Dynamic Stochastic Control Applications in Finance and Insurance

by

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Declaration

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Abstract

Control theory has gained a widespread use in almost every area of decision making problems. In this thesis, we seek to construct a premium setting strategy and an asset allocation strategy of a non-life insurance company whose goal is to maximise a metric of her utility function.

As insurance companies do not have perfect insight into future market and cannot assume any given scenario with certainty, stochasticity is introduced to model the market conditions and the risk processes that the running of the insurance business is subject to.

The problem is formulated as a continuous time and continuous space control problem where the state process is controlled continuously in a way to achieve the target. Bellman optimality principle in a stochastic environment is used to reduce the continuous time decision problem into a fixed point decision problem under the umbrella of Hamilton-Jacobi-Bellman equation.

We also consider the pricing of financial derivative products written on catastrophe losses. Since the market of catastrophe insurance is incomplete, we make use of the concept of indifference of utility theory of a market participant to derive the so-called affordable price.

List of Symbols

- β discount rate
- α_t controlled variable at time t
- $\mathcal{A}$ set of dmissible set, control space
- $\mathbf{X}_t$ multidimentional state variables at time t
- $J(t, \mathbf{x})$ Cost function at time t
- $V(t, \mathbf{x})$ Value function at time t
- B_t standard Brownian motion at time t
- μ drift of a Brownian motion
- σ volatility of a Brownian motion
- τ stopping time
- $\Pi(t)$ Space-time Poisson process at time t
- $\aleph(dt, dy)$ Poisson measure
- π Expected claim size
- N_t Poisson process with rate λ
- $\hat{p}_t$ Optimal premium
- G Growth rate of the exposure
- W_t Wealth process at time t
- q_t Exposures at time t

- Θ_t Optimal risky asset allocation at time t
- $\psi(w, q)$ Survival probability with inial wealth w and initial exposures q
- $\phi(w, q)$ Ruin probability with inial wealth w and initial exposures q
- R_t Risk process at time t
- $U(w)$ Utility of wealth w
- L_t Cumulative loss index at time t
- $\mathcal{F}_t$ Filtration at time t
- $\mathbb{E}\Big[X \mid \mathcal{F}_t\Big]$ conditional expectation of a random variable X given the filtration at time t

Contents

Chapter 1

Introduction

1.1 Background

The aim of the present thesis is to apply a variety of techniques and tools to solve optimal dynamic decision problems related to finance and insurance. Many optimisation problems are formulated in terms of continuous time and or continuous space control problems where a variable (*the control variable*) is controlled in such a way to achieve a specific goal, usually maximising or minimising a function. As opposed to traditional actuarial pricing and financial optimisation methods which deal with one period optimisation decisions, the dynamic stochastic control produces for an optimisation starting at a time t, the decision to be taken at each time $u > t$.

Until recently, the mean-variance analysis developed by Markowitz (1952), Markowitz (1959) and Tobin (1958) was the main approach adopted in finance. In the mean-variance framework, an efficient frontier is constructed on which the investor's portfolio allocation decision is based upon her utility function. The mean-variance approach for the case where the investor can allocate resources to a risk free asset and a set of risky assets yields two-fund separation result. Due to its simplicity to implement and the closed form solution it affords, this optimisation approach is widely used. However, the mean-variance optimisation framework suffers a heavy shortcoming in that it is cast in a one period horizon maximisation setting. When decisions have to be taken over time in order to rebalance the investment portfolio, a different framework is required to tackle the problem. We consider therefore in this thesis a framework in which the investor can sequentially make decisions given the current state of the portfolio. The mathematics needed comes under the umbrella of stochastic control theory whereby the risky assets are modelled as Markov processes and the optimisation problem is to maximise an expected future metric of the portfolio value. Usually, the metric is taken to be a utility function of wealth. An example is the optimal investment problem of Merton (1969) where an agent investing some part of her wealth in risky and non risky assets whilst consuming the rest aims to maximise her expected utility (a function of his consumption and wealth).

In the insurance field, Actuaries have long considered the premium setting process in a one period setting. The pure premium is evaluated as the expectation of the claim where usually a loading factor, dependent on the risk parameter of the claim size is added to form the true premium. The introduction of control theory in the insurance literature started with de Finetti (1957) who constructed a predefined level of control action for the surplus reserve of an insurance company; this was extended by Borch (1967) who proposed a more general solution of the problem. Balzer and Benjamin (1980) presented a clearer application of control theory in insurance pricing by proposing a model of the form

$$p_t = \theta \mathbb{E}\left[C_t\right] - \epsilon S_{t-1} \tag{1.1}$$

where C_t, S_t are respectively the claims and the surplus value at the end of year t, θ and ϵ are positive constants with $\theta > 1$, $\epsilon \in [0, 1]$ and p_t is the premium. This model has been significantly improved by Martin-Löf (1983) who approached the problem in a stochastic fashion by using the dynamic programming techniques introduced by Bellman (1957). From there, dynamic programming techniques have gained enormous interest in the control theory literature and different settings have been investigated including state processes involving diffusion and jumps.

Common to all the control problems is the need to make sequential decisions i.e. one has to make decisions continuously up to a certain time horizon which can be known in advance, be infinite or be random. The Bellman principle, which is the key principle governing dynamic programming, states that the optimal decision between today and a future time is equivalent to the optimal decision from today to tomorrow given that the decision from tomorrow to the future time is optimally taken. As simple as it is, the Bellman optimality principle is the corner stone of the dynamic control theory in finance and insurance. The stochastic control methodology often reduces the sequential decision taking problem to a partial differential equation (PDE) which is not usually analytically solvable. The methodology involves modelling the state variable as a Markov process whose evolution is governed by the control variable. In the case of a state variable following a diffusion process, the stochastic control methodology reduces to a Hamilton-Jacobi-Bellman (HJB) equation expressed as a second order partial differential equation whereas in the case of a state variable following a jump process, which is especially relevant to insurance applications, the problem reduces to a partial integro-differential equations (PIDE). The solutions of the PDE or the PIDE if they exist are determined by the boundary conditions imposed by the optimisation problem. Due to the difficulty in getting analytical solutions of the differential equations (DE), numerical based solutions are usually called for.

Different techniques have been elaborated by many authors. One approach is to solve the PDE using numerical methods by discretising the different orders of the PDE equations.

This usually results in creating artificial boundary conditions since the natural boundary conditions do not always provide a way to the solution.
The *Markov chain approximation* method developed by Kushner and Dupuis (1993) has for long been the most parsimonious method for solving systematically stochastic control problems, see Kushner (1977), Kushner (1984) and Kushner (1990). The idea inherent in the approximation relies on the local properties of the chain encapsulated by the *local consistency condition* which merely states that the mean and the second moments of the increment of the discretised chain must tend to the mean and second moment of the increment of the original state variable as the discretisation method becomes finer. An application of the method to Merton's problem can be found in Munk (2003).

Most of the numerical solutions methods including the Markov chain approximation and grid based methods in general suffer from the curse of dimensionality. This mean that the time and space required to find a solution grow exponentially as the dimension increases. Another grid based method addressing the exponential curse of high dimension is the *Quantization algorithm* method for multi-dimensional stochastic control problems. This method projects a time-discretised version of the continuous time stochastic process onto an optimal grid in the sense that the error of projection is minimised for the optimal grid. See Pages, Pham, and Printems (2004).

Although still considered by many practitioners as a no go area, Malliavin Calculus (Nualart (1995)) has gained interest in relation to financial optimal control problems. Though it has been considered for many years as highly theoretical and technical, it has gained support in that it provides numerical based solutions that do not suffer from the curse of dimensionality. The integration by parts property of the Malliavin derivative provide a rather nice expectation framework in which many stochastic pricing problems such as the computation of the greeks for a European style plain vanilla option, (see Fournie, Lasry, Lebuchoux, Lions, and Touzi (1999)) or a stochastic control problem that maximises inter temporal and terminal utility functions (see (Detemple, Garcia, and Rindisbacher (2003)), fall.

The previous numerical and analytical approaches make the assumption of regularity on the value function. Analytical solutions methods are often subject to some regularity conditions on the value function which are unknown in advance. There are many stochastic control problems in which the value function does not satisfy the regularity condition of the Hamilton Jacobian Bellman equation in the classical sense. The notion of a *viscosity solution* introduced by Crandall and Lions (1983) provides a framework in which such irregular value functions can be constructed. The constructed solutions are coherent with the classical solutions for value function that are regular. The construction of such solutions is not part of the present thesis; interested readers may refer to Fleming and Soner (1993), Crandall and Lions (1983) and Crandall, Ishii, and Lions (1992).

Although it is not trivial to obtain an analytical solution to a randomly selected stochastic control problem, the literature provides many examples of known solved control problems in the finance arena as well as in the actuarial field. The next section is aimed to expose the different areas in finance and actuarial studies in which control theory is applied.

1.2 Literature Review

Since the seminal papers of Merton (1969) and Merton (1971), stochastic control has gained a large audience in the investment and risk management community. The books by Øksendal (1998), Fleming and Rishel (1975), Fleming and Soner (1993), Øksendal and Sulem (2005) and Karatzas and Shreve (1986) cover most of today's problems and methods in this field. The optimal investment and consumption problem of Merton has revolutionised the way investment and portfolio managers make decision over time in contrast to the Markowitz paradigm which is equilibrium based. Most analysis of portfolio selection as in the case of Markowitz-Tobin mean variance, maximises over one period. In his paper Merton (1969), the author examines the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous time model where income is generated by returns on assets that are stochastic. The problem of choosing optimal portfolio selection and consumption rules can be formulated as follows. Given an initial wealth $W_0 = w$ and a time horizon T, Merton derived the optimal consumption and asset allocation rule that will maximise the inter temporal utility of consumption for an investor as well as the terminal bequest function which is a function of terminal wealth. Thus the aim is to solve for π_t, $c(t)$ and $V(x)$ defined as

$$V(W_0) = \sup_{c_t, \pi_t} \mathbb{E}\left[\int_0^T e^{-\beta t} u(c(t))dt + B(T, W_T) \right]$$

where c_t is the consumption function, $B(T, .)$ is the bequest function at T , $u(.)$ is the running utility function, π is the vector of the proportion of assets and W_t is the wealth at time t which evolves stochastically. The particular case of two assets (one risky and one risk free) was examined in detail by Merton. An analytical solution was provided for the special case of constant relative risk aversion and also for infinite time horizon case with no bequest function. A similar problem was considered by Samuelson (1969) who treats the optimisation in a discrete time setting. Similar problems are treated by Merton in Merton (1971), Merton (1973) and Merton (1975). Since then, dynamic optimal asset allocation has become widely researched with many different versions of the problem being investigated by academics.

In Blanchet-Scalliet, Karoui, Jeanblanc, and Martellini (2003), the authors extend the optimal investment problem of Merton (1969) by allowing the conditional distribution of an agent's time horizon to be stochastic and correlated to returns on risky assets. For a constant relative risk aversion utility function, the authors found that the allocation strategy is seriously affected by the dynamics of the exit time which is modelled as an Ito process. In Korn (1998), the author relaxes the usual assumption of no trading costs in continuous time portfolio optimisation by introducing the impulse control technique in order to prevent the trading from having an infinite variation. In the paper, the author allows the investor to change his portfolio only finitely often in finite time intervals and derives a nontrivial asymptotically optimal solution for the problem of exponential utility maximisation. Cvitanic and Karatzas (1992), Xu and Shreve (1992), and He and Pearson (1993) all consider optimal allocation with constraints on the strategies while in Korn and Trautmann (1999), the authors consider a problem using a terminal wealth constraint. In the majority of the literature, risky assets are modelled as a diffusion process and the control problem is simplified into solving a PDE. The problem of optimal consumption and portfolio problem in a jump diffusion market consisting of a bank account and a stock is considered by Framstad, Øksendal, and Sulem (1998) where price is modelled as a Levy process. The results generate a very similar solution to the original case under pure diffusion process. In Browne (1995), the author adopts a more conservative, actuarial approach by putting more weight on the event of an investor being ruined rather than making superior wealth. The investment's objective is to minimise the probability of ruin in the presence of a liability process rather than maximising a function of wealth.

Although the use of control theory in finance has been applied in a complete market context, it has recently gained ground in an incomplete market setting using *Equivalent utility theory.* The method attempts to price contingent claims by considering two scenarios in which an investor is indifferent, see Musiela and Zariphopoulou (2004), Young and Zariphopoulou (2001), Young and Zariphopoulou (2002) Zariphopoulou (2001a), Young (2004), Shouda (2005), Sicar and Zariphopoulou (2005), Rouge and Karoui (2000), Mania and Schweizer (2005), Grasselli and Hurd (2004), Grasselli and Hurd (2005), and Lim

(2005).

In the insurance literature, it took until 1994 for the first lecture to address the dynamic optimisation problem, see Martin-Löf (1994). Since then, a rapid development in stochastic control theory among actuarial academics has grown. In Taksar and Asmussen (1997) the authors considered a classical actuarial insurance risk process and aimed to maximise the discounted dividend payments which are paid whenever the reserve is positive. The authors modelled the reserve r_t at time t as a diffusion process following the stochastic differential equation

$$dr_t = (\mu - a_t)dt + \sigma dB_t$$

where B_t is the standard Brownian motion μ and σ are constants with $\sigma \geq 0$ and a_t is the rate of dividend payment at time t. The objective is to maximise the total discounted dividend payment

$$J(x) = \mathbb{E}\left[\int_0^{\tau} e^{-\beta t} a_t dt\right] \tag{1.2}$$

where β is the constant force of interest and τ is the ruin time i.e.

$$\tau = \inf\left\{t \geq 0, r(t) = 0\right\}.$$

In the control framework, the objective is to evaluate at each time the policy a_t and therefore compute the function V defined as

$$V(x) = \sup_{\{a_t\}_{t\geq 0}} J(x) = \sup_{\{a_t\}_{t\geq 0}} \mathbb{E}\left[\int_0^{\tau} e^{-\beta t} a_t dt\right] \tag{1.3}$$

The authors solved the problem analytically considering two scenarios:

(a) When the dividend rate is bounded by a constant $a_0 \leq \infty$, then if a_0 is smaller than some critical value α, the optimal strategy is to always pay the maximal dividend rate a_0 otherwise the optimal policy prescribes to pay nothing when the reserve is below some critical level m and to pay the maximal dividend rate a_0 when the reserve is above m.

(b) In the second case, when the dividend rate is unbounded then the optimal strategy prescribes to pay out whatever amount exceeds some critical level m but not to pay out dividends when the reserve is below m i.e. $a_t = \max\{r_t - m, 0\}$.

In the actuarial pension framework, control theory has also gained a lot of interest. Gerrard, Haberman, and Vigna (2004) investigated the annuity risk faced by a member of a defined contribution pension scheme and introduced the income drawdown option whereby the member of the scheme is allowed to choose when to convert the final capital into pension within a certain period after retirement. The objective was to look for the optimal investment strategy to be adopted after retirement whilst allowing for periodic fixed withdrawal from the fund. A quadratic loss function was introduced to penalise deviation from a target at each time. Recently, due to changes in regulation of pension schemes across Europe, control theory serves as a mean of tackling Liability Driven Investment (LDI) in which liability streams are as important or more important to be met than making a substantial return on the pension fund. See Detemple, Garcia, and Rindisbacher (2003)

In the wider actuarial area, Paulsen and Gjessing (2004) considered a typical insurance business in which premium income is deterministically perturbed by a diffusion process with the random claim payout following a jump process. The authors sought to maximise the terminal expected utility by controlling the asset allocation and the proportion of business ceded to a re-insurer company. Taksar and Højgaard (1999) extended the work of Taksar and Asmussen (1997) to control for the proportion of insurance business that is ceded to re-insurers while Amussen, Højgaard, and Taksar (2000) included a control for investment strategies. Using a traditional risk process framework, Højgaard

(2001) let the premium rate be dependent on the size of the business with the objective to maximise the dividends pay-outs.

Another popular area in the Actuarial literature where control theory has gained significant ground especially among German researchers is the minimisation of ruin probabilities. Among them, Hipp and Plum (2000), Hipp and Plum (2003), Hipp and Vogt (2003), Schmidli (2001), Schmidli (2002), Gaier and Grandits (2002), Gaier and Grandits (2003) and Gaier and Grandits (2003) represent important contributions to this field of research.

Finally, Browne (1995) considers the asset allocation problem of Merton without consumption in the presence of an uncontrollable stochastic liability flow. The author adopts a more conservative actuarial approach in the objective function by minimising the probability of ruin. He showed that in the absence of the risk free asset, the optimal strategy is equivalent to maximising the terminal utility of wealth for a firm having an exponential utility function.

1.3 Overview of the thesis

The thesis is organised as follows:

In **Chapter** 2, we introduce the mathematical background needed to solve stochastic control problems in the framework of HJB. We present the necessary conditions for soluble problems, the derivation of the HJB differential equation both in the case of the state variable following a diffusion problem as well as a Jump diffusion problem and finally we show how the verification theorem can be used to validate the solution of the HJB .

In **Chapter** 3, we examine the optimal premium setting strategy for a firm operating in a monopolistic market regulated by a demand function. A choice of demand function is made to describe how customers react to the level of the premium. A high premium is unattractive which results in the company having lower business volume. A low premium is very attractive but results in a higher likelihood that the company will not be able to meet its liabilities. The objective of the firm is to maximise utility of wealth by setting an attractive premium to policyholders given that the firm is the sole provider of the insurance policy in the market. An analytical solution is obtained which is validated by the verification theorem.

In **Chapter** 4, we present the asset allocation problem in the context of insurance risk process. We consider an insurance company operating in a market in which it has no control over the price of the premium for a particular non life policy. The aim of the firm is to optimally allocate between risky and risk free assets while receiving a premium income according to the law of demand and supply. For a firm adopting an exponential utility function, an analytical solution is obtained and validated by the verification theorem.

In **Chapter** 5, we present the common instruments used in Catastrophe insurance and highlight the difficulty in pricing in such an incomplete market. We walk through the literature in the field and comment on the deficiencies in current pricing methodologies.

In **Chapter** 6, we present a new method based on indifference utility theory that we adopt and adapt to the pricing of catastrophe insurance. We derive a general formula for what we denote by as *minimum affordable price* which is expressed in terms of expectations under the physical probability measure. Though the primary goal of the thesis is to produce analytical solutions to the problems it attempts to solve, a translated gamma distribution approximation to the aggregate loss index at maturity and a Monte-Carlo simulation are performed to produce two sets of numerical based solutions.

In **Chapter** 7, we conclude the thesis and expose the possible extensions to the problems solved.

In the next chapter we will look at the mathematics and the framework needed to solve stochastic control problems.

Chapter 2

Mathematical Background of Stochastic Control Theory

2.1 Introduction

The aim of this chapter is to present the basic concepts and results needed in modelling events that are random and evolve with time (*Stochastic events*) and events that are random and evolve with time on which one has a control over their evolution (*Stochastic control events*). Since there are several books which give a detailed account of the former, we refer the reader to the introductory books Brzéźniak and Zastawniak (2000), Øksendal (1998), and Karatzas and Shreve (1986). In this chapter we will concentrate solely on the mathematics of the latter which underline the core of stochastic control problems and hence of this thesis.

Among the different classes of stochastic processes, this thesis will concentrate on *Diffusion processes* that are driven by a *Brownian motion* (Appendix B.1) and *Jump processes* (Appendix C.2) that are driven by *Poisson processes* (Appendix C.1). The Brownian motion and the Poisson process form the basis of the *Jump-diffusion processes* (

Appendix C.4) that are the driving processes adopted in this thesis. The two processes are continuous-time stochastic processes, which basically means that they are continuous time dependent random-variables. They are also *Markovian* (Appendix A.0.1) in the sense that the future value taken by the processes depends only on the current value and not by the whole history of the values that they have taken. We will in the next section present the Brownian motion and the Poisson process in the context of control theory with some properties and results that are relevant to this thesis.

Ordinary calculus is based on functions that are continuous, differentiable and continuously differentiable. However, many of the models for Markov processes do not have such nice analytical properties. Since Poisson processes are discontinuous (Appendix A.0.2), and Gaussian processes are not smooth (not differentiable with respect to time) (Appendix A.0.2), there is a need to review and revise the standard calculus so as to include these essential properties. Most of the results needed for the two classes of stochastic processes without control over the evolution of the process are presented in Appendices (A,B,C). Before putting the control problem into a mathematical framework, we will show some examples of practical control problems.

2.2 Example of Control problems

Optimal problems in a stochastic environment may in general be divided into two classes that share similar features in the dynamic programming framework: *the optimal stopping time* problems such as the time to exercise an American style option in which the decision variable is the time, and the *optimal control problem* such as Merton's investment and consumption problem in which the control variable acts on the state variable. We investigate each in turn.

2.2.1 Optimal Stopping time problems

When to sell an asset We consider an investor holding a stock or an asset (house) that he is willing to sell. The asset evolves according to the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.1}$$

where $\mu(t, X_t)$ is the drift component of the asset and $\sigma(t, X_t)$ represents its volatility. At the time of sale, a fixed transaction cost $a > 0$ is paid. If the agent opts to sell the asset at time t, the realised value (asset value less the transaction cost) is $e^{-\beta t}(X_t - a)$ where β is the rate of discount. The optimisation problem is to find the stopping time τ which maximises the expected realised value of the asset i.e.

$$\sup_{\tau} \mathbb{E}\left[e^{-\beta\tau}(X_\tau - a)\right],$$

and to calculate the expected realised value.

When to invest in a project We consider an agent who wishes to invest in a project, for example in a company. From the beginning of the investment, the agent receives a dividend which evolves as in (2.1). The capital invested is C. If the agent invests at time t, the long term profit realised is $\int_t^\infty e^{-\beta s} X_s ds - e^{-\beta t}C$. The optimisation problem is to find the optimal time τ that maximises the expected profit

$$\sup_{\tau} \mathbb{E}\left[\int_\tau^\infty e^{-\beta s} X_s ds - e^{-\beta\tau}C\right].$$

Natural ressource extraction We consider the price of a natural resource (gas, oil) evolving as a geometric Brownian motion of the form

$$dP_t = P_t(\mu dt + \sigma dB_t).$$

We assume that at each time t the quantity of resources Q_t decays exponentially according to the deterministic differential equation

$$dQ_t = -\lambda Q_t dt$$

where $\lambda > 0$ is the rate of decay. We also assume the time unit cost of extraction to be $K > 0$. If we stop the extraction at time t, the profit from the extraction is $\int_0^t e^{-\beta s}(P_t - K)dQ_s + e^{-\beta t}B(P_t, Q_t)$ where $B(p,q)$ is the value of the extraction site when the price of the resource is p and the remaining quantity is q. The objective is to find the stopping time τ that maximises the expected profit, i.e.

$$\sup_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-\beta s}(P_t - K)dQ_s + e^{-\beta\tau}B(P_\tau, Q_\tau)\right].$$

In the next section, we consider problems in which the evolution of the state variable is controlled.

2.2.2 Stochastic control problems

In this section, we consider examples in which the evolution of the state variable is influenced by a control variable. The control variable is a process $(\alpha_t)_{t>0}$ whose value is based on the history of the state variable up to the current time. Let

$$dX_t = \mu(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dB_t. \tag{2.2}$$

The most famous example of this type of problem is the following:

Merton's Investment and consumption problem We consider an investor who has a finite time horizon with an initial wealth x. At each time t, the investor consumes an amount c_t of her wealth and invests a proportion π_t in a risky asset while the remaining proportion is invested in a risk free asset. The optimisation problem is to optimally allocate the proportion between the two classes of investment asset in order to maximise her inter-temporal utility function. We will discuss this problem in greater detail in the next section.

Irreversible investment In this example, we consider a firm whose profit depends on capital Y_t and a stochastic parameter Z_t which may be considered in this particular example as the demand function which evolves according to the stochastic differential equation

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dB_t.$$

We denote by $\Omega(y, z)$, the profit function when the demand is z and the capital is y. We also assume that the capital of the firm never decreases but grows at a rate $u_t > 0$ such that

$$dY_t = u_t dt$$

and the unit cost of increase in capital is $c(Y_t)$. The optimisation problem is to find the rate of expansion of the firm's capital that will maximise the pure profit of the firm i.e.

$$\sup_u \mathbb{E}\left[\int_0^T \left(\Omega(y_t, z_t) - c(y_t)u_t\right) dt\right]$$

Cost of surreplication in an uncertain volatility model We consider the price of a stock evolving according to the differential equation

$$dX_t = \sigma_t X_t dB_t$$

where the volatility at time t σ_t is random. Given a terminal claim option written on the stock with payoff $B(X_T)$ at expiry, the problem is to calculate the surreplication cost given by

$$\sup_\sigma \mathbb{E}\left[B(X_T)\right]$$

In the following section, we set up the control problem in a mathematical framework and provide the methodology to the solutions.

2.3 Stochastic Control background

We consider the evolution of a mathematical system according to the differential law

$$d\mathbf{X}_t = f(t, \mathbf{X}_t), \quad t \geq 0,$$

where the function f describes the dynamic of the system and $\mathbf{X}_t$ is the vector value of its parameters $\mathbf{X}$ at time t. If the function f can be precisely measured and completely determined, no stochastic theory will be needed in which case we will be faced with a deterministic control problem. However, if f varies randomly with time or if the errors of measuring f is not negligible, stochasticity will be needed. In this section, we consider the process $\mathbf{X}_t$ as a multidimensional diffusion process governed by the stochastic differential equation

$$d\mathbf{X}_s = b(s, \mathbf{X}_s)ds + \sigma(s, \mathbf{X}_s)d\mathbf{B}_s. \tag{2.3}$$

If, in addition the coefficients b and or σ in equation (2.3) depend on some control parameters subject to vary over the course of time according to our will, we have a *controlled diffusion process* which is the solution to an equation of the type :

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t b(s, \mathbf{X}_s, \alpha_s)ds + \int_0^t \sigma(s, \mathbf{X}_s, \alpha_s)d\mathbf{B}_s \tag{2.4}$$

where α_s is the value of the controlled parameter α at time s. If we adopt the terminology of control theory, the process $\mathbf{X}_t$ is called the *controlled state process*, and the process α_t is called the *control variable*.

We now consider the dynamics of the state variable in the Euclidean space $\mathbb{R}^n$,

$$d\mathbf{X}_t = b(t, \mathbf{X}_t, \alpha_t)dt + \sigma(t, \mathbf{X}_t, \alpha_t)d\mathbf{B}_t \tag{2.5}$$

where $\mathbf{X}_t \in \mathbb{R}^n$ and $\mathbf{B}_t$ is a $d-$dimensional Brownian motion, and assume that the process $\mathbf{X}_t$ is completely observable, that is, we can choose the process α_t on the basis of the values of the controlled state process $\mathbf{X}_t$. We also suppose that we live in the same time scale as $\mathbf{X}_t$ does. Thus, up to time t we can see the trajectory of $\mathbf{X}_s$ only for $s \in [0, t]$, and our decision about the value of the control parameter α at time t can be only based on the trajectory

$$\mathbf{X}_{[0,t]} : \mathbf{X}_{[0,t]}(w) := \{(s, \mathbf{X}_s) : s \in [0, t]\}$$

which can be simply written as a function $\alpha_t = \alpha(\mathbf{X}_{[0,t]})$.

In control theory terminology, such functions are called *policies* or *strategies* and are denoted

$$\alpha = \Big\{\alpha(\mathbf{X}_{[0,t]}), t > 0\Big\}.$$

Looking at the dynamics of (2.5), and considering the state process at time t, it is clear that the move to the next state at time $t + h$ does not depend on the whole trajectory taken by the state from the initial time to t but only on the value of the current state, thus $\mathbf{X}_t$ is a *Markov process* on which all the analysis that we will develop in the next chapters will be based. Equation (2.5) is a time inhomogeneous Markov process since the functions b and σ are explicitly dependent on time. A particular case where $b = b(\alpha_t, \mathbf{X}_t)$ and $\sigma = \sigma(\alpha_t, \mathbf{X}_t)$ results in a time homogeneous Markov process for which the optimal decision α is independent of time for an infinite horizon problem and or for a class of cost functions. If we assume that the control variable α varies in a set $\mathcal{A}$ of so-called *admissible* controls and we choose appropriately the random process α_s with values in $\mathcal{A}$, we can obtain various solutions to equation (2.5). This gives rise to the questions as to whether there exists a solution to equation (2.5) for a chosen process α_t and whether the solution is unique when it occasionally exists.

Generally, the control problem requires maximising a function of the state variable by controlling the process α_t. Let us formulate this problem by maximising the function $\mathbb{E}[J(t, \mathbf{X}_t)]$ given the trajectory of the controlled process up to and including time t where

$$J(t, \mathbf{X}_t) = \int_t^{\Upsilon} f(s, \mathbf{X}_s, \alpha_s) ds + B(\mathbf{X}_{\Upsilon}) \tag{2.6}$$

and where Υ can be taken as:

- $\Upsilon = T$ (Finite time horizon problem)
- $\Upsilon = \infty$ (Infinite time horizon problem)
- $\Upsilon = \tau$ (Stopping time in infinite horizon problem)
- $\Upsilon = \tau \wedge T$ (A stopping time in a finite horizon T.)

At any time t, our decision α_t must take into account all the relevant information available up to time t (the trajectory of $\mathbf{X}_t$); it does not depend on the future behaviour of the processes $\mathbf{X}_t$. Mathematically, this can be expressed by setting a framework that accounts for the dependency on the past and not the future.

We start by considering a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the σ algebra on (Ω, P) and $\mathcal{F}_t$ is the filtration generated by the system of Brownian motions that govern the diffusion processes. In this setting, the processes driven by the Brownian motion are $\mathcal{F}_t$ adapted i.e. $\sigma(B_s : 0 \leq s \leq t) \subseteq \mathcal{F}_t$. An immediate consequence of this setting is the optimal control $\hat{\alpha}_t$ is $\mathcal{F}_t$ adapted. The control problem therefore turns to looking for a function $V(t, \mathbf{X})$ defined as:

$$V(t, \mathbf{x}) = \max_{\substack{\alpha_s \\ t \le s \le \Upsilon}} \mathbb{E}\left[J(t, \mathbf{X}) \mid \mathcal{F}_t\right] = \max_{\substack{\alpha_s \\ t \le s \le \Upsilon}} \mathbb{E}\left[\int_t^{\Upsilon} f(s, \mathbf{X}_s, \alpha_s) ds + B(\mathbf{X}_\Upsilon) \mid \mathcal{F}_t\right]. \quad (2.7)$$

The function $f(t, \mathbf{X}_t, \alpha_t)$ is called the *running cost function* whereas $V(t, \mathbf{x})$ is called the *value function* and describes fully the solution of the control problem. Due to the Markov property of the state variable, the whole history represented by the filtration $\mathcal{F}_t$ at time t is not needed, the current value of the state process $\mathbf{X}_t$ is as informative as the filtration. Therefore in the rest of the thesis, we shall be using the current state $\mathbf{X}_t = \mathbf{x}$ instead of considering the filtration at time t to exhibit the Markov property of the state variable, i.e.

$$\begin{aligned} V(t, \mathbf{x}) &= \max_{\substack{\alpha_s \\ t \le s \le \Upsilon}} \mathbb{E}\left[\int_t^{\Upsilon} f(s, \mathbf{X}_s, \alpha_s) ds + B(\mathbf{X}_\Upsilon) \mid \mathcal{F}_t\right] \\ &= \max_{\substack{\alpha_s \\ t \le s \le \Upsilon}} \mathbb{E}\left[\int_t^{\Upsilon} f(s, \mathbf{X}_s, \alpha_s) ds + B(\mathbf{X}_\Upsilon) \mid \mathbf{X}_t = \mathbf{x}\right] \end{aligned}$$

Before getting into the derivation of the equation satisfied by $V(t, \mathbf{x})$, we need to ensure the existence of the solution of the stochastic differential equation (2.5) and the existence of the integral $J(t, \mathbf{X})$ when a choice of the control variable α is made.

2.3.1 Existence condition of a controlled SDE

We consider Equation (2.5) where $\mathbf{X}_t \in \mathbb{R}^n$, $\alpha = (\alpha_t) \in \mathcal{A} \subset \mathbb{R}^m$ and $\mathbf{W}_t$ is a d-dimensional Brownian motion. Let the functions $b : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{A} \to \mathbb{R}^{n \times d}$ of Equation (2.5) satisfy the linear growth condition i.e.

$$\begin{aligned} &\exists\, C \ge 0, \forall\, t \in \mathbb{R}^+, \forall\, \mathbf{x}_t \in \mathbb{R}^n, \forall\, \alpha_t \in \mathcal{A}\,, \\ &\left|b(t, \mathbf{x}_t, \alpha_t)\right| + \left|\sigma(t, \mathbf{x}_t, \alpha_t)\right| \le C(1 + |\mathbf{x_t}| + |\alpha_t|). \end{aligned} \quad (2.8)$$

Let us also assume that b and σ satisfy the *Lipschitz* condition uniformly in $\mathcal{A}$ i.e.

$$\begin{aligned}&\exists\, C \geq 0, \forall\, t \in \mathbb{R}^+, \forall\, \mathbf{x}_t, \mathbf{y}_t \in \mathbb{R}^n, \forall\, \alpha_t \in \mathcal{A},\\ &\left|b(t,\mathbf{x}_t,\alpha_t) - b(t,\mathbf{y}_t,\alpha_t)\right| + \left|\sigma(t,\mathbf{x}_t,\alpha_t) - \sigma(t,\mathbf{y}_y,\alpha_t)\right| \leq C\left|\mathbf{x}_t - \mathbf{y}_t\right|,\end{aligned} \tag{2.9}$$

then the SDE (2.5) has a unique strong solution in $\mathbf{R}^n$ with initial value $\mathbf{X}_0 = \mathbf{x}$. Moreover,

$$\mathbb{E}\left[\sup_{t\leq s\leq \Upsilon} |\mathbf{X}_s|^2\right] \leq +\infty \tag{2.10}$$

where $|\mathbf{X}|$ is the L^2 norm. (See Pham (2002)).

2.3.2 Existence of the cost function

Let consider again the cost function

$$J(t,\mathbf{X}_t) = \mathbb{E}\left[\int_t^{\Upsilon} f(s,\mathbf{X}_s,\alpha_s)ds + B(\mathbf{X}_\Upsilon)\right] \tag{2.11}$$

in a finite time horizon problem where $(\Upsilon = T)$, if B is bounded and f satisfies the quadratic growth condition, i.e.

$$|f(t,\mathbf{x}_t,\alpha_t)| \leq C\left(1 + |\mathbf{x}_t|^2 + |\alpha_t|^2\right),\ \forall\ (t,\mathbf{x}_t,\alpha_t) \in [0,T] \times \mathbb{R}^n, \alpha_t \in \mathcal{A}, \tag{2.12}$$

then the definite integral $J(t,\mathbf{x}_t)$ in Equation (2.11) converges .

In infinite time horizon $(\Upsilon = \infty)$ it is not clear whether the indefinite integral converges but if we adopt a cost function in the financial mathematics literature where the cost at each time is discounted to the original time i.e. $f(t,\mathbf{X}_t,\alpha_t) = e^{-\beta t} f_\beta(\mathbf{X}_t,\alpha_t)$, a sufficient condition for the convergence is achieved when the discounting rate β is large enough together with the running cost $f_\beta(\mathbf{X}_t,\alpha_t)$, satisfying the quadratic growth condition. The proof in the two cases follows from the fact that when a solution exists for the SDE (2.5), equation (2.10) is verified which together with the quadratic growth condition, ensures the convergence of the integral.

2.4 Derivation of the HJB equation

In this section, we describe how the dynamic principle of *Bellman* allows us to characterise the value function in terms of a partial differential equation so-called the *Hamilton-Jacobi-Bellman* equation.

We consider the case of a finite horizon problem. Let $0 < h < T - t$ where T is the finite horizon and t is the current time, let us also assume that we are applying a control α_s on the interval $[t, t+h]$. At time $t+h$, the state of the system becomes $\mathbf{X}_{t+h}$ and we observe it at time $t+h$. Let us assume that we know the optimal policy to apply from time $t+h$ onward i.e. we know the control α_s^*, $t+h \leq s \leq T$ such that

$$\begin{aligned} V(t+h, \mathbf{X}_{t+h}) &= J(t+h, \mathbf{X}_{t+h}, \alpha^*) \\ &= \mathbb{E}\left[\int_{t+h}^{T} f(s, \mathbf{X}_s, \alpha_s^*)\, ds + B(\mathbf{X}_T) \mid \mathcal{F}_{t+h}\right]. \end{aligned} \tag{2.13}$$

Consider the control:

$$\tilde{\alpha}_s = \begin{cases} \alpha_s, & t \leq s \leq t+h \\ \alpha_s^* & t+h \leq s \leq T, \end{cases} \tag{2.14}$$

we obtain by the law of iterated conditional expectations

$$\begin{aligned} J(t, \mathbf{x}, \tilde{\alpha}) &= \mathbb{E}\left[\int_{t}^{t+h} f(s, \mathbf{X}_s, \alpha_s)\, ds + \int_{t+h}^{T} f(s, \mathbf{X}_s, \alpha_s^*)\, ds + B(\mathbf{X}_T) \mid \mathbf{X}_t = \mathbf{x}\right] \\ &= \mathbb{E}\left[\int_{t}^{t+h} f(s, \mathbf{X}_s, \alpha_s)\, ds + \mathbb{E}\left[\int_{t+h}^{T} f(s, \mathbf{X}_s, \alpha_s^*)\, ds + B(\mathbf{X}_T) | \mathbf{X}_{t+h}\right] \mid \mathbf{X}_t = \mathbf{x}\right] \\ &= \mathbb{E}\left[\int_{t}^{t+h} f(s, \mathbf{X}_s, \alpha_s)\, ds + v(t+h, \mathbf{X}_{t+h}) \mid \mathbf{X}_t = \mathbf{x}\right]. \end{aligned} \tag{2.15}$$

The Bellman optimality principles states that if we choose the decision α_s in the interval of time $[t, t+h]$ in order to maximise $J(t, \mathbf{X}, \tilde{\alpha})$, then we obtain the optimal control over the whole interval $[t, T]$. This implies that the optimal control in $[t, T]$ can be decomposed in $\alpha_s^*, s \in [t, t+h]$ and $\alpha_s^*, s \in [t+h, T]$ where the latter is the optimal policy for a problem starting at time $t+h$ where the system is in the state $\mathbf{X}_{t+h}$. Considering equation (2.15), it is evident that

$$V(t, \mathbf{x}) = \sup_{\alpha} \mathbb{E}\left[\int_t^{t+h} f(s, \mathbf{X}_s, \alpha_s)\, ds + V(t+h, \mathbf{X}_{t+h}) \mid \mathbf{X}_t = \mathbf{x}\right]. \quad (2.16)$$

We can now formally derive the Hamilton-Jacobi-Bellman equation using equation (2.16). We first consider the constant control $\alpha_s = a \in \mathcal{A}$ on the interval $[t, t+h]$. Using equation (2.16), we obtain

$$V(t, \mathbf{x}) \geq \mathbb{E}\left[\int_t^{t+h} f(s, \mathbf{X}_s, a)\, ds + V(t+h, \mathbf{X}_{t+h}) \mid \mathbf{X}_t = \mathbf{x}\right]. \quad (2.17)$$

If we assume that $V \in C^{1,2}([0, T[, \mathbb{R}^n)$, we obtain using *Ito's* formula between t to $t+h$:

$$\begin{aligned} V\left(t+h, \mathbf{X}_{t+h} \mid \mathbf{X}_t = \mathbf{x}\right) &= V(t, \mathbf{x}) + \int_t^{t+h} \left[\frac{\partial V}{\partial s} ds + \sum_{i=1}^n \frac{\partial V}{\partial \mathbf{X}^i} d\mathbf{X}^i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 V}{\partial \mathbf{X}^i \partial \mathbf{X}^j} d\mathbf{X}^i d\mathbf{X}^j\right] \\ &= V(t, \mathbf{x}) + \int_t^{t+h} \left(\frac{\partial V}{\partial s} ds + \mathbf{b}(s, \mathbf{X}, a).\nabla_{\mathbf{X}} V \right. \\ &\left. + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \left(\sigma\, \sigma^T\right)_{i,j}(s, \mathbf{X}_s, a) \frac{\partial^2 V}{\partial \mathbf{X}^i \partial \mathbf{X}^j} d\mathbf{X}^i d\mathbf{X}^j\right) \\ &+ \int_t^{t+h} (\nabla_{\mathbf{X}} V)^T \sigma(s, \mathbf{X}, a) d\mathbf{W}_s. \end{aligned} \quad (2.18)$$

Taking the expectation of equation (2.18) after defining

$$\mathcal{L}^a(V) = \mathbf{b}(s, \mathbf{X}, a).\nabla_{\mathbf{X}} V + \frac{1}{2} tr(\sigma(s, \mathbf{X}, a)\sigma'(s, \mathbf{X}, a) D^2 V)$$

we obtain

$$\mathbb{E}\left[V\left(t+h, \mathbf{X}_{t+h}\right) \mid \mathbf{X}_t = \mathbf{x}\right] = V(t, \mathbf{x}) + \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial V}{\partial s} + \mathcal{L}^a(V)\right) ds\right]$$

which after substitution in equation (2.17) yields:

$$\mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial V}{\partial s} + \mathcal{L}^a(V) + f(s, \mathbf{X}, a)\right) ds\right] \leq 0. \tag{2.19}$$

We proceed to the derivation by dividing equation (2.19) by h and tending h to 0, this yields

$$\frac{\partial V}{\partial t} + \mathcal{L}^a(V) + f(t, \mathbf{X}, a) \leq 0. \tag{2.20}$$

Equation (2.20) is valid for every $a \in \mathcal{A}$ therefore it is valid on the sup of $\mathcal{A}$:

$$\frac{\partial V}{\partial t} + \sup_{a \in \mathcal{A}} \left[\mathcal{L}^a(V) + f(t, \mathbf{X}, a)\right] \leq 0. \tag{2.21}$$

If we assume that α^* is an optimal control, then using equation (2.16), we obtain

$$V(t, \mathbf{x}) = \mathbb{E}\left[\int_t^{t+h} f(s, \mathbf{X}_s^*, \alpha_s^*)\, ds + V(t+h, \mathbf{X}_{t+h}^*) \;\middle|\; \mathbf{X}_t = \mathbf{x}\right]. \tag{2.22}$$

where $\mathbf{X}^*$ is the state of the system and the solution to equation (2.5) given that at time t, $\mathbf{X}_t = \mathbf{x}$ and the control α^* is applied.

By a similar argument and with regularity conditions assumed to be satisfied on V we obtain:

$$\frac{\partial V}{\partial t} + \mathcal{L}^{\alpha^*}(V) + f(t, \mathbf{X}, \alpha^*) = 0 \tag{2.23}$$

which combined with equation (2.21) proves that V satisfies:

$$\frac{\partial V}{\partial t} + \sup_{a\in\mathcal{A}} \left[\mathcal{L}^a(V) + f(t, \mathbf{X}, a)\right] = 0 \quad \forall (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n. \tag{2.24}$$

Equation (2.24) is a second-order parabolic partial differential equation known as the *Hamilton-Jacobi-Bellman equation* that characterises the solution of the value function. For the particular value function, solution to the control problem is determined by the boundary condition

$$V(T.\mathbf{x}) = B(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$$

which results from the definition of the value function. The dynamic programming principal of optimality suggests that if we find a control α^* such that

$$\sup_{a\in\mathcal{A}} \left[\mathcal{L}^a(V) + f(t, \mathbf{X}, a)\right] = \mathcal{L}^{\alpha^*(t,\mathbf{x})}(V) + f(t, \mathbf{X}, \alpha^*(t, \mathbf{x})) \tag{2.25}$$

i.e.

$$\alpha^*(t, \mathbf{x}) = \arg\max_{a\in\mathcal{A}} \left[\mathcal{L}^a(V)(t, \mathbf{x}) + f(t, \mathbf{X}, a)\right] \tag{2.26}$$

then we have

$$\frac{\partial V}{\partial t} + \mathcal{L}^{\alpha^*(t,\mathbf{x})}(V) + f(t, \mathbf{X}, \alpha^*(t, \mathbf{x})) = 0 \tag{2.27}$$

and

$$V(t, \mathbf{x}) = \mathbb{E}\left[\int_t^T f(s, \mathbf{X}^*_s, \alpha^*_s)\, ds + g(\mathbf{X}^*_T) \;\middle|\; \mathbf{X_t} = \mathbf{x}\right]. \tag{2.28}$$

where $\mathbf{X}^*$ is a solution to the SDE:

$$\begin{aligned} d\mathbf{X}^*_s &= b(s, \mathbf{X}^*_s, \alpha^*_s) + \sigma(s, \mathbf{X}^*_s, \alpha^*_s) d\mathbf{B}_s \quad t \le s \le T \\ \mathbf{X}^*_s &= \mathbf{x}. \end{aligned} \tag{2.29}$$

2.4.1 Infinite time horizon

An interesting case of the infinite time horizon problem arises when the state variable is time homogeneous Markov and when the cost is discounted back to the initial time. Let us assume that the evolution of the state is governed by the SDE

$$\begin{aligned} d\mathbf{X}_s &= b(\mathbf{X}_s, \alpha_s) + \sigma(\mathbf{X}_s, \alpha_s)d\mathbf{B}_s \\ \mathbf{X}_t &= \mathbf{x}. \end{aligned} \tag{2.30}$$

Let the cost function be rewritten in terms of the discount factor as:

$$V(t, \mathbf{x}) = \sup_{\alpha} \mathbb{E}\left[\int_t^{\infty} e^{-\beta s} f(\mathbf{X}_s, \alpha_s) ds \mid \mathbf{X}_t = \mathbf{x}\right]. \tag{2.31}$$

If we define $V(\mathbf{x})$ as

$$V(\mathbf{x}) = \sup_{\alpha} \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} f(\mathbf{X}_s, \alpha_s) ds \mid \mathbf{X}_0 = \mathbf{x}\right], \tag{2.32}$$

then it is clear from the time homogeneous Markov property of $\mathbf{X}_t$ that

$$V(t, \mathbf{x}) = e^{-\beta t} V(0, \mathbf{x}) = e^{-\beta t} V(\mathbf{x}). \tag{2.33}$$

The proof is given in (3.3.1). The dynamic programming problem turns to the search of the function

$$V(\mathbf{x}) = \sup_{\substack{\alpha_s \\ 0 \leq s \leq h}} \mathbb{E}\left[\int_0^{h} e^{-\beta s} f(\mathbf{X}_s, \alpha_s) ds + e^{-\beta h} V(\mathbf{x}_h) \mid \mathbf{X}_0 = \mathbf{x}\right]. \tag{2.34}$$

If we replace (2.33) in (2.24), we obtain the appropriate Bellman equation:

$$-\beta V(\mathbf{x}) + \sup_{\alpha \in \mathcal{A}} \left[\mathcal{L}^{\alpha} V(\mathbf{x}) + f(\mathbf{x}, \alpha)\right] = 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.35}$$

which does not involve any terminal condition and is free of the time variable.

2.4.2 Optimal stopping time problems

Optimal stopping time problems are similar to control problems; the objective is to stop the process at an appropriate time which is best among all possible stopping time in order to optimise a certain goal.

Formulation of the problem

We assume that the state variable is governed by the stochastic differential equation

$$dX_s = b(X_s)ds + \sigma(X_s)\mathbf{dB}_s, \tag{2.36}$$

where $\mathbf{X}_s \in \mathbb{R}^n$ and W is a d-Brownian motion adapted to the filtration $\mathcal{F}_t$. We also assume that the function $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times d}$ satisfy the usual linear growth conditions and the Lipschitz condition that guarantee the existence and uniqueness of (2.36) given an initial condition. Let $\{\mathbf{X}_s^x, s \geq 0\}$ be the solution of (2.36) which takes the value $\mathbf{X}_t = \mathbf{x}$ at time t. We denote as $\mathcal{T}$ the set of all stopping times i.e. the set of all positives random variables such that

$$\forall t \geq 0, \text{ the event } \{\tau \leq t\} \in \mathcal{F}_t.$$

We now consider

$$V(\mathbf{x}) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-\beta s} f(\mathbf{X}_s^x)ds + e^{-\beta\tau} B(\mathbf{X}_\tau^x)\right] \tag{2.37}$$

where β is the discounting factor and f and B are defined from $\mathbb{R}^n$ to $\mathbb{R}$ with B being bounded. We adopt here the convention that $e^{-\beta\tau}B(\mathbf{X}_\tau^x)$ is equal to zero at every $\omega \in \Omega$

where $\tau(\omega) = \infty$. We also assume that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s}\left|f(\mathbf{X}_s^x)\right| ds\right] < \infty, \tag{2.38}$$

which ensures that (2.37) is well defined. The objective is to find the value function V and to determine the optimal stopping time i.e. the stopping time $\tau^\star$ such that

$$V(\mathbf{x}) = \mathbb{E}\left[\int_0^{\tau^\star} e^{-\beta s} f(\mathbf{X}_s^x) ds + e^{-\beta\tau^\star} B(\mathbf{X}_{\tau^\star}^x)\right]. \tag{2.39}$$

The variational inequality

In this subsection, we describe formally the dynamic programming principle that allows us to obtain the properties satisfied by the value function $\mathbf{V}$.

Consider the time $t = 0$ where the state variable has value $\mathbf{X}_0 = \mathbf{x}$. Let $h > 0$ and let us assume that the process is not stopped in the time interval $[0, h]$. We can then capitalise a total discounted cost $\int_0^h e^{-\beta s} f(\mathbf{X}_s^x) ds$ and the state variable at time h is $\mathbf{X}_h^x$. If we apply at h the optimal strategy, we obtain $V(\mathbf{X}_h^x)$ which discounted back to time $t = 0$ gives $e^{-\beta h} V(\mathbf{X}_h^x)$. Along these lines, without stopping the process in $[0, h]$, we obtain the expected cost function

$$\mathbb{E}\left[\int_0^h e^{-\beta s} f(\mathbf{X}_s^x) ds + e^{-\beta h} V(\mathbf{X}_h^x)\right] \tag{2.40}$$

By definition of the value function V which is the expected cost function maximised over all stopping times, we have:

$$V(\mathbf{x}) \geq \mathbb{E}\left[\int_0^h e^{-\beta s} f(\mathbf{X}_s^x) ds + e^{-\beta h} V(\mathbf{X}_h^x)\right]. \tag{2.41}$$

For $h > 0$ and small enough, we obtain the following approximation

$$\mathbb{E}\left[\int_0^h e^{-\beta s} f(\mathbf{X}_s^x) ds + (1 - \beta h) V(\mathbf{X}_h^x) - V(\mathbf{x})\right] + o(h) \leq 0.$$

Applying Ito's formula between 0 and h after assuming that $V \in C^2(\mathbb{R}^n)$, we obtain

$$V(\mathbf{X}_h) = V(\mathbf{x}) + \int_0^h \mathcal{L}V(\mathbf{X}_s^x)ds + \int_0^h \nabla_{\mathbf{x}} V(\mathbf{X}_s^x)' \sigma'(\mathbf{X})dW_s, \tag{2.42}$$

where

$$\mathcal{L}(V) = \mathbf{b}(\mathbf{X}).\nabla_{\mathbf{X}} V + \frac{1}{2} tr(\sigma(\mathbf{X})\sigma'(\mathbf{X})D^2 V).$$

By taking the expectation of (2.42) we obtain

$$\mathbb{E}\left[V(\mathbf{X}_h)\right] = V(\mathbf{x}) + \mathbb{E}\left[\int_o^h \mathcal{L}V(\mathbf{X}_s^x)ds\right]. \tag{2.43}$$

Substitute (2.43) in (2.40) and divide by h yields

$$\mathbb{E}\left[\frac{1}{h}\int_0^h e^{-\beta s} f(\mathbf{X}_s^x)ds + \frac{1}{h}\int_0^h \mathcal{L}V(\mathbf{X}_s^x) - \beta V(\mathbf{X}_h^x)\right] \leq 0 \tag{2.44}$$

and by tending h to zero, we obtain

$$\beta V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) - f(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.45}$$

Moreover, using (2.37) while taking $\tau = 0$ in the supremum of $\mathcal{T}$ we have

$$V(\mathbf{x}) \geq B(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{2.46}$$

Now, let us assume that at $t = 0$, $V(\mathbf{x}) > B(\mathbf{x})$. Let $\tau^\star$ be the stopping time,

$$\tau^\star = \inf\left\{s \geq 0 : V(\mathbf{X}_s^{\mathbf{x}}) = B(\mathbf{X}_s^{\mathbf{x}})\right\}. \tag{2.47}$$

On the infinitesimal time interval $[0, h \wedge \tau^\star)$, we have $V(\mathbf{X}_s^{\mathbf{x}}) > B(\mathbf{X}_s^{\mathbf{x}})$. This suggests that it is not optimal to stop the process on $[0, h \wedge \tau^\star)$ since by doing so, we will receive a cost $B(\mathbf{X}_s^{\mathbf{x}})$ whereas we could do better by receiving $V(\mathbf{X}_s^{\mathbf{x}}) > B(\mathbf{X}_s^{\mathbf{x}})$. This in turn implies the equality in equation (2.41)

$$V(\mathbf{x}) \geq \mathbb{E}\left[\int_0^{h\wedge\tau^\star} e^{-\beta s} f(\mathbf{X}_s^x) ds + e^{-\beta(h\wedge\tau^\star)} V(\mathbf{X}_{h\wedge\tau^\star}^x)\right]. \tag{2.48}$$

By a similar token together with the estimation of $P[\tau^\star \leq h]$ in $O(h)$, and by applying Ito lemma, we obtain by tending h to zero,

$$\beta V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) - f(\mathbf{x}) = 0, \text{ whenever } \quad V(\mathbf{x}) > B(\mathbf{x}). \tag{2.49}$$

Combining equations (2.45), (2.46) and (2.49) , the value function therefore satisfies:

$$\min\left(\beta V - \mathcal{L}V - f, V - B\right) = 0. \tag{2.50}$$

The method used to derive (2.50) indicates that it is never optimal to stop the process when $V(\mathbf{X}_s^\mathbf{x}) > B(\mathbf{X}_s^\mathbf{x})$ but whenever $V(\mathbf{X}_s^\mathbf{x}) = B(\mathbf{X}_s^\mathbf{x})$ we can obtain a cost $V(\mathbf{X}_s^\mathbf{x})$ by stopping at that time. This suggests that the stopping time defined as

$$\tau^\star = \inf\left\{s \geq 0 : V(\mathbf{X}_s^\mathbf{x}) = B(\mathbf{X}_s^\mathbf{x})\right\}$$

with the natural convention that $\inf\{\emptyset\} = \infty$ is optimal for the problem (2.37).
Let us now introduce the open interval on $\mathbb{R}^n$ defined as:

$$D = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) > B(\mathbf{x})\}$$

which we will refer to as the continuity region since whenever the process $\mathbf{X}_s^\mathbf{x}$ is in D, it is not optimal to stop therefore we let the process continue. The stopping time $\tau^\star$ is therefore the smallest stopping time and can also be defined as the first exit time from the open interval D i.e.

$$\tau^\star = \inf\left\{s \geq 0 : \ \mathbf{X}_s^\mathbf{x} \notin D\right\}.$$

Let $B \in C^2(\mathbb{R}^n)$ and define the set U by

$$U = \{\mathbf{x} \in \mathbb{R}^n : \beta B(\mathbf{x}) - \mathcal{L}B(\mathbf{x}) - f(\mathbf{x}) < 0\},$$

where

$$U \subset D.$$

The present formulation suggests that it is never optimal to stop the process before it leaves the region U.

From (2.49), we obtain:

$$\beta V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) - f(\mathbf{x}) = 0, \ \ \forall \mathbf{x} \in D. \tag{2.51}$$

Moreover by the definition of D, we have

$$V(\mathbf{x}) = B(\mathbf{x}), \ \ \forall \mathbf{x} \in \partial D \tag{2.52}$$

where ∂D is the boundary of D.

When the domain of D is known, the problem (2.51)-(2.52) is a Dirichlet problem and under some regularity conditions on the frontier ∂D, coupled with the ellipticity conditions on the diffusion coefficient $\sigma(\mathbf{x})$, it has a regular solution (see Friedman 1975). Usually D is unknown and therefore we need some extra conditions on the frontier ∂D to identify the domain of D and the value function V. This condition called the *Smooth fit* (see Shiryayev(1978), Jacka (1993)) expresses the continuity of the gradient of the value function on the frontier i.e.

$$\nabla_{\mathbf{x}} V(\mathbf{x}) = \nabla_{\mathbf{x}} B(\mathbf{x}) \ \ \forall \mathbf{x} \in \partial D. \tag{2.53}$$

2.5 Solution of the PDE: The Verification theorem

As mentioned in the introduction, there is no systematic method for solving the Hamilton-Jacobi-Bellman equation. The only method in the literature is based on guessing the solution and proving that the solution does indeed verify the PDE. This method relies on the so-called *verification theorem* below.

Theorem 2.5.1 (Verification: Finite horizon) .

Let $\Psi \in C^{1,2}\left([0,T] \times \mathbb{R}^n\right)$ satisfy the Hamilton-Jacobi-Bellman equation with boundary condition i.e.

$$\Psi(T,.) = B \quad \text{and} \quad \forall (t,\mathbf{x}) \in \left([0,t) \times \mathbb{R}^n\right),\ \exists \alpha^*(t,\mathbf{x}) \in \mathcal{A}$$

$$\text{such that}$$

$$\frac{\partial \Psi}{\partial t}(t,\mathbf{x}) + \sup_{\alpha \in \mathcal{A}} \left[\mathcal{L}^\alpha \Psi(t,\mathbf{x}) + f(t,\mathbf{x},\alpha)\right] = \frac{\partial \Psi}{\partial t}(t,\mathbf{x}) + \mathcal{L}^{\alpha^*}(t,\mathbf{x}) + f(t,\mathbf{x},\alpha^*(t,\mathbf{x})) = 0,$$

then $\Psi = V$ is the value function on $\left([0,T] \times \mathbb{R}^n\right)$ and α_* is a Markovian Optimal control.

It is also true that when Ψ is obtained, then it is unique. If furthermore Ψ satisfies the quadratic growth condition i.e.

$$|\Psi(t,\mathbf{x},\alpha)| \leq C\left(1 + |\mathbf{x}^2|\right) \ \forall (t,\mathbf{x},\alpha) \in \left([0,t[\times \mathbb{R}^n \times \mathcal{A}\right),$$

then Ψ is unique in the space of quadratic growth functions.

Theorem 2.5.2 (Verification: Infinite horizon) .

Let $\Psi \in C^2(\mathbb{R}^n)$ satisfy the quadratic growth condition and verify:

$$\beta \Psi(\mathbf{x}) + \sup_{\alpha \in \mathcal{A}} \left[\mathcal{L}^\alpha \Psi(\mathbf{x}) + f(\mathbf{x},\alpha)\right] = \beta \Psi(\mathbf{x}) + \mathcal{L}^{\alpha^*}(\mathbf{x}) + f(\mathbf{x},\alpha^*(\mathbf{x})) = 0,$$

where $\alpha^*(\mathbf{x}) \in \mathcal{A}$, then Ψ is a unique solution of the Hamilton-Jacobi-Bellman equation.

In the derivation of the Hamilton-Jacobi-Bellman equation and in the verification theorems, we have assumed the existence of a $C^{1,2}$ and C^2 value function respectively depending on the finite or infinite time horizon. These are purely assumptions and the verification theorem is by no means an existence result. Now, we will state an existence result for $C^{1,2}$ value functions in the finite time horizon. The existence in the infinite time horizon follows logically.

Theorem 2.5.3 (Existence of a $C^{1,2}$ value function) . *Let assume that*

- *The control space $\mathcal{A}$ is compact*
- *b, σ and $f \in C^{1,2}$ and are bounded*
- *The terminal cost $B \in C^3$ is bounded*
- $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \exists C \geq 0$ *such that* $\forall (t, \mathbf{x}, \alpha) \in [0, T] \times \mathbb{R}^n \times \mathcal{A}, \;\; \mathbf{y}'\sigma\sigma'(t, \mathbf{x}, \alpha)\,\mathbf{y} \geq C\,|\epsilon|^2$,

Then the Hamilton-Jacobi-Bellman equation admits a unique bounded solution $V \in C^{1,2}([0, T] \times \mathbb{R}^n)$. (See (Pham 2002))

The above theorem is constructed as a sufficient condition of the existence of $C^{1,2}$ value functions. When those conditions are not satisfied, $C^{1,2}$ functions could exist under necessary conditions that require more complicated mathematics. Different theorems according to the nature of the control problem allow us to check if necessary conditions are indeed met. In the event that the control problem fails to meet the necessary conditions, one has to look for non regular solutions by considering the theory of *viscosity solutions*, see Crandall and Lions (1983), and Crandall, Ishii, and Lions (1992).

2.6 The jump and diffusion process

Let $\Pi(t)$ be the space-time Poisson process with non state dependent amplitude $h(t, y)$. We can represent $d\Pi(t)$ as

$$d\Pi(t) = \int_{\Gamma} h(t, y)\aleph(dt, dy) \tag{2.54}$$

where the Poisson measure $\aleph(dt, dy)$ is merely a short hand notation for $\aleph([t, t+dt], [y, y+dy])$ and $y \in \Gamma$ represents the size of the jump with distribution function $F_Y(y)$. In this section, we will develop and derive the HJB equation related to state process that evolves not only according to a diffusion process but also with a jump process. The section completes the generalisation of Markov noise in continuous time, by including space-time Poisson noise with a randomly distributed amplitude conditioned on a Poisson jump time. Properties of Poisson process are in appendix (C).

2.6.1 Ito formula for The Jump and Diffusion Process

Let $\mathbf{X}_t$ be the state process with values in $\mathbb{R}^n$ evolving according to the stochastic differential equation

$$\begin{aligned} d\mathbf{X}_t &= b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{W}_t + d\Pi(t) \\ &= b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{W}_t + \int_{\Gamma} h(t, y)\aleph(dt, dy) \end{aligned}$$

where α_t as in the diffusion case is the value of the control parameter α at time t. We assume here that we have no control over the jump amplitude so that $h(t, y)$ is independent of α.

The change in value of the state process $\mathbf{X}_t$ can be decomposed into a continuous component and a jump component. See appendix (C). Using Ito's formula, we can derive the change in a function $G(t, \mathbf{X})$ as

$$\begin{aligned} dG(t, \mathbf{X}) = &\left[\frac{\partial G}{\partial t} + b(t, \alpha_t, \mathbf{X}_t)\nabla_{\mathbf{X}} G + \frac{1}{2} tr\left(\sigma(t, \mathbf{X}_t, \alpha_t)\sigma^{'}(t, \mathbf{X}_t, \alpha_t) D^2 G\right)\right] dt \\ &+ (\nabla_{\mathbf{X}} V)^T \sigma(t, \mathbf{X}_t, \alpha_t) d\mathbf{W}_t + \int_\Gamma [G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)] \aleph(dt, dy). \end{aligned} \tag{2.55}$$

2.6.2 The HJB equation

Let $\mathcal{F}_t^\star$ be the history of the Brownian motion $\mathbf{W}_t$ and the marked Poisson process $d\Pi(t)$. Following the argument in section (2.4), the value function for the stochastic problem

$$V(t, \mathbf{x}) = \max_{\substack{\alpha_s \\ t \leq s \leq \Upsilon}} \mathbb{E}\left[\int_t^\Upsilon f(s, \mathbf{X}_s, \alpha_s) ds + B(\mathbf{X}_\Upsilon) \mid \mathcal{F}_t^\star\right] \tag{2.56}$$

verifies the equation

$$\frac{\partial V}{\partial t} + \mathcal{L}^{\alpha^*(t,\mathbf{x})}(V) + f(t, \mathbf{X}, \alpha^*(t, \mathbf{x})) + \mathbb{E}_{\mathcal{F}_t^\star}\left[\int_\Gamma [G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)] \aleph(dt, dy)\right] = 0 \tag{2.57}$$

where

$$\alpha^*(t, \mathbf{x}) \in \arg\max_{\alpha \in \mathcal{A}} [\mathcal{L}^\alpha(V)(t, \mathbf{x}) + f(t, \mathbf{X}, \alpha)] \tag{2.58}$$

and

$$\mathcal{L}^\alpha(V) = \mathbf{b}(t, \mathbf{X}, \alpha).\nabla_{\mathbf{X}} V + \frac{1}{2} tr(\sigma(t, \mathbf{X}, \alpha)\sigma^{'}(t, \mathbf{X}, \alpha) D^2 V).$$

Since

$$\begin{aligned} &\mathbb{E}_{\mathcal{F}_t^\star}\left[\int_\Gamma [G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)] \aleph(dt, dy)\right] \\ &\qquad = \lambda(t) dt \int_\Gamma [G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)] f_Y(y) dy \\ &\qquad\qquad = \lambda(t) dt \mathbb{E}_Y [G(t, \mathbf{X}_t + h(t, Y)) - G(t, \mathbf{X}_t)], \end{aligned} \tag{2.59}$$

the HJB equation for the problem (2.56) turns to

$$\frac{\partial V}{\partial t} + \mathcal{L}^{\alpha^*(t,\mathbf{x})}(V) + f(t, \mathbf{X}, \alpha^*(t, \mathbf{x})) + \lambda(t)\mathbb{E}_Y\left[G(t, \mathbf{X}_t + h(t, Y)) - G(t, \mathbf{X}_t)\right] = 0. \quad (2.60)$$

2.6.3 Existence and Uniqueness of solutions to the state jump diffusion process

Let us consider the general case where the jump amplitude depends also on the state variable and the controlled parameter α i.e. when a jump occurs at time t, it is of size $h(t, \mathbf{X}_t, y)$. The state variable evolves according to the law

$$d\mathbf{X}_t = b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{W}_t + \int_\Gamma h(t, y, \mathbf{X}_t)\aleph(dt, dy). \quad (2.61)$$

To ensure the existence of the stochastic integrals and the existence of a solution to (2.61), we will need the following conditions, to which we will refer to as *the general existence conditions.*

Let $\alpha \in \mathcal{A}$, and assume that the following conditions hold :

There exists constants C and L, $\in \mathbb{R}$ such that $\forall\ \mathbf{x}_1\ \mathbf{x}_2 \in \mathbb{R}^n$

$$\begin{aligned}
&\int_0^\Upsilon |b(t, \mathbf{X}, \alpha)|^2\, dt < \infty \\
&\int_0^\Upsilon \int_\Gamma |h(t, y, \mathbf{X}_t)|^2 f_Y(y)dydt < \infty \\
&|b(t, \mathbf{X}, \alpha)|^2 + |\sigma(t, \mathbf{X}, \alpha)|^2 + \lambda(t)\int_\Gamma |h(t, y, \mathbf{X}_t)|^2 f_Y(y)dy \le C\left(1 + |\mathbf{x}|^2\right) \\
&|b(t, \mathbf{x}_1, \alpha) - b(t, \mathbf{x}_2, \alpha)| \le L\,|\mathbf{x}_1 - \mathbf{x}_2| \\
&|\sigma(t, \mathbf{x}_1, \alpha) - \sigma(t, \mathbf{x}_2, \alpha)|^2 + \lambda(t)\int_\Gamma |h(t, y, \mathbf{x}_1) - h(t, y, \mathbf{x}_2)|^2 f_Y(y)dy \le L^2\,|\mathbf{x}_1 - \mathbf{x}_2|^2.
\end{aligned} \quad (2.62)$$

Theorem 2.6.1 *If the functions $b(t, \mathbf{X}, \alpha)$, $\sigma(t, \mathbf{X}, \alpha)$ and $h(t, \mathbf{X}, Y)$ are linearly bounded* *by the constant* C *and satisfy the uniform Lipschitz-condition with the constant* L *i.e. they satisfy the general conditions then the stochastic differential equation (2.61) has for every* $\alpha \in \mathcal{A}$ *a unique solution* $\mathbf{X}_t \in \mathbb{R}^n$ *adapted to the combined filtration* $\mathcal{F}_t^\star$ *with sample paths that are continuous from the right with left-hand limits. Moreover,*

$$\mathbb{E}\left[|\mathbf{X}_t|^2\right] < \infty \quad \forall\, t. \tag{2.63}$$

See Gihman and Skorohod (1979), Theorem 3.4p.138 & p.156 for the proof.

2.7 The Martingale Optimality Principle

In this section, we will show a different technique that will enable us to verify if our guessed value is optimal. In section (2.5) we show the assumptions and sufficient conditions for the guessed value function to be optimal under the so call *verification Theorem* for state diffusion processes. We will now elaborate on a different technique called *The Martingale Optimality Principle* that is applicable not only to diffusion processes but also to processes with jumps. This technique is mostly applicable when the objective function is to optimise a terminal finite function. Again, we consider the stochastic control problem:

$$V(t,\mathbf{x}) = \max_{\substack{\alpha_s \\ t\le s\le \Upsilon}} \mathbb{E}\left[\int_t^\Upsilon f(s,\mathbf{X}_s,\alpha_s)ds + B(\mathbf{X}_\Upsilon) \mid \mathcal{F}_t\right].$$

Theorem 2.7.1 *Assume that we have found a control strategy* $\{\alpha_s^*\}_{t\le s\le\Upsilon}$ *and a function* $M(t,\mathbf{X})$ *such that*

1) $M^\star(t,\mathbf{x}) = \mathbb{E}\left[\int_t^\Upsilon f(s,\mathbf{X}_s,\alpha_s^\star)ds + B(\mathbf{X}_\Upsilon) \mid \mathcal{F}_t\right]$

2) $M^\star(t,\mathbf{X})$ *is a Martingale with respect to* $\mathcal{F}_t$

3) For any other control strategy $\{\alpha_s\}_{t\le s\le\Upsilon}$, $M(t,\mathbf{X})$ *is a Supermartingale with respect* *to $\mathcal{F}_t$, then*

a) $\{\alpha^*_s\}_{t\leq s\leq \Upsilon}$ *is the optimal control strategy for the problem, and*

b) For all possible initial states, $M^\star(t,\mathbf{x})$ *coincides with the value function i.e.*

$$\forall\ (t,\mathbf{x}),\ M^\star(t,\mathbf{x}) = V(t,\mathbf{x}). \tag{2.64}$$

The proof is simple and relies on the Markov property of the chain together with assumptions (1), (2) and (3), see Korn (2003).

Chapter 3

Pricing in a Monopoly Insurance Market

3.1 Introduction

In this chapter, we will analyse the behaviour of an insurance company operating in a purely monopolistic market where the company has total control over the market. At any given time t, the insurance company can set a premium p_t for each policy and this premium cannot be challenged by any customer or by any other insurance company or by any law. Under these circumstances, the customer has two courses of action. She either purchases the policy or turns it down.

A customer who decides to take up the policy will be paying a premium continuously at the prevailing rate p_t set at time t by the insurance company and will be covered for as long as she continues to pay the prevailing premium. Also, at any time t, a customer is free to withdraw from the policy and therefore stops paying the premiums and hence stop being covered. There is no penalty charged at withdrawal to the customer and no benefit is paid if the customer has not made any claim for the random period T that she has been covered for. This is analogous to the classical risk theory in which the premium is paid at a constant rate and the insured is covered for as long as premium is paid. We also consider the fact that the structure of the insurance market imposes some conditions on the premium set by the insurance company through the number of customers willing to buy the policy at a given price. It is evident that in a true monopoly market, the demand of a good is a decreasing function of price. The objective is to elaborate on an optimal strategy that will dictate to the company the premium to be charged in order to achieve a specific goal. We will first look at maximising the discounted wealth of the company and then evaluate the risk of bankruptcy under the optimal strategy. In the next section, we will focus on the mathematical modelling of the market and set the framework for the optimisation problem.

3.2 The market structure

3.2.1 The demand function

We denote by q_t the number of policy-holders at time t . It is evident that at a future time $t + dt$, this number could go up due to new business (sale of new policies) or go down due to withdrawal. This suggests a model by which in a small interval of time dt, there could be an increase or decrease in the number of policy-holders. A candidate model for the dynamic of q_t would be a *birth and death process* . We assume that for any contract subscribed, the contract remains in force for an exponential time T with parameter κ i.e. $T \sim exp(\kappa)$. The probability that a contract that has been in force for a time t is no longer in force for an extra interval of time h is

$$P\left[T < t + h \mid T > t\right] = P[T \leq h] = 1 - e^{\kappa h}.$$

We may obtain the withdrawal rate as a limit of the probability.

$$\lim_{h \to dt} P[T \leq h] = \kappa dt + o(dt) \approx \kappa dt. \tag{3.1}$$

Above is the death rate of a single contract so that the total rate while q_t contracts are in force is κq_t. The birth process will be modelled according to the way the company generates new business. If we denote n_t as the rate of generating of new business, a first equation for the dynamic of the business size may be written as :

$$dq_t = -\kappa q_t dt + n_t dt. \tag{3.2}$$

An important question about n_t is how new business is generated with time. One trivial and partial answer to the question is to allow n_t to depend on the reputation of the company. Since reputation at time t is positively correlated to the number of policy-holders at time t $(q_t,)$ we make n_t depend on q_t through an increasing function of q_t. say $h(.)$. It is also true that a company that charges higher premiums will attract fewer policy-holders. Consequently, n_t will be a decreasing function of the premium charged, say $g(p_t)$. These two points are expressed in terms of the following equation:

$$n_t = h(q_t)g(p_t) \tag{3.3}$$

where $h(.)$ increases with q_t and $g(.)$ decreases with respect to p_t.
We will be calling g the demand function. For simplicity, we take h as an identity function. This leaves us with the full dynamic of q_t as

$$dq_t = q_t\bigg(g(p_t) - \kappa\bigg)dt. \tag{3.4}$$

The next section will be devoted to the choice of the demand function $g(p_t)$ that describes the reaction of customers as premium changes.

3.2.2 Choice of the demand function

In section (3.2.1), we denoted as $g(p_t)$ the demand function of the monopoly market which mathematically represents the growth rate of the exposure in the absence of withdrawals. In this section, we consider some desirable features of the demand function and make an appropriate choice. One would wish that the demand function should prevent the company from charging a negative premium in order to build exposure which we will denote as the *non-negative premium condition.* A candidate function will therefore be a function which achieves its maximum value at zero.

Since there are a finite number of customers in the insurance market, we will also like our demand function to have a finite number of policy-holders when the premium is set at its lowest level i.e. $p_t = 0$ which we denote as the *non-diverging condition* when premium is at lowest value. This outlaws the set of functions g such that

$$\lim_{p_t \to 0} g(p_t) = \infty.$$

One feature that we have mentioned for the demand function in section (3.2.1) is its decreasing aspect with respect to the premium. We will also wish to ensure that the demand function prevents the insurance company from making excessive expected profits by charging more and more on the premium.

Our goal therefore in the choice of the demand function is to assess how the expected profit changes as a function of the price charged.

We propose a quadratic candidate function which satisfies the non everywhere increasing profit condition, the non diverging condition when the premium is at minimum and the non-negative premium condition:

$$g(p_t) = \begin{cases} l - np_t^2 & \text{if} \quad p_t < \sqrt{\frac{l}{n}} \\ 0 & \text{if} \quad p_t \geq \sqrt{\frac{l}{n}} \end{cases} . \tag{3.5}$$

Together with the withdrawal rate κ, we can express the exposure dynamic as

$$dq_t = q_t(m - np_t^2)dt \quad \text{where} \quad m = l - \kappa. \tag{3.6}$$

In the control theory literature and most importantly in the risk theory literature, few attempt in incorporating a dynamic demand function has been successful. In this thesis, some demand functions coming from the economics literature have been explored unsuccessfully. The exponential demand function which also satisfies the eligibility criteria appears to be analytical intractable in solving the dynamic programming problem whereas a linear demand function yields the uninteresting bang-bang control solution.

3.2.3 The claim size and claim severity

We model the aggregate claims as a compound Poisson process with a randomly distributed jump amplitude conditioned on a Poisson jump in time. If we consider the policy of a customer say i, a claim occurs with rate λ per unit time and when the claim occurs, it is of size Y_i where Y_i is a random variable assumed to be independent and identically distributed i.e. $Y_i = Y$. At any time t, the exposure of the company denoted by q_t is a function of time; this implies that the rate at which the claim of size Y occur is λq_t which depends on time by solving Equation (3.6).

Denote by S_t the cumulative claim size up to time t, and by N_t the cumulative number of claims, we can represent the aggregate claim amount S_t at time t as a marked Poisson process with a time dependent rate $\lambda(t) = \lambda q_t$ and amplitude Y on a sample space Γ.

Hence

$$S_t = \int_0^t \int_\Gamma y \aleph(dt, dy), \tag{3.7}$$

where $\aleph(dt, dy)$ is the Poisson random measure, on the product space of the marked space Γ and the time ($R_+ * \Gamma$). The measure $\aleph(dt, dy)$ assigns unit mass (y, t) if a mark y arrives at time t. It can be decomposed into the measure of the jump amplitude and the Poisson measure since the jump process N_t and the jump amplitude are independent. This produces

$$\aleph(dy, dt) = \lambda q_t dt f(y) dy \tag{3.8}$$

where $f(y)$ is the density function of the claim size distribution. Combining equations (3.7) and (3.8), we may compute the expected change in the aggregate claim in a small interval of time $[t, t + dt]$ as

$$\mathbb{E}\left[dS_t \ |\mathcal{F}_t\right] = \int_\Gamma y\lambda q_t dt f(y) dy. \tag{3.9}$$

In the Risk processes literature, the aggregate process S_t is represented as

$$S_t = \sum_{i=1}^{N_t} Y_i \quad \text{and} \quad dS_t = Y dN_t \quad \text{where} \quad N_t \sim Po(\lambda q_t).$$

3.2.4 The wealth process

We assume that the company starts with wealth $W_0 = x \geq 0$ at time $t = 0$. We also assume that the share holders of the company requires a return of rate α. Changes in the company's wealth in a small interval of time dt are brought about by:

- Excess return on capital required by shareholders i.e. the interest element on shareholders investment expressed as $\alpha W_t dt$ in a small interval of time dt.

- Premium income paid by policy-holders $q_t p_t dt$ in a small interval of time dt

- And by the payment of claims $dS_t = YdN_t$ occurred during the time interval dt

Thus the dynamic of the wealth is described as

$$dW_t = -\alpha W_t dt + p_t q_t dt - YdN_t. \tag{3.10}$$

This is decomposed in the continuous change and the jump change as

$$dW_t = dW_t^{cont} - dW_t^{jump} \quad \text{where} \quad dW_t^{jump} = \int_\Gamma y \; dN_t f(y)dy. \tag{3.11}$$

3.3 Derivation of the HJB equation

In this section, we consider $\mathcal{F}$ to be the σ algebra generated by the marked Poisson process $S_{\{t>0\}}$ and $\mathcal{F}_t$ to be the history of the process S_t up to time t. The company starts at time $t = 0$ with initial wealth $W_0 = x$ aiming to maximise the expected discounted wealth J_0 in an infinite horizon.

Let us denote by $\mathbf{X}_t$ the vector of state variables at time t, $\mathbf{X}_t = [W_t, q_t]$. The objective is to maximise the discounted wealth of the company up to time ∞ by controlling the premium charged p_t. Let

$$\tilde{J}(t, \mathbf{X}_t) = \mathbb{E}\left[\int_t^\infty e^{-\beta s} W_s ds \;\Big|\; \mathcal{F}_t\right] \quad \text{and} \quad \tilde{V}(t, \mathbf{X}_t) = \max_{\substack{p_s \\ t \leq s \leq \infty}} \tilde{J}(t, X_t), \tag{3.12}$$

the value function can be split up into the disjoint intervals $[t, t+dt]$ and $[t, \infty]$:

$$\begin{aligned}
\tilde{V}(t, \mathbf{X}_t) &= \max_{\substack{p_s \\ t \leq s \leq \infty}} \mathbb{E}\left[\int_t^\infty e^{-\beta s} W_s ds \;\Big|\; \mathcal{F}_t\right] \\
&= \max_{\substack{p_s \\ t \leq s \leq \infty}} \mathbb{E}\left[\int_t^{t+dt} e^{-\beta s} W_s ds + \int_{t+dt}^\infty e^{-\beta s} W_s ds \;\Big|\; \mathcal{F}_t\right] \\
&= \max_{\substack{p_s \\ t \leq s \leq \infty}} \mathbb{E}\left[\int_t^{t+dt} e^{-\beta s} W_s ds + \mathbb{E}\left[\int_{t+dt}^\infty e^{-\beta s} W_s ds \;\Big|\; \mathcal{F}_{t+dt}\right] \;\Bigg|\; \mathcal{F}_t\right].
\end{aligned}$$

Applying the principle of dynamic programming, we can can decompose the maximisation into disjoint intervals $[t, t+dt]$ and $[t+dt, \infty]$:

$$\begin{aligned}
\tilde{V}(t, \mathbf{X}_t) &= \max_{\substack{p_s \\ t \leq s \leq t+dt}} \mathbb{E}\left[\int_t^{t+dt} e^{-\beta s} W_s ds + \max_{\substack{p_s \\ t+dt \leq s \leq \infty}} \mathbb{E}\left[\int_{t+dt}^{\infty} e^{-\beta s} W_s ds \;\middle|\; \mathcal{F}_{t+dt}\right] \;\middle|\; \mathcal{F}_t\right] \\
&= \max_{\substack{p_s \\ t \leq s \leq t+dt}} \mathbb{E}\left[\int_t^{t+dt} e^{-\beta s} W_s ds + \tilde{V}(t+dt, \mathbf{X}_{t+dt}) \;\middle|\; \mathcal{F}_t\right] \\
&= \max_{\substack{p_s \\ t \leq s \leq t+dt}} \mathbb{E}\left[\int_t^{t+dt} e^{-\beta s} W_s ds + \tilde{V}(t, \mathbf{X}_t) + d\tilde{V} \;\middle|\; \mathcal{F}_t\right].
\end{aligned} \tag{3.13}$$

Taking the limit when dt tends to zero of both sides yields :

$$\max_{p_t} \left\{ e^{-\beta t} W_t dt + \mathbb{E}\left[d\tilde{V} \mid \mathcal{F}_t\right] \right\} = 0. \tag{3.14}$$

We can now apply the Itô formula for jump processes to obtain

$$\begin{aligned}
\mathbb{E}\left[d\tilde{V} \mid \mathcal{F}_t\right] &= \frac{\partial \tilde{V}}{\partial t} dt + \frac{\partial \tilde{V}}{\partial q_t} \mathbb{E}\left[dq_t \mid \mathcal{F}_t\right] + \frac{\partial \tilde{V}}{\partial W_t} \mathbb{E}\left[W_t \mid \mathcal{F}_t\right] \\
&\quad + \lambda q_t dt \int_{\Gamma} \Big(\tilde{V}[t, W_t - y, q_t] - \tilde{V}[t, W_t, q_t]\Big) f_Y(y) dy.
\end{aligned} \tag{3.15}$$

In order to proceed to the optimisation with less variables, we make a change of the value function:

$$V(t, \mathbf{X}_t) = e^{\beta t} \tilde{V}(t, \mathbf{X}_t) \tag{3.16}$$

and continue the maximisation of the transformed value function $V(t, \mathbf{X}_t)$. The derivatives and difference equations according to equation (3.16) yield

$$\begin{aligned}
&\frac{\partial \tilde{V}}{\partial t} = -\beta e^{-\beta t} V(t, X_t) + e^{-\beta t} \frac{\partial V}{\partial t}, \qquad \frac{\partial \tilde{V}}{\partial q_t} = e^{-\beta t} \frac{\partial V}{\partial q_t}, \qquad \frac{\partial \tilde{V}}{\partial W_t} = e^{-\beta t} \frac{\partial V}{\partial W_t} \\
&\tilde{V}[t, W_t - y, q_t] - \tilde{V}[t, W_t, q_t] = e^{-\beta t}\Big(V[t, W_t - y, q_t] - V[t, W_t, q_t]\Big).
\end{aligned} \tag{3.17}$$

Substituting equation (3.15) and the different expressions of equation (3.17) into equation (3.14) after multiplying it by $e^{\beta t}$, we obtain :

$$\max_{p_t} \left\{ \begin{array}{l} W_t dt - \beta V dt + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial q_t} \mathbb{E}\left[dq_t \mid \mathcal{F}_t\right] + \frac{\partial V}{\partial W_t} \mathbb{E}\left[dW_t \mid \mathcal{F}_t\right] \\ + \lambda q_t dt \int_\Gamma \Big(V[t, W_t - y, q_t] - V[t, W_t, q_t] \Big) f_Y(y) dy \end{array} \right\} = 0. \quad (3.18)$$

We now show that $V(t, \mathbf{X}_t)$ has less variables than $\tilde{V}(t, \mathbf{X}_t)$.

Theorem 3.3.1 *$V(t, \mathbf{X}_t)$ is independent of t.*

Proof The proof relies on showing that $V(t, \mathbf{X}_t = \mathbf{x}) = V(0, \mathbf{x})$ for all t, which will mean that $V(t, \mathbf{X}_t = \mathbf{x})$ is independent of t and therefore $\frac{\partial V}{\partial t} = 0$. The argument will be based on comparison by first making a one to one correspondence between the control $(t, \mathbf{x})$ and those for $(0, \mathbf{x})$ using a translation.

Let us denote by a any control for $(0, \mathbf{x})$, where $a \in A$ and $a(s)$ is defined for $s \geq 0$ and let z be its corresponding path with $z(0) = \mathbf{x}$.

Now consider $\tilde{a}(u) = a(u-t)$ and $\tilde{z}(u) = z(u-t)$ for $u \geq t$ with $z(t) = \mathbf{x}$, where z and $\tilde{z}$ solve the same differential equation. If we had started with $\tilde{a}$ and $\tilde{z}$, we would have obtained a and z by inverse translation. And the performance by a and $\tilde{a}$ are just the same, i.e.

$$\begin{aligned} J(t, \mathbf{X}_t) &= e^{-\beta t} \int_t^\infty e^{-\beta u} h(\tilde{z}(u), \tilde{a}(u)) du \\ &= \int_t^\infty e^{-\beta(u-t)} h(\tilde{z}(u), \tilde{a}(u)) du \\ &= \int_0^\infty e^{-\beta s} h(\tilde{z}(s+t), \tilde{a}(s+t)) ds \quad \text{by taking} \quad u = s + t \\ &= \int_0^\infty e^{-\beta s} h(z(s), a(s)) ds. \end{aligned}$$

Now taking maximum over a for $(0, \mathbf{x})$, and $\tilde{a}$ for $(0, \mathbf{x})$ respectively, we obtain

$$\begin{aligned} V(t, \mathbf{X}_t = \mathbf{x}) &= \max_{\tilde{a} \in A} \int_t^\infty e^{-\beta(u-t)} h(\tilde{z}(u), \tilde{a}(u)) du \\ &= \max_{a \in A} \int_t^\infty e^{-\beta s} h(z(s), a(s)) ds \\ &= V(0, \mathbf{x}) \end{aligned}$$

Therefore $\frac{\partial V}{\partial t} = 0$. □

This in turn simplifies equation (3.18) which after substitution of the state variables dynamic yields

$$\max_{p_t} \left\{ \begin{array}{l} W_t - \beta V - \frac{\partial V}{\partial W_t}(\alpha W_t - p_t q_t) + \frac{\partial V}{\partial q_t} q_t \big[g(p_t) - \kappa\big] \\ \\ +\lambda q_t \int_\Gamma \big[V(W_t - y, q_t) - V(W_t, q_t)\big] f(y) dy \end{array} \right\} = 0. \qquad (3.19)$$

Equation (3.19) is a first order PIDE with no boundary condition. It is a fact that there is no systematic method to obtain solution of PIDE equations derived from the jump stochastic control problem or control problems with jump and diffusion. We will therefore seek a solution through a guess method and verify that the solution indeed satisfies the requirements imposed on the state dynamic and the value function. The next section will be devoted to a search for a solution to the PIDE.

3.4 Solution to the HJB Equation

The approach to solve the PIDE (3.19) is based on a trial of possible functions of two variables. A candidate function is proposed as the solution and the verification theorem is applied to prove that it is indeed the solution of the problem. Due to the nature of the cost function which is linear in the wealth process, we suggest a transformed value function of the form

$$V = AW_t + Bq_t \quad \text{where} \quad A \in \mathbb{R} \ \text{ and } \ B \in \mathbb{R}. \tag{3.20}$$

Given the proposed value function (3.20), after substituting the different partial derivatives, the PIDE (3.19) turns to:

$$W_t\left[1 - A(\alpha + \beta)\right] + q_t \max_{p_t}\left\{ -\beta B + Ap_t - \lambda A\pi + B\big[g(p_t) - \kappa\big]\right\}, \tag{3.21}$$

where π is the expected value of the claim distribution function i.e.

$$\pi = \int_\Gamma y f(y)dy.$$

A value function of the form (3.20) makes the PIDE separable in W_t and q_t regardless of the demand function. Notice that the maximisation is carried out only on the second part of equation (3.21). In the rest, we will consider the quadratic demand function of section (3.2.1) that describes at best the reaction of customers as the premium changes. We substitute the quadratic demand function in (3.21) which yields

$$1 - A(\beta + \alpha) = 0 \quad \Longrightarrow A = \frac{1}{\alpha + \beta} \tag{3.22}$$

and

$$\max_{p_t}\left\{ -\beta B - \lambda\pi A + Ap_t + B[l - \kappa - np_t^2]\right\} = 0. \tag{3.23}$$

3.4.1 Optimal premium and conditionality

We obtain the candidate optimal premium by applying the first order maximisation condition to equation (3.23).

First Order Condition

Differentiating the expression in the brackets of (3.23) with respect to p_t and setting it to be equal to zero yields the optimal premium,

$$\hat{p_t} = \frac{A}{2nB}. \tag{3.24}$$

Second order condition

The second order condition imposes the condition that the second derivative of the bracketed expression in (3.23) is negative to make the optimal value (3.24) a maximum. This is translated as

$$-2nB < 0 \Longrightarrow B > 0. \tag{3.25}$$

It is trivial to see that since $A \geq 0$, the optimal premium is positive if the second order condition is met.

To solve for B, we replace the value of the premium by the value of the optimal premium from expression (3.24) into equation (3.23) and remove the maximisation. B then solves:

$$B(m - \beta) - \lambda\pi A + A\hat{p_t} - nB\hat{p_t}^2 = 0,$$

which after substitution turns to the quadratic equation

$$(m - \beta)B^2 - \lambda\pi AB + \frac{A^2}{4n} = 0. \tag{3.26}$$

The discriminant Δ of (3.26) will impose a condition on the admissible set where:

$$\Delta = \lambda^2\pi^2A^2 - \frac{A^2}{n}(m - \beta) = A^2\left[\lambda^2\pi^2 - \frac{m - \beta}{n}\right] = A^2\Delta'. \tag{3.27}$$

3.4.2 Analysis of the possible roots

We are looking for positive real roots given by the second order condition. This implies that the discriminant must be positive to obtain a solution. It is also true that from the properties of quadratic equations, if there are real roots B_1 and B_2, their product is

$$\frac{A^2}{4n(m-\beta)}.$$

This in turn suggests that if there are two real roots then they are of different sign if $(m-\beta) < 0$ and of the same sign if $(m-\beta) > 0$. We will first look at the possibility of a positive root for a sufficient condition ensuring the positiveness of the discriminant.

Case $(m-\beta) < 0$

Assuming $m - \beta < 0 \implies \beta > m$,

the discriminant is non negative. Two different real roots B_1 and B_2 are solutions to (3.26) :

$$B_1 = \frac{\lambda\pi A - A\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}}{2(m-\beta)} \tag{3.28}$$

and

$$B_2 = \frac{\lambda\pi A + A\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}}{2(m-\beta)}. \tag{3.29}$$

Under the condition $(m-\beta) < 0$, $B_2 < 0$ and $B_1 > 0$. Consequently the only candidate solution for the HJB equation satisfying the second order condition is B_1.

Case $m - \beta = 0$

When $m - \beta = 0$, the quadratic equation (3.26) simplifies to a linear equation with solution

$$B = \frac{A}{4n\lambda\pi} > 0$$

Case $m - \beta > 0$

In the case where $m - \beta > 0 \quad \Longrightarrow \quad \beta < m$,

the discriminant is not always positive. To ensure the existence of real roots, restrictions are needed.

The analysis will therefore be made in the interval

$0 < m - \beta \leq n\lambda^2\pi^2 \Longleftrightarrow m - n\lambda^2\pi^2 \leq \beta \leq m$.

This ensures that $m - \beta > 0$ and the discriminant is positive. Solving for B in the above range yields B_1 and B_2 as in equations (3.28) and (3.29). Using again the product and sum of real roots of a quadratic equation if they exist, we obtain:

$$B_1 B_2 = \frac{A^2}{4n(m-\beta)} > 0 \quad \text{and} \quad B_1 + B_2 = \frac{\lambda\pi A}{(m-\beta)} > 0 \tag{3.30}$$

Expression (3.30) implies that $B_1 > 0$ and $B_2 > 0$ thus both B_1 and B_2 satisfy the second order condition. In the following sections, a choice will be made as to which of B_1 and B_2 is the solution to the control problem. This requires the verification theorem under the optimal strategy which will be the subject of the next section.

3.5 Verification Theorem

This section is concerned with the different tools that enable us to verify if our candidate function is indeed the value function. The analysis is based on the verification theorem applied to the infinite horizon discounted running cost problem. Through this analysis, we will be able to discard the value of B that does not satisfy the verification theorem and the result of the analysis will provide the value function if it is of the form expressed in equation (3.20). We start with the fact that the solution $V(t, \mathbf{X}_t) = e^{\beta t}\tilde{V}(t, \mathbf{X}_t)$ of the HJB equation exists and the first and second order condition in the admissibility region $\beta \leq \beta_{min}$ are met. In order for $\tilde{V}(t, \mathbf{X}_t)$ to be the value function, it needs to satisfy the convergence to zero condition which is the subject of the next section.

3.5.1 Convergence condition of the value function

We assume that there exists a value B_i for $i = 1, 2$ for the optimal control problem under the optimal strategy which predicts to charge the constant premium $\hat{p}_t = \frac{A}{2nB_i}$ with value function

$$Z_i(t, \hat{W}, \hat{q}) = e^{-\beta t}(A\hat{W}_t + B_i\hat{q}_t).$$

We would like to assess the conditions under which

$$\lim_{t\to\infty} \mathbb{E}\left[Z_i(t, \hat{W}, \hat{q})\right] = 0$$

where $\hat{W}_t$ and $\hat{q}_t$ represent respectively the wealth process and the exposure under the optimal strategy. Under the optimal strategy the exposure q_t evolves as :

$$d\hat{q}_u = \hat{q}_u g(\hat{p}_u)du. \tag{3.31}$$

Since the optimal premium is constant, the solution of the differential equation is given by

$$\hat{q}_t = q_0 \exp\left(t\left(m - \frac{1}{4n}\left(\frac{A}{B_i}\right)^2\right)\right) \tag{3.32}$$

The dynamic of the wealth process under optimal strategy evolves as :

$$d\hat{W}_u = -\alpha\hat{W}_u du + \hat{p}_u\hat{q}_u du - dS_u. \tag{3.33}$$

Equation (3.33) is an ordinary differential equation that may be solved using the integrating factor:

$$\hat{W}_t e^{\alpha t} = W_0 + \frac{A}{2nB_i}\int_0^t q_u e^{\alpha u}du - \int_0^t YdN_u e^{\alpha u}.$$

The claim size and the claim severity are independent i.e.

$$\mathbb{E}\left[Yd\hat{N}_u\right] = \mathbb{E}\left[Y\right]\mathbb{E}\left[d\hat{N}_u\right] = \pi\lambda\hat{q_u}du.$$

Taking the expectation of both sides while interchanging the order of integration yields

$$\mathbb{E}\left[\hat{W}_t\right] = e^{-\alpha t}\left[W_0 + \left(\frac{A}{2nB_i} - \lambda\pi\right)\int_0^t \hat{q_u}e^{\alpha u}du\right].$$

For simplicity of the notation, we let $G_i = m - \frac{1}{4n}(\frac{A}{B_i})^2$.

We can now substitute $\hat{q_u}$ by expression (3.32) which yields

$$\mathbb{E}\left[\hat{W}_t\right] = e^{-\alpha t}\left[W_0 - \left(\frac{A}{2nB_i} - \lambda\pi\right)\frac{q_0}{G_i + \alpha}\right] + \frac{q_0}{G_i+\alpha}e^{tG_i}. \tag{3.34}$$

Using (3.32), we compute the expectation of the value function

$$\mathbb{E}\left[e^{-\beta t}(A\hat{W}_t + B\hat{q_t})\right] = Ae^{-(\alpha+\beta)t}\left[W_0 - \left(\frac{A}{2nB_i} - \lambda\pi\right)\frac{q_0}{G_i+\alpha}\right] + q_0\left[\frac{A}{G_i+\alpha} + B\right]e^{(G_i-\beta)t}. \tag{3.35}$$

The first term of (3.35) tends to zero when t tends to ∞, therefore

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-\beta t}(A\hat{W}_t + B_i\hat{q_t})\right] = 0 \iff G_i - \beta < 0.$$

3.5.2 Optimal premium under the convergence condition

In this section, we will use the result of section (3.5.1) to choose the correct value of B_i that completely defines the optimal premium $\hat{p_t}$ in the admissibility region.

Theorem 3.5.1 *In the set of admissibility region i.e. $\Delta \geq 0$, B_1 is solution to the quadratic equation satisfying the second order condition and the convergence condition of the value function.*

Proof The proof is split up in the different two intervals of the admissibility set. We recall that the discriminant is positive when

$$m - \beta \leq n\lambda^2\pi^2 \text{ i.e. } \beta > \beta_{min} = m - n\lambda^2\pi^2.$$

We will therefore prove the theorem in the three different cases

a) $m - \beta < 0$

b) $m - \beta = 0$

c) $0 < m - \beta \leq n\lambda^2\pi^2$.

a) Case $m - \beta < 0$

In section (3.4.2), we proved that when $m - \beta < 0$ there is only one candidate solution B_1 for equation (3.26) and satisfying the second order condition for the control problem. We only need to show that the convergence condition is also satisfied :

$$G_1 - \beta = (m - \beta) - \frac{1}{4n}(\frac{A}{B_1})^2 < 0 \quad \text{since} \quad m - \beta < 0.$$

Therefore the value function converges to zero and B_1 is indeed the solution.

b) Case $m - \beta = 0$

For $m = \beta$, we obtained in section (3.4.2) a linear equation in B with solution

$$B = \frac{A}{4n\lambda\pi}$$

$$G - \beta = m - \beta - \frac{1}{4n}(\frac{A}{B})^2 = -\frac{1}{4n}(\frac{\lambda\pi}{4n})^2 < 0.$$

Therefore B is the solution .

c) Case $0 < m - \beta \leq n\lambda^2\pi^2$

In this case, there are two candidate solutions B_1 and B_2 (see section (3.4.2)) that satisfy the second order condition. We will therefore show that only B_1 satisfies the convergence to zero condition.

Let

$$B_\epsilon = \frac{\lambda\pi A + \epsilon A\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}}{2(m-\beta)} \quad \text{where} \quad \epsilon = \begin{cases} -1 \implies B_\epsilon = B_1 \\ 1 \implies B_\epsilon = B_2 \end{cases} .$$

The objective is to prove that

$$G_\epsilon - \beta \begin{cases} < 0 & \text{when} \quad \epsilon = -1 \\ > 0 & \text{when} \quad \epsilon = 1 \end{cases} . \tag{3.36}$$

Proof

$$\begin{aligned} G_\epsilon - \beta &= m - \beta - \frac{1}{4n}(\frac{A}{B_\epsilon})^2 = m - \beta - \frac{1}{n}\frac{(m-\beta)^2}{(\lambda\pi + \epsilon\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}})^2} \\ &= \frac{m-\beta}{(\lambda\pi + \epsilon\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}})^2}\left[(1+\epsilon^2)\left[\lambda^2\pi^2 - \frac{m-\beta}{n}\right] + 2\lambda\pi\epsilon\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}\right] \end{aligned}$$

Since $m - \beta > 0$, $G_\epsilon - \beta$ takes the sign of

$$I_\epsilon = (1+\epsilon^2)\left[\lambda^2\pi^2 - \frac{m-\beta}{n}\right] + 2\lambda\pi\epsilon\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}.$$

***** $\epsilon = 1 \iff B_\epsilon = B_2$

$I_1 > 0$ therefore B_2 does not satisfy the convergence to zero condition.

* $\epsilon = -1 \Longleftrightarrow B_\epsilon = B_1$

$I_{-1} = 2(m-\beta)\left(\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}\right)\left[\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}} - \lambda\pi\right] < 0$

therefore B_1 satisfies the convergence to zero condition. □

Sections **a)** and **c)** show that when $\Delta \geq 0$ i.e. $\beta \geq \beta_{min} = m - n\lambda^2\pi^2$ there is only one root solution B_1 of equation (3.26) satisfying the second order optimisation condition and the convergence condition of the verification theorem under optimal strategy. □

Corollary 3.5.2

$$For \qquad \beta \geq \beta_{min}, \quad \hat{p}_t = \frac{A}{2nB_1} = \lambda\pi + \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}} \tag{3.37}$$

and the value function of the control problem is

$$\tilde{V}(t,w,q) = e^{-\beta t}(Aw + Bq) \tag{3.38}$$

where

$$A = \frac{1}{\alpha+\beta} \quad and \quad B = \begin{cases} \frac{\lambda\pi A - A\sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}}{2(m-\beta)} & if\ \beta \neq m \\ \frac{A}{4n\lambda\pi} & if\ \beta = m \end{cases}.$$

Proof For $\beta > \beta_{min}$ and $\beta \neq m$,

$$\hat{p}_t = \frac{A}{2nB_1} = \frac{m-\beta}{n}\frac{1}{\lambda\pi - \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}}. \tag{3.39}$$

Multiplying the top and bottom by $\lambda\pi - \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}$ yields the result .

For $\beta = m$,

$$\hat{p}_t = \frac{A}{2nB} = 2\lambda\pi = \lambda\pi + \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}} \quad \text{for} \quad \beta = m.$$

The rest of the proof follows from the consequence of the verification theorem. □

3.6 Optimality in the non admissible region

The objective in this section is to assess the existence of a strategy that outperforms other strategies in the non admissible region $\beta < \beta_{min}$.
We showed in section (3.4.1) that an optimal policy if it exists predicts to charge a constant premium. In the rest of this section, we will prove that there is no constant premium strategy that is optimal in the non admissible region.

3.6.1 Integrability under a constant premium strategy

In this section, we would like to assess the conditions under which a constant premium strategy $\tilde{p}$ diverges the value function.

Proposition 3.6.1 *For a constant premium strategy $\tilde{p}$, the resulting value function $\tilde{V}$ diverges when $m - n\tilde{p}^2 - \beta > 0$*

Proof Under a constant premium strategy $\tilde{p}$, the exposure at time $u > t$ is computed as

$$\tilde{q_u} = \tilde{q_t} e^{\int_t^u (g(\tilde{p}) - \kappa)ds} = \tilde{q_t} e^{(m - n\tilde{p}^2)(u-t)}. \tag{3.40}$$

The wealth at any time $s > u > t$ follows

$$\tilde{W_s} e^{-\beta s} = e^{-(\alpha+\beta)s} \left[\tilde{W_t} e^{\alpha t} + \tilde{p} \int_t^s q_u e^{\alpha u} du - \int_t^s Y dN_u e^{\alpha u} \right].$$

Applying Fubini's theorem on the conditional expectation yields,

$$\mathbb{E}\left[\tilde{W_s} e^{-\beta s} | \mathcal{F}_t \right] = e^{-(\alpha+\beta)s} \left[\tilde{W_t} e^{\alpha t} + \left(\tilde{p} - \lambda\pi \right) \int_t^s \tilde{q_u} e^{\alpha u} du \right].$$

Substituting expression (3.40) yields

$$\mathbb{E}\left[\tilde{W_s} e^{-\beta s} | \mathcal{F}_t \right] = e^{-(\alpha+\beta)s} \left[\tilde{W_t} e^{\alpha t} + \left(\tilde{p} - \lambda\pi \right) \tilde{q_t} e^{-(m-n\tilde{p}^2)t} \int_t^s e^{(m-n\tilde{p}^2+\alpha)u} du \right].$$

Taking $\tilde{G} = m - n\tilde{p}^2$ and rearranging we obtain,

$$\mathbb{E}\left[\tilde{W}_s e^{-\beta s}|\mathcal{F}_t\right] = e^{\alpha t}\left[\tilde{W}_t - \frac{\tilde{q}_t(\tilde{p}-\lambda\pi)}{\tilde{G}+\alpha}\right]e^{-(\alpha+\beta)s} + e^{-t\tilde{G}}\frac{\tilde{q}_t(\tilde{p}-\lambda\pi)}{\tilde{G}+\alpha}e^{(\tilde{G}-\beta)s}. \tag{3.41}$$

Integrating both sides of (3.41) from t to ∞ while considering the order of the integration yields

$$\tilde{V}(t,q,w) \geq \left[\tilde{W}_t - \frac{\tilde{q}_t(\tilde{p}-\lambda\pi)}{\tilde{G}+\alpha}\right]\frac{e^{-\beta t}}{\alpha+\beta} + e^{-t\tilde{G}}\frac{\tilde{q}_t(\tilde{p}-\lambda\pi)}{\tilde{G}+\alpha}\int_t^{\infty} e^{(\tilde{G}-\beta)s}ds,$$

from which we deduce that $\tilde{V}(t,q,w)$ will diverge if $\tilde{G}-\beta > 0$. □

3.6.2 No optimal strategy in the non admissible region $\beta \in [0, \beta_{min})$

In this section, we want to effectively prove that when the discriminant is not positive i.e. $\beta < \beta_{min}$, there is no constant optimal strategy. We will proceed by constructing a family of policies p_ϵ that leads to a value function V_ϵ that diverges to infinity with reference to Proposition (3.6.1).

Proposition 3.6.2 *For $0 \leq \beta < \beta_{min}$, there is no constant optimal policy.*

Proof Let's consider again the dynamic of the wealth process and the exposure under any policy that predicts charging a constant premium p_ϵ, using the results in section (3.5), and denoting by W_t^ϵ and q_t^ϵ the respective wealth and exposure under the constant policy strategy ϵ, we obtain :

$$q_u^\epsilon = q_t^\epsilon e^{(m-np_\epsilon^2)(u-t)}$$

and

$$W_s^\epsilon e^{-\beta s} = e^{-(\alpha+\beta)s}\left[W_t^\epsilon e^{\alpha t} + p_\epsilon \int_t^s q_u^\epsilon e^{\alpha u}du - \int_t^s YdN_u^\epsilon e^{\alpha u}\right].$$

Taking $G^\epsilon = m - n\bar{p}_\epsilon^2$, yields

$$V^\epsilon(t,q,w) = \left[W_t^\epsilon - \frac{q_t^\epsilon(p_\epsilon - \lambda\pi)}{G^\epsilon + \alpha}\right]\frac{e^{-\beta t}}{\alpha+\beta} + e^{-tG^\epsilon}\frac{q_t^\epsilon(p^\epsilon - \lambda\pi)}{G^\epsilon + \alpha}\int_t^\infty e^{(G^\epsilon-\beta)s}ds. \tag{3.42}$$

Denoting as

$$J_\epsilon = \int_t^\infty e^{[G^\epsilon - \beta]s}ds,$$

our objective is to construct a policy p_ϵ under a force of interest $\beta \in [0, \beta_{min})$ such that

a) $(p_\epsilon - \lambda\pi)$ is non negative,

b) $G^\epsilon + \alpha$ is positive, and

c) J_ϵ diverges to ∞.

Proposition 3.6.3 *The family of policies*

$$P_\epsilon = \lambda\pi + \epsilon \textit{ where } \quad \epsilon \in \left[0, \sqrt{\frac{m-\beta}{n}} - \lambda\pi\right) \quad \textit{and} \quad \beta \in [0, \beta_{min})$$

diverges the value function V_ϵ.

Proof Firstly we will show that under the constraint $\beta < \beta_{min}$, the upper bound of ϵ is well defined i.e. positive.

$$\begin{aligned}\beta < \beta_{min} &\implies \frac{m-\beta}{n} > \lambda^2\pi^2 \quad \text{(The expression in the root is positive)}\\ &\implies \sqrt{\frac{m-\beta}{n}} - \lambda\pi > 0 \quad \text{(The upper bound of } \epsilon \text{ is non negative)}\end{aligned}$$

By construction of the domain of ϵ, $p_\epsilon - \lambda\pi = \epsilon \geq 0$.

We now consider $G^\epsilon + \alpha = m + \alpha - np_\epsilon^2$ and show that it is positive.

$$\lambda\pi \leq p_\epsilon \leq \sqrt{\frac{m-\beta}{n}} \implies \beta - m \leq -np_\epsilon^2.$$

Adding α to both sides and rearranging gives

$$\alpha + \beta \leq m + \alpha - np_\epsilon^2 = G^\epsilon + \alpha \geq 0.$$

Lastly, we consider $G^\epsilon - \beta = m - \beta - np_\epsilon^2$ and show that it is positive.

$$\begin{aligned} \lambda\pi \leq p_\epsilon < \sqrt{\frac{m-\beta}{n}} \quad &\Longrightarrow \quad \beta - m < -np_\epsilon^2 \\ &\Longrightarrow \quad 0 < m - \beta - np_\epsilon^2 = G^\epsilon - \beta. \end{aligned}$$

Using the result in section (3.6.1), and since $G_\epsilon - \beta > 0$, the indefinite integral J_ϵ diverges. The three proofs demonstrate that under the restriction $\beta < \beta_{min}$, the value function V_ϵ diverges when policy p_ϵ is applied. □

Since $\epsilon \in \left[0, \sqrt{\frac{m-\beta}{n}} - \lambda\pi\right)$, there are infinitely many strategies p_ϵ indexed by ϵ which provide infinite wealth for the company. Consequently there is no optimal strategy. □

3.7 Optimal Premium and Exposure's Growth

3.7.1 The premium

Under the optimal strategy in the admissibility region $\beta \geq \beta_{min}$, the optimal premium may be simplified to

$$\hat{p}_t = \frac{A}{2nB_1} = \lambda\pi + \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}. \tag{3.43}$$

If we take $\hat{p}_t = \hat{p}_t(\lambda, \pi, \beta)$, as the claim severity or claim size increases, so is the optimal premium $\hat{p}_t$ as expected. It is also an increasing function of the discount rate β. The minimum value of the optimal premium is attained when the force of interest rate is at minimum i.e.

$$\beta = \beta_{min} \Longrightarrow \hat{p}(\lambda, \pi, \beta_{min}) = \lambda\pi. \tag{3.44}$$

Equation (3.44) gives the value of the pure premium which is just sufficient on average to pay off the claims. In the actuarial literature, the second term of the premium

$$\varpi = \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}}$$

may be considered as the loaded premium over the pure premium and increases in line with the risk free rate β. Consequently, the optimal premium increases in line with the risk free rate β through the loading factor ϖ .

Theorem 3.7.1 *Starting with a positive wealth, the future expected wealth is positive under the optimal strategy.*

Proof First notice that $\hat{p}_t - \lambda\pi = \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}} \geq 0$.
We recall the dynamic of the wealth under the optimal strategy as :

$$d\hat{W}_t = -\alpha\hat{W}_t dt + \hat{p}_t\hat{q}_t dt - dS_t. \tag{3.45}$$

The conditional expectation of the change in wealth given the history at time t, is given by

$$\mathbb{E}\left[d\hat{W}_t \mid \mathcal{F}_t\right] = \left[-\alpha\hat{W}_t + \hat{q}_t\left(\hat{p}_t - \lambda\pi\right)\right]dt. \tag{3.46}$$

Equation (3.46) means that if there is any profit made in selling the insurance i.e. $\hat{p}_t \geq \lambda\pi$ then a part of the profit is transferred to the investors as a proportion of the current wealth i.e. $\alpha\hat{W}_t$. The expected wealth at any future time s is given by

$$\mathbb{E}\left[\hat{W}_s \mid \mathcal{F}_t\right] \;=\; e^{-\alpha s}\left[\hat{W}_t e^{\alpha t} + \left(\hat{p}_t - \lambda\pi\right)\hat{q}_t e^{-t\hat{G}} \int_t^s e^{u\hat{G}+\alpha} du\right].$$

Since $\hat{p}_t - \lambda\pi \geq 0$ then $\mathbb{E}\left[\hat{W}_s \mid \mathcal{F}_t\right] \geq 0$ whenever $\hat{W}_t \geq 0$. □

3.7.2 The growth rate of the exposure under optimal strategy

In this section we will look at the growth rate of the exposure and assess what factors if any influence it under the optimal strategy. Let us recall the dynamic of the exposure:

$$dq_t = q_t(g(p_t) - \kappa)dt = q_t(m - np_t^2)dt. \tag{3.47}$$

Under the optimal strategy which predicts charging a constant premium rate, a solution to equation (3.47) given the initial exposure q_0 may be expressed as

$$q_t = q_0 e^{t(m - n\hat{p}_t^2)}. \tag{3.48}$$

Let us also denote by $G = m - n\hat{p}_t^2$ the growth rate of the exposure. Since the premium charged depends on the risk free rate β, and the growth rate G depends on the premium charged, we would like to assess how the growth rate depends directly on the risk free rate by assessing the function

$$G = G(\beta) \quad \text{for} \quad \beta > \beta_{min}.$$

At the minimum value of the risk free rate $\beta = \beta_{min}$, the optimal premium is $p_t = \lambda\pi$ and the growth rate is β_{min} which is the maximum value of the exposure growth rate :

$$\beta = \beta_{min} \implies p_t = \lambda\pi \quad \text{and} \quad G(\beta_{min}) = \beta_{min} = G_{max},$$

where G_{max} represents the maximum value of the exposure growth rate.

One should also notice that since the risk free rate is positive then $G_{max} > 0$. As the risk free rate increases, the premium increases and the Growth rate decreases.

There exists a turning point rate $\bar{\beta} > \beta_{min}$ such that

a) If $\beta \leq \bar{\beta}$ then $G(\beta) > 0$ i.e. the exposure grows exponentially at rate G.

b) If $\beta \geq \bar{\beta}$ then $G(\beta) < 0$ i.e the exposure decays exponentially at rate G.

The turning rate $\bar{\beta}$ is solution to the equation :

$$\hat{p}(\bar{\beta}) = \lambda\pi + \sqrt{\lambda^2\pi^2 - \frac{m - \bar{\beta}}{n}} = \sqrt{\frac{m}{n}}, \tag{3.49}$$

which solves to :

$$\bar{\beta} = 2n\sqrt{\frac{m}{n}}\Big[\sqrt{\frac{m}{n}} - \lambda\pi\Big]. \tag{3.50}$$

3.8 Significance of results

Equation (3.43) expresses the premium as an increasing function of the discount rate β. In section (3.7.2), we discovered that as the premium increases through the average claim per unit or through the discount rate, the exposure decreases .

The discount rate β can therefore be considered as a measure of short term aim for the insurance company operating under an infinite time horizon. For a high value of the discount rate β, the positive expected profit will be higher since the premium charged will be higher. Since each profit made is discounted, the profit made in the distant future will be discounted with a higher discount factor than in the near future, therefore the present value of distant future profit will be less significant than the present value of near future profit. On the exposure side, the exposure will be decreasing therefore distant future total business will be less valuable than near future business. The insurance company will consequently have preference for short term aim than long term aim. If the discount rate is low. the company will be much more interested in building the exposure in the near future and charge the constant premium to the high number of policy-holders.

We also discover that for an optimal policy to exist, the insurance company must apply at least the minimum discounting rate β_{min} which yields the pure premium just sufficient to cover the claims. Interestingly, when the minimum premium is charged, the exposure's growth rate is identical to the discounting rate β_{min}.

3.9 Ruin Probability under optimal strategy

In this section, we focus on the likelihood of ruin under the optimal strategy that predicts charging constant premium. Insurance risk has always been a main concern in the actuarial contexts. The literature in this field is extensive both in the classical risk theory as well as in the the controlled Risk theory.
In Sundt and Josef (1995), the authors provide a numerical solution scheme to the differential equation that is satisfied by the infinite time ruin probability in continuous time where the aggregate claim process is a compound process with an exponential claim size distribution, and the premium rate as well as the interest rate are both assumed constant. In Albrecher and Kainhofer (2000), ruin probability was computed under a classical risk process in the presence of a non-linear dividend barrier. Simulation techniques were presented in Yuanjiang, Xucheng, and Xhang (2003) for a general claim size distribution. Other authors investigate the probability of ruin in the framework of control theory. Hipp and Taksar (2000) investigate the control of new business in the context of general risk processes in order to minimise the ruin probability, and Schmidli (2000) investigates the optimal excess of loss reinsurance policy to minimise the ruin probability. The interested reader is referred to Ma and Sun (2000), Albrecher, Reinhold, and Tichy (2000), Schmidli (1991) and Grandell (1990) for more details.
Unlike most ruin functions, which depend only on wealth, our probability of ruin will depend also on the exposure explicitly. We will in the rest of this section derive the integro-differential equation satisfied by the ruin probability and evaluate the ruin probability numerically.

3.9.1 Derivation of the Integro-differential equation

The objective is to compute the probability of ruin $\phi(w, q)$ when the state variables at time t are $\hat{W}_t = w$ and $\hat{q}_t = q$. We recall that under the optimal strategy, the growth rate is constant and is denoted by $\hat{G} = m - n\hat{p}_t^2$. Starting at time t with the initial wealth w and the initial exposure q, the ruin probability at time $t + dt$ depends on the change in the wealth and the exposure added to the occurrence or not of a claim in the small interval of time dt. If a claim does not occur in the interval $[t, t + dt]$, the ruin probability at time $t + dt$ will be $\psi(w + d\hat{w}_t^{cont}, q + d\hat{q}_t)$ and the probability that no claim occurs in $[t, t + dt]$ is $1 - \lambda q dt$.

If a claim occurs in $[t, t + dt]$, and is of size y, the ruin probability at time $t + dt$ will be $\psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t)$ and the probability that a claim occurs in $[t, t + dt]$ is $\lambda q dt$. Combining these two scenarios, we obtain

$$\begin{aligned}\psi(w, q) &= (1 - \lambda q dt)\psi(w + d\hat{w}_t^{cont}, q + d\hat{q}_t) \\ &+ \lambda q dt \left[\int_0^\infty \psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t) f_Y(y) dy\right].\end{aligned} \tag{3.51}$$

But

$$\psi(w + d\hat{w}_t^{cont}, q + d\hat{q}_t) = \psi(w, q) + \psi_w d\hat{w}_t^{cont} + \psi_q d\hat{q}_t \quad \text{where} \quad d\hat{q}_t = q\hat{G}dt. \tag{3.52}$$

Substitute (3.52) in (3.51) yields

$$\begin{aligned}\psi(w, q) &= (1 - \lambda q dt)\left[\psi(w, q) + \psi_w d\hat{w}_t^{cont} + \psi_q d\hat{q}_t\right] + \\ &\lambda q dt \left[\int_0^{w + d\hat{w}_t^{cont}} \psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t) f_Y(y) dy + \right. \\ &\left. \int_{w + d\hat{w}_t^{cont}}^\infty \psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t) f_Y(y)\right].\end{aligned} \tag{3.53}$$

For $\quad y > w + d\hat{w}_t^{cont}, \quad \psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t) = 1,$

therefore

$$0 = \psi_w(-\alpha w + \hat{p}q) + \psi_q q\hat{G} - \lambda q \left[\psi(w,q) + \psi_w d\hat{w}_t^{cont} + \psi_q dq_t\right] + \lambda q \left[\int_0^{w+d\hat{w}_t^{cont}} \psi(w + d\hat{w}_t^{cont} - y, q + d\hat{q}_t) f_Y(y) dy + 1 - F_Y(w + d\hat{w}_t^{cont})\right].$$

Tending the small change of time dt incorporated in the change of the wealth and the change of the exposure to zero bearing in mind that

$$\lim_{dt\to 0} d\hat{w}_t^{cont} = \lim_{dt\to 0} d\hat{q}_t = 0.$$

yields

$$0 = \psi_w(-\alpha w + \hat{p}q) + \psi_q q\hat{G} - \lambda q\psi(w,q) + \lambda q \left[\int_0^w \psi(w-y,q) f_Y(y) dy + 1 - F_Y(w)\right]$$

Let us consider the survival probability $\phi = 1 - \psi$. With the first derivatives

$$\phi_q = -\psi_q \quad \text{and} \quad \phi_w = -\psi_w,$$

we may express the Integro-differential equation in terms of the survival probability:

$$\lambda q\phi - q\hat{G}\phi_q + (\alpha w - \hat{p}q)\phi_w - \lambda q \left[\int_0^w \phi(w-y,q) f_Y(y) dy\right] = 0 \qquad (3.54)$$

with boundary conditions

$$\lim_{w\to\infty} \phi(w,q) = 1, \quad \lim_{w\to 0} \phi(w,q) = \phi^0(q) \quad \text{and} \quad \lim_{q\to\infty} \phi(w,q) = \phi_\infty(w).$$

where $\phi^0(q)$ and $\phi_\infty(w)$ are to be determined.

The analytical solution of the differential equation (3.54) constitutes a challenge in the case of the general claim size distribution and also in the case of an exponential claim distribution. We will proceed using Monte Carlo simulation to compute the probability of ruin.

3.9.2 Monte Carlo Simulation of the Ruin probability

This section is concerned with a simulation approach of the wealth process under optimal strategy. We explain the logic underlying the choice of the parameter values.
We first make a reasonable assumption that the demand function is in such a way that charging a pure premium $\hat{p} = \lambda\pi$ is attractive and as result the exposure grows at a rate $a > 0$. This implies that

$$G_{max} = m - n(\lambda\pi)^2 = a. \tag{3.55}$$

We also assume that the implied premium that leaves the exposure constant i.e. $G = 0$ is higher than the pure premium by a loading factor $b > 0$ i.e.

$$G = m - n\hat{p}_t^2 = 0 \implies \hat{p}_t = \sqrt{\frac{m}{n}} \quad \text{and} \quad \sqrt{\frac{m}{n}} = (1+b)\lambda\pi. \tag{3.56}$$

Combining equations (3.55) and (3.56) yields

$$n = \frac{a}{b(2+b)(\lambda\pi)^2}. \tag{3.57}$$

In the simulation , we choose $a = 15\%$, $b = 10\%$ and n and m respectively as in (3.57) and (3.55).

3.10 Sensitivity analysis on ruin probability

Figure (3.2) shows the result of sensitivity analysis on the parameters affecting the finite time probability of ruin. The time horizon is taken to be 100 units.

Increase in Initial wealth

The north-west graph on figure (3.2) shows the relationship between the ruin probability and the initial wealth. The graph shows a decrease in the probability of ruin as the initial wealth increases which is conform with the classical theoretical result for distribution function having a closed form solution for the ruin probability. As initial wealth increases, there are increasing reserve to handle bankruptcy hence the likelihood of ruin decreases. The actual probability of ruin are tabulated in table (3.1)

Wealth	60	110	160	210	260	310	360	410	460	510
Prob(Ruin)	0.33	0.27	0.23	0.20	0.14	0.13	0.11	0.10	0.07	0.07

Table 3.1: **Probability of ruin as wealth increases**

Shift in the Demand function

The north-east graph on figure (3.2) shows as increase in the ruin probability as the demand function shifts upward. In the simulation, an upward shift in the demand function is carried out by increasing $G_{max} = a$ of equation (3.55). In fact, a policyholder makes a subscription decision based on the premium set and not on her utility function. An upward shift in the demand function means the policy is more attractive to the policy holders and since the claim size and claim severity are constant, the premium diminishes through the loading factor. Under the simulation conditions, the loading factor may be computed as

$$\varpi = \sqrt{\lambda^2\pi^2 - \frac{m-\beta}{n}} = \sqrt{\lambda^2\pi^2 - \frac{a+n(\lambda\pi)^2-\beta}{n}} = \sqrt{\frac{-a+\beta}{n}} = \sqrt{b(2+b)(\lambda\pi)^2(-1+\frac{\beta}{a})} \tag{3.58}$$

which decreases as the demand shift upward hence a decrease in the ruin probability.

Increase in the average claim size and claim severity

The south-west graph on figure (3.2) shows a decreasing trend of the probability of ruin as a function of the mean claim size. This may be explained by the fact that ceteris paribus, an increase in the average claim size increases the loading factor and hence diminishes the likelihood of bankruptcy, whereas the south-east graph shows an inconclusive trend about the probability of ruin as the claim severity increases. This may be explained by the fact that though the loading factor increases with the claim severity, the benefit can be fully, partially or under offset by the potential of a high number of claims hence the inconclusive effect.

3.11 Summary

In this chapter, we have considered the pricing of non-life insurance premium in a monopolistic market driven by a demand function. The obtention of an analytical solution to the problem has been made possible due to the form of the demand function considered as the aggregate of the withdrawal of policies and the new business generation. The optimal premium rate obtained using dynamic programming principle is time independent; this suggests that the problem could have been solved deterministically using the calculus of variation or by following the following steps:

1) Consider the premium as constant

2) Compute the expected wealth for any time s

3) Compute the expected running cost by integrating the discounted expected wealth from t to ∞ which produce an equation similar to equation (3.42)

4) Differentiate the expected total running cost obtained with respect to the constant premium p_t for $\beta > \beta_{min}$ yields the expected answer.

We also notice that the discount rate β can be considered as a measure of short term aim of the company. The solvency criteria imposes a minimum discounting rate equal to the maximum exposure growth rate and which when adopted yield the pure premium. Under the optimal strategy, the probability of ruin decreases as initial wealth or claim size increases, increases as the demand increases and is inconclusive as the claims severity changes.

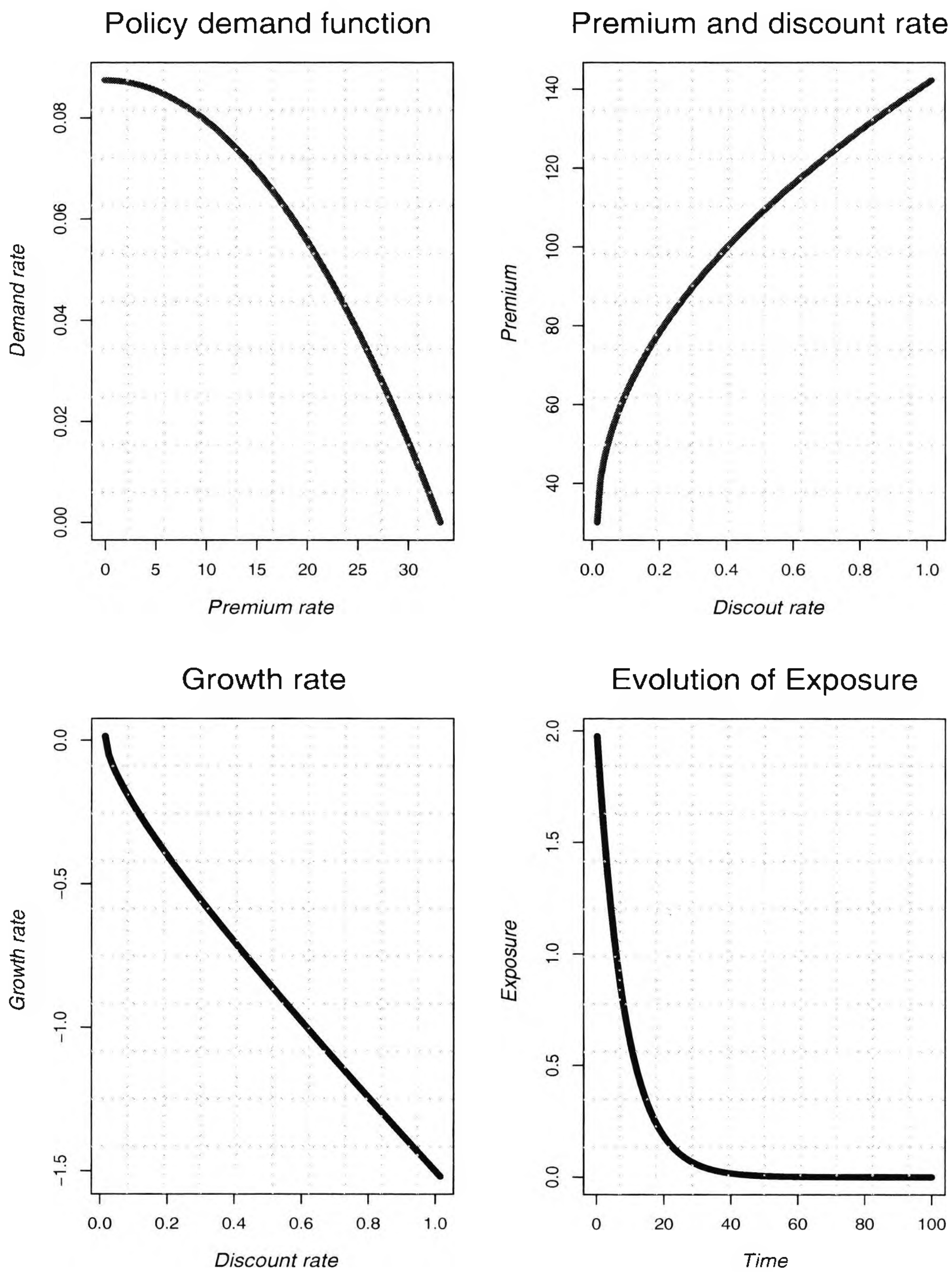

Figure 3.1: **Optimal Premium**

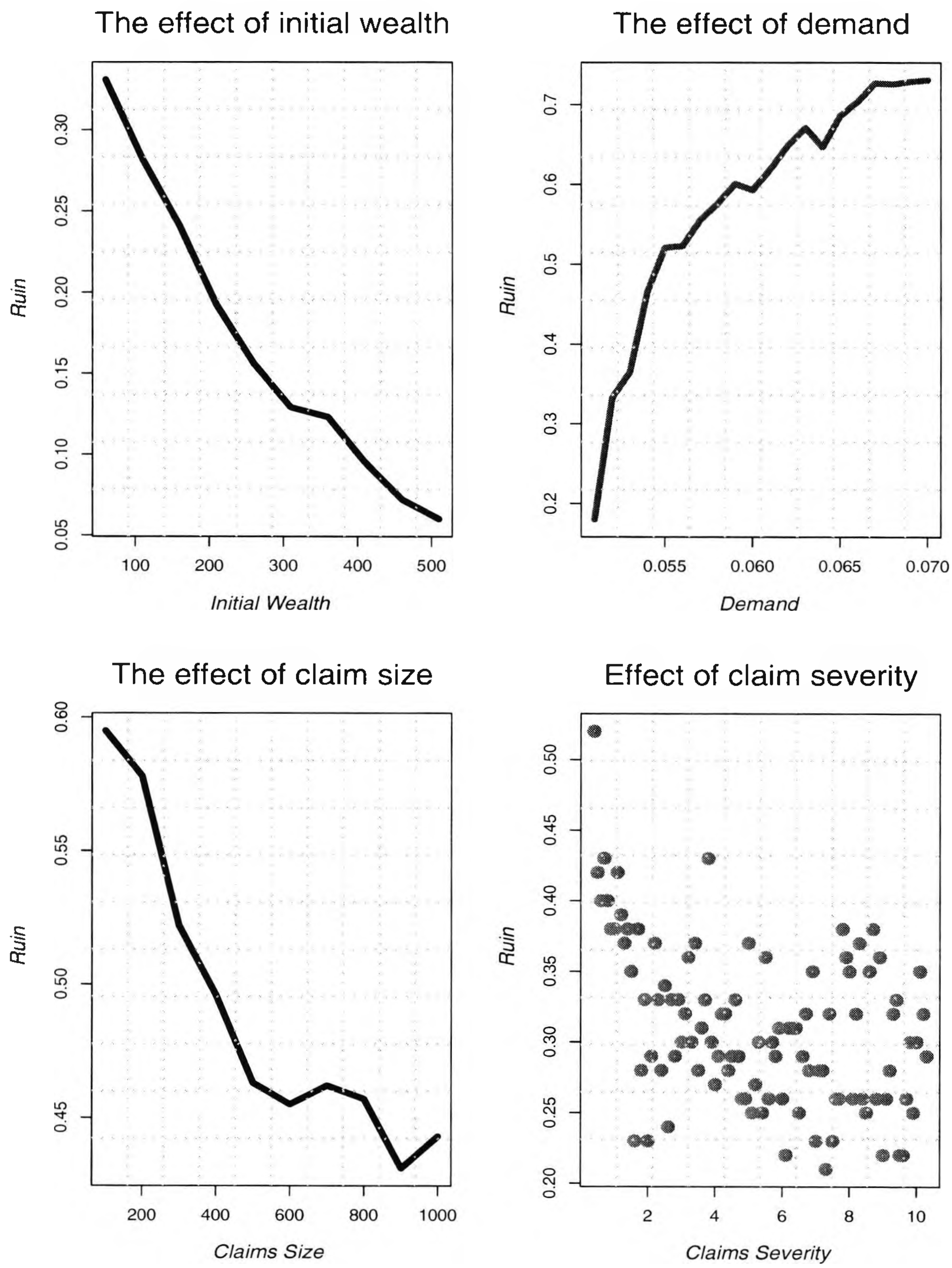

Figure 3.2: **Sensitivity of the ruin probability**

Chapter 4

Optimal Asset Allocation in the Context of Insurance Risk Process

4.1 Introduction

In this chapter, we consider the problem of an insurance company operating in a regime where the price of the insurance product is set by the forces of demand and supply. Higher prices will generally result in a lower demand. We also consider the fact that the insurance company has no control over the price of the product. The company is faced with an uncontrollable stochastic flow which we denote in the insurance literature by a risk process. The objective is to maximise the terminal utility function of the company by implementing an asset allocation strategy. The capital market is composed of two assets: a risky asset and a risk free asset.

Among the different methods used to solve stochastic control problems, two main approaches are widely used in the asset allocation problem. The traditional stochastic control approach and the martingale approach which originated from the paper Harrison and Kreps (1979b) and Harrison and Kreps (1981). In this work, we will be using the traditional approach pioneered by Merton which relies on the stochastic control approach .

Asset allocation is a classical problem on which many works have already been done. The problem originated from financial economics where an economic agent seeks to achieve a goal such as minimising the probability of ruin or maximising a function of wealth by allocating optimally her wealth in different assets.

Consider the pioneering Investment and Consumption problem of Merton (1971) where an agent with initial wealth W_0 seeks to maximise her utility function on a final time horizon T while investing an amount Θ_t in a risky asset and consuming an amount c_t. Merton derives an analytical solution for a class of utility functions for finite time and for infinite time horizon. In this chapter, we consider solving Merton's problem with no consumption, with income arriving deterministically whilst the outgoing are stochastic. The analytical solution of the optimal policy and the value function are obtained and the verification theorem is applied to validate them.

4.2 The Model

We assume that there is only one risky asset whose price is denoted as S_t and one risk free asset that is priced as C_t. The evolution of the risky asset follows a geometric Brownian motion which satisfies the stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dB_t) \tag{4.1}$$

where B_t is a standard Brownian motion, μ is a positive constant (the drift) and σ is a positive constant (the volatility). The risk free asset evolves deterministically according to the differential equation

$$dC_t = rC_t dt \tag{4.2}$$

where r is the risk free interest rate on all bonds.

The company operating in the insurance industry receives a constant premium p per policy for as long as the policy is in force and pays a claim of amount Y if a claim occurs. This is the classical Risk process on which extensive work has been done.
Particularly in the simple case where the volume of business is taken to be unity, there exist an extensive literature dealing with different objectives functions.

In Browne (1995), the author considered a firm facing an uncontrollable stochastic flow modelled as a diffusion process. The firm's wealth is invested in the risky and risk free asset. Under exponential utility, Browne derived an analytical solution for the optimal investment in the risky asset with the firm objective to maximise the terminal utility function as well as minimising the probability of ruin which is defined as the first time the firm's wealth reaches 0.
In Browne (1999), the author derived the optimal investment in the risky asset with the objective function to maximise the probability of reaching a certain goal (as an example, terminal wealth reaching a certain higher value within a given time).

Schmidli (2003) considered a classical risk model with the possibility of reinsurance and the possibility of investment in a risky asset. The objective was to control the risky investment allocation and the proportional reinsurance to be taken. For a small claim distribution, he proves that the optimal strategy converges to the asymptotically optimal strategy as the capital increases to infinity.

In this work we also consider the volume of business q_t which represents the number of policyholders at time t. We consider the natural dependency of the business volume on the premium charged p through the demand function

$$dq_t = q_t g(p)dt \quad \text{thus} \quad q_t = q_0 e^{tg(p)} \tag{4.3}$$

where $g(p)$ is a decreasing function of the premium.

Though the literature in risk processes is very large, there are few works that consider the volume of business as a variable of the process. This is to my knowledge the first attempt to include the volume of business implied by a demand function in the risk process under the dynamic stochastic control framework. If we denote by R_t the risk process, R_t is best described by its dynamic

$$dR_t = pq_t dt - \int_\Gamma y \aleph(dt, dy) \tag{4.4}$$

where $\aleph(dt, dy)$ is a marked point process on $\mathbb{R}_+ \times \Gamma$ and N_t is the claim counting process at time t when business volume is q_t. We assume that N_t is a Poisson process with rate λq_t where λ is a positive constant representing the claim frequency of each individual policy. Consequently, $\aleph(dt, dy)$ is a marked Poisson process with jump amplitude $h(t, y) = y$. Since the claim severity in the model depends on the business volume, the more policies we have in our portfolio, the more claims we are likely to pay out but also the more premium income we surely receive from the policyholders.

A more general question arising in the premium pricing context is to determine if the premium income is sufficient to cover the claim payout. If we denote the expected claim amount by $\pi = \mathbb{E}[Y]$, the safety and profit principle recommends to load the theoretical premium in order to decrease the chance of ruin and to allow for profit, which is translated into $p > \pi$ where p is the premium charged.

In this work, we do not impose such condition which is usually regarded as a static profit maximisation condition. Instead, we adopt a dynamic view on a fixed time horizon while considering the demand function that dictates the premium to be set giving a number of policies.

The insurance company is financed by shareholders who receive a dividend payout continuously with time. The dividend paid at any time is proportional to the wealth of the company with a constant of proportionality α. The company invests an amount Θ_t in the risky asset and the remaining $W_t - \Theta_t$ in the risk free asset. The wealth process therefore evolves as:

$$\begin{aligned} dW_t &= -\alpha W_t dt + \Theta_t \frac{dS_t}{S_t} + (W_t - \Theta_t) r dt + dR_t \\ &= -\alpha W_t dt + \Theta_t(\mu dt + \sigma dB_t) + (W_t - \Theta_t) r dt + p q_t dt - \int_\Gamma y \aleph(dt, dy) \\ &= dW_t^{cont} - \int_\Gamma y \aleph(dt, dy). \end{aligned}$$

The objective of the investor is to maximise the finite time horizon utility by optimally allocating the wealth between risky and non risky assets. While the choice of utility function is subjective, there are specific utility functions that have many objective criteria associated with them. We assume that the investor has an exponential utility function on the form

$$U(w) = \gamma - \eta e^{-\nu w}$$

where $\gamma > 0$, $\eta > 0$ and $\nu > 0$. The exponential utility has a constant absolute risk aversion parameter ν . This is deduced from the fact that $-\frac{U''(w)}{U'(w)} = \nu$, see Pratt (1964). It can be seen that the exponential utility function plays a prominent role in insurance mathematics and actuarial practise since they are the only utility functions under which the principle of "zero utility" gives a fair premium that is independent of the level of the reserves or the wealth of an insurance company (see Gerber (1979)) .

The company is not allowed to short the stock, i.e. we do not allow the optimal investment in the risky asset to be negative. Since the terminal utility of wealth is a function of the terminal wealth W_T which is itself random, we will consider maximising the expected utility of wealth giving the current wealth W_t. Mathematically, we would like to maximise

$$J(t, \mathbf{X}) = \mathbb{E}\left[U(W_T) \,\middle|\, \mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the filtration up to time t generated by both the Brownian motion and the marked Poisson process. We will approach the problem by deriving the differential equation satisfied by the value function

$$V(t, \mathbf{X}) = \max_{\Theta_t} J(t, \mathbf{X}) = \max_{\Theta_t} \mathbb{E}\left[U(W_T) \,\middle|\, \mathcal{F}_t\right]$$

using the dynamic programming technique. The partial differential equation is on the integro-differential form due to the jump process and will be solved in the subsequent sections analytically.

4.3 The HJB EQUATION

In this section, we will derive the integro-differential equation satisfied by the value function with the conditions under which a solution of the equation exists.

Let

$$\begin{aligned} V(t, \mathbf{X}) &= \max_{\substack{\Theta_s \\ t \le s \le T}} \mathbb{E}\left[U(W_T) \,\middle|\, \mathcal{F}_t\right] \\ &= \max_{\substack{\Theta_s \\ t \le s \le T}} \mathbb{E}\left[U(W_T) \,\middle|\, \mathcal{F}_{t+dt}\right| \left.\mathcal{F}_t\right]. \end{aligned} \tag{4.5}$$

Using the principle of dynamic programming, we decompose the maximisation over disjoint interval $[t, t+dt]$ and $[t+dt, T]$:

$$\begin{aligned} V(t,\mathbf{X}) &= \max_{\substack{\Theta_s \\ t\leq s\leq t+dt}} \mathbf{E}\left[\max_{\substack{\Theta_s \\ t+dt\leq s\leq T}} \mathbb{E}\left[U(W_T) \,\Big|\, \mathcal{F}_{t+dt}\right] \,\Big|\, \mathcal{F}_t\right] \\ &= \max_{\substack{\Theta_s \\ t\leq s\leq t+dt}} \mathbb{E}\left[V(t+dt, X_{t+dt}) \,\Big|\, \mathcal{F}_t\right] \\ &= \max_{\substack{\Theta_s \\ t\leq s\leq t+dt}} \mathbb{E}\left[V(t, X_t) + dV \,\Big|\, \mathcal{F}_t\right] \end{aligned} \tag{4.6}$$

Taking the limit when dt tends to zero of both sides yields :

$$\max_{\Theta_t}\left\{\mathbb{E}\left[dV \,\Big|\, \mathcal{F}_t\right]\right\} = 0.$$

An application of Ito' s formula for Jump and diffusion processes, see (Appendix C) yields

$$\begin{aligned} \mathbb{E}\left[dV \,\Big|\, \mathcal{F}_t\right] &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial q_t}\mathbb{E}\left[dq_t \,\Big|\, \mathcal{F}_t\right] + \frac{\partial V}{\partial W_t}\mathbb{E}\left[dW_t^{cont} \,\Big|\, \mathcal{F}_t\right] + \frac{\partial^2 V}{\partial^2 W_t}\mathbb{E}\left[dW_t^{cont^2} \,\Big|\, \mathcal{F}_t\right] \\ &+ \eta q_t dt \int_\Gamma \Big(V[t, W_t - y, q_t] - V[t, W_t, q_t]\Big) f_Y(y)dy. \end{aligned} \tag{4.7}$$

where dW_t and dq_t follow respectively the differential equations (4.5) and (4.3). Thus the HJB equation is

$$\begin{aligned} &\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial W_t}\left[\Big(r-\alpha\Big)W_t + pq_t\right]dt + \max_{\Theta_t}\left\{(\mu - r)\Theta_t\frac{\partial V}{\partial W_t} + \frac{\sigma^2\Theta_t^2}{2}\frac{\partial^2 V}{\partial^2 W_t}\right\}dt \\ &+ \frac{\partial V}{\partial q_t}q_t g(p)dt + \lambda q_t dt\int_\Gamma \Big(V[t, W_t - y, q_t] - V[t, W_t, q_t]\Big) f_Y(y)dy = 0. \end{aligned} \tag{4.8}$$

Let us assume that the HJB equation (4.8) has a classical solution. To obtain the optimum, we use the first order condition by differentiating (4.8) with respect to Θ_t and setting the derivatives to zero. This yields

$$\hat{\Theta}_t = -\frac{(\mu - r)}{\sigma^2}\frac{\frac{\partial V}{\partial W_t}}{\frac{\partial^2 V}{\partial^2 W_t}}. \tag{4.9}$$

The second order condition of optimality requires $\frac{\partial^2 V}{\partial^2 W_t} < 0$.

We will therefore be looking for a solution V to (4.8) satisfying the second order condition together with the no short sale condition imposed on the amount of wealth invested on the risky investment. This latter constraint can be translated into the optimal value by imposing a further constraint on the first derivative of the value function i.e. $\frac{\partial V}{\partial W_t} > 0$ which is true since the risky investment drift is greater than the risk free rate ($\mu \geq r$). In fact, if

$$\mu \geq r \qquad \frac{\partial V}{\partial W_t} > 0 \qquad \text{and} \qquad \frac{\partial^2 V}{\partial^2 W_t} < 0, \tag{4.10}$$

the optimal value of the risky asset investment is positive. In the set of solutions of the HJB equation, we will be looking for solution that has a positive first derivative and a negative second derivative with respect to the wealth together with the boundary condition $V(W_T, q, p) = U(W_T)$. Substituting (4.9) in (4.8) yields after simplification:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial q_t} q_t g(p) + \frac{\partial V}{\partial W_t}\left[\Big(r - \alpha\Big)W_t + pq_t\right] - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 \frac{\left(\frac{\partial V}{\partial W_t}\right)^2}{\frac{\partial^2 V}{\partial^2 W_t}}$$

$$+ \lambda q_t \int_\Gamma \Big(V[t, W_t - y, q_t] - V[t, W_t, q_t]\Big) f_Y(y)dy = 0$$

$$\text{such that}$$

$$\frac{\partial V}{\partial W_t} > 0, \quad \frac{\partial^2 V}{\partial^2 W_t} < 0 \text{ and } \; V(T, W, p, q) = U(W_T). \tag{4.11}$$

Equation (4.11) together with the conditions is equivalent to a nonlinear integro-differential Cauchy problem for the value function V.

4.3.1 Analytical solution to the HJB equation

The Cauchy integro-differential problem will be solved analytically by the usual method of guess. The verification theorem will be used in the next section to prove that the solution obtained is indeed the correct solution.

We will be inspired by the previous work of Pliska (1986). The Pliska problem is similar to the present problem; the difference is that in Pliska (1986) there is no dividend payment and no risk process involved in the wealth process. The author found that the optimal investment in the risky asset is independent of the wealth process which is a feature of the exponential utility adopted. A much more similar problem is the optimal investment problem in Browne (1995) in which there is no dividend payment to shareholders but the risk process is involved in the wealth process. In Browne (1995) the risk process was approximated by a diffusion process, correlated with the Brownian motion underlying the risky asset process whose drift is constant i.e. the business volume is constant up to the final time horizon T. The HJB in Browne (1995) is simply a Cauchy non linear differential equation but not integro-differential. The author shows that the optimal risky asset allocation depends on the correlation between the risk process and the risky asset but the value function does not.

In the present work, we propose a solution of the form

$$V(t,w,q,p) = \gamma - \eta \exp\left\{ -\nu w e^{(r-\alpha)(T-t)} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) + q_t \times h(t,p) \right\} \quad (4.12)$$

where $h(t,p)$ is a suitable function of the constant premium charged and time. We first note that the proposed solution (4.12) satisfies

$$\frac{\partial V}{\partial t} = \left[\nu (r-\alpha) W_t e^{(r-\alpha)(T-t)} + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + q_t \frac{\partial h(t,p)}{\partial t} \right] [V - \gamma], \quad (4.13)$$

$$\frac{\partial V}{\partial q_t} = h(t,p)\left[V-\gamma\right] \quad , \frac{\partial V}{\partial W} = -\nu e^{(r-\alpha)(T-t)}\left[V-\gamma\right], \tag{4.14}$$

and

$$\frac{\partial^2 V}{\partial^2 W} = \nu^2 e^{2\times(r-\alpha)(T-t)}\left[V-\gamma\right]. \tag{4.15}$$

The discontinuous change in the value function produces,

$$V[t,W-y,q,p] - V[t,W,q,p] = \left[e^{y\nu e^{(r-\alpha)(T-t)}} - 1\right]\left[V-\gamma\right] \quad \text{hence}$$

$$\int_\Gamma \Big(V[t,W_t-y,q_t] - V[t,W_t,q_t]\Big) f_Y(y)dy = \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-t)}} - 1\right]\left[V-\gamma\right]. \tag{4.16}$$

Substituting equations (4.13) , (4.14) , (4.15) and (4.16) into (4.11) yields

$$\begin{aligned} &\nu(r-\alpha)W_t e^{(r-\alpha)(T-t)} + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 + q_t\frac{\partial h(t,p)}{\partial t} + q_t g(p)h(t,p) \\ &-\nu e^{(r-\alpha)(T-t)}\Big[(r-\alpha)W_t + pq_t\Big] + \lambda q_t \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-t)}} - 1\right] \\ &-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 = 0 \end{aligned} \tag{4.17}$$

which after simplification yields

$$q_t\left\{\frac{\partial h(t,p)}{\partial t} + g(p)h(t,p) - \nu p e^{(r-\alpha)(T-t)} + \lambda\mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-t)}} - 1\right]\right\} = 0. \tag{4.18}$$

Since $q_t \neq 0$, we are left with the first order differential equation

$$\frac{\partial h(t,p)}{\partial t} + g(p)h(t,p) - \nu p e^{(r-\alpha)(T-t)} + \lambda\mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-t)}} - 1\right] = 0 \tag{4.19}$$

with boundary condition $h(T,p) = 0$.

Equation (4.19) is a first order differential equation in the time variable t that can be solved using the integrating factor techniques:

$$\begin{aligned}
\frac{\partial h(t,p)}{\partial t} + g(p)h(t,p) &= \nu p e^{(r-\alpha)(T-t)} - \lambda \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-t)}} - 1\right] \\
\int_t^T d\left(e^{sg(p)}h(s,p)\right) &= \nu p \int_t^T e^{(r-\alpha)(T-s)+sg(p)}ds \\
&\quad -\lambda \int_t^T \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-s)}} - 1\right] e^{sg(p)}ds.
\end{aligned}$$

Since $h(T,p) = 0$, we obtain

$$h(t,p) = \lambda \int_t^T \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-s)}} - 1\right] e^{g(p)(s-t)}ds - \nu p \int_t^T e^{(r-\alpha)(T-s)+g(p)(s-t)}ds \quad (4.20)$$

and hence under the optimal strategy, the risky investment is simplified as

$$\hat{\Theta}_t = \left(\frac{\mu - r}{\nu\sigma^2}\right) e^{-(r-\alpha)(T-t)}. \quad (4.21)$$

4.3.2 The verification theorem

The Value function of the partial integro-differential equation was obtained through a method of guess. We will now verify through the Martingale optimality principle that equation (4.12) is indeed the value function of the Cauchy problem.

Theorem 4.3.1 (Optimality) *Under the optimal strategy, V is a martingale with respect to the filtration generated by the Brownian motion underlying the risky asset process and the marked Poisson process underlying the Risk process.*

Proof The proof is driven by the dynamic of the wealth process under the optimal strategy. We recall that under optimal strategy,

$$\hat{\Theta}_t = \left(\frac{\mu - r}{\nu\sigma^2}\right) e^{-(r-\alpha)(T-t)}.$$

If we replace the risky asset allocation under the optimal strategy into the dynamic of the wealth process, we obtain:

$$\begin{aligned} dW_u - (r-\alpha)W_u du &= pq_u du - \int_\Gamma y\aleph(du,dy) + \hat{\Theta}_u\left[(\mu-r)du + \sigma dB_u\right] \\ &= pq_u du - \int_\Gamma y\aleph(du,dy) \\ &+ \left(\frac{\mu-r}{\nu\sigma^2}\right) e^{-(r-\alpha)(T-u)}\left[(\mu-r)du + \sigma dB_u\right] \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} d\left(W_u e^{-(r-\alpha)u}\right) &= pq_u e^{-(r-\alpha)u} du - e^{-(r-\alpha)u}\int_\Gamma y\aleph(du,dy) \\ &+ \left(\frac{\mu-r}{\nu\sigma^2}\right) e^{-(r-\alpha)T}\left[(\mu-r)du + \sigma dB_u\right]. \end{aligned} \tag{4.23}$$

Integrating (4.23) over the range t to $t+s$ for any $s \geq 0$ yields

$$\begin{aligned} W_{t+s}e^{-(r-\alpha)(t+s)} &= W_t e^{-(r-\alpha)t} - \int_t^{t+s}\int_\Gamma e^{-(r-\alpha)u}\, y\,\aleph(du,dy) \\ &+ p\int_t^{t+s} q_u e^{-(r-\alpha)u} du + \frac{s}{\nu}\left(\frac{\mu-r}{\sigma}\right)^2 e^{-(r-\alpha)T} \\ &+ \left(\frac{\mu-r}{\nu\sigma}\right) e^{-(r-\alpha)T}\int_t^{t+s} dB_u. \end{aligned} \tag{4.24}$$

In line with equation (4.12), the value function at any time $t+s$ is

$$\begin{aligned} V(t+s, W_{t+s}, q_{t+s}, p) = \gamma - \eta \exp\Bigg\{ &-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t-s) \\ &+ q_{t+s}h(t+s,p) - \nu W_{t+s}e^{(r-\alpha)(T-t-s)}\Bigg\}. \end{aligned} \tag{4.25}$$

The first objective is to prove that V is a martingale, i.e.

$$\mathbf{E}\left[V(t+s, W_{t+s}, q_{t+s}, p) \,\middle|\, \mathcal{F}_t\right] = V(t, W_t, q_t, p).$$

Substituting (4.23) into (4.25), we obtain

$$\begin{aligned} V(t+s, W_{t+s}, q_{t+s}, p) &= \gamma - \eta \exp\Bigg\{ -\nu e^{(r-\alpha)(T-t)} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) \\ &\quad + q_{t+s} h(t+s, p) - \nu p \int_t^{t+s} q_u e^{(r-\alpha)(T-u)} du \\ &\quad + \nu \int_t^{t+s} \int_\Gamma e^{(r-\alpha)(T-u)}\, y\, \aleph(du, dy) \\ &\quad - \frac{s}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - \left(\frac{\mu - r}{\sigma}\right) \int_t^{t+s} dB_u \Bigg\}. \end{aligned}$$

Using equation (4.20) and (4.3), yields

$$\begin{aligned} h(t+s, p) &= e^{-sg(p)} h(t, p) + \nu p e^{-(t+s)g(p)} \int_t^{t+s} e^{(r-\alpha)(T-u) + ug(p) du} \\ &\quad - \lambda e^{-(t+s)g(p)} \int_t^{t+s} \mathbf{E}\left[e^{Y\nu e^{(r-\alpha)(T-u)}} - 1 \right] e^{ug(p)} du \end{aligned}$$

so that

$$\begin{aligned} h(t+s, p) q_{t+s} = h(t, p) q_t + \nu p \int_t^{t+s} q_u e^{(r-\alpha)(T-u)} du \\ - \lambda \int_t^{t+s} q_u \mathbf{E}\left[e^{Y\nu e^{\left(r-\alpha\right)(T-u)}} - 1 \right] du. \end{aligned} \tag{4.26}$$

Substituting (4.26) into (4.26) and after simplification, we obtain

$$\begin{aligned} V(t+s, W_{t+s}, q_{t+s}, p) = \gamma - \eta \exp\Bigg\{ -\nu e^{(r-\alpha)(T-t)} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) \\ + \int_t^{t+s} \int_\Gamma \nu e^{(r-\alpha)(T-u)}\, y\, \aleph(du, dy) - \frac{s}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \\ - \int_t^{t+s} \eta q_u \mathbf{E}\left[e^{Y\nu e^{\left(r-\alpha\right)(T-u)}} - 1 \right] du \\ - \left(\frac{\mu - r}{\sigma}\right) \int_t^{t+s} dB_u + q_t \times h(t, p) \Bigg\}. \end{aligned} \tag{4.27}$$

We may substitute in (4.27) the value of $V(t, W_t, q_t, p)$ which in turn yields after taking the expectation:

$$\mathbf{E}\left[V(t+s,W_{t+s},q_{t+s},p)-\gamma \;\Big|\; \mathcal{F}_t\right] = \left[V(t,W_t,q_t,p)-\gamma\right] \times \exp\left\{-\frac{s}{2}\left(\frac{\mu-r}{\sigma}\right)^2 - \int_t^{t+s} \lambda q_u \mathbf{E}\left[e^{Y\nu e^{\left(r-\alpha\right)(T-u)}}-1\right] du\right\} \times \mathbf{E}\left[\exp\left\{\int_t^{t+s}\int_\Gamma \nu y e^{(r-\alpha)(T-u)}\,\aleph(du,dy)\right\} \;\Big|\; \mathcal{F}_t\right] \times \mathbf{E}\left[\exp\left\{\left(\frac{\mu-r}{\sigma}\right)\int_t^{t+s} dB_u\right\} \;\Big|\; \mathcal{F}_t\right]. \tag{4.28}$$

But

$$\mathbf{E}\left[\exp\left\{\int_t^{t+s}\int_\Gamma \nu y e^{(r-\alpha)(T-u)}\,\aleph(du,dy) \;\Big|\; \mathcal{F}_t\right]\right\} = \exp\left\{\int_t^{t+s} \eta q_u \mathbf{E}\left[e^{y\nu e^{\left(r-\alpha\right)(T-u)}}-1\right]\right\} du \tag{4.29}$$

and

$$\mathbf{E}\left[\exp\left\{\left(\frac{\mu-r}{\sigma}\right)\int_t^{t+s} dBu\right\} \;\Big|\; \mathcal{F}_t\right] = \exp\left\{\frac{s}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right\}. \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.28), yields the expected result:

$$\mathbf{E}\left[V(t+s,W_{t+s},q_{t+s},p) \;\Big|\; \mathcal{F}_t\right] = \gamma + \left[V(t,W_t,q_t,p)-\gamma\right] = V(t,W_t,q_t,p) \tag{4.31}$$

Therefore the value function under optimal strategy is a martingale. □

Theorem 4.3.2 (Supermartingale) *For any policy $(b_t)_{0<t<T}$ different from the presumed optimal policy $(\Theta_t)_{0<t<T}$, the implied value function V_b is a strict supermartingale with respect to the filtration generated by the Brownian motion underlying the risky asset process and the marked Poisson process underlying the Risk process.*

Proof We first consider the proposed solution

$$V_b(t,w,q,p) = \gamma - \eta \exp\left\{ -\nu w e^{(r-\alpha)(T-t)} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) + q_t h(t,p) \right\}$$

which we rewrite for simplicity as

$$V_b(t,w,q,p) = \gamma - \eta \exp\Big(F(t,w_t,q_t) \Big).$$

The objective is to prove that for any policy $b_t \neq \Theta_t$,

$$\begin{aligned} \mathbf{E}\left[V_b(t+s, W_{t+s}, q_{t+s}, p) \;\middle|\; \mathcal{F}_t \right] \;&\leq\; V_b(t, W_t, q_t, p) \quad \text{i.e.} \\ \gamma - \eta \mathbf{E}\left[\exp\left\{ F(t+s, w, q) \right\} \;\middle|\; \mathcal{F}_t \right] \;&<\; \gamma - \eta \exp\left\{ F(t,w,q) \right\} \end{aligned} \tag{4.32}$$

which equivalently results to

$$\mathbf{E}\left[\exp\left\{ F(t+s, w_{t+s}, q_{t+s}) - F(t,w,q) \right\} \;\middle|\; \mathcal{F}_t \right] \geq 1. \tag{4.33}$$

Considering any policy $(b_t)_{t\geq 0}$, the wealth at time $t+s$ may be computed as

$$\begin{aligned} W_{t+s} e^{-(r-\alpha)(t+s)} = W_t e^{-(r-\alpha)t} &+ p \int_t^{t+s} q_u e^{-(r-\alpha)u} du \\ &- \int_t^{t+s} \int_\Gamma e^{-[r-\alpha]u} \, y \, \aleph(du, dy) \\ &+ \int_t^{t+s} b_u e^{-(r-\alpha)} u \left[(\mu - r) du + \sigma dB_u \right]. \end{aligned} \tag{4.34}$$

Combining (4.20) and (4.34), yields

$$
\begin{aligned}
F(t+s, w_{t+s}, q_{t+s}) - F(t,w,q) = {} & \frac{s}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - (\mu - r)\int_t^{t+s} \nu e^{(r-\alpha)(T-u)} b_u du \\
& - \sigma \int_t^{t+s} \nu e^{(r-\alpha)(T-u)} b_u dB_u \\
& + \int_t^{t+s}\int_\Gamma \nu y e^{(r-\alpha)(T-u)} \aleph(du, dy) \\
& - \exp\left\{\int_t^{t+s} \lambda q_u \mathbf{E}\left[e^{Y\nu e^{\left(r-\alpha\right)(T-u)}} - 1\right]\right\} du.
\end{aligned} \tag{4.35}
$$

Let $f_u = \nu e^{(r-\alpha)(T-u)} b_u$. Using equation (4.30) together with the Ito Isometry yields

$$
\mathbf{E}\left[e^{F(t+s,w_{t+s},q_{t+s})-F(t,w,q)} \;\middle|\; \mathcal{F}_t\right] = \exp\left\{\frac{s}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + \int_t^{t+s}\left[-(\mu - r) f_u + \frac{\sigma^2}{2} f_u^2\right] du\right\}. \tag{4.36}
$$

In the calculus of variations framework, the Euler-Lagrange necessary condition for an extremal yields

$$
f_u = \frac{\mu - r}{\sigma^2} \quad \Rightarrow \quad b_u = \left(\frac{\mu - r}{\nu\sigma^2}\right) e^{-(r-\alpha)(T-u)} = \hat{\Theta}_u. \tag{4.37}
$$

The second order condition shows that f_u is actually a minimum since $\sigma^2 > 0$. Substituting (4.37) into (4.36) yields the derived expected equality for the martingale case obtained from (4.31) i.e.

$$
\mathbf{E}\left[e^{F(t+s,w_{t+s},q_{t+s})-F(t,w,q)} \;\middle|\; \mathcal{F}_t\right] = 1. \tag{4.38}
$$

Since Θ_u is the minimum, any policy $b_u \neq \Theta$ will result in

$$
\mathbf{E}\left[e^{F(t+s,w_{t+s},q_{t+s})-F(t,w,q)} \;\middle|\; \mathcal{F}_t\right] > 1. \tag{4.39}
$$

□

4.4 Interpreting the optimal policy

The optimal risky asset investment $\hat{\Theta}_t$ depends on the time but not on the current wealth W_t due to the constant aversion to risk utility function. Whatever the risk process R_t appears to be, the optimal strategy is independent of it. This is based on the fact that we have assumed that the source of randomness in the risk process and in the investment process are independent. This is by contrast to Browne (1995) in which the optimal policy depends also on the risk process as a consequence of the correlation between the risk process and the investment process . It is also interesting to look at the interpretation of the optimal policy:

- If we assume that the company capital is fully raised by shareholders, we can interpret α as the dividend rate of the shareholders which is in fact the interest rate the shareholders obtain by investing in the company. By the no arbitrage principle, the dividend rate should be higher than the risk free rate hence $\alpha > r$. This shows that as we approach the finite time horizon T, the optimal strategy predicts to invest less in the risky asset which contrasts with Browne (1995) that predicts to invest more in the risky asset as we approach the deadline. If on the other hand we assume that the company is not completely owned by shareholders i.e. there is a possibility of dividend rate being lesser than the risk free rate ($\alpha < r$), the optimal strategy radically changes and dictates to invest more in the risky asset as we approach the terminal horizon (Browne's result). In general, the result of Browne can be derived if we set the shareholders dividend rate to be zero ($\alpha = 0$).

- The optimal policy is inversely proportional to ν. This is as expected since ν is the degree of aversion to risk using Pratt measure. As ν gets larger, the individual is more averse to risk and therefore will invest less in the risky asset as shown by the result.

- We also notice that the optimal policy is proportional to the Sharpe ratio and inversely proportional to the volatility of the risky access. In fact, the higher the Sharpe ratio, the higher the market price of risk and therefore the more attractive will be the risky investment versus risk free investment. At the same time, for a constant Sharpe ratio, a more volatile risky investment will be less attractive.

- Though the value function depends on the demand function, the optimal allocation in the risky asset does not. In fact the analysis is based on a broad class of demand functions whose growth rate depends on the premium. This has the advantage that the solution is portable from one market to another.

These results are supported on figure (4.1) with the following starting parameters:
The drift and the volatility of the risky asset are $\mu = 0.08$ and $\sigma = 0.13$ per year
The risk free rate is constant with value $r = 0.04$ per year
The risk aversion parameter is set at $\nu = 0.6$
The dividend payable to shareholder is set to $\alpha = 0.07$
The time horizon is set to be $T = 10$ years.

4.5 Summary

The optimal asset allocation problem analytically solved in this chapter can be considered as an extension of Merton's investment problem in the context of insurance risk process, driven by a deterministic demand function and a Poisson process. The resulting optimal strategy which is time dependent changes depending on the level of the dividend rate compared to the risk free rate. An extension of the problem is possible, see Chapter (7). Maringale optimality principle is used to validate the analytical solution.

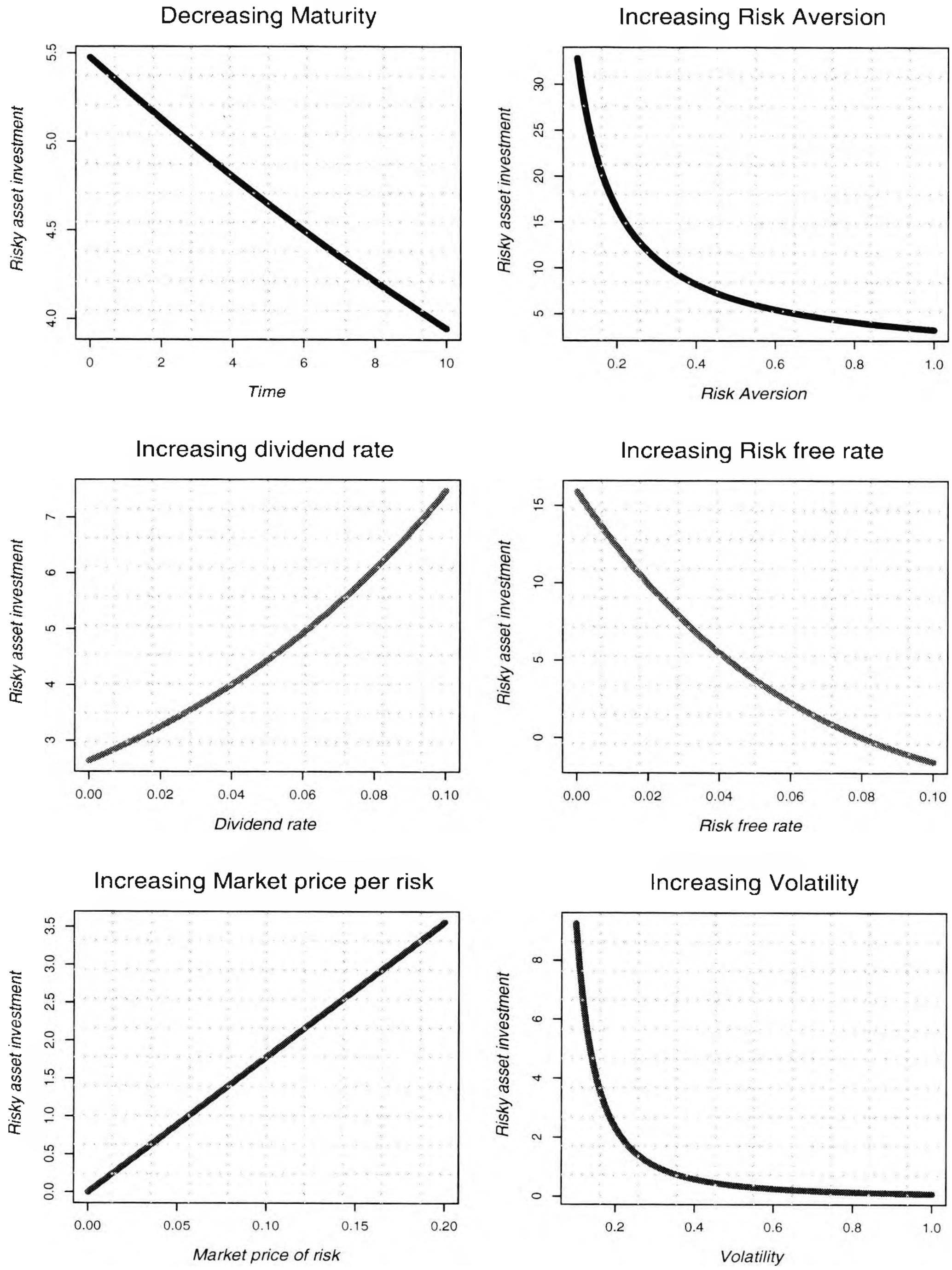

Figure 4.1: **Optimal Asset allocation**

Chapter 5

Catastrophe Insurance

5.1 Introduction

For many years, the insurance industry has suffered from many catastrophes losses mainly due to natural disasters, and in recent times also due to terrorism, of which insurance premiums only cover a small part. Many of the catastrophe premiums are too high so that there is no room to increase capacity in the catastrophe insurance market. Until 1993, the only method available to insurance companies to hedge their underwriting risk was through reinsurance contracts but reinsurance contracts do not always benefit the insurer specially for catastrophe cover, which usually requires a large premium. Both the insurance and the reinsurance industries have become increasingly concerned about the concentration of exposures linked to a single event. The experience of major natural catastrophes in the nineties such as Hurricane Andrew in 1992, followed by Northridge California earthquake in 1994 resulted in great concerns that insurance and reinsurance companies may not have enough capital to meet their liabilities. This provoked in the industry, the need to increase capacity not only within the traditional insurance system but also in transferring them into the more liquid financial markets. On December 11, 1992, The Chicago Board of Trade (CBOT) made the first attempt to introduce a financial instrument for the catastrophe reinsurance market, and ever since, the market has grown according to investors'needs. The first instrument proposed by CBOT was the catastrophe insurance futures contracts (CAT) which started trading at the beginning of 1993, followed by options and call spreads which proved very successful. Unlike financial markets in which products such as stocks on which futures contracts and options are written, the insurance market has no continuously updated underlying cash price. The solution chosen by CBOT was the loss ratio index, calculated by the Insurance Services Office (ISO) for CAT products. The main idea underlying trading insurance related risk on capital market is twofold. It allows insurance and reinsurance companies to dynamically adjust their exposure to natural catastrophic risk through hedging with standardised financial contracts at low cost and additionally attracts new capital linked to natural catastrophic risk from investors for whom those derivatives provide an excellent opportunity for portfolio diversification. A question raised by the introduction of insurance derivatives is the one that addresses how those financial contracts should be valued in order to preserve the valuation of existing insurance contracts. In a general insurance market, information about the risk underwritten is reflected in the insurance premium through the expected claim and loading valuation principle. As the same risk underlies insurance derivatives, one has to come up with a price that depends not only on the underlying risk being underwritten but also and mostly that rules out the arbitrage possibility that may arise from trading in both insurance and financial contract markets. This suggests that both traditional actuarial valuation skills as well as financial mathematics skills are indispensable to produce a price that will be accepted by all participants in the market.

In a Black Scholes world, derivatives are written on the underlying asset that are tradable and for which their price does not involve any unhedgeable element such as insurance claim size. These features of the asset together with the no arbitrage principle produce a unique price for all derivatives through the completeness of the market which in turn is equivalent to the existence of a unique equivalent measure under which the discounted price of assets is a martingale, see Harrison and Pliska (1981).

In the context of catastrophe insurance market, the valuation of derivatives appears much more problematic compared to the Black Scholes world for two reasons. First there is no tradable catastrophe asset since the indices on which the contracts are written are not tradable; this precludes the possibility of pricing according to no-arbitrage market for that the no-arbitrage argument, can only hold when all underlying assets are explicitly defined. Secondly, the mathematical description of the indexes incorporates a Marked Poisson process with a stochastic jump size which makes the catastrophe derivatives market incomplete. For these two reasons, it is therefore not possible to perfectly replicate the movements and therefore the payoffs of those derivatives by continuously trading in other securities. As a consequence, the price processes of catastrophe derivatives cannot be uniquely determined solely on the basis of excluding arbitrage opportunities.

The objective of this chapter is to to review the different models used in the modelling of catastrophe risk introduced by CBOT and the pricing of the financial derivatives written on the indexes derived from catastrophic losses. The losses and hence the indexes will be modelled as compound Poisson processes where the amplitude of the jump will reflect the size of the catastrophe arising unpredictably with time. In the next section, we will explain the first derivatives introduced by CBOT, the reasons for their unpopularity and their improvement over time.

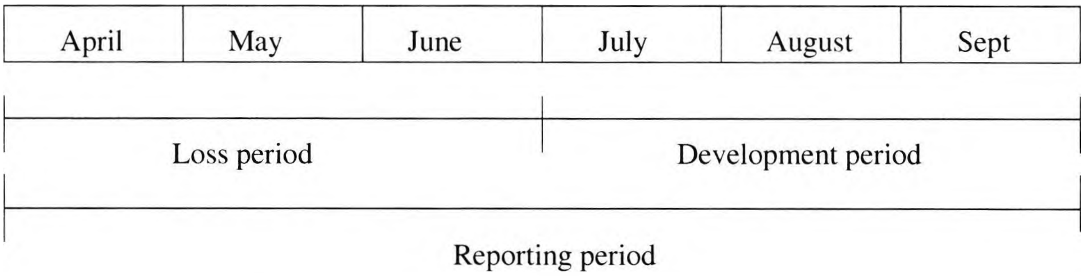

Figure 5.1: **Illustration of a CAT contract**

5.2 CAT futures contract

The first derivative introduced by the CBOT was the CAT futures contract which is available for trading in March, June, September and December. If we consider the March 1994 contract, it trades from the beginning of January until the end of June; the period between January 1 and March 31 is called the *event quarter*, the period between January and the end of June is the *reporting period*. The underlying asset is the *ISO index*. Each quarter, approximately 100 American insurance companies report property loss data to the Insurance Service Office (ISO). ISO then selects a pool of at least 10 of these companies on basis of size, diversity of business, and quality of reported data. The ISO index is then calculated as the loss ratio of the pool:

$$\text{ISO Index} = \frac{\text{reported incurred losses}}{\text{earned premiums}}.$$

The list of companies included in the pool is announced by the CBOT prior to the beginning of the trading period for the contract. The premium volume of the companies participating to the pool is also announced prior to the start of trading period. Thus, the premium in the pool is known as a positive constant so that change in the loss ratio is due to change in the market expectation of losses.

Example The June contract (figure 5.1) covers losses from events occurring in April, May and June as reported to the participating companies by the end of September.

If we let L_T denote the sum of the selected losses incurred during the quarter corresponding to the contract and reported at the end of the following quarter (T) and let π denote the announced premiums earned during the three months exposure period, then the future settlement value F_T is given by :

$$F_T = \$25,000 \times \min\left(\frac{L_T}{\pi}, 2\right). \tag{5.1}$$

Futures prices are quoted in points (US \$250) and tenths of a point . Equation (5.1) shows that the settlement value is capped at 200% which is a desirable feature in insurance, ensuring than the party who is short on the contract has a limited loss. The maximum loss of the long position is simply the agreed price of the futures contract.

At some point $t \in [0, T]$, the price F_t will reflect the market's expectation of the terminal price. If we consider an insurer who is willing to hedge against a possible excess of losses over a given quarter, one option she might consider is to buy futures contract. Assuming that the contract is bought at time $t = 0$, at an agreed price F_0 which will be paid at time T, if severe catastrophes occur, the insurer will gain $F_T - F_0$ which will offset the heavy losses incurred in her underwriting. The more correlated the insurer's losses are with the ISO index, the better the hedge, see Geman (1994). Let denote by $(F_t)_{0 \leq t \leq T}$ the insurance future's price, a position at time t means the commitment to pay the random amount $F_T - F_t$ at time T. A holder of a long position will thus receive $F_T - F_t$ at time T. There are theoretically no cash-flows before time T, although in reality, the CBOT requires certain payment from agents before time T. Equation (5.1) can be rewritten as

$$F_T = \$25,000 \times \left(\frac{L_T}{\pi} - \max\left(\frac{L_T}{\pi} - 2, 0\right)\right).$$

In finance terminology, the settlement of a catastrophe futures is equivalent to a long position in the loss ratio and a short position in a European call option with maturity T and strike price 2 where the underlying is the loss ratio. Depending on the assumption used to model the aggregate loss process L_t, an analytical solution may be found for the futures price F_t. In Geman and Cummins (1994), the instantaneous claim process is defined as S_t so that in a small interval of time $[t, t+dt]$, the amount of claim to be reported in the ISO index is equal to dS_t. The value of the aggregate claim L_T at the end of the reporting period T is

$$L_T = \int_0^T dS_s.$$

Since the reporting of the claims is continuous and claims are always positive, the process S_t is considered in Geman and Cummins (1994) as a geometric Brownian motion with some jumps describing the random claims occurring during the event period. The time to maturity is divided into two intervals, $I_1 = [0, \frac{T}{2}]$ represents the event period and $I_2 = [\frac{T}{2}, T]$ represents the development period.

For $t \in I_1$, the claim process S_t follows the jump and diffusion process

$$dS_t = S_{t_-} \left(\mu dt + \sigma dB_t\right) + k dN_t,$$

where,

B_t is a standard Brownian motion,

μ and σ are positive constants representing the continuous part parameters of the instantaneous claims,

k is a positive constant representing the claim size when a loss is incurred, and N_t is a Poisson process orthogonal to the Brownian motion B_t, with intensity λ.

For $t \in I_2$, S_t is simply a geometric Brownian motion

$$dS_t = S_t \left(\mu' dt + \sigma' dB_t \right).$$

Although catastrophe claims related to insurance cannot be bought and traded as an asset, it is nevertheless true that the Futures on catastrophes can be regarded as a tradable asset. In Geman (1994), futures are viewed as financial instrument. Furthermore, since the claim size is assumed to be constant in the model, the insurance market is therefore regarded as complete. This in turn allows the application of the consequence of the no arbitrage assumption introduced in Harrison and Kreps (1979a) and Harrison and Pliska (1981) which in turn implies the existence of a probability measure Q under which the discounted price of all assets is a martingale. Under Q, the claim process behaves similarly as in the physical probability with different parameters, i.e. under Q,

$$dS_t = \begin{cases} S_{t^-} \left(\varrho dt + \sigma d\hat{B}_t \right) + k d\hat{N}_t \quad \text{where} \quad \mu = \varrho + \rho\sigma \quad \text{for} \quad t \in I_1 \\ \\ S_t \left(\varrho' dt + \sigma' d\hat{B}_t \right) \quad \text{where} \quad \mu' = \varrho' + \rho\sigma' \quad \text{for} \quad t \in I_2 \end{cases} . \tag{5.2}$$

The parameter ρ represents as in (Shimiko 1992) the equilibrium market price of a claim level risk assumed to be constant in $[0, T]$. Under the above setting and assumptions, see Geman (1989) and (Jamshidian 1989), the future price is a martingale under Q i.e.

$$\begin{aligned} F_t &= \mathbb{E}_Q \left[F_T \,\middle|\, \mathcal{F}_t \right] \\ &= \frac{\$25,000}{\pi} \times \left[\mathbb{E}_Q \left[L_T \,\middle|\, \mathcal{F}_t \right] - \mathbb{E}_Q \left[\max\left(L_T - 2\pi, 0 \right) \,\middle|\, \mathcal{F}_t \right] \right] \end{aligned} \tag{5.3}$$

An analytical solution was obtained depending on whether the price is being computed in the event period or development period.

For $t \in I_2$,

$$F_t = \frac{\$25,000}{\pi}\left\{\int_0^t S_s ds + \frac{S_t}{\varrho'}\left[e^{\varrho'(T-t)} - 1\right] + T \times \mathbb{E}_Q\left[\max\left(\frac{1}{T}\int_0^T S_s ds - \frac{2\pi}{T}, 0\right) \Big| \mathcal{F}_t\right]\right\}$$

where the second term can be viewed as the value of an Asian option with strike price $\frac{2\pi}{T}$, see Geman (1994) for the detailed calculations and an explicit formula for the Laplace transform of the Asian option.

For $t \in I_1$, the pricing is much more challenging due to the jump in the the dynamic of the claim process. The process containing the jump in (5.2) is recognised as the Doléans-Dade exponential. The futures price F_t is given by

$$\begin{aligned} F_t &= \int_0^t S_s ds + S_t\left[\frac{e^{\varrho(T/2-t)} - 1}{\varrho}\right] + k\lambda\left[\frac{e^{\varrho(T/2-t)} - \varrho(T/2-t) - 1}{\varrho^2}\right] \\ &\quad + S_0 e^{\varrho'(T/2-t)}\left[\frac{e^{\varrho' T/2} - 1}{\varrho'}\right] + k\lambda\left(\frac{e^{\varrho(T/2-t)} - 1}{\varrho}\right)\left(\frac{e^{\varrho' T/2} - 1}{\varrho'}\right). \end{aligned} \tag{5.4}$$

Using the same dynamic of the loss process with the assumption of constant claim size in Geman (1994), the price of call option written on futures of the ISO index was derived. While the assumption of completeness of the market in Geman (1994) is convenient in the pricing, the assumption of a constant claim size is questionable because of the nature of catastrophe claims which can vary.

A more realistic view was taken in Aase (1999) in which the loss index was modelled as a compound Poisson process with random claim size. The author investigated the price of futures and derivatives written on the futures. As in Geman (1994), the futures are traded in the market, and therefore the incompleteness of the market does not arise from the fact that the underlying asset is not traded but from the fact that the size of the claims is random. This problem was circumvented by specifying the preference of market participants by a utility function which determines a unique price processes within the framework of partial equilibrium theory under uncertainty. An analytical solution was obtained under the assumptions of exponential utility function and the aggregate loss following a gamma distribution.

A more convenient derivative called a CAT call spread has proved more popular than futures and plain vanilla call on futures. A CAT call spread involves buying a call at a strike price k_1 and selling a call at a higher spread k_2 with both calls having the same maturity T. The underlying asset on the CAT spread call is the futures contract. The payoff at maturity T of a CAT call spread with loss ratio attachment k_1 and k_2 is :

$$C_{spread} = \min\left\{ \max\left(F_T - k_1, 0\right), k_2 - k_1 \right\}$$

Call spreads are much more desirable by the market participants than the futures since the amount of risk at stake is much more limited for the short position and the long position. Although the CAT call spread has proved successful, the CAT products in general lost viability among markets participants for many reasons. One of the reasons was that the ISO index I_t was published only prior to the settlement date (just after the end of the reporting period). This means that a company participating to the pool could be aware of part of the data used to compute the index whereas companies not in the pool cannot. This situation created an information asymmetry in the market therefore preventing people from entering the market.

Other reasons are linked to the moral hazard behaviours that a company which has a short position in the pool can adopt as a strategy. In fact, a company in the index pool with a short position in futures contract can manipulate the index by delaying the report of big losses that it incurs, therefore reducing the payoff of the future contract that it is subject to pay to the long position holder. This problem occurred with the Northbridge earthquake which was a late quarter catastrophe of the March 1994 contract. The settlement value was too low and did not entirely represent the real accumulated losses of the industry. Since then, a new generation of financial instruments were created to fix the bugs in the CAT products. A more standardised product was created in 1995 on a new index called the PCS (Property Claims Service) index on which options are written.

5.3 PCS OPTIONS

PCS options were introduced at CBOT in September 1995. PCS options are standardised, exchange-traded contracts with only cash contracts available. The underlying assets of a PCS option are PCS indices on which, call, put and call spreads are written; futures are no longer available as opposed to CAT contracts. Most of the trading activities occur in call spread options because of the limits on losses that their payoff provides to both parties in the long position as well as in the short position. CBOT lists PCS options both as "small cap" contracts, which limit the amount of aggregate industry losses that can be included under the contract to $20 billion, and as "large contracts" which track losses from $20 billion to $50 billion.

As CAT products, the *loss period* is the time during which a catastrophic event must occur so that resulting losses can be included in a particular index. During the loss period, PCS provides loss estimates as catastrophes occur. Most of the options have a quarterly loss period with contracts lists for March, June, September and December. Others have annual loss periods and are only available as annual contracts. The last day of the loss period is the calendar day of the quarter or year. Losses from catastrophes starting in one quarter or year and ending in the next will be included in the quarter or year in which the catastrophe started.

The *development period* is the time after the loss period during which the PCS continues to estimate and re-estimate losses from catastrophes occurring during the loss period. Market participants can choose either a six-month or twelve-month development period. The development period begins immediately after the loss period ends. The PCS index value at the end of the chosen development period will be used for settlement purposes, even though PCS loss estimates may continue to change. PCS-options settle in cash on the last business day of the development period. The settlement value L_T for each index represents the sum of the current PCS insured loss estimates provided and revised over the loss and development periods. Table (5.5) clarifies the time structure of the insurance contracts.

Contract Month	**Loss Period**	**Development period**		**Settlement date**	
		Six Month	*Twelve Month*	*Six Month*	*Twelve Month*
March	$Jan - Mar$	$Apr1 - Sep30$	$Apr1 - Mar31$	$Sep30$	$Mar31$
June	$Apr - Jun$	$Jul1 - Dec31$	$Jul1 - Jun30$	$Dec31$	$Jun30$
September	$Jul - Sep$	$Oct1 - Mar31$	$Oct1 - Sep30$	$Mar31$	$Sep30$
December	$Oct - Dec$	$Jan1 - Jun30$	$Jan1 - Dec31$	$Jun30$	$Dec31$
Annual	$Jan - Dec$	$Jan1 - Jun30$	$Jan1 - Dec31$	$Jun30$	$Dec31$

(5.5)

Each PCS loss index represents the sum of then-current PCS estimates for insured catastrophic losses in the area and loss period divided by $100 million. The indices are quoted in points and tenths of a point and each index point equals $200 cash value as indicated in table (5.6).

PCS Loss index value	PCS Options cash equivalent	Industry loss equivalent
0.1	\$20	\$10 *million*
1.0	\$200	\$100 *million*
50.0	\$10,000	\$5 *billion*
200.0	\$40,000	\$20 *billion(small cap limit)*
250.0	\$50,000	\$25 *billion*
350.0	\$70,000	\$35 *billion*
500.0	\$100,000	\$50 *billion(large cap limit)*

(5.6)

Strike values are listed in integral multiples of five points. For a small cap contract, strike values range from 5 to 195. For large cap contracts, strike values range from 200 to 495. PCS options are European options, which means that they can only be exercised on the expiration date i.e at the end of the development period. The payoff of a PCS call option at expiration day T, exercise price K_1 and cap value K_2 is illustrated in in figure (5.2). The value of the option at maturity is expressed as

$$PCS_{call} = \min \left\{ \max \left(L_T - K_1, 0 \right), K_2 - K_1 \right\}. \tag{5.7}$$

The indices are computed through different phases. At the announcement of a catastrophe, PCS generally provides a *flash* estimate of the anticipated industry insured property losses from such event. The flash estimates are generally based on PCS's initial meteorological or seismological information and information from industry personnel and public officials. The estimates are expressed in terms of a range of estimated total insured property losses. These flash loss estimates give the insurers and re-insurers an initial perspective on the catastrophe's severity but are not included in the indices calculated

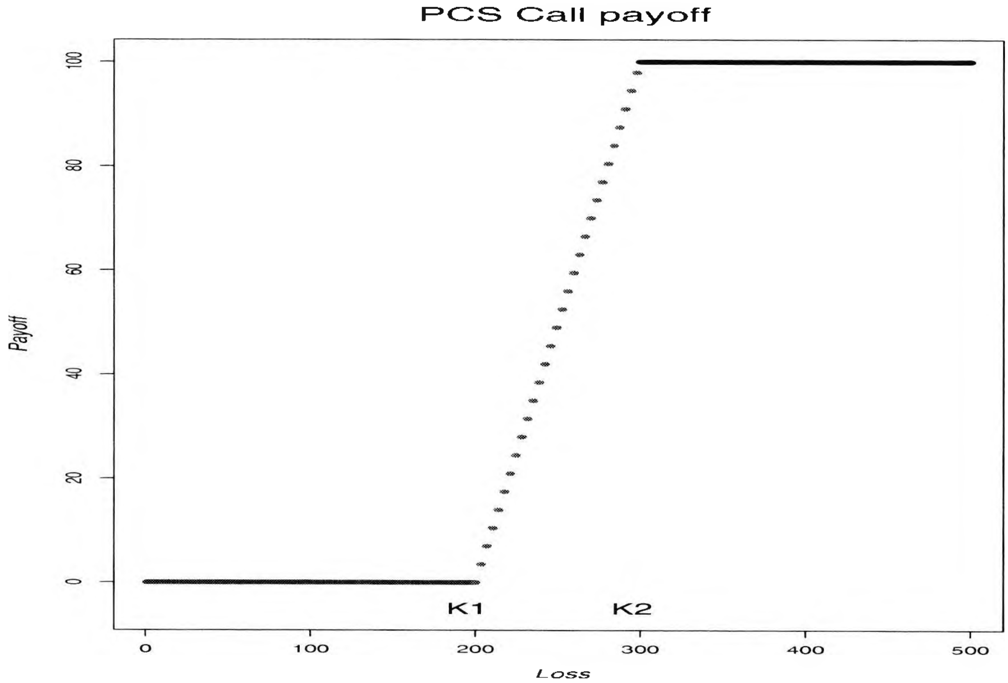

Figure 5.2: Illustration of a PCS call option

for the CBOT.

The indices compiled for the CBOT comprise *Preliminary* loss estimates and are adjusted according to *Resurvey* loss estimates. The preliminary loss estimate of anticipated insured property losses is typically prepared and released within several days to two weeks after the occurrence of a PCS identified catastrophe. If a catastrophe is large enough, PCS will continue to survey loss information to determine whether its preliminary estimate should be adjusted. PCS releases the initial Resurvey estimate to subscribers approximately 60 days after the Preliminary Estimate is issued and may continue the resurvey process and publish additional Resurvey estimates approximately every 60 days until it believes that the loss has been reasonably approximated. The advantage of PCS products over CAT products relies on the fact that no market participant, nor any agency can release any information on the losses before being officially published. This means that all market participants including companies in the index pool receive the same information at the same time thereby eliminating the problem of information asymmetry. Also, when PCS estimates the loss indices, they conduct surveys of the market. These surveys are confidential and they are not used directly in the estimation of the indices. So it is quasi-impossible for insurance companies to affect the indices, and therefore the moral hazard problem is eliminated. The construction of the PCS options also eliminates the problem by late occurring catastrophes. The PCs index does not directly depend on a number of reported claims and the time from the end of the event period to the time the index is settled is also longer for the PCS products than it was for CAT products. The problem of the PCS options arises in the pricing of the derivatives written on the indices.

5.4 Pricing of PCS options

PCS options are based on an underlying loss index that is not traded itself hence the payoff at maturity T of the option is directly related to the aggregate claims index which becomes the important element in the valuation of the option. The market is therefore incomplete even with constant jump sizes of the underlying index. Different authors in the literature have considered different frameworks for valuing the price of terminal claim options written on the index.

In Geman and Yor (1994), just like in Geman (1994), the underlying index is directly modelled as a geometric Brownian motion added to a Poisson process with a constant jump amplitude. Although the loss index is a non traded asset, the pricing has been possible by basing the no-arbitrage argument on the existence of a class of layers of reinsurance with different attachment points to guarantee completeness of the insurance derivative market.

If we consider the loss period to be $[0, T_1]$ and the reporting period to be $[T_1, T_2]$, the dynamic of the aggregate reported claim evolves as :

$$dL_t = \begin{cases} dS_t + kdN_t & \text{for} \quad t \in [0, T_1] \\ dS_t & \text{for} \quad t \in [T_1, T_2] \end{cases}$$

where k represents the jump size assumed to be constant, S_t is a geometric Brownian motion that represent the adjustment made to the claim and N_t represents the number of claims of size k made by time t. The authors investigated the pricing of a call option with strike price K with the assumption of a constant force of interest during the life of the option $[0, T]$. Based on the completeness of the market, the price of the derivative is computed as

$$C(t) = e^{-r(T-t)} \mathbb{E}_Q \left[(L_T - K)^+ \mathcal{F}_t \right],$$

where the measure Q is the particular martingale measure that best fits the prices of reinsurance layers quoted by major reinsurance companies among the class of martingale measures. The authors then used an Asian approach to obtain a semi-analytical solution for the value of $C(t)$ in form of the Laplace transform.

In Murman (2001) the author took a different approach and considered the aggregate loss index as a compound Poisson process with no adjustment made to the claims. In this framework, we consider the aggregate index claim of the form

$$L_t = \sum_{k=1}^{N_t} Y_k$$

where N_t is a Poisson process with constant claim severity λ and $Y_k = Y$ is the size of the claim. With G denoting the distribution of Y, the author constructs an equivalent probability measure Q under which the index process L_t preserves the compound Poisson structure.

If $(\lambda, dG(x))$ denotes the characteristic of the compound Poisson process L_t, under the original measure P, a probability measure Q under which L_t preserves the compound Poisson structure is equivalent to P if and only if there exists a non negative constant κ and a measurable function $v : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

$$\int_0^\infty v(y)dG(y) = 1,$$

such that the associated density process $\xi_t = \mathbb{E}_P[\xi_T \mid \mathcal{F}_t]$ of the Radon-Nikodym derivative derived from the probability measure Q, $(\xi_T = \frac{dQ}{dP})$ is given by

$$\begin{aligned} \xi_t &= \left(\prod_{k=1}^{N_t} \kappa v(Y_k)\right). \exp\left(\int_0^t \int_0^\infty (1 - \kappa v(y))\lambda dG(y)ds\right) \\ &= exp\left(\sum_{k=1}^{N_t} \ln(\kappa v(Y_k)) + \lambda(1-\kappa)t\right). \end{aligned} \tag{5.8}$$

Under the new measure Q index by the constant κ, and the function v, the index loss process L_t has characteristic $(\lambda^Q, dG^Q(y)) = (\lambda\kappa, v(y)dG(y))$ and is associated to the couple $(\kappa, v(y))$; moreover, the set of equivalent probability measures $P^{\kappa,v}$ is one-to-one. For details of the proof, see Delbaen and Haezendonck (1989).

The author proposed two different methods based on the no-arbitrage assumption for the valuation of a terminal claim option $G(L_T)$ under the assumption of a constant force of interest r.

The first method is based on computing the price of the derivative through the Fourier transformation. In a no-arbitrage market, the price of the derivative is computed as

$$\pi_t^Q(t, L) = \mathbb{E}_Q\left[e^{-r(T-t)}G(L_T) \mid \mathcal{F}_t\right] = e^{-r(T-t)}f^Q(t, L)$$

where Q is the equivalent martingale measure. Assuming that there exists a constant k such that $G(.) - k \in L^2(\mathbb{R})$ then the price $f^Q(t, L)$ can be represented as

$$f^Q(t, L) = \int_{-\infty}^{\infty} e^{iuL} L^Q_{T-t}(u)\varphi(u)du + k \tag{5.9}$$

where $\varphi(.)$ is the inverse Fourier transform of $G(.) - k$ and $L^Q_{T-t}(.)$ is the characteristic function of L_{T-t} under the probability measure Q i.e.

$$L^Q_{T-t}(u) = exp\left(\lambda^Q \left(\int_0^{\infty} e^{iuy} dG^Q(y) - 1\right)(T-t)\right).$$

This yields the derivative price on the form

$$\pi^Q_t(t, L) = e^{-r(T-t)} \int_{-\infty}^{\infty} e^{iuL} exp\left(\lambda^Q \left(\mathbb{E}_Q[e^{iuY}] - 1\right)(T-t)\right)\varphi(u)du + k.$$

The second method exploits the fact that the discounted price processes in the insurance market are martingales under an equivalent measure. Infinitesimal generator with a Markov Process $L = (L_t)_{0 \le t \le T}$ on the set of price function $f : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is applied, from which the author deduced that the process $M = (M_t)_{0 \le t \le T}$ with

$$M_t = f(t, L_t) - f(0, L_0) - \int_0^t \mathcal{A}(f)(s, L_s)ds$$

is a martingale under the measure Q where

$$\mathcal{A}(f^Q(t, L)) = \frac{\partial f^Q(t, L)}{\partial t} + \lambda^Q \cdot \int_0^{\infty} \left(f^Q(t, L + y) - f^Q(t, L)\right) dG^Q(y).$$

The author proves that for a choice of the parameters (κ, v) and for any constant $k \in \mathbb{R}$ such that $G(.) - k \in L^2(\mathbb{R})$, $f^Q(t, L)$ satisfies the partial integro-differential equation

$$\frac{\partial f^Q(t, L)}{\partial t} = \lambda^Q \cdot f^Q(t, L) - \lambda^Q \int_0^\infty f^Q(t, L + y))dG^Q(y)$$

with boundary condition $f^Q(T, L) = G(L)$ and has a unique solution satisfied by (5.9). In both methods, the choice of the parameters (κ, v) can be made to reflect the structure of the market or the preference of an agent. In a liquid insurance derivative market, it is therefore possible to obtain the market prices of frequency risk κ and jump size $v(.)$ as implied parameters from observed derivative prices. In the context of a market with a representative agent, market price of frequency and jump size risk are determined by the preferences of the representative agent. The principle of utility maximisation thus determines the unique price process of the insurance.

A different approach is considered in Christensen (2000) where the author considered the aggregate index to be modelled as an exponential Levy process

$$L_t = L^0 exp(Z_t),$$

where Z_t is a Levy process defined differently depending on whether the process evolves in the loss period $[0, T_1]$ or in the development period $[T_1, T_2]$ according to :

$$X_t = \sum_{i=1}^{N_t} Y_i \quad \text{for} \quad t \in [0, T_1]$$

where N_t is a Poisson process with parameter λ_1 and Y_i is exponentially distributed with parameter ϱ implying that $L_t - L^0 \sim Pa(\varrho, L^0)$. In the development period, X_t is expressed as a random sum of Normal distribution according to

$$X_t = X_{T_1} + \sum_{i=1}^{\tilde{N}_{t-T_1}} \tilde{Y}_i \quad \text{for} \quad t \in [T_1, T_2],$$

where $\tilde{N}_t$ is a Poisson process with parameter λ_2 and $\tilde{Y}_i \sim N(0, \sigma^2)$. The author priced a capped call option with cap K and strike price A whose payoff is

$$C(T_2, L_{T_2}) = \min\left(\max\left(L_{T_2} - A, 0\right), K - A\right).$$

A no-arbitrage assumption is again made, which results in the price at time t being:

$$C(t, L) = e^{-r(T_2 - t)} \mathbb{E}_Q \left[C(T_2, L_{T_2}) | \mathcal{F}_t\right]. \tag{5.10}$$

But since L_t is not tradable, the author considered the process $\frac{L_t}{P_t}$ as the traded asset where P_t is the deterministic premium paid up to time t. The author defined the Esscher transform

$$M(z, t, h) = \int_{-\infty}^{\infty} e^{zx} F(dx, t; h) = \frac{M(z + h, t)}{M(h, t)}$$

where

$$F(dx, t, h) = \frac{e^{hx} F(dx, t)}{M(h, t)}$$

and

$$M(z, t) = \mathbb{E}[e^{zX_t}] = \int_{-\infty}^{\infty} e^{zx} F(dx, t).$$

The idea in Gerber and Cummins (1996) is then used to obtain the risk neutral Esscher Measure Q under which the discounted price of the process $\frac{L_t}{P_t}$ is a martingale. The

Radon-Nikodym derivative for the risk neutral Esscher measure is characterised by

$$\frac{\partial Q}{\partial P}|\mathcal{F}_t = \begin{cases} \frac{e^{h_l X_t}}{M_l(h_l,t)} & \text{for} \quad t \in [0, T_1] \\ \\ \frac{e^{h_l X_{T_1}}}{M_l(h_l,T_1)} \frac{e^{h_d(X_t - X_{T_1})}}{M_d(h_d,t-T_1)} & \text{for} \quad t \in [T_1, T_2] \end{cases} \tag{5.11}$$

where h_l and h_d are such that

$$\mathbb{E}_Q\left[e^{-rt}\frac{L_t}{P_t}\right] = \frac{L^0}{P_0} = 1.$$

For $t \in [0, T_1]$, h_l is obtained as the solution of the equation

$$\mathbb{E}_Q\left[e^{-rt}\frac{L_t}{P_t}\right] = 1 \Longleftrightarrow M(1, t, h_l) = e^{rt}P_t \quad \text{whereas}$$

For $t \in [0, T_1]$, h_d is obtained as the solution of the equation

$$\begin{aligned} &\mathbb{E}_Q\left[e^{-rt}\frac{L_t}{P_t}\right] = 1. \\ &\Longrightarrow \mathbb{E}_Q\left[e^{\tilde{X}_t}\right] = e^{rt}P - t \\ &\Longrightarrow \mathbb{E}_Q\left[e^{\tilde{X}_{T1}}e^{\tilde{X}_t - \tilde{X}_{T1}}\right] = e^{rt}P_{T_1}\tilde{P}_{t-T_1} \\ &\Longrightarrow Ml(1, T_1, h_l)M(1, t - T_1, h_d) = e^{rt}P_{T_1}\tilde{P}_{t-T_1}. \end{aligned}$$

Although there is more than one equivalent measure, the author used the Esscher transform to produce a unique equivalent measure Q defined as in Equation (5.11) under which the price of the derivative is provided as in Equation (5.10).

In the next chapter, the *principle of equivalent Utility* for the pricing of dynamic risk, pioneered by Young and Zariphopoulou is applied to calculate the indifference price of a catastrophe option where the payoff depends only on the value of the index at the term of the option.

Chapter 6

Pricing of Catastrophe Insurance via Indifference Utility Theory

6.1 Introduction to Equivalent utility theory

The Black-Scholes pricing theory is based on the fact that one can create and construct a portfolio that can accurately replicate the payoff of a contingent claim. The risk associated with the financial product is thereby completely eliminated or *hedged* therefore one can argue that the value of the product is the cost of creating the hedging portfolio. One of the assumption of the Black-Scholes model is the completeness of the derivatives market. In the case of a catastrophe insurance derivatives market, the market is *incomplete* and there is no universal theory to date that successfully produces a unique value of the contingent claim that all market agents agree on.

In an incomplete market with unhedgeable risk present, there are a variety of ideas on what should be the right notion of pricing. Various methods including the ones in chapter (5) can be used. The approach that we will adopt in the pricing is based on an expected utility argument built around investors' preferences towards the risks that cannot be eliminated. The risk preference of an investor is quantified through her utility function of wealth. To fully determine the price of the option for an investor given her utility function, one compares her maximum expected utility of wealth when the investor takes up the option therefore paying a price and her maximum expected utility of wealth without taking up the option. The indifference price is the price that makes the investor indifferent between the two investment opportunities. The fundamental idea underlying the technique is based on the principle of *certainty equivalence* drawn from economic theory but modified in order to accommodate the dynamic nature of the market. The principle was introduced in Hodges and Neuberger (1999) and was extended in Panas, Davis, and Zariphopoulou (1993). Since then, many authors, see Young and Zariphopoulou (2001), Young and Zariphopoulou (2002), Zariphopoulou (2001a) and Zariphopoulou (2001b) have applied the utility based technique to the pricing of financial options and insurance products.

In Shouda (2005), the author considered the indifference price of defaultable bonds whose recovery values are not predictable. A partial integro-differential equation was derived via a backward stochastic differential equation for an investor with exponential utility function. As a result of the incompleteness of the market, the author proposed a natural risk measure to compute the indifference price of the bond.

In the indifference pricing framework. different methodologies may be used. One method is to work with the diffusion or jump-diffusion model of the state variables and derive the PDE or PIDE for the indifference price.

In Musiela and Zariphopoulou (2004), the authors considered an incomplete market where there are tradable assets and non tradable assets with correlated Brownian motions and derived a nonlinear PDE for the indifference price of a contingent claim. In Sicar and Zariphopoulou (2005), the authors considered a stochastic volatility model and showed that the indifference price is bounded by a risk neutral price and the certainty equivalent price.

Another popular indifference pricing method is to use the duality method in Fritelli (2000) and Delbaen, Grandits, Rheinlander, Samperi, Schweizer, and Stricker (2002) to derive a backward stochastic differential equation (BSDE) for the indifference price process. In Rouge and Karoui (2000), a diffusion model with a constrained trading opportunity was considered and the author derived the BSDE for the indifference price whereas in Mania and Schweizer (2005) the authors considered a general semi-martingale filtration, derived the BSDE for the indifference price and studied the asymptotic results with respect to the risk aversion coefficient.

In the absence of an analytical solution of the indifference price which is a solution to a PDE or PIDE, numerical solutions are called for.
Grasselli and Hurd (2004) proposed a Monte Carlo algorithm for the exponential hedging problems which is computationally expensive. In Grasselli and Hurd (2005), they authors studied the numerical methods for indifference pricing for a stochastic volatility model when the volatility process follows a reciprocal CIR process and showed the result for the volatility claims only, leaving out the claims on the stock. In Lim (2005), a numerical solution was proposed to compute the indifference price and risk monitoring strategy of a contingent claim in an incomplete market with exponential utility function. Using the duality between exponential optimal investment problem and the relative entropy problem, the author recasts the option writer's optimal investment problem as a minimax problem and derive the complete procedure of finding the numerical solution of the indifference price.

The common ground in the use of the principle of equivalent utility in the pricing of contingent claims is the fact that the risks to be priced are related to uncertainties that do not correspond to fluctuations of a tradable asset making the market incomplete. An example of the risk in this thesis is the claim size of an insurance policy that is random and cannot be hedged using financial products. The principle of equivalence can also be applied in a complete market i.e. if the contingent claim is written on a tradable asset. In Young and Zariphopoulou (2002), the authors proved that for a contingent claim written on a tradable asset, the equivalent utility based method yields a price that is utility independent and identical to the Black-Scholes price.

The principle of indifference utility is based on the definition given by Hodges and Neuberger:

Definition The indifference price of the European claim $G = G(L_T)$, is defined as the function $h = h(t, x, L)$ such that the investor is indifferent towards the following two scenarios: optimise the expected utility without the derivative therefore not facing any contingent claim and not receiving any compensation (premium); and optimise it with the derivative taking into account, on one hand the liability $G = G(L_T)$ at expiration T to be faced as well as the compensation $h(t, x, L)$ received at the time of inception t.

If we denote by $V(t, x)$ the maximum expected utility without using the derivative where x is the initial wealth of the agent and denote by $U(t, x, L)$ the maximum expected utility of the agent having to face a contingent claim at the time of expiration T where L represents the current value of the asset on which the contingent claim is written, then the indifference price is the value of the function $\hat{h}$ such that

$$V(t, x) = U(t, x + \hat{h}(t, x, L), L). \tag{6.1}$$

The next section will be the application of equation (6.1) in pricing Catastrophe insurance.

6.2 Pricing via indifference utility Theory

As the pricing process involves the maximum expected utility of two investment scenarios, one needs to solve two optimal stochastic problems and extract the parameter price that makes their value functions equal under the umbrella of equivalence utility principle. In the subsequent analysis, we will adopt the exponential utility function on the form

$$u(w) = -\eta e^{-\nu w} \tag{6.2}$$

where w represents the wealth of an investor. We assume that the contingent claim is a terminal claim and is written on an insurance claim index L_t which evolves in a manner similar to a PCS option. The term of the option (reporting period) $[0, T]$ is divided into the loss period $[0, T_1]$ and the development period $[T_1, T]$. Due to the nature of the aggregate claim, we represent its dynamic in the loss period as a compound Poisson process with a constant claim severity i.e.

$$L_t = \sum_{i=1}^{N_t} Y_i \quad \text{for} \quad t \in [0, T1],$$

where N_t is the Poisson counting process of the claims with rate λ, and Y_i are the claim sizes assumed to be independent and identically distributed. The change in the aggregate claim index is expressed as

$$dL_t = Y dN_t \quad \text{for} \quad t \in [0, T_1]. \tag{6.3}$$

In the development period $[T_1, T]$, the aggregate claim process is represented as a geometric Brownian motion with parameters μ_L and σ_L. Thus L_t evolves according to

$$dL_t = L_t(\mu_L dt + \sigma_L dB_t^L) \quad \text{for} \quad t \in [T_1, T]$$

where B_t^L is a standard Brownian motion. Thus at any point $t \in [T_1, T]$,

$$L_t = L_{T_1} \exp\left((\mu_L - \frac{\sigma_L^2}{2})(t - T_1) + \sigma_L B_t^T - B_{T_1}^T\right)$$

where L_{T_1} is the aggregate claim at the end of the loss period. The parameter μ_L expresses the upward revaluation of the claim in monetary terms which we consider above the rate of inflation whereas σ_L expresses its volatility.

The objective is to determine at time $t \in [0, T_1]$ the price of an insurance contingent claim $G = G(L_T)$ given the value of the aggregate index claim at time t . We also assume that investors have the opportunity to trade between a risky asset S_t and a risk-free asset C_t. The risky asset evolves dynamically as

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where $\mu \geq 0$ is the drift of the asset and $\sigma \geq 0$ its volatility. The risk free asset evolves deterministically as

$$dC_t = rC_t dt$$

where $r \geq 0$ is the risk free interest rate. At time t, the investor has capital $W_t = w$ and is allowed to trade dynamically between the risky asset and risk free asset at no cost.

If we let Θ_t denote the amount invested in the risky asset and Θ_t^0 the amount invested in the risk free asset, the total current wealth satisfies the budget constraint $W_t = \Theta_t + \Theta_t^0$ with the flexibility of the company to short sell stocks or borrow cash at the risk free rate. Thus the wealth follows the dynamics

$$dW_s = rW_s ds + \Theta_s(\mu - r)ds + \sigma\Theta_s dB_s \quad t \leq s \leq T. \tag{6.4}$$

In the absence of a liability and therefore of a premium received, the investor equipped with an increasing utility function therefore preferring more to less will maximise the expected utility of terminal wealth

$$V(t,w) = \sup_{\substack{\Theta_s \\ t\le s\le T}} \mathbb{E}\left[u(W_T) \mid W_t = w\right] \tag{6.5}$$

where Θ_s satisfies the usual integrability condition $\mathbb{E}\left[\int_t^T \Theta_s^2 ds < \infty\right]$. By the same token as in chapter (4), the solution of (6.5) satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial V}{\partial t} + \max_\Theta \left[(\mu - r)\Theta \frac{\partial V}{\partial w} + \frac{1}{2}\sigma^2\Theta^2 \frac{\partial^2 V}{\partial^2 w}\right] + rw\frac{\partial V}{\partial w} = 0 \\ V(T,w) = u(w) \end{cases} . \tag{6.6}$$

The optimal solution is obtained as a feedback control

$$\hat{\Theta}(t,w) = -\frac{(\mu - r)}{\sigma^2}\frac{V_w}{V_{ww}} \tag{6.7}$$

in which V solves (6.6):

$$\frac{\partial V}{\partial t} + rwV_w - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V_w^2}{V_{ww}} = 0. \tag{6.8}$$

From chapter (4) which assumes an exponential utility function of the form (6.2), we can derive the value function solution to (6.6) and the optimal risky asset investment by setting the return on shareholders investment to null ($\alpha = 0$), and the Risk process to be identically null ($p = 0$ and $\lambda = 0$). This yields:

$$V(t,w) = -\eta e^{-\nu w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T-t)} \tag{6.9}$$

and the optimal investment in the risky asset is computed as

$$\Theta_t = \frac{(\mu - r)^2}{\nu\sigma^2} e^{-r(T-t)}. \tag{6.10}$$

The optimal investment is independent of the state variables and decreases as the risk aversion of the investor measured by the constant ν increases. It predicts to invest less in the risky asset as we approach the final time horizon T.

In the case where liability induced by a premium is involved, the principle of equivalence utility will be assessed depending on whether the investor is in a short or long position on the contingent claim. An insurer who is short on the option will receive some premium $h(t, w, L_t)$ that will be added to her current wealth and be invested in the assets. She will have to face the contingent claim at the term of the option. The associated value function is therefore

$$U(t, w, L_t) = \sup_{\substack{\Theta_s \\ t \leq s \leq T}} \mathbb{E}\left[u(W_T - G(L_T)) \,\Big|\, W_t = w + h_{Short}(t, w, L_t)\right]. \tag{6.11}$$

Her indifference price will therefore be h_{Short} such that her terminal expected utility remains the same if she had not sold the policy i.e.

$$U(t, w + h_{Short}(t, w, L_t), L_t) = V(t, w). \tag{6.12}$$

Similarly, we can think of the option as a hedging strategy for an insurer who is concerned about big claims. Depending on her utility function, she will be willing to pay a price h_{Long} to cover a potential big loss in the future. The policy-holder has two courses of action. She either purchases the option to hedge the risk, paying an amount h_{Long} at inception t and invest the remaining of her wealth optimally in the assets or choose not to buy the policy at h_{Long} and invest her wealth w in the assets optimally in which case she will be facing the terminal claim at time T. Her indifference price will be the h_{Long} such that she is indifferent between hedging the potential big loss and not hedging it; thus h_{Long} is the solution of

$$V(t, w - h_{Long}) = U(t, w, L_t) \tag{6.13}$$

In both cases, the technique results in equating the expected terminal utility of wealth of not insuring the risk with the value when the risk in insured. The price h_{Short} can be thought of as the minimum price the party on the short position is willing to accept for taking the risk and this price will inevitably depend on her utility function in an incomplete market and particularly through the degree of aversion to risk . Similarly, the price h_{Long} can be considered as the maximum price the party on the long position is willing to pay for transferring the terminal risk to a risk taker. Unlike the Black-Scholes models in which both parties agree on the same price, the two prices do not coincide in an incomplete market even if the two parties have the same risk preference. This is due to the non linearity of (6.12) and (6.13). In real life the assumptions of identical utility functions for the two parties is not true since the short party is less averse to risk than the long party ($\nu_{short} \geq \nu_{long}$) which furthermore increases the gap between the prices. We will in the next section consider the price h_S to be set by a party in the long position on the terminal claim option with payoff $G = G(L_T)$.

6.3 Construction of the Solution

We will proceed in this section by determining the value function of the optimal policy in the two intervals of time $[T_1, T]$ and $[0, T_1]$.

Value function in $[T_1, T]$

Let $\tilde{U}$ be the value function in the development period $[T_1, T]$, $\tilde{U}$ satisfies

$$\begin{cases} \tilde{U}(t, w, L) = \sup_{\substack{\Theta_s \\ T_1 \leq t \leq T}} \mathbb{E}\left[u(W_T - G(L_T)) \,\middle|\, W_t = w\right] \\ \\ \tilde{U}(T, W_T, L_T) = u(W_T - G(L_T)) \end{cases} \quad \text{for} \quad t \in [T_1, T]. \qquad (6.14)$$

Using the dynamic programming technique, the optimal wealth invested in the risky asset is obtained as a feedback control

$$\hat{\Theta}(t,w) = -\frac{(\mu - r)}{\sigma^2}\frac{\tilde{U}_w}{\tilde{U}_{ww}}. \tag{6.15}$$

Following the same steps as in chapter (4), $\tilde{U}$ satisfies the partial differential equation

$$\tilde{U}_t dt - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\tilde{U}_w^2}{\tilde{U}_{ww}} dt + rw\tilde{U}_w dt + \tilde{U}_L \mathbf{E}\left[dL_t \mid \mathcal{F}_t\right] + \frac{1}{2}\tilde{U}_{LL}\mathbf{E}\left[dL_t^2 \mid \mathcal{F}_t\right] = 0 \tag{6.16}$$

with boundary condition $\tilde{U}(T,w,L) = u\left(W_T - G(L_T)\right)$ where

$$dL_t = L_t\left(\mu_L dt + \sigma_L dB_t^L\right) \quad \text{for} \quad t \in [T_1, T].$$

Taking the conditional expectation, yields

$$\begin{cases} \tilde{U}_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\tilde{U}_w^2}{\tilde{U}_{ww}} + rw\tilde{U}_w + \tilde{U}_L L_t \mu_L + \frac{1}{2}\sigma_L^2 L_t^2 \tilde{U}_{LL} = 0 \\ \\ \tilde{U}(T,w,L) = u(W_T - g(L_T)) \end{cases} . \tag{6.17}$$

By the nature of the utility function which has a constant aversion to risk, we propose a solution of the form

$$\tilde{U}(t,w,L) = V(t,w) \times \tilde{F}(t,L). \tag{6.18}$$

Using (6.18), we obtain

$$\tilde{U}_t = V_t\tilde{F} + V\tilde{F}_t, \qquad \tilde{U}_w = V_w\tilde{F}$$

$$\tilde{F}_{ww} = V_{ww}\tilde{F}, \qquad \tilde{U}_L = V\tilde{F}_L$$

$$\text{and} \qquad \tilde{U}_{LL} = V\tilde{F}_{LL}.$$

Substituting (6.19) into (6.17), yields

$$\tilde{F}\left[\frac{\partial V}{\partial t}+rwV_w-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\frac{V_w^2}{V_{ww}}\right]+V\left[\tilde{F}_t+\mu_L L_t\tilde{F}_L+\frac{\sigma_L^2L_t^2}{2}\tilde{F}_{LL}\right]=0. \qquad (6.19)$$

Due to (6.8), the first term of (6.19) vanishes leading to

$$\tilde{F}_t+\mu_L L_t\tilde{F}_L+\frac{\sigma_L^2L_t^2}{2}\tilde{F}_{LL}=0,$$

with boundary condition derived from the equality

$$\begin{aligned}\tilde{U}(T,w,L_T) &= u\left(w-G(L_T)\right)=V(T,w)\times\tilde{F}(T,L_T)\\ &= -\eta e^{-\nu w+\nu G(L_T)}=-\eta e^{-\nu w}\times\tilde{F}(T,L_T)\\ \Longrightarrow\ \tilde{F}(T,L_T) &= e^{\nu G(L_T)}=F(L_T).\end{aligned}$$

$\tilde{F}$ then verifies

$$\begin{cases}\tilde{F}_t+\mu_L L_t\tilde{F}_L+\frac{\sigma_L^2L_t^2}{2}\tilde{F}_{LL}=0\\ \\ \tilde{F}(T,L)=e^{\nu G(L)}=F(L)\end{cases}. \qquad (6.20)$$

The Feynman-Kac formula gives us the following representation of $\tilde{F}$:

$$\tilde{F}(t,L)=\mathbb{E}\left[e^{\nu G(L_T)}\,\middle|\,\mathcal{F}_t\right]\quad\text{for}\quad t\in[T_1,T], \qquad (6.21)$$

thus a candidate value function in $[T_1,T]$ is

$$\tilde{U}(t,w,L)=V(t,w)\mathbb{E}\left[e^{\nu G(L_T)}\,\middle|\,\mathcal{F}_t\right]\quad\text{for}\quad t\in[T_1,T]. \qquad (6.22)$$

Value function in $[0, T_1]$

In the loss period, the aggregate loss index evolves dynamically according to equation (6.3). If we let U be the value function for $t \in [0, T_1]$ i.e.

$$U(t, w, L) = \sup_{\substack{\Theta_s \\ 0 \le s \le T_1}} \mathbb{E}\left[u(W_T)\right],$$

using again the dynamic programming principle, we obtain a feedback control of the optimal risky asset allocation as in (6.15) where $\tilde{U}$ is replaced by U. The HJB equation satisfied by U follows the equation

$$U_t dt - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{U_w^2}{U_{ww}} dt + rwU_w dt + \lambda dt \int_0^{\infty} \Big[U(t, w, L + y) - U(t, w, L)\Big] f_Y(y) dy = 0,$$

where f_Y denotes the density function of the claim size distribution. The continuity of the value function with the time variable at $t = T_1$ as the result of the principle of dynamic programming imposes the natural condition $U(T_1, w, L) = \tilde{U}(T_1, w, L)$ which in turn constitutes the boundary condition for the problem in the loss period $[0, T_1]$. The complete HJB equation with its boundary condition turns to :

$$\begin{cases} U_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{U_w^2}{U_{ww}} + rwU_w + \lambda \mathbb{E}\Big[U(t, w, L + y) - U(t, w, L)\Big] = 0 \\ \\ U(T_1, w, L) = \tilde{U}(T_1, w, L) = V(T_1, w)\mathbb{E}\left[e^{\nu G(L_T)} \,|\mathcal{F}_{T_1}\right] \end{cases} . \qquad (6.23)$$

We again assume that U is of the form

$$U(t, w, L) = V(t, w) \times F(t, L). \qquad (6.24)$$

The partial derivatives and change in U are calculated as:

$$U_t = V_t F + F_t V, \quad U_w = V_w F, \quad U_{ww} = V_{ww} F \quad \text{and}$$
$$U(t, w, L + Y) - U(t, w, L) = V\left[F(t, L + Y) - F(t, L)\right]. \qquad (6.25)$$

Substituting (6.24) into (6.23) while accounting for (6.25) yields

$$F\left[\frac{\partial V}{\partial t} + rwV_w - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V_w^2}{V_{ww}}\right] + V\left[F_t + \lambda\mathbb{E}\Big[F(t, L+Y) - F(t, L)\Big]\right] = 0.$$

The first term vanishes due to equation (6.8) therefore F satisfies

$$\begin{cases} F_t + \lambda\mathbb{E}\Big[F(t, L+x) - F(t, L)\Big] = 0 \\ \\ F(T_1, L) = \tilde{F}(T_1, L) = \mathbb{E}\left[e^{\nu G(L_T)} \,|\mathcal{F}_{T_1},\, L_{T_1} = L\right] = \xi(L) \end{cases} . \qquad (6.26)$$

The solution of (6.26) is a path sum of possible histories or trajectories of the aggregate claim index process of the form :

$$F(t, L) = \mathbb{E}\left[\xi(L_{T_1 - t}) \;\Big|\; L_0 = L\right].$$

But since the process L_t is time homogeneous, this can be simply represented as

$$F(t, L) = \mathbb{E}\left[\xi(L_{T_1}) \;\Big|\; L_t = L\right],$$

and given that

$$\xi(L_{T_1}) = \mathbb{E}\left[e^{\nu G(L_T)} \;\Big|\; \mathcal{F}_{T_1}\right],$$

we obtain after simplification using the Tower property,

$$F(t, L) = \mathbb{E}\Big[\;\mathbb{E}\left[e^{\nu G(L_T)} \;\Big|\; \mathcal{F}_{T_1}\right] \;\Big|\; L_t = L\Big] = \mathbb{E}\left[e^{\nu G(L_T)} \;\Big|\; L_t = L\right]. \qquad (6.27)$$

This shows that for $t \in [0, T_1]$ where the option can be bought, a candidate value function is of the form

$$U(t, w, L) = V(t, w)\, F(t, L) \qquad (6.28)$$

where V is according to (6.9) and F according to (6.27).

6.3.1 Verification via the martingale optimality condition

In this section, we will use the martingale optimality theorem of section (2.7) to show that $\tilde{U}$ and U respectively described by (6.22) and (6.28) are indeed the value functions. The proof follows from the fact that U and $\tilde{U}$ can be written in a factorised form, see (6.18) and (6.24). Following (6.15), the optimal risky asset allocation during the whole life of the option is therefore independent of the index loss and is exactly as described by equation (6.10). We now consider the candidate value function $\tilde{U}$ in the development period $[T_1, T]$,

Theorem 6.3.1 *$\tilde{U}$ is martingale when the optimal control is applied, and is a sub-martingale otherwise.*

Proof

$$\begin{aligned}\mathbb{E}\left[\tilde{U}(t+s, W_{t+s}, L_{t+s}) \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[V(t+s, W_{t+s}) \times \tilde{F}(t+s, L_{t+s}) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[V(t+s, W_{t+s}) \,\middle|\, \mathcal{F}_t\right] \times \mathbb{E}\left[\tilde{F}(t+s, L_{t+s}) \,\middle|\, \mathcal{F}_t\right].\end{aligned}$$

But $\tilde{F}$ is a martingale i.e.

$$\mathbb{E}\left[\tilde{F}(t+s, L_{t+s}) \,\middle|\, \mathcal{F}_t\right] = \tilde{F}(t, L_t).$$

Also, since V is the value function of the stochastic control problem with no loss index and since the dynamic of the loss index does not involve the control variable, V is a martingale when the optimal strategy is applied and a sub-martingale when it is not. This fact together with (6.29) proves that $\tilde{U}$ is a martingale when the optimal strategy is adopted and a sub-martingale whenever it is not applied. Consequently, $\tilde{U}$ is indeed the value function in $[T_1, T]$. By the same token, we can easily show that U is indeed the value function in $[0, T_1]$. □

We are now ready to derive a closed form formula for the equivalent price of the option for the long position as well as for the short position.

6.3.2 The general price of an option and some properties of the price

We derive in this section the price of any terminal claim option.

Theorem 6.3.2 *The price of the long position h_{Long} is equal to the price of the short position h_{Short} and are given by the formula*

$$h_{Long} = h_{Short} = \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L\right] = h(\nu). \tag{6.29}$$

Proof Applying the principle of utility equivalence to the short position, we obtain

$$U(t, w + h_{Short}, L) = V(t, w)$$

which yields:

$$V(t, w + h_{Short}) * F(t, L) = V(t, w)$$

hence

$$h_{Short} = \frac{e^{-r(T-t)}}{\nu} \ln F(t, T) = \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L\right].$$

For the long position price, we equate

$$U(t, w, L) = V(t, w - h_{Long})$$

which is equivalent to

$$V(t, w) * F(t, L) = V(t, w) \exp\left[\nu e^{r(T-t)} h_{Long}\right]$$

hence

$$h_{Long} = \frac{e^{-r(T-t)}}{\nu} \ln F(t, T) = h_{Short} = \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\Big|\, L_t = L\right] = h(\nu), \qquad (6.30)$$

completing the proof. □

The above formula brings out some important ingredients of the utility based valuation approach. First and foremost, we remark that by contrast to existing methods for incomplete markets, the price of the option is given as the expectation of a function of the payoff and the degree of aversion to risk $g(\nu, x) = e^{\nu x}$ which is similar to risk neutral pricing. But unlike risk neutral pricing in which the price of option is computed as the discounted value of the payoff under the the equivalent martingale probability measure, we notice that the expectation in the formula of $h(\nu)$ is with respect to the physical probability measure. The fact that the expectation is taken on a function of the payoff and the degree of aversion to risk constitutes a direct consequence of the utility based approach that distorts the original payoff and replaces it by a derived function of the utility function . Although the derived function is in general nonlinear with respect to the payoff, the price of the option can be obtained easily by using Monte Carlo simulations methods that will be explored in the subsequent sections in lieu of solving partial differential equations.

Theorem 6.3.3 *As the degree of aversion to risk increases, the price of the option $h(\nu)$ increases.*

Proof The proof relies on the Lyapunov inequality on the class of random variables in L_p. We first consider the price of the option as a function of the degree of aversion to risk,

$$\begin{aligned} h(\nu) &= \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L\right] \\ &= e^{-r(T-t)} \ln \left(\mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L\right] \right)^{\frac{1}{\nu}} \\ &= e^{-r(T-t)} \ln \Big(I(\nu) \Big). \end{aligned}$$

Let us consider the Hölder Inequality on the functions F and G,

$$\mathbb{E}\left[\left|FG\right|\right] \leq \left(\mathbb{E}\left[|F|^p\right]\right)^{\frac{1}{p}} \left(\mathbb{E}\left[|G^q|\right]\right)^{\frac{1}{q}} \quad \text{where} \quad p \geq 1\ , q \geq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Taking $F = X^p$, $G = 1$ where $X = e^{G(L_T)}|L_t = L$, we obtain

$$\mathbb{E}\left[|X^\nu|\right] \leq \mathbb{E}\left[|X^\nu|^p\right] = \mathbb{E}\left[\left|X^{\nu p}\right|\right] \quad \text{for} \quad \nu >= 0$$

$$\Longrightarrow \left(\mathbb{E}\left[\left|X^\nu\right|\right]\right)^{\frac{1}{\nu}} \leq \left(\mathbb{E}\left[\left|X^{\nu p}\right|\right]\right)^{\frac{1}{\nu p}} = \left(\mathbb{E}\left[\left|X^{\varkappa}\right|\right]\right)^{\frac{1}{\varkappa}} \quad \text{where} \quad \varkappa = \nu p. \tag{6.31}$$

Since $p \geq 1$ and $\varkappa \geq \nu$, $I(\nu) = \left(\mathbb{E}\left[\left|X^\nu\right|\right]\right)^{\frac{1}{\nu}} = \left(\mathbb{E}\left[e^{\nu G(L_T)}s \,\middle|\, L_t = L\right]\right)^{\frac{1}{\nu}}$ increases in ν .

□

The price can be seen as a function of the aversion to risk from the buyer's point of view as well as from the seller point of view. If the short position holder has a high aversion to risk, she will charge a higher premium. On the other hand, if the buyer has a higher aversion to risk, she will be prepared to pay a higher premium to have the risk passed to the seller.

Equation (6.30) shows that if the two parties have the same utility function and the same aversion to risk i.e. $\nu_{Long} = \nu_{Short}$, then both parties agree on the same universal price on a terminal contingent claim option with payoff $G(L_T)$ which is a particular feature in Black-Scholes pricing theory. In reality, and specially in insurance risk hedging, the party on the short position has lower aversion to risk than the party on the long position so that her risk aversion parameter is lower i.e. $\nu_{Short} \leq \nu_{Long}$; consequently, the price of the short position is lower i.e. $h_{Short} \leq h_{Long}$. This suggests that in reality, if the two parties adopt the same utility function of wealth, the buyer of the option will be able to afford the price of the seller.

Another important point is to assess how different the computed price $h(\nu)$ based on the equivalent utility theory is from the traditional actuarial perspective. To do that, we will first make use of Jensen's inequality. Since the function exponential is convex, applying Jensen inequality shows that

$$h(\nu) = \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\Big|\, L_t = L\right] \geq e^{-r(T-t)} \mathbb{E}\left[G(L_T) \,\Big|\, L_t = L\right] = h_A(r). \quad (6.32)$$

In equation (6.32), the price $h_A(r)$ can be considered as the traditional actuarial price of the option purely based on the expectation and not on the aversion of the party on the long or short position, using the risk free rate as a discounting rate. The inequality in prices, brought about by the application of Jensen's inequality shows that for a risk averse investor who adopts an exponential utility function, regardless of her degree of aversion to risk, the price she is willing to pay will always be above the actuarial price if the actuarial discounting rate of return is the risk free rate. This result as one might think is surprising since it goes against the principle of no-arbitrage in the financial market. In fact, equation (6.32) is not surprising because the actuarial discount rate δ is usually different from the risk free rate of return r. On the contrary, the result shows that there is an implied actuarial discounting rate with regard to the level of risk under the derivative contract and the risk aversion of the seller of the contract such that the actuarial price equates the indifference price $h(\nu) = h_A(\delta)$ where δ denotes the no-arbitrage actuarial discounting rate which is computed as

$$\delta = r - \frac{1}{T-t} \ln \left[\frac{\ln \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L \right]}{\mathbb{E}\left[\nu G(L_T) \,\middle|\, L_t = L \right]} \right] = r - \rho. \tag{6.33}$$

Equations (6.33) shows that the no-arbitrage actuarial discounting rate is lower than the risk free rate of return . In practise, δ is not chosen by equating the indifference price to the actuarial price since the indifference price is unknown in the actuarial pricing. Instead, δ is chosen subjectively in order to reflect the level of risk in the financial contract as well as the degree of aversion to risk to the seller of the contract; this subjectivity element in the choice does not guarantee that the two prices will be equal and therefore can create an arbitrage in the market if one can exactly compute $h(\nu)$. This example is one of the many including the article of Andrew Smith (Smith 1996) which claim that the traditional actuarial approach on its own is not appropriate for pricing financial derivatives. In the following, we show that if the seller is completely neutral to risk, she will charge the actuarial price based on the risk free rate.

Theorem 6.3.4 *The lower bound of the option is attained when the party on the short position is risk neutral i.e. $\nu = 0$ and the consequent price is the actuarial price with a risk free discounting rate.*

Proof The proof relies on the fact that the price of the option can be written as a cumulant function. Observe that

$$h(\nu) = \frac{e^{-r(T-t)}}{\nu} \ln \left(\mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L \right] \right) = \frac{e^{-r(T-t)}}{\nu} \ln \left(M_{G(L_T)}(\nu) \right) = e^{-r(T-t)} \frac{K_{G(L_T)}(\nu)}{\nu}$$

where $M_{G(L_T)}(\nu) = \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L \right]$ is the conditional moment generating function of the payoff evaluated at the risk aversion parameter ν and $K_{G(L_T)}(\nu)$ is the corresponding cumulant function.

A series expansion of e^x at $x_0 = 0$ may at this point be used to rewrite $M_{G(L_T)}(\nu)$ as

$$M_{G(L_T)}(\nu) = 1+\sum_{i=1}^{\infty}\frac{\nu^i}{i!}\mathbb{E}\left[G(L_T)^i \;\middle|\; L_t = L\right] = 1+\nu\sum_{i=1}^{\infty}\frac{\nu^{i-1}}{i!}\mathbb{E}\left[G(L_T)^i \;\middle|\; L_t = L\right] = 1+\nu S$$

where

$$S = \sum_{i=1}^{\infty}\frac{\nu^{i-1}}{i!}\mathbb{E}\left[G(L_T)^i \;\middle|\; L_t = L\right]. \tag{6.34}$$

The cumulant function $K_{G(L_T)}(\nu) = \ln(1 + \nu S)$ may also be expanded using the series expansion of $\ln(1 + x)$ at $x_0 = 0$. This yields

$$K_{G(L_T)}(\nu) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\nu^j S^j = \nu S + \sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{j}\nu^j S^j = \nu S\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k+1},\nu^k S^k\right]$$

from which we deduce that

$$\frac{K_{G(L_T)}(\nu)}{\nu} = S\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k+1}\nu^k S^k\right].$$

From (6.34), we compute the limit of S,

$$\lim_{\nu\to 0} S = \mathbb{E}\left[G(L_T)\;\middle|L_t = L\right]$$

and conclude therefore that

$$\lim_{\nu\to 0} h(\nu) = e^{-r(T-t)}\mathbb{E}\left[G(L_T)\;\middle|\; L_t = L\right] = h_A(r).$$

□

The result interestingly shows that not only the actuarial price of the option ($h_A(r)$) when the rate of discount is the risk free rate is the cheapest in the market but also that when the seller of the option is completely neutral to risk ($\nu = 0$), she does not care about the volatility of the payoff. This finding suggests that the fair actuarial premium pricing approach with risk free rate is based on the risk neutrality of the policy seller.

6.4 Pricing of some classical reinsurance options

In this section, we consider the major types of insurance options. The price of any option with payoff $G(L_T)$ is given by :

$$h(t, L, \nu) = \frac{e^{-r(T-t)}}{\nu} \ln \mathbb{E}\left[e^{\nu G(L_T)} \,\middle|\, L_t = L\right] \tag{6.35}$$

where for $t < T_1$,

$$L_T \mid L_t = \left(L_t + \sum_i^{N_T - N_t} Y_i\right) e^{(\mu_L - \frac{\sigma_L^2}{2})(T-T_1) + \sigma B^L_{T-T_1}}. \tag{6.36}$$

At terminal horizon T, the aggregate loss function can be decomposed as a sum of the product of a log-normal distribution and the compound Poisson process. The price which depends on the expectation of the exponential of the payoff is analytically challenging even in the case of the simplest payoff $G(L_T) = L_T$ due to the presence of the moment generating function of a log-normal distribution. Depending on the form of the payoff, different methods will be applied to compute the price of the option. The advantage in having the price in the conditional form (6.35) is that in the absence of any simplification due to the structure of the payoff, Monte Carlo simulation may be used to obtain the equivalent price. On figure (6.1) is the histogram of the aggregate loss index at the terminal horizon using Monte Caro simulations.

6.4.1 PCS call option

The PCS call option with strike price K_1 and cap value K_2 provides at the term of the contract, a payoff

$$G(L_T) = \min\left\{ \max\left(L_T - K_1, 0\right), K_2 - K_1 \right\} = (L_T - K_1)1_{\{L_T \geq K_1\}} - (L_T - K_2)1_{\{L_T \geq K_2\}}.$$

Since $G(.)$ is not linear, an analytical form of the price is doomed to failure. We will proceed by numerical solution using Monte Carlo techniques. Figure (6.2) shows the distribution of the simulated payoff and table (6.1) shows the resulting price.

6.4.2 Stop Loss

The stop loss is a non proportional type of reinsurance and works similarly to excess-of-loss reinsurance. While excess of loss reinsurance is related to single loss amounts, either per risk or per event, stop loss covers are related to the total amount of claims for a given time period net of an agreed retention amount K. The re-insurer pays the excess of the total claim amount above the retention amount . The payoff is similar to a European call option with strike price K with the difference that the underlying variable of the option is not the stock but the aggregate claim. The payoff at maturity may be written as

$$G(L_T) = (L_T - K)1_{\{L_T \geq K\}}.$$

The price of such option maybe also be computed using Monte Carlo simulations. Figure (6.2) shows the distribution of a Stop loss payoff and table (6.1) shows the resulting price.

6.4.3 Proportional reinsurance

The index L_t can also be taken as the aggregate claims of an insurance company which is concerned about the aggregate claim from running the business in a given interval of time $[0, T]$. The company could, giving its aversion to risk, decide or not to purchase a reinsurance that protects against large amount of aggregate claims at the end of the term of the business. A common option available to the insurance company is to cede a proportion of whatever the aggregate claims appear to be to a re-insurer company: *the proportional insurance.* We consider a company ceding a proportion κ of the aggregate risk to a re-insurer. Proportional reinsurance in the framework of equivalent utility is suitable for a company or a reinsurance company whose losses are heavily weighted in the loss index. Giving the risk aversion ν of the re-insurer, the price in (6.35) represents the minimum price the re-insurer is willing to charge to retain a proportion κ of the aggregate loss where

$$G(L_T) = \kappa L_T.$$

In contrast to the traditional expected liability actuarial pricing approach that does not take into consideration the aversion of the insurer. The payoff may be explicitly written as

$$G(L_T) = \kappa L_T = \left(\kappa L + \kappa \sum_{i}^{N_T - N_t} Y_i\right) e^{(\mu_L - \frac{\sigma_L^2}{2})(T-T_1) + \sigma B_{T-T_1}^L}.$$

A close form solution of the price of the price $\mathbb{E}\left[e^{\nu G(L_T)} | L_t = L\right]$ may only be obtained for claim size distributions whose moment generating function are everywhere finite. Since real life claim size distribution do not satisfy the condition, we will proceed via a Monte Carlo simulations. The resulting distribution of the payoff is shown on figure (6.2) and table (6.1) shows the price of the option.

6.5 Parametric Approximation of the Loss Index

In section (6.4.3), we made a distributional assumption on the claim size and evaluate the price of the option as the mean of the simulations across all paths. In the absence of a parametric assumption on the claim size distribution, it is of actuarial practise to approximate the maturity time aggregate loss index L_T by a distribution that preserves some key properties of the aggregate claim. Approximate calculations are desirable in the industry as time is a crucial factor. We will in this section, approximate the aggregate loss index at time T by a translated gamma distribution with parameter $\Upsilon \sim \gamma(\phi, \psi)$ translated by Δ units i.e. $L_T | L_t = \Delta + \Upsilon$ such that the mean, the variance and the skewness coefficient of the translated gamma match with those of the aggregate loss i.e.

$$\mathbb{E}[L_T\,|L_t] = \Delta + \frac{\phi}{\psi}, \qquad \mathbb{V}ar[L_T\,|L_t] = \frac{\phi}{\psi^2} \quad \text{and} \quad \mathbb{S}k[L_T\,|L_t] = \frac{2}{\sqrt{\phi}} \tag{6.37}$$

where $\mathbb{S}k$ is the coefficient of skewness. Knowing the first three moments of the aggregate loss, the parameters of the translated gamma can be deduced as

$$\phi = \frac{4}{\mathbb{S}k^2}, \qquad \psi = \frac{\mathbb{E}[L_T\,|L_t]}{\mathbb{V}ar} \quad \text{and} \quad \Delta = \mathbb{E}[L_T\,|L_t] - \frac{4\mathbb{V}ar}{\mathbb{E}[L_T\,|L_t] \times \mathbb{S}k^2}. \tag{6.38}$$

PCS

At this point, we can rewrite the payoff of a PCS call option in terms of the value taken by the gamma distribution Υ,

$$G(L_T) = \begin{cases} 0 & \text{if} \quad \Upsilon < K_1 - \Delta \\ \Upsilon + \Delta - K_1 & \text{if} \quad K_1 - \Delta < \Upsilon < K_2 - \Delta \\ K_2 - K_1 & \text{if} \quad \Upsilon > K_2 - \Delta \end{cases}. \tag{6.39}$$

We are now ready to compute the approximated price of the PCS option. Let us evaluate the conditional expectation based on the approximation of the aggregate loss.

$$\begin{aligned}
\mathbb{E}\left[e^{\nu G(L_T)}\,|L_t\right] &= \int_0^{K_1-\Delta} f_\Upsilon(y)dy + \int_{K_1-\Delta}^{K_2-\Delta} e^{\nu(y+\Delta-K_1)} f_\Upsilon(y)dy + e^{\nu(K_2-K_1)} \int_{K_2-\Delta}^{\infty} f_\Upsilon(y)dy \\
&= P\Big[\Upsilon < K_1 - \Delta\Big] + e^{\nu(\Delta-K_1)} \int_{K_1-\Delta}^{K_2-\Delta} e^{\nu y} \frac{\psi^\phi}{\Gamma(\phi)} y^{\phi-1} e^{-\psi y} dy \\
&\quad + e^{\nu(K_2-K_1)} P\Big[\Upsilon > K_2 - \Delta\Big] \\
&= P\Big[\Upsilon < K_1 - \Delta\Big] + e^{\nu(\Delta-K_1)} \left(\frac{\psi}{\psi-\nu}\right)^\phi \times \int_{K_1-\Delta}^{K_2-\Delta} \frac{(\psi-\nu)^\phi}{\Gamma(\phi)} y^{\phi-1} e^{-(\psi-\nu)y} dy \\
&\quad + e^{\nu(K_2-K_1)} P\left[\Upsilon > K_2 - \Delta\right] \\
&= P\Big[\Upsilon < K_1 - \Delta\Big] + e^{\nu(\Delta-K_1)} \left(\frac{\psi}{\psi-\nu}\right)^\phi \times P\Big[K_1 - \Delta < \Upsilon' < K_2 - \Delta\Big] \\
&\quad + e^{\nu(K_2-K_1)} P\Big[\Upsilon > K_2 - \Delta\Big],
\end{aligned}$$

where $\Upsilon' \sim \gamma(\phi, \psi - \nu)$. If we denote by χ_n the chi-square with n degrees of freedom, using transformation techniques, we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{\nu G(L_T)}\,|L_t\right] &= e^{\nu(\Delta-K_1)}\left(\frac{\psi}{\psi-\nu}\right)^{\phi} \times P\Big[2(\psi-\nu)(K_1-\Delta) < 2(\psi-\nu)\Upsilon' < 2(\psi-\nu)(K_2-\Delta)\Big] \\
&+ P\Big[2\psi\Upsilon < 2\psi(K_1-\Delta)\Big] + e^{\nu(K_2-K_1)}P\Big[2\psi\Upsilon > 2\psi(K_2-\Delta)\Big] \\
&= e^{\nu(\Delta-K_1)}\left(\frac{\psi}{\psi-\nu}\right)^{\phi} \times P\Big[2(\psi-\nu)(K_1-\Delta) < \chi_{2\phi} < 2(\psi-\nu)(K_2-\Delta)\Big] \\
&\quad + P\Big[\chi_{2\phi} < 2\psi(K_1-\Delta)\Big] + e^{\nu(K_2-K_1)}P\Big[\chi_{2\phi} > 2\psi(K_2-\Delta)\Big]. \qquad (6.40)
\end{aligned}$$

Denoting by $F_{\chi_{2\phi}}$ the distribution function of $\chi_{2\phi}$, the price of the PCS option becomes

$$\begin{aligned}
h(t, L, \nu) &= \frac{e^{-r(T-t)}}{\nu}\ln\Bigg[F_{\chi_{2\phi}}(a_1) + e^{\nu(\Delta-K_1)}\left(\frac{\psi}{\psi-\nu}\right)^{\phi} \times \left(F_{\chi_{2\phi}}(b_2) - F_{\chi_{2\phi}}(b_1)\right) \\
&\quad + e^{\nu(K_2-K_1)}\Big(1 - F_{\chi_{2\phi}}(a_2)\Big)\Bigg] \qquad (6.41)
\end{aligned}$$

where

$a_1 = 2\psi(K_1-\Delta)$, $b_2 = 2(\psi-\nu)(K_2-\Delta)$, $b_1 = 2(\psi-\nu)(K_1-\Delta)$ and $a_2 = 2\psi(K_2-\Delta)$.

Stop Loss

Proceeding to the same methodology as in section (6.5) we can find an approximated value of the price of a stop loss option using the translated gamma approximation technique to the aggregate loss. $L_T \mid L_t = \Delta + \Upsilon$. The payoff is rewritten in terms of Υ

$$G(L_T) = \begin{cases} 0 & \text{if } \Upsilon < K - \Delta \\ \Upsilon + \Delta - K & \text{if } \Upsilon \geq K - \Delta \end{cases} \qquad (6.42)$$

and the price of the option is approximated by

$$h(t, L, \nu) = \frac{e^{-r(T-t)}}{\nu}\ln\left[F_{\chi_{2\phi}}(2\psi(K-\Delta) + e^{\nu(\Delta-K)}\left(\frac{\psi}{\psi-\nu}\right)^{\phi} \times \Big(1 - F_{\chi_{2\phi}}\Big(2(\psi-\nu)(K-\Delta)\Big)\Big)\right]. \qquad (6.43)$$

Proportional reinsurance

By the same token as above, the price of a proportional reinsurance may be derived as

$$\begin{aligned} h(t, L, \nu) &= \frac{e^{-r(T-t)}}{\nu}\left[\ln \mathbb{E}\left[e^{\kappa\nu(\Delta+Y)}\right]\right] \\ &= \frac{e^{-r(T-t)}}{\nu}\left[\kappa\nu\Delta + \phi \ln\left(\frac{\psi}{\psi-\kappa\nu}\right)\right] \end{aligned} \tag{6.44}$$

6.5.1 Computation of the parameters of the translated gamma distribution

Equation (6.38) provides the parameters of the translated gamma knowing the different moments of the aggregate loss. In this section, we will evaluate the different moments of the aggregate loss at time T. Let us consider equation (6.36), and decompose the conditional terminal loss $L_T \,|L_t = L$ as

$$L_T \mid L_t = A \times S \times Z$$

where $A = e^{(\mu_L - \frac{\sigma_L^2}{2})(T-T_1)}$, $\quad S = L_t + \sum_i^{N_T-N_t} Y_i \quad$ and $Z = e^{\sigma B^L_{T-T_1}}$.

We first compute the first three moments of the conditional loss in terms of the moments of its components:

$$\text{Moments of } L_T \,\Big|\, L_t \begin{cases} \mathbb{E}[L_T \,|L_t] = A\mathbb{E}[S]\mathbb{E}[Z] \\ \mathbb{V}ar[L_T \,|L_t] = A^2\Big[\mathbb{E}[S^2]\mathbb{E}[Z^2] - \mathbb{E}^2[S]\mathbb{E}^2[Z]\Big] \\ \mathbb{S}k[L_T \,|L_t] = A^3\frac{\left[\mathbb{E}[S^3]\mathbb{E}[Z^3] - 3\mathbb{E}[S^2]\mathbb{E}[Z^2]\mathbb{E}[S]\mathbb{E}[Z] + 2\mathbb{E}^3[S]\mathbb{E}^3[Z]\right]}{\mathbb{V}ar[L_T \,|L_t]^{\frac{3}{2}}} \end{cases} . \tag{6.45}$$

We obtain the first three moments of S through its moment generating function :

$$M_S(\theta) = e^{\theta L + \lambda(T_1-t)[M_Y(\theta)-1]}$$

where Y represents the claim size, and $M_Y(\theta)$ represents its moment generating function. By first, second and third differentiating $M_S(\theta)$ and evaluating at $\theta = 0$, we respectively obtained

$$\text{Moments of } S \begin{cases} \mathbb{E}[S] = \lambda(T_1 - t)\mathbb{E}[Y] + L \\ \mathbb{E}[S^2] = \lambda(T_1 - t)\Big[\mathbb{E}[Y^2] + \mathbb{E}[Y]\mathbb{E}[S]\Big] + L\mathbb{E}[S] \\ \mathbb{E}[S^3] = \lambda(T_1 - t)\Big[\mathbb{E}[Y^3] + 2\mathbb{E}[Y^2]\mathbb{E}[S] + \mathbb{E}[Y]\mathbb{E}[S^2]\Big] + L\mathbb{E}[S^2] \end{cases} \tag{6.46}$$

The Z component of the loss is a log-normal variable whose moments may be derived through the transformation $Z = e^N$ where $N \sim \mathcal{N}\Big(0, \sigma'^2(T - T_1)\Big)$. This yields

$$\text{Moments of } Z \begin{cases} \mathbb{E}[Z] = e^{\frac{\sigma'^2(T-T_1)}{2}} \\ \mathbb{E}[Z^2] = e^{2\sigma'^2(T-T_1)} \\ \mathbb{E}[Z^3] = e^{\frac{9}{2}\sigma'^2(T-T_1)} \end{cases} \tag{6.47}$$

6.5.2 Monte Carlo simulations parameters

```
Claim distribution is Gamma with parameter 10 and 2

Expected claimsize = 5  millions

Claim process is Poisson with rate  0.5  per month

Cap K1= 10 Millions      Cap K2 = 20 Millions  (PCS)

Strike K = 15 (Stop Loss)

Retention = 0.4 (Proportional)

Loss event period = 6 months        Development period = 3 months

Effective monthly free rate = 0.05/12

Drift of claims adjustment in development period = 0.03/12 per month

Volatility of claims adjustment in development period = 0.13/12 per month

Initial Loss = 10 Billions

Risk aversion parameter =.2

retention = 0.4
```

6.5.3 Results

Figure (6.1) shows a comparison between the aggregate Index Loss distribution when a Monte Carlo simulation and a gamma approximation are applied. The Aggregate loss Index is positively skewed which justifies the approximation. The parameters of the implied gamma distribution obtained in the approximation are:

$$\Delta = -5.89 \quad \phi = 9.363 \quad \text{and} \quad \psi = 0.301 \quad \text{such that} \quad L_T \mid L_t = \Delta + \Upsilon(\phi, \psi). \quad (6.48)$$

In table (6.1) we compare the first 3 moments of the approximated gamma distribution to the Monte Carlo aggregate Index claim. The moments of the Gamma approximation are obtained using formula (6.37) with the computed values of ϕ, ψ and Δ in (6.48). Table (6.1) also shows a comparison of the prices using Monte Carlo simulation and gamma approximation. Aside the Stop Loss option, the discrepancy in price of the two methods is not very significant. This suggests that the gamma approximation which is computationally advantageous has a merit to be exploited since it does not involve simulations in the pricing. The significant discrepancy in the price of Stop loss is due to the fact the tail of the approximated gamma is slightly longer than in the Monte carlo simulation(see the last column of Table (6.1)).

If we look at the prices using the Monte Carlo method, we notice that the Stop Loss has the highest price. This is explained by the fact that the payoff is not bounded above hence the higher price.

The proportional option has a retention of $\kappa = 0.4$. Jensen's inequality shows the price of the option is higher than the discounted value of

$$\mathbb{E}[G(L_T)] = \kappa \times \mathbb{E}[L_T] = 0.4 \times 25.21 = 10.08$$

using the risk free rate as the discounting rate. But since the discounting factor is itself less than 1, the price of the option must be greater than $\mathbb{E}[G(L_T)]$ which is satisfied in table (6.1).

	Aggregate Loss			Prices			Prob
	Mean	Variance	Skewness	PCS	Stop	Prop	$P(L_T \geq 60)$
Monte Carlo	25.21	83.25	0.65	8.74	22.33	11.17	0.0014
Gamma Approx	25.26	103.65	0.67	8.66	28.88	11.63	0.0033

Table 6.1: **Monte Carlo and Gamma approximation prices**

6.6 Summary

In this chapter, we have shown that using the indifference utility theory, one can price in an incomplete market. Although the resulting differential equation from the HJB is an integro-differential equation, a quasi-analytical solution has been obtained in terms of physical expectation. We also show that in the presence of exponential utility function, the two parties agree on the same indifference price which increase as the aversion to risk increases. The minimum indifference price equals to the actuarial price which is obtained when we are neutral to risk. We also notice that due to the fact that the aggregate terminal loss is positively skewed, a translated gamma approximation provides a quick and simple way to the pricing of the options. Martingale optimality principal has been used to validate the solution of the partial integro-differential equation.

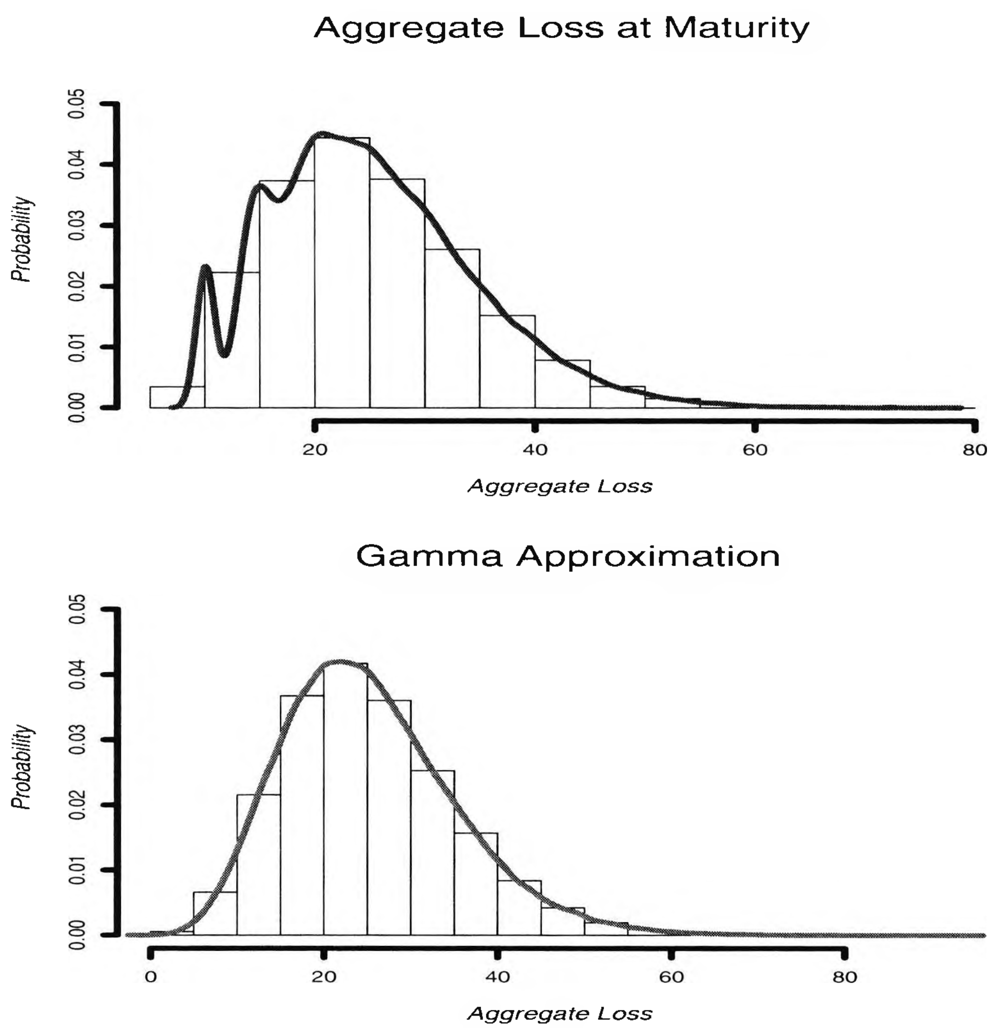

Figure 6.1: **Distribution of the Aggregate Index Loss**

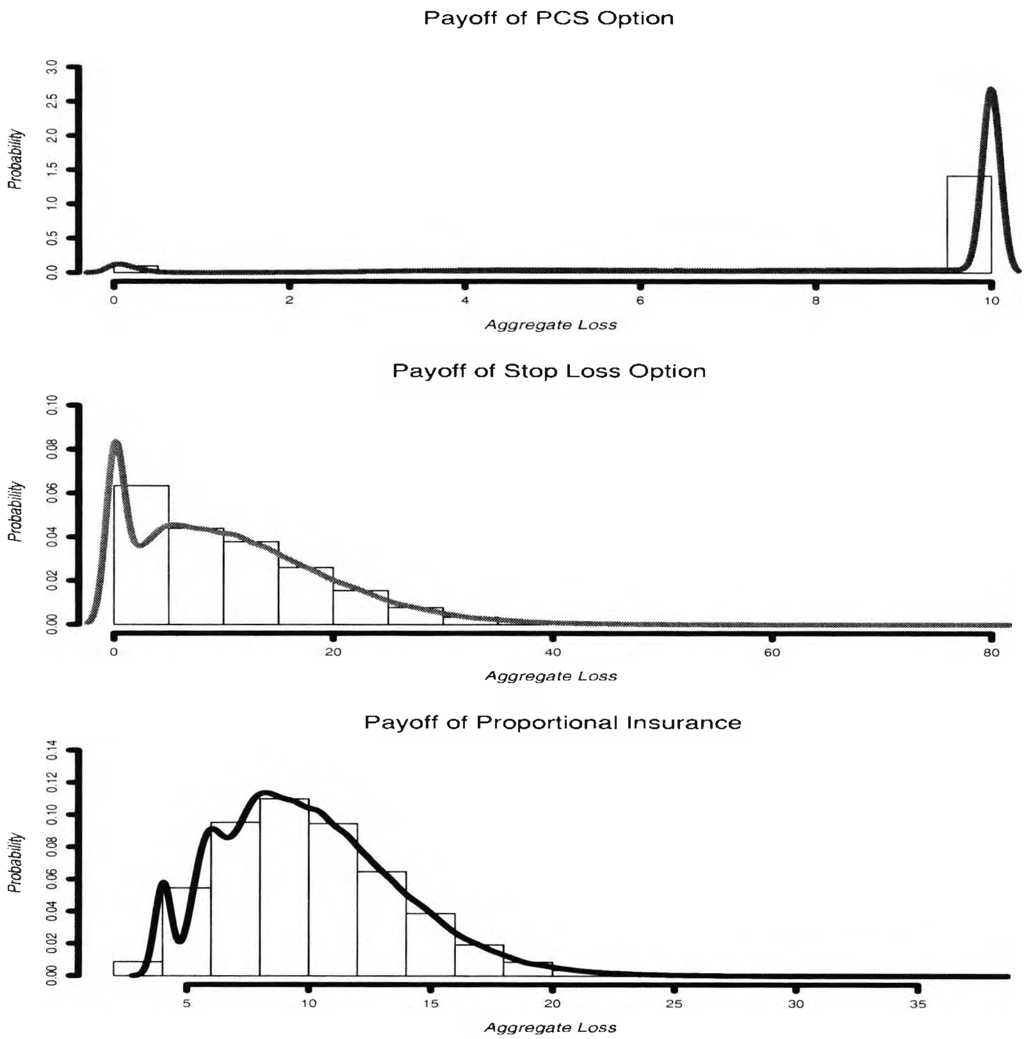

Figure 6.2: **Distribution of the Payoff**

Chapter 7

Conclusion and Future Research

In this thesis, we have presented the use of stochastic control theory to solve dynamic control problems and have shown how a real life problem can be mathematically modelled and solved using differential equations and probability theory results. Extensive use of the dynamic programming principle has been extensively applied to transform the all time optimisation problem to a single time optimisation in HJB framework.

In Chapter 3, we have solved analytically, the problem of optimal premium setting of a non-life insurance policy in a monopolistic market. The optimal strategy predicts to charge a constant premium rate which satisfies the premium solvency criteria. We also noted the correspondence between minimum discount rate and solvency criteria. In the results, the utility discount rate has been interpreted as a measure of short aim of the insurance company. We also examined the probability of ruin under optimal strategy. The partial integro-differential equation derived was a mathematical challenge to be solved analytically due to the boundary conditions that are not obvious to be determined. A Monte Carlo simulation was performed which yields a decreasing probability of ruin as the initial wealth or the claim size increases, an increasing probability of ruin as the demand of policies shifts upward and an inconclusive trend of the ruin probability as the claim severity changes. An extension of this work can be made by solving the PIDE satisfied by the probability of ruin and by assuming other demand functions that are influenced by exogenous factors so that the market can be extended to oligopoly market.

In Chapter 4, we considered an asset allocation problem in the context of insurance risk process. An analytical solution of the problem is obtained and validated using the Martingale optimality principle. The optimal asset strategy when the risk free rate is less than the dividend rate dictates to invest less in the risky asset as we approach the finite horizon and to invest more if the risk free rate is higher than the dividend rate. An extension of the problem to be explored in a future research is to make the dividend rate a control variable or to make the payment of the dividend contingent on the level of the wealth being above a certain floor value. We further noticed that the optimal asset allocation is inversely proportional to the risk aversion parameter, proportional to the Sharpe ratio, independent of the demand function and of the insurance risk process. A logical further extension will be to make the insurance risk process be correlated with the risky asset and to assume many risky assets.

In Chapter 6, we elaborated a framework for pricing options in an incomplete market based on the indifference utility theory. Catastrophe insurance index loss was considered as the underlying option variable. We observed that by adopting an exponential utility function, the long and the short position agree on the same price which clears the market. Our results also showed that the price of the option increases as the risk aversion parameter increases with the minimum price reflecting a neutrality to risk. A semi-closed form solution of the price in terms of physical probability expectation is obtained. The right skewness of the terminal aggregate loss index and the expectation semi-closed form solution of the option price allowed us to respectively approximate the terminal aggregate index loss by a translated gamma distribution and to use Monte Carlo simulations in computing the price of the derivatives. The result shows that the discrepancy in the two prices is not significant. In the loss period, claims are not inflated. A possible extension of the model will be to inflate any claim incurred from the random time of occurrence up to the end of the loss period.

Appendix A

Stochastic Processes and Properties

Measurable space If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \implies F^C \in \mathcal{F}$, where $F^C = \Omega \backslash F$ is the complement of F in Ω

(iii) $A_1, A_2, \cdots \in \mathcal{F} \implies A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ is called a measurable space .

Probability measure A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function

$$P : \mathcal{F} \to [0, 1] \quad \text{such that}$$

(a) $P(\emptyset) = 0$, $P(\Omega) = 1$

(b) if $A_1, A_2, \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple $(\Omega, \mathcal{F}. P)$ is called a *probability space* and the subsets F of Ω which belong do $\mathcal{F}$ are called *$\mathcal{F}$-measurable sets* which in the context of probability are called *events* with the interpretation that

$$P(F) = \text{"the probability that the event } F \text{ occurs"}$$

Measurable function If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $\mathbf{X} : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$\mathbf{X}^{-1}(U) := \{\omega \in \Omega; \mathbf{X}(\omega) \in U\} \in \mathcal{F}$$

for all open subsets $U \in \mathbb{R}^n$

Stochastic Processes

A *stochastic process* is a parametrised collection of random variables

$$\{\mathbf{X}_t\}_{t \in T}\ , \quad T \subset \mathbb{R}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ with values assumed in $\mathbb{R}^n$. In this thesis, we will be dealing with continuous time stochastic processes therefore the parameter set T will be taken as the time variable i.e. $T = \mathbb{R}^+$.

For each fixed $t \in T$, we have a random variable $\omega \to \mathbf{X}_t(\omega); \quad \omega \in \Omega$, and for every $\omega \in \Omega$ we obtain the function $t \to \mathbf{X}_t(\omega); \quad t \in T$ which is called the *path* of the process $\mathbf{X}_t$

Filtration A family $\mathcal{F}_t$ of σ-algebra on ω parametrised by $T \subset \mathbb{R}$ is called a filtration if

$$\forall\ s, t \in T \ \text{ such that } \ s \leq t, \quad \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

We can think of $\mathcal{F}_{t_1}$ as the history of the process $\mathbf{X}$ from time $t = 0$ to time $t = t_1$

Some Properties of stochastic processes

Martingales A stochastic process $\mathbf{X}_t$ parametrised by $t \in T$ is called a *martingale (sub-martingale, supermartingale)* with respect to the filtration $\mathcal{F}_t$ if

1) $\mathbf{X}_t$ is integrable for each $t \in T$ i.e. $\forall\, t \in T \quad \int_\omega |\mathbf{X}_t|\, dP < \infty$

2) $\mathbf{X}_t$ is $\mathcal{F}_t$-measurable for each $t \in T$ (in which case we say that $\mathbf{X}_t$ is adapted to $\mathcal{F}_t$)

3) $\forall\, s \geq 0$, $\forall\, t \in T$, $\mathbb{E}\left[\mathbf{X}_{t+s} \mid \mathcal{F}_t\right] = \mathbf{X}_t$ (respectively, $\leq$ and $\geq$)

In the thesis, martingale and supermartingale are used to carry out some verification proofs through the *Martingale optimality principle.*

A.0.1 Markov processes

An important class of stochastic processes is the *Markov Processes.* The characteristic property of such processes is that it retains no memory of where it has been in the past. This means that only the current state of the process can influence where it goes next called the *Markov property.* This property offers many advantages in the analysis of the behaviour of these processes.

Markov process A stochastic process $\mathbf{X}_t$ for $t \geq 0$ in continuous time is a *Markov process* on a space $\mathbb{R}^n$ if for any $t \geq 0$, $\Delta t \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$,

$$Prob\left[\mathbf{X}_{t+\Delta t} \leq \mathbf{x} \,\middle|\, \mathcal{F}_t\right] = Prob\left[\mathbf{X}_{t+\Delta t} \leq \mathbf{x} \,\middle|\, \mathbf{X}_t\right]$$

where $\mathcal{F}_t$ represents the history of the process $\mathbf{X}_t$ up to time t. Thus the Markov process can be called a memory-less process. This memory-less property of Markov processes leads immediately to the independent increments property of Markov processes. In this thesis, we have only dealt with processes that are Markovian therefore substituting the filtration of a process to its current value in our calculation.

Independent increment and stationarity of Markov processes

If $\mathbf{X}_t$ is a Markov process, then the state increment $\Delta\mathbf{X}_t = \mathbf{X}_{t+\Delta t} - \mathbf{X}_t$ is independent of $\Delta\mathbf{X}_s = \mathbf{X}_{s+\Delta s} - \mathbf{X}_s$ if the time interval are disjoint i.e. $s + \Delta s \leq t$ or $t + \Delta t \leq s$. A Markov process is called *stationary or time-homogeneous* if the probability distribution depends only on the time difference i.e.

$$Prob\left[\mathbf{X}_{t+\Delta t} - \mathbf{X}_t \leq \mathbf{x}\right] = Prob\left[\Delta\mathbf{X}_t \leq \mathbf{x}\right]$$

depends only on $\Delta t \geq 0$ and is independent of $t \geq 0$.

A.0.2 Continuity and Smoothness

Continuity .

- A Process X_t is a *continuous process* at t_0 if $\lim_{\Delta t \to 0} X_{t_0+\Delta t} = X_{t_0}$ provided that the limit exists, else X_t is *discontinuous* at t_0.

- The process X_t has a *jump discontinuity* at t_0 if $\lim_{\substack{\Delta t \to 0 \\ |\Delta t|>0}} X_{t_0+\Delta t} \neq X_{t_0}$, provided that the limit exist, i.e. the limit from the left does not agree with the limit from the right. This is represented as

$$X_{t_0^-} = \lim_{\Delta t \to 0^+} X_{t_0 - \Delta t} \neq X_{t_0^+} = \lim_{\Delta t \to 0^+} X_{t_0+\Delta t}$$

where $\Delta t \to 0^+$ means $\left(\Delta t \to 0, \Delta t > 0\right)$.
We say that X_t has a *jump* at $t = t_0$. The corresponding jump at the jump discontinuity is defined as $[X](t_0) \equiv X_{t_0} - X_{t_0^-}$.

- The process X_t is *right-continuous* at t_0 if

$$\lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} X_{t_0+\Delta t} = X_{t_0}$$

such that the jump of X at t_0 is defined as $[X](t_0) \equiv X_{t_0} - X_{t_0^-}$. since $X_{t_0^+} = X_{t_0}$. *Left-continuous processes* are defined similarly.

Smoothness .

- The process X_t is *smooth* at t_0 if

$$\lim_{\Delta t \to 0} \frac{\Delta X_{t_0}}{\Delta t}$$

exists, i.e. X_t is differentiable at t_0; else the process X_t is *non-smooth.*

Appendix B

Diffusion Processes and Ito Formula

B.1 Brownian motion

Brownian Motion A continuous time stochastic process B_t, $t \in \mathbb{R}^+$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ if :

1) $B_0 = 0$ (almost surely)

2) The sample path of B_t is continuous i.e. $t \to B_t$ is continuous a.s.

3) B_t has independent(a) and stationary(b) increment, i.e.

   a) $\forall\ s, t,\ s \leq t,\ B_t - B_s$ is independent of $\mathcal{F}_s$

   b) $\forall\ s, t,\ s \leq t,\ B_t - B_s$ and B_{t-s} have the same distribution

4) The increment $B_t - B_s$ is normally distributed with mean 0 and variance $t-$ i.e. the density function of B_t is :

$$f_{B_{t-s}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{t-s}} \quad \text{for} \quad x \in \mathbb{R}$$

B.1.1 Some properties of Brownian motion

- The sample paths are nowhere differentiable
- Brownian motion is a martingale

B.2 Stochastic integral

When attempting to define calculus for Brownian motion and other diffusions, one has to face the fact that their sample path, though they are continuous are nowhere differentiable. A direct approach to stochastic integral like

$$\int_0^t f(s,\omega)dB_s(\omega) \quad \text{where} \quad f:[0,\infty)\times\Omega \longrightarrow \mathbb{R} \tag{B.1}$$

is doomed to failure. The expression (B.1) is called an *Ito integral* and will be the subject of this section. It is distinguished by the fact that we are integrating with respect to Brownian motion (as can be seen from the dB_s), not with respect to time. There is an extensive literature that deals with Ito integral, see Brzézniak and Zastawniak (2000), Øksendal (1998), and Karatzas and Shreve (1986). Only few properties and results of Ito integral will be presented here.

Ito Integral Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t,\omega):[0,\infty)\times\Omega \to \mathbb{R}$ such that

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the *Borel* σ-algebra on $[0,\infty)$

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted

(iii) $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$

then, for any function $f \in \mathcal{V}$, the Ito integral $\mathcal{I}[f](\omega) = \int_S^T f(t,\omega)dB_t(\omega)$ satisfies:

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty}\int_S^T \phi_n(t,\omega)dB_t(\omega) \quad \left(\lim \text{ in } L^2(P)\right)$$

where $\{\phi_n\}$ is a sequence of elementary functions adapted to $\mathcal{F}_t$ such that

$$\mathbb{E}\left[\int_S^T (f(t,\omega)-\phi_n(t,\omega))^2\,dt\right] \to 0 \quad \text{as } n\to\infty$$

B.2.1 properties of Ito Integral

(i) $\mathbb{E}\left[\int_S^T f(t,\omega)dB_t(\omega)\right] = 0$

(ii) $\mathbb{E}\left[\left(\int_S^T f(t,\omega)dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(t,\omega)dt\right]$ for all $f \in \mathcal{V}(S,T)$

(iii) $\mathcal{I}[f](\omega) = \int_S^T f(t,\omega)dB_t(\omega)$ is a martingale with respect to $\mathcal{F}$

(iv) If f is deterministic i.e. $f(t,\omega) = f(t)$, then

$$\int_S^T f(t)dB_t(\omega) \sim N\left(0, \int_S^T f^2(t)dt\right)$$

where N denotes the normal distribution

B.3 The general Ito formula for diffusion processes

Let

$$d\mathbf{X}(t) = \mathbf{u}dt + \mathbf{v}d\mathbf{B}(t)$$

where

$$\mathbf{X}(t) = \left\{\begin{array}{c}\mathbf{X}_1(t)\\ \vdots\\ \mathbf{X}_n(t)\end{array}\right\},\quad \mathbf{u} = \left\{\begin{array}{c}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{array}\right\},\quad \mathbf{v} = \left\{\begin{array}{ccc}\mathbf{v}_{11} & \cdots & \mathbf{v}_{1m}\\ \vdots & & \vdots\\ \mathbf{v}_{n1} & \cdots & \mathbf{v}_{nm}\end{array}\right\},\quad d\mathbf{B}(t) = \left\{\begin{array}{c}d\mathbf{B}_1(t)\\ \vdots\\ d\mathbf{B}_m(t)\end{array}\right\}$$

be an n-dimensional Ito process. Let $g(t, \mathbf{x}) = (g_1(t, \mathbf{x}), \cdots, g_p(t, \mathbf{x}))$ be a C^2 map from $[0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process

$$Y(t, \omega) = g(t, \mathbf{X}(t))$$

is again an Ito process, whose component number k, Y_k is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t, \mathbf{X})dt + \sum_i \frac{\partial g_k}{\partial \mathbf{x}_i} d\mathbf{X}_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g^k}{\partial \mathbf{x}_i \partial \mathbf{x}_j}(t, \mathbf{X}) d\mathbf{X}_i d\mathbf{X}_j$$

where $d\mathbf{B}_i dt = dt d\mathbf{B}_i = 0$ and $d\mathbf{B}_i \times d\mathbf{B}_j = \delta_{ij} dt$

Appendix C

Jump and Diffusion processes

C.1 Poisson Processes

A Poisson process is a non deceasing continuous time counting process with values in $\mathbb{N}$. Since a Poisson process suffers from positive jumps of integer magnitude, it is therefore discontinuous. which makes the differentiability problems of the Poisson process of second importance. Thus the analytical problems are even more severe than for the Brownian motion.

Homogeneous Poisson process A process $N_t \in \mathbb{N}$ is a *time homogeneous* Poisson process with intensity $\lambda > 0$ if

- $N_0 = 0$
- N_t has independent and stationary increments
- For $h \geq 0$ and h small,

$$P(N_h = k) = \begin{cases} \lambda h + o(h) & \text{for } k = 1 \\ 1 - \lambda h + o(h) & \text{for } k = 0 \\ o(h) & \text{otherwise} \end{cases}$$

In this thesis, we have allowed the intensity to depend on time as a consequence of the business volume that is time dependent. This gives birth to a *non-homogeneous* Poisson process with rate $\lambda(t)$. The increments are still independent for any non-overlapping intervals and are Poisson distributed.

$$N_{t+s} - N_t \sim Poisson\left(\int_t^{t+s} \lambda(u)dy\right) \tag{C.1}$$

and

$$P\left(dN_{t+dt} = N_{t+dt} - N_t = k\right) = \begin{cases} \lambda(t)dt + o(dt) & \text{for } k = 1 \\ 1 - \lambda(t)dt + o(dt) & \text{for } k = 0 \\ o(dt) & \text{otherwise} \end{cases}$$

In contrast to an homogeneous Poisson process, a non-homogeneous Poisson process does not have a stationary increment. This can be seen from the increment distribution (C.1) for which the rate depends on s and not only on $s - t$.

Non-homogeneous Poisson processes can always be converted to an homogeneous Poisson process via a time transformation.

Let's

$$N_t \sim Poisson\left(\Lambda(t)\right)$$

The conversion from regular time t to modified time s is accomplished through

$$s = \Lambda(t) = \int_0^t \lambda(u)du$$

where $\lambda(t) = \frac{d\Lambda(t)}{dt}$ is the rate of the non-homogeneous process. Then the new process in the new time is $\tilde{N}_s = N_t$ and $\left\{\tilde{N}_s\right\}_{s\geq 0}$ is time homogeneous with rate 1.

C.1.1 Properties of a Poisson processes

Most of Poisson processes results will be reviewed in section (C.2) when dealing with the Poisson mark measure. We will just state some results about the first two moments of a Poisson processes.

Let's

$$N_t \sim Poisson\left(\int_0^t \lambda(u)du\right),$$

- ***Expectation***

$$\mathbb{E}\left[dN_t\right] = \lambda(t)dt$$

- ***Variance***

$$\mathbb{E}\left[dN_t^2\right] = \lambda(t)^2 dt \implies \mathbb{V}ar\left[dN_t\right] = \lambda(t)dt$$

C.2 Space-Time Poisson process

The space time Poisson process is a generalisation of the Poisson process with the difference that the amplitude of the jump when it occurs at time t is not deterministic as opposed to a Poisson process but random. We denote as $h(t, Y, \mathbf{X})$ the amplitude of the jump when it occurs where Y is a random variable on a support Γ and $\mathbf{X}_t$ is the state process at time t. The distribution function of Y is denoted by $F_Y(y)$. The space-time Poisson process is equivalent to a Poisson process when the jump amplitude is taken as 1 i.e. $h(t, y, \mathbf{x}) = 1$. In the thesis, we limit ourselves to jump amplitude that does not depend on the state variable and also on a state variable that depends on only one source of jump. Let $\Pi(t)$ be the space-time Poisson process with non state dependent amplitude $h(t, y)$, we can represent $d\Pi(t)$ as

$$d\Pi(t) = \int_\Gamma h(t, y)\aleph(dt, dy) \tag{C.2}$$

where the Poisson measure $\aleph(dt, dy)$ is merely a short hand notation for $\aleph([t, t+dt], [y, y+dy])$

C.3 Some properties of Marked Poisson process

In the different proofs of the stochastic control problem, we have used the below results of the marked Poisson process.

Independent increment

$\aleph$ has an independent increments on non-overlapping intervals in time t and marks y i.e. , $\aleph_{i,k} = \aleph([t_i, t_i + \Delta t_i), [y_k, y_k + \Delta y_k))$ is independent of $\mathcal{N}_{j,l} = \aleph([t_j, t_j + \Delta t_j), [y_l, y_l + \Delta y_l))$, provided that the time interval $[t_i, t_i + \Delta t_i)$ does not overlap with $[t_j, t_j + \Delta t_j)$ and the mark interval $[y_k, y_k + \Delta y_k)$ has no overlap with $[y_l, y_l + \Delta y_l)$. Below are some results used in the thesis.

The expectation of $d\Pi(t)$

$$\mathbb{E}\left[d\Pi(t)\right] = \lambda(t)dt \int_\Gamma h(t, y) f_Y(y)dy = \lambda(t)dt\mathbb{E}_{\mathbb{Y}}\left[h(t, Y)\right] \tag{C.3}$$

The Variance of $d\Pi(t)$

$$\mathbb{V}ar[d\Pi(t)] = \lambda(t)dt \int_\Gamma h(t, y)^2 f_Y(y)dy = \lambda(t)dt\mathbb{E}_{\mathbb{Y}}\left[h(t, Y)^2\right] \tag{C.4}$$

The expectation of the exponential of $\Pi(t)$

$$\begin{aligned}
\mathbb{E}\left[e^{\int_{t_0}^{t_1} d\Pi(t)}\right] &= \mathbb{E}\left[\exp\left\{\int_{t_0}^{t_1}\int_\Gamma h(t, y)\aleph(dt, dy)\right\}\right] \\
&= \exp\left\{\int_{t_0}^{t_1}\int_\Gamma \left(e^{h(t,y)} - 1\right) f_Y(y)dy\lambda(s)ds\right\} \\
&= \exp\left\{\int_{t_0}^{t_1} \mathbb{E}_{\mathbb{Y}}\left[e^{h(t,y)} - 1\right]\lambda(s)ds\right\}
\end{aligned} \tag{C.5}$$

see (Hanson and Westman 2003) for the proof of (C.3),(C.4) and (C.5).

C.4 The jump diffusion process

In this section, we show how Ito formula can be applied to jump and diffusion processes by separating the jump change (discontinuous part) to the continuous change.

C.4.1 Ito formula for The Jump and Diffusion Process

Let $\mathbf{X}_t$ be the state process with values in $\mathbb{R}^n$ evolving according to the stochastic differential equation

$$\begin{aligned} d\mathbf{X}_t &= b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{B}_t + d\Pi(t) \\ &= b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{B}_t + \int_\Gamma h(t, y)\aleph(dt, dy) \end{aligned} \tag{C.6}$$

where α_t as in the diffusion case is the value of the controlled parameter α at time t. The change in value of the state process $\mathbf{X}_t$ can be decomposed into:
The continuous changes reflected by the deterministic change $b(t, \alpha_t, \mathbf{X}_t)dt$ and the random change $\sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{B}_t$ so that

$$d\mathbf{X}_t^{cont} = b(t, \alpha_t, \mathbf{X}_t)dt + \sigma(t, \alpha_t, \mathbf{X}_t)d\mathbf{B}_t \tag{C.7}$$

and

The discontinuous change or jump change brought about the mark Poisson process i.e.

$$d\mathbf{X}_t^{jump} = \int_\Gamma h(t, y)\aleph(dt, dy), \tag{C.8}$$

thus

$$d\mathbf{X}_t = d\mathbf{X}_t^{cont} + d\mathbf{X}_t^{jump} \tag{C.9}$$

Let us assume that the jump amplitude $h(t, y)$ is independently distributed from the Brownian motion $\mathbf{W}_t$. Conditioning on the Poisson event occurring where the Poisson process N_t underlying the mark Poisson process is distributed as :

$$P(dN_t = k) = \left\{ \begin{array}{ll} 1 - \lambda dt & if \;\; k = 0 \\ \lambda dt & if \;\; k = 1 \\ 0 & otherwise \end{array} \right\} + o(dt) \tag{C.10}$$

the total change in the state will be explained by the change caused by the continuous change and the change caused by the discontinuous change. If we consider a composite function $G(t, \mathbf{X}_t)$ of the state which is assumed to be at least twice continuously differentiable in $\mathbf{x}$ and once in t, the total change in G can be decomposed into :

The continuous change

$$\begin{aligned} dG(t,\mathbf{X})^{cont} &= \frac{\partial G}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial G}{\partial \mathbf{X}_i} d\mathbf{X}_i^{cont} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 G}{\partial \mathbf{X}_i \partial \mathbf{X}_j} d\mathbf{X}_i^{cont} d\mathbf{X}_j^{cont} \\ &= \left[\frac{\partial G}{\partial t} + b(t, \alpha_t \mathbf{X}_t)\nabla_{\mathbf{X}} G + \frac{1}{2} tr\left(\sigma(t, \mathbf{X}_t, \alpha_t)\sigma^{'}(t, \mathbf{X}_t, \alpha_t) D^2 G\right)\right] \\ &\quad + (\nabla_{\mathbf{X}} V)^T \sigma(t, \mathbf{X}_t, \alpha_t) d\mathbf{B}_t \end{aligned} \tag{C.11}$$

Jump change

$$dG(t,\mathbf{X})^{jump} = \int_{\Gamma} \left[G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)\right] \aleph(dt, dy) \tag{C.12}$$

The Ito formula therefore allows for the continuous change and the jump change if it occurs therefore :

$$\begin{aligned} dG(t,\mathbf{X}) = &\left[\frac{\partial G}{\partial t} + b(t, \alpha_t \mathbf{X}_t)\nabla_{\mathbf{X}} G + \frac{1}{2} tr\left(\sigma(t, \mathbf{X}_t, \alpha_t)\sigma^{'}(t, \mathbf{X}_t, \alpha_t) D^2 G\right)\right] dt \\ &+ (\nabla_{\mathbf{X}} V)^T \sigma(t, \mathbf{X}_t, \alpha_t) d\mathbf{B}_t + \int_{\Gamma} \left[G(t, \mathbf{X}_t + h(t,y)) - G(t, \mathbf{X}_t)\right] \aleph(dt, dy). \end{aligned} \tag{C.13}$$

Bibliography

Aase, K. (1999). An equilibrum model of catastrophe insurance futures and spreads. *Geneva papers on Risk and Insurance Theory 24*, 69–96.

Albrecher, H. and R. Kainhofer (2000). Risk theory with a non-linear dividend barrier. *Department of Mathematics, Graz University of technology*.

Albrecher, H., K. Reinhold, and R. Tichy (2000). Efficient simulation techniques for a generalised ruin model. *Department of Mathematics, Graz University of technology*.

Amussen, S., B. Højgaard, and M. Taksar (2000). Optimal risk control and dividend distribution policies. example of excess-of-loss reinsurance for an insurance corporation. *Finance & Stochastics 4(3)*, 299–324.

Balzer, L. A. and S. Benjamin (1980). Dynamic response of insurance systems with delayed profit/loss sharing feedback to isolated unpredicted claims. *Journal of the Institute of Actuaries 109*, 285–316.

Bellman, R. E. (1957). *Dynamic Programming.*

Blanchet-Scalliet, C., N. E. Karoui, M. Jeanblanc, and L. Martellini (2003). Optimal investment and consumption decisions when time-horizon is uncertain. *Techical Report*.

Borch, K. (1967). The theory of risk. *Journal of the Royal statistical society 29*, 432–452.

Browne, S. (1995). Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research 20*(4), 937–958.

Browne, S. (1999). Reaching goals by a deadline: Digital options and continuous active portfolio management. *Advanced Applied Probability 31*, 551–577.

Brzézniak, Z. and T. Zastawniak (2000). *Basic Stochastic Processes.* Springer.

Christensen, C. V. (2000). A new model for pricing catastrophe insurance derivatives. *Mathematics subject Classification 62P05, University of Aarhus, Denmark.*

Crandall, M. G., H. Ishii, and P. L. Lions (1992). Users guide to viscosity solutions of second order partial differential equations. *Bulletin of Amer. Math. Soc. 27*, 1–67.

Crandall, M. G. and P. L. Lions (1983). Viscosity solutions of hamilton-jacobi equations. *Trans. Amer. Math. Soc.*, 1–42.

Cvitanic, J. and I. Karatzas (1992). Convex duality in constrained portfolio optimisation. *Annals of Applied probability.*

de Finetti, B. (1957). Su una impostazione alternativa della teoria colletiva del rischio. *Transations of the 15th International Congres of Actuaries.*

Delbaen, F., P. Grandits, T. Rheinlander, D. Samperi, M. Schweizer, and C. Stricker (2002). Exponential hedging and entropic penalties. *Mathematical Finance.*

Delbaen, F. and J. Haezendonck (1989). A martingale approach to premium calculation principles in arbitrage-free market. *Insurance: Mathematics and Economics.*

Detemple, J. B., R. Garcia, and M. Rindisbacher (2003). A monte-carlo method for optimal portfolios. *The Journal of Finance 58*(1), 401–446.

Fleming, W. H. and R. W. Rishel (1975). *Deterministic and Stochastic Optimal Control.* Springer.

Fleming, W. H. and H. M. Soner (1993). *Controlled Markov Processes and Viscosity Solutions.* Springer.

Fournie, E., J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi (1999). Applications of malliavin calculus to monte carlo methods in finance. *Finance abd Stochastics 3*(391-412).

Framstad, N. C., B. Øksendal, and A. Sulem (1998). Optimal consumption and portfolio in a jump diffusion market. *Workshop on Mathematical Finance, INRIA, Paris*, 9–20.

Fritelli, M. (2000). The minimal entropy martingale measure and the valuation problem in incomplete market. *Mathematical Finance.*

Gaier, J. and P. Grandits (2002). Ruin probabilities in the presence of regularly varying tails and optimal investment. *Insurance: Mathematics and Economics 30*, 211–217.

Gaier, J. and P. Grandits (2003). Ruin probabilities and investment under interest force in the presence of regularly varying tails. *Technical report. Vienna University of Technology..*

Geman, H. (1989). The importance of the forward neutral probability measure in a stochastic approach of interest rates. *Working paper, ESSEC, Cergy-Pontoise, France.*

Geman, H. (1994). Changes of numeraire and correlation risk management. *Risk conference on Correlation, London.*

Geman, H. and J. D. Cummins (1994). An asian option approach to valuation of insurance futures contracts. *Review of Futures Markets.*

Geman, H. and M. Yor (1994). Stochastic time changes in catastrophe option pricing. *Insurance: Mathematics and Economics.*

Gerber, H. (1979). An introduction to mathematical risk theory. *Huebner Foundation Monograph.*

Gerber, H. U. and J. D. Cummins (1996). Actuarial bridges to dynamic hedging and option pricing. *Insurance: Mathematics and Economics*.

Gerrard, R., S. Haberman, and E. Vigna (2004). Optimal investment choices post-retirement in a defined contribution pension scheme. *Insurance: Mathematics & Economics 35*, 321–342.

Gihman, I. and A. V. Skorohod (1979). *The Theory of Stochastic Processes*. Springer-Verlag.

Grandell, J. (1990). *Aspects of Risk Theory*. Springer Verlag, Berlin.

Grasselli, M. R. and T. R. Hurd (2004). A monte carlo method of exponential hedging of contingents claims. *Working paper*.

Grasselli, M. R. and T. R. Hurd (2005). Indifference price and hedging under stochastic volatility models. *Working paper*.

Hanson, F. B. and J. J. Westman (2003). *Applied Stochastic Processes and Control for Jump-Diffusions: Modelling, Analysis and Computation*. Society for Industrial and Applied Mathematics.

Harrison, J. M. and D. Kreps (1979a). Martingale and arbitrage in multiperiod securities markets. *Journal of Economic Theory*.

Harrison, J. M. and S. R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications*, 215–260.

Harrison, M. and D. Kreps (1979b). Martingales and arbitarge in multiperiod security markets. *Journal of Economics Theory*.

Harrison, M. and D. Kreps (1981). Martingales and stochastic integrals in the theory of continuous trading. *Northwestern University, Center for Mathematical Studies in Economics and Management Science* (454).

He, H. and N. Pearson (1993). Consumtion and portfolio policies with incomplete nmarkets and short-sale constraints: The infinite dimensional case. *Journal of Economics*

Theory 54, 259–304.

Hipp, C. and M. Plum (2000). Optimal investment for insurers. *Insurance Mathematics & Economics 27*, 215–228.

Hipp, C. and M. Plum (2003). Optimal investment for investors with state dependent income and for insurers. *Finance and Stochastics 7*.

Hipp, C. and M. Taksar (2000). Stochastic control for optimal new business. *Insurance Mathematics & Economics*.

Hipp, C. and M. Vogt (2003). Optimal dynamic xl reinsurance. *ASTIN Bulletin 33*, 193–208.

Hodges, S. D. and A. Neuberger (1999). Optimal replication of contingent claims under transaction costs. *Review of Futures markets*.

Højgaard, B. (2001, May). Optimal dynamic premium control in non -life insurance maximixing dividend pay-outs. Technical report, Aalborg University, Dep of Mathematical sciences.

Jamshidian, F. (1989). Bond and options valuation in the gaussian interest rate model. *Merrill Lynch Capital Markets, New York*.

Karatzas, I. and S. E. Shreve (1986). *Brownian Motion and stochastic Calculus*. Springer.

Korn, R. (1998). Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance and Stochastics 2*, 85–114.

Korn, R. (2003). The martingale optimality principle in finance: The best you can is good enough. Technical report, WILMOTT.

Korn, R. and S. Trautmann (1999). Optimal control of option portfolios. *Johannes Gutenberg-University Mainz, OR-Spektrum 21*, 123–146.

Kushner, H. and P. Dupuis (1993). *Numerical Methods for Stochastic Control prroblems in Continuous Time.* Springer.

Kushner, H. J. (1977). Probability methods for approximations in stochastic control and for elliptic equations. *Academic Press, New York.*

Kushner, H. J. (1984). Approximation and weak convergence methods for random-process with application to stochastic system theory. *MIT Press.*

Kushner, H. J. (1990). Numerical methods for stochastic control problems in continuous time. *SIAM J. Control and Optimization 28*(999-1048).

Lim, J. (2005). A numerical algorithm for indifference pricing in incomplete markets. *Separtment of Mathematics, University of Texas at Austin.*

Ma, J. and X. Sun (2000). Ruin probabilities for insurance models involving investments. *Department of pure mathematics, Purdue University, West la Fayette.*

Mania, M. and M. Schweizer (2005). Dynamic exponential untility indifference valuation. *The Annals of Applied Probability 3.*

Markowitz, H. (1952). Portfolio selection. *Journal of Finance 7*, 77–91.

Markowitz, H. (1959). Portfolio selection: Efficient diversification of investments. *Wiley, New York..*

Martin-Löf, A. (1983). Premium control in an insurance system, an approach using linear control theory. *Scandinavian Actuarial Journal.*

Martin-Löf, A. (1994). Lectures on the use of control theory in insurance. *Scandinavian Actuarial Journal.*

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *Review of Economics and Statistics 51*, 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous time. *Journal of Economic Theory 3*, 373–374.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica 41*, 867–888.

Merton, R. C. (1975). The theory of finance from the perspective of continuous time. *Journal of Financial and Quantitative Analysis 10*, 659–674.

Munk, C. (2003). The markov chain approximation approach for numerical solution of stochastic control problems: experiences from mertons problem. *Applied Mathematics and Computation 136*(1), 47–77.

Murman, A. (2001). Pricing catastrophe insurance derivatives. *The Wharton Business School, PA 19104-6218*.

Musiela, M. and T. Zariphopoulou (2004). An exemple of indifference prices under exponential preferences. *Finance and Stochastic*.

Nualart, D. (1995). *The Malliavin Calculus and Related Topics*. Springer-Verlag.

Øksendal, B. (1998). *Stochastic Differential Equations*. Springer.

Øksendal, B. and A. Sulem (2005). *Applied Stochastic Control of Jump Diffusions*. springer.

Pages, G., H. Pham, and J. Printems (2004). An optimal markovian quantization algorithm for multidimensional stochastic control problems. *Stochastics and Dynamics 4*(4), 501–545.

Panas, V. G., M. H. A. Davis, and T. Zariphopoulou (1993). European option pricing with transaction costs. *Journal of Control and Optimization*.

Paulsen, J. and H. Gjessing (2004). Optimal control of a risk exposure, reinsurance and investments for insurance portfolios. *Insurance: Mathematics & Economics 35*, 21–51.

Pham, H. (2002). Contrôle optmmal stochastique et applications en finance. Technical report, Université Paris 7, Laboratoire de Probabilités et Modéles alèatoires.

Pliska, S. R. (1986). A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operational Research 3*, 371–382.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica 32*, 122–136.

Rouge, R. and N. E. Karoui (2000). Pricing via utility maximization and entropy. *Mathematical Finance 10.*

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics 51*, 239–246.

Schmidli, H. (1991). On minimising the ruin probability by investment and reinsurance. *Laboratory of Actuarial Mathematics, University of Copenhagen.*

Schmidli, H. (2000). On cramer-lundberg approximations for ruin probabilities under optimal excess of loss reinsurance. *Laboratory of Actuarial Mathematics, University of Copenhagen.*

Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scandinavian Actuarial Journal 1*, 51–68.

Schmidli, H. (2002). On minimising the ruin probability by investment and reinsurance. *Annals of Applied probability 12*, 890–907.

Schmidli, H. (2003). Asymptotic of ruin probability for risl processes under the optimal reinsurance policies:the small claom case. *Laboratory of Actuarial Mathematics, University of Copenhagen Research paper, 60F10.*

Shimiko, D. (1992). The valuation of multiple claims insurance contracts. *Journal of Financial and Quantitative Analysis.*

Shouda, T. (2005). The indifference price of a defaultable bonds with unpredictable recovery and their risk premiums. *Graduate School of International Corporate Strategy, Hitotsubashi University.*

Sicar, R. and T. Zariphopoulou (2005). Bounds and asymptotic approximations for utility prices when volatility is random. *SIAM Journal of control and optimizations.*

Smith, A. (1996). How actuaries can use financial economics. *British Actuarial Journal 2*(1057).

Sundt, B. and T. Josef (1995). Ruin estimates under interest force. *Insurance Mathematics & Economics 16.*

Taksar, M. and S. Asmussen (1997). Controlled diffusion models for optimal dividend pay-out. *Insurance Mathematics & Economics 20.*

Taksar, M. and B. Højgaard (1999). Controlling risk exposure and dividents payout scheme: insuranmce company example. *Mathematical Finance 9*, 153–182.

Tobin, J. (1958). Liquidity preference as behavior towards risk. *The Review of Economic Studies 25*, 65–86.

Xu, G. L. and S. E. Shreve (1992). A duality method for optimal consumtion and investment under short-selling prohibitions: General market coefficients. *Annals of Applied probability 2*, 87–112.

Young, R. V. and T. Zariphopoulou (2001). Pricing insurance via stochastic control: Optimal consumption and terminal wealth. *Working paper, School of Business, University of Wincousin-Madison.*

Young, R. V. and T. Zariphopoulou (2002). Pricing dynamic insurance risks using the principle of equivalence utility. *Scandinavian Actuarial Journal.*

Young, V. R. (2004). Pricing in an incomplete market wiath an affine term structure. *Mathematical Finance.*

Yuanjiang, H., L. Xucheng, and J. Xhang (2003). Some results of ruin probability for the classical risk process. *Journal of applied mathematics and decisions sciences 7.*

Zariphopoulou, T. (2001a). Investment-consumption models with transaction fees and market parameters. *SIAM Journal on Control and Optimization.*

Zariphopoulou, T. (2001b). Stochastic control methods in asset pricing. *Handbook of Stochastic and applications.*