Demystifying generic beliefs on jump models

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In option pricing theory, assuming log-returns which are normally distributed equates to assuming they are driven by a Brownian motion, something we know not being close to reality since the celebrated Black-Scholes formula (Black and Scholes, 1973) itself. Indeed, this very same result offers the way to invalidate the assumption it is based on, once we use it to extract implied volatilities. The smile/smirk shape of the implied volatility across moneyness is a very tangible signal of non-zero skewness and excess kurtosis in the log-returns distribution.

Generalizations of the Black-Scholes paradigm started to appear in the literature relatively soon after the publication of the 1973 result. In the quest of more agile distributions, many authors have stumbled upon jump processes such as Lévy processes or more general constructions like time changed Lévy processes. In spite of the convincing results shown in a literature spanning across several decades (see Eberlein and Keller, 1995, Madan et al., 1998, Kou, 2002, Carr et al., 2002, Carr and Wu, 2004, Ballotta and Rayée, 2022, just to mention some key contributions), jump based models are often met by scepticism. Typical criticisms include the limited tractability of these models, and the fact that they lead to incomplete markets. Furthermore, for many the word jump is synonym of market crash, i.e. an event sufficiently rare not to justify the added mathematical challenges that jumps bring into the modelling.

The idea of this article is to 'demystify' these claims and argue about the validity of jump processes for financial modelling. Let me start from the last of the above mentioned criticisms.

A quick look at the Oxford dictionary and we learn that the word jump means 'to push oneself off a surface and into the air by using the muscles in one's legs and feet'. Therefore a jump spans from small hops such as those we make when jogging, to somewhat bigger ones such as those we make when we jump-rope towards a healthier us, to the jump of a polevaulter over a 6 meters bar - not such a frequent occurrence. In other words, jumps are much more than just big leaps.

Similarly, in mathematics jump means much more than just a rare event of significant severity such as the ones that can be portrayed by a compound Poisson process. If chosen appropriately, a jump process can model movements of any size and frequency, and therefore portray what we observe in the financial markets: small jumps most of the time and the occasional larger movement due to the 'arrival of important new information' (Merton, 1976).

We can see this ability in Figure 1, which shows the time-1 densities of the CGMY process (with drift) of Carr et al. (2002) with characteristic function

$$\phi_{CGMY}(u) = exp(t\ (\mathrm{i}uD + C\Gamma(-Y)((G+\mathrm{i}u)^Y - G^Y + (M-\mathrm{i}u)^Y - M^Y))), \tag{1}$$

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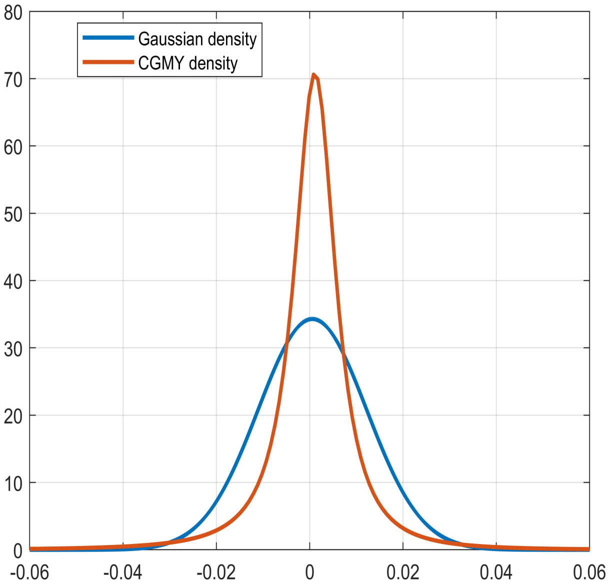

Figure 1: A comparison of densities: CGMY versus normal distribution. CGMY distribution parameters: $D = 0.0012$, $C = 0.0017$, $G = 23.0756$, $M = 32.6267$, $Y = 1.0142$. Normal distribution parameters: $\mu = 0.0006$, $\sigma = 0.0116$.

and the Brownian motion with the same mean and standard deviation. What this plot tells us is that the CGMY process, in spite being a pure jump process, has a distribution which is more peaked at the origin than the normal one, meaning that the process generates tiny movements with higher frequency than the Brownian motion. Furthermore, the CGMY density slims down in the flanks, whilst the tails are considerably thicker than what is predicted by the normal distribution, signalling that, although middle sized movements occur with a lower probability than in the case of the Brownian motion, big changes occur with much higher frequency.

This pattern of events offered by the CGMY distribution, as well as by the class of generalized hyperbolic distributions and its subclasses, does resemble what we observe empirically in the financial market. This is exemplified in Figure 2, which shows the daily log-returns of the S&P500 (observed from January 2016 to November 2021) and the best fitting normal (left-hand side panel) and CGMY (right-hand side panel) distribution. The parameters used in Figure 1 are the ones obtained from this fitting exercise. The more realistic performance of alternative distributions is well worth the added layer of mathematical and computational sophistication.

The second criticism to jump models discussed in this note is the one of limited tractability. Similarly to the large majority of diffusion-based stochastic volatility models currently used, jump models are described by a characteristic function, as seen before. In practical terms, this means that we can recover the densities and the moments of the corresponding distributions, and we can price derivative instruments such as options, even when these quantities are not known in an explicit form. All we need is a robust Fourier based numerical scheme (such as Carr and Madan, 1999, Eberlein et al., 2010, Fang and Oosterlee, 2008). The characteristic function can also be used to develop Monte Carlo simulation algorithms such as the ones proposed by Broadie and Kaya (2006) for the Heston model and Ballotta and Kyriakou (2014) for the CGMY process. Furthermore, all of the above helps us with the interpretability of the models and understand which parameter controls primarily which feature.

For an illustration, let us consider again the CGMY process X_t described by the characteristic

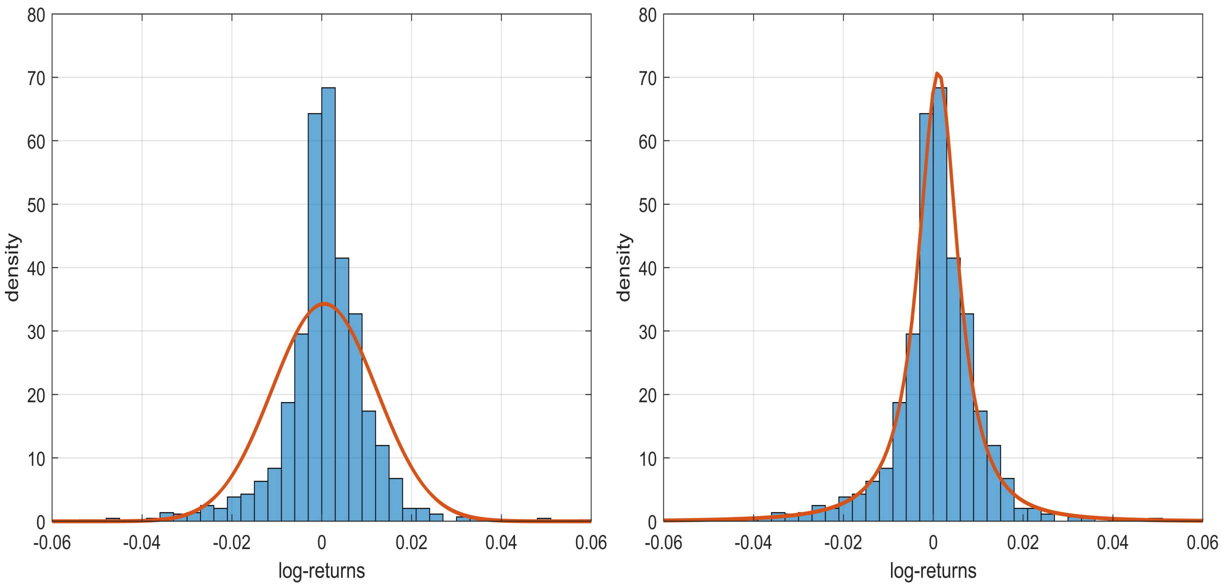

Figure 2: Best fitting CGMY and normal distributions: Maximum Likelihood Estimation (MLE) to S&P500 daily log-returns. Observation period: 01/2016 - 11/2021 (source: yahoo! finance). Left-hand side panel: the normal distribution - MLE parameters: $\mu = 0.0006$, $\sigma = 0.0116$. Right-hand side panel: the CGMY distribution - MLE parameters: $D = 0.0012$, $C = 0.0017$, $G = 23.0756$, $M = 32.6267$, $Y = 1.0142$. CGMY MLE based on the COS method performed for the recovery of the density function.

function (1). By repeated differentiation of the natural logarithm of this characteristic function, i.e. the so-called characteristic exponent, it follows that

$$\begin{aligned}\mathbb{E}(X_t) &= Dt + C\Gamma(1-Y)(M^{Y-1} - G^{Y-1})t \\ \mathbb{V}ar(X_t) &= C\Gamma(2-Y)(M^{Y-2} + G^{Y-2})t,\end{aligned}$$

whilst the indices of skewness and excess kurtosis are

$$sk(X_t) = \frac{C\Gamma(3-Y)(M^{Y-3} - G^{Y-3})}{(C\Gamma(2-Y)(M^{Y-2} + G^{Y-2}))^{3/2}\sqrt{t}}, \qquad ek(X_t) = \frac{C\Gamma(4-Y)(M^{Y-4} + G^{Y-4})}{(C\Gamma(2-Y)(M^{Y-2} + G^{Y-2}))^2 t}$$

respectively. Thus, the drift parameter $D \in \mathbb{R}$ enters only in the mean, whilst the model parameters C, G, M and Y all contribute to all moments of the distribution. The role of D is to avoid that the mean and the skewness have necessarily the same sign. Indeed, bearing in mind that by construction $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$, we can deduce that G and M control the sign of the skewness - a fact from which we can also deduce that the CGMY process can capture different shapes of the implied volatility.

This is illustrated in Figure 3 in which three different values of the parameters G and M are considered. In the first case, $G < M$ and consequently the distribution is negatively skewed and the resulting implied volatility (extracted from call options) is decreasing. In the second case, $G > M$ which originates a positively skewed distribution and an increasing implied volatility. The last case is the one in which $G = M$: the distribution is symmetric and the implied volatility is almost flat. The CGMY density is not known explicitly but we can recover it numerically (in this specific example the COS method of Fang and Oosterlee, 2008 is used), as we can compute numerically also the option prices. The implied volatilities are then obtained, as usual, by inversion of the Black-Scholes formula.

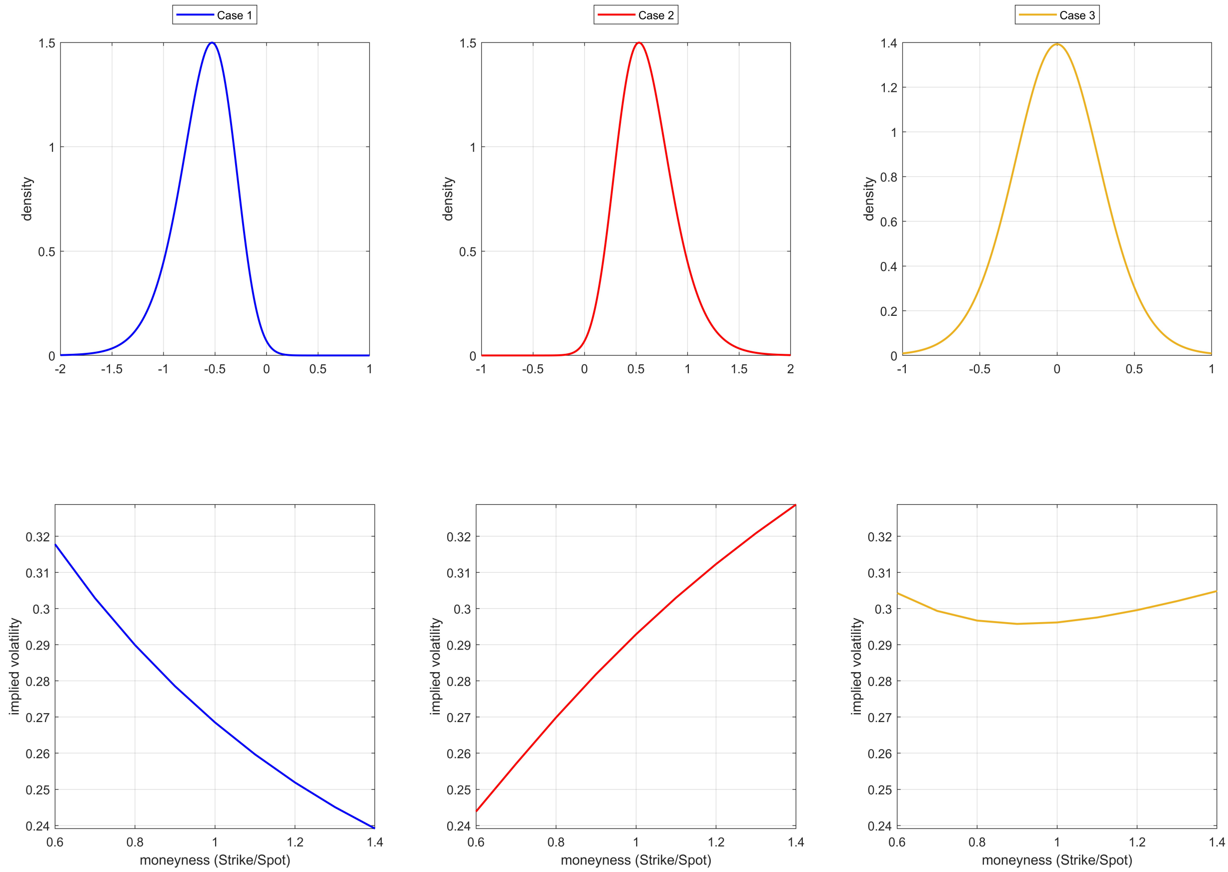

Figure 3: Understanding the role of G and M in the CGMY model. Top row: CGMY densities. Bottom row: corresponding implied volatilities. Parameters: $C = 0.6509$, $Y = 0.8$ - Case 1: $G = 5.853$, $M = 32.27$; Case 2: $G = 32.27$, $M = 5.853$; Case 3: $G = M = 8.6297$. Maturity: 1 year. Option prices obtained using the COS method.

Another example showing the interpretability of a jump model is offered by Wystup (2021), who by means of the Kou process (Kou, 2002) explains the reasons behind reductions in option prices when the probability of an upward jump increases.

It is worth noticing that when a jump model is adopted, the option price can still be written as a linear combination of the underlying spot price S and the discounted value of the strike K, i.e. for a call option

$$C = S\ P_1(S_T \geq K) - e^{-rT} K\ P_2(S_T \geq K). \tag{2}$$

The weights $P_1(\cdot)$ and $P_2(\cdot)$ are the probabilities generated by the chosen model that the option is exercised at maturity, with $P_1(\cdot)$ defined under the measure using the spot price as numéraire (the spot measure), and $P_2(\cdot)$ defined instead with respect to the risk neutral measure. The result is a straightforward consequence of the change of numéraire technique of (Geman et al., 1995, Theorem 2), and it is also pointed out by Eberlein and Keller (1995), Madan et al. (1998) and Ballotta and Fusai (2015) for example.

Thus, as in the case of the Black-Scholes formula, we can read from (2) a hedging strategy based on the delta portfolio. Differently from the Black-Scholes formula, the delta is no longer given by the normal distribution, but by the distribution of the chosen stochastic process under the spot measure. Indeed, given that the price of the call option with maturity T is

$$C = \mathbb{E}\left(e^{-rT}\left(S_T - K\right)^+\right),$$

with $S_t = Se^{X_t}$ for a chosen stochastic process X_t with density function $f_{X_t}(\cdot)$, the above is equivalent to

$$C = e^{-rT} \int_{\ln \frac{K}{S}}^{\infty} (Se^x - K) f_{X_T}(x) dx.$$

By differentiation under the integral sign using the Leibniz integral rule, it follows that

$$\Delta = \frac{\partial C}{\partial S} = e^{-rT} \int_{\ln \frac{K}{S}}^{\infty} e^x f_{X_T}(x) dx;$$

therefore by changing the numéraire, we obtain that $\Delta = P_1(S_T \geq K)$.

The reality of the financial markets shows us that we can hedge the small, high frequency changes in the price of the underlying using the delta; big jumps cannot be fully hedged, also because the theoretically perfect Black-Scholes hedge ignores them by construction. However, using a more realistic distribution can help to partially take them into account thanks to skewness and excess kurtosis.

The final criticism moved against jump models discussed here is the one related to the incompleteness of the market. I have left this one for last, although it is in general the first argument used in any conversation about jump models. Based on the fundamental theorem of asset pricing, jump models indeed lead to an incomplete market, i.e. a market characterized by (infinitely) many risk neutral probability measures, and in which contingent claims might be unattainable. The same consideration though holds for (diffusion based) stochastic volatility models. As a matter of fact it also holds for the Black-Scholes model as the volatility parameter is not only not directly observable, but is also not unique: the exercise of extracting the implied volatility shows that we require one value of the volatility parameter for each combination of strike and time to maturity. Model calibration is the answer to this issue; as a matter of fact calibration is a way of 'completing' the market with traded options. Reality is that financial markets are incomplete. Thus, better equipping ourselves accordingly with a model which is flexible enough. The 'incompleteness' of the jump models should better be considered as a property of richness rather than a drawback.

In the examples used in this article, I refer exclusively to jump models built on Lévy processes, i.e. processes with independent and stationary increments. These processes perform very satisfactorily in reproducing implied volatilities along the moneyness dimension; however, the same cannot be said when we consider the time to maturity dimension. More sophisticated jump models built out of Lévy processes and equipped with stochastic volatility features are available in the literature: from time inhomogenous Lévy processes, i.e. processes with just independent increments, as the ones adopted by Eberlein and Kluge (2005) for example, to affine processes as in Kallsen (2006) or time changed Lévy processes as in Carr and Wu (2004) and Ballotta and Rayée (2022). Regardless of how we build these more general models, the considerations offered in this article hold: the models are realistic, tractable and can be calibrated. Therefore they represent an excellent choice for financial applications.

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