by shifting some probability mass of the distribution  $G(t)$  to the left implies increasing the expected utility of the risk-averse market maker. All risk-averse agents prefer  $\tilde{w}_A$  to  $\tilde{w}_B$ , and thus would like to make the markets for bonds with shorter expected waiting time. The market maker's inventory holding cost, in turn, decreases in the expected waiting time, holding the price volatility constant. This is consistent with the observations in the corporate bond market that when taking up a long position from a trader who wants to sell an issue at a certain price, a market maker would be more than happy to offer a better price to the seller if she were sure that she could find a potential buyer to whom she could quickly unload the position.

Under Second Order Stochastic Dominance in Lemma 1, the market maker who is risk-averse can be characterized by a monotonically increasing and concave utility function  $U$ . Let  $S$  denote the market maker's reservation price (spread) of buying or selling one unit of the asset. Now we assume that

- (J) The spread  $S$  is determined in such a way that the market maker's expected utility remains unchanged after each transaction.





**Figure 5.1:** Distributions of arrival time  $\tilde{t}$ .

The distribution of the random arrival time  $\tilde{t}$  is characterized by the distribution function  $G(t)$ .  $\tilde{t}_A$  has the distribution function  $G_A$  and  $\tilde{t}_B$  the distribution function  $G_B$ .

That is, the expected utility of the market maker is kept constant throughout transactions. Hereafter, we give the precise definition of  $S$  under the utility indifference argument. Let  $w_0$  be the initial wealth of the market maker before a purchase or a sell, and  $\tilde{w}$  be the change of his wealth during the course of waiting to unwind the position (i.e. over the random waiting time  $\tilde{t}$ ).

**Definition 4.** *The reservation price (spread)  $S$ , for the purchase or the sale of one unit of the bond by the market maker, is defined as the solution of the equation*

$$\mathbb{E}[U(w_0)] = \mathbb{E}[U(w_0 \pm \tilde{w} + S)], \quad (5.3)$$

where  $\pm$  depends on whether the transaction is a purchase or a sell.

The reservation price  $S$  is essentially the spread at which the market maker is indifferent (in the sense that his expected utility under competition is unchanged) between doing nothing and receiving  $S$  now to have the position in his inventory. Alternatively stated,  $S$  can be interpreted as the minimum amount that the market maker wishes to receive to buy or sell one unit of the bond. Under Assumption (G), the zero profit condition forces the market maker to post the spread  $S$  which is exactly equal to the inventory holding cost in absence of information issues.

### 5.2.4 Brownian Motion and Poisson Arrivals

Now we will give an example of our general model, specified by a Brownian Motion for asset prices and a homogeneous Poisson process for order arrivals. Moreover, with some specific assumptions including the assumption about the market maker's utility function, the spread  $S$  can be quantified. We assume that

- (K) The efficient price  $P$  of a bond under the 'risk neutral' measure moves as an arithmetic Brownian Motion. That is,  $dP(s) = \sigma(s)dB(s)$ , where  $B(s)$  is a standard Brownian Motion. For expositional ease,  $\sigma(s)$  is assumed to be time-invariant.

- (L) Except on the offering date 0, traders come to the market to buy or sell one unit of the assets at a time. The arrivals of orders on a bond are characterized by an independent homogeneous Poisson process, with the arrival rate  $\mu$ . Then, the random waiting time  $\tilde{t}$  follows an exponential distribution with the expected waiting time  $1/\mu$ . The trading intensity of each Type 1 trader is exogenously given and characterized by another Poisson hazard rate, denoted by  $\lambda$ .
- (M) The mean arrival rates from both sides (i.e. buying and selling) are approximately equal so that the market is clear on average.<sup>1</sup>

Trading is performed via a monopolistic competitive market maker who stands ready to trade. The market maker bridges the time gaps between the arrivals of buyers and sellers to the market, and is compensated by charging the spread, which is competitively set. Assumption (M) implies that the market maker is seeking out the market spread that equilibrates buying and selling pressures. Furthermore, we assume

- (N) The risk-averse market maker has a negative exponential utility function, i.e.  $U(w) = -e^{-\gamma w}$ , where  $\gamma$  is the absolute risk aversion of the market maker and  $w$  is her wealth.

The negative exponential utility function combined with the above assumptions on asset prices and order arrivals enable us to calculate the spread analytically (as a function of the fundamental volatility and the order arrival rate).<sup>2</sup>

**Proposition 1.** *The reservation price (spread)  $S$  for buying and selling one unit of a bond is equal to  $\frac{\gamma\sigma^2}{2\mu}$ , where  $\gamma$  is the absolute risk aversion of the market maker,  $\sigma$  is the fundamental volatility, and  $\mu$  is the arrival rate of orders on the bond.*

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<sup>1</sup>From the perspective of the market maker, trading commences from the interim date 1.

<sup>2</sup>Roger (1999) also derives the bid-ask spread in the utility indifference manner. However, his derivation does not involve the order arrival rate or waiting time.

*Proof.* By the definition of the reservation price  $S$ , Equation (5.3) can be written as:

$$-e^{-\gamma w_0} = \mathbb{E}[-e^{-\gamma(w_0 \pm \tilde{w} + S)} | t], \quad (5.4)$$

where  $w_0$  is the market maker's initial wealth,  $t$  is the waiting time until the offsetting order, and  $\tilde{w}$  is the change of the market maker's wealth over the waiting time  $t$ . Rearranging the terms in Equation (5.4) yield:

$$e^{\gamma S} = \mathbb{E}[e^{\pm \gamma \tilde{w}} | t], \quad (5.5)$$

Since under Assumption (K) and (N),  $\tilde{w}|t \sim N(0, \sigma^2 t)$  and  $U(w) = -e^{-\gamma w}$ , we have:

$$\mathbb{E}[U(\tilde{w})|t] = -e^{\gamma^2 \sigma^2 t/2}, \quad (5.6)$$

where  $\cdot|t$  means being conditional on  $t$  and  $\mathbb{E}[\cdot|t]$  is the conditional expectation.

Subsequently,

$$e^{\gamma S} = e^{\gamma^2 \sigma^2 t/2}. \quad (5.7)$$

Under Assumption (L) the expected waiting time is  $\frac{1}{\mu}$ , thus the spread  $S$  is obtained as:

$$S = \frac{\gamma \sigma^2}{2\mu}. \quad (5.8)$$

□

Hence, the spread increases while the price risk  $\sigma^2$  increases, and decreases while the order arrival rate  $\mu$  increases. As the market becomes more volatile, the spread widens; as the orders arrive more frequently, the spread narrows. This result is consistent with [Stoll \(1978a\)](#) and [Bollen \*et al.\* \(2004\)](#). [Stoll \(1978a\)](#) shows that the relative bid-ask spread includes an inventory-holding-costs term which equals the product of the return volatility and the expected time the market maker

expects the position to be open. [Bollen \*et al.\* \(2004\)](#) show that the inventory holding costs are positively related to the expected value of the square root of the time between offsetting trades. If the market is perfectly liquid, that is  $\mu \rightarrow \infty$ , the spread vanishes (i.e. there is no transaction costs).

The expected waiting time is supposed to be far shorter than the maturity of a bond. Intuitively, the market maker will not make a market for a bond, if the waiting time of the offsetting order on this bond is too long. Alternatively speaking, there is a probability that the market maker gets left with an inventory of bonds on date 2. However, this probability is bound to be very small as the expected waiting time of an offsetting order is much shorter than the maturity of the bond. Moreover, the risk of getting left with an inventory is compensated by charging bid-ask spreads. Therefore, we have

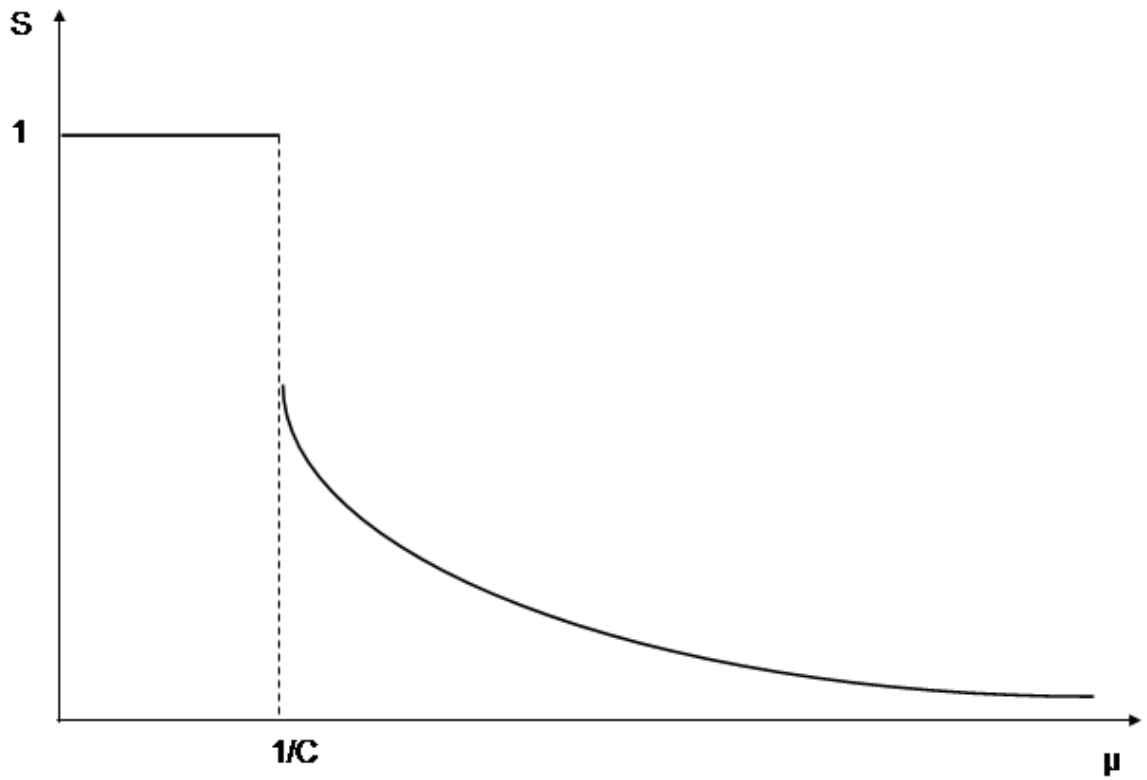
- (O) A bond with the expected waiting time of orders  $1/\mu > C$  is considered to be illiquid, where  $C$  is the cutoff value which is a constant. That is, if a bond with the expected waiting time that is greater than  $C$ , the market maker will choose not to make a market for this bond. The spread for an illiquid bond is assumed to be 1 (that is  $S(\mu) = 1$ ,  $1/\mu > C$ , as in [Figure 5.2.](#)).

In the next section we consider the decisions made by Type 1 traders. As we will see later, the equilibrium is mostly determined by their actions.

### 5.2.5 The Investors' Decisions

Recall that, on date 0 there are  $M$  Type 1 traders who make investment choices out of a pool of  $N$  bonds. The total supply of the  $N$  bonds is  $K$ , with the supply of Bond  $j \in \mathcal{N} = \{1, 2, \dots, N\}$  in terms of dollar value, denoted  $k_j$  (that is,  $\sum_{j=1}^N k_j = K$ ,  $0 < k_j$ ).

Under Assumption (E) and (L), each Type 1 trader with an exogenously given trading intensity is subject to the risk of liquidating some of her assets on date 1. More precisely, we assume that



**Figure 5.2:** Spread

This figure shows that the spread is a function of the arrival rate:

$$S(\mu) = \begin{cases} \frac{\gamma\sigma^2}{2\mu}, & \frac{1}{\mu} < C, \\ 1, & \frac{1}{\mu} \geq C, \end{cases},$$

where  $\mu$  is the order arrival rate,  $\gamma$  is the market maker's absolute risk aversion coefficient,  $\sigma$  is the volatility of the price dynamics of the bond, and  $C$  is the cutoff value which is a constant.

(P) Each trader  $i \in \mathcal{M} = \{1, 2, \dots, M\}$  with a trading intensity  $\lambda_i$  on date 0 purchases bonds with a total face value of  $X_i + Y_i$ , where  $X_i$  is the total tradable shares that she prepares for possible early liquidation on date 1 and  $Y_i$  represents the total non-tradable shares that she will hold until maturity (date 2). Let  $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{iM})$  and  $\mathbf{Y}_i = (Y_{i1}, Y_{i2}, \dots, Y_{iM})$ , with  $X_i = \sum_{j \in \mathcal{N}} X_{ij}$  and  $Y_i = \sum_{j \in \mathcal{N}} Y_{ij}$ , where  $X_{ij}$  is the tradable shares of bond  $j$  held by trader  $i$ , and  $Y_{ij}$  is the non-tradable shares of bond  $j$  held by trader  $i$ .

Assumption (P) also implies that each trader  $i$  is essentially allocating her trading intensity  $\lambda_i$  across the bonds.

**Definition 5.** *An allocation of the trading intensity of trader  $i$  is defined as  $\mathbf{h}_i = (h_{i1}, h_{i2}, \dots, h_{iN})$ , where  $h_{ij}$  the the normalized trading intensity of trader  $i$  for bond  $j$ , with  $\sum_{j=1}^N h_{ij} = 1$  and  $h_{ij} \in [0, 1]$ .*

Therefore, the arrival rate of orders on bond  $j \in \mathcal{N}$  can be expressed as

$$\mu_j = \sum_{i \in \mathcal{M}} \mu_{ij} = \sum_{i \in \mathcal{M}} \lambda_i h_{ij}, \quad (5.9)$$

where  $\mu_{ij}$  is the arrival rate of orders on bond  $j$  from trader  $i$ . Moreover, we assume that

(Q) The tradable shares  $X_{ij}$  of bond  $j$  held by trader  $i$  is proportional to her trading allocation. That is,  $X_{ij} = h_{ij} X_i$ .

Assumption (Q) basically means that, the more likely the trader will trade a bond, the more tradable shares of the bond she holds.

Now we proceed to derive the initial bond price in the presence of illiquidity. The introduction of transaction costs could change equilibrium prices. However, it should be possible to identify equilibrium prices in a frictional cost economy as being a function of the prices that prevail in a corresponding frictionless economy plus some extra factor, and obviously the factor should depend upon the future illiquidity of assets. Thus, price discounts or the extra yields are very likely to

be associated with the future illiquidity of assets. In addition, equilibrium prices may not be uniquely identifiable, but must preclude arbitrage opportunities.

In the presence of illiquidity or transaction costs, as previous stated, upon purchasing a security a rational investor needs to take into account both the likelihood and the potential costs of reselling the security back to the market at a later point in time.

Let  $\mathcal{L} \subseteq \mathcal{N}$  and  $\mathcal{J} \subseteq \mathcal{N}$  represent the sets of liquid and illiquid bonds, respectively.  $\mathcal{L} \cup \mathcal{J} = \mathcal{N}$  and  $\mathcal{L} \cap \mathcal{J} = \Phi$ , where  $\Phi$  is the empty set. That is, a bond must be either a liquid bond or an illiquid one, but not both at the same time. On date 0, the price of a liquid bond  $P_j^0$ ,  $j \in \mathcal{L}$  is determined as

$$P^0(\mu_j) = 1 - [1 - e^{-\bar{\mu}_j}] \cdot S(\mu_j), \quad \frac{1}{\mu_j} < C, \quad (5.10)$$

where  $\bar{\mu}_j = \frac{1}{M}\mu_j$  is the average arrival rate, and  $S(\mu_j)$  is the spread, competitively set by the market maker, which depends exclusively on  $\mu_j$ .<sup>1</sup>

The price of a liquid bond in Equation 5.10 expressed as a discount of the face value is derived using a risk-neutral pricing framework similar to the one by [Duffie & Singleton \(1999\)](#) where the price of a defaultable bond is associated with the default intensity as well as a factor which is related to liquidity effects or carrying cost. [Amihud & Mendelson \(1986\)](#) and [Vayanos & Vila \(1999\)](#) show that in order to induce the marginal investor to buy the asset, the price has to fall by the present value of the transaction costs that he will incur.<sup>2</sup> Therefore, the discounted value which is the second term in Equation 5.10 is equal to the probability of early liquidation (the term in the squared brackets) times the spread charged by the market maker. The probability of early liquidation is a function of the average arrival rate which could be interpreted as the trading intensity of the marginal investors. The bond price  $P_j^0$  is then equal to the marginal investors' willingness to pay. In other words, risk-neutral investors are willing to pay a

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<sup>1</sup>For expositional ease, we ignore the time value of interest or the risk-free interest rate as it is essentially the same across the bonds.

<sup>2</sup>In contrast, [Vayanos \(1998\)](#) shows that the PV term overstates the effect of transaction costs on the risky stock price.



premium on assets that help them when liquidity is needed.<sup>1</sup>

Since illiquid bonds are generally inferior to liquid ones, they should have lower prices than those of liquid ones. Thus, we assume that:

**(R)** The price of the illiquid bond  $P_j^0$ ,  $j \in \mathcal{J}$  is determined as

$$P^0(\mu_j) = 1 - \frac{\gamma\sigma^2}{2M}, \quad \frac{1}{\mu_j} \geq C. \quad (5.11)$$

The price of an illiquid bond in Equation 5.11 can be obtained by taking the limit as  $\mu$  approaches to zero in Equation 5.10. Therefore, bond prices across the whole spectrum of liquid and illiquid bonds are expressed in terms of arrival rates as shown in Figure 5.3.

With a subset of information, a reduced-form approach of modelling decision making by pension funds or insurance companies is adopted. For Type 1 traders' decisions on date 0, we assume that

**(S)** Each trader  $i \in \mathcal{M}$ , with an endowment  $e_i$  and a trading intensity  $\lambda_i$  solves the following problem:

$$\max_{X_i, \mathbf{Y}_i, \mathbf{h}_i} e_i - \sum_{j \in \mathcal{N}} P_j^0(h_{ij}X_i + Y_{ij}), \quad (5.12)$$

subject to:

1. total investment constraint:

$$X_i + Y_i \geq F_i, \quad (5.13)$$

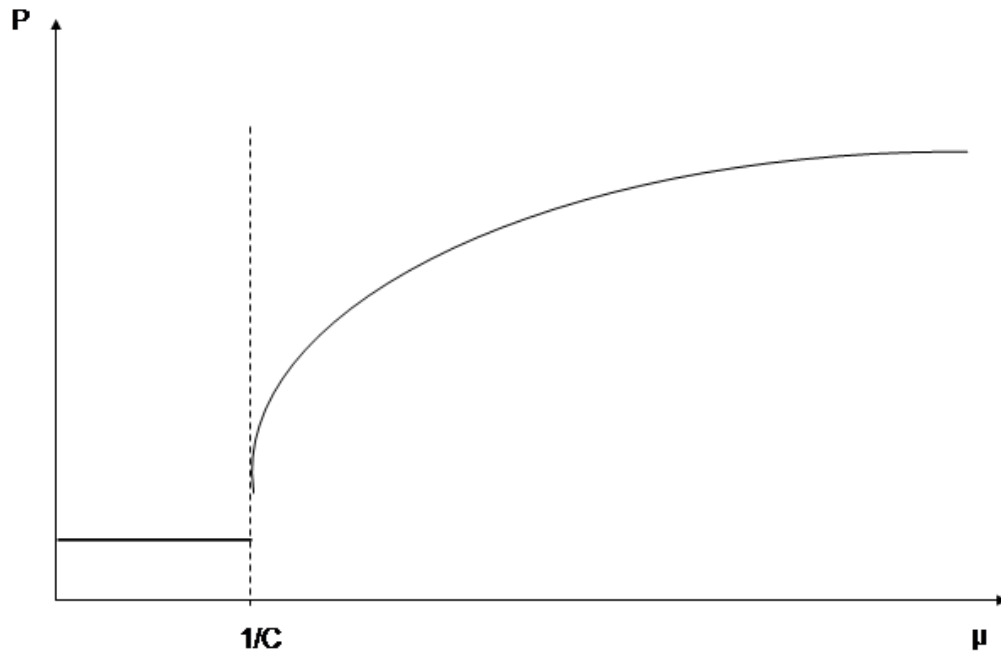
where  $F_i$  is the total face value required at maturity  $T$ .

2. pre-mature liquidation constraint:

$$\sum_{j \in \mathcal{N}} [1 - S(\mu_j)] h_{ij} X_i \geq V_i, \quad (5.14)$$

---

<sup>1</sup>If we assume that traders are homogenous, in equilibrium every trader is essentially the marginal investor.



**Figure 5.3:** Bond price

This figure shows that the price is a function of the arrival rate:

$$P(\mu) = \begin{cases} 1 - [1 - e^{-\frac{\mu}{M}}]S(\mu), & \frac{1}{\mu} < C, \\ 1 - \frac{\gamma\sigma^2}{2M}, & \frac{1}{\mu} \geq C, \end{cases},$$

where  $\mu$  is the order arrival rate,  $\gamma$  is the market maker's absolute risk aversion coefficient,  $\sigma$  is the volatility of the dynamics of the bond, and  $M$  is the number of Type 1 traders.

where  $V_i$  is the value required in the event of early liquidation,

3. market clearing constraints:

$$\sum_{i \in \mathcal{M}} h_{ij} X_i + Y_{ij} = k_j, \quad j \in \mathcal{N}, \quad (5.15)$$

4. and nonnegativity constraints:

$$X_i \geq 0, \quad (5.16)$$

$$Y_{ij} \geq 0, \quad (5.17)$$

$$\sum_{j \in \mathcal{N}} h_{ij} = 1, \quad 0 \leq h_{ij} \leq 1. \quad (5.18)$$

The second term  $\sum_{j \in \mathcal{L}} P_j^0(h_{ij} X_i + Y_{ij})$  in the objective function (Equation 5.12) is the total investment, and thus, the objective function represents the post-investment residual surplus, that is, the cash left over for trader  $i$  after making her investment. In other words, the goal of each trader is to choose her portfolio holding as well as her trading intensity allocation such that her residual surplus is maximized.

The interpretation of the total investment constraint could be that pension funds or insurance companies have long term liabilities which are hedged by buying bonds. As being subject to the risk of pre-mature selling, a minimum value in the event of early liquidation is required by each investor. It is the minimum value that sellers expect to receive upon liquidation. This is similar to the concept of the recovery value in the event of default in the credit risk literature (e.g. [Duffie & Singleton \(1999\)](#)).

There are links between and restrictions on the values of the endowment  $e_i$ , the total face value  $F_i$  and the minimum value required upon liquidation  $V_i$ .  $e_i$  is greater than or equal to the (date 0) present value of  $F_i$  such that the investor gets left with a non-negative surplus on the offering date. The sum of  $F_i$  is equal

to the total supply (i.e. all the bonds are sold out.).  $V_i$  obviously can not be greater than the (date 1) present value of  $F_i$  as the investor can only liquidate what he holds.

Writing

$$f(X_i, \mathbf{Y}_i, \mathbf{h}_i) = e_i - \sum_{j=1}^N P_j^0(h_{ij}X_i + Y_{ij}), \quad (5.19)$$

$$g_1(X_i, Y_i) = X_i + Y_i, \quad (5.20)$$

$$g_2(X_i, \mathbf{h}_i) = \sum_{j=1}^N [1 - S(\mu_j)] h_{ij} X_i, \quad (5.21)$$

$$g_3(X_i, Y_{ij}, h_{ij}) = \sum_{i=1}^M (h_{ij}X_i + Y_{ij}), \quad j = 1, \dots, N \quad (5.22)$$

$$g_4(\mathbf{h}_i) = \sum_{j=1}^N h_{ij}, \quad (5.23)$$

$$g_5(h_{ij}) = 1 - h_{ij}, \quad j = 1, \dots, N, \quad (5.24)$$

where

$$P^0(\mu_j) = \begin{cases} 1 - [1 - e^{-\frac{\mu_j}{M}}] S(\mu_j), & \frac{1}{\mu_j} < C, \\ 1 - \frac{\gamma\sigma^2}{2M}, & \frac{1}{\mu_j} \geq C, \end{cases} \quad (5.25)$$

$$S(\mu_j) = \begin{cases} \frac{\gamma\sigma^2}{2\mu_j}, & \frac{1}{\mu_j} < C, \\ 1, & \frac{1}{\mu_j} \geq C, \end{cases} \quad (5.26)$$

and  $\mu_j = \sum_{i=1}^M \lambda_i h_{ij}$ , one can reformulate the same problem to make it more suitable for exposition and calculation:

$$\text{maximize } f(X_i, \mathbf{Y}_i, \mathbf{h}_i), \quad \forall \quad i \in \mathcal{M}, \quad (5.27)$$

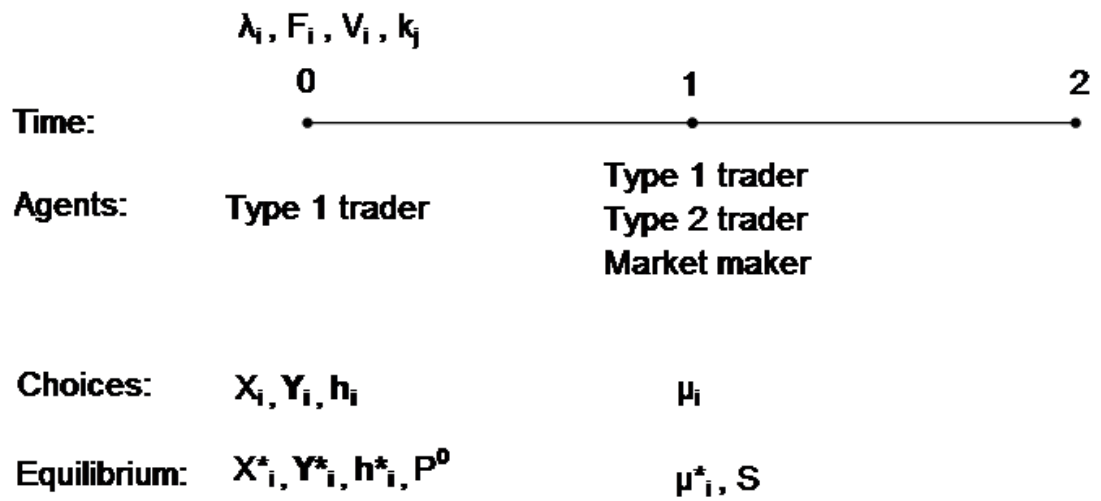
$$\begin{aligned}
\text{subject to } & g_1(X_i, Y_i) \geq F_i, \\
& g_2(X_i, \mathbf{h}_i) \geq V_i, \\
& g_3(X_i, Y_{ij}, h_{ij}) = k_j, \quad j = 1, \dots, N, \\
& g_4(\mathbf{h}_i) = 1, \\
& g_5(h_{ij}) \geq 0, \quad j = 1, \dots, N, \\
& X_i \geq 0, \\
& Y_{ij} \geq 0, \quad j = 1, \dots, N, \\
& h_{ij} \geq 0, \quad j = 1, \dots, N.
\end{aligned} \tag{5.28}$$

### 5.2.6 Time Line

For the economy defined above, we now summarize the sequence of events, agents' actions, and the corresponding equilibria.

On date 0, each Type 1 trader given her endowment  $e_i$  and trading intensity  $\lambda_i$  maximizes her post-investment residual surplus by strategically choosing her portfolio holding  $\{X_i, \mathbf{Y}_i\}$  and her trading intensity allocation  $\{\mathbf{h}_i\}$ , subject to some external constraints (i.e. the total investment constraint  $F_i$ , pre-mature liquidation constraint  $V_i$ , and market clearing constraints  $k_j$ ). The initial equilibrium is reached at the optimal portfolio holdings  $\{X_i^*, \mathbf{Y}_i^*\}$ , the optimal trading intensity allocations  $\{\mathbf{h}_i^*\}$ , and the equilibrium bond prices  $P^0(\mu_j^*)$ .

On date 1, Type 1 traders subject to early liquidation sell some of their assets back to the marketplace. Since date 1 (but before date 2), Type 2 traders randomly come to the market to buy bonds. The market maker stands ready to bridge the gap between the sellers and the buyers by seeking the spreads which equilibrate the market. Under Assumption (M), the subsequent equilibria are reached at the market spreads  $S(\mu_j^*)$ .



**Figure 5.4:** Time Line

This figure illustrates the time line of the economy. Above the time line are the exogenous variables and below the time line are the appearance of the agents and endogenous variables.

## 5.3 Characterization of Equilibrium

In this section, we describe how to obtain a characterization of the initial equilibrium. The objective of each investor  $i$  is to maximize the residual surplus given in (5.27) by choosing her portfolio holding  $\{X_i, \mathbf{Y}_i\}$ , and her trading pattern  $\{\mathbf{h}_i\}$  and subject to some external constraints.

Now we can define the initial equilibrium:

**Definition 6.** *An equilibrium (on date 0) consists of a collection of investors' portfolio holdings  $\{\{X_i^*, \mathbf{Y}_i^*\}_{i \in \mathcal{M}}\}$ , trading patterns  $\{\{\mathbf{h}_i^*\}_{i \in \mathcal{M}}\}$ , and a collection of bond prices  $\{\{P^0(\mu_j^*)\}_{j \in \mathcal{N}}\}$ , such that the maximum in (5.27) is reached for all  $i$  and the constraints in (5.28) are also satisfied.*

The Lagrangian for the objective function in (5.27) that is to be maximized is:

$$\begin{aligned}
 L(X_i, \mathbf{Y}_i, \mathbf{h}_i) &= f(X_i, \mathbf{Y}_i, \mathbf{h}_i) + \nu(g_1(X_i, Y_i) - F_i) + \omega(g_2(X_i, \mathbf{h}_i) - V_i) + \sum_{j=1}^N \phi_j g_5(h_{ij}) \\
 &\quad - \sum_{j=1}^N \eta_j(g_3(X_i, Y_{ij}, h_{ij}) - k_j) - \xi(g_4(\mathbf{h}_i) - 1) \\
 &\quad + \epsilon X_i + \sum_{j=1}^N \theta_j Y_{ij} + \sum_{j=1}^N \varphi_j h_{ij}.
 \end{aligned} \tag{5.29}$$

Based on the above Lagrangian, the first order conditions implies that: the equalities are

$$\frac{\partial L}{\partial X_i} = 0 \tag{5.30}$$

$$\frac{\partial L}{\partial Y_{ij}} = 0, \quad j = 1, \dots, N, \tag{5.31}$$

$$\frac{\partial L}{\partial h_{ij}} = 0, \quad j = 1, \dots, N, \tag{5.32}$$

### 5.3 Characterization of Equilibrium

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together with the  $3N + 3$  complementary slackness conditions

$$\nu(g_1(X_i, Y_i) - F_i) = 0 \quad (5.33)$$

$$\omega(g_2(X_i, \mathbf{h}_i) - V_i) = 0 \quad (5.34)$$

$$\phi_j g_5(h_{ij}) = 0, \quad j = 1, \dots, N, \quad (5.35)$$

$$\epsilon X_i = 0 \quad (5.36)$$

$$\theta_j Y_{ij} = 0, \quad j = 1, \dots, N, \quad (5.37)$$

$$\varphi_j h_{ij} = 0, \quad j = 1, \dots, N, \quad (5.38)$$

and the equality constraints

$$\eta_j(g_3(X_i, Y_{ij}, h_{ij}) - k_j) = 0, \quad j = 1, \dots, N, \quad (5.39)$$

$$\xi(g_4(\mathbf{h}_i) - 1) = 0. \quad (5.40)$$

This gives  $6N + 5$  equations for  $2N + 1$  variables  $X_i, Y_{ij}, j = 1, \dots, N, h_{ij}, j = 1, \dots, N$ , and  $4N + 4$  Lagrange multiplier  $\nu, \omega, \phi_j, \epsilon, \theta_j, \varphi_j, \eta_j, \xi, j = 1, \dots, N$ . These equality conditions go along with  $6N + 6$  inequalities

$$\nu \geq 0 \quad ; \quad g_1(X_i, Y_i) \geq F_i \quad (5.41)$$

$$\omega \geq 0 \quad ; \quad g_2(X_i, \mathbf{h}_i) \geq V_i \quad (5.42)$$

$$\phi_j \geq 0 \quad ; \quad g_5(h_{ij}) \geq 0, \quad j = 1, \dots, N, \quad (5.43)$$

$$\epsilon \geq 0 \quad ; \quad X_i \geq 0 \quad (5.44)$$

$$\theta_j \geq 0 \quad ; \quad Y_{ij} \geq 0, \quad j = 1, \dots, N, \quad (5.45)$$

$$\varphi_j \geq 0 \quad ; \quad h_{ij} \geq 0, \quad j = 1, \dots, N. \quad (5.46)$$

One can use these first-order conditions to characterize the equilibrium. In other words, one can identify the equilibrium by solving simultaneously the set of non-linear first-order conditions, equality and inequality constraints for all agents.



## 5.4 Quantitative Implications

To study the quantitative implications of our model, we consider the case where the asset menu available to two homogeneous agents (e.g. with respect to the constraints), who seek to maximize their residual surplus after the initial investments on date 0, consists of three bonds (e.g. bond 1, 2, and 3). In addition to Assumption (A), (B) and (C), we assume that each of the three bonds are in supply 100 dollars (i.e.  $k_1 = k_2 = k_3 = \$100$ ).

Moreover, we use the following parameter values. We assume that on the offering date 0 the investors, each with an endowment of 150 dollars (i.e.  $e_i = 150$ ,  $i = 1, 2$ ) and a trading intensity of 10 (i.e.  $\lambda_i = 10$ ,  $i = 1, 2$ ), seek to buy the bonds to form their investment portfolios. For the total investment constraint, we assume that each investor is required to realize a total face value of 150 dollars at the maturity (i.e.  $F_i = 150$ ,  $i = 1, 2$ ). For the early liquidation constraint, we vary the ratio of the minimum value required in the event of early liquidation to the required total face value from 0 up to near 100 percent (i.e.  $\frac{V_i}{F_i} \in [0, 100\%)$ ,  $i = 1, 2$ ). We will refer to it as the early liquidation ratio and for short the ratio hereon.

We consider the setup where the competitive market maker has an exponential utility with the absolute risk aversion coefficient  $\gamma = 1$ . The cutoff value  $C$  is assumed to be equal to 1.<sup>1</sup> This implies that the market maker is only willing to make a market for a bond which they expect to be able to unwind before the bond matures (i.e. the expected waiting time is less than 1 which is the time interval between the interim date 1 and the maturity date 2.). For the price dynamics of the bonds, we assume that the volatility  $\sigma$  is equal to 0.5.

### 5.4.1 Trading Behavior

We first consider the trading behaviors of the two agents in the equilibrium; one can imagine this to be the case where each trader needs to allocate her trading intensity among the three bonds.

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<sup>1</sup>Altering the cutoff value will not change the results qualitatively.

## 5.4 Quantitative Implications

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In Figure 5.5, we present the equilibrium allocations of the normalized trading intensity of one of the agents. As we can see in Figure 5.5, when the early liquidation ratio is below 33.10%, trading is concentrated almost on a single bond whose trading intensity is 18. The trading intensities of both the other two bonds are equal to 1 which is defined as the cutoff value of a bond being illiquid. Correspondingly, Figure 5.6 shows that there exists only one tradable bond when the ratio is below 33.10%.

When the quantity of bonds traded implies that there will be some tradable and non-tradable bonds (e.g. the case described above), the identity of the ones which are tradable could be the result of a ‘Sunspot’ equilibrium, where it is optimal for traders to randomly label a subset of the bonds as the ‘liquid’ ones and concentrate trading on them. A ‘Sunspot’ equilibrium is characterized as a randomization over multiple certainty equilibria. In other words, the bonds become liquid only because traders believe they will be liquid. It need not be anything fundamental about the condition of those bonds. Anything that causes them to anticipate a bond to be liquid will lead to concentration of trading in that bond. A concentration of trading in liquid bonds results in liquidity externalities. Bringing traders together creates liquidity externalities because the additional traders arriving in the marketplace reduce trading costs for all traders.

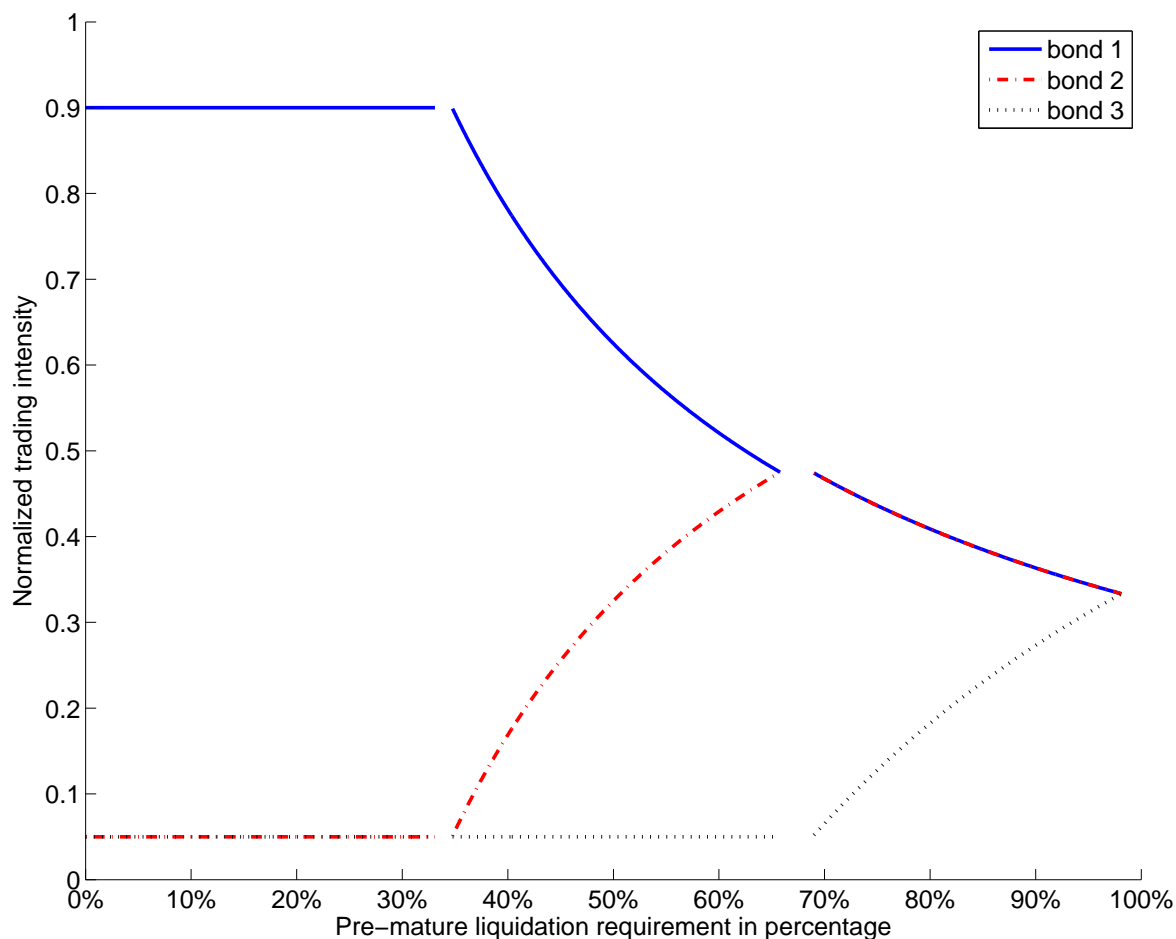
There is extensive evidence that investors benefit from a liquidity externality when they concentrate their trades in liquid assets. Amihud & Mendelson (1991) compare the yields on short-term U.S. Treasury notes and bills both of which are short-term single-payment instruments with the same maturities of 6 months or less, generating the same cash flows and having identical risk. However, their liquidity is different. Duffie (1996) mentions that it is common for traders to roll a proportion of their positions into the successive current issues. Similar to our argument, the tendency for reduced liquidity over time is to some extent a self-fulfilling prophecy, since expecting lower liquidity will themselves lower liquidity. Krishnamurthy (2002) find that variations in the off-the-run/on-the-run bond spread are driven by aggregate factors related to investors’ preference for liquid assets, although Goldreich *et al.* (2005) point out that the spread can also be related to the expected future liquidity. Recently, Krishnamurthy & Vissing-Jorgensen (2012) examine the spread between the interest rates customers receive

on 6-month Federal Deposit Insurance Corporation (FDIC) insured certificates of deposit (CDs) and 6-month Treasury bills, and find that the spread can only be affected by liquidity.

Without loss of generality, when there is only one bond that is traded, bond 1 is labeled as the liquid one. Subsequently, when only two bonds are tradable, bond 1 and 2 are labeled as the liquid ones. While the ratio increases from 33.10% up to 65.79%, the normalized trading intensity of bond 1 decreases from 0.9 to 0.475, and the normalized trading intensity of bond 2 increases from 0.05 to 0.475. At the same time the normalized trading intensity of bond 3 remains unchanged at 0.05. Figure 5.6 shows that the number of tradable bonds becomes two when the ratio is in between 33.10% and 65.79%.

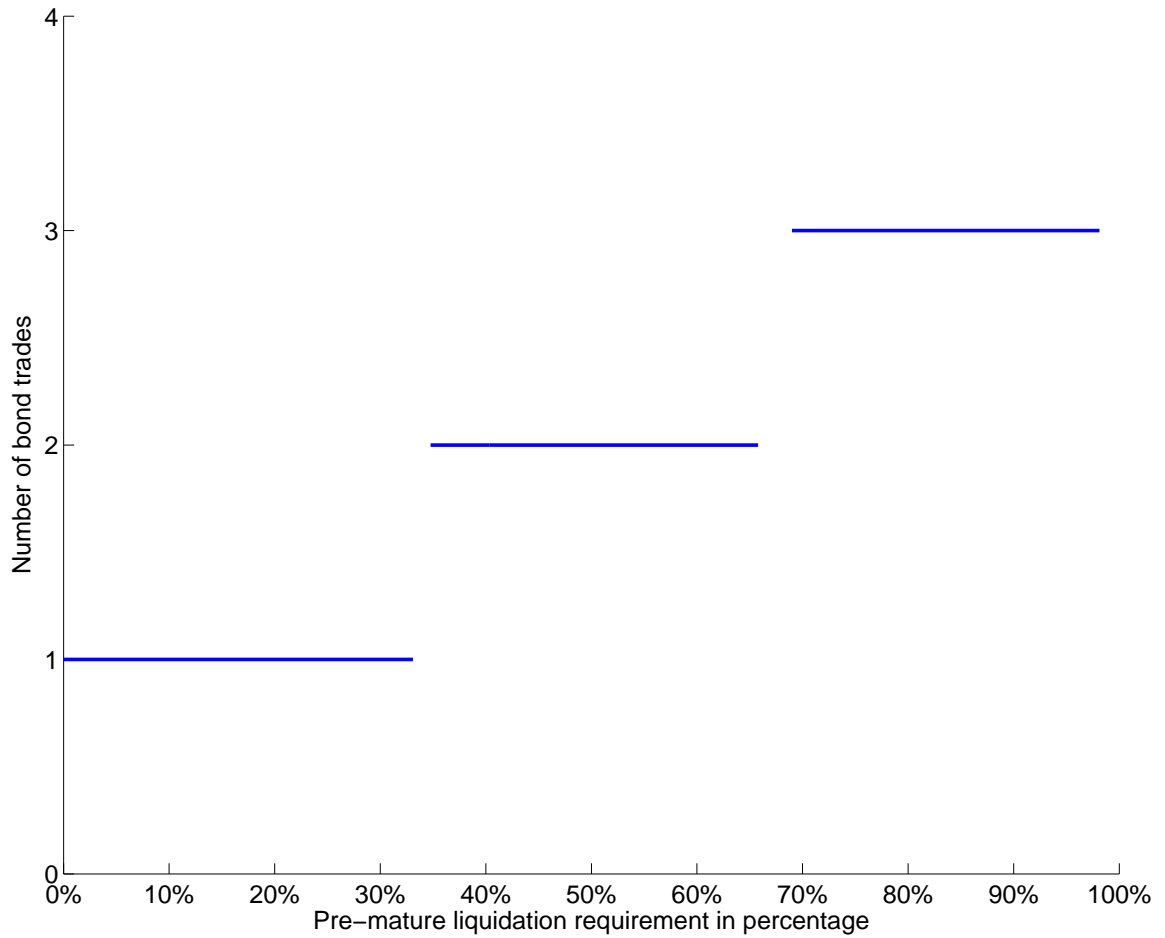
As the ratio goes beyond 65.79%, the normalized trading intensities for both bond 1 and 2 decrease from 0.475 to  $1/3$ , whereas the normalized trading intensity of bond 3 increases from 0.05 to  $1/3$ . That means that, with  $\mu_1 = \mu_2 = \mu_3 = 20/3$ , investors can achieve the highest early liquidation ratio. As shown in Figure 5.6, all the three bonds are tradable or liquid when the ratio is above 65.79%.

In summary, for low early liquidation ratios, investors have strong preference for concentrating trading on a small number of bonds. While the ratio increases, investors start to spread trading over more bonds. Figure 5.6 shows that as the ratio increases, the number of tradable bonds increases stepwise. Furthermore, Figure 5.7 indicates that by concentrating trading on a limited number of bonds investors could gain more surplus, whereas spreading trading over more issues reduces their residual surplus and achieves higher liquidation ratio. The distance between the blue solid line and the black dashed line is essentially the extra surplus that investors gain by concentrating liquidity. The result is partially consistent with Economides & Siow (1988) who show that liquidity consideration will limit the number of markets in a competitive economy. Investors may prefer fewer markets so that liquidity is enhanced in those markets, even though they will have fewer assets to choose from.



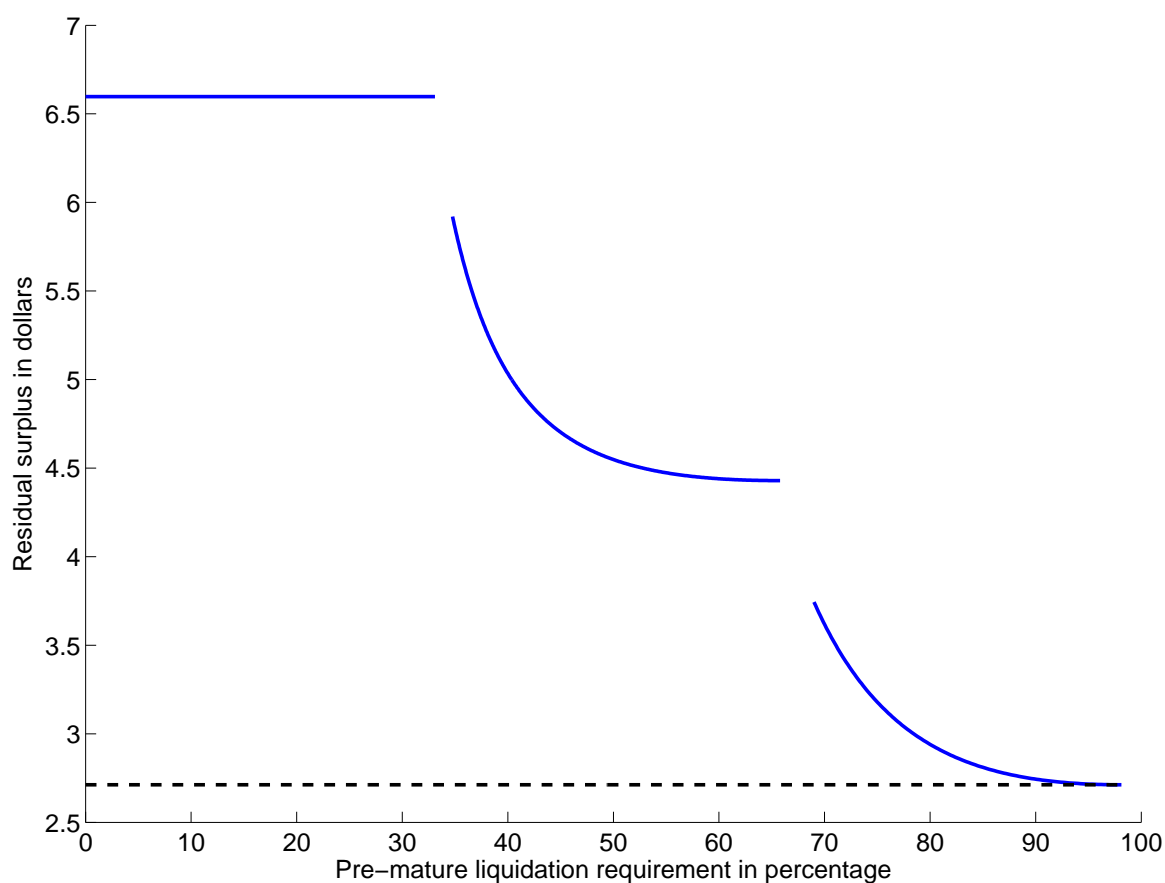
**Figure 5.5:** Trading Pattern

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. The blue solid line refers to the normalized trading intensity of agent  $i$ ,  $i = 1, 2$ , for bond 1, the red dash-dot line refers to the normalized trading intensity of agent  $i$ ,  $i = 1, 2$ , for bond 2, and the dotted line refers to the normalized trading intensity of agent  $i$ ,  $i = 1, 2$ , for bond 3. On the x-axis is the early liquidation ratio.



**Figure 5.6:** Number of Bond Trades

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. The blue solid line refers to the number of tradable bonds. On the x-axis is the early liquidation ratio.



**Figure 5.7:** Liquidity Externality

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. The blue solid line refers to the post-investment residual surplus (i.e. cash saving from investing partly in cheaper illiquid bonds); the black dashed line refers to the post-investment residual surplus if agents choose to spread trading equally across bonds. On the x-axis is the early liquidation ratio.

### 5.4.2 Portfolio Holdings

In Figure 5.8, we present the tradable shares held by one of the agents. As the early liquidation ratio starts to increase, the tradable shares of bond 1 increase sharply, whereas a very large proportion of the other two bonds remains illiquid.<sup>1</sup> When the ratio passes 33.10%, the non-tradable shares of bond 1 decrease to zero. Meanwhile more tradable shares are being supplied by bond 2. In the end, investors need to keep all their portfolio holdings tradable in order to fulfill the early liquidation constraint. For instance, in the top panel of Figure 5.9, when the ratio is equal to 33.10%, investors are better off by keeping only bond 1 tradable and locking away the other two bonds. Then, in the middle panel where the ratio is increased to 65.79%, in order to fulfill a higher early liquidation constraint, all the shares of bond 1 and 2 need to be tradable. Finally, in the bottom panel investors need to have all their holdings of bond 1, 2 and 3 tradable so that they can achieve an even tougher constraint, which means that they anticipate to unwind all their holdings prior to maturity.

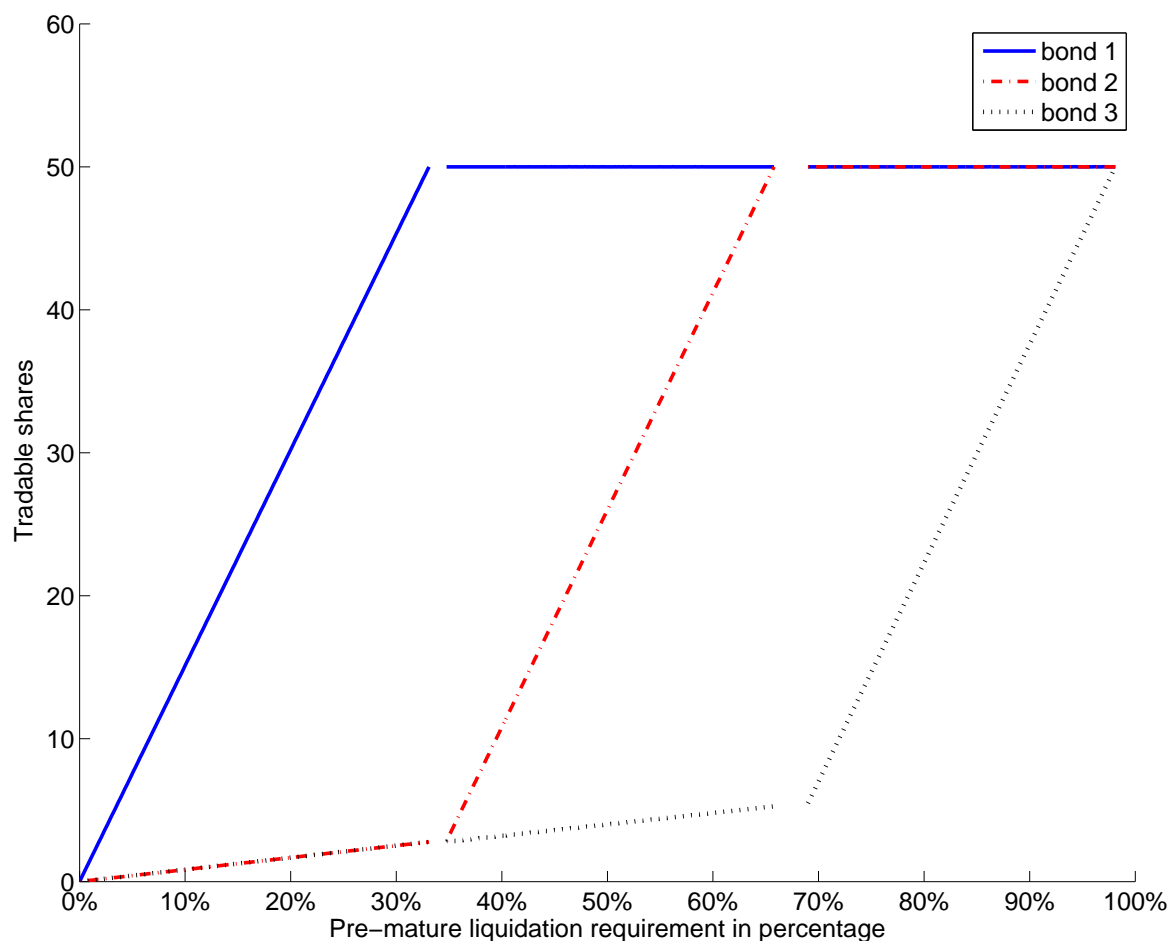
In summary, the aggravated tradable (non-tradable) shares increase (decrease) as the early liquidation ratio increases. Intuitively, for low early liquidation ratios, investors have a certain degree of freedom to fill a large part of their investment portfolios with cheap and illiquid bonds. For high early liquidation ratios, in order to avoid high transaction costs investors need to prepare themselves with more liquid bonds. Therefore, our model predicts that economic conditions, or funding conditions of investors or firms may affect liquidity demands. The equilibrium results from the behaviors of the forward looking investors who increase demand for liquid assets ahead of economic downturns. This is partially consistent with Holmstrom & Tirole (2001) show that leverage ratios and capital adequacy requirements affect the corporate demand for liquid assets.

### 5.4.3 Bond Prices

In Figure 5.10, we present the equilibrium bond prices on date 0. As we know from the model, the bond prices are exclusively determined by the trading intensities

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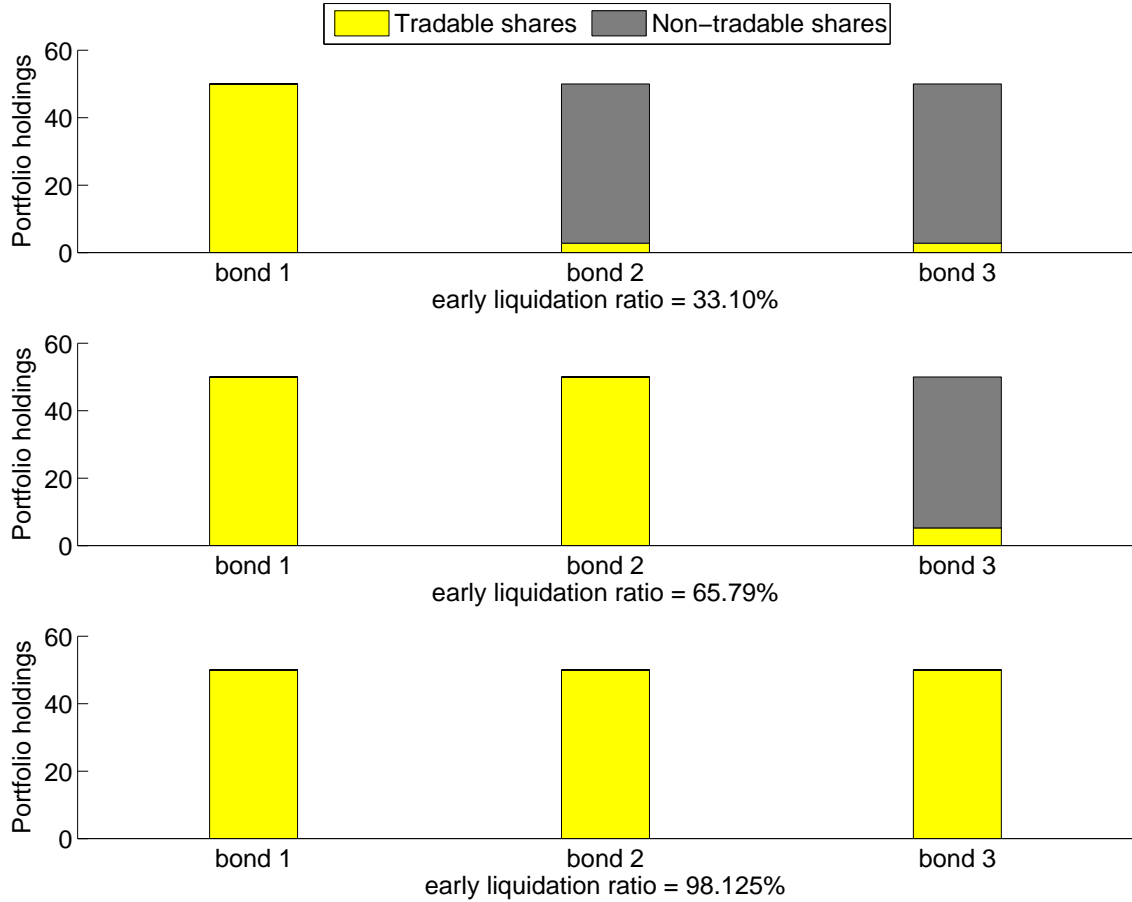
<sup>1</sup>The resulting tradable shares correspond to the equilibrium with equality pre-mature liquidation constraint.



**Figure 5.8:** Tradable Shares

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. The blue solid line refers to the tradable shares of bond 1 held by agent  $i$ ,  $i = 1, 2$ , the red dash-dot line refers to the tradable shares of bond 2 held by agent  $i$ ,  $i = 1, 2$ , and the dotted line refers to the tradable shares of bond 2 held by agent  $i$ ,  $i = 1, 2$ . On the x-axis is pre-mature liquidation requirement which is expressed in terms of the ratio of the required early liquidation amount to the required total face value.





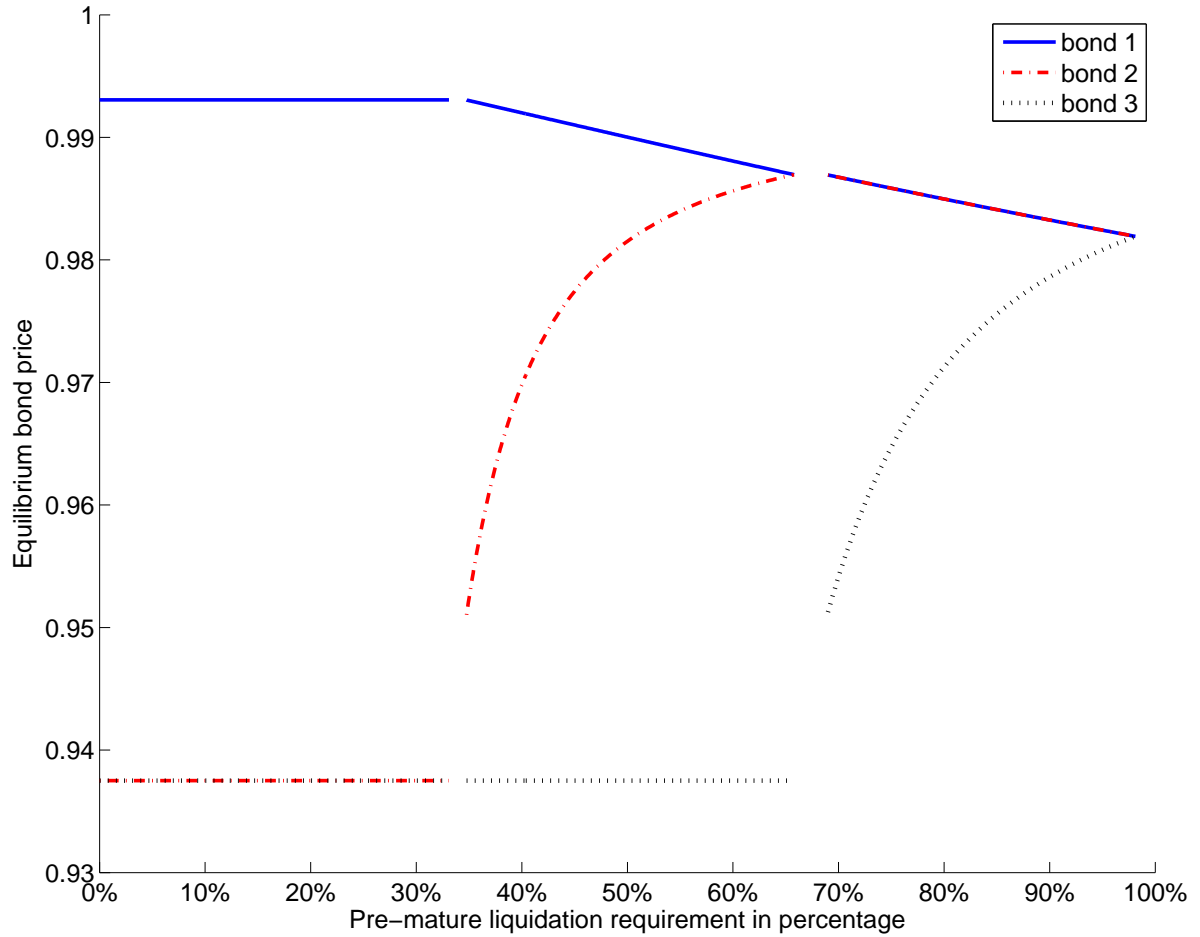
**Figure 5.9:** Examples

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. Colorful bars represent the portfolio holdings of agent  $i$ ,  $i = 1, 2$ . The top panel shows the portfolio holdings in the case where the early liquidation ratio is equal to 33.10%, the middle panel shows the portfolio holdings in the case where the ratio is 65.79%, and finally the bottom panel shows the portfolio holdings in the case where the ratio is 98.125%. Bright yellow area represents the tradable shares held by the agent, and the dark gray area represents the non-tradable shares.

or the arrival rates of the bonds. When the early liquidation ratio is below 33.10%, the equilibrium price of bond 1 is much higher than those of the other two bonds. As the ratio increases from 33.10% to 65.79%, the price of bond 2 increases steeply, while the price of bond 1 decreases very slowly. When the ratio goes beyond 65.79%, the prices of the three bonds finally converge.

The feature, that bond prices increase faster than that they decrease, is the ultimate reason that investors have strong preference for concentrating trading. As investors spread trading over more bonds, the surplus they lose from making more bonds tradable is much higher than the surplus gained by reducing the prices of the perviously liquid bonds. Therefore, investors have an incentive to concentrate trading, subject to economic or funding conditions.

In addition, one could imagine that agents with high early liquidation ratios are short-term investors, and agents with low early liquidation ratios are long-term investors. Long-term investors prefer high-yield illiquid bonds to low-yield liquid ones, whereas short-term investors having in mind future transaction costs would like to have more liquid bonds in their portfolios. This clientele effect is well-documented in the literature (See [Amihud & Mendelson \(1986\)](#), [Vayanos & Vila \(1999\)](#), and [Vayanos & Wang \(2007\)](#)).



**Figure 5.10:** Equilibrium Bond Price

This figure provides selected results from solving the equilibrium on date 0. The economy has two homogeneous agents maximizing their post-investment residual surplus subject to some external constraints. The blue solid line refers to the equilibrium price of bond 1, the red dash-dot line refers to the equilibrium price of bond 2, and the dotted line refers to the equilibrium price of bond 3. On the x-axis is pre-mature liquidation requirement which is expressed in terms of the ratio of the required early liquidation amount to the required total face value.

## **5.5 Summary and Conclusion**

Theoretical studies normally impose ex ante assumptions on asset liquidity. Our study takes a step forward in bridging such a gap by developing an equilibrium model in which trading concentration is created endogenously.

We show that (otherwise identical) bonds may differ in liquidity. Investors have strong preference for trading concentration, subject to liquidity constraints. We also show that the identity of which bonds do trade could be a result of a ‘Sunspot’ equilibrium. Changing liquidity constraints can lead to different equilibrium configurations where trading is spread over bonds. By capitalizing the concepts of stochastic dominance and utility indifference price, and some specific assumptions including the assumptions on the asset value and the order arrival rate, the equilibrium prices and bid-ask spreads can be quantified.

# Chapter 6

## Further Research

There are potentially a few directions towards which we can further our current studies. We will briefly discuss them in this section.

### 6.1 Higher Frequency and More Bonds

In our current study we analyze the corporate bond liquidity at a daily frequency, it would be interesting to look at how liquidity and liquidity premium behave at a higher frequency level, for instance, at a trade-by-trade frequency.

Moreover, one of the main limits of our extended Roll model is that the variance of the nonpreservable efficient price is time-invariant. In the literature of estimating realized variance, [Bandi & Russell \(2006\)](#) use sample moments of high frequency returns to identify both the time-varying variance of the unobservable efficient returns and the variance of the microstructure noise which in our model is referred as the half spread. [Hansen & Lunde \(2006\)](#) study the implications of the market microstructure noise in high-frequency data for the realized variance under a general specification for the noise. [Bandi & Russell \(2008\)](#) derive a MSE optimal high-frequency sampling theory to reduce the microstructure noise-induced bias. Although the focus of these papers is on the realized variance, combining the time-varying component and high frequency data with a more realistic microstructure component (e.g. discreteness, different type of trades, and correlation between

the underlying price process and the microstructure noise) will be producing a cleaner estimation of the bid-ask spread.

Our empirical study only focuses on bonds issued by a specific company. [Blanco \*et al.\* \(2005\)](#) use daily mid-market quotes of CDS and daily interpolated bond yields on 33 reference entities to study the empirical relation between CDSs and credit spreads. The sample used by [Longstaff \*et al.\* \(2005\)](#) consists of CDS premia for 5-year contracts and corporate bond prices for 68 firms (Aaa through BB). [Bao \*et al.\* \(2011\)](#) estimate their illiquidity measure by using either trade-by-trade prices or end of the day prices of 1032 bonds (Aaa through C). Although the availability and accuracy of high frequency transaction data of both CDSs and corporate bonds significantly reduce the potential dataset, it is still possible to apply our methodology to a larger sample set by including bonds with different ratings or bonds issued by firms in different countries or industries. With a larger set of data, we will be able to examine liquidity both at the market level and at the individual bond level. We can analyse liquidity variations across rating categories and sectors.

## 6.2 Heterogenous agents

In our third study, we analyze bond liquidity in an equilibrium economy with multiple agents who are homogeneous (i.e. with respect to liquidity constraints). Other studies also consider agents who are heterogeneous with respect to endowments or preference. These include [Bhamra & Uppal \(2010\)](#) and [Buss \*et al.\* \(2011\)](#) where agents have time-additive preferences or a recursive utility. In our risk neutral approach, to allow agents with heterogeneous liquidity constraints, which could lead to the case where agents with different trading allocations, might be more challenging than the homogeneous case which is already complicated enough.

# Chapter 7

## Conclusions

In this thesis we study liquidity in bond markets from both the empirical and theoretical point of view. In the following, we recapitulate the key findings and conclusions of our research.

### 7.1 Summary of Major Findings

#### 7.1.1 An Extended Model of Estimating Effective Bid-ask Spread

In this study we present an extended model for the estimation of the effective bid-ask spread that allows trades to execute either inside or outside the quoted bid-ask spread. We extend the Roll model by adding an extra parameter  $\lambda$ , the so-called ‘spread multiplier’, which is constructed to separate spreads of different magnitudes. In other words, we generalize Roll’s spread estimator (a scalar) to include a vector of spreads with associated probabilities.

The extended model is estimated by a Bayesian Gibbs approach based on only transaction data. Since we treat the models with different values of  $\lambda$  as competitive models, the value of  $\lambda$  is determined via a Bayesian model selection method by comparing marginal likelihoods evaluated from the Gibbs outputs.

The new model together with the model estimation and selection procedures offers an alternative way to more accurately estimate the effective bid-ask spread

and the underlying return variance.

### 7.1.2 Non-default Yields Spreads and Illiquidity of Corporate Bonds

In this study, we use the data from both credit default swap and corporate bond markets to study the relationships between different liquidity proxies and the non-default component of corporate bond spreads as well as our illiquidity measure.

Firstly, we separate the non-default component of bond spreads from the default one by applying a non-parametric reduced-form credit risk model to simultaneously price credit default swaps and corporate bonds.

We then apply a Kalman filter extension of the original Roll model to estimate the unobservable non-default bond residuals as well as our illiquidity measure.

By using the panel data during 2006 - 2010, we examine the relationships between different liquidity proxies and the non-default component of corporate bond yield spreads as well as the illiquidity measure via panel regressions. The empirical results that the illiquidity measure is related to both the bond characteristics and the liquidity proxies based on the transaction-level data, therefore reflecting the current level of market liquidity. We also find that the non-default component of the yield spreads has a nonlinear and positive relationship with the time-to-maturity and a positive relationship with the illiquidity measure as well as the default risk, therefore reflecting the future expected liquidity.

### 7.1.3 An Equilibrium Model of Liquidity in Bond Markets

This study provides an alternative theory of liquidity, exclusively based on inventory holding risk, to understand why there exist heterogeneous levels of liquidity among otherwise identical bonds.

We develop an equilibrium model in which the identity of which bonds are liquid, and how large their spreads and liquidity discounts (or liquidity premium) are, are determined endogenously. We assume that in the economy a number of (otherwise identical) bonds are exchanged among the market participants: a



## 7.1 Summary of Major Findings

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set of risk neutral investors with exogenously determined cash flow and liquidity constraints form their investment portfolios by maximizing their post-investment residual surplus on the offering date; another set of risk neutral traders come later on to buy these bonds; a risk averse competitive market maker acts as the whole financial intermediaries by bridging the gaps between buyers and sellers and charges spreads for bearing the inventory-holding risk.

By utilizing the concepts of stochastic dominance, utility indifference pricing and the assumptions on the dynamics of bond prices and the order arrival rate, the equilibrium bond prices and bid-ask spreads can be quantified.

Investors have strong preference for trading concentration, subject to liquidity constraints. In particular, when the quantity of bonds which do trade implies that there will be some tradable and non-tradable bonds, the identity of the ones which do trade could be the result of a ‘Sunspot’ equilibrium. Traders with tougher liquidity constraints need to spread trading over more bonds and have more tradable shares in their portfolios. Therefore, asset holdings and trading patterns are sensitive to liquidity constraints.

# Appendix A

## Figures and Tables

**Table A.1:** Regression of the Illiquidity on Liquidity and Credit Proxies with Sub-periods.

	Coefficients on equations						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Cons	2.22**	2.26**	2.01**	2.01**	1.94**	2.02**	1.92**
	[3.40]	[3.23]	[3.02]	[3.05]	[3.17]	[3.16]	[2.76]
Exp(-Maturity)		-1.05**	-1.02**	-0.89**	-0.96**	-0.94**	-0.97**
		[-3.48]	[-4.35]	[-3.41]	[-4.33]	[-3.40]	[-3.85]
Log(Issuance)	-0.27**		-0.23**	-0.23**	-0.23**	-0.21**	-0.21**
	[-3.12]		[-2.60]	[-2.64]	[-2.87]	[-2.55]	[-2.27]
Turnover							
Log(Trade Size)							
Log(# Trades)							
Volatility	0.33**	0.32**		0.33**	0.34**	0.33**	0.33**
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Coefficients on equations							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	[6.52]	[6.83]		[6.81]	[6.99]	[6.96]	[6.86]
CDS Spread	0.21**	0.12**	0.17**		0.20**	0.15**	0.16**
	[6.21]	[5.04]	[5.97]		[6.71]	[4.92]	[6.30]
Exp(-Maturity) I	-3.77**						
	[-2.93]						
Exp(-Maturity) II	-1.97**						
	[-2.05]						
Exp(-Maturity) III	-1.85**						
	[-3.75]						
Exp(-Maturity) IV	-0.99**						
	[-5.24]						
Log(Issuance) I		-0.28**					
		[-3.00]					
Log(Issuance) II		-0.27**					
		[-2.91]					
Log(Issuance) III		-0.24**					
		[-2.59]					
Log(Issuance) IV		-0.26**					
		[-2.93]					
Volatility I			-0.26**				
			[-2.78]				
Volatility II			0.17**				
			[3.85]				
Volatility III			0.36**				
			[6.11]				
Volatility IV			0.09				
			[0.82]				
Continued on next page							

	Coefficients on equations						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
CDS Spread I				-0.68**			
				[-2.34]			
CDS Spread II				0.10**			
				[3.31]			
CDS Spread III				0.17**			
				[6.06]			
CDS Spread IV				0.11**			
				[4.05]			
Turnover I					-2.55		
					[-0.99]		
Turnover II					-6.05**		
					[-3.72]		
Turnover III					-12.08**		
					[-2.74]		
Turnover IV					-6.58*		
					[-1.83]		
Log(Trade Size) I						-0.05**	
						[-5.09]	
Log(Trade Size) II						-0.04**	
						[-5.55]	
Log(Trade Size) III						-0.01	
						[-0.77]	
Log(Trade Size) IV						-0.04**	
						[-3.65]	
Log(# Trades) I							-0.08**
							[-3.79]
Log(# Trades) II							-0.05**

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Coefficients on equations							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
							[-4.16]
Log(# Trades) III							-0.01
							[-0.58]
Log(# Trades) IV							-0.05**
							[-2.04]
Obs	10196	10196	10196	10196	10183	10184	10195
$R^2$	20.36%	21.01%	20.87%	20.85%	21.07%	21.52%	20.68%

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**Table A.2:** Regression of the Non-default Yield Spread on Liquidity and Credit Proxies with Sub-periods.

	Coefficients on equations				
	(1)	(2)	(3)	(4)	(5)
Cons	0.52** [2.94]	1.06** [4.39]	0.43** [2.46]	0.84** [4.87]	0.67** [4.07]
Exp(-Maturity)		-2.09** [-12.17]	-1.92** [-16.93]	-1.94** [-16.66]	-2.31** [-21.84]
Log(Issuance)	0.05** [2.16]		0.10** [4.51]	0.08** [3.67]	0.02 [0.75]
Turnover					
Log(Trade Size)					
Log(# Trades)					
Bid-ask Spread	0.06** [17.83]			0.06** [18.18]	
Coupon					
Volatility	-0.00 [-0.63]	0.00 [0.91]		-0.01* [-1.69]	-0.00 [-0.91]
CDS Spread	0.17** [20.89]				0.22** [29.72]
Exp(-Maturity) I	-16.74** [-9.66]				
Exp(-Maturity) II	-0.84 [-1.34]				

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	Coefficients on equations				
	(1)	(2)	(3)	(4)	(5)
Exp(-Maturity) III	-0.74**				
	-3.04				
Exp(-Maturity) IV	-2.90**				
	[-24.15]				
Log(Issuance) I		-0.07**			
		[-2.01]			
Log(Issuance) II		0.01			
		[0.42]			
Log(Issuance) III		0.13**			
		[4.12]			
Log(Issuance) IV		-0.03			
		[-0.82]			
Volatility I			-0.60**		
			[-10.84]		
Volatility II			-0.05**		
			[-1.98]		
Volatility III			0.08**		
			[14.99]		
Volatility IV			-0.14**		
			[-8.28]		
CDS Spread I				-6.64**	
				[-24.57]	
CDS Spread II				-0.29**	
				[-8.89]	
CDS Spread III				0.11**	
				[13.51]	
CDS Spread IV				-0.33**	

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	Coefficients on equations				
	(1)	(2)	(3)	(4)	(5)
				[-20.36]	
Illyq I					-0.07**
					[-3.54]
Illyq II					0.03*
					[1.90]
Illyq III					0.13**
					[25.23]
Illyq IV					-0.07**
					[-6.93]
Obs	10196	10214	10214	10196	10196
$R^2$	34.62%	31.60%	24.28%	41.29%	34.44%

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**Table A.3:** Summary Statistics.  
The illquidity measure in dollars; the non-default yield spread in percentage

	observation	the illquidity measure			the non-default yield spread		
		mean	std dev	obs	mean	std dev	
bond01	1158	0.4066	0.4671	1162	0.9688	1.1594	
bond02	1148	1.0097	1.1911	1152	1.0505	0.7878	
bond03	1146	0.5457	0.8848	1150	1.0423	1.0378	
bond04	1154	0.8139	1.1600	1158	1.1150	0.9792	
bond06	811	0.4469	0.7750	815	0.8868	1.2992	
bond07	845	0.3569	0.5356	847	0.7459	1.3778	
bond08	1141	0.4620	0.6794	1145	0.9112	1.1579	
bond11	778	0.6043	0.8447	780	1.0267	1.2267	
bond12	766	0.4837	0.5865	771	1.1629	1.2200	
bond13	765	0.8689	0.9092	768	1.1551	0.5676	
bond14	520	0.6888	0.7870	522	1.2457	1.1775	
bond15	538	0.6257	0.6845	540	1.3681	1.1935	
bond16	536	0.8922	1.2912	538	1.2890	0.6172	
bond17	537	0.4867	0.5015	539	1.3567	0.8885	
bond18	535	0.6376	0.5715	537	1.1995	0.5454	
bond19	531	0.6123	0.6448	533	1.3229	0.8488	
bond20	290	0.3504	0.3193	291	0.9508	0.2565	
average		0.6054	0.7549		1.1058	0.9612	

# Appendix B

## Useful Information

### B.1 Stochastic Dominance

Stochastic dominance provides a rather general framework in which to model the market maker's decisions. It will also be sufficient for our purposes. Before applying it we will briefly summarize the key results that we will need.

In comparing two prospects, stochastic dominance measures to what extent either has higher probabilities associated with higher payoffs and lower probabilities associated with lower payoffs. Any rational investor (who is an expected utility maximizer) would prefer prospect  $A$  over prospect  $B$  if both have the same possible payoffs, but  $A$  has higher probabilities associated with higher-valued payoffs and lower probabilities associated with lower-valued payoffs.

First-order stochastic dominance (FSD), originally developed by [Quirk & Saposnik \(1962\)](#), attempts to order uncertain prospects assuming only that investors' utility functions are monotonically increasing with respect to wealth. Specifically, prospect  $A$  first-order stochastically dominates Prospect  $B$ , if and only if there exist some  $\xi$  such that  $X_B \stackrel{d}{=} X_A + \xi$  where  $\xi$  is a non-positive random variable<sup>1</sup>. In other words, investors prefer  $A$  to  $B$ , regardless of their risk

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<sup>1</sup>  $\stackrel{d}{=}$  means "is equivalent in distribution to".

preferences, as long as their utility function is monotonically increasing, i.e., more wealth is preferred to less.

Second-order stochastic dominance (SSD) assumes, in addition to monotonic utility functions, weak global risk aversion (i.e., diminishing marginal utility for wealth). SSD was developed by [Hadar & Russell \(1969\)](#). In terms of distribution functions  $F_A$  and  $F_B$ , prospect  $A$  is second-order stochastically dominant over prospect  $B$  if and only if  $\int_{-\infty}^x [F_B(t) - F_A(t)] dt \geq 0$  for all  $x$ , with strict inequality for some  $x$ . In other words, risk-averse investors (who are expected utility maximizers with monotonically increasing and concave utility function) prefer  $A$  to  $B$  if and only if the above integral condition holds on their distribution functions.

## B.2 Mean-Preserving Spreads

A mean-preserving spread (MPS) is a change from one probability distribution  $F_A$  to another probability distribution  $F_B$ , where  $F_B$  is formed by spreading out one or more portions of  $F_A$ 's probability density function while leaving the mean unchanged. As such, the concept of mean-preserving spreads provides a stochastic ordering of equal-mean probability distributions. SSD is closely related to mean-preserving spread through Rothschild and Stiglitz Theorem.

Prospect  $A$  second-order stochastically dominates prospect  $B$ , if and only if there exist some  $\xi$  and  $\epsilon$  such that  $X_B \stackrel{d}{=} X_A + \xi + \epsilon$ , with  $\xi \leq 0$ , and with  $\mathbb{E}(\epsilon | X_A + \xi) = 0$  for all values of  $X_A + \xi$  ( $\epsilon$  is a fair game with respect to  $X_A + \xi$ ).

Therefore, by the Rothschild and Stiglitz Theorem ([Rothschild \(1970\)](#)), if  $A$  and  $B$  have the same mean (so that the random variable  $\xi$  degenerates to the fixed number 0), then  $B$  is second-order stochastically dominated by  $A$  if  $X_B$  differs from  $X_A$  by the addition of mean-preserving spreads  $\epsilon$ .

## B.3 Sum of Poisson Random Variables

Let  $\tilde{t}_1$  and  $\tilde{t}_2$  be independent Poisson random variables where  $\tilde{t}_i$  has a  $\text{Poisson}(\lambda_i)$  distribution for  $i = 1, 2$ . Then,  $\tilde{t}_1 + \tilde{t}_2$  has a Poisson distribution with  $\lambda_1 + \lambda_2$ .

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## B.4 Derivatives in the first order conditions

*Proof.* According to the convolution formula for discrete random variables,  $\tilde{t} = \tilde{t}_1 + \tilde{t}_2$  has probability distribution

$$P(\tilde{t}_1 + \tilde{t}_2 = t) = f_{\tilde{t}}(t) = \sum_{x=0}^t f_{\tilde{t}_1}(x) f_{\tilde{t}_2}(t-x). \quad (\text{B.1})$$

so,

$$f_{\tilde{t}}(t) = \sum_{x=0}^t \frac{\lambda_1^x}{x!} e^{-\lambda_1} \frac{\lambda_2^{t-x}}{(t-x)!} e^{-\lambda_2} \quad (\text{B.2})$$

$$= e^{-(\lambda_1 + \lambda_2)} \sum_{x=0}^t \frac{\lambda_1^x}{x!} \frac{\lambda_2^{t-x}}{(t-x)!} \quad (\text{B.3})$$

$$= e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^t}{t!}. \quad (\text{B.4})$$

□

Let  $\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_M$  be independent Poisson random variables where  $\tilde{t}_i$  has a  $\text{Poisson}(\lambda_i)$  distribution for  $i = 1, 2, \dots, M$ . Therefore,  $\tilde{t}_1 + \tilde{t}_2 + \dots + \tilde{t}_M$  has a  $\text{Poisson}(\lambda_1 + \lambda_2 + \dots + \lambda_M)$  distribution.

## B.4 Derivatives in the first order conditions

$$\begin{aligned} S'_j &= \left( \frac{\gamma \sigma^2}{2 \sum_{i=1}^M \lambda_i h_{ij}} \right)' \\ &= -\frac{\gamma \sigma^2 \lambda_i}{2 \mu_j^2}, \quad \frac{1}{\mu_j} < 1, \end{aligned} \quad (\text{B.5})$$

$$S'_j = 0, \quad \frac{1}{\mu_j} \geq 1, \quad (\text{B.6})$$

$$\begin{aligned}
P'_j &= \left(1 - [1 - e^{-\frac{\mu_j}{N}}]S(\mu_j)\right)' \\
&= -\left(1 - e^{-\frac{\mu_j}{N}}\right) \left(\frac{\gamma\sigma^2}{2\sum_{i=1}^M \lambda_i h_{ij}}\right)' - \left(1 - e^{-\frac{\mu_j}{N}}\right)' \left(\frac{\gamma\sigma^2}{2\sum_{i=1}^M \lambda_i h_{ij}}\right) \\
&= \left(1 - e^{-\frac{\mu_j}{N}}\right) \frac{\gamma\sigma^2 \lambda_i}{2\mu_j^2} - \frac{\lambda_i}{N} e^{-\frac{\mu_j}{N}} \frac{\gamma\sigma^2}{2\mu_j} \\
&= S_j \frac{\lambda_i}{\mu_j} - S_j \frac{\lambda_i}{\mu_j} e^{-\frac{\mu_j}{N}} - S_j \frac{\lambda_i}{N} e^{-\frac{\mu_j}{N}} \\
&= S_j \lambda_i \left[ \frac{1}{\mu_j} - \left(\frac{1}{\mu_j} + \frac{1}{N}\right) e^{-\frac{\mu_j}{N}} \right], \quad \frac{1}{\mu_j} < 1,
\end{aligned} \tag{B.7}$$

$$P'_j = 0, \quad \frac{1}{\mu_j} \geq 1, \tag{B.8}$$

$$\frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial X_i} = -\sum_{j=1}^N h_{ij} P_j^0 \tag{B.9}$$

$$\frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial Y_{ij}} = -P_j^0, \quad j = 1, \dots, N \tag{B.10}$$

$$\frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial h_{ij}} = -P_j^0 X_i - P'_j (h_{ij} X_i + Y_{ij}), \quad j = 1, \dots, N \tag{B.11}$$

$$\frac{\partial g_1(X_i, \mathbf{Y}_i)}{\partial X_i} = 1 \tag{B.12}$$

$$\frac{\partial g_1(X_i, \mathbf{Y}_i)}{\partial Y_{ij}} = 1, \quad j = 1, \dots, N \tag{B.13}$$

$$\frac{\partial g_2(X_i, \mathbf{h}_i)}{\partial X_i} = \sum_{j=1}^N [1 - S_j] h_{ij} \tag{B.14}$$

## B.4 Derivatives in the first order conditions

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$$\frac{\partial g_2(X_i, \mathbf{h}_i)}{\partial h_{ij}} = [1 - S_j]X_i - S'_j h_{ij} X_i, \quad j = 1, \dots, N \quad (\text{B.15})$$

$$\frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial X_i} = h_{ij}, \quad j = 1, \dots, N \quad (\text{B.16})$$

$$\frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial Y_{ij}} = 1, \quad j = 1, \dots, N \quad (\text{B.17})$$

$$\frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial h_{ij}} = X_i, \quad j = 1, \dots, N \quad (\text{B.18})$$

$$\frac{\partial g_4(\mathbf{h}_i)}{\partial h_{ij}} = 1, \quad j = 1, \dots, N \quad (\text{B.19})$$

$$\frac{\partial g_5(h_{ij})}{\partial h_{ij}} = -1, \quad j = 1, \dots, N \quad (\text{B.20})$$

$$\frac{\partial L}{\partial X_i} = \frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial X_i} + \nu \frac{\partial g_1(X_i, \mathbf{Y}_i)}{\partial X_i} + \omega \frac{\partial g_2(X_i, \mathbf{h}_i)}{\partial X_i} - \sum_{j=1}^N \eta_j \frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial X_i} + \epsilon = 0 \quad (\text{B.21})$$

$$\frac{\partial L}{\partial Y_{ij}} = \frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial Y_{ij}} + \nu \frac{\partial g_1(X_i, \mathbf{Y}_i)}{\partial Y_{ij}} - \eta_j \frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial Y_{ij}} + \theta_j = 0, \quad j = 1, \dots, N \quad (\text{B.22})$$

$$\frac{\partial L}{\partial h_{ij}} = \frac{\partial f(X_i, \mathbf{Y}_i, \mathbf{h}_i)}{\partial h_{ij}} + \omega \frac{\partial g_2(X_i, \mathbf{h}_i)}{\partial h_{ij}} + \phi_j \frac{\partial g_5(h_{ij})}{\partial h_{ij}} - \eta_j \frac{\partial g_3(X_i, Y_{ij}, h_{ij})}{\partial h_{ij}} - \xi \frac{\partial g_4(\mathbf{h}_i)}{\partial h_{ij}} + \varphi_j = 0, \quad j = 1, \dots, N \quad (\text{B.23})$$

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