INTEGRAL EQUATION FORMULATIONS OF EXTERIOR ACOUSTIC SCATTERING PROBLEMS

by

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DEDICATION

This thesis is mainly dedicated to my long departed father, Shaikh Bodiuzzaman, who was a constant source of inspiration and encouragement throughout my intellectual development, and to my mother, Begum Homaira Zaman, who reared me and nurtured me during those formative years of my childhood and has supported me with her infinite patience throughout this work.

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Thanks to all my family and relatives, both here and abroad, for their constant support.

And last but not least, my infinite gratitude to *ALLAH* the almighty for everything I have been able to achieve.

...and the things on this earth which he has multiplied in varying colours (and qualities), verily in this is a sign for men who celebrate the praises of God (in gratitude)...

Al-Quran, XVI.13

ABSTRACT

The work presented in this thesis is particularly concerned with a robust integral equation formulation of acoustic scattering and radiation problems, which are essentially exterior Neumann boundary-value problems. Both layer theory and the Helmholtz formula, used in the classical formulation (pre-1968), result in a non-uniqueness problem. This non-uniqueness is purely mathematical and has no bearing on the actual physical problem. Various workers over the past two decades or so developed alternative formulations, which resolve the problem of non-uniqueness but also suffer from computational drawbacks.

Kussmaul (1969) developed a formulation involving the superposition of a simple-layer potential and a double-layer potential, combined by a coupling parameter. Kussmaul also presented a uniqueness proof valid for all wave-numbers. However his formulation involves an integral operator which has a hypersingular kernel. This creates computational difficulties. My thesis presents a new integral equation formulation which involves the superposition of a layer potential generated by simple sources on the given boundary, plus a layer potential generated by dipole sources located on an interior boundary similar and similarly situated to the given boundary. These two potentials are also combined by a coupling parameter. However, unlike the Kussmaul formulation, this avoids the integral operator containing the hypersingular kernel. An argument towards uniqueness is presented. Some test radiation problems and some scattering problems are investigated. Numerical results are given which show that the new formulation gives excellent agreement with the analytical results.

The thesis also presents a derivation of wave-functions via layer potentials generated by a uniform distribution of sources on a spherical surface. This is utilized in the discussion of the hypersingular kernel of a certain integral operator, and the analysis is used to verify Terai's (1980) result for a hypersingular integral on a flat plate.

CONTENT

PART III: INTEGRAL EQUATION FORMULATIONS

PART IV: NUMERICAL TREATMENTS AND RESULTS

INTRODUCTION

The formulation of acoustic scattering and radiation problems presents formidable difficulties. The problems entail solving a partial differential equation in an infinite domain with given conditions on an internal boundary and at infinity. Progress is only possible by first recasting the partial differential equation as an integral equation. This enables effective numerical solutions to be achieved using the Boundary Element Method (BEM). This method has developed over the last two decades as an alternative to the more traditional numerical methods i.e. the Finite Element Method (FEM) and the Finite Difference Method (FDM). As regards acoustic problems BEM has advantages over the traditional methods both in terms of applications and accuracy.

The essential difference between the BEM and FEM/FDM is immediately evident, in that the latter require the discretisation of a full domain, in some cases an infinite domain, whereas the former requires the discretisation of the boundary only. Thereby BEM reduces the dimension of the domain of operation by one. Over the years texts by Brebbia [8], Jaswon and Symm [36] and Bannerjee and Butterfield [20] have given a very useful and extensive account of BEM. Although Jaswon and Symm [36] deal only with harmonic and biharmonic problems, the numerical treatment therein can equally well be applied to wave problems (Helmholtz problems).

The BEM can be applied equally well to both interior and exterior boundary-value problems. For exterior problems both FEM and FDM face complications in regard to the discretisation of an infinite exterior domain. These may be applied to such problems by assuming a fictitious boundary in the far-field, discretising the domain bounded by the given boundary and the outer boundary, and then applying the condition at infinity to the outer boundary. Generally, these can only be achieved with great difficulty. For such problems the BEM is a more effective tool as only the boundary needs be discretised.

For some interior problems the BEM provides a competitive numerical method of solution to FEM or to FDM. The performance of each method depends on the particular partial differential equation, the nature of the surface, the boundary condition and the method of discretisation. Symm [88] gives a useful comparative analysis of numerical methods applied to a particular problem. He found that FEM and certainly FDM can be awkward to apply in irregularly shaped domain for which the BEM emerges as a more appropriate method.

Acoustic scattering or radiation problems are essentially exterior boundary-value problems subject to Dirichlet or Neumann boundary conditions. The central theme of this thesis is to present a new robust BEM formulation, the Adapted Kussmaul Formulation (AKF), for the exterior Neumann problem. The thesis divides naturally into four parts.

Part I (chapters 1-4) introduces the Helmholtz equation and its fundamental solutions, so opening the way to the theory of simple-layer and double-layer Helmholtz potentials. We show how these potentials may be utilized to formulate boundary integral equations, and also point out that the classical integral equation formulation for exterior problems breaks down at a certain spectrum of "critical" wave-numbers. These wave-numbers may be identified as the eigenfrequencies (eigenvalues) of the corresponding interior problem. The eigenfrequencies are widely separated at low values of the frequency (i.e.

low wave-numbers) but become more and more closely bunched together as the frequency increases. It is well understood that this breakdown feature is purely mathematical and has no bearing on the actual physical problem, for which a unique solution always exists. Details of the equivalence between exterior and interior problems are discussed in chapter 4.

In part II (chapters 5, 6), we generate interior and exterior Helmholtz potentials from uniform source distribution on a spherical surface. These enable us to verify boundary properties of Helmholtz potentials at any Liapunov surface. Also, these help us to understand alternative definitions of the hypersingular operator N_k (defined in chapter 2 p25), and to choose the one which appears to be most physically significant. It may be noted that the evaluation of N_k presents the greatest computational difficulties involved in the numerical solution.

Burton [13], and also Kleinmann and Roach [42], give well-documented reviews of various alternative BEM formulations of the exterior acoustic problem. In part III (chapters 7, 8), we briefly review the most significant formulations to date, which attempt to resolve the breakdown problem discussed in part I. Most important of these are that of Burton and Miller [17], of Schenck [82] and that of Kussmaul [48]. Burton and Miller [17] utilize SHE (Surface Helmholtz Equation) in combination with its normal derivative equation, in order to ensure a unique solution even at critical values of the wave-number k. However the N_k problem remains. Burton [13, 14] and Terai [93] present ways of circumventing this. The former suggests a regularisation process (originally propounded by Panich [70]), for which the numerical implementiation is very cumbersome. The latter presents a contour integration method, but only applicable to planar boundary elements. Schenck

[82] avoids N_k by combining SHE with a Helmholtz relation at selected interior points. However, since the resulting system of equations is over-determined, he only achieves a solution by an optimisation procedure. This optimisation procedure is wholly dependent on a judicious selection of the so called "good" interior points which provide the Helmholtz interior relation. As yet no consensus has been reached as to how one selects these points.

Kussmaul's [48] formulation involves a superposition of a simple-layer potential and a double-layer potential, both generated by the same continuous source distribution on a given surface. This provides a unique solution. However, due to the presence of the dipole potential, a highly singular kernel appears in the boundary integral equation i.e. it too involves the N_k difficulty. This motivated the development of a new formulation AKF (see earlier paragraph) which also utilizes a superposition of layer potentials except that - unlike Kussmaul - the dipole potential is now generated from an interior surface similar (and similarly situated) to the given boundary. This avoids the N_k operator but still provides a unique solution. Kussmaul proved uniqueness within a 2-D context. We provide (in chapter 8) an alternative proof which holds for 3-D problems and which can also be adapted for the AKF.

Over the last two decades, Burton and Miller [17], Schenck [82] and to a certain extent Kussmaul [48] have been in the forefront. More recently, Jin [40] has developed a formulation which involves an integral equation of the first kind. He makes use of Sobolov space in order to provide a unique solution. But the numerical implementation still remains to be reported.

In part IV (chapters 9, 10), we describe our numerical methods for the AKF and apply them to achieve numerical solutions of some exterior Neumann and acoustic scattering problems. Naturally, the surface of the scatterer is assumed to be rigid, as prescribed by the Neumann boundary condition. The scattering pattern of an incident wave on a sphere, and the pressure distribution around the sphere, are computed. As part of our test problems, we find the radiation pattern of a pulsating sphere and that of an oscillating sphere. All our results have been displayed in terms of graphs.

To summarise, the most original material of the thesis appears in chapter 8 plus its applications in chapters 9 and 10. Also, much of the analysis in chapters 5 and 6, in particular the analysis of the hypersingular operator N_k, appears to be original.

As regards future research, I aim to apply the AKF method to hard acoustic scattering by a set of prolate spheroids of axial ratios varying from $b/a = 1$ (i.e. a sphere) to $b/a = 5$, where a is the radius of the central cross-section of the prolate spheroid and b is its semi-length. I hope that my results will provide benchmark solutions for future methods of attack.

Part of this thesis has been published [104], under the title " A new BEM formulation of acoustic scattering problems", in Proceedings of BEM XV, Vol.I, 1993.

PART I

BACKGROUND THEORY

Chapter 1
Helmholtz Equation

In this chapter we give a brief discussion of the Helmholtz equation, which is the equation satisfied by an acoustic wave in a homogeneous continuum. For a given obstacle in the medium, an analysis of the behaviour of a wave function on the boundary of the obstacle and at infinity requires a study of the boundary features i.e. conditions on the boundary and at infinity. These are also discussed later in this chapter.

1.1 Equations of the acoustic medium

Supposing a region in space is filled with a homogeneous acoustic medium of density ρ and speed of sound c. A small amplitude acoustic wave propagates through this given homogeneous medium according to the linear wave equation [60]

$$\nabla^2\Phi(p,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi(p,t) \tag{1.1}$$

where Φ plays the role of a scalar velocity potential at any point p in the medium at time t. This means that the particle velocity, denoted by $\underline{v}$, is defined by

$$\underline{v} = \nabla\Phi \tag{1.2}$$

Let the acoustic pressure at time t be denoted by P. Then from Newton's equation of motion we have

$$\rho \frac{\partial \underline{v}}{\partial t} = -\nabla P$$

which becomes

$$\rho \frac{\partial}{\partial t}[\nabla\Phi] = -\nabla P$$

From which

$$P = -\rho \frac{\partial \Phi}{\partial t} \tag{1.3}$$

Specialising Φ into the form

$$\Phi(p,t) = \phi(p)e^{-i\omega t} \tag{1.4}$$

we find that ϕ satisfies the reduced wave equation (or more commonly the Helmholtz equation)

$$\nabla^2\phi(p) + k^2\phi(p) = 0 \tag{1.5}$$

at any field point p in the medium. In relation (1.4), $e^{-i\omega t}$ represents the time harmonic dependence i.e. the representation (1.4) is specialised for steady monochromatic waves. Here

$$\omega = 2\pi\nu$$

where ν denotes the frequency of propagation in hertz. Also

$$k = \omega c^{-1} = 2\pi\lambda^{-1}$$

is the acoustic wave number, where λ denotes the wave-length.

We consider the propagation of small-amplitude acoustic waves in a medium of negligible viscosity. From the relation (1.3) and (1.4) we have

$$P(p,t) = i\omega\rho\phi(p)e^{-i\omega t} \tag{1.6}$$

at any field point p in the medium. Although both P and $\underline{v}$ satisfy the wave equation (1.1), it is convenient to work in terms of a single function ϕ from which the scalar velocity potential Φ may be computed using the specialised form (1.4) with the time-harmonic part having been incorporated. Likewise the excess pressure and the particle velocity are computed using the same ϕ. Usually the time-harmonic component is omitted in the calculation and the pressure is found as

$$P(p) = i\omega\rho\phi(p) \tag{1.7}$$

at any field point p, where ϕ satisfies the Helmholtz equation as noted before.

1.2 Liapunov smoothness

The boundary features of an obstacle in an acoustic medium play an essential role in any boundary-value problem (details in chapter 4). In this thesis we restrict ourselves to boundaries which satisfy a certain smoothness condition formulated by Liapunov [31].

In fig.1 ∂B denotes the boundary of an obstacle in an acoustic medium, n_q, n_o, n_l denote normals at q, q_o, q_l respectively on the surface ∂B. A surface ∂B is said to be Liapunov if it satisfies the following three conditions (see Smirnov [79], Günter [31]):

(i) A tangent plane and a normal exists at every point p on ∂B.

(ii) $\forall q \in \partial B$, $\exists$ a single fixed number $\varepsilon>0$ such that the (see fig.1) neighbourhood surface

$$\{p \in \mathbb{R}^3 : |p-q| < \varepsilon\} \cap \partial B \tag{1.8}$$

intersects lines parallel to the normal at q (i.e. n_q) in at most one point only. It is clear that if the above property holds for any given value $\varepsilon_0 > 0$ it also holds for any other smaller values i.e. for

$$\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots > 0$$

(iii) If θ is the acute angle formed by the normals n_o, n_l at two points (fig.1) q_o, q_l, then θ satisfies the Hölder condition [65], [25]

$$\theta \leq D\,|q_0 - q_1|^{\alpha}$$

where D, α are constants and $D>0$, $0<\alpha\leq 1$.

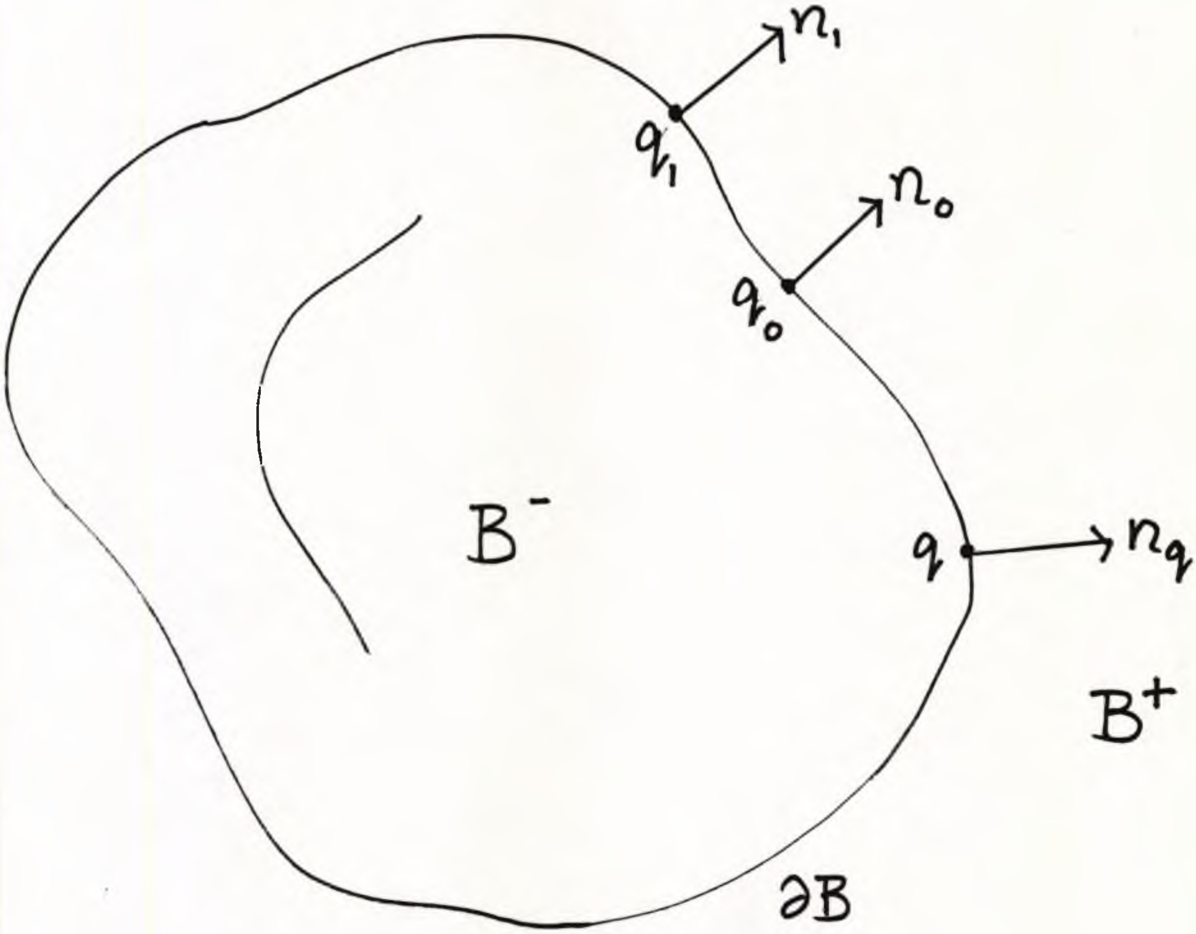

Fig.1 Boundary features for a general Liapunov surface; B^-, B^+ are respectively the regions interior and exterior to the surface ∂B .

A consequence of condition (ii) is that the neighbourhood surface in (1.8) may be represented in terms of a tangent-normal system of coordinates centred around the point q, with ζ -axis coinciding with the normal at q, and ξ,η being two orthogonal axes in the tangent plane such that the neighbourhood surface in (1.8) has the equation

$$\zeta = \zeta(\xi,\eta)$$

where ζ is single-valued at least over part of the tangent plane, and by conditions (i) and (iii) ζ possesses first derivatives.

It may be mentioned that Kellogg in his classic work [41] introduced a somewhat more general smoothness condition than Liapunov. However, in this thesis we find that Liapunov smoothness condition is sufficient for our requirement. Indeed restriction to Liapunov surfaces is necessary for formulations of boundary-value problems of acoustics with the aid of potential theory.

In this thesis we shall be concerned with a closed boundary (denoted by ∂B), in a linear isotropic homogeneous continuum. This boundary encloses an interior domain denoted by B^- and it also forms an internal boundary of an infinite exterior domain denoted by B^+, see fig.1. Note that

$$B^- \cup \partial B \cup B^+ = \mathbb{R}^3$$

More generally we may assume that ∂B consists of a number of sub-surfaces ∂B_i such that

$$\partial B = \bigcup_i \partial B_i \quad (1 \leq i \leq n) \quad , \quad \partial B_i \cap \partial B_j = \emptyset \quad for \quad i \neq j$$

where each sub-surfaces ∂B_i is assumed to satisfy Liapunov smoothness conditions as detailed in the previous section.

1.3 Conditions on the boundary

Any boundary-value problem of finding the unknown function (in the acoustics situation it is the wave function ϕ) which satisfies a certain equation (in this case the Helmholtz equation (1.5)) in the interior domain B^-, requires the knowledge of the behaviour of the unknown function on the boundary ∂B , and in the case of the exterior domain B^+, one also needs to know the behaviour of the function at infinity.

<u>Boundary conditions</u>

Any of the following classical boundary conditions may be assumed on ∂B :

(*a*) *Dirichlet condition,* $\phi(p) = f(p) \quad , \quad p \in \partial B$ (1.9)

(*b*) *Neumann condition ,* $\dfrac{\partial \phi}{\partial n}(p) = f(p) \quad , \quad p \in \partial B$ (1.10)

(*c*) *Robin condition ,* $\dfrac{\partial \phi}{\partial n}(p) + h(p)\phi(p) = f(p) \; , \; p \in \partial B$ (1.11)

where f,h are given functions. However, in this thesis we primarily consider Neumann problems i.e. condition (b) only.

Radiation condition

This condition is in fact the boundary condition at infinity i.e. on the boundary of a sphere with infinite radius and completely encloses the obstacle or the structure with boundary ∂B . In fact this condition ensures a regular behaviour of the functions at infinity. Also, this is in keeping with the requirement that all scattered or radiated acoustic waves are out-going at infinity. The so called radiation condition was first formulated by Sommerfeld [80] though later works by Kupradze [46], Atkinson [6], Rellich [72], and Wilcox [96] sharpened its formulation. In 3-D it has the form

$$\lim_{r\to\infty} [r\{\frac{\partial\phi}{\partial r}(r) + ik\phi(r)\}] = 0 \qquad (1.12)$$

where $r=|p|$, the distance of a general field point p from a fixed origin within B^{-}. Any function which satisfies (1.5) and (1.12) is known as a radiating wave function.

According to a fundamental theorem (see Smirnov [79]), the solution of a wave function in the exterior domain satisfying the Helmholtz equation (1.5) in that domain and subject to the radiation condition (1.12) and one of the boundary conditions (1.9),(1.10) or (1.11) is unique, provided that

$$\Re(k)>0\,,\ \Im(k)\geq 0$$

where k denotes,as before,the acoustic wave-number. This thesis only deals with k real and $k>0$ i.e. $\Im(k)=0$ as dictated by the real situation where the wave number k is the ratio of two real quantities i.e. $k=\omega c^{-1}$. This implies that the exterior Helmholtz problems under consideration always have unique solutions.

Chapter 2
Helmholtz Potentials

The integral equation method is widely used in solving boundary-value problems of acoustic scattering and radiation. The main advantage of this method is the fact that numerical procedures refer only to the boundary (2-D surface) of the domain (finite or infinite) under consideration, thereby reducing the domain of discretisation by one dimension. To embark on integral equation formulations for Helmholtz boundary-value problems, we first of all define layer potentials and examine their properties.

2.1 Simple-layer Potential

A simple-layer potential has the form

$$L_k\sigma(p) = \int_{\partial B} g_k(p,q)\,\sigma(q)\,dq \quad ; \quad p \in \mathbb{R}^3 \quad . \tag{2.1}$$

This is a Helmholtz potential generated at a field point p by a continuous distribution of simple sources extending over a Liapunov surface ∂B , and of surface density $\sigma(q)$ at q on the surface ∂B which is assumed to be Hölder continuous over ∂B (see section 1.2 in chapter 1). Here dq is a surface differential (or an area element) at q, see fig.2.

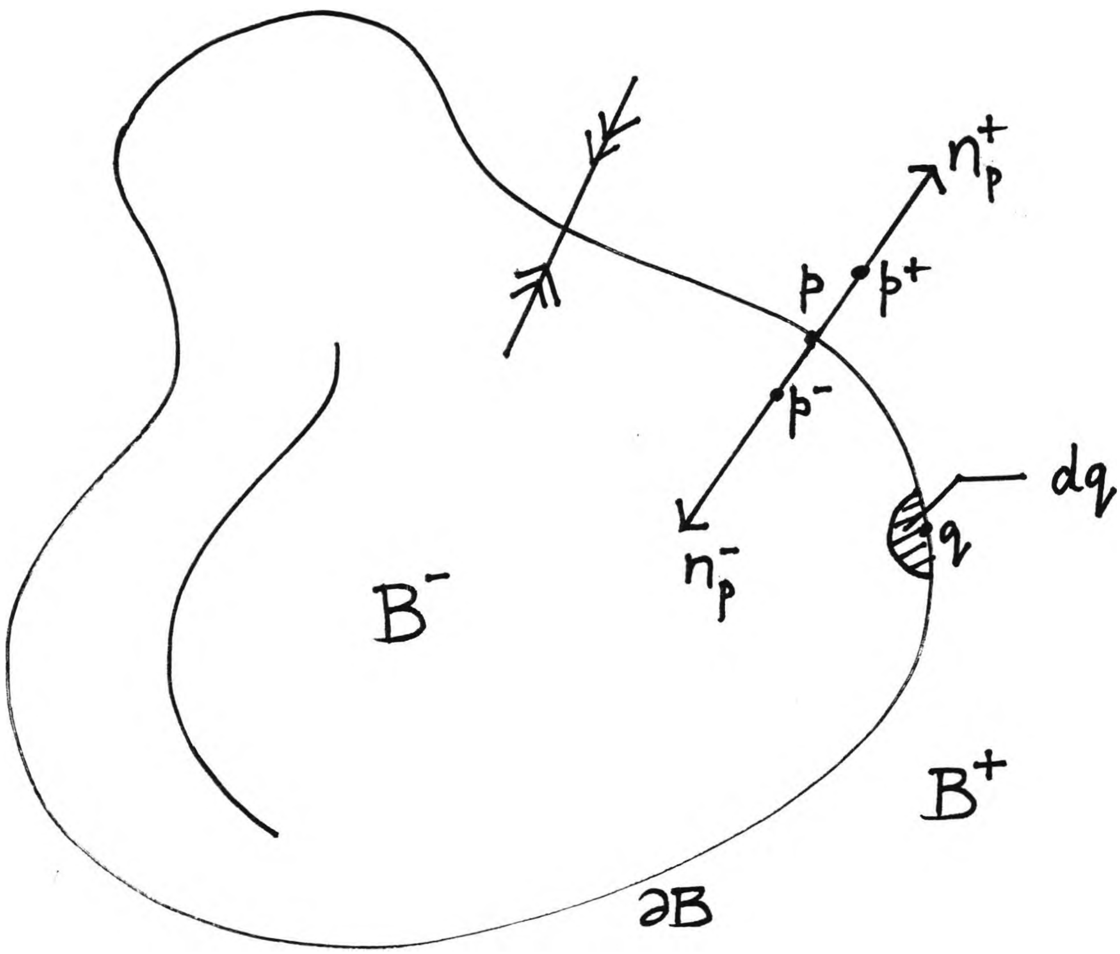

Fig.2 Boundary features for a surface with continuous distribution of simple sources generating a simple-layer potential at $p^{\pm}$; the double arrow signifies the continuity of the simple-layer potential across the surface ∂B .

In 3-D

$$g_k(p,q) = \frac{e^{-ik|p-q|}}{|p-q|} \quad . \tag{2.2}$$

This is the free-space Green's function for the Helmholtz equation which, from the physical point of view, represents the potential at p generated by a unit point source at q (and vice-versa). This potential is a continuous function of p, differentiable to all orders, and it satisfies the Helmholtz equation

$$\nabla^2 g_k(p,q) + k^2 g_k(p,q) = 0 \tag{2.3}$$

everywhere except at the source point q. Formally g_k satisfies (in both variables)

$$\nabla^2 g_k(p,q) + k^2 g_k(p,q) = -4\pi\,\delta(|p-q|) \tag{2.4}$$

everywhere i.e. it is a fundamental solution of the Helmholtz equation, where δ is the Dirac delta function centred upon q, see Dirac [28], Lighthill [54], Jones [38].

Clearly g_k also satisfies the radiation condition (chapter 1, 1.12). Note that g_k is a close generalisation of the free-space Green's function

$$g(p,q) = \frac{1}{|p-q|} \tag{2.5}$$

for Laplace's equation; in particular they have the same order of singularity as $|p-q| \to 0$, as may be seen by expanding the functions for small $|p-q|$. In 3-D we find that

$$g_k(p,q) \simeq \frac{1}{|p-q|} + O(1) \simeq g(p,q)\ ,\ as\ |p-q| \to 0 \quad . \quad (2.6)$$

An immediate consequence is that the jump and continuity properties of the Helmholtz potential at ∂B parallel those of the Laplace potential at ∂B. Apart from the property (2.6) the existence of the integral (2.1) depends on the smoothness of layer density σ and the smoothness properties of the surface ∂B (see section 1.2 in chapter 1). The smoothness condition on ∂B may be expressed by writing

$$\partial B = \bigcup_i \partial B_i \quad , \quad (1 \le i \le n\ ,\ n \in \mathbb{N})$$

as detailed in chapter 1 ; and requiring that each ∂B_i is a Liapunov surface. We also require that σ be Hölder continuous over ∂B (as already mentioned) i.e. σ satisfies the following inequality

$$|\sigma(q_1) - \sigma(q_2)| < A|q_1 - q_2|^{\beta} \quad ; \quad 0<\beta\le 1\ ,\ A>0 \quad (2.7)$$

for any two distinct $q_1, q_2 \in \partial B$.

Subject to these conditions the simple-layer potential L_k has the following principal properties:

(i) It exists, and it is continuous and differentiable, everywhere in $B^+ \cup B^-$ and satisfies the Helmholtz equation i.e.

$$\nabla^2 L_k\sigma(p) + k^2 L_k\sigma(p) = 0 \quad ; \quad \forall\, p \in \mathbb{R}^3\backslash\partial B$$

and it also satisfies the radiation condition (1.12).

(ii) It exists and is continuous on ∂B despite the singularity (2.6), since this is essentially a weak singularity. Also, its value at p in ∂B is continuous with its neighbouring values (see fig.2) in B^+ and in B^- i.e.

$$\lim_{p^+\to p} L_k\sigma(p^+) = L_k\sigma(p) = \lim_{p^-\to p} L_k\sigma(p^-) \quad ; \qquad p\in\partial B \quad , (2.8)$$

where $p^+\in B^+, p^-\in B^-$

(iii) There exist two distinct normal derivatives $\partial L_k/\partial n_p^{\;+}$, $\partial L_k/\partial n_p^{\;-}$ at p in ∂B , one on each side of ∂B . We adopt the convention that these two derivatives have equal status in the sense that the relevant variables $n_p^{\;+}, n_p^{\;-}$ both increase moving away from ∂B (see fig.2). At any point $p^\pm\in B^+\cup B^-$ on the normal line through p in ∂B , other than the initial point p, we have

$$\frac{\partial L_k\sigma}{\partial n_{p^\pm}}(p^\pm) = \int_{\partial B} g_k^{\prime}(p^\pm,q)\,\sigma(q)\,dq \tag{2.9}$$

where $g_k^{\prime}(p^\pm,q)$ signifies the derivative of g_k in the direction of the normal passing through $p^\pm$ keeping q fixed, see Jaswon and Symm [36]. But at the initial point, i.e. p,

$$\frac{\partial L_k\sigma}{\partial n_p}(p) = -2\pi\,\sigma(p) + \int_{\partial B} g_k'(p,q)\,\sigma(q)\,dq \quad ; \quad p\in\partial B \quad . \qquad (2.10)$$

It is often convenient to replace (2.10) by either the form

$$\frac{\partial L_k\sigma}{\partial n_p^{\,-}}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_p^{\,-}}(p,q)\,\sigma(q)\,dq \quad ; \quad p\in\partial B \qquad (2.11)$$

or

$$\frac{\partial L_k\sigma}{\partial n_p^{\,+}}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_p^{\,+}}(p,q)\,\sigma(q)\,dq \quad ; \quad p\in\partial B \quad . \qquad (2.12)$$

Superposing (2.11) and (2.12) yields the jump,

$$\frac{\partial L_k\sigma}{\partial n_p^{\,+}}(p) + \frac{\partial L_k\sigma}{\partial n_p^{\,-}}(p) = -4\pi\,\sigma(p) \quad ; \quad p\in\partial B \quad . \qquad (2.13)$$

in the normal derivative $\partial L_k/\partial n_p$ at ∂B . This follows since $g_k(p,q)$ remains continuous as p crosses the surface ∂B so that

$$\frac{\partial g_k}{\partial n_p^{\,+}}(p,q) + \frac{\partial g_k}{\partial n_p^{\,-}}(p,q) = 0 \quad . \qquad (2.14)$$

Note that (2.13) is a consequence of the singular behaviour (2.6).

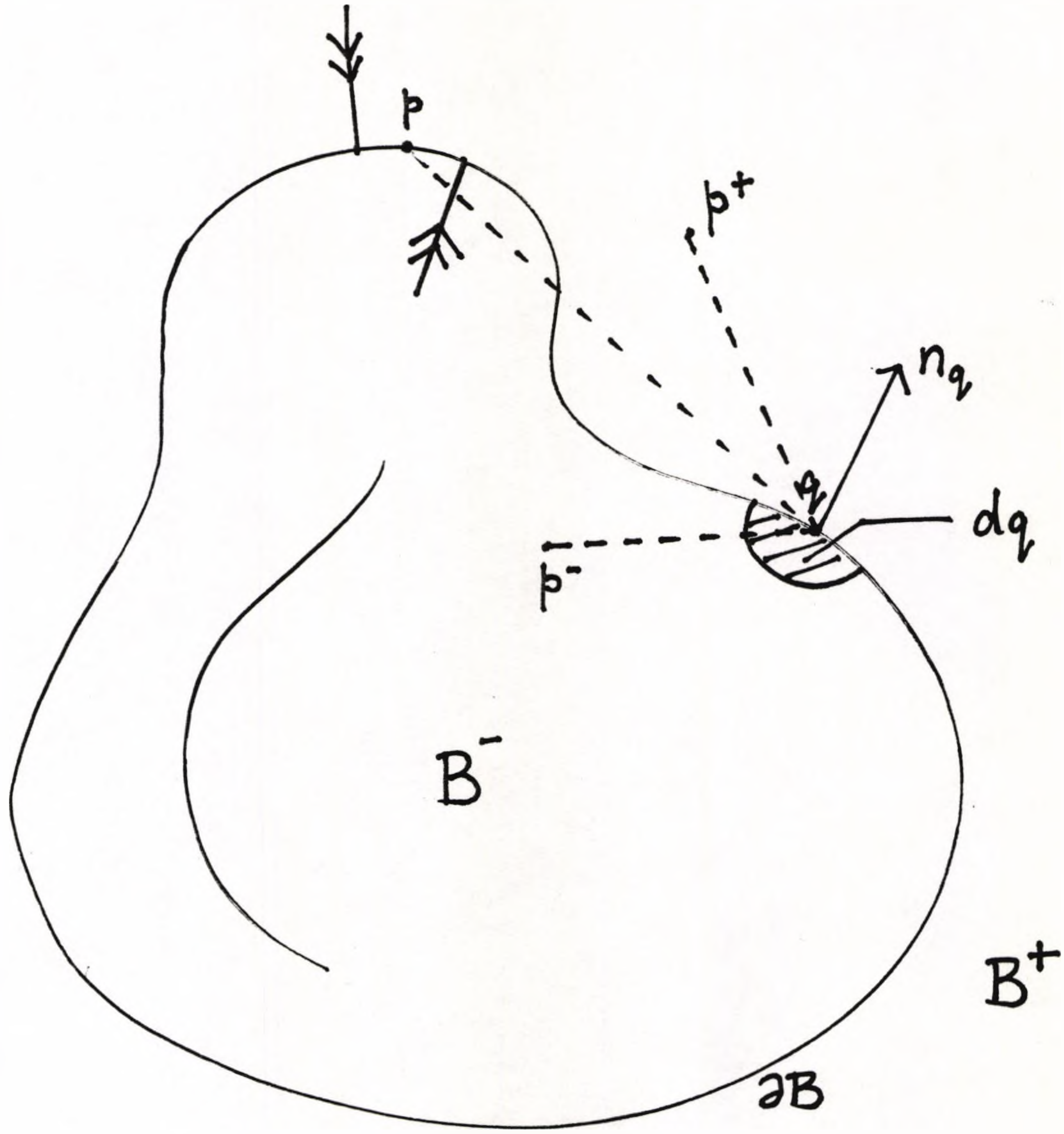

Fig.3 Boundary features for a surface with continuous distribution of dipole sources generating a double-layer potential at $p^{\pm}$; the double arrows signify the discontinuity of the double-layer potential across the surface ∂B .

2.2 Double-layer potential

We introduce a dipole source at $q \in \partial B$, again following Jaswon and Symm [36], defined by

$$g_k(p,q)' = \frac{\partial g_k}{\partial n_q}(p,q) = \frac{\partial}{\partial n_q}[\frac{e^{-ik|p-q|}}{|p-q|}] \tag{2.15}$$

where n_q (normal at q) points into the domain under consideration. From one point of view it can be regarded as the normal derivative of $g_k(p,q)$ at q keeping p fixed. On the other hand, we find it convenient to regard it as the potential at p generated by a unit dipole source at q, see fig.3. A useful formula for computing $g_k(p,q)'$ is to write

$$\frac{\partial}{\partial n_q}[\frac{e^{-ik|p-q|}}{|p-q|}] = \nabla_q[\frac{e^{-ik|p-q|}}{|p-q|}]\cdot \hat{n}_q = [1 + ik|p-q|]\frac{e^{-ik|p-q|}}{|p-q|^3}(\vec{p}-\vec{q}) \cdot \hat{n}_q$$

where $\hat{n}_q$ denotes the unit normal vector at q, and $\vec{p}, \vec{q}$ denote the position vectors of the points p,q respectively.

Note that $g_k(p,q)'$ satisfies in p the Helmholtz equation

$$\nabla^2 g_k(p,q)' + k^2 g_k(p,q)' = 0$$

everywhere except at the source point q. Formally $g_k(p,q)'$ satisfies

$$\nabla^2 g_k(p,q)' + k^2 g_k(p,q)' = -4\pi\,\delta(|p-q|') \quad ; \quad \forall p \tag{2.16}$$

where the prime on δ function denotes the normal derivative at q. This shows that it is a second fundamental solution of the Helmholtz equation (1.3). Also, this potential is a continuous function of p and is differentiable to all orders. As in the simple-layer case, $g_k(p,q)'$ is also a close generalisation of $g(p,q)'$, which is the second fundamental solution of Laplace's equation

$$g(p,q)' = \frac{\partial g}{\partial n_q}(p,q) = \frac{\partial}{\partial n_q}[|p-q|^{-1}] \; . \qquad (2.17)$$

Also, $g_k(p,q)'$ behaves in the same manner as $g(p,q)'$ as $|p-q|\to 0$, as may be seen by expanding this function for small $|p-q|$. In 3-D we find that

$$g_k(p,q)' \simeq -\frac{1}{|p-q|^2}[\frac{\partial r}{\partial n_q}] + O(1) \simeq g(p,q)' \quad , \quad as\ r=|p-q|\to 0 \quad . \qquad (2.18)$$

We may now introduce a layer potential using $g_k(p,q)'$ as the Green's function i.e.

$$M_k\mu(p) = \int_{\partial B} g_k(p,q)'\mu(q)\,dq \qquad . \qquad (2.19)$$

This is a Helmholtz double-layer potential generated at a field point p by a continuous distribution of dipole sources extending over a Liapunov surface ∂B and surface density $\mu(q)$ at $q\in\partial B$. Here, as before, dq is a surface differential (i.e. an area element) at q.

weak since for points on a Liapunov surface, see Mikhlin [61],

$$\left|\frac{\partial r}{\partial n_q}\right| \leq \eta r^{\gamma} \quad ; \quad 0<\gamma\leq 1$$

where η, γ are constants. Consequently, for small $r=|p-q|$ we find for 3-D that [25], [65],

$$|g_k(p,q)'| \leq \frac{\eta}{|p-q|^{2-\epsilon}} \quad , \quad \epsilon>0$$

which demonstrates that the singularity in the kernel of the double-layer potential is weak (details in chapter 3), indeed the resulting integral operator M_k is compact on $L^2(\partial B)$ (see chapter 3). This singularity (2.18) is responsible for the jump and continuity properties of the double-layer potential at ∂B , which parallel those of the Laplace double-layer potential.

As before, the existence of the integral in (2.19) depends on the layer density μ , the singularity of the kernel function $g_k(p,q)'$ (noted before) and the smoothness properties of ∂B . We make the same assumptions on the layer density μ and the surface ∂B as in the simple-layer case.

Subject to these conditions M_k has the following properties:

(i) It exists, and is continuous and differentiable everywhere in B^+ and B^- and satisfies the Helmholtz equation and satisfies the Helmholtz equation

$$\nabla^2 M_k \mu(p) + k^2 M_k \mu(p) = 0 \quad ; \quad \forall p \in \mathbb{R}^3 \setminus \partial B$$

and in B^+ it satisfies the radiation condition (1.12).

(ii) It exists on ∂B even though there is a singularity as $|p-q| \to 0$,which is essentially a weak singularity (as noted before) and consequently for $p \in \partial B$

$$M_k \mu(p) = \int_{\partial B} g_k(p,q)' \mu(q) dq < \infty .$$

However its value at p in ∂B is not continuous (see fig.3) with its neighbouring values in $B^+ \cup B^-$ i.e.

$$\lim_{p^+ \to p} M_k \mu(p^+) = 2\pi \mu(p) + M_k \mu(p) \quad ; \quad p \in \partial B \tag{2.20}$$

and

$$\lim_{p^- \to p} M_k \mu(p^-) = -2\pi \mu(p) + M_k \mu(p) \quad ; \quad p \in \partial B \tag{2.21}$$

i.e. the jump as it crosses the surface is

$$\lim_{p^- \to p} M_k \mu(p^-) - \lim_{p^+ \to p} M_k \mu(p+) = -4\pi \mu(p) \quad ; \quad p \in \partial B \tag{2.22}$$

(iii) Normal derivatives are continuous i.e. denoting by

$$\frac{\partial}{\partial n_p^-} = \lim_{p^- \to p} \frac{\partial}{\partial n_p}$$

$$\frac{\partial}{\partial n_p^+} = \lim_{p^+ \to p} \frac{\partial}{\partial n_p}$$

we have

$$\frac{\partial M_k \mu}{\partial n_p^-}(p^-) + \frac{\partial M_k \mu}{\partial n_p^+}(p^+) = 0 \quad . \tag{2.23}$$

We denote the normal derivatives of L_k and M_k by M_k^T and N_k respectively i.e.

$$\frac{\partial L_k \sigma}{\partial n_p}(p) = M_k^T \sigma(p) = \int_{\partial B} \frac{\partial g_k}{\partial n_p}(p,q)\, \sigma(q)\, dq \tag{2.24}$$

and

$$\frac{\partial M_k \mu}{\partial n_p}(p) = N_k \mu(p) = \frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\, \mu(q)\, dq \quad . \tag{2.25}$$

Note that M_k^T is the transpose of M_k, hence the symbol. The kernel of M_k^T has the same weak singularity as that of M_k. The derivative with respect to n_p in (2.25) can not be taken inside the integral sign because

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) \simeq \frac{1}{|p-q|^3} + O(1) \quad ; \quad |p-q| \to 0 \quad , \tag{2.26}$$

consequently the kernel of N_k becomes hypersingular (to be discussed in chapter 3) i.e. non-integrable. Some integral equation formulations involve integral operators with such hyper-singular kernels. This poses difficulties in numerical treatment and requires special techniques. This can also be interpreted in the sense of Hadamard's finite-part integration [33]. Rêgo Silva et al [77, 78] exploit this in their treatment of the hypersingular kernel.

Chapter 3

Basic Integral Equation Theory

In this chapter we review some of the classical theory relating to Fredholm integral equation of the type

$$(-\lambda I + K)\psi = f \tag{3.1}$$

where λ is a complex constant parameter, K is a linear integral operator and I denotes the identity operator. Most of the integral equations which arise from the formulation of acoustic problems are of the type (3.1), which is usually referred to as a Fredholm integral equation of the second kind. Integral equation formulations of acoustic scattering or radiation problems via layer potentials involve questions of the existence and uniqueness of solutions. In order to resolve these questions we have to examine the properties of the integrals in some detail, in particular the nature of the kernels involved in the operator K . Mikhlin [61] deals with this subject within the framework of completely continuous operators, otherwise known as compact operators, a class of integral operators which contains most of the usual integral operators of potential theory. Roach has given a clear review of the subject in one of his papers [76]. In the light of these discussions we shall later examine the features of integral operators arising from the layer definition. The theory may then be utilized to deduce whether or not the solution to an integral equation exists and whether it is unique if it does exist. In the case of the Helmholtz equation the operator $\mathbf{K}$ is a function of the wave-number k and we shall find that solutions do not exist for a spectrum of denumerable values of k whilst a unique solution exists for the remaining values of k. The results that follow are classical results whose proofs may be found in [7], [25], [68].

3.1 Compact operators

A compact operator $\mathbf{K} : X \to Y$ is defined to be a linear operator such that it maps any bounded set in X into a compact set in Y.

A set is said to be compact if every sequence in the set contains a convergent subsequence.

We mention below some of the principal results from the classical theory concerning compact operators which impinge upon our present discussion:

(i) All compact operators are bounded.

(ii) Any linear combination of compact operators is compact.

(iii) The product of two bounded operators is compact if one of the operators is compact.

Let Ω be a set in 3-dimensional Euclidean space. Let $p, q \in \Omega$. A function $\kappa(p,q)$ defined on $\Omega \times \Omega$ for which

$$\int_{\Omega}\int_{\Omega} |\kappa(p,q)|^2 dp\, dq < \infty \tag{3.2}$$

is known as Fredholm kernel (or Hilbert-Schmidt kernel). The integral operator generated by a Fredholm kernel is called a Fredholm operator, i.e.

$$\mathrm{K}u(p) = \int_{\Omega} \kappa(p,q)\, u(q)\, dq \quad . \tag{3.3}$$

It may be proved that the operator in (3.3) is compact in $L^2(\Omega)$ (see Smithies [81]) i.e. the space of Lebesgue squared integrable functions. All the operators with which we will be concerned with have the form (3.3), where $u \in X$ and $\mathrm{K}: X \to Y$.

The <u>Transpose</u> of K is given by

$$(\mathrm{K}^T u)(p) = \int_{\Omega} \kappa(q,p)\, u(q)\, dq \quad ; \quad u \in X \quad . \tag{3.4}$$

The <u>Adjoint</u> of K is given by

$$(\mathrm{K}^* u)(p) = \int_{\Omega} \overline{\kappa(q,p)}\, u(q)\, dq \quad ; \quad u \in X \tag{3.5}$$

where $\overline{\kappa(q,p)}$ denotes the complex conjugate of $\kappa(q,p)$.

An integral operator K is said to be <u>symmetric</u> if

$$\kappa(p,q) = \kappa(q,p) \quad , \quad \forall\, p,q \in \Omega$$

and is said to be <u>self-adjoint</u> or <u>Hermitian</u> (see Dirac [28]) if

$$\kappa(p,q) = \overline{\kappa(q,p)} \quad , \quad \forall\, p,q \in \Omega \quad .$$

Clearly, L_k and N_k (see chapter 2) are symmetric operators and M_k^T is the transpose of M_k as noted before. Next let us recall an important result from classical Fredholm theory concerning the existence and uniqueness of the solution of (3.1).

3.2 Fredholm theory for compact integral operators

Let us suppose that K is a compact integral operator (as defined in (3.3)), which is defined on the complex Hilbert space $L^2(\Omega)$ where Ω is bounded. The inner product of two functions ψ, φ of this space is defined as follows

$$\langle \psi, \varphi \rangle = \int_\Omega \psi(q) \overline{\varphi(q)} \, dq$$

where bar denotes complex conjugate.

The adjoint operator K^* is related to the operator K by

$$\langle \psi, K\varphi \rangle = \langle K^*\psi, \varphi \rangle \quad . \tag{3.6}$$

Let us now consider the equation (3.1) and its adjoint equation

$$(-\bar{\lambda} I + K^*)\varphi = g \quad \cdot \tag{3.7}$$

As a first step towards examining the equation (3.1) we look at the corresponding homogeneous integral equation

$$(-\lambda I + K)\psi = 0 \; . \tag{3.8}$$

The main features of the Fredholm theory are contained in the following theorem:

<u>Fredholm alternative</u> [61], [47]

<u>Either</u> the homogeneous equation (3.8) has only the trivial solution i.e. (3.8) is satisfied only for $\psi \equiv 0$ *, in which case the inhomogeneous equation (3.1) has a unique solution for any choice of* $f \in L^2(\Omega)$ *;*
<u>or</u> the homogeneous equation (3.8) has a finite number of linearly independent non-trivial solutions $\{\psi_i \; ; \; i=1,2,\ldots,n\}$ *; which gives rise to two possibilities:*

(I) the orthogonality condition (3.10) is satisfied, yielding a non-unique solution of (3.1).

(II) the orthogonality condition (3.10) is <u>not</u> satisfied in which case the equation (3.1) does not have a solution at all.

If the first alternative holds then the adjoint homogeneous equation

$$(-\bar{\lambda}I + K^*)\varphi = 0 \tag{3.9}$$

also has only the trivial solution,i.e. $\varphi \equiv 0$, in which case the inhomogeneous adjoint equation (3.7) has a unique solution for any choice of $g \in L^2(\Omega)$. Whilst if the second alternative holds then the adjoint homogeneous equation (3.9) also has a finite number of linearly independent non-trivial solutions $\{\varphi_i \; ; \; i=1,2,\ldots,n\}$.

Any value of λ for which the homogeneous equation (3.8) has a non-trivial solution is called an <u>eigenvalue</u> or a <u>characteristic value</u> of the equation and the corresponding solutions are called the <u>eigenfunctions</u> or the <u>charecteristic functions</u>.

If λ is an eigenvalue of (3.8), with corresponding eigenfunctions $\{\psi_i\ ; i=1,2,...,n\}$, then the integral equation (3.1) only has a solution provided the inhomogeneous term f is orthogonal to all the eigenfunctions $\{\varphi_i\ ; i=1,2,...,n\}$ corresponding to the eigenvalue $\bar{\lambda}$ of the homogeneous adjoint equation (3.9) i.e. if

$$\langle f, \varphi_i \rangle = 0 \quad , \qquad i=1,2,...,n \ . \tag{3.10}$$

However,this solution is not unique since any linear combination of $\{\psi_i\}$ would satisfy (3.8).

According to Fredholm theory the equation (3.8) (and hence (3.9)) has either a finite or a countable set of eigenvalues. If the set is countable it has no finite limit point.

For our purpose we can restate the Fredholm alternative in terms of the transpose $\mathbf{K}^T$, by taking the complex conjugate of the equations involving $\mathbf{K}^*$ i.e. equation (3.7) takes the form

$$(-\lambda \mathrm{I} + \mathrm{K}^T)\varphi = g \ , \tag{3.7a}$$

and the equation (3.9) takes the form

$$(-\lambda \mathrm{I} + \mathrm{K}^T)\varphi = 0 \ \cdot \tag{3.9a}$$

The orthogonality condition takes the form

$$\langle f, \overline{\varphi_i} \rangle = 0 \quad , \qquad i=1,2,\ldots,n \quad . \tag{3.11}$$

So the theorem may be restated in the same form by incorporating equations (3.7a),(3.9a), which replaces equations (3.7) and (3.9) respectively, and (3.11).

The wave numbers k are related to the quantities λ and so the uniqueness of the solution of the integral equation is dependent on certain categories of the wave-numbers k.

Before we look at the situations which give rise to weak and hyper-singularity of the kernel functions, we must review the properties of the kernel functions of the integral operator involved in this thesis.

3.3 Properties of the kernel functions

The results mentioned in this section may be found in [13] and [14]. Henceforth, g_o will denote the free space Green's function for the Laplace equation, as defined in chapter 2.

3.3.1 $g_k(p,q)$ and its derivatives with respect to $r=|p-q|$

In 3-dimensions we have

$$g_k(p,q) = \frac{e^{-ik|p-q|}}{|p-q|} \tag{3.12}$$

$$\frac{\partial g_k}{\partial r}(p,q) = \frac{e^{-ikr}}{r^2}(-1-ikr) \tag{3.13}$$

$$\frac{\partial^2 g_k}{\partial r^2}(p,q) = \frac{e^{-ikr}}{r^3}(2 + 2ikr - k^2r^2) \ . \tag{3.14}$$

Putting $k=0$ in (3.12), (3.13) and (3.14) we obtain the analogous situations for the free space Green's function of the Laplace equation.

3.3.2 Expressions for the normal derivatives of g_k

$$\frac{\partial g_k}{\partial n_q}(p,q) = \frac{\partial g_k}{\partial r}(p,q)\frac{\partial r}{\partial n_q} \tag{3.15}$$

$$\frac{\partial g_k}{\partial n_p}(p,q) = \frac{\partial g_k}{\partial r}(p,q)\frac{\partial r}{\partial n_p} \tag{3.16}$$

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) = \frac{\partial g_k}{\partial r}\frac{\partial^2 r}{\partial n_p \partial n_q} + \frac{\partial^2 g_k}{\partial r^2}\frac{\partial r}{\partial n_p}\frac{\partial r}{\partial n_q} \ , \tag{3.17}$$

where

$$\frac{\partial r}{\partial n_p} = \frac{\vec{r}\cdot\vec{n}_p}{r} \tag{3.18}$$

$$\frac{\partial r}{\partial n_q} = -\frac{\vec{r}\cdot\vec{n}_q}{r} \tag{3.19}$$

$$\frac{\partial^2 r}{\partial n_p \partial n_q} = -\frac{1}{r}\{(\vec{n}_p\cdot\vec{n}_q) + \frac{\partial r}{\partial n_p}\frac{\partial r}{\partial n_q}\} \quad . \tag{3.20}$$

Substituting for (3.13),(3.14),(3.18),(3.19) and (3.20) in (3.17) we get

$$\begin{aligned}\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) = \frac{e^{-ikr}}{r^3}[&(1 + ikr)(\vec{n}_p\cdot\vec{n}_q) \\ &- \frac{1}{r^2}(\vec{r}\cdot\vec{n}_p)(\vec{r}\cdot\vec{n}_q)(1 + ikr - k^2r^2)]\end{aligned} \tag{3.21}$$

Putting $k=0$ in (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) we find the analogous situations for the free space Green's function of Laplace's equation.

3.3.3 Behaviour near sigularity

In the following, $p,q \in \Omega$, where Ω is smooth at p

$$g_k(p,q) = g_0(p,q) + O(1) = O(|p-q|^{-1}) + O(1) \quad , \quad |p-q| \to 0 \quad , \tag{3.22}$$

$$g_k(p,q)' = O(|p-q|^{-2}) + O(1) = g_0(p,q)' \quad , \quad |p-q| \to 0 \tag{3.23}$$

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) = O(|p-q|^{-3}) + O(1) \quad , \quad |p-q| \to 0 \quad . \tag{3.24}$$

Moreover

$$[g_k(p,q) - g_0(p,q)] = O(1) \quad ; \quad as \ |p-q| \to 0 \ , \tag{3.25}$$

$$[\frac{\partial g_k}{\partial n_p}(p,q) - \frac{\partial g_0}{\partial n_p}(p,q)] = O(1) \quad ; \quad as \ |p-q| \to 0 \ , \tag{3.26}$$

$$[\frac{\partial g_k}{\partial n_q}(p,q) - \frac{\partial g_0}{\partial n_q}(p,q)] = O(1) \quad ; \quad as \ |p-q| \to 0 \ , \tag{3.27}$$

$$\frac{\partial}{\partial n_p}[\frac{\partial g_k}{\partial n_q}(p,q) - \frac{\partial g_0}{\partial n_q}(p,q)] = O(|p-q|^{-1}) + O(1) \ ; \ as \ |p-q| \to 0 \ . \tag{3.28}$$

Clearly from (3.25) and (3.28)

$$\lim_{|p-q| \to 0} \frac{\partial^2}{\partial n_p \partial n_q}[g_k - g_0] \neq \frac{\partial^2}{\partial n_p \partial n_q} \lim_{|p-q| \to 0} [g_k - g_0] \ .$$

3.4 Weak singularity

Let Ω be a bounded set in 2-dimensional Euclidean space. An integral operator K with a kernel function $\kappa(p,q)$, which satisfies

$$|\kappa(p,q)| \leq M|p-q|^{\epsilon-2} \quad , \quad \forall\, p,q \in \Omega \ , \ p \neq q \ , \tag{3.29}$$

where $M>0$ and $0<\epsilon\leq 2$, is said to have a weakly singular kernel. Some examples are as follows.

(i) The kernel function of L_k:

$$\kappa(p,q) = g_k(p,q)$$

where g_k is as defined in (3.12). From which we get

$$|g_k(p,q)| = \frac{1}{|p-q|}$$

Clearly this kernel function satisfies the condition (3.25), therefore it is weakly singular.

(ii) The kernel function of M_k:

$$\kappa(p,q) = g_k(p,q)' = -(1 + ik|p-q|)\frac{e^{-ik|p-q|}}{|p-q|^2}\frac{\partial r}{\partial n_q} \ , \tag{3.30}$$

where $g_k(p,q)'$ is as defined in chapter 2.

Now

$$|\frac{\partial r}{\partial n_q}| < \eta\,|p-q|^{\gamma} \quad , \quad 0<\gamma\leq 1 \ , \ \eta>0 \ ,$$

for a Liapunov surface [61], from which

$$|g_k(p,q)'| \leq \delta\,|p-q|^{\gamma-2} \quad , \quad as \quad |p-q|\to 0 \ ,$$

where $\delta>0$. This clearly satisfies the condition (3.29). Hence the kernel is weakly singular.

In a similar way it can be shown that the kernel function of the operator M_k^T is also weakly singular.

(iii) The kernel function of N_k:

$$\kappa(p,q) = \frac{\partial}{\partial n_p}[g_k(p,q)'] \ .$$

From (3.21) and also,(3.24)

$$\left|\frac{\partial}{\partial n_p}[g_k(p,q)']\right| = \left|\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\right| \approx \frac{1}{|p-q|^3} \qquad as \quad |p-q|\to 0$$

i.e. $M=1$ but $\epsilon=-1$, clearly showing that it does not satisfy the

condition (3.25), therefore the kernel fails to be weakly singular. Moreover, Maue [63] and later Mitzner [65] have shown by a complicated series of vector transformations that N_k operator may be expressed in the form

$$N_k\mu = \frac{\partial}{\partial n_p}\int_{\partial B}\frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq = \int_{\partial B}\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,\mu(q)\,dq$$

$$= \int_{\partial B}\{(n_q\times\nabla_q\mu(q))\cdot(n_p\times\nabla_p g_k) + k^2(n_p\cdot n_q)g_k\mu(q)\}dq \ . \quad (3.31)$$

Analysing the integrand in (3.31) we find that the singularity is of Cauchy type operating on tangential derivative of the layer density and so N_k is not a weakly singular operator. In fact the integral in (3.31) is non-integrable.

However, it can be shown that

$$\frac{\partial^2}{\partial n_p \partial n_q}\{g_k(p,q) - g_0(p,q)\} \simeq \frac{1}{|p-q|} + O(1) \qquad as \quad |p-q| \to 0 \ . \qquad (3.32)$$

Consequently the operator $(N_k\text{-}N_o)$ becomes weakly singular if treated as a single operator. This is utilized in some of the singularity treatments of integral equation formulations where the hyper-singular kernel of N_k arises, see chapter 7.

Chapter 4
Boundary Value Problems and Classical Formulations

In this chapter we review the classical formulations of Helmholtz problems. There are two methods of approach. One is indirect which makes use of the layer theory and the other involves the unknown wave function directly.

4.1 Boundary value problems

We have seen how time-harmonic wave problems in acoustics and also in electro-magnetism (not considered in this thesis) can be reduced to finding solutions of the Helmholtz equation, and we have also mentioned some important boundary conditions. Many other branches of mathematical physics (e.g. hydrodynamics, elasticity) give rise to the same equation with similar boundary conditions. In acoustic scattering or radiation problems, we seek wave functions in the exterior domain B^+ which satisfy specified boundary conditions at the scattering or radiating surface, and also the radiation condition at infinity. In the scattering case, the total wave is expressed as the superposition of the incident wave and scattered wave i.e.

$$\phi(p) = \phi_{inc}(p) + \phi_{sc}(p) \quad ; \quad p \in B^+ \quad . \tag{4.1}$$

The mathematical formulation of scattering problems now reads as follows:

(i) Solve $\nabla^2\phi(p) + k^2\phi(p) = 0 \quad ; \quad \forall\, p \in B^+$ (4.2a)

subject to

(ii) either $\phi(p) = 0 \quad ; \quad p \in \partial B$ (*Dirichlet condition*)

or $\dfrac{\partial\phi}{\partial n}(p) = 0 \quad ; \quad p \in \partial B$ (*Neumann condition*) (4.2b)

(iii) where ϕ_{sc} satisfies the radiation condition (1.12), (4.2c)

Sometimes it is convenient and appropriate to work in terms of the scattered wave rather than the total wave, in which case the scattering problem becomes as follows:

(i) Solve $\nabla^2\phi_{sc}(p) + k^2\phi_{sc}(p) = 0 \quad ; \quad \forall\, p \in B^+$ (4.3a)

subject to

(ii) either $\phi_{sc}(p) = -\phi_{inc}(p) \quad ; \quad p \in \partial B$ (*Dirichlet condition*)

or $\dfrac{\partial\phi_{sc}}{\partial n}(p) = -\dfrac{\partial\phi_{inc}}{\partial n}(p) \quad ; \quad p \in \partial B$ (*Neumann condition*) (4.3b)

(iii)where ϕ_{sc} satisfies the radiation condition (1.12). (4.3c)

Our radiation problem concerns a fictitious test source enclosed by a (mathematical) surface ∂B within an infinite domain. In this case both ϕ_{rad} and $\partial\phi_{rad}/\partial n$ are known on the surface ∂B . We may now formulate a test Dirichlet problem for which ϕ_{rad} is given on ∂B or a test Neumann problem for which $\partial\phi_{rad}/\partial n$ is given on ∂B . In either case the unknown data may be calculated from the given data for the purposes of a comparative analysis. Of course the test problems may be interior or exterior depending on the direction chosen for $\partial\phi_{rad}/\partial n$. We always choose the exterior case in line with the scattering problem which means that ϕ_{rad} becomes analogous to ϕ_{sc} of the exterior scattering problem. As a result the solution behaves like a scattered wave satisfying the radiation condition, and therefore the problem can be expressed in the form (4.3a,b,c). The incident wave terms appearing in (4.3b) are in this case replaced by given functions.

4.2 Indirect formulations

Since both $L_k\sigma$ and $M_k\sigma$ are radiating wave functions in B^+, it would be convenient to express the exterior solution of the Helmholtz equation by means of layer potentials. By enforcing the appropriate boundary condition on the surface ∂B we obtain integral equations for the unknown density function.

Exterior Dirichlet problem (EDP)

We seek a solution in the form of a simple-layer potential

$$\phi(p^+) = L_k\sigma(p^+) \quad ; \quad p \in B^+ \quad . \qquad (4.4)$$

Taking the limit as $p^+ \to p \in \partial B$, using the continuity of L_k at ∂B and applying the Dirichlet boundary condition, we immediately obtain the boundary relation

$$f(p) = L_k\sigma(p) \quad ; \quad p \in \partial B \qquad (4.5a)$$

i.e.

$$f(p) = \int_{\partial B} g_k(p,q)\,\sigma(q)\,dq \quad ; \qquad p \in \partial B \ ,$$

where f is the prescribed boundary value of ϕ . This is a Fredholm integral equation of the first kind for the surface density σ in terms of $f(p)$.

Alternatively, we may seek a solution in the form of a double-layer potential

$$\phi(p^+) = M_k\mu(p^+) \qquad ; \qquad p \in B^+ \ . \qquad (4.6)$$

Taking the limit as $p^+ \to p \in \partial B$, using the jump properties and applying the Dirichlet condition, we obtain the boundary relation

$$f(p) = 2\pi\,\mu(p) + M_k\mu(p) \qquad ; \qquad p \in \partial B \qquad (4.7a)$$

i.e.

$$f(p) = 2\pi\,\mu(p) + \int_{\partial B} g_k(p,q)'\,\mu(q)\,dq \qquad ; \qquad p \in \partial B \ . \qquad (4.8)$$

This is a Fredholm integral equation of the second kind for μ in terms of f.

Exterior Neumann problem (ENP)

We first seek a solution in the form (4.4). Differentiating in the direction of the normal at p passing through p^+, taking the limit as $p^+ \to p \in \partial B$ and applying the Neumann boundary condition (1.10) yields

$$\frac{\partial \phi}{\partial n_p}(p) = -2\pi\,\sigma(p) + M_k^T\sigma(p) \qquad ; \qquad p \in \partial B \ . \qquad (4.9a)$$

i.e.

$$f(p) = -2\pi\,\sigma(p) + \int_{\partial B} g_k'(p,q)\,\sigma(q)\,dq \quad ; \qquad p \in \partial B \ ,$$

which is a Fredholm integral equation of the second kind for σ in terms of f.

On the other hand, seeking a solution in the form (4.6), then differentiating in the direction of the normal at p passing through p^+ and taking the limit as $p^+ \to p \in \partial B$, yields the boundary relation

$$\frac{\partial\phi}{\partial n_p}(p) = N_k \mu(p) \quad ; \qquad p \in \partial B \ , \tag{4.10a}$$

where N_k has been defined in chapter 2. On applying the Neumann boundary condition (1.10) this becomes the integral equation

$$f(p) = \frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq \quad ; \qquad p \in \partial B \ ,$$

which is a (non-Fredholm) equation of the first kind for μ in terms of f. It is usual to write

$$\frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq = \int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,\mu(q)\,dq \ ,$$

even though the right-hand integral is well known to be hypersingular. Such integrals can only be given a meaning by using specialised methods, in particular, utilizing Hadamard's finite-part integration [33,77]. A formulation on these lines has recently been proposed by C.Jin [40].

Once σ or μ has been computed, (4.4) or (4.6) may be used to generate $\phi(p^+)$. For a solution valid in the interior, we proceed in a similar way except that the solution obviously need not satisfy the radiation condition (1.12). Brief details follow.

<u>Interior Dirichlet problem (IDP)</u>

Seeking a solution in the simple layer form

$$\phi(p^-) = L_k\sigma(p^-) \ ,$$

and using the Dirichlet boundary condition, yields the boundary integral equation

$$f(p) = L_k\sigma(p) \quad ; \qquad p\in\partial B \ , \tag{4.5b}$$

for σ in terms of f.

Also the double-layer formulation yields

$$\phi(p^-) = M_k\mu(p^-) \ ,$$

which gives the boundary integral equation

$$f(p) = -2\pi\mu(p) + M_k\mu(p) \quad ; \qquad p\in\partial B \ , \tag{4.7b}$$

for μ in terms of f.

Interior Neumann problem (INP)

The Neumann boundary condition applied to a simple-layer formulation yields the integral equation

$$
\begin{aligned}
f(p) &= 2\pi\,\sigma(p) + M_k^T\sigma(p) \quad ; \qquad p \in \partial B \\
&= 2\pi\,\sigma(p) + \int_{\partial B} g_k'(p,q)\,\sigma(q)\,dq \quad ; \; p \in \partial B
\end{aligned}
\tag{4.9b}
$$

Also, the Neumann condition applied to a double-layer formulation yields the boundary integral equation

$$
\begin{aligned}
f(p) &= N_k\mu(p) \quad ; \qquad p \in \partial B \\
&= \int_{\partial B} g_k'(p,q)'\,\mu(q)\,dq \quad ; \; p \in \partial B
\end{aligned}
\tag{4.10b}
$$

Note that (4.9a) which refers to ENP, is the transpose of (4.7b) which refers to IDP. Also note that (4.7a) which refers to EDP, is the transpose of (4.9b) which refers to INP. This interior and exterior connection will be detailed in section 4.4 .

4.3 Direct formulations

This method is based on Green's second theorem

$$
\begin{aligned}
&\int_{\partial B} \{\phi_1(q)\frac{\partial \phi_2}{\partial n}(q) - \phi_2(q)\frac{\partial \phi_1}{\partial n}(q)\}\,dq \\
&= \int_B \{\phi(q)_1\nabla^2\phi_2(q) - \phi_2\nabla^2\phi_1(q)\}\,dV
\end{aligned}
\tag{4.11}
$$

where B is a domain enclosed by a boundary ∂B , not necessarily simply-connected. Applying Green's second theorem (4.11) to an exterior wave function ϕ^+ in B^+ which satisfies the radiation condition (1.12), and to the free space Green's function g_k , we obtain

$$\int_{\partial B}\{\phi^+(q)\frac{\partial g_k}{\partial n_q}(p,q) - g_k(p,q)\frac{\partial \phi^+}{\partial n_q}(q)\}dq = 4\pi\,\phi^+(p) \quad ; \qquad p\in B^+ \ . \tag{4.12a}$$

The first integral on the l.h.s. of (4.12a) is seen to be a double-layer potential generated by a source density ϕ^+ on ∂B ; also, the second integral is seen to be a simple-layer potential generated by a source density $\partial\phi^+/\partial n_q$ on ∂B . The second integral remains continuous as $p^+\to p\in\partial B$ but the first integral jumps by $2\pi\phi^+$ as $p^+\to p\in\partial B$. Accordingly the formula now becomes

$$\int_{\partial B}\{\phi^+(q)\frac{\partial g_k}{\partial n_q}(p,q) - g_k(p,q)\frac{\partial \phi^+}{\partial n_q}(q)\}dq = 2\pi\,\phi^+(p) \quad ; \qquad p\in \partial B \ . \tag{4.12b}$$

By contrast with (4.12a), this is a functional relation between ϕ^+ and $\partial\phi^+/\partial n$ on ∂B which constrains one in terms of the other i.e. it provides boundary integral equations for ϕ^+ or $\partial\phi^+/\partial n$ on ∂B .

As we move from $p\in\partial B$ to $p^-\in B^-$ a further jump of $2\pi\phi^+$ occurs, yielding the identity

$$\int_{\partial B}\{\phi^+(q)\frac{\partial g_k}{\partial n_q}(p,q) - g_k(p,q)\frac{\partial \phi^+}{\partial n_q}(q)\}dq = 0 \quad : \qquad p\in B^- \ . \tag{4.12c}$$

The boundary equation (4.12b) is known as the Surface Helmholtz Equation (SHE), which in operator form becomes

$$M_k \phi^+(p) - L_k[\frac{\partial \phi^+}{\partial n_q}](p) = 2\pi \phi^+(p) \quad , \quad p \in \partial B \; . \qquad (4.14)$$

For the Dirichlet boundary condition we have

$$L_k[\frac{\partial \phi^+}{\partial n_q}](p) = M_k \phi^+(p) - 2\pi \phi^+(p) \quad , \quad p \in \partial B \quad , \qquad (4.15)$$

i.e. a Fredholm integral equation of the first kind for $\partial \phi^+ / \partial n_q$ in terms of ϕ^+ given on ∂B . For the Neumann boundary condition we have

$$-2\pi \phi^+(p) + M_k \phi^+(p) = L_k[\frac{\partial \phi^+}{\partial n_q}](p) \quad , \quad p \in \partial B \quad , \qquad (4.16)$$

i.e. a Fredholm integral equation of the second kind for ϕ^+ in terms of $\partial \phi^+ / \partial n_q$ given on ∂B .

A Similar application of Green's second theorem to an interior wave function $\phi^- \in \partial B$ yields the following:

$$\int_{\partial B} \{ \phi^- \frac{\partial g_k}{\partial n_q} - g_k \frac{\partial \phi^-}{\partial n_q} \} dq = \quad 0 \qquad ; \quad p \in B^+ \; , \qquad (4.13a)$$

$$= -2\pi \phi^-(p) \qquad ; \quad p \in \partial B \; , \qquad (4.13b)$$

$$= -4\pi \phi^{-}(p) \qquad ; \quad p \in B^{-} \,. \qquad (4.13c)$$

4.4 Interior - exterior connection

Integral equation formulations for Helmholtz problems in the exterior region have an implicit relation to the corresponding problems in the interior region. Although there is no physical connection between the two regions, yet the formulations for the two regions have equivalent integral operators. Burton [13] has given a very clear analysis of this equivalence.

As already noted, whenever the wave-number k equals certain discrete values, the interior problem with homogeneous boundary condition has a non-trivial solution. These values are the <u>eigenvalues</u> of the problem and the corresponding non-trivial solutions are the <u>eigenfunctions</u>. It may be shown [13] that the eigenvalues of the interior homogeneous problem must be real. At these eigenvalues (see chapter 3), the corresponding integral equation governing the exterior case breaks down i.e. it may either fail to yield a unique solution or it may be insoluble for the given inhomogeneous term (cf. chapter 3 section 3.2).

In the subsections that follow we demonstrate the connection between the interior and the exterior problems i.e. we show that the Neumann/Dirichlet formulation (ENF/EDF) of the exterior problem breaks down whenever the wave-number k equals the eigenvalue of the interior Dirichlet/Neumann problem. First we consider the integral equations governing the interior problems.

4.4.1 Equations derived from the Helmholtz formulae

Let the infinite set of wave-numbers, for which the interior homogeneous Dirichlet problem has non-trivial solutions, be denoted by K_D. Likewise let K_N denote the infinite set of wave-numbers for which the interior homogeneous Neumann problem has non-trivial solution.

<u>Homogeneous Dirichlet case</u>

Inserting the boundary condition $\phi=0$ into the boundary equation (4.13b), and writing the results in integral operator form, we find that $\partial\phi/\partial n$ satisfies

$$L_k[\frac{\partial\phi}{\partial n}](p) = 0 \quad ; \qquad p\in\partial B \ . \tag{4.17}$$

Now differentiating the boundary equation (4.13b) with respect to the normal at $p\in\partial B$, inserting the same boundary condition and writing the result in operator form, we get

$$(-2\pi I + M_k^T)[\frac{\partial\phi}{\partial n}](p) = 0 \quad ; \qquad p\in\partial B \ . \tag{4.18}$$

Clearly the boundary values of $\partial\phi/\partial n$ satisfy both (4.17) and (4.18) simultaneously. Now, when $k\in K_D$, the interior homogeneous Dirichlet problem has non-trivial solutions, which means $\phi\neq0$ in B^- even though $\phi\equiv0$ on ∂B . Therefore it follows from (4.13c) that $\partial\phi/\partial n\neq0$ on ∂B , implying that (4.17) and (4.18) have non-trivial solutions. This result has an important implication to the exterior Neumann formulation.

Homogeneous Neumann case

We insert the boundary condition $\partial\phi/\partial n = 0$ into the boundary equation (4.13b). Then, writing the result in operator notation we get

$$(2\pi I + M_k)[\phi](p) = 0 \quad ; \qquad p \in \partial B \ . \tag{4.19a}$$

Now differentiating (4.13b) with respect to the normal at $p \in \partial B$, inserting the same boundary condition and writing the result in the operator form, we get

$$N_k[\phi](p) = 0 \quad ; \quad p \in \partial B \ , \tag{4.19b}$$

where the operator N_k is as defined in chapter 2. Now, when $k \in K_N$, the interior homogeneous Neumann problem has non-trivial solutions, which means $\phi \neq 0$ in B^- even though $\partial\phi/\partial n = 0$ on ∂B . Therefore, it follows from (4.13c) that $\phi \neq 0$, $p \in \partial B$, implying that (4.19a,b) have non-trivial solutions.

corresponding interior problem. This result has an important implication to the exterior Dirichlet formulation.

4.4.2 Equations derived from layer theory

Homogeneous Dirichlet case

Inserting the boundary condition $\phi = 0$ (i.e. $f = 0$) into (4.5b) and into (4.7b), and writing the equations in operator form we get

$$L_k[\sigma](p) = 0 \quad ; \qquad p \in \partial B \ , \tag{4.20}$$

$$(-2\pi \mathrm{I} + M_k)[\mu](p) = 0 \quad ; \qquad p \in \partial B \ . \tag{4.21}$$

We readily see that (4.20) is mathematically identical to (4.17) but that (4.21) is the transpose equivalent of (4.18). This means that (4.20) and (4.21) have non-trivial solutions whenever (4.17) and (4.18) respectively do.

Homogeneous Neumann case

Inserting the boundary condition $\partial\phi/\partial n = 0$ into (4.9b) and into (4.10b) we get

$$(2\pi \mathrm{I} + M_k^T)[\sigma](p) = 0 \ , \quad p \in \partial B \ , \tag{4.22}$$

$$N_k[\mu](p) = 0 \quad , \qquad p \in \partial B \ . \tag{4.23}$$

Again we readily see that (4.23) is mathematically identical to (4.19b) but that (4.22) is the transpose equivalent of (4.19a). This means that (4.22) and (4.23) have non-trivial solutions whenever (4.19a) and (4.19b) respectively do.

4.4.3 Connection with exterior problems

We seek a solution of the exterior Neumann problem in the form of a simple-layer potential

$$\phi(p) = L_k[\sigma](p) \quad , \qquad p \in B^+ \ .$$

Taking the normal derivative and introducing the Neumann condition, we get

$$(-2\pi \mathrm{I} + M_k^T)[\sigma](p) = f(p) \quad , \qquad p \in \partial B \ . \tag{4.24}$$

This boundary equation has a unique solution if the homogeneous equation

$$(-2\pi I + M_k^T)[\sigma](p) = 0 \quad , \qquad p \in \partial B \ , \tag{4.25}$$

has only the trivial solution (cf. chapter 3). However, since (4.25) is mathematically identical to (4.18), it has non-trivial solutions for wave-numbers $k \in K_D$, i.e. for these wave-numbers ENF fails. Arguing similarly one may show that the EDF fails to give unique solution if $k \in K_N$.

Using the Helmholtz formula we arrive at similar conclusion, i.e. ENF breaks down when $k \in K_D$. Likewise the EDF breaks down when $k \in K_N$. Note that in these situations an essential distinction may be drawn between the layer formulation and the Helmholtz formulation, i.e. the inhomogeneous equation (4.24) is insoluble because the free term f is not orthogonal to the eigenfunctions of the homogeneous adjoint equation; by contrast with the Helmholtz formulation which yields non-unique solutions since the free term satisfies the compatibility condition (cf. chapter 3).

See Appendix **V** for an illustration, with reference to a sphere, of the connection between the interior and the exterior problem.

PART II

ANALYSIS FOR A SPHERICAL BOUNDARY

Chapter 5

Wave functions via layer-potentials

In this chapter we verify the properties of the layer potentials, detailed in chapter 2, with reference to a sphere.

5.1 Simple-layer case

We consider a sphere of radius a. The source density at any q in ∂B, the surface of the sphere, is denoted by $\sigma(q)$, see Fig.4. Then the simple-layer Helmholtz potential, generated at a field point $p \in \mathbb{R}^3$ by a continuous distribution of source points q in ∂B, is given by

$$\int_{\partial B} g_k(p,q)\,\sigma(q)\,dq \quad ; \quad p \in \mathbb{R}^3 \ , \tag{5.1}$$

where dq is the area element at q and

$$g_k(p,q) = \frac{e^{-ik|p-q|}}{|p-q|} \ .$$

If we assume a uniform distribution of sources over the spherical surface i.e.

$$\sigma(q) = \sigma_0 \ , \quad \forall q \in \partial B \ ,$$

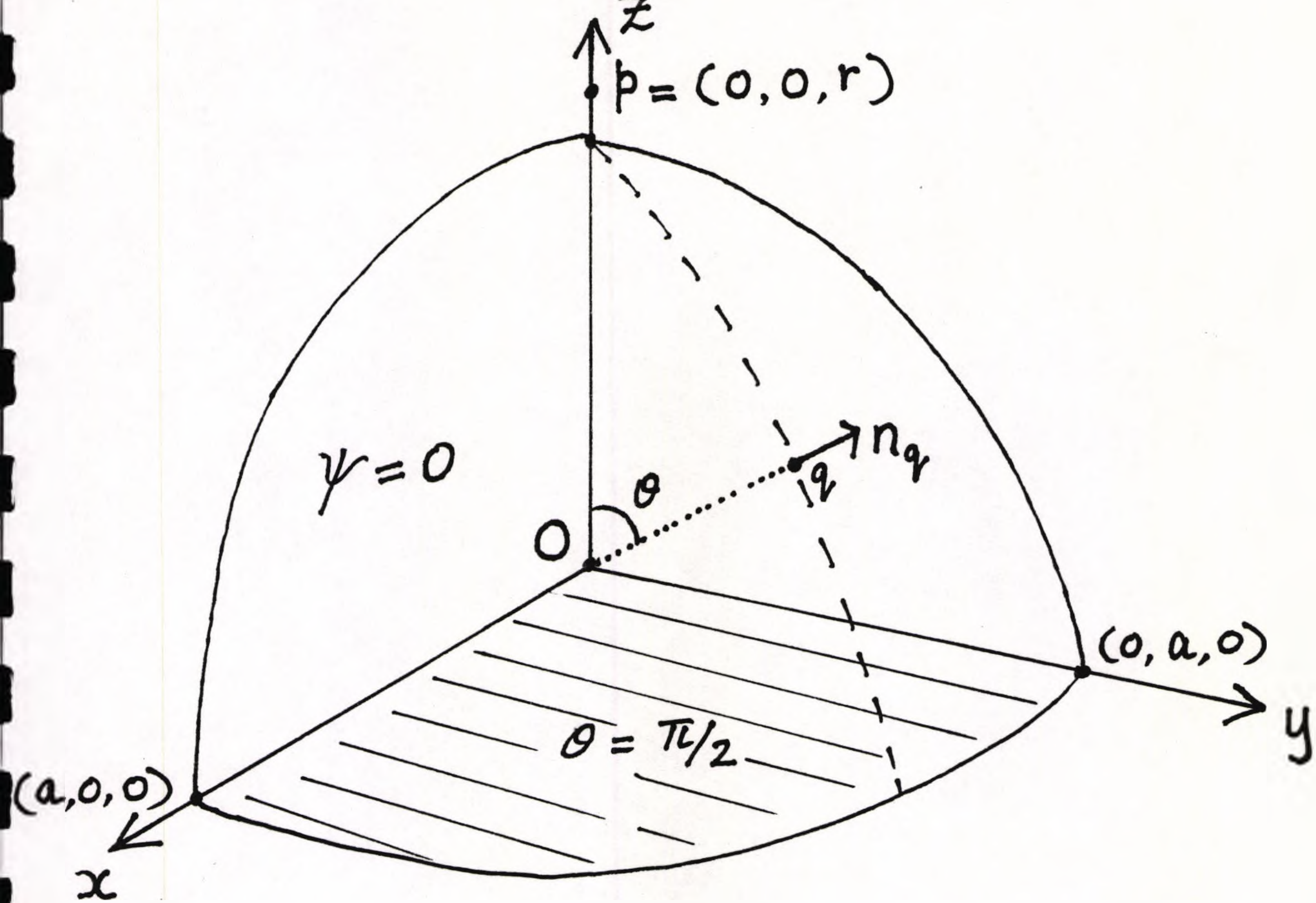

Fig.4 One octant of the surface of a sphere of radius a displaying cartesian coordinates of points on it. Note that point p is in the exterior or in the interior depending on whether $r > a$ or $r < a$.

then this generates the wave-function (or potential)

$$\phi(p) = \sigma_0 \int_{\partial B} g_k(p,q)\, dq \quad ; \quad p \in \mathbb{R}^3 \, . \tag{5.2}$$

It is possible to integrate (5.2) analytically by choosing the field point $p=(0,0,r)$, utilizing spherical polar coordinates with $\theta=0, \psi=0$ (see Fig.4), involving no loss of generality.

Interior wave-function

For this let $r<a$. An arbitrary boundary point q has the form

$$q = (\, a \sin\theta \cos\psi,\ a \sin\theta \sin\psi,\ a \cos\theta \,) \ , \tag{5.3a}$$

in which case

$$|p-q| = (a^2 + r^2 - 2ra\cos\theta)^{1/2} \ , \tag{5.3b}$$

$$dq = a^2 \sin\theta \, d\theta \, d\psi \ . \tag{5.3c}$$

If so, then (5.2) yields the interior wave-function

$$\phi^{-}(p) = \sigma_0 \int_0^{2\pi} \int_0^{\pi} \frac{e^{-ik(a^2+r^2-2ra\cos\theta)^{1/2}}}{(a^2+r^2-2ra\cos\theta)^{1/2}} \, a^2 \sin\theta \, d\theta \, d\psi \quad ; \quad |p| = r < a \ , \tag{5.3}$$

and substitution yields

$$= \frac{a}{r} \sigma_0 \int_0^{2\pi} \int_{a-r}^{r+a} e^{-ik\beta} d\beta \, d\psi \quad ; \quad r<a \ ,$$

where

$$\beta = (a^2 + r^2 - 2ra\cos\theta)^{1/2}$$

which on integration (see App.I) gives

$$\phi^-(p) \;=\; -\frac{2\pi a\sigma_0}{ikr}[e^{-ik(r+a)}-e^{-ik(a-r)}] \quad ; \quad |p|=r<a$$

$$= 4\pi a\sigma_0 e^{-ika}\,\frac{\sin kr}{kr} \quad ; \quad |p|=r<a \quad . \qquad (5.4)$$

This is radially symmetric as expected.

<u>Exterior wave function</u>

A similar integration (replacing a-r by r-a as the lower limit) shows that the corresponding exterior wave function is

$$\phi^+(p) \;=\; 4\pi a\sigma_0 \sin ka[\frac{e^{-ikr}}{kr}] \quad ; \quad |p|=r>a \quad . \qquad (5.5)$$

Again this is radially symmetric as expected. A simpler line of argument is to note that

$$|p-q| \simeq |p| - |q|\cos\theta \quad , \quad as \ |p|\rightarrow\infty$$

$$|p-q|^{-1} \simeq |p|^{-1} \quad , \quad as \ |p|\rightarrow\infty$$

in which case the asymptotic behaviour of $g_k(p,q)$ is as follows

$$g_k(p,q) \;\simeq\; [e^{ik|q|\cos\theta_{pq}}]\frac{e^{-ik|p|}}{|p|} \quad , \quad as \ |p|\rightarrow\infty \ , \qquad (5.5a)$$

where q is any surface point and θ_{pq} is the angle between the position vectors

p and q. In the case of a sphere the angle θ_{pq} is simply the polar angle θ . Substituting for p and q, and integrating (5.5a) with respect to q we have

$$\int_{\partial B} g_k(p,q)\,\sigma_o\,dq \;=\; 2\pi a^2 \frac{e^{-ikr}}{r} \int_0^{\pi} e^{ika\cos\theta} \sin\theta\, d\theta \;\;,$$

which yields (5.5).

Note that (5.4) and (5.5) give the common boundary function

$$\phi(p) \;=\; 4\pi a\,\sigma_0 e^{-ika} \frac{\sin ka}{ka} \quad ; \quad |p| = r = a \;\; . \tag{5.6}$$

Clearly

$$\lim_{p^- \to p} \phi^-(p^-) \;=\; \phi(p) \;=\; \lim_{p^+ \to p} \phi^+(p^+) \quad ; \quad p \in \partial B \;\;, \tag{5.7}$$

where p^-, p^+ denote interior and exterior field points of the sphere respectively. The relation (5.7) clearly confirms the continuity at ∂B of this simple-layer potential despite the weak singularity (cf. 2.1) of the integral operator in (5.2) as $q \to p \in \partial B$. These results conform with the general theory of Helmholtz potentials described in section 2.1 in chapter 2. It may be noted from (5.4) and (5.5) that both the interior and the exterior functions vanish on the boundary when the wave-number $ka = n\pi\,,\, n \in \mathbb{N}$; also,that the exterior function vanishes everywhere at these wave-numbers.

<u>Normal derivatives</u>

The normal derivatives of these wave-functions are as follows:

$$\frac{\partial\phi^-}{\partial n} = [-\frac{\partial\phi^-}{\partial r}]_{r=a} = 4\pi\sigma_0 e^{-ika}[\frac{\sin ka}{ka} - \cos ka] \ , \tag{5.8a}$$

$$\frac{\partial\phi^+}{\partial n} = [\frac{\partial\phi^+}{\partial r}]_{r=a} = -4\pi\sigma_0 \frac{\sin ka}{ka} e^{-ika}[ika+1] \ , \tag{5.8b}$$

where each normal points into the region concerned. Note that (see App.II)

$$\frac{\partial\phi^-}{\partial n} + \frac{\partial\phi^+}{\partial n} = -4\pi\sigma_0 \ , \tag{5.9}$$

which confirms the jump at ∂B of the normal derivatives of a simple-layer potential,in line with the general theory of Helmholtz potentials, Smirnov [79]. Putting $k=0$ in (5.4), (5.5) and (5.6) we retrieve the analogous static results i.e.

$$\begin{aligned} \phi^-(p) &= 4\pi a\sigma_0 \ ; \qquad |p| = r<a \\ \phi^+(p) &= 4\pi a^2\sigma_0 r^{-1} \ ; \qquad |p| = r>a \\ \phi(p) &= 4\pi a\sigma_0 \ ; \qquad |p| = r = a \ , \end{aligned} \tag{5.10}$$

in which case also

$$\frac{\partial\phi^-}{\partial n} + \frac{\partial\phi^+}{\partial n} = -4\pi\sigma_0 \ ,$$

which may alternatively be obtained by putting $k=0$ in (5.8a), (5.8b) and superposing the normal derivatives.

Now let us consider the right-hand side of (5.2) with $\sigma_0=1$ i.e.the integral

$$\int_{\partial B} g_k(p,q)\,dq \quad ; \quad p \in \mathbb{R}^3 \ ,$$

where ∂B denotes the surface of a sphere of radius a. We may show that

$$\frac{\partial}{\partial n_p^-}\left[\int_{\partial B} g_k(p,q)\,dq\right] = -2\pi + \int_{\partial B} \frac{\partial g_k}{\partial n_p^-}(p,q)\,dq \quad ; \quad p \in \partial B \ , \qquad (5.11a)$$

$$\frac{\partial}{\partial n_p^+}\left[\int_{\partial B} g_k(p,q)\,dq\right] = -2\pi + \int_{\partial B} \frac{\partial g_k}{\partial n_p^+}(p,q)\,dq \quad ; \quad p \in \partial B \ , \qquad (5.11b)$$

where n_p^-, n_p^+ denote the normals at $p \in \partial B$ pointing respectively into the interior and the exterior regions. To compute the integrals on the right hand sides of (5.11), choose $p=(0,0,r)_{r=a}$, and note that

$$\frac{\partial}{\partial n_p^{\pm}} = \left[\pm\frac{\partial}{\partial r}\right]_{r=a} \ .$$

Therefore

$$\frac{\partial g_k}{\partial n_p^{\pm}}(p,q) = \frac{\partial}{\partial n_p^{\pm}}\left[\frac{e^{-ik|p-q|}}{|p-q|}\right]$$

$$= \pm\frac{\partial}{\partial r}\left[\frac{e^{-ik(a^2+r^2-2ar\cos\theta)^{1/2}}}{(a^2+r^2-2ar\cos\theta)^{1/2}}\right]_{r=a}$$

$$= \pm\frac{1}{2a}\left[-ik - \frac{1}{\eta}\right]e^{-ik\eta}$$

where

$$\eta = 2a\sin(\theta/2) \ .$$

Using (5.3c), now yields

$$\int_{\partial B} \frac{\partial g_k}{\partial n_p^{\pm}}(p,q)dq = \pm\frac{1}{2a}\int_0^{2\pi}\int_0^{2a}[-ik\eta - 1]e^{-ik\eta}d\eta\, d\psi \ ,$$

which can be evaluated to give

$$\int_{\partial B} \frac{\partial g_k}{\partial n_p^{-}}(p,q)dq = 2\pi[\frac{i}{ka}\{e^{-i2ka} - 1\} - e^{-i2ka}] \quad ; \qquad p\in\partial B \ , \tag{5.12a}$$

$$\int_{\partial B} \frac{\partial g_k}{\partial n_p^{+}}(p,q)dq = 2\pi[e^{-i2ka} - \frac{i}{ka}\{e^{-i2ka} - 1\}] \quad ; \qquad p\in\partial B \ . \tag{5.12b}$$

The left hand side of (5.11a,b) may be obtained by putting $\sigma_0=1$ in (5.8a,b). Relations (5.11a,b) follow at once by using (5.8) and (5.12) in the respective equations. An immediate deduction from (5.11a) and (5.11b) is that

$$\frac{\partial}{\partial n_p^{-}}[\int_{\partial B} g_k(p,q)dq] + \frac{\partial}{\partial n_p^{+}}[\int_{\partial B} g_k(p,q)dq] = -4\pi \quad ; \qquad p\in\partial B \ , \tag{5.13}$$

since it readily follows from (5.12a) and (5.12b) that

$$\int_{\partial B} \frac{\partial g_k}{\partial n_p^{-}}(p,q)dq + \int_{\partial B} \frac{\partial g_k}{\partial n_p^{+}}(p,q)dq = 0 \quad ; \qquad p\in\partial B \ .$$

Note that (5.13) can be obtained by simply putting $\sigma_0=1$ in (5.9). The formulae (5.11a) and (5.11b) are particular cases of the more general normal derivative formulae

$$\frac{\partial \phi^-}{\partial n_p^-}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_p^-}(p,q)\,\sigma(q)\,dq \quad ; \quad p \in \partial B \;, \quad (5.14a)$$

$$\frac{\partial \phi^+}{\partial n_p^+}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_p^+}(p,q)\sigma(q)\,dq \quad ; \quad p \in \partial B \;, \quad (5.14b)$$

which readily gives

$$\frac{\partial \phi^-}{\partial n_p^-}(p) + \frac{\partial \phi^+}{\partial n_p^+}(p) = -4\pi\,\sigma(p) \quad ; \quad p \in \partial B \;,$$

since

$$\int_{\partial B} \frac{\partial g_k}{\partial n_p^-}(p,q)\,dq = -\int_{\partial B} \frac{\partial g_k}{\partial n_p^+}(p,q)\,dq \quad ; \quad p \in \partial B \;,$$

where ∂B is a general closed Liapunov surface.

5.2 Double-layer case

As discussed in chapter 2, a second fundamental solution of the Helmholtz equation is obtained by differentiating the free-space Green's function g_k in the normal direction at $q \in \partial B$,i.e.

$$\frac{\partial g_k}{\partial n_q}(p,q) = \frac{\partial}{\partial n_q}\left[\frac{e^{-ik|p-q|}}{|p-q|}\right] \quad ; \quad p \in \mathbb{R}^3$$

which is the dipole potential generated at p by a unit dipole source at $q \in \partial B$, where the normal points into the exterior region. As before we consider a sphere of radius a with the source density $\mu(q)$ at any $q \in \partial B$, see Fig.4. Then the double-layer Helmholtz potential generated at the field point $p \in \mathbb{R}^3$ by a continuous distribution of source points is given by

$$\int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq \quad ; \qquad p \in \mathbb{R}^3 \ . \tag{5.15}$$

If we assume a uniform distribution of dipole sources over the spherical surface i.e.

$$\mu(q) = \mu_0 \quad , \qquad \forall q \in \partial B \ ,$$

then this generates a wave-function (or potential)

$$\Phi(p) = \mu_0 \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,dq \quad ; \qquad p \in \mathbb{R}^3 \backslash \partial B \ . \tag{5.16}$$

As before, write without loss of generality

$$p = p^{\pm} = (0,0,r) \quad , \quad r \neq a \ ,$$

also now in place of (5.3a) write

$$q = (R \sin\theta \cos\psi, R \sin\theta \sin\psi, R\cos\theta)_{R=a} \ .$$

For the sphere

$$\frac{\partial g_k}{\partial n_q}(p,q) = \left[\frac{\partial g_k}{\partial R}\right]_{R=a} = \frac{\partial}{\partial R}\left[\frac{e^{-ik|p-q|}}{|p-q|}\right]_{R=a} \ .$$

Therefore the interior wave-function takes the form

$$\Phi^-(p) = \mu_0 \int_0^{2\pi}\int_0^{\pi} \frac{\partial}{\partial R}\left[\frac{e^{-ik(R^2+r^2-2Rr\cos\theta)^{1/2}}}{(R^2+r^2-2Rr\cos\theta)^{1/2}}\right]_{R=a} a^2 \sin\theta\, d\theta\, d\psi \qquad (*)$$

$$= \mu_0 \int_0^{2\pi}\int_0^{\pi} \left[-\frac{ik}{\rho^2} - \frac{1}{\rho^3}\right] \qquad (**)$$

$$\times\, e^{-ik\rho}(a - r\cos\theta) a^2 \sin\theta\, d\theta\, d\psi \quad ; \qquad |p| = r < a$$

where

$$\rho = (a^2 + r^2 - 2ar\cos\theta)^{1/2} \ ,$$

Note that the term enclosed by the parenthesis [ ] in (*) is given in (5.3) with a replaced by R. Detailed integration of (**) gives

$$\Phi^-(p) = -4\pi\mu_0(ika + 1)e^{-ika}\frac{\sin kr}{kr} \quad ; \qquad |p| = r < a \ . \quad (5.17a)$$

A similar analyses give the following:

$$\Phi^+(p) = 4\pi\mu_0[ka\cos ka - \sin ka]\frac{e^{-ikr}}{kr} \quad ; \qquad |p| = r > a \ , \quad (5.17b)$$

$$\Phi(p) = 2\pi\mu_0 e^{-i2ka} - 4\pi\mu_0 e^{-ika}\frac{\sin ka}{ka} \quad ; \qquad |p| = r = a \ . \quad (5.17c)$$

Note that (5.17b) may be obtained by utilizing the asymptotic behaviour of $\partial g_k/\partial n_q$. This is obtained by differentiating the asymptotic behaviour (5.5a) of $g_k(p,q)$ in the normal direction at q i.e.

$$\frac{\partial g_k}{\partial n_q}(p,q) \simeq \frac{\partial}{\partial R}\{[e^{ikR\cos\theta}]\frac{e^{-ik|p|}}{|p|}\}_{R=a}$$

$$\simeq [ik\cos\theta\, e^{ik|q|\cos\theta}]\frac{e^{-ik|p|}}{|p|} \quad , \quad as \ |p|\to\infty \tag{5.17d}$$

where q is any surface point. Substituting for p and q, and integrating (5.17d) with respect to q we get

$$\int_{\partial B}\frac{\partial g_k}{\partial n_q}(p,q)\,\mu_o dq = 2\pi a^2\mu_o ik\frac{e^{ikr}}{r}\int_0^\pi \cos\theta\, e^{ika\cos\theta}\sin\theta\, d\theta \ ,$$

which yields (5.17b).

Putting $k=0$ in (5.17a),(5.17b) and (5.17c) yields the corresponding statics results (see Jaswon and Symm [36]):

$$\begin{aligned} \Phi(p) &= -4\pi\mu_0 \ ; && |p|=r<a \\ &= -2\pi\mu_0 \ ; && |p|=r=a \\ &= \ 0 \ ; && |p|=r>a \end{aligned} \tag{5.18}$$

From (5.17a) and (5.17b), note the following limiting values as we approach the boundary:

$$\Phi^-(p) \to -4\pi\mu_0(ika+1)e^{-ika}\frac{\sin ka}{ka} \ , \quad as \ |p|=r\to a$$

$$\Phi^+(p) \to 4\pi\mu_0(a\cos ka - \frac{\sin ka}{k})\frac{e^{-ika}}{a} \ , \quad as \ |p|=r\to a \tag{5.19}$$

showing clearly how the double-layer potential jumps at ∂B i.e.

$$[\Phi^-(p) - \Phi^+(p)] \to -4\pi\mu_0 \;, \quad as \;\; |p| \to a \;.$$

Limiting values in (5.19) are related to the boundary values of Φ given in (5.17c), by the following formulae:

$$\Phi^+(p) \to \Phi(a) + 2\pi\mu_0 \;, \quad as \;\; |p| = r \to a$$
$$\Phi^-(p) \to \Phi(a) - 2\pi\mu_0 \;, \quad as \;\; |p| = r \to a \tag{5.20}$$

where $\Phi(a)$ denotes the boundary value of Φ given in (5.17c). Note that (5.20) is a particular case of the more general jump relation of the double-layer potential i.e.

$$\Phi^+(p) \to \Phi(p_0) + 2\pi\mu(p_0) \;, \quad as \;\; p \to p_0 \in \partial B$$
$$\Phi^-(p) \to \Phi(p_0) - 2\pi\mu(p_0) \;, \quad as \;\; p \to p_0 \in \partial B \tag{5.20a}$$

Normal derivatives

Normal derivatives of the wave-functions given by (5.17a) and (5.17b) may be obtained by noting that for a sphere

$$\frac{\partial\Phi}{\partial n_p^{\pm}}(p) = \pm\left[\frac{\partial\Phi}{\partial r}(p)\right]_{r=a} \;; \qquad p \in \partial B \;,$$

where

$$p = (0,0,r)_{r=a} .$$

Therefore we have the following:

$$\frac{\partial \Phi^-}{\partial n_p^-} = -\frac{\partial}{\partial r}[-4\pi(ika+1)e^{-ika}\frac{\sin kr}{kr}]_{r=a}$$

$$= 4\pi\frac{(ika+1)}{a}e^{-ika}[\cos ka - \frac{\sin ka}{ka}] \quad , \quad (5.21a)$$

$$\frac{\partial \Phi^+}{\partial n_p^+} = \frac{\partial}{\partial r}[4\pi(ka\cos ka - \sin ka)\frac{e^{-ikr}}{kr}]_{r=a}$$

$$= -4\pi\frac{(ika+1)}{a}e^{-ika}(\cos ka - \frac{\sin ka}{ka}) \quad . \quad (5.21b)$$

From which we see that

$$[\frac{\partial \Phi^-}{\partial n_p^-}(p) + \frac{\partial \Phi^+}{\partial n_p^+}(p)]_{|p|=a} = 0 \quad , \quad (5.22)$$

where the notations n_p^+, n_p^- carry the same meaning as before. This shows clearly the continuity of the normal derivative of a double-layer potential despite its jump at ∂B . So we have verified a general property (see chapter 2) of the Helmholtz double-layer potential with respect to a Liapunov surface. For $k=0$ the right-hand sides of (5.21a,b) become zero as expected. The result in (5.20) have been verified for a sphere of radius a. The boundary behaviour of a double-layer potential for $k=0$ is depicted in Fig.5, which graphs the magnetostatic potential generated by a uniform magnetic shell having the form of a circle transverse to the ξ -axis. This graph is due to Jaswon and El-Damanawi [37] for their work on dislocation. See Appendices III, IV for full details of the calculations described so far.

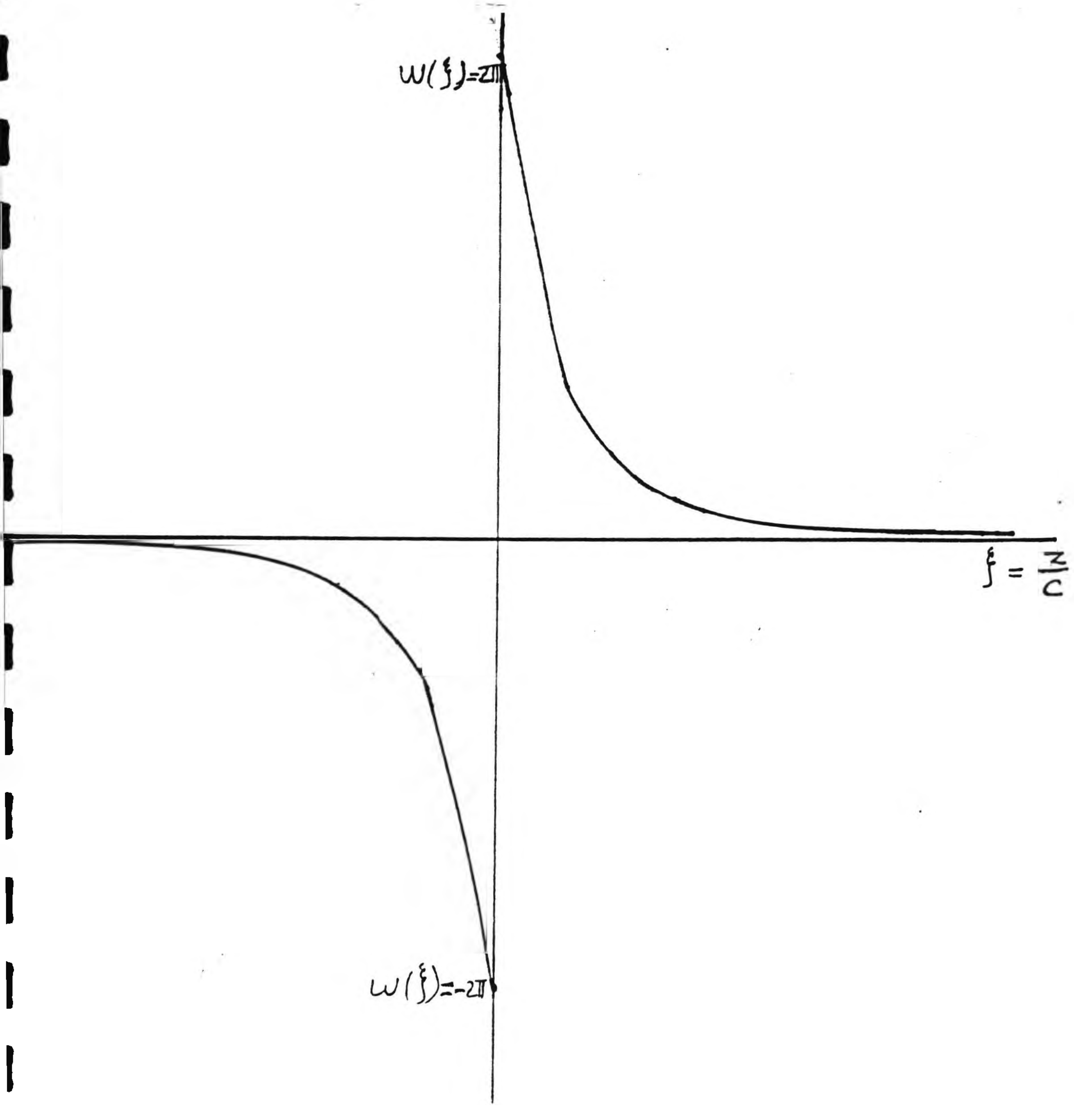

Fig.5 Magnetostatic potential W, generated by a uniform magnetic shell having circular shape oriented transverse to the ξ -axis.

Chapter 6

Discussion of integral operator N_k

In this chapter we look at the integral

$$\text{(I)} \qquad N_k(p) = \int_{\partial B} \lim_{p \to p_0} \left[\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) \right] dq \quad ; \quad p \notin \partial B \ , \quad p_0 \in \partial B \ , \qquad (6.1a)$$

which plays a decisive role in various BEM formulations such as Burton and Miller [17], Kussmaul [48] and Jin [40]. In (6.1a) p is a point sufficiently close to the surface on the normal at $p_0 \in \partial B$. Note here that both normal derivatives are to be performed, followed by the limiting process, before the integration. The following singularity behaviour has been noted in chapter 3:

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) = O(|p-q|^{-3}) \quad ; \qquad as \quad |p-q| \to 0 \ .$$

Consequently the integrand in (6.1a) becomes hypersingular as $q \to p$ and as a result the integral fails to yield a finite value.

Some effective alternative definitions of N_k are as follows:

$$\text{(II)} \qquad N_k(p) = \lim_{p \to p_0} \left[\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\, dq \right] \quad ; \quad p \notin \partial B \ , \quad p_0 \in \partial B \ , \qquad (6.1b)$$

where p has the same status as in (6.1a) except that the limit is taken after the integration has been performed;

$$\text{(III)} \qquad N_k(p) = \lim_{p \to p_0} \Big[\frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\, dq \Big] \quad ; \quad p \notin \partial B \ , \quad p_0 \in \partial B \ , \qquad (6.1c)$$

where here p is initially kept fixed within the integral, after which the normal derivative operation with respect to p is performed, followed by the limit as $p \to p_0 \in \partial B$.

The definition **(III)** is physically plausible since it provides the normal derivative of a double-layer potential at ∂B (see chapter 5). In all these operations we use the convention that the normals on the surface are pointing outward. Note that the definitions **(II)** and **(III)** are mathematically equivalent (which always yield finite values) i.e. the integral (6.1c) is a continuous and differentiable function of p everywhere except at ∂B, therefore we may differentiate under the integral sign to obtain **(II)** from **(III)**. In particular, for a sphere we shall verify that

$$\lim_{p \to p_0} \Big[\frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\, dq \Big] = \lim_{p \to p_0} \Big[\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\, dq \Big] \quad ; \quad p \notin \partial B \ , \quad p_0 \in \partial B \ .$$

However **(III)** is clearly simpler to evaluate than **(II)** and we therefore adopt **(III)** as the standard definition of N_k in this thesis. Also, note that **(III)** (and **(II)**) necessarily remains continuous across ∂B , as mentioned in chapter 2 (section 2.2) and chapter 5. These properties and features of the definitions may be verified with reference to a spherical surface in the following section.

6.1 Tests of definition (II) and definition (III) for a sphere

We first look at the definition (II). Using spherical polar coordinates for a sphere of radius a write

$$p = (0,0,r)_{r \to a} \ , \quad p_0 = (0,0,a) \ ,$$

$$q = (R\sin\theta\cos\psi, R\sin\theta\sin\psi, R\cos\theta)_{R \to a} \ ,$$

which gives

$$|p-q| = (R^2 + r^2 - 2Rr\cos\theta)^{1/2}{}_{R \to a, r \to a} \ .$$

Now for a sphere

$$\frac{\partial}{\partial n_q} = [\frac{\partial}{\partial R}]_{R \to a}$$

$$\frac{\partial}{\partial n_p} = [\frac{\partial}{\partial r}]_{r \to a}$$

Therefore

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q} = \frac{\partial}{\partial r}[\frac{\partial}{\partial R}(\frac{e^{-ik(R^2 + r^2 - 2Rr\cos\theta)^{1/2}}}{(R^2 + r^2 - 2Rr\cos\theta)^{1/2}})] \ ; \qquad R \to a \ , \quad r \to a \ .$$

$$= [\frac{ik\cos\theta}{\xi^2} + \frac{\cos\theta - k^2\zeta_1\zeta_2}{\xi^3} + \frac{3ik\zeta_1\zeta_2}{\xi^4} + \frac{3\zeta_1\zeta_2}{\xi^5}] e^{-ik\xi}{}_{R \to a, r \to a} \ , \quad (6.2)$$

where

$$\xi = |p-q| = (R^2 + r^2 - 2Rr\cos\theta)^{1/2}$$

$$\zeta_1 = R - r\cos\theta \tag{6.2a}$$

$$\zeta_2 = r - R\cos\theta$$

It is now necessary to take the limit as $R \to a$, since the integration in (II) is to be performed with respect to $q \in \partial B$. If so then the form of (6.2) remains the same except that now

$$\xi = (a^2 + r^2 - 2ar\cos\theta)^{1/2}$$

$$\zeta_1 = a - r\cos\theta \tag{6.2b}$$

$$\zeta_2 = r - a\cos\theta$$

Taking p just inside the surface, i.e. $r<a$, using (5.3c) and integrating gives

$$\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq = \int_0^{2\pi}\int_0^{\pi}\Big[\frac{ik\cos\theta}{\xi^2} + \frac{\cos\theta - k^2\zeta_1\zeta_2}{\xi^3}$$

$$+ \frac{3ik\zeta_1\zeta_2}{\xi^4} + \frac{3\zeta_1\zeta_2}{\xi^5}\Big]e^{-ik\xi}a^2\sin\theta\, d\theta\, d\psi \quad ; \quad r<a \tag{6.2c}$$

where ξ, ζ_1, ζ_2 refer to the expressions given in (6.2b). Substitution of these into (6.2c) gives

$$\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q} dq = \frac{1}{4r^2} \int_0^{2\pi} \int_{a-r}^{r+a} [2(a^2+r^2)\{\frac{ik}{\xi}+\frac{1}{\xi^2}\}$$

$$+ \{1 + ik\xi - k^2\xi^2\} + (a^2-r^2)^2\{\frac{k^2}{\xi^2} - \frac{3ik}{\xi^3} - \frac{3}{\xi^4}\}]e^{-ik\xi} d\xi d\psi \quad ; r<a$$

i.e.

$$\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q} dq = -4\pi e^{-ika}(1+ika)[\cos kr - \frac{\sin kr}{kr}](\frac{1}{r}) \quad ; \quad |p| = r<a \ .$$

Now taking limit as $p \to p_0$ (i.e. as $r \to a$), it follows at once that

$$\lim_{p\to p_0} \int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) dq = -4\pi e^{-ika}\frac{(1+ika)}{a}[\cos ka - \frac{\sin ka}{ka}] \ . \qquad (6.3)$$

For the case $r>a$, definition **(II)** gives the same value as in (6.3), as may be expected from the identification of **(II)** with **(III)**.

To test the definition **(III)** we need only recall (5.17) and (5.21) from chapter 5, from which we readily get the following for the case $r<a$:

$$\frac{\partial}{\partial n_p} \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q) dq = \frac{\partial}{\partial r}[-4\pi(ika+1)e^{-ika}\frac{\sin kr}{kr}]_{r\to a}$$

$$(6.4a)$$

$$= -4\pi(ika+1)e^{-ika}[\cos kr - \frac{\sin kr}{kr}]\frac{1}{r}_{r\to a}$$

also for the case $r>a$ we get

$$\frac{\partial}{\partial n_p}\int_{\partial B}\frac{\partial g_k}{\partial n_q}(p,q)\,dq = \frac{\partial}{\partial r}\left[4\pi(ka\cos ka-\sin ka)\frac{e^{-ikr}}{kr}\right]_{r\to a}$$

$$=-4\pi(ka\cos ka-\sin ka)\frac{(ikr+1)}{r}\frac{e^{-ikr}}{kr}\Bigg|_{r\to a} \quad . \quad (6.4b)$$

It readily follows from (6.4a), (6.4b) that

$$\lim_{p\to p_0^{\pm}}\left[\frac{\partial}{\partial n_p}\int_{\partial B}\frac{\partial g_k}{\partial n_q}(p,q)\,dq\right] = -4\pi(ika+1)e^{-ika}\left(\cos ka-\frac{\sin ka}{ka}\right)\frac{1}{a}\ , \quad (6.5)$$

where the superscripts indicate the approach to the surface point from the inside(-) and from the outside(+). We have thus recovered (6.3), using **(III)**, with much less effort.

The preceding discussion clearly shows for a spherical surface that as long as the point p is not initially on the surface and only coincides with the surface after the integration has been performed, then the operator N_k in (6.1a) could be given a meaning. Next we look at the failure of **(I)** with reference to a spherical surface when the point p is already on the boundary.

6.2 Test of definition (I) for a sphere

We test N_k with reference to the sphere with the same forms for p, p_o, q as in section 6.1 . Putting $r=a$ i.e. letting $p\to p_0\in\partial B$ in (6.2), and simplifying the terms, yield the following:

$$\int_{\partial B} \lim_{p \to p_0 \in \partial B} \left[\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) \right] dq$$

$$= \int_0^{2\pi}\int_0^{\pi} [k^2 a^2 \sin^2(\theta/2) - \frac{1}{2} ik\sin(\theta/2) - \frac{1}{4a} \tag{6.6}$$

$$-\frac{1}{2} ik \frac{1}{\sin(\theta/2)} - \frac{1}{4a\sin^2(\theta/2)}] e^{-ik2a\sin(\theta/2)} \cos(\theta/2)\, d\theta\, d\psi$$

Note that the first three terms of the integrand on the right hand side of (6.6) are readily seen to be integrable, whereas the last two terms are clearly singular when $\theta = 0$. Therefore, letting $0 < \varepsilon \leq \theta \leq \pi$ for these terms, and integrating, we find

$$\int_{\varepsilon}^{\pi} [-\frac{1}{2} ik \frac{1}{\sin(\theta/2)} - \frac{1}{4a\sin^2(\theta/2)}] e^{-ik2a\sin(\theta/2)} \cos(\theta/2)\, d\theta$$

$$= e^{-ik2a}(2a)^{-1} - e^{-ik2a\sin(\varepsilon/2)} (2a\sin(\varepsilon/2))^{-1} \; ; \quad \varepsilon > 0$$

which clarifies the behaviour of the integral as $\varepsilon \to 0$. This demonstrates the failure of **(I)** as a definition of N_k.

6.3 Surface Helmholtz Equation (SHE)

Confirmations of **(III)** (and by the same token **(II)**) as advantageous definitions of N_k may be obtained by exploiting the Surface Helmholtz Equation (SHE) applied to a sphere. We therefore recall the Surface

Helmholtz Equation (equation (4.14) in chapter 4),

$$\int_{\partial B}[\ \phi(q)\frac{\partial g_k}{\partial n_q}(p,q)\ -\ \frac{\partial \phi}{\partial n_q}(q)g_k(p,q)\ ]\ dq\ =\ 2\pi\,\phi(p)\quad ;\quad p\in\partial B\ ,(6.7a)$$

where n_q points into the exterior region. Operating with $\partial/\partial n_p$ on SHE yields

$$\int_{\partial B}[\ \phi(q)\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\ -\ \frac{\partial \phi}{\partial n_q}(q)\frac{\partial g_k}{\partial n_p}(p,q)\ ]dq\ =\ 2\pi\frac{\partial \phi}{\partial n_p}(p)\ ;\quad p\in\partial B\ .(6.7b)$$

Now introducing the simple exterior wave function

$$\phi(p)\ =\ \frac{a}{r}e^{-ik(r-a)}\quad ;\qquad |p|=r\geq a\ .$$

We note that

$$\phi(p)\ =\ 1\qquad ;\qquad p\in\partial B$$

$$\frac{\partial \phi}{\partial n_p}(p)\ =\ -\frac{1}{a}(ika\ +\ 1)\quad ;\qquad p\in\partial B \tag{6.8}$$

Substitution of (6.8) into (6.7b) gives

$$\int_{\partial B}\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq\ =\ -\frac{(ika+1)}{a}[\,2\pi\ +\ \int_{\partial B}\frac{\partial g_k}{\partial n_p}(p,q)\,dq\,]\quad ;\quad p\in\partial B\ .(6.9)$$

Note that (see chapter 5) for a spherical surface

$$\int_{\partial B}\frac{\partial g_k}{\partial n_p}(p,q)\,dq\ =\ 2\pi\,[\,\frac{e^{-i2ka}-1}{ika}\ +\ e^{-i2ka}\,]\quad ;\quad p\in\partial B\ ,$$

which is substituted for the integral on the right-hand side of (6.9) to obtain

$$\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq = -4\pi\left[\frac{(ika+1)}{a^2}\right]\left[a\cos ka - \frac{\sin ka}{k}\right]e^{-ika} \;; \quad p \in \partial B \;.$$

Accordingly we see that it would be consistent to define N_k by referring to (6.5) i.e. **(III)**.

In the section that follows we test the definitions **(I)**, **(II)** and **(III)** for a plate.

6.4 Tests of N_k for a flat plate

Consider a flat surface J, see Fig.6, with boundary contour equation given by

$$r = r(\theta) \;,$$

i.e.

$$J = \{(r,\theta)\,;\quad 0 \le r \le r(\theta), \quad 0 \le \theta \le 2\pi\} \;,$$

where a cylindrical coordinate system is used i.e.

$$\begin{aligned} p &= (0,0,z)_{z\to 0} \\ p_0 &= (0,0,0) \qquad\qquad , \\ q &= (r\cos\theta, r\sin\theta, h)_{h\to 0} \end{aligned}$$

$$|p-q| = [r^2 + (z-h)^2]^{1/2} \quad ; \quad z\to 0 \;,\; h\to 0 \;.$$

Now

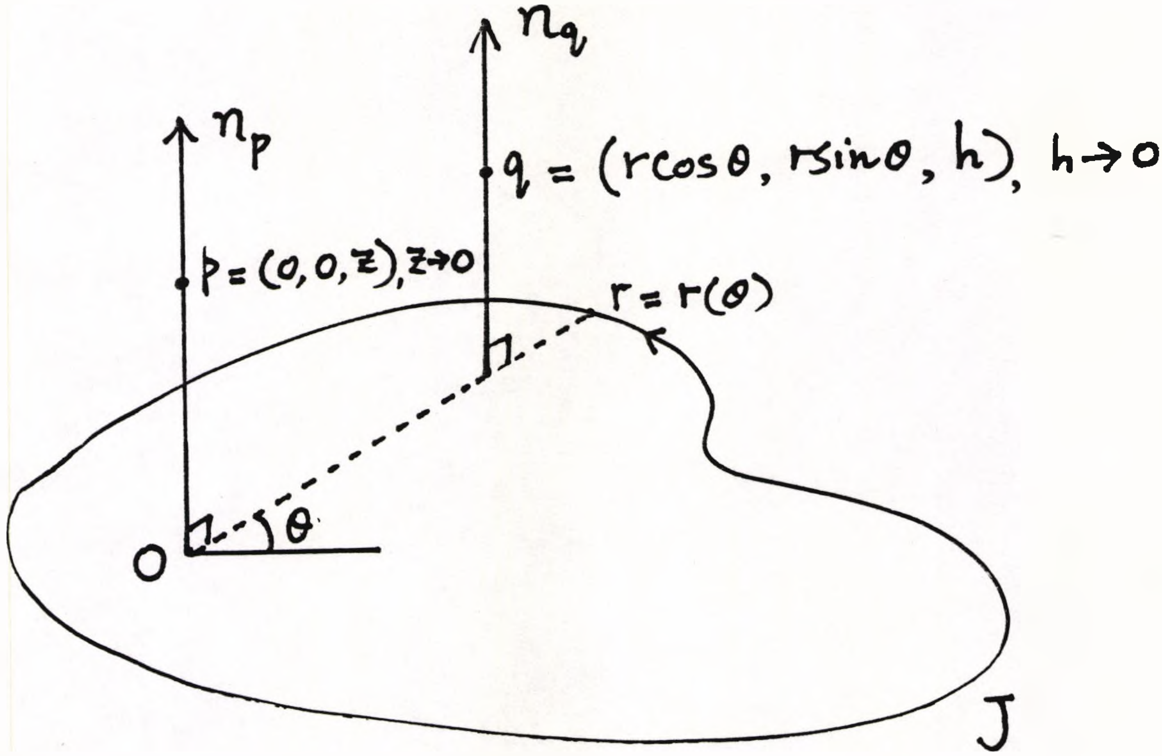

Fig.6 Surface features of a flat plate *J*.

$$\frac{\partial}{\partial n_q} = [\frac{\partial}{\partial h}]_{h \to 0}$$

$$\frac{\partial}{\partial n_p} = [\frac{\partial}{\partial z}]_{z \to 0}$$

and therefore

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q} = \frac{\partial}{\partial z}[\frac{\partial g_k}{\partial h}] \quad ; z \to 0 \ , \ h \to 0 \ ;$$

However to test **(II)**, we put $h=0$ and keep $z>0$, i.e.

$$\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q) = \frac{\partial}{\partial z}[\frac{\partial g_k}{\partial h}{}_{h=0}]_{z>0}$$

$$= [\frac{ik}{\gamma^2} + \frac{1+k^2z^2}{\gamma^3} - \frac{3ikz^2}{\gamma^4} - \frac{3z^2}{\gamma^5}]e^{-ik\gamma} \quad ; \quad z>0 \tag{6.10}$$

where

$$\gamma = (r^2 + z^2)^{1/2} \ .$$

Therefore we have:

$$\int_J [\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)]_{p \notin J} \ dq = \int_0^{2\pi}\int_0^{r(\theta)} [\frac{\partial}{\partial z}(\frac{\partial g_k}{\partial h})_{h=0}]_{z>0} \ r dr d\theta$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} [\frac{ik}{\gamma^2} + \frac{1+k^2z^2}{\gamma^3} - \frac{3ikz^2}{\gamma^4} - \frac{3z^2}{\gamma^5}] e^{-ik\gamma} \ r dr d\theta_{z>0} \ ,$$

$$= \int_0^{2\pi}\int_z^{R(\theta)} [\frac{ik}{\gamma} + \frac{1+k^2z^2}{\gamma^2} - \frac{3ikz^2}{\gamma^3} - \frac{3z^2}{\gamma^4}]e^{-ik\gamma} \; d\gamma \, d\theta \quad ,(6.11)$$
$$z>0$$

where

$$R(\theta) = \sqrt{r(\theta)^2+z^2} \qquad ; \quad z>0 \; .$$

Since the contour function $r(\theta)$ is not prescribed it is not possible to integrate (6.11) exactly. However, writing the integral in (6.11) in the form

$$\int_0^{2\pi}\int_z^{R(\theta)} [\frac{ik}{\gamma}+\frac{1}{\gamma^2}]e^{-ik\gamma}d\gamma\, d\theta + \int_0^{2\pi}\int_z^{R(\theta)} [\frac{k^2}{\gamma^2}-\frac{3ik}{\gamma^3}-\frac{3}{\gamma^4}]$$
$$\times\; z^2 e^{-ik\gamma}d\gamma\, d\theta \qquad ; z>0$$

and integrating with respect to γ give

$$\int_0^{2\pi}[\frac{e^{-ikz}}{z}-\frac{e^{-ikR(\theta)}}{R(\theta)}]d\theta + z^2\int_0^{2\pi}[\frac{ik}{R(\theta)^2}+\frac{1}{R(\theta)^3}]e^{-ikR(\theta)}d\theta$$
$$- \int_0^{2\pi}[ik+\frac{1}{z}]e^{-ikz}d\theta$$
$$= -2\pi ike^{-ikz} - \int_0^{2\pi}\frac{e^{-ikR(\theta)}}{R(\theta)}d\theta$$
$$+ z^2\int_0^{2\pi}[\frac{ik}{R(\theta)^2}+\frac{1}{R(\theta)^3}]e^{-ikR(\theta)}d\theta \qquad ; \quad z>0$$

Note that for very small z the last integral in the above is negligible and consequently, taking limits

$$\lim_{p \to p_0 \in J} \int_J [\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)]dq = -2\pi ik - \lim_{z \to 0} \int_0^{2\pi} \frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}} d\theta \ . \quad (6.12)$$

Note that the integral on the right hand side of (6.12) represents a simple-layer potential generated at $p=(0,0,z)$ by a continuous distribution of unit sources on the contour $r=r(\theta)$. Therefore we may write

$$\lim_{z \to 0} \int_0^{2\pi} \frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}} d\theta = \int_0^{2\pi} \lim_{z \to 0} [\frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}}] d\theta = \int_0^{2\pi} \frac{e^{-ikr(\theta)}}{r(\theta)} d\theta \ ,(6.13)$$

Therefore for a given contour equation $r=r(\theta)$ of the boundary of a flat plate, definition **(II)** gives

$$\lim_{z \to 0} \int_J \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)dq = -2\pi ik - \int_0^{2\pi} \frac{e^{-ikr(\theta)}}{r(\theta)} d\theta \ . \qquad (6.14)$$

To test definition **(III)** for the flat plate J we write

$$\int_J \frac{\partial g_k}{\partial n_q}(p,q)dq = \int_0^{2\pi} \int_0^{r(\theta)} \frac{\partial}{\partial h} [\frac{e^{-ik\sqrt{r^2+(z-h)^2}}}{\sqrt{r^2+(z-h)^2}}]_{h=0} \ rdrd\theta \ ; \qquad z>0 \ ,$$

where p, q have the same form as previously. The right-hand integral after some manipulations becomes

$$= z\int_0^{2\pi}\int_z^{\sqrt{r(\theta)^2+z^2}}\left[\frac{ik}{\gamma} + \frac{1}{\gamma^2}\right]e^{-ik\gamma}\,d\gamma\,d\theta \quad ; \quad z>0 \;, \qquad (6.15)$$

where γ has already been defined. Performing the integration with respect to γ , the right-hand side of (6.15) becomes

$$= \int_0^{2\pi}\left[e^{-ikz} - \frac{z e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}}\right]d\theta \quad ; \quad z>0 \;,$$

i.e.

$$= 2\pi e^{-ikz} - z\int_0^{2\pi}\left[\frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}}\right]d\theta \quad ; \quad z>0 \;.$$

Then operating with $\partial/\partial z$ yields

$$\frac{\partial}{\partial n_p}\int_J \frac{\partial g_k}{\partial n_q}dq$$

$$= -2\pi ike^{-ikz} - \left[1 + z\frac{\partial}{\partial z}\right]\int_0^{2\pi}\frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}}\,d\theta \quad ; \quad |p| = z>0 \;. \qquad (6.16)$$

For $z=0$, i.e.$p=p_o$ it follows at once that the third term of (6.16) vanishes, so that we recover (6.12) i.e.

$$\lim_{p\to p_0\in J}\frac{\partial}{\partial n_p}\int_J \frac{\partial g_k}{\partial n_q}dq = -2\pi ik - \lim_{z\to 0}\left[\int_0^{2\pi}\frac{e^{-ik\sqrt{r(\theta)^2+z^2}}}{\sqrt{r(\theta)^2+z^2}}d\theta\right] \;.$$

We therefore obtain

$$\lim_{p \to p_0 \in J} \frac{\partial}{\partial n_p} [\int_J \frac{\partial g_k}{\partial n_q}(p,q)\, dq] = -2\pi ik - \int_0^{2\pi} \frac{e^{-ikr(\theta)}}{r(\theta)} d\theta \ . \qquad (6.17)$$

Therefore **(III)** gives an identical result to that of **(II)** as expected.Note that the right hand sides of (6.17), (6.14) are identical, as expected. Note that putting $k=0$ in the formula (6.17) gives the expected magnetostatic result. We remark that Formula (6.17) has also been obtained by Terai [93], using a method based upon Hadamard's limiting procedure [33]. See Appendix **VI** for an alternative viewpoint of this Terai result i.e. the formula in (6.17).

Finally, to test the usual definition **(I)** for the flat surface J, put $z=0$, i.e. $p=p_o$, in (6.11), which yields

$$\int_J \lim_{p \to p_0 \in J} [\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)] dq = \int_0^{2\pi} \int_0^{r(\theta)} [\frac{ik}{r} + \frac{1}{r^2}] e^{-ikr} dr d\theta \ . \qquad (6.18)$$

Clearly the right-hand integral of (6.18) does not yield a finite value since the interval of r contains the origin.

PART III

INTEGRAL EQUATION FORMULATIONS

Chapter 7

Review of improved formulations

In chapter 4 it was noted that the straightforward classical integral equation formulations of the exterior Helmholtz problem are defective, in that these formulations fail at certain critical values of the wave-number k, which are the eigenvalues of the corresponding homogeneous interior problem. This difficulty arises whether one utilizes layer-potentials or Helmholtz formula. As mentioned before this is purely a mathematical complication and has no bearing on the physical problem. This phenomenon has been known for at least fifty years [55]. A considerable amount of work has been done over the last three decades in developing formulations valid for all wave-numbers. Burton [13] and Harris [34] give very useful surveys of improved formulations which were proposed before 1973. In the ensuing sections we present a brief review of the most significant formulations.

7.1 Brundrit

Brundrit in his paper [18] utilized a layer potential i.e. indirect formulation (see chapter 4), and developed a numerical method which did not suffer from the effect of critical wave-numbers. For the problem of hard acoustic scattering by a fixed obstacle of boundary ∂B , he proposed that the scattered wave be represented as

$$\phi(p) = L_k\sigma(p) \quad ; \quad p \in B^+ \;, \tag{7.1}$$

where B^+ is the infinite region outside the obstacle. Applying the Neumann boundary condition, one notes that the source density σ satisfies the boundary integral equation

$$\frac{\partial\phi}{\partial n}(p) = [-2\pi I + M_k^T]\,\sigma(p) \quad ; \quad p \in \partial B \;. \tag{7.2}$$

However the homogeneous equation

$$0 = [-2\pi I + M_k^T]\,\sigma(p) \quad ; \quad p \in \partial B \tag{7.3}$$

has non-trivial solutions σ_j^* at any failing (characteristic) values of k which are also the eigenvalues of the corresponding interior Dirichlet problem. One can therefore write the general solution of (7.2) in the form

$$\sigma_o + \sum_j c_j \sigma_j^* \;,$$

at any choice of k, where σ_o is a particular solution, and c_j are arbitrary constants. Brundrit noted that the non-trivial eigensolutions of (7.3) are the boundary-values of $\partial\phi/\partial n$, for the interior Dirichlet eigenfunctions, which satisfy (see equations (4.17), (4.18) in chapter 4)) simultaneously

$$L_k[\frac{\partial\phi}{\partial n}](p) = 0 \qquad ; \quad p \in \partial B$$

$$,$$

$$(-2\pi I + M_k^T)[\frac{\partial\phi}{\partial n}](p) = 0 \qquad ; \quad p \in \partial B$$

as a result of using Helmholtz formula, which yield

$$L_k \sum_j c_j \sigma_j^*(p) = \sum_j c_j L_k[\sigma_j^*](p) = 0 \qquad ; \quad p \in \partial B \ ,$$

i.e.

$$\begin{aligned} & L_k[\sigma_o + \sum_j c_j \sigma_j^*](p) \\ & = L_k[\sigma_o](p) + \sum_j c_j L_k[\sigma_j^*](p) \\ & = L_k[\sigma_o](p) \qquad ; \ p \in \partial B \end{aligned}$$

Brundrit further noted that by continuity of a simple-layer potential and by uniqueness, the wave-function generated by the eigensolutions of (7.3) must vanish identically in the exterior, i.e.

$$L_k[\sum_j c_j \sigma_j^*](p) = \sum_j c_j L_k[\sigma_j^*](p) = 0 \qquad ; \quad p \in B^+ \ .$$

So he argued that σ_j^* component of a general solution of (7.2) would make no contribution in the exterior.

Alternatively, one can utilize the non-trivial eigensolutions σ_j^* to generate an exterior potential i.e.

$$V(p) = L_k \sigma_j^*(p) \qquad ; \quad p \in B^+ \ ,$$

for each j. Then, using (7.3)

$$\frac{\partial V}{\partial n_+}(p) = [-2\pi I + M_k^T]\sigma_j^*(p) = 0 \qquad ; \quad p \in \partial B \ ,$$

where the (+) signifies that the normal points into the exterior. Now Since $\partial V/\partial n_+=0$ for each j, it follows, by uniqueness, that V vanishes identically in the exterior i.e.

$$L_k\sigma_j^*(p) = 0 \qquad ; \quad p\in B^+ ,$$

and by continuity

$$V_+(p) = L_k\sigma_j^*(p) = 0 \qquad ; \quad p\in\partial B$$

for each j. Therefore, as noted by Brundrit, the σ_j^* component of the general solution does not make any contribution in the exterior.

However, in this proposal Brundrit overlooks the fact that the boundary integral equation (7.2) has no solution whenever k coincides with any of these critical values, since the eigenfunctions of the corresponding adjoint equations of (7.3) are not orthogonal to the inhomogeneous term in (7.2). Note that Brundrit's approach can also be formulated via the Helmholtz formula which yields a consistent non-unique solution.

7.2 Panich

Utilizing layer potentials, Panich [70] initially proposed

$$\phi(p) = [L_k + \mu M_k]\sigma(p) \quad ; \quad p\in B^+ , \tag{7.4}$$

where μ is a complex constant, chosen to be $+i$ if $\Re(k)\geq 0$, see Burton [14], [15]. Therefore σ satisfies

$$\frac{\partial\phi}{\partial n}(p) = [-2\pi I + M_k^T + \mu N_k]\sigma(p) \quad ; \quad p\in\partial B . \tag{7.5}$$

Since (7.5) contains the hypersingular operator N_k, which is not compact (see chapter 4), Panich [70] proposed replacing (7.4) by the hybrid layer potential

$$\phi(p) = [L_k + \mu M_k L_0]\sigma(p) \quad ; \quad p \in B^+ \ , \tag{7.6}$$

where the subscript 0 denotes putting $k=0$ in the definition of L_k. Therefore σ satisfies

$$\frac{\partial\phi}{\partial n}(p) = [-2\pi I + M_k^T + \mu N_k L_0]\sigma(p) \quad ; \quad p \in \partial B \ . \tag{7.7}$$

Writing

$$N_k L_0 = (N_k - N_0)L_0 + N_0 L_0 \ ,$$

and using the identity

$$N_0 L_0 = (M_0^T + 2\pi I)(M_0^T - 2\pi I) \ ,$$

we get

$\partial\phi/\partial n =$

$$[-2\pi I + M_k^T + \mu\{(N_k - N_0)L_0 + (M_0^T)^2 - 4\pi^2 I\}]\sigma(p) \quad ; \quad p \in \partial B \ . \tag{7.8}$$

Here all the operators are weakly singular and hence compact. This is one of the earliest examples of the regularisation technique for weakening the singularity of the kernel. By using Fredholm theory, Colton and Kress [25] deduced that (7.8) has a unique solution for all non-negative real k provided $\Im(\mu) > 0$. However numerical implementation is very expensive and cumbersome due to the involvement of the product of matrix approximations to the operators.

7.3 Copley

Copley [26] proposed applying the Helmholtz formula (see equation (4.12c) chapter 4) at points in the interior region i.e.

$$\int_{\partial B}[\phi(q)\frac{\partial g_k}{\partial n_q}(p,q) - g_k(p,q)\frac{\partial \phi}{\partial n_q}(q)]dq = 0 \quad ; \quad p \in B^- .$$

So avoiding the singularity problem. This idea was originated by Kupradze [46]. In operator notation

$$M_k\phi(p) = L_k(\frac{\partial \phi}{\partial n})(p) \quad ; \quad p \in B^- , \tag{7.9}$$

which is a functional relation between ϕ and $\partial\phi/\partial n$ on ∂B, involving the non-singular kernel g_k. This becomes an integral equation of the first kind for ϕ in terms of $\partial\phi/\partial n$. Copley calls it the interior Helmholtz integral relation. He proves two uniqueness theorem:

I. *if (7.9) holds* $\forall p \in B^-$ *, the solution of the functional equation is unique.*

II. *if the surface* ∂B *is axisymmetric and (7.9) holds for all p along the axis of symmetry in* B^-*, the solution is unique.*

Copley [26] proposed a scheme for enforcing (7.9) at a finite number of points $\{p_1,p_2p_3,...\}$, denoted by P, in B^-. However a bad choice of p could cause an interference with the interior eigenfunctions $\hat{\phi}$, which satisfies

$$[4\pi I + M_k]\hat{\phi}(p) = L_k(\frac{\partial\hat{\phi}}{\partial n})(p) \quad ; \quad p \in B^- . \qquad (7.10)$$

A typical bad choice is when p lies on a nodal surface of eigenfunctions $\hat{\phi}$. These nodal surfaces are such that $\hat{\phi}(p)=0$ i.e. (7.10) is the same as (7.9). If this is true for the choice P then the eigenfunctions $\hat{\phi}$ will also satisfy (7.9), thereby causing an interference. Cunefare et al [27] suggested a linear combination of (7.9) and its normal derivative, but as the points in P do not lie on ∂B the normal derivative operation can not be well defined. It is also noted that any numerical implementation of this fails as the number of such points increases, see Schenck [82]. Copley solved some axisymmetric radiation problems but noted that his method fails to give accurate results for bodies which are not sufficiently smooth. Fenelon [29] obtained results for acoustic radiation by a finite cylinder.

7.4 Schenck

Schenck [82] proposed a method using the SHE (Surface Helmholtz Equation) and supplementing it by the interior Helmholtz relation i.e.

$$[-2\pi I + M_k]\phi(p) = L_k(\frac{\partial\phi}{\partial n})(p) \quad ; \quad p \in \partial B , \qquad (7.11a)$$

$$M_k\phi(p) = L_k(\frac{\partial\phi}{\partial n})(p) \quad ; \quad p \in B^- . \qquad (7.11b)$$

Schenck showed that for any k there is only one solution to (7.11a) which also satisfies (7.11b) for all points in the interior region. Discretisation of (7.11a) and (7.11b) for only few points in B^- gives a non-square system of linear equations, which is solved by a least-square method. This method is better known as the Combined Helmholtz Integral Equation Formulation (CHIEF). For lower ranges of values of k, successful applications of CHIEF have been reported [4], [84]. Seybert et al [84, 85, 86, 87] have used CHIEF successfully to model radiation and scattering problems using quadratic isoparametric elements. However,the major drawback of this method is that we do not know how many and where best to locate the interior points. A point selected on the nodal surface of an interior eigenfunction gives rise to a non-unique problem, as noted in the preceding section. No criterion has yet been reported which enables one to isolate these "bad" interior points.

7.5 Kussmaul

Kussmaul in his paper [48] proposes for a scattered wave

$$\phi(p) = [L_k + \mu M_k]\sigma(p) \quad ; \quad p \in B^+ \ . \tag{7.12}$$

Note that it is essentially a superposition of a simple and double-layer distribution over the boundary. The method we propose in the next chapter is an adaptation of this. The hard boundary condition on (7.12) yields

$$\frac{\partial \phi}{\partial n}(p) = [-2\pi I + M_k^T + \mu N_k]\sigma(p) \quad ; \quad p \in \partial B \ . \tag{7.13}$$

This proposal is analogous to the mixed potential used by Brakhage and Werner [19], Leis [57, 58, 59] in their treatment of the Dirichlet case. The constant μ is chosen to be $+i$ if $\Re(k) \geq 0$ and $-i$ if $\Re(k) < 0$ $(\Im(k) \geq 0)$. This choice ensures the uniqueness of the solution of the equation (7.13). As the operator N_k is non-compact, the operator on the right hand side of (7.13) also becomes non-compact, and as a result, Fredholm theory can not be applied to deduce whether or not a solution exists. Kussmaul uses a regularisation technique which, by eliminating the non-compactness, enables him to prove the existence of solution. Due to the complexity of the regularisation technique few numerical implementations have been reported. Kirkup and Henwood [51] used this method to compute solutions of some acoustic radiation problems.

7.6 Burton and Miller

Burton and Miller [17] proposed a method which is the direct analogue of the indirect formulation of Panich [70]. They proposed a linear combination of the SHE (see eq.4.14) and its normal derivative form, i.e.

$$[-2\pi I + M_k + \alpha N_k]\phi(p) = [L_k + \alpha(2\pi I + M_k^T)]\frac{\partial\phi}{\partial n}(p) \ ; \quad p \in \partial B \ ,(7.14)$$

where α is an arbitrary coupling parameter. It is proved, see [17],[92], that, if α is chosen to be a complex constant with the $\Im(\alpha)>0$, then the homogeneous equation

$$[-2\pi I + M_k + \alpha N_k]\phi = 0 \ , \qquad (7.15)$$

has only the trivial solution $\phi \equiv 0$ for real values of k, ensuring that (7.14) has a unique solution for each wave number. Harris [34] in a similar way to Tzu-chu Lin [92] extends the uniqueness proof to the case where α is taken to be a function of p i.e. provided $\Im(\alpha(p))>0, \forall p \in \partial B$, then (7.15) has only the trivial solution. This result is used in the study of the conditioning of the formulation.

Though the Burton and Miller proposal is undoubtedly elegant analytically, it suffers from a numerical drawback due to the involvement of the non-compact (i.e. hypersingular) integral operator N_k. To resolve this problem Burton [15], following Panich [70], regularises N_k (see section 7.2) to obtain the Regularised Burton and Miller formulation (RBM):

$$[-2\pi I + M_k + \alpha(p)\{L_0(N_k - N_0) + M_0^2 - 4\pi^2 I\}]\phi(p)$$

$$= [L_k + \alpha(p)L_0(2\pi I + M_k^T)]\frac{\partial\phi}{\partial n}(p) \quad ; \qquad p\in\partial B ,$$

where all the integral operators are weakly singular and compact. The choice of $\alpha(p)$ is such that $\Im(\alpha(p))>0$ for $k > 0$. The main drawback of RBM is having to compute the product of operators (by multiplying their matrix approximations) which proves to be very expensive and cumbersome. the reason for the regularised version is twofold. Firstly to establish the uniqueness of solution using classical Fredholm theory, and secondly to make it amenable to numerical treatment despite its complexities. However it seems possible to compute a direct approximation to N_k. Terai [93] (see chapter 6) gives an approximation to N_k for a flat surface, and Stallybrass [83] introduced a pointwise variational principle to approximate N_k, which Meyer et al [66, 67] utilized in order to show that

$$\int_{\partial B}\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq = k^2\int_{\partial B} g_k(p,q)\, n_p\cdot n_q\, dq \ .$$

which may be used in

$$N_k\phi(p) = \int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_q}[\phi(q) - \phi(p)]\,dq \; + \; k^2\phi(p)\int_{\partial B} g_k(p,q)\,\boldsymbol{n}_p \cdot \boldsymbol{n}_q\,dq \;\; ; \;\; p \in \partial B \; .$$

The second integral on the right hand side is weakly singular and the first integral can be interpreted in the sense of Cauchy principal value.

Amini and Harris [4] give a comparison between the RBM and the direct Burton and Miller form. Reut [75] has developed a Burton and Miller formulation which he calls Composite Outward Normal Derivative Overlapping Relation (CONDOR), based in part upon the work of Terai [93].

7.7 Ursell and Jones

Ursell [94, 95] noted that $g_k(p,q)$ can be replaced by any fundamental solution to the Helmholtz equation in the exterior domain B^+ which satisfies the radiation condition, i.e. a modified Green's function

$$G(p,q) \; = \; g_k(p,q) \; + \; \Gamma(p,q) \;\; , \qquad (7.16)$$

is used where $\Gamma(p,q)$ is an analytic wave function in B^+. Using this in the classical formulations (layer potential or Helmholtz formula) with (7.16) replacing the classical free-space Green's function, and applying the hard boundary condition, one obtains (see chapter 4) the boundary equations

$$(Layer) \qquad \frac{\partial\phi}{\partial n}(p) \; = \; -2\pi\,\sigma(p) \; + \int_{\partial B} \frac{\partial G}{\partial n_p}(p,q)\,\sigma(q)\,dq \;\; ; \qquad p \in \partial B \;\; , \qquad (7.17)$$

$$\int_{\partial B} G(p,q)\frac{\partial\phi}{\partial n_q}(q)\,dq$$

(Helmholtz formula) . (7.18)

$$= -2\pi\,\phi(p) + \int_{\partial B}\frac{\partial G}{\partial n_q}(p,q)\,\phi(q)\,dq \;\; ; \;\; p\in\partial B$$

Now if

$$\frac{\partial\Gamma}{\partial n}(p,q) + \gamma\Gamma(p,q) = 0 \quad ; \quad p\in S_R\subset B^- \ ,$$

where γ is a complex constant such that $\Im(\gamma)>0$ and S_R is a sphere enclosed completely within ∂B , then Ursell [94] shows that the solutions to (7.17) and (7.18) are unique. Ursell [94, 95] noted that $\Gamma(p,q)$ may be chosen to be an infinite series of spherical wave functions, which converges quickly for small k but slowly as k increases.

Jones [39] proposed a modification to this by replacing the infinite series with a finite one. He suggested

$$\Gamma(p,q) = \sum_{m=0}^{M}\sum_{n=0}^{m} b_{mn}\,\psi_{mn}(p)\Psi_{mn}(q) \ , \tag{7.19}$$

where

$$\psi_{mn}(p) = h_m(kr_p)P_m^{\,n}(\cos\theta_p)\cos\phi_p$$

$$\Psi_{mn}(p) = h_m(kr_p)P_m^{\,n}(\cos\theta_p)\sin\phi_p$$

The point p has spherical polar coordinates (r_p, θ_p, ϕ_p) . The functions $h_m(kr_p)$ are spherical Hankel functions of the first kind and $P_m^{\,n}$ are associated Legendre functions. Supposing

$$K = \{ k_i \mid 0<k_1<k_2<k_3<\ldots \} ,$$

Jones [39] showed that if $k < k_{M+2}$ then (7.17) and (7.18) have unique solutions provided the b_{mn}'s are real and non-zero. Kleinmann and Kress [43] suggested that b_{mn} be chosen so as to minimise the condition number of the resulting integral operator. The major drawback of this method is that for an arbitrary surface one does not know how large to take M since K is usually unknown. Also it is known [15] that the number of elements in K less than a given k is proportional to k^3, and so for a moderately large k we have to take a considerably large number of terms. Consequently $\Gamma(p,q)$ becomes very costly. As such there has been little or no report of its implementation in practical problems.

7.8 Piaszczyk and Klosner

Piaszczyk and Klosner [71] proposed a method similar to CHIEF, but using the exterior Helmholtz relation

$$\phi(p) = [M_k \phi](p) - [L_k \frac{\partial \phi}{\partial n}](p) \quad ; \qquad p \in B^+ , \tag{7.19}$$

in addition to the SHE equation. Naturally ϕ is not known for points in B^+, and for hard scattering $\partial\phi/\partial n$ only is known on ∂B . Therefore it is supposed [71] that there exists some function $Z(p)$ which gives a simple impedence relationship

$$\phi(p) = Z(p)\frac{\partial\phi}{\partial n}(p) \quad ; \qquad p \in \partial B \ . \tag{7.20}$$

Using (7.20) in (7.19), one can compute ϕ for a few points in B^+. Utilizing these values and a suitable discretisation of SHE, an overdetermined system for ϕ in ∂B can be formed. Using a least-squares procedure one can solve this system of linear equations and find ϕ on ∂B . And then one can recompute ϕ in B^+ and repeat the process until a convergence criterion is satisfied. The proposal is analytically sound since a uniqueness proof is given [71]. However this method also suffers from a major drawback in that it requires the solution of an over-determined system at each iteration, which is very costly from a computational point of view. Also the rate of convergence depends on the choice $Z(p)$ and it is doubtful whether or not the iteration process will converge at all [27].

Chapter 8

A new integral equation formulation of exterior acoustic scattering problem

The classical integral equation method viz. [23], [24], gives unreliable results for all but a relatively low range of wave-numbers, (see section 7.7 chapter 7). They are generally reliable only for wave-numbers satisfying $kD<6$, where D is a characteristic dimension of the surface. In the preceding chapter we reviewed some of the significant proposals which attempt to resolve this complication. Of these, Schenck [82], and Burton and Miller [17] have been widely tested and implemented using various numerical techniques. Recently, Rego Silva et al [77], following the Burton and Miller approach, evaluated the hypersingular integral numerically by an algorithm of Guiggiani et al [32] building essentially upon Hadamard's finite part integration. Both Harris [34] and Liu and Rizzo [56] have used the Burton and Miller approach, but the hypersingular integral is first weakened by a regularisation technique. Wu et al [98] followed on the line of Schenck [82], and by using supercomputers, proposed an improvement on CHIEF. Kirkup and Henwood [51], following Kussmaul, presented some computational solutions of acoustic radiation problems. Here the hypersingular integral was evaluated by the method of Terai [93], (see chapter 6). All these formulations have been implemented numerically, exploiting BEM techniques, but they involve computational difficulties, mainly arising from the presence of the hypersingular operator N_k, (see chapter 6). Although Schenck [82] avoids the N_k operator, it requires a set of interior points to be chosen. This choice greatly influences the performance of the method. However, little guidence is given in the literature on how to isolate the so called "bad" interior points. Also, at higher wave-numbers more interior points are required for an accurate result and so adversely affecting the numerical conditioning.

Burton and Miller [17] require a free coupling parameter η to be chosen. Apart from the requirement that $\Im(\eta)\neq 0$, no consensus has yet arisen as to the precise choice of this parameter. Also, this method involves the difficulty of having to discretise the N_k operator or at least its regularised form. Documented analysis of the boundary integral equation suggests that this approach is potentially reliable across the wave-number range. In fact Reut [75] reported that his BEM approach, code named CONDOR (Composite Outward Normal Derivative Overlapping Relation) based upon Burton and Miller, is superior to that of CHIEF.

There remains the Kussmaul [48] approach for consideration. Unlike Schenck and, Burton and Miller he uses layer potentials. This is conceptually simple and provides a unique solution but suffers from the presence of the N_k operator. In the ensuing sections we describe this approach and show how this may be improved by eliminating the N_k.

8.1 Kussmaul's approach

Recalling from chapter 7, it was suggested by Kussmaul [48] (within a 2-D context) that an alternative representation of any solution of the Helmholtz equation in the infinite exterior domain B^+ would be

$$\phi(p) = [L_k + \eta M_k]\sigma(p) \quad ; \qquad p \in B^+ \ , \qquad (8.1)$$

where η is a coupling parameter and the operator notations carry the same meaning as before (see ch.4 and ch.7). Note that (8.1) is a superposition of a simple and a double-layer potential generated by the same source density σ on the surface ∂B . The hard scattering boundary condition gives

$$\frac{\partial\phi}{\partial n}(p) = [-2\pi I + M_k^T + \eta N_k]\sigma(p) \quad ; \qquad p \in \partial B \ , \qquad (8.2)$$

where $\partial\phi/\partial n$ is supposed known i.e. (8.2) is an integral equation of the second kind for σ in terms of $\partial\phi/\partial n$. Kussmaul proved an existence and uniqueness theorem within a 2-D context. Here we present a uniqueness proof from a different point of view.

Consider the corresponding homogeneous equation

$$0 = [-2\pi I + M_k^T + \eta N_k]\sigma \ , \qquad (8.3a)$$

and its associated transpose equation

$$0 = [-2\pi I + M_k + \eta N_k]\nu \ . \qquad (8.3b)$$

These equations are related, in that the existence of a non-trivial solution of the one implies a corresponding non-trivial solution of the other.

Suppose there exists a non-trivial solution $\nu = \hat{\nu} \neq 0$ of (8.3b) i.e.

$$0 = [-2\pi I + M_k + \eta N_k]\hat{\nu} \ . \qquad (8.3c)$$

We can use this $\hat{\nu}$ to generate a double-layer potential in the interior i.e.

$$W(p^-) = M_k \hat{v}(p^-) \quad ; \qquad p^- \in B^- \ ,$$

from which

$$W_i(p) = [-2\pi I + M_k]\,\hat{v}(p) \quad ; \qquad p \in \partial B \ ,$$

where W_i signifies the limit of W from a point in the interior region just off the boundary. Also

$$\frac{\partial W}{\partial n_i}(p) = N_k \hat{v}(p) \quad ; \quad p \in \partial B \ ,$$

where n_i denotes the normal into the interior region. Utilizing (8.3c) we find that

$$W_i(p) + \eta \frac{\partial W}{\partial n_i}(p) = 0 \quad ; \qquad p \in \partial B \ , \tag{8.4a}$$

with complex conjugate

$$W_i^*(p) + \eta^* \frac{\partial W^*}{\partial n_i}(p) = 0 \quad ; \qquad p \in \partial B \ . \tag{8.4b}$$

Now apply Green's second theorem to W, W^* in $B^- \cup \partial B$. Omitting the variables we readily obtain

$$\int_{\partial B} \{ W_i \frac{\partial W^*}{\partial n_i} - W_i^* \frac{\partial W}{\partial n_i} \} dq = (k^2 - k^{*2}) \int_{B^-} |W|^2 \, dB \ . \tag{8.5}$$

Since k is assumed to have only real values (see chapter 1), the right-hand side of (8.5) vanishes yielding, from (8.4a), (8.4b)

$$\left(\frac{1}{\eta} - \frac{1}{\eta^*}\right) \int_{\partial B} |W_i(q)|^2 \, dq = 0 \ ,$$

i.e.

$$-\frac{2i\,\Im(\eta)}{|\eta|^2} \int_{\partial B} |W_i(q)|^2 \, dq = 0 \ ,$$

where $\Im$ signifies the imaginary part of a complex number. If $\Im(\eta) \neq 0$ then clearly

$$W_i(p) = 0 \quad ; \quad p \in \partial B \ . \tag{8.6a}$$

Hence, by using (8.4a),

$$\frac{\partial W}{\partial n_i}(p) = 0 \quad ; \quad p \in \partial B \ . \tag{8.6b}$$

Using the continuity properties of the normal derivative of a double-layer potential we find

$$\frac{\partial W}{\partial n_e}(p) + \frac{\partial W}{\partial n_i}(p) = 0 \quad ; \quad p \in \partial B \ ,$$

which by (8.6b) gives

$$\frac{\partial W}{\partial n_e}(p) = 0 \quad ; \quad p \in \partial B \ , \tag{8.6c}$$

where n_e denotes the normal pointing into the exterior region. Accordingly, by a fundamental existence and uniqueness theorem [79] for a well-behaved exterior solution of the Helmholtz equation, we conclude that

$$W(p) \equiv 0 \quad ; \quad p \in B^+ \ ,$$

i.e.

$$W_e(p) = 0 \quad ; \quad p \in \partial B \ . \tag{8.6d}$$

Now from the jump properties of a double-layer potential we note that

$$W_i(p) - W_e(p) = -4\pi \hat{\nu}(p) \quad ; \quad p \in \partial B \ ,$$

where W_e signifies the limit of W from a point in the exterior just off the boundary. Noting (8.6a), (8.6d) we have

$$4\pi \hat{\nu}(p) = 0 \quad ; \quad p \in \partial B \ , \tag{8.7}$$

which contradicts the assumption $\hat{\nu} \neq 0$. It follows that the transpose equation (8.3b), and consequently the equation (8.3a), has only the trivial solution. As a result the Kussmaul boundary equation (8.2) has a unique solution provided $\Im(\eta) \neq 0$.

Alternatively one can argue as follows:

Suppose there exists a non-trivial solution $\hat{\sigma}$ of the homogeneous boundary equation (8.3a), which we rewrite in the form

$$-2\pi \hat{\sigma}(p) + [M_k{}^T \hat{\sigma}](p) = (-\eta)[N_k \hat{\sigma}](p) \quad ; \quad p \in \partial B$$

i.e.

$$\frac{\partial}{\partial n_e}[L_k \hat{\sigma}](p) = \frac{\partial}{\partial n_e}[(-\eta)M_k \hat{\sigma}](p) \quad ; \quad p \in \partial B \ . \tag{8.7a}$$

The left-hand side of (8.7a) represents the normal derivative of a simple-layer potential generated by $\hat{\sigma}$ on ∂B , whereas the right-hand side represents the normal derivative of a double-layer potential generated by the same source density on ∂B (incorporating $-\eta$ into the right-hand side integral). As the normal derivatives of these potentials are equal on the same boundary ∂B , they must define the same wave function in the exterior region by virtue of the exterior Neumann uniqueness theorem i.e.

$$[L_k\hat{\sigma}](p) = (-\eta)[M_k\hat{\sigma}](p) \quad ; \quad p \in B^+ .$$

But this is not possible, since for instance in the case of a sphere of radius a we may generate the following potentials from a uniform source density $\hat{\sigma}_o$ on the surface:

$$[L_k\hat{\sigma}_o](p) = 4\pi a\hat{\sigma}_o \sin ka \frac{e^{-ik|p|}}{k|p|} \quad ; \quad |p|>a$$

$$(-\eta)[M_k\hat{\sigma}_o](p) = -4\pi\eta\,\hat{\sigma}_o[ka\cos ka - \sin ka]\frac{e^{-ik|p|}}{k|p|} \quad ; \quad |p|>a$$

Note that (i) the coefficients differ considerably,

(ii) normal derivatives at the spherical surface ∂B could not be equal everywhere on ∂B for any real choice of k, irrespective of the choice of η , provided $\Im(\eta)\neq 0$.

This contradiction can only be resolved by choosing $\hat{\sigma}_o = 0$ which implies that (8.7a) is possible only when $\hat{\sigma}_o \equiv 0$.

8.2 Adapted Kussmaul formulation (AKF)

The formulation (8.1) may be improved by introducing an internal fictitious surface ∂B^* (see fig.7 on the following page), similar and similarly situated to the given surface ∂B, i.e. for each $q \in \partial B$ there corresponds a $q^* \in \partial B^*$ such that

$$q^* = \vartheta q \; ,$$

where ϑ is a real constant such that $0<\vartheta<1$. According to our computations, it is found that $\vartheta \approx 0.5$ gives the best results.

We now introduce a continuous distribution of dipole sources on ∂B^* characterised by the source density

$$\sigma(q^*) = \sigma(q) \quad ; \quad q^* \in \partial B^* \; , \quad q \in \partial B \; .$$

This generates a double-layer potential

$$\int_{\partial B^*} \frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\sigma(q^*)\,dq^* \quad ; \quad p \in B^+ \cup \partial B \; .$$

Thus we define

$$\Gamma_k^{\vartheta}(p) = \vartheta \int_{\partial B} \frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\sigma(q^*)\,dq \quad ; \quad p \in B^+ \cup \partial B \; , \quad q^* \in \partial B^* \; ,$$

for each given $\vartheta \in (0,1]$. Note that the integrand

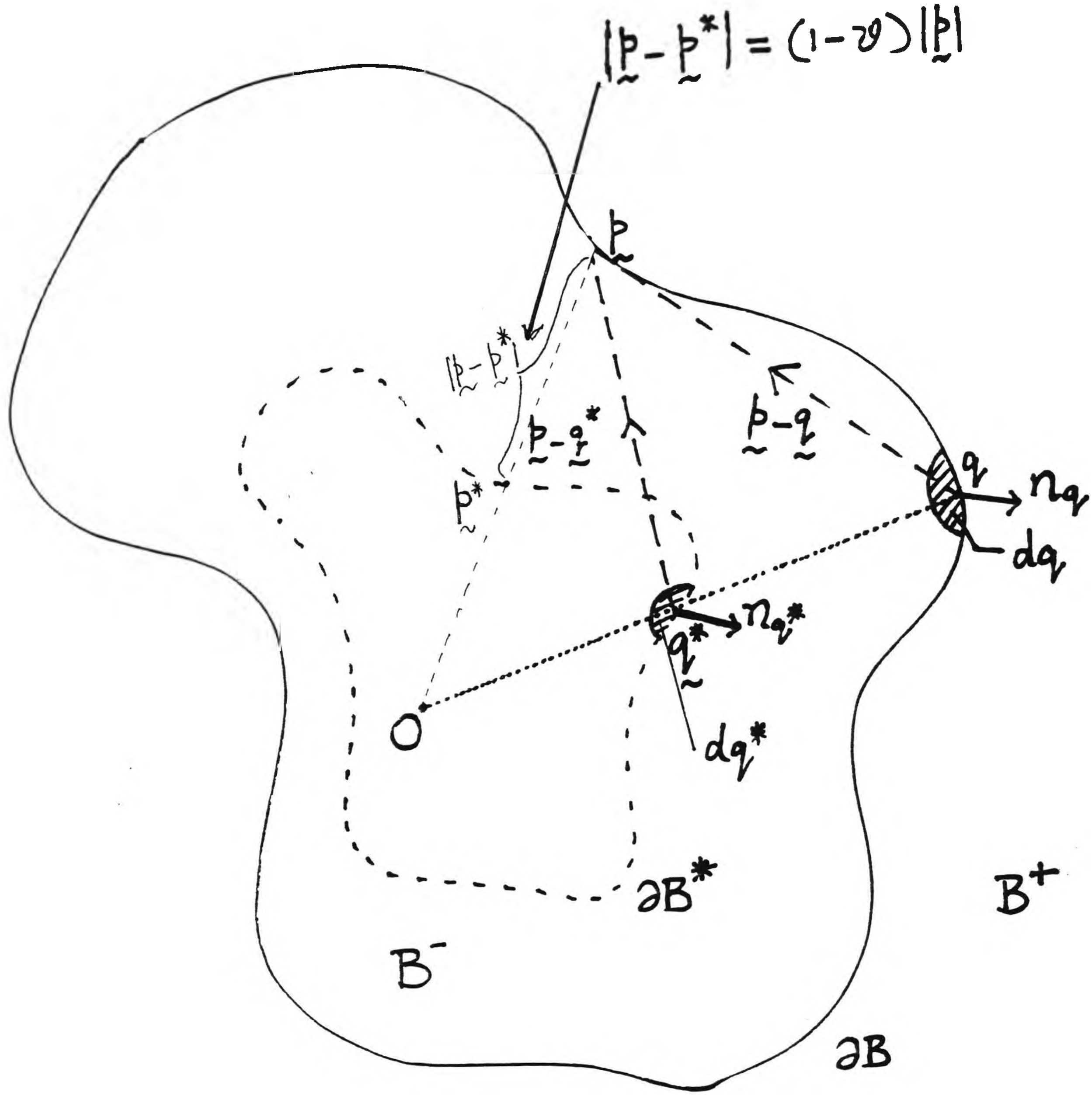

Fig.7 Boundary features for the Adapted Kussmaul Formulation (AKF). Note that the interior region B^- includes the whole of the region interior to the given boundary ∂B .

$$\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\sigma(q^*)$$

remains regular as $q \to p \in \partial B$ since $q^* \notin \partial B$. Consequently this potential, denoted by Γ_k^{ϑ} , remains finite and differentiable at any $p \in \partial B$, with a readily evaluated normal derivative

$$N_k^{\vartheta}(p) = \frac{\partial \Gamma_k^{\vartheta}}{\partial n}(p) = \vartheta \int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)\,\sigma(q)\,dq \;; \quad p \in \partial B,\; q^* \notin \partial B \;. \tag{8.8}$$

Unlike the N_k operator as found in the Kussmaul boundary equation, the kernel of the operator N_k^{ϑ} in (8.8) remains completely regular over the surface since the dipoles are no longer located on the given surface. Note that for $\vartheta = 1$, Γ_k^{ϑ} reduces to the usual double-layer potential generated by a continuous distribution of sources on the boundary ∂B and the normal derivative operation becomes the usual N_k operator.

Now we represent any solution of the Helmholtz equation in the infinite region exterior to the surface ∂B , by combining a simple-layer potential with the newly defined double-layer potential Γ_k^{ϑ} with a coupling parameter η

$$\phi(p) = [L_k + \eta\,\Gamma_k^{\vartheta}]\,\sigma(p) \quad ; \qquad p \in B^+ \cup \partial B \;, \quad 0 < \vartheta \leq 1 \;, \tag{8.9}$$

where we choose $\eta = i\vartheta^{-1}$. This choice ensures that the coupling parameter between the layer-potentials eventually takes the form $+i$. This formulation we call the Adapted Kussmaul Formulation (AKF). This is a hybrid potential distribution except that the dipole components are located away from the surface. It may be regarded as a classical simple-layer potential with the free-space Green's function g_k replaced by the modified Green's function

$$G_k(p,q) = g_k(p,q) + \eta \frac{\partial g_k}{\partial n_{q^*}}(p,q^*) ,$$

where G_k satisfies in p the equation

$$\nabla^2 G_k(p,q) + k^2 G_k(p,q) = -4\pi\,\delta(|p-q|) .$$

Note that G_k has the same singularity behaviour as that of g_k when $|p-q| \to 0$. The hard acoustic boundary (Neumann) condition applied to (8.9) yields

$$\frac{\partial \phi}{\partial n}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial G_k}{\partial n_p}(p,q)\,\sigma(q)\,dq \quad ; \quad p \in \partial B , \tag{8.10}$$

where

$$\frac{\partial G_k}{\partial n_p}(p,q) = \frac{\partial g_k}{\partial n_p}(p,q) + \eta \frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*) \quad ; \quad p \in \partial B,\ q^* \in \partial B^* .$$

Relation (8.10) is a Fredholm integral equation of the second kind for σ in terms of the given term $\partial\phi/\partial n$ on the left hand side. Note that in (8.10), the only sigularity in the kernel is the weak singularity of $\partial g_k/\partial n_p$, which presents no computational difficulties.

8.3 Uniqueness argument

Since the formulation (8.9) is mathematically similar to the original Kussmaul formulation (8.1) with the exception that the dipole contributions are now generated from a fictitious surface ∂B^* within the given surface ∂B , it is expected that (8.9) also has a unique solution. However we present an argument similar to the one presented at the end of section 8.1 where we dealt with the uniqueness of the Kussmaul formulation.

In order to establish the uniqueness of solution of (8.9) it is sufficient to show that the corresponding homogeneous equation

$$-2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial G_k}{\partial n_p}(p,q)\,\sigma(q)\,dq = 0 \quad ; \quad p \in \partial B \; , \qquad (8.11)$$

has only the trivial solution. Suppose there exists a non-trivial solution $\hat{\sigma} \neq 0$ of (8.11). This means that (8.11) may be rewritten in the form

$$-2\pi\,\hat{\sigma} + \int_{\partial B} \frac{\partial g_k}{\partial n_p}(p)\,\hat{\sigma}(q)\,dq$$

$$= (-\eta\,\vartheta)\int_{\partial B} \frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)\,\hat{\sigma}(q^*)\,dq \quad ; \quad p \in \partial B,\; q^* \in \partial B^*$$

i.e.

$$\frac{\partial}{\partial n_p}\int_{\partial B} g_k(p,q)\,\hat{\sigma}(q)\,dq$$

$$= \frac{\partial}{\partial n_p}\int_{\partial B} [-\eta\,\vartheta]\,\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\hat{\sigma}(q^*)\,dq \;;\; p \in \partial B,\; q^* \in \partial B^* \qquad .(8.12)$$

Now the left hand side of (8.12) provides the normal derivative on ∂B of a simple-layer potential. While the right hand side provides the normal derivative on ∂B of a double-layer potential generated from a dipole distribution on ∂B^* . Since their normal derivatives are equal on ∂B (by virtue of the equation (8.12)), they must define the same unique solution of the Helmholtz equation in the infinite region exterior to ∂B i.e.

$$\int_{\partial B} g_k(p,q)\,\hat{\sigma}(q)\,dq = [-\eta\,\hat{\vartheta}]\int_{\partial B} \frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\hat{\sigma}(q^*)\,dq \quad ; \quad p \in B^+ , \tag{8.13}$$

by virtue of a fundamental existence and uniqueness theorem [79]. However since the asymptotic behaviour of a dipole potential must differ from that of a simple-layer potential generated from same sources, it follows that the two solutions can not be equal. For instance, in the case of a sphere of radius a (cf. chapter 5) one can obtain an exterior wave function as the simple-layer potential generated from a uniform distribution of source densities $\sigma = 1$ i.e.

$$\begin{aligned}
&\int_{\partial B} g_k(p,q)\,\sigma(q)\,dq \\
&= \int_0^{2\pi}\int_0^{\pi} g_k(p(r,\theta,\psi),q(a,\theta,\psi))\,a^2\sin\theta\,d\theta\,d\psi \qquad (8.14) \\
&= 4\pi a \sin ka \frac{e^{-ikr}}{kr} \quad ; \qquad r = |p|
\end{aligned}$$

for any point p in the infinite region exterior to the sphere or on the boundary. For the same sphere, one can also obtain an exterior wave function as the potential $\Gamma_k^{\,\theta}$ generated from the same uniform source densities as before i.e.

$$(-\eta\vartheta)\int_{\partial B}\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\,\hat{\sigma}(q^*)dq$$

$$= (-\eta\vartheta)\int_0^{2\pi}\int_0^{\pi}\frac{\partial g_k}{\partial R}_{R=\lambda a}(p(r,\theta,\psi),q^*(R,\theta,\psi))\,a^2\sin\theta\, d\theta\, d\psi \tag{8.15}$$

$$= -4\pi\eta\,[ka\cos\vartheta ka - \frac{\sin\vartheta ka}{\vartheta}]\frac{e^{-ikr}}{kr} \quad ; \qquad r = |p|$$

for the same point p in the exterior region. Clearly the coefficients of (8.14) and (8.15) are not equal. Therefore the only way (8.13) could hold is when $\hat{\sigma} \equiv 0$. Hence (8.11) could only permit trivial solution which means (8.9) has unique solution.

See Appendix **VII** for an alternative argument for the uniqueness. For this purpose, an adjoint of the homogeneous boundary equation (8.11) is considered therein. It is argued that the adjoint equation can only admit the trivial solution, analogous to the argument in 8.1 .

PART IV

NUMERICAL TREATMENTS AND RESULTS

Chapter 9
Numerical Treatments

For the purpose of testing the viability of the new integral equation formulation, code named AKF, some numerical test radiation problems have been performed. AKF has also been utilized for the analysis of the acoustic scattering by a spherical surface. These problems will be addressed in the next chapter. For ease of computation, all the problems that we consider are axially symmetric in terms of the boundary and the radiation-scattering field. BEM has been used which reduces the domain problem into a boundary problem thereby reducing the dimension of the problem by one. A first step towards this is the discretisation process.

9.1 Discretisation process

First we recall, from the previous chapter, the AKF boundary integral equation for a given Neumann condition (equivalent to a hard scattering surface)

$$\begin{aligned}\frac{\partial\phi}{\partial n}(p) &= -2\pi\,\sigma(p) + \int_{\partial B}\Big[\frac{\partial g_k}{\partial n_p}(p,q) \\ &+ i\,\frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)\Big]\,\sigma(q)\,dq \quad ; \quad p\in\partial B,\ q^*\in\partial B^*, q^*=\vartheta q\end{aligned} \quad , \qquad (9.1)$$

where the bold $\boldsymbol{i}=\sqrt{-1}$ denotes the complex imaginary number. The left hand side of (9.1) represents the boundary values of the normal derivative of the unknown wave function which satisfies the Helmholtz equation in the infinite exterior domain and also the radiation condition. The equation (9.1) is an integral equation of the second kind for the unknown source density σ in term of the given $\partial\phi/\partial n$.

In this thesis we consider problems involving a sphere, cube and cylinder. In these cases the boundary ∂B is simply divided into M sub-intervals which are the boundary elements S_j, $j=1,2,3,\ldots$,M, so that

$$S_k \cap S_l = \varnothing, \textit{for } k \neq l \quad ; \qquad \bigcup_{j=1}^{M} S_j = \partial B \ ,$$

which give the preliminary discretisation

$$\begin{aligned} \frac{\partial\phi}{\partial n}(p) = -2\pi\,\sigma(p) + \sum_{j=1}^{M} \int_{S_j} [\frac{\partial g_k}{\partial n_p}(p,q) \\ + \boldsymbol{i}\frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)]\,\sigma(q)dq \ ; \quad p \in \partial B, \quad q^* \in S_j^* \subset \partial B^* \end{aligned} \qquad , \qquad (9.2)$$

where S_j^* is the boundary element on ∂B^* corresponding to the element S_j on ∂B . Although the original boundary is retained throughout, the unknown source densities and the boundary values are to be approximated on each element. Now the boundary values of $\partial\phi/\partial n$ and the unknown source densities σ are approximated over each boundary element through a constant interpolation function χ i.e. the approximate source densities $\tilde{\sigma}$ can be represented by

$$\tilde{\sigma}(q) = \sum_{j=1}^{M} \chi_j(q)\,\sigma_j \quad ; \qquad q \in \partial B \quad , \tag{9.3}$$

and the approximate boundary values of $\partial\phi/\partial n$ can be represented by

$$\frac{\partial\tilde{\phi}}{\partial n}(q) = \sum_{j=1}^{M} \chi_j(q)\,\frac{\partial\phi^j}{\partial n} \quad ; \qquad q \in \partial B \ , \tag{9.4}$$

where χ is a characteristic function with the properties

$$(i) \qquad \chi_j(q) = 1 \ , \quad \textit{if } q \in S_j$$

$$= 0 \ , \quad \textit{if } q \notin S_j$$

$$(ii) \qquad \sum_{j=1}^{M} \chi_j(q) = 1 \quad , \qquad \forall q \in \partial B$$

here σ_j is the unknown uniform source density on S_j and $\partial\phi^j/\partial n$ is a constant boundary value on S_j defined by the given Neumann data, i.e.

$$\frac{\partial\phi^j}{\partial n} = \frac{\partial\phi}{\partial n}(q_j) \quad , \quad \textit{for a convenient location } q_j \in S_j \ .$$

We now use (9.3) and (9.4) in (9.2) to derive the discretised approximated version of (9.2)

$$\sum_{j=1}^{M} \chi_j(p) \frac{\partial \phi^j}{\partial n} = -2\pi \sum_{j=1}^{M} \chi_j(p)\, \sigma_j + \sum_{j=1}^{M} [\int_{S_j} \{(\frac{\partial g_k}{\partial n_p}(p,q)$$

$$+ i \frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)) \sum_{l=1}^{M} \chi_l(q)\, \sigma_l \} dq] \;;\quad p \in \partial B,\ q^* \in S_j^* \quad ,$$

which, on noting that

$$\sum_{l=1}^{M} \chi_l(q)\, \sigma_l = \sigma_j \ , \mathit{for}\ q \in S_j \ ,$$

yields

$$\begin{aligned} \sum_{j=1}^{M} \chi_j(p) \frac{\partial \phi^j}{\partial n} &= -2\pi \sum_{j=1}^{M} \chi_j(p)\, \sigma_j \\ &+ \sum_{j=1}^{M} \sigma_j \int_{S_j} [\frac{\partial g_k}{\partial n_p}(p,q) + i \frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)] dq \;;\quad p \in \partial B,\ q^* \in S_j^* \end{aligned} \tag{9.5}$$

Note that (9.5) is a linear equation with M unknowns, $\{\sigma_j : 1 \le j \le M\}$. To solve it we require a system of M linear equations satisfied by these same unknowns. This can be obtained by collocating the equation at chosen collocation (pivotal) points on the surface. This choice is dependent on the particular boundary element configuration. Before considering the collocation points we look at the residual methods.

9.2 Weighted residual methods

The residual function for the AKF formulation is defined to be the residual in approximation (9.5), i.e.

$$\tilde{R}(\underline{\sigma};p) = -2\pi\sum_{j=1}^{M}\chi_j(p)\sigma_j + \sum_{j=1}^{M}\sigma_j\int_{S_j}\left[\frac{\partial g_k}{\partial n_p}(p,q)\right.$$

$$\left. + i\frac{\partial^2 g_k}{\partial n_p \partial n_{q^*}}(p,q^*)\right]dq - \sum_{j=1}^{M}\chi_j(p)\frac{\partial\phi^j}{\partial n} \quad ; \quad p\in\partial B \tag{9.6}$$

where $\underline{\sigma}$ is the M-vector representing the unknown approximate source density. Imposition of conditions on the residual function (9.6) gives rise to various weighted residual methods. For instance, through minimisation over $\underline{\sigma}$ of the norm of the residual vector, defined by

$$\|\tilde{R}(\underline{\sigma};p)\|_{w,\alpha} \equiv \{\int_{\partial B} w(p)\,|\tilde{R}(\underline{\sigma};p)|^{\alpha}\,dp\}^{1/\alpha} \quad ,$$

where $w(p) > 0$ for p on the boundary, gives rise to the least-square method and other related methods. Typically the least-square method takes $w(p)=1$ for p on the boundary and $\alpha=2$. The Galerkin method, the Collocation method and other related methods arise by finding the $\underline{\sigma}$ which satisfies

$$\int_{\partial B}\tilde{R}(\underline{\sigma};p)\,\zeta_i(p)\,dp = 0 \quad ; \quad for \quad i=1,2,\ldots M \quad ,$$

where $\{\zeta_1,\zeta_2,\ldots\zeta_M\}$ is a set of M linearly independent functions defined on the boundary ∂B . These functions are known as the test (basis) functions. The Galerkin method is obtained by putting $\zeta_i(p)=\chi_i(p)$, for $i=1,2,\ldots M$ and the collocation method is obtained by putting $\zeta_i(p)\equiv\delta(p-p_i)$,for $i=1,2,\ldots,M$;

where δ is the Dirac delta distribution. In this thesis we follow the collocation method, i.e. we seek $\underline{\sigma}$ so that

$$\tilde{R}(\underline{\sigma};p_i) = 0 \ ,$$

where the p_i's are selected surface collocation points on each boundary element i.e. $p_i \in S_i$ for each $i=1,2,\ldots,M$.

9.3 Collocation points

Convenient collocation points $\{p_i \in S_i : 1 \le i \le M\}$ are selected at each boundary element S_i. Its location in S_i is purely determined by the optimum computational accuracy i.e. it is to some extent user-defined. These points are usually taken at the centroid of each boundary element.

Collocating the equation (9.5) at these M selected points gives

$$\frac{\partial \phi^i}{\partial n} = -2\pi\sigma_i + \sum_{j=1}^{M} \sigma_j \int_{S_j} [\frac{\partial g_k}{\partial n_{p_i}}(p_i,q) + i \frac{\partial^2 g_k}{\partial n_{p_i} \partial n_{q^*}}(p_i,q^*)]\,dq \quad ; \qquad 1 \le i \le M \tag{9.7}$$

where for each i

$$\frac{\partial \phi^i}{\partial n} = \frac{\partial \phi}{\partial n}(p_i) \quad ; \quad p_i \in S_i \ ,$$

and n_{p_i} denotes the normal at p_i. Note that each collocation point determines a linear equation with M unknowns i.e. (9.7) determines a system of M linear equations with M unknowns. Defining

$$A_{ij} = -2\pi\,\delta_{ij} + \int_{S_j} [\frac{\partial g_k}{\partial n_{p_i}}(p_i,q) + i\frac{\partial^2 g_k}{\partial n_{p_i}\partial n_{q^*}}(p_i,q^*)]\,dq \ , \qquad (*)$$

where

$$\begin{aligned} \delta_{ij} &= 1 \ , \quad \textit{if } i=j \\ &= 0 \ , \quad \textit{if } i \neq j \end{aligned} \ ,$$

one can rewrite (9.7)

$$\frac{\partial\phi^i}{\partial n} = \sum_{j=1}^{M} A_{ij}\sigma_j \ ; \qquad \textit{for } 1 \le i \le M \ ,$$

where each i determines a boundary value of $\partial\phi/\partial n$ on the boundary element S_i and also the coefficients A_{ij}, referring to the collocation point p_i with integral over the boundary element S_j, thereby yielding a linear equation with M unknowns. Consequently we have a system of M equations with M unknowns which may be expressed in the matrix form

$$\underline{A}\,\underline{\sigma} = \underline{v} \ , \qquad (9.8a)$$

where

$$\begin{aligned} \underline{A} &= (A_{ij}) \ , \ 1 \le i,j \le M \\ \underline{\sigma} &= [\sigma_1, \sigma_2, \ldots, \sigma_M]^T \\ \underline{v} &= [\frac{\partial\phi^1}{\partial n}, \frac{\partial\phi^2}{\partial n}, \ldots, \frac{\partial\phi^M}{\partial n}]^T \end{aligned} \ ,$$

i.e.

$$\begin{vmatrix} \frac{\partial\phi^1}{\partial n} \\ \frac{\partial\phi^2}{\partial n} \\ \cdot \\ \cdot \\ \frac{\partial\phi^M}{\partial n} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1M} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2M} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MM} \end{vmatrix} \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \sigma_M \end{vmatrix} . \quad (9.8b)$$

The system (9.8b) can be solved by utilizing the Gaussian elimination procedure. But first the matrix elements A_{ij} are to be computed, which involve both non-singular and singular integrals depending on the location of the matrix element.

9.4 Numerical Integrations

Each non-singular integral in the matrix approximation is replaced by a quadrature formula

$$\int_S F(q)\,dq \simeq \sum_{j=1}^{n} w_j F(q_j) \quad ,$$

where $q_1, q_2, \ldots, q_n$ are the quadrature points and $w_1, w_2, \ldots, w_n$ are the quadrature weights.

Having established the matrix equation form (9.8b) corresponding to the system of equations determined by (9.7) one is left to compute the integrals in the matrix elements as defined in (*) in the preceding section. For distinct i and j (i.e. the non-diagonal elements) the matrix elements contain only regular integrals for which standard Gaussian quadrature is sufficient; otherwise (i.e. for the diagonal elements) the first integral in (*) becomes singular, for which special treatment is necessary.

Case when i and j are distinct

For this situation standard Gaussian quadrature is utilized. First the following coordinate transformation is used

$$\begin{aligned} q_1 &= q_1(\xi_1,\xi_2) \\ q_2 &= q_2(\xi_1,\xi_2) \end{aligned} ,$$

so that both $|\xi_1|,|\xi_2| \leq 1$ and (q_1,q_2) are the coordinates of the surface at q. Therefore, (*) becomes

$$\int_{-1}^{1}\int_{-1}^{1}[\frac{\partial g_k}{\partial n_{p_i}} + i\frac{\partial^2 g_k}{\partial n_{p_i}\partial n_{q^*}}]J(\xi)d\xi_1 d\xi_2 \ , \qquad (9.9)$$

where $J(\xi)=J(\xi_1,\xi_2)$ is the Jacobian of the transformation. Since the integral in (9.9) is also non-singular, the standard Gauss-Legendre procedure can be employed to compute it.

Case when i coincides with j

This is the situation for the diagonal elements. In this situation the elements contain a singular term i.e.

$$\int_{S_i} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq \quad , \tag{9.10}$$

As noted in chapter 3, the singularity is of the weak type since its behaviour is of the order $O(|p_i - q|^{-2})$ as $|p_i - q| \to 0$. These singular integrals are computed by using a treatment depending on the surface configuration, which we address in section 9.5.

Having computed all the elements of the matrix $\underline{A}$ and the known terms $\underline{v} = (\partial\phi^i/\partial n)$ we can apply the Gaussian elimination process to solve for $\underline{\sigma} = (\sigma_j)$. Note that these discrete values give the approximate source density over the surface, as given by (9.3). Utilizing these discrete values of the approximate source density $\tilde{\sigma}$, we can generate an approximate quasi-hybrid potential $\tilde{\phi}$ at any point in the exterior or on the surface, i.e. following the integral representation of (8.9) we have

$$\tilde{\phi}(p) = \int_{\partial B} [g_k(p,q) + \eta \frac{\partial g_k}{\partial n_{q^*}}(p,q^*)]\,\tilde{\sigma}(q)\,dq \quad ; \qquad p \in B^+ \cup \partial B \ , \quad q^* \in \partial B^*, q^* = \vartheta q$$

Using the boundary elements S_j's and (9.3) for the approximate σ , we get

$$\tilde{\phi}(p) = \sum_{j=1}^{M} \int_{S_j} \{[g_k(p,q) + i\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)]$$

$$\times \sum_{i=1}^{M} \chi_i(q)\sigma_i\}dq \quad ; \quad p \in B^+ \cup \partial B, \; q^* \in S_j^*$$

which yields

$$\tilde{\phi} = \sum_{j=1}^{M} \sigma_j \int_{S_j} [g_k(p,q) + i\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)]dq \quad ; \quad p \in B^+ \cup \partial B \,, \quad q^* \in S_j^* \quad .(9.11)$$

The singular integrals which occur in the discretisations (9.7) and (9.11) will be dealt with in the following section.

9.5 Treatment of singularities

For the test problems and other related problems that are considered, the boundary is chosen so that it has an axis of rotation and also the boundary conditions are such that they are symmetric about that axis. It is natural to build the axisymmetry into the BEM, thereby reducing the number of elements required to obtain a satisfactory result. In this thesis we have considered three types of surfaces viz. spherical, cuboid and cylindrical. The integrals occurring in the diagonal elements of the matrix $\underset{\sim}{A}$ involve a weak singularity. Let us recall the diagonal elements

$$A_{ii} = -2\pi + \int_{S_i} [\frac{\partial g_k}{\partial n_{p_i}}(p_i,q) + i\frac{\partial^2 g_k}{\partial n_{p_i}\partial n_{q^*}}(p_i,q^*)]dq \quad ; \quad 1 \le i \le M \quad .(9.12)$$

The second term under the integral is non-singular as

$$|p_i - q^*| = |p_i - \vartheta q| \neq 0 \quad , \quad \forall\, q \in S_i \, ; \, 0 < \vartheta < 1 \; .$$

Therefore standard Gauss-Legendre quadrature rule can be applied to compute this integral. However, the first term under the integral sign has a singularity as $q \to p_i$. This singularity is of weak type, as noted before (chapter 3), hence special singularity treatments are required which we detail in the folloing subsections.

9.5.1 Spherical Surface

For a spherical surface ∂B of radius a, following Jaswon and Symm [36], we determine the singular integral in the diagonal term (9.12) as follows:

$$\int_{S_i} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq = \int_{\partial B} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq - \sum_{\substack{j=1\\ j\neq i}}^{M} \int_{S_j} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq \; . \qquad (9.13)$$

It readily follows that since $p_i \notin S_j$ the integrals under the summation are non-singular. As for the integral over the whole of the boundary we write

$$\int_{\partial B} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq = \int_0^{2\pi}\int_0^{\pi} \left[\frac{\partial g_k}{\partial \rho}(\rho,\theta,\psi)\right]_{\rho=a} a^2 \sin\theta\, d\theta\, d\psi \; , \qquad (9.14)$$

where by utilizing spherical polar coordinates (ρ,θ,ψ)

$$q = q(a,\theta,\psi) \; ,$$

and due to spherical symmetry one can assume

$$p_i = (0,0,a) \ .$$

This yields by direct integration (cf. chapter 5) the following:

$$\int_{\partial B} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq = 2\pi e^{-ika}[e^{-ika} - \frac{2\sin ka}{ka}] \ .$$

Note that the above relation is independent of the position of p_i on the surface ∂B , hence the generality. We therefore have

$$\int_{S_i} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq = 2\pi e^{-ika}[e^{-ika} - \frac{2\sin ka}{ka}]$$

$$- \sum_{\substack{j=1 \\ j\neq i}}^{M} \int_{S_j} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq \qquad ; \quad 1 \leq i \leq M$$

The computed approximate source densities $\{\sigma_i\}$ which are uniform on each boundary element S_i, enable us to calculate the approximate acoustic potential at any point p. Clearly when $p \in \partial B$ a singular integral is involved (although of the weak type). The approximate potential on the boundary is given by

$$\tilde{\phi}(p) = \sum_{i=1}^{M} \sigma_i \int_{S_i} [g_k(p,q) + i\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)]\,dq \quad ; \quad p \in \partial B \ .$$

Note that the first integral on the right hand side is singular when p is on the ith boundary element. For this we write

$$\int_{S_i} g_k(p,q)\,dq = \int_{\partial B} g_k(pq)\,dq - \sum_{\substack{j=1\\ j\neq i}}^{M} \int_{S_j} g_k(p,q)\,dq \quad ; \quad p \in S_i \ .$$

Note that the integrals under the summation are all non-singular and are amenable to standard quadrature rule. As for the first integral, one can again make use of the radial symmetry of the point p i.e. without loss of generality one can assume

$$\begin{aligned} p &= (0,0,a) \\ q &= q(a,\theta,\psi) \end{aligned}$$

where θ,ψ are the spherical polar coordinates and a is the radius of the sphere. We have

$$\int_{\partial B} g_k(p,q)\,dq = \int_0^{2\pi}\int_0^{\pi} g_k(p,q)\,a^2 \sin\theta\, d\theta\, d\psi$$

$$= 4\pi a \sin ka \frac{e^{-ika}}{ka}$$

9.5.2 Cylindrical Surface

Test problems are performed on cylinder of cross section radius a with flat ends hence, only the singular integrals which are taken on the curved surface need be treated as the singular integrals on the flat ends are zero due to the orthogonality of the normal to the radial vectors which are parallel to the flat planes. The singular integrals are written

$$\int_{S_i} \frac{\partial g_k}{\partial n_{p_i}}(p_i,q)\,dq = \int_{S_i} \left[\frac{\partial g_k}{\partial n_{p_i}} - \frac{\partial g_o}{\partial n_{p_i}}\right](p_i,q)\,dq + \int_{S_i} \frac{\partial g_o}{\partial n_{p_i}}(p_i,q)\,dq \quad ; \quad p_i \in S_i \ .$$

Note that the first integral on the right hand side is non-singular since for small $|p_i - q|$ the integrand behaves like $O(1)$, (cf. Chapter 3). As for the second integral on the right hand side we make use of the elliptic integrals. the subscript o denotes the free space Green's function for the Laplace equation. Kirkup [50] gives a useful discussion of the elliptic integrals to which we refer here.

Elliptic Integrals

The elliptic integrals $\mathscr{K}(j;m)$ are defined as follows

$$\mathscr{K}(j;m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{j-1/2}\, d\theta \quad ; \quad 0 \le m < 1 \ ,$$

where j=-2,-1,0,1, using the notation used by Jaswon and Symm [36] and by Banarjee and Butterfield [20]. $\mathscr{K}(1;m)$ and $\mathscr{K}(0;m)$ are available in subroutine libraries. Abramowitz and Stegun [1] gives a method of approximation. $\mathscr{K}(-1;m)$ and $\mathscr{K}(-2;m)$ can be computed from the $\mathscr{K}(1;m)$ and $\mathscr{K}(0;m)$ using the following relation, see Gradshteyn and Ryzhik [100],

$$\mathscr{K}(-1;m) = \frac{\mathscr{K}(1;m)}{1-m} \quad ; \quad 0 \le m < 1$$

$$\mathscr{K}(-2;m) = \frac{-\mathscr{K}(0;m)}{3(1-m)} + \frac{(4-2m)\mathscr{K}(1;m)}{3(1-m)^2} \quad ; \quad 0 \le m < 1$$

Now define

$$\mathscr{F}(j;A,B) = \int_0^{\pi} (A \pm B\cos\theta)^{j-1/2} d\theta \quad ; \quad A > B \geq 0 \ ,$$

where as before $j=-2,-1,0,1$. This can be written as

$$\mathscr{F}(j;A,B) = 2(A+B)^{j-1/2} \mathscr{K}(j;2B/(A+B)) \ ,$$

for $j=-2,-1,0,1$. Introduce the following function for convenience

$$\omega(A,B,C,D) = \int_0^{\pi} \frac{C+D\cos\theta}{(A-B\cos\theta)^{3/2}} d\theta \ ,$$

which gives

$$\omega(A,B,C,D) = \frac{1}{B}[D\mathscr{F}(0;A,B) + (BC-AD)\mathscr{F}(-1;A,B)] \ .$$

For small B we use the asymptotic formula

$$\omega \approx C\pi(A-B)^{-3/2} \ .$$

Now consider the singular integral

$$\int_{S_i} \frac{\partial g_0}{\partial n_{p_i}}(p_i,q)\, dq \ .$$

For the ease of computation p_i's are taken on the line $\theta=0$ i.e.

$$p_i = (a,0,z^i) \quad ; \quad 1 \leq i \leq M_c$$

$$q = q(a,\theta,z)$$

$$n_{p_i} = (1,0,0)$$

where θ, z are the cylidrical coordinates, a is the radius of the cylinder and M_c is the number of boundary elements on the curved surface. These give

$$\int_{S_i} \frac{\partial g_0}{\partial n_{p_i}}(p_i,q)\,dq = -\int_{z_{i_1}}^{z_{i_2}} [\,\omega(A,B,C,D) + \omega(A,-B,C,-D)\,]\,dz \ , \quad (9.15)$$

where

$$A = 2a^2 + (z'-z)^2$$

$$B = 2a^2$$

$$C = a^2$$

$$D = -a^2$$

A standard Gaussian quadrature method is applied on the right hand side of (9.15) provided the quadrature points do not coincide with the z^j's.

A singular integral is also involved when calculating the approximate acoustic potential at any point $p=(0,0,b)$ on a flat end, where b is the semi-length of the cylinder and the cylinder radius is a. The relevant integral is

$$\int_S g_k(p,q)\,dq \qquad ; \quad p \in S \subset \partial B \ ,$$

where S is an element on a flat end, see Fig.8 on the following page, which encloses the point p. Our cylinder test problems are such that we need only divide the boundary into axisymmetric elements i.e. S signifies a circular element on the flat surface where z=b. Utilizing cylindrical polar coordinates we write

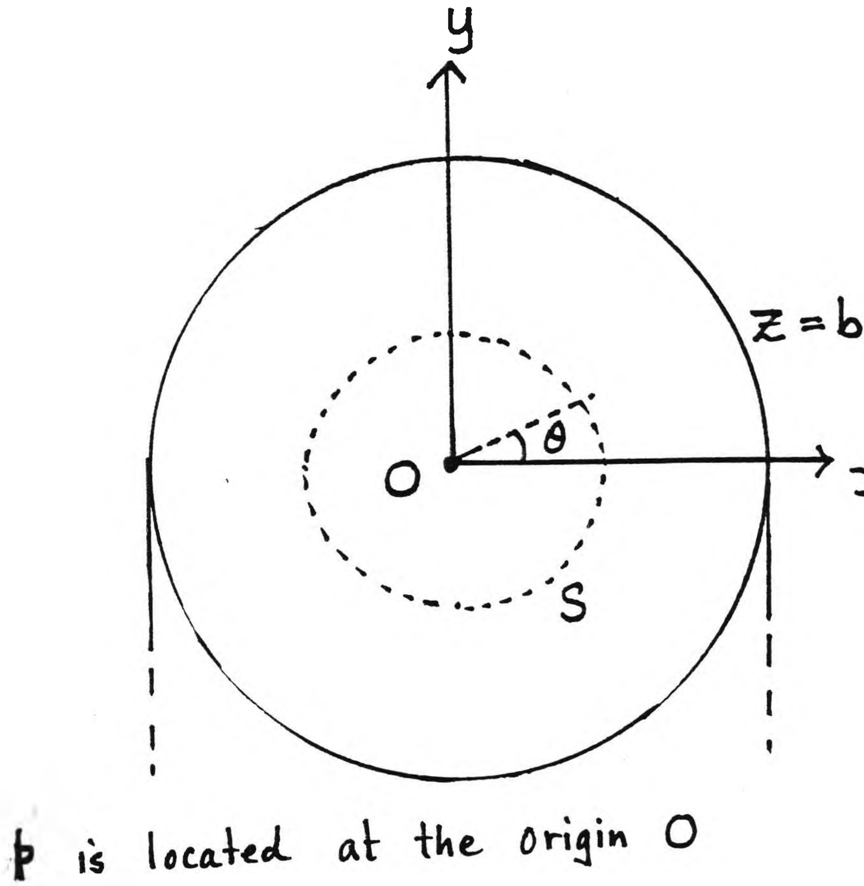

Fig.8 Circular elements on the flat surfaces of a cylinder. The singularity is located at the origin.

$$q = q(\rho,\theta,b)$$

$$\rho = |p-q| \qquad ,$$

$$S = \{ (\rho,\theta) : \ 0 \le \rho \le \alpha \le a \ , \ \ 0 \le \theta \le 2\pi \ \}$$

where α is a constant. By taking p to be the origin we can write

$$\int_{S_i} g_k(p,q)\,dq = \int_0^{2\pi}\int_0^{\alpha} \frac{e^{-ik\rho}}{\rho}\,\rho\,d\rho\,d\theta = \int_0^{2\pi}\int_0^{\alpha} e^{-ik\rho}\,d\rho\,d\theta \ ,$$

which is then evaluated directly.

9.5.3 Cubical Surface

An experiment was performed on a cube of side *2a*. The singular integrals here do not pose any problem since

$$\frac{\partial g_k}{\partial n_{p_i}}(p_i,q) = \nabla_q g_k \cdot n_{p_i} = 0 \quad , \qquad \forall q \in S_i \ ,$$

as n_{p_i} is orthogonal to $\nabla_q g_k$ on the planar elements S_i. As for the calculation of the approximate acoustic potential at any p in the element S on the cube surface, again a singular integral arises. A polar transformation, (see Fig.9 on the following page), yields

$$\int_S g_k(p,q)\,dq = \sum_{i=1}^{4} \int_{\theta_i}^{\theta_{i+1}} \int_0^{\rho_i(\theta)} e^{-ik\rho(\theta)}\,d\rho\,d\theta \quad ; \qquad p \in S \ ,$$

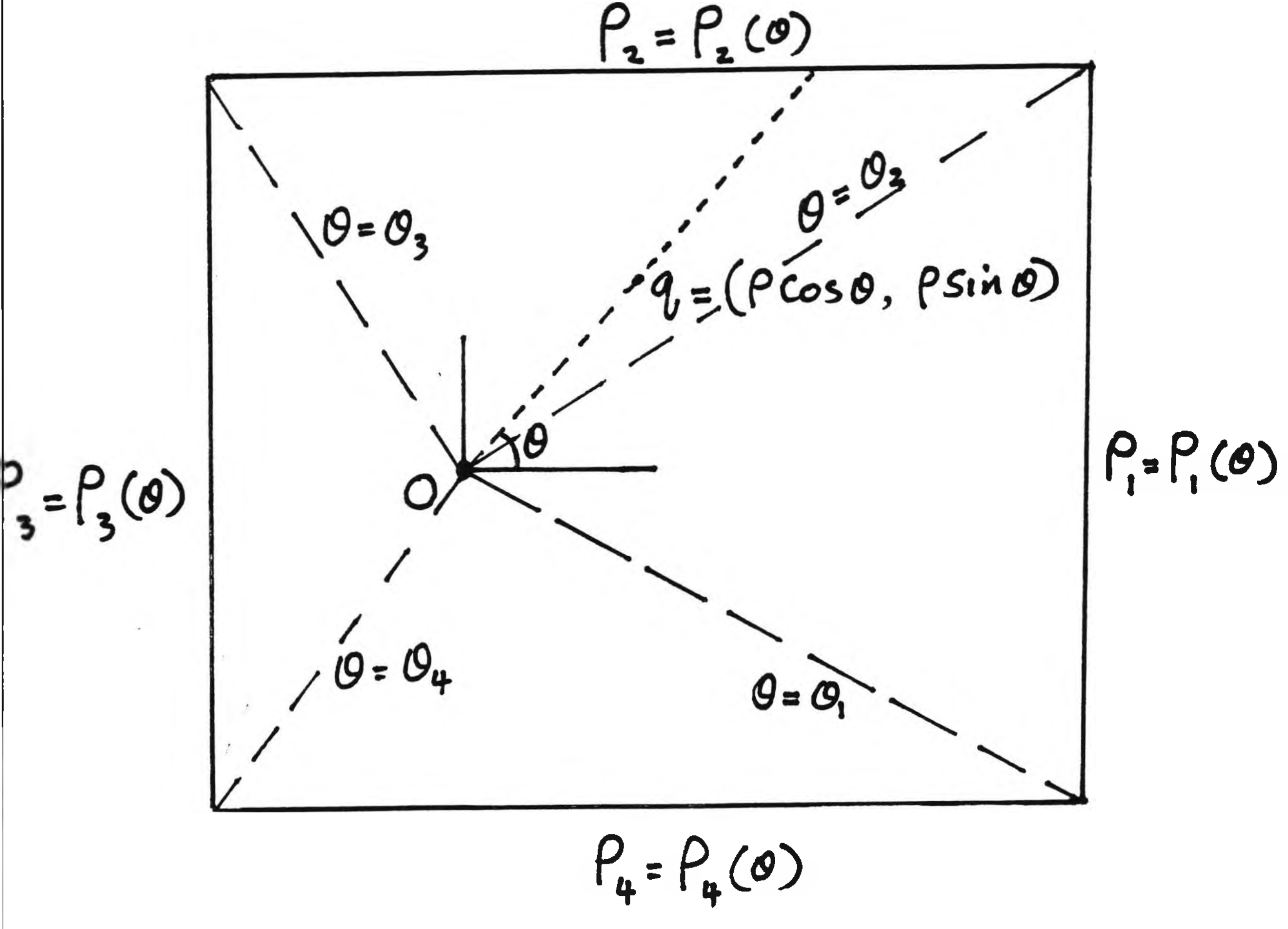

Fig.9 Flat element on the side of a cube. The singularity is located at the origin.

where $\rho=|p-q|$, θ_5 coincides with θ_1 and p is taken as the origin and the element S is such that

$$S \ : \quad \theta_i \leq \theta \leq \theta_{i+1} \ , \quad 0 \leq \rho \leq \rho_i(\theta) \ , \quad 1 \leq i \leq 4 \ .$$

here $\rho=\rho_i(\theta)$, $1 \leq i \leq 4$ make up the outer contour of the element S. Standard Gaussian quadrature is then applied to evaluate the integrals under the summation.

Chapter 10
Numerical Results

For the purpose of testing the new formulation code named AKF some test problems on a sphere, cube and cylinder are to be performed. The test problems considered here are Neumann problems defined by the prescribed boundary values of the normal derivative of the unknown acoustic potential.

In particular we consider the radiation effect at the boundary of a sphere, a cylinder and a cube, produced by a point source within the given boundary. We also consider the radiation pattern from a pulsating sphere and from an oscillating sphere. Analytical results for the radiation patterns are known in both the cases. We also examine the scattering pattern of a plane wave incident on a sphere. Here too the analytical results for the scattered acoustic pressure per incident wave pressure are known.

All these test problems in this thesis are studied for a given range of wave-numbers.

10.1 Sphere test problem

Consider a sphere of radius a, see Fig.10 on the following page, centre O, enclosing a point source located at Q having cartesian coordinates

$$Q = (0,0,a/2) \; ,$$

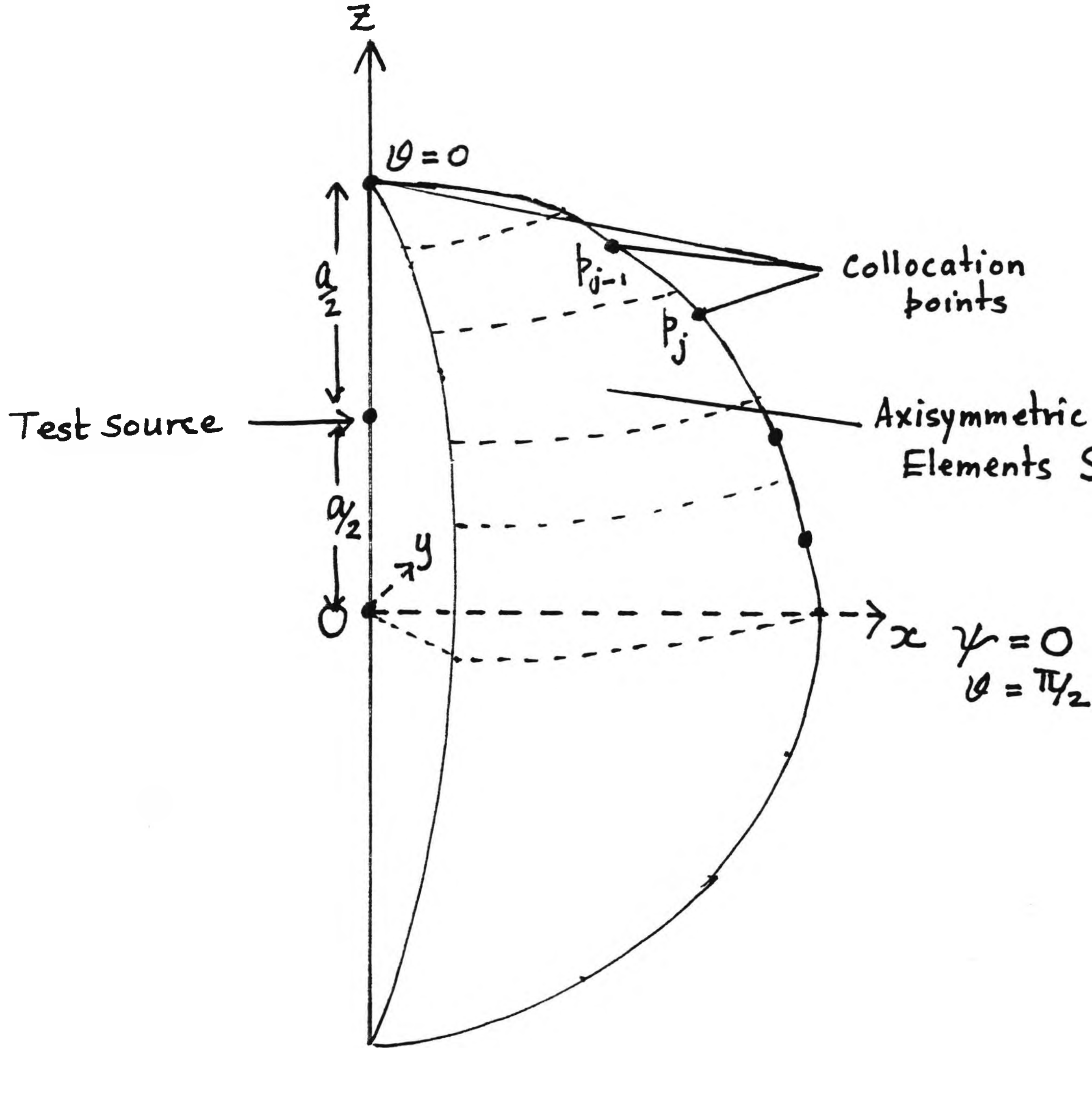

Fig.10 Test source and the axisymmetric elements on a sphere.

relative to a coordinate system centred at O. The potential ϕ generated at any point p by this source is given by

$$\phi(p) = \frac{e^{-ik|p-Q|}}{|p-Q|} \quad ; \quad p \in \mathbb{R}^3 \backslash \{Q\} \ , \tag{10.1}$$

which possesses known values at every point of the surface, denoted by ∂B , of the sphere. This potential has an associated normal derivative at p , given by

$$\frac{\partial \phi}{\partial n}(p) = -\frac{1 + ik|p-Q|}{|p-Q|^3}(p-Q)\cdot n_p e^{-ik|p-Q|} \quad ; \quad p \in \partial B \ , \tag{10.2}$$

where the unit normal vector n_p at p points into the region B^+ exterior to the boundary ∂B . Writing (10.2) in spherical polar coordinates, note that

$$\begin{aligned} p &= p(r,\theta,\psi)_{r=a} \\ \rho = |p-Q| &= (r^2 + a^2/4 - ra\cos\theta)^{1/2}{}_{r=a} \\ \frac{\partial \phi}{\partial n}(p) = \frac{\partial \phi}{\partial r}_{r=a} &= -\frac{1 + ik\rho}{2\rho^3} e^{-ik\rho} a[2 - \cos\theta] \ ; \quad r=a \end{aligned} \tag{10.3}$$

it is clear from (10.3) that $\partial\phi/\partial n$ is symmetric about the z-axis since it depends only on the polar angle θ . We utilize this fact in our BEM discretisation. We now formulate the following Neumann problem:

> *Given the values of $\partial\phi/\partial n$ on ∂B calculate the approximate values of ϕ which we denote by $\hat{\phi}$, at any point on the boundary or in the exterior region.*

In order to apply the BEM discretisation process as described in the previous chapter, the sphere surface is subdivided into (see Fig.10) M axisymmetric boundary intervals $S_j;j=1,2,...M$, defined by

$$S_j = \{(a,\theta,\psi) \ : \quad 0 \leq \psi \leq 2\pi \ , \ \ (j-1)\pi/M \leq \theta \leq j\pi/M \ \} \ ; \quad 1 \leq j \leq M \ ,$$

as depicted in Fig.10, where θ,ψ are the spherical polar coordinates. Henceforth, the term "boundary element" will be used to mean the boundary intervals just defined. A convenient collocation point p_j is chosen in S_j as follows:

$$p_j \ : \quad \psi = 0 \ , \quad \theta = \frac{\pi(2j-1)}{2M} \ ; \quad 1 \leq j \leq M \ .$$

Utilizing these collocation points in the BEM discretisation process and using treatments as described in the chapter 9, we arrive at a system of M linear equations with M unknowns $\{\sigma_j\}$. These σ_j 's are the unknown approximate source densities at collocation points p_j's. Using (10.3) the boundary value at each collocation point is calculated. The approximate acoustic potential at surface point $(0,0,a)$ is computed (cf. chapter 9) utilizing the approximate source densities σ_j 's, and the amplitudes $|\hat{\phi}|$ of the approximate potential are compared with the exact potential calculated using (10.1) and that of the classical method, for a given range of wave-numbers. Fig.14 shows an excellent agreement of the AKF with the exact solution, whereas the classical formulation fails at wave-numbers viz. 4.5, 5.7, 6.9 etc. In fact, the wave-number 4.5 is known to be a characteristic eigenfrequency of the interior Dirichlet problem. We also compute the approximate potentials for $ka=50.$ at each surface collocation point and compare them with the exact potential (see Figs.15,15a,16). At this large wave-number AKF agrees quite well with the exact whereas the classical result fails at all but few collocation points.

10.2 Cylinder test problem

A cylinder of aspect ratio *a:b=1:5* is considered. Here *a* is the radius of the cylinder and *b* is its semi-length. The two ends are flat, see Fig.11. The Neumann boundary condition is that which is produced by a fictitious point source at

$$\hat{p} = (0,0,0) \ .$$

The exact potential generated at any point p by this source is given by

$$\phi(p) = \frac{e^{-ik|p-\hat{p}|}}{|p-\hat{p}|} = \frac{e^{-ik|p|}}{|p|} \quad ; \quad p\in\mathbb{R}^3\backslash\{(0,0,0)\} \ .$$

This generates the Neumann boundary conditions

$$\frac{\partial\phi}{\partial n}(p) = -\frac{1 + ik|p|}{|p|^3}(p\cdot n_p)e^{-ik|p|} \quad ; \quad p\in\partial B \ , \qquad (10.4)$$

where the normal n_p at p points into the region exterior to the cylinder and ∂B here denotes the boundary of the cylinder. Writing (10.4) in cylindrical polar coordinates, denoting the flat surfaces by Γ_+ and Γ_- represented by the equations $z=+b$ and $z=-b$ respectively, and denoting the curved surface by Γ_c i.e.

$$\partial B = \Gamma_c \cup \Gamma_+ \cup \Gamma_- \ ,$$

we get

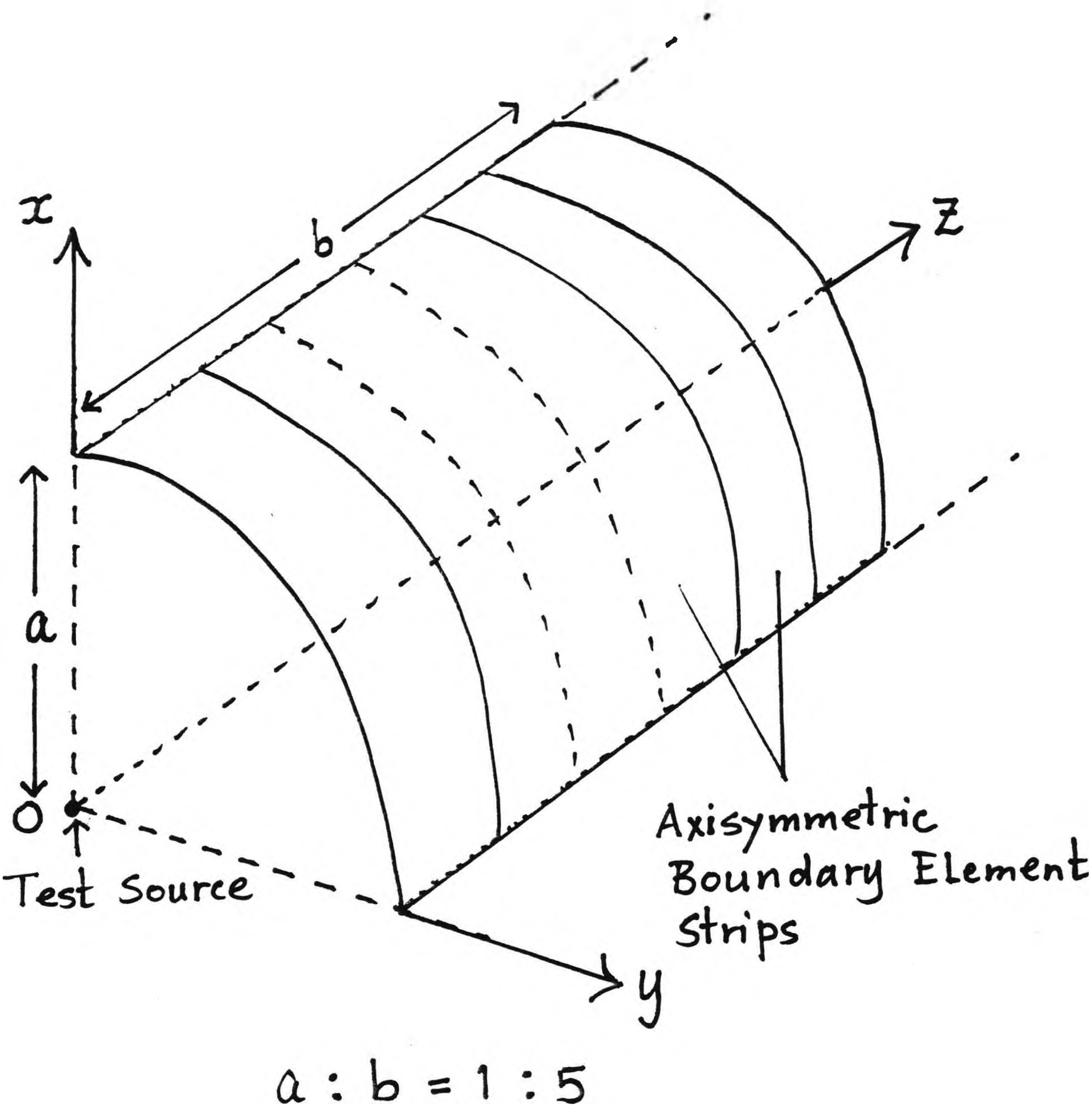

Fig.11 Test source and the axisymmetric elements on a cylinder of aspect ratio 1:5.

$$p = p(a,\theta,z)_{|z|\le b}\ ,\quad p\in\Gamma_c$$

$$p = p(r,\theta,\pm b)_{r\le a}\ ,\quad p\in\Gamma_+\cup\Gamma_-$$

$$\frac{\partial\phi}{\partial n}(p) = \frac{\partial\phi}{\partial z}_{z=\pm b} = -\frac{1+ik|p|}{|p|^3}\,b\,e^{-ik|p|}\quad ;\quad p\in\Gamma_+\cup\Gamma_-$$

$$\frac{\partial\phi}{\partial n}(p) = \frac{\partial\phi}{\partial r}_{r=a} = -\frac{1+ik|p|}{|p|^3}\,a\,e^{-ik|p|}\quad ;\quad p\in\Gamma_c$$

Due to the rotational symmetry of the boundary condition only the top flat surface and the top curved surface need be subdivided. Utilizing the cylindrical polar coordinates (r,θ,z), divide the top flat surface Γ_+ into M_+ elements and the top curved surface into M_c elements so that

$$S_j = \begin{cases} \{(r,\theta,b) : \dfrac{a(j-1)}{M_+}\le r\le\dfrac{aj}{M_+}\ ,\ 0\le\theta\le 2\pi\}\ ; & 1\le j\le M_+ \\ \{(a,\theta,z) : b-\dfrac{b(j-M_+)}{M_c}\le z\le b-\dfrac{b(j-M_+-1)}{M_c}\ ,\ 0\le\theta\le 2\pi\}; & M_++1\le j\le M \end{cases}$$

where M is the total number of elements so that $M=M_++M_c$ and $M_+{:}M_c=1{:}5$, in line with the aspect ratio. Using cartesian coordinates the collocation points are chosen as follows:

$$p_j = \begin{cases} (\dfrac{a(2j-1)}{2M_+},0,b) & ; \quad 1\le j\le M_+ \\ (a,0,b-\dfrac{b(2(j-M_+)-1)}{2M_c}) & ; \quad M_++1\le j\le M \end{cases}$$

Having computed the approximate source densities $\{\sigma_j\}$ by solving the system of M linear equations using the Gaussian elimination process, we calculate the approximate potential $\hat{\phi}$ at the surface point $(0,0,b)$ and compare, (see Fig. 18), the amplitudes $|\hat{\phi}|$ of the approximate potential at this point with that of the analytical (i.e. the exact) and the classical results, for a given range of wave-numbers. The figure clearly demonstrates the agreement of the AKF results with that of the exact result.

10.3 Cube test problem

We consider a cube of side $2a$ centred at the origin. The Neumann boundary condition is that which is produced by a (fictitious) test point source at

$$\hat{p} = (0,0,0) \ .$$

The exact potential generated at any point $p=(x_1,x_2,x_3)$ in cartesian coordinates is given by

$$\phi(p) = \frac{e^{-ik(x_1^2+x_2^2+x_3^2)^{1/2}}}{(x_1^2+x_2^2+x_3^2)^{1/2}} \quad ; \quad (x_1,x_2,x_3) \neq (0,0,0) \ .$$

This generates the Neumann boundary condition

$$\frac{\partial \phi}{\partial n}(p) = -\frac{1+ik\rho}{\rho^3} x_s e^{-ik\rho} \quad ; \quad s=1,2,3$$

$$\rho = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

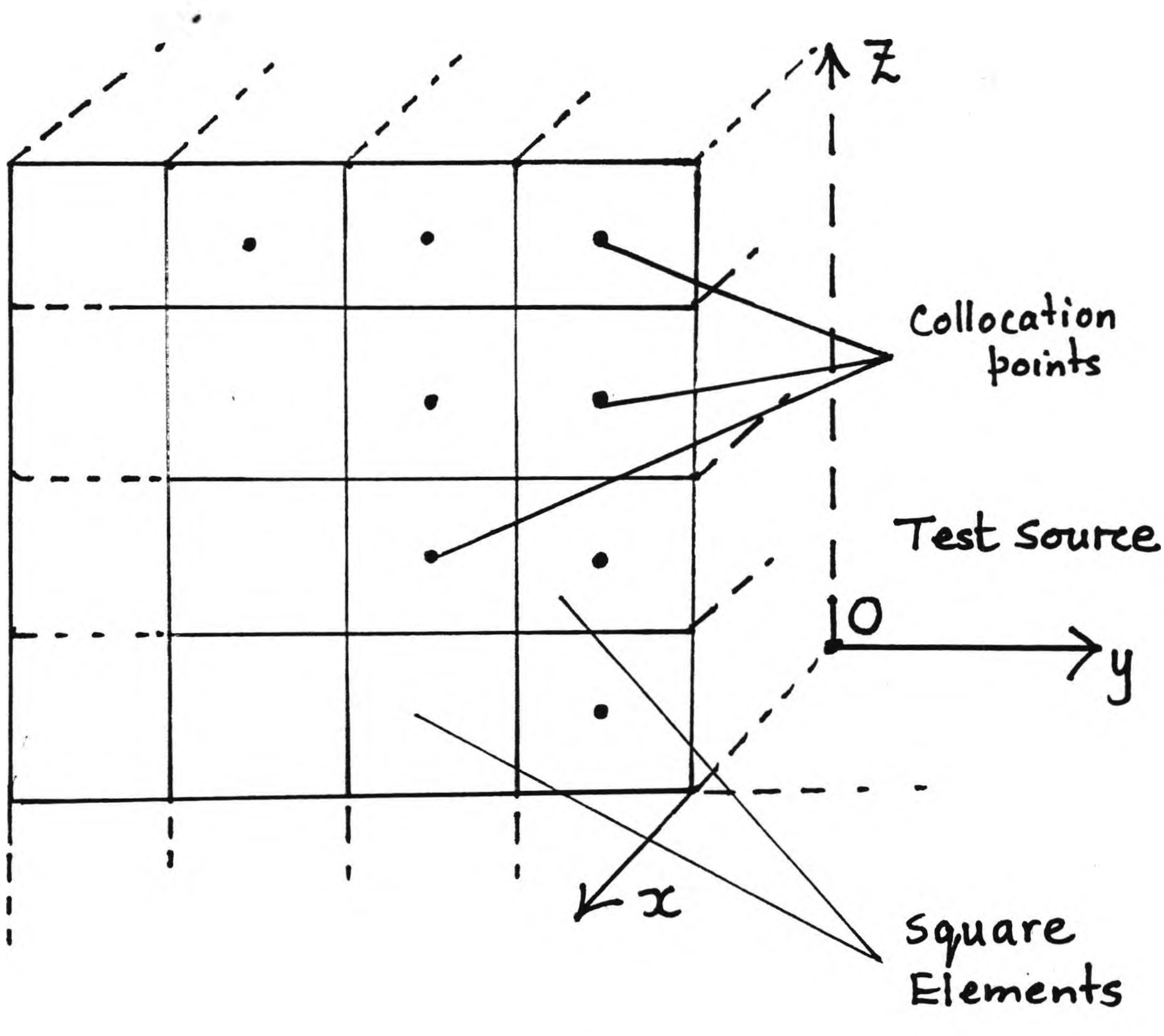

Fig.12 Test source and the square elements on a quarter of a cube side.

where the vector $\boldsymbol{n}$ is parallel to the x_s-axis for each s. For a point p on any side of the cube, for which $|x_s|=a$, for some $s=1,2,3$, we have

$$\frac{\partial\phi}{\partial n}(p) = \left[\pm\frac{\partial\phi}{\partial x_s}\right]_{x_s=\pm a} = -\frac{1+ik\rho}{\rho^3}ae^{-ik\rho} \ .$$

Due to the symmetry of the boundary condition only one side of the cube need be subdivided. Without loss of generality we subdivide the side $x_3=a$, see Fig.12. Divide this cube-side into $M(=m\times m)$ square elements i.e.

$$S_j = \{(x_1,x_2,a) \ : \quad -a+\frac{2a(j_t-1)}{m} \le x_t \le -a+\frac{2aj_t}{m} \ , \quad t=1,2\} \ ; \quad 1\le j_t \le m \ ,$$

where $j=j(j_1,j_2)$ such that $1\le j\le M$. Collocation points are chosen as

$$p_j = (-a+\frac{a(2j_1-1)}{M}, -a+\frac{a(2j_2-1)}{M}) \ ; \quad 1\le j_1,j_2\le m \ , \quad j=j_1+m(j_2-1) \ .$$

Having solved for the unknown $\{\sigma_j\}$ from the system of M^2 linear equations by using Gaussian elimination process we compute the approximate potential $\hat{\phi}$ at the surface point $p=(0,0,a)$ and compare the amplitude $|\hat{\phi}|$ with the exact and that of the classical solution. Fig.17 shows clear agreement of the AKF with the exact.

10.4 Pulsating Sphere

Sound waves radiating by a pulsating sphere were looked at. The exact analytical solution for the acoustic pressure P at distance r from the centre of the sphere of radius a, pulsating with a uniform radial velocity U_o is ([101], [60]) given by

$$P(r) = U_o z_o \frac{ika}{1+ika} \frac{e^{-ika[(r/a)-1]}}{(r/a)} ,$$

where z_o is the acoustic characteristic impedence of the medium such that $z_o = \rho c$. In order to calculate the acoustic potential on the surface of the sphere for a given range of wave-numbers ka, the radius a, the normal surface velocity U_o and z_o are all assumed to be unity i.e. in this case

$$P(r) = \frac{ik}{1+ik} r^{-1} e^{-ik(r-1)} \quad ; \quad |r|>0 .$$

Using the relation, see references [100], [60],

$$P(r) = -ik\phi(r) \quad ; \quad |r|>0 , \tag{10.5}$$

the exact acoustic potential at distance r from the origin is

$$\phi(r) = -\frac{1}{1 + ik} r^{-1} e^{-ik(r-1)} \quad ; \quad |r|>0 .$$

The same BEM analysis is executed as in the sphere test problem (cf. 10.1). For prescribed boundary values of the normal derivative of the potential ϕ , we compute the approximate source densities and then calculate the amplitude of the approximate potentials $|\hat{\phi}|$, $\Re(\hat{\phi})$ and $\Im(\hat{\phi})$. We then calculate the same for the pressure. These are compared with the analytical for a range of wave-numbers, see Figs.19,20.

10.5 Oscillating Sphere

Radiation from an oscillating sphere of radius a is also investigated. We have taken $\theta=0$ to be the direction of oscillation of the sphere, where θ is the polar angle. The exact analytical solution, see [101], [60], for the acoustic pressure P at distance r from the centre of the sphere oscillating is given by

$$P(r) = (a/r)^2 U_o \cos\theta\, z_o \frac{ika(1+ikr)}{2(1+ika)-k^2a^2} e^{-ik(r-a)} \quad ; \quad |r|>0 \ , \quad (10.6)$$

where $U_o\cos\theta$ is the radial velocity and z_o is the acoustic characteristic impedence. For the purpose of our test a, U_o and z_o are assumed to be unity. The exact acoustic potential is computed using the relation (10.6) i.e. the exact acoustic potential takes the form

$$\phi(r) = -\frac{\cos\theta}{2(1+ik)-k^2}(1+ikr)r^{-2}e^{-ik(r-1)} \quad ; \quad |r|>0 \ .$$

Hence the normal derivative of the potential is given by

$$\frac{\partial\phi}{\partial n}(p) = \left.\frac{\partial\phi}{\partial r}\right|_{r=1} = \cos\theta \quad ; \quad 0\le\theta\le\pi \ , \quad p=p(r,\theta,\psi) \ ,(10.7)$$

where (r,θ,ψ) are spherical polar coordinates. The same BEM analysis is executed as in the case of the test (radiation) problem on a sphere. By prescribing the boundary values obtained by using (10.7) a system of M linear equation is solved by using the standard Gaussian elimination process, which yields the values of the unknown source densities $\{\sigma_j\}$. This is utilized to compute the values of the approximate potential $\hat{\phi}$ at the surface point (0,0,1). We then compute the pressure at the same point. The real part and the imaginary part of the pressure are calculated and compared with the exact and that of the classical formulation for a range of wave-numbers. Figs.21,22 show a marked failure of the classical and an excellent agreement of the AKF, particularly at the wave-number 4.5 which is a known eigenfrequency of the corresponding Dirichlet interior problem. Fig.23 shows the normalised far field pressure distribution.

10.6 Scattering from a sphere

A case study of scattering of a plane wave from a rigid sphere is also looked at. The incident plane wave is assumed to be travelling in the positive z-direction, see Fig.13. A sphere of radius a is positioned with the centre at the origin i.e. $\theta=\pi$ is the nearest point on the sphere facing the plane wave, where θ is the polar angle. The incident wave (function) potential is given by

$$\phi_{in}(p) = e^{-ik|p|\cos\theta} \quad ; \qquad p \in \mathbb{R}^3 \ .$$

The total acoustic potential ϕ at any point is the superposition (cf. chapter 4) of the scattered potential and the incident potential at that point i.e.

$$\phi(p) = \phi_{in}(p) + \phi_{sc}(p) \quad ; \qquad p \in B^{+} \cup \partial B \ ,$$

where the subscript sc signifies the scattered potential, B^{+} the region exterior to the sphere and ∂B the boundary. Hence the boundary values of the normal derivative of the total potential is

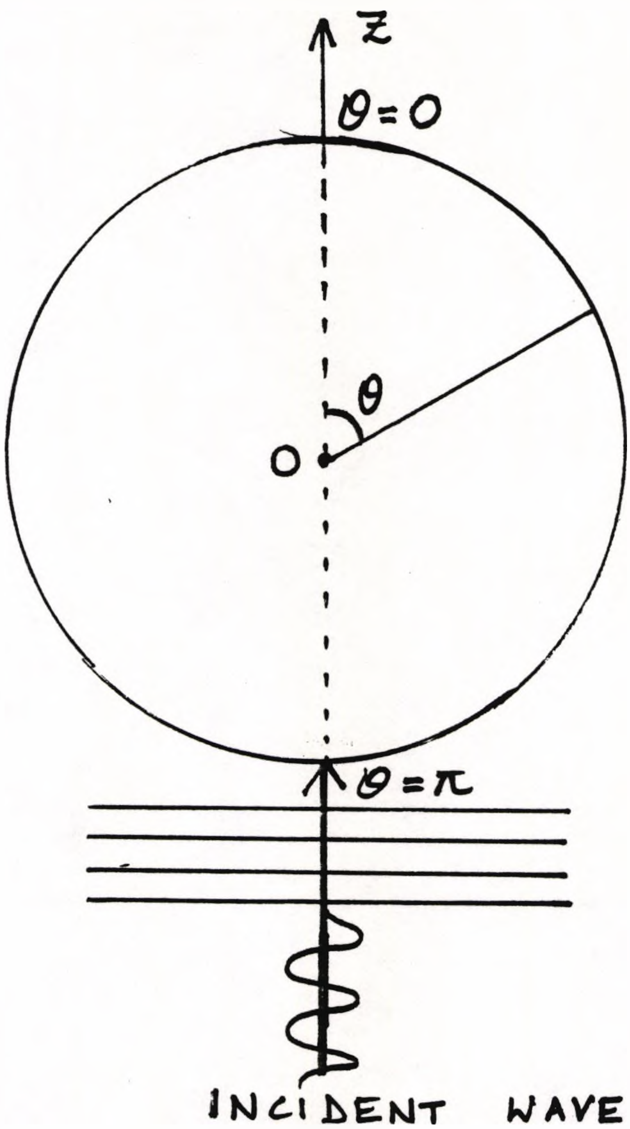

Fig.13 Scattering of plane waves incident on a sphere. Plane waves are moving in the direction of positive z-axis where θ is the polar angle. Note that $\theta=\pi$ is the first point on the sphere facing the incident wave.

$$\frac{\partial \phi}{\partial n}(p) = \frac{\partial \phi_{in}}{\partial n}(p) + \frac{\partial \phi_{sc}}{\partial n}(p) \quad ; \quad p \in \partial B \ . \qquad (10.8)$$

For scattering from a rigid boundary the normal component of the velocity potential vanishes

$$\nabla \phi \cdot n_p = 0 \quad ; \quad p \in \partial B$$

$$i.e. \quad \frac{\partial \phi}{\partial n}(p) = 0 \quad ; \quad p \in \partial B$$

i.e.

$$\frac{\partial \phi_{sc}}{\partial n}(p) = -\frac{\partial \phi_{in}}{\partial n}(p) \quad ; \quad p \in \partial B \ . \qquad (10.9)$$

Here the incident wave is known. We therefore solve the following Neumann problem:

Given a wave ϕ_{in} *incident on a spherical obstacle with the known* $\partial \phi_{in}/\partial n$ *on the surface, calculate the scattered wave* ϕ_{sc} *at any point on the surface or in the exterior.*

BEM features are the same as in the sphere test problem i.e. boundary elements S_j's and the collocation points p_j's are defined as in 10.1 . We now apply the procedure described in chapter 9 by prescribing the boundary values at each collocation point using (10.9) and calculate the unknown source densities σ_j 's at p_j's. These computed approximate source densities σ_j are utilized to compute the unknown approximate scattered potential $\hat{\phi}_{sc}$ which gives the approximate total potential

$$\hat{\phi}(p) = \phi_{in}(p) + \hat{\phi}_{sc}(p) \quad ; \quad p \in \partial B \cup B^{+} \ . \tag{10.10}$$

The analytical solution, see [101], for the scattered potential at any point p is given by

$$\phi(p) = \sum_{m=0}^{\infty} i_{2m+1} L_m(\cos\theta) \sin\delta_m \, e^{i\delta_m} h_m^{(2)}(k|p|) \quad ; \quad p \in \partial B \cup B^{+} \ , \tag{10.11}$$

where i is the imaginary number and

L_m : Legendre Polynomials

δ_m : Phase angle for a given wave-number

$h_m^{(2)}$: Complex conjugate of Hankel function of order m

Approximate acoustic pressure of the scattered wave is calculated using the relation

$$\hat{P}_{sc}(p) = -ik\rho c \hat{\phi}_{sc}(p) \quad ; \quad p \in \partial B \cup B^{+} \ ,$$

where ρc signifies the characteristic acoustic impedence of the medium, which is assumed to be unity. Incident acoustic pressure P_{in} is calculated similarly by using the incident potential. Clearly the approximate total acoustic pressure is the superposition of $\hat{P}_{sc}$ and P_{in} at any point. The analytical solution of the total acoustic pressure at any point is given by [101],

$$P(p) = P_{in} e^{-i\omega t} (ka)^{-2} \sum_{m=0}^{\infty} \frac{2m+1}{B_m} L_m(\cos\theta) \, e^{-i(\delta_m - \frac{\pi m}{2})} \quad ; \ p = p(r,\theta,\psi) \ , \tag{10.12}$$

where (r,θ,ψ) are the spherical polar coordinates and B_m is the phase amplitude for a given wave-number. Figs.24,25,26,27 show a comparison of the angular distributions of the amplitude of the scattered acoustic pressure per unit incident pressure, i.e. $|P_{sc}|/|P_{in}|$, for both AKF, which gives the approximate, and the analytical. These are plotted for wave-numbers $ka=0.1$, 1.0 at field points $r=5a$ and for wave-numbers $ka=\pi, 4.493$ at field point $r=3a$, where a is the radius of the sphere. Fig.28 shows a graph which is plotted for a range of wave-number of the pattern of the amplitude of the total acoustic pressure per unit incident pressure at the point $\theta=\pi$ which is the nearest point on the surface facing the incident wave i.e. $|P|/|P_{in}|$ is computed using the AKF and is compared with the analytical and that of the classical for the given range of wave-numbers. Note that the graph in Fig.28 clearly shows that for wave-numbers which are very small the total pressure amplitude almost equals that of the incident wave at that point i.e. for small wave-number the wave-lengths of the incident waves are large compared to the dimension of the obstacle, hence the occurrence of this phenomenon, which is as expected [60], [101]. But as the wave-numbers increase, thereby decreasing the wave-lengths of the incident waves, causing a variation in the pressure amplitudes at that point. The AKF method demonstrates a very good agreement with the analytical solution, especially near the wave-numbers $ka=4.5$, 5.7, 7.2, $9.$. As noted before, the wave-number 4.5 is known to be situated near an eigenfrequency (in this case 4.493) of the interior Dirichlet problem, for which the classical method fails.

GRAPHS

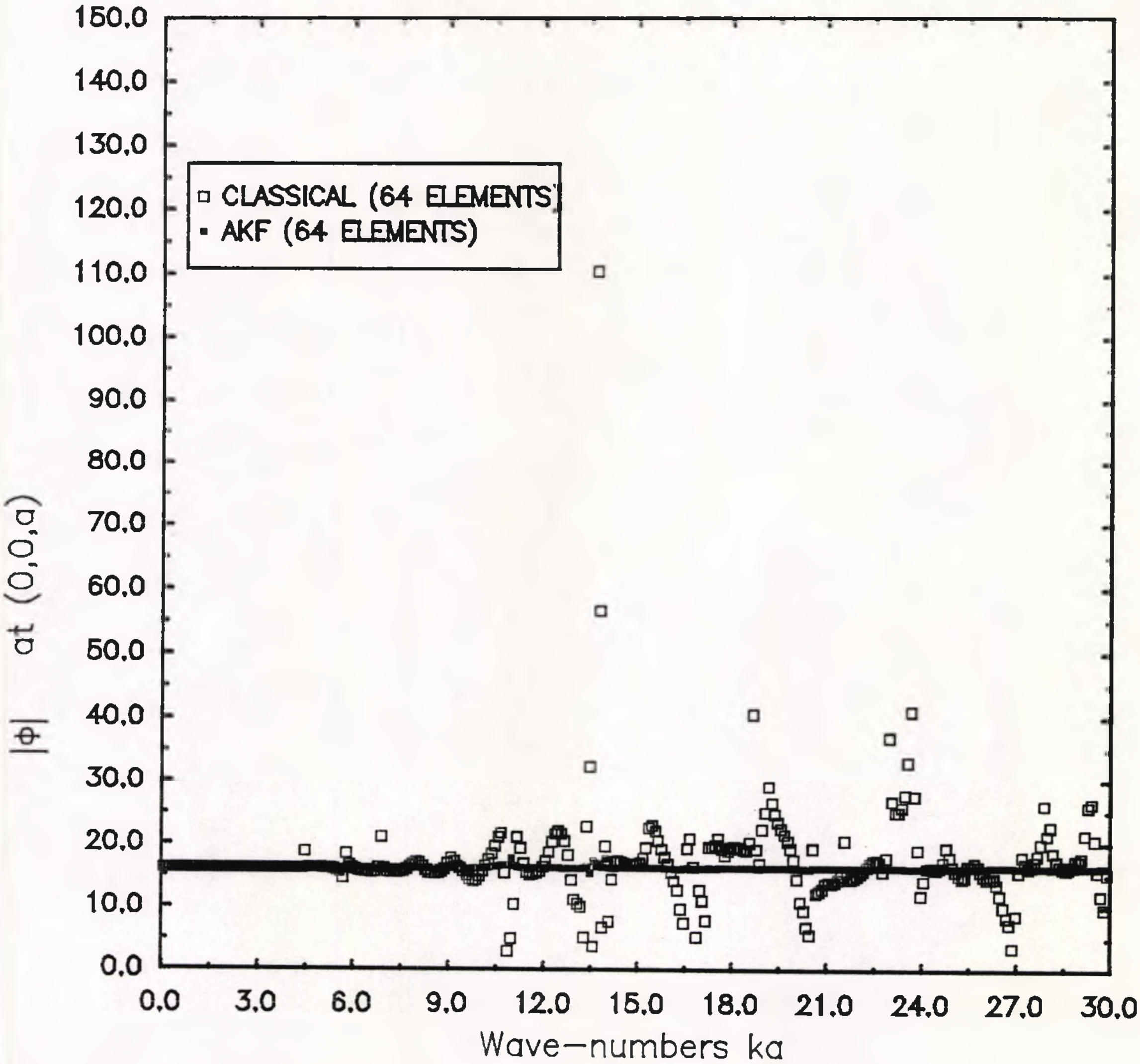

FIG. 14. Computed $|\Phi|$ at $(0,0,a)$ on a sphere of radius a, $a=0.12$; for a range of wave-numbers. The exact potential=16.6. Test source=$(0,0,a/2)$

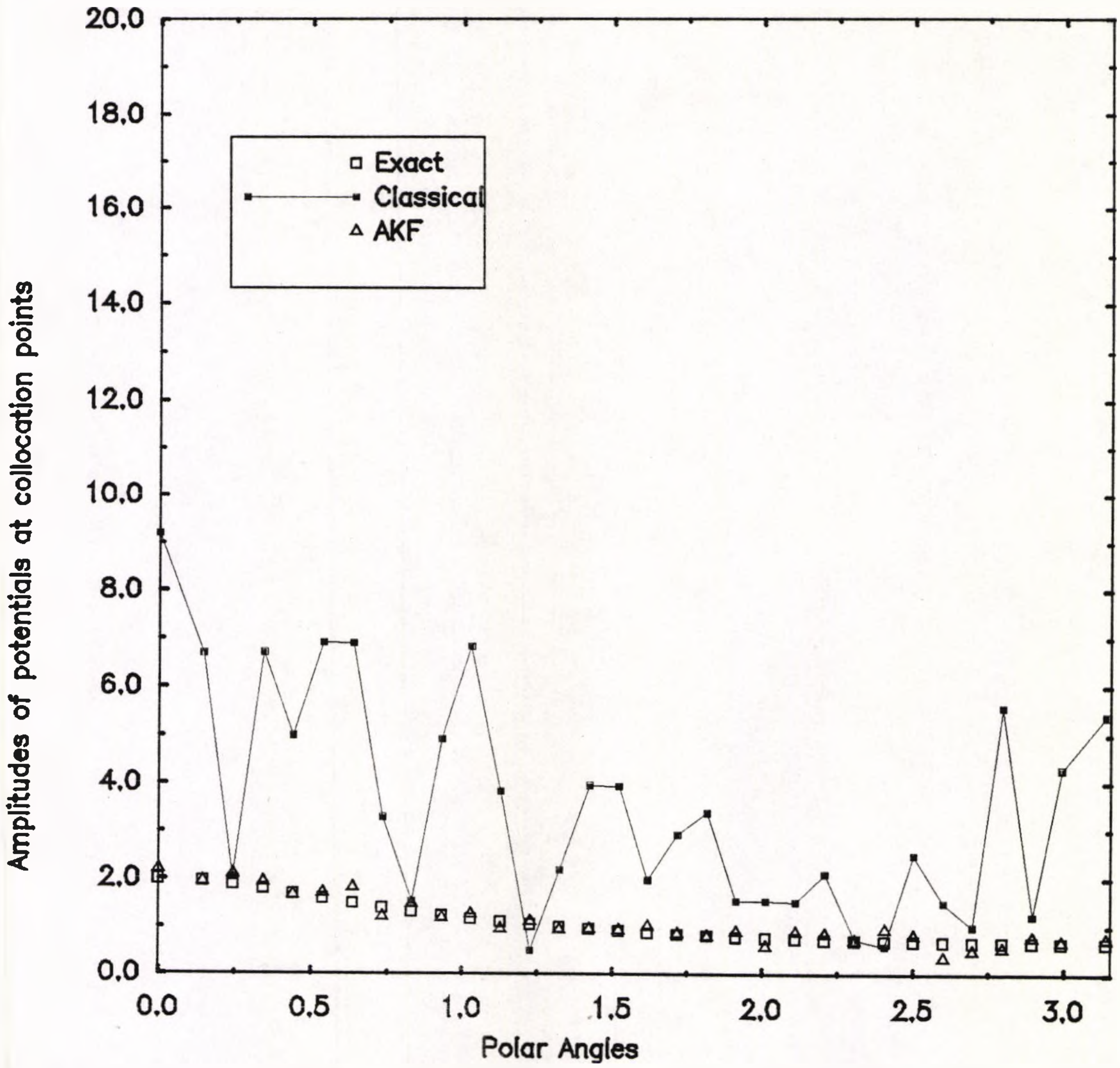

FIG. 15 Angular distribution of potentials at the collocation points on a sphere of radius a, where a=1, for wave-number ka=50., test source at (0,0,a/2), with 32 axially symmetric elements

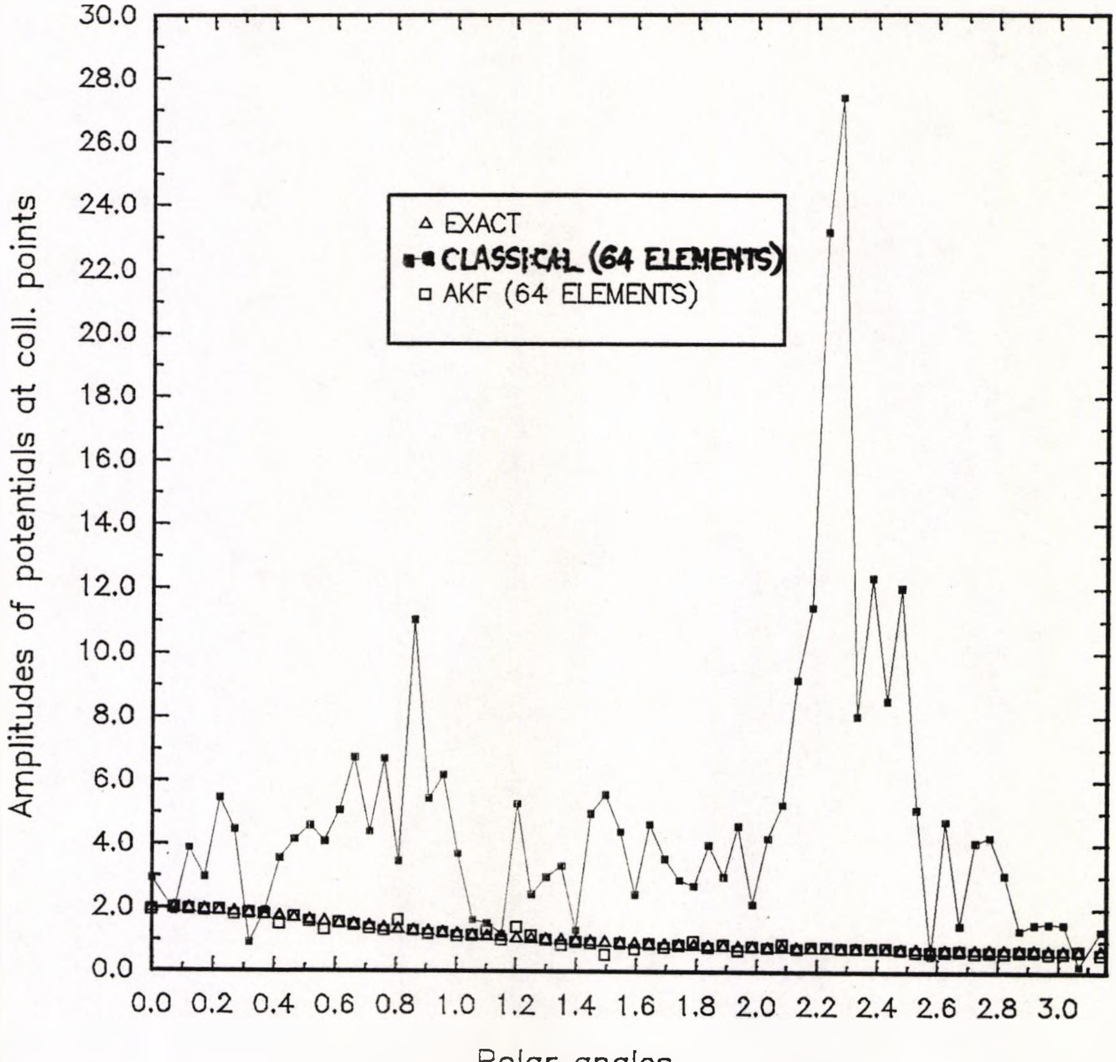

FIG.15a. Angular distribution of potentials at the collocation points on a sphere of radius a, where a=1. ; for wave–number ka=50.., test source at (0,0,a/2)

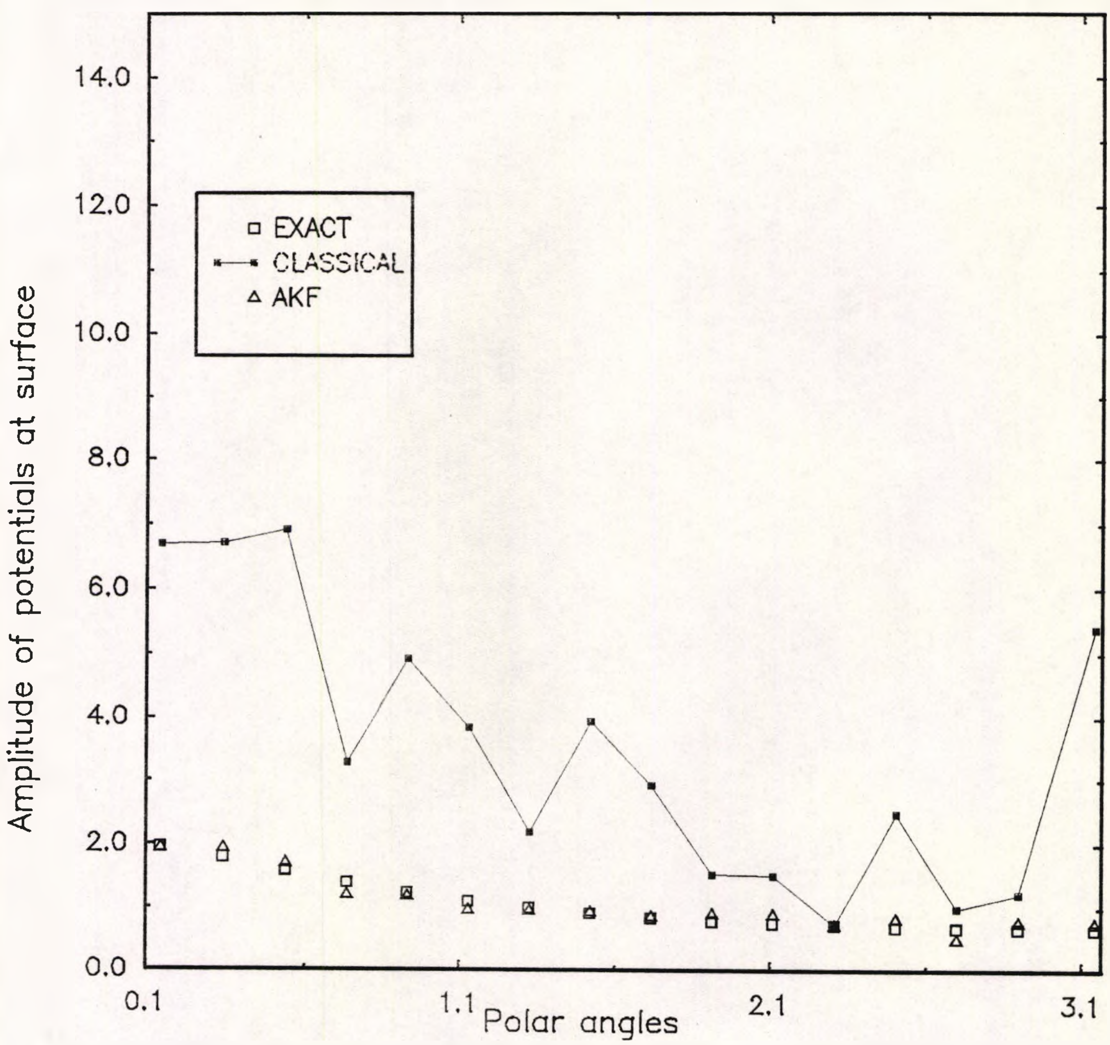

FIG. 16. Angular distribution of potentials at surface points of a sphere of radius, a for wave–number ka=50., test source=(0,0,a/2), 32 axially–symmetric elements.

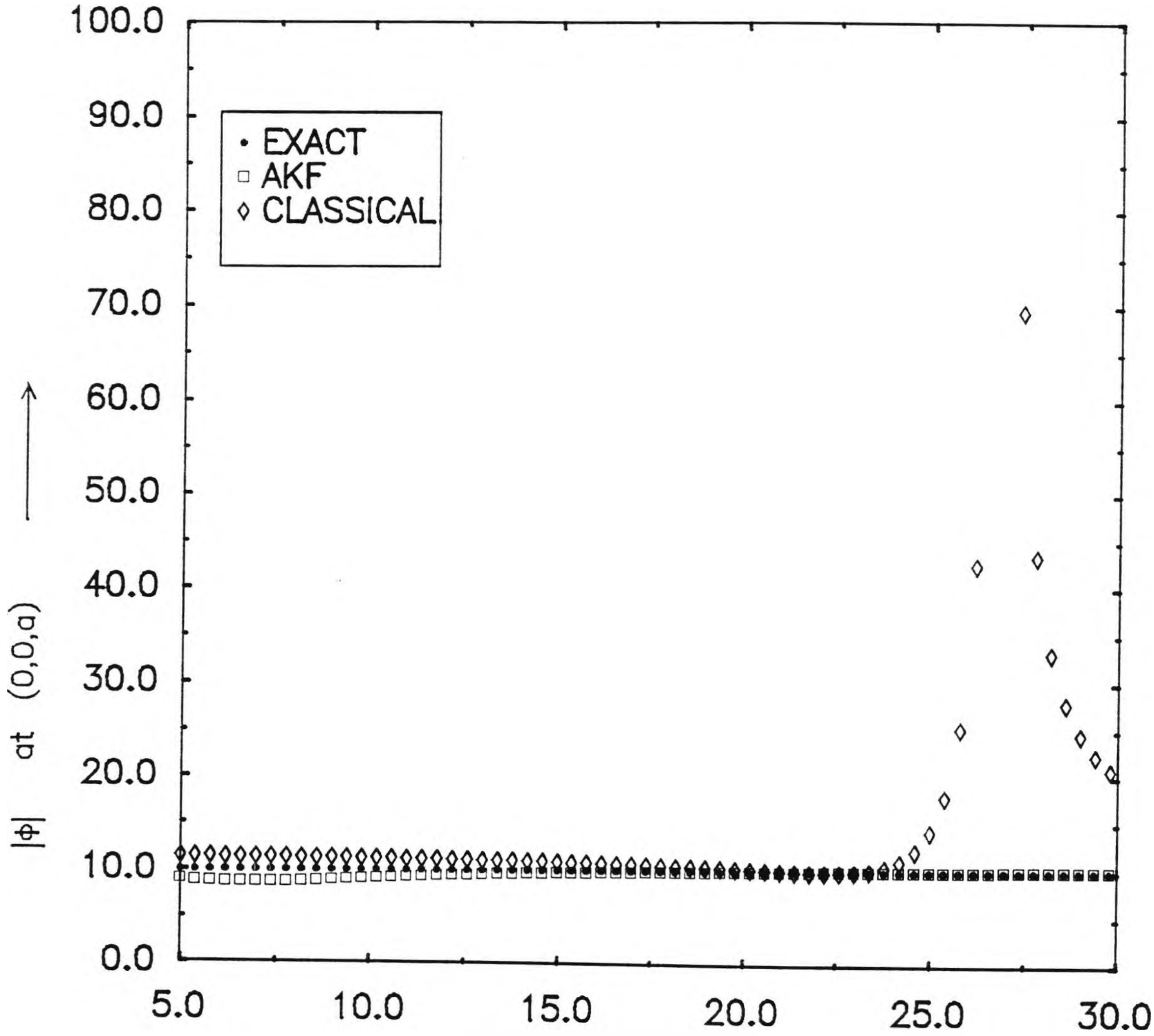

Fig.17. Amplitudes of ϕ at (o,o,a) on a cube of side 2a. Here a=0.1, with test source at (0,0,0), 150 Boundary elements

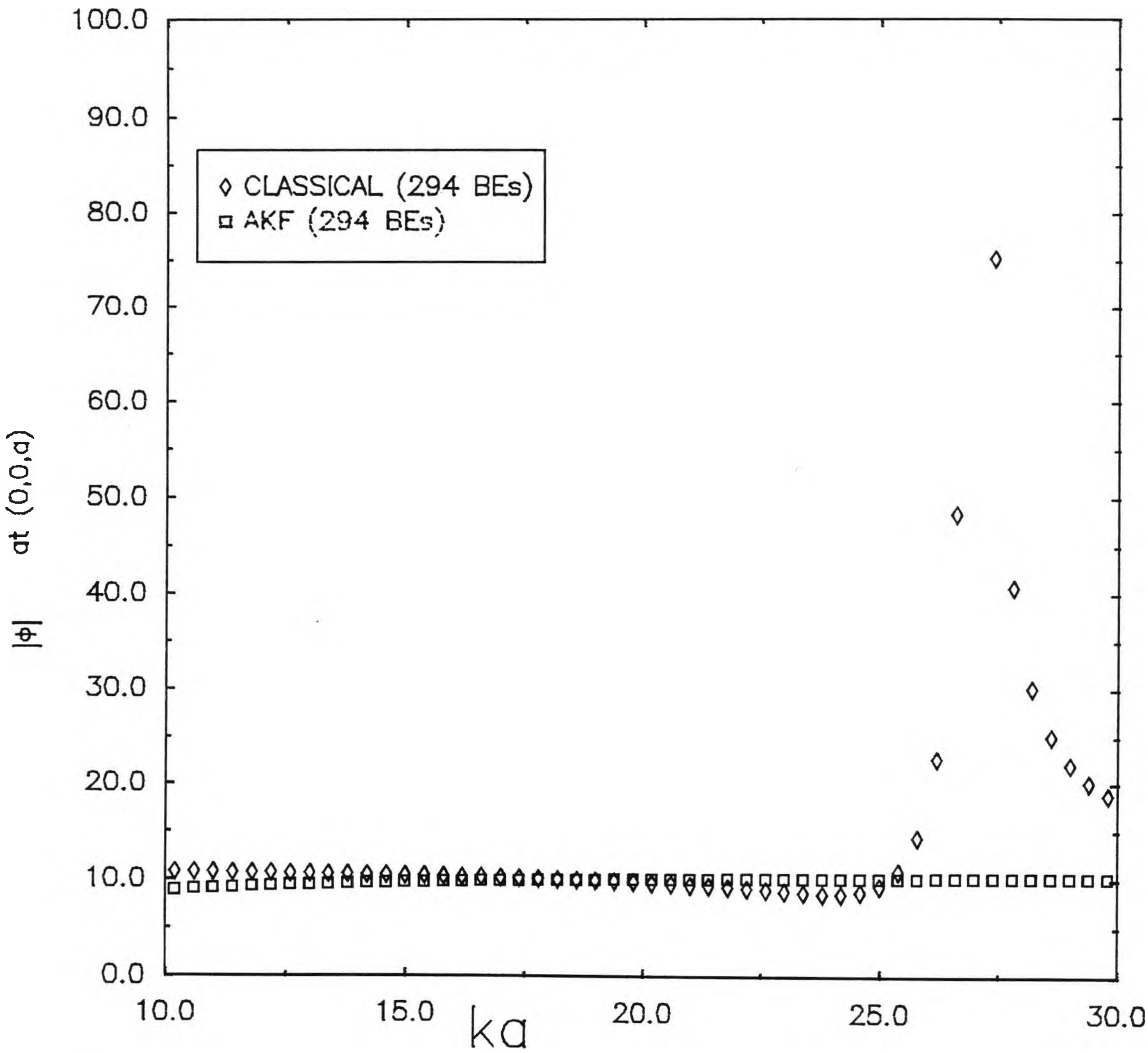

Fig.17a. Amplitudes of |Φ| at (0,0,a) on a cube of side 2a. Here a=0.1, with test source at (0,0,0). The exact solution is 10.0 .

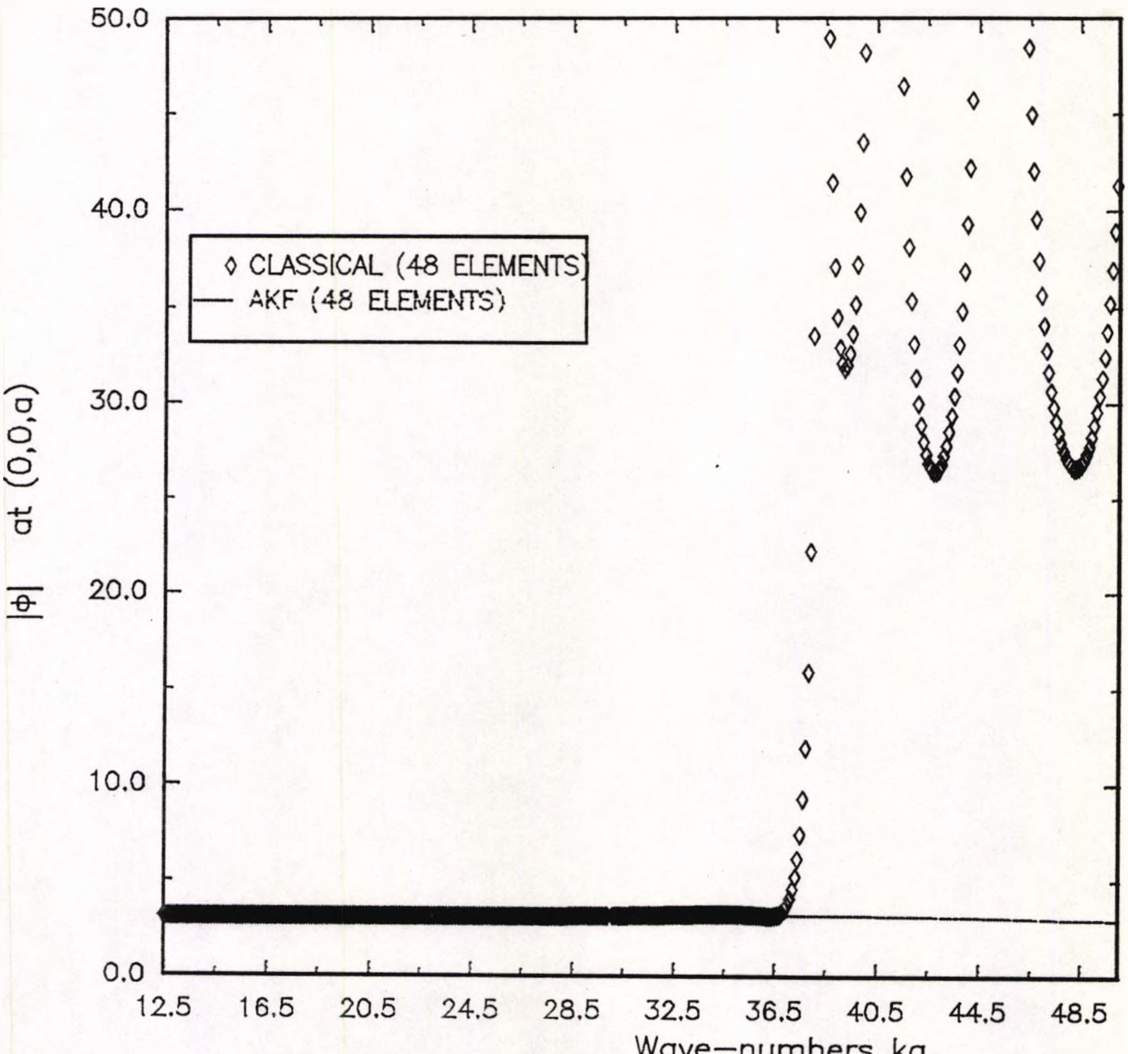

FIG.18. Computed $|\Phi|$ at (0,0,a) on a cylinder of aspect ratio 1:5. Here radius is a/5 and a=0.3, so that the exact potential is 3.333

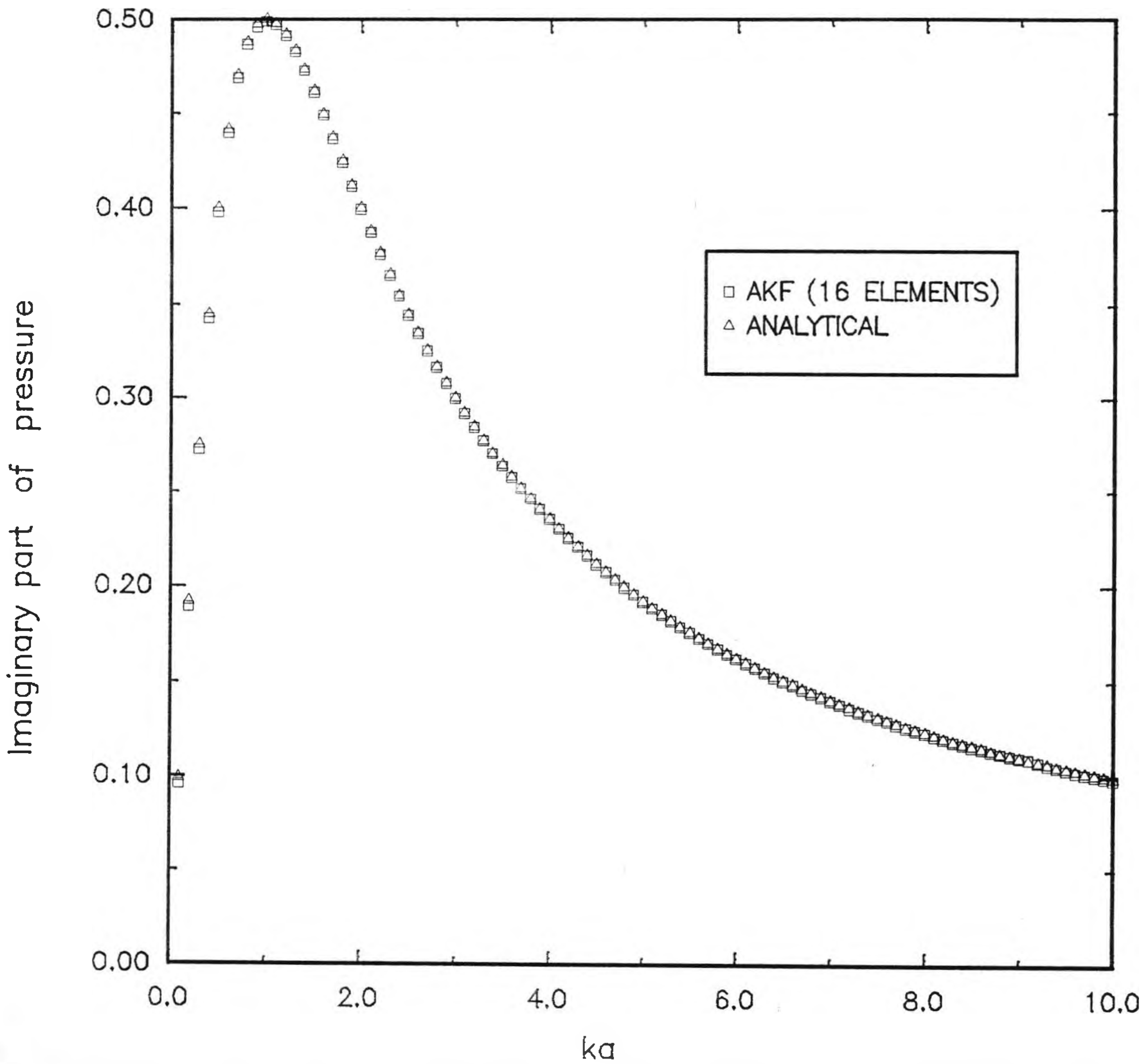

FIG.19. Imaginary part of the pressure on the surface of a pulsating sphere of radius a=1..Sphere is pulsating with a uniform radial velocity.

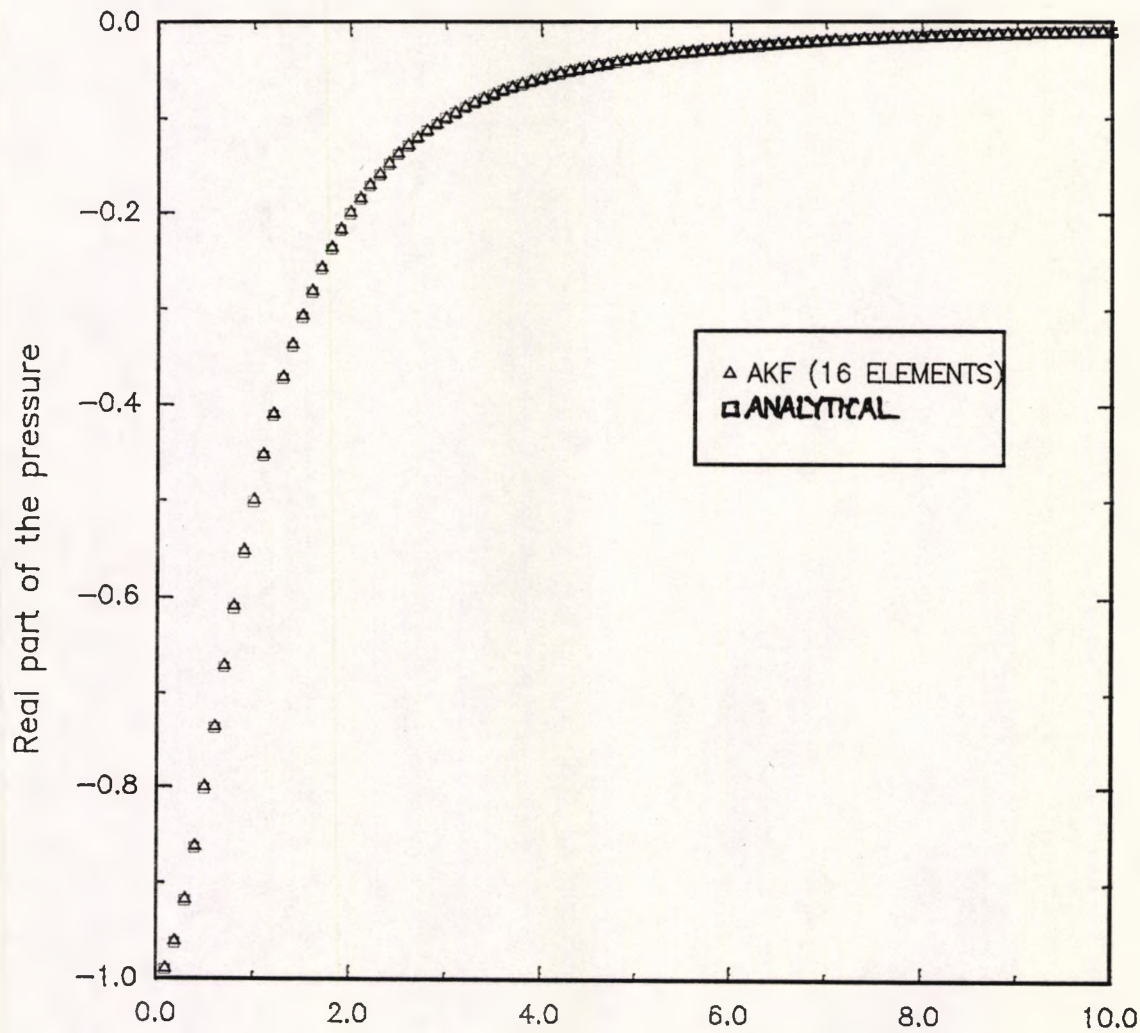

FIG.20. Real part of the pressure on the surface of pulsating sphere of radius a=1. . Sphere is pulsating with a uniform radial velocity

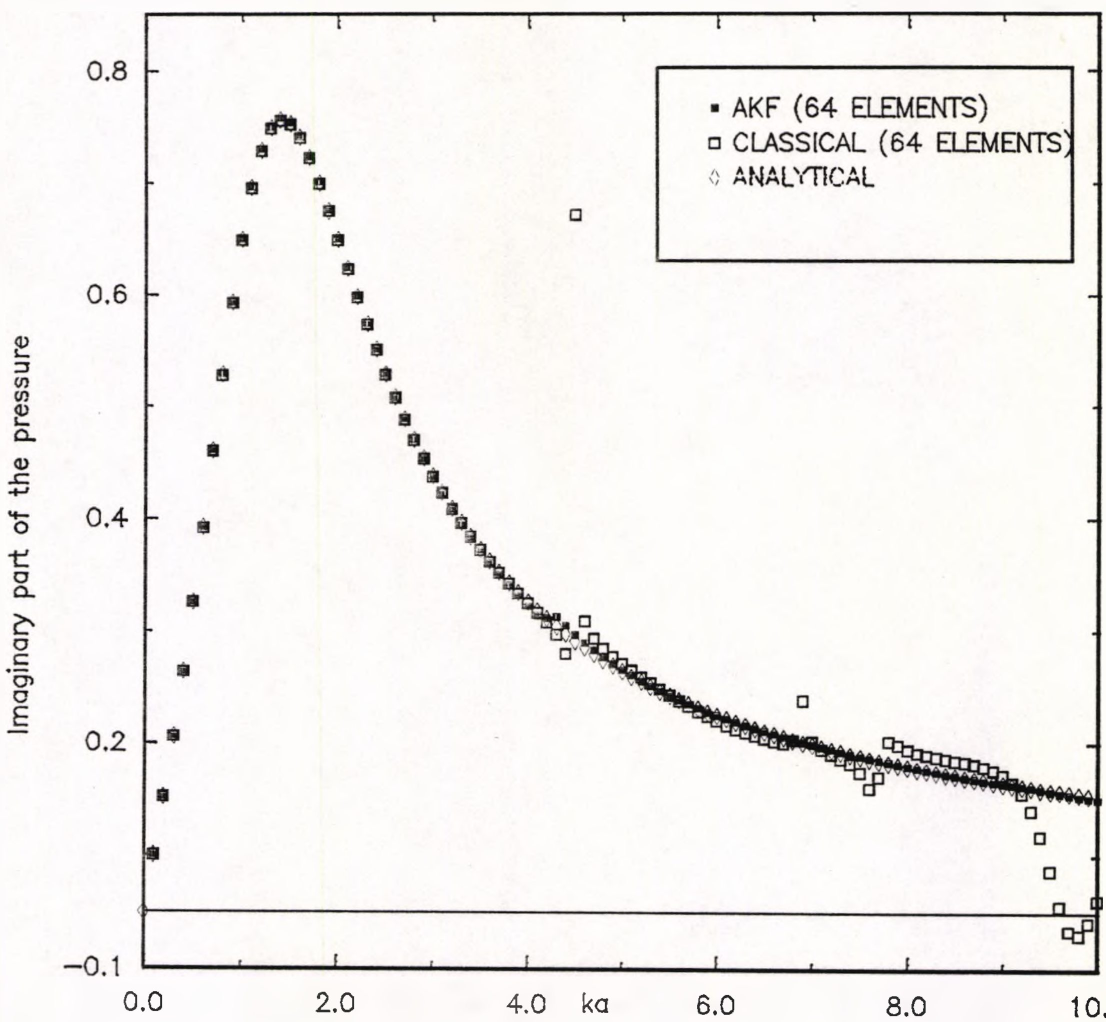

FIG.21. Imaginary part of the pressure on the surface of an oscillating sphere of radius=1. . The sphere is oscillating with a radial velocity of cos ϑ where ϑ is the polar angle. Oscillation of the sphere is in the direction ϑ=0.

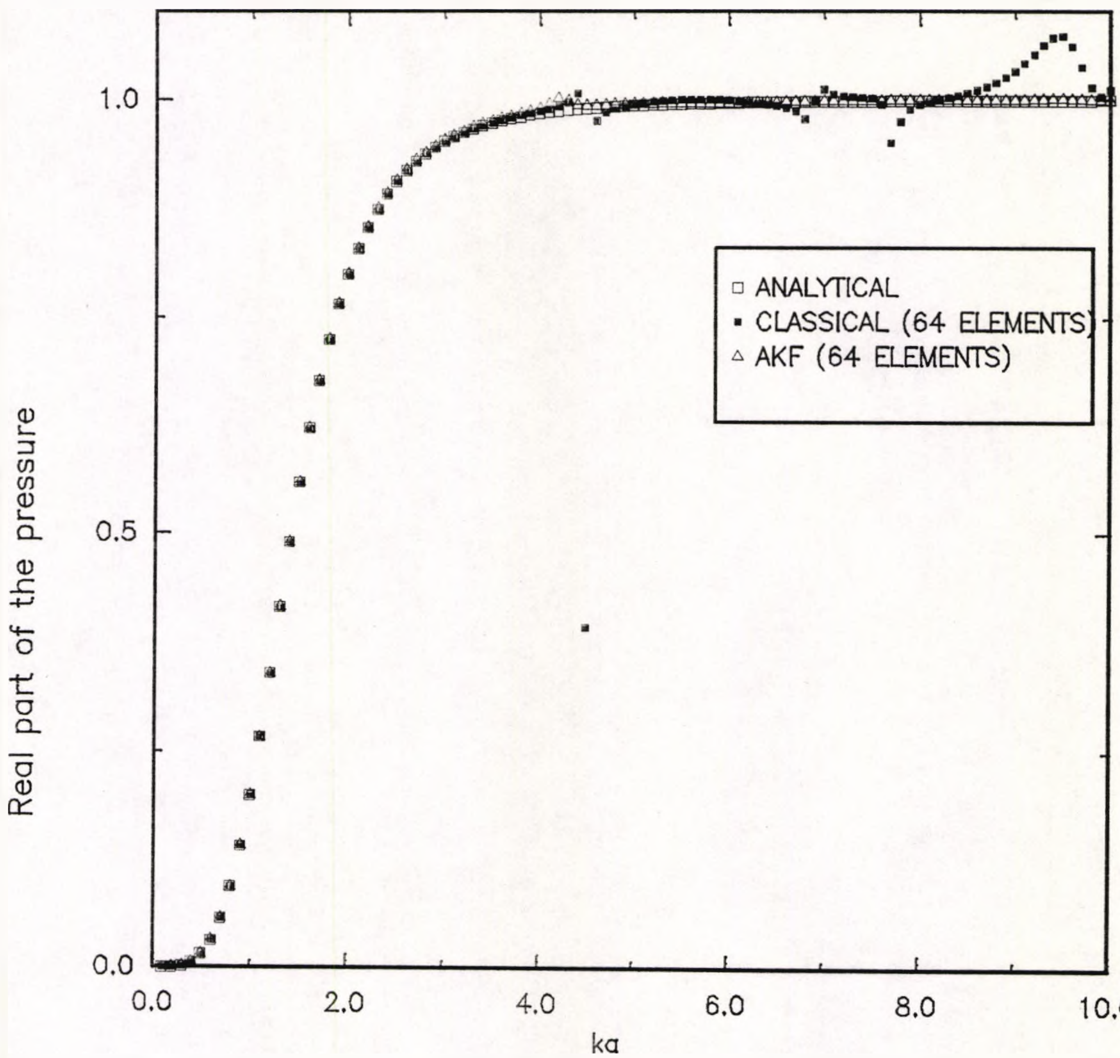

FIG.22. Real part of the pressure on the surface of an oscillating sphere of radius=1. .The sphere is oscillating with a radial velocity of $\cos\vartheta$ where ϑ is the polar angle. Oscillation is in the direction, $\vartheta=0$

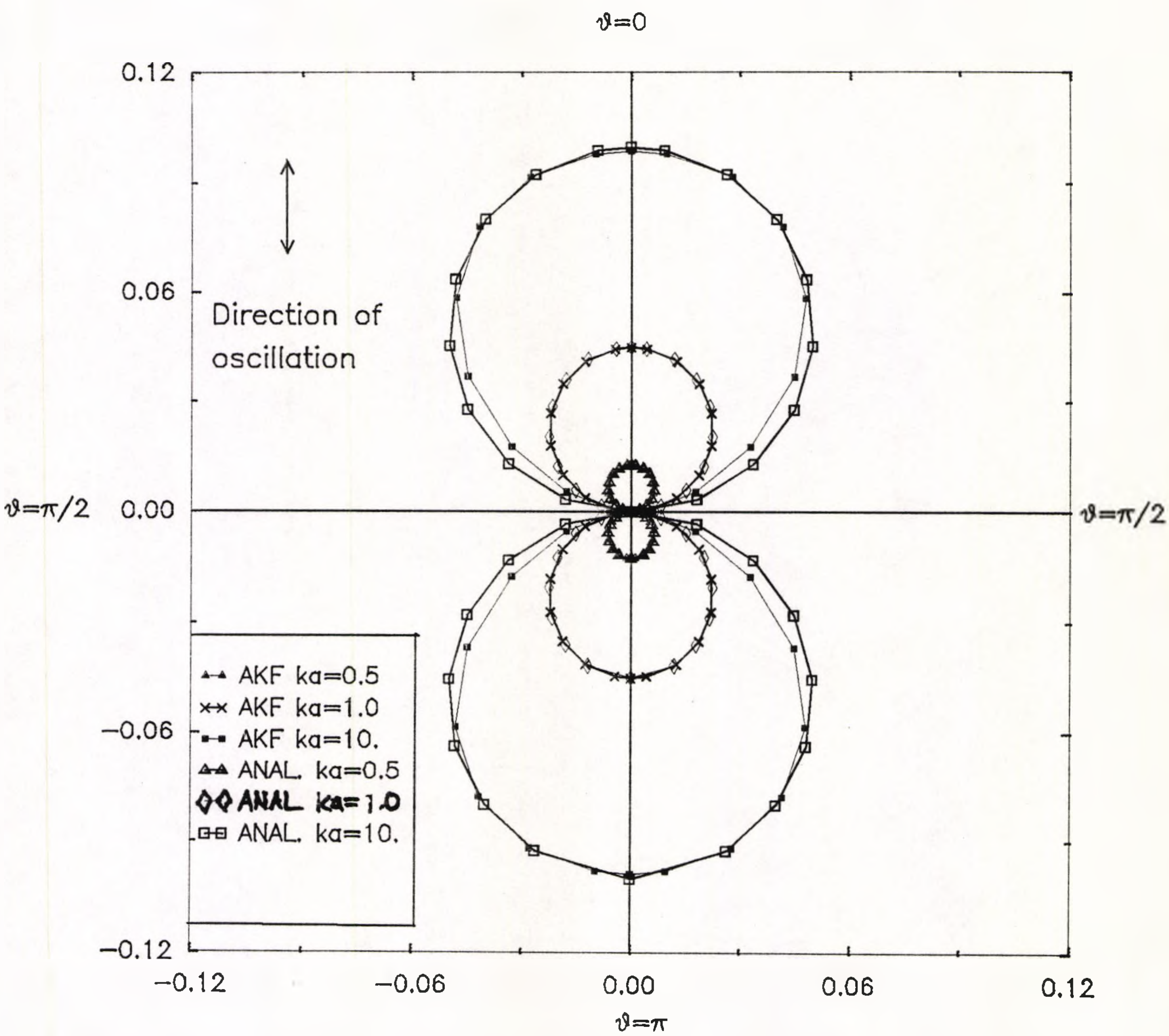

FIG.23. Normalised far field pressure distribution for oscillating sphere at r=10a for wave–numbers ka=0.5,1.0,10. .Here a is the radius of the sphere and ϑ is the polar angle. 64 Elements are considered.

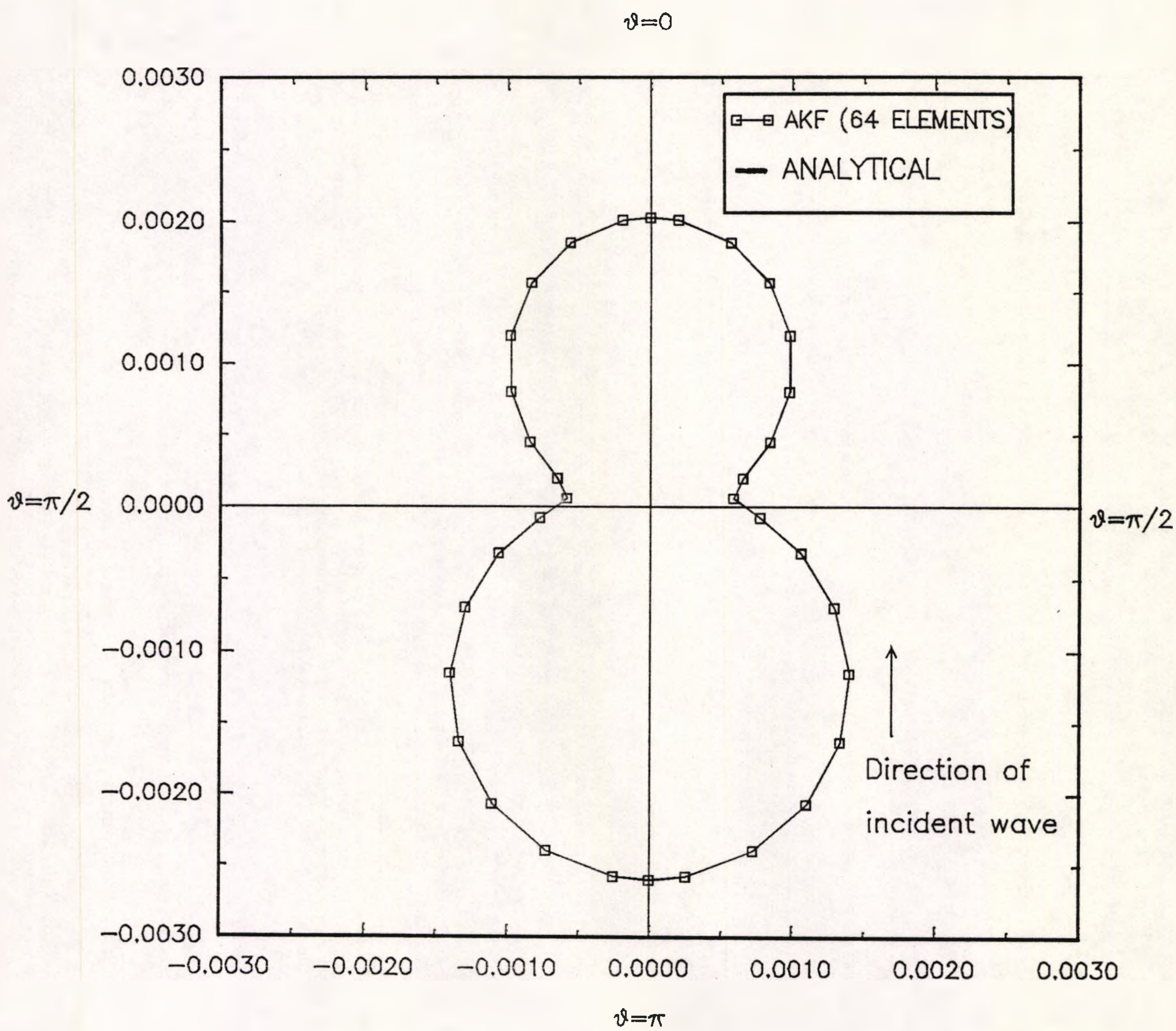

FIG..24. Angular distribution of the pressure amplitude of the scattered sound per unit incident pressure, for ka=0.1, r=5a, where a is the radius of the sphere.

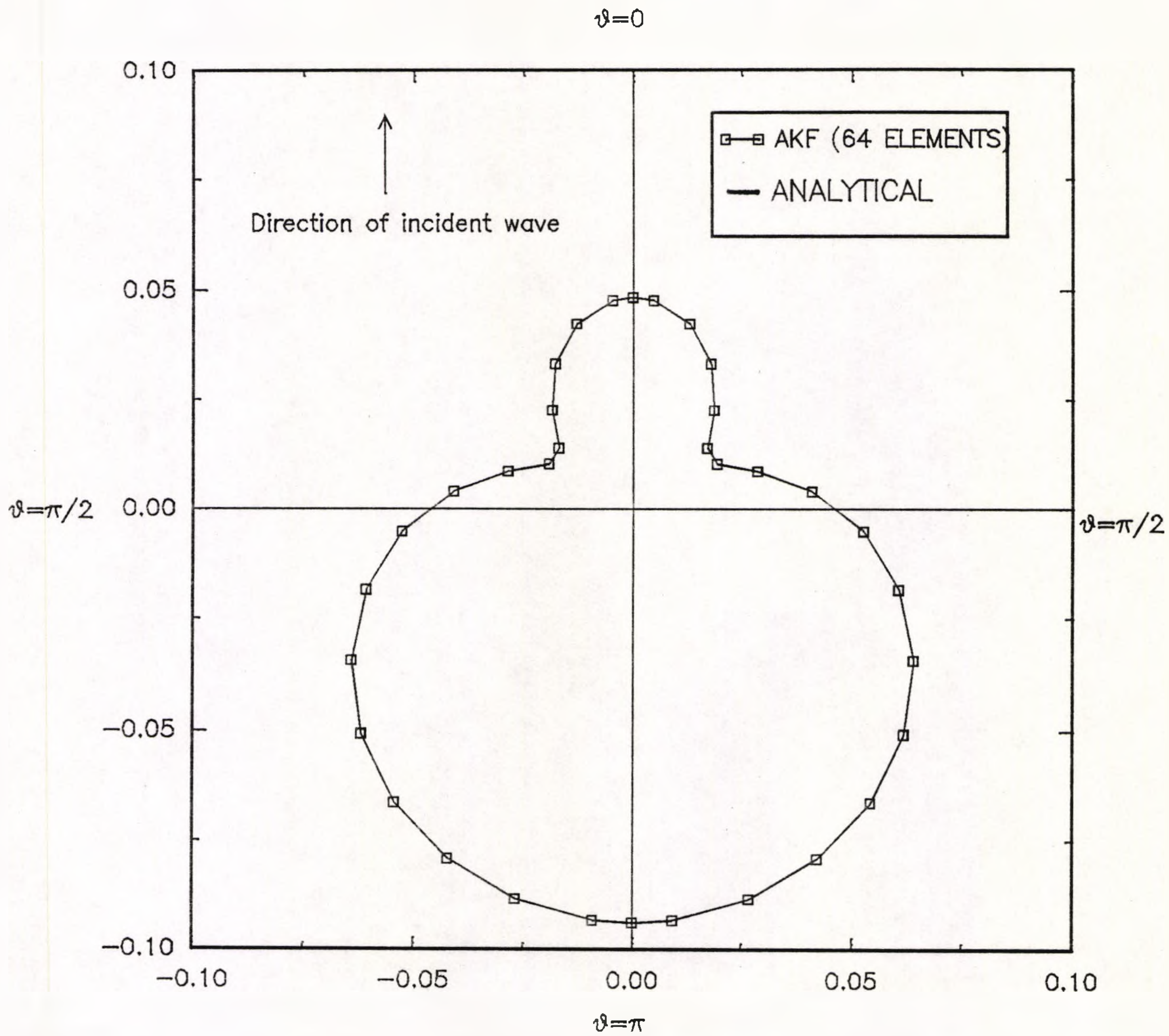

FIG.25. Angular distribution of the pressure amplitude of the scattered sound per unit incident pressure for $ka=1.0$, $r=5a$, where a is the radius of the sphere.

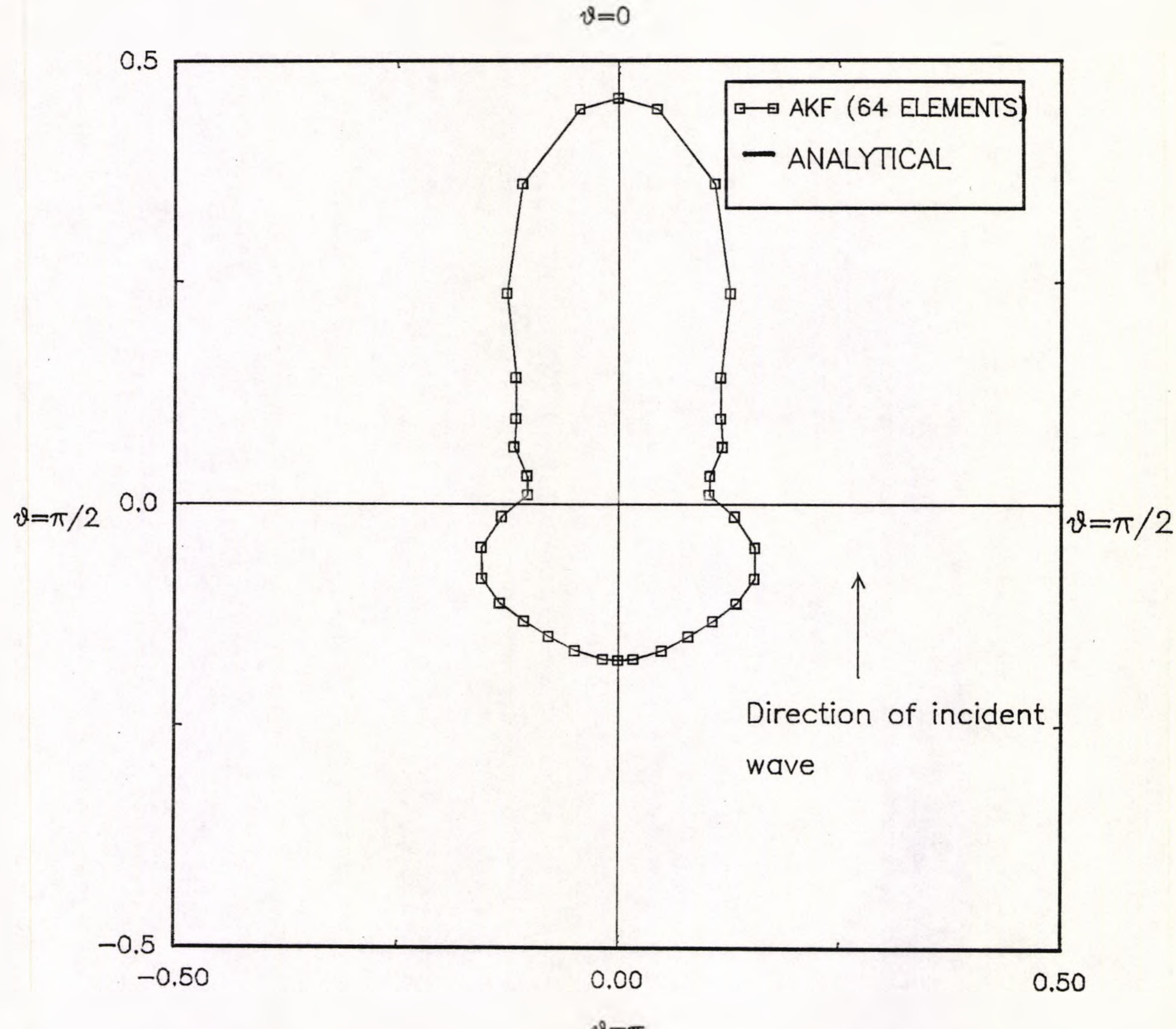

FIG.26. Angular distribution of the pressure amplitude of the scattered sound per unit incident pressure for ka= π ,r=3a, where a is the radius of the sphere.

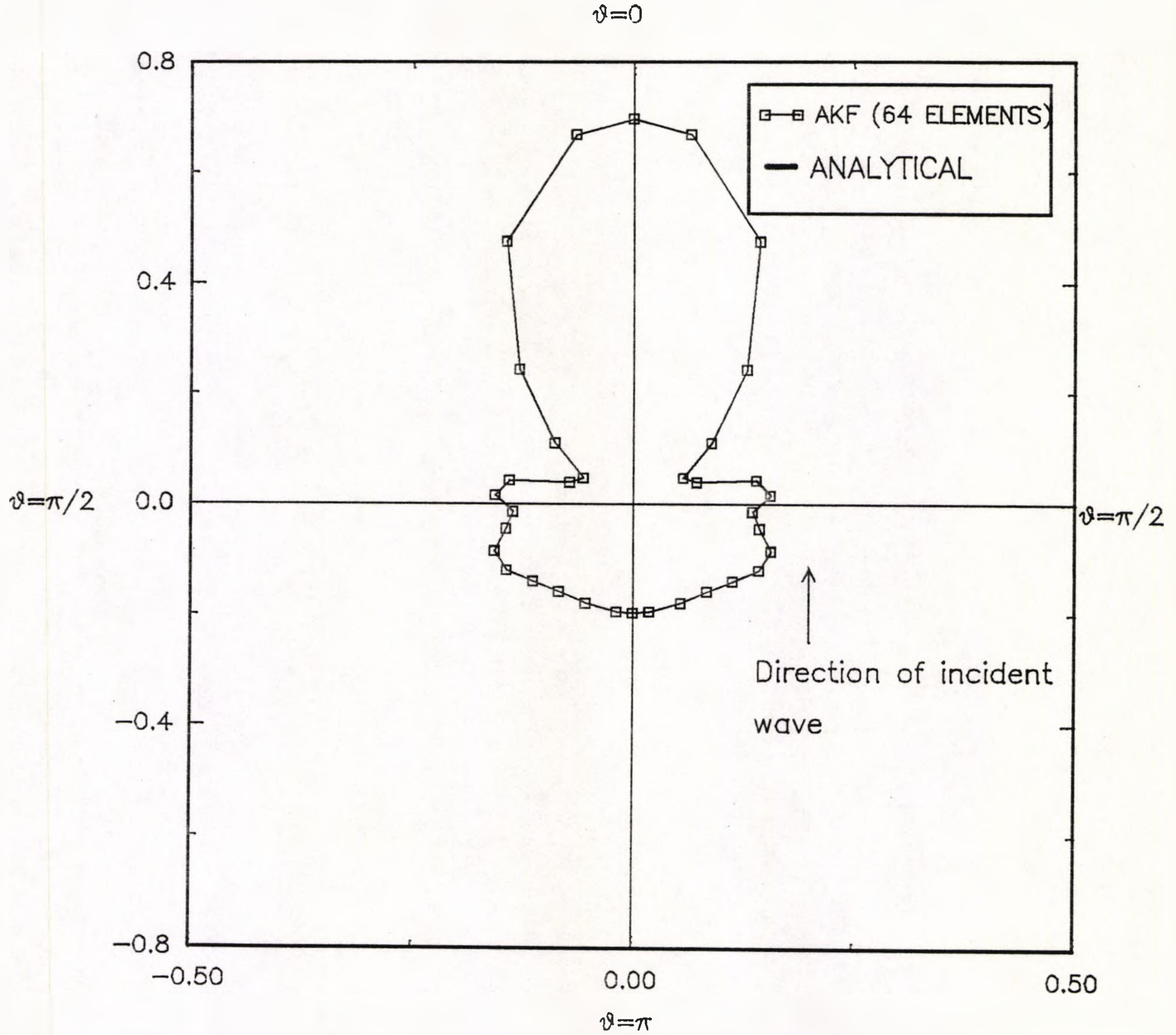

FIG.27. Angular distribution of the pressure amplitude of the scattered sound per unit incident pressure, for ka=4.493, r=3a, where a is the radius of the sphere.

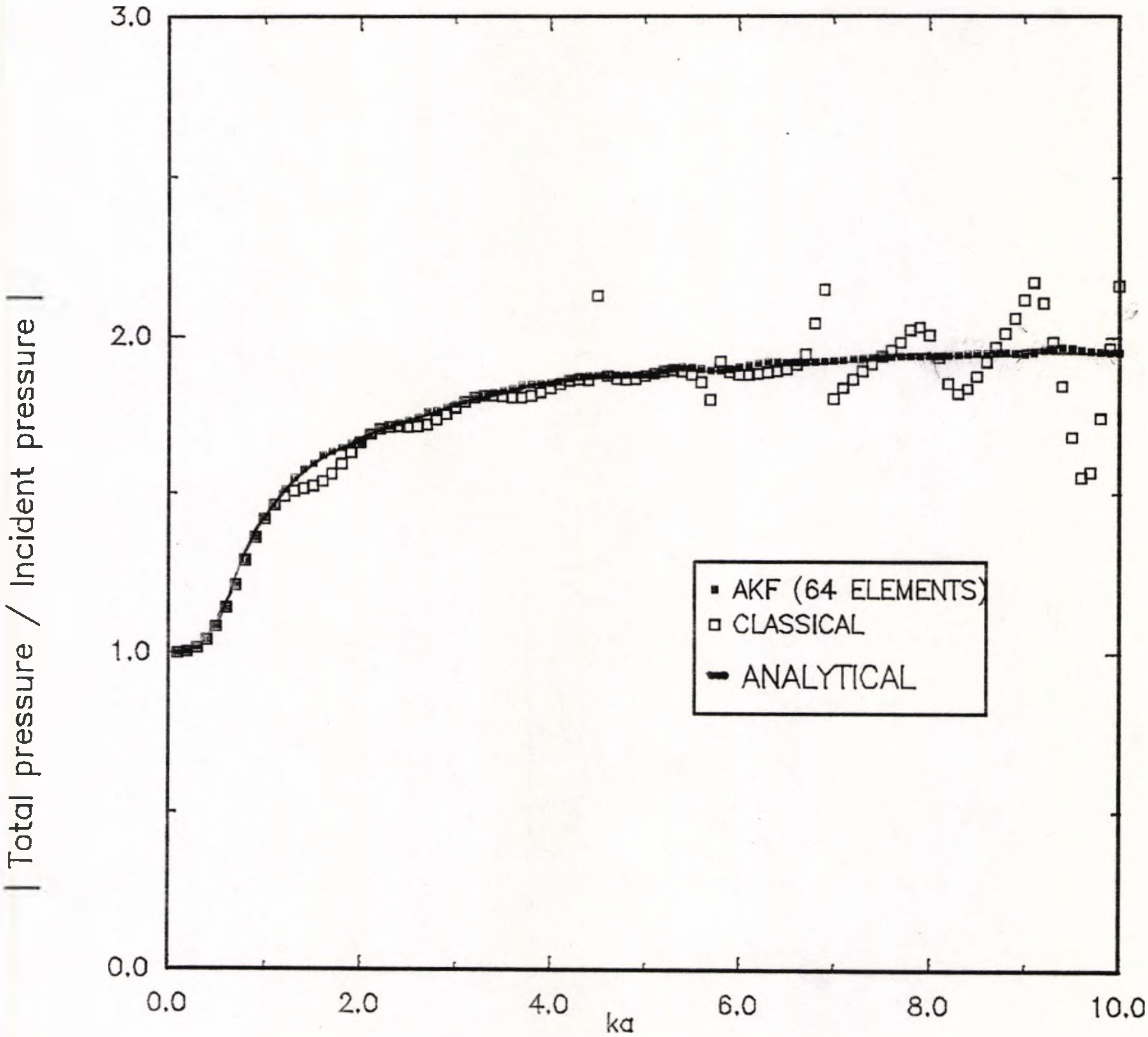

FIG.28. Amplitude of the total acoustic pressure per unit incident pressure on the surface of a sphere at $\vartheta=\pi$. This is the nearest point on the sphere facing the incident wave.

APPENDICES
&
BIBLIOGRAPHY

APPENDICES

I. Interior wave-functions generated by simple-layer potentials

Sphere of radius a is considered. The field point $p = (0,0,r)$ and the source points using spherical coordinates are

$$q = (a\sin\theta\cos\psi,\ a\sin\theta\sin\psi,\ a\cos\theta) \quad ; \quad 0 \le \theta \le \pi \quad , \quad 0 \le \psi \le 2\pi \ ,$$

so that

$$\rho = |p-q| = (a^2+r^2-2ar\cos\theta)^{1/2} \ .$$

For $r<a$, i.e. p inside the sphere and uniform source density $\sigma = \sigma_o$

$$\phi(p) = \int_{\partial B} \frac{e^{-ik|p-q|}}{|p-q|}\sigma_o dq$$

$$= \sigma_o \int_0^{2\pi}\int_0^{\pi} \frac{e^{-ik\rho}}{\rho} a^2 \sin\theta \, d\theta \, d\psi$$

$$= \frac{a\sigma_o}{r}\int_0^{2\pi}\int_{a-r}^{a+r} e^{-ik\rho} d\rho \, d\psi$$

$$= -\frac{2\pi\sigma_o a}{ikr}[e^{-ik(a+r)} - e^{-ik(a-r)}]$$

$$= 2\pi\sigma_o a e^{-ika}\frac{[e^{ikr} - e^{-ikr}]}{ikr} \ .$$

Since

$$e^{ikr} - e^{-ikr} = 2i\sin kr \ ,$$

therefore

$$\phi(p) = 4\pi\sigma_o a e^{-ika}\frac{\sin kr}{kr} \quad ; \quad |p| = r < a \ .$$

Similar treatment for the case $r > a$ gives the result as mentioned in chapter.5.

II. Normal derivative properties of the simple-layer integral

Assuming a uniform source density $\sigma = \sigma_o$ over the surface ∂B of a sphere of radius a and taking the derivative of g_k at $p=(0,0,r)_{r=a}$ in the normal direction pointing into the region exterior to ∂B , we get

$$\int_{\partial B}\frac{\partial g_k}{\partial n_p}(p,q)\sigma_o dq = \sigma_o\int_{\partial B}\frac{\partial g_k}{\partial r}(p,q)\Big|_{r=a} dq \quad ; \quad p\in\partial B \ ,$$

where the expression for q is as before. This integral now takes the form

$$-\sigma_o\int_0^{2\pi}\int_0^{\pi}\frac{[2ika\sin^2(\theta/2)+\sin(\theta/2)]}{4a^2\sin^2(\theta/2)}e^{-2ika\sin(\theta/2)}a^2\sin\theta\, d\theta\, d\psi$$

$$= -\sigma_o\int_0^{2\pi}\int_0^{\pi}\Big[\frac{ik}{2a}+\frac{1}{4a^2\sin(\theta/2)}\Big]e^{-2ika\sin(\theta/2)}a^2\sin\theta\, d\theta\, d\psi \qquad .$$

Substituting

$$u = \sin(\theta/2) \quad , \quad 2du = \cos(\theta/2)d\theta$$

gives

$$-8\pi\sigma_o a^2\int_0^1[\frac{ik}{2a} + \frac{1}{4a^2u}]ue^{-2ikau}du$$

$$= -2\pi\sigma_o\int_0^1[2ikau + 1]e^{-2ikau}du$$

$$= -2\pi\sigma_o[(-u - \frac{1}{ika})e^{-2ikau}]_{u=0}^{u=1}$$

$$= -4\pi\sigma_o\frac{\sin ka}{ka}e^{-ika} + 2\pi e^{-2ika} \quad .$$

But

$$\frac{\partial}{\partial n_p}\int_{\partial B}g_k(p,q)\sigma_o dq = \frac{\partial}{\partial r}[4\pi a\sigma_o \sin ka\frac{e^{-ikr}}{kr}]_{r=a} \quad ; \quad |p|=r\geq a$$

$$= 4\pi\sigma_0 e^{-ika}\frac{\sin ka}{ka}[-ika - 1] \quad ,$$

showing that

$$\frac{\partial}{\partial n_p}\int_{\partial B}g_k(p,q)\sigma_o dq - \int_{\partial B}\frac{\partial g_k}{\partial n_p}(p,q)\sigma_o dq = -2\pi\sigma_o \quad ; \quad p\in\partial B \quad ,$$

as is expected. A similar treatment may be applied to the approach of the normal derivative from the interior which would show the expected jump.

III. Exterior wave-functions generated by double-layer potentials

Without loss of generality, let $p=(0,0,r)$, $r>a$, be a general point in the exterior of ∂B , the surface of a sphere of radius a. Assume a uniform source distribution $\mu=\mu_o$ over ∂B and take

$$q = (R\sin\theta\cos\psi, R\sin\theta\sin\psi, R\cos\theta)_{R=a} \quad ; \quad 0 \le \theta \le \pi \quad , \quad 0 \le \psi \le 2\pi \quad ,$$

at a general point on the surface, so that

$$\rho = |p-q| = (r^2 + R^2 - 2rR\cos\theta)^{1/2}{}_{R=a} \quad ; \quad \rho = \rho(R,\theta) \ , \ r>a \ .$$

If so, then

$$\phi(p) = \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu_o dq = \mu_o \int_0^{2\pi}\int_0^{\pi} \frac{\partial}{\partial R}\Big[\frac{e^{-ik\rho}}{\rho}\Big]_{R=a} a^2 \sin\theta\, d\theta\, d\psi \tag{*}$$

$$= \mu_o \int_0^{2\pi}\int_0^{\pi} \frac{[-ik\rho - 1]}{\rho^3} e^{-ik\rho}[a - r\cos\theta]a^2\sin\theta\, d\theta\, d\psi \quad .$$

Putting $R=a$ in ρ and substituting the following in (*), we get

$$\rho = [r^2 + a^2 - 2ra\cos\theta]^{1/2}$$

$$\rho d\rho = ra\sin\theta\, d\theta$$

$$\frac{\rho^2 + a^2 - r^2}{2a} = a - r\cos\theta \quad ,$$

which gives

$$\phi(p) = \frac{\pi}{r}\int_{(r-a)}^{(r+a)} \frac{[-ik\rho - 1]}{\rho^2} e^{-ik\rho}[\rho^2 + a^2 - r^2]d\rho$$

$$= \frac{\pi}{r}\int_{(r-a)}^{(r+a)} \{[-ik\rho - 1]e^{-ik\rho} + (a^2 - r^2)[-\frac{ik}{\rho} - \frac{1}{\rho^2}]e^{-ik\rho}$$

$$= \frac{\pi}{r}\Big[(\rho + \frac{2}{ik} + \frac{a^2 - r^2}{\rho})e^{-ik\rho}\Big]_{\rho=r-a}^{\rho=r+a}$$

$$= 4\pi\,\mu_o[ka\cos ka - \sin ka]\frac{e^{-ikr}}{kr} \quad ; \quad |p| = r > a \ .$$

Similar analysis for the case $r<a$ gives the result as stated in chapter 5.

IV. Surface values of the double-layer potential

Here $p=(0,0,a)$, with q as before and

$$\rho \;=\; |p-q| \;=\; (a^2+R^2-2aR\cos\theta)^{1/2}{}_{R=a} \quad ; \quad \rho=\rho(R,\theta) \;.$$

So we have

$$\int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu_o\,dq \quad ; \quad p\in\partial B$$

$$= \mu_o \int_0^{2\pi}\int_0^{\pi} \frac{\partial}{\partial\rho}\Big[\frac{e^{-ik\rho}}{\rho}\Big]\frac{\partial\rho}{\partial R}_{R=a} \quad a^2\sin\theta\, d\theta\, d\psi$$

$$= \mu_o a^3 \int_0^{2\pi}\int_0^{\pi} \frac{[-ik\rho-1]}{\rho^3} e^{-ik\rho}[1-\cos\theta]\sin\theta\, d\theta\, d\psi \;;\; \rho=\rho(a,\theta)$$

Now writing

$$\rho=2a\sin(\theta/2) \quad ; \quad \frac{\rho^2}{2a^2} = 1-\cos\theta \qquad ,$$

$$d\rho = a\cos(\theta/2)\,d\theta \qquad ,$$

the above integral becomes

$$2\pi\,\mu_o a^3 \int_0^{2a} \frac{[-ik\rho-1]}{\rho^3} e^{-ik\rho} \frac{\rho^2}{2a^2}\,\frac{\rho}{a}\,\frac{d\rho}{a}$$

$$= \frac{\pi\mu_o}{a}\int_0^{2a}[-ik\rho-1]e^{-ik\rho}\,d\rho \quad .$$

Hence

$$\int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu_o\,dq \;=\; \frac{\pi\,\mu_o}{a}\Big[\{\rho + \frac{2}{ik}\}e^{-ik\rho}\Big]_{\rho=0}^{\rho=2a}$$

$$= \frac{\pi\,\mu_o}{a}\Big[\{2a + \frac{2}{ik}\}e^{-2ika} - \frac{2}{ik}\Big]$$

$$= 2\pi\,\mu_o e^{-2ika} - 4\pi\,\mu_o \frac{\sin ka}{ka} e^{-ika}\;.$$

We also have the following:

$$\lim_{r\to a}\int_{\partial B} \frac{\partial}{\partial n_q}\Big[\frac{e^{-ik|p^+-q|}}{|p^+-q|}\Big]\mu_o\,dq\;; \qquad |p^+| = r > a$$

$$= 4\pi\,\mu_o[ka\cos ka - \sin ka]\frac{e^{-ika}}{ka} \qquad ,$$

so demonstrating the jump property

$$\lim_{p^+\to p}\int_{\partial B} \frac{\partial g_k}{\partial n_q}(p^+,q)\,\mu_o\,dq \;-\; \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu_o\,dq \;=\; 2\pi\,\mu_o\;; \qquad p\in\partial B\;.$$

Similar treatment for the case $r<a$ yields the expected result as stated in chapter 5.

V. The interior/exterior correspondence with respect to a sphere.

Exterior Neumann Problem (ENP) requires the construction of an exterior wave-function, usually represented by a simple-layer potential generated by a continuous source distribution over the boundary ∂B , i.e.

$$\phi(p) = \int_{\partial B} g_k(p,q)\,\sigma(q)\,dq \quad ; \qquad p \in B^+ \cup \partial B$$

$$\frac{\partial\phi}{\partial n}(p) = -2\pi\,\sigma(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_p}(p,q)\,\sigma(q)\,dq \quad ; \quad p \in \partial B\ .$$

Normal vectors here and throughout are pointing into the region concerned. Applying the above normal derivative equation to a uniform source distribution $\sigma = \sigma_o$ on a sphere of radius a, we get from **(II)**

ENP $$\frac{\partial\phi}{\partial n}(p) = 4\pi\,\sigma_o \frac{\sin ka}{ka} e^{-ika}[-ika-1] \quad ; \quad |p| = a\ . \qquad \text{(i)}$$

The corresponding Interior Dirichlet Problem (IDP) requires the construction of a wave-function, usually represented by a double-layer potential generated by a continuous source distribution on the same boundary, i.e.

$$\phi(p) = \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq \qquad ; \qquad p \in B^-$$

$$\phi(p) = -2\pi\,\mu(p) + \int_{\partial B} \frac{\partial g_k}{\partial n_q}(p,q)\,\mu(q)\,dq \quad ; \qquad p \in \partial B\ .$$

Applying the above boundary relation to a uniform source distribution $\mu = \mu_o$ on the sphere, we get from **(IV)**

IDP $$\phi(p) = 4\pi\,\mu_o \frac{\sin ka}{ka} e^{-ika}[ika+1] \qquad ; \qquad |p| = a\ . \qquad \text{(ii)}$$

Similar treatments for the Exterior Dirichlet Problem (EDP) and the corresponding Interior Neumann Problem (INP) yield the following relations respectively

$$EDP \qquad \phi(p) = 4\pi\mu_o[ka\cos ka - \sin ka]\frac{e^{-ika}}{ka} \quad ; \quad |p| = a \ , \quad \text{(iii)}$$

$$INP \qquad \frac{\partial\phi}{\partial n}(p) = 4\pi\sigma_o[\sin ka - ka\cos ka]\frac{e^{-ika}}{ka} \quad ; \quad |p| = a \ . \quad \text{(iv)}$$

Notice that the equations (i) and (ii) are identical, as are the equations (iii) and (iv), which clearly show that the corresponding pairs of homogeneous boundary equations are equivalent. And therefore that a breakdown of the one at any characteristic frequency implies a breakdown of the other.

VI. An alternative view of the N_k operator for a flat plate.

It has been shown (see chapter 6) that

$$\int_{\partial B}\frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq = -2\pi ik - \int_0^{2\pi}\frac{e^{-ik\rho(\theta)}}{\rho(\theta)}d\theta \quad ; \quad p,q\in\partial B \ , \qquad (*)$$

where $\rho=\rho(\theta)$ is the polar equation, relative to the singular point $p=(0,0,0)$, of the contour bounding a flat element ∂B . If the element becomes a disc of radius R then the above integral evaluates to

$$-2\pi[ik + \frac{e^{-ikR}}{R}] \ . \qquad (**)$$

This may also be obtained by considering the integral (*) defined on a spherical cap Ω_β of extent defined by a polar angle β , i.e.

$$\Omega_\beta = \{(a,\theta,\psi) \ : \quad 0\le\theta\le\beta \ , \ 0\le\psi\le 2\pi\} \ .$$

Writing $p=(0,0,z)$ and q as before, we obtain

$$q = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)_{r=a}$$

$$\varrho = |p-q| = (a^2+z^2-2az\cos\theta)^{1/2} \qquad ,$$

and therefore

$$\int_{\Omega} \frac{\partial^2 g_k}{\partial n_p \partial n_q}(p,q)\,dq$$

$$= \int_0^{2\pi}\int_0^{\beta} [\frac{ik\cos\theta}{\varrho^2} + \frac{\cos\theta - k^2\tau}{\varrho^3} + \frac{3ik\tau}{\varrho^4} + \frac{3\tau}{\varrho^5}]e^{-ik\varrho}a^2\sin\theta\, d\theta\, d\phi \; ; \qquad (***)$$

$$\tau = (a-z\cos\theta)(z-a\cos\theta) \qquad .$$

Substitution of

$$\cos\theta = \frac{a^2+z^2-\varrho^2}{2az} \quad ; \quad A(\beta) = (a^2+z^2-2az\cos\beta)^{1/2}{}_{z=a}$$

$$\sin\theta\, d\theta = \frac{\varrho\, d\varrho}{az}$$

$$\tau = \frac{\varrho^4-(a^2-z^2)^2}{4az} \quad ,$$

into the integral yields

$$\frac{\pi}{2z^2}\int_{z-a}^{A(\beta)} \{2(a^2+z^2)[\frac{ik}{\varrho}+\frac{1}{\varrho^2}] + (a^2-z^2)^2[\frac{k^2}{\varrho^2}-\frac{3ik}{\varrho^3}-\frac{3}{\varrho^4}]$$

$$+ [1+ik\varrho-k^2\varrho^2]\}e^{-ik\varrho}d\varrho \quad ; \quad z>a \quad ,$$

now integrating, putting in the limit values and letting $z=a$, we obtain

$$\pi\{\frac{-2e^{-ikA(\beta)}}{A(\beta)} - \frac{1}{2a^2}[\frac{4}{ik}+3A(\beta)+ikA(\beta)^2]e^{-ikA(\beta)} - 2ik+\frac{2}{ika^2}\} \ .$$

Note that the a^{-2} terms cancel each other out, so the above takes the form

$$-2\pi\{ik + \frac{e^{-ikA(\beta)}}{A(\beta)}\} + O(\beta) \ ,$$

therefore, as $\beta\to 0$ i.e as the cap reduces to an approximate flat disc, the above behaves as

$$-2\pi[ik+\frac{e^{-ikA(\beta)}}{A(\beta)}] \quad ; \quad A(\beta) = 2a\sin(\beta/2)\simeq a\beta \ .$$

This agrees with the result in (**), since for a given a and for $\beta\approx 0$ we may write $A(\beta)=R$.

VII. An alternative argument for uniqueness of AKF

Recall the AKF formulation in the form

$$\phi(p) = [L_k\sigma](p) + \eta\int_{\partial B^*}\frac{\partial g_k}{\partial n_{q^*}}(p,q^*)\sigma(q^*)dq^* \quad ; \quad p\in B^+ \cup \partial B \ .$$

Applying the Neumann boundary condition, the homogeneous boundary equation takes the form

$$0 = -2\pi\sigma(p)+M_k^T\sigma(p)+\eta\int_{\partial B^*}\frac{\partial^2}{\partial n_p\partial n_{q^*}}(p,q^*)\sigma(q^*)dq^* \quad ; \quad p\in\partial B \ .$$

The homogeneous adjoint equation becomes

$$0 = -2\pi\,\mu(p) + [M_k\mu](p) + \eta \int_{\partial B} \frac{\partial^2}{\partial n_p \cdot \partial n_q}(q,p^*)\,\mu(q)\,dq$$
,

$$0 = -2\pi\,\mu(p) + [M_k\mu](p) + \eta\,[N_k\mu](p^*) \quad , p \in \partial B \quad , p^* = \vartheta p$$

where $0 < \vartheta < 1$. Suppose there is a non-trivial solution $\mu = \hat{\mu}$ of the above equation. Using this source density, generate a double-layer potential

$$W(p) = [M_k\hat{\mu}](p) \qquad ; \; p \in B^- \; ,$$

which satisfies

$$W(p) + \eta \frac{\partial W}{\partial n}(p^*) = 0 \quad ; \; p \in \partial B \quad , p^* = \vartheta p \; .$$

By letting p^* approach p, we find for the same $\hat{\mu}$ that W also satisfies

$$W(p) + \eta \frac{\partial W}{\partial n}(p) = 0 \quad ; \; p \in \partial B \; ,$$

as $\partial W / \partial n$ is continuous across the boundary as it is the normal derivative of a double-layer potential. Now applying the Green's identity on W and W^* analogous to the uniqueness proof in chapter 8 of Kussmaul formulation one can show that $\hat{\mu}$ can only be trivial.

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