Stochastic Approach to Pension Funding, allowing for the pension accrual density function

by

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Thesis submitted for the degree of

DOCTOR OF PHILOSOPHY

CASS BUSINESS SCHOOL

FACULTY OF ACTUARIAL SCIENCE AND STATISTICS

MARCH 2003

DECLARATION

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Acknowledgements

Upon the completion of this work, I wish to thank the following people who have helped me considerably:

I would like to thank my supervisor Professor Steven Haberman for the help and encouragement he provided me continuously during these years. I also thank him greatly for his support, valuable advice and comments.

I would like to thank Professor Haralampos Papageorgiou (National and Kapodistrian University of Athens) who gave me the right direction in order to start this research. He was always willing to answer my questions.

I am also grateful to my good friend, Angela Fassoulis, who was so willing to make time in her busy schedule, to help me with my English.

Last but not least I wish to thank my family who also helped and encouraged me during this period. Especially, my daughter Angela for whom this work is dedicated to. Thank you for being so patient with me all these years.

ABSTRACT

The thesis introduces a different insight into the traditional methods of pension funds by implementing two ideas:

(a) To consider the age of the plan participant in the development of the pension scheme as an additional factor that affects the growth of the Fund and the determination of the Contribution rates. This is attempted by the pension purchase density function 'm(x)' which is viewed as a probability density function. ' New Cost methods ' are defined based on the statistical distributions: Power function, Truncated Pareto and Truncated Exponential.

(b) To consider the parameter 'λ', that determines how quickly the Unfunded Liabilities are covered, as a random variable; 'λ' takes values around a fixed value zone, which is considered as the expected value of the random variable.

We build a theoretical model, independent of the distributional assumptions, that has run on the fundamental bases, where either each of the rates of investment return and 'λ' or both are random variables. The first and second moments of the Fund and Contribution rates are calculated, as well as their ultimate values as time 't' tends to infinity.

A simulation analysis is performed assuming that either or both parameters (i(t),λ(t)) are random with a Log Normal distribution. In addition, for either case, we assumed that the pension plan is implemented based on a different pension accrual density function each time. On the basis of the simulated data, the 3^{rd} , 4^{th} moments and the percentile values of the Fund and Contribution levels are calculated.

Conclusions derived for both the Actuarial liability level under the 'New Cost Methods' and the growth of the Fund on each basis of the theoretical model. In particular we show that a) the development of Normal Cost follows the pattern of the accrual function 'm(x)' and b) the Actuarial Liability is higher under the density function that allocates larger proportions of the benefit at younger ages. We also specify an 'optimal region', m^*, for the number of years, m, over which the unfunded liability is spread. We show that for m greater than a particular value m^* the variances of both the fund and the contribution are increasing functions of m.The conclusions are confirmed by the simulation data.

The results raise questions such as the important issue of dependency between the rates of investment return and the spread parameter. These questions imply the extension of this work allowing for further steps in the future.

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CHAPTER 1

INTRODUCTION

Pension systems started in United Kingdom in the early years of the 18^{th} century through several schemes that were implemented to provide pensions for widows and orphans as well as for the poor old people in parishes.

Today, they are one of the major social achievements of our time. They have successfully reduced the risk of poverty in old age, and they are also an important feature of modern economies and financial markets.

For this reason, we thought that it would be important to provide an insight into the basic ideas that have formed pension theory throughout these years and describe the different models introduced over time for the implementation and management of the pension schemes. We also point out the main conclusions drawn from their application to pension plans.

Since this is the introductory part, we include the aims of the thesis, the methodology followed, and the outline of each chapter. At the end of chapter one, we give a list of symbols, the basic actuarial functions and the basic pension cost concepts to which we will refer throughout the thesis.

1.1 ACTUARIAL LITERATURE ON PENSION FUNDING

We consider the period from the start of the 18th century up to the start of the 20th century. The main contributions are as follows:

Maseres, 1772, an eminent lawyer in United Kingdom, puts to Parliament the first proposals for the provision of old age pensions, for enabling parishes to grant life annuities to poor persons. Around that time, Richard Price, the author of the Treatise on Reversionary payments, calculates extensive tables of contributions for the parliamentary bill, which are found as an appendix to Maseres's 'The Principles of the Doctrine of Annuities', 1783.

Farr, 1853, in the 'Twelfth Annual Report of the Registrar-General of Births, Deaths, and Marriages in England' published in the United Kingdom, shows how to use salary scales where the clerks could progress from one salary scale through three others. Salary scale calculations are unusual in actuarial literature, despite all efforts of Farr, until after:

Manly's paper, 1901, constitutes almost a text book on the subject of valuing staff pension funds giving exact and mathematical solutions of problems which at that time had been considered by some as being beyond the scope of accurate calculation; insoluble except by general methods of approximation. As the pioneer though of the main principles of pension fund valuation Hardy is mentioned, but Hardy did not publish his methods and only a small circle of colleagues knew of them.

Meanwhile in 1889, Germany becomes the first country in history that introduces old age pensions for its population.

M' Lauchlan, 1908, sets out in the form of questions and answers, a careful analysis of the characteristics of a model pension fund at various times in its existence. Subjects as 'fairness', 'what exactly is purchased by contributions in respect of new entrants at various ages', 'the effect of different career salary patterns' are discussed. The question of drawing up a pension scale graduated according to the age at entry is pointed out.

In the years followed, pension-funding theory was developing at a slow pace. As a main reason is that during these years Social Security was strong enough to provide pensions for people who leaving service, since the ratio between active workers and retirees was high. Getting closer to the 1950's, the actuaries realized that the high increase in birth rates would result to a high number of retirees. They also realised the improvement in life expectancy. During these years questions as 'prefunded social security plans are demographically immune', 'social security is best offered as defined benefit plan or a defined contribution plan' arise. More structured information and efforts to establish funding methods and formulae to estimate the Fund and Contributions level then become necessary.

Since 1950 the main contributions are as follows:

Trowbridge, 1952, clarifies the basic principles underlying the funding methods employed by pension actuaries, by utilizing a simple mathematical pension plan model operating in a stationary population.

To classify funding methods, Trowbridge uses the equation of maturity:

$C + d*F = B$, where C, F, B are all constants and d is the rate of discount.

Trowbridge classifies the funding methods in ascending order of F, (or descending order of C), dividing them into six classes.

Class I, that describes 'pay as you go' funding, according to which no contributions are made to the plan beyond those immediately necessary to meet benefit payments falling due.

Class II, that describes 'terminal funding' according to which no funding is contemplated for active lives. The present value of future pension benefits is contributed for each life as it reaches retirement.

Class III, that introduces the first method that funds in any respect for employees not yet retired, 'Unit Credit'. Unit Credit funding is based on the principle that the pension provided at retirement age will be divided into as many 'units' as the active membership years, with one unit assigned to each year.

Class IV, that includes four well known funding methods classed together because once the ultimate condition has been reached they produce identical contributions and build up identical reserves: Entry Age Normal, Initial Level Premium Funding, Aggregate Funding, and Attained Age Normal.

Classes V and VI, that include funding methods which produce higher eventual reserves and lower eventual contributions than any one of the methods discussed in the others. Class V includes "initial funding", where an employee's benefits are fully funded as soon as he is hired. Class VI is considered in order to point out the extreme case of heavy funding, called "complete funding". Under "complete funding", the present value of future benefit payments is fully paid off as the plan members reach retirement. Trowbridge uses discrete functions (see chapter 2) for their description.

Nesbitt also in 1952, presents a continuous time formulation (see chapter 2). Continuous time methods, offered certain advantages for the purposes of exploring the pension funding theory. In addition, after a few changes in assumptions and notation, continuous function formulae parallel to the discrete ones are obtained.

Cooper and Hickman, 1967, working with the continuous time formulation, introduce the pension purchase density function m(x) $a \leq x \leq r$. For a stationary population they define m(x), $a \leq x \leq r$, as the rate at which retirement income is purchased at age x for a member of the pension group, where a and r are the entry and retirement age respectively. Their discussion is centered on pension purchase density functions, (p.p.d.f.), m(x), that fulfill the following properties: m(x) $\succ 0$, $a \leq x \leq r$ and $\int_a^r$ m(x) dx = 1. (Note the similarity in form between the p.p.d.f. and the probability density function for an absolutely continuous random variable.) They also describe the cumulative pension purchase function denoted by M(x) as:

$$M(x) = \begin{cases} \int_a^x m(t)dt, & a \leq x \prec r \\ 1, & x \geq r \end{cases}$$

which is identical with the accrual function.

Bowers et al,1976, introduce the mathematical principles applicable to pension funding under dynamic conditions of population growth, inflation, and automatic adjustment of benefits. From their paper we point out the equation: NC(t)+δ*AL (t) - B(t) = $\frac{dAL}{dt}$ that shows that the rate of change of the plan accrued liability AL(t) is expressed by the rates of inflow of plan normal cost NC(t) and assumed interest less the rate of outflow of pension payments B(t). It also generalizes the equation of maturity stated by Trowbridge for a model plan in a stationary condition to a model plan subject to growth factors. This equation has similarities to Thiele's differential equation in life insurance mathematics.

In 1979, they introduce a new function x(θ), calculated from the equation

$e^{\theta * x(\theta)} = \int_a^r e^{\theta * x} * m(x)dx$, where x(θ) is named as the average age of normal cost payment associated with the actuarial cost method defined by m(x) and the combination of interest (δ), population and salary forces (τ), $\theta = \delta - \tau$.

Winklevoss ,1977, considers the gap between the mathematical and numerical aspects of the pension cost literature, (existing at that time), in his attempt to develop a computer model to investigate pension costs; combining in this way, computer science and mathematics. His book, published in 1977, serves the needs of anyone interested in the dynamics of pensions. It provides actuarial notation and develops methods of quantifying known pension costs, concepts and procedures.

O' Brien, 1985, sees pension funding as a problem in stochastic control. Hence, choosing the form of the contribution function, he introduces a stochastic dynamic pension fund model via a stochastic differential equation in the variable, 'X_t' representing the fund ratio. That process X_t is analysed by Lyapunov type methods and its first and second moments are computed. O' Brien, suggests also a linear function of the present value of future benefits and of the fund, as pension contribution function, in place of the one-parameter family of funding methods. The one-parameter family, is defined by the contribution formula: $C_t = (k+d)*(V_t - F_{t-1})$ given in Trowbridge (1963), where F_{t-1} is the fund value at time t-1, V_t the present value of benefits at time t, d the discount rate and k a positive number less than 1.

In 1986, O'Brien provides some theoretical justification for such a method by showing, in a simplified model, that the optimal solution of a stochastic control problem yields, as contribution function, an affine function of the present value of future benefits and of the fund.

The transition from the deterministic model discussed by many of the previous authors to the stochastic one is made by Dufresne, 1986, under the assumption that the rates of return are independent, identically distributed random variables. The stochastic development of the rates of return allowed greater flexibility, while the calculation of the limiting values of the second moments of fund and contribution levels sets the limits for the underlying parameters. The latter is of great importance.

Working in a stationary population, Dufresne calculates the first and second moments of F(t) and C(t) and their ultimate values, showing that under certain conditions:

$\lim_{t\to\infty} EF(t) = AL$, $\lim_{t\to\infty} EC(t) = NC$ and $\lim_{t\to\infty} VarF(t) = \sigma^2(i(t))\dfrac{(EF(\infty))^2}{(1-k)*u^2}$,

$\lim_{t\to\infty} VarC(t) = \dfrac{VarF(\infty)}{(\ddot{a}_{\overline{m|}})^2}$, k<1 and $2 \leq m \leq \infty$. He also calculates the covariance

between F(t) and F(t+h), between C(t) and C(t+h) and between F(t) and C(t+h), $h \geq 0$

proving that: $Cov(F(t),F(t+h)) = q^h * VarC(t)$, $Cov(F(t), C(t+h)) = -q^h * \dfrac{VarF(t)}{\ddot{a}_{\overline{m|}}}$ and

given that k<1, as $t \to \infty$ $Cov(F(t),F(t+h)) \to q^h$, $Cov(C(t),C(t+h)) \to q^h$, $Cov(F(t),C(t+h)) \to -q^h$

Dufresne deals with both the discrete and the continuous cases. For the latter, he uses the Stochastic Differential Equations defining the instantaneous rates of return as $\gamma(t) = \gamma + \sigma(i(t))^* \frac{dW(t)}{dt}$ where W(t) is the Wiener process and $\frac{dW(t)}{dt}$ is what is known as 'white noise'. He also shows that by working in a stationary population and using the spread method the conclusions reached are the same as the ones under the discrete case.

Dufresne specifies an 'optimal region', m^*, for the number of years, m, over which the unfunded liability is spread. He shows that for m greater than a particular value m^*, the variances of both the fund and the contribution are increasing functions of m. Thus the 'optimal' values of m are $1 \leq m \leq m^*$.

Dufresne, 1988, shows that there is a trade off between Var F(t) and Var C(t) in the limits as $t \to \infty$ (and for finite t under certain conditions), and in some sense there is an optimal region for the choice of the spread period, m. Specifically, if $E(1+i(t))^2 > 1$ then both VarF(∞) and VarC(∞) become infinite for some finite m and there exists a value m^* such that a) for $m \leq m^*$ VarF(∞) increases and VarC(∞) decreases with increasing m and b) for $m \geq m^*$ both VarF(∞) and VarC(∞) increase with increasing m.

Dufresne, 1989, derives the first two moments of the contribution and fund levels at time t considering the actuarial loss experienced during the period (t-1,t), L_t, where L_t is defined as the difference between the unfunded liability at time t and the corresponding one if all actuarial assumptions had been realized. In particular Dufresne obtains the moments of F and C and their limits, as $t \to \infty$, from those of the L's. He shows that there is a trade off between Var F(t) and Var C(t) in the limits as $t \to \infty$ and that there is not an optimal region for the choice of the spread period, m.

In the presence of a simple stochastic model for real investment returns and for a defined benefit pension scheme Haberman, 1992, extends the Dufresne model and studies the fund and contribution rate level considering that the contribution rate is fixed, relative to the fund level but with a time delay. He sets up formulae for studying mathematically the progression with time of the expected values, variances and covariances of contributions and fund levels, exploring as well their behavior both for finite time and in the limiting case as t $\to \infty$.

Haberman, 1993, investigates the variability of pension contributions and fund levels, for the case of different time intervals between valuations. In particular, under annual and triennial valuations, choosing a mathematical model to represent the behavior of a defined benefit pension scheme, he looks at the spread period for amortizing valuation surpluses and deficiencies. He also presents how the produced results may be generalized to apply to valuations every n years where n is an integer. Haberman thus extends the results produced by Dufresne (1988) for the annual valuation, (in the annual valuation the optimal spread period is denoted by M^*), by demonstrating an optimal spread period, M_1^*, and implementing formulae for the limiting variance of the fund level, F(t), and the contribution rate, C(t), as t $\to \infty$ in the triennial valuations.

Zimbidis and Haberman, 1993, study the variability of contribution rates and fund levels, when there are time delays and feedback into a pension fund. The time delay of q years (q = 0,1,2,3,…) is introduced at each time t, when the contribution rate is fixed by considering the information or the actual data for the pension fund at time t-q. Zimbidis and Haberman, extend the results produced by Dufresne (1988) when q=0 and Haberman (1992) when q=1, to the case of q = 2 and for large values of q, i.e. q$\geq$3.

When the delays in information are long they show that the pension scheme may become unstable.

Owadally and Haberman, 1999, compare three methods of adjusting the Normal Cost as gains or losses arise: a) the spread method, b) the method of amortizing unfunded liability and c) the method of amortizing unfunded liability and spreading subsequent surpluses/deficits accordingly. When gains / losses are amortized over finite terms, they establish that there exists a range of amortization periods, $[1,\ m_a^*]$, wider than $[1,\ m_s^*]$ that is optimal and for which there is a trade off between ultimate variabilities of the fund and contribution levels; where the subscripts s and a refer to methods a and b respectively. Based on the criterion of minimizing variabilities of both the fund and contribution levels ultimately, they conclude that spreading surpluses / deficits may be regarded as more efficient than amortizing gains/losses. For equal spread and amortization periods, amortization of gains / losses methods, achieves greater fund security than does spreading surpluses/deficits.

A further body of recent literature has considered the case where the real rates of return follow either an autoregressive or a moving average process.

Haberman, 1993, obtains recursive formulae for the expectations and the variability of fund and contribution levels for finite t.

Haberman, 1994, explores the detailed properties of the first two moments of the fund and contribution rates level at time t. These properties lead to consideration of the "optimal" range of values of the spread period, M. He shows that for certain values of φ the implied optimal values of M are consistent with those shown in Dufresne (1988) which would correspond approximately to the case $\varphi = 0$. Haberman, extending his analysis, derives as well the first two moments for the case where the earned real rates of return follow a second order autoregressive model.

Gerrard and Haberman, 1996, produce recursive formulae for the expected actuarial loss in a given year and for the expectation of the square of this quantity. They prove that for suitable values of the parameters, these expectations converge in time to limits which can be found by a simple numerical procedure in any given case.

Mandl and Mazurova, 1996, investigate the case where the numbers of entrants randomly fluctuate, using spectral decomposition of stationary random sequences. They derive formulae for the variances of the fund level and the discounted cash flows in the case of direct defrayal of pension cost, and obtain similar results to those of Haberman (1994).

Cairns and Parker, 1997, examine how the fund levels and contribution rates are affected through the valuation basis and the amortization period changes. They introduce an efficient frontier as a means of choosing an optimal funding strategy. Cairns and Parker, extend Dufresne results (1988,1989) to cover the more general cases where the valuation and mean long-term rates of interest are not equal.

Haberman and Wong, 1997, derive expressions for the moments of the contribution and fund level, assuming (for both moments) the moving average processes of order 1 and order 2, (MA(1), MA(2)).They show that there are maximum values of M, M_1, and M_2, (where M_2 is much smaller than M_1), respectively for which convergence holds. Comparing their results with those of Haberman, 1994, they show that the structure of the equations obtained under MA(1) is close to that obtained for the autoregressive AR(1) process. However, the results are exact rather than approximate as for the AR(1) case. The limiting value of EF(t) though, is much simpler for the AR(2) model than for the MA(2), due to the simpler covariance function obtained.

Bedard, 1999, based on the bilinear Markovian representation, finds explicit expressions for the moments of the fund level and the value of the total contribution. Bedard produces results similar to those of Haberman and Wong.

Owadally and Haberman, 2003, compare two methods for the amortization of asset gains (losses): a) direct amortization over a fixed term, and b) indirect and proportional form of amortization which 'spreads' forward pension fund surpluses and deficits over a moving term. They show that the first method leads to more variable fund and contribution levels. Through simulations they prove that: when rates of return are dependent, following simple autoregressive AR(1) and moving average MA(1) processes, spreading surpluses and deficits remains more efficient at achieving secure funding levels and stable contribution rates. Owadally and Haberman, proceeding to an approximate analysis for more general and weakly autocorrelated AR(p) logarithmic rates of return, reach the conclusions of Dufresne (1988) that 'under modern economic conditions, m $\in$ $[1, m_s^*]$ is an efficient range over which to spread surpluses and deficits', where the subscript s stands for method a.

A further theme in the literature follows in from O' Brien (1985) and introduces control theory, in the presence of a simple stochastic model for real investment returns and for a defined benefit pension scheme.

Haberman and Sung, 1994 determine the optimal contribution strategy. They discuss two types of risk, concerned respectively with the stability and security of funding: the 'contribution rate' risk and the 'solvency risk'. They introduce an objective function, to allow their simultaneous minimization. Haberman and Sung, determine the optimal contribution strategy in the presence of a deterministic model.

Haberman, 1997, considers the 'contribution rate risk' and the criterion of minimizing a particular measure of this risk, based on the variance of the present value of future contributions. Haberman shows, that there is an optimal value for the choice of spread period used for the elimination of valuation deficiencies and surpluses.

Chang, 2000, following Haberman and Sung, 1994, sees also pension plan funding as a dynamic control process. He introduces two performance measures to evaluate the effectiveness of plan contributions: The cost induced performance measure (CIPM) and the ratio-induced performance measure (RIPM). With the objective of minimizing the performance measure, he determines the optimal contributions. Comparing these measures, Chang concludes that RIPM produces more stable results than CIPM.

Cairns, 2000, proposes a continuous time stochastic pension fund model, in which there are n risky assets plus the risk-free asset as well as randomness in the level of benefit outgo. He considers Marcov control strategies, that optimize over the contribution rate and over the range of possible asset – allocation strategies. Cairns, introduces a general quadratic form which provides an explicit solution for the optimal contribution and asset allocation strategies. These solutions, independent of the level of uncertainty in the level of benefit outgo, suggested that small schemes should operate in the same way as large ones. Cairns, also discusses the effects of constraints on contribution and asset allocation strategies, proceeding as well to a comparison between the optimal and dynamic control strategies.

Haberman et al, 2000, deal with the simultaneous minimization of the 'contribution rate risk' and the 'solvency risk'. They introduce a performance criterion using the fraction of the unfunded liability paid off (k) or the spread period (M) as the control variable. Their results lead to practical conclusions about the optimal funding strategy and, hence, about the optimal choice of the contribution rate, subject to the constraints needed for the convergence of the performance criterion.

Finally, we quote a new approach to the cost allocation for the unfunded liability.

Chang and Chen, 2002, study the cost allocation for the unfunded liability, and generalize the constant value assumption in cost amortization, by modeling the returns and valuation rates simultaneously. By approximating the conditional and unconditional moments of the plan contribution and fund size level through Taylor series expansion, they estimate stability under different allocation periods.

Summarizing the above we reach the following conclusions:

Pension funding literature may be divided in three periods:

The first period starts from the 18th century and lasts up to 1950. During these years the necessity of old age pensions is formulated and, as an example, Maseres, 1772, puts to the UK Parliament the first proposal for the provision of old age pensions. The structure for the actuarial calculations is set down by Manly in 1901 who gives exact mathematical solutions to certain actuarial problems. In 1908, M' Lauchlan discusses the analysis of the characteristics of a model pension plan and introduces the new concepts of 'fairness', 'new entrants', 'career salary patterns'. Meanwhile, in Germany in 1889, the first national old age pensions system is introduced.

The second period starts from 1950 and lasts up to the early 1980's. During these years the pension funding theory is developed. The basic principles underlying the funding methods employed by pension actuaries are clarified. Actuarial notation is provided and methods of quantifying known pension costs, concepts and procedures are determined. Trowbridge classifies funding methods; Cooper and Hickman introduce the pension purchase density function; and Bowers et al, introduce the mathematical principles applicable to pension funding under dynamic conditions of population growth, inflation and automatic adjustment of benefits.

The third period starts from mid 1980's and lasts up to the present time. During these years research in pension funding area is much developed. The effect of the rates of investment return being random on the Fund and Contribution rates level is investigated: Dufresne shows that there is an optimal spread period $(1,m^*)$, below which there is a trade off between VarF(t) and VarC(t). Haberman extends these results for the cases of both different time intervals between valuations and where the rates of investment return follow either an autoregressive or a moving average process. The control theory is introduced and Haberman and Sung determine optimal contribution strategy in the presence of a deterministic model. The effect of the different investment strategies on the Fund level is examined and Owadally and Haberman reach the conclusion that "under modern economic conditions there is an efficient range over which to spread surpluses and deficits". The treatment of the Unfunded Liability is further discussed and Owadally and Haberman compare three methods of adjusting the Normal Cost as gains or losses arise. In 2002 a new approach to the cost allocation for the Unfunded Liability is proposed by Chang and Chen according to which the constant value assumption in cost amortization is generalized by modeling the returns and valuation rates simultaneously.

It is important to mention that under these different approaches to pension funding, the results concerning the evolution over time of the Fund and Contribution rates are similar.

1.2 AIM and OUTLINE of the THESIS

Our aim, throughout this research, is to extend pension-funding theory by adding some new ideas on how a pension plan could be managed. We have both ideas a) to view the actuarial cost methods as "processes" where time is considered and b) to view the fund and contribution rates as "processes" with respect to time by modeling the parameter that amortizes the Unfunded Liability. Our work has been implemented so that the thesis is mainly connected with the body of the literature where investment returns form an i.i.d sequence of random variables. In addition, we have adopted the new approach of allowing the parameter in the cost amortization of the unfunded liability to have a non-constant value.

1.2.1 OBJECTIVES

The aim of this study is to provide an insight into the traditional methods of pension funding through an attempt:

a) to analyze the model pension plan development where the pension purchase density function m(x) corresponds to a probability density function f(x). Under this assumption, the age of the member is an additional factor that affects the building up of the Fund and the determination of the Contributions. In practice, age is a factor that increases Service Cost, in the sense that as the scheme participant becomes older he/she is getting closer to retirement. Actuaries apply such Cost Methods mainly in defined benefit schemes. We note that in defined contribution plans, the scheme sponsor may offer to the plan member the choice to purchase his/her benefit by contributing higher or lower amounts according to his/her age. The scheme sponsor has also the flexibility, according to his /her financial plans, to contribute in this way as well. In defined benefit schemes, the actuary may choose either to build up the Fund at a higher pace by purchasing higher amounts of benefits at the younger ages, or at a lower pace by purchasing higher amounts at the older ones. The actuary also avoids the unfunded liability risk, caused in the case where the average age of the new entrants in the scheme is different from the average age, which he has assumed for the valuation. In addition, the normal retirement age may need to be changed at a later stage in the plan[1]; and through applying such cost methods the actuary may estimate the extra or the lower cost for the benefit purchased annually and adjust the Normal Cost accordingly.

b) to specify the Fund and Contribution levels using a parameter 'λ' which determines how quickly Unfunded Liabilities are covered. The amortization of the Unfunded Liability is a major concern for the actuary, and λ is viewed as a control parameter with the objective of controlling the dynamic behaviour of C(t) and F(t) over time. Since it is often the case that actuarial assumptions are not in agreement with experience, setting λ as a constant throughout a long period of years is restrictive. For that reason, we consider λ as a random variable, acquiring values around a fixed value zone, which is considered as the expected value of the random variable. Given that λ varies in that zone, we then analyze the development of the Fund against the Actuarial Liability. In practice, actuaries are faced more frequently by schemes with unfounded liabilities for a number of reasons, including the following: the scheme sponsor has paid lower amount of contributions than that required; the scheme has not been valuated on a regular basis; high amounts of benefits may have been paid to the senior management upon their retirement; the acquisition by the sponsor of a company

[1] Recently, due to the law of equality between both sexes many companies had to revise the normal retirement age in their policy documents.

with an associated unfunded liability in its pension scheme or for which benefit levels are to be maintained; and benefit improvements. Also, in practice, the pace with which the actuary amortizes gains or losses heavily depends on the scheme sponsor's forecast of financial results for his/her business over the next years. Therefore, we consider that it is important to give the actuary the flexibility to propose a pattern according to which the amortized amounts may occasionally either increase or decrease. Apart from the above cases when the actuary designs the plan, he/she may consider it desirable to use a variable parameter in order to prevent high amounts of unfunded liability. Also, in practice, when designing a pension plan, the method of amortization of the assumed unfunded liability may be taken into account by the actuary if the assumptions are not sufficiently conservative.

The model for b) is run on three fundamental bases:

i) where the rates of investment return 'i(t)' are random variables and λ is constant

ii) where λ is a random variable and the rates of investment return constant

iii) where both λ and rates of investment return are random variables

The study then focuses on the following:

1. the comparison between the 'new' pension functions and the 'traditional' ones
2. the derivation of the first and second moments of the Fund and Contribution levels in the stochastic model through a discrete time formulation.
3. the calculation of skewness and kurtosis of the Fund and Contribution levels in the stochastic model on the basis of simulated data.
4. the calculation of the percentiles values of the Fund and Contribution levels in the stochastic model on the basis of simulated data.

Emphasis is given to the development of the Fund and Contribution levels over time t rather than at specific points in time.

1.2.2 METHODOLOGY

We consider the financial structure of a defined benefit pension scheme, as represented by a simple mathematical model, using a stationary population. We also consider individual funding methods, which both determine an actuarial liability and a normal cost at every valuation date and amortize inter-valuation gains or losses over a fixed number of years. In the light of the above, we focus on the effect of applying the questions a) and b) discussed in 1.2.1. on the contribution rates and fund level for the scheme.

For the model pension plan, entry is fixed at age a and retirement at age r. Only retirement benefits are considered, i.e. no account is taken of death, disability and withdrawal benefits. Furthermore, it is assumed that the annual rate of salary for a participant age x at time t is s_x, $a \leq x \leq r$, and that s_x increases exponentially at a rate τ[2].

For each different probability density function considered, we both run the model on a deterministic basis and on a stochastic basis via simulation. We proceed through computer simulations, as an effort to provide results that could lead to practical conclusions.

These results are presented in detail and discussed in chapter 6.

When the actuary considers the effect of the age of the scheme participant upon the implementation of the plan, he/she estimates with a higher accuracy the service cost of

[2] Inflation is included

the scheme participants; thus providing the scheme sponsor with more accurate information on the amount he/she has to include in the pension cost for the next period. Also, when the actuary has the flexibility to vary the parameter that amortizes unfunded liability he/she may either strengthen or not the Fund value at certain time intervals according to the realized gains or losses.
The results show the effect on the fund and contribution rate of introducing further variability through frequent changes to the amortization parameter.
It is thus hoped that pension actuaries may find some practical benefit from the outcome of this research, which aims to offer a new perspective on the pension funding development.

1.2.3 OUTLINE

Each chapter starts with an introduction where the content of the chapter is described. An analysis is then quoted, and, when necessary, numerical illustrations placed at the end. From all mathematical proofs, those of the main results form part of the analysis, while the others are set out in the Appendices. A review of literature is also included to the extent that it is relevant to the chapter. Focusing on each part separately, we may point out the following:

Chapter 2 includes the funding methods traditionally used in Europe, Canada and the U.S.A.. They are presented under both a discrete and a continuous time formulation, for the individual and the aggregate methods separately. The major focal point is the comparison between them. Chapter 2 provides a necessary background for chapter 3, since the already known funding methods are compared with the ‘new’ ones that have been developed.

Chapter 3 introduces the new idea of how we may build the fund by considering the age of the scheme member in the model development. The main hypothesis now focuses on the different pension accrual density functions m(x), such that m(x) $\succ 0$, $a \leq x \leq r$ and $\int_a^r$ m(x) dx = 1. The major issues considered are:

a) The definition of each one of m(x) and its corresponding accrual function M(x).

b) The categorization of m(x)s according to their relationship with either an accelerating or a decelerating cost method.

c) The comparison between the various density and accrual functions according to their development.

Chapter 4 introduces the pension funding functions Normal Cost, Actuarial Liability and Fund Value, which are calculated on the basis of the density functions. These calculations are made using a continuous time formulation on a deterministic basis. Considering the fact that when facing reality the economic assumptions rarely coincide with the actuarial ones, in the model development Normal Cost is adjusted taking into account the difference between the Actuarial Liability and the value of the Fund, (unfunded liability). This adjustment is made by using the 'Spread Method', according to which we divide the Unfunded Liability by the present value of an annuity certain. The major issues considered are:

a) The calculation of the Actuarial Liability and Normal Cost under each m(x).

b) The development of F(t) based on the formula:

$\frac{dF(t)}{dt}$ = (δ-λ)*F(t) + NC(t) – B(t) + λ*AL(t), where the parameter 'λ' is initially considered constant, and is used to control the size of F in relation to AL.

c) The comparison of the Actuarial Liability between two different choices of m(x) after examining the difference of their corresponding accrual functions M(x)s.
d) The calculation and the comparison of the Normal Costs based on the m(x)s used in (c) in order to examine the Actuarial Liability development against Normal Cost.
e) The Normal Cost development if the parameter 'λ' is calculated on the basis of the pension density function.
f) The comparison between the 'new' and the traditional cost methods.

Chapter 5 deals with the Stochastic Model. The main hypothesis is that the rates of investment return and the parameter 'λ' follow a Log Normal distribution. The theoretical model is built independently of the distribution assumptions. Specifically, the following cases are studied:

a) The development of the Fund and the Contribution level where the rates of investment return are independent, identically distributed random variables, and the parameter 'λ' is constant. The results are an extension of those of Dufresne (1988) and Owadally and Haberman (1999).
b) The development of the Fund and the Contribution level where the parameter 'λ' is random variable and the rates of investment return 'i(t)' are constant.
c) The development of the Fund and of the Contribution level where both the rates of investment return are independent, identically distributed random variables, and the parameter 'λ' is a random variable, given also that i(t) and λ(t) are mutually independent.

The first and second moments of the Fund and Normal Cost have been calculated for all the above cases on the basis of a discrete time formulation.

Chapter 6 extends the ideas of Chapter 5 and focuses in more detail on the simulation results. As mentioned above, the thesis relies heavily on simulations, since this is a convenient way of obtaining results and the application of the results is considered to be of importance.

The model is used to perform a simulation for each different probability density function considered and for each one of the above cases (a)-(c) of chapter 5. The simulation results concern the growth of the Fund and the determination of the Contributions level. The major points studied are:

a) the goodness of fit between the model and the simulations, comparing the simulation results and the results obtained with the theoretical model.
b) the sensitivity of results to changes in parameters, calculating the percentile values (the 1%, 5%, 25%, 50%, 75%, 95% and 99% percentiles) over the years, on the basis of the simulated data.
c) the 3rd and 4th moments (skewness, kurtosis) of the distribution of the Fund and Contribution rates.

1.3 NOTATION

1.3.1 LIST OF SYMBOLS

a : Entry Age

r : Retirement age

ω: The age that terminates the mortality table

i(t): rate of investment return

i: E(i(t)), Valuation rate of investment return

d: $\frac{i}{1+i}$

δ: Valuation force of Investment return

γ: Price inflation rate

p_r: Promotion rate

$\tau' = \gamma + p_r$: Annual rate of salary increase

τ: Valuation force of salary rate of increase; $1+\tau' = e^{\tau}$

β': Pension adjustment rate

β: Valuation force of pension adjustment rate ; $1+\beta' = e^{\beta}$

l_x: Number of survivors at age x obtainable from a service table[3]

m: Length of the spread period (in years)

$\lambda = \frac{1}{\ddot{a}_{\overline{m}|}}$: Spread parameter

$\lambda(t)$: The process of amortizing U(t)

b: The proportion of the Final or Career Average Salary

m(x): The pension accrual density function

s_x : The annual salary rate for a participant aged x at time t = 0, $a \le x \le r$.

W : The initial entry salary at age a, i.e. s_a = W

NC_x : The Normal Cost of a plan member at age x

AL_x : The Actuarial Liability of a plan member at age x

NC(t): The annual rate of plan Normal Cost at time t.

AL(t): The accrued Liability of the plan at time t.

B(t): The annual rate of pension outgo at time t.

U(t): The Unfunded Liability of the plan at time t.

C(t): The Contribution rate at time t.

F(t): The Fund level at time t.

1.3.2 BASIC ACTUARIAL FUNCTIONS

- ${}_np_x$ = probability that a person aged x will survive for n years = $\frac{l_{x+n}}{l_x}$
- ${}_nq_x$ = probability that a person aged x will die within n years = 1 - $\frac{l_{x+n}}{l_x}$

For simplicity purposes, we have considered a single decrement environment[4] and a stationary population.

In this study, we are looking for the major drivers of pension fund dynamics; rates of investment return and the spread parameter. Assuming both a single decrement environment and a stationary population, we develop a simple model since we maintain the scheme population stable during the years. We have decided to work with a simple model because such a model allows a detailed study of these factors.

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[3] Service table: A service table shows the number of employees out of an original group who survive to its future attained age.

[4] Single decrement environment: The most important decrement prior to retirement age is withdrawal. However, for simplicity purposes, for the stationary population we assumed that the pension plan participants are exposed to death only. Other contingencies such as withdrawals, ill, health and early retirements are not considered.

In practice, most pension plans also have additional benefits such as death, disability and withdrawal benefits, in accordance with the plan document. Our focus on a pension plan providing only age retirement benefits is supported by the observation that the bulk of the liability in defined benefit pension plans comes from these benefits rather than the ancillary benefits (Lee, 1986). If these restrictions were relaxed, we could develop a more complex model. Death benefits are usually different both before and after retirement (greater if there is a spouse / children). The disability monthly benefit usually ceases at the earliest of death, recovery or retirement. The withdrawal benefit depends on the vesting; sometimes the plan pays a deferred monthly pension and sometimes a lump sum at termination. We understand that as models become more complex we input more and more factors and find that more detail comes in the input from each simulation. It then becomes very difficult to identify why certain effects are evident. However, simple models provide the backup for the analysis of more complex models and give pointers to what we should be investigating.

The cases of a new or ageing scheme are left to future research.

- pension adjustment function, $\beta(x)$

It is used to denote the adjustment of the initial pension at age of retirement r, of a retiree age x, $x > r$, $\beta(x) = e^{\beta^*(x-r)}$

- Pension incurrence density function, h(t).

The symbol h(t) represents the density at time t of the amount of newly incurred age r pensions. Thus $h(10) = 1{,}000$ implies that in the moment $(10, 10 + dt)$ the amount of new age r pensions which come into effect under the model plan is approximately 1,000dt.

Our population is stationary and thus the density of new retirees at time t + r-x from participants aged x at time t is l_r. Each of these will, at time t + r-x, have annual salary rate $e^{\tau*(t+r-1-x)}* s_a$.

Then according to the definition above, the density of new pensions incurred, i.e. entering benefit status at time t + r-x, for those who at time t are aged x (x<r), or who at time t +r –x were aged r ($x \geq r$), can be expressed as: $h(t + r\text{-}x) = e^{\tau*(t+r-1-x)}* s_a * l_r$.

For x = r, h(t + r-x) becomes $h(t) = e^{\tau*(t-1)}* s_a * l_r$ and represents the density of new pensions incurred at time t.

- salary function, g(t)

It is used to estimate the future salaries of the plan members in case their benefits and / or their contributions are tied to them.

Throughout the chapter, growth in salaries over time will be represented by means of a function g(t), defined as : $g(t) = e^{\tau t}$, $0 \leq t < r - a$

At entry age a, t = 0. The salary at entry age, s_a, remains as a base factor for $t \geq 0$.

The cumulative salary from the entry age a, up to, but not including, age x is denoted by S_x. Thus, for $x > a$ we have: $S_x = \sum_{t=a}^{x-1} s_t$

The estimated employee's salary at age x and the cumulative salary, both based on his salary at age a, are given respectively by the following formulae:

a) Discrete time formulation

$$s_x = s(a) * e^{\tau(x-a)} = W * e^{\tau(x-a)},\ S_x = \sum_{t=a}^{x-1} * s_t = W * s^{\tau}_{\overline{x-a|}} ,$$

b) Continuous time formulation

$$s_x = s(a) * e^{\tau(x-a)} = W e^{\tau(x-a)},\ S_x = \int_a^x W * e^{\tau(y-a)}\, dy = W * \bar{s}^{\tau}_{\overline{x-a|}}$$

- Accrual function, M(x)

The cumulative pension purchase function is denoted by M(x), and defined by:

$$M(x) = \begin{cases} 0 \quad , & x \prec a \\ \int_a^x m(t)dt, & a \leq x \prec r \\ 1 \quad , & x \geq r \end{cases}$$

M(x) represents that fraction of the actuarial value of future pensions accrued as an actuarial liability at age x under the actuarial cost method.

- Benefit function, B_x

This is used to determine the amount of benefits to be paid at retirement.

The most common type of benefit formula used in pension plans, is the so called unit benefit formula that provides a unit of benefit for each year of credited service. There are three basic formulae associated with defined benefit plans: flat benefit, career average, and either final average or final salary. We do not consider the first type, since it is rarely used.

  - The Career Average Salary formula

The second type of benefit formula, provides a pension benefit that is defined in terms of some stipulated percentage, of the scheme participant's career average salary; i.e. b percent of each year's current salary.

Denoting by b_x the annual benefit accrual during the year of age x to x+1 for an age a entrant, then $b_x = b * s_x$. The accrued benefit function, i.e. the sum of each attained age accrual up to but not including age x, B_x is equal to: $B_x = \sum_{t=a}^{x-1} b_t$.

- The Final Salary formula

The third type of benefit formula provides a given percentage of the scheme participant's salary nearest retirement, s_{r-1}.

Denoting by b_x the annual benefit accrual during the year of age x to x+1 for an age a entrant, then $b_x = b * s_{r-1}$. The accrued benefit function, i.e. the sum of each attained age accrual up to but not including age x, B_x is equal to: $B_x = \sum_{t=a}^{x-1} b_t = b^* (x - a) * s_{r-1}$.

- Salary based Annuity

This is a life annuity reflecting the fact that the contributions during a member's working lifetime tend to depend on the member's salary.

We define ${}^{s}\ddot{a}_{x:\overline{r-x}|} = \sum_{t=0}^{r-x-1} \frac{s_{x+t}}{s_x} e^{-\delta * t} * {}_tp_x$

- Stationary population

This is a population with a constant size age distribution. In particular, in this population, the density of deaths at a certain age x and greater, equals the density of the number of lives attaining age x, at any time t. For a pension plan we may simply consider that under a stationary population, when a life age x leaves the age group of the plan either by death, or by retirement, its place is simultaneously taken by a new entrant at age a.

1.3.3 BASIC PENSION COST CONCEPTS

- Present Value of future Benefits (PVFB)

It is the liability associated with the future benefits of all existing plan members.

For an employee currently age x, having entered the plan at age a and retiring at age r, for each unit of initial pension from age r, PVFB is given by:

$$PVFB_x = \sum_{u=r-x-1}^{\omega} e^{-\delta * u} * {}_u p_x * \beta(x+u) = \frac{D_r^{(\delta)}}{D_x^{(\delta)}} * \ddot{a}_r^{(\delta-\beta)},$$

For a retiree, $x \geq r$, it is given by: $PVFB_x = \sum_{u=0}^{\omega} e^{-\delta * u} * {}_u p_x * \beta(x+u) = e^{\beta*(x-r)} * \ddot{a}_r^{(\delta-\beta)}$

- Normal Cost (NC)

It is a measure of the suggested level of funding based on the actuarial cost method which has been adopted. In general, it is designed to fund the present value of future benefits over the employee's working life time, the pattern of amortization payments being specified by the particular actuarial cost method.

- Actuarial Liability (AL)

It is the portion of the present value of future benefits theoretically amortised by the age x of the participant, with whom it is associated, exclusive of the Normal Cost then due.

- Equivalence Principle

The equivalence principal (of life insurance mathematics) states that the present value of future benefits at age x, $a \leq x < r$, equals to the actuarial liability at age x plus the present value of future normal costs yet to be made.

Thus, $(PVFB)_x = AL_x + (PVFNC)_x$

Substituting $(PVFNC)_x$ by : $(PVFNC)_x = \sum_{u=x}^{r-1} NC_u * {}_{u-x}p_x * e^{-\delta*(u-x)}$,

we may write: $(PVFB)_x = AL_x + \sum_{u=x}^{r-1} NC_u * {}_{u-x}p_x * e^{-\delta*(u-x)}$

- Unfunded Liability (UL)

It is the difference between the plan's total actuarial liability (with respect to the active and non-active members) and the fund allocated to active members at time t.

CHAPTER 2

TRADITIONAL COST METHODS

2.1 INTRODUCTION

Cost Methods are methods implemented in pension schemes for the funding of the future liabilities. They are applied to 'build up' gradually the necessary funds needed to cover the appropriate pension benefits at the retirement age of each participant.

In this chapter, we describe the actuarial methods most commonly applied by actuaries in the European Community, Canada and the USA for calculations relating to private retirement provision. They are presented separately, based on a deterministic model. For their presentation we use the discrete time formulation (the continuous formulation may be implemented in a similar way), and we divide them into the following two major categories described below:

Individual Cost methods: The actuarial formulae relate to each plan member and the total plan cost equals the sum of each individual cost.

Aggregate Cost methods: The actuarial formulae for deriving the plan cost relate to the aggregate of all plan members.

2.2 INDIVIDUAL COST METHODS

For the presentation of the Individual Cost Methods we allocated them into two main categories, Accrued and Projected methods, as in Winklevoss, 1977.

2.2.1 ACCRUED BENEFIT COST METHODS

An Accrued Benefit Cost Method is a method that explicitly considers the benefits that accrue (or are assumed to accrue) in the current year, and the cumulative benefits that have accrued (or are assumed to have accrued) to date. The cost of benefits accruing in the current year represents the normal cost of the method, while the value of all benefits accrued to date represents the actuarial liability.

Under this method, we describe:

The Current (or Traditional) Unit Credit Cost Method, aiming to maintain a fund equal to the value of accrued benefits by reference to their amount as at the calculation date.

The Projected Unit Credit Cost Method, aiming to maintain a fund equal to the value of accrued benefits by reference to their projected amount at the date of retirement.

The distinction between Current and Projected Unit Credit is not strict. However, in order to make a distinction, we thought to consider as "Current Unit Credit" the method where the units are defined as a flat percentage benefit of the current salary and as "Projected Unit Credit" the method where the units are defined as the projected benefit at retirement divided by the number of active years before retirement.

2.2.1.1 Current Unit Credit Cost Method

The main areas of use in Europe (Collinson, 1993) are Belgium (minimum funding purposes) and The Netherlands.

It is most often used with pension plans that provide a flat pension benefit allocated throughout the working years in equal portions, b_x

Assuming that the pension benefit depends on the employee's current salary at age x, s_x, for each one of the l_x members entering the scheme, the Normal Cost and the Actuarial Liability are given by the following formulae:

Discrete time formulation

$$NC_x = b_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} = b*s_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}, \quad AL_x = B_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} = b*S_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$$

$$\frac{AL_x}{NC_x} = \frac{b_x}{B_x} = \frac{s_x}{S_x}$$

and for a population with l_x persons aged x:

$$NC = \sum_{x=a}^{r-1} l_x * b_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} = \ddot{a}_r^{(\delta-\beta)} D_r^{(\delta)} * b * \sum_{x=a}^{r-1} e^{\delta * x} * s_x \qquad \textbf{(2.1)}$$

$$AL = \sum_{x=a}^{r-1} l_x * B_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} + B_r * \sum_{x=r}^{\omega} l_x * \ddot{a}_x^{(\delta-\beta)} * e^{\beta*(x-r)} \text{ or}$$

$$AL = \ddot{a}_r^{(\delta-\beta)} D_r^{(\delta)} * \sum_{x=a}^{r-1} e^{\delta * x} * B_x + B_r * e^{-\beta * r} \sum_{x=r}^{\omega} l_x * \ddot{a}_x^{(\delta-\beta)} * e^{\beta * x} \qquad \textbf{(2.2)}$$

Continuous time formulation

$$NC_x = b*s_x * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)}, \quad AL_x = b * S_x * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)}$$

and for a population with l_x persons aged x:

$$NC = \bar{a}_r^{(\delta-\beta)} \; D_r^{(\delta)} * b * \int_a^{r-1} e^{\delta * x} * s_x \, dx \qquad \textbf{(2.3)}$$

$$AL = \bar{a}_r^{(\delta-\beta)} \; D_r^{(\delta)} * \int_a^{r-1} e^{\delta * x} * B_x \, dx + B_r * \frac{1}{\delta} * l_r \left(-\bar{a}_r^{(\delta-\beta)} + \int_r^{\omega} e^{\beta*(x-r)} * {}_{x-r}p_r \, dx\right) \qquad \textbf{(2.4),}$$

(see Appendix 1)

2.2.1.2 Projected Unit Credit Cost Method

The main areas of use (Collinson. 1993) are the United Kingdom, Ireland, Belgium, Spain and Portugal.

It is most often used with pension plans that provide a pension benefit linked to the employee's salary throughout his/her entire career or part thereof.

The current salary is projected (through a salary scale) to the date of retirement. For each one of the l_x members entering the scheme, the Normal Cost and the Actuarial Liability are given by the following formulae:

Discrete time formulation

$$NC_x = b * s_{r-1} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}, \quad AL_x = b * (x - a) * s_{r-1} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)},$$

$$\frac{AL_x}{NC_x} = x - a$$

and for a population with l_x persons aged x:

$$NC = b * s_{r-1} * \sum_{x=a}^{r-1} \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} \, l_x = \ddot{a}_r^{(\delta-\beta)} * D_r^{(\delta)} * b * s_{r-1} * s^{\delta}_{\overline{r-a}|} \qquad \textbf{(2.5)}$$

$$AL = \ddot{a}_r^{(\delta-\beta)} * D_r^{(\delta)} * b * (r-a) * s_{r-1} * s^{\delta}_{\overline{r-a}|} + (r-a) * s_{r-1} * e^{-\beta * r} * b * \sum_{x=r}^{\omega} l_x * \ddot{a}_x^{(\delta-\beta)} * e^{\beta * x} \qquad \textbf{(2.6)}$$

The continuous time formulation follows in the same way as the discrete time formulation.

2.2.2 PROJECTED BENEFIT COST METHODS

With a projected benefit cost method, the Normal Cost associated with plan membership at age x is the independent variable and the corresponding benefit accrual under this method, which must be derived from the Normal Cost function, is the dependent variable.

The Entry Age Normal Cost Method, aims to establish the level contribution rate which, when payable over the active lifetime of the employee, is sufficient to finance the benefits being provided.

The main areas of use in Europe (Collinson, 1993) are Germany (Book reserved pension plans), United Kingdom and Spain.

It is most often used with pension plans which amortize, at the entry age, the present value of future benefits over the participant's total years of credited service.

Assuming that the pension benefit depends on the plan participant's salary for each one of the members entering the scheme, the Normal Cost and the Actuarial Liability are given by the following formulae:

Discrete time formulation

$NC_x = b*s_x$,

The percentage, b, can be derived after equating the participant's future salary (time b) to the present value of his/her projected benefit, as follows:

$$b * s_a * {}^{s}\ddot{a}_{\overline{a:r-a|}} = B_r * \frac{D_r}{D_a} * \ddot{a}_r^{(\delta-\beta)} \Rightarrow b = \frac{B_r}{s_a * {}^{s}\ddot{a}_{\overline{a:r-a|}}} * \frac{D_r}{D_a} * \ddot{a}_r^{(\delta-\beta)} \qquad \textbf{(2.7)}$$

$$NC_x = \frac{B_r}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_a} * \ddot{a}_r^{(\delta-\beta)} * s_x \qquad \textbf{(2.8)}$$

$$AL_x = B_r * \frac{D_r}{D_x}\ \ddot{a}_r^{(\delta-\beta)} - b * s_x\ {}^s\ddot{a}_{x:\overline{r-x}|} = B_r * \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} \qquad \textbf{(2.9),}$$

(see Appendix 2)

$$\frac{AL_x}{NC_x} = {}^s\ddot{s}_{a:\overline{x-a}|}, \qquad \text{(see Appendix 3)}$$

For either case, Career Average Salary, $B_r = b * S_r$, or Final Salary, $B_r = b*(r-a)*s_{r-1}$ the total Normal Cost and the total Actuarial Liability for a population of l_x at age x are given by the following formulae:

$$NC = \sum_{x=a}^{r-1} l_x * b * s_x \text{ or}$$

$$NC = \ddot{a}_r^{(\delta-\beta)} * \frac{B_r}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_a} * \sum_{x=a}^{r-1} l_x * s_x \qquad \textbf{(2.10)}$$

$$AL_x = B_r * \sum_{x=a}^{r-1} \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}\ l_x + B_r * \sum_{x=r}^{\omega} l_x * \ddot{a}_x^{(\delta-\beta)} * e^{\beta*(x-r)} \qquad \textbf{(2.11)}$$

The continuous time formulation follows in the same way as the discrete time formulation.

2.3 FORMULAE CONNECTING SUCCESSIVE ACTUARIAL LIABILITIES

The equations:

$(AL_x + NC_x)*l_x*e^{\delta} = AL_{x+1}*\ l_{x+1},\ a \le x \le r$ **(2.12)**

and $(AL_x - B_x)*l_x*e^{\delta} = AL_{x+1}*\ l_{x+1},\ x \ge r$ **(2.13)**

which were introduced by Trowbridge, hold under any Individual Cost method.

At this point, we will quote these for the Unit Credit method, using the discrete time formulation.

Let $a \le x \le r$, then

$$(AL_x + NC_x)*l_x*e^{\delta} = (S_x + s_x) * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} *l_x*e^{\delta} =$$

$$= (\sum_{t=a}^{x-1} s_t + s_x) * l_r*e^{-\delta*r} * \frac{1}{l_x * e^{-\delta*x}} * l_x*e^{\delta} * \ddot{a}_r^{(\delta-\beta)} = S_{x+1}* \frac{l_r * e^{-\delta*r}}{l_{x+1} * e^{-\delta(x+1)}} * \ddot{a}_r^{(\delta-\beta)} * l_{x+1}$$

$$\Rightarrow (AL_x + NC_x)*l_x*e^{\delta} = AL_{x+1}*\ l_{x+1}$$

Let $x \ge r$, then

$$(AL_x - B_x)*l_x*e^{\delta} = S_r * e^{\beta*(x-r)}*(\ddot{a}_x^{(\delta-\beta)} - 1)*l_x*e^{\delta}$$

$$= S_r * e^{\beta*(x-r)}* \ddot{a}_{x+1}^{(\delta-\beta)} * e^{-(\delta-\beta)}* \frac{l_{x+1}}{l_x} *l_x*e^{\delta} = S_r * e^{\beta*(x-r+1)}* \ddot{a}_{x+1}^{(\delta-\beta)} *l_{x+1} =$$

$$= AL_{x+1} * l_{x+1}$$

2.4 COMPARISONS ON THE BASIS OF INDIVIDUAL COST METHODS

Among the different cost methods, the Unit Credit methods are the most convenient from the perspective of the actuary. A main advantage is that under both, the Current or Projected, the actuary estimates the Normal Cost on the basis of the actual salary of the plan members.

Under the Entry Age Normal method, the actuary has to use a hypothetical entry age for all scheme participants and a hypothetical starting salary, neither of which may be appropriate as conditions change. For example, in order to calculate the latter would require assumptions about past salary increases for current scheme members and could lead to having to store information for members with the same starting age but different starting salaries. Furthermore, historical salaries are often not available, and where they are available they may not be appropriate for obtaining a level normal cost where the participant has had a promotion.

Unit Credit could be considered preferable from the perspective of the scheme sponsor as well. This conclusion is derived implicitly from the fact that this method is mostly used in countries where occupational pension schemes are well developed such as UK and the Netherlands[5]. In addition it is the only one accepted by the International Accounting Standards and USGAAP. Hence, if the scheme sponsors either plan or are obliged to apply international standards in the future, they are likely to prefer to apply Unit Credit from the start of the plan so that to avoid any actuarial loss (gain) due to the change to the Unit Credit at a later stage.

[5] According to the official report of the Commisions of the European Communities held in Brussels in 2002, the second pillar of occupational pension schemes in the Netherlands is more developed than any where else in the EU.

Numerical results are quoted to illustrate the development of the different cost methods. In the illustrative examples, we calculate the Normal Cost and the Actuarial Liability, assuming that the benefit B_r is one unit. For comparison purposes, we consider the same amount of benefit B_r for all methods, and define b_x accordingly.

The formulae used are presented in table 2.1:

***Table 2.1**: Formulae used in the illustrative examples*

Actuarial Cost Method	Normal Cost, NC_x	Actuarial Liability, AL_x
Current Unit Credit $b_x = b * s_x$, $B_x = b * S_x$	$\frac{1}{S_r} * s_x * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{1}{S_r} * S_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$
Projected Unit Credit $b_x = b * s_{r-1}$, $B_x = b * (x - a) * s_{r-1}$	$\frac{1}{(r-a)} * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{x-a}{r-a} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$
Entry Age Normal $B_x = b * (r - a) * s_{r-1}$	$\frac{s_x}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_x}{D_a} * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$

In all the formulae of table 2.1, the salary, s_x, increases exponentially at a force of rate τ, while the other parameters take the following values:

$a = 30$, $r = 65$, $i = 0.05$, $\beta' = 0.015$, $\tau' = 0.03$, $\gamma = 0$, $s_a = W = 1$ unit

As service table, (assuming $x \geq 30$), the illustrative life table quoted in Bowers et al (1986) has been used.

2.4.1 COMPARISON IN TERMS OF NORMAL COST

Normal Cost is calculated on the basis of the Unit Credit and Entry Age Normal methods, for a plan participant who enters the scheme at age 30 and retires at age 65. The results at specific ages, under each method, are presented in table 2.2, and depicted in figure 2.1.

Table 2.2: *Normal cost under the various cost methods for an age 30 entrant who retires at age 65*

age	**C.U.C.**	**P. U.C.**	**E.A.N.**
30	0.03	0.05	0.07
35	0.04	0.06	0.08
40	0.06	0.08	0.09
45	0.09	0.11	0.11
50	0.14	0.14	0.13
55	0.22	0.18	0.15
60	0.34	0.24	0.17
64	0.50	0.32	0.19

We make the following remarks on table 2.2:

In the Projected Unit Credit, Normal Cost is lower at younger ages than the corresponding one under Entry Age Normal, as a result of the discounting factor $e^{-\delta(r-x)} * {}_{r-x}p_x$.

Although the CUC Normal Cost is 60% of the PUC Normal Cost at entry age '30' it is almost 1.4 times higher by age 60 and 1.6 times higher by age 64 (1 year before retirement). This result may easily be derived by observing the relationship between ${}^{PUC}b_x = \frac{1}{r-a}$ and ${}^{CUC}b_x = \frac{1}{S_r} * s_x$. Under the Projected Unit Credit b_x remains constant throughout the years of service, while under the Current Unit Credit b_x is a strictly increasing function of s_x.

At the younger ages ${}^{PUC}b_x \leq {}^{CUC}b_x$, while around age 50 they approximate each other, and thereafter the above inequality is reversed.

Under the Entry Age Normal, Normal Cost is higher than the corresponding one under Current Unit Credit up to age of 50, and decreases thereafter. This result is expected, since the annuity coefficient $\frac{1}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} \frac{D_r}{D_a}$ at the older ages becomes smaller than the corresponding one under Current Unit Credit in both cases, i.e. $\frac{1}{S_r} * \frac{D_r}{D_x}$ and $\frac{1}{s_{r-1} * (r-a)} \frac{D_r}{D_x}$ since $D_a \geq D_x \Rightarrow \frac{D_r}{D_a} \leq \frac{D_r}{D_x}$. At the older ages D_x is significantly lower than D_a and, as a consequence, $\frac{D_r}{D_x}$ is significantly higher than $\frac{D_r}{D_a}$.

As far as the other coefficient component is concerned, $s_a * {}^s\ddot{a}_{a:\overline{r-a}|}$, it is lower than S_r and s_{r-1} * (r- a). Both inequalities $\frac{1}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} > \frac{1}{S_r}$ and $\frac{1}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} > \frac{1}{s_{r-1} * (r-a)}$ are reversed at the older ages due to the D_a and D_x components.

The results of table 2.2 are described in figure 2.1 below.

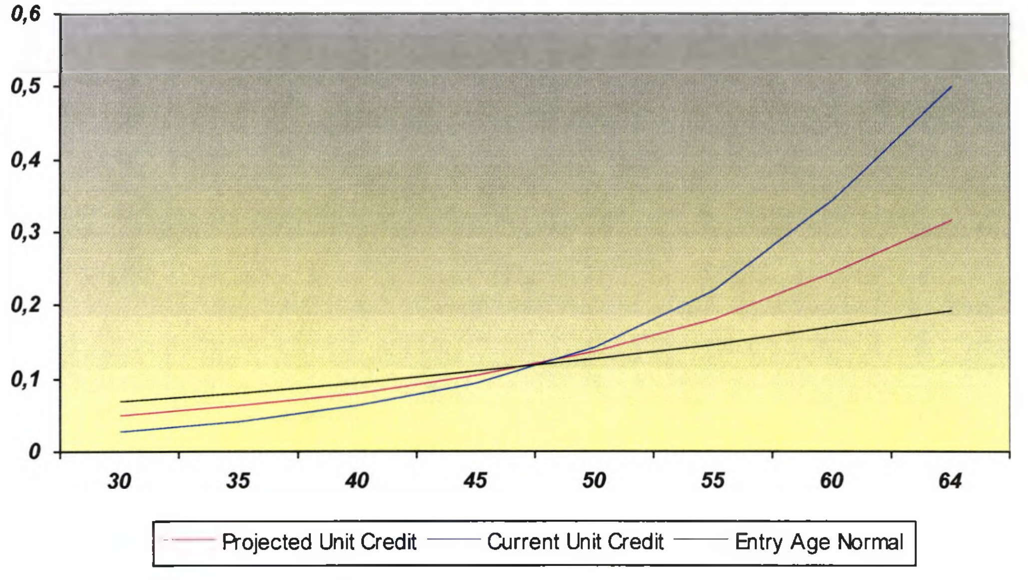

Figure 2.1: Normal Cost under various cost methods

The effect of the different methods on the pension scheme implementation is better described when we calculate Normal Cost as a percentage of salary. Table 2.2 then becomes:

Table 2.2[a]: Normal Cost as a percentage of salary under various cost methods, for an age 30 entrant who retires at age 65

age	C.U.C.	P.U.C.	E.A.N.
35	3.64%	5.42%	7.00%
40	4.70%	6.04%	7.00%
45	6.09%	6.75%	7.00%
50	7.96%	7.61%	7.00%
55	10.52%	8.68%	7.00%
60	14.17%	10.09%	7.00%
64	18.36%	11.61%	7.00%

The above results point out quite clearly that the Unit Credit normal cost is a strictly increasing function of salary. The normal cost under EAN is a constant percentage of salary throughout the period.

2.4.2 COMPARISON IN TERMS OF ACTUARIAL LIABILITY

The Actuarial Liability is calculated on the basis of the Unit Credit and Entry Age Normal methods, for a plan participant who enters the scheme at age 30 and retires at age 65. The results at specific ages, under each method, are presented in table 2.3, and depicted in figure 2.3.

Table 2.3: *Actuarial liability under the various cost methods for an age 30 entrant who retires at age 65*

age	C.U.C.	P.U.C.	E.A.N.
30	0.00	0.00	0.00
35	0.19	0.31	0.43
40	0.54	0.81	1.06
45	1.13	1.58	1.96
50	2.14	2.75	3.24
55	3.84	4.54	5.08
60	6.74	7.35	7.77
64	10.60	10.78	10.91
65	11.89	11.89	11.89

The Actuarial Liability values show clearly that the methods discussed, although varying in approach, are only different because of the effect of the actuarial assumptions on the timing of the annual costs. In particular, the magnitude by which the actuarial liability of each cost method exceeds (or is exceeded by) any one of the corresponding other methods can be reflected by the coefficient 'C' of the Present Value of Future Benefits under each one of the above cases.

Specifically for the coefficients below, at age x

${}^{PUC}C_x = \frac{x-a}{r-a}$, ${}^{CUC}C_x = \frac{s^{\tau}_{\overline{x-a|}}}{s^{\tau}_{\overline{r-a|}}}$, and ${}^{EAN}C_x = \frac{{}^{s}\ddot{a}_{a:\overline{x-a|}}}{{}^{s}\ddot{a}_{a:\overline{r-a|}}}$, it holds:

$$0 \leq \frac{s^{\tau}_{\overline{x-a|}}}{s^{\tau}_{\overline{r-a|}}} \leq \frac{x-a}{r-a} \leq \frac{{}^{s}\ddot{a}_{a:\overline{x-a|}}}{{}^{s}\ddot{a}_{a:\overline{r-a|}}} \leq 1$$

This is described in figure 2.2, below:

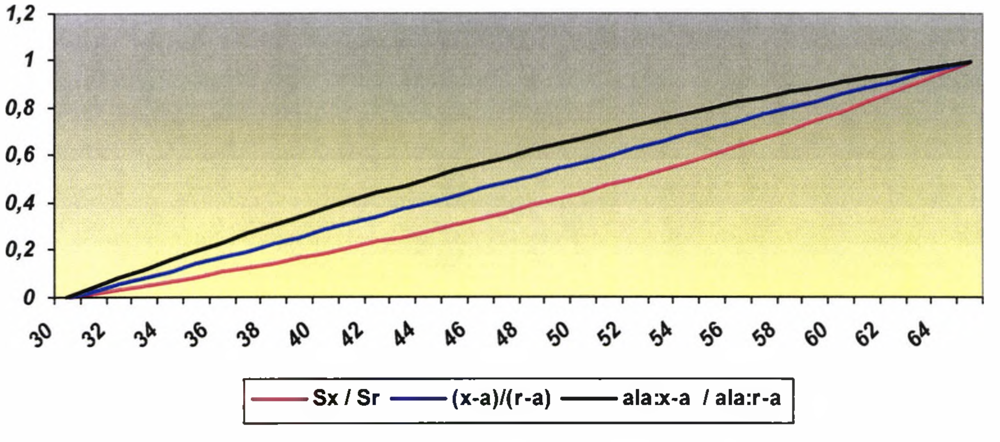

Figure 2.2: Coefficients of PVFB

Because of the above inequality, we have the following ranking of the Actuarial Liability under each cost method, presented as well in figure 2.3:

$${}^{CUC}AL_x < {}^{PUC}AL_x < {}^{EAN}AL_x \qquad \textbf{(2.14)}$$

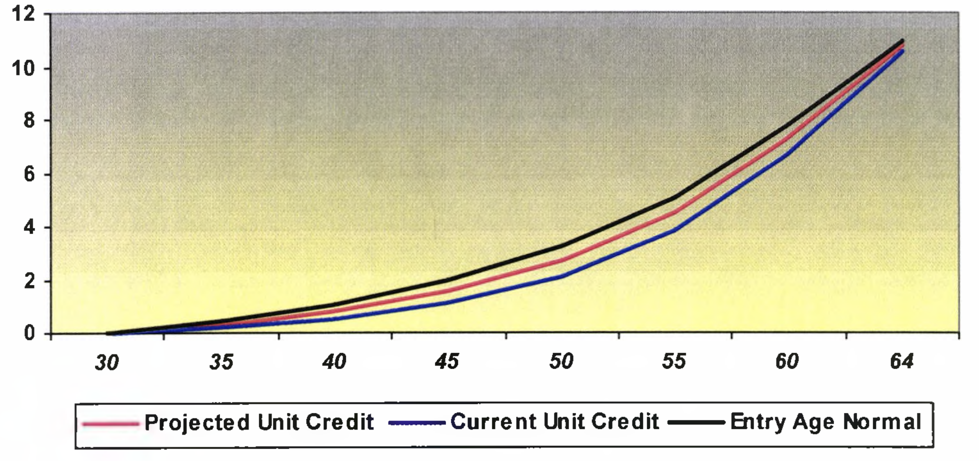

Figure 2.3: Actuarial Liability under various cost methods

At this point, it is of importance to mention that:

The 'security' associated with any particular method depends on the relationship between the required amount of the standard fund held at any time and the corresponding amount of liabilities.

In our attempt to 'quantify' the security level, we may link it to the level of the Fund value, considering that if a funding method allows for high Fund values the level of security increases.

The results derived by the previous comparison show that:

Under the Entry Age Normal method, the Actuarial Liability is higher than under the other two methods (P.U.C and C.U.C) and as a consequence we may expect the same result for the value of the Fund. Under the Projected Unit Credit method, the Actuarial Liability is higher than for the Current Unit Credit method and thus the Fund value is also expected to be higher.

Based on these conclusions we may attempt to list the funding methods in order of ascending security as follows:

(a) Current Unit Credit , (b) Projected Unit Credit , (c) Entry Age Normal

Collinson, 1993, listed also those according to the above order of ascending security.

This is indeed the case under the "classical" assumptions where the salary earns a positive real return, $\delta - \gamma > 0$, and the function l_x is decreasing with x; i.e. $f(x) = \frac{D_x^{(\delta-\gamma)}}{D_a^{(\delta-\gamma)}}$ is a strictly decreasing function, defined as the expected present value at the entry age a of a unit amount payable on survival to the age x.

However, Dufresne, 1986, considers the more general case where $\delta - \gamma > 0$ may not apply. Thus, in particular, Dufresne, 1986, proves mathematically that under certain conditions the ultimate value of the actuarial liability under Projected Unit Credit might be either higher or lower than the corresponding one under Entry Age Normal. According also to the assumptions used under those conditions, the ultimate value of the Normal Cost under Projected Unit Credit might be either higher or lower than the corresponding one under Entry Age Normal. Dufresne's results may described as follows:.

<u>If f(x) is strictly decreasing, then</u> : ${}^{PUC}AL < {}^{EAN}AL$,

If $\delta-\gamma > 0 \Rightarrow {}^{PUC}NC > {}^{EAN}NC$

If $\delta-\gamma < 0 \Rightarrow {}^{PUC}NC < {}^{EAN}NC$

<u>If f(x) is strictly increasing, then:</u> ${}^{PUC}AL > {}^{EAN}AL$,

If $\delta-\gamma > 0 \Rightarrow {}^{PUC}NC < {}^{EAN}NC$

If $\delta-\gamma < 0 \Rightarrow {}^{PUC}NC > {}^{EAN}NC$

Our results and those of Collinson, are the same with the results of Dufresne for the case we and Collinson have assumed; i.e. where f(x) is strictly decreasing and $\delta-\gamma>0$.

2.5 AGGREGATE COST METHODS

From all Aggregate Cost Methods in this section we discuss the most commonly used, according to our experience: the Aggregate and the Frozen Initial Liability.
The approach we follow is that of Aitken (1994).

2.5.1 AGGREGATE

The philosophy of the Aggregate cost method, is that the future Normal Costs will pay for the value of the benefits in excess of the funds on hand (Aitken, 1994). Thus, having U(t) = 0, we have:

$$NC * \frac{\sum_{x=a}^{r-1} {}^{s}\ddot{a}_{x:\overline{r-x|}} * b_x * l_x}{\sum_{x=a}^{r-1} b_x * l_x} = \sum_{x=a}^{\omega} B_r * PVFB_x - F(t) \qquad \textbf{(2.15)}$$

The key difference between the Aggregate and Frozen Initial Liability Method (described below) is the treatment of the Unfunded Liability. In particular, under the Aggregate Method, the amount of the unfunded liability is used to increase (or decrease) the future contribution rate. Pension schemes under the Aggregate Method are considered as funded, regardless of the difference between the value of the actuarial liability and the value of the assets; the amount of the difference between them affects future contributions that are revised accordingly. This is a reason why scheme sponsors may prefer the Aggregate method: if a pension plan has assets lower than the corresponding actuarial liability value, then the amount of difference between them is not registered in the Balance Sheet so that the scheme sponsor is not charged with any unfunded liability amount which has to be amortized during a certain period. In countries like Greece where second pillar occupational pensions are still not widespread, the Aggregate method is much preferable. In particular, in Greece most of the defined benefit group policy contracts are issued at the initiative of the scheme sponsor[6] and in their majority, either do not allow for employees' contributions or allow for a very low percentage. Under the Aggregate method, scheme sponsors are not obliged to amortize any part of the unfunded liability and thus they have a reduced exposure to liquidity problems.

2.5.2 FROZEN INITIAL LIABILITY

The philosophy of this method is that the supplemental costs amortize the FIL, and the future normal costs provide for the value of the benefits in excess of the unamortized portion of the FIL and the funds on hand.

The unfunded liability is determined annually and it is defined by the actuarial liability less the market value of the assets of the plan. The actuarial liability is calculated after summing all the individual counterparts determined either through the Entry Age Normal (EAN method) or the Current Unit Credit (considering the employee' s attained age; Attained Age method). The Normal Cost is computed based on either the Entry Age or the Current Unit Credit, after considering the weighted average of the entry or attained annuity value running from entry or attained age to the age of retirement. Specifically:

[6] The majority of them are issued by subsidiaries of well known foreign companies and by the Greek Banks.

2.5.2.1. Entry Age Normal

Under this method, the initial Unfunded Actuarial Liability (at entry age a, time 0) described as the Frozen Initial Liability, is equal to:

$$AL(0) = \sum_{x=a}^{r-1} B_r * PVFB_a - NC(0) * \frac{\sum_{x=a}^{r-1} {}^{s}\ddot{a}_{a:\overline{r-a}|} * s_x * l_x}{\sum_{x=a}^{r-1} s_x * l_x} = FIL = U(0),$$

where both NC(0) and AL(0) are computed by the Entry Age Normal Cost method, b_x is a level percent of salary, $b_x = b * s_x$, and no retired lives are covered by the plan at inception.

At each valuation date we have:

$$NC(t) * \frac{\sum_{x=a}^{r-1} {}^{s}\ddot{a}_{a:\overline{r-a}|} * s_x * l_x}{\sum_{x=a}^{r-1} s_x * l_x} = \sum_{x=a}^{\omega} B_r * PVFB_a - U(t) - F(t) \qquad \textbf{(2.16),}$$

(Aitken 1994)

2.5.2.2 Attained Age Normal

This method differs from the Entry Age Normal as regards the computation of the Frozen Initial Liability that is performed under the Current Unit Credit.

FIL is based on the accrued pension benefit and at each valuation date we have:

$$NC(t) * \frac{\sum_{x=a}^{r-1} {}^{s}\ddot{a}_{x:\overline{r-x}|} * s_x * l_x}{\sum_{x=a}^{r-1} s_x * l_x}{}^{7} = \sum_{x=a}^{\omega} B_r * PVFB_x - U(t) - F(t) \qquad \textbf{(2.17),}$$

(Aitken 1994)

The Attained Age method may be preferable to the actuary since it describes better the funded status of the plan. In particular, under this method, the total benefits are divided into past service and future service components. Past service benefits correspond to the assets and future service to the future contributions[8]. As a consequence, it can be clearly seen whether the value of assets is adequate to cover the accrued rights of the scheme participants and the future contributions are sufficient to cover the future service liability.

2.5.2.3 FIL, Comparison between Entry and Attained Age Normal

Based on the Normal Cost formulae, (2.16) and (2.17), we calculate the Normal Cost under each one of the above FIL methods. We use a stationary population, based on the same assumptions as the ones used for the different parameters (for example entry age, rate of interest, and so on) to compare the development of Normal Cost under the Individual Cost Methods in section 2.4. For illustration purposes the Frozen Initial Liability is amortized over a 10 years period. The Unfunded Liability each year equals

to: $U(t) = U(0) * \frac{{}^{\delta}\ddot{a}_{\overline{10-t}|}}{{}^{\delta}\ddot{a}_{\overline{10}|}}$, t = 1,2,3,4,5,6,7,8,9,10

[7] $b_x = b * s_x$

[8] These are compared in terms of expected present values.

On the basis of the results obtained, we observe that the unfunded actuarial liability is smaller under the Attained age method; hence the normal and supplemental costs, throughout the employees' active years of service, follow the trajectories presented in figures 2.4 and 2.5:

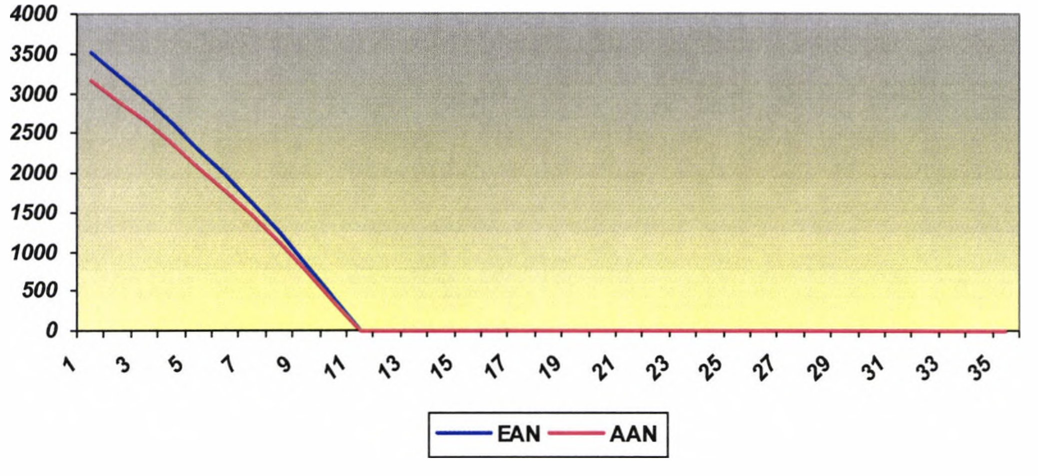

Figure 2.4 Supplemental Cost under FIL Entry Age and FIL Attained Age

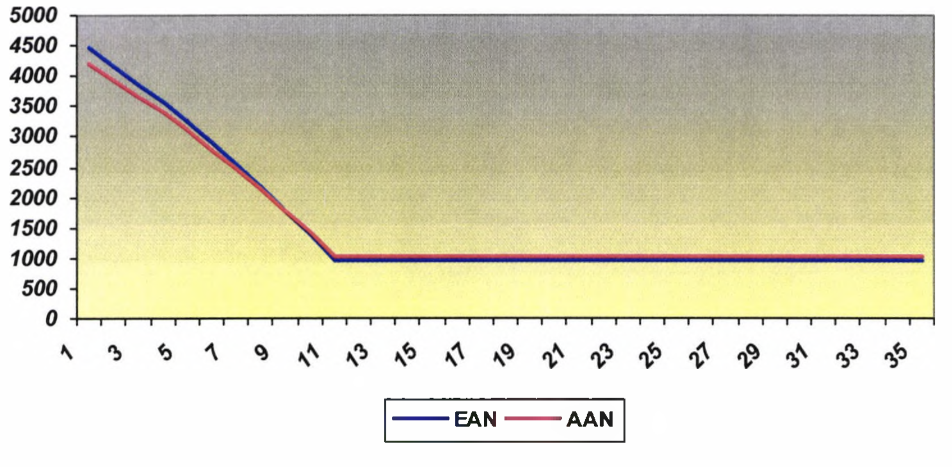

Figure 2.5: Normal Cost under FIL Entry Age and FIL Attained Age

Figure 2.4 shows that the resulting Supplemental Cost is smaller under the Attained Age than under Entry Age Normal, due to the smaller unfunded actuarial liability of the former. For the same reason, figure 2.5 shows that the resulting normal cost is higher under the Attained Age Method than under Entry Age Normal.

2.5.3 MAIN DIFFERENCE BETWEEN INDIVIDUAL and AGGREGATE COST METHODS

By implementing the aggregate methods on the same basis as the one used for the individual ones, it is possible for the basic principle underlying an aggregate and an individual cost method to be clearly observed ; see formulae **(2.18)** and **(2.19)** below:

For an Aggregate Cost Method the Normal Cost is obtained under the following formula:

$$NC(t) = \left[\sum_{x=a}^{\omega} B_r * PVFB_x - U(t) - F(t) \right] * \frac{\sum_{x=a}^{r-1} l_x * b_x}{\sum_{x=a}^{r-1} {}^{s}\ddot{a}_{x:\overline{r-x}|} * l_x * b_x} \qquad \textbf{(2.18)}$$

For the Individual Cost methods, for a population with l_x persons aged x, Normal Cost is obtained as:

$$NC(t) = \sum_{x=a}^{r-1} l_x * b_x * PVFB_x - U(t) - F(t) \qquad \textbf{(2.19)}$$

We define as average working life annuity: the weighted average of the members' annuity values running from their attained age to the age of retirement, being multiplied by the number of active participants at that age and their annual benefit b_x.

From the above, the basic principle underlying an aggregate and an individual cost method may be summarized as follows:

With the individual methods we obtain the total normal cost by making calculations for each active participant and summing the individual results, while with the aggregate methods by making calculations for the total group after computing the average working life annuity.

Winklevoss in 1977, allocated the aggregate methods into the same categories as the individuals i.e. into Accrued and Projected Aggregate methods. These are presented below:

2.5.3.1 Accrued Benefit Cost Method

Despite the fact that this method is not commonly used, we have decided to adopt the definition of Normal cost as cited by Winklevoss (1977) in order to show the one-to-one correspondence between the methods implemented using the same philosophy, both on an individual and on an aggregate basis.

Discrete time formulation

$$\text{NC} = \sum_{x=a}^{r-1} l_x * b_x * \left[\frac{\sum_{x=a}^{r-1} l_x * B_r * (PVFB)_x}{\sum_{x=a}^{r-1} l_x * B_x} \right] \qquad \textbf{(2.20)}$$

2.5.3.2 Projected Benefit Cost Method

<u>Frozen Initial Liability</u>

a) Entry Age Normal

In line with the principle stated earlier for an aggregate method, we will define the Normal Cost as:

Discrete time formulation

$$NC(t) = \sum_{x=a}^{r-1} l_x * s_x * \left[\frac{\sum_{x=a}^{\omega} l_x * B_r * PVFB_a - U(t) - F(t)}{\sum_{x=a}^{r-1} l_x * {}^{s}\ddot{a}_{a:\overline{r-a}|}} \right] \quad (2.21)$$

b) Attained Age Normal

In practice, it is the same with a), after replacing the Entry Age Normal by the Current Unit Credit in the Initial Liability calculation.

The Normal Cost formula is formed as follows:

Discrete time formulation

$$NC(t) = \sum_{x=a}^{r-1} l_x * s_x \left[\frac{\sum_{x=a}^{\omega} l_x * B_r * PVFB_x - F(t) - U(t)}{\sum_{x=a}^{r-1} l_x * {}^{s}\ddot{a}_{x:\overline{r-x}|}} \right] \quad (2.22)$$

The actuarial liability is defined by the Actuarial Liability under each method's individual counterpart.

2.6 COMPARISON BETWEEN 'PAY AS YOU GO' and ACTUARIAL FUNDING

The 'Pay as you go' method is a financing system that does not involve advance funding and under which, ideally, contributions are assessed at a sufficient level to meet the current benefit payments of the scheme – in our discussion this means benefits payable to retirees.

Since Social Security relies on the above method, the example below offered by Brown (1996) is quoted in order to point out the difference between this method and Actuarial Funding.

- Actuarial Funding

Suppose an individual currently age 30 with an annual income of one unit, payable continuously, wishes to retire at age 65.

The required contributions are determined after setting the present value of all contributions equal to the present value of all retirement income benefits at a defined age.

i.e. $C = \int_0^{35} e^{-\delta * t} * \frac{l_{30+t}}{l_{30}} dt = e^{-35*t} * \frac{l_{65}}{l_{30}} * \int_0^{\infty} e^{-\delta * t} * \frac{l_{65+t}}{l_{65}} dt \Rightarrow$

$$\Rightarrow C = C(\delta) = \frac{\int_{65}^{\infty} e^{-\delta x} * l_x dx}{\int_{30}^{65} e^{-\delta x} * l_x dx} \qquad \textbf{(2.23)}$$

- Pay as you go funding

Suppose a 'pay as you go' plan wishes to pay annual income of one unit payable continuously to all retirees alive age 65 and over.

Contributions will be made by all working employees age 30 to 65 inclusive.

In a 'pay as you go' system contribution income is immediately distributed as benefit outgo.

Thus, in a stationary population the contribution formula is :

$$C * (T_{30} - T_{65}) = T_{65} \Rightarrow C = \frac{\int_{65}^{\infty} l_x dx}{\int_{30}^{65} l_x dx} \qquad \textbf{(2.24)}$$

It is clearly seen that (2.24) is a special case of (2.23) when $\delta = 0$, since in a pay as you go funding contributions are immediately distributed as benefits outgo.

Commenting on 2.23 and 2.24,

Haberman pointed out that a factor, which affects contribution rate in (2.23), is the rate of investment return δ. It can be shown that $\frac{dC(\delta)}{d\delta} \prec 0$.

A factor which affects contribution rate in (2.24), pointed out by Brown as well, is the population included in T_{30} and T_{65} respectively. It becomes clear, that the social security funding depends only on the ratio of beneficiaries to workers and is therefore sensitive to demographic shifts.

Actuarial funding is also sensitive to demographic shifts. However, contributions are determined before the benefit outgo and if they are closely monitored, the effect of an unfunded liability due to those shifts may be mitigated.

APPENDIX 1

$$B_r * \int_r^{\omega} e^{\beta*(x-r)} * l_x * \bar{a}_x^{(\delta-\beta)} \, dx = B_r * \frac{1}{\delta} * l_r * (- \bar{a}_r^{(\delta-\beta)} + \int_r^{\omega} e^{\beta*(x-r)} * {}_{x-r}p_r \, dx)$$

Proof

$$\int_r^{\omega} e^{\beta*(x-r)} * l_x * \bar{a}_x^{(\delta-\beta)} \, dx = \int_r^{\omega} e^{\beta*(x-r)} * l_x * \frac{\bar{N}_x^{(\delta-\beta)}}{e^{-(\delta-\beta)*x} * l_x} \, dx = \int_r^{\omega} e^{\delta x - \beta r} * \bar{N}_x^{(\delta-\beta)} \, dx =$$

$$= \frac{1}{\delta} * e^{\delta x - \beta r} * \bar{N}_x^{(\delta-\beta)} \Big|_r^{\omega} + \frac{1}{\delta} * \int_r^{\omega} e^{\delta x - \beta r} e^{-(\delta-\beta)x} * l_x \, dx =$$

$$= - \frac{1}{\delta} * e^{(\delta-\beta)r} * \bar{N}_r^{(\delta-\beta)} + \frac{1}{\delta} * l_r * \int_r^{\omega} e^{\beta*(x-r)} * {}_{x-r}p_r \, dx =$$

$$= \frac{1}{\delta} * l_r *(- \bar{a}_r^{(\delta-\beta)} + \int_r^{\omega} e^{\beta*(x-r)} * {}_{x-r}p_r \, dx) \qquad \textbf{(1)}$$

From (1) $\Rightarrow$

$$B_r * \int_r^{\omega} e^{\beta*(x-r)} * l_x * \bar{a}_x^{(\delta-\beta)} \, dx = B_r * \frac{1}{\delta} * l_r * (- \bar{a}_r^{(\delta-\beta)} + \int_r^{\omega} e^{\beta*(x-r)} * {}_{x-r}p_r \, dx)$$

APPENDIX 2

$$AL_x = B_r * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} - b * s_x * {}^{s}\ddot{a}_{x:\overline{r-x|}} = B_r * \frac{{}^{s}\ddot{a}_{a:\overline{x-a|}}}{{}^{s}\ddot{a}_{a:\overline{r-a|}}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$$

Proof

$$B_r * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} - b * s_x * {}^{s}\ddot{a}_{x:\overline{r-x|}} =$$

$$= B_r * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)} - \frac{B_r}{s_a * {}^{s}\ddot{a}_{a:\overline{r-a|}}} * \frac{D_r}{D_a} * \ddot{a}_r^{(\delta-\beta)} * s_x \, {}^{s}\ddot{a}_{x:\overline{r-x|}} =$$

$$= B_r * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)} * \left(1 - \frac{D_x}{D_a} * \frac{s_x}{s_a} * \frac{{}^s\ddot{a}_{x:\overline{r-x}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}}\right) = B_r * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)} * \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}}$$

APPENDIX 3

$$NC_x = \frac{B_r}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_a} * \ddot{a}_r^{(\delta-\beta)} * s_x \text{ , } AL_x = B_r * \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$$

$$\frac{AL_x}{NC_x} = {}^s\ddot{s}_{a:\overline{x-a}|}$$

<u>Proof</u>

$$\frac{AL_x}{NC_x} = {}^s\ddot{a}_{a:\overline{x-a}|} * \frac{D_x}{D_a} * \frac{s_x}{s_a} = {}^s\ddot{a}_{a:\overline{x-a}|} * \frac{1}{{}_a^sE_{x-a}} = {}^s\ddot{s}_{a:\overline{x-a}|}$$

CHAPTER 3

PENSION FUNDING USING THE ACCRUAL DENSITY FUNCTION m(x)

3.1 INTRODUCTION

This chapter introduces the idea of implementing a model pension plan, after considering the age of the insured. For this purpose, the accrual density function m(x) is used, defined so as to correspond to a probability density function f(x):

$$m(x) \succ 0, \; a \leq x \leq r \text{ and } \int_a^r m(x)\, dx = 1$$

The different pension accrual density functions m(x) are defined along with their corresponding accrual functions M(x).

These functions are examined in terms of whether they are associated with an accelerating or a decelerating cost method, and compared in terms of their development. As chapter 2, chapter 3 is also based on a deterministic model.

3.2 CHOICE of m(x) s

Since m(x) corresponds to a probability density function, the range of all possible choices for m(x) is extensive. Accordingly, a large number of cost methods may be defined. However, a criterion for choosing m(x) may be derived from its mathematical properties and its age profile.

The categorization of m(x) with respect to its association with an accelerating or a decelerating cost method has been defined by Cooper and Hickman, 1967, as follows:

Let $m'(x)$ the first derivative of m(x).

If $M''(x) = m'(x) < 0,\ a \leq x \leq r$, the actuarial cost method defined by m(x) results in decelerating funding at age x, and the actuarial cost method associated with m(x) is a decelerating actuarial cost method.

If $M''(x) = m'(x) > 0,\ a \leq x \leq r$, the actuarial cost method defined by m(x) results in accelerating funding at age x, and the actuarial cost method associated with m(x) is an accelerating actuarial cost method.

The pension accrual density functions m(x) discussed in the thesis are the ones developed according to the following distributions which we have chosen as being possible candidates for application to pension funding methods:

- Power function
- Uniform
- Truncated Pareto
- Truncated Exponential

For our decision, we were based on the observation that the m(x) and M(x) under a uniform distribution, coincide with the benefit accrued under the Normal Cost and Actuarial Liability respectively, for the Projected Unit Credit method. Since Uniform distribution is the special case of the Power function when p =1, we thought to examine the general case where p $\neq 1$.

Extending our choice, we also decided to consider Truncated Pareto and Truncated Exponential since both are related to the Power function distribution. In particular: if X has a Power function distribution X^{-1} has a Pareto distribution; if X has a Pareto distribution with parameters p, x_0 then $\ln(\frac{X}{x_0})$ has an Exponential distribution with parameter p; if X has Uniform distribution defined in [0,1] then –lnX has an Exponential distribution with parameter 1 (see Appendix 4).

We use truncated distributions because with the random variable we refer to the age of the plan member and thus it has always to be a nonnegative number, that lies in a specified interval.

According to the definitions supplied by Johnson, et al (1995):

3.2.1 POWER FUNCTION DISTRIBUTION

Suppose X ~ Power function, then

$$m(x) = \begin{cases} p * \dfrac{(x-a)^{p-1}}{(r-a)^{p}} & , a \le x \le r \ , \ p \succ 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)^p}{(r-a)^p} & , a \le x \le r \ , \ p \succ 0 \\ 1 & , x \succ r \end{cases}$$

Obviously, M(a) = 0 and M(r)=1, i.e. M(x) has the properties of an accrual function.

3.2.2 UNIFORM

In case p = 1, X ~ Uniform distribution and

$$m(x) = \begin{cases} \dfrac{1}{(r-a)} & , a \le x \le r \\ 0 & , \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)}{(r-a)} & , a \le x \le r \ , \\ 1 & , x \succ r \end{cases}$$

Obviously, M(a) = 0 and M(r)=1, i.e. M(x) has the properties of an accrual function.

As we may observe, under a uniform distribution m(x), M(x) coincides with the benefit accrued under the Normal Cost and Actuarial Liability, respectively, for the projected Unit Credit method. Based on this result, in the following chapters, the uniform distribution will be considered as the link between these cost methods and the traditional ones.

3.2.3 TRUNCATED PARETO

Suppose X ~ Truncated Pareto distribution, then

$$m(x) = \begin{cases} \dfrac{\frac{k}{a} * (\frac{a}{x})^{k+1}}{1-(\frac{a}{r})^{k}}, & a \le x \le r,\ k \succ 0 \\ 0, & \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1-(\frac{a}{x})^{k}}{1-(\frac{a}{r})^{k}}, & a \prec x \prec r\ ,\ k \succ 0 \\ 1 & ,\quad x \ge r \end{cases}$$

Obviously, M(a) = 0 and M(r)=1, i.e. M(x) has the properties of an accrual function.

3.2.4 TRUNCATED EXPONENTIAL

Suppose X ~ Truncated Exponential distribution, then

$$m(x) = \begin{cases} \frac{1}{\sigma} * \dfrac{1}{1-e^{(-\frac{r-a}{\sigma})}} * e^{-\frac{x-a}{\sigma}}, & a \prec x \prec r,\ \sigma \succ 0 \\ 0, & \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1-e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}} & , a \prec x \prec r \ , \sigma \succ 0 \\ 1 & , \quad x \ge r \end{cases}$$

Obviously, M(a) = 0 and M(r)=1, i.e. M(x) has the properties of an accrual function.

3.2.5 REMARKS

We have mentioned that a criterion for choosing m(x) may be derived from both its mathematical properties and its age profile. We have also mentioned that an important factor in choosing an acceptable pension accrual function is the interval range of the underlying parameters. Because the age of the plan member has to be a non-negative number that lies in a specified interval, we must use a truncated distribution for m(x). As we have already mentioned, since m(x) corresponds to a probability density function, the range of all possible choices for m(x) is extensive. Accordingly, a large number of cost methods may be defined. However, we cannot ignore another factor important for our choice; the utility of the accrual function from the perspective of the actuary. Before we selected the set of different distributions used throughout the thesis, we also studied both the Truncated Gamma and Doubly Truncated Normal distributions. The Gamma is related to the family of beta and exponential distributions which we have used. In particular, the exponential density function is a special case of that of gamma. However, in practice Gamma is not easy to apply. According to its probability density function (Johnson et al 1994),

$$m(x) = \begin{cases} \dfrac{(x-1)^{z-1} * e^{-(x-a)*k}}{\int_a^r (x-1)^{z-1} * e^{-(x-a)*k} dx} & , \quad z>0,\ k>0,\ a \le x \le r \\ 0 & , \quad \text{otherwise} \end{cases}$$

so that the actuary needs to define two parameters: z and k; we have seen that the behaviour of m(x) is very sensitive to the z and k values. In addition, from the resulting form of $m'(x)$, we cannot determine if m(x) is associated with either an accelerating or a decelerating cost method.

The Doubly Truncated Normal has also been considered, simply because the Normal distribution holds a central position in statistics. According to its probability density function (Johnson, et al 1994),

$$m(x) = \begin{cases} \dfrac{\frac{1}{\sqrt{2*\pi*\sigma}} * e^{\frac{-(x-\xi)^2}{2*\sigma^2}}}{\int_a^r \frac{1}{\sqrt{2*\pi*\sigma}} * e^{\frac{-(t-\xi)^2}{2*\sigma^2}} dt} & , \ a \le x \le r \\ 0 & , \text{otherwise} \end{cases}$$

Here, the actuary has to define both parameters ξ, and σ.

We point out that the value of ξ determines if m(x) is associated with either an accelerating or a decelerating cost method. Specifically, from the definition of $m'(x)$=m(x)*($-\frac{x-\xi}{\sigma^2}$) if x> ξ, then $m'(x)$ is associated with a decelerating cost method and vice versa. We observed that as the age of the scheme participant is less than ξ, he / she purchases a higher portion of benefit along with his / her age increase up to ξ; while thereafter, i.e. as the age becomes higher than ξ, the opposite hold. The latter is of importance since the mean value of m(x) allows the actuary to build up the fund at a higher pace in the early years and at a lower pace thereafter. As for the Gamma, the Truncated Normal is also less convenient for the actuary than those we have chosen for application to pension funding methods in the sense that the estimation of 2 parameters is needed.

Since our principal aim is to focus on the pension scheme implementation when the age of the plan member is taken into account (and not to consider exhaustively all the different pension accrual functions that may be suitable), we have thus decided to include neither the Gamma nor the Normal distribution in this study. However, this does not exclude the possibility of extending our work in the future by considering other distributions, studying those more thoroughly and comparing the corresponding results with those already derived.

3.3 CATEGORIZATION of m(x)

- Power function distribution

$$m'(x) = p^*(p-1)^* \frac{(x-a)^{p-2}}{(r-a)^p} \ , \ a \leq x \leq r \ , \ p \succ 0$$

If $p \prec 1$, then $m'(x) \prec 0$ which implies that the actuarial cost method associated with m(x) is a decelerating actuarial cost method.

If $p \succ 1$, then $m'(x) \prec 0$ which implies that the actuarial cost method associated with m(x) is an accelerating actuarial cost method.

At this point we have to mention the special property of the Power function that as a result of the parameter p value m(x) may be categorized as either a decelerating or an accelerating cost method. Also, when p =1, i.e. under Uniform distribution, the actuarial cost method is characterized by $m'(x) = 0$. This result shows that it is neither a decelerating nor an accelerating cost method.

- Truncated Pareto

$$m'(x) = -(k+1)* \frac{\frac{k}{a}*(\frac{a}{x})^{k+1}}{1-(\frac{a}{r})^{k}} \quad , a \leq x \leq r\, , \; k \succ 0$$

i.e. $m'(x) \prec 0$ which implies that the actuarial cost method associated with m(x) is a decelerating actuarial cost method.

- Truncated Exponential

$$m'(x) \;=\; -\frac{1}{\sigma^{2}}*\frac{e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}}, \; a \prec x \prec r\, , \; \sigma \succ 0$$

i.e. $m'(x) \prec 0$ which implies that the actuarial cost method associated with m(x) is a decelerating actuarial cost method.

3.4 COMPARISON of m(x) DEVELOPMENT

3.4.1 COMPARISON WITHIN THE SAME DISTRIBUTION

Before we discuss the properties of m(x) under the different distributions, a comparison within the same distribution is conducted in order to examine how the development of m(x) is affected by the parameter value of each one.

Specifically: for each parameter, different values were chosen in an increasing order and the m(x) pattern between the entry age and the age of retirement is observed.

We point out, that for a certain distribution, we have tried different parameter values in order to examine the trend of m(x) along with the age increase and thus to conclude on the magnitude of the portion of the benefit purchased.

Setting $a = 30$, r = 65, we obtain the following results:

- Power function

For the parameter, p, we set the values p = 0.3, p = 0.8, p = 1, (Uniform distribution), p = 1.5, and p =1.8.The results, at specific ages, are presented in tables 3.1 and 3.2:

Table 3.1 *m(x) development under Power function*

age	p = 0.3	p = 0.8	p = 1	p = 1.5	p = 1.8
32	0.064	0.041	0.029	0.010	0.005
33	0.048	0.037	0.029	0.013	0.007
34	0.039	0.035	0.029	0.014	0.009
35	0.033	0.034	0.029	0.016	0.011
40	0.021	0.029	0.029	0.023	0.019
45	0.016	0.027	0.029	0.028	0.026
50	0.013	0.026	0.029	0.032	0.033
55	0.011	0.024	0.029	0.036	0.039
60	0.01	0.024	0.029	0.04	0.045
65	0.009	0.023	0.029	0.043	0.051

__Table 3.2__ M(x) development under Power function

age	p = 0.3	p = 0.8	p = 1	p = 1.5	p = 1.8
35	0.558	0.211	0.143	0.054	0.030
40	0.687	0.367	0.286	0.153	0.105
45	0.776	0.508	0.429	0.281	0.218
50	0.845	0.639	0.571	0.432	0.365
55	0.904	0.764	0.714	0.604	0.546
60	0.955	0.884	0.857	0.794	0.758
65	1.000	1.000	1.000	1.000	1.000

The above tables illustrate the conclusion derived in the previous section, m(x) is associated with either a decelerating or an accelerating actuarial cost method.

When $p<1$, the values of m(x) s are high at the younger ages and low at the older ones.

When $p>1$, the values of m(x) s are low at the younger ages and high at the older ones.

In any case:

at the older ages, the corresponding m(x) values increase significantly as the value of p increases

the M(x) development reflects the m(x) trend. In table 3.2 we may observe that M(x) increases at a faster pace when $p<1$ and at a slower pace when $p>1$.

Under Uniform distribution $p = 1$, m(x) is constant throughout the age interval, since it is independent of the age of the insured; accordingly, the M(x) development is a linear function of age.

The results of tables 3.1 and 3.2 are described in figures 3.1 and 3.2 respectively:

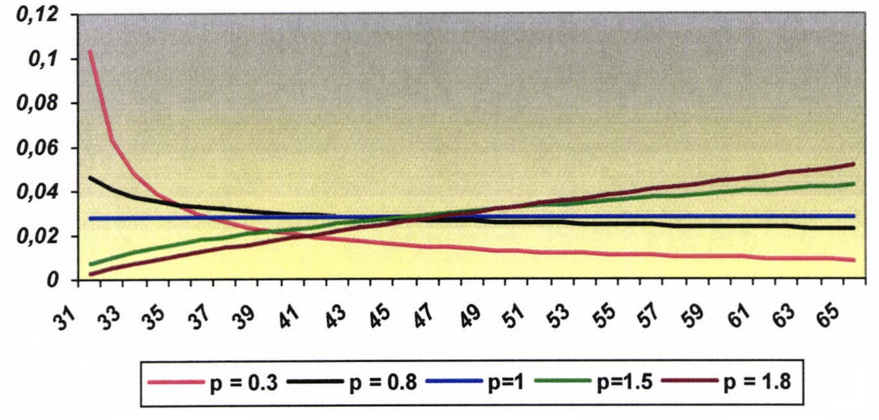

Figure 3.1 m(x) s development under the different Power function parameter values

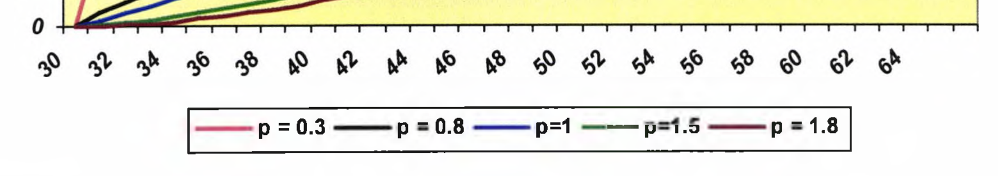

Figure 3.2 M(x) s development under the different Power function parameter values

- Truncated Pareto

For the parameter 'k' we set the values k = 0.3, k = 0.8 and k = 1.5. The results, at specific ages, are presented in tables 3.3 and 3.4:

Table 3.3 *m(x) development under Truncated Pareto*

age	**k = 0.3**	**k = 0.8**	**k = 1.5**
35	0.040	0.044	0.050
40	0.033	0.034	0.035
45	0.029	0.028	0.026
50	0.025	0.023	0.020
55	0.022	0.019	0.016
60	0.020	0.017	0.013
65	0.018	0.014	0.011

Table 3.4 *M(x) development under Truncated Pareto*

age	**k = 0.3**	**k = 0.8**	**k = 1.5**
35	0.218	0.252	0.301
40	0.399	0.446	0.511
45	0.553	0.601	0.664
50	0.686	0.727	0.780
55	0.803	0.833	0.870
60	0.907	0.923	0.942
65	1.000	1.000	1.000

Since m(x) is associated with a decelerating cost method, the values of m(x) in table 3.3 are higher at the younger ages and lower at the older ones. Accordingly, as k increases, the values at the younger ages become higher, and therefore the values at the older ages become lower. Since m(x) integrates to 1 over the age range this effect follows naturally from this feature.

The M(x) development in table 3.4 reflects the m(x) s trend.

The results of tables 3.3 and 3.4 are described in the figures 3.3 and 3.4 respectively:

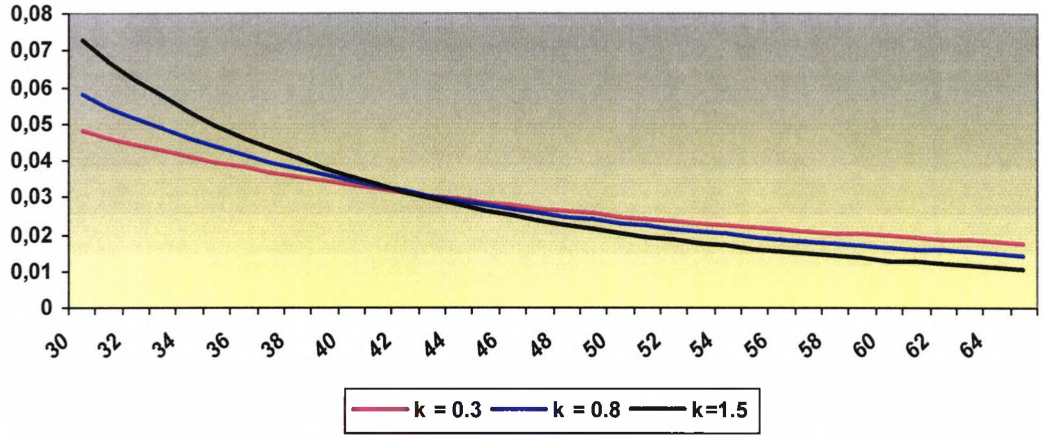

Figure 3.3 m(x) s development under the different Truncated Pareto parameter values

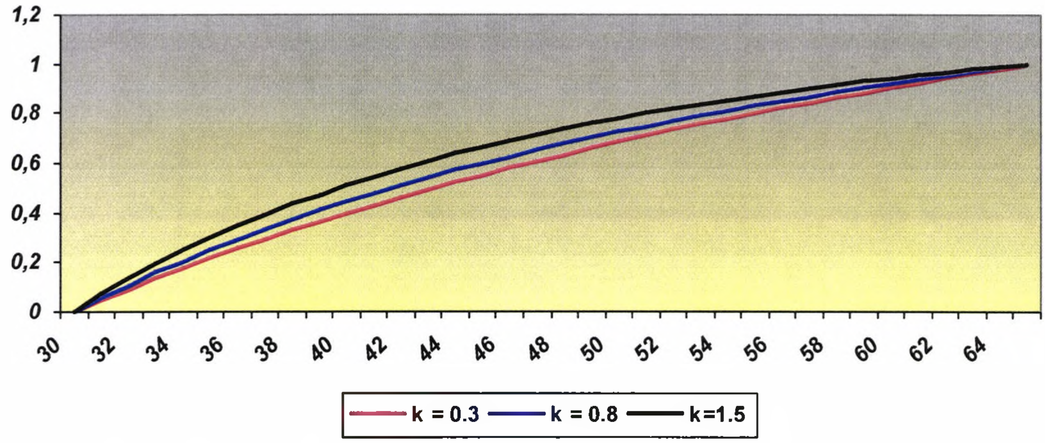

Figure 3.4 M(x) s development under the different Truncated Pareto parameter values

- Truncated Exponential

For the parameter, σ, we set the values σ = 30, σ = 40 and σ = 50.

The results, at specific ages, are presented in the following tables, 3.5 and 3.6:

Table 3.5 *m(x) development under Truncated Exponential*

age	**σ = 30**	**σ = 40**	**σ = 50**
35	0.041	0.038	0.036
40	0.035	0.033	0.033
45	0.029	0.029	0.029
50	0.025	0.026	0.027
55	0.021	0.023	0.024
60	0.018	0.020	0.022
65	0.015	0.018	0.020

Table 3.6 *M(x) development under Truncated Exponential*

age	**σ = 30**	**σ = 40**	**σ = 50**
35	0.223	0.202	0.189
40	0.412	0.379	0.360
45	0.571	0.536	0.515
50	0.707	0.675	0.655
55	0.821	0.797	0.782
60	0.918	0.905	0.896
65	1.000	1.000	1.000

Since m(x) is associated with a decelerating cost method, the values of m(x) in table 3.5 are higher at the younger ages and lower at the older ones. Accordingly, as σ increases, the values at the younger ages become lower and therefore the values at the older ages become higher.

The M(x) development in table 3.6 reflects the m(x) s trend.

The results of tables 3.5 and 3.6 are described in figures 3.5 and 3.6 respectively:

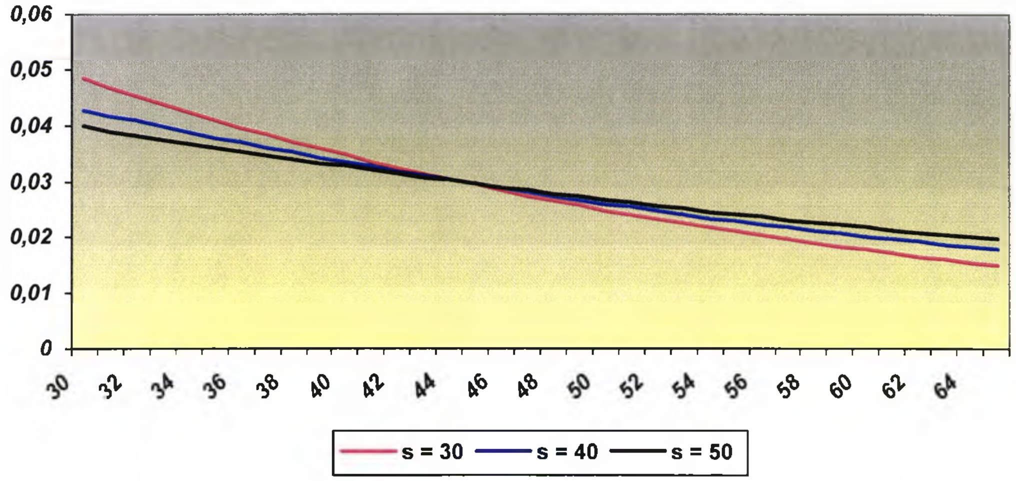

Figure 3.5 m(x) s development under the different Truncated Exponential parameter values

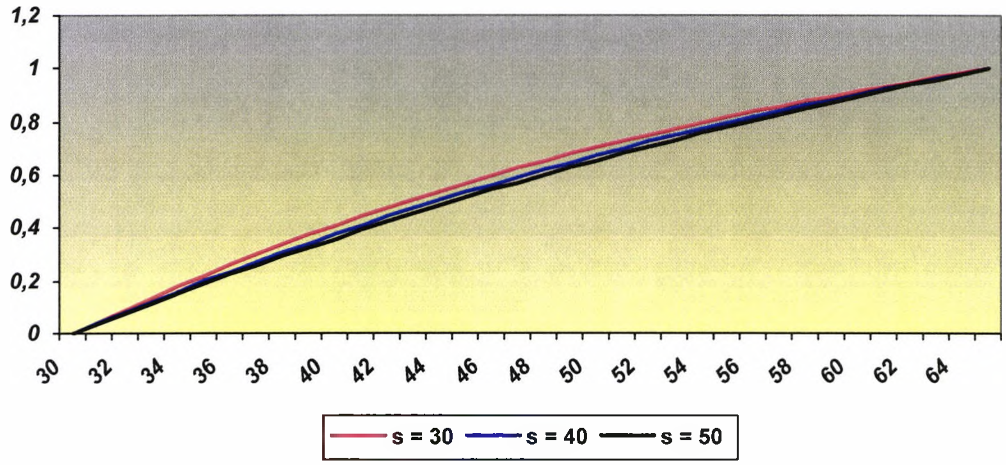

Figure 3.6 M(x) s development under the different Truncated Exponential parameter values

3.4.2 COMPARISON UNDER DIFFERENT CHOICES of m(x)

The comparison of m(x) s under different distributions was made for the age interval

$[a = 30, r = 65]$.

As far as the parameter values of Power function p, Truncated Pareto k, and Truncated Exponential σ are concerned, they have been set equal to:

Power function: p = 0.3, 0.8

Truncated Pareto: k = 0.3, 0.8, 1.5

Truncated Exponential : $\sigma = 30, 40, 50$

Since, for the Truncated Pareto and the Truncated Exponential, the m(x) are categorized as decelerating actuarial cost methods, for comparison purposes we choose the Power function parameter values to be less than 1.

In tables 3.7 and 3.8, the values of m(x) s and M(x) s are presented, at specific ages:

Table 3.7 *m(x) development under different distributions*

age	Power p = 0.3	Power p = 0.8	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	Pareto k = 1.5
35	0.033	0.034	0.041	0.038	0.036	0.040	0.044	0.050
40	0.021	0.029	0.035	0.033	0.033	0.033	0.034	0.035
45	0.016	0.027	0.029	0.029	0.029	0.029	0.028	0.026
50	0.013	0.026	0.025	0.026	0.027	0.025	0.023	0.020
55	0.011	0.024	0.021	0.023	0.024	0.022	0.019	0.016
60	0.010	0.024	0.018	0.020	0.022	0.020	0.017	0.013
65	0.009	0.023	0.015	0.018	0.020	0.018	0.014	0.011

In table 3.7, we observe that the lowest values of the m(x) s associated with a decelerating cost method are calculated under the Power function p =0.3. Truncated Pareto, k = 1.5, gives very low m(x) values after age 50 and considerably higher values up to age 40. m(x) values under Truncated Exponential $\sigma = 40$ and Truncated Pareto k = 0.3 are very close to each other. As expected, since $\int_a^r m(x)\,dx = 1$, the low values of m(x) s at the older ages imply that M(x) s rapidly accumulate to 1, as it is observed below, in table 3.8. This is clearly seen in the case of Power function under p = 0.3, since the values of m(x) s are significantly lower than those under all other distribution functions from age 35 and thereafter .

The results of table 3.7 are also described below in figure 3.7

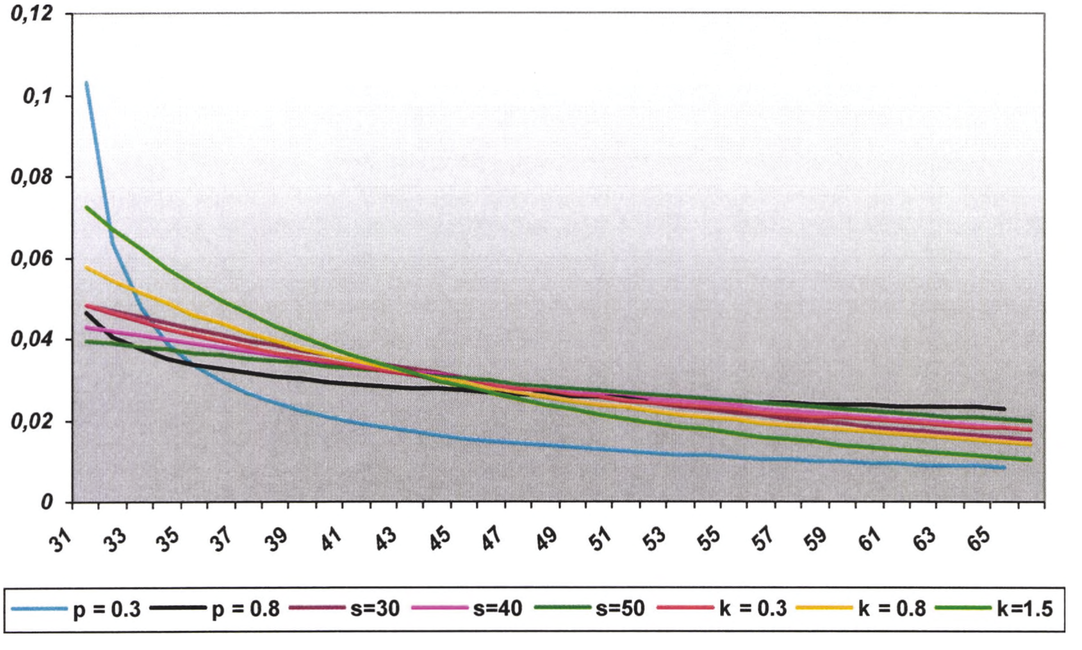

Figure 3.7 m(x) s development under the different distributions parameter values

__Table 3.8__ M(x) development under different distributions

age	Power p = 0.3	Power p = 0.8	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	Pareto k = 1.5
35	0.558	0.211	0.223	0.202	0.189	0.218	0.252	0.301
40	0.687	0.367	0.412	0.379	0.360	0.399	0.446	0.511
45	0.776	0.508	0.571	0.536	0.515	0.553	0.601	0.664
50	0.845	0.639	0.707	0.675	0.655	0.686	0.727	0.780
55	0.904	0.764	0.821	0.797	0.782	0.803	0.833	0.870
60	0.955	0.884	0.918	0.905	0.896	0.907	0.923	0.942
65	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

The results of table 3.8 are also described below in figure 3.8

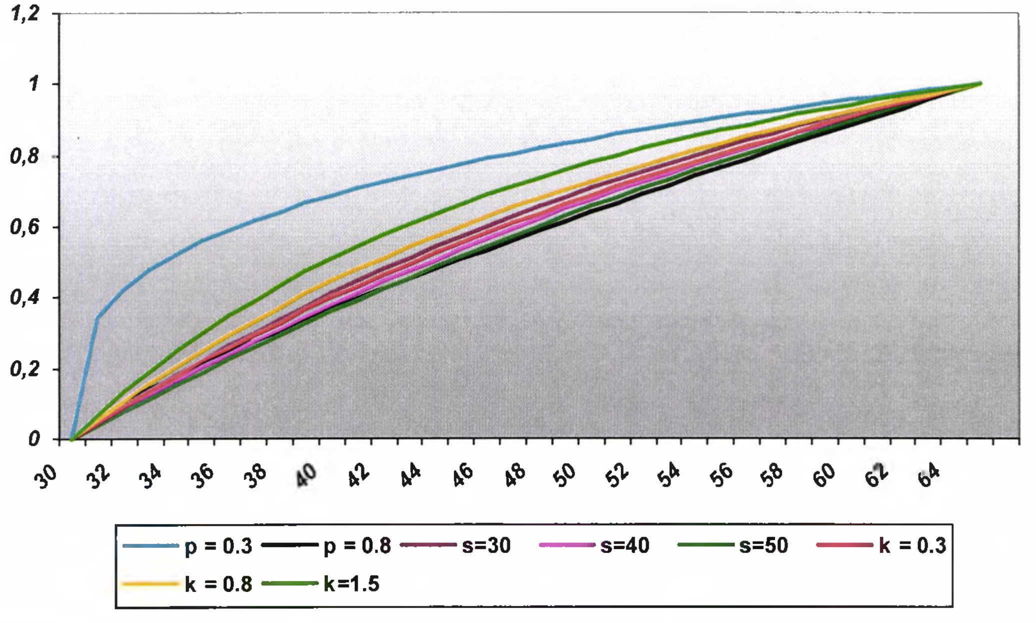

Figure 3.8 M(x) s development under the different distributions parameter values

The actuary has to decide how quickly to build up the fund (in the absence of variations from expected experience) i.e. at a lower or at a higher pace on the basis of the annual portion of benefit purchased. If, for example, the average age of the scheme members is young, he/she may prefer to build up the fund at a lower pace during the early years and at a higher thereafter as they get closer to retirement. The choice of the distribution depends on that decision. Thus, the set of different distributions associated with either cost method gives the actuary the flexibility to choose, among them, the one that increases the fund level at a higher pace. Between, for example, the pension accrual functions that are associated with a decelerating cost method, he/she knows in advance that the fund will build up at a higher pace if the Truncated Pareto is applied than if the Power function, $p<1$ is applied. The parameter of the distribution chosen affects as well the pace of the fund. If, for example, the Truncated Pareto is considered, the actuary knows in advance that the fund will build up at a higher pace if its parameter value is equal to 1.5 (or higher) than if it is equal to 0.3 (or lower).

Summarizing the above, we conclude that the different accrual functions as well as the different sets of parameters applied allow the actuary some flexibility regarding the pace of building up the fund within the restrictions he/she has already set on the development of its level.

In tables 3.1-3.6, we have focused on the different distributions which we have considered for the pension plan implementation; in tables 3.7 – 3.8 we have compared these. The comparison between the traditional and new funding methods is presented in section 4.9.

3.5 CONCLUSIONS

The age of the plan participant may be considered in the implementation of the plan through the pension accrual function m(x). On that basis, the actuary has the flexibility to build up the fund either at a higher or at a lower pace.

We think that it is useful to clarify that, in practice, when we propose a new structure for the plan implementation our aim is to give more tools to the actuary to proceed. The actuary's main task is to keep the right balance between a sufficiently funded scheme and the sponsor's financial plans. This balance is difficult to be kept today due to the flexible forms of employment and job mobility. From their respective viewpoints, scheme sponsors care for low costs and scheme participants care for their benefit payments when due. However, we cannot ignore that the modernisation of pension schemes implies more transparency. On that basis, it will become possible to provide better information to the scheme participants about what they can expect from their pension system and what additional effort may be required to achieve the desired living standard after retirement. If, as a consequence, the scheme participants were to come to the position of deciding on a cost method, then their choice for the method that better fits to their age and expectations will depend on the benefits provided by the plan benefit rules.

If the benefit is a pension, then young participants usually prefer to buy higher portions of benefit as they become older since it is likely that they will change jobs and thus they may not be willing to pay high contributions while starting their career[9]. Old participants may prefer to buy lower portions of benefits as their age increases. One reason behind this is the fact that, as they get older, and thus closer to retirement, their salaries increases and as a consequence so do their contributions. If they also decide to buy higher proportion of benefits, then they increase their contributions even more; even if they could afford to pay more, they may prefer to avoid very high amounts. Another reason may result from plan benefits rules which often allow older members with high levels of past service in the plan to acquire both vested rights and a percentage of their retirement benefits, if they were to retire earlier, paid immediately upon the date of the early retirement. Hence, older members prefer to accrue, in advance, higher portions of their benefits for the case where they exercise one of these rights.

Sometimes, according to the plan rules, the choice of a lump sum payment is provided at retirement. This may provide some assistance in the case of need in order to meet some unexpected expenses. On this basis, young participants may prefer to pay higher amount of contributions along as their age increases.

Summarizing, we conclude that there is not a certain rule that either the actuary or the scheme participants should follow for the choice of the cost method which heavily depends on the benefits provided by the pension plan, the sponsor's financial plans, and the economic environment.

[9] Some people may argue that, even if young participants pay high amounts, in the case that they move to an other job, the accrued amount of their contributions is returned to them with interest. However, this is not always the case since very often, according to the plan benefit rules less than 100% of the contributions are returned to them and not the whole amount. This strengthens their choice to pay low contributions at the start.

From all distributions chosen, the Power function has the special property that allows m(x) to be associated with either a decelerating or an accelerating cost method, on the basis of the adjustment of the parameter 'p' value.

The comparison between the different methods is on the basis of the portion of benefit purchased throughout the participant's active years. The different distributions (and the set of parameters investigated) show how the portion of benefit varies along with the age of the scheme participant. If it is considered more appropriate to increase the contribution rate along with the age of the plan participant, the Power function with $p>1$ has to be used. Otherwise, we may select either the Power function with $p<1$ or one of Truncated Pareto or Truncated Exponential.

We conclude that, if it becomes necessary to change the level of contributions in a pension scheme, then there is not a unique process; that of applying uniformly to all scheme participants the corresponding rate that either increases or decreases normal cost. The actuary may apply different rates on the basis of the age of the participant determined by his/her choice of the 'appropriate' distribution and the 'appropriate' parameter values. However, this is not always the case in the sense that occasionally there are various reasons that prevent the actuary from proceeding with the 'right choice'. The actuary then is obliged to proceed with the 'best choice' i.e. with the method that is closest to that initially considered as the most appropriate. The analysis in terms of appropriate accrual functions and appropriate parameters that we have presented gives him / her the tools to proceed accordingly.

APPENDIX 4

- If X ~ Power function (p, k) then X^{-1} ~ Pareto (p , $\frac{1}{k}$)

<u>Proof</u>:

Let V = X^{-1}

$$f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv} P\{V \le v\} = \frac{d}{dv} P\{X^{-1} \le v\} = \frac{d}{dv} P\{X \ge \frac{1}{v}\} =$$

$$= \frac{d}{dv}\left[1 - F_X\left(\frac{1}{v}\right)\right] = - f_X\left(\frac{1}{v}\right) * \left(\frac{1}{v}\right)' = \begin{cases} \left(\frac{1}{v}\right)^2 * p * \frac{\left(\frac{1}{v}\right)^{p-1}}{k^p}, & p>0,\ 0 \le \frac{1}{v} \le k \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} p * v^{-(p+1)} * (x_0)^p, & p>0,\ v \ge x_0 = \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

- If X ~ Pareto (p , x_0), then ln($\frac{X}{x_0}$) ~ Exponential (p)

<u>Proof</u>

Let V = ln($\frac{X}{x_0}$)

$$f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv} P\{V \le v\} = \frac{d}{dv} P\{\ln\left(\frac{X}{x_0}\right) \le v\} = \frac{d}{dv} P\{\frac{X}{x_0} \le e^v\} =$$

$$= \frac{d}{dv} P\{X \le e^v * x_0\} =$$

$$= \frac{d}{dv}[F_X(e^v * x_0)] = f_X(e^v * x_0) * (e^v * x_0)' = \begin{cases} p*(e^v * x_0)^{-(p+1)} * e^v * x_0 * (x_0)^p & p>0, v \geq x_0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} p * e^{-v*p} & p>0, v \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If X ~ U[0,1] then –lnX ~ Exponential (1)

Proof:

Let V = -lnX

$$f_V(v) = \frac{d}{dv}F_V(v) = \frac{d}{dv}P\{V \leq v\} = \frac{d}{dv}P\{-\ln X \leq v\} = \frac{d}{dv}P\{\ln X \geq -v\} = \frac{d}{dv}P\{X \geq e^{-v}\} =$$

$$= \frac{d}{dv}[1 - F_X(e^{-v})] = -f_X(e^{-v}) * (e^{-v})' = \begin{cases} 1*e^{-v}, & 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-v}, & v \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

CHAPTER 4

A NEW PENSION PLAN MODEL

4.1 INTRODUCTION

This section includes the 'new cost methods' that are implemented through the pension density functions defined in chapter 3. These methods are regarded as individual. Based on a deterministic model, the pension funding functions are described and compared for each of the p.d.fs. using the continuous time formulation.

The behavior of the Actuarial Liability, the Normal Cost and the fund value F(t) over time are examined. In this regard, an important equation is the following:

$$\frac{dF(t)}{dt} = (\delta-\lambda)*F(t) + NC(t) - B(t) + \lambda*AL(t)$$

4.2 DERIVATION OF THE PENSION FUNDING FUNCTIONS

In the model pension scheme, we consider an active group which extends over the ages a to r, with all new entrants coming in at age a and all retirements occurring at age r; and only retirement benefits are considered. For both the active and the retired participants, survivorship is in accordance with the function l_x, which does not depend on the time variable t. Salaries at time t = 0 are represented by the function s(x), and thereafter they increase exponentially; this assumption establishes a year-of-experience pattern of growth for salaries. Initial pensions are a fixed percentage, b, of final salaries and increase during retirement by a factor β(x) which is used to denote the adjustment of the initial pension at age of retirement r, of a retiree age x; x > r ,$\beta(x) = e^{\beta*(x-r)}$. For $a \leq x \prec r$, the density of new pensions to be incurred at time t +r-x in respect to the survivors of participants aged x at time t is given by the function h(t + r – x). For x $\geq$ r, h(t + r – x) is the density of new pensions incurred at time t for those who were then aged r and who may or may not be surviving at age x at time t.

The Normal Cost rate NC_x in regard to a participant aged x for B_r units of initial pension from age r is given in the continuous case by the following formula:

$$NC_x = \begin{cases} B_r * m(x) * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} & , a \leq x \prec r \\ 0 & , x > r \end{cases} \qquad \textbf{(4.1)}$$

The accrued liability in regard to a participant aged x for B_r units of initial pension from age r is given in the continuous case by the following formula:

$$AL_x = \begin{cases} B_r * M(x) * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)}, & a \le x \le r \\ B_r * \bar{a}_x^{(\delta-\beta)} * e^{\beta*(x-r)}, & x > r \end{cases} \quad \textbf{(4.2)}$$

The annual rate of plan Normal Cost at time t, equals to:

$$NC(t) = \int_a^r h(t+r-x)*m(x) * e^{-\delta*(r-x)} * \bar{a}_r^{(\delta-\beta)}\, dx \quad \textbf{(4.3),}$$

Assuming for the model plan that pensions are a flat percentage, b, of final salary, then: $h(t+r-x) = e^{\tau*(t+r-1-x)}*l_r * b*s_a$. Rewriting formula (4.3) based on (4.1) $\Rightarrow$

$$\Rightarrow NC(t) = \int_a^r h(t+r-x) * \frac{l_x}{l_r} *(\frac{1}{B_r})* NC_x\, dx = e^{\tau*t} * NC(0) \quad \textbf{(4.4)}$$

The annual rate of plan Accrued Liability at time t, equals to:

$$AL(t)= \int_a^r h(t+r-x)*M(x)*e^{-\delta*(r-x)}* \bar{a}_r^{(\delta-\beta)}\, dx + \int_r^{\omega} h(t+r-x)* \frac{l_x}{l_r} * \bar{a}_x^{(\delta-\beta)}\, e^{\beta*(x-r)}dx \quad \textbf{(4.5),}$$

where the second term represents the value of future pension payments for those participants already retired at time t.

Given that M(x) = 1, for $x \ge r$, then using (4.2),

$$AL(t) = \int_a^{\omega} h(t+r-x)* \frac{l_x}{l_r} * (\frac{1}{B_r})* AL_x\, dx = e^{\tau*t} * AL(0) \quad \textbf{(4.6)}$$

The density of pensions for participants aged x at time t, $x \ge r$, is determined: a) by the density of new pensions at time t + r – x, b) by survivorship from age r to age x, and c) by the adjustment factor β(x). Hence this density is given by the formula: $h(t+r-x)*_{x-r}p_r*\beta(x)$. It then follows that the annual rate of pension outgo is given in the continuous case by the following formula:

$$B(t) = \int_r^{\omega} h(t+r-x) * {}_{x-r}p_r * \beta(x)\, dx = e^{\tau*t} * B(0) \quad \textbf{(4.7)}$$

The Unfunded Liability at time t, equals to:

$U(t) = AL(t) - F(t)$ **(4.8)**

The Contribution rate at time t, equals to:

$C(t) = NC(t) + \lambda(t)*U(t)$ **(4.9)**

As far as the Fund value at time t is concerned, Bowers et al (1976) show that

$$\frac{dF(t)}{dt} = NC(t) + (\delta-\lambda) * F(t) + \lambda * AL(t) - B(t) \quad \textbf{(4.10)}$$

Solving the differential equation (4.10) we have:

$$F(t) = F(0)*e^{(\delta-\lambda)*t} + e^{\tau*t}\, \overset{-(\lambda-\delta+\tau)}{\bar{a}_{\bar{t}|}}\ (NC(0) + \lambda*AL(0) - B(0))$$ **(4.11)** (see Appendix 5)

Setting the initial Fund value at time t=0, F(0), equal to 0, (4.11) becomes:

$$F(t) = e^{\tau*t} * \overset{-(\lambda-\delta+\tau)}{\bar{a}_{\bar{t}|}} *(NC(0) + \lambda*AL(0) - B(0)) \quad \textbf{(4.12)}$$

Let $r_C = (NC(0) + \lambda*AL(0) - B(0))$.

At time t = 0, the pension scheme is funded by r_c which represents the contributions made at that time, $(NC(0) + \lambda*AL(0))$, minus the benefits paid also at that time. Equation (4.12) shows that the Fund value at time t equals to the present value of an annuity certain payable continuously for t years, at an interest rate equal to $\lambda-\theta$, where θ allows for the difference between the force of investment return and salary increase.

From equation (4.11) we have:

$$F(t) - F(0) * e^{(\delta-\lambda)*t} * e^{-\tau*t} = \overset{-(\lambda-\delta+\tau)}{\bar{a}_{\bar{t}|}} *(NC(0) + \lambda*AL(0) - B(0)) \Rightarrow$$

$$\Rightarrow \frac{F(t) - F(0)*e^{(\delta-\lambda)t}}{e^{\tau t} * AL(0)} = \overset{-(\lambda-\delta+\tau)}{\bar{a}_{\bar{t}|}} * \left(\frac{NC(0) - B(0)}{AL(0)} + \lambda\right) \text{ or}$$

$$\frac{F(t) - F(0)*e^{(\delta-\lambda)t}}{AL(t)} = \overset{-(\lambda-\delta+\tau)}{\bar{a}_{\bar{t}|}} * \left(\frac{NC(0) - B(0)}{AL(0)} + \lambda\right) \quad \textbf{(4.13)}$$

Setting the initial Fund value at time t=0, F(0), equal to 0, (4.13) becomes:

$$\frac{F(t)}{AL(t)} = \bar{a}_{\overline{t}|}^{(\lambda-\delta+\tau)} * \left(\frac{NC(0)-B(0)}{AL(0)} + \lambda\right) \quad \textbf{(4.14)}$$

Equation (4.14) shows that the ratio of the Fund value over the Actuarial Liability at time t equals to the present value of an annuity certain payable continuously for t years, (at an interest rate equal to $\lambda - \theta$, $\theta = (\delta - \tau)$), such that: the total amount paid during each year equals to the sum of: The ratio of Normal Cost at time t=0 minus any benefits due at that time over the Actuarial Liability at time t=0, and the rate of amortizing the Unfunded Liability.

4.3 INTRODUCTION of the FUNCTION x(θ)

Bowers et al (1979) define the annual rate of terminal funding[10] cost for the plan at time t, as the rate at which, at time t, the actuarial present value of future pensions for members reaching age r is incurred. It could then serve as a building block and as a standard of comparison for other contribution patterns.

On the basis of this rate, Bowers et al display the function that, for a continuous model, allocates the actuarial present value of future pension benefits to the various valuation times in a participant's active life. In particular, assuming that an actuarial cost method with an accrual function M(x) is selected, they calculate the Normal Cost on the basis of the annual rate of terminal funding cost. Applying this result for the case where salaries are changing exponentially and pensions are also adjusted exponentially, they show that the annual Normal Cost rate at time t, is sufficient with interest to provide the terminal funding cost for r - $x(\theta)$ years later.

[10] Under the terminal funding methods, pensions are not funded by contributions during active membership. Instead, single contributions. Instead single contributions are made to the fund at the time of retirement, see section 1.1.

Hence, the function thus $x(\theta)$ is introduced, and it is interpreted as an average age of Normal Cost payment associated with the Actuarial Cost Method defined by m(x) and the combination of interest, population and salary forces $\theta = \delta - \tau$.

In the exponential growth case $x(\theta)$ is calculated from the equation:

$$e^{\theta * x(\theta)} = \int_a^r e^{\theta * x} * m(x)\,dx .$$

For the exponential growth case, Bowers et al proposed an amortization process on the basis of $x(\theta)$ by defining a mean temporary annuity value $\bar{a}(t)$ equal to the ratio of the present value at time t of future Normal Costs of the plan over the annual rate of Normal Cost of the plan at time t. Finding also a more specific expression for $\bar{a}(t)$ in the exponential case, they simplified $\bar{a}(t)$ to a function independent of t and equal to:

$$\bar{a}(t) = \frac{\int_a^r e^{(\delta-\tau)^* x} * (1 - M(x))dx}{\int_a^r e^{(\delta-\tau)^* x} * m(x)dx}$$

In that process $\lambda = \dfrac{1}{\bar{a}(t)} = \dfrac{\int_a^r e^{(\delta-\tau)^* x} * (1 - M(x))dx}{\int_a^r e^{(\delta-\tau)^* x} * m(x)dx} = \dfrac{1}{\bar{a}^{-\theta}_{\overline{x(\theta)-a}|}}$ **(4.15)**

where $\theta = \delta - \tau$,(see Appendix 6)

In the special case, where $\theta = 0$, $\lambda = \dfrac{1}{\mu - a}$ **(4.16)**

where $\mu = \int_a^r x * m(x)\,dx$, (see Appendix 6)

Since we have considered the exponential growth case, we considered the range for the parameter that amortizes the unfunded liability under the selected set of the accrual functions. In addition, the number of years m, over which the unfunded liability could spread is calculated using the equation $\lambda = \dfrac{1}{\bar{a}_{\overline{m}|}}$. We consider this approach as the first step for estimating the pace of the amortization of the Actuarial Liability on the basis of different λ values from a specified range interval. Illustrative examples are presented in Appendix 7; λ is calculated with respect to all pension density functions we have discussed in chapter 3.

Considering λ as above, the equation of the fund at time t, F(t) is then adjusted accordingly. In particular, replacing λ[11] in each one of the equations 4.11 – 4.14, we have:

$$F(t) = F(0) * e^{(\delta-\lambda)*t} + e^{\tau*t} * \overset{-(\lambda-\theta)}{a_{\bar{t}|}} * (NC(0) + AL(0) * \frac{1}{\overset{-\theta}{a}_{\overline{x(\theta)-a}|}} - B(0)) \quad \textbf{(4.17)}$$

$$\frac{F(t) - F(0) * e^{(\delta-\lambda)t}}{AL(t)} = \overset{-(\lambda-\delta+\tau)}{a_{\bar{t}|}} * (\frac{NC(0) - B(0)}{AL(0)} + \frac{1}{\overset{-\theta}{a}_{\overline{x(\theta)-a}|}}) \quad \textbf{(4.18)}$$

Setting the initial Fund value at time t=0, F(0), equal to 0, then

$$F(t) = e^{\tau*t} * \overset{-(\lambda-\theta)}{a_{\bar{t}|}} * (NC(0) + AL(0) * \frac{1}{\overset{-\theta}{a}_{\overline{x(\theta)-a}|}} - B(0)) \quad \textbf{(4.19)}$$

$$\frac{F(t)}{AL(t)} = \overset{-(\lambda-\delta+\tau)}{a_{\bar{t}|}} * (\frac{NC(0) - B(0)}{AL(0)} + \frac{1}{\overset{-\theta}{a}_{\overline{x(\theta)-a}|}}) \quad \textbf{(4.20)}$$

[11] λ value is not shown neither in the annuity rate nor in the exponent, for presentation purposes.

4.4 APPLICATION of the DENSITY FUNCTIONS m(x)

Since we have considered the exponential growth case, we have decided to revisit the results of Bowers et al, after applying the selected set of the accrual functions of the new cost methods. For each distribution function, F(t) is then specified as follows:

- Power function

Given that under the Power distribution:

$$\bar{a}^{\theta}_{\overline{x(\theta)-a|}} = \begin{cases} \frac{1}{\theta} * (1 - e^{a*\theta} * \frac{(r-a)^p}{p * \int_a^r e^{\theta x} * (x-a)^{p-1} dx}) , & \theta \neq 0 \\ p * \frac{r-a}{p+1} , & \theta = 0 \end{cases}$$

(see Appendix 4.7)

equation (4.18) becomes:

$$\frac{F(t) - F(0) * e^{(\delta-\lambda)t}}{AL(t)} = \bar{a}^{(\lambda-\delta+\tau)}_{\overline{t|}} * (\frac{p *^P S_{NC} - {}^P R * S_B}{{}^P S_{AL} + {}^P R * (\frac{\bar{a}^{-(\tau-\beta)}_{r} - \bar{a}^{-(\delta-\beta)}_{r}}{\delta - \tau})} + $$

$$+ (\frac{1}{\theta} * (1 - e^{\alpha\theta} * \frac{(r-a)^p}{p * \int_a^r e^{\theta x} * (x-a)^{p-1} dx}))^{-1}), \theta \neq 0 \quad \textbf{(4.21) or}$$

$$\frac{F(t) - F(0) * e^{(\delta-\lambda)t}}{AL(t)} = \bar{a}^{(\lambda-\delta+\tau)}_{\overline{t|}} * (\frac{p *^P S_{NC} - {}^P R * S_B}{{}^P S_{AL} + {}^P R * (\frac{\bar{a}^{-(\tau-\beta)}_{r} - \bar{a}^{-(\delta-\beta)}_{r}}{\delta - \tau})} + (p \frac{r-a}{p+1})^{-1}), \theta = 0 \quad \textbf{(4.22)}$$

where, ${}^{P}S_{NC} = \int_0^{r-a} e^{(\delta-\tau)*y} * y^{p-1}\,dy$, ${}^{P}S_{AL} = \int_0^{r-a} e^{(\delta-\tau)*y} * y^{p}\,dy$,

$S_B = \int e^{\tau*(r-x)}\,{}_{x-r}p_r * \beta(x)\,dx$ and ${}^{P}R = e^{-(\tau-\delta)*(r-1-a)} * \frac{(r-a)^p}{\bar{a}_r^{-(\delta-\beta)}}$

- Uniform

In the case p = 1, X ~ Uniform distribution

Given that under Uniform distribution:

$$\bar{a}^{\theta}_{\overline{x(\theta)-a|}} = \begin{cases} \frac{1}{\theta} * (1 - \frac{r-a}{\bar{s}^{\theta}_{\overline{r-a|}}}),\ \theta \neq 0 \\ \frac{r-a}{2}, \quad \theta = 0 \end{cases}$$ (see Appendix 7)

equation (4.18) becomes:

$$\frac{F(t)-F(0)*e^{(\delta-\lambda)t}}{AL(t)} = \bar{a}^{(\lambda-\delta+\tau)}_{\overline{t|}} \left(\frac{\bar{a}^{-(\delta-\tau)}_{\overline{r-a|}} - {}^{U}R * S_B}{{}^{U}S_{AL} + {}^{U}R * (\frac{\bar{a}_r^{-(\tau-\beta)} - \bar{a}_r^{-(\delta-\beta)}}{\delta-\tau})} + \left(\frac{1}{\theta}(1-\frac{r-a}{\bar{s}^{\theta}_{\overline{r-a|}}})\right)^{-1}\right) \theta \neq 0 \quad \textbf{(4.23)}$$

or $$\frac{F(t)-F(0)*e^{(\delta-\lambda)t}}{AL(t)} = \bar{a}^{(\lambda-\delta+\tau)}_{\overline{t|}} * \left(\frac{\bar{a}^{-(\delta-\tau)}_{\overline{r-a|}} - {}^{U}R * S_B}{{}^{U}S_{AL} + {}^{U}R * (\frac{\bar{a}_r^{-(\tau-\beta)} - \bar{a}_r^{-(\delta-\beta)}}{\delta-\tau})} + (\frac{r-a}{2})^{-1}\right) \theta=0 \quad \textbf{(4.24)}$$

where, ${}^{U}S_{AL} = (r-a) - \bar{a}^{-(\delta-\tau)}_{\overline{r-a|}} * \frac{1}{\delta-\tau}$, $S_B = \int e^{\tau*(r-1-x)} * {}_{x-r}p_r * \beta(x)\,dx$, ${}^{U}R = \frac{r-a}{\bar{a}_r^{-(\delta-\beta)}}$

- Truncated Pareto

Given that under Truncated Pareto:

$$a_{\overline{x(\theta)-a|}}^{-\theta} = \begin{cases} \frac{1}{\theta}*(1-e^{\alpha\theta}*\frac{1-(\frac{a}{r})^{k}}{k*a^{k}*\int_{a}^{r} e^{\theta*x}*x^{-(k+1)}dx}),\ \theta \neq 0 \\ \frac{k*a^{k}*(r^{1-k}-a^{1-k})}{(1-k)*(1-(\frac{a}{r})^{k})} - a,\ \theta = 0 \end{cases}$$

(see Appendix 7)

equation (4.18) becomes:

$$\frac{F(t)-F(0)*e^{(\delta-\lambda)t}}{AL(t)} = a_{\overline{t|}}^{-(\lambda-\delta+\tau)}*(\frac{k*{}^{TP}S_{NC}-{}^{TP}R*S_{B}}{{}^{TP}S_{AL}+Q+{}^{TP}R*\frac{a_{r}^{-(\tau-\beta)}-a_{r}^{-(\delta-\beta)}}{\delta-\tau}}+$$

$$+(\frac{1}{\theta}*(1-e^{\alpha\theta}*\frac{1-(\frac{a}{r})^{k}}{k*a^{k}*\int_{a}^{r} e^{\theta*x}*x^{-(k+1)}dx}))^{-1}),\theta \neq 0 \qquad \textbf{(4.25)}$$

or $$\frac{F(t)-F(0)*e^{(\delta-\lambda)t}}{AL(t)} = a_{\overline{t|}}^{-(\lambda-\delta+\tau)}*(\frac{k*{}^{TP}S_{NC}-{}^{TP}R*S_{B}}{{}^{TP}S_{AL}+Q+{}^{TP}R*\frac{a_{r}^{-(\tau-\beta)}-a_{r}^{-(\delta-\beta)}}{\delta-\tau}}+$$

$$+(\frac{k*a^{k}*(r^{1-k}-a^{1-k})}{(1-k)*(1-(\frac{a}{r})^{k})} - a)^{-1}), \quad \theta=0 \qquad \textbf{(4.26)}$$

where ${}^{TP}S_{NC} = \int_{a}^{r} e^{a*(\delta-\tau)*y}*y^{-(k+1)}dy$, ${}^{TP}S_{AL} = \int_{a}^{r} e^{a*(\delta-\tau)*y}*y^{-k}dy$

$S_B = \int^{w} e^{\tau*(r-x)}*{}_{x-r}p_r*\beta(x)\,dx$, ${}^{TP}R = e^{-(\tau-\delta)*r}\frac{1-(\frac{a}{r})^{k}}{a_{r}^{-(\delta-\beta)}*a^{k}}$, $Q = e^{-(\tau-\delta)*r}*a^{-k}*a_{\overline{r-a|}}^{-(\delta-\tau)}(1-(\frac{a}{r})^{k})$

- Truncated Exponential

Given that under Truncated Exponential:

$$\frac{1}{\theta} * (1 - e^{\alpha\theta} * \frac{\sigma * (1 - e^{-\frac{r-a}{\sigma}})}{e^{\frac{a}{\sigma}} * \int_a^r e^{\theta x} * e^{-\frac{x}{\sigma}} dx}) = \frac{1}{\theta} * (1 - \frac{a_{\overline{r-a}|}^{-(\frac{1}{\sigma})}}{s_{\overline{r-a}|}^{-(\theta-\frac{1}{\sigma})}}) \text{ , } \theta \neq 0$$

$$a_{\overline{x(\theta)-a}|}^{-\theta} = \Bigg\{$$

$$\sigma - \frac{r-a}{e^{\frac{r-a}{\sigma}} - 1}, \theta = 0$$ (see Appendix 7)

equation (4.18) becomes:

$$\frac{F(t) - F(0) * e^{(\delta-\lambda)t}}{AL(t)} = a_{\overline{t}|}^{-(\lambda-\delta+\tau)} * \Bigg(\frac{\sigma^{-1} * a_{\overline{r-a}|}^{-(\delta-\tau-\frac{1}{\sigma})} + {}^{TE}R * S_B}{(s_{\overline{r-a}|}^{-(\delta-\tau)} - a_{\overline{r-a}|}^{-(\delta-\tau-\frac{1}{\sigma})}) + {}^{TE}R * \frac{a_{\overline{r-a}|}^{-(\tau-\delta)} - a_{\overline{r-a}|}^{-(\delta-\beta)}}{\delta-\tau}} +$$

$$+ \Big(\frac{1}{\theta} * (1 - e^{\alpha\theta} * \frac{\sigma * (1 - e^{-\frac{r-a}{\sigma}})}{e^{\frac{a}{\sigma}} * \int_a^r e^{\theta x} * e^{-\frac{x}{\sigma}} dx}) \Big)^{-1} \Big), \theta \neq 0 \qquad \textbf{(4.27)}$$

or $$\frac{F(t) - F(0) * e^{(\delta-\lambda)t}}{AL(t)} = a_{\overline{t}|}^{-(\lambda-\delta+\tau)} * \Bigg(\frac{\sigma^{-1} * a_{\overline{r-a}|}^{-(\delta-\tau-\frac{1}{\sigma})} + {}^{TE}R * S_B}{(s_{\overline{r-a}|}^{-(\delta-\tau)} - a_{\overline{r-a}|}^{-(\delta-\tau-\frac{1}{\sigma})}) + {}^{TE}R * \frac{a_{\overline{r-a}|}^{-(\tau-\delta)} - a_{\overline{r-a}|}^{-(\delta-\beta)}}{\delta-\tau}} +$$

$$+ (\sigma - \frac{r-a}{e^{\frac{r-a}{\sigma}} - 1})^{-1}), \theta = 0 \qquad \textbf{(4.28)}$$

where, $S_B = \int e^{\tau*(r-1-x)} * {}_{x-r}p_r * \beta(x)\, dx$, ${}^{TE}R = \frac{s_{\overline{r-a}|}^{-(\sigma^{-1})} * \sigma^{-1}}{a_r^{-(\delta-\beta)}}$, We conclude that under the selected distribution functions we have derived exact mathematical solutions for the ratio of the fund value over the Actuarial Liability at time t. In particular, it is equal to a constant amount payable continuously for t years that can be exactly calculated for each one of the 'new cost methods'.

4.5 COMPARISON IN TERMS of NORMAL COST and ACTUARIAL LIABILITY

In section 2.4, we compared the traditional cost methods, for a participant aged x, on the basis of the Normal Cost and Actuarial Liability. In this section we will proceed to the same comparison for the 'new cost methods' as these are defined under the selected accrual functions discussed in chapter 3. As in the traditional methods, our calculations concern a participant who enters in the scheme at age 30 and retires at age 65. By implementing the formulae for the Normal Cost and Actuarial Liability according to the equations (4.1) and (4.2) for $a \leq x \leq r$ we obtain the NC_x and AL_x values presented in tables 4.1 and 4.2 for selected ages:

Table 4.1: *Normal Cost under the various cost methods*

age	Power p = 0.8	Power p = 1	Power p = 1.5	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	Pareto k = 1.5
35	0.07	0.06	0.04	0.09	0.08	0.08	0.09	0.10	0.11
40	0.08	0.08	0.07	0.10	0.09	0.09	0.09	0.10	0.10
45	0.10	0.11	0.10	0.11	0.11	0.11	0.11	0.10	0.10
50	0.12	0.14	0.16	0.12	0.13	0.13	0.12	0.11	0.10
55	0.16	0.18	0.23	0.13	0.15	0.15	0.14	0.12	0.10
60	0.20	0.24	0.34	0.15	0.17	0.19	0.17	0.14	0.11
64	0.26	0.32	0.47	0.17	0.20	0.22	0.20	0.16	0.12

Table 4.2: *Actuarial Liability under the various cost methods*

age	Power p = 0.8	Power p = 1	Power p = 1.5	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	Pareto k = 1.5
35	0.46	0.31	0.12	0.49	0.44	0.42	0.48	0.55	0.66
40	1.04	0.81	0.43	1.17	1.08	1.02	1.13	1.27	1.45
45	1.87	1.58	1.03	2.10	1.98	1.90	2.04	2.21	2.44
50	3.08	2.75	2.08	3.40	3.25	3.15	3.30	3.50	3.75
55	4.86	4.54	3.84	5.22	5.07	4.97	5.11	5.30	5.53
60	7.58	7.35	6.80	7.87	7.75	7.68	7.77	7.91	8.07
64	10.84	10.78	10.63	10.93	10.90	10.88	10.90	10.94	10.98
65	11.89	11.89	11.89	11.89	11.89	11.89	11.89	11.89	11.89

Commenting on tables 4.1 and 4.2 we point out:

Normal Cost development follows the development of m(x). Hence, the higher the m(x) the higher the Normal Cost, the higher the M(x) and hence the Actuarial Liability. But eventually, at the age of retirement, r, AL(r) should all be equal as M(r) =1.

As we have discussed in chapter 3, the different accrual functions, as well as the different sets of parameters applied, allow the actuary some flexibility regarding the pace of building up the fund within the restrictions he/she has already set on the development of its level. In this section, we have shown that, in practice, he/she might proceed to a further step by calculating the Normal Cost and Actuarial Liability values. Thus, in the absence of variations from expected experience, the actuary could estimate at the start of the plan the expected Contributions and Fund level, on the basis of the average age of the scheme members.

The results of both parts 2.4 and 4.5 will be considered in section 4.9, for the comparison between the traditional and the new cost methods.

4.6 COMPARISON IN TERMS of the ACCRUED LIABILITY at TIME t

Bowers et al (1986) prove the following proposition:

Proposition : Consider two actuarial functions $M_I(x)$,$M_{II}(x)$. If $D(x) = M_I(x) - M_{II}(x)$ is such that $D'(a) > 0$ and $D'(x) = 0$ has exactly one solution, $a \prec x \prec r$, then $AL_I(t) > AL_{II}(t)$.

Application 1: Comparison between the Actuarial Liability development under the Truncated Exponential (AL_I) and under the Uniform (AL_{II}) distribution function

Let X ~ Truncated Exponential distribution, then

$$M_I(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1-e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}} & , a \prec x \prec r \ , \sigma \succ 0 \\ 1 & , \quad x \ge r \end{cases} \quad \textbf{(4.29)}$$

Let X ~ Uniform distribution, then

$$M_{II}(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)}{(r-a)} & , a \le x \le r \ , \\ 1 & , x \ge r \end{cases} \quad \textbf{(4.30)}$$

From (4.41) and (4.42) $\Rightarrow D(x) = M_I(x) - M_{II}(x) =$

$$= \frac{1-e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}} - \frac{(x-a)}{(r-a)} \Rightarrow D'(x) = \frac{1}{\sigma} * \frac{e^{-(x-a)/\sigma}}{1-e^{-\frac{r-a}{\sigma}}} - \frac{1}{r-a} \Rightarrow$$

$$\Rightarrow D'(a) = \frac{1}{\sigma} * \frac{1}{1-e^{-\frac{r-a}{\sigma}}} - \frac{1}{r-a} \quad \textbf{(4.31)}$$

Since : $\dfrac{r-a}{\sigma} > 0 \Rightarrow e^{-\frac{r-a}{\sigma}} > 1 - \dfrac{r-a}{\sigma}$

From (4.43) $\Rightarrow \sigma * (1 - e^{-\frac{r-a}{\sigma}}) < r - a$, and $\dfrac{1}{\sigma} * \dfrac{1}{1-e^{-\frac{r-a}{\sigma}}} > \dfrac{1}{r-a}$ **(4.32)**

i.e. $D'(a) > 0$

$$D'(r) = \frac{1}{\sigma} * \frac{1}{e^{\frac{r-a}{\sigma}} - 1} - \frac{1}{r-a}$$

Since: $\dfrac{r-a}{\sigma} > 0 \Rightarrow e^{\frac{r-a}{\sigma}} > 1 + \dfrac{r-a}{\sigma}$

From (4.44) $\Rightarrow \sigma * (e^{\frac{r-a}{\sigma}} -1) > r - \alpha$, and $\frac{1}{\sigma} * \frac{1}{e^{\frac{r-a}{\sigma}} - 1} < \frac{1}{r-a}$

i.e. $D'(r) < 0$

$D''(x) < 0$, where $D''(x) = -\frac{1}{\sigma^2} * \frac{e^{-(x-a)/\sigma}}{1 - e^{-\frac{r-a}{\sigma}}}$

Hence: $D'(x)$ =0 for exactly one value of x, $a \prec x \prec r$.

As a result from the above $AL_I(t)_{(\text{Truncated Exponential})} > AL_{II}(t)_{(\text{Uniform})}$, as shown by the numerical examples in table 4.2.

Application 2: Comparison between the Actuarial Liability development under the Uniform (AL_I) and under the Power (AL_{II}) distribution functions.

Let X ~ Uniform distribution, then

$$M_I(x) = \begin{cases} 0 & , x \prec a \\ \frac{(x-a)}{(r-a)} & , a \le x \le r \\ 1 & , x \ge r \end{cases} \qquad \textbf{(4.33)}$$

Let X ~ Power function, then

$$M_{II}(x) = \begin{cases} 0 & , x \prec a \\ \frac{(x-a)^p}{(r-a)^p} & , a \le x \le r \ , \ p \succ 0 \\ 1 & , x \succ r \end{cases} \qquad \textbf{(4.34)}$$

From (4.33) and (4.34) $\Rightarrow D(x) = M_I(x) - M_{II}(x) = \frac{(x-a)}{(r-a)} - \frac{(x-a)^p}{(r-a)^p} \Rightarrow$

$D'(x) = \frac{1}{r-a} - p* \frac{(x-a)^{p-1}}{(r-a)^p} \Rightarrow D'(a) = \frac{1}{r-a}$ i.e. $D'(a) > 0$

$$D'(r) = \frac{1}{r-a} - \frac{p}{r-a} \quad : \qquad \underline{\text{if } p > 1,}\ D'(r) < 0$$

$$D''(x) = -\ p*(p-1)* \frac{(x-a)^{p-2}}{(r-a)^{p}}$$

Hence: $D'(x) = 0$ for exactly one value of x, $a \prec x \prec r$.

As a result $AL_I(t)_{(Uniform)} > AL_{II}(t)_{(Power\ function,\ p>1)}$ as shown by the numerical examples in table 4.2.

<u>Application 3:</u> Comparison between the Actuarial Liability development under the Truncated Pareto (AL_I) and under the Power (AL_{II}) distribution functions.

Let X ~ Truncated Pareto, then

$$M_I(x) = \begin{cases} 0 & , x \leq a \\ \dfrac{1-(\frac{a}{x})^k}{1-(\frac{a}{r})^k} & , a \prec x \prec r\ ,\ k \succ 0 \\ 1 & , x \geq r \end{cases} \qquad \textbf{(4.35)}$$

Let X ~ Power function, then

$$M_{II}(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)^p}{(r-a)^p} & , a \leq x \leq r\ ,\ p \succ 0 \\ 1 & , x \succ r \end{cases} \qquad \textbf{(4.36)}$$

From (4.35) and (4.36) $\Rightarrow D(x) = M_I(x) - M_{II}(x) =$

$$= \frac{1-(\frac{a}{x})^k}{1-(\frac{a}{r})^k} - \frac{(x-a)^p}{(r-a)^p} \Rightarrow D'(x) = \frac{(\frac{a}{x})^k * \frac{k}{x}}{1-(\frac{a}{r})^k} - p* \frac{(x-a)^{p-1}}{(r-a)^p} \Rightarrow$$

$$D'(a) = \frac{\frac{k}{a}}{1-(\frac{a}{r})^k} \quad \text{i.e. } D'(a) > 0$$

$$D'(r) = \frac{(\frac{k}{r})*(\frac{a}{r})^k}{1-(\frac{a}{r})^k} - \frac{p}{r-a}$$

We write $\frac{(\frac{k}{r})*(\frac{a}{r})^k}{1-(\frac{a}{r})^k} = \frac{(\frac{k}{r})*d^k}{1-d^k}$ and d = $\frac{a}{r}$.

Then we have proved that in the case: $0 \prec k \prec 1$, k < $\frac{p}{d}$ the following holds:

$$\frac{(\frac{k}{r})*(\frac{a}{r})^k}{1-(\frac{a}{r})^k} < \frac{p}{r-a} \qquad ,a \prec x \prec r \ , \ p \succ 0, \ k \succ 0$$ (see Appendix 9)

$$D''(x) = -\frac{a^k * k * (k+1) * (\frac{1}{x})^{k+2}}{1-(\frac{a}{r})^k} - p * (p-1) * \frac{(x-a)^{p-2}}{(r-a)^p}$$

Hence: $D'(x) = 0$ for exactly one value of x, $a \prec x \prec r$

As a result $AL_I(t)_{(Trunc.Pareto)}^{k<1, k<\frac{p}{d}} > AL_{II}(t)_{(Power\ fn)}$ as shown by the numerical examples in table 4.2.

4.7 COMPARISON of RESULTS

Summarizing the above results, we have:

The Actuarial Liability is higher under the Truncated exponential distribution when compared with the Power function choice for m(x), given that $p \geq 1$:

$$AL(t)_{(\text{Truncated Exponential})} > AL(t)_{(\text{Uniform})} > AL(t)_{(\text{Power fn}, p>1)} \quad \textbf{(4.37)}$$

The Actuarial Liability is higher under the Truncated Pareto distribution when compared with the Power function choice for m(x), given that : $k<1$ and $k < \frac{p}{d}$:

$$AL(t)^{k<1, k<\frac{p}{d}}_{(\mathit{Trunc.Pareto})} > AL(t)_{(\text{Power fn})} \quad \textbf{(4.38)}$$

The Actuarial Liability is higher under the Uniform distribution when compared with the Power function choice for m(x), given that $p > 1$:

$$AL(t)_{(\text{Uniform})} > AL(t)_{(\text{Power fn}, p>1)} \quad \textbf{(4.39)}$$

The density function associated with a decelerating method has a higher Actuarial Liability than the density function associated with an accelerating method. This is reasonable, since in chapter 3 we have shown that in the decelerating methods the accrued function, M(x), rapidly accumulates to 1.

When we compare the Actuarial Liability under two functions both associated with a decelerating method, we observe, that the Actuarial Liability is higher under that density function that allocates larger proportions of the benefit at younger ages. This is to be expected, since, for this case, we have shown (chapter 3) that the accrued function, M(x), accumulates to 1 at a faster pace.

4.8 RELATIONSHIP BETWEEN ACTUARIAL LIABILITY and NORMAL COST

The level of correlation between the Normal Cost and the Actuarial Liability is confirmed through the following proposition (Trowbridge, 1952):

In a pension plan mathematical model where:

a) the rate of investment is allowed to change with time, while the other actuarial factors remain constant,

b) as time, t, changes new entrants are allowed to enter the population in such a way that the population is stationary,

c) the interest rate assumption for valuation purposes is fixed,

d) the contribution income and benefit outgo occur at the start of each inter-valuation period and

e) the valuations are carried out at annual intervals,

the Actuarial Liability AL satisfies the equilibrium equation:

$$AL = (1+i) * (AL + NC - B) \quad \textbf{(4.40)}$$

Then given that AL_I, AL_{II}, denote the Actuarial Liability development under methods I and II respectively, (4.40) in each case can be written as:

$AL_I = (1+i) * (AL_I + NC_I - B) \Rightarrow NC_I - B = - d * AL_I$

and $AL_{II} = (1+i) * (AL_{II} + NC_{II} - B) \Rightarrow NC_{II} - B = - d * AL_{II}$

Considering that : $\frac{B - NC_I}{AL_I} = \frac{B - NC_{II}}{AL_{II}} = d$, then given that:

$AL_I > AL_{II} \Rightarrow NC_I < NC_{II}$ and vice versa.

Hence, the inequalities (4.37) – (4.39), imply that:

$$NC(t)_{(\text{Truncated Exponential})} < NC(t)_{(\text{Uniform})} < NC(t)_{(\text{Power fn}, \, p>1)} \quad \textbf{(4.41)}$$

$$NC(t)_{(\text{Trunc.Pareto})}^{k<1, k<\frac{p}{d}} < NC(t)_{(\text{Power fn})} \quad \textbf{(4.42)}$$

$NC(t)_{(Uniform)} < NC(t)_{(Power\ fn,\ p>1)}$ **(4.43)**

4.9 COMPARISON BETWEEN the 'NEW' and the TRADITIONAL COST METHODS

For the comparison between the traditional and the 'new' methods, assuming that the benefit at the age of retirement is equal to 1 unit, we observe the following at the outset, as far as the traditional methods are concerned:

- Current Unit Credit

In this case, career average salary benefit, it is assumed that the benefit of a plan member with entry age a, after r- a years of service, is equal to a percentage of the total payroll throughout his / her career: $b_x = b * s_x$, $B_x = b * S_x$

Setting b = $\frac{1}{S_r}$, the Normal Cost part allocated to a participant aged x is equal to:

m(x) = $\frac{1}{S_r} * s_x$, and m(x) clearly sums to 1 over the range x=a to r-1.

Also, we have that:

$$M(x) = \begin{cases} 0, & x \le a \\ \frac{1}{S_r} * S_x, & a \prec x \prec r \\ 1, & x \ge r \end{cases}$$

- Projected Unit Credit

In this case, final salary benefit, it is assumed that the benefit of a plan member with entry age a, after r- a years of service, equals to a percentage of his / her final salary for each year of service : $b_x = b * s_{r-1}$, $B_x = b * (x - a) * s_{r-1}$

Setting b = $\frac{1}{s_{r-1} * (r-a)}$, the Normal Cost part allocated to a participant aged x is equal to m(x) = $\frac{1}{r-a}$ and m(x) clearly sums to 1 over the range x = a to r-1.

Also we have that:

$$M(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{r-a} & a \prec x \prec r \\ 1, & x \ge r \end{cases}$$

- Entry Age Normal

As in the final salary benefit considered above, it is assumed that the benefit of a plan member with entry age a, after r- a years of service, is equal to a percentage of his / her final salary for each year of service : $B_r = b*(r-a)*s_{r-1}$

Setting b = $\frac{1}{s_{r-1} * (r-a)}$, the Normal Cost part allocated to a participant aged x is equal to m(x) = $\frac{1}{s_a * {}^s\ddot{a}_{a:\overline{r-a}|}}$ * s_x * $\frac{D_x^\delta}{D_a^\delta}$ and m(x) clearly sums to 1 over the range x = a to r-1.

Also we have that:

$$M(x) = \begin{cases} 0, & x \le a \\ \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} & a \prec x \prec r \\ 1, & x \ge r \end{cases}$$

On the basis of the above, we observe that the functions used for the Normal Cost allocation and for the accrued benefit have the properties of a probability distribution for a discrete random variable. As mentioned in chapter 3, when the Normal Cost and

Accrued Liability are based on the Uniform distribution, then their development coincides with that under the traditional Unit Credit Cost method, where the benefit is equally allocated throughout the working years.

In the light of the above we shall proceed to compare the traditional Cost methods with the proposed new ones.

4.9.1 COMPARISON IN TERMS of NORMAL COST and ACTUARIAL LIABILITY at AGE x

In section 2.4, formula 2.14 shows that ${}^{CUC}AL_x < {}^{PUC}AL_x < {}^{EAN}AL_x$.

In section 4.6, we show that:

$$\left\{ \begin{array}{l} AL_{x(\text{Trunc Exponential})} > AL_{x\,(\text{Uniform})} > AL_{x\,(\text{Power fn},\ p>1)} \\ AL_{x\,(\underset{Trunc.Pareto}{k<1,k<\frac{p}{d}})} > AL_{x\,(\text{Power fn})} \\ AL_{x\,(\text{Uniform})} > AL_{x\,(\text{Power fn},\ p>1)} \end{array} \right. \qquad \textbf{(4.44)}$$

Combining (2.14) and (4.44) we conclude:

$${}^{CUC}AL_x < {}^{PUC}AL_x \equiv AL_{x\,(\text{Uniform})} < AL_{x\,(\text{Truncated Exponential})} \qquad \textbf{(4.45)}$$

$${}^{CUC}AL_x < {}^{PUC}AL_x \equiv AL_{x\,(\text{Uniform})} < AL_{x\,(\underset{Trunc.Pareto}{k<1,k<\frac{p}{d}})} \qquad \textbf{(4.46)}$$

$${}^{EAN}AL_x > {}^{PUC}AL_x \equiv AL_{x\,(\text{Uniform})} > AL_{x\,(\text{Power},\ p>1)} \qquad \textbf{(4.47)}$$

The above inequalities (4.45) – (4.47) are illustrated in the table 4.3:

***Table 4.3**: Actuarial Liability under the new & traditional cost methods*

Age	CUC	PUC ≡ ≡Uniform	Power p = 1.5	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	E.A.N
35	0.19	0.31	0.12	0.49	0.44	0.42	0.48	0.55	0.43
40	0.54	0.81	0.43	1.17	1.08	1.02	1.13	1.27	1.06
45	1.13	1.58	1.03	2.10	1.98	1.90	2.04	2.21	1.96
50	2.14	2.75	2.08	3.40	3.25	3.15	3.30	3.50	3.24
55	3.84	4.54	3.84	5.22	5.07	4.97	5.11	5.30	5.08
60	6.74	7.35	6.80	7.87	7.75	7.68	7.77	7.91	7.77
64	10.60	10.78	10.63	10.93	10.90	10.88	10.90	10.94	10.91
65	11.89	11.89	11.89	11.89	11.89	11.89	11.89	11.89	11.89

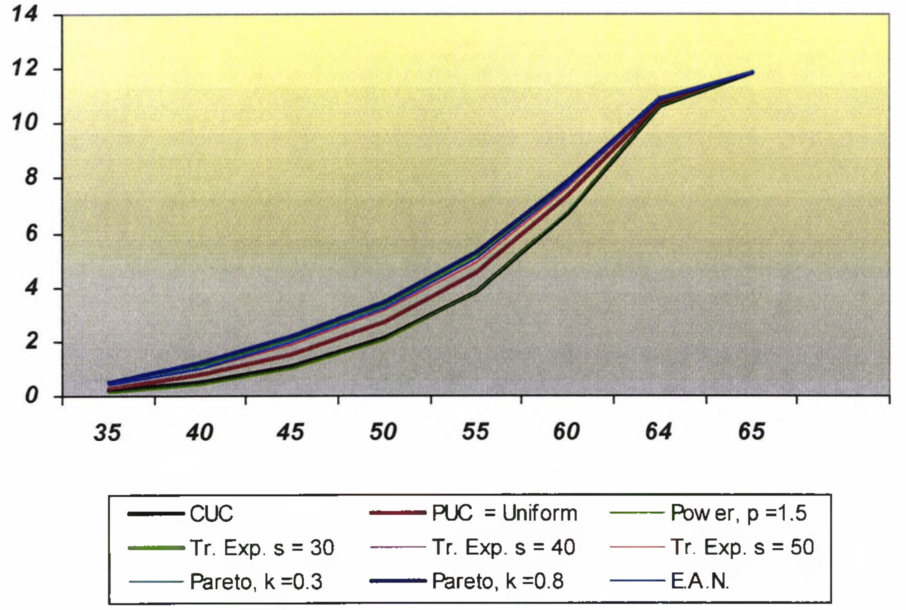

Figure 4.1 Actuarial Liability under the new & traditional cost methods

The Normal Cost at specific ages is presented in the following table 4.4:

Table.4.4: *Normal Cost under the new & traditional cost methods*

age	CUC	PUC ≡ ≡Uniform	Power p = 1.5	Tr. Exp. σ = 30	Tr. Exp. σ = 40	Tr. Exp. σ = 50	Pareto k = 0.3	Pareto k = 0.8	E.A.N
35	0.04	0.06	0.04	0.09	0.08	0.08	0.09	0.10	0.08
40	0.06	0.08	0.07	0.10	0.09	0.09	0.09	0.10	0.09
45	0.09	0.11	0.10	0.11	0.11	0.11	0.11	0.10	0.11
50	0.14	0.14	0.16	0.12	0.13	0.13	0.12	0.11	0.13
55	0.22	0.18	0.23	0.13	0.15	0.15	0.14	0.12	0.15
60	0.34	0.24	0.34	0.15	0.17	0.19	0.17	0.14	0.17
64	0.50	0.32	0.47	0.17	0.20	0.22	0.20	0.16	0.19
65	0.55	0.34	0.51	0.18	0.21	0.23	0.21	0.17	0.20

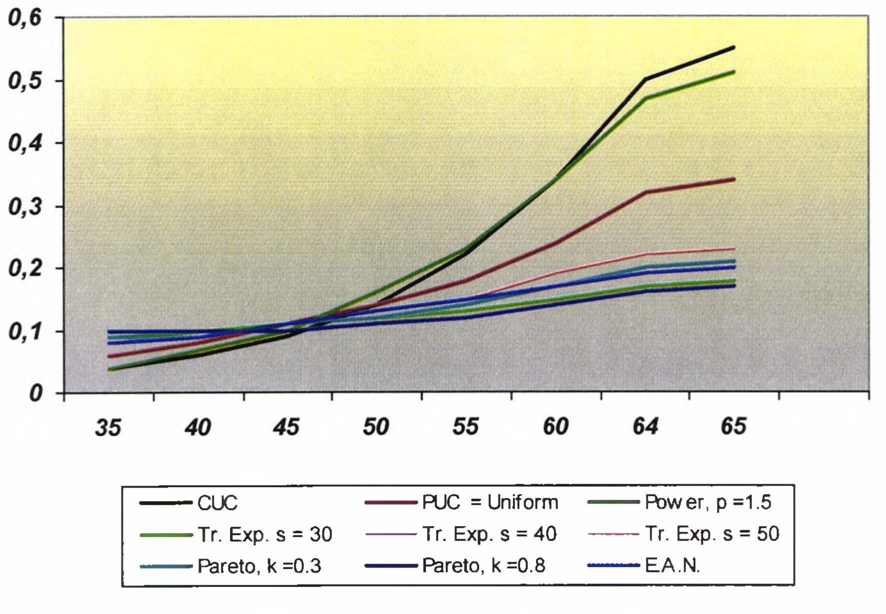

Figure 4.2: Normal Cost under the new & traditional cost methods

We make the following remarks on tables 4.3 and 4.4 and Figures 4.1 and 4.2:

In table 4.4 the figures provided by the Current Unit Credit approach those under the Power function when p>1. For this reason we may consider that they form one group. This result is to be expected since under both methods, the benefit is allocated in higher proportions as age increases.

In table 4.4 the figures provided under Entry Age Normal are lower than the corresponding ones provided under the Truncated Exponential and Truncated Pareto. We may also consider that Entry Age Normal, Truncated Exponential and Truncated Pareto form one group, where the benefit is allocated in lower proportions as age increases.

Setting as AL_I, AL_{II} the Actuarial Liability and NC_I, NC_{II} the Normal Cost under methods I and II respectively after age 50 we observe that if $AL_I > AL_{II}$ (table 4.3) then $NC_I < NC_{II}$ (table 4.4) and vice versa; a result that is to be expected.

As a general remark, after the comparison of the two groups, defined as above, we may repeat the one stated in section 2.4: 'The progress with time of both the Normal Cost and Actuarial Liability is determined (ceteris paribus) by the choice of the function, m(x); the Normal Cost follows the trend that m(x) follows, while the higher is the value of this function at younger ages, the higher is the Actuarial Liability'.

In order to examine the sensitivity of the results obtained, we have tested the underlying parameters, changing consecutively the entry age, the age of retirement and the valuation rate of investment return.

In particular, we first assumed that the participant enters in the scheme at age 25 and retires at age 65. From the results we observe that the trend followed by the Normal Cost and Actuarial Liability throughout the participant's active years is kept the same with the corresponding one calculated assuming that the entry age is 30. As expected though, the Normal Cost values are lower because the cost is spread over 40 instead of 35 years; as a consequence, the Actuarial Liability increases slightly above age 50, to the age of retirement. On the other hand, when we assume that the participant enters the scheme at age 40, the Normal Cost increases since it is spread over 25 years and the Actuarial Liability decreases slightly up to the age of retirement.

The difference between the entry and retirement age is the key assumption that affects their values. Specifically, given that the age of retirement is kept constant, the higher the number of years in service, the lower the Normal Cost and the higher the liability as the plan members approach retirement.

Testing the development of the Liability and Normal Cost assuming that the participant enters in the scheme at age 30 and retires at age 70, we also observe that it follows the same pattern as that of the case where the participant enters at age 30 and retires at age 65. We point out that, as the retirement age increases, the Normal Cost and Actuarial Liability values decrease due to the change in post-retirement life expectancy.

The Normal Cost and Actuarial Liability patterns remain also unchanged if the valuation rate of investment return either increases or decreases. The effect of its change is focused on the cost and liability values, which as expect increase as it decreases, and vice versa.

Summarizing the above we may conclude that the way that the Normal Cost and Actuarial Liability progress with time does not change significantly along with the change of either one of the entry age and the age of retirement or the valuation rate of investment return.

4.9.2 COMPARISON IN TERMS of the ACCRUED LIABILITY and NORMAL COST at TIME t

We now apply the Bowers et al proposition in order to compare the development of the Accrued Liability at time t between the traditional Cost methods:

Application 1: Comparison between the development of the Actuarial Liability under the Projected Unit Credit , (AL_I), and under the Current Unit Credit where the pension benefit is linked to the employee's salary (AL_{II})

Under the Projected Unit Credit

$$M_I(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)}{(r-a)} & , a \le x \le r , \\ 1 & , x \ge r \end{cases} \tag{4.48}$$

Under the Current Unit Credit

$$M_{II}(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1}{S_r} * S_x & , a \prec x \prec r , \\ 1 & , x \ge r \end{cases} \tag{4.49}$$

From (4.48) and (4.49) $\Rightarrow D(x) = M_I(x) - M_{II}(x) =$

$$= \frac{(x-a)}{(r-a)} - \frac{1}{S_r} * S_x = \frac{(x-a)}{(r-a)} - \frac{e^{(x-a)\tau}-1}{e^{(r-a)\tau}-1} \Rightarrow D'(x) = \frac{1}{r-a} - \frac{\tau * e^{(x-a)\tau}}{e^{(r-a)\tau}-1} \Rightarrow$$

$$\Rightarrow D'(a) = \frac{1}{r-a} - \frac{\tau}{e^{(r-a)\tau}-1} = \frac{1}{r-a} - \frac{1}{\bar{s}^{\tau}_{\overline{r-a|}}} \text{, providing } \tau>0$$

$$\text{Since : } r-a < \bar{s}^{\tau}_{\overline{r-a|}} \Rightarrow \frac{1}{r-a} > \frac{1}{\bar{s}^{\tau}_{\overline{r-a|}}} \Rightarrow D'(a)>0$$

$$D'(r) = \frac{1}{r-a} - \frac{\tau * e^{(r-a)\tau}}{e^{(r-a)\tau}-1} = \frac{1}{r-a} - \frac{1}{\overset{-\tau}{a}_{\overline{r-a}|}}, \text{ providing } \tau>0$$

$$\text{Since : } r-a > \overset{-\tau}{a}_{\overline{r-a}|} \Rightarrow \frac{1}{r-a} < \frac{1}{\overset{-\tau}{a}_{\overline{r-a}|}} \Rightarrow D'(r)<0$$

$$D''(x)<0, \text{ where } D''(x) = -\ \tau * \frac{\tau * e^{(x-a)\tau}}{e^{(r-a)\tau}-1} < 0$$

Hence: $D'(x)=0$ for exactly one value of x, $a \prec x \prec r$.

As a result from the above $AL_I(t)_{P.U.C.} > AL_{II}(t)_{C.U.C.}$ **(4.50)**

Application 2: Comparison between the development of the Actuarial Liability under the Entry Age Normal, where the pension benefit is linked with the employee's salary (AL_I), and under the Projected Unit Credit (AL_{II}).

Under the Entry Age Normal

$$M_I(x) = \begin{cases} 0 & , x \leq a \\ \dfrac{{}^{s}\ddot{a}_{a:\overline{x-a}|}}{{}^{s}\ddot{a}_{a:\overline{r-a}|}} & , a \prec x \prec r , \\ 1 & , x \geq r \end{cases} \qquad \textbf{(4.51)}$$

Under the Projected Unit Credit

$$M_{II}(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)}{(r-a)} & , a \leq x \leq r , \\ 1 & , x \geq r \end{cases} \qquad \textbf{(4.52)}$$

From (4.51) and (4.52) $\Rightarrow D(x) = M_I(x) - M_{II}(x) =$

$$= \frac{{}^{s}\ddot{a}_{a:\overline{x-a}|}}{{}^{s}\ddot{a}_{a:\overline{r-a}|}} - \frac{(x-a)}{(r-a)} = \frac{N_a^{(\delta-\tau)} - N_x^{(\delta-\tau)}}{N_a^{(\delta-\tau)} - N_r^{(\delta-\tau)}} - \frac{(x-a)}{(r-a)} \Rightarrow$$

$$\Rightarrow D'(x) = \frac{D_x^{(\delta-\tau)}}{N_a^{(\delta-\tau)} - N_r^{(\delta-\tau)}} - \frac{1}{r-a}$$

$$D'(a) = \frac{D_a^{(\delta-\tau)}}{N_a^{(\delta-\tau)} - N_r^{(\delta-\tau)}} - \frac{1}{r-a} = \frac{1}{{}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|}} - \frac{1}{r-a}$$

Since : r- $a >$ ${}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|} \Rightarrow \frac{1}{r-a} < \frac{1}{{}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|}} \Rightarrow D'(a)>0$, providing $\delta - \tau>0$

$$D'(r) = \frac{D_r^{(\delta-\tau)}}{N_a^{(\delta-\tau)} - N_r^{(\delta-\tau)}} - \frac{1}{r-a} = \frac{D_r^{(\delta-\tau)}}{D_a^{(\delta-\tau)}} \frac{1}{{}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|}} - \frac{1}{r-a}$$

Since : r- $a < \frac{D_a^{(\delta-\tau)}}{D_r^{(\delta-\tau)}} \, {}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|} \Rightarrow \frac{1}{r-a} > \frac{D_r^{(\delta-\tau)}}{D_a^{(\delta-\tau)}} \frac{1}{{}^{s}\ddot{a}^{(\delta-\tau)}_{a:\overline{r-a}|}} \Rightarrow D'(r)<0$, providing $\delta - \tau>0$

$D''(x) < 0$, where $D''(x) = - \frac{D_x^{(\delta-\tau)} * (\mu_x + \delta)}{N_a^{(\delta-\tau)} - N_r^{(\delta-\tau)}} < 0$

Hence: $D'(x)=0$ for exactly one value of x, $a \prec x \prec r$

As a result from the above shown also by Bowers et al (1986),

$AL_I(t)_{EAN} > AL_{II}(t)_{P.U.C.}$ **(4.53)**

From (4.50) and (4.53) we have:

${}^{CUC}AL(t) < {}^{PUC}AL(t) < {}^{EAN}AL(t)$ **(4.54)**

Combining the inequalities (4.37) – (4.39) with (4.54) we conclude:

${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)_{(Truncated\ Exponential)}$ **(4.55)**

${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)^{k<1, k<\frac{p}{d}}_{(Trunc\ Pareto)}$ **(4.56)**

$AL(t)_{(Power\ ,\ p>1)} < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < {}^{EAN}AL(t)$ **(4.57)**

These inequalities show that the Actuarial Liability of the accelerating cost methods (i.e. Current Unit Credit, Power function, $p > 1$) is less than the Actuarial Liability of the decelerating cost methods (i.e. Truncated Exponential, Truncated Pareto, Entry Age Normal).

The conclusions derived from the comparison in terms of the Accrued Liability at time t, are the same with the ones derived from the comparison in terms of the Actuarial Liability at age x, (table 4.3). These conclusions, verify the proposition proved by Bowers et al, according to which: " if m(x) is associated with a decelerating cost method ($m'(x)<0$) and $m_1(x)$ is associated with an accelerating cost method ($m_1'(x)>0$), then $M(x) > M_1(x)$, $a<x<r$ "

The inequalities (4.55) – (4.57) imply:

$$^{CUC}NC(t) > {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} > NC(t)_{(Truncated\ Exponential)} \tag{4.58}$$

$$^{CUC}NC(t) > {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} > NC(t)_{\left(\substack{k<1, k<\frac{p}{d} \\ Trunc.Pareto}\right)} \tag{4.59}$$

$$NC(t)_{(Power\ ,\ p>1)} > {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} > {}^{EAN}NC(t) \tag{4.60}$$

4.10 CALCULATION of AL(t), NC(t) and B(t) WHEN $\tau = 0$

When $\tau = 0$, the formulae of NC(t), AL(t), and B(t) may revised as follows:

$$NC(t)= \int_a^r e^{\tau*(t+r-1-x)} * l_r * s_a * b * e^{-\delta(r-x)} * \bar{a}_r^{(\delta-\beta)} * m(x)dx =$$

$$= \int_a^r l_r * s_a * b * e^{-\delta(r-x)} * \bar{a}_r^{(\delta-\beta)} * m(x) \Rightarrow$$

$$\Rightarrow NC(t)= l_r * s_a * b * e^{-\delta * r} * \bar{a}_r^{(\delta-\beta)} * \int_a^r e^{\delta * x} * m(x)dx \qquad \mathbf{(4.61)}$$

$$AL(t) = \int_a^r e^{\tau*(t+r-1-x)} * l_r * s_a * b * M(x) * e^{-\delta*(r-x)} * \bar{a}_r^{(\delta-\beta)} dx +$$

$$+ \int_r^w e^{\tau*(t+r-1-x)} * l_x * s_a * b \; \bar{a}_x^{(\delta-\beta)} \; e^{\beta*(x-r)} dx =$$

$$= l_r * s_a * b * \bar{a}_r^{(\delta-\beta)} * e^{-\delta*r} * \int_a^r M(x) * e^{\delta*x} dx + b * s_a * e^{-\beta*r} * \int_r^w l_x * \bar{a}_x^{(\delta-\beta)} * e^{\beta*x} dx$$

$$\Rightarrow AL(t)=l_r * s_a \, b \, \bar{a}_r^{(\delta-\beta)} \; e^{-\delta*r} \int_a^r M(x)e^{\delta*x} dx + \frac{b * s_a}{\delta} \left[\int_r^w l_x * e^{\beta*(x-r)} dx - l_r \bar{a}_r^{(\delta-\beta)} \right] \qquad \mathbf{(4.62)},$$

following the same mathematical argument as in Appendix 8.

The integral in formula 4.62 is approximated considering either Simpson's or Rectangular rule. We set the value of AL(t) equal to the average of the values that result after applying each method of approximation (see Appendix 10).

For the calculation of B(t) given that t = 0, the equilibrium equation is applied i.e.

$$B = NC + d*AL \quad \Leftrightarrow \quad B(0) = NC(0) + d*AL(0) \qquad \mathbf{(4.63)}$$

It is clearly seen, from formulae (4.4), (4.6) and (4.7) stated in section 4.2, that the values of NC(t), AL(t) and B(t) when $\tau = 0$ do not depend on t. They remain constant throughout the years, equal to NC(0), AL(0) and B(0) respectively.

These values, are presented in the tables below, (4.5) – (4.7), for the distributions being considered regarding the choices of the valuation of the rates of investment which will be used as expected values for the simulations we discuss in chapter 6:

Table 4.5: *AL(0) , NC(0) and B(0) when E(i(t)) = 0.03*

Distribution fn	AL(0)	NC(0)	B(0)
Power p=0.8	263.55	5.98	13.66
Power p=1	252.33	6.31	13.66
Power p=1.5	231.13	6.93	13.66
Tr. Expon. σ=30	272.56	5.71	13.65
Tr. Expon. σ=40	267.73	5.85	13.65
Tr. Expon. σ=50	264.75	5.94	13.65
Tr. Pareto k=0.3	269.79	5.79	13.65
Tr. Pareto k=0.8	276.20	5.59	13.63
Tr. Pareto k=1.5	284.86	5.32	13.62

Table 4.6: *AL(0) , NC(0) and B(0) when E(i(t)) = 0.05*

Distribution fn	AL(0)	NC(0)	B(0)
Power p=0.8	204.65	3.78	13.53
Power p=1	197.84	4.11	13.53
Power p=1.5	184.63	4.76	13.55
Tr. Expon. σ=30	210.5	3.49	13.51
Tr. Expon. σ=40	207.51	3.64	13.52
Tr. Expon. σ=50	205.65	3.73	13.52
Tr. Pareto k=0.3	208.69	3.58	13.52
Tr. Pareto k=0.8	212.63	3.39	13.52
Tr. Pareto k=1.5	217.75	3.10	13.47

Table 4.7: *AL(0) , NC(0) and B(0) when E(i(t)) = 0.07*

Distribution fn	AL(0)	NC(0)	B(0)
Power p=0.8	165.82	2.55	13.40
Power p=1	161.49	2.84	13.40
Power p=1.5	152.84	3.43	13.43
Tr. Expon. σ=30	169.83	2.28	13.39
Tr. Expon. σ=40	167.89	2.41	13.39
Tr. Expon. σ=50	166.68	2.49	13.39
Tr. Pareto k=0.3	168.56	2.36	13.39
Tr. Pareto k=0.8	171.09	2.19	13.38
Tr. Pareto k=1.5	174.37	1.97	13.38

The results of tables 4.5 – 4.7 (and those presented in the appendix 10) show that the effect of the rate of the investment return does not depend on the accrual pension function used.

As we present in figure 4.3, the Actuarial Liability decreases as the expected value of the rates of investment return increases. This result is discussed in detail in section 6.6.

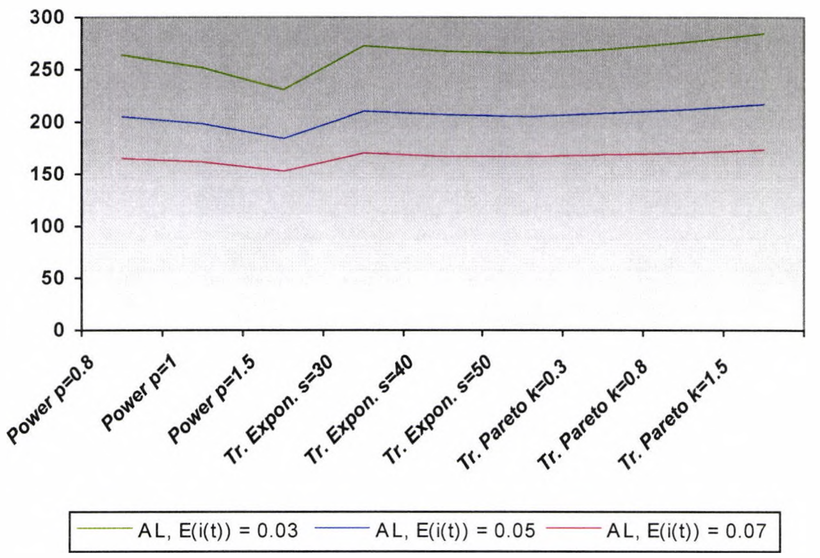

Figure 4.3: Actuarial Liability at time t=0, under different values of the rate of investment return

Both the Actuarial Liability and Normal Cost values satisfy equations (4.55) – (4.60)

$$ {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)_{(Truncated\ Exponential)} \Rightarrow $$

$$ \Rightarrow {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} > NC(t)_{(Truncated\ Exponential)} \quad \textbf{(4.64)} $$

$$ {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)^{k<1, k<\frac{p}{d}}_{(Trunc.Pareto)} \Rightarrow $$

$$ \Rightarrow {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} > NC(t)^{k<1, k<\frac{p}{d}}_{(Trunc.Pareto)} \quad \textbf{(4.65)} $$

$$ AL(t)_{(Power\ ,\ p>1)} < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} \Rightarrow $$

$$ \Rightarrow NC(t)_{(Power\ ,\ p>1)} > {}^{PUC}NC(t) \equiv NC(t)_{(Uniform)} \quad \textbf{(4.66)} $$

For the Power function and the Truncated Exponential, the Actuarial Liability level decreases as the parameters, p, σ increase. For the Power function distribution this is straightforward because when p becomes higher than 1 the actuarial cost method which up to that value (p=1) is associated with a decelerating cost method, becomes associated with an accelerating cost method (section3.3). For the Truncated Pareto, the Actuarial Liability level increases, along with k.

Within the same distribution, comparing the Actuarial Liability and Normal Cost under the different parameter values, say AL, AL' and NC, NC' respectively, we conclude that when AL > AL' then NC< NC' and vice versa.

The illustrative examples when $\tau = 0$ show the range interval of the Actuarial Liability and Normal Cost under the new cost methods. By considering in advance how the level of the fund varies on the basis of the accrual function which we use, we improve our understanding of the effect of the different distribution functions and their underlying parameters on the pension scheme implementation.

4.11 CONCLUSIONS

The development of Normal Cost follows the pattern of the accrual function 'm(x)'.

The Actuarial Liability level depends on the benefit parts allocated to the plan members on yearly basis. In particular, it is higher when the density function allocates higher proportions of benefit at younger ages.

Comparing the 'new' with the traditional cost methods, we may conclude that: When the benefit is allocated in higher proportions as age increases, the Normal Cost values are very similar when they are calculated either under the Current Unit Credit method or using the Power function. When the benefit is allocated in lower proportions as age increases, the Normal Cost values are very similar when they are calculated either under Entry Age Normal method or using one of the Truncated Exponential or Truncated Pareto methods.

The following inequalities hold for the accrued liability at time t as well as for the Actuarial Liability at age x

a) ${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < AL(t)_{(\text{Truncated Exponential})}$

b) ${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < AL(t)_{(\mathit{Trunc.Pareto})}^{k<1, k<\frac{p}{d}}$, and

c) $AL(t)_{(\text{Power}, p>1)} < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < {}^{EAN}AL(t)$, These inequalities show that in practice, among the different accrual functions, a lower Actuarial Liability is expected from those that are associated with an accelerating cost method than from the ones associated with a decelerating cost method.

Remark:

The link for the comparison between the 'new' and traditional cost methods, is the Power function distribution that coincides with Projected Unit Credit when p = 1. As far as the Current Unit Credit is concerned, according to its definition, the portion of benefit purchased annually is a constant percentage of the participant's salary. In order for the power function to be equivalent with the Current Unit Credit, the p value needs to approach 1. As p $\rightarrow 1$, $(x-a)^{p-1} \rightarrow 1$ and the fraction $\frac{(x-a)^{p-1}}{(r-a)^{p}}$ becomes independent of the age x.

APPENDIX 5

$$\frac{dF(t)}{dt} = NC(t) + (\delta-\lambda) * F(t) + \lambda * AL(t) - B(t) \Rightarrow$$

$$\Rightarrow F(t) = F(0) * e^{(\delta-\lambda)} + e^{\tau * t} * \bar{a}_{\overline{t|}}^{(\lambda-\delta+\tau)} * (NC(0) + \lambda * AL(0) - B(0))$$

<u>Proof</u>

$$\frac{dF(t)}{dt} = NC(t) + (\delta-\lambda) * F(t) + \lambda * AL(t) - B(t) \Rightarrow$$

$$\Rightarrow \frac{dF(t)}{dt} - (\delta-\lambda) * F(t) = NC(t) + \lambda * AL(t) - B(t) \Rightarrow$$

$$\Rightarrow \frac{d}{dt}[F(t) * e^{-(\delta-\lambda)*t}] = (NC(t) + \lambda * AL(t) - B(t)) * e^{-(\delta-\lambda)*t} \Rightarrow$$

$$\Rightarrow F(s) * e^{-(\delta-\lambda)*s} \Big|_{s=0}^{s=t} = \int_0^t (NC(s) + \lambda * AL(s) - B(s)) * e^{-(\delta-\lambda)*s}\, ds \Rightarrow$$

$$\Rightarrow F(t) * e^{-(\delta-\lambda)*t} - F(0) = \int_0^t (NC(s) + \lambda * AL(s) - B(s)) * e^{-(\delta-\lambda)*s}\, ds \Rightarrow$$

$$\Rightarrow F(t) = F(0) * e^{(\delta-\lambda)*t} + e^{(\delta-\lambda)*t} \int_0^t (NC(s) + \lambda * AL(s) - B(s)) * e^{-(\delta-\lambda)*s}\, ds \qquad \textbf{(1)}$$

$$\int_0^t (NC(s)+\lambda*AL(s) - B(s)) * e^{-(\delta-\lambda)*s}\, ds = (NC(0) +\lambda*AL(0) - B(0)) * \int_0^t e^{(\tau-\delta+\lambda)s} ds \qquad \textbf{(2)}$$

since we know that $NC(s) + \lambda*AL(s) - B(s) = (NC(0) +\lambda*AL(0) - B(0)) * e^{\tau*s}$

$$(2) \Rightarrow \int_0^t (NC(s) + \lambda * AL(s) - B(s)) * e^{-(\delta-\lambda)*s}\, ds =$$

$$\bar{s}_{\overline{t|}}^{(\lambda-\delta+\tau)} *(NC(0)+\lambda*AL(0)-B(0)) = e^{(\lambda-\delta+\tau)*t} * \bar{a}_{\overline{t|}}^{(\lambda-\delta+\tau)} *(NC(0) + \lambda*AL(0) - B(0)) \qquad \textbf{(3)}$$

$$\overset{(3)}{\Rightarrow} F(t) = F(0) * e^{(\delta-\lambda)*t} + e^{(\delta-\lambda)*t} * e^{(\lambda-\delta+\tau)*t} * \bar{a}_{\overline{t|}}^{(\lambda-\delta+\tau)} *(NC(0) + \lambda*AL(0) - B(0)) =$$

$$= F(0) * e^{(\delta-\lambda)*t} + e^{\tau*t} * \bar{a}_{\overline{t|}}^{(\lambda-\delta+\tau)} *(NC(0) + \lambda*AL(0) - B(0))$$

APPENDIX 6

$$\lambda(t) = (\overline{a}^{\theta}_{\overline{x(\theta)-a|}})^{-1}$$

Proof:

$$(\lambda(t))^{-1} = \frac{\int_a^r e^{(\delta-\tau)x} * (1-M(x))dx}{\int_a^r e^{(\delta-\tau)x} * m(x)dx} = \frac{[\frac{1}{\delta-\tau} * e^{(\delta-\tau)x}]_a^r - [\frac{1}{\delta-\tau} * e^{(\delta-\tau)x} * M(x)]_a^r}{\int_a^r e^{(\delta-\tau)x} * m(x)dx} +$$

$$+ \int_a^r e^{(\delta-\tau)} * m(x)dx * \frac{1}{(\delta-\tau) * \int_a^r e^{(\delta-\tau)} * m(x)dx}$$

$$= \frac{1}{\delta-\tau} * \frac{e^{(\delta-\tau)r} - e^{(\delta-\tau)a} - e^{(\delta-\tau)r} * 1 + \int_a^r e^{(\delta-\tau)x} m(x)dx}{\int_a^r e^{(\delta-\tau)x} m(x)dx} = \frac{1}{\theta} * \frac{e^{\theta * x(\theta)} - e^{\theta a}}{e^{\theta x(\theta)}} =$$

$$= \frac{1 - e^{\theta a} * (e^{\theta x(\theta)})^{-1}}{\theta}, \text{ where } \delta - \tau = \theta$$

$$\text{then}: (\lambda(t))^{-1} = (\frac{1 - e^{\theta(\alpha - x(\theta))}}{\theta})^{-1} = (\overline{a}^{\theta}_{\overline{x(\theta)-a|}})^{-1} \text{ or } \quad \lambda(t) = \frac{1}{\overline{a}^{\theta}_{\overline{x(\theta)-a|}}}$$

If $\theta = 0$, then $\overline{a}^{\theta}_{\overline{x(\theta)-a|}} = \mu - a$

Proof :

$$\lim_{\theta\to 0} \overline{a}^{\theta}_{\overline{x(\theta)-a|}} = \lim_{\theta\to 0} \frac{1 - e^{-\theta(x(\theta)-a)}}{\theta} \overset{l'\,Hopital\ rule}{=} \lim_{\theta\to 0} \frac{\frac{d}{d_\theta} * (1 - e^{-\theta(x(\theta)-a)})}{\frac{d}{d_\theta}\theta} \qquad \textbf{(1)}$$

$$\frac{d}{d_\theta} e^{-\theta*(x(\theta)-a)} = \frac{d}{d_\theta} * e^{a\theta} * e^{-\theta*x(\theta)} = a * e^{a\theta} * e^{-\theta*x(\theta)} + e^{a\theta} * \frac{d}{d_\theta} * e^{-\theta*x(\theta)} =$$

$$= a * e^{a\theta} * e^{-\theta * x(\theta)} - e^{a\theta} * \int_a^r x * e^{\theta x} * m(x)dx * [\int_a^r e^{\theta x} * m(x)dx]^{-2} =$$

$$= a * e^{a\theta} * [\int_a^r e^{\theta x} * m(x)dx]^{-1} - e^{a\theta} * \int_a^r x * e^{\theta x} * m(x)dx * [\int_a^r e^{\theta x} * m(x)dx]^{-2}$$

When $\theta \to 0$,

$$\frac{d}{d_\theta} e^{-\theta*(x(\theta)-a)} = a * [\int_a^r m(x)dx]^{-1} - \int_a^r x * m(x)dx - [\int_a^r m(x)dx]^{-1} = a - \mu \qquad \textbf{(2)}$$

$$\text{From (1),(2)} \Rightarrow \lim_{\theta \to 0} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = \lim_{\theta \to 0} \frac{1 - e^{-\theta(x(\theta)-a)}}{\theta} = \lim_{\theta \to 0} \frac{\frac{d}{d_\theta} * (1 - e^{-\theta(x(\theta)-a)})}{\frac{d}{d_\theta}\theta} = \frac{-(a-\mu)}{1}$$

$$\Rightarrow \lim_{\theta \to 0} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = \mu - a$$

APPENDIX 7

Illustrative Examples:

The following examples show the values of λ under the different distributions when they are calculated based on the function $x(\theta)$; i.e. $\lambda = \frac{1}{\overset{-\theta}{a}_{\overline{x(\theta)-a|}}}$.

For the calculations we have assumed that:

$i = 0.05$, $\tau' = 0.03$

Entry age $a = 30$, age of retirement $r = 65$.

The results are presented in the following table, 4.8:

Table 4.8: *λ values calculated based on $x(\theta)$*

Distributions	**λ value when θ ≠ 0**	**λ value when θ = 0**
Power fn, p = 0.8	0.070	0.064
Uniform dn, p = 1	0.064	0.057
Power fn, p = 1.5	0.056	0.048
Trunc. Pareto, k =0.3	0.074	0.068
Trunc. Pareto, k =0.8	0.080	0.074
Trunc. Pareto, k =1.5	0.11	0.083
Trunc. Exponential, σ = 30	0.076	0.071
Trunc. Exponential, σ = 40	0.073	0.067
Trunc. Exponential, σ = 50	0.071	0.065

In table (4.8) we observe that λ acquires the highest values under the Truncated Pareto when k = 0.3 and the lowest ones under the Power function when p = 1.5.

We also observe that under the Power function and the Truncated Exponential, the λ values decrease as the parameters p and σ increase while under the Truncated Pareto they increase as k increases.

The λ values that are calculated based on the function x(θ), may be used as an indication for the length, in years 'm', of the amortization period. Calculating 'm' based on the equation $\lambda = \frac{1}{\bar{a}_{\overline{m}|}}$ we obtain the following result:

$m = -\frac{1}{\delta} * \ln(1 - \frac{1}{\lambda} * \delta)$, 'm' is determined if $(1 - \frac{1}{\lambda} * \delta) > 0$.

In particular, substituting the λ values presented in table 4.8 in the equation $\lambda = \frac{1}{\bar{a}_{\overline{m}|}}$ and solving with respect to 'm', the number of years 'm' indicated are presented in table 4.9, below:

Table 4.9: *Length of the amortization period based on λ values*

Distributions	**length of the amortization period (in years), θ ≠ 0**	**length of the amortization period (in years), θ = 0**
Power fn, p = 0.8	24	29
Uniform dn, p = 1	29	40
Power fn, p = 1.5	42	n/a[12]
Trunc. Pareto, k =0.3	22	26
Trunc. Pareto, k =0.8	19	22
Trunc. Pareto, k =1.5	12	18
Trunc. Exponential, σ = 30	21	24
Trunc. Exponential, σ = 40	23	27
Trunc. Exponential, σ = 50	24	28

[12] The value of m is not available, since $(1-(\lambda/\delta)) < 0$

Limiting values of λ

For each distribution we calculate the limiting values of λ as the relevant parameter value (p, k, σ), tends either to infinity or to zero.

Power function

$$\bar{a}^{\theta}_{\overline{x(\theta)-a|}} = \begin{cases} \frac{1}{\theta} * (1 - e^{a\theta} * \dfrac{(r-a)^p}{p * \int_a^r e^{\theta x} * (x-a)^{p-1} dx}) , & \theta \neq 0 \\ p * \dfrac{r-a}{p+1} , & \theta = 0 \end{cases}$$

When θ ≠ 0, consider

$$A = e^{a\theta} * \frac{(r-a)^p}{p * \int_a^r e^{\theta x} * (x-a)^{p-1} dx} = \frac{(r-a)^p}{p * \int_a^r e^{\theta(x-a)} * (x-a)^{p-1} dx} = \frac{(r-a)^p}{\int_0^{r-a} e^{\theta*(x-a)} * d(x-a)^p dx} =$$

$$\frac{(r-a)^p}{(x-a)^p * e^{\theta*(x-a)}\Big|_0^{r-a} - \theta * \int_0^{r-a} e^{\theta*t} * t^p dt} = \frac{(r-a)^p}{(r-a)^p * e^{\theta*(r-a)} - \theta * \int_0^{r-a} e^{\theta*t} * t^p dt} =$$

$$= \frac{1}{e^{\theta*(r-a)} - \theta * \int_0^{r-a} e^{\theta*t} * (\frac{t}{r-a})^p dt}$$

$$\lim_{p\to\infty} (\frac{t}{r-a})^p = 0 \Rightarrow \lim_{p\to\infty} A = e^{-\theta*(r-a)} \quad \textbf{(1)}$$

$$\lim_{p\to 0} (\frac{t}{r-a})^p = 1 \Rightarrow \lim_{p\to 0} A = \frac{1}{e^{\theta*(r-a)} - \theta * \frac{1}{\theta} * (e^{\theta*(r-a)} - 1)} = 1 \quad \textbf{(2)}$$

$$\lim_{p\to\infty} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = \overset{-\theta}{a}_{\overline{r-a|}} \Rightarrow \lim_{p\to\infty} \lambda = (\overset{-\theta}{a}_{\overline{r-a|}})^{-1}$$

From **(1)** , **(2)** $\Rightarrow$ $\{$

$$\lim_{p\to 0} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = 0 \Rightarrow \lim_{p\to 0} \lambda = \infty$$

When θ = 0

$$\lim_{p\to\infty} \mathrm{p} * \frac{r-a}{p+1} = \lim_{p\to\infty} \frac{r-a}{1+\frac{1}{p}} = \mathrm{r}-a \qquad \textbf{(3)}$$

$$\lim_{p\to 0} \mathrm{p} * \frac{r-a}{p+1} = \lim_{p\to 0} \frac{r-a}{1+\frac{1}{p}} = 0 \qquad \textbf{(4)}$$

$$\lim_{p\to\infty} \lambda = \frac{1}{r-a}$$

From **(3)** , **(4)** $\Rightarrow$ $\{$

$$\lim_{p\to 0} \lambda = \infty$$

<u>Truncated Pareto:</u>

$$\frac{1}{\theta} * (1 - \mathrm{e}^{\alpha\theta} * \frac{1-(\frac{a}{r})^k}{k * a^k * \int_a^r e^{\theta * x} * x^{-(k+1)} dx}) \text{ , } \theta \neq 0$$

$$\overset{-\theta}{a}_{\overline{x(\theta)-a|}} = \{$$

$$\frac{k * a^k * (r^{1-k} - a^{1-k})}{(1-k) * (1-(\frac{a}{r})^k)} - a \text{ , } \theta = 0$$

When θ ≠ 0, consider

$$A = \frac{1-(\frac{a}{r})^k}{k*a^k*\int_a^r e^{\theta*x}*x^{-(k+1)}dx} = \frac{1-(\frac{a}{r})^k}{-a^k*\int_a^r e^{\theta*x}*dx^{-k}} =$$

$$= \frac{1-(\frac{a}{r})^k}{-a^k*(x^{-k}*e^{\theta*x}\Big|_a^r - \theta*\int_a^r e^{\theta*x}*x^{-k}dx)} =$$

$$= \frac{1-(\frac{a}{r})^k}{-(\frac{a}{r})^k*e^{\theta*r}+e^{\theta*a}+\theta*\int_a^r e^{\theta*x}*(\frac{a}{x})^k dx}$$

$$a \le x \le r \Rightarrow \lim_{k\to\infty}(\frac{a}{r})^k \Rightarrow \lim_{k\to\infty} A = e^{-a*\theta} \qquad \textbf{(5)}$$

$$\lim_{k\to 0}(\frac{a}{r})^k = 1 \Rightarrow \lim_{k\to 0} A \overset{l'\,Hopital\ rule}{=} \lim_{k\to 0}\frac{-(\frac{a}{r})^k*\ln(\frac{a}{r})}{-(\frac{a}{r})^k*\ln(\frac{a}{r})*e^{\theta*r}+\theta*\int_a^r e^{\theta*x}*(\frac{a}{x})^k*\ln(\frac{a}{x})dx} =$$

$$= \frac{-\ln(\frac{a}{r})}{-\ln(\frac{a}{r})*e^{\theta*r}+\theta*\int_a^r e^{\theta*x}*\ln(\frac{a}{x})dx} =$$

$$\frac{-\ln(\frac{a}{r})}{-\ln(\frac{a}{r})*e^{\theta*r}+e^{\theta*x}*\ln(\frac{a}{x})\Big|_a^r-\int_a^r e^{\theta*x}*(-\frac{a}{x^2})*\frac{1}{\frac{a}{x}}dx} = \frac{-\ln(\frac{a}{r})}{\int_a^r e^{\theta*x}*\frac{1}{x}dx} \qquad \textbf{(6)}$$

$$\lim_{k\to\infty} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = 0 \Rightarrow \lim_{k\to\infty}\lambda = \infty$$

From **(5)** , **(6)** ⇒ {

$$\lim_{k\to 0} \overset{-\theta}{a}_{\overline{x(\theta)-a|}} = \lim_{k\to 0}\lambda = (\frac{1}{\theta}*(1-e^{a*\theta}*\frac{-\ln(\frac{a}{r})}{\int_a^r e^{\theta*x}*\frac{1}{x}dx})^{-1}$$

Remark: The integral $\int_a^r e^{\theta * x} * \frac{1}{x} dx$ may be approximated either after analyzing $e^{\theta * x}$ as follows: $\int_a^r e^{\theta * x} * \frac{1}{x} dx = \ln(\frac{r}{a}) + \sum_{k=1}^{\infty} \frac{(\theta * r)^k}{k! * k} - \sum_{k=1}^{\infty} \frac{(\theta * a)^k}{k! * k}$ or by using an approximate method of numerical integration (for example, Simpson's rule or the trapezoidal rule).

When $\theta = 0$

$$\lim_{k \to \infty} \frac{k * a^k * (r^{1-k} - a^{1-k})}{(1-k) * (1 - (\frac{a}{r})^k)} - a = \lim_{k \to \infty} \frac{1}{\frac{1}{k} - 1} * \frac{r * (\frac{a}{r})^k - a}{1 - (\frac{a}{r})^k} - a = 0 \qquad \textbf{(7)}$$

$$\lim_{k \to 0} \frac{k * a^k * (r^{1-k} - a^{1-k})}{(1-k) * (1 - (\frac{a}{r})^k)} - a \overset{l'Hopital rule}{=} \lim_{k \to 0} * \frac{r * (\frac{a}{r})^k - a + k * \ln(\frac{a}{r}) * r * (\frac{a}{r})^k}{-(1 - (\frac{a}{r})^k) + (1-k) * (-\ln(\frac{a}{r}) * (\frac{a}{r})^k)} - a =$$

$$= \frac{r - a}{-\ln(\frac{a}{r})} - a \qquad \textbf{(8)}$$

From **(7)** , **(8)** $\Rightarrow$

$$\begin{cases} \lim_{k \to \infty} \lambda = \infty \\ \lim_{k \to 0} \lambda = \left(\frac{r-a}{-\ln(\frac{a}{r})} - a\right)^{-1} \end{cases}$$

<u>Truncated Exponential:</u>

$$a_{\overline{x(\theta)-a}|}^{-\theta} = \begin{cases} \frac{1}{\theta} * \left(1 - \frac{a_{\overline{r-a}|}^{-(\frac{1}{\sigma})}}{s_{\overline{r-a}|}^{-(\theta - \frac{1}{\sigma})}}\right), \theta \neq 0 \\ \sigma - \frac{r-a}{e^{\frac{r-a}{\sigma}} - 1}, \theta = 0 \end{cases}$$

When θ ≠ 0, consider

$$\lim_{\sigma\to\infty} {}^{-(\theta-\frac{1}{\sigma})}s_{\overline{r-a}|} = \lim_{\sigma\to\infty} \frac{e^{(\theta-\frac{1}{\sigma})*(r-a)}-1}{\theta-\frac{1}{\sigma}} = {}^{-\theta}s_{\overline{r-a}|}$$

$$A=\frac{{}^{-(\frac{1}{\sigma})}a_{\overline{r-a}|}}{{}^{-(\theta-\frac{1}{\sigma})}s_{\overline{r-a}|}} \Rightarrow \lim_{\sigma\to\infty} A = \lim_{\sigma\to\infty} \frac{{}^{-(\frac{1}{\sigma})}a_{\overline{r-a}|}}{{}^{-(\theta-\frac{1}{\sigma})}s_{\overline{r-a}|}} = \frac{\lim_{\sigma\to\infty}\frac{1-e^{-(r-a)*\frac{1}{\sigma}}}{\frac{1}{\sigma}}}{{}^{-\theta}s_{\overline{r-a}|}} = \frac{r-a}{{}^{-\theta}s_{\overline{r-a}|}} \qquad \textbf{(9)}$$

$$\lim_{\sigma\to 0} A = \lim_{\sigma\to 0} \frac{{}^{-(\frac{1}{\sigma})}a_{\overline{r-a}|}}{{}^{-(\theta-\frac{1}{\sigma})}s_{\overline{r-a}|}} = \lim_{\sigma\to 0} \frac{\frac{1-e^{-(r-a)*\frac{1}{\sigma}}}{\frac{1}{\sigma}}}{\frac{e^{(r-a)*(\theta-\frac{1}{\sigma})}-1}{(\theta*\sigma-1)*\frac{1}{\sigma}}} = 1 \qquad \textbf{(10)}$$

From **(9)** , **(10)** ⇒

$$\begin{cases} \lim_{\sigma\to\infty} {}^{-\theta}a_{\overline{x(\theta)-a}|} = \frac{1}{\theta}\left(1-\frac{r-a}{{}^{-\theta}s_{\overline{r-a}|}}\right) \Rightarrow \lim_{\sigma\to\infty} \lambda = \theta * \left(1-\frac{r-a}{{}^{-\theta}s_{\overline{r-a}|}}\right)^{-1} \\ \lim_{\sigma\to 0} {}^{-\theta}a_{\overline{x(\theta)-a}|} = 0 \Rightarrow \lim_{\sigma\to 0} \lambda = \infty \end{cases}$$

When θ = 0

$$\lim_{\sigma\to\infty} \sigma - \frac{r-a}{e^{\frac{r-a}{\sigma}}-1} = \lim_{\sigma\to\infty} \sigma * \left(1-(r-a)*\frac{1}{{}^{-\frac{1}{\sigma}}s_{\overline{r-a}|}}\right) = \infty \qquad \textbf{(11)}$$

$$\lim_{\sigma\to 0} \sigma - \frac{r-a}{e^{\frac{r-a}{\sigma}}-1} = \lim_{\sigma\to 0} \sigma * \left(1-(r-a)*\frac{1}{{}^{-\frac{1}{\sigma}}s_{\overline{r-a}|}}\right) = 0 \qquad \textbf{(12)}$$

$$\lim_{\sigma \to \infty} \lambda = 0$$

From **(11)**, **(12)** $\Rightarrow$ {

$$\lim_{\sigma \to 0} \lambda = \infty$$

The above results are summarised in the following tables, 4.10, 4.11:

Table 4.10: *Limiting values for λ, based on $x(\theta)$, as either parameter $\to \infty$*

Distributions	**λ value when $\theta \neq 0$**	**λ value when $\theta = 0$**
Power function, $\lim_{p \to \infty} \lambda$	$(a^{\frac{-\theta}{r-a}})^{-1}$	$\frac{1}{r-a}$
Trunc. Pareto, $\lim_{k \to \infty} \lambda$	∞	∞
Trunc. Exponential, $\lim_{\sigma \to \infty} \lambda$	$\theta * (1 - \frac{r-a}{s^{\frac{-\theta}{r-a}}})^{-1}$	0

Table 4.11: *Limiting values for λ, based on $x(\theta)$, as either parameter $\to 0$*

Distributions	**λ value when $\theta \neq 0$**	**λ value when $\theta = 0$**
Power function, $\lim_{p \to 0} \lambda$	∞	∞
Trunc. Pareto, $\lim_{k \to 0} \lambda$	$\left(\frac{-\ln(\frac{a}{r})}{-\ln(\frac{a}{r}) * e^{\theta * r} + \theta * \int_a^r e^{\theta * x} * \ln(\frac{a}{x}) dx}\right)^{-1}$	$\left(\frac{r-a}{-\ln(\frac{a}{r})} - a\right)^{-1}$
Trunc. Exponential, $\lim_{\sigma \to 0} \lambda$	∞	∞

Remarks:

Tables 4.10 and 4.11 show that λ value is bounded under Power function and Truncated Exponential as either parameter (p,σ) tends to infinity. As either p or σ tends to zero, λ tends to infinity. In the Truncated Pareto the opposite is observed; λ value is bounded as k tends to zero while it tends to infinity as k tends to infinity.

APPENDIX 8

$$\int_{r}^{\omega} e^{-\tau(x-r)} * \frac{l_x}{l_r} * \bar{a}_x^{\,\delta-\beta} * e^{\beta*(x-r)} dx = \frac{l_r}{\delta-\tau} * \left(\bar{a}_r^{\,(\tau-\beta)} - \bar{a}_r^{\,(\delta-\beta)} \right)$$

Proof:

$$\int_{r}^{w} e^{\tau*(r-x)} * l_x * \bar{a}_x^{\,(\delta-\beta)} * e^{\beta(x-r)} dx = \int_{x=r}^{w} e^{-(\tau-\beta)*(x-r)} * l_x * \int_{u=0}^{w} e^{-(\delta-\beta)u} * {}_u p_x \, du \, dx$$

$$= \int_{x=r}^{w} e^{-(\tau-\beta)*(x-r)} * \int_{u=0}^{w} e^{-(\delta-\beta)u} * l_{x+u} \, du \, dx \qquad \textbf{(1)}$$

Setting x+u = y, then : r < x < y < w , u = y – x, and (1) becomes:

$$\int_{x=r}^{w} e^{-(\tau-\beta)*(x-r)} * \int_{y=x}^{w} e^{-(\delta-\beta)*(y-x)} * l_y \, dy \, dx = \int_{y=r}^{w} l_y \int_{x=r}^{y} e^{-(\tau-\beta)*(x-r)} * e^{-(\delta-\beta)*y} * e^{(\delta-\beta)*x} \, dx \, dy =$$

$$= \int_{y=r}^{w} l_y * e^{-(\delta-\beta)*y} * e^{(\tau-\beta)*r} * \int_{x=r}^{y} e^{(\delta-\tau)*x} \, dx \, dy =$$

$$= \int_{y=r}^{w} l_y * e^{-(\delta-\beta)*y} * e^{(\tau-\beta)*r} * \left[\frac{1}{\delta-\tau} * \left(e^{(\delta-\tau)*y} - e^{(\delta-\tau)*r} \right) \right] dy =$$

$$= \frac{1}{\delta-\tau} * \left[\int_{y=r}^{w} l_y * e^{-(\delta-\beta)*y} * e^{(\tau-\beta)*r} * e^{(\delta-\tau)*y} \, dy - \int_{y=r}^{w} l_y * e^{-(\delta-\beta)*y} * e^{(\tau-\beta)*r} * e^{(\delta-\tau)*r} \, dy \right] =$$

$$= \frac{1}{\delta-\tau} * \left[\int_{y=r}^{w} l_y * e^{-(\tau-\beta)*(y-r)} \, dy - \int_{y=r}^{w} l_y * e^{-(\delta-\beta)*(y-r)} \, dy \right] \qquad \textbf{(2)}$$

Setting y – r = t, then 0 < t < w-r , and (2) becomes:

$$\frac{1}{\delta-\tau} * \left[\int_{t=0}^{w-r} l_{r+t} * e^{-(\tau-\beta)*t} \, dt - \int_{t=0}^{w-r} l_{r+t} * e^{-(\delta-\beta)*t} \, dt \right] =$$

$$= \frac{l_r}{\delta-\tau} * \left[\int_{t=0}^{w-r} {}_t p_r * e^{-(\tau-\beta)*t} \, dt - \int_{t=0}^{w-r} {}_t p_r * e^{-(\delta-\beta)*t} \, dt \right] = \frac{l_r}{\delta-\tau} * \left(\bar{a}_r^{\,(\tau-\beta)} - \bar{a}_r^{\,(\delta-\beta)} \right)$$

APPENDIX 9

$$D'(r) = \frac{(\frac{k}{r})*(\frac{a}{r})^k}{1-(\frac{a}{r})^k} - \frac{p}{r-a} \qquad , a \prec x \prec r \ , p \succ 0 \ , \ 0 \prec k \prec 1$$

We need to find the necessary and sufficient conditions so that D'(r)< 0 and $0 \prec k \prec 1$

$$D'(r)<0 \Rightarrow k < \frac{p}{d}$$

Let $\frac{a}{r}$ = d, d<1, then: $\frac{(\frac{k}{r})*d^k}{1-d^k} - \frac{\frac{p}{r}}{1-d} < 0 \Leftrightarrow \frac{k*d^k}{1-d^k} - \frac{p}{1-d} < 0$

$$-d^k < -d \Rightarrow 1 - d^k < 1 - d \Rightarrow \frac{1}{1-d^k} > \frac{1}{1-d}$$

$$d^k > d \Rightarrow \{$$

$$k*d < k*d^k$$

$$\frac{kd}{1-d^k} - \frac{p}{1-d} < \frac{k*d^k}{1-d^k} - \frac{p}{1-d} < 0 \Rightarrow$$

$$\Rightarrow \frac{kd}{1-d} - \frac{p}{1-d} < \frac{kd}{1-d^k} - \frac{p}{1-d} < \frac{k*d^k}{1-d^k} - \frac{p}{1-d} < 0$$

$$\Rightarrow \frac{kd}{1-d} - \frac{p}{1-d} < 0 \Rightarrow kd < p \Rightarrow k < \frac{p}{d}$$

<u>Inverse</u>

$$k < \frac{p}{d} \Rightarrow D'(r) < 0$$

Let $k < \frac{p}{d} \Rightarrow kd < p \Rightarrow k*d^k < pd^{k-1} \Rightarrow \frac{k*d^k}{1-d^k} < \frac{pd^{k-1}}{1-d^k} \Rightarrow$

$$\Rightarrow \frac{k*d^k}{1-d^k} < \frac{p*d^{k-1}}{1-d}\frac{1-d}{1-d^k} \Rightarrow \frac{k*d^k}{1-d^k} - \frac{p}{1-d} < \frac{p}{1-d}*(d^{k-1}*\frac{1-d}{1-d^k} - 1) \qquad \textbf{(1)}$$

$$d^{k-1} * \frac{1-d}{1-d^k} - 1 = - \frac{1-d^{k-1}}{1-d^k} < 0 \qquad \textbf{(2)}$$

From (1) and (2) $\Rightarrow$ $\frac{k * d^k}{1-d^k} - \frac{p}{1-d} < 0$

APPENDIX 10

Approximation of the integral value $\int^{\omega} l_x * e^{\beta*(x-r)}\, dx = \int_{0}^{\omega-r} l_{r+t} * e^{\beta*t}\, dt$

Applying Simpson's rule

Setting $l_{r+t} * e^{\beta*t} = f(t)$, the approximation using Simpson's rule is obtained by dividing

$[0,\omega\text{-}r]$ into an even number of equal intervals, of length $\Delta_t = 1 = \frac{\omega - r}{\omega - r}$, and

approximating f(t) by a quadratic through 3 successive points corresponding to t_0, t_1, t_2, t_3, t_4,..., t_{n-2}, t_{n-1}, t_n. Geometrically, this replaces the curve y = f(t) by a set of approximating parabolic arcs.

According to the above, the integral value is approximated as follows:

$$\int_{0}^{\omega-r} l_{r+t} * e^{\beta*t}\, dt \approx \frac{\Delta_x}{3} * \{ y_0 + 4* y_1 + 2*y_2 + 4*y_3 + \ldots +2* y_{n-2} + 4* y_{n-1} + y_n \},$$

where $y_i = l_{r+i} * e^{\beta*i}$, $i = 0,\ldots,\omega\text{-}r$

Applying Rectangular rule

The approximation using Rectangular rule was obtained by dividing $[0,\omega\text{-}r]$ into a

number of equal intervals, of length $\Delta_t = 1 = \frac{\omega - r}{\omega - r}$.

The integral value is approximated as follows:

$$\int_{0}^{\omega-r} l_{r+t} * e^{\beta*t}\, dt \approx \Delta_t * \{y_0 + y_1 + y_2 + y_3 + \ldots + y_{n-2} + y_{n-1} \} \textbf{ or } \Delta_t * \{ y_1 + y_2 + y_3 + \ldots + y_{n-1} + y_n \},$$

where $y_i = l_{r+i} * e^{\beta*i}$, $i = 0,\ldots,\omega\text{-}r$

CHAPTER 5

STOCHASTIC PENSION FUND MODELING

5.1 INTRODUCTION

In practice, actuarial assumptions are set on the basis of the historical experience provided by the pension plan over the years. However, most of times these assumptions are not exactly realized since important new issues keep arising in the market and/or the economic environment. As a consequence, often these assumptions need to be revised in order to fulfill the new requirements.

Since actuaries have to estimate in advance the possible range of deviations between the assumptions set and reality, they have to implement methods to deal with these deviations.

In this chapter, such methods are discussed, focusing on the effect of varying either investment returns or the spread parameter since both may determine, at a high level, the value of the Fund. Considering either one of these or both as random variables, we are taking into account their deviation from a fixed value zone; the expected value of each parameter. On this basis the deviation of the Fund value from its expected value is calculated.

The idea behind this approach is to allow for a range of different parameter values that will influence positively or negatively the Fund level over time. Based on this approach we can measure the sensitivity of Fund values and Contribution rates over a long year period to changes in assumptions.

We shall consider a simple theoretical model which is constructed to be independent of the distribution assumptions; which has been described in chapter 4 by the following pair of equations (4.8) and (4.9):

$C(t) = NC(t) + \lambda(t)*UL(t)$, and $UL(t) = AL(t) - F(t)$, where $t \geq 0$.

The model is applied to a defined benefit pension plan, and a regular valuation is carried out. We calculate the first and second moments of the Fund and Contribution rates, considered as random variables, based on the discrete time formulation.

The following cases are considered:

a) where the rates of investment return are independent, identically distributed random variables and the spread parameter 'λ' is constant.

b) where the spread parameter is a random variable and the rates of investment return '$i(t)$' are constant.

c) where both the rates of investment return '$i(t)$' and the spread parameter '$\lambda(t)$' are independent identically distributed random variables given also that they are mutually independent.

5.2 PENSION FUNDING WHERE the RATES of INVESTMENT RETURN are RANDOM VARIABLES

The rate of investment return has a significant effect in pension funding.
As mentioned in section 1.1, Dufresne (1986) made the transition from the deterministic model to the stochastic one assuming that the rates of investment return are independent identically distributed random variables. Thereafter, Haberman (1992,1993), Owadally and Haberman (1999) and others have implemented pension models according to that approach.
We will also follow this assumption considering a defined benefit pension model where the salary function includes salary increases at a rate τ.
C(t) and F(t) are estimated every year, based on the membership of the scheme at that time. As t changes, however, new entrants are allowed to the membership so that the population remains stationary.
For the calculations we assume that the rate of investment return earned on the fund during the period (t, t+1) is i(t+1), where E(i(t+1)) = i. We further define
σ_i^2 = Var(i(t+1)).

The moments of F(t) and C(t) are estimated on the basis that both contribution income and benefit outgo occur at start of each scheme year, implying along with the previous stated assumptions that the equation of equilibrium holds with annual valuations; i.e. AL = (1+i) * (AL + NC – B).

5.2.1 MOMENTS of F(t) and C(t)

Assuming that the spread parameter 'λ' is a constant, and that i(t) and F(t) are independent, we calculate the first and second moments of F(t), C(t)
i.e. E (F(t)), Var (F(t)) and E (C(t)), Var (C(t))

5.2.1.1 Expected Value

Calculation of EF(t)

$$E(F(t)) = E(i(t)+1) * (E(F(t-1)) + \lambda * (AL(t-1) - E(F(t-1))) + NC(t-1) - B(t-1)) \quad \textbf{(5.1)}$$

$$(5.1) \overset{(4.4)-(4.7)}{\Rightarrow} E(F(t)) = (1+i) *((1-\lambda) * E(F(t-1)) + e^{\tau*(t-1)} (NC(0) + \lambda * AL(0) - B(0)) \quad \textbf{(5.2)}$$

Setting $q=(1+i)*(1-\lambda)$ and based on the equation of equilibrium, (5.2) becomes:

$$E(F(t)) = q* E(F(t-1)) + e^{\tau*(t-1)} *(1-q)*AL(0) \quad \textbf{(5.3)}$$

Setting EF(t) = F(0) at time t = 0,

When t = 1, $E(F(1)) = q * F(0) + (1-q) * AL(0)$ **(5.4)**

Based on (5.4), (5.3) can be rewritten as:

$$EF(t)=q^t*F(0)+(1-q)*AL(0) *(q^{t-1} + q^{t-2} * e^{\tau} + q^{t-3} * e^{2*\tau} +..+ q * e^{(t-2)*\tau} + e^{(t-1)*\tau}) \quad \textbf{(5.5)}$$

$$\Rightarrow EF(t) = q^t*F(0) + (1-q)*AL(0) * \sum_{n=0}^{t-1} q^{t-(1+n)} * e^{\tau*n} =$$

$$= q^t * F(0) + (1-q)*AL(0) *q^{t-1} * \sum_{n=0}^{t-1} q^{-n} * e^{\tau*n} = q^t * F(0) + (1-q)*AL(0)* \frac{q^t - e^{\tau*t}}{q - e^{\tau}} \quad \textbf{(5.6)}$$

Setting as $(1-q)*AL(0)* \frac{1}{q - e^{\tau}} = c$, (5.6) is finally written as:

$$EF(t) = q^t*(F(0) + c) - e^{\tau*t} *c \quad \textbf{(5.7)}$$

Calculation of EC(t)

$$EC(t) = E(NC(t) + \lambda*(AL(t) - F(t)) \overset{(4.4)-(4.7)}{\Rightarrow} EC(t)= e^{\tau*t} *(NC(0) + \lambda*AL(0)) - \lambda * EF(t)$$

$$\overset{(5.7)}{\Rightarrow} EC(t)= e^{\tau*t} *(NC(0) + \lambda*AL(0)) - \lambda * (q^t * (F(0) + c) - e^{\tau*t} * c) \Rightarrow$$

$$\Rightarrow EC(t)= e^{\tau*t} *(NC(0) + \lambda*(AL(0) + c)) - \lambda * q^t * (F(0) + c) \quad \textbf{(5.8)}$$

These results ((5.7) and (5.8)) show that when the salary function is considered, the results of Dufresne (1988) and Owadally and Haberman (1999) can be extended, after including the salary rate of increase. When $\tau = 0$, our results are identical to those of Dufresne (1988) and Owadally and Haberman (1999).

The salary rate of increase is a key factor that affects both EF(t) and EC(t). A key assumption affecting the level of the Actuarial Liability is the difference between the rate of increase of salaries and the discount rate. In an economic environment where $i(t) > -1$ with a high probability, high values of τ lead to low values of EF(t) with a corresponding increase of EC(t) and vice versa. During periods of high inflation with consequent large salary increases this could represent an onerous provision for the scheme sponsor.

This is clearly observed from the illustrative examples presented in the table below, where our results ($\tau \neq 0$) are compared with those of Dufresne and Owadally & Haberman ($\tau = 0$), for the case where $E(i(t)) = 0.05$ [13]

Comparison of the first moments EF(t)when $\tau = 0$ and $\tau \neq 0$

year	$\tau = 0.00$	$\tau = 0.03$	$\tau = 0.05$
1	9.48	9.20	9.02
5	43.22	39.65	37.47
10	77.32	66.57	60.49
20	125.43	97.26	83.32
30	155.36	111.40	91.94
50	185.57	120.93	96.42

[13] $\lambda = \frac{1}{\ddot{a}_{\overline{15|}}^{(0.05)}}$

Limiting Values:

In order to examine the limiting values we will consider the ratios $\frac{EF(t)}{e^{\tau * t}}$ and $\frac{EC(t)}{e^{\tau * t}}$ so that to deal with 'real' values that allow for inflation.

Considering λ as a penal rate of interest charged on the unfunded liability (as suggested by Dufresne (1988)),q = (1+i) * (1-λ) may be seen as the rate of return earned during a year in excess of the amortization charge. Also $\frac{q}{e^{\tau}} = e^{-\tau} * (1+i) * (1-\lambda)$ may be seen as the net rate of return earned during a year over the salary rate of increase, in excess of the amortization charge.

Convergence in each model for the first moments (EF(t), EC(t)) is obtained, if the interest earned during a year, in excess of the amortization charge, is lower than the salary rate of increase. In particular, under this assumption:

$$\frac{EF(t)}{e^{t\tau}} = (F(0) + c) * \left(\frac{q}{e^{\tau}}\right)^{t} - c \qquad \textbf{(5.9)}$$

It can be seen that if i>-1, $\frac{q}{e^{\tau}} < 1$. Then, as $t \rightarrow \infty \Rightarrow$

$$\frac{EF(\infty)}{e^{\tau * t}} \rightarrow -c = AL(0) * \frac{q-1}{q-e^{\tau}} \qquad \textbf{(5.10)}$$

i.e. the ultimate value of the expected fund level equals the actuarial liability increased by salary growth.

$$\frac{EC(t)}{e^{\tau * \alpha}} = NC(0) + \lambda * (AL(0) + c) - \lambda * (F(0) + c) * \left(\frac{q}{e^{\tau}}\right)^{t} \qquad \textbf{(5.11)}$$

as $t \rightarrow \infty \Rightarrow \frac{EC(\infty)}{e^{\tau * \alpha}} \rightarrow NC(0) + \lambda * (AL(0) + c) =$

$$= NC(0) + \lambda * AL(0) * \frac{1-e^{\tau}}{q-e^{\tau}} \qquad \textbf{(5.12)}$$

i.e. the ultimate value of the expected contribution level equals the sum of the normal cost and the amortized part of the actuarial liability, which is increased by salary growth.

In the case $\tau = 0$, $c = -AL(0)$ and the formulae (5.7) and (5.8) become :

$$EF(t) = q^t * (F(0) - AL(0)) + AL(0) \quad \textbf{(5.13)}$$

$$EC(t) = NC(0) - \lambda * q^t *(F(0) - AL(0)) \quad \textbf{(5.14)}$$

as derived by Dufresne (1986,1988).

<u>Limiting Values</u>:

It can be seen that if $i>-1$, $q < 1$. Then as $t \to \infty \Rightarrow$

$$EF(\infty) \to AL(0) \quad \textbf{(5.15)}$$

and $EC(\infty) \to NC(0)$ **(5.16)**

From these formulae, ((5.15) and (5.16)), we may expect that given no growth on salaries over time, as $t \to \infty$, the initial unfunded liability will be completely amortized and the fund and contribution levels will reach their target values.

When $\tau = 0$, the results are the same as those of Dufresne (1988) and Owadally and Haberman (1999).

5.2.1.2 Variance

<u>Calculation of VarF(t)</u>

$$VarF(t) = E(F(t))^2 - (EF(t))^2 \quad \textbf{(5.17)}$$

$$E(F(t))^2 = E\{ (1+i(t)) * (F(t-1) + \lambda * (AL(t-1) - F(t-1)) + NC(t-1) - B(t-1)) \}^2 =$$

$$= E\ (1+i(t))^2 * E\{(1-\lambda)* F(t-1) + e^{\tau(t-1)} * \frac{1-q}{1+i} *AL(0) \}^2$$

$$= E\ (1+i(t))^2 * E\{\frac{q}{1+i} * F(t-1) + e^{\tau(t-1)} * \frac{1-q}{1+i} * AL(0)\}^2$$

$$= E(1+i(t))^2 * (\frac{1}{1+i})^2 *$$

$$* E\{(q^2 * (F(t-1)^2 + e^{2\tau(t-1)} * (1-q)^2 * AL(0)^2 + 2 * q * (1-q) * F(t-1) * e^{\tau(t-1)} * AL(0)\}$$

$$\Rightarrow E[F(t)]^2 = E(1 + i^2(t) + 2*i(t)) * (\frac{1}{1+i})^2 *$$

$$* E\{(q^2 * (F(t-1)^2 + e^{2\tau(t-1)} * (1-q)^2 * AL(0)^2 + 2 * q * (1-q) * F(t-1) * e^{\tau(t-1)} * AL(0)\}$$

$$\Rightarrow E[F(t)]^2 = (1 + \sigma_i^2 + i^2 + 2*i) * (\frac{1}{1+i})^2 *$$

$$* \{q^2 * E(F(t-1))^2 + e^{2\tau(t-1)} * (1-q)^2 * AL(0)^2 + 2 * q * (1-q) * E(F(t-1)) * e^{\tau(t-1)} * AL(0)\} \Rightarrow$$

$$E[F(t)]^2 = (1 + \frac{\sigma_i^2}{(1+i)^2}) * \{q^2 * VarF(t-1) + (E\ F(t))^2\} \quad \textbf{(5.18)}$$

$$(5.17) \overset{(5.18)}{\Rightarrow} VarF(t) = E(F(t))^2 - (EF(t))^2 =$$

$$= (1 + \frac{\sigma_i^2}{(1+i)^2}) * q^2 * VarF(t-1) + \frac{\sigma_i^2}{(1+i)^2} * [EF(t)]^2 \quad \textbf{(5.19)}$$

This is equivalent to the result of Owadally and Haberman (1999).

Substituting in (5.19) the value of EF(t) from (5.7), we have:

$$VarF(t) = (1 + \frac{\sigma_i^2}{(1+i)^2}) * q^2 * VarF(t-1) + \frac{\sigma_i^2}{(1+i)^2} * (q^t * (F(0)+c) - e^{\tau * t} * c)^2 \quad \textbf{(5.20)}$$

Setting at time t = 0 VarF(0) = 0, then at time t = 1,

$$VarF(1) = \frac{\sigma_i^2}{(1+i)^2} * (q * (F(0)+c) - e^{\tau} * c)^2 \quad \textbf{(5.21)}$$

Based on (5.21), (5.20) can be rewritten as:

$$VarF(t) = [(1 + \frac{\sigma_i^2}{(1+i)^2}) * q^2]^{t-1} * \frac{\sigma_i^2}{(1+i)^2} * (q * (F(0)+c) - e^{\tau} * c)^2 +$$

$$+ [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{t-2} * \frac{\sigma_i^2}{(1+i)^2} * (q^2*(F(0)+c) - e^{\tau*2} * c)^2 +$$

$$+ \ldots \frac{\sigma_i^2}{(1+i)^2} * (q^t*(F(0)+c) - e^{\tau*t} * c)^2 \Rightarrow$$

$$\Rightarrow VarF(t) = \frac{\sigma_i^2}{(1+i)^2} \sum_{n=0}^{t-1} [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{(t-1)-n} *(q^{n+1}*(F(0)+c) - e^{\tau*(n+1)}*c)^2 \quad \textbf{(5.22)}$$

Working on the summation part of (5.22) we obtain the following:

$$\sum_{n=0}^{t-1} [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{(t-1)-n} *(q^{n+1}*(F(0)+c) - e^{\tau*(n+1)}*c)^2 =$$

$$= [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{t-1} *\{(F(0)+c)^2 * q^2 * \sum_{n=0}^{t-1} \left(\frac{q^2}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2}\right)^n +$$

$$+ c^2e^{2*\tau} \sum_{n=0}^{t-1} \left(\frac{e^{2*\tau}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2}\right)^n - 2*c*(F(0)+c) * e^\tau *q* \sum_{n=0}^{t-1} \left(\frac{e^\tau * q}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2}\right)^n \} =$$

$$=[(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{t-1}*\{(F(0)+c)^2 q^2 * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - q^{2*t}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2 - q^2} *[(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{1-t} +$$

$$+ c^2e^{2*\tau} * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - e^{2*\tau*t}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2 - e^{2*\tau}} * [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{1-t} -$$

$$2*c*(F(0)+c) * e^\tau *q* \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - (e^\tau * q)^t}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2 - e^\tau * q} * [(1+\frac{\sigma_i^2}{(1+i)^2})*q^2]^{1-t} \} =$$

$$= (F(0)+c)^2 q^2 * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - q^{2*t}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2 - q^2} + c^2e^{2*\tau} * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - e^{2*\tau*t}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2 - e^{2*\tau}}$$

$$- 2*c*(F(0)+c) * e^{\tau} *q* \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t-(e^{\tau}*q)^t}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2-e^{\tau}*q} \qquad (5.23)$$

$$(4.22) \overset{(4.23)}{\Rightarrow} \text{VarF(t)} = (F(0)+c)^2 * [((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - q^{2*t}] +$$

$$+ c^2e^{2*\tau} * \frac{\sigma_i^2}{(1+i)^2} * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t - e^{2*\tau*t}}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2-e^{2*\tau}} -$$

$$- 2*c*(F(0)+c) * e^{\tau} *q* \frac{\sigma_i^2}{(1+i)^2} * \frac{((1+\frac{\sigma_i^2}{(1+i)^2})*q^2)^t-(e^{\tau}*q)^t}{(1+\frac{\sigma_i^2}{(1+i)^2})*q^2-e^{\tau}*q} \qquad (5.24)$$

Setting:

$\Psi = - (F(0)+c)^2$

$$\Theta = c^2e^{2*\tau} \frac{\sigma_i^2}{(1+i)^2} * \frac{1}{e^{2*\tau}-(1+\frac{\sigma_i^2}{(1+i)^2})*q^2} = c^2 * \frac{\sigma_i^2}{(1+i)^2} * \frac{e^{2*\tau}}{e^{2*\tau}-(1-\lambda)^2*[(1+i)^2+\sigma_i^2]}$$

$$\Omega = - 2*c* \frac{\sigma_i^2}{(1+i)^2} * (F(0)+c) * e^{\tau} * q \frac{1}{e^{\tau}*q-(1+\frac{\sigma_i^2}{(1+i)^2})*q^2} =$$

$$= - 2*c* \frac{\sigma_i^2}{(1+i)^2} * (F(0)+c) * \frac{e^{\tau}}{e^{\tau}-\frac{1-\lambda}{1+i}*[(1+i)^2+\sigma_i^2]}$$

(5.24) is rewritten as:

$$\text{VarF(t)} = \Theta * e^{2*\tau*t} + \Psi * ((1-\lambda)*(1+i))^{2*t} + \Omega * (e^{\tau}*(1-\lambda)*(1+i))^t -$$

$$- (\Theta+\Psi+\Omega) * ((1-\lambda)^2*((1+i)^2+\sigma_i^2)^t \qquad (5.25)$$

Calculation of VarC(t)

$$VarC(t) = Var(NC(t)+ \lambda*(AL(t) - F(t)) = \lambda^2 * VarF(t) \Rightarrow$$

$$\overset{(5.25)}{\Rightarrow} VarC(t) = \lambda^2 * \{ \Theta * e^{2*\tau*t} + \Psi * ((1-\lambda)*(1+i))^{2*t} + \Omega*(e^{\tau} *(1-\lambda)*(1+i))^{t} -$$

$$- (\Theta+\Psi+\Omega) * ((1-\lambda)^2*((1+i)^2 + \sigma_i^2)\} \quad \textbf{(5.26)}$$

These results ((5.25),(5.26)) show that when the salary function is considered, the results of Dufresne (1988) and Owadally and Haberman (1999) can be extended, after including the salary rate of increase. When $\tau = 0$, our results are identical to those of Dufresne (1988) and Owadally and Haberman (1999).

As with the first moments, we also expect a decrease for the second moments along with the decrease of the difference between the salary rate of increase and the rate of investment return (formula 5.17). Thus, as with the first moments, we have also calculated some illustrative examples. In particular, we have calculated the standard deviation of the Fund at time t, t>0, as a percentage of the expected value of the Fund as at that time, for the cases where $E(i(t)) = 0.05$ & $\sigma_i = 0.025$, $E(i(t)) = 0.05$ & $\sigma_i = 0.05$ and $E(i(t)) = 0.05$ & $\sigma_i = 0.15$[14]. The forthcoming results show that the level of this ratio changes quite significantly, as the standard deviation of i(t) acquires high values, since, for high values of σ_i, the Fund becomes less stable.

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[14] $\lambda = \dfrac{1}{\ddot{a}_{\overline{15|}}^{(0.05)}}$

Comparison of the Standard deviations in percent of EF(t), $\frac{(Var(F(t)))^{\frac{1}{2}}}{EF(t)}$ when $\tau = 0$

and $\tau \neq 0$

year	$\sigma_i = 0.025$			$\sigma_i = 0.05$			$\sigma_i = 0.15$		
	$\tau = 0.00$	$\tau = 0.03$	$\tau = 0.05$	$\tau = 0.00$	$\tau = 0.03$	$\tau = 0.05$	$\tau = 0.00$	$\tau = 0.03$	$\tau = 0.05$
1	2.38%	2.38%	2.38%	4.76%	4.76%	4.76%	14.29%	14.29%	14.29%
5	3.46%	3.41%	3.38%	6.92%	6.82%	6.76%	20.92%	20.63%	20.44%
10	4.44%	4.30%	4.20%	8.89%	8.60%	8.42%	27.16%	26.25%	25.65%
20	5.70%	5.30%	5.05%	11.44%	10.63%	10.13%	35.57%	32.92%	31.30%
30	6.48%	5.80%	5.42%	13.02%	11.65%	10.87%	41.13%	36.46%	33.83%
50	7.32%	6.18%	5.63%	14.73%	12.42%	11.30%	47.64%	39.30%	35.38%

Limiting Values:

In order to examine the limiting values we will consider the ratios $\frac{Var(F(t))}{e^{2*\tau*t}}$ and

$\frac{Var(C(t))}{e^{2*\tau*t}}$ so that to deal with 'real' values that allow for inflation.

Considering λ as a penal rate of interest charged on the unfunded liability, as before, $q = (1+i) * (1-\lambda)$ may be seen as the rate of return earned during a year in excess of the amortization charge. Also $\frac{q}{e^{\tau}} = e^{-\tau} *(1+i) *(1-\lambda)$ may be seen as the net rate of return earned during a year over the salary rate of increase, in excess of the amortization charge.

$$\frac{Var(F(t))}{e^{2*\tau*t}} = \Theta + \Psi * \left(\frac{q}{e^{\tau}}\right)^{2*t} + \Omega * \left(\frac{q}{e^{\tau}}\right)^{t} - (\Theta + \Psi + \Omega) * \left(\frac{(1-\lambda)^2 * ((1+i)^2 + \sigma_i^2)}{e^{2*\tau}}\right)^{t} \quad \textbf{(5.27)}$$

It can be seen that if $-1 < i < \left(\frac{e^{2*\tau}}{(1-\lambda)^2} - \sigma_i^2\right)^{\frac{1}{2}} - 1$, $\lambda < 1$, $\frac{q}{e^{\tau}} < 1$, and

$\frac{(1-\lambda)^2 * ((1+i)^2 + \sigma_i^2)}{e^{2*\tau}} < 1$. Then, as $t \to \infty \Rightarrow$

$$\Rightarrow \frac{Var(F(\infty))}{e^{2*\tau*\alpha}} \rightarrow \Theta \qquad \textbf{(5.28)}$$

$$\text{and } \frac{Var(C(\infty))}{e^{2*\tau*\alpha}} \rightarrow \lambda^2 * \frac{Var(F(\infty))}{e^{2*\tau*\alpha}} \rightarrow \lambda^2 * \Theta \qquad \textbf{(5.29)}$$

Limiting results show that, after a long period of the scheme being continued, both the Variance of the Fund and the Variance of the Contribution rates, are stabilized.

The convergence criteria for the second moments provide restrictions on the set of parameter values. In particular: Based on the inequality i < $(\frac{e^{2*\tau}}{(1-\lambda)^2} - \sigma_i^2)^{\frac{1}{2}}$ -1, we may specify an 'optimal region' m^* for the number of years m over which the unfunded liability is spread. In particular, we show that convergence is obtained, when m is less than

$$m^* = \ln\left(1 - d * \frac{1}{1 - \frac{e^{\tau}}{((1+i)^2 + \sigma_i^2)^{\frac{1}{2}}}}\right) * \frac{1}{\ln(\frac{1}{1+i})}.$$

When τ = 0, c = - AL(0) and the constants Ψ, Ω, Θ become:

$$\Psi = -(F(0)-AL(0))^2$$

$$\Theta = AL(0)^2 * \frac{\sigma_i^2}{(1+i)^2} * \frac{1}{1-(1-\lambda)^2 * [(1+i)^2 + \sigma_i^2]}$$

$$\Omega = 2 * AL(0) * \frac{\sigma_i^2}{(1+i)^2} * (F(0)-AL(0)) * \frac{1}{1 - \frac{1-\lambda}{1+i} * [(1+i)^2 + \sigma_i^2]}$$

Hence (5.25) and (5.26) are written as:

$$VarF(t) = \Theta + \Psi*((1-\lambda)*(1+i))^{2*t} + \Omega((1-\lambda)(1+i))^t - (\Theta+\Psi+\Omega)*((1-\lambda)^2*((1+i)^2+\sigma_i^2))^t \qquad \textbf{(5.30)}$$

$$\text{and } VarC(t) = \lambda^2 * VarF(t) \qquad \textbf{(5.31)}$$

Limiting Values:

It can be seen that if $-1 < i < (\frac{1}{(1-\lambda)^2} - \sigma_i^2)^{\frac{1}{2}} - 1$, $\lambda<1$, $q<1$, and

$(1-\lambda)^2 * ((1+i)^2 + \sigma_i^2) < 1$. Then, as $t \to \infty \Rightarrow$

$$\text{Var } F(\infty) \to \Theta \tag{5.32}$$

and $$\text{Var } C(\infty) \to \lambda^2 * \Theta \tag{5.33}$$

When $\tau = 0$, $m^* = \ln\left(1 - d * \frac{1}{1 - \frac{1}{((1+i)^2 + \sigma_i^2)^{\frac{1}{2}}}}\right) * \frac{1}{\ln(\frac{1}{1+i})}$.

When $\tau = 0$, the results are the same to those of Owadally and Haberman (1999).

5.2.2 ILLUSTRATIVE EXAMPLES

For the case where τ =0, we calculate the number of years m^* below which we obtain convergence for the following sets of assumptions:

Table of Assumptions

E(i(t)) = 0.03	E(i(t)) = 0.05	E(i(t)) = 0.07
$\sigma_\iota = 0.025$	$\sigma_\iota = 0.025$	$\sigma_\iota = 0.025$
$\sigma_\iota = 0.05$	$\sigma_\iota = 0.05$	$\sigma_\iota = 0.05$
$\sigma_\iota = 0.1$	$\sigma_\iota = 0.1$	$\sigma_\iota = 0.1$
$\sigma_\iota = 0.15$	$\sigma_\iota = 0.15$	$\sigma_\iota = 0.15$

We have decided that our choice of the expected value of the rates of investment return should reflect, under the current economic environment, the rate at which the liabilities could effectively be settled. Thus, we have used values that approach the rates of return on high quality fixed income investments of the appropriate maturity. The long-term Greek Government bonds as at 31/12/2001 had a yield of about 6% per year and about 5% per annum as at 31/12/2003. We expect that this decreasing trend will continue in the forthcoming years, and we expect that this rate will be higher than the long term rate of inflation that is considered to be about 2.5%.

The value of $\sigma(i(t))$ has then been decided on the basis of the possible fluctuations we may expect within a year.

According to the above assumptions, we obtain the following table for m^*

Table 5.1: *Values of m^* when i(t) are random*

σ_I	E(i(t)) = 0.03	E(i(t)) = 0.05	E(i(t)) = 0.07
0.025	156.76	106.14	82.05
0.05	110.88	78.10	61.75
0.1	67.76	51.10	41.99
0.15	45.82	36.64	31.15

Table 5.1 shows that as either the expected value or the standard deviation of i(t) increases, the number of years below which we obtain convergence decreases significantly. Specifically: When E(i(t)) is equal to 0.03 and σ_i is equal to 0.025, $m^* = 156.76$ while when E(i(t)) is equal to 0.07 and σ_i is equal to 0.15, $m^* = 31.15$. This is an expected result, since, when either one of E(i(t)) and σ_i or both increase, the Fund becomes less stable and thus the Unfunded Liability should be amortized at a higher pace.

Setting $\lambda = \frac{1}{\ddot{a}_{\overline{15}|}^{(0.05)}}$, we calculate the expected value and the variance for the Fund and Contribution rates; (formulae 5.2, 5.8 and 5.30,5.31).

For the calculations we considered the Actuarial Liability, Normal Cost and Pension Outgo values presented in tables 4.5 –4.7 at time t=0. The fund value at time t = 0, is set equal to F(0) = 0.

Their limiting values are compared to the Actuarial Liability and Normal Cost at time t = 0.

In the tables below, 5.2 and 5.3 the ultimate values, as $t \to \infty$, of EF(t) and EC(t) are presented:

Table 5.2: *EF(t) at time t ≥ 100*

Distribution fn	**E(i(t)) = 0.03**	**E(i(t)) = 0.05**	**E(i(t)) = 0.07**
Power p=0.8	263.55	204.65	165.82
Power p=1	252.33	197.84	161.49
Power p=1.5	231.13	184.63	152.84
Tr. Expon. σ=30	272.56	210.5	169.83
Tr. Expon. σ=40	267.73	207.51	167.89
Tr. Expon. σ=50	264.75	205.65	166.68
Tr. Pareto k=0.3	269.79	208.69	168.56
Tr. Pareto k=0.8	276.20	212.63	171.09
Tr. Pareto k=1.5	284.86	217.75	174.37

Table 5.3: *EC(t) at time t ≥ 100*

Distribution fn	**E(i(t)) = 0.03**	**E(i(t)) = 0.05**	**E(i(t)) = 0.07**
Power p=0.8	5.98	3.78	2.55
Power p=1	6.31	4.11	2.84
Power p=1.5	6.93	4.76	3.43
Tr. Expon. σ=30	5.71	3.49	2.28
Tr. Expon. σ=40	5.85	3.64	2.41
Tr. Expon. σ=50	5.94	3.73	2.49
Tr. Pareto k=0.3	5.79	3.58	2.36
Tr. Pareto k=0.8	5.59	3.39	2.19
Tr. Pareto k=1.5	5.32	3.10	1.97

The results in tables 5.2, 5.3, calculated for $t \geq 100$, verify the theoretical results obtained in section 5.1.1. as $t \to \infty$;this is clearly seen if we compare these results with the corresponding ones in tables 4.5 – 4.7.

We have to mention though, that in practice, as E(i(t)) increases, the limiting values of EF(t) and EC(t) are reached at a significantly lower pace. Specifically: EF(t) approaches its limiting value, i.e. $\frac{EF(t)}{\lim\limits_{t\to\infty} EF(t)} = 99\%$, at about 70 years when E(i(t))=0.03, and at about 160 years when E(i(t)) = 0.07. Importantly, when E(i(t))=0.03, $\frac{EF(t)}{\lim\limits_{t\to\infty} EF(t)}$ =50% at about 10 years while when E(i(t)) = 0.07; $\frac{EF(t)}{\lim\limits_{t\to\infty} EF(t)}$ = 50% after 24 years.

Contribution rates are adjusted accordingly; $\frac{\lim\limits_{t\to\infty} EC(t)}{EC(t)}$ = 99% at about 90 years when E(i(t)) = 0.03 and at about 220 years when E(i(t))=0.07, $\frac{\lim\limits_{t\to\infty} EC(t)}{EC(t)}$ = 50% at about 20 years when E(i(t)) = 0.03 and at about 60 years when E(i(t))=0.07.

The above practical implications are examined in detail in section 6.6.3, where the simulation results when the rates of investment return are random variables are discussed.

We may also observe that for two different EF(t) values, say $EF_1(t)$, $EF_2(t)$, as $t \to \infty$ if $EF_1(t) > EF_2(t)$ then $EC_1(t) < EC_2(t)$, where $EC_1(t)$, $EC_2(t)$ result from the $EF_1(t)$, $EF_2(t)$ values. This result may theoretically proved as follows:

<u>Proposition</u>: Let $EF_1(t)$, $EF_2(t)$, $EF_1(t) \neq EF_2(t)$, and the corresponding to those values contribution rates $EC_1(t)$, $EC_2(t)$.

Then as $t \to \infty$ if $EF_1(t) > EF_2(t) \Rightarrow EC_1(t) < EC_2(t)$

<u>Proof:</u>

Let $AL_1(t)$, $AL_2(t)$ the actuarial liability values based on which the $EF_1(t)$, $EF_2(t)$ are calculated. Then clearly, if $EF_1(t) > EF_2(t) \Rightarrow AL_1(t) > AL_2(t)$

From formula 5.8, $EC_1(t) = B - \lambda * q^t * F(0) + AL_1(t) * (\lambda * q^t - d)$ and

$EC_2(t) = B - \lambda * q^t * F(0) + AL_2(t) * (\lambda * q^t - d)$

Then: $EC_1(t) - EC_2(t) = (\lambda * q^t - d) * (AL_1(t) - AL_2(t))$

Assuming $i > -1$, as $t \to \infty$, $\lambda * q^t - d \to -d \Rightarrow EC_1(t) < EC_2(t)$

Remark:

As $t \to 0$, then $\lambda * q^t - d \to \lambda - d \Rightarrow EC_1(t) > EC_2(t)$

When $t = \ln(\frac{\lambda}{d} - q)$ then $EC_1(t) = EC_2(t)$

In Appendix 12, we may observe the development of EF(t) and EC(t) when

$F(0) = 0$ and $F(0) \neq 0$, $F(0) > AL(0)$[15] .

In Appendix 11, the ultimate values as $t \to \infty$, of VarF(t) and VarC(t) are presented (tables 5.6 – 5.11), for all cases discussed in the table of the assumptions: The theoretical results obtained in section 5.1.1, are verified, i.e.

$$VarF(\infty) \to AL(0)^2 * \frac{\sigma_i^2}{(1+i)^2} * \frac{1}{1-(1-\lambda)^2 * [(1+i)^2 + \sigma_i^2]}, \quad VarC(\infty) \to \lambda^2 * VarF(\infty).$$

In order to examine how the above results are affected as the constant λ changes, we undertake some sensitivity tests on the basis of the equation $\lambda = \frac{1}{\ddot{a}_{\overline{15|}}^{(0.05)}}$, regarding the valuation rate and the number of years. In particular, we calculate λ assuming that the valuation rate is equal consecutively to 0.03 and to 0.07, keeping m unchanged and equal to 15; and assuming that the number of years m is equal consecutively to 10 and 20, keeping the valuation rate unchanged and equal to 0.05.

[15] When $F(0) \neq 0$, $F(0) < AL(0)$ F(t), C(t) are developed as in case where $F(0) = 0$.

From the results on the next page we observe that as either the valuation rate decreases or the number of years increases, λ decreases and the Fund reaches its limiting value at a lower pace; this is as expected due to the lower pace at which the Unfunded Liability is amortized. The sensitivity of the results to changing the constant λ, is described in the following table where the value of EF(t) as a percentage of $\lim_{t\to\infty} EF(t)$ and the value of EC(t) as a percentage of $\lim_{t\to\infty} EC(t)$ are presented.

For the presentation we choose Power function, p = 0.08 and we calculate EF(t) and EC(t) assuming that E(i(t)) = 0.05.

Sensitivity results

$\lambda = \frac{1}{\ddot{a}_{\overline{m}\vert}^{(i)}}$	$\frac{1}{\ddot{a}_{\overline{15}\vert}^{(0.05)}}$	$\frac{1}{\ddot{a}_{\overline{15}\vert}^{(0.03)}}$	$\frac{1}{\ddot{a}_{\overline{15}\vert}^{(0.07)}}$	$\frac{1}{\ddot{a}_{\overline{10}\vert}^{(0.05)}}$	$\frac{1}{\ddot{a}_{\overline{20}\vert}^{(0.05)}}$
$\frac{EF(t)}{\lim_{t\to\infty} EF(t)} = 99\%$	97 yrs	128 yrs	78yrs	56 yrs	150 yrs
$\frac{\lim_{t\to\infty} EC(t)}{EC(t)} = 99\%$	132 yrs	171 yrs	108 yrs	80 yrs	198 yrs
$\frac{EF(t)}{\lim_{t\to\infty} EF(t)} = 50\%$	15 yrs	19 yrs	12 yrs	9 yrs	23 yrs
$\frac{\lim_{t\to\infty} EC(t)}{EC(t)} = 50\%$	34 yrs	42 yrs	29 yrs	23 yrs	47 yrs

It is clearly seen that the number of years over which the Unfunded Liability is spread affects significantly the development of the fund; these results are discussed in detail in section 6.7. The valuation rate also affects the level of the fund, but not as much as the spread period. In addition, we point out that if we increase the valuation rate used to calculate λ, i.e. if E(i(t)) = 0.05 and we calculate λ assuming a higher value than 0.05, the λ value increases and the fund level reaches its expected value more quickly and vice versa.

5.3 PENSION FUNDING WHERE the PARAMETER λ is a RANDOM VARIABLE

The spread parameter 'λ' determines the level of the amortization of the Unfunded Liability and its value depends on the actuary's selection of the length of the amortization period.

But to consider λ as though it were constant during a long period restricts our flexibility of strengthening or not the Fund value at certain time intervals according to the realized experience gains or losses. Since in real life, the fund fluctuates above and below its target level, if these fluctuations are not dealt with, (i.e. if the contribution rate remains fixed), then the fund will ultimately either runs out of assets from which to pay the benefits or grows exponentially out of control. In this part of chapter 5, we will consider the variability in funding levels and contribution rates when there are fluctuations in the spread parameter and how these can be controlled. As has already mentioned in section 1.2.1, in practice, actuaries are faced more frequently by schemes with unfunded liabilities. Also in practice, the pace with which the actuary amortizes gains or losses heavily depends on the scheme sponsor's financial plans. In our opinion it is important to give the actuary the flexibility to propose a pattern according to which the amortized amounts may occasionally either increase or decrease.

In chapter 4, we calculated λ based on the $x(\theta)$ function. This calculation suggested that $x(\theta)$ could be considered as a basis of approximating the years to amortize the unfunded liability; see Appendix 7. Another approach, adopted as well by the International Accounting Standards and US GAAP, is to amortize the unfunded liability, at each measurement date, over the expected future working lifetime of the existing employees. We understand that then, the λ value would change on a yearly basis according to the changes in the membership profile. Following those approaches, we thereafter extend our thoughts on defining a fixed value as a basis and allow λ varying around a mean value; in other words to treat λ as a random variable.

Treating λ as a random variable, we can calculate in advance the expected Fund and Contribution level as well as the maximum and minimum Fund and Contribution values resulting from the corresponding λ's. Thus, we become aware for the best and the least favorable performance of the Fund that may occur, during a long year period. As a consequence, we can decide for either an increase or a decrease of the λ value with some safety, on the basis of the progress of the estimated Fund level with time.

As a random variable, the spread parameter 'λ' acquires either very low or very high values not applicable in real life. To solve this problem, we may define an upper and a lower bound so that to exclude the extreme, non-desirable results. In other words, we may include the idea that the spread parameter 'λ' is a truncated random variable by introducing a range over which it varies; i.e. we may 'regard' it as random, with some safety margins.

Thinking of 'λ' as a penal interest rate, we allow it to take very high values up to the upper bound, in order to balance a possible high level of the unfunded liability. On the other hand, the lower bound may be thought as a value that sets 'security loadings' when the Unfunded Liability level approaches zero.

λ(t) as random variable is studied in the same way as i(t) being a random variable; i.e.

a) we consider a defined benefit pension model that allows for the salary function, with a salary rate of increase 'τ'

b) the model, estimates on a yearly basis C(t) and F(t) based on the membership of the scheme at that time. As t changes, however, we allow for new entrants to the membership so that the population remains stationary.

c) for the calculations we assume that the $\lambda(t)$ rate set for the unfunded liability amortization during the period (t, t+1) is $\lambda(t+1)$, where $E(\lambda(t+1)) = \lambda$ and we further define $\sigma_\lambda^2 = Var(\lambda(t+1))$.

d) the moments of F(t) and C(t) are estimated on the basis that both contribution income and benefit outgo occur at start of each scheme year, implying along with the previous stated assumptions that the equation of equilibrium holds with annual valuations: i.e. AL = (1+i) * (AL + NC – B).

5.3.1 MOMENTS of F(t) and C(t)

Assuming that the value of the rate of investment return is fixed, and that $\lambda(t)$ and F(t) are independent, we calculated the first and second moments of F(t), C(t); i.e. E(F(t)), Var(F(t)) and E (C(t)), Var (C(t)).

5.3.1.1 Calculation of EF(t) and EC(t)

Both EF(t) and EC(t) are exactly the same as the corresponding ones derived when i(t) are random in section 5.2, and as a consequence all the results produced for the first moments hold in this case as well. We have already shown in section 5.2.2 that as λ acquires high values, the unfunded liability is amortized at a higher pace, than if it acquires low values, and as a consequence the expected value of the Fund increases.

5.3.1.2 Variance

Calculation of VarF(t)

$$VarF(t) = E(F(t))^2 - (EF(t))^2 \quad \textbf{(5.34)}$$

$$E(F(t))^2 = (1+i)^2 * E[\,F(t-1) + \lambda(t) * (AL(t-1) - F(t-1)) + NC(t-1) - B(t-1)\,]^2 =$$

$$= E\,[(1+i) * (1 - \lambda(t)) * (\,F(t-1) - e^{\tau(t-1)} * AL(0)\,) + e^{\tau(t-1)} * AL(0)]^2 =$$

$$=(q^2+\sigma_\lambda^2 *(1+i)^2)*VarF(t-1)+\sigma_\lambda^2 *(1+i)^2*[E(F(t-1))-e^{\tau(t-1)}*AL(0)]^2 +[E(F(t))]^2 \quad \textbf{(5.35)}$$

where $q = (1+i)*(1-\lambda)$.

Based on (5.35), (5.34), becomes:

$$VarF(t)=(q^2+\sigma_\lambda^2 (1+i)^2)*VarF(t-1)+\sigma_\lambda^2 *(1+i)^2*[E(F(t-1)) - e^{\tau(t-1)}*AL(0)]^2 \Rightarrow$$

$$\Rightarrow VarF(t)=(q^2+\sigma_\lambda^2 (1+i)^2)*VarF(t-1)+\sigma_\lambda^2 *(1+i)^2*[E\,(U(t-1)]^2 \quad \textbf{(5.36)}$$

$$E\,(U(t-1)]= (F(t-1)) - e^{\tau(t-1)} *AL(0) \overset{(5.7)}{=} q^{t-1}* (F(0)+c) - e^{\tau(t-1)} *AL(0) *\left(\frac{1-e^\tau}{q-e^\tau}\right) \quad \textbf{(5.37)}$$

Setting also $c' = AL(0) *\left(\frac{1-e^\tau}{q-e^\tau}\right)$, (5.36) becomes:

$$VarF(t) = (q^2 +\sigma_\lambda^2 * (1+i)^2)*VarF(t-1) +\sigma_\lambda^2 *(1+i)^2*[\,q^{t-1}*(F(0)+c)-e^{\tau(t-1)} *c'\,]^2 \quad \textbf{(5.38)}$$

Setting at time $t = 0$ $VarF(0) = 0$, then at time $t = 1$,

$$VarF(1) = \sigma_\lambda^2 *(1+i)^2 *[q*(F(0)+c) - c'\,]^2 \quad \textbf{(5.39)}$$

Based on (5.39), (5.38) can be rewritten as:

$$VarF(t) = \sigma_\lambda^2 *(1+i)^2 * \sum_{n=0}^{t-1} [(q^2+\sigma_\lambda^2 *(1+i)^2]^{(t-1)-n} *(\,q^n (F(0)+c) - e^{\tau*n}*c')^2 \quad \textbf{(5.40)}$$

Working on the summation part of (5.40) we obtain the following:

$$\sum_{n=0}^{t-1} [(q^2+\sigma_\lambda^2 *(1+i)^2]^{(t-1)-n} *(\,q^n (F(0)+c) - e^{\tau*n}*c')^2 =$$

$$[(q^2+\sigma_\lambda^2 *(1+i)^2]^{t-1} *\{((F(0)+c)^2 * \sum_{n=0}^{t-1} \left(\frac{q^2}{(q^2+\sigma_\lambda^2 *(1+i)^2}\right)^n +$$

$$+ c'^2 \sum_{n=0}^{t-1} \left(\frac{e^{2*\tau}}{q^2+\sigma_\lambda^2*(1+i)^2}\right)^n - 2*c'*(F(0)+c)* \sum_{n=0}^{t-1} \left(\frac{q*e^\tau}{(q^2+\sigma_\lambda^2*(1+i)^2}\right)^n \} =$$

$$= [q^2+\sigma_\lambda^2*(1+i)^2]^{t-1}*\{(F(0)+c)^2 * \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - q^{2*t}}{(q^2+\sigma_\lambda^2*(1+i)^2)-q^2} * [q^2+\sigma_\lambda^2*(1+i)^2]^{1-t} +$$

$$+ c'^2 * \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - e^{2*\tau*t}}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^{2*\tau}} * [q^2+\sigma_\lambda^2*(1+i)^2]^{1-t}$$

$$2*c'*(F(0)+c)* \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - (e^\tau*q)^t}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^\tau*q} *[q^2+\sigma_\lambda^2*(1+i)^2]^{1-t}\} =$$

$$= (F(0)+c)^2 * \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - q^{2*t}}{\sigma_\lambda^2*(1+i)^2} + c'^2 * \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - e^{2*\tau*t}}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^{2*\tau}} -$$

$$- 2*c'*(F(0)+c)* \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - (e^\tau*q)^t}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^\tau*q} \qquad \textbf{(5.41)}$$

$$(5.38) \overset{(5.41)}{\Rightarrow} \text{VarF(t)} = (F(0)+c)^2 *(((q^2+\sigma_\lambda^2*(1+i)^2)^t - q^{2*t}) +$$

$$+ \sigma_\lambda^2*(1+i)^2*c'^2* \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - e^{2*\tau*t}}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^{2*\tau}} -$$

$$- \sigma_\lambda^2*(1+i)^2*2*c'*(F(0)+c)* \frac{((q^2+\sigma_\lambda^2*(1+i)^2)^t - (e^\tau*q)^t}{(q^2+\sigma_\lambda^2*(1+i)^2)-e^\tau*q} \qquad \textbf{(5.42)}$$

Setting:

$$\Psi' = - (F(0)+c)^2$$

$$\Theta' = \sigma_\lambda^2*(1+i)^2*c'^2* \frac{1}{e^{2*\tau}-(q^2+\sigma_\lambda^2*(1+i)^2)} =$$

$$= \sigma_\lambda^2*(1+i)^2*c'^2* \frac{1}{e^{2*\tau}-(1+i)^2*[(1+\lambda)^2+\sigma_\lambda^2]}$$

$$\Omega' = - \sigma_\lambda^2*(1+i)^2*2*c'*(F(0)+c)* \frac{1}{e^\tau*q-(q^2+\sigma_\lambda^2*(1+i)^2)} =$$

$$= - \sigma_\lambda^2*(1+i)^2*2*c'*(F(0)+c)* \frac{1}{e^\tau*q-(1+i)^2*[(1+\lambda)^2+\sigma_\lambda^2]}$$

(5.42) is rewritten as: $\text{VarF(t)}= \Theta' * e^{2*\tau*t} + \Psi' * ((1-\lambda)*(1+i))^{2*t} +$

$+ \Omega'* (e^{\tau}*(1-\lambda)*(1+i))^t - (\Theta'+\Psi'+\Omega')*((1+i)^2*((1-\lambda)^2+\sigma_\lambda^2))^t$ **(5.43)**

This result is very similar to (5.25), when the rates of investment return are random.

Calculation of VarC(t)

$VarC(t) = Var(NC(t)+\lambda(t)*(AL(t)-F(t)) = Var(\lambda(t)*(e^{\tau*t}AL(0) - F(t))) =$

$= e^{2\tau}*(AL(0))^2*\sigma_\lambda^2 + Var(\lambda(t)*F(t)) - 2*Cov(\lambda(t)*e^{\tau*t}AL(0), \lambda(t)*F(t)) =$

$= e^{2\tau}*(AL(0))^2\ \sigma_\lambda^2 + E(\lambda(t)*F(t))^2 - (E(\lambda(t)*F(t)) -$

$2*[E(\lambda(t)*e^{\tau*t}AL(0)*\lambda(t)*F(t)) - E(\lambda(t)*e^{\tau*t}AL(0))*E(\lambda(t)*F(t))] =$

$= e^{2\tau}(AL(0))^2\ \sigma_\lambda^2 + (\sigma_\lambda^2+\lambda^2)*VarF(t) + \sigma_\lambda^2*(E(F(t))^2 - 2*\sigma_\lambda^2\ *e^{\tau*t}AL(0)*EF(t) \Rightarrow$

$\Rightarrow VarC(t) = (\sigma_\lambda^2+\lambda^2)*VarF(t) + \sigma_\lambda^2*(EF(t) - e^{\tau*t}AL(0))^2$ **(5.44)**

$\overset{(5.7),(5.44)}{\Rightarrow} VarC(t) = (\sigma_\lambda^2+\lambda^2)*\{\Theta'*e^{2*\tau*t} + \Psi'*((1-\lambda)*(1+i))^{2*t} + \Omega'*(e^{\tau}*(1-\lambda)*(1+i))^t -$

$- (\Theta'+\Psi'+\Omega')*((1+i)^2*((1-\lambda)^2+\sigma_\lambda^2))^t\} + \sigma_\lambda^2*(q^t*(F(0)+c)-e^{\tau*t}*c')^2$ **(5.45)**

where $c' = AL(0)*(\dfrac{1-e^{\tau}}{q-e^{\tau}})$

Limiting Values:

In order to examine the limiting values we will consider the ratios $\dfrac{Var(F(t))}{e^{2*\tau*t}}$ and $\dfrac{Var(C(t))}{e^{2*\tau*t}}$ so that to deal with 'real' values that allow for inflation.

Considering λ as a penal rate of interest charged on the unfunded liability, $q = (1+i)*(1-\lambda)$ may be seen as the rate of return earned during a year in excess of the amortization charge. Also $\dfrac{q}{e^{\tau}} = e^{-\tau}*(1+i)*(1-\lambda)$ may be seen as the net rate of return earned during a year, over the salary rate of increase in excess of the amortization charge.

$$\frac{Var(F(t))}{e^{2*\tau*t}} = \Theta' + \Psi' * (\frac{q}{e^{\tau}})^{2*t} + \Omega'(\frac{q}{e^{\tau}})^{t} - (\Theta' + \Psi' + \Omega') * (\frac{(1+i)^2 * ((1-\lambda)^2 + \sigma_\lambda^2)}{e^{2*\tau}})^{t} \quad \textbf{(5.46)}$$

It can be seen that if $1-(\frac{e^{2*\tau}}{(1+i)^2} - \sigma_\lambda^2)^{\frac{1}{2}} < \lambda < 1$, $-1 < i$, then $\frac{q}{e^{\tau}} < 1$,

and $\frac{(1+i)^2 * ((1-\lambda)^2 + \sigma_\lambda^2)}{e^{2*\tau}} < 1$. Then as $t \to \infty \Rightarrow$

$$\frac{Var(F(\infty))}{e^{2*\tau*\alpha}} \to \Theta' \quad \textbf{(5.47)}$$

$$\frac{Var(C(t))}{e^{2*\tau*t}} = (\sigma_\lambda^2 + \lambda^2) * \frac{Var(F(t))}{e^{2*\tau*t}} + \sigma_\lambda^2 * (\frac{EF(t)}{e^{\tau*t}} - AL(0))^2 \quad \textbf{(5.48)}$$

$$\text{as } t \to \infty \Rightarrow \frac{Var(C(\infty))}{e^{2*\tau*\alpha}} \to (\sigma_\lambda^2 + \lambda^2) * \Theta' + \sigma_\lambda^2 * (AL(0) * \frac{1-e^{\tau}}{q-e^{\tau}})^2 \quad \textbf{(5.49)}$$

Limiting results show that after a long period of the scheme being continued, both the Variance of the Fund and the Variance of the Contribution rates, become stabilized. The convergence criteria for the second moments provide restrictions for the set of parameter values. In particular: based on the inequality $1-(\frac{e^{2*\tau}}{(1+i)^2} - \sigma_\lambda^2)^{\frac{1}{2}} < \lambda < 1$, we may specify an 'optimal region' m^* for the number of years m over which the unfunded liability is spread. In particular, we show that convergence is obtained, when m is less than

$$m^* = \ln(1 - d * \frac{1}{1-(\frac{e^{2*\tau}}{(1+i)^2} - \sigma_\lambda^2)^{\frac{1}{2}}}) * \frac{1}{\ln(\frac{1}{1+i})}$$

When $\tau = 0$, $c' = 0$ and the constants Ψ', Ω', Θ' become:

$\Psi' = -(F(0)-AL(0))^2$, $\Theta' = 0$ and $\Omega' = 0$.

Hence (5.43) and (5.45) are written as:

$$VarF(t) = \Psi' * ((1-\lambda)*(1+i))^{2*t} - \Psi' * ((1+i)^2*((1-\lambda)^2 + \sigma_\lambda^2)^{t} \quad \textbf{(5.50)}$$

$$VarC(t) = (\sigma_\lambda^2 + \lambda^2) * \Psi' * [((1-\lambda)*(1+i))^{2*t} - ((1+i)^2((1-\lambda)^2 + \sigma_\lambda^2)] +$$

$$+ \sigma_{\lambda}^{2} * q^{t} * (EF(0)-AL(0))^{2} \qquad \textbf{(5.51)}$$

Limiting Values:

It can be seen that if $1-\left(\frac{1}{(1+i)^{2}}-\sigma_{\lambda}^{2}\right)^{\frac{1}{2}} < \lambda < 1$, $-1 < i$, $q < 1$ and $((1+i)^{2}*((1-\lambda)^{2}+\sigma_{\lambda}^{2}) < 1$

Then, as $t \to \infty$ $\mathrm{VarF}(\infty) \to 0$ **(5.52)**

and $\mathrm{VarC}(\infty) \to 0$ **(5.53)**

Limiting results show again that the rate of salary increase is a key assumption. When we assume that salaries are adjusted to the inflation rate, and given that there are neither any material changes to the benefits provided nor any changes in membership profile then the Variance of the Fund and Contribution rates both follow a decreasing trend; and after a long period for the scheme they tend to zero.

When $\tau = 0$, 'the optimal region', m^{*}, becomes:

$$m^{*} = \ln\left(1-d*\frac{1}{1-\left(\frac{1}{(1+i)^{2}}-\sigma_{\lambda}^{2}\right)^{\frac{1}{2}}}\right)\frac{1}{\ln\left(\frac{1}{1+i}\right)}$$

5.3.2 ILLUSTRATIVE EXAMPLES

For the case where $\tau = 0$ and assuming that the valuation rate is equal to 0.05, we calculate the number of years m^* below which we obtain convergence, for the following set of assumptions for the standard deviation of λ:

Table of Assumptions

σ_λ	0.05
σ_λ	0.1
σ_λ	0.15

We would like to mention that for comparison purposes, we chose σ_λ values equal to those of σ_i; thinking also that it is reasonable for λ to vary on the basis of these figures. The effect of $E(\lambda(t))$ is considered implicitly through the m^* values; since $E(\lambda(t))$ is calculated using the equation $E(\lambda(t)) = \frac{1}{\ddot{a}^{(i)}_{\overline{m}|}}$ the optimal region determines the range interval above which we may chose $E(\lambda(t))$ value.

According to the above assumptions, we obtain the following table for m^*.

Table 5.4: *Values of m^* when $\lambda(t)$ is random variable*

σ_λ	m^*	$E(\lambda^*(t)) = \frac{1}{\ddot{a}^{(i)}_{\overline{m^*}|}}$
0.05	74.15	0.0489
0.1	47.29	0.0529
0.15	33.01	0.0595

Table 5.4 shows that as the standard deviation of $\lambda(t)$ increases, the number of years m^* below which we obtain convergence decreases. This is mainly due to the fact that as σ_λ increases, the pace of amortizing the unfunded liability may vary significantly on a yearly basis and as a consequence the fund level becomes less stable. We have thus to increase the average pace of amortising the actuarial liability, in order to balance the low values that $\lambda(t)$ may take throughout the years because of σ_λ. Table 5.4 shows the values of $E(\lambda(t))$, above which we obtain convergence, taking into account the standard deviation of $\lambda(t)$.

Comparing m^* values of table 5.1 with those of table 5.4, we may conclude that m^* is more sensitive to changes in the mean and variance of $\lambda(t)$. This is to be expected since the pace of the amortization of the unfunded liability has a higher effect on the level of the fund.

We calculate the expected value and the variance for the Fund and Contribution rates; (formulae 5.2, 5.8 and 5.30,5.31). For the calculations of the Actuarial Liability, Normal Cost and Pension Outgo values the rate of investment return is assumed equal to 0.05 and the mean spread period equal to $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{5}|}^{(0.05)}}$. The fund value at time t=0, is set equal to $F(0) = 0$. We also assume, for the calculation of the second moments, that $\sigma_\lambda = 0.1$. The limiting values of the Fund and Contribution rates are compared to the Actuarial Liability and Normal Cost at time $t = 0$.

As far as the Expected value of the Fund and Contribution rates is concerned, we come up with the same conclusions as in section 5.2.2 In tables (5.12), (5.13) of Appendix 11, we present the values of the second moments of the Fund and Contribution rates at time t equal to 1, 10,20,30, 40, 50, 80 and 100 years. The theoretical results obtained in section 5.3.1.2, are verified; i.e. as $t \to \infty$, $VarF(t) \to 0$, $VarC(t) \to 0$.

5.4 PENSION FUNDING WHERE BOTH the RATES of INVESTMENT RETURN and the PARAMETER λ are RANDOM VARIABLES

As it has been discussed both the rates of investment return and the spread parameter determine the growth of the Fund, ultimately.

In the previous sections 5.1, 5.2 we considered one of those as a random variable, while the other was fixed. In this chapter, we will consider both as random.

In practice, the actuary may wish either to increase or to decrease the spread parameter value according to the fluctuations in the rates of investment return, considering that $i(t)$ and $\lambda(t)$ are negatively correlated. A higher than expected rate of investment return increases the value of the assets; as a consequence we obtain a lower unfunded liability level than expected. The actuary thus is allowed to decrease the value of λ. On the other hand, when the rates of investment return are kept at a low level, the value of assets is lower than expected; as a consequence, we obtain a higher unfunded liability than expected. A high value of λ could then be considered to amortize the unfunded liability.

However, as a first attempt at approaching this problem, we will assume that they are mutually independent, so that analytical results can be derived. These results may then hold only approximately, for the more realistic case of dependency. At this stage, we thought that it would be easier, as a first step, to see which factor has a stronger effect on the development of the fund and contribution rates. We leave the second step, allowing for correlation between these parameters, as an open question for the future.

We consider the pension model of sections 5.2 and 5.3; i.e.

a) we are working on a defined benefit pension plan model that allows for the salary function, with a salary rate of increase 'τ'

b) the model estimates on a yearly basis C(t) and F(t) based on the membership of the scheme at that time. As t changes, new entrants are allowed to the membership so that the population remains stationary

c) the rate of investment return earned on the fund during the period (t, t+1) is i(t+1), where E(i(t+1)) = i. We further define σ_i^2 = Var(i(t+1))

d) the spread parameter rate set for the unfunded liability amortization during the period (t, t+1) is λ(t+1), where E(λ(t+1)) = λ. We further define σ_λ^2 = Var(λ (t+1))

e) the moments of F(t) and C(t) are estimated on the basis that both contribution income and benefit outgo occur at start of each scheme year, implying along with the previous stated assumptions that the equation of equilibrium holds with annual valuations: i.e. AL = (1+i) * (AL + NC – B).

5.4.1 MOMENTS of F(t) and C(t)

Assuming that λ(t), F(t) are independent, and i(t), λ(t), F(t) are also independent, we derive the first and second moments of F(t), C(t); i.e. E (F(t)), Var(F(t)) and E (C(t)), Var (C(t)).

5.4.1.1 Calculation of EF(t) and EC(t)

Both EF(t) and EC(t) are exactly the same as the corresponding ones derived when i(t) are random in section 5.1, and as a consequence all the results produced for the first moments hold in this case as well. As we have already mentioned in sections 5.2 and 5.3, the rate of salary increase, which we consider, affects significantly the level of the fund; as it increases, the expected fund value decreases considerably. We have also mentioned that the number of years over which the unfunded liability is spread affects significantly the development of the fund; more than the valuation rate. These results are discussed in detail in section 6.9.

5.4.1.2 Variance

<u>Calculation of VarF(t)</u>

$$VarF(t) = E(F(t))^2 - (EF(t))^2 \qquad \textbf{(5.54)}$$

$$E(F(t))^2 = E\ \{(1+i(t))^2 * [\ F(t-1) + \lambda(t) * (AL(t-1) - F(t-1)) + NC(t-1) - B(t-1)\ ]^2\} =$$

$$= (\sigma_i^2 + (1+i)^2) * (1+i)^{-2} * E\ [(1+i)*(1-\lambda(t))*(F(t-1) - e^{\tau(t-1)} * AL(0)\ ) + e^{\tau(t-1)} * AL(0)]^2 =$$

$$\overset{(5.35)}{=} (\sigma_i^2 * (1+i)^{-2} + 1) * \{\ (q^2 + \ \sigma_\lambda^2 * (1+i)^2\ ) * VarF(t-1) + \sigma_\lambda^2 * (1+i)^2 *$$

$* [E(F(t-1)) - e^{\tau(t-1)} * AL(0)]^2 + (EF(t))^2 \}$ **(5.55)**

where $q = (1+i)*(1-\lambda)$.

Based on (5.55), (5.54), becomes:

$$VarF(t) = (\sigma_i^2 *(1+i)^{-2} +1)* \{ (q^2 + \sigma_\lambda^2 * (1+i)^2)* VarF(t-1) + \sigma_\lambda^2 * (1+i)^2 *$$

$$* [E(F(t-1)) - e^{\tau(t-1)} * AL(0)]^2 \} + \sigma_i^2 *(1+i)^{-2} * (EF(t))^2 \quad \textbf{(5.56)}$$

Let $\sigma_i^2 *(1+i)^{-2} +1 = z$, $q^2 + \sigma_\lambda^2 * (1+i)^2 = z_1$, $\sigma_\lambda^2 * (1+i)^2 = z_2$.

Then, setting at time $t = 0$ $VarF(0) = 0$, at time $t = 1$,

$$VarF(1) = z * (z_2 * (F(0) + c - c')^2) + \sigma_i^2 *(1+i)^{-2} * (EF(1))^2 \quad \textbf{(5.57)}$$

Based on (5.57), (5.56) can be rewritten as:

$$VarF(t) = z_2 * z * \sum_{n=0}^{t-1} (z*z_1)^{(t-1)-n} * (q^n * (F(0) +c) - e^{\tau*n} * c')^2 +$$

$$+ \sigma_i^2 *(1+i)^{-2} \sum_{n=1}^{t} (z*z_1)^{t-n} * (EF(n))^2 \quad \textbf{(5.58)}$$

Working on the summation part of (5.58) we obtain the following:

$$VarF(t) = z_2 * z * \{ (F(0) + c)^2 * \frac{(z*z_1)^t - q^{2*t}}{z*z_1 - q^2} + (c')^2 * \frac{(z*z_1)^t - e^{2*\tau*t}}{z*z_1 - e^{2*\tau}} -$$

$$- 2* c'*(F(0) +c)* \frac{(z*z_1)^t - (q*e^\tau)^t}{z*z_1 - q*e^\tau} \} +$$

$$+ \sigma_i^2 *(1+i)^{-2} * \{ (F(0) + c)^2 * q^2 * \frac{(z*z_1)^t - q^{2*t}}{z*z_1 - q^2} + c^2 * e^{2*\tau} * \frac{(z*z_1)^t - e^{2*\tau*t}}{z*z_1 - e^{2*\tau}} -$$

$$- 2* c*(F(0) +c)* q*e^\tau * \frac{(z*z_1)^t - (q*e^\tau)^t}{z*z_1 - q*e^\tau} \} \quad \textbf{(5.59)}$$

(5.59) after replacing z_1, z_2, z_3 becomes:

$$Var\,F(t) = (F(0) + c)^2 * \{ [((1+i)^2 + \sigma_i^2) * ((1-\lambda)^2 + \sigma_\lambda^2)]^t - q^{2*t} \} +$$

$$+(\sigma_\lambda^2 * ((1+i)^2+\sigma_i^2)* c'^2 + \sigma_i^2 *(1+i)^{-2} * c^2 * e^{2*\tau})*$$

$$\frac{[(1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)]^t - e^{2*\tau*t}}{((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2) - e^{2*\tau}} -$$

$$- 2* (F(0)+c) * (\sigma_\lambda^2 *((1+i)^2+\sigma_i^2)*c'+\sigma_i^2 (1+i)^{-} c*q *e^{\tau} *$$

$$* \frac{[((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)]^t - (qe^{\tau})^t}{((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2) - qe^{\tau}} \quad \textbf{(5.60)}$$

Setting:

$$\Psi'' = - (F(0)+c)^2$$

$$\Theta'' =(\sigma_\lambda^2 * ((1+i)^2+\sigma_i^2)* c'^2 + \sigma_i^2 *(1+i)^{-2} * c^2 * e^{2*\tau})*$$

$$\frac{1}{e^{2*\tau} - ((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}$$

$$\Omega'' = -2* (F(0)+c)(\sigma_\lambda^2 *((1+i)^2+\sigma_i^2)*c'+\sigma_i^2 (1+i)^{-1}*c*q*e^{\tau})$$

$$* \frac{1}{q*e^{\tau} - ((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}$$

(5.60) is rewritten as:

$$\mathrm{VarF}(t)= \Theta'' * e^{2*\tau*t} + \Psi'' * ((1-\lambda)*(1+i))^{2*t} +$$

$$+ \Omega''* (e^{\tau} *(1-\lambda)*(1+i))^t - (\Theta''+\Psi''+\Omega'') * [((1+i)^2 + \sigma_i^2) * ((1-\lambda)^2 + \sigma_\lambda^2)]^t \quad \textbf{(5.61)}$$

This result is very similar to (5.25), when the rates of investment return are random and the spread parameter is constant.

<u>Calculation of VarC(t)</u>

$$\mathrm{VarC}(t) = \mathrm{Var}(NC(t)+ \lambda(t)*(AL(t) - F(t)) \overset{(5.44)}{\Rightarrow}$$

$$\Rightarrow \mathrm{VarC}(t) =(\sigma_\lambda^2+\lambda^2)*\mathrm{VarF}(t) + \sigma_\lambda^2 *(EF(t) - e^{\tau*t} AL(0))^2 \quad \textbf{(5.62)}$$

$$\overset{(5.61)}{\Rightarrow} \mathrm{VarC}(t) = (\sigma_\lambda^2+\lambda^2)* \{\Theta'' * e^{2*\tau*t} + \Psi'' * ((1-\lambda)*(1+i))^{2*t} +$$

$+ \Omega'' (e^{\tau} *(1-\lambda)*(1+i))^t$ -($\Theta''+\Psi''+\Omega''$)*$[((1+i)^2 + \sigma_i^2) * ((1-\lambda)^2 + \sigma_\lambda^2)]^t$ } +

$+ \sigma_\lambda^2 * (q^t * (F(0)+c)-e^{\tau*t}*c')^2$ **(5.63)**

Limiting values

In order to examine the limiting values we will consider the ratios $\frac{Var(F(t))}{e^{2*\tau*t}}$ and $\frac{Var(C(t))}{e^{2*\tau*t}}$ so that to deal with 'real' values that allow for inflation.

Considering λ as a penal rate of interest charged on the unfunded liability, q=(1+i)(1-λ), may be seen as the rate of interest earned during a year in excess of the amortization charge. Also $\frac{q}{e^{\tau}} = e^{-\tau} *(1+i)*(1-\lambda)$ may be seen as the net rate of return earned during a year, over the salary rate of increase in excess of the amortization charge.

$$\frac{Var(F(t))}{e^{2*\tau*t}} = \Theta'' + \Psi'' * (\frac{q}{e^{\tau}})^{2*t} + \Omega'' * (\frac{q}{e^{\tau}})^t -$$

$$- (\Theta''+\Psi''+\Omega'')*(\frac{((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}{e^{2*\tau}})^t \quad \textbf{(5.64)}$$

It can be seen that if $1- (\frac{e^{2*\tau}}{(1+i)^2+\sigma_i^2} - \sigma_\lambda^2)^{\frac{1}{2}} < \lambda < 1$, $-1 < i < (\frac{e^{2*\tau}}{(1-\lambda)^2+\sigma_\lambda^2} - \sigma_i^2)^{\frac{1}{2}}$, then $\frac{q}{e^{\tau}} < 1$, and $\frac{((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}{e^{2*\tau}} < 1$. Then, as $t \to \infty \Rightarrow$

$$\frac{Var(F(\infty))}{e^{2*\tau*t}} \to \Theta'' \quad \textbf{(5.65)}$$

$$\frac{Var(C(t))}{e^{2*\tau*t}} = (\sigma_\lambda^2 + \lambda^2)* \frac{Var(F(t))}{e^{2*\tau*t}} + \sigma_\lambda^2 * (\frac{EF(t)}{e^{\tau*t}} - AL(0))^2 \quad \textbf{(5.66)}$$

as $t \to \infty \Rightarrow \frac{Var(C(\infty))}{e^{2*\tau*t}} \to (\sigma_\lambda^2 + \lambda^2)* \Theta'' + \sigma_\lambda^2 * (c')^2$ **(5.67)**

As in the case where either one of the rates of investment return or the spread parameter is random variable, the limiting results show that, when the pension scheme is implemented on the basis where both i(t) and λ(t) are random variables and mutually independent, after a long period of time, both the Variance of the Fund and the Variance of the Contribution rates, become stabilized.

Based on the above parameter restrictions, we may specify an 'optimal region' m^* for the number of years m over which the unfunded liability is spread. In particular, we show that convergence is obtained, when m is less than

$$m^* = \ln\left(1 - d * \frac{1}{1-\left(\frac{e^{2*\tau}}{(1+i)^2+\sigma_i^2} - \sigma_\lambda^2\right)^{\frac{1}{2}}}\right) * \frac{1}{\ln\left(\frac{1}{1+i}\right)}$$

When τ = 0, c' = 0, c = -AL(0) and the constants Ψ'', Ω'', Θ'' become:

$$\Psi'' = -\,(F(0)-AL(0))^2,$$

$$\Theta'' = \sigma_i^2 * (1+i)^{-2} * (AL(0))^2 * \frac{1}{1-((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}$$

$$\Omega'' = -\,2 * (F(0)-AL(0)) * \sigma_i^2\,(1+i)^{-1} * (-AL(0))*q) * \frac{1}{q-((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}$$

Hence (5.61) and (5.63) are written as:

$$VarF(t) = \Theta'' + \Psi'' * ((1-\lambda)*(1+i))^{2*t} +$$

$$+\ \Omega'' * ((1-\lambda)*(1+i))^t - (\Theta''+\Psi''+\Omega'') * [((1+i)^2+\sigma_i^2) * ((1-\lambda)^2+\sigma_\lambda^2)]^t \qquad \textbf{(5.68)}$$

and

$$VarC(t) = (\sigma_\lambda^2 + \lambda^2) * \{\Theta'' + \Psi'' * ((1-\lambda)*(1+i))^{2*t} +$$

$$+\ \Omega'' * ((1-\lambda)*(1+i))^t - (\Theta''+\Psi''+\Omega'') * [((1+i)^2+\sigma_i^2) * ((1-\lambda)^2+\sigma_\lambda^2)]^t\} +$$

$$+\ \sigma_\lambda^2 * (q^t * (F(0)-AL(0)))^2 \qquad \textbf{(5.69)}$$

Limiting Values:

It can be seen that if 1- $\left(\frac{1}{(1+i)^2+\sigma_i^2}-\sigma_\lambda^2\right)^{\frac{1}{2}}<\lambda<1$, and $-1<i<\left(\frac{1}{(1-\lambda)^2+\sigma_\lambda^2}-\sigma_i^2\right)^{\frac{1}{2}}-1$

q< 1, and $((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)<1$. Then, as $t\to\infty \Rightarrow$

$$\mathrm{VarF}(\infty)\to\sigma_i^2(1+i)^{-2}(AL(0))^2\frac{1}{1-((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)} \quad \textbf{(5.70)}$$

and $$\mathrm{VarC}(\infty)\to(\sigma_\lambda^2+\lambda^2)*\frac{\sigma_i^2*(1+i)^{-2}*(AL(0))^2}{1-((1+i)^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)} \quad \textbf{(5.71)}$$

When $\tau=0$, $m^*=\ln\left(1-d*\frac{1}{1-\left(\frac{e^{2*\tau}}{(1+i)^2+\sigma_i^2}-\sigma_\lambda^2\right)^{\frac{1}{2}}}\right)*\frac{1}{\ln(\frac{1}{1+i})}$

5.4.4 ILLUSTRATIVE EXAMPLES

For the case where $\tau=0$, and E(i(t)) = 0.05 we calculate the number of years m^* below which we obtain convergence for the following set of assumptions concerning σ_i and σ_λ:

Table of Assumptions

$\sigma_i = 0.05$	$\sigma_i = 0.1$	$\sigma_i = 0.15$
$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$
$\sigma_\lambda = 0.1$	$\sigma_\lambda = 0.1$	$\sigma_\lambda = 0.1$
$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$

We calculate the ‘optimal region’ considering the changes in the standard deviations and not the changes in the expected values because the effect of changing E(i(t)) has been discussed in section 5.2, while the effect of changing E(λ(t)) has been shown in section 5.3. Since λ(t) and i(t) are mutually independent, the effect of changing simultaneously both parameters is measured by adding the effect of changing each one of i(t) and λ(t).

According to the above assumptions we obtain the following table for m^*

***<u>Table 5.5</u>**: Values of m^* when both the i(t) and λ(t) are random variables*

σ_λ	m^*, $\sigma_i = 0.05$	m^*, $\sigma_i = 0.1$	m^*, $\sigma_i = 0.15$
0.05	62.30	46.12	34.45
0.1	43.86	36.63	29.40
0.15	31.58	28.07	23.88

Table 5.5 shows that as either standard deviation increases (σ_i or σ_λ), the number of years m^* below which we obtain convergence decreases. Specifically when σ_i=σ_λ= 0.05 m^* = 62.30 while when $\sigma_i = \sigma_\lambda = 0.15$, $m^* = 23.88$. As we have also mentioned in sections 5.2 and 5.3, this is mainly due to the fact that, as either one of σ_λ or σ_i increases, the fund level becomes less stable.

We calculate the expected value and the variance for the Fund and Contribution rates; (formulae 5.2, 5.8 and 5.30,5.31). Their limiting values are compared to the Actuarial Liability and Normal Cost at time t = 0. For the calculations we consider the Actuarial Liability, Normal Cost and Pension Outgo values presented in table 4.6 at time t=0,

where $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{15|}}^{(0.05)}}$, and F(0) = 0.

For the calculation of the second moments we assume that $\sigma_\lambda = \sigma_i = 0.1$.

As far as the Expected value of the Fund and Contribution rates is concerned, we come up with the same conclusions as in section 5.2.2

In the tables (5.14), (5.15) of Appendix 11 the values of the second moments of the Fund and Contribution rates at time t equal to 1, 10,20,30, 40, 50, 80, 100 are presented

The theoretical results obtained in section 5.4.1.1 are verified; i.e.as t $\to \infty$,

$$\text{VarF(t)} \to \sigma_i^2 (1+\text{i})^{-2} (\text{AL}(0))^2 \frac{1}{1-((1+\text{i})^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)} \text{ and}$$

$$\text{VarC(t)} \to (\sigma_\lambda^2 + \lambda^2)* \frac{\sigma_i^2 *(1+i)^{-2}*(AL(0))^2}{1-((1+\text{i})^2+\sigma_i^2)*((1-\lambda)^2+\sigma_\lambda^2)}.$$

For the case where both i(t) and λ(t) are random variables we also examine the standard deviations of F(∞) and C(∞), expressed in percentages of Actuarial Liability, regarding different values for the parameter λ(t) while E(i(t)) is kept constant and equal to 0.05. VarF(∞) and VarC(∞) are calculated assuming $\sigma_i=\sigma_\lambda=0.05$ and $\sigma_i = \sigma_\lambda = 0.15$. We choose these assumptions since they correspondingly reflect both: the case with a low deviation from the expected value of each parameter and the case with a high deviation effect. For the presentation we choose the Power function, p = 0.08; the conclusions are the same for each one of the distribution functions defined in chapter 3. The results are presented below, in table 5.5a:

***Table 5.5a**: Standard deviations of $F(\infty)$ and $C(\infty)$, in percent of AL, for different values of λ*

	$\sigma_i = \sigma_\lambda = 0.05$		$\sigma_i = \sigma_\lambda = 0.15$	
λ	$\frac{(Var(F(\infty)))^{\frac{1}{2}}}{AL}$	$\frac{(Var(C(\infty)))^{\frac{1}{2}}}{AL}$	$\frac{(Var(F(\infty)))^{\frac{1}{2}}}{AL}$	$\frac{(Var(C(\infty)))^{\frac{1}{2}}}{AL}$
0.06	32.9707%	2.5751%	n/a	n/a
0.07	23.3688%	2.0103%	349.0556%	57.7790%
0.08	19.1296%	1.8047%	95.2646%	16.1950%
0.09	16.6104%	1.7101%	68.8318%	12.0407%
0.10	14.8964%	1.6655%	56.7190%	10.2252%
0.11	13.6351%	1.6475%	49.4156%	9.1918%
0.12	**12.6579%**	**1.6455%**	44.4077%	8.5304%
0.1201	12.6493%	1.6456%	**44.3653%**	**8.5251%**
0.2	8.8421%	1.8228%	28.3073%	7.0768%
0.21	8.5858%	1.8534%	27.3627%	7.0615%
0.22	8.3535%	1.8846%	**26.5170%**	**7.0607%**
0.221	8.3315%	1.8878%	26.4373%	7.0613%
0.30	7.0534%	2.1452%	21.9537%	7.3635%
0.40	6.1505%	2.4793%	18.9271%	8.0857%
0.50	5.6081%	2.8180%	17.1553%	8.9553%
0.60	5.2572%	3.1653%	16.0252%	9.9110%
0.70	5.0257%	3.5269%	15.2857%	10.9429%
0.80	4.8778%	3.9099%	14.8161%	12.0594%
0.90	4.7951%	4.3223%	14.5543%	13.2795%
1.00	4.7685%	4.7745%	14.4700%	14.6319%

Table 5.5a shows that the standard deviation of $C(\infty)$ is not a monotonic function of λ; up to a certain value of λ there is a 'trade off' between the standard deviation of $F(\infty)$ and the standard deviation of $C(\infty)$. When $\sigma_i = \sigma_\lambda = 0.05$, $C(\infty)$ is an increasing function of λ for $\lambda = \lambda^* = 0.12 < \lambda < 1$ but a decreasing one when $\lambda < \lambda^*$. When $\sigma_i = \sigma_\lambda = 0.15$ $C(\infty)$ is an increasing function of λ, for $\lambda = \lambda^* = 0.22 < \lambda < 1$ but a decreasing one when $\lambda < \lambda^*$. As σ_i ,σ_λ increase, λ^* increases as well.

The "trade off" between the standard deviation of F and the standard deviation of C for different values of λ, has been shown by Dufresne (1989) for the case when the rates of investment return are i.i.d random variables. We have extended this result, for the case where both the rates of investment return and the spread parameter λ are mutually independent random variables.

On the basis of the above assumptions, we also examine the standard deviations of $F(\infty)$ and $C(\infty)$, expressed respectively as percentages of the Actuarial Liability and Normal Cost, regarding different values of the spread period, m.

The results are presented below in tables 5.5b and 5.5c:

Table 5.5b: *Standard deviations of $F(\infty)$ and $C(\infty)$, in percent of AL,NC for different values of the spread period, $\sigma_i = \sigma_\lambda = 0.05$.*

m	$\frac{(Var(F(\infty)))^{\frac{1}{2}}}{AL}$	$\frac{(Var(C(\infty)))^{\frac{1}{2}}}{NC}$
1	4.77%	258.33%
5	8.35%	101.97%
8	10.81%	91.05%
10	12.38%	89.13%
11	**13.15%**	**89.01%**
20	20.36%	100.63%
25	24.91%	113.32%
30	30.17%	129.96%
35	36.48%	151.39%
40	44.41%	179.52%
45	55.05%	218.30%

Table 5.5c: *Standard deviations of F(∞) and C(∞), in percent of AL,NC for different values of the spread period,* $\sigma_i = \sigma_\lambda = 0.15$

m	$\frac{(Var(F(\infty)))^{\frac{1}{2}}}{AL}$	$\frac{(Var(C(\infty)))^{\frac{1}{2}}}{NC}$
1	14.47%	791.70%
2	16.99%	490.63%
3	20.22%	416.40%
4	23.40%	389.43%
5	**26.52%**	**382.04%**
6	29.64%	385.29%
7	32.81%	395.37%
8	36.08%	410.44%
10	43.06%	452.42%
15	66.13%	629.19%

Tables 5.5b and 5.5c show that when the spread period is below a certain number of years, there is a 'trade off' between the standard deviation of F(∞) and the standard deviation of C(∞); as F(∞) increases, C(∞) decreases. We may clearly see that, for the case of high standard deviations for the rates of investment return and the spread parameter, because the level of the Fund becomes less stable, the spread period below which we observe the "trade off" between the standard deviations of F(∞) and C(∞) decreases significantly.

The "trade off" between the standard deviation of F and the standard deviation of C for different values of m, has been shown by Dufresne (1988) for the case when the rates of investment return are i.i.d random variables. We have extended this result, for the case where both the rates of investment return and the spread parameter λ are random variables and mutually independent.

As shown also by Dufresne (1989), for the case where the rates of investment return are i.i.d. random variables, in the neighborhood of zero as the standard deviation of F(∞) increases, the standard deviation of C(∞) decreases. As presented in table 5.5d, this result is extended, for the case where both the rates of investment return and the spread parameter λ are random variables.

Table 5.5d: *Standard deviations of F(∞) and C(∞), as i(t) → 0*

	$\sigma_i = \sigma_\lambda = 0.15$		$\sigma_i = \sigma_\lambda = 0.15$	
i(t)	$(Var(F(\infty)))^{\frac{1}{2}}$	$(Var(C(\infty)))^{\frac{1}{2}}$	$(Var(F(\infty)))^{\frac{1}{2}}$	$(Var(C(\infty)))^{\frac{1}{2}}$
$1*10^{-6}$	24.779702	2.589318727	84.00960732	14.77205672
$1*10^{-15}$	24.7796068	2.589308752	84.00915816	14.77197774

5.5 CONCLUSIONS

In this chapter, we studied the development of the Fund and Contribution rates level based, consecutively, on the following stochastic models:

a) where the rates of investment return, i(t) are random variables and the spread parameter is constant.

b) where the spread parameter λ(t) is random variable and i(t) are constant.

c) where both i(t), λ(t) are random variables.

In each model, the first and second moments of the Fund and Normal Cost have been calculated on the basis of a discrete time formulation.

The respective results produced by Dufresne (1988) and Owadally and Haberman (1999) have been extended after including the salary rate of increase. We show that the salary rate of increase is the key assumption for the Actuarial Liability level and thus for the expected value of the level of the fund and the contribution rates.

In an economic environment, where i(t) > -1, convergence in each model for the first moments (EF(t), EC(t)) is obtained, if the interest earned during a year in excess of the amortization charge, is lower than the salary rate of increase.

Convergence is also observed for the second moments, after restrictions are applied to the set of the parameter values. According to the parameter restrictions an 'optimal region', m^*, is specified for the number of years, m, over which the unfunded liability is spread. We show that for m greater than a particular value m^*, the variances of both the fund and the contribution are increasing functions of m; the 'optimal' values of m are

$1 \leq m \leq m^*$. After a long period for the scheme being continued, limiting results show that both the Variance of the fund and the contribution rates are stabilized.

In practice, as either E(i(t)) increases or E(λ(t)) decreases, both the fund and contribution rates reach their ultimate values, Actuarial Liability and Normal Cost, at a lower pace.

The stability of the fund decreases, under high standard deviations for either the rates of investment return or the spread parameter; as a consequence the 'optimal period' below which we obtain convergence, is significantly reduced.

This effect is shown clearly when we consider λ(t) as random variable since m^* is more sensitive to change in the mean and variance of λ(t).

Assuming that the salary rate of increase is equal to zero, when both i(t), λ(t) are random variables, illustrative examples show that:

- the ultimate value of the standard deviation of the contribution rates is not a monotonic function of λ(t). Up to a certain value of λ, there is a 'trade off' between the ultimate value of the standard deviation of the fund and the ultimate value of the standard deviation of the contribution rates.
- when the spread period is below a certain number of years, there is a 'trade off' between the standard deviation of $F(\infty)$ and the standard deviation of $C(\infty)$; as $F(\infty)$ increases, $C(\infty)$ decreases.

- as i(t) value approaches zero, then we observe that the ultimate value of standard deviation of the fund increases, while the ultimate value of standard deviation of the contribution rate decreases.

Dufresne (1989) has showed the above results for the case where the rates of investment return are i.i.d. random variables. In this work, Dufresne's results have been extended for the case where both i(t) and λ(t) are mutually independent random variables.

APPENDIX 11

Table 5.6: *E(i(t))=0.03, VarF(t) at time t ≥ 100*

Distribution fn	σ_i = 0.025	σ_i = 0.05	σ_i = 0.1	σ_i = 0.15
Power p=0.8	329.10	1,332.99	5,614.85	13,858.98
Power p=1	301.69	1,221.94	5,147.11	12,704.47
Power p=1.5	253.11	1,025.21	4,318.40	10,658.99
Tr. Expon. Σ=30	352.00	1,425.75	6,005.58	14,823.39
Tr. Expon. σ=40	339.63	1,375.63	5,794.49	14,302.33
Tr. Expon. σ=50	332.09	1,345.12	5,665.96	13,985.12
Tr. Pareto k=0.3	344.87	1,396.87	5,883.96	14,523.19
Tr. Pareto k=0.8	361.45	1,464.03	6,166.84	15,521.43
Tr. Pareto k=1.5	384.46	1,557.22	6,559.36	16,190.27

Table 5.7: *E(i(t))=0.03 VarC(t) at time t ≥ 100*

Distribution fn	σ_i = 0.025	σ_i = 0.05	σ_i = 0.1	σ_i = 0.15
Power p=0.8	2.77	11.22	47.27	116.68
Power p=1	2.54	10.29	43.33	106.96
Power p=1.5	2.13	8.63	36.36	89.74
Tr. Expon. σ=30	2.96	12.00	50.56	124.79
Tr. Expon. σ=40	2.86	11.58	48.78	120.41
Tr. Expon. σ=50	2.79	11.32	47.70	117.74
Tr. Pareto k=0.3	2.90	11.76	49.54	122.27
Tr. Pareto k=0.8	3.04	12.32	51.92	128.15
Tr. Pareto k=1.5	3.24	13.11	55.22	136.04

Table 5.8: *E(i(t))=0.05, VarF(t) at time t ≥ 100*

Distribution fn	**σ_i = 0.025**	**σ_i = 0.05**	**σ_i = 0.1**	**σ_i = 0.15**
Power p=0.8	263.75	1,073.42	4,616.51	11,875.22
Power p=1	246.49	1,003.21	4,314.55	11,098.48
Power p=1.5	214.66	873.64	3,757.28	9,664.99
Tr. Expon. σ=30	279.05	1,135.69	4,884.33	12,564.13
Tr. Expon. σ=40	271.16	1,103.61	4,746.35	12,209.19
Tr. Expon. σ=50	266.33	1,083.93	4,661.68	11,991.42
Tr. Pareto k=0.3	274.28	1,116.29	4,800.88	12,349.48
Tr. Pareto k=0.8	284.71	1,158.75	4,983.48	12,819.19
Tr. Pareto k=1.5	298.58	1,215.21	5,226.29	13,443.77

Table 5.9: *E(i(t))=0.05, VarC(t) at time t ≥ 100*

Distribution fn	**σ_i = 0.025**	**σ_i = 0.05**	**σ_i = 0.1**	**σ_i = 0.15**
Power p=0.8	2.22	9.04	38.87	99.98
Power p=1	2.08	8.45	36.23	93.44
Power p=1.5	1.81	7.36	31.63	81.37
Tr. Expon. σ=30	2.35	9.56	41.12	105.78
Tr. Expon. σ=40	2.28	9.29	39.96	102.79
Tr. Expon. σ=50	2.24	9.13	39.25	100.95
Tr. Pareto k=0.3	2.31	9.39	40.42	103.97
Tr. Pareto k=0.8	2.39	9.76	41.96	107.92
Tr. Pareto k=1.5	2.51	10.23	43.99	113.18

Table 5.10: *E(i(t))=0.07, VarF(t) at time t ≥ 100*

Distribution fn	**σ_i = 0.025**	**σ_i = 0.05**	**σ_i = 0.1**	**σ_i = 0.15**
Power p=0.8	272.69	1,122.33	5,076.38	14,604.93
Power p=1	258.64	1,064.46	4,814.62	13,851.85
Power p=1.5	231.67	953.45	4,312.50	12,407.23
Tr. Expon. σ=30	286.05	1,177.26	5,324.84	15,319.76
Tr. Expon. σ=40	279.55	1,150.51	5,203.84	14,971.65
Tr. Expon. σ=50	275.54	1,134.02	5,129.27	14,757.10
Tr. Pareto k=0.3	281.78	1,159.70	5,245.42	15,091.27
Tr. Pareto k=0.8	290.31	1,194.83	5,404.30	15,548.38
Tr. Pareto k=1.5	301.54	1,241.03	5,613.26	16,149.55

Table 5.11: *E(i(t))=0.07, VarC(t) at time t ≥ 100*

Distribution fn	**σ_i = 0.025**	**σ_i = 0.05**	**σ_i = 0.1**	**σ_i = 0.15**
Power p=0.8	2.29	9.45	42.74	122.96
Power p=1	2.18	8.96	40.53	116.62
Power p=1.5	1.95	8.03	36.31	104.45
Tr. Expon. σ=30	2.41	9.91	44.83	128.98
Tr. Expon. σ=40	2.35	9.69	43.81	126.04
Tr. Expon. σ=50	2.32	9.55	43.18	124.24
Tr. Pareto k=0.3	2.37	9.76	44.16	127.05
Tr. Pareto k=0.8	2.44	10.06	45.49	130.90
Tr. Pareto k=1.5	2.54	10.45	47.26	135.96

Table 5.12: *Var F(t), where* $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{5}|}^{(0.05)}}$, $\sigma_\lambda = 0.1$

Distr. fn / years	**1**	**10**	**20**	**30**	**40**	**50**	**80**	**100**
Power p=0.8	461.75	136.80	5.49	0.17	0.00	0.00	0.00	0.00
Power p=1	431.55	127.85	5.13	0.16	0.00	0.00	0.00	0.00
Power p=1.5	375.81	111.34	4.47	0.13	0.00	0.00	0.00	0.00
Tr. Expon. σ=30	488.54	144.74	5.81	0.18	0.00	0.00	0.00	0.00
Tr. Expon. σ=40	474.74	140.65	5.65	0.17	0.00	0.00	0.00	0.00
Tr. Expon. σ=50	466.27	138.14	5.55	0.17	0.00	0.00	0.00	0.00
Tr. Pareto k=0.3	480.19	142.27	5.71	0.17	0.00	0.00	0.00	0.00
Tr. Pareto k=0.8	498.46	147.68	5.93	0.18	0.00	0.00	0.00	0.00
Tr. Pareto k=1.5	522.74	154.87	6.22	0.19	0.01	0.00	0.00	0.00

Table 5.13: *Var C(t), where* $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{5}|}^{(0.05)}}$, $\sigma_\lambda = 0.1$

Distr. fn / years	**1**	**10**	**20**	**30**	**40**	**50**	**80**	**100**
Power p=0.8	307.91	15.71	0.46	0.01	0.00	0.00	0.00	0.00
Power p=1	287.77	14.69	0.43	0.01	0.00	0.00	0.00	0.00
Power p=1.5	250.60	12.79	0.38	0.01	0.00	0.00	0.00	0.00
Tr. Expon. σ=30	326.66	17.35	0.62	0.05	0.02	0.02	0.02	0.02
Tr. Expon. σ=40	317.43	16.86	0.60	0.05	0.02	0.02	0.02	0.02
Tr. Expon. σ=50	311.77	16.56	0.59	0.05	0.02	0.02	0.02	0.02
Tr. Pareto k=0.3	321.08	17.05	0.61	0.05	0.02	0.02	0.02	0.02
Tr. Pareto k=0.8	333.29	17.70	0.63	0.05	0.02	0.02	0.02	0.02
Tr. Pareto k=1.5	349.53	18.56	0.66	0.05	0.02	0.02	0.02	0.02

Table 5.14: *Var F(t), where* $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{15}|}^{(0.05)}}$, $E(i(t)) = 0.05$, $\sigma_\lambda = \sigma_i = 0.1$

Distr. fn / years	1	10	20	30	40	50	80	100
Power p=0.8	466.75	2,381.03	2,778.02	3,054.80	3,453.44	3,891.83	4,840.50	5,122.14
Power p=1	436.22	2,225.29	2,596.31	2,854.99	3,227.55	3,637.27	4,523.89	4,787.11
Power p=1.5	379.88	1,937.87	2,260.97	2,486.24	2,810.68	3,167.48	3,939.58	4,168.80
Tr. Expon. σ=30	493.83	2,519.16	2,939.18	3,232.02	3,653.78	4,117.61	5,121.31	5,419.29
Tr. Expon. σ=40	479.88	2,447.99	2,856.15	3,140.71	3,550.56	4,001.29	4,976.64	5,266.20
Tr. Expon. σ=50	471.32	2,404.33	2,805.20	3,084.69	3,487.23	3,929.91	4,887.87	5,172.26
Tr. Pareto k=0.3	485.39	2,476.12	2,888.96	3,176.80	3,591.36	4,047.26	5,033.82	5,326.71
Tr. Pareto k=0.8	503.86	2,570.30	2,998.85	3,297.63	3,727.96	4,201.20	5,225.28	5,529.31
Tr. Pareto k=1.5	528.41	2,695.53	3,144.96	3,458.30	3,909.59	4,405.89	5,479.87	5,798.71

Table 5.15: *Var C(t), where* $E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{15}|}^{(0.05)}}$, $E(i(t)) = 0.05$, $\sigma_\lambda = \sigma_i = 0.1$

Distr. fn / years	**1**	**10**	**20**	**30**	**40**	**50**	**80**	**100**
Power p=0.8	389.50	205.99	113.93	80.56	73.02	75.32	89.37	94.38
Power p=1	364.02	192.52	106.48	75.30	68.24	70.40	83.52	88.20
Power p=1.5	317.01	167.65	92.73	65.57	59.43	61.31	72.73	76.81
Tr. Expon. σ=30	412.10	217.94	120.54	85.24	77.25	79.69	94.55	99.85
Tr. Expon. σ=40	400.45	211.78	117.14	82.83	75.07	77.44	91.88	97.03
Tr. Expon. σ=50	393.31	208.01	115.05	81.35	73.73	76.06	90.24	95.30
Tr. Pareto k=0.3	405.05	214.22	118.48	83.78	75.93	78.33	92.94	98.14
Tr. Pareto k=0.8	420.46	222.37	122.99	86.97	78.82	81.31	96.47	101.88
Tr. Pareto k=1.5	440.95	233.20	128.98	91.21	82.66	85.27	101.17	106.84

APPENDIX 12

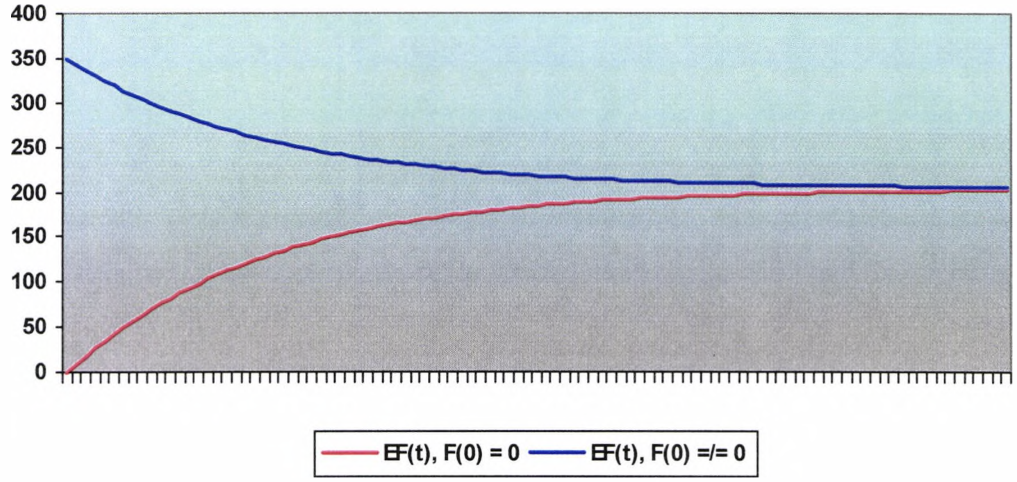

Figure 5.1: EF(t) development when F(0) = 0 and F(0) ≠ 0, F(0) > AL(0)

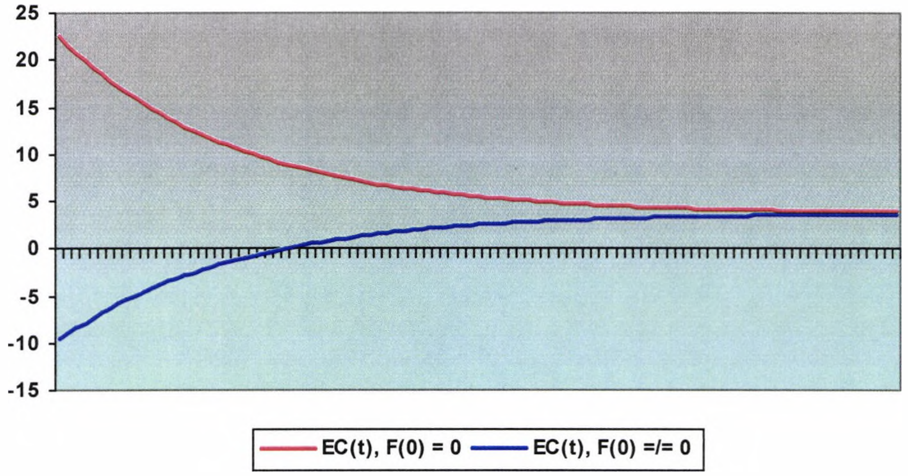

Figure 5.2: EC(t) development when F(0) = 0 and F(0) ≠ 0, F(0) > AL(0)

CHAPTER 6:

SIMULATION ANALYSIS

6.1 INTRODUCTION

In this chapter we investigate if the theoretical model proposed in Chapter 5 can be applied in real life situation and use the results to measure the risks associated with funding.

For this reason we have proceeded with simulation analysis assuming that either of the investment rates of return and 'λ' or both variables are random with a Log Normal distribution. For our choice we have focused on those distributions that have elegant properties and thus are convenient for evaluating the value of the fund. Both the Normal and Log Normal distributions have been considered by many others researchers for the case where the i(t) are assumed to be random variables. For λ the stochastic approach is new; we have investigated the Log Normal distribution for reasons of consistency, and also thinking about potential future work, which would involve examining i(t) and $\lambda(t)$ assuming dependency. For the latter case, we thought that it would be very helpful to apply the same distribution for both random variables. However, this does not exclude the possibility of extending our work in the future by considering other distributions, for example Normal distribution.

For either case, we assume that the pension plan is implemented based on a different pension accrual density function each time, using those defined in section 3.2. We may proceed as above, because the theoretical model is independent of any density function, which we may consider for the pension plan implementation. Therefore, all simulation results provided by the different functions are comparable with the corresponding ones calculated by the theoretical formulae in chapter 5.

The effect of each parameter being stochastic is examined separately, (Model I and Model II respectively), assuming that the other parameter is a constant. In addition, the effect when both parameters are stochastic is considered (Model III).

6.2 THE MODEL

All simulations have been performed using the StatGraph software. We select the Log Normal distribution setting each time the expected value and the standard deviation of the parameter that is assumed as the random variable; for each choice, we derive 1,000 simulated paths over a period of 100 years. For the variance and estimation of tail percentiles the number of paths required was higher than 1,000; they reached the number of 10,000 while discussing the second model. For these calculations we have used SPLUS software.

Based on these, we calculated the values for both the fund and the contribution rates according to the formulae described in section 4.2. Then we calculated the first and second moments using the formulae described in sections 5.2-5.4, where either each of the rate of investment return and the spread parameter or both are random variables. For the illustrative examples, we choose the case where $\tau = 0$. We have decided to use a simple model, because of convenience but also because such a model does allow detailed study of the important factors, which contribute to the qualitative behavior of the Actuarial Liability and Contribution rates. These factors are the expected value and the standard deviation of the parameter assumed as the random variable.

6.3 DATA CHECK

When all simulated paths were completed, and before proceeding to the calculations of the results, we check the data produced by the random number generation. Data checks took place on the basis of the Normal distribution, for which the expected number of values that fall under each one of the following intervals is known: $(0,\mu-4.4*\sigma)$[16], $(\mu-4.4*\sigma,\mu-4*\sigma)$, $(\mu-4*\sigma,\mu-3*\sigma)$, $(\mu-3*\sigma,\mu-2*\sigma)$, $(\mu-2*\sigma,\mu-\sigma)$, $(\mu-\sigma,\mu)$, $(\mu,\mu+\sigma)$, $(\mu+\sigma,\mu+2*\sigma)$, $(\mu+2*\sigma,\mu+3*\sigma)$, $(\mu+3*\sigma,\mu+4*\sigma)$, $(\mu+4*\sigma,\mu+4.4*\sigma)$, $(\mu+4.4*\sigma,\infty)$[17]. We use the χ^2 statistic to test whether sample data indicate that the particular model for the parameter distribution fits the data, comparing the observed number of the simulated figures allocated in those intervals with the expected one. We consider that the parameter distribution fits the data, when the chi square test performed higher either than 97.5% at the 0.025 percentile, $\chi^2_{0.025}$ or higher than 95% at the 0.05 percentile, $\chi^2_{0.05}$.

6.4 CALCULATION OF THE RESULTS

In each one of the 100-year periods, from the simulated data we record:

The 1%, 5%, 25%, 50%, 75%, 95% and 99% percentiles: We focus on the percentile values throughout the years in order to discuss, under each model, the sensitivity of results to changes in parameters (in our case expected value and

[16] μ, σ are the mean and the standard deviation of the Normal distribution.
[17] See the Normal distribution tables.

standard deviation). This is an empirical analysis in which the growth level of the Fund as well as the Contribution rates level is indicated under each choice of the expected value and for each standard deviation of the parameter assumed as random. On the basis of the percentile values, the skewness and kurtosis of the distribution of the fund and the contribution rates are examined.

Descriptive Measures of the Sample: Average, Variance and Standard Deviation. We briefly present, for comparison purposes, these results and the corresponding ones derived by the theoretical model.

With the simulation approach, we are able to analyze the growth of the Fund under the chosen variables and thus to become aware of its development over time as well as the highest and lowest values we may expect about its level throughout the years and a range of key percentiles. Thus, in our opinion, by using the stochastic simulation approach, we gain knowledge of the experience as to how the theoretical results might be applied in practice as well as information about the measurement of the risk taken for the level of the Fund and Contribution, under the different parameter values.

6.5 PRESENTATION OF RESULTS

As we have also discussed in section 4.10, the level of Actuarial Liability a) tends to decrease slightly when the plan is implemented using either the Power or the Exponential pension density function as the parameters p, σ respectively increase b) tends to increase along with the increase of the parameter k when the plan is implemented under the Truncated Pareto. This trend is also followed, in all models, when we examine the level of difference between the sample average and the expected value provided by the theoretical formula as well as between the sample and the theoretical variance. It is followed as well when we examine the level of the percentile values.

Similar conclusions are derived for different values of p as well as when instead of the Power function, for the implementation of the Fund, we use either Truncated Exponential or Truncated Pareto

For the above reasons, we decided to present the results calculated based on the Power function distribution, p= 0.8.

The results provided under each Model are discussed separately and overall conclusions are given at the end of the chapter.

6.6 MODEL I, RANDOM RATES of INVESTMENT RETURN

6.6.1 ASSUMPTIONS

For Model I we consider the following different cases:

Case III, E(i(t)) = 0.07	Case I, E(i(t)) = 0.03	Case II, E(i(t)) = 0.05
$\sigma_i = 0.025$	$\sigma_i = 0.025$	$\sigma_i = 0.025$
$\sigma_i = 0.05$	$\sigma_i = 0.05$	$\sigma_i = 0.05$
$\sigma_i = 0.10$	$\sigma_i = 0.10$	$\sigma_i = 0.10$
$\sigma_i = 0.15$	$\sigma_i = 0.15$	$\sigma_i = 0.15$

We have applied these values for the calculation of NC(t), AL(t), F(t) and C(t) of the following distributions:

Power function (p=0.8, p=1, p=1.5)

Truncated Pareto (k=0.3, k=0.8, k=1.5)

Truncated Exponential (σ=30, σ=40, σ=50)

The spread period, defined as the number of years, over which some extra contribution should be paid so that the unfunded liability turns to zero, for all cases, equals to 15 years. Hence, the amortization spread parameter equals to $\lambda = \frac{1}{\ddot{a}_{\overline{15}|}^{(0.05)}}$.

6.6.2 DESCRIPTIVE MEASURES of the SAMPLE: AVERAGE, VARIANCE and STANDARD DEVIATION

6.6.2.1 Sample Average

Simulations verify the expected result that the sample average, $\bar{F}(t)=\sum_{k=1}^{1000}\frac{F_k(t)}{1000}$,

$\bar{C}(t)=\sum_{k=1}^{1000}\frac{C_k(t)}{1000}$ approximate significantly the theoretical expected value EF(t), EC(t).

EF(t), EC(t) provided by the theoretical model, remain unchanged as the standard deviation of the rates of investment return changes. The sample average changes, since the standard deviation value affects the simulated data.

For the cases where σ_i is equal to either 0.1 or 0.15, we increase the number of simulations to 5,000 paths over a 100 year period, because for high standard deviations we need more simulated figures to reach conclusions.

6.6.2.2 Variance, Standard Deviation[18]

The simulation results verify that the sample variance $s_F^2=\sum_{k=1}^{1000}\frac{(F_k(t)-\bar{F}(t))^2}{1000-1}$,

$s_C^2=\sum_{k=1}^{1000}\frac{(C_k(t)-\bar{C}(t))^2}{1000-1}$ are close to the expected one provided by the theoretical model, VarF(t), VarC(t).

Both VarF(t) and VarC(t) provided by the theoretical model change as the standard deviation value of the rates of investment return changes. The sample variance also changes, since the standard deviation value affects the simulated data.

[18] The positive square root of the sample variance, s_F, s_C, gives the sample standard deviation.

In figures 6.1 and 6.2 below the sample variance and the theoretical variance for the Fund are plotted for the cases where E(i(t))=0.05 and $\sigma_i = 0.05$, E(i(t))=0.05 and $\sigma_i = 0.15$[19] ; the latter is based on 8,000 simulated paths. The differences between the simulated and analytical results are due to sampling error only.

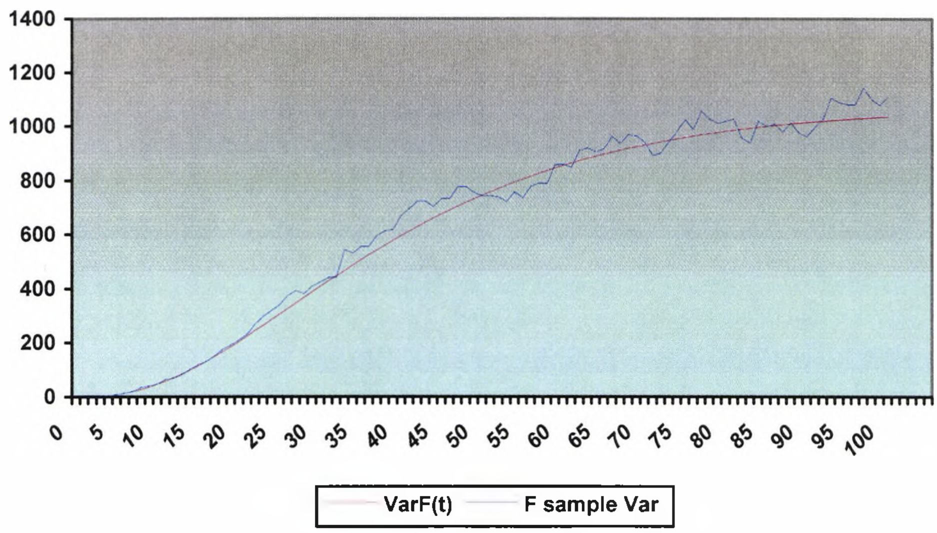

Figure 6.1: VarF(t) vs Sample Variance, E(i(t)) = 0.05, σ_i = 0.05

[19] The pattern followed by the sample variance of the Contribution rates when sketched against the theoretical variance VarC(t), is identical to that followed by the sample variance of the Fund, when sketched against the theoretical variance VarF(t).

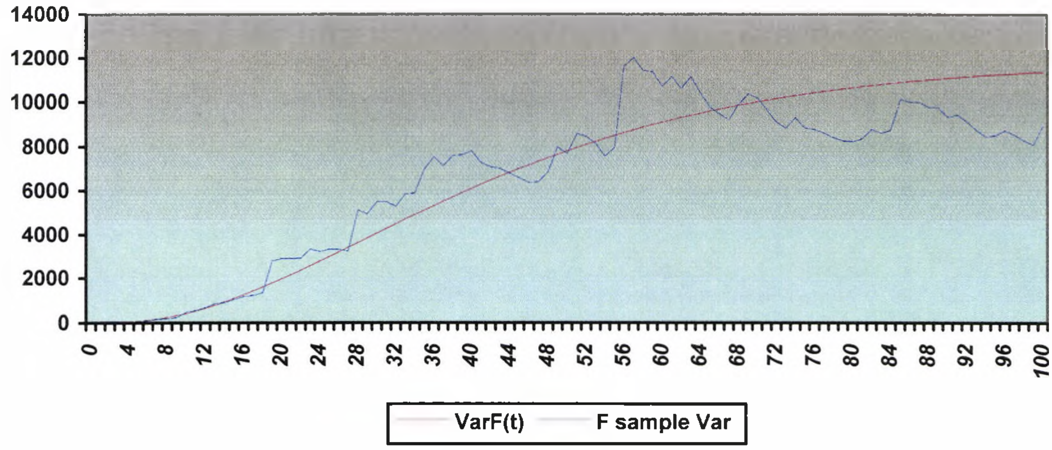

Figure 6.2: VarF(t) vs Sample Variance, E(i(t)) = 0.05, σ_i = 0.15

6.6.3 PERCENTILES

For each one of the standard deviations of i(t), the 1%, 5%, 25%, 50%, 75%, 95% and 99% percentile values of the Fund and Contribution rates are calculated.

The percentiles indicate the sensitivity of the Fund and Contribution rates level becoming either low or high as a result of changes in parameters for the proposed Log Normal model. The theoretical model does not enable us to calculate percentiles easily since we need to specify a distribution for the random inputs.

The concentration of the percentile values around the median is also examined, taking their difference from the median on a yearly basis. This process deals with the measurement of both skewness and kurtosis of the simulated data.

The results produced verify that both the expected value and the standard deviation of i(t) have a high impact on the growth level of the Fund. In particular, the variability of the Fund increases along with the increase of the standard deviation.

In addition, the results show that as time "t" increases the difference between the percentiles and the median tends to stabilize at a certain level, for both the Fund and

Contribution rate distributions; this is expected since the parameter values we consider fulfill the requirements for convergence as $t \rightarrow \infty$.

According to the formulae used in section 4.2, the Contribution rates depend directly on the growth of the Fund. Specifically, the higher is the value of the Fund the lower is the value of the contribution rates and vice versa. As a consequence, the pattern of the Contribution rates percentiles is affected directly by the Fund.

6.6.3.1 Fund Percentiles

In tables 6.1 and 6.2 the 1% and 99% percentile Fund values are shown at specific points of time, as a percentage of the Actuarial Liability. We chose those, since they reflect two extreme cases respectively: 1% percentile shows the value below which 1% of cases occur for the Fund level and the 99% percentile the value above which 1% of cases occur.

We assume that σ_i equals either to 0.025 or 0.05. For the case where $E(i(t))=0.05$ we also quote the corresponding values when σ_i equals to 0.1 and 0.15, based on 5,000 simulated paths, in order to show the effect of changing the standard deviation.

Table 6.1: *Model I, Random Rates of investment return, 1% percentile Fund values as a percentage of the Actuarial Liability*

E (i(t))	σ_i	1% percentile Fund values as a percentage of Actuarial Liability								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.03	0.025	0.27	0.45	0.58	0.67	0.78	0.83	0.86	0.88	0.88
	0.05	0.27	0.44	0.56	0.64	0.73	0.78	0.80	0.82	0.82
0.05	0.025	0.20	0.35	0.46	0.55	0.67	0.74	0.78	0.82	0.83
	0.05	0.19	0.33	0.43	0.51	0.61	0.66	0.70	0.74	0.74
	0.1	0.19	0.32	0.41	0.47	0.55	0.60	0.62	0.65	0.65
	0.15	0.19	0.31	0.39	0.45	0.52	0.56	0.58	0.61	0.61
0.07	0.025	0.12	0.23	0.31	0.38	0.50	0.58	0.63	0.74	0.77
	0.05	0.12	0.21	0.29	0.35	0.44	0.51	0.54	0.64	0.63

Table 6.2: *Model I, Random Rates of investment return, 99% percentile Fund values as a percentage of the Actuarial Liability*

E (i(t))	σ_i	99% percentile Fund values as a percentage of the Actuarial Liability								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.03	0.025	0.32	0.56	0.73	0.87	1.03	1.11	1.16	1.19	1.19
	0.05	0.35	0.63	0.85	1.00	1.24	1.43	1.42	1.52	1.43
0.05	0.025	0.23	0.43	0.58	0.70	0.89	1.03	1.09	1.18	1.19
	0.05	0.26	0.49	0.69	0.83	1.08	1.23	1.36	1.52	1.57
	0.1	0.30	0.63	0.88	1.13	1.43	1.64	1.89	2.17	2.23
	0.15	0.38	0.78	1.13	1.50	2.02	2.22	2.34	2.87	2.62
0.07	0.025	0.15	0.28	0.40	0.51	0.67	0.82	0.93	1.13	1.18
	0.05	0.17	0.31	0.46	0.58	0.80	1.00	1.14	1.51	1.45

We make the following remarks on tables 6.1 and 6.2:

The lower values are achieved when E(i(t))=0.07. This is due to the fact that the expected rate of investment return is also considered in the calculation of the Actuarial Liability, which decreases as E(i(t)) increases.. If Actuarial Liability remained unchanged, then we could observe that an increase of the rate of investment return results to a higher than expected level of the fund Thus, it is linked with a higher risk in terms of a less favorable performance of the growth of the Fund than that expected. This is a result also observed in chapter 5, section 5.2.2, where as table 5.2 shows, the expected value of the Fund decreases as E(i(t)) increases; i.e. the distribution is shifted to the left.

As the standard deviation increases the distribution is more spread out. In terms of the 1% percentile, a significant decrease in the Fund level is observed along side an increase in the standard deviation.

None of the 1% percentiles in Tables 6.1 and 6.2 are zero. We may conclude that the risk of a deficit at any time is less than 1%. However, at this point, we have to take into account the fact that the starting value of the Fund level, F(0), has been assumed equal to zero. An open question for the future is to examine the Fund level status, when its starting value, F(0), is different than zero.

Table 6.3 shows the median values of the Fund in the years 5,20 and 100, and their difference from the corresponding percentiles (percentile value – median value) for the case where σ_i equals 0.025. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.3 *: Model I, Random Rates of investment return, Difference between the Fund percentiles and the Median ,* σ_i*=0.025*

E(i(t))	years	median	Difference between the median and the respective percentile of fund					
			1%	99%	5%	95%	25%	75%
0.03	5	74.16	-3.08	9.49	-2.39	5.81	-1.28	1.59
	20	192.09	-15.94	36.93	-11.54	21.96	-5.72	8.17
	100	259.90	-28.91	53.42	-22.02	36.42	-10.29	13.01
0.05	5	43.08	-2.42	4.52	-1.93	2.96	-1.52	0.99
	20	124.25	-12.14	19.67	-9.27	13.38	-7.19	5.14
	100	201.25	-31.33	42.67	-22.72	28.32	-18.49	10.33
0.07	5	21.98	-1.31	2.20	-0.97	1.46	-0.44	0.50
	20	71.68	-7.85	12.17	-5.60	7.43	-2.61	2.98
	100	154.22	-26.58	41.08	-20.05	26.61	-8.10	10.49

We make the following remarks on table 6.3:

For each value of E(i(t)) we note that, the lower percentile has a smaller difference between it and the median than the corresponding upper percentile, and that this gap increases as the percentiles become more outlying. As E(i(t)) increases, this gap tends to be reduced; a few exceptions may observed in the outlying percentiles.

When E(i(t)) equals 0.03 the difference between the median and the 25% percentile in the 20^{th} year is 5.72 (in absolute values) and for the 75% percentile is 8.17. The upper percentile difference is therefore 43% larger than the lower percentile. For the 1% percentile and 99% percentile the corresponding difference for the upper percentile is 132% larger than for the lower percentile. When E(i(t)) equals 0.07 the difference for the 75% percentile is 14% larger than that for the 25% percentile and the difference for the 99% percentile is 55% larger than that for the 1% percentile. This is an indication of the skewness of the distribution of Fund. In particular, the Fund appears to become skewed to the right. Since the distance of the percentiles from the median tends to increase with time, an increase in kurtosis may be expected over time (see sections 6.6.4.1 and 6.6.4.2).

The effect of changing the standard deviation, is presented in the following table 6.4 for the years 5,20 and 100 for the case where E(i(t)) equals 0.05. For the cases where E(i(t)) = 0.05, $\sigma_i = 0.1$ and E(i(t)) = 0.05, $\sigma_i = 0.15$, the results are based on 5,000 simulated paths. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.4: *Model I, Random Rates of investment return, Difference between the Fund percentiles and the Median, E(i(t))=0.05*

Standard deviation	years	median	Difference between the median and the respective percentile of fund					
			1%	99%	5%	95%	25%	75%
0.025	5	43.08	-2.42	4.52	-1.93	2.96	-1.52	0.99
	20	124.25	-12.14	19.67	-9.27	13.38	-7.19	5.14
	100	201.25	-31.33	42.67	-22.72	28.32	-18.49	10.33
0.05	5	42.57	-3.07	-2.48	-1.21	1.98	6.19	10.14
	20	122.40	-17.73	-14.64	-7.20	10.00	31.45	48.26
	100	197.33	-46.42	-35.32	-17.41	22.24	61.09	123.38
0.1	5	41.62	-3.17	20.29	-2.69	9.81	-1.51	2.67
	20	118.32	-21.94	113.45	-18.13	52.23	-9.66	14.41
	100	185.50	-52.02	270.21	-42.99	137.31	-22.35	35.60
0.15	5	40.91	-2.85	35.94	-2.51	13.84	-1.45	2.85
	20	113.71	-21.67	192.27	-18.14	81.48	-10.04	17.29
	100	175.85	-50.94	360.06	-42.38	166.37	-22.86	42.45

As table 6.4 indicates when the standard deviation is higher than 0.05 the gap between the upper and lower percentile is very high, especially when the percentiles become more outlying. This is an indication of skewness of the distribution of Fund. In particular, it appears to become skewed to the right (see sections 6.6.4.1 and 6.6.4.2). This is an expected result since, as discussed earlier, with higher standard deviations, the distribution becomes more spread out, as a consequence of the very high or very low rates of investment return that may occur and may influence the progress of the Fund over time.

6.6.3.2 Contribution Rates percentiles

High values for the Fund lead to low values for the Contribution rates and vice versa. However, we will discuss the Contribution rates percentiles because it is important to investigate how low / high these values become in response to fluctuations in the level of the Fund. In tables 6.5 and 6.6 the 1% and 99% percentile Contribution rates values are shown at specific points of time, as a percentage of Normal Cost. We chose those, since they reflect two extreme cases respectively. From the 1% percentile we get the value of the Contribution rates below which 1% of cases occur while from the 99% percentile the value above which 1% of cases occur, when an unfavorable performance for the Fund is considered. The results are presented for the cases where σ_i equals to 0.025 and σ_i equals to 0.05. When E(i(t))=0.05 we also present the cases where σ_i equals to 0.1 and σ_i equals to 0.15, based on 5,000 simulated paths, in order to show the standard deviation effect.

Table 6.5: *Model I, Random Rates of investment return, 1% percentile Contribution rates values as a percentage of Normal Cost*

E (i(t))	σ_i	1% percentile Contribution rates values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.03	0.025	3.76	2.79	2.09	1.53	0.87	0.56	0.34	0.23	0.24
	0.05	3.65	2.50	1.61	1.02	0.02	-0.72	-0.68	-1.11	-0.76
0.05	0.025	4.81	3.85	3.07	2.47	1.55	0.86	0.56	0.11	0.05
	0.05	4.68	3.51	2.55	1.82	0.60	-0.13	-0.80	-1.57	-1.82
	0.1	4.46	2.83	1.59	0.34	-1.13	-2.17	-3.41	-4.81	-5.09
	0.15	4.10	2.07	0.35	-1.46	-4.07	-5.06	-5.65	-8.29	-7.04
0.07	0.025	6.10	5.32	4.61	3.95	2.95	2.10	1.44	0.25	-0.06
	0.05	5.98	5.11	4.23	3.50	2.17	0.99	0.15	-2.02	-1.69

Table 6.6: *Model I, Random Rates of investment return, 99% percentile Contribution rates values as a percentage of Normal Cost*

E (i(t))	σ_i	99% percentile Contribution rates values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.03	0.025	3.96	3.21	2.69	2.34	1.91	1.70	1.58	1.50	1.50
	0.05	3.97	3.26	2.80	2.46	2.08	1.90	1.82	1.73	1.75
0.05	0.025	4.98	4.25	3.69	3.24	2.64	2.29	2.07	1.87	1.84
	0.05	5.01	4.32	3.82	3.43	2.92	2.70	2.47	2.28	2.30
	0.1	5.03	4.40	3.95	3.63	3.22	3.00	2.88	2.73	2.73
	0.15	5.04	4.43	4.02	3.73	3.37	3.17	3.07	2.95	2.93
0.07	0.025	6.23	5.62	5.12	4.67	4.02	3.51	3.20	2.56	2.38
	0.05	6.26	5.70	5.24	4.88	4.33	3.95	3.73	3.17	3.18

The Contribution rates values presented in tables 6.5 and 6.6 show, as expected, a significant decrease (table 6.5) as the Fund level increases (table 6.2) and a significant increase (table 6.6) as the Fund level decreases (table 6.1).

In the extreme case of the 1% percentile, the values may turn to be negative as either the expected value, E(i(t)), or the standard deviation, σ_i, increases. Negative values imply that, following a very good investment performance of the Fund, we may expect a Fund surplus and no more contributions are necessary, and indeed, contribution refunds occur.

Table 6.7 shows the median values of the Contribution rates in the years 5,20 and 100, and their difference from the corresponding percentiles (percentile value – median value) for the case where σ_i equals 0.025. The percentiles are grouped in corresponding pairs for ease of comparison.

Tables 6.7 *: Model I, Random Rates of investment return, Difference between the Contribution percentiles and the Median,* $\sigma_i = 0.025$

E(i(t))	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
0.03	5	23.35	-0.87	0.28	-0.53	0.22	-0.15	0.12
	20	12.53	-3.39	1.46	-2.01	1.06	-0.75	0.52
	100	6.31	-4.90	2.65	-3.34	2.02	-1.19	0.94
0.05	5	18.61	-0.42	0.22	-0.27	0.18	-0.19	0.08
	20	11.16	-1.81	1.11	-1.23	0.85	-0.99	0.38
	100	4.10	-3.92	2.87	-2.60	2.08	-1.96	0.99
0.07	5	15.74	-0.20	0.12	-0.13	0.09	-0.05	0.04
	20	11.18	-1.12	0.72	-0.68	0.51	-0.27	0.24
	100	3.61	-3.77	2.44	-2.44	1.84	-0.96	0.74

We make the following remarks on table 6.7:

As expected, for the Contribution rates, the opposite relationship holds between the lower and the upper percentiles from that of the Fund percentiles (section 6.6.3.1.); i.e. the lower percentile of each pair has the higher difference between it and the median than the upper percentile. This is an indication of the skewness of the distribution of Contribution rates, which is the opposite side of that of the Fund. In particular, the distribution of the Contribution rates appears to become skewed to the left. Since the distance of the percentiles from the median tends to increase over time, an increase in kurtosis may be expected as well (see sections 6.6.4.1 and 6.6.4.1).

The effect of changing the standard deviation is presented in the following table 6.8 for the years 5,20 and 100, for the case where E(i(t)) equals 0.05. For the cases where E(i(t)) = 0.05, σ_i = 0.1 and E(i(t)) = 0.05, σ_i = 0.15 the percentile values are based on 5,000 and 8,000 simulated paths respectively. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.8: *Model I ,Random Rates of investment return, Difference between the Contribution percentiles and the Median, E(i(t)) = 0.05*

Standard deviation	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
0.025	5	18.61	-0.42	0.22	-0.27	0.18	-0.19	0.08
	20	11.16	-1.81	1.11	-1.23	0.85	-0.99	0.38
	100	4.10	-3.92	2.87	-2.60	2.08	-1.96	0.99
0.05	5	18.65	-0.93	0.28	-0.57	0.23	-0.18	0.11
	20	11.33	-4.43	1.63	-2.89	1.34	-0.92	0.66
	100	4.45	-11.32	4.26	-5.61	3.24	-2.04	1.60
0.10	5	18.74	-1.86	0.29	-0.90	0.25	-0.24	0.14
	20	11.70	-10.41	2.01	-4.79	1.66	-1.32	0.89
	100	5.54	-24.79	4.77	-12.60	3.94	-3.27	2.05
0.15	5	18.80	-3.18	0.26	-1.21	0.23	-0.26	0.13
	20	12.11	-16.88	1.99	-7.16	1.66	-1.56	0.91
	100	6.40	-33.74	4.71	-15.61	3.95	-3.71	2.17

Table 6.8 indicates that as the standard deviation increases the gap between the upper and lower percentile increases along with time, especially when the percentiles become

more outlying. The lower percentile of each pair has a higher difference between it and the median than the upper percentile and, as a consequence skewness to the left side is expected (see sections 6.6.4.1 and 6.6.4.2).

6.6.4 SKEWNESS[20] and KURTOSIS

6.6.4.1 Skewness in Model I

From the definition of skewness for this model, skewF = -skewC.

Skewness is calculated for the years 1,5,10,20,30,50,100.

Tables 6.9 and 6.10 confirm the conclusions derived in section 6.6.3, as far as both the Fund and Contribution rate values are concerned. In particular, table 6.9 shows that as the expected value increases, the distribution becomes skewed to the right for the Fund; table 6.10 shows that as σ_i increases, skewness values increase as well. We point out that skewness of symmetric distribution is 0 by definition.

Because of the definition stated above, the skewness values for the Contribution rates are not presented.

<u>Table 6.9</u> *Model I, Random Rates of investment return: Fund skewness,* σ_i*=0.025*

E(i(t))	Fund Skewness					
	5^{th} yr	10^{th} yr	20^{th} yr	30^{th} yr	50^{th} yr	100^{th} yr
0.03	1.57	1.38	1.26	0.93	1.08	0.95
0.05	0.84	0.65	0.57	0.72	0.71	0.48
0.07	0.77	0.45	0.51	0.43	0.45	0.55

[20] in Appendix 13 the definitions are quoted

Table 6.10 *Model I, Random Rates of investment return: Fund skewness, E(i(t))= 0.05*

σ_i	Fund Skewness					
	5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.025	0.84	0.65	0.57	0.72	0.71	0.48
0.05	1.92	2.49	1.51	1.42	1.52	1.76
0.1	3.56	4.22	2.87	3.24	3.20	5.47
0.15	8.49	11.81	18.97	13.02	6.75	7.15

6.6.4.2 Kurtosis in Model I

Kurtosis is calculated for the years 1,5,10,20,30,50,100. It is clear from the definition of kurtosis in Appendix 13, that the kurtosis is the same for both fund value and contribution rate.

Table 6.11 and 6.12 values below, confirm the conclusions derived in section 6.6.3, as far as both the Fund and Contribution rate values are concerned:

Specifically, as presented in table 6.11, when the expected value increases a peaked distribution of the sample values is formed in the latter years.

Table 6.11 *Model I, Random Rates of investment return: Fund / Contribution rates kurtosis, σ_i=0.025*

E(i(t))	Fund / Contribution rates kurtosis, in specific years					
	5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.03	3.67	3.34	3.41	1.45	2.17	1.60
0.05	1.00	0.99	0.26	0.92	1.09	0.41
0.07	1.29	0.30	0.30	0.13	0.40	0.73

In table 6.12, we present the values of kurtosis when the standard deviation of i(t) increases; as σ_i increases, kurtosis increases as well.

Table 6.12 *Model I, Random Rates of investment return: Fund / Contribution rates kurtosis, E(i(t))= 0.05*

σ_i	Fund / Contribution rates kurtosis, in specific years					
	5^{th} yr	10^{th} yr	20^{th} yr	30^{th} yr	50^{th} yr	100^{th} yr
0.025	1.00	0.99	0.26	0.92	1.09	0.41
0.05	6.78	15.35	4.64	3.71	4.44	6.84
0.1	16.11	62.91	10.72	12.59	49.28	62.14
0.15	117.54	281.48	627.23	309.61	97.53	105.72

6.7 MODEL II, RANDOM SPREAD PARAMETER 'λ'

6.7.1 ASSUMPTIONS

For Model II we considered the following different cases:

Case I	Case II	Case III	Case IV	Case V
amort. period 5 years	amort. period 10 years	amort. period 15 years	amort. period 20 years	amort. period 25 years
$E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{5}\vert}}$	$E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{10}\vert}}$	$E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{15}\vert}}$	$E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{20}\vert}}$	$E(\lambda(t)) = \frac{1}{\ddot{a}_{\overline{25}\vert}}$
$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$	$\sigma_\lambda = 0.05$
$\sigma_\lambda = 0.10$	$\sigma_\lambda = 0.10$	$\sigma_\lambda = 0.10$	$\sigma_\lambda = 0.10$	$\sigma_\lambda = 0.10$
$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$	$\sigma_\lambda = 0.15$

We applied these values for the calculation of NC(t), AL(t), F(t) and C(t) of the following distributions:

Power function (p=0.8, p=1, p=1.5)

Truncated Pareto (k=0.3, k=0.8, k=1.5)

Truncated Exponential (σ=30, σ=40, σ=50)

In all cases, the rates of investment return i(t) are assumed equal to 0.05, $\forall \ t \geq 0$

Since the spread period parameter is assumed to be random, we consider that the spread period of 5,10,15,20 and 25 years is the value that corresponds to the mean of the distribution for the parameter λ. Hence, throughout this section we refer to the mean spread period.

6.7.2 DESCRIPTIVE MEASURES of the SAMPLE: AVERAGE, VARIANCE and STANDARD DEVIATION

6.7.2.1 Sample Average

Simulations verify the expected result that the sample average, $\bar{F}(t) = \sum_{k=1}^{1000} \frac{F_k(t)}{1000}$,

$\bar{C}(t) = \sum_{k=1}^{1000} \frac{C_k(t)}{1000}$ approximate significantly the theoretical expected value EF(t), EC(t).

As in Model I, section 6.6.2, the expected values EF(t) and EC(t) provided by the theoretical model remain unchanged as the standard deviation of the spread parameter changes. The sample averages change, since the value of the standard deviation affects the simulated data.

The assumption that the spread period parameter is a random variable implies that the period of amortizing the unfunded liability changes on a yearly basis according to λ. In other words, based on the equation $\lambda = \frac{1}{\ddot{a}_{\overline{m}|}}$, as λ changes, the number of years, 'm' that correspond to the value of λ, changes as well. As a consequence, the Contribution rates in year t, depend not only on the growth of the Fund in year t-1,(resulted by all λ values up to that time i.e. λ(1),λ(2), ...,λ(t-1)), but also on the number of years determined by the value of λ(t), specified through simulations, for year t .

The effect of λ on the Contribution rates becomes significant as σ_λ increases, since the distribution becomes more spread out. In particular when σ_λ is higher than 0.05, deviations are observed between the sample average and the theoretical expected value, EC(t), for the case where the mean spread period is longer than 10 years. When the mean spread period equals either 5 or 10 years, the sample average approaches the expected value provided by the theoretical formula. As it is extended between 10 and 20 years, discrepancies are observed when σ_λ equals to 0.15. These discrepancies become more significant when σ_λ equals either to 0.1 or 0.15 and the mean spread period equals 25 years. For these cases, we have increased the number of simulated paths. In particular for the case where the mean spread period equals 25 years and σ_λ 0.1 we have obtained convergence after considering 5,000 simulated paths over a 100 year period .The same result is also observed, when σ_λ increases to 0.15[21].

6.7.2.2 Variance, Standard Deviation[22]

Both of the values VarF(t) and VarC(t) provided by the theoretical model change along with the standard deviation value of the spread parameter. The sample variance also changes, since the standard deviation value affects the simulated data.

For the case where the mean spread period equals 25 years, and σ_λ 0.15, convergence between the sample average, s_F^2, and VarF(t) is achieved, after increasing the number of simulated paths to 3,000.

The sample variance for the contribution rates when the mean spread period is extended, for the cases where $\sigma_\lambda = 0.1$ or $\sigma_\lambda = 0.15$ deviates significantly from the theoretical one, VarC(t). Focusing on the case where the mean spread period equals 25 years and σ_λ 0.1 we have proceeded as follows[23]: As a first step, we increase the number of simulated paths to 5000 over a 100 year period; the convergence, between the sample variance and the theoretical one, is not at a desired level. Based on this result

[21] It generally holds that if the sample average approaches significantly the theoretical expected value when a long spread period of amortizing the unfunded liability is assumed, then, the same outcome will definitely hold for a shorter one.

[22] the positive square root of the sample variance, s_F, s_C, gives the sample standard deviation.

[23] The case where the spread period equals to 25 years and $\sigma_\lambda = 0.15$ was not considered since the variability of the Contribution rates sample was very high and a desired result would not be achievable.

and because both VarF(t) and VarC(t) (as determined by the theoretical formula) reach their long term values at some time, t>100 with a long spread period we extend the 100 year period to 250, and increase the number of simulated paths to 2,000. The results show convergence between the sample average and the theoretical expected value. However, the skewness of the distributions indicates that the number of simulated paths should increase to achieve stability. For this reason, we test 10,000 paths over 250 years. Checking the behavior of the results provided by those data, we observe a higher level of convergence, especially as t is extended to 250 years, where the sample variance coincides with the theoretical one. This result, (based on 10,000 paths over 250 years) is described in figure 6.3 where the sample variance is plotted against the theoretical one, VarC(t). The differences between the simulated and analytical results are due to sampling error which is reduced as we increase the number of simulations.

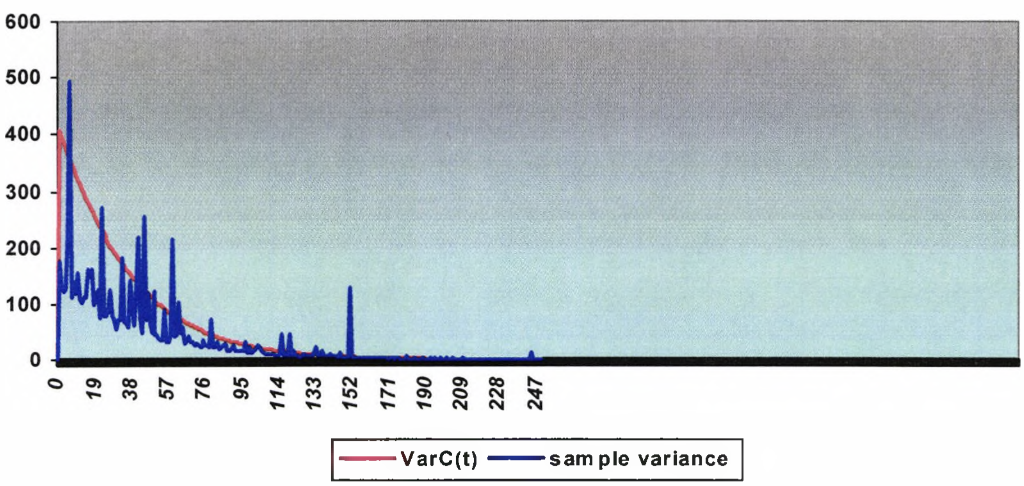

Figure 6.3: VarC(t) vs Sample Variance, spread period = 25 years, σ_λ = 0.1

In addition, we have shown (table 5.4) that the 25 years period when $\sigma_\lambda = 0.15$, is very close to the m^*=33 years period, below which we obtain convergence.

6.7.3 PERCENTILES

For each one of the standard deviations of λ ' s, the 1%, 5%, 25%, 50%, 75%, 95% and 99% percentile values of the Fund and Contribution rates are calculated.

The percentiles indicate the sensitivity of the Fund and Contribution rates level becoming either low or high as a result of changes in parameters for the proposed Log Normal model. The theoretical model does not enable us to calculate percentiles easily since we need to specify a distribution for the random inputs.

The concentration of the percentile values around the median is also examined, taking their difference from the median on a yearly basis. This process deals with the measurement of both the skewness and the kurtosis of the simulated data.

We observe that, for both the Fund and Contribution rates the main components affecting the magnitude of the percentiles and their concentration around the median are the length of the mean of the distribution and the standard deviation of $\lambda(t)$. In particular as either the value of the spread period that corresponds to the distribution mean or the standard deviation of $\lambda(t)$ increases, the distribution becomes more spread out and the percentiles distance from the median increases.

As in Model I, section 6.6.3 the results show that as time, t, increases, both the spread of the percentiles of the Fund and Contribution rates' around the median tends to become stabilized. This is expected since the parameter values we consider fulfill the requirements for convergence as $t \to \infty$.

6.7.3.1 Fund percentiles

In tables 6.13 and 6.14 the 1% and 99% percentile Fund values are shown at specific points of time, as a percentage of the Actuarial Liability. We chose those, since they reflect two extreme cases respectively. 1% percentile shows the value below which 1% of cases occur for the Fund level while the 99% percentile the value above which 1% of cases occur.

In order to compare the growth of the Fund level under the lowest and the highest standard deviation of lamda we assume that σ_λ equals either to 0.05 or 0.15

Table 6.13: *Model II,Random spread parameter: 1% percentile Fund values as a percentage of the Actuarial Liability*

am period	σ_λ	1% percentile Fund values as a percentage of the Actuarial Liability								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
5ys	0.05	0.51	0.78	0.92	0.96	0.99	1.00	1.00	1.00	1.00
	0.15	0.29	0.61	0.82	0.92	0.98	1.00	1.00	1.00	1.00
10ys	0.05	0.16	0.37	0.54	0.69	0.84	0.92	0.96	1.00	1.00
	0.15	0.00	0.07	0.20	0.34	0.61	0.77	0.88	0.98	1.00
15ys	0.05	0.02	0.13	0.26	0.37	0.55	0.70	0.79	0.94	0.97
	0.15	0.00	0.00	0.00	0.00	0.00	0.19	0.36	0.73	0.89
20ys	0.05	0.00	0.00	0.05	0.12	0.26	0.42	0.52	0.78	0.85
	0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.23	0.38
25ys	0.05	0.00	0.00	0.00	0.00	0.06	0.20	0.28	0.55	0.67
	0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 6.14: *Model II, Random spread parameter: 99% percentile Fund values as a percentage of the Actuarial Liability*

am period	σ_λ	99% percentile Fund values as a percentage of the Actuarial Liability								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
5ys	0.05	0.76	0.92	0.98	0.99	1.00	1.00	1.00	1.00	1.00
	0.15	1.01	1.03	1.02	1.01	1.00	1.00	1.00	1.00	1.00
10ys	0.05	0.55	0.73	0.84	0.90	0.96	0.99	0.99	1.00	1.00
	0.15	1.09	1.38	1.20	1.19	1.09	1.06	1.03	1.00	1.00
15ys	0.05	0.48	0.66	0.75	0.83	0.90	0.95	0.97	0.99	1.00
	0.15	1.05	1.27	1.40	1.31	1.31	1.24	1.18	1.08	1.05
20ys	0.05	0.40	0.57	0.66	0.74	0.84	0.89	0.93	0.98	0.99
	0.15	1.26	1.60	1.76	1.87	2.12	1.93	1.86	1.55	1.24
25ys	0.05	0.42	0.55	0.63	0.70	0.79	0.86	0.90	0.96	0.98
	0.15	1.28	1.47	1.64	1.78	1.84	1.85	1.87	1.69	1.58

We make the following remarks on tables 6.13 and 6.14:

High values for the mean spread period (extended longer than 10 years) and the standard deviation are linked with a high variability. For the extreme case of the 1% percentile, there is a deficit; i.e. the scheme has insufficient assets. Hence, an unfavourable performance of the Fund at a certain time cannot be excluded. However, as in Model I, section 6.6.3.1, we also have to mention that these results are produced assuming that the starting value of the Fund level, F(0), equals zero. As an open question for the future we leave to examine the Fund level status when the starting value, F(0), is not equal to zero.When the mean spread period is low, the variability of the Fund is also at a low level, even as the standard deviation σ_λ increases. This is due to the fact that the unfunded liability is amortized quickly.

In table 6.15, the relationship between the distance of the F(t) percentile values from the median (percentile value – median value) is presented, at time t = 5,20 and 100 years assuming that $\sigma_\lambda = 0.05$. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.15: *Model II, Random spread parameter: Difference between the Fund percentiles and the Median σ_λ=0.05*

Mean spread period	years	median	Difference between the median and the respective percentile of the Fund					
			1%	99%	5%	95%	25%	75%
5ys	5	130.00	-25.75	25.64	-18.12	17.99	-7.69	6.89
	20	200.98	-3.51	1.99	-2.27	1.52	-0.79	0.71
	100	204.65	0.00	0.00	0.00	0.00	0.00	0.00
10ys	5	69.40	-37.39	42.47	-26.90	29.03	-12.27	11.36
	20	166.15	-25.92	19.03	-17.18	14.63	-7.49	6.55
	100	204.61	-0.12	0.03	-0.06	0.03	-0.02	0.02
15ys	5	40.68	-37.13	57.04	-26.68	37.32	-12.04	13.86
	20	125.69	-49.79	43.48	-33.67	31.35	-14.14	13.78
	100	203.10	-3.89	1.18	-2.21	1.00	-0.66	0.51
20ys	5	25.75	-35.70	55.72	-28.53	35.91	-12.90	13.63
	20	94.32	-70.16	56.94	-49.88	38.62	-19.09	16.24
	100	196.33	-21.43	6.50	-11.27	5.39	-3.54	2.69
25ys	5	17.62	-35.73	68.77	-27.77	47.82	-12.60	15.99
	20	70.93	-72.38	71.79	-55.50	52.21	-23.74	21.06
	100	182.46	-44.71	17.22	-28.18	14.92	-9.36	7.71

Table 6.15 shows that for small values of t (example 5) and low values of the mean spread period (example 5), the lower percentile of each pair has a higher difference between it and the median than the upper percentile. When the mean spread period is longer than 10 years this is observed to occur at a higher value of t, which is expected since, as discussed earlier, stability is reached at a later time point when mean spread periods are increased.

As the value of the mean spread period increases, the gap between the percentiles and the median is higher, especially when the percentiles become more outlying. For example, when it equals 5 years, the difference between the median and 1% percentile in the 20th year is 3.51, in absolute values, and in the 5th year is 25.75, and between the median and the 99% percentile is 1.99 and 25.64 respectively; while when the mean spread period equals 25 years the corresponding figures in years 20 and 5 are equal to 72.38, in absolute values, and 35.73 for the 1% percentile and 71.79 and 25.64 for the 99% percentile; i.e. when the mean spread period is extended to 25 years in the 5th year the 1% percentile absolute value (18.11) is almost equal to the median value while the 99% percentile value is almost 5 times higher than the median value. In year 20 the 1% percentile absolute value (1.45) is almost 49 times less than the median value while the 99% percentile value is equal to twice the median value. When the mean spread period equals 5 years, in year 20 the percentile values are very close to the median while in year 5 there is some distance. In addition, when the mean spread period equals 5 years, in both cases the percentiles are at equal distance from the median. For the cases where the mean spread period is extended to 20 or to 25 years, in year 100, the distance of the percentiles from the median is, as expected, still significant since the long term value of the Fund is reached at a later stage than 100 years. This is an indication of the skewness in the distribution of Fund. In particular, it appears to become skewed to the left. Kurtosis could also be expected in the later years (see sections 6.7.4.1 and 6.7.4.2).

The effect of the standard deviation is described in Table 6.16 where the distance of the values of the fund percentiles from the median is calculated for different values of the standard deviation for the case where the mean spread period equals 5 years. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.16: *Model II, Random spread parameter, Difference between the Fund percentiles and the Median, spread period equals 5 years*

standard deviation	years	median	Difference between the median and the respective percentile of the Fund					
			1%	99%	5%	95%	25%	75%
0.05	5	130.00	-25.75	25.64	-18.12	17.99	-7.69	6.89
	20	200.98	-3.51	1.99	-2.27	1.52	-0.79	0.71
	100	204.65	0.00	0.00	0.00	0.00	0.00	0.00
0.10	5	127.70	-47.27	52.28	-32.73	39.40	-14.03	15.06
	20	201.28	-7.41	2.92	-4.51	2.40	-4.51	1.28
	100	204.65	0.00	0.00	0.00	0.00	0.00	0.00
0.15	5	127.07	-68.19	79.19	-49.61	52.88	-21.32	23.20
	20	201.89	-14.02	4.31	-8.72	2.86	-2.82	1.57
	100	204.65	0.00	0.00	0.00	0.00	0.00	0.00

We make the following remarks on table 6.16:

The lower percentile of each pair has the higher difference between it and the median than the upper percentile. In particular, the distribution of the Fund appears to become skewed to the left (see sections 6.7.4.1 and 6.7.4.2).With a standard deviation higher than 0.05, this outcome is observed, as expected, at a later stage, since, for the case where standard deviation value is high, stability is achieved later. Both kurtosis and skewness will increase along with the standard deviation increase (see sections 6.7.4.1 and 6.7.4.2).

As discussed earlier, when the mean spread period is low, the unfunded liability is amortized during a short period of time, and the long term values of the Fund are also attained within a short spread period. This is clearly observed in table 6.16 as the distance of the percentiles from the median is at a considerable level in year 5 and significantly decreases by year 20.

6.7.3.2 Contribution Rates percentiles

High values for the Fund lead to low values for the Contribution rates and vice versa. However, we will discuss the Contribution rates percentiles because it is important to investigate how low / high these values become in response to fluctuations of the level of the Fund.

In tables 6.17 and 6.18 the 1% and 99% percentile Contribution rates values are shown at specific points of time, as a percentage of Normal Cost. We chose those, since they reflect two extreme cases respectively. From the 1% percentile we get the value of the Contribution rates below which 1% of cases occur while from the 99% percentile the value above which 1% of cases occur, when an unfavorable performance for the Fund is considered. In order to compare the Contribution rates level under the lowest and the highest standard deviation of lamda we assume that σ_λ equals either to 0.05 or 0.15.

Table 6.17: *Model II, Random spread parameter, 1% percentile Contribution rates values as a percentage of Normal Cost*

am period	σ_λ	1% percentile Contribution values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
5ys	0.05	3.48	1.83	1.26	1.09	1.01	1.00	1.00	1.00	1.00
	0.15	0.72	0.71	0.71	0.89	0.98	1.00	1.00	1.00	1.00
10ys	0.05	2.82	2.06	1.62	1.40	1.17	1.07	1.03	1.00	1.00
	0.15	0.22	-2.17	0.00	-0.50	0.35	0.67	0.84	0.97	0.99
15ys	0.05	2.11	1.85	1.63	1.51	1.20	1.16	1.09	1.02	1.00
	0.15	0.72	0.30	-2.04	-1.14	-0.47	0.38	0.46	0.77	0.79
20ys	0.05	1.80	1.68	1.50	1.49	1.30	1.20	1.15	1.04	1.02
	0.15	0.42	-0.47	-1.50	-3.56	-2.82	-1.56	-1.88	-0.14	0.11
25ys	0.05	1.53	1.48	1.43	1.35	1.30	1.18	1.15	1.07	1.04
	0.15	0.26	-1.55	-0.99	-2.15	-1.18	-1.67	-1.37	-0.91	-1.60

Table 6.18: *Model II, Random spread parameter, 99% percentile Contribution rates values as a percentage of Normal Cost*

am period	σ_λ	99% percentile values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
5ys	0.05	7.37	3.80	2.11	1.43	1.07	1.01	1.00	1.00	1.00
	0.15	10.35	6.00	3.07	2.04	1.20	1.03	1.00	1.00	1.00
10ys	0.05	9.56	7.02	5.27	3.81	2.42	1.65	1.30	1.03	1.01
	0.15	13.90	10.78	7.79	7.14	4.06	2.72	1.72	1.09	1.02
15ys	0.05	10.83	8.32	6.96	6.20	4.61	3.04	2.44	1.38	1.17
	0.15	14.59	13.07	12.03	11.26	9.47	6.48	4.61	2.43	1.69
20ys	0.05	11.08	8.62	8.04	7.67	6.04	5.21	3.91	2.29	1.77
	0.15	15.25	15.06	13.71	12.60	11.14	10.77	7.52	4.96	3.08
25ys	0.05	11.00	11.14	9.28	9.58	7.95	6.69	5.81	3.94	2.72
	0.15	14.72	14.35	15.00	14.46	12.72	13.60	12.07	8.30	7.94

We make the following remarks on tables 6.17 and 6.18:

When the mean spread period equals 5 years, the 1% and 99% percentile coincide after year 30. This result is to be expected, since the Unfunded Liability is amortized during a short period of time. When the mean spread period exceeds 5 years, and as standard deviation increases, the contribution rates become more volatile.

As the mean spread period is extended, the 1% percentile values decrease approaching each other throughout all years. As shown earlier when discussing the 1% percentile Fund values, the risk of possible losses increases along with the increase in the number of years over which the unfunded liability is amortized. Combining those results, we may conclude, that when a long mean spread period corresponds to the mean of the distribution of λ, the frequency with which valuations are performed in the pension scheme should be increased, in order to check the adequacy of the Fund at close time intervals.

Table 6.19 shows the difference between the C(t) percentile values and the median, (percentile value – median value), in years 5,20 and 100 under each value of the mean spread period we assume and for the case where σ_λ equals to 0.05. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.19: *Model II, Random spread parameter, Difference between the Contribution percentiles and the Median* $\sigma_\lambda = 0.05$

mean spread period	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
5	5	20.01	-6.83	7.86	-4.75	5.77	-2.24	2.33
	20	4.56	-0.44	0.83	-0.32	0.55	-0.16	0.19
	100	3.78	0.00	0.00	0.00	0.00	0.00	0.00
10	5	19.20	-8.54	16.97	-6.73	11.83	-3.43	7.72
	20	8.17	-2.86	6.24	-2.37	4.05	-1.23	1.45
	100	3.79	0.00	0.00	0.00	0.00	0.00	0.00
15	5	17.45	-9.45	23.50	-7.59	14.15	-4.19	4.78
	20	10.10	-4.38	13.33	-3.77	7.90	-2.04	2.73
	100	3.90	-0.10	0.51	-0.09	0.51	-0.05	0.09
20	5	15.79	-8.98	26.12	-7.87	15.66	-3.97	4.85
	20	10.46	-4.84	18.54	-4.15	11.97	-2.23	4.02
	100	4.32	-0.47	2.37	-0.39	1.33	-0.22	0.35
25	5	13.18	-7.41	28.40	-6.14	14.92	-2.90	4.77
	20	10.87	-5.75	25.37	-4.90	13.07	-2.42	4.47
	100	4.99	-1.07	5.31	-0.96	2.95	-0.52	0.89

We make the following remarks on table 6.19:

The lower percentile of each pair has a smaller difference between it and the median than the upper percentile and this gap increases as the percentiles become more outlying. In addition, this gap becomes more prominent as the mean spread period increases. For example, when it equals 5 years, in the 20th year the difference of the 75% percentile from the median (0.19) is 24% larger than that for the 25% percentile (0.16, in absolute values). For the 1% and 99% percentiles, the corresponding difference for the upper percentile is 90% larger than that for the lower percentile. Considering a mean spread period of 25 years, the difference for the 75% percentile is 84% larger than that for the 25 percentile and the difference for the 99% percentile is 341% than that of the 1% percentile. This is an indication of the skewness of the distribution of Contribution rates. In particular, it appears to become skewed to the right (see sections 6.7.4.1 and 6.7.4.2).

The standard deviation effect is described in Table 6.20 where the distance of the C(t) percentile values from the median is presented by value of the standard deviation, for the case where the mean spread period equals 5 years. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.20: *Model II, Random spread parameter, Difference between the Contribution percentiles and the Median, spread period equals 5 years*

standard deviation	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
0.05	5	20.01	-6.83	7.86	-4.75	5.77	-2.24	2.33
	20	4.56	-0.44	0.83	-0.32	0.55	-0.16	0.19
	100	3.78	0.00	0.00	0.00	0.00	0.00	0.00
0.10	5	18.83	-10.16	16.34	-7.72	11.57	-3.75	3.99
	20	4.45	-0.56	1.84	-0.48	1.19	-0.26	0.4
	100	3.78	0.00	0.00	0.00	0.00	0.00	0.00
0.15	5	17.11	-14.39	22.04	-9.75	15.43	-5.00	6.30
	20	4.28	-0.93	3.44	-0.52	2.08	-0.29	0.56
	100	3.78	0.00	0.00	0.00	0.00	0.00	0.00

We make the following remarks on table 6.20:

The lower percentile of each pair has a lower difference between it and the median than the upper percentile. In particular, the distribution of the Contribution rates appears to become skewed to the right (see sections 6.7.4.1 and 6.7.4.2). As observed in table 6.16, section 6.7.3.1, when the mean spread period is low, the distance of the percentiles from the median are at a considerable level at low values of t, (for example, year 5), while significantly decrease thereafter (for example, year 20).

6.7.4 SKEWNESS[24] and KURTOSIS

6.7.4.1 Skewness in Model II

Skewness is calculated for the years 1,5,10,20,30,50,100.

Tables 6.21 and 6.22 confirm the conclusions derived in section, 6.7.3, as far as both the Fund and Contribution rate values are concerned:

Specifically as presented in table 6.21, when the mean spread period is extended, the distribution becomes skewed to the left for the Fund and skewed to the right for the Contribution rates values.

We present skewness for both the Fund and Contribution rates since for model II the relationship between skewF and SkewC, i.e. skewF = - SkewC, does not hold as it holds for model I.

[24] in Appendix 13 the definitions are quoted

Table 6.21 *Model II, Random spread parameter: Fund / Contribution rates skewness,* $\sigma_\lambda = 0.05$

amort period		Skewness					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
5ys	F(t)	-0.01	-0.46	-0.82	-1.12	-1.31	-1.73
	C(t)	0.25	0.61	0.90	0.99	1.69	1.84
10ys	F(t)	0.21	-0.22	-0.42	-0.61	-0.96	-2.25
	C(t)	0.70	0.86	0.93	1.27	1.61	2.48
15ys	F(t)	0.61	0.23	-0.12	-0.49	-0.85	-1.81
	C(t)	1.15	1.01	1.24	1.59	1.61	2.54
20ys	F(t)	0.51	0.22	-0.24	-0.52	-0.77	-1.79
	C(t)	1.20	1.09	1.45	1.68	1.96	2.62
25ys	F(t)	0.81	0.38	0.00	-0.34	-0.55	-1.09
	C(t)	1.95	1.79	1.93	2.45	2.31	2.32

As presented in table 6.22, when the standard deviation increases the distribution becomes skewed to the left for the Fund and skewed to the right for the Contribution rates values; in the later years it is clear that as σ_λ increases, skewness values increase as well.

Table 6.22 *Model II, Random spread parameter: Fund / Contribution skewness, mean spread period value = 5 years*

σ_λ		Skewness					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F skew	-0.01	-0.46	-0.82	-1.12	-1.31	-1.73
	C skew	0.25	0.61	0.90	0.99	1.69	1.84
0.1	F skew	0.22	-0.50	-1.06	-1.69	-2.40	-5.09
	C skew	0.55	0.94	1.53	1.98	2.88	5.98
0.15	F skew	0.31	-0.39	-1.28	-2.37	-4.05	-7.50
	C skew	0.10	1.04	1.41	2.29	3.84	7.56

6.7.4.2 Kurtosis in Model II

Kurtosis is calculated for the years 1,5,10,20,30,50,100.Table 6.23 and 6.24 values below, confirm the conclusions derived in section 6.7.3, as far as both the Fund and Contribution rate values are concerned:

Specifically as presented in table 6.23, for small values of t and low values of the mean spread period kurtosis is small; thus the distribution is close to the bell-shaped normal distribution. As kurtosis is higher than zero, a peaked distribution of the sample values is formed. In specific years, kurtosis takes negative values so that the distribution appears to be flat.

Table 6.23 *Model II, Random spread parameter: Fund / Contribution rates kurtosis*

$\sigma_\lambda = 0.05$

Amort period		Fund / Contribution rates kurtosis					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
5ys	F(t)	0.14	0.58	1.22	2.66	2.62	3.76
	C(t)	-0.10	0.56	1.50	1.53	5.33	4.50
10ys	F(t)	0.05	-0.02	0.18	0.38	1.19	10.22
	C(t)	0.23	0.93	1.19	2.14	4.39	10.13
15ys	F(t)	0.64	0.08	-0.26	0.22	0.93	5.56
	C(t)	1.93	1.35	1.99	3.88	3.52	11.19
20ys	F(t)	0.40	0.08	-0.06	0.53	0.65	5.57
	C(t)	1.72	1.38	2.52	4.50	6.35	11.26
25ys	F(t)	0.77	0.19	-0.19	-0.16	0.12	1.56
	C(t)	5.95	4.70	5.65	9.61	8.71	8.96

In table 6.24, we present the values of kurtosis when the standard deviation of $\lambda(t)$ increases; as σ_λ increases, kurtosis increases as well.

Table 6.24 *Model II, Random spread parameter: Fund / Contribution rates kurtosis, mean spread period value = 5 years*

σ_λ		Fund / Contribution rates kurtosis, in specific years					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F kurt	0.14	0.58	1.22	2.66	2.62	3.76
	C kurt	-0.10	0.56	1.50	1.53	5.33	4.50
0.1	F kurt	-0.12	0.74	1.36	3.85	7.39	42.10
	C kurt	0.22	1.11	3.71	5.95	11.49	53.86
0.15	F kurt	0.83	1.51	3.09	9.18	24.24	69.76
	C kurt	2.34	1.36	5.92	7.35	21.34	75.63

6.8 MODEL III, RANDOM RATES of INVESTMENT RETURN & RANDOM SPREAD PARAMETER

6.8.1 ASSUMPTIONS

For Model III we considered the following different cases:

Case I	Case II	Case III
$E(\lambda) = \frac{1}{\ddot{a}_{\overline{15}\|}}$, $E(i(t)) = 0.05$	$E(\lambda) = \frac{1}{\ddot{a}_{\overline{15}\|}}$, $E(i(t)) = 0.05$	$E(\lambda) = \frac{1}{\ddot{a}_{\overline{15}\|}}$, $E(i(t)) = 0.05$
$\sigma_i = \sigma_\lambda = 0.05$	$\sigma_i = 0.10$, $\sigma_\lambda = 0.05$	$\sigma_i = 0.15$, $\sigma_\lambda = 0.05$
$\sigma_i = 0.05$, $\sigma_\lambda = 0.10$	$\sigma_i = \sigma_\lambda = 0.10$	$\sigma_i = 0.15$, $\sigma_\lambda = 0.10$
$\sigma_i = \sigma_\lambda = 0.15$	$\sigma_i = \sigma_\lambda = 0.15$	$\sigma_i = \sigma_\lambda = 0.15$

We applied these values for the calculation of NC(t), AL(t), F(t) and C(t) of the following distributions:

Power function (p=0.8, p=1, p=1.5)

Truncated Pareto (k=0.3, k=0.8, k=1.5)

Truncated Exponential (σ=30, σ=40, σ=50)

Since the spread period parameter is assumed as random we consider that the spread period of 15 years is the value that corresponds to the mean of the distribution for the parameter λ . Hence, throughout this section we refer to the mean spread period.

6.8.2 DESCRIPTIVE MEASURES of the SAMPLE: AVERAGE, VARIANCE and STANDARD DEVIATION

6.8.2.1 Sample Average

Simulations verify the expected result that the sample average, $\bar{F}(t)=\sum_{k=1}^{1000}\frac{F_k(t)}{1000}$,

$\bar{C}(t)=\sum_{k=1}^{1000}\frac{C_k(t)}{1000}$ approximate significantly the theoretical expected value EF(t), EC(t).

Both the expected values, EF(t) and EC(t), provided by the theoretical model, remain unchanged as the standard deviation of the rates of investment return and/or the spread parameter changes. The sample average changes, since the standard deviation value affects the simulated data.

As σ_λ and σ_i reach high levels, the variability of the Contribution rates values increases significantly and we have to consider a higher number than 1,000 of simulated paths to obtain convergence. Specifically, for the cases where σ_i=0.1, σ_λ=0.15 and σ_i=σ_λ=0.15 we increased the number of simulated paths to 4,000 and 3,000 respectively.

6.8.2.2 Variance, Standard deviation

Both the variance values, VarF(t) and VarC(t), provided by the theoretical model, change as the standard deviation value of the rates of investment return and/or the spread parameter changes. The sample variances also change, since the standard deviation value affects the simulated data.

As σ_λ increases, the variability of the Fund and Contribution rates also increases especially, when σ_i value lies at a high level. This is an expected result, obtained as well under Model I and Model II under high standard deviation values, (see also appendix

11). Under Model III, the distribution of the Fund and Contribution rates is more spread out than under either Model I or Model II, noting that, in addition, the random variables, i(t), $\lambda(t)$ are mutually independent.As an open question for the future we leave the case of examining the Fund and Contribution rates level when i(t) and $\lambda(t)$ are dependent random variables.

In table 5.5 we have shown that as either standard deviation increases (σ_i or σ_λ) the number of years m^* below which we obtain convergence decreases significantly.

For the case where $\sigma_i = \sigma_\lambda = 0.15$, we have shown (table 5.5) that convergence cannot be obtained because the mean spread period, m^*, has to be less than 23.88 years.

Convergence between the sample variance of the Contribution rates and the one provided by the theoretical model is difficult to obtain even after increasing the number of simulated paths. For the case where $\sigma_i = 0.1$, $\sigma_\lambda = 0.15$, the sample variance seems to approach the variance provided by the theoretical model after increasing the simulated paths, over a 100 year period, to 10,000[25], (figures 6.6, 6.7). In figures 6.4 – 6.7 below, the sample variance of the Fund and the Contribution rates are plotted against the theoretical VarF(t) and VarC(t) respectively, for the cases where $\sigma_i = \sigma_\lambda = 0.05$ and $\sigma_i = 0.1$, $\sigma_\lambda = 0.15$. Differences between the simulated and analytical results in figures 6.6 and 6.7 are due to sampling error; and these are reduced as we increase the number of simulations.

[25] λ outliers are omitted.

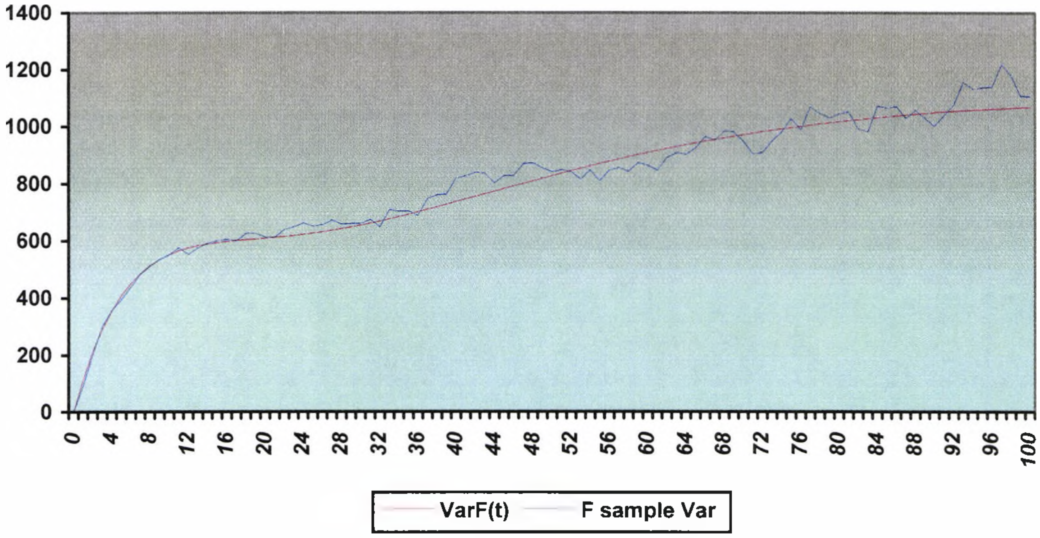

Figure 6.4: VarF(t) vs Sample Variance, $\sigma_i = \sigma_\lambda = 0.05$

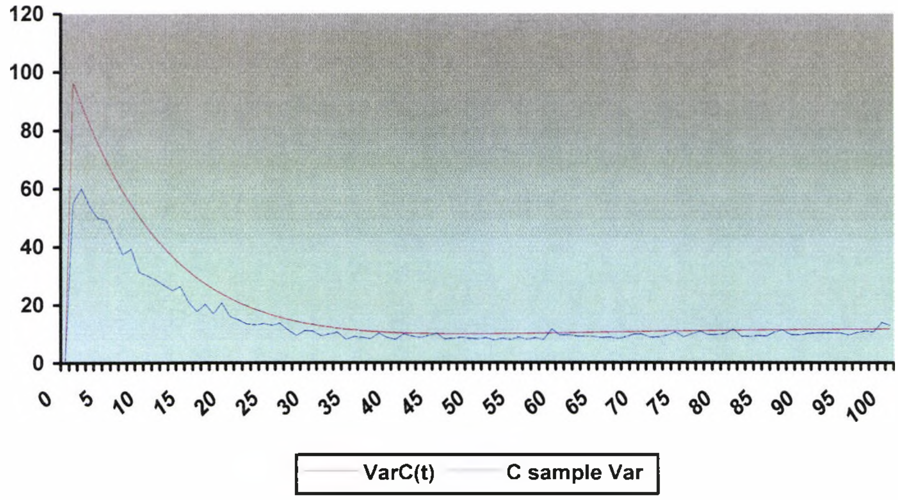

Figure 6.5: VarC(t) vs Sample Variance, $\sigma_i = \sigma_\lambda = 0.05$

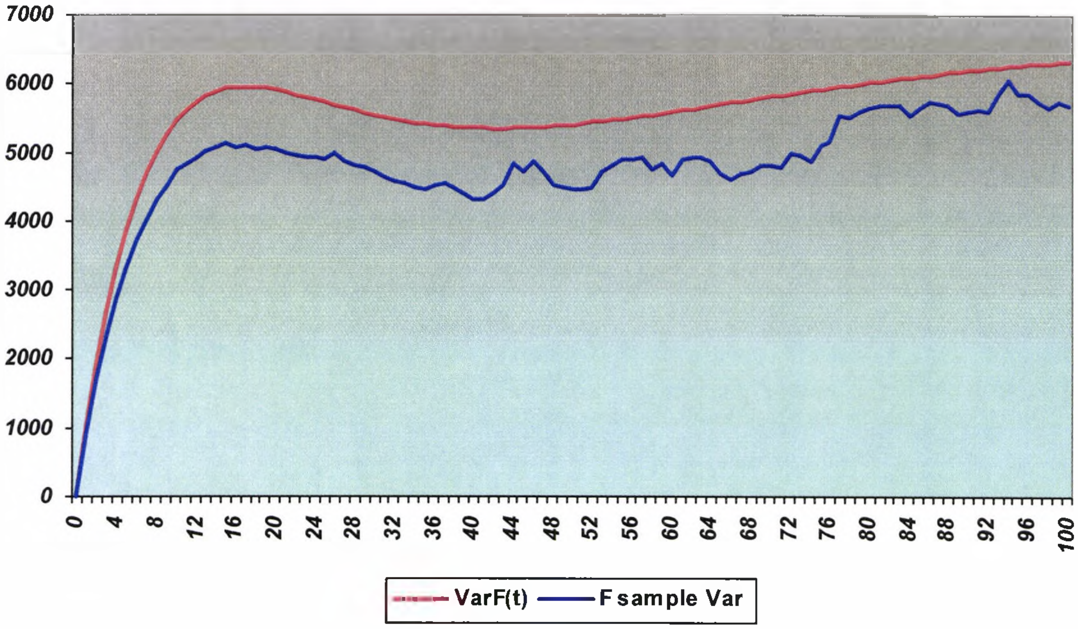

Figure 6.6: VarF(t) vs Sample Variance, $\sigma_i = 0.1$ and $\sigma_\lambda = 0.15$

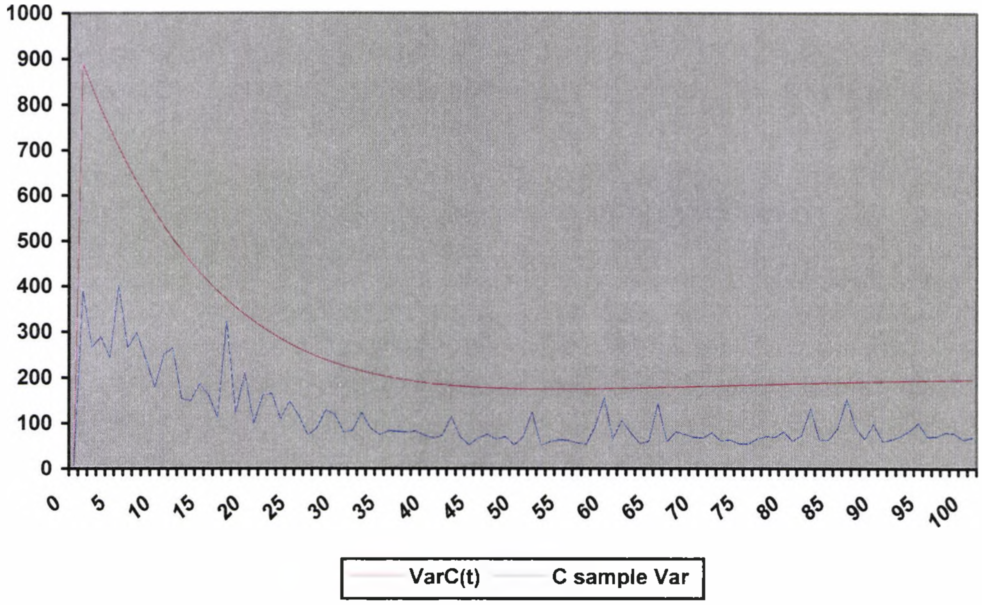

Figure 6.7: VarC(t) vs Sample Variance, $\sigma_i = 0.1$ and $\sigma_\lambda = 0.15$

6.8.3 PERCENTILES

For each one of the standard deviations of $\lambda(t)$ and per each standard deviation of $i(t)$, the 1%, 5%, 25%, 50%, 75%, 95% and 99% percentile values of the Fund and Contribution rates are calculated.

The percentiles indicate the sensitivity of the Fund and Contribution rates level becoming either low or high as a result of changes in parameters for the proposed Log Normal model. The theoretical model does not enable us to calculate percentiles easily since we need to specify a distribution for the random inputs.

The concentration of the percentile values around the median is also examined, taking their difference from the median on a yearly basis. This process deals with the measurement of both the skewness and the kurtosis of the simulated data.

The results show that the higher effect on the growth level of the Fund comes from the randomness of the spread parameter when compared with that of the randomness of the rates of investment return. In addition, the results verify that a) the standard deviation assumed for either parameter, has a high impact on the growth level of the Fund and b) as time "t" increases, the distance of the percentiles of the distribution of the Fund and the Contribution rates from the median tends to stabilize; this is expected since the parameter values we consider fulfill the requirements for convergence as $t \to \infty$.

6.8.3.1 Fund percentiles

In the tables 6.25 and 6.26 the 1% and 99% percentile values are shown at specific points of time, as a percentage of the Actuarial Liability. We chose those, since they reflect two extreme cases respectively. 1% percentile shows the value below which 1%

of cases occur for the Fund level while the 99% percentile the value above which 1% of cases occur.

Table 6.25: *Model III, Random rates of investment return, random spread parameter: 1% percentile Fund values as a percentage of the Actuarial Liability*

σ_i	σ_λ	1% percentile Fund values as a percentage of the Actuarial Liability								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.05	0.05	0.01	0.13	0.25	0.34	0.50	0.58	0.64	0.69	0.72
	0.1	0.00	0.00	0.01	0.11	0.24	0.40	0.51	0.67	0.68
	0.15	0.00	0.00	0.00	0.00	0.00	0.14	0.33	0.55	0.62
0.1	0.05	0.01	0.12	0.23	0.31	0.44	0.50	0.55	0.59	0.63
	0.1	0.00	0.00	0.01	0.10	0.21	0.36	0.43	0.55	0.57
	0.15	0.00	0.00	0.00	0.00	0.00	0.13	0.26	0.45	0.50
0.15	0.05	0.01	0.12	0.22	0.32	0.41	0.48	0.53	0.56	0.57
	0.1	0.00	0.00	0.01	0.08	0.21	0.38	0.43	0.49	0.50
	0.15	0.00	0.00	0.00	0.00	0.00	0.09	0.25	0.39	0.41

Table 6.26: *Model III, Random rates of investment return, random spread parameter: 99% percentile Fund values as a percentage of the Actuarial Liability*

		99% percentile Fund values as a percentage of the Actuarial Liability								
σ_i	σ_λ	5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.05	0.05	0.46	0.68	0.80	0.92	1.07	1.29	1.33	1.45	1.57
	0.1	0.86	1.01	1.10	1.16	1.28	1.32	1.36	1.53	1.45
	0.15	1.05	1.20	1.31	1.32	1.43	1.50	1.45	1.56	1.70
0.1	0.05	0.47	0.73	0.92	1.14	1.37	1.55	1.70	2.02	2.15
	0.1	0.84	1.01	1.11	1.35	1.48	1.68	1.74	2.26	2.44
	0.15	1.17	1.43	1.51	1.62	1.79	1.78	1.95	2.24	2.25
0.15	0.05	0.54	0.82	1.11	1.36	1.67	2.58	2.30	2.54	2.78
	0.1	0.88	1.13	1.38	1.63	1.75	2.54	2.38	2.68	3.13
	0.15	1.11	1.58	1.58	1.71	1.84	2.66	2.54	2.69	2.76

The 1% and 99% percentile values presented in tables 6.25 and 6.26 as a percentage of the Actuarial liability, are at a similar level with those presented in tables 6.13 and 6.14 for Model II, section 6.7.3.1 for the case where the mean spread period equals 15 years. The extreme case of the 1% percentile, shows that an unfavourable performance of the Fund, that may result in a Fund deficit, cannot be excluded. As in Models I and II sections 6.6.3.1, 6.7.3.1, we must also mention at this point that the starting value of the Fund level, F(0), has been assumed to be equal to zero. An open question for the future is to examine the Fund level status when its starting value, F(0), is different than zero. In the following tables 6.27 and 6.28 the F(t) percentile values distance from the median is presented, in the years 5,20 and 100. The effect of the standard deviation increase is shown for each parameter separately, keeping the other constant at the level of 0.05. The results are compared with those presented in Table 6.6, Model I section 6.6.3.1, and in Table 6.15, Model II section 6.7.3.1 for the case where the mean spread period equals 15 years. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.27: *Model III, Random rates of investment return, random spread parameter: Difference between the Fund percentiles and the Median,* $\sigma_i = 0.05$

σ_λ	years	Difference between the median and the respective percentile						
		median	1%	99%	5%	95%	25%	75%
0.05	5	39.89	-37.22	54.61	-26.92	35.99	-12.46	13.58
	20	121.86	-51.55	65.92	-38.09	43.33	-16.78	16.61
	100	192.76	-45.40	129.04	-34.96	59.50	-16.77	21.67
0.10	5	32.44	-51.25	143.37	-40.08	79.70	-19.30	27.54
	20	122.04	-100.10	114.45	-73.90	73.87	-31.04	31.25
	100	193.89	-54.35	103.77	-40.73	60.08	-16.40	22.12
0.15	5	25.65	-58.81	188.79	-49.04	113.13	-24.75	37.09
	20	118.27	-153.66	151.66	-119.84	96.61	-41.46	42.37
	100	193.82	-67.78	154.47	-44.92	67.73	-18.72	22.41

Table 6.28: *Model III, Random rates of investment return, random spread parameter: Difference between the Fund percentiles and the Median,* $\sigma_\lambda=0.05$

σ_i	years	Difference between the median and the respective percentile						
		median	1%	99%	5%	95%	25%	75%
0.05	5	39.89	-37.22	54.61	-26.92	35.99	-12.46	13.58
	20	121.86	-51.55	65.92	-38.09	43.33	-16.78	16.61
	100	192.76	-45.40	129.04	-34.96	59.50	-16.77	21.67
0.10	5	39.49	-37.06	57.50	-26.90	36.46	-12.23	13.77
	20	117.13	-52.67	115.33	-37.32	56.59	-16.49	20.97
	100	181.72	-52.28	257.83	-42.81	125.66	-20.82	33.81
0.15	5	38.88	-36.37	71.30	-26.43	39.73	-12.21	14.24
	20	113.88	-49.20	164.04	-38.16	70.53	-17.69	19.56
	100	170.88	-54.80	398.80	-43.90	170.65	-22.75	41.36

We make the following remarks on tables 6.27 and 6.28:

The upper percentile of each pair has a higher difference between it and the median than the lower percentile. In particular, it appears that the distribution exhibits skewness to the right (see sections 6.8.4.1 and 6.8.4.2). When σ_λ equals 0.15, (Table 6.27), the distribution of the Fund is spread out and this trend is not kept throughout all years. Specifically we observe[26] that around year 20 and for some period of time thereafter, the lower percentile has a higher difference between it and the median than the upper percentile.

The effect of an increase in σ_λ as σ_i remains unchanged (equal to 0.05) is not similar to that observed for Model II, table 6.15, in terms that in model II the distribution is

[26] The number of simulated paths is increased up to 5,000.

shifted to the left (skewness to the left), while in Model III it appears skewed to the right.

The effect of an increase in σ_i as σ_λ remains unchanged (equal to 0.05) is similar to that observed in Model I, table 6.6, in terms that under both models, the distribution appears skewed to the right (see sections 6.8.4.1 and 6.8.4.2).

6.8.3.2 Contribution Rates percentiles

The 1% and 99% percentile Contribution rate values as a percentage of Normal Cost, are shown at specific points of time in the following tables, 6.29 and 6.30. As it has also been mentioned in sections 6.6 3.2 and 6.7.3.2 we think it is important to investigate how low / high these values become in response to fluctuations of the level of the Fund.

Table 6.29: *Model III, Random rates of investment return, random spread parameter: 1% percentile Contribution rates values as a percentage of Normal Cost*

σ_i	σ_λ	1% percentile Contribution rates values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.05	0.05	2.11	1.84	1.58	1.27	0.58	-0.21	-0.64	-1.44	-2.16
	0.1	1.21	0.89	0.68	0.14	-0.74	-0.96	-1.51	-1.09	-1.44
	0.15	0.80	0.39	-1.49	-0.31	-1.14	-1.08	-2.04	-1.41	-3.23
0.1	0.05	2.12	1.71	1.37	0.45	-0.34	-1.49	-2.81	-4.22	-4.56
	0.1	1.22	0.96	0.63	-0.27	-1.75	-1.38	-2.32	-5.31	-7.88
	0.15	0.36	0.19	-2.85	-1.51	-2.33	-1.73	-3.29	-4.01	-7.51
0.15	0.05	2.07	1.63	0.31	-1.12	-2.10	-5.71	-5.78	-6.09	-8.93
	0.1	1.20	0.54	0.01	-1.28	-4.55	-6.69	-5.68	-6.42	-8.85
	0.15	0.12	-0.12	-3.89	-2.36	-3.59	-4.37	-7.75	-6.11	-8.43

Table 6.30: *Model III, Random rates of investment return, random spread parameter: 99% percentile Contribution rates values as a percentage of Normal Cost*

σ_i	σ_λ	99% percentile Contribution rates values as a percentage of Normal Cost								
		5th yr	10th yr	15th yr	20th yr	30th yr	40th yr	50th yr	80th yr	100th yr
0.05	0.05	10.83	8.54	7.14	6.42	5.08	3.83	3.46	2.97	3.22
	0.1	14.37	11.96	10.91	8.96	7.23	5.76	4.64	3.40	3.91
	0.15	15.23	12.96	12.23	11.88	9.81	6.73	6.14	3.89	4.70
0.1	0.05	11.07	8.23	7.23	6.65	5.44	4.61	4.16	3.53	3.95
	0.1	14.35	11.97	10.90	9.13	7.62	6.16	5.68	4.85	4.94
	0.15	14.95	12.99	12.61	12.23	9.90	7.81	6.82	6.37	5.65
0.15	0.05	11.11	8.67	7.18	6.86	5.73	4.82	4.58	4.20	4.35
	0.1	14.40	12.10	10.82	9.63	7.70	7.34	6.37	5.30	5.75
	0.15	14.90	13.68	12.56	11.80	10.05	8.34	7.64	5.79	6.23

As in table 6.7 in Model I, in table 6.29 we observe that in the extreme case of the 1% percentile, the values may turn to be negative as the standard deviation, σ_i, increases. A result that implies that a refund of contribution may occur with a reasonable level of probability. However, since, as discussed earlier, the $\lambda(t)$ effect appears to be higher than that of i(t) and given that as σ_λ increases in the extreme case of the 1% percentile the Fund level turns to be negative, we have to point out that when such values are obtained the adequacy of the Fund should be closely monitored .

The distance of the Contribution rates percentile values from the median is presented in tables 6.31 and 6.32 in the years 5,20 and 100. The effect of the standard deviation increase is shown for each parameter separately, keeping the other constant at the level of 0.05. The results are compared with those presented in tables 6.8, Model I section

6.6.3.2, and 6.19, Model II section 6.7.3.2 for the case where the mean spread period equals 15 years. The percentiles are grouped in corresponding pairs for ease of comparison.

Table 6.37: *Model III, Random rates of investment return, random spread parameter: Difference between the Contribution percentiles and the Median, σ = 0.05*

σ_λ	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
0.05	5	17.61	-9.63	23.35	-7.73	14.24	-4.27	4.85
	20	10.23	-5.44	14.04	-4.03	9.19	-2.05	3.18
	100	4.65	-12.83	7.54	-5.25	4.14	-1.56	1.53
0.1	5	13.59	-9.00	40.77	-7.67	26.02	-4.31	8.47
	20	8.48	-7.95	25.41	-4.50	15.66	-2.56	4.54
	100	4.33	-9.76	10.46	-4.62	5.46	-1.09	1.33
0.15	5	11.51	-8.50	46.08	-6.85	28.48	-4.31	8.13
	20	7.25	-8.43	37.70	-3.72	20.68	-2.27	5.33
	100	4.07	-16.28	13.70	-4.65	5.81	-0.66	1.17

Table 6.38: *Model III, Random rates of investment return, random spread parameter: Difference between the Contribution percentiles and the Median,* $\sigma_\lambda = 0.05$

σ_i	years	median	Difference between the median and the respective percentile of the Contribution rates					
			1%	99%	5%	95%	25%	75%
0.05	5	17.61	-9.63	23.35	-7.73	14.24	-4.27	4.85
	20	10.23	-5.44	14.04	-4.03	9.19	-2.05	3.18
	100	4.65	-12.83	7.54	-5.25	4.14	-1.56	1.53
0.1	5	17.53	-9.51	24.32	-7.80	14.43	-4.21	4.77
	20	10.41	-8.71	14.76	-4.71	9.17	-2.30	3.39
	100	5.40	-22.64	9.54	-10.64	5.95	-2.40	2.03
0.15	5	17.62	-9.80	24.42	-8.07	14.32	-4.36	4.79
	20	10.85	-15.09	15.09	-5.53	9.51	-2.71	3.30
	100	6.15	-39.91	10.31	-13.97	6.85	-2.89	2.29

In tables 6.31 and 6.32, we observe that the distance of the percentiles from the median does not follow a stable pattern, especially as the percentiles become more outlying. In the first years the upper percentile of each pair appears to have the higher difference between it and the median than the lower percentile, while thereafter the opposite is observed. This is an indication of skewness to the right in the first years, as in Model II section 6.7.3.1, and to the left thereafter, as in Model I section 6.6.3.1 (see sections 6.8.4.1 and 6.8.4.2).

6.8.4 SKEWNESS[27] and KURTOSIS

6.8.4.1 Skewness in Model III

Skewness is calculated for the years 1,5,10,20,30,50,100.

Tables 6.33 and 6.34 confirm the conclusions derived in section, 6.8.3, as far as both the Fund and Contribution rate values are concerned. In particular, as either standard deviation remains unchanged while the other increases skewness to the right appears for the distribution of the Fund. For the Contribution rate distribution, skewness to the right appears in the first years and to the left thereafter.

Table 6.33 *Model III, Random Rates of investment return, Random Spread parameter: Fund / Contribution rates skewness,* $\sigma_i = 0.05$

σ_λ		Fund / Contribution rates skewness, in specific years					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F kurt	0.63	0.32	0.38	1.00	1.22	1.61
	C kurt	1.17	1.01	1.14	0.51	-0.58	-2.46
0.1	F kurt	1.63	0.67	0.32	-0.04	0.45	2.04
	C kurt	0.99	1.43	1.43	-8.92	-12.16	-0.32
0.15	F kurt	3.15	1.44	0.75	0.42	0.30	2.24
	C kurt	-23.36	-23.48	-26.61	-12.10	-5.62	-3.83

[27] in Appendix 13 the definitions are quoted

Table 6.34 *Model III, Random Rates of investment return, Random Spread parameter*

Fund / Contribution rates skewness, $\sigma_\lambda = 0.05$

σ_i		Fund / Contribution rates skewness, in specific years					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F kurt	0.63	0.32	0.38	1.00	1.22	1.61
	C kurt	1.17	1.01	1.14	0.51	-0.58	-2.46
0.1	F kurt	0.79	0.93	1.61	2.19	3.66	7.48
	C kurt	1.18	0.91	0.58	-0.14	-4.08	-11.82
0.15	F kurt	1.11	1.86	10.52	14.57	9.47	9.11
	C kurt	1.14	0.77	-9.59	-11.50	-4.60	-6.03

6.8.4.2 Kurtosis in Model III

Kurtosis is calculated for the years 1,5,10,20,30,50,100.

Table 6.35 and 6.36 values below, confirm the conclusions derived in section 6.8.3, as far as both the Fund and Contribution rate values are concerned; when either standard deviation remains unchanged while the other increases, kurtosis values are very high.. In particular, assuming that σ_i does not change as σ_λ increases, for Contribution rates no kurtosis is observed, while Fund kurtosis is obtained for some period of time. When σ_λ remains unchanged as σ_i increases, for both the Contribution rates and the Fund, as the number of years increases, no kurtosis is observed.

Table 6.35 *Model III, Random Rates of investment return, Random Spread parameter: Fund / Contribution rates kurtosis,* $\sigma_i = 0.05$

σ_λ		Fund / Contribution rates kurtosis, in specific years					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F kurt	0.77	0.29	0.62	4.51	3.37	5.45
	C kurt	2.03	1.29	1.82	5.59	4.53	17.54
0.1	F kurt	5.97	1.11	0.90	0.72	1.12	11.76
	C kurt	3.61	2.03	5.03	200.81	284.76	9.88
0.15	F kurt	20.35	5.82	4.77	4.87	3.15	12.93
	C kurt	612.56	676.75	797.09	297.68	74.89	50.20

Table 6.36 *Model III, Random Rates of investment return, Random Spread parameter: Fund / Contribution rates kurtosis,* $\sigma_\lambda = 0.05$

σ_i		Fund / Contribution rates kurtosis, in specific years					
		5th yr	10th yr	20th yr	30th yr	50th yr	100th yr
0.05	F kurt	0.77	0.29	0.62	4.51	3.37	5.45
	C kurt	2.03	1.29	1.82	5.59	4.53	17.54
0.1	F kurt	1.59	2.54	6.50	10.30	31.95	90.36
	C kurt	2.06	1.43	2.72	4.95	39.31	228.46
0.15	F kurt	2.94	9.65	201.45	313.62	161.60	144.97
	C kurt	1.89	1.81	205.96	241.71	40.58	66.23

6.9 CONCLUSIONS

After the detailed analysis of the simulated data assuming that either or both parameters are random variables with Log Normal distribution, we have reached the conclusions below:

Under any pension accrual density function given that the expected value and standard deviation of the variable parameter are the same, similar conclusions are obtained for both the progress with time of the Fund and Contribution rate.

Simulations verify the expected result that the sample average, and the sample variance approximate significantly the corresponding values obtained by the theoretical model.

Analyzing the sensitivity of results to changes in parameters we observe that:

Both the level of the expected value of the random variable parameter and its standard deviation value have a high impact on the growth of the Fund and, as a consequence, on the Contribution rates level.

As either the expected value of the rates of investment return increases or the expected value of the spread parameter decreases (i.e. the number of years the unfunded liability is amortised increases), there is a greater probability that the scheme will at some time have insufficient assets.

As the standard deviation of either parameter increases, the distribution of the Fund as well as the distribution of the Contribution rates becomes more spread out.

When rates of investment return are random and with an initial funding level of 0%, the 1% percentile of the Fund distribution shows that there is not a possibility of deficit.

When either the spread parameter or both parameters are assumed as random variables and with an initial funding level of 0%, there is a non zero probability of deficit for the scheme; this probability appears to increase when standard deviation values are increased and the mean spread period is increased.

From the definition of skewness, in model I it holds that: skewF = -skewC.

In each model, skewness and low kurtosis values are observed over time, when the standard deviation values of either parameter ($i(t)$ or $\lambda(t)$) are low.

APPENDIX 13

Skewness

The sample skewness, denoted by $\hat{\gamma}_3$, characterizes the degree of asymmetry of the sample of all possible Fund values around the sample average.

Sample skewness is calculated through the following formula:

$$\hat{\gamma}_3 = \frac{n^{\frac{1}{2}} * \sum_{i=1}^{1000} (F_i - \bar{F}(t))^3}{\left(\sum_{i=1}^{1000} (F_i - \bar{F}(t))^2\right)^{\frac{3}{2}}}$$, where n = number of sample values

and the outcome values compared with the conclusions derived when the distance of the percentiles from the median is examined.

Kurtosis

The sample kurtosis, denoted by $\hat{\gamma}_4$, characterizes the relative peakedness or flatness of a sample, compared with the normal distribution[28].

Sample kurtosis is calculated through the following formula:

$$\hat{\gamma}_4 = \frac{n * \sum_{i=1}^{1000} (F_i - \bar{F}(t))^4}{\left(\sum_{i=1}^{1000} (F_i - \bar{F}(t))^2\right)^2} - 3$$, where n = number of sample values

The values are compared with the conclusions derived when the distance of the percentiles from the median was examined.

[28] According to the definition above, the kurtosis for any normal distribution equals to zero.

CHAPTER 7

CONCLUSIONS and EXTENSIONS

7.1 CONCLUSIONS

In our study we view the actuarial cost methods as 'processes' where time is considered, adding as another factor, in the pension scheme implementation, 'the age of the plan participant'. In particular, we introduce the accrual density function m(x), so as to correspond to a probability density function f(x).

We have chosen the following distributions as being possible candidates for application to pension funding methods: Power function, Truncated Exponential and Truncated Pareto. The criteria set for our choice are the mathematical properties of m(x), its age profile and the interval range of its underlying parameters. In addition, we have taken into account the utility of the accrual function from the perspective of the actuary.

The comparison between the different methods is based on the portion of benefit purchased throughout the participant's active years. The different distributions (and the set of parameters investigated) show how the portion of benefit varies along with the age of the scheme participant. Among the distributions chosen, we identify the Power function for its special property that allows m(x) to be associated with either a decelerating or an accelerating cost method, on the basis of the adjustment of the parameter p value.

We conclude that there is no definite rule that either the actuary or the scheme participants should follow for the choice of the cost method, since this depends on the benefits provided by the pension plan, the sponsor's financial plans, and the economic environment. However, if the actuary thinks that it would be more appropriate to increase the contribution rate along with the age of the plan participant, the Power function with $p>1$ has to be used. Otherwise either the Power function with $p<1$ or one of Truncated Pareto or Truncated Exponential may be selected.

The pension funding functions have been calculated with the use of the density functions defining 'New Cost methods', which were compared with the 'Traditional'. According to the results derived we have reached the following general conclusions:

The development of Normal Cost follows the pattern of the accrual function 'm(x)'.

On the basis of the benefit allocation along with age increase there are two groups for the Normal Cost value. In the first group, the benefit is allocated in higher proportion as age increases and Normal Cost values are very similar when they are calculated either under the Current Unit Credit method or using the Power function, $p>1$. In the second group, the benefit is allocated in lower proportion as age increases and Normal Cost values are very similar when they are calculated either under the Entry Age Normal method or using one of the Truncated Exponential and the Truncated Pareto.

The Actuarial Liability is higher under the density function that allocates larger proportions of the benefit at younger ages. In particular, the following inequalities hold for the accrued liability at time t as well as for the Actuarial Liability at age x

a) ${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)_{(Truncated\ Exponential)}$

b) ${}^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < AL(t)_{(Trunc.Pareto)}^{k<1, k<\frac{p}{d}}$, and

c) $AL(t)_{(Power\ ,\ p>1)} < {}^{PUC}AL(t) \equiv AL(t)_{(Uniform)} < {}^{EAN}AL(t)$,

These inequalities show that in practice, among the different accrual functions, a lower Actuarial Liability is expected from those that are associated with an accelerating cost method than from the ones associated with a decelerating cost method.

An idea, which we have also developed, is to view the fund and contribution rates as 'processes' with respect to time by modeling the parameter that amortizes the unfunded liability. Using a simple theoretical stochastic model built independently of the distribution assumptions, we have obtained theoretical results for the first two moments of the Fund and Contribution rates distributions. We have proceeded through a discrete time formulation for the cases where: a) i(t) are random variables and λ(t) is constant, b) λ(t) is a random variable and the rates of investment return 'i(t)' constant, c) both λ(t) and i(t) are random and mutually independent. Independence is assumed between the Fund value, 'F(t)' and the random parameter.

The respective results produced by Dufresne (1988) and Owadally and Haberman (1999) have been extended after including the salary rate of increase. We show that the salary rate of increase is a key assumption for the Actuarial Liability level and thus for the expected value of the level of the fund and the contribution rates.

For each one of the above models:

- Convergence for the first moments is obtained in an economic environment, where $i > -1$ and if the interest earned during a year in excess of the amortization charge is lower than the salary rate of increase.
- Convergence for the second moments is also observed, after restrictions are applied to the set of the parameter values. According to the parameter restrictions an 'optimal region', m^*, is specified for the number of years, m, over which the unfunded liability is spread. We show that for m greater than a particular value m^*, the variances of both the fund and the contribution are increasing functions of m; the 'optimal' values of m are $1 \leq m \leq m^*$.

Assuming that the salary rate of increase is equal to zero, when both i(t), λ(t) are random variables, illustrative examples show that:

- The ultimate value of the standard deviation of the contribution rates is not a monotonic function of λ(t). Up to a certain value of λ, there is a 'trade off' between the ultimate value of the standard deviation of the fund and the ultimate value of the standard deviation of the contribution rates.
- When the spread period is below a certain number of years, there is a 'trade off' between the standard deviation of $F(\infty)$ and the standard deviation of $C(\infty)$; as $F(\infty)$ increases, $C(\infty)$ decreases.

- As i(t) value approaches zero, the ultimate value of standard deviation of the fund increases, while the ultimate value of standard deviation of the contribution rate decreases.

Dufresne (1988) has showed the above results for the case where the rates of investment return are i.i.d. random variables. In this work, Dufresne's results have been extended for the case where both i(t) and λ(t) are mutually independent random variables.

Simulations performed assuming that either or both parameters are random and distributed as Log Normal random variables. For our choice, we have taken into account the fact that the Log Normal is convenient for evaluating the value of the fund. We have also considered that it has been used by many others researchers for the case where the i(t) are assumed to be random variables. For λ, the stochastic approach is new; we have investigated the Log Normal distribution for reasons of consistency, and also thinking about potential future work, which would involve examining i(t) and λ(t) assuming dependency.

For either one of the above cases, we also assumed that the pension plan is implemented based on a different pension density function each time with a 0% initial funding level.

According to the results derived, we have reached the following general conclusions:

- Under any pension density function given that the expected value and standard deviation of the variable parameter are the same, similar conclusions derived for both the Fund and Contribution rates progress with time. The level of difference between either the sample average and the theoretical expected value or the sample and the theoretical variance in all models: a) tends to decrease slightly when the plan is implemented using either the Power or the Exponential density function as the parameters p, σ respectively increase; b) tends to increase along with the increase of the parameter k when the plan is implemented under the Truncated Pareto. The same hold when comparing the level of the percentile values.

- Analysing the sensitivity of results to changes in parameters:

Both the level of the expected value of the random variable parameter and its standard deviation value have a high impact on the growth of the Fund and, as a consequence, on the Contribution rates level.

As either the expected value of the rates of investment return increases or the expected value of the spread parameter decreases (i.e. the number of years the unfunded liability is amortised increases), there is a greater probability that the scheme will at some time have insufficient assets.

As the standard deviation of either parameter increases, the distribution of the Fund as well as the distribution of the Contribution rates becomes more spread out.

In each model, skewness and low kurtosis values are observed over time, when the standard deviation values of either parameter ($i(t)$ or $\lambda(t)$) are low.

7.2 FUTURE WORK

Throughout the implementation of our work, questions relating to the models used and assumptions made have arisen suggesting further, future steps in our approach to pension funding. In particular:

We have assumed that the Fund value at start 'F(0)' equals zero. Future work could investigate Fund performance when the initial funding level is different than 0%.

We are also aware that our analysis disregards the important issue of dependency between the rates of investment return 'i(t)' and the spread parameter 'λ(t)'. This issue though raises deeper questions and clearly needs much further research. We would propose two Log Normal models with a correlation coefficient $\rho \neq 0$.

Dependency as well could be investigated between λ(t) and F(t) noting that the fund level heavily depends on the assumption concerning λ(t).

Besides the above, more general questions on the basis of our work could be investigated. A few ideas for further research are the following:

For Models I and III we have assumed that the rates of investment return are i.i.d. random variables. Future work could investigate as well the effect of introducing time series models of i(t).

Future work, could also consider other risk measures for both F(t) and C(t), for example, measures based on conditional mean shortfall.

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