Studies in Vector Potential Theory

By

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ABSTRACT

This thesis studies the theory of simple-layer and double-layer vector potentials. The connection with Somigliana's formula is brought out and throws light upon the behaviour of such potentials. Our analysis provides an easy route to the construction of Volterra dislocations.

CONTENTS

Introduction

The theory of simple-layer and double-layer vector potentials was first systematically given by Kupradze (1965). He was guided by the well known corresponding theory of simple-layer and double-layer scalar potentials, e.g. Kellogg (1929). Kupradze showed that the displacement fields of classical linear elastostatics, Knops&Payne (1971), could be represented by vector potentials, and he used his results to formulate vector boundary-integral equations covering all the main boundary-value problems of elastostatics. However these formulations were not favoured by applied mathematicians and theoretical engineers for three main reasons:

1. The potentials were generated by hypothetical vector sources which have no clear physical significance;
2. They led to vector integral equations which involved highly singular kernels over curved boundaries, so precluding any simple mathematical analysis, Smithies(1958),
3. Numerical solutions were out of the question because of the absence of adequate discretisation procedures and fast digital computers.

Some years before Kupradze's treatise much experience was gained in the discretisation and numerical solution of scalar boundary integral equations. This development led to a powerful numerical method of attack upon certain important problems of classical potential theory e.g. computation of electrostatic capacitance, Symm (1963), torsional rigidity, Jaswon&Ponter (1963) and potential fluid motion, Hess&Smith (1967). It also opened the way for the numerical solution of certain biharmonic boundary-value problems, in particular those arising from the bending or stretching of thin plates. This was achieved by exploiting Almansi's representation of a biharmonic function in terms of two harmonic functions, and then representing these harmonic functions as potentials, Jaswon & Symm (1977). However, a more far-reaching development was Rizzo's exploitation of Somigliana's formula on the boundary. This provided a functional relation between boundary displacements and tractions, which immediately yielded vector integral equations covering all the main boundary-value problems of the elastostatics. It may be shown, e.g. Jaswon & Symm (1977), that the Kupradze boundary formulations are mathematically equivalent to those of Rizzo. However Rizzo's formulations became widely acceptable because they involved directly the quantities of immediate engineering interest, i.e. the boundary displacements and tractions. In its discretised version, coupled with suitable software packages e.g. Brebbia (1978), Rizzo's approach has been developed into the BEM technique as we know it today.

Much of the mathematical foundation for BEM had in fact been already laid down by Kupradze, since Somigliana's formula involves the superposition of a simple-layer and double-layer vector potential. It therefore seems of interest to look closely at Kupradze's potentials by reference to some simple elastostatic fields having qualitatively distinct behaviours at infinity. No particular difficulty arises in representing any of these fields by a simple-layer potential $\underline{V}$. This is because $\underline{V} \longrightarrow O(r^{-1})$ as $r \longrightarrow \infty$, in line with the general behaviour of a regular elastostatic field $\underline{\phi}$. Physically speaking, $O(r^{-1})$ behaviour at infinity implies the existence of a resultant force acting on the boundary, an inherent feature of $\underline{V}$ since this is generated by a distribution of point-forces on the boundary.

Considerable difficulty arises with the representation of $\underline{\phi}$ by a double-layer vector potential $\underline{W}$, because $\underline{W} \longrightarrow O(r^{-2})$ as

$r \longrightarrow \infty$ whilst in general $\underset{\sim}{\phi} \longrightarrow O(r^{-1})$ as $r \longrightarrow \infty$. It has been suggested by Jaswon & Symm (1977) that we may write $\underset{\sim}{\phi} = \underset{\sim}{W}$ if $\underset{\sim}{\phi} \longrightarrow O(r^{-2})$ as $r \longrightarrow \infty$, but a closer analysis shows that such fields fall into two main classes :

1. $\underset{\sim}{\phi}$ provides a resultant moment acting on the boundary , in which case $\underset{\sim}{\phi} \neq \underset{\sim}{W}$, since $\underset{\sim}{W}$ provides no resultant moment acting on the boundary Jaswon & Symm (1977) ;

2. $\underset{\sim}{\phi}$ provides a null resultant moment acting on the boundary , in which case we may write $\underset{\sim}{\phi} = \underset{\sim}{W}$.

In case (1) we may supplement $\underset{\sim}{W}$ by suitable resultant-moment producing terms. More generally we may always supplement $\underset{\sim}{W}$ by resultant-force and resultant-moment producing terms to achieve a representation of any regular $\underset{\sim}{\phi}$. Examples will be given later.

The representation of $\underset{\sim}{\phi}$ by $\underset{\sim}{V}$ yields vector boundary-integral equations of the first kind for the relevant source-density distribution $\underset{\sim}{\sigma}$. Unique solutions always exist, but do not seem to have been achieved numerically . The representation of $\underset{\sim}{\phi}$ by $\underset{\sim}{W}$ yields vector boundary-integral equations of the second kind for the source-density distribution $\underset{\sim}{\mu}$. A solution may not exist. However a set of non-unique solutions always exists if $\underset{\sim}{W}$ is suitably supplemented, so allowing some flexibility in the choise of $\underset{\sim}{\mu}$ for generating $\underset{\sim}{W}$. These issues will be exploited by reference to specific problems.

Vector double-layer potentials offer an easy route to the theory of Volterra dislocations . This is a sheet in the elastic continuum across which the displacement jumps by a rigid-body component , the strains and stress remaining continuous . Such jumps may be ensured by introducing the double-layer distribution $\underline{\mu} = \underline{a} + \underline{b} \wedge \underline{r}$ over the sheet, where $\underline{a}$, $\underline{b}$ are constant vectors. If $\underline{b} = \underline{0}$ we obtain the vector analogue of a uniform magnetic shell or vortex-equivalent sheet . The analysis is given for a uniform magnetic shell . The field of a Volterra dislocation on a circular sheet is compared with that of a uniform magnetic shell on the sheet . As expected, the two fields have similar qualitative features .

This thesis divides naturally into three main parts. Part I summarises Kupradze's vector potential theory with a view to later applications. A new analysis is given for displacement fields having $O(r^{-2})$ behaviour as $r \longrightarrow \infty$. We show how to complete the double-layer vector potential so as to represent an arbitrary regular elastostatic displacement field.The connection with Somigliana's formula is brought out and helps to throw light upon Kupradze's representation

Part II uses the Papkovich-Neuber formula to construct some representative displacement fields in the infinite domain exterior to a spherical cavity, and it shows how to represent these fields by vector potentials.The potentials can not in general be evaluated exactly,but their asymptotic equivalence to the fields is verified.

Part III applies Kupradze's double-layer vector potentials to construct the field of Volterra dislocations. This brings out the analogy with the theory of a uniform magnetic shell and it also helps to connect Volterra dislocations with crystal dislocations, Pearson (1959).

Part of this thesis has been embodied in three published papers of which copies are attached at the end.

PART I

INTRODUCTORY ANALYSIS

This provides a summary of vector potential theory in a form suitable for subsequent applications.

Chapter 1

Vector Potential Theory

1.0 Introduction

It was Kupradze who first introduced vector potentials into the theory of elastostatics. He was very much guided by the role of scalar potentials in the theory of harmonic functions. Corresponding to the scalar simple-layer potentials there exist vector simple-layer potentials. Corresponding to the scalar double-layer potentials there exist vector double-layer potentials. Green's formula parallels Somigliana's formula. Corresponding to harmonic functions there exist displacement fields. Corresponding to the normal derivatives of a harmonic function there exist the traction vectors associated with a displacement field . Corresponding to a uniform harmonic function there exists a rigid-body displacement field . Corresponding to the scalar integral equations there exist vector integral equations.

1.1 Vector simple-layer potential

Corresponding to the scalar potential we introduce the vector potential

$$\underline{V}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy \quad ; \ \underline{y}\subset\partial B \ ; \ \underline{x}\subset B_i, B_e \ . \tag{1.1.1}$$

Here ∂B is a closed Liapunov smooth surface Jaswon & Symm (1977), $\underline{y}$ signifies a point of ∂B, dy signifies the area element at $\underline{y}$, $\underline{x}$

signifies any point of space i.e. within the interior domain B_i bounded by ∂B or within the infinite exterior domain B_e bounded internally by ∂B or on ∂B itself. Also $\underline{\underline{g}}(\underline{x},\underline{y})$ signifies the displacement dyadic:

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \begin{bmatrix} g(\underline{x}_1,\underline{y}_1) & g(\underline{x}_1,\underline{y}_2) & g(\underline{x}_1,\underline{y}_3) \\ g(\underline{x}_2,\underline{y}_1) & g(\underline{x}_2,\underline{y}_2) & g(\underline{x}_2,\underline{y}_3) \\ g(\underline{x}_3,\underline{y}_1) & g(\underline{x}_3,\underline{y}_2) & g(\underline{x}_3,\underline{y}_3) \end{bmatrix} \qquad (1.1.2)$$

where, in the isotropic continuum:

$$g(\underline{x}_\alpha,\underline{y}_\beta) = \frac{1-\kappa}{\mu\rho}\delta_{\alpha\beta} + \frac{\kappa}{\mu}\frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{\rho^3}$$

$$= \frac{1-\kappa}{\mu\rho}\delta_{\alpha\beta} + \frac{\kappa}{\mu\rho}\frac{\partial\rho}{\partial x_\alpha}\frac{\partial\rho}{\partial x_\beta}$$

$$= \frac{1}{\mu\rho}\delta_{\alpha\beta} - \frac{\kappa}{\mu}\frac{\partial^2\rho}{\partial x_\alpha \partial x_\beta} \quad ; \rho = |\underline{x}-\underline{y}| \;;\; \alpha,\beta = 1,2,3, \qquad (1.1.3)$$

where μ is the shear modulus, ν is Poisson's ratio $(0 < \nu \leq \frac{1}{2})$ and $\kappa^{-1} = 4(1-\nu)$. This is Kelvin's solution for the displacement component in the α - direction at $\underline{x}$ generated by a unit point-force acting in the β-direction at $\underline{y}$. Clearly column 1 defines the displacement vector at $\underline{x}$ generated by a unit point-force acting in the 1-direction at $\underline{y}$, etc. By virtue of the symmetry property::

$$g(\underline{x}_\alpha,\underline{y}_\beta) = g(\underline{y}_\beta,\underline{x}_\alpha) \;, \qquad (1.1.4)$$

we see that row 1 defines the displacement vector at $\underline{y}$ generated by a unit point-force acting in the 1-direction at $\underline{x}$. Finally $\underline{\sigma}$ signifies a vector source density with components $\underline{\sigma} = < \sigma_1, \sigma_2, \sigma_3 >$; and $\sigma_1 dy$ provides the magnitude of the point-force acting in the 1-direction at $\underline{y}$ etc. Expressed in terms of components, (1.1.1) appears as

$$V_\alpha(\underline{x}) = \int_{\partial B} g(\underline{x}_\alpha, \underline{y}_\beta)\, \sigma_\beta(\underline{y})\, dy \quad ; \ \underline{x} \subset B_i, B_e; \ \underline{y} \subset \partial B \ ; \ \alpha, \beta = 1,2,3 \qquad (1.1.5)$$

It has been proved by Kupradze (1965) that $\underline{V}$ has properties entirely analogous to those of scalar simple-layer potentials. These have been listed by Jaswon (1984).

1.2 Traction vector

Associated with $\underline{\underline{g}}(\underline{x}, \underline{y})$ we may compute the fundamental traction dyadic of the medium:

$$\underline{\underline{g}}^*(\underline{x},\underline{y}) = \begin{bmatrix} g^*(\underline{x}_1,\underline{y}_1) & g^*(\underline{x}_1,\underline{y}_2) & g^*(\underline{x}_1,\underline{y}_3) \\ g^*(\underline{x}_2,\underline{y}_1) & g^*(\underline{x}_2,\underline{y}_2) & g^*(\underline{x}_2,\underline{y}_3) \\ g^*(\underline{x}_3,\underline{y}_1) & g^*(\underline{x}_3,\underline{y}_2) & g^*(\underline{x}_3,\underline{y}_3) \end{bmatrix} \qquad (1.2.1)$$

where

$$g^*(\underline{x}_\alpha, \underline{y}_\gamma) = \frac{2\nu-1}{2(1-\nu)} \frac{1}{\rho^2} \Big[\frac{\partial \rho}{\partial x_\alpha} n_\gamma - \frac{\partial \rho}{\partial x_\gamma} n_\alpha + \frac{\partial \rho}{\partial n} \Big\{ \delta_{\alpha\gamma} + \frac{3}{1-2\nu} \frac{\partial \rho}{\partial x_\alpha} \frac{\partial \rho}{\partial x_\gamma} \Big\} \Big] ;$$

$$\alpha, \gamma = 1,2,3; \underline{x} \subset \partial B \qquad (1.2.2)$$

Column 1 of (1.2.1) signifies the traction vector at $\underline{x}$ generated by a unit point-force acting in the 1-direction at $\underline{y}$ etc. Kupradze (1965) proved the important formula:

$$\underline{V}^*(\underline{x}) = \int_{\partial B} \underline{\underline{g}}^*(\underline{x},\underline{y}) \cdot \underline{\sigma}(\underline{x})dy - 2\pi\underline{\sigma}(\underline{x}) \quad ; \underline{x}, \underline{y} \subset \partial B, \qquad (1.2.3)$$

for the traction vector at $\underline{x}$ associated with $\underline{V}$, corresponding with that for the normal derivative $\underline{V}'(\underline{x})$ of scalar potential theory Kellog (1929) . Following the sign convention by Jaswon & Symm (1977), we replace (1.2.3) by the formulae:

$$\underline{V}_i^*(\underline{x}) = \int_{\partial B} \underline{\underline{g}}_i^*(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy - 2\pi\underline{\sigma}(\underline{x}) \quad ; \underline{x}, \underline{y} \subset \partial B \qquad (1.2.4)$$

$$\underline{V}_e^*(\underline{x}) = \int_{\partial B} \underline{\underline{g}}_e^*(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy - 2\pi\underline{\sigma}(\underline{x}) \quad ; \underline{x}, \underline{y} \subset \partial B, \qquad (1.2.5)$$

referring to the traction at ∂B acting upon B_i, B_e respectively. Since

$$\underline{\underline{g}}_i^*(\underline{x},\underline{y}) + \underline{\underline{g}}_e^*(\underline{x},\underline{y}) = 0, \qquad (1.2.6)$$

owing to the continuity of $\underline{\underline{g}}^*(\underline{x},\underline{y})$ at $\underline{x}$ for a fixed $\underline{y}$, it follows that:

$$\underline{V}_i^*(\underline{x}) + \underline{V}_e^*(\underline{x}) = -4\pi\underline{\sigma}(\underline{x}). \qquad (1.2.7)$$

This useful result has a simple physical interpretation. Imagine the area element dx as a thin elastic strip sandwiched between the sides of ∂B, which is subject to a resultant force $\underline{\sigma}dx(4\pi)$ generated from its interior and balanced by a resultant force $(\underline{V}_i^* + \underline{V}_e^*)dx$ applied over the boundary.

I.3 Vector double-layer potential

A second, equivalent, traction dyadic associated with $\underline{\underline{g}}(\underline{x},\underline{y})$ is

$$\underline{\underline{g}}(\underline{x},\underline{y})^* = \begin{bmatrix} g(\underline{x}_1,\underline{y}_1)^* & g(\underline{x}_1,\underline{y}_2)^* & g(\underline{x}_1,\underline{y}_3)^* \\ g(\underline{x}_2,\underline{y}_1)^* & g(\underline{x}_2,\underline{y}_2)^* & g(\underline{x}_2,\underline{y}_3)^* \\ g(\underline{x}_3,\underline{y}_1)^* & g(\underline{x}_3,\underline{y}_2)^* & g(\underline{x}_3,\underline{y}_3)^* \end{bmatrix} \quad (1.3.1)$$

constructed by interchanging $\underline{x}$, $\underline{y}$ in (1.2.1). It may be shown Jaswon & Symm (1977), that column 1 of (1.3.1) signifies an elastostatic field, i.e. that generated by a unit traction source acting in the 1-direction at $\underline{y}$, etc. This field corresponds with the scalar field generated by a dipole source at $\underline{y}$ and has analogous properties. In particular it allows us to construct the vector double-layer potential:

$$\underline{W}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^* . \underline{\mu}(\underline{y}) dy \qquad ; \ \underline{x} \subset B_i, B_e; \ \underline{y} \subset \partial B. \qquad (1.3.2)$$

Here $\underline{\mu}$ signifies a vector source-density with components $\underline{\mu} = \langle \mu_1, \mu_2, \mu_3 \rangle$; $\mu_1 dy$ provides the magnitude of the traction acting in the 1-direction at $\underline{y}$, etc.. Expressed in terms of components (1.3.2) appears as:

$$W_\alpha(\underline{x}) = \int_{\partial B} g(\underline{x}_\alpha, \underline{y}_\beta)^* \mu_\beta(\underline{y}) dy \quad ; \ \underline{x} \subset B_i, B_e; \ \underline{y} \subset \partial B \ ; \quad \alpha, \beta = 1,2,3. \qquad (1.3.3)$$

$\underline{W}$ has properties entirely analogous to those of the scalar double-layer potential W.

In particular it defines an elastostatic displacement field everywhere except at ∂B, and it jumps at ∂B according to the formula:

$$\lim_{\underline{x}_i \to \underline{x}} \underline{W}(\underline{x}_i) = \underline{W}(\underline{x}) + 2\pi\underline{\mu}(\underline{x}) \quad ; \ \underline{x} \subset \partial B. \qquad (1.3.4)$$

$$\lim_{\underline{x}_e \to \underline{x}} \underline{W}(\underline{x}) = \underline{W}(\underline{x}) - 2\pi\underline{\mu}(\underline{x}) \quad ; \ \underline{x} \subset \partial B, \qquad (1.3.5)$$

as we pass from B_i or B_e to ∂B.

It will be noted that row 1 of (1.2.1) defines an elastostatic displacement field at $\underline{y}$, i.e. that generated by a unit traction force acting in the 1-direction at $\underline{x}$. Also, row 1 of (1.3.1) defines the traction vector at $\underline{y}$ generated by a unit point force acting in the 1-direction at $\underline{x}$.

Chapter 2

Representation of Elastostatic Displacement Fields by Vector Potentials

2.0 Introduction

In chapter 1 we noted that vector simple-layer and vector double-layer potentials are displacement fields under broad conditions. In this chapter we investigate the representation of an arbitrary displacement field by such potentials. We also show how the theory of single-layer potential representations can be based upon Somigliana's formula.

2.1 Somigliana's formula

Let $\underline{\phi}$ be a displacement field in B_i which assumes a given set of boundary values on ∂B. Regarding $\underline{\phi}(\underline{y})$ as a vector double-layer source density at $\underline{y} \subset \partial B$, it generates the vector double-layer potential

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})\overset{*}{\underset{i}{\cdot}}\underline{\phi}(\underline{y})dy \quad ; \qquad \underline{x} \subset B_i \ , \ \underline{y} \subset \partial B. \tag{2.1.1}$$

Also $\underline{\phi}$ has an associated traction vector $\underline{\phi}_i^*(\underline{y})$ at ∂B. Regarding this as a vector simple-layer source density, it generates the simple-layer potential:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})\cdot\underline{\phi}_i^*(\underline{y})dy \quad ; \quad \underline{x} \subset B_i \quad ; \ \underline{y} \subset \partial B. \tag{2.1.2}$$

Superposing (2.1.1) and (2.1.2) gives the identity:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i \cdot \underline{\phi}(\underline{y})dy - \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\phi}^*_i(\underline{y})dy$$

$$= 4\pi\underline{\phi}(\underline{x}) \quad ; \; \underline{x} \subset B_i \; , \quad \underline{y} \subset \partial B, \tag{2.1.3}$$

valid for a harmonic function $\underline{\phi}$ in B_i. This is Somigliana's formula, Smirnov —— (1964). This formula provides a fundamental link between the theory of elastostatic displacement fields and vector potential theory. When $\underline{x}$ lies on ∂B, (2.1.3) becomes:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i \cdot \underline{\phi}(\underline{y})dy - \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\phi}^*_i(\underline{y})dy$$

$$= 2\pi\underline{\phi}(\underline{x}) \quad ; \; \underline{x} \; , \quad \underline{y} \subset \partial B, \tag{2.1.4}$$

because the integral (2.1.2) remains continuous as $\underline{x}$ passes from B_i to ∂B whilst (2.1.1) jumps by $-2\pi\underline{\phi}$. Formula (2.1.4) provides a functional relation between $\underline{\phi}$ and $\underline{\phi}^*$ on ∂B, which has been used to generate boundary integral equations covering all the boundary-value problems of elastostatics.

When $\underline{x}$ passes from ∂B into B_e there occurs a further jump in the integral (2.1.1), giving Betti's identity:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i \cdot \underline{\phi}(\underline{y})dy - \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\phi}^*_i(\underline{y})dy$$

$$= 0 \quad ; \quad \underline{x} \subset B_e \; , \; \underline{y} \subset \partial B. \tag{2.1.5}$$

All exterior equations carry the same signs as their interior counterparts. For a regular displacement field $\underline{f}$ in B_e, which assumes continuous boundary values $\underline{f}(\underline{y})$ and continuous boundary tractions $\underline{f}^*_e(\underline{y})$ at $\underline{y} \subset \partial B$, Somigliana's formula yields the corresponding exterior formulae:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x}.\underline{y})^*_i . \underline{f}(\underline{y})dy - \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) . \underline{f}^*_e(\underline{y})dy$$

$$= 4\pi \underline{f}(\underline{x}) \quad ; \quad \underline{x} \subset B_e \quad , \underline{y} \subset \partial B \tag{2.1.6}$$

$$= 2\pi \underline{f}(\underline{x}) \quad ; \quad \underline{x} \ , \ \underline{y} \subset \partial B \tag{2.1.7}$$

$$= 0 \quad ; \quad \underline{x} \subset B_i \quad , \ \underline{y} \subset \partial B \tag{2.1.8}$$

where

$$\underline{f} = -\frac{1}{4\pi} |\underline{x}|^{-1} \int_{\partial B} \underline{f}^*_e(\underline{y})dy + O(|\underline{x}|^{-2}) \quad ; \quad |\underline{x}| \longrightarrow \infty. \tag{2.1.9}$$

2.2 Extension of Somigliana's formula

Given a displacement field ϕ in B_i defined by Somigliana's formula (2.1.3), we may generalise the formula by superposing upon it the identity (2.1.8) where $\underline{f}$ is an arbitrary regular exterior displacement field, Jaswon & Bhargava (1961):

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . [\underline{\phi}(\underline{y}) - \underline{f}(\underline{y})]dy$$

$$- \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) . [\underline{\phi}^*_i(\underline{y}) + \underline{f}^*_e(\underline{y})]dy$$

$$= 4\pi \underline{\phi}(\underline{x}) \quad ; \quad \underline{x} \subset B_i \quad , \ \underline{y} \subset \partial B. \tag{2.2.1}$$

We now consider two distinct possibilities for $\underline{f}$:

(i) $\underline{f} = \underline{\phi}$ over ∂B providing the vector simple-layer representation:

$$-\frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) : [\underline{\phi}_i^*(\underline{y}) + \underline{f}_e^*(\underline{y})] dy$$

$$= \underline{\phi}(\underline{x}) \qquad ; \qquad \underline{x} \subset B_i \; , \; \underline{y} \subset \partial B \qquad (2.2.2)$$

generated by the source density:

$$\underline{\sigma} = -\frac{1}{4\pi}(\underline{\phi}_i^* + \underline{f}_e^*). \qquad (2.2.3)$$

This construction assumes the existence of a unique regular $\underline{f}$ in B_e, which satisfies:

$$\underline{f} = \underline{\phi} \qquad \text{at } \partial B \; , \qquad (2.2.4)$$

ensured by the exterior Dirichlet-uniqueness theorem of elastostatics.

(ii) $\underline{f}_e^* = -\underline{\phi}_i^*$ over ∂B providing the vector double-layer representation:

$$\frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_i^* . [\underline{\phi}(\underline{y}) - \underline{f}(\underline{y})] dy$$

$$= \underline{\phi}(\underline{x}) \qquad ; \qquad \underline{x} \subset B_i \; , \; \underline{y} \subset \partial B, \qquad (2.2.5)$$

generated by the source density:

$$\underline{\mu} = \frac{1}{4\pi}(\underline{\phi} - \underline{f}). \qquad (2.2.6)$$

This construction assumes the existence of a unique regular $\underline{f}$ in B_e which satisfies:

$$\underline{f}^*_e = -\underline{\phi}^*_i \qquad \text{on } \partial B, \qquad (2.2.7)$$

ensured by the exterior Neumann existence-uniqueness theorem of elastostatics. These are fundamental existence-uniqueness theorems which are entirely analogous to those for harmonic functions in exterior domains.

2.3 Exterior representations

Formulae (2.2.2),(2.2.5) refer to $\underline{\phi}$ in B_i. Somigliana's formula also holds for $\underline{\phi}$ in B_e subject to a suitable restriction on the behaviour of $\underline{\phi}$ at infinity, i.e. $\underline{\phi} = O(r^{-1})$ as $r \longrightarrow \infty$. If so we may always write:

$$\underline{\phi}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy \quad ; \quad \underline{x} \subset B_e \, , \; \underline{y} \subset \partial B. \qquad (2.3.1)$$

where

$$\underline{\sigma} = -\frac{1}{4\pi}(\underline{\phi}^*_e + \underline{f}^*_i), \qquad (2.3.2)$$

assuming the existence of a unique $\underline{f}$ in B_i which satisfies (2.2.4), i.e. ensured by the interior Dirichlet existence-uniqueness theorem for elastostatics.

Under more restrictive conditions (see below) we may write:

$$\underline{\phi}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . \underline{\mu}(\underline{y})dy \;; \quad \underline{x} \subset B_e \;, \; \underline{y} \subset \partial B, \tag{2.3.3}$$

where

$$\underline{\mu} = \frac{1}{4\pi} (\underline{\phi} - \underline{f}), \tag{2.3.4}$$

assuming the existence of a field $\underline{f}$ in B_i which satisfies:

$$\underline{f}^*_i = - \underline{\phi}^*_e \qquad \text{on } \partial B. \tag{2.3.5}$$

This field is subject to the interior Neumann existence-theorem for elastostatics. Even if $\underline{f}$ exists, it is not unique since equation (2.3.5) has the class of solutions:

$$\underline{f} = \underline{f}_o + (\underline{a} + \underline{b} \wedge \underline{x}) \;, \text{ in } B_i \;, \tag{2.3.6}$$

where $\underline{f}_o$ defines a particular solution and $\underline{a} + \underline{b} \wedge \underline{x}$ defines an arbitrary rigid-body displacement field. Substituting (2.3.6) into (2.3.3) we find

$$\underline{\phi}(\underline{x}) = \frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . \underline{\phi}(\underline{y})dy - \frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . (\underline{a}+\underline{b} \wedge \underline{y})dy - \frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . \underline{f}_o(\underline{y})dy ; \; \underline{x} \subset B_e \tag{2.3.7}$$

$$= \frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . \underline{\phi}(\underline{y})dy - \frac{1}{4\pi}\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_i . \underline{f}_o(\underline{y})dy,$$

since the second integral of (2.3.7) is zero by putting $\underline{\phi} = \underline{a}+\underline{b}\wedge\underline{y}, \underline{\phi}^* = \underline{0}$ into (2.1.5). It follows that the class of solutions (2.3.6) all generate the same $\underline{\phi}$ in B_e.

The field $\underline{f}$ could only exist in B_i provided the traction $\underline{f}^*_i$ on ∂B produce neither a resultant force nor resultant moment. These conditions may be expressed by writing:

$$\int_{\partial B} \underline{f}_i^*(\underline{y})dy = 0, \tag{2.3.8}$$

$$\int_{\partial B} \underline{y} \wedge \underline{f}_i^*(\underline{y})dy = 0, \tag{2.3.9}$$

respectively which imply from (2.3.5) that

$$\int_{\partial B} \underline{\phi}_e^*(\underline{y})dy = 0, \tag{2.3.10}$$

$$\int_{\partial B} \underline{y} \wedge \underline{\phi}_e^*(\underline{y})dy = 0. \tag{2.3.11}$$

Accordingly we may only write $\underline{\phi} = \underline{W}$ in B_e provided the traction $\underline{\phi}_e^*$ on ∂B produces neither a resultant force nor a resultant moment acting on B_e. This of course could only be known if $\underline{\phi}_e^*$ were known on ∂B.

Condition (2.3.10) implies $\underline{\phi} = O(r^{-2})$ as $r \longrightarrow \infty$, since a resultant force produces $O(r^{-1})$ behaviour as follows from Kelvin's point force solution (1.1.3). It might be supposed that condition (2.3.11) implies $\underline{\phi} = O(r^{-3})$ as $r \longrightarrow \infty$, since a resultant moment generally produces $O(r^{-2})$ behaviour, e. g. see solution (4.2.10) for a twist nucleus. However there exist certain special fields characterised by $O(r^{-2})$ behaviour which do not produce a resultant moment, e. g. the field defined by $\underline{W}$ in B_e, Jaswon & Symm (1977).

Another example will be given in the next chapter. Of course $\underline{\phi} = O(r^{-3})$ as $r \longrightarrow \infty$ always implies a null resultant force and a null resultant moment.

Chapter 3

Boundary Integral Equations

3.0 Introduction

In this chapter we utilise the preceding theory to formulate boundary integral equations. These supplement the theory and allow us to complete the vector double-layer representation for exterior fields.

3.1 Formulation by vector simple-layer potential

The representation:

$$\underline{\phi}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy \quad ; \quad \underline{x} \subset B_i, B_e, \ \underline{y} \subset \partial B, \qquad (3.1.1)$$

remains continuous as $\underline{x}$ approaches ∂B whether from B_i or B_e. Accordingly (3.1.1) provides the boundary relation:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy = \underline{\phi}(\underline{x}) \quad ; \quad \underline{x}, \ \underline{y} \subset \partial B, \qquad (3.1.2)$$

which is a vector integral equation of the first kind for $\underline{\sigma}$ in terms of $\underline{\phi}$. This has a unique solution given by (2.3.2). However this theoretical solution would not generally be available, and it would be necessary to solve the equation directly for $\underline{\sigma}$. Direct solutions of (3.1.1) whether analytical or numerical, do not seem to have been attempted.

An interesting choice of ϕ is $\phi = \underline{a} + \underline{b} \wedge \underline{x}$ in B_i , where $\underline{a}$, $\underline{b}$ are constant vectors which provide $\phi_i^* = 0$. It is convenient to break down $\underline{a} + \underline{b} \wedge \underline{x}$ into the six independent vectors:

$$\left.\begin{array}{ll} \underline{d}_1 = \langle 1 , 0 , 0 \rangle & \underline{d}_2 = \langle 0 , 1 , 0 \rangle \\ \underline{d}_3 = \langle 0, 0 , 1 \rangle & \underline{d}_4 = \langle 1 ,0 , 0 \rangle \wedge \underline{r} \\ \underline{d}_5 = \langle 0 , 1 , 0 \rangle \wedge \underline{r} & \underline{d}_6 = \langle 0 , 0 , 1 \rangle \wedge \underline{r} \end{array}\right\} , \qquad (3.1.3)$$

so yielding the six independent equations:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\lambda}_s(\underline{y})dy = \underline{d}_s(\underline{x}) \quad ; \underline{x} , \underline{y} \subset \partial B ; s=1,\ldots,6, \qquad (3.1.4)$$

with the corresponding six solutions $\underline{\lambda}_s$; $s = 1,\ldots,6$. These equations are the vector analogues of Symm's equation for electrostatic capacitance, Symm (1963 ,1964).

Computing the traction vector for each side of (3.1.4) we find:

$$\int_{\partial B} \underline{\underline{g}}_i^*(\underline{x},\underline{y}) \cdot \underline{\lambda}_s(\underline{y})dy - 2\pi \underline{\lambda}_s = 0 \; ; \underline{x} , \underline{y} \subset \partial B \; ; s=1,\ldots,6 \qquad (3.1.5)$$

which is a homogeneous vector integral equation of the second kind for $\underline{\lambda}_s$. Clearly this has the six independent non-trivial solutions $\underline{\lambda}_s$; $s = 1,\ldots,6$. Accordingly, assuming that classical Fredholm theory applies, the adjoint equations:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_i^* \cdot \underline{d}_s(\underline{y})dy - 2\pi \underline{d}_s(\underline{y}) = 0 \; ; \underline{x} , \underline{y} \subset \partial B ; s=1,\ldots,6, \qquad (3.1.6)$$

have corresponding non-trivial solutions $\underline{d}_s$. These solutions may be confirmed by substituting $\underline{\phi} = \underline{d}_s$, $\underline{\phi}_i^* = (\underline{d}_s^*)_i = 0$ into Somigliana's boundary formula (2.1.4).

Operating upon both sides of (3.1.2) by the integral operator $\int_{\partial B} \underline{\lambda}_s(\underline{x}) \ldots dx$, we have

$$\int_{\partial B} \underline{\lambda}_s(\underline{x}) \cdot \underline{\phi}(\underline{x}) dx = \int_{\partial B} \underline{\lambda}_s(\underline{x}) \cdot \left[\int_{\partial B} \underline{\underline{g}}(\underline{x}, \underline{y}) \cdot \underline{\sigma}(\underline{y}) dy \right] dx \; ; \qquad \underline{x}, \underline{y} \subset \partial B. \quad (3.1.7)$$

Assuming that the order of integration may be inverted (Fubini's theorem), and interchanging $\underline{x}$, $\underline{y}$ in (3.1.4), we find:

$$\int_{\partial B} \underline{\lambda}_s(\underline{x}) \cdot \underline{\phi}(\underline{x}) dx = \int_{\partial B} \left[\int \underline{\underline{g}}(\underline{y}, \underline{x}) \cdot \underline{\lambda}_s(\underline{x}) dx \right] \cdot \underline{\sigma}(\underline{y}) dy$$

$$= \int_{\partial B} \underline{d}_s(\underline{y}) \cdot \underline{\sigma}(\underline{y}) dy \quad ; \qquad s = 1, \ldots, 6. \qquad (3.1.8)$$

Now substituting:

$$\underline{\sigma} = -\frac{1}{4\pi} (\underline{V}_e^* + \underline{V}_i^*)$$

$$= -\frac{1}{4\pi} (\underline{\phi}_e^* + \underline{\phi}_i^*), \qquad (3.1.9)$$

into (3.1.8), yields:

$$\int_{\partial B} \underline{\lambda}_s(\underline{x}) \cdot \underline{\phi}(\underline{x}) dx = -\frac{1}{4\pi} \int_{\partial B} \underline{\phi}_e^*(\underline{x}) \cdot \underline{d}_s(\underline{x}) dx. \qquad (3.1.10)$$

This last integral gives the components of the resultant force (s = 1,2,3) and the resultant moment (s = 4,5,6) associated with ϕ_e^* on ∂B. Therefore the left-hand integral in (3.1.10) provides these components if ϕ is given on ∂B instead of ϕ_e^*.

3.2 Formulation by vector double-layer potential

Corresponding to (1.1.1) we may always write:

$$\phi(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_i^* \cdot \underline{\mu}(\underline{y})dy \quad ; \quad \underline{x} \subset B_i \ ; \ \underline{y} \subset \partial B. \tag{3.2.1}$$

This integral jumps at ∂B, so providing the vector boundary integral equation:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_i^* \cdot \underline{\mu}(\underline{y})dy + 2\pi\underline{\mu}(\underline{x}) = \phi(\underline{x}) \ ; \ \underline{x} \ , \ \underline{y} \subset \partial B \tag{3.2.2}$$

for $\underline{\mu}$ in terms of ϕ. There exists a unique solution given by (2.3.4). In practice, of course, it would be necessary to solve (3.2.2) directly for $\underline{\mu}$; however neither an analytical nor a numerical solution seems as yet to have been attempted.

The exterior equation corresponding with (3.2.2) is:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_e^* \cdot \underline{\mu}(\underline{y})dy + 2\pi\underline{\mu}(\underline{x}) = \phi(\underline{x}) \ ; \ \underline{x} \ , \ \underline{y} \subset \partial B. \tag{3.2.3}$$

This requires analysis by vector Fredholm theory since the associated homogeneous equation:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})_e^* \cdot \underline{\mu}(\underline{y})dy + 2\pi\underline{\mu}(\underline{x}) = 0 \ ; \ \underline{x} \ , \ \underline{y} \subset \partial B, \tag{3.2.4}$$

has the six independent non-trivial solutions $\underline{\mu}_s = \underline{d}_s$; s=1,...,6. Note that (3.2.4) becomes (3.1.6) on writing $\underline{\underline{g}}(\underline{x},\underline{y})^*_e = -\underline{\underline{g}}(\underline{x},\underline{y})^*_i$. Therefore the adjoint equation to (3.2.4):

$$\int_{\partial B} \underline{\underline{g}}^*_e(\underline{x},\underline{y}).\underline{\lambda}(\underline{y})dy + 2\pi\underline{\lambda}(\underline{x}) = 0 \; ; \quad \underline{x} , \quad \underline{y} \subset \partial B, \qquad (3.2.5)$$

has the corresponding six independent non-trival solution $\underline{\lambda} = \underline{\lambda}_s$; s = 1,...,6. Assuming that vector Fredholm theory applies, equation (3.2.2) only has a solution subject to the orthogonality conditions:

$$\int_{\partial B} \underline{\phi}(\underline{x}).\underline{\lambda}_s(\underline{x})dx = 0 \; ; \qquad s = 1,\ldots,6. \qquad (3.2.6)$$

By virtue of the equality (3.1.10), these express a null resultant force and a null resultant moment produced by $\underline{\phi}^*_e$ acting upon ∂B, so confirming the conditions (2.3.10), (2.3.11) obtained directly on physical grounds. If (3.2.6) holds, a general solution exists and may be written:

$$\underline{\mu} = \sum_{s=1}^{6} a_s \underline{d}_s + \underline{d}_o \qquad (3.2.7)$$

where $\underline{d}_o$ is any particular solution and a_s are arbitrary scalar coefficients providing an arbitrary rigid-body displacement, in accordance with the previously obtained general solution (2.3.6).

Clearly the representation:

$$\underline{\phi}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_e.\underline{\mu}(\underline{y})dy \; ; \quad \underline{x} \subset B_e , \quad \underline{y} \subset \partial B \qquad (3.2.8)$$

is incomplete since μ may be not always exist. This can also be seen directly, since $\underline{W} = O(r^{-2})$ as $r \longrightarrow \infty$ whilst in general $\phi = O(r^{-1})$ as $r \longrightarrow \infty$. Physically interpreted, $\underline{W}$ provides no resultant force at infinity, by contrast with the general behaviour of ϕ. However, even if $\phi = O(r^{-2})$ at infinity, it may not necessarily be represented by $\underline{W}$. This is because $\underline{W}$ provides no resultant moment at infinity, by contrast with the general ϕ having $O(r^{-2})$ behaviour as $r \longrightarrow \infty$.

Accordingly we extend the representation (3.2.1) by writing:

$$\phi(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^{*}_{e}.\underline{\mu}(\underline{y})dy + \underline{a}.\underline{\underline{g}}(\underline{x},\underline{y})_{\underline{y}=0}$$

$$+\underline{b} \wedge \nabla.\underline{\underline{g}}(\underline{x},\underline{y})_{\underline{y}=0}; \quad \underline{x} \subset B_e, \quad \underline{y} \subset \partial B. \qquad (3.2.9)$$

where $\underline{a}$, $\underline{b}$ are constant vectors to be determined:

$$\underline{a} = \langle a_1, a_2, a_3 \rangle ; \quad \underline{b} = \langle b_1, b_2, b_3 \rangle \equiv \langle a_4, a_5, a_6 \rangle. \qquad (3.2.10)$$

This representation yields the boundary integral equation:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^{*}_{e}.\underline{\mu}(\underline{y})dy + 2\pi\underline{\mu}(\underline{x})$$

$$= \phi(\underline{x}) - [\underline{a}.\underline{\underline{g}}(\underline{x},\underline{y}) + \underline{b} \wedge \nabla.\underline{\underline{g}}(\underline{x},\underline{y})]_{\underline{y}=0}. \qquad (3.2.11)$$

It will be noted that $\underline{a}.\underline{\underline{g}}$ has $O(r^{-1})$ behaviour as $r \longrightarrow \infty$, giving the resultant force without moment (see below) produced by ϕ; also $\underline{b} \wedge \nabla.\underline{\underline{g}}(\underline{x},\underline{y})$ has $O(r^{-2})$ behaviour giving the resultant moment without force(see below) produced by ϕ.

In terms of components these vectors appears as:

$$\underline{a}.\underline{\underline{g}}(\underline{x},\underline{y}) = \sum_{\beta=1}^{3} a_\beta g(\underline{y}_\beta, \underline{x}_\alpha) \;; \qquad \alpha = 1,2,3$$

$$\underline{b} \wedge \nabla.\underline{\underline{g}}(\underline{y},\underline{x}) = \sum_{\beta=1}^{3} (\underline{b} \wedge \nabla)\, \underline{\underline{g}}(\underline{y}_\beta, \underline{x}_\alpha) \;; \quad \alpha = 1,2,3 \quad . \tag{3.2.12}$$

Operating upon both sides of (3.2.11) by the integral operator $\int_{\partial B} \underline{\lambda}_s(\underline{x})\ldots dx$ and interchanging the order of integration we note that:

$$0 = \int_{\partial B} \underline{\lambda}_s(\underline{x}) \;.\Big[\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^*_e.\; \underline{\mu}(\underline{y})dy \;+\; 2\pi\underline{\mu}(\underline{x})\Big]dx$$

$$= \int_{\partial B} \underline{\lambda}_s(\underline{x}).[\underline{\phi}(\underline{x}) - \underline{a}.\underline{\underline{g}}(\underline{y},\underline{x}) - \underline{b} \wedge \nabla.\underline{\underline{g}}(\underline{y},\underline{x})]_{\underline{y}=0} dx \tag{3.2.13}$$

i.e.

$$\left.\begin{aligned} \int_{\partial B} \underline{\lambda}_s(\underline{x}).\underline{\phi}(\underline{x})dx &= \underline{a}.\underline{d}_s(\underline{y}) = a_s \;; \quad s=1,2,3 \\ &= 0 \;; \quad s=4,5,6 \end{aligned}\right\} \tag{3.2.14}$$

showing the absence of a resultant moment. Also:

$$\left.\begin{aligned} \int_{\partial B} \underline{\lambda}_s(\underline{x}).\underline{\phi}(\underline{x})dx &= \underline{b}\wedge \nabla.\underline{d}_s(\underline{y}) = 2b \;; \quad s=4,5,6 \\ &= 0 \;; \quad s=1,2,3 \end{aligned}\right\} \tag{3.2.15}$$

showing the absence of a resultant force.

With these values of $\underline{a}$, $\underline{b}$ the integral equation (3.2.11) always has a solution of the form:

$$\underline{\mu} = \underline{d}_o + \sum_{s=1}^{6} a_s \underline{d}_s \tag{3.2.16}$$

where $\underline{d}_o$ is a particular solution and a_s ; $s=1,...,6$ are arbitrary scalar coefficients.

PART II

INTEGRAL REPRESENTATIONS

Using the Papkovich-Neuber formula, we construct three simple but qualitatively distinct elastostatic fields in the infinite domain exterior to a spherical cavity, and we represent these fields by both simple-layer and double-layer vector potentials.

Chapter 4

Exterior Sphere Problems

4.0 Introduction

In this chapter we construct three simple but qualitatively distinct elastostatic displacement fields in the infinite region B_e exterior to a spherical cavity, utilising the Papkovich-Neuber formula. We also calculate the tractions associated with these fields. This paves the way for vector integral representations in the following chapters.

4.1 Papkovich-Neuber formula

In an isotropic linear elastic continuum, the elastic displacement vector $\underset{\sim}{\phi}$ satisfies the Cauchy-Navier equation, Sommerfeld (1964):

$$\mu\nabla^2\underset{\sim}{\phi} \quad + \quad (\lambda + \mu)\nabla(\nabla.\underset{\sim}{\phi}) \quad = 0 \qquad (4.1.1)$$

in the absence of body force, where λ, μ are Lame's elastic constants. This equation is preferably written:

$$\nabla^2\underset{\sim}{\phi} \quad + \frac{1}{1-2\nu} \quad \nabla(\nabla.\underset{\sim}{\phi}) = 0 \quad ; \nu = \frac{\lambda}{2(\lambda+\mu)} \; . \qquad (4.1.2)$$

A general solutions to equation (4.1.2) has been given by Papkovich (1932) and Neuber (1934) in the form:*

* Eubanks & Sternberg (1956)

$$\underset{\sim}{\phi} = \langle h_1, h_2, h_3 \rangle - \kappa\nabla(x_1h_1 + x_2h_2 + x_3h_3 + f);$$

$$\nabla^2 \langle h_1, h_2, h_3 \rangle = 0; \quad \nabla^2 f = 0; \quad \kappa^{-1} = 4(1-\nu) \tag{4.1.3}$$

where $\underset{-}{h}$ is a harmonic vector function and f is a harmonic scalar function. It is often possible to solve problems quickly by guessing a suitable choice of $\underset{-}{h}$, f, $(r \geq a)$ as will be seen below.

4.2 Construction of displacement fields

We first construct an elastostatic field in $B_e(r \geq a)$ subject to the following two requirements:

(i) $\underset{\sim}{\phi} = \langle 0, 0, t_3 \rangle$ on $r=a$; t_3 = a constant , (4.2.1)

i.e. the spherical boundary of radius a is given a uniform rigid-body translation of amount t_3 in the 3-direction.

(ii) $\underset{\sim}{\phi} = O(r^{-1})$ as $r \longrightarrow \infty$. (4.2.2)

An efficient way of calculating $\underset{\sim}{\phi}$ in $r \geq a$ is to use the Papkovich-Neuber formula (4.1.3). In this case we try:

$$h_1 = 0 \ , \ h_2 = 0 \ , \ h_3 = \frac{\alpha}{r} \ , \quad f = \beta \frac{\partial r^{-1}}{\partial x_3} \ , \tag{4.2.3}$$

where α , β are constants to be determined. If so, the required field is

$$\underset{\sim}{\phi} = \langle 0 \ , 0 \ , \frac{\alpha}{r} \rangle - \kappa \nabla\left(\frac{\alpha x_3}{r} + \beta \frac{\partial r^{-1}}{\partial x_3}\right) \ ; \quad r \geq a \tag{4.2.4}$$

$$\kappa^{-1} = 4(1-\nu)$$

This field clearly satisfies conditions (4.2.1) and (4.2.2) provided that (App. I):

$$\alpha = \frac{3at_3}{3-2\kappa} \quad ; \quad \beta = \frac{\alpha a^2}{3} \tag{4.2.5}$$

hence,

$$\phi = \gamma < \frac{x_1 x_3}{r^3} - \frac{a^2 x_1 x_3}{r^5} , \frac{x_2 x_3}{r^3} - \frac{a^2 x_2 x_3}{r^5} ,$$

$$\frac{1-\kappa}{\kappa r} + \frac{x_3^2}{r^3} + \frac{a^2}{3r^3} - \frac{a^2 x_3^2}{r^5} > ; \gamma = \frac{3at_3 \kappa}{3-2\kappa} . \quad r>a. \tag{4.2.6}$$

We also construct a field satisfing the following two requirements:

(i) $\phi = <0, 0, \omega_3> \wedge < x_1 , x_2 , x_3 > = \omega_3 <-x_2, x_1, 0>$

on $r = a$; ω_3 = a constant (4.2.7)

i.e. the spherical boundary is given a uniform rigid-body rotation of amount ω_3 about the 3-direction,

(ii) $\phi = O(r^{-1})$ as $r \longrightarrow \infty$. (4.2.8)

In (4.1.3) we choose:

$$h_1 = -\frac{a^2 \omega_3 x_2}{r^3} , \quad h_2 = \frac{a^3 \omega_3 x_1}{r^3} , \quad h_3 = 0 , \quad f = 0 \tag{4.2.9}$$

which yields:

$$\phi = -\frac{a^3 \omega_3}{r^3} < x_2 , - x_1 , 0 > \quad ; \; r \geqslant a, \tag{4.2.10}$$

and this clearly satisfies conditions (4.2.7) and (4.2.8).

Finally we construct a field satisfing the following two requirements:

(i) $\phi(a) = h$ on $r = a$, h = a constant (4.2.11)

i.e. the spherical boundary is given a uniform radial displacement of amount h.

(ii) $\phi = O(r^{-1})$ as $r \longrightarrow \infty$. (4.2.12)

In (4.1.3) we choose:

$$h_1 = h_2 = h_3 = 0 \quad ; f = \frac{L}{r} \; ; \; L = \text{a constant}, \tag{4.2.13}$$

which yields:

$$\phi = - L \kappa \, \nabla\left(\frac{1}{r}\right) = L \kappa < \frac{x_1}{r^3}, \frac{x_2}{r^3}, \frac{x_3}{r^3} > \quad . \tag{4.2.14}$$

The radial component of (4.2.14) is:

$$\phi_r = L \kappa < \frac{x_1}{r^3} , \frac{x_2}{r^3} , \frac{x_3}{r^3} > . < \frac{x_1}{r} , \frac{x_2}{r} , \frac{x_3}{r} >$$

$$= \frac{L \kappa}{r^2} , \qquad \text{i.e.} \quad \frac{L \kappa}{a^2} = h, \tag{4.2.15}$$

on using (4.2.13). If so (4.2.14) becomes:

$$\phi = \frac{h a^2}{r^3} < x_1 , x_2 , x_3 > . \tag{4.2.16}$$

4.3 Calculation of tractions

To calculate the traction vector on r=a associated with the field (4.2.6) we first compute the local dilatation:

$$\nabla . \phi = \frac{\partial \phi_\alpha}{\partial x_\alpha} = \gamma' \frac{x_3}{r^3} \quad ; \qquad \gamma' = \frac{3at_3(2\kappa - 1)}{3 - 2\kappa} \tag{4.3.1}$$

where ϕ_α ; α =1,2,3 are the components of ϕ.

Clearly $\nabla.\phi$ is a harmonic function in $r \geqslant a$. Next we compute the stress tensor $\Phi_{\alpha\beta}$, by using the stress-strain relation:

$$\Phi_{\alpha\beta} = \mu \left(\frac{\partial\phi_\alpha}{\partial x_\beta} + \frac{\partial\phi_\beta}{\partial x_\alpha} \right) + \nabla.\phi \, \delta_{\alpha\beta} \; ; \; \alpha, \beta = 1,2,3, \qquad (4.3.2)$$

where,

$$\delta_{\alpha\beta} = \begin{cases} 0 & ; \alpha \neq \beta \\ 1 & ; \alpha = \beta \end{cases} \qquad (4.3.3)$$

The stress components are calculated to be:

$$\left.\begin{aligned}
\Phi_{11} &= 2\mu\gamma \left(\frac{x_3}{r^3} - \frac{3x_1^2 x_3}{r^5} - \frac{a^2 x_3}{r^5} + \frac{5a^2 x_1^2 x_3}{r^7} \right) + \lambda\gamma' \frac{x_3}{r^3} \\
\Phi_{22} &= 2\mu\gamma \left(\frac{x_3}{r^3} - \frac{3x_2^2 x_3}{r^5} - \frac{a^2 x_3}{r^5} + \frac{5a^2 x_2^2 x_3}{r^7} \right) + \lambda\gamma' \frac{x_3}{r^3} \\
\Phi_{33} &= 2\mu\gamma \left(\frac{2\kappa-1}{\kappa} \frac{x_3}{r^3} - \frac{3x_3^3}{r^5} - \frac{3a^2 x_3}{r^5} + \frac{5a^2 x_3^3}{r^7} \right) + \lambda\gamma' \frac{x_3}{r^3} \\
\Phi_{12} &= \Phi_{21} = \frac{4\mu\gamma x_1 x_2 x_3}{r^5} \\
\Phi_{13} &= \Phi_{31} = \mu\gamma \left(\frac{\kappa-1}{\kappa} \frac{x_1}{r^3} - \frac{x_1 x_3^2}{r^5} - \frac{a^2 x_1}{r^5} + \frac{5a^2 x_1 x_3^2}{r^7} \right) \\
\Phi_{23} &= \Phi_{32} = \mu\gamma \left(\frac{2\kappa-1}{\kappa} \frac{x_2}{r^3} - \frac{6x_2 x_3^2}{r^5} - \frac{2a^2 x_2}{r^5} + \frac{10a^2 x_2 x_3^2}{r^7} \right)
\end{aligned}\right]_{r>a.} \qquad (4.3.4)$$

On the boundary, (4.3.4) becomes:

$$\left.\begin{aligned}
\Phi_{11} &= \lambda\gamma' \frac{x_3}{a^3} + 4\mu\gamma \frac{x_1^2 x_3}{a^5} \\
\Phi_{22} &= \lambda\gamma' \frac{x_3}{a^3} + 4\mu\gamma \frac{x_2^2 x_3}{a^5} \\
\Phi_{33} &= \lambda\gamma' \frac{x_3}{a^3} + 2\mu\gamma \left(- \frac{x_3}{\kappa a^3} + \frac{2x_3^2}{a^5}\right) \\
\Phi_{12} &= \Phi_{21} = 4\mu\gamma \frac{x_1 x_2 x_3}{a^5} \\
\Phi_{13} &= \Phi_{31} = \mu\gamma \left(- \frac{x_1}{\kappa a^2} + \frac{4x_1 x_3^2}{a^5}\right) \\
\Phi_{23} &= \Phi_{32} = \mu\gamma \left(- \frac{x_2}{\kappa a^3} + \frac{4x_2 x_3^2}{a^5}\right)
\end{aligned}\right]_{r=a.} \qquad (4.3.5)$$

We now compute the traction vector on r=a acting upon B_e, by substituting from (4.3.5) into :

$$\underline{\phi}_e^* = \begin{bmatrix} \phi_1^* \\ \phi_2^* \\ \phi_3^* \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad (4.3.6)$$

where:

$$n_1 = \frac{x_1}{a} \quad , \quad n_2 = \frac{x_2}{a} \quad , \quad n_3 = \frac{x_3}{a}, \tag{4.3.7}$$

i.e.

$$\phi^*_e = \langle -H\frac{x_1x_3}{a^4} \, , \, -H\frac{x_2x_3}{a^4} \, , \, -H\frac{x_3^2}{a^4} - \frac{\mu\gamma}{\kappa a^2} \rangle$$

$$= \langle 0 \, , \, 0 \, , \, -\frac{\mu\gamma}{\kappa a^2} \rangle \; ; \; H = \lambda\gamma' + 4\mu\gamma - \frac{\mu\gamma}{\kappa} = 0. \tag{4.3.8}$$

This yields the resultant force:

$$\int_{\partial B} \phi^*_e dy = \langle 0, \, 0, \, -\frac{4\mu\gamma}{\kappa} \rangle \, , \tag{4.3.9}$$

and the resultant moment:

$$\int_{\partial B} \underset{\sim}{y} \wedge \phi^*_e dy = 0, \tag{4.3.10}$$

as expected acting on the boundary B_e.

Following these steps for the rotation field (4.2.10) we find $\nabla.\phi = 0$ in $r \geq a$ as expected . If so (4.3.2) readily gives:

$$\Phi_{11} = \frac{6\mu\omega_3 x_1 x_2 a^3}{r^5}$$

$$\Phi_{22} = \frac{-6\mu\omega_3 x_1 x_2 a^3}{r^5} \quad , \quad \Phi_{33} = 0$$

$$\Phi_{12} = \Phi_{21} = \frac{3\mu\omega_3 a^3 (x_2^2 - x_1^2)}{r^5}$$

$$\Phi_{13} = \Phi_{31} = \frac{3\mu\omega_3 x_2 x_3 a^3}{r^5} \qquad (4.3.11)$$

$$\Phi_{23} = \Phi_{32} = -\frac{3\mu\omega_3 x_1 x_3 a^3}{r^5} \qquad r \geqslant a.$$

On the boundary this becomes

$$\Phi_{11} = \frac{6\mu\omega_3 x_1 x_2}{a^2}$$

$$\Phi_{22} = -\frac{6\mu\omega_3 x_1 x_2}{a^2}, \qquad \Phi_{33} = 0$$

$$\Phi_{12} = \Phi_{21} = \frac{3\mu\omega_3 (x_2^2 - x_1^2)}{a^2} \qquad (4.3.12)$$

$$\Phi_{13} = \Phi_{31} = \frac{3\mu\omega_3 x_2 x_3}{a^2}$$

$$\Phi_{23} = \Phi_{32} = -\frac{3\mu\omega_3 x_1 x_3}{a^2} \qquad r \geqslant a.$$

Now substituting from (4.3.12) and (4.3.7) into (4.3.6) we obtain:

the traction vector:

$$\phi_e^* = \frac{3\mu\omega_3}{a} < x_2 , - x_1 , 0 >, \qquad (4.3.13)$$

This yields the resultant force:

$$\int_{\partial B} \phi_e^*(y)dy = < 0 , 0 , 0 >, \qquad (4.3.14)$$

as expected, and the resultant moment:

$$\int_{\partial B} y \wedge \phi_e^*(y)dy = \frac{3\mu\omega_3}{a} \int_{\partial B} < x_1x_3, x_2x_3, -x_1^2-x_2^2 > dy$$

$$= < 0 , 0 , -8 a^3 \mu\omega_3 >. \qquad (4.3.15)$$

For the displacement field (4.2.16) we note that $\nabla.\phi = 0$ in $r \geq a$. If so (4.3.2) gives:

$$\left.\begin{aligned}
\Phi_{11} &= 2\mu h a^2 \left(\frac{1}{r^3} - \frac{3x_1^2}{r^5}\right) \\
\Phi_{22} &= 2\mu h a^2 \left(\frac{1}{r^3} - \frac{3x_2^2}{r^5}\right) \\
\Phi_{33} &= 2\mu h a^2 \left(\frac{1}{r^3} - \frac{3x_3^2}{r^5}\right) \\
\Phi_{12} &= \Phi_{21} = -\frac{6\mu h a^2 x_1 x_2}{r^5} \\
\Phi_{13} &= \Phi_{31} = -\frac{6\mu h a^2 x_1 x_3}{r^5}
\end{aligned}\right\} \qquad (4.3.16)$$

$$\Phi_{23} = \Phi_{32} = -\frac{6\mu h a^2 x_2 x_3}{r^5} \quad \Bigg\rfloor_{r>a.} \qquad (4.3.16)$$

On r=a (4.3.16) becomes:

$$\left.\begin{aligned}
\Phi_{11} &= 2\mu h\left(\frac{1}{a} - \frac{3x_1^2}{a^3}\right) \\
\Phi_{22} &= 2\mu h\left(\frac{1}{a} - \frac{3x_2^2}{a^3}\right) \\
\Phi_{33} &= 2\mu h\left(\frac{1}{a} - \frac{3x_3^2}{a^3}\right) \\
\Phi_{12} &= \Phi_{21} = -\frac{6\mu h x_1 x_2}{a^3} \\
\Phi_{13} &= \Phi_{31} = -\frac{6\mu h x_1 x_3}{a^3} \\
\Phi_{23} &= \Phi_{32} = -\frac{6\mu h x_2 x_3}{a^3}
\end{aligned}\right\rfloor_{r \geq a.} \qquad (4.3.17)$$

Substituting from (4.3.17) into (4.3.6) yields:

$$\underset{\sim}{\phi}_e^* = -\frac{4\mu h}{a^2} \langle x_1, x_2, x_3 \rangle, \qquad (4.3.18)$$

which gives the radial traction:

$$\underset{e}{\phi^*}.\underline{n} = -\frac{4\mu h}{a^2} < x_1, x_2, x_3 > . < \frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a} >$$

$$= -\frac{4\mu h}{a^3} a^2 = -\frac{4\mu h}{a}, \tag{4.3.19}$$

acting on the boundary. Also (4.3.18) yields the resultant force:

$$\int_{\partial B} \underset{e}{\phi^*}(\underline{y})dy = -\frac{4\mu h}{a}\int_{\partial B} <y_1 , y_2 , y_3 >dy$$

$$= < 0 , 0 , 0 > \tag{4.3.20}$$

and the resultant moment:

$$\int_{\partial B} \underline{y} \wedge \underset{e}{\phi^*}(\underline{y})dy = \frac{4\mu h}{a}\int_{\partial B} < 0 , 0 , 0 >dy$$

$$= < 0 , 0 , 0 >, \tag{4.3.21}$$

as expected.

For the interior field we choose :

$$\underline{h} = L< x_1, x_2, x_3>, \qquad f = 0 , L = \text{a constant}, \tag{4.3.22}$$

which yields:

$$\underline{\phi} = (1-2\kappa)L<x_1, x_2, x_3>. \tag{4.3.23}$$

By condition (4.2.11) it follows that:

$$L = \frac{h}{(1-2\kappa)a} \tag{4.3.24},$$

giving the interior field:

$$\underset{\sim}{\phi} = \frac{h}{a} < x_1, x_2, x_3 >. \tag{4.3.25}$$

The associated traction vector is:

$$\underset{\sim i}{\phi^*} = - \frac{(3\lambda + 2\mu)h}{a^2} < x_1, x_2, x_3 >, \tag{4.3.26}$$

which yields the radial traction:

$$\underset{\sim i}{\phi^*} . \underset{\sim}{n} = - \frac{(3\lambda + 2\mu)h}{a^2} < x_1, x_2, x_3 > . < \frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a} >$$

$$= - (3\lambda + 2\mu)h/a, \qquad r = a. \tag{4.3.27}$$

Chapter 5

Representation by Vector Simple-Layer Potentials

5.0 Introduction

We now construct vector simple-layer potentials which represent the fields of chapter 4. It seems impossible to evaluate the potentials exactly, but we obtain their asymptotic equivalence to the fields and also their equivalence at $r = 0$, Jaswon & El-Damanawi (1986).

5.1 Integral representation: translation problem

To represent the field (4.2.6) by a vector simple-layer potential, we use the vector source-density formula:

$$\underline{\sigma} = -\frac{1}{4\pi}(\underline{\phi}_e^* + \underline{\phi}_i^*). \tag{5.1.1}$$

Here $\underline{\phi}_e^*$ has been computed in (4.3.8), and $\underline{\phi}_i^* = 0$, since $\underline{\phi} = \langle 0, 0, t_3 \rangle$ in $r = a$, so (5.1.1) becomes:

$$\underline{\sigma} = \langle 0, 0, \frac{\mu \gamma}{4\pi k a^2} \rangle. \tag{5.1.2}$$

Substituting from (5.1.2) into:

$$\underline{V}(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy; \quad \underline{y} \subset \partial B \; ; \; \underline{x} \subset B_e, B_i, \tag{5.1.3}$$

we should identically obtain ϕ for any choice of $\underset{-}{x}$. In practice it seems impossible to evaluate this integral exactly, but its asymptotic behaviour can be examined as follows:

First we note that:

$$\underline{g}(\underline{x},\underline{y}) \longrightarrow \underline{g}(\underline{x},\underline{0}) \text{ as } \underline{x} \longrightarrow \infty, \tag{5.1.4}$$

$$\underline{V}(\underline{x}) \longrightarrow \underline{g}(\underline{x},\underline{0}).\int_{\partial B} \underline{\sigma}(\underline{y})dy; \quad \underline{x} \longrightarrow \infty, \tag{5.1.5}$$

where:

$$\underline{g}(\underline{x},\underline{0}) = \begin{bmatrix} \frac{1-\kappa}{\mu r} + \frac{\kappa}{\mu}\frac{x_1^2}{r^3} & \frac{\kappa}{\mu}\frac{x_1x_2}{r^3} & \frac{\kappa}{\mu}\frac{x_1\,x_3}{r^3} \\ \frac{\kappa}{\mu}\frac{x_1x_2}{r^3} & \frac{1-\kappa}{\mu r} + \frac{\kappa x_2^2}{r^3} & \frac{\kappa}{\mu}\frac{x_3x_2}{r^3} \\ \frac{\kappa}{\mu}\frac{x_1x_3}{r^3} & \frac{\kappa}{\mu}\frac{x_2x_3}{r^3} & \frac{1-\kappa}{\mu r} + \frac{\kappa x_3^2}{r^3} \end{bmatrix} \tag{5.1.6}$$

$$|\underline{x}-\underline{0}| = r.$$

The integral of (5.1.2) gives:

$$\int_{\partial B} \underline{\sigma}(\underline{y})dy = \left\langle 0\ ,\ 0\ ,\ \int_{\partial B} \frac{\mu\gamma}{4\pi\kappa a^2}dy \right\rangle$$

$$= \left\langle 0\ ,\ 0\ ,\ \frac{\mu\gamma}{\kappa} \right\rangle. \tag{5.1.7}$$

Substituting from (5.1.6) and (5.1.7) into (5.1.5) we obtain:

$$\underline{v}(\underline{x}) = \langle v_1(\underline{x}) , v_2(\underline{x}) , v_3(\underline{x}) \rangle$$

$$\rightarrow \gamma \langle \frac{x_1 x_3}{r^3} , \frac{x_2 x_3}{r^3} , \frac{1-\kappa}{\kappa r} + \frac{x_3^2}{r^3} \rangle, \tag{5.1.8}$$

which agrees exactly with the asymptotic components of $\underline{\phi}$ as given in (4.2.6). Physically speaking the asymptotic field is that generated by a point force of magnitude $\frac{4\pi\mu\gamma}{\kappa}$ acting in the 3-direction located at $\underline{y} = 0$.

We remark that the integral (5.1.3) can be evaluated exactly at the centre of the sphere, i.e. putting $\underline{x} = 0$ in (5.1.3) yields:

$$\underline{v}(\underline{0}) = \int_{\partial B} \underline{\underline{g}}(\underline{0},\underline{y}).\underline{\sigma}(\underline{y})dy$$

$$= \langle 0 , 0 , t_3 \rangle , \tag{5.1.9}$$

as expected.

5.2 Integral representation:rotation problem

To represent the field (4.2.10) by a vector simple-layer potential, we use (5.1.1) where $\underline{\phi}^*_e$ has been computed in (4.3.13). Also $\underline{\phi}^*_i = 0$ since $\underline{\phi} = \omega_3 \langle -x_2, x_1, 0 \rangle$ on $r = a$, so that (5.1.1) becomes:

$$\underline{\sigma} = -\frac{3\mu\omega_3}{4\pi a} \langle y_2 , -y_1, 0 \rangle. \tag{5.2.1}$$

The integration of (5.2.1) gives:

$$\int_{\partial B} \underline{\sigma}(\underline{y})dy = \langle 0 , 0 , 0 \rangle. \qquad (5.2.2)$$

Substituting from (5.1.6) and (5.2.2) into (5.1.5):

$$\underline{V}(\underline{x}) = \langle 0 , 0 , 0 \rangle, \qquad (5.2.3)$$

i.e. the first-order asymptotic approximation to $\underline{\underline{g}}(\underline{x},\underline{y})$ gives a null result. Using the second approximation, Jaswon & Symm (1977):

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \underline{\underline{g}}(\underline{x},\underline{0}) + \underline{y}.\nabla\, \underline{\underline{g}}(\underline{x},\underline{y})_{\underline{y}=0} , \qquad (5.2.4)$$

where $\nabla\underline{\underline{g}}(\underline{x},\underline{y})$ denotes the gradient vector at $\underline{y} = \underline{0}$ associated with each component of $\underline{\underline{g}}(\underline{x},\underline{y})$, so:

$$\underline{y}.\left[\nabla\underline{\underline{g}}(\underline{x},\underline{y})\right]_{\underline{y}=0} = \frac{1-\kappa}{\mu r^3}\begin{bmatrix} \underline{x}.\underline{y} & 0 & 0 \\ 0 & \underline{x}.\underline{y} & 0 \\ 0 & 0 & \underline{x}.\underline{y} \end{bmatrix}$$

$$- \frac{2\kappa}{\mu r^3}\begin{bmatrix} x_1y_1 & 0 & 0 \\ 0 & x_2y_2 & 0 \\ 0 & 0 & x_3y_3 \end{bmatrix}$$

$$+\frac{\kappa}{\mu}\begin{bmatrix} \frac{3x_1^2\underline{x}\cdot\underline{y}}{r^5} & \frac{-x_2y_1 - y_2x_1}{r^3} + \frac{3x_1x_2\underline{x}\cdot\underline{y}}{r^5} & \frac{-x_3y_1 - y_3x_1}{r^5} + \frac{3x_1x_3\underline{x}\cdot\underline{y}}{r^5} \\ \frac{-x_2y_1 - x_1y_2}{r^3} + \frac{3x_1x_2\underline{x}\cdot\underline{y}}{r^5} & \frac{3x_2^2\underline{x}\cdot\underline{y}}{r^5} & \frac{-x_3y_2 - x_2y_3}{r^5} + \frac{3x_2x_3\underline{x}\cdot\underline{y}}{r^5} \\ \frac{-x_3y_1 - x_1y_3}{r^3} + \frac{3x_1x_3\underline{x}\cdot\underline{y}}{r^5} & \frac{-x_3y_2 - x_2y_3}{r^5} + \frac{3x_2x_3\underline{x}\cdot\underline{y}}{r^5} & \frac{3x_3^2\underline{x}\cdot\underline{y}}{r^5} \end{bmatrix} \qquad (5.2.5)$$

Substituting from (5.1.6) and (5.2.5) into (5.1.5), and noting the null result from the first approximation, we obtain:

$$\underline{V}(\underline{x}) = -\frac{a^3\omega_3}{r^3} < x_2 \, , \, - x_1 \, , \, 0 >. \qquad (5.2.6)$$

This asymptotic field exactly agrees with ϕ as given by (4.2.10). Physically speaking the asymptotic field is that generated by a point couple of moment $-8\pi\mu a^3\omega_3$ about the 3-axis located at $\underline{y} = \underline{0}$.

The integral (5.1.3) can be evaluated exactly at the centre of the sphere i.e. putting $\underline{x} = \underline{0}$ in (5.1.3) yields:

$$\underline{V}(\underline{0}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy = \omega_3 < -x_2 \, , \, x_1 \, , \, 0 >$$

$$= < 0 \, , \, 0 \, , \, 0 > . \qquad (5.2.7)$$

5.3 Integral representation: pressure problem

To represent the field (4.2.16) we use (5.1.1) where ϕ_e^* and ϕ_i^* have been computed in (4.3.18) and (4.3.26), so that:

$$\underline{\sigma} = \frac{3(\lambda + 2\mu)h}{4\pi a^2} \langle y_1, y_2, y_3 \rangle. \tag{5.3.1}$$

Now, substituting from (5.1.6), (5.2.5) and (5.3.1) into (5.1.3) up to the second approximation yields:

$$\underline{V}(\underline{x}) = \frac{3h(\lambda+\mu)}{8\pi\mu a^2} \int_{\partial B} \begin{bmatrix} \frac{\lambda+3\mu}{(\lambda+\mu)r} + \frac{x_1^2}{r^3} & \frac{x_1x_2}{r^3} & \frac{x_1x_3}{r^3} \\ \frac{x_1x_2}{r^3} & \frac{\lambda+3\mu}{(\lambda+\mu)r} + \frac{x_2^2}{r^3} & \frac{x_2x_3}{r^3} \\ \frac{x_1x_3}{r^3} & \frac{x_2x_3}{r^3} & \frac{\lambda+3\mu}{(\lambda+\mu)r} + \frac{x_3^2}{r^3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} dy$$

$$+ \frac{3h(\lambda+3\mu)}{8\pi\mu a^2 r^3} \int_{\partial B} \begin{bmatrix} \underline{x}\cdot\underline{y} & 0 & 0 \\ 0 & \underline{x}\cdot\underline{y} & 0 \\ 0 & 0 & \underline{x}\cdot\underline{y} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \underline{y} \end{bmatrix} dy$$

$$
-\frac{6h(\lambda+\mu)}{8\pi\mu a^2 r^3}\int_{\partial B}\begin{bmatrix} x_1y_1 & 0 & 0 \\ 0 & x_2y_2 & 0 \\ 0 & 0 & x_3y_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} dy
$$

$$
+\frac{3h(\lambda+\mu)}{8\pi\mu a^2 r^3}\int_{\partial B}\begin{bmatrix} \frac{3x_1^2\,\underline{x}\cdot\underline{y}}{r^2} & -x_2y_1-x_1y_2+\frac{3x_1x_2\,\underline{x}\cdot\underline{y}}{r^2} & -x_3y_1-x_1y_3+\frac{3x_1x_3\,\underline{x}\cdot\underline{y}}{r^3} \\ -x_2y_1-x_1y_2+\frac{3x_1x_2\,\underline{x}\cdot\underline{y}}{r^2} & \frac{3x_2^2\,\underline{x}\cdot\underline{y}}{r^2} & -x_3y_2-x_2y_3+\frac{3x_2x_3\,\underline{x}\cdot\underline{y}}{r^2} \\ -x_3y_1-x_1y_3+\frac{3x_1x_3\,\underline{x}\cdot\underline{y}}{r^2} & -x_3y_2-x_2y_3+\frac{3x_2x_3\,\underline{x}\cdot\underline{y}}{r^2} & \frac{3x_3^2\,\underline{x}\cdot\underline{y}}{r^2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} dy \tag{5.3.2}
$$

The first integral gives zero, the others give:

$$
\underline{V}(\underline{x}) = ha^2 \left\langle \frac{x_1}{r^3}, \frac{x_2}{r^3}, \frac{x_3}{r^3} \right\rangle, \tag{5.3.3}
$$

which is the exact field (4.2.16). Clearly the integral of (5.1.3) is zero at the centre of the sphere as expected.

5.4 The general rigid-body displacement

The boundary displacements (4.2.1) , (4.2.7) are particular cases of the general rigid-body displacement:

$$\underline{\phi} = \underline{t} + \underline{w} \wedge \underline{r}, \tag{5.4.1}$$

where $\underline{t} = \langle t_1, t_2, t_3 \rangle$, $\underline{w} = \langle w_1, w_2, w_3 \rangle$ are constant vectors. We replace (5.4.1) by the six independent vectors:

$$\left.\begin{aligned} t_1\underline{d}_1 &= \langle t_1, 0, 0 \rangle; & t_2\underline{d}_2 &= \langle 0, t_2, 0 \rangle; \\ t_3\underline{d}_3 &= \langle 0, 0, t_3 \rangle; & w_1\underline{d}_4 &= \langle w_1, 0, 0 \rangle \wedge \underline{r}; \\ w_2\underline{d}_5 &= \langle 0, w_2, 0 \rangle \wedge \underline{r}; & w_3\underline{d}_6 &= \langle 0, 0, w_3 \rangle \wedge \underline{r} \end{aligned}\right\} \tag{5.4.2}$$

The exterior fields corresponding with $t_3\underline{d}_3, w_3\underline{d}_6$ have been determined in (4.2.6),(4.2.10) and therefore by symmetry we may immediately write down the exterior fields corresponding with $t_1\underline{d}_1$, $t_2\underline{d}_2$, $w_1\underline{d}_4$ and $w_2\underline{d}_5$. These provide the general formula:

$$\underline{\phi} = \frac{3a\kappa}{3-2\kappa}\left[\left(\frac{1-\kappa}{\kappa}\frac{1}{r} + \frac{a^2}{3r^3}\right)\underline{t} + \frac{\langle x_1, x_2, x_3\rangle}{r^3}\left(1-\frac{a^2}{r^2}\right)\langle t_1x_1 + t_2x_2 + t_3x_3\rangle\right] + \frac{a^3}{r^3}\underline{w}\wedge\underline{r}; \quad r \geq a. \tag{5.4.3}$$

The vector source-densities $\underline{\sigma}_s$; s =1,...,6* generate the exterior fields, $t_i \underline{d}_i$; i = 1,2,3, $w_i \underline{d}_i$; i = 4,5,6. Introducing the normalised source-densities,(3·1·3):

$$\left.\begin{aligned}
\underline{\lambda}_1 &= \frac{\underline{\sigma}_1}{t_1} = \left\langle \frac{3\mu}{4\pi a(3-2\kappa)}, 0, 0 \right\rangle \\
\underline{\lambda}_2 &= \frac{\underline{\sigma}_2}{t_2} = \left\langle 0, \frac{3\mu}{4\pi a(3-2\kappa)}, 0 \right\rangle \\
\underline{\lambda}_3 &= \frac{\underline{\sigma}_3}{t_3} = \left\langle 0, 0, \frac{3\mu}{4\pi a(3-2\kappa)} \right\rangle \\
\underline{\lambda}_4 &= \frac{\underline{\sigma}_4}{w_1} = \frac{3\mu}{4\pi a} \langle 1, 0, 0 \rangle \wedge \underline{r} \\
\underline{\lambda}_5 &= \frac{\underline{\sigma}_5}{w_2} = \frac{3\mu}{4\pi a} \langle 0, 1, 0 \rangle \wedge \underline{r} \\
\underline{\lambda}_6 &= \frac{\underline{\sigma}_6}{w_3} = \frac{3\mu}{4\pi a} \langle 0, 0, 1 \rangle \wedge \underline{r}
\end{aligned}\right] \quad (5.4.4)$$

we obtain the six vector integral equations:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\lambda}_s(\underline{y}) dy = \underline{d}_s(\underline{x}); \qquad \underline{x} \subset \partial B \ ; \ s = 1,\dots,6, \quad (5.4.5)$$

where $\underline{d}_s$; s = 1,...,6 have been defined in (3.1.3).

We may now verify the important equations

$$\int_{\partial B} \underline{\phi} \cdot \underline{\lambda}_s(\underline{y}) dy = -\frac{1}{4\pi} \int_{\partial B} \underline{\phi}_e^* \cdot \underline{d}_s(\underline{y}) dy; \ s = 1,\dots,6, \quad (5.4.6)$$

* (5·1·2) & (5·2·1).

which hold for a general Liapounov-surface ∂B.

First, we verify (5.4.6) for the field (4.2.6):

The left-hand side of (5.4.6) yields:

$$\int_{\partial B} \underline{\phi} \cdot \underline{\lambda}_3(\underline{y})dy = \int_{\partial B} \langle 0, 0, t_3 \rangle . \langle 0, 0, \frac{3\mu}{4\pi a(3-2\kappa)} \rangle dy$$

$$= \frac{3\mu a t_3}{3-2\kappa} = \frac{\mu\gamma}{\kappa}, \quad (5.4.7)$$

and the right-hand side yields:

$$-\frac{1}{4\pi}\int_{\partial B} \underline{\phi}_e^* \cdot \underline{d}_3(\underline{y})dy = -\frac{1}{4\pi}\int_{\partial B} \langle 0, 0, -\frac{3\mu a t_3}{a^2(3-2\kappa)} \rangle .$$

$$\langle 0, 0, 1 \rangle dy = \frac{3\mu a t_3}{3-2\kappa} \quad (5.4.8)$$

as expected.

Secondly, for the field (4.2.7), the left-hand side of (5.4.6) provides:

$$\int_{\partial B} \underline{\phi} \cdot \underline{\lambda}_6(\underline{y})dy = \frac{3\mu w_3}{4\pi a}\int_{\partial B} \langle -x_2, x_1, 0 \rangle . \langle -x_2, x_1, 0 \rangle dy$$

$$= \frac{3\mu w_3}{4\pi a}\int_{\partial B} (x_1^2 + x_2^2)dy = 2\mu w_3 a^3, \quad (5.4.9)$$

and the right-hand side provides:

$$-\frac{1}{4\pi}\int_{\partial B} \underline{\phi}_e^* \cdot \underline{d}_6(\underline{y})dy = \frac{3\mu w_3}{4\pi a}\int_{\partial B} \langle -x_2, x_1, 0 \rangle . \langle -x_2, x_1, 0 \rangle dy$$

$$= 3\mu w_3 a^3. \quad (5.4.10)$$

Chapter 6

Representation by Vector Double-layer Potentials

6.0 Introduction

In this chapter we represent the previous elastostatic fields by double-layer vector potentials. The determination of double-layer vector source-densities proves to be considerally more complicated than that of vector simple-layer source-densities. Three qualitatively distinct problems are considered below. Jaswon & El-Damanawi (1987).

6.1 Vector double-layer source-densities

As already noted in (4.2.6), the exterior displacement field for the translation problem is:

$$\underline{\phi}_I = \frac{\gamma}{r^3}\langle x_1x_3,\ x_2x_3,\ \frac{1-\kappa}{\kappa}r^2 + x_3^2\rangle - \frac{\gamma a^2}{r^5}\langle x_1x_3,\ x_2x_3, -\frac{1}{3}r^2 + x_3^2\rangle; \quad \gamma = \frac{3at\kappa}{3-2\kappa}. \qquad (6.1.1)$$

This has $O(r^{-1})$ behaviour as $r \longrightarrow \infty$, so that it can not be represented by a vector double-layer potential. Slightly adapting the complete representation (3.2.11), we write:

$$\underline{\phi}_I = \underline{W}(\underline{x}) + a\langle g(\underline{x}_1,\underline{y}_3), g(\underline{x}_2,\underline{y}_3), g(\underline{x}_3,\underline{y}_3)\rangle_{\underline{y}=0}$$

$$+ \underline{b} \wedge \nabla . \underline{\underline{g}}(\underline{x}, \underline{y}) \Big|_{\underline{y}=0}, \tag{6.1.2}$$

where

$$\left.\begin{aligned} a = a_3 &= \int_{\partial B} \underline{\lambda}_3(\underline{x}) . \underline{\phi}_I(\underline{x}) dx = \frac{\mu\gamma}{\kappa}, \\ b_s &= \int_{\partial B} \underline{\lambda}_s(\underline{x}) . \underline{\phi}_I(\underline{x}) dx = 0; \quad s=4,5,6, \\ \underline{b} &= \langle b_1, b_2, b_3 \rangle \equiv \langle b_4, b_5, b_6 \rangle = \langle 0, 0, 0 \rangle. \end{aligned}\right] \tag{6.1.3}$$

on using (3.2.14) and (5.4.7). So we have identified a field:

$$\underline{\phi}_{II} = \frac{\mu\gamma}{\kappa} \langle g(\underline{x}_1, \underline{y}_3), g(\underline{x}_2, \underline{y}_3), g(\underline{x}_3, \underline{y}_3) \rangle$$

$$= \frac{\gamma}{r^3} \langle x_1 x_3, x_2 x_3, \frac{1-\kappa}{\kappa} r^2 + x_3^2 \rangle, \tag{6.1.4}$$

which provides the $O(r^{-1})$ component of $\underline{\phi}_I$, yielding a new field:

$$\underline{\phi}_{III} = \underline{\phi}_I - \underline{\phi}_{II}$$

$$= -\frac{\gamma a^2}{r^5} \langle x_1 x_3, x_2 x_3, -\frac{1}{3} r^2 + x_3^2 \rangle \tag{6.1.5}$$

i.e. $\underline{\phi}_{III} =$

$$= -\frac{\gamma}{a^3} \langle x_1 x_3, x_2 x_3, -\frac{a^2}{3} + x_3^2 \rangle \Big|_{r=a} \tag{6.1.6}$$

having $O(r^{-3})$ behaviour as $r \longrightarrow \infty$. This field may be represented by $\underline{W}$ for a suitable choise of $\underline{\mu}$.

There is no difficulty in calculating the stress components and the traction vector associated with $\underline{\phi}_{III}$. Clearly $\nabla.\underline{\phi}_{III}=0$ everywhere, so we readily compute the exterior traction vector:

$$\underline{\phi}^*_{III} = -\frac{2\mu\gamma}{a^4} \langle 3x_1x_3,\ 3x_2x_3,\ x_1^2+x_2^2-2x_3^2 \rangle, \tag{6.1.7}$$

which yields the resultant force:

$$\int_{\partial B} \underline{\phi}^*_{III} dy = \langle 0 , 0 , 0 \rangle, \tag{6.1.8}$$

and the resultant moment:

$$\int_{\partial B} \underline{y} \wedge \underline{\phi}^*_{III} dy = \langle 0 , 0 , 0 \rangle. \tag{6.1.9}$$

These are both null as may be expected for an $O(r^{-3})$ displacement field. Accordingly $\underline{\phi}_{III}$ could be represented by $\underline{W}$ for a suitable choice of vector source-density $\underline{\mu}$:

$$\underline{\mu} = \frac{1}{4\pi} (\underline{\phi}_{III} - \underline{f}), \qquad r = a, \tag{6.1.10}$$

where $\underline{f}$ is an interior field subject to the condition:

$$\underline{f}^* = - \underline{\phi}^*_{IIIe}. \tag{6.1.11}$$

Now, from (6.1.7) and (6.1.11)

$$\underline{f}^* = \frac{2\mu\gamma}{a^4} \langle 3x_1x_3,\ 3x_2x_3,\ x_1^2 + x_2^2 - 2x_3^2 \rangle, \tag{6.1.12}$$

which yields through fairly straightforward calculations (App.II) the boundary field:

$$\underline{f} = \frac{\gamma}{(3\lambda+2\mu)a^3} \langle (2\lambda+8\mu)x_1x_3,\ (2\lambda+8\mu)x_2x_3,$$

cf (II.19)

$$-(4\lambda+6\mu)(x_1^2+x_2^2) - (2\lambda-2\mu)x_3^2\rangle. \qquad (6.1.13)$$

However an arbitrary rigid- body displacement:

$$\frac{5\gamma(\lambda+2\mu)}{(3\lambda+2\mu)a^3} \langle 0,\ 0\ ,\ -\frac{2\mu}{3(\lambda+2\mu)}\ a^2\rangle, \qquad (6.1.14)$$

may be added to (6.1.13). If so (6.1.10) becomes:

$$\underline{\mu} = -\frac{5\gamma(\lambda+2\mu)}{4\ (3\lambda+2\mu)a^3} \langle x_1x_3,\ x_2x_3,\ -a^2+x_3^2\rangle \qquad (6.1.15)$$

As regards the pressure problem, similar analysis from (4.3.18),(4.3.16) and (6.1.10) yields:

$$\underline{\mu} = \frac{3(\lambda+2\mu)h}{4\ (3\lambda+2\mu)a} \langle x_1,x_2,x_3\rangle. \qquad (6.1.16)$$

6.2 Integral representation:translation problem

To represent the field (6.1.2) by a vector double-layer potential, we use the vector source-density (6.1.15). Also:

$$g(\underline{x}_\alpha,\underline{y}_\beta)^* = \frac{2\nu-1}{2(1-\nu)}\ \frac{1}{\rho^2}\left[\frac{\partial\rho}{\partial y_\beta}\, n_\alpha - \frac{\partial\rho}{\partial y_\alpha}\, n_\beta\right.$$

$$+ \frac{\partial \rho}{\partial n} \left(\delta_{\alpha\beta} + \frac{3}{1-2\nu} \frac{\partial \rho}{\partial y_\beta} \frac{\partial \rho}{\partial y_\alpha} \right)] ; \quad \alpha, \beta = 1,2,3 \quad (6.2.1)$$

where

$$\frac{\partial \rho}{\partial n} \simeq \frac{\underline{x}\cdot\underline{y}}{ar} + O(r^{-1}); n_\alpha = \frac{x_\alpha}{r}; \alpha = 1,2,3. \quad (6.2.2)$$

This provides the asymptotic expansion:

$$g(\underline{x},\underline{y})^* \simeq \frac{1-2\kappa}{ar^3} \begin{bmatrix} \underline{x}\cdot\underline{y} & -x_1y_2+x_2y_1 & -x_1y_3+x_3y_1 \\ -x_2y_1+x_1y_2 & \underline{x}\cdot\underline{y} & -x_2y_3+x_3y_2 \\ -x_1y_3+x_3y_1 & x_2y_3-x_3y_2 & \underline{x}\cdot\underline{y} \end{bmatrix}_{I}$$

$$+ \frac{6\kappa\,\underline{x}\cdot\underline{y}}{ar^5} \begin{bmatrix} (x_1-y_1)^2 & (x_1-y_1)(x_2-y_2) & (x_1-y_1)(x_3-y_3) \\ (x_1-y_1)(x_2-y_2) & (x_2-y_2)^2 & (x_1-y_1)(x_3-y_3) \\ (x_1-y_1)(x_3-y_3) & (x_2-y_2)(x_3-y_3) & (x_3-y_3)^2 \end{bmatrix}_{II+III}$$

Where $I = O(r^{-2})$ & $II + III = O(r^{-3})$.

$$
\simeq \frac{\mu}{a(\lambda+2\mu)r^3}\begin{bmatrix} \underline{x}\cdot\underline{y} & -x_1y_2+x_2y_1 & -x_1y_3+x_3y_1 \\ -x_1y_2+x_2y_1 & \underline{x}\cdot\underline{y} & -x_2y_3+x_3y_2 \\ x_1y_3-x_3y_1 & x_2y_3-x_3y_2 & \underline{x}\cdot\underline{y} \end{bmatrix}_{I}
$$

$$
+\frac{(3\lambda+2\mu)\,\underline{x}\cdot\underline{y}}{a(\lambda+2\mu)r^5}\begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix}_{II}
$$

$$
-\frac{3\lambda+2\mu}{(\lambda+2\mu)ar^5}\begin{bmatrix} 2x_1x_3y_1y_3 & x_1x_3y_2y_3 & x_1x_3(y_1^2+y_3^2) \\ x_2x_3y_2y_3 & 2x_2x_3y_2y_3 & x_2x_3(y_2^2+y_3^2) \\ (x_1^2+x_3^2)y_1y_3 & (x_2^2+x_3^2)y_2y_3 & 2x_3^2y_3^2 \end{bmatrix}_{III} \qquad (6.2.3)
$$

$$
+\,O(r^{-4}) \quad ,\text{as } r \longrightarrow \infty .
$$

Substituting from (6.1.15) and (6.2.3) into (2.3.3) gives:

$$\phi(\underline{x}) = \int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})*.\underline{\mu}(\underline{y})dy$$

$$= -\gamma a^2 < \frac{x_1 x_3}{r^5}, \frac{x_2 x_3}{r^5}, -\frac{1}{3r^3} + \frac{x_3^2}{r^5} >. \qquad (6.2.4)$$

Dyadics I and II integrate to zero and dyadic III gives (6.2.4), which equals the exact field (6.1.5).

6.3 Integral representation:rotation problem

The rotation field (4.2.10) has $O(r^{-2})$ behaviour as $r \longrightarrow \infty$. The associated traction (4.3.13) produces a null resultant force (4.3.14) but a resultant moment (4.3.15). This resultant-moment generates the entire field, leaving no provision for a contribution by $\underline{W}$. More generally the effect of point -couple would be accounted for by the terms involving $\underline{a}, \underline{b}$ in the complete representation (3.2.9).

6.4 Integral representation:pressure problem

Substituting from (6.1.16) and (6.2.3) into (2.3.3) yields:

$$\phi(\underline{x}) = ha^2 < \frac{x_1}{r^3}, \frac{x_2}{r^3}, \frac{x_3}{r^3} >. \qquad (6.4.1)$$

Also dyadic III gives zero,and dyadics I,II give (6.4.1) which equals the field (5.3.3).

Chapter 7

Exact Evaluation of Vector Integrals

7.0 Introduction

We have represented elastostatic displacement fields by means of integrals evaluated asymptotically in the infinite region exterior to the spherical cavity r = a . In this chapter we evaluate the integrals exactly at the particular field point $\underline{x} = \langle 0, 0, z \rangle$.

7.1 Exact simple-layer integral:translation problem

In the general dyadic:

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \frac{1-\kappa}{\mu\rho}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\kappa}{\mu\rho^3}\begin{bmatrix} (x_1-y_1)^2 & (x_1-y_1)(x_2-y_2) & (x_1-y_1)(x_3-y_3) \\ (x_1-y_1)(x_2-y_2) & (x_2-y_2)^2 & (x_2-y_2)(x_3-y_3) \\ (x_1-y_1)(x_3-y_3) & (x_2-y_2)(x_3-y_3) & (x_3-y_3)^2 \end{bmatrix} \tag{7.1.1}$$

let $\underline{x} = \langle 0 , 0 , z \rangle$. If so (7.1.1) becomes:

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \frac{1-\kappa}{\mu\rho}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+\frac{\kappa r}{\mu\rho^3}\begin{bmatrix} r^2\sin^2\theta\cos^2\psi & r^2\sin^2\theta\sin\psi\cos\psi & -r\sin\theta\cos\psi(z-r\cos\theta) \\ r^2\sin^2\theta\cos\psi\sin\psi & r^2\sin^2\theta\cos\psi & -r\sin\theta\sin\psi(z-r\cos\theta) \\ r\sin\theta\cos\psi(z-r\cos\theta) & r\sin\theta\cos\psi(z-r\cos\theta) & (z-r\cos\theta)^2 \end{bmatrix}$$

(7.1.2))

where:

$$\rho^2 = \sum_{i=1}^{3}(x_i-y_i)^2 = (z^2+r^2-2rz\cos\theta). \qquad (7.1.3)$$

There are distinct cases:

(i) $z>a$: Substituting from (5.1.2) and (7.1.2) into (5.1.3)

we find **(see the derivation attached at the end)**:

$$\int_{\partial B}\underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{x})dy$$

$$= \left\langle 0 , 0 , \int_{\partial B}\left(\frac{1-\kappa}{\mu\rho} + \frac{\kappa(z-r\cos\theta)^2}{\mu\rho^3}\right)\frac{\mu\gamma}{4\pi\kappa r^2}.r^2\sin\theta d\theta d\psi\right\rangle_{r=a}$$

$$= \left\langle 0, 0 , \gamma\left(\frac{1}{\kappa z} - \frac{2a^2}{3z^3}\right)\right\rangle, \qquad (7.1.4)$$

i.e.the same value as (4.2.6) for z>a.

(ii) z = a: In this the dyadic (7.1.1) becomes:

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \frac{1-\kappa}{2\mu a\sin(\theta/2)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \frac{\kappa}{8\mu a\sin^3(\theta/2)}\begin{bmatrix} \sin^2\theta\cos^2\psi & \sin^2\theta\cos^2\psi & -\sin\theta\cos\psi\sin^2(\theta/2) \\ \sin^2\theta\sin\psi\cos\psi & \sin^2\cos^2\psi & -2\sin\theta\sin\psi\sin^2(\theta/2) \\ \{\sin\theta\cos\psi\}\cdot\{-2\sin^2(\theta/2)\} & \{\sin\theta\sin\psi\}\cdot\{-2\sin^2(\theta/2)\} & \sin^4(\theta/2) \end{bmatrix}$$

(7.1.5)

Substituting from (5.1.2) and (7.1.5) into (5.1.3) we find:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) \cdot \underline{\sigma}(\underline{x})\,dy$$

$$= \left\langle 0\ ,\ 0\ ,\ \iint_{\partial B}\left\{\left(\frac{1-\kappa}{2a\mu\sin(\theta/2)} + \frac{4\kappa a^2\sin^4(\theta/2)}{8\mu a^3\sin^3(\theta/2)}\right)\cdot \frac{\mu\gamma}{4\pi\kappa a}\right\}a^2\sin\theta\,d\theta\,d\psi\right\rangle$$

$$= \left\langle 0\ ,\ 0\ ,\ \frac{\gamma(3-2\kappa)}{3a\kappa}\right\rangle = \langle 0\ ,\ 0\ ,\ t_3\rangle \qquad (7.1.6)$$

i.e.the same value as (4.2.6) for z=a.

(iii) z<a. In this case we obtain:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\underline{\sigma}}(\underline{x})dy = \langle 0 , 0 , t_3 \rangle \qquad (7.1.7)$$

as expected, using a similar analysis to that of (i).

7.2 Exact simple-layer integral:rotation problem

As before there are three distinct cases:

(i) z>a: Substituting from (7.1.2) and (5.2.1) into (5.1.3) we find:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\underline{\sigma}}(\underline{x})dy = -\frac{3\mu\omega_3}{4\pi a}\Big\langle 0 , 0 , \iint_{\partial B}\Big\{-\frac{r^2\sin^2\theta\cos\psi(z-r\cos\theta)}{\mu\rho^3} + \frac{r^2\sin^2\theta\sin\psi\cos\theta(z-r\cos\theta)}{\mu\rho^3}\Big\}a^2\sin\theta d\theta d\psi\Big\rangle$$

$$= \langle 0 , 0 , 0 \rangle, \qquad (7.2.1)$$

i.e. the same value as (4.2.10) at $\underline{x} = \langle 0 , 0 , z \rangle$.

(ii) z = a: in this case substituting from (7.1.5) and (5.2.1) into (5.1.3) yields:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy = -\frac{3\mu w_3}{4\pi a} \Big\langle 0\ ,\ 0\ ,$$

$$\iint_{\partial B}\Big\{-\frac{\kappa a^2\sin^2\theta\sin\psi\cos\psi(a-a\cos\theta)}{8a^3\mu\sin^3(\theta/2)} +$$

$$+\frac{\kappa a^2\sin^2\theta\sin\psi\cos\psi(a-a\cos\theta)}{8a^3\mu\sin^3(\theta/2)}\Big\}\ a^2\sin\theta d\theta d\psi \Big\rangle$$

$$= \langle\ 0\ ,\ 0\ ,\ 0\ \rangle, \qquad (7.2.2)$$

i.e. the same value as (4.2.10) for $\underline{x} = \langle\ 0\ ,\ 0\ ,\ a\ \rangle$.

(iii) $z<0$: Using similar analysis to that of (i):

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy = \langle\ 0\ ,\ 0\ ,\ 0\ \rangle, \qquad (7.2.3)$$

as expected.

7.3 Exact simple-layer integral:pressure problem

As before there are three distinct cases:

(i) $z>a$: Substituting from (5.3.1) and (7.1.2) into (5.1.3) yields:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}).\underline{\sigma}(\underline{y})dy$$

$$= -\frac{3(\lambda+2\mu)}{4\pi a^2}\ \langle\ 0\ ,\ 0\ ,\iint_{\partial B}\Big\{\frac{\kappa a^2\sin^2\theta\cos^2\psi(z-a\cos\theta)}{\mu\rho^3} -$$

$$- \frac{\kappa a^2 \sin^2\theta \sin^2\psi (z - a\cos\theta)}{\mu \rho^3} + \frac{(1-\kappa) a\cos\theta}{\mu \rho} +$$

$$+ \frac{\kappa a\cos\theta (z - a\cos\theta)^2}{\mu \rho^3} \Big\} a^2 \sin\theta d\theta d\psi >$$

$$= - \frac{3(\lambda+2\mu)h}{4\pi a} < 0 , 0 , \int\int_{\partial B} \Big\{ \frac{\kappa (za^2\cos^2\theta - a^2 z)}{\mu \rho^3} + \frac{a\cos\theta}{\mu \rho} \Big\} a^2 \sin\theta d\theta d\psi >$$

$$= \frac{ha^2}{z^2} < 0 , 0 , 1 >, \qquad (7.3.1)$$

i.e. the same value as (4.2.16) at $\underline{x} = < 0 , 0 , z >$.

(ii) z=a : Substituting from (5.3.1) and (7.1.5) into (5.1.3):

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y}) . \underline{\sigma}(\underline{y}) dy$$

$$= - \frac{3(\lambda+2\mu)h}{4\pi a^2} < 0 , 0 , \int\int_{\partial B} \Big\{ - \frac{\kappa a^3 \sin^2\theta (1-\cos\theta)}{8\mu a^3 \sin^3(\theta/2)} +$$

$$+ \frac{(1-\kappa) a\cos\theta}{2\mu a \sin(\theta/2)} + \frac{\kappa a^3 \cos\theta (1-\cos\theta)^2}{8\mu a^3 \sin^3(\theta/2)} \Big\} a^2 \sin\theta d\theta d\psi >$$

$$= h < 0 , 0 , 1 >, \qquad (7.3.2)$$

i.e. the same value as (4.2.16) at $\underline{x} = < 0 , 0 , a >$.

(iii) z<a : Using similar analysis to that of (i).

7.4 Exact double-layer integral: translation problem

In the general traction dyadic (6.2.3) let $\underline{x} = < 0 , 0 , z >$. If so we obtain:

$$
\underline{\underline{g}}(\underline{x},\underline{y})^*_e = \frac{(1-2\kappa)}{a\rho^3}\begin{bmatrix} \underline{x}\cdot\underline{y} & 0 & zy_1 \\ 0 & \underline{x}\cdot\underline{y} & zy_2 \\ -zy_1 & -zy_2 & \underline{x}\cdot\underline{y} \end{bmatrix}
$$

$$
+\frac{6\kappa\underline{x}\cdot\underline{y}}{a\rho^5}\begin{bmatrix} y_1^2 y & y_1y_2 & -y_1(z-y_3) \\ y_1y_2 & y_2^2 & -y_2(z-y_3) \\ -y_1(z-y_3) & -y_2(z-y_3) & (z-y_3)^2 \end{bmatrix} \quad (7.4.1)
$$

On the boundary, i.e. $\underline{x} = \langle 0, 0, a \rangle$, we find:

$\rho = 2a\sin(\theta/2)$, $\rho' = -\dfrac{\underline{x}\cdot\underline{y}}{a\rho} = \sin(\theta/2)$. If so (7.4.1) becomes:

$$
\underline{\underline{g}}(\underline{x},\underline{y})^* = \frac{(1-2\kappa)}{4a^2\sin^2(\theta/2)}\begin{bmatrix} -1 & 0 & \cos(\theta/2)\cos\phi \\ 0 & 1 & \cos(\theta/2)\sin\phi \\ \dfrac{\cos(\theta/2)\cos\phi}{\sin(\theta/2)} & \dfrac{\cos(\theta/2)\sin\phi}{\sin(\theta/2)} & \sin^2(\theta/2) \end{bmatrix}
$$

$$
-\frac{3\kappa}{2a^2\sin(\theta/2)}
\begin{bmatrix}
\cos^2(\theta/2)\cos^2\psi & \cos^2(\theta/2)\sin\psi\cos\psi & -\cos(\theta/2)\cos\psi\sin(\theta/2) \\
\cos^2(\theta/2)\sin\psi\cos\psi & \cos^2(\theta/2)\sin^2\psi & -\cos(\theta/2)\sin\psi\sin(\theta/2) \\
-\cos(\theta/2)\cos\psi\sin(\theta/2) & -\cos(\theta/2)\sin\psi\sin(\theta/2) & \sin^2(\theta/2)
\end{bmatrix}
$$

(7.4.2)

As before, three cases must be distinguished in each problem:

(i) $z > a$: Substituting from (6.1.15) and (7.4.1) into (3.2.8):

$$
\int_{\partial B} \underline{\underline{g}}(\underline{x}.\underline{y})^{*}.\underline{\mu}(\underline{x})dy = \left\langle 0\ ,\ 0\ ,\ 2\pi \iint_{\partial B} \left\{ -\frac{(1-2\kappa)zy_3(y_1^2+y_2^2)}{a\rho^3} + \frac{6\kappa(a-z\cos\theta)(z-y_3)y_3(y_1^2+y_2^2)}{\rho^5} - \frac{(1-2\kappa)(a-z\cos\theta)y_3^2}{\rho^3} + \frac{6\kappa(a-z\cos\theta)(z-y_3)^2y_3^2}{\rho^5} \right\} a^2\sin^2\theta d\theta d\psi \right\rangle
$$

$$
= -\frac{2\gamma a^2}{3z^3} < 0\ ,\ 0\ ,\ 1 >, \qquad (7.4.3)
$$

i.e. the same value as (6.1.5) at $\underline{x} = < 0\ ,\ 0\ ,\ z >$.

(ii) $z = a$: Substituting from (6.1.15) and (7.4.2) into (3.2.3):

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^{*}_{e}.\underline{\mu}(\underline{x})dy \;+\; 2\pi\underline{\mu}(\underline{x})$$

$$= \left\langle 0 \;,\; 0 \;,\; + \frac{5\gamma(\lambda+2\mu)}{4\,(3\lambda+2\mu)a^3} \int_{\partial B}\left\{ \frac{(1-2\,\kappa)\cos(\theta/2)\sin\theta\cos\theta}{4\,\sin^2(\theta/2)} + \frac{3\,\kappa\cos(\theta/2)\sin\theta\cos\theta}{2} - \left(\frac{1-2\,\kappa}{4a^2\sin(\theta/2)} + \frac{3\,\sin(\theta/2)}{2a^2}\right)\left(-\frac{3\lambda+2\mu}{3(\lambda+2\mu)}\,a^2 + a^2\cos^2\theta\right)\right\} 2\,a^2\sin\theta d\theta \right\rangle - \frac{5\gamma\mu}{3a(3\lambda+2\mu)} \langle\, 0 \;,\; 0 \;,\; 1 \,\rangle$$

$$= -\frac{2\gamma}{3a} \langle\, 0 \;,\; 0 \;,\; 1 \,\rangle, \qquad (7.4.4)$$

i.e. the same value as (6.1.5) for $\underline{x} = \langle\, 0 \;,\; 0 \;,\; a \,\rangle$.

(iii) $z < a$: A similar analysis applies as for (i).

7.5 Exact double-layer integral:pressure problem

(i) $z > a$: Substituting from (6.1.16) and (7.4.1) into (3.2.1): $i \rightarrow e$

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})^{*}_{e}.\underline{\mu}(\underline{x})dy = \Big\langle 0 \;,\; 0 \;,$$

$$- \frac{3(\lambda+2\mu)h}{4\pi a(3\lambda+2\mu)} \int_{\partial B}\left\{ \frac{(1-2\,\kappa)z(y_1^2+y_2^2)}{a\,\rho^3} + \right.$$

$$+\frac{6\kappa(a-z\cos\theta)(z-y_3)(y_1^2+y_3^2)}{\rho^5}+\frac{(1-2\kappa)(a-z\cos\theta)y_3}{\rho^3}+$$

$$\left.\frac{6\kappa(a-z\cos\theta)(z-y_3)^2y_3}{\rho^5}\right\} a^2\sin\theta d\theta d\psi \Big\rangle$$

$$= \frac{ha^2}{z^2} < 0 , 0 , 1 >, \qquad (7.5.1)$$

i.e. the same value as (5.3.3) for $\underline{x} = < 0 , 0 , z >$.

(ii) $z = a$: Substituting from (6.1.16) and (7.4.2) into (3.2.3):

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})*\underline{\mu}(\underline{y})dy + 2\pi\underline{\mu}(\underline{x})$$

$$= \Big\langle 0 , 0 , 2\pi\frac{3(\lambda+2\mu)h}{4\pi a(3\lambda+2\mu)} \int_{\partial B}\Big\{-(1-2\kappa)\left[1-\sin^2(\theta/2)\right]+$$

$$6\kappa\left[\sin^2(\theta/2)-\sin^4(\theta/2)\right]-\frac{1}{2}(1-2\kappa)$$

$$\left[1-2\sin^2(\theta/2)\right]-$$

$$\left.3\kappa\left[\sin^2(\theta/2)-2\sin^4(\theta/2)\right]\cos(\theta/2)\right\} d\theta \Big\rangle +$$

$$\frac{(3\lambda-2\mu)h}{2(3\lambda+2\mu)} < 0 , 0 , 1 >$$

$$= h < 0 , 0 , 1 >, \qquad (7.5.2)$$

i.e. the same value of (5.3.3) at $\underline{x} = \langle 0 , 0 , a \rangle$.

(iii) $z < a$: A similar analysis applies as for (i).

PART III

VOLTERRA DISLOCATIONS

A Volterra dislocation is the elastostatic analogue of a uniform magnetic shell or vortex-equivalent sheet. Just as these may be regarded mathematically as uniform dipole sheets, so dislocations may be regarded as specialised traction sheets. This model is briefly explained and connected up with the theory of Taylor dislocations in a crystal.

Chapter 8

Volterra Dislocations

8.0 Introduction

A Volterra (1907) dislocation is a sheet within the linear elastic continuum, across which the displacement field jumps by a rigid-body translation or rotation without impairing the continuity of the strain and stress components. Some simple dislocation models may be constructed with a hollow cut cylinder as exhibited in fig 1.
We may regard the sheet as a specialised distribution of traction sources, and these generate a vector double-layer potential which may be identified as the elastostatic field of the dislocation. The bounding contour of the sheet, i.e. the dislocation line, plays an important physical role in the theory of crystal dislocations, Taylor (1934); Nabarro (1967).

Clearly a dislocation sheet is the vector analogue of a uniform magnetic shell or vortex equivalent sheet, which are particular examples of a uniform dipole sheet. This generates a scalar double-layer potential, eg. a magnetostatic potential or velocity potential, which jumps by a uniform amount on crossing the sheet without impairing the continuity of the magnetostatic field or fluid velocity. Just as in the dislocation case, the bounding contour of the sheet plays an important physical role, being identified as an electric current or vortex line as the case may be, Pearson (1959).

In his original treatment, Volterra utilised Somigliana's formula, which is the fundamental formula of vector potential theory, Jaswon & Symm (1977).

However, since this involves the superposition of vector simple-layer and double-layer potentials, it obscures the useful dipole analogy. We exploit the analogy to calculate the field of a dislocation having the form of a circular disc.

8.1 Scalar double-layer potentials

A continuous distribution of dipoles over a sheet S contained by a contour ∂S generates the potential:

$$W(\underline{x}) = \int_S g(\underline{x},\underline{y})' \mu(\underline{y}) dy; \quad \underline{y} \subset S , \quad \underline{x} \not\subset S , \qquad (8.1.1)$$

where $\mu(\underline{y})$ signifies the dipole source-density at $\underline{y}$ and dy signifies the element of area at $\underline{y}$. Also

$$g(\underline{x},\underline{y}) = |\underline{x}-\underline{y}|^{-1} , \qquad (8.1.2)$$

and

$$g(\underline{x},\underline{y})' = g'(\underline{y},\underline{x}) = \frac{\partial}{\partial n_y} g(\underline{y},\underline{x}) \qquad (8.1.3)$$

i.e. $g(\underline{x},\underline{y})'$ is the normal derivative of $g(\underline{x},\underline{y})$ at $\underline{y}$ keeping $\underline{x}$ fixed. Physically expressed, $g(\underline{x},\underline{y})$ signifies the potential at $\underline{x}$ generated by a unit dipole source at $\underline{y}$. It is well established, Kellogg (1929), that W has the following general properties:

(i) W is continuous and differentiable at least to the second order, and satisfies $\nabla^2 W = 0$, everywhere except at S, i.e. W defines a harmonic function of $\underline{x}$ everywhere except at S.

(ii) $W = O(r^{-2})$ as $r \longrightarrow \infty$.

(iii) $[W] = 4\pi\mu(\underline{x})$ at $\underline{x} \subset S$, where [ ] signifies the jump in a quantity on crossing S.

(iv) $[\frac{dW}{dn}] = 0$, i.e. the normal derivative (but not necessarily the tangential derivatives) of W remains continuous on crossing S.

If $\mu(=m)$ is uniform over S, then W has the following additional properties:

(v) $[\nabla W] = 0$, i.e. both the normal and tangential derivatives of W remain continuous on crossing S.

(vi) $$[W] = \int_{\gamma} \nabla W . d\underline{\gamma} = 4\pi m$$

for any circuit γ which loops ∂S (fig.2)

These two properties characterise a uniform magnetic shell or vortex-equivalent sheet, focusing attention upon ∂S as the physically significant entity i.e. seat of an electric current or of fluid vorticity as the case may be.

To fix the ideas we choose S to be a circular disc of radius c in the y_1, y_2 plane with centre at $y_1 = 0$, $y_2 = 0$. If so

$$\underline{y} = \langle y_1, y_2, y_3 \rangle = \langle r\cos\theta, r\sin\theta, h \rangle_{h=0}$$

$$d\underline{y} = r\,dr\,d\theta \quad . \qquad (8.1.4)$$

Also , for ease of integration , we consider only

$$\underline{x} = \langle x_1, x_2, x_3 \rangle = \langle 0, 0, z \rangle \quad ; \; z > 0 \qquad (8.1.5)$$

$$g(\underline{x},\underline{y}) = [\, r^2 + (z-h)^2]^{-1/2}_{h=0} \, ,$$

$$g(\underline{x},\underline{y})' = \frac{d}{dh} [r^2 + (z-h)^2]^{-1/2}_{h=0}$$

$$= \frac{z}{(r^2+z^2)^{3/2}} \quad ; \qquad z > 0. \qquad (8.1.6)$$

Inserting this with $\mu(\underline{y}) = m$, into the integral (1) gives:

$$W = W(z) = 2\pi m \int_{r=0}^{r=c} \frac{z r\,dr}{(r^2+z^2)^{3/2}}$$

$$= 2\pi m \left(1 - \frac{z}{(z^2+c^2)^{1/2}}\right) \qquad ; \; z > 0 \qquad (8.1.7)$$

$$= 2\pi m\,(1 - \cos\alpha\,) \, ; \; \alpha = \cos^{-1} \frac{z}{(z^2+c^2)^{1/2}} . \qquad (8.1.8)$$

This is of course a well known classical result usually obtained by the method of solid angles, Collatz (1966)

Note that:

(i) $W = O(z^{-2})$ as $z \longrightarrow \infty$ as follows from (8.1.7)

(ii) $W \longrightarrow 2\pi m$ as $z \longrightarrow 0$ as also follows from (8.1.7)

(iii) $W = 0$ for $z = 0$ as follows from (8.1.7) and also directly from the fact that $g(\underline{x},\underline{y})' = 0$ for $\underline{x} \subset S$.

Referring to the integral (8.1.1) these last two results appear respectively as, Burkill (1970):

$$\text{(ii)} \quad \lim_{z \to 0} \int_S g(\underline{x},\underline{y})' \, m dy = 2\pi m , \qquad (8.1.9)$$

$$\text{(iii)} \quad \int_S \lim_{z \to 0} g(\underline{x},\underline{y})' \, m dy = 0 , \qquad (8.1.10)$$

with a jump which arises from the non-uniform convergence of the function, Ferrar (1938):

$$U_z(r) = \frac{zr}{(z^2+r^2)^{3/2}} \quad , \text{as } z \longrightarrow 0. \qquad (8.1.11)$$

Since $U_z(r)$ is anti-symmetric with respect to z, W is also anti-symmetric with respect to z, i.e.

$$W(z) = -2\pi m\left(1 - \frac{|z|}{(z^2+r^2)^{1/2}}\right) ; \; z < 0 \qquad (8.1.12)$$

so yielding:

$$[W] = 4\pi m , \; \left[\frac{dW}{dz} \right] = 0 , \qquad (8.1.13)$$

in line with general theory. A graph of W(z) appears in fig 3 (m=1).

8.2 Vector double-layer potentials

Corresponding with W, we introduce the vector double-layer potential:

$$\underset{\sim}{W}(\underset{\sim}{x}) = \int_S \underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})^* \cdot \underset{\sim}{\mu}(\underset{\sim}{y})dy \quad ; \quad \underset{\sim}{y} \subset S \quad \underset{\sim}{x} \not\subset S \quad (8.2.1)$$

where $\underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})^*$ signifies the fundamental traction dyadic of the medium and $\underset{\sim}{\mu}(\underset{\sim}{y})$ signifies a vector source-density.* In terms of components:

$$\underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})^* = \begin{bmatrix} g(\underset{\sim}{x}_1,\underset{\sim}{y}_1)^* & g(\underset{\sim}{x}_1,\underset{\sim}{y}_2)^* & g(\underset{\sim}{x}_1,\underset{\sim}{y}_3)^* \\ g(\underset{\sim}{x}_2,\underset{\sim}{y}_1)^* & g(\underset{\sim}{x}_2,\underset{\sim}{y}_2)^* & g(\underset{\sim}{x}_2,\underset{\sim}{y}_3)^* \\ g(\underset{\sim}{x}_3,\underset{\sim}{y}_1)^* & g(\underset{\sim}{x}_3,\underset{\sim}{y}_2)^* & g(\underset{\sim}{x}_3,\underset{\sim}{y}_3)^* \end{bmatrix} \quad (8.2.2)$$

where $g(\underset{\sim}{x}_\alpha,\underset{\sim}{y}_\beta)^*$ provides the β-component of traction at $\underset{\sim}{y}$ generated by a unit point-force acting along the α- direction at $\underset{\sim}{x}$. Clearly row 1 of (8.2.2) defines the traction vector at $\underset{\sim}{y}$ generated by a unit point-force acting along the 1-direction at $\underset{\sim}{x}$, etc. Also column 1 of (8.2.2) defines an elastostatic displacement field, i.e. that generated by a unit traction-source acting along the 1-direction at $\underset{\sim}{y}$, etc. This means that $\underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})^*$ plays the role of a vector dipole potential corresponding with the scalar dipole potential $g(\underset{\sim}{x},y)'$ .Writing $\underset{\sim}{\mu} = \langle\mu_1,\mu_2,\mu_3\rangle$,(8.2.1)appears in component form as:

* This has already been introduced in 1.3 for a closed surface B whilst here we are concerned with an open surface .

$$W(\underset{\sim}{x}) = \int_S g(\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\beta)^* \mu_\beta(\underset{\sim}{y})dy \quad ; \quad \alpha, \beta = 1,2,3 \qquad (8.2.3)$$

assuming the summation convention for dummy subscripts.

To evaluate (8.2.2) we must first compute the fundamental displacement dyadic of the medium:

$$\underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{y}) = \begin{bmatrix} g(\underset{\sim}{x}_1, \underset{\sim}{y}_1) & g(\underset{\sim}{x}_1, \underset{\sim}{y}_2) & g(\underset{\sim}{x}_1, \underset{\sim}{y}_3) \\ g(\underset{\sim}{x}_2, \underset{\sim}{y}_1) & g(\underset{\sim}{x}_2, \underset{\sim}{y}_2) & g(\underset{\sim}{x}_2, \underset{\sim}{y}_3) \\ g(\underset{\sim}{x}_3, \underset{\sim}{y}_1) & g(\underset{\sim}{x}_3, \underset{\sim}{y}_2) & g(\underset{\sim}{x}_3, \underset{\sim}{y}_3) \end{bmatrix} \qquad (8.2.3$$

where $g(\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\beta)$ provides the β-component of displacement at $\underset{\sim}{y}$ generated by a unit point-force acting along the α-direction at $\underset{\sim}{x}$. Alternatively, since $g(\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\beta) = g(\underset{\sim}{y}_\beta, \underset{\sim}{x}_\alpha)$, it also provides the α-component of displacement at $\underset{\sim}{x}$ generated by a unit point-force acting along the β-direction at $\underset{\sim}{y}$. Clearly both row 1 and column 1 of (8.2.3) define elastostatic displacement vectors, etc. For an infinite linear isotropic elastic continuum, the dyadic components are nothing more than Kelvin's point-force solution, Love (1927), written systematically in subscript notation.

It has been shown by Kupradze (1965) that $\underset{\sim}{W}$ has the following properties in a linear isotropic elastic continuum:

(i) $\underline{W}$ is continuous and differentiable at least to the second order, and satisfies the Cauchy-Navier equation, everywhere except at S, i.e. $\underline{W}$ defines an elastostatic displacement field everywhere except at S.

(ii) $\underline{W} = O(r^{-2})$ as $|\underline{x}| = r \longrightarrow \infty$.

(iii) $[\underline{W}] = 4\pi\underset{\sim}{\mu}(\underline{x})$ at $\underline{x} \subset S$.

If $\underset{\sim}{\mu}(\underline{y}) = \underline{b} + \underline{\omega} \wedge \underline{y}$, where $\underline{b}$, $\underline{\omega}$ are constant vectors, i.e. $\underline{\mu}$ varies as a rigid-body displacement over S, then $\underline{W}$ has the following additional property analogous to $[\nabla \underline{W}] = 0$ in the scalar case:

(iv) $$\left[\frac{\partial W_\alpha}{\partial x_\beta} + \frac{\partial W_\beta}{\partial x_\alpha} \right] = 0 \; ; \qquad \alpha, \beta = 1,2,3,$$

i.e. the strains associated with $\underline{W}$ remain continuous on crossing S. This means that the stresses and therefore the tractions remain continuous on crossing S so identifying the sheet as a Volterra dislocation.

If $\underline{\omega} = 0$, i.e. no rotational jump, then $\underline{W}$ has the following additional property which replaces (iii) above:

(v) $$[\underline{W}] = \oint_{\gamma} \nabla \underline{W} . d\underline{\gamma} = 4\pi \underline{b} \; ; \quad 4\pi\underline{b} = \text{Burger's vector},$$

for any circuit which loops the dislocation line ∂S. Here $4\pi\underline{b}$ is the Burger's vector of the dislocation line as defined in the theory of crystal dislocations (see section 8.5)

8.3 Circular dislocations

Choosing a circular sheet of radius c as before, and again writing

$$\underline{x} = \langle o,o,z\rangle\ ,\quad \underline{y} = \langle y_1,y_2,h\rangle_{h=0} \tag{8.3.1}$$

we compute the components of $\underline{\underline{g}}(\underline{x},\underline{y})^*$ from the known components of $\underline{\underline{g}}(\underline{x},\underline{y})$, using a similar analysis to that of Section 6.2.

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \frac{1-\kappa}{\mu\rho}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\kappa}{\mu\rho^3}\begin{bmatrix} y_1^2 & y_1y_2 & -y_3z \\ y_1y_2 & y_2^2 & -y_2z \\ -y_1z & -y_2z & z^2 \end{bmatrix}, \tag{8.3.2}$$

$$g(x,y)^* = \frac{(1-2\kappa)z}{\mu\rho^3}\begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & -y_2 \\ y_1 & y_2 & 1 \end{bmatrix} + \frac{6\kappa}{\mu\rho^5}\begin{bmatrix} y_1^2 & y_1y_2 & y_1z \\ y_1y_2 & y_2^2 & y_2z \\ y_1 & y_2 & -z^2 \end{bmatrix}. \tag{8.3.3}$$

The integral (8.2.3) may then be evaluated exactly for the six independent rigid-body displacements:

$$\underset{\sim}{b} = b_1 \langle 1 , 0 , 0\rangle , b_2\langle 0, 1 , 0 \rangle , b_3\langle 0 , 0 , 1\rangle$$

$$\omega_1\langle 1,0,0\rangle\wedge\underset{\sim}{y} , \omega_2\langle 0,1,0\rangle\wedge\underset{\sim}{y} \quad \omega_3\langle 0,0,1\rangle\wedge\underset{\sim}{y}$$

8.4 Two-dimensional continuum dislocations

Problems of two-dimensional linear isotropic elastostatics, in the absence of body forces, may be conveniently formulated through a stress function χ which satisfies the biharmonic equation:

$$\nabla^4\chi = \nabla^2 (\nabla^2\chi) = 0 ; \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{8.4.1}$$

The displacement components associated with χ are given by the formulae, Muskhelishvili (1953$_{a,b}$):

$$2\mu W_1 = (1-\nu)H - \frac{\partial\chi}{\partial x_1} , \quad 2\mu W_2 = (1-\nu)H^* - \frac{\partial\chi}{\partial x_2} \tag{8.4.2}$$

where H, H^* are conjugate harmonic functions defined by

$$\frac{\partial H}{\partial x_1} = \frac{\partial H^*}{\partial x_2} = \nabla^2\chi \tag{8.4.3}$$

and μ, ν denote the shear modulus and Poisson's ratio respectively. These formulae simplify somewhat by introducing the Almansi (1897) representation for χ:

$$\chi = x_1\phi + \psi \;(\text{or} \;\; x_2\phi + \psi) \tag{8.4.4}$$

where ϕ , ψ are harmonic functions, since then, Brown(1973)

$$\nabla^4 \chi = 2 \frac{\partial \phi}{\partial x_1} \quad , \quad H = 2\phi \; , \quad H^* = 2\phi^* \qquad (8.4.5)$$

so enabling us to replace (8.4.2) by

$$2\mu W_1 = 2(1-\nu)\phi - \frac{\partial \chi}{\partial x_1} \quad , \quad 2\mu W_2 = 2(1-\nu)\phi^* - \frac{\partial \psi}{\partial x_2} \; .(8.4.6)$$

Note that the functional equation $x_1\phi + \psi = 0$ has the two independent non-trivial solutions, Laskar(1971):

$$\phi = 1 \quad \psi = -x_1 \quad ; \quad \phi = x_2 \quad , \psi = -x_1 x_2 \qquad (8.4.7)$$

showing that an arbitrary rigid-body displacement may be superposed upon W_1, W_2 keeping χ invariant.

Formulae (8.4.7) point to the dislocation solution (omitting dimensional coefficients)

$$\chi = x_1 \log r \quad ; \quad \phi = \log r \; , \; \phi^* = \theta \; , \; \psi = 0 \qquad (8.4.8)$$

yielding the translation jumps:

$$[W_1] = 0, \; [W_2] = \frac{2\pi}{\mu}(1-\nu) \qquad (8.4.9)$$

for any complete circuit about the origin. They also point to a second, independent, solution:

$$\chi = x_2 \log r \; : \; \phi = -\theta \; , \; \phi^* = \log r, \quad \psi = (x_2 \log r + x_1 \theta) \qquad (8.4.10)$$

yielding the translation jumps:

$$[W_1] = -\frac{2\pi}{\mu}(1-\nu), \quad [W_2] = 0. \qquad (8.4.11)$$

Here the dislocation line coincides with the x_3-axis as exhibited in fig 4 , so identifying the dislocation sheet as the half-plane $x_2 = 0$, $x_1 < 0$. These are purely mathematical models. Physical models could only be constructed by making the body multiply-connected, i.e. replacing the dislocation line by a hollow tube or core which in general has the form of a torus enclosing ∂S. We then cut through the material so as to intersect the core, rigidly translate one side of the cut relative to the other, and weld the sides together again in the new configuration. Six independent dislocations can be constructed across the cut, of which two examples have appeared in fig. 1

8.5 Crystal dislocations

The atomistic structure of an edge dislocation is modelled in fig. 5 ,which depicts a section of the crystal at right angles to the dislocation line. This provides a crystalline version of the continuum dislocation modelled in fig. 4 . Here the straight lines numbered 1,2,...,6 mark the traces of crystal planes at right angles to the slip direction, i.e. that of the translation jump(8.4.11). Fig. 5 (a) refers to the perfect crystal . Figure 5(b) shows the crystal severed into two halves across the slip plane, fig. 5(c) shows an extra, Frank (1949), half-plane, denoted p, inserted symmetrically between the upper half-planes 3 and 4 . In fig 5 (d) the two halves of the crystal have been stitched together by re-introducing the atomic forces, matching as far as possible half- planes of the same number, and thereby leaving the central half-plane without a partner.This operation requires the upper half-plane to be compressed and the lower half-plane to be extended.The lower edge of p, identified as the x_3-axis of fig4 marks the edge dislocation line. Foreman,Jaswon,Wood (1951).

The dislocation lies at the centre of a small region of misfit bounded by the almost perfectly matched half-planes 1 and 6, beyond which the crystal is perfect. Since the misfit also falls off vertically the region is preferably pictured as a cylindrical domain, sometimes termed the dislocation core,and indicated by the circled area in fig 5(d). In 5(e) the dislocation has effectively jumped forward by one inter-atomic spacing to the right compared with 5 (d), as the central spot now falls between the lower half-planes 4,5 instead of between 3,4. This jump does not imply any movement of matter: p still remains the neighbour of the upper half-plane 3 (being now labelled 4') but its lower part deviates slightly to the right,thereby becoming aligned with the lower half- plane4.The upper half-plane 4 is left without a partner,to assume the role formerly held by p (being now labelled p'). The dislocation thus propagates very much like a travelling wave or disturbance,instantaneously separating the slipped from the unslipped regions of perfect crystal.It eventually becomes blocked at some particular point,or passes right out of the crystal as shown in fig.5(f). Since the configurations 5(d) and 5(e) have the same energy,the dislocation,to a first approximation moves under a vanishingly small stress.This provides the essential mechanism of plastic deformation.

The locked-up stress field generated by the continuum dislocation provides a very good approximation to that of a crystal dislocation outside the region of the dislocation core. Within the latter region, the strains are so large that classical elasticity can no longer be applied and a direct calculation of atomic displacements becomes necessary.

Since dislocations are singularities in stress fields, they interact with other dislocations, and more generally, with other geometrical imperfections. For instance two edge dislocations in the same slip plane repel or attract each other, according to whether their signs are like or unlike. If they are on different slip planes the situation becomes more complex, but the general possibility arises of dislocations blocking or locking each other by virtue of their mutual interactions, an effect which provides the essential mechanism in all theories of work hardening.

FIGURES

Fig.1 Volterra dislocations in a hollow cut cylinder.

(a) cut cylinder

(b) edge dislocation : sides of cut relatively displaced in direction at right angles to cylinder axis

(c) screw dislocation : sides of cut relatively displaced in direction paralled to cylinder axis

Fig.2 Irreducible circuit γ around a contour ∂S modelling a dislocation line, vortex or electric current.

Fig.3 Graph of W in terms of the non-dimensional parameter $\zeta = z/c$, showing the jump in W and continuity of $dW/d\zeta$ at $\zeta = 0$ (for choice $m = 1$).

Fig.4 Model of 2-dimensional continuum dislocation.

(a) section at right angles to dislocation line exhibiting the origin as a singularity in the field

(b) 3-dimensional picture of dislocation line bounding the infinite sheet $-0 < x_1 < 0$, $x_2 = 0$

Fig.5 Model of 2-dimensional crystal dislocation (following Taylor). This is an edge dislocation since it propagates in direction of slip. Note that the extra half-plane p becomes successively identified with the upper half-planes 4,5,6....., eventually reaching the boundary of the crystal.

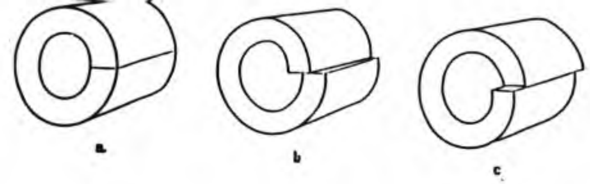


figure 1

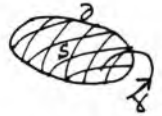


figure 2

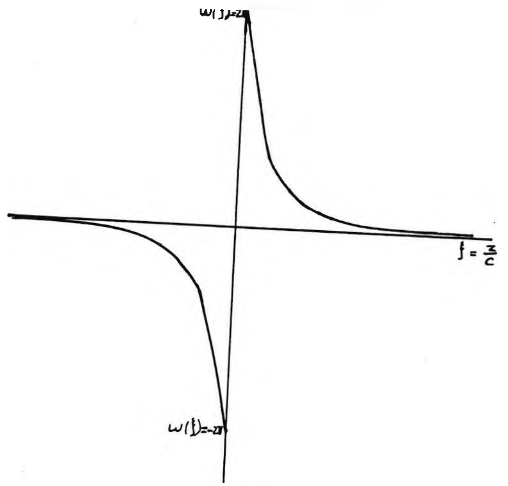


figure 3

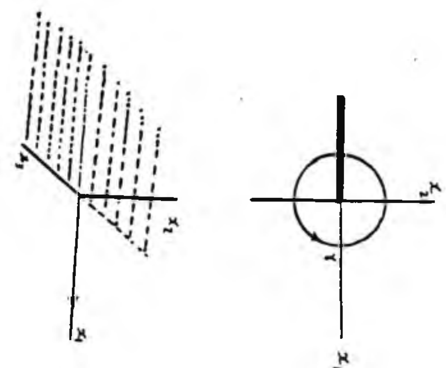


figure 4

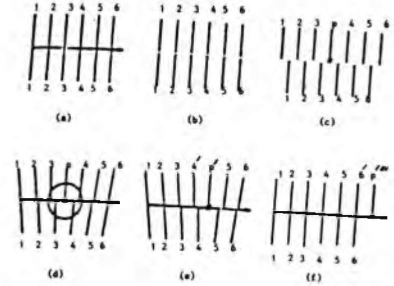
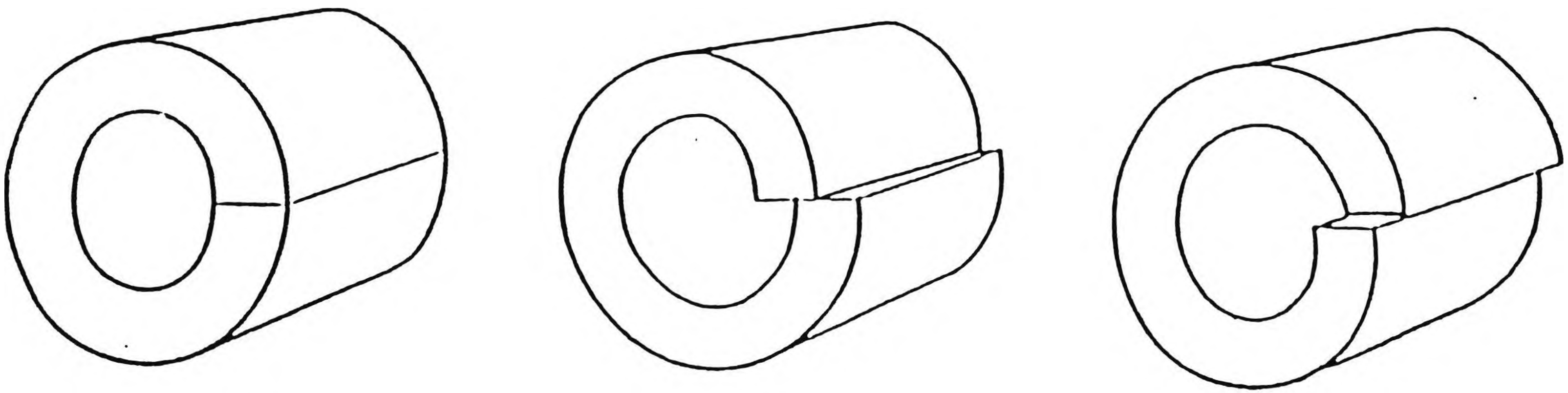


figure 5



a b c

figure 1

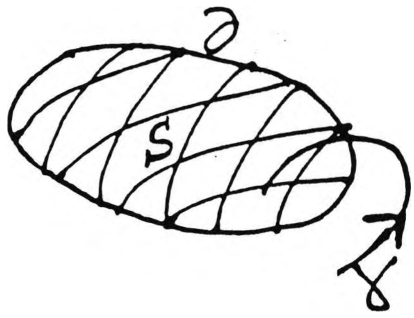


figure 2

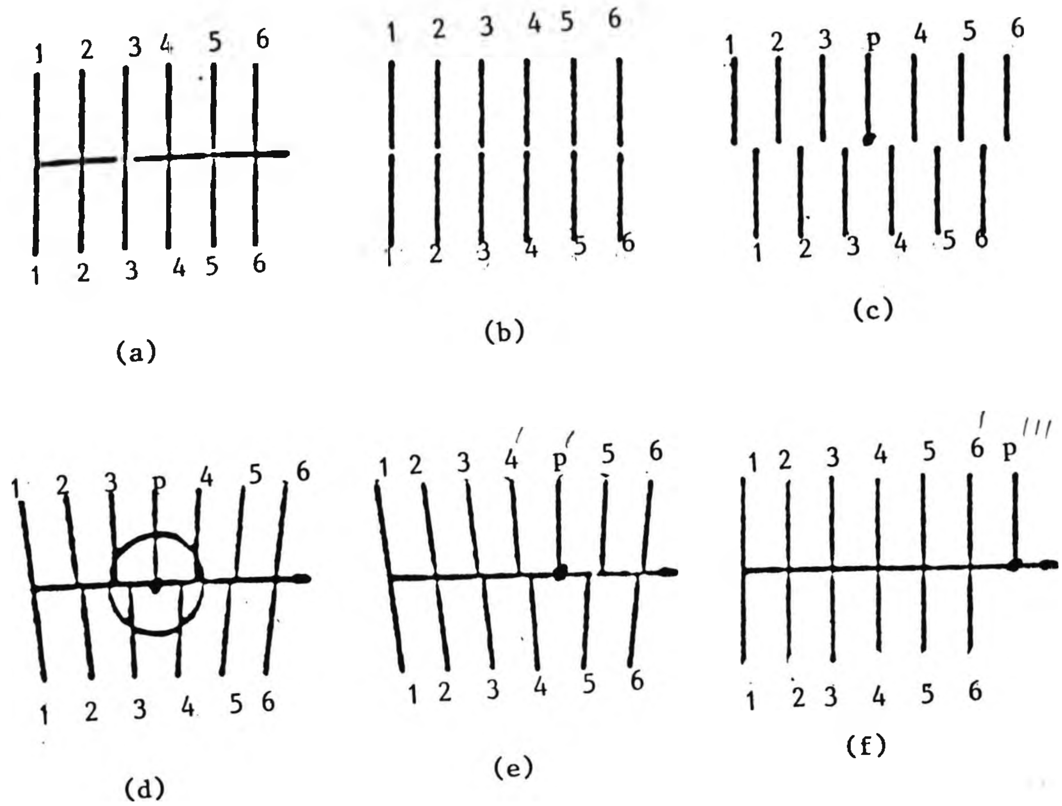


figure 5

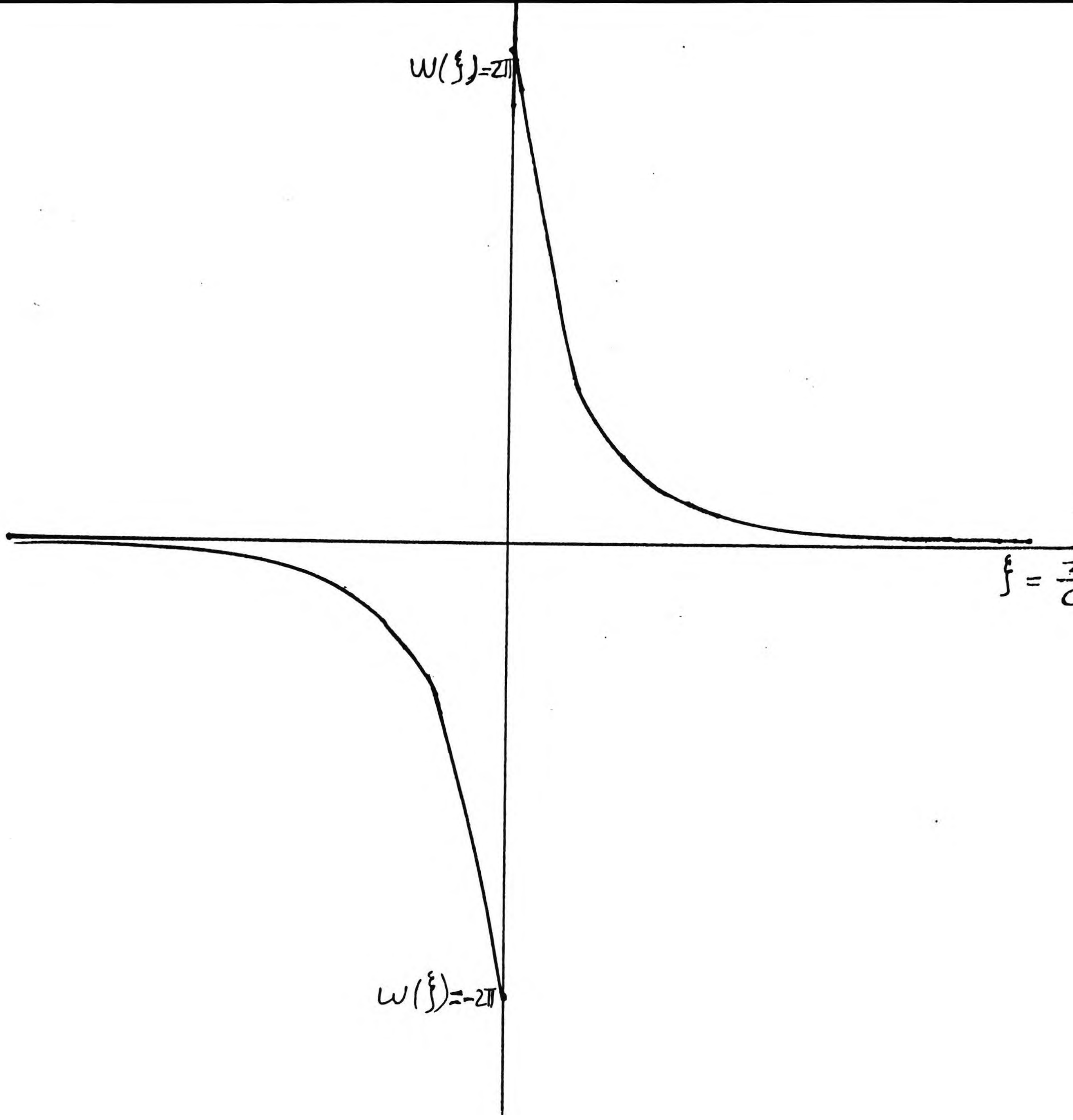


figure 3

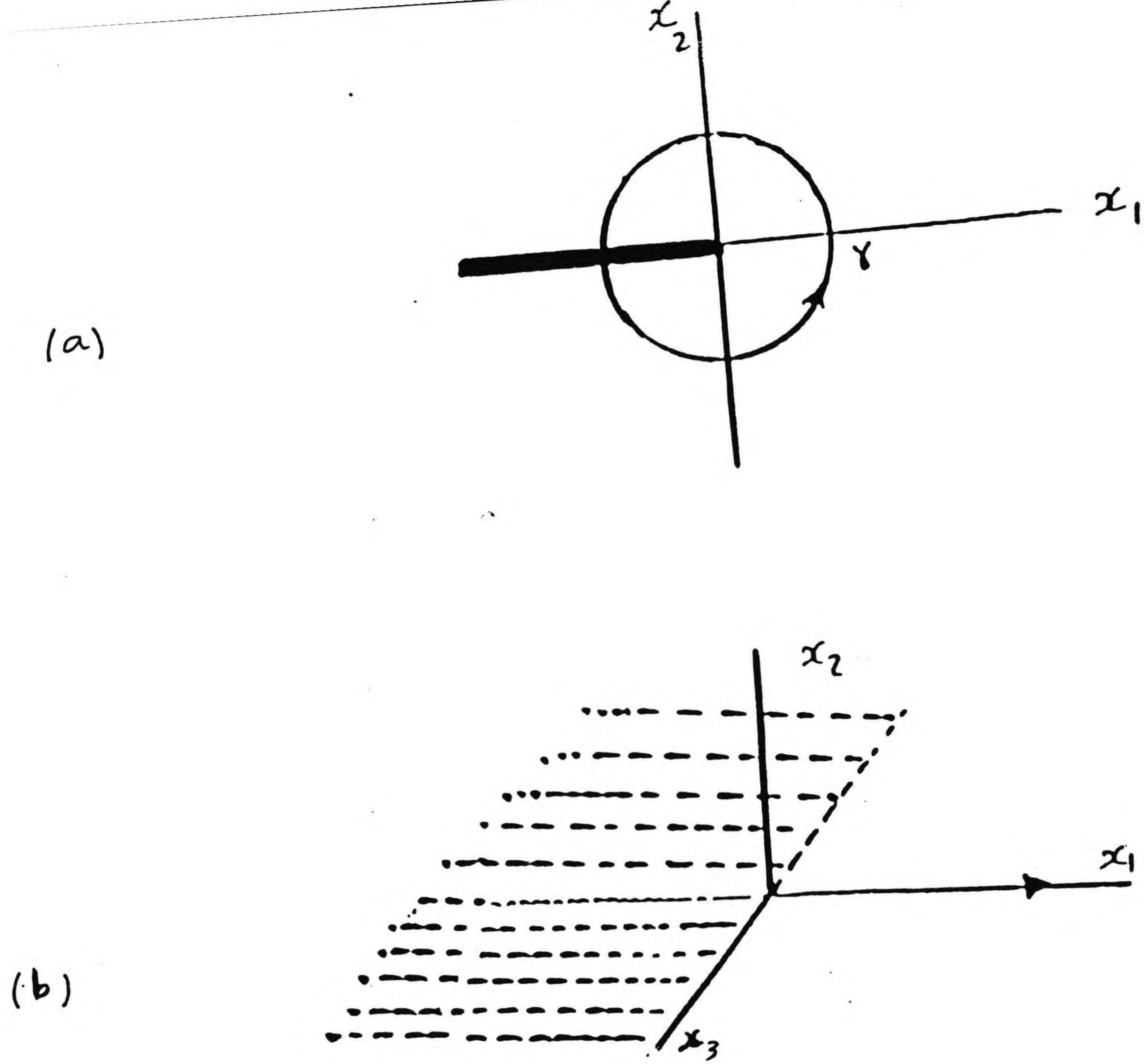


figure 4

Appendices

Sphere:translation problem

Given $\underset{\sim}{\Phi} = \langle o,o,t_3\rangle$ on the boundary r= a of a spherical cavity where t_3 is a constant,i.e. $\underset{\sim}{\Phi}$ is a rigidy-body translation,find $\underset{\sim}{\Phi}$ in the infinite region $r \geqslant a$ such that

(i) $\underset{\sim}{\Phi} = O(r^{-1})$ as $r \longrightarrow \infty$;

(ii) $\underset{\sim}{\Phi} = \langle o,o, t_3\rangle$ on $r = a$.

(iii) $\underset{\sim}{\Phi}$ =satisfies Cauchy-Navier equation:

We choose:

$$h = \langle o, o , \alpha/r\rangle, \; f = \beta \partial r^{-1}/\partial x_3 \; , \tag{I.1}$$

in the Papkovich-Neuber representation,which yields:

$$\underset{\sim}{\Phi} = \langle o ,o, \alpha/r\rangle - K\nabla(\alpha x_3/r - \beta x_3/r^3) \tag{I.2}$$

where α ,β are constants to be determind. In components (I.2) becomes

$$\begin{aligned} \Phi_1 &= K(\alpha x_1 x_3/r^3 - 3\beta x_1 x_3/r^5) \\ \Phi_2 &= K(\alpha x_2 x_3/r^3 - 3\beta x_2 x_3/r^5) \\ \Phi_3 &= (1-K)\alpha/r + K(\alpha x_3^2/r^3 + \beta/r^3 - 3\beta x_3^2/r^5) \end{aligned} \tag{I.3}$$

Now applying the boundary condition (ii):

$\Phi_1 = 0$, $\Phi_2 = 0$(give the same value)

$$\alpha x_1 x_3/ a^3 = 3\beta x_1 x_3/a^5 \quad \text{i.e. } \beta = \alpha a^2/3. \tag{I.4}$$

Also $\Phi_3 = t_3$ gives:

$$\alpha/a(1-K) + K\alpha/3a = t_3 \text{ , i.e. } \alpha = 3at_3/3-2K, \qquad (I.5)$$

Substituting from (I.4),(I.5) into (I.3) gives the field ($\gamma = K\alpha$)

$$\underset{\sim}{\Phi} = \gamma\,(x_1x_3/r^3 - a^2x_1x_3),\; x_2x_3/r^3 - a^2x_2x_3/r^5,$$

$$(1/K - 1)/r + x_3^2/r^3 + a^2/3r^3 - a^2x_3^2/r^5, \quad r \geq a$$

(I.6)

Now we differentiate (I.6) with respect to x_i; i = 1,2,3 to obtain:

$$\Phi_{1/1} = \gamma\,(x_3/r^3 - 3x_1^2x_3/r^5 - a^2x_3/r^5 + 5a^2x_1^2x_3/r^7)$$

$$\Phi_{1/2} = \gamma\,(-3x_1x_2x_3/r^5 + 5a^2x_1x_2x_3/r^7)$$

$$\Phi_{1/3} = \gamma\,(x_1/r^3 - 3x_1x_3^2/r^5 - a^2x_1/r^5 + 5a^2x_1x_3^2/r^7) \qquad (I.7)$$

$$\Phi_{2/2} = \gamma\,(x_3/r^3 - 3x_2^2x_3/r^5 - a^2x_3/r^5 + 5a^2x_2^2x_3/r^7)$$

$$\Phi_{2/1} = \gamma\,(-3x_1x_2x_3/r^5 + 5a^2x_1x_2x_3/r^7)$$

$$\Phi_{2/3} = \gamma\,(x_2/r^3 - 3x_2x_3^2/r^5 + 5a^2x_2x_3^2)$$

$$\Phi_{3/1} = \gamma\,((1-1/K)x_1/r^3 - 3x_1x_3^2/r^5 - a^2x_1/r^5 + 5a^2x_1x_3^2/r^7)$$

$$\Phi_{3/3} = \gamma\,((3-1/K)x_3/r^3 - 3x_3^2/r^5 - a^2x_2/r^5 + 5a^2x_2x_3^2/r^7)$$

$$\Phi_{3/2} = \gamma\,((1-1/K)\,x_2/r^3 - 3x_2x_3^2/r^5 - a^2x_2/r^5 + 5a^2x_2x_3^2/r^7)$$

(I.7)

These expressions readily provide the dilatation:

$$\Delta = \partial\Phi_1/\partial x_1 + \partial\Phi_2/\partial x_2 + \partial\Phi_3/\partial x_3 = \Phi_{1/1} + \Phi_{2/2} + \Phi_{3/3}, \qquad (I.8)$$

$$\Delta = \gamma\,(x_3/r^3 - 3x_1^2x_3/r^5 - a^2x_3/r^5 + 5a^2x_1^2x_3/r^7$$

$$+x_3/r^3 - 3x_2^2x_3/r - a^2x_3/r^5 + 5a^2x_2^2x_3/r^7$$

$$+(3-1/\kappa)x_3/r^3 - 3x_3^3/r^5 - 3a^2x_3/r^5 + 5a^3x_3^3/r^7)$$

$$= \gamma\,(2-1/\kappa)x_3/r^3 = 3at_3(2\ -1)x_3/(3-2\kappa)r^3$$

$$= \gamma' x_3/r^3 = \gamma' x_3/a^3;\ r=a\ ,\ \gamma' = 3at_3(2\kappa-1)/3-2\kappa \qquad (I.9)$$

which is seen to be a harmonic function.

The accompanying strain tensor at r = a is

$$\tfrac{1}{2}(\Phi_{\alpha/\beta} + \Phi_{\beta/\alpha})_{r=a}\ ;\alpha,\beta = 1,2,3 \qquad (I.10)$$

$$\Phi_{11} \equiv \Phi_{1/1} = 2\gamma x_1^2x_3/a^5 \qquad \Phi_{22} = \Phi_{2/2} = 2\gamma x_2^2x_3/a^5$$

$$\Phi_{33} = \Phi_{3/3} = \gamma\,(-x_3/\kappa a^3 + 2x_3^3/a^5)$$

$$\Phi_{12} = \Phi_{21} = 2\gamma x_1x_2x_3/a^5$$

$$\Phi_{13} = \Phi_{31} = \gamma(-x_1/2\kappa a^3 + 2x_1x_3^2/a^5)$$

$$\Phi_{23} = \Phi_{32} = \gamma \, (-x_2/2\kappa a^3 + 2x_2 x_3^2/a^5) \qquad \text{(I.10)}$$

From the stress-strain relations:

$$\bar{\Phi}_{\alpha\beta} = \mu \, (\Phi_{\alpha|\beta} + \Phi_{\beta|\alpha}) + \lambda \, \nabla \cdot \underset{\sim}{\Phi} \, \delta_{\alpha\beta}; \quad \alpha, \beta = 1,2,3 \qquad \text{(I.11)}$$

we calculate the stress components at r = a:

$$\begin{aligned}
\bar{\Phi}_{11} &= \lambda \gamma' x_3/a^3 + 4\mu\gamma x_1^2 x_3/a^5 \\
\bar{\Phi}_{22} &= \lambda \gamma' x_3/a^3 + 4\mu\gamma x_2^2 x_3/a^5 \\
\bar{\Phi}_{33} &= \lambda \gamma' x_3/a^3 + 2\mu\gamma x_3/a^3 \\
\bar{\Phi}_{12} &= \bar{\Phi}_{21} = 4\mu\gamma \, x_1 x_2 x_3/a^5 \\
\bar{\Phi}_{13} &= \bar{\Phi}_{31} = \mu\gamma(-x_1/\kappa a^3 + 4x_1 x_3^2/a^5) \\
\bar{\Phi}_{23} &= \bar{\Phi}_{32} = \mu\gamma \, (-x_2/\kappa a^3 + 4x_2 x_3^2/a^5)
\end{aligned} \qquad \text{(I.12)}$$

Finally we obtain the traction vector :

$$\underset{\sim}{\Phi}^{*} = \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \\ \Phi_3^* \end{bmatrix} = \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} \\ \bar{\Phi}_{12} & \bar{\Phi}_{22} & \bar{\Phi}_{23} \\ \bar{\Phi}_{13} & \bar{\Phi}_{23} & \bar{\Phi}_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$
= \begin{bmatrix}
x_3/a^3 + 4\mu x_1^2 x_3/a^5 & 4\mu x_1 x_2 x_3/a^5 & -x_1/a^3 + 4x_1 x_3^2/a^5 \\
4\mu x_1 x_2 x_3/a^5 & x_3/a^3 + 4\mu x_2^2 x_3/a^5 & -x_2/a^3 + 4x_1 x_3^2/a^5 \\
(-x_1/a^3 + 4x_1 x_3^3/a^5) & (-x_2/a^3 + 4x_2 x_3^2/a^5) & -2x_3/a^3 + 4x_3^3/a^5 + 2\mu x_3/a^3
\end{bmatrix}
\begin{bmatrix} x_1/a \\ x_2/a \\ x_3/a \end{bmatrix}
$$

$$
= \langle Hx_1 x_3/a^4 \ , \ Hx_2 x_3/a^4 \ , \ Hx_3^2/a^4 - \mu/\kappa a^2 \rangle
$$

$$
= \langle 0, 0, -\mu/\kappa a^2 \rangle; \qquad \text{(I.13)}
$$

where

$$
H = \lambda \gamma' + 4\mu\gamma - \mu\gamma/\kappa = 0 \ . \qquad \text{(I.14)}
$$

Appendix II

Vector double-layer source density

Given the boundary traction

$$\underset{\sim}{\Phi}^{*} = -2\mu\gamma/a^{4} < 3x_1x_3,\ 3x_2x_3,\ -x_1^2 - x_2^2 + 2x_3^2 >, \qquad \text{(II.1)}$$

on a spherical cavity r = a , we determine the corresponding boundary displacement field $\underset{\sim}{\Phi}$. It is clear that $\underset{\sim}{\Phi}^{*}$ satisfies the equilibrium conditions:

$$\int_{r=a} \Phi^{*}(\underset{\sim}{x})dx = 0, \qquad \int_{r=a} \underset{\sim}{x} \wedge \Phi^{*}(\underset{\sim}{x})dx = 0. \qquad \text{(II.2)}$$

In (4.1.3) we choose the second-degree harmonic functions:

$$\left.\begin{aligned} h_1 &= a_1x_1x_2 + a_2x_1x_3 + a_3x_2x_3 + a_4(x_2^2 - x_1^2) + a_5(x_1^2 - x_3^2), \\ h_2 &= b_1x_1x_2 + b_2x_1x_3 + b_3x_2x_3 + b_4(x_2^2 - x_1^2) + b_5(x_1^2 - x_3^2), \\ h_3 &= c_1x_1x_2 + c_2x_1x_3 + c_2x_2x_3 + c_4(x_2^2 - x_1^2) + c_5(x_1^2 - x_3^2); \end{aligned}\right]_{r \lessgtr a} \qquad \text{(II.3)}$$

where $a_i, b_i, c_i; i=1,.,.,5$ are constants to be determined. If so, the Papkovich-Neuber formula gives:

$$\left.\begin{aligned} \underset{\sim}{\Phi} = &< a_1x_1x_2 + a_2x_1x_3 + a_3x_2x_3 + a_4(x_2^2 - x_1^2) + a_5(x_1^2 - x_3^2), \\ &b_1x_1x_2 + b_2x_1x_3 + b_3x_2x_3 + b_4(x_2^2 - x_1^2) + b_5(x_1^2 - x_3^2), \\ &c_1x_1x_2 + c_2x_1x_3 + c_3x_2x_3 + c_4(x_2^2 - x_1^2) + c_5(x_1^2 - x_3^2)> \\ &-\kappa\nabla(a_1x_1^2x_2 + a_2x_1^2x_3 + a_3x_1x_2x_3 + a_4x_1x_2^2 - a_5x_1^3 + a_5x_1^3 - a_5x_1x_3^3 \end{aligned}\right|_{r \lessgtr a}$$

$$+b_1x_1x_3^2+ b_2x_1x_2x_3+ b_3x_2^2x_3+ b_4x_2^3-b_4x_2x_1^2 +b_5x_2x_1^2 +b_5x_2x_3^2 +$$

$$+c_1x_1x_2x_3+c_2x_1x_3^2+c_3x_2x_3^2+c_4x_3x_2^2-c_4x_3x_1^2+c_5x_1^2x_3- c_5x_1^3). \quad \text{(II.4)}$$

To calculate the traction vector on r = a associated with (II.4), we use the traction components formula, Love (1927):

$$a\underset{\sim}{\phi}_i/\mu = \lambda/\mu x_i\Delta+ \partial\chi/\partial x_i + x_i\partial h_i/\partial x_i - h_i;$$
$$\chi = x_1h_1+ x_2h_2+x_3h_3. \quad \text{(II.5)}$$

First we compute the local dilatation:

$$\Delta=(-6a_4+6a_5+3b+3c_2)x_1+ (3a_1+ 6b_4+ 3c_3)x_2$$
$$+(3a_2+3b_3-6c_5)x_3. \quad \text{(II.6)}$$

Also

$$\chi =4 a_1x_1^2x_2+4a_2x_1^2x_3+4a_3x_1x_2x_3+4a_4x_1x_2^3 -4a_2x_1^3$$
$$+4a_5x_1^3-4a_5x_1x_3^2+4b_1x_1x_2^2+4b_2x_1x_2x_3+4b_3x_2^2x_3+4b_4x_2^3$$
$$-4b_4x_2x_1^2 +4b_5x_2x_1^2-4b_5x_2x_3^2+4c_1x_1x_2x_3+4c_3x_2x_3^2+2c_4x_2^2x_3$$
$$-4c_4x_1^2x_3+4c_5x_1^2x_3-4c_5x_3^3. \quad \text{(II.7)}$$

$$\partial\chi/\partial x_1=(8a_1-8b_4+8b_5)x_1x_2+(8a_2-8c_4+8c_5)x_1x_3$$
$$+(4a_3+4b_2+4c_1)x_2x_3+(-12a_4+12a_5)x_1^2$$
$$+(4a_4+4b_1)x_2^2+(-4a_5+4c_2)x_3^2. \quad \text{(II.8)}$$

$$\partial\chi/\partial x_2=(6a_4+6b_1)x_1x_2+(4a_3+4b_2+4c_1)x_1x_3$$
$$+(8b_3+6c_4)x_2x_3+(4a_1-4b_4+4b_5)x_1^2$$
$$+ 12b_4x_2^2 + (-4b_5+4c_3)x_3^2. \quad \text{(II.9)}$$

$$\partial X/\partial x_3 = (4a_3 + 4b_2 + 4c_1)x_1x_2 + (-8a_5 + 8c_2)x_1x_3$$

$$+ (-8b_5 + 8c_3)x_2x_3 + (4a_2 - 4c_4 + 4c_5)x_1^2$$

$$+ (4b_3 + 4c_4)x_2^2 - 12c_5x_3^2 . \qquad \text{(II.10)}$$

Finally

$$x_1\partial h_1/\partial x_1 - h_1 = (-2b_4 + 3a_1 + 2b_5)x_1x_2 + (-2c_4 + 2c_5 + 3a_2)x_1x_3$$

$$+ (2a_3 + b_2 + c_1)x_2x_3 + (-4a_4 + 4a_5)x_1^2$$

$$+ b_1x_2^2 + (-2a_5 + c_2)x_3^2 . \qquad \text{(II.11)}$$

$$x_2\partial h_2/\partial x_2 - h_2 = (2a_4 + 3b_1)x_1x_2 + (c_1 + a_3 + 2b_2)x_1x_3$$

$$+ (2c_4 + 3b_3)x_2x_3 + (a_1 - 2b_4 + 2b_5)x_1^2$$

$$+ 4b_4x_2^2 + (-2b_5 + c_3)x_3^2 , \qquad \text{(II.12)}$$

$$x_3\partial h_3/\partial x_3 - h_3 = (2c_1 + a_3 + b_2)x_1x_2 + (3c_2 - 2a_5)x_1x_3$$

$$+ (3c_3 - 2b_5)x_2x_3 + (a_2 - 2c_4 + 2c_5)x_1^2$$

$$+ (b_3 + 2c_4)x_2^2 - 3c_5x_3^2 . \qquad \text{(II.13)}$$

Now substituting from (II.6),(II.8) and (II.11) into (II.5) we find .

$$a\phi_1^*/\mu = [\lambda/\lambda+2\mu(a_1+2b_4+c_3)+(3a_1-2b_4+2b_5)-\kappa(8a_1-8b_4+8b_4)]x_1x_2$$

$$+[\lambda/\lambda+2\mu(a_2+b_4+2c_5)\ +(3a_2-2c_4+2c_5)-\ (8a_2-8c_4+8c_5)]x_1x_3$$

$$+[(2a_3+b_2+c_1)-\ \kappa(4a_3+4b_2+4c_1)]x_2x_3$$

$$+[\lambda/\lambda+2\mu(-2a_3+2a_5+b_1+c_2)\ +(-4a_4+4a_5)-\kappa(-12a_4+12a_5)]x_1^2$$

$$+[(2a_4+b_1)-\kappa(4a_4+4b_1)]x_2^2+[(-2a_5+c_2)-\kappa(-4a_5+4c_2)]x_3^2. \quad (II.14)$$

$$a\phi_2^*/\mu = [\lambda/\lambda+2\mu(-2a_4+2a_5b_1+c_2)\ +(2a_4+3b_1)-\kappa(6a_4+6b_1)]x_1x_2$$

$$+[\lambda/\lambda+2\mu(a_2+b_3-2c_5)+(3b_3+2c_4)-\kappa(8b_3+8c_4)]x_2x_3$$

$$+[(a_3+2b_2c_1)-\kappa(4a_3+4b_2+4c_1)]x_1x_3$$

$$+[(a_1-2b_4+2b_5)-\kappa(4a_1-4b_4+4B_5)]x_1^2$$

$$+[\lambda/\lambda+2\mu(a_1+2b_4+c_3)+4b_4-12\ b_4]x_2^2$$

$$+[(-2b_5+c_3)\ -\kappa(-4b_5+4c_3)]x_3^2. \quad (II.15)$$

$$a\phi_3^*/\mu = [(a_3+b_2+2c_1)-\ \kappa(4a_3+4b_2+c_1)]x_1x_2$$

$$+[\lambda/\lambda+2\mu(-2a_4+2a_5+b_1+c_2)\ -2a_5+3c_2\ -\kappa(-8a_5+8c_2)]x_1x_3$$

$$+[\lambda/\lambda+2\mu(a_1+2b_4+c_3)+(-2b_5+3c_3)-\kappa(-8b_5+8c_3)]x_2x_3$$

$$+[(a_2-2c_4+2c_5)-\kappa(4a_2-4c_4+4c_5)]x_1^2$$

$$+[\ (b_3+2c_4)-\kappa(4b_3+4c_3)]x_2^2$$

$$+[\lambda/\lambda+2\mu(a_2+b_3-2c_5)-4c_5+12\kappa c_5]x_3^2. \qquad \text{(II.16)}$$

Now the three components of (II.1) are (II.14),(II.15)and (II.16) respectively which give 15 unknowns in 15 equations. By solving these 15 equations we get:

$$a_1=a_3=a_4=a_5=0, \quad b_1=b_2=b_4=b_5=0, \quad c_1=c_2=c_3=0,$$

$$a_2=b_3=3(\lambda+2\mu)/(3\lambda+2\mu), \quad c_5=-2(\lambda+2\mu)/(3\lambda+2\mu)=2c_4 \qquad \text{(II.17)}$$

Substituting from (II.17) into (II.3) gives:

$$\left.\begin{aligned}
h_1 &= -6\gamma(\lambda+2\mu)/a^4(3\lambda+2\mu)x_1x_3 \\
h_2 &= -6\gamma(\lambda+2\mu)/a^4(3\lambda+2\mu)x_2x_3 \\
h_3 &= -2\gamma(\lambda+2\mu)/a^4(3\lambda+2\mu)(-x_1^2-x_2^2+2x_3^2) \\
f &= 0
\end{aligned}\right] \qquad \text{(II.18)}$$

yielding the boundary displacement field:

$$\Phi = \left[2\gamma(\lambda + 4\mu)/a^3(3\lambda + 2\mu)\right]\left\langle x_1 x_3,\quad x_2 x_3,\right.$$

$$-\left[(2\lambda + 3\mu)/(\lambda + 4\mu)\right]\left[(x_1^2 + x_2^2)\right]$$

$$\left. + \left[(\mu - \lambda)/(\lambda + 4\mu)\right] x_3^2 \right\rangle . \qquad \text{(II.19)}$$

An arbitrary rigid-body displacement can be superposed on (II.19).

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THE DERIVATION OF THE EQUATION (7.1.4)

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$$\int_{\partial B} \underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y}).\underset{\sim}{\sigma}(\underset{\sim}{y})dy = \int_0^{\pi}\int_0^{2\pi} 1-\kappa/\mu\rho \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \mu\gamma/4\pi\kappa a^2 \end{bmatrix} a^2\sin\theta d\theta d\psi$$

$$+\int_0^{\pi}\int_0^{2\pi} \kappa/\mu\rho^3 \begin{bmatrix} a^2\sin^2\theta\cos^2\psi & a^2\sin^2\theta\sin\psi\cos\psi & -a\sin\theta\cos\psi(z-a\cos\theta) \\ a^2\sin^2\theta\cos\psi\sin\psi & a^2\sin^2\theta\cos\psi & -a\sin\theta\sin\psi(z-a\cos\theta) \\ a\sin\theta\cos\psi(z-a\cos\theta) & a\sin\theta\cos\psi(z-a\cos\theta) & (z-a\cos\theta)^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \mu\gamma/4\pi\kappa a^2 \end{bmatrix} a^2\sin\theta d\theta d\psi ,$$

$$z > a$$

$$= (\mu\gamma a^2/4\pi\kappa a^2)\int_0^{\pi}\int_0^{2\pi}\Big\{ \big< -\kappa a\sin\theta\cos\psi(z-a\cos\theta)/\mu\rho^3 ,\ -\kappa a\sin\theta\sin\psi(z-a\cos\theta)/\mu\rho^3 ,$$

$$(1-\kappa/\mu\rho + \kappa(z-a\cos\theta)^2/\mu\rho^3 \big> \Big\} \sin\theta d\theta d\psi \qquad (1)$$

It is clear that the first and second integrals components are zero, therefore (1) gives:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})\cdot\underline{\sigma}(\underline{y})\,dy = (\mu\gamma/4\pi K)\left\langle 0\ ,\ 0\ ,\ \int_0^{\pi}\int_0^{2\pi}\left\{(1-\kappa)/\mu\rho + \kappa(z-a\cos\theta)^2/\mu\rho^3\right\}\sin\theta\, d\theta\, d\psi\right\rangle$$

$$=(\mu\gamma/2K)\left\langle 0\ ,\ 0\ ,\ \int_0^{\pi}\left[\left\{(1-\kappa)/(z^2+a^2-2az\cos\theta)^{\frac{1}{2}}+\right.\right.\right.$$

$$\left.\left.\left.+\kappa(z^2-2az\cos\theta+a^2\cos^2\theta)/(z^2+a^2-2az\cos\theta)^{\frac{3}{2}}\right\}\sin\theta\,d\theta\right]\right\rangle$$

$$=(\gamma/2K)\left\langle 0\ ,\ 0\ ,\ \int_0^{\pi}\left[\left\{(1-\kappa)/(z^2+a^2-2az\cos\theta)^{\frac{1}{2}}+\right.\right.\right.$$

$$\left.\left.\left.+\kappa\left[(z^2-2az\cos\theta+a^2)+(a^2\cos^2\theta-a^2)\right]/(z^2+a^2-2az\cos\theta)^{\frac{3}{2}}\right\}\sin\theta\,d\theta\right]\right\rangle$$

$$=(\gamma/2K)\left\langle 0\ ,\ 0\ ,\ \int_0^{\pi}\left\{1/(z^2+a^2-2az\cos\theta)^{\frac{1}{2}}-\kappa a^2(1-\cos^2\theta)/(z^2+a^2-2az\cos\theta)^{3/2}\right\}\sin\theta\,d\theta\right\rangle,$$

$$z>a \qquad (2)$$

Put $\cos\theta = u$ $\sin\theta d\theta = -du$, $-1\leq u\leq 1$,

in which case (2) becomes:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})\cdot\underline{\sigma}(\underline{y})dy = (\gamma/2\kappa)\left\langle 0 \;,\; 0 \;,\; \int_{1}^{-1} - 1/(z^2+a^2-2azu)^{\frac{1}{2}}\, du \right\rangle +$$

$$+(\gamma/2\kappa)\left\langle 0 \;,\; 0 \;,\; \int_{1}^{-1} \kappa a^2\left\{(1-u^2)/(z^2+a^2-2azu)^{3/2}\right\}du\right\rangle;$$

$z > a \;, a/z = v,$

$$= (\gamma/2\kappa)\left\langle 0 \;,\; 0 \;,\; \int_{1}^{-1} -\left\{1/z(1+v^2-2uv)^{\frac{1}{2}}\right\}du\right\rangle$$

$$+(\gamma/2\kappa)\left\langle 0 \;,\; 0 \;,\; \int_{1}^{-1} \kappa a^2\left\{(1-u^2)/z^3(1+v^2-2uv)^{\frac{3}{2}}\right\} du\right\rangle$$

$$= (\gamma/2\kappa)\left\langle 0 \;,\; 0 \;,\; (-1/z)I_1 + (\kappa a^2/z^3)I_2 \right\rangle, \qquad (3)$$

where

$$I_1 = \int_{1}^{-1}\left\{1/(1+v^2-2uv)^{\frac{1}{2}}\right\}du = -(1/v)\left[(1+v^2-2uv)^{\frac{1}{2}}\right]_{1}^{-1}$$

$$= -(1/v)\left[(1+v)-(1-v)\right] = -2 \;, \qquad (4)$$

$$I_2 = \int_{1}^{-1}\left\{(1-u^2)/(1+v^2-2uv)^{3/2}\right\}du = \left[((1-u^2)/v)(1+v^2-2uv)^{-\frac{1}{2}}\right]_{1}^{-1} + \int_{1}^{-1}\left\{2u/v(1+v^2-2uv)^{\frac{1}{2}}\right\}du$$

$$= 0 - \left[2u(1+v^2-2uv)^{\frac{1}{2}}/v^2\right]_{1}^{-1} + \int_{1}^{-1}\left\{(2/v^2)(1+v^2-2uv)^{\frac{1}{2}}\right\}du$$

$$= 4/v^2 - (2/3v^3)\left[(1+v)^2-(1-v^2)^3\right] = -4/3 \qquad (5)$$

Substituting from (4) and (5) into (3) gives:

$$\int_{\partial B} \underline{\underline{g}}(\underline{x},\underline{y})\cdot\underline{\sigma}(\underline{y})dy = \gamma/2\kappa\left\langle 0 \;,\; 0 \;,\; 2/z - 4\kappa a^2/3z^3\right\rangle$$

$$= \gamma\left\langle 0 \;,\; 0 \;,\; 1/\kappa z - 2a^2/3z^3\right\rangle, \quad z > a \;. \qquad (6)$$

Vector Potential Theory

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ABSTRACT

Linear elastostatic displacement fields may be represented by vector potentials analogous to the representation of harmonic functions by scalar potentials. We introduce fields which vanish at infinity and provide rigid-body displacements on the surface of a spherical cavity in an infinite linear elastic continuum. The asymptotic identity of the vector potential with the field is demonstrated for each problem. This work carries an interesting significance in the theory of vector boundary integral equations.

SIMPLER-LAYER POTENTIALS

It is interesting to represent elastostatic displacement fields by vector potentials. These have properties closely analogous to those of scalar potentials. Thus, given a closed Liapunov-smooth surface ∂B, let $\sigma(\underset{\sim}{q})$ be the source density at a point $\underset{\sim}{q} \subset \partial B$. If dq denotes the surface element at $\underset{\sim}{q} \subset \partial B$, then $\sigma(\underset{\sim}{q})dq$ is the source strength at $\underset{\sim}{q}$. This provides a potential $g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq$ at any point $\underset{\sim}{p}$ inside or outside ∂B or lying on ∂B. Here

$$g(\underset{\sim}{p},\underset{\sim}{q}) = g(\underset{\sim}{q},\underset{\sim}{p}) = |\underset{\sim}{p}-\underset{\sim}{q}|^{-1} \quad , \tag{1}$$

denotes the Newtonian potential at $\underset{\sim}{p}$ generated by a Unit source at $\underset{\sim}{q}$ or vice versa. Superposing the contributions from all over ∂B, we obtain the simple-layer potential,

$$V(\underset{\sim}{p}) = \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \; \underset{\sim}{p} \subset B_i \quad , \quad B_e \subset \partial B \tag{2}$$

where B_i denotes the interior domain bounded by ∂B and B_e

denotes the infinite exterior domain internally bounded by ∂B. If σ is Hölder-continuous on δB, then $V(\underset{\sim}{p})$ has properties which may be summarized as follows, Kellogg (1929):

(i) V is continuous and differentiable to any order in B_i, B_e. Also

$$\frac{\partial V(\underset{\sim}{p})}{\partial p_1} = \int_{\partial B} \frac{\partial}{\partial p_1} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \; \underset{\sim}{p} \subset B_i, B_e \tag{3}$$

where $\underset{\sim}{p} = \langle p_1, p_2, p_3 \rangle$. In some equations we write $\underset{\sim}{p} = \langle x_1, x_2, x_3 \rangle$.

(ii) V satisfies Laplace's equation in B_i, B_e i.e.

$$\nabla^2 V(\underset{\sim}{p}) = \int_{\partial B} \nabla^2 g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = 0; \; \underset{\sim}{p} \subset B_i, B_e \tag{4}$$

since $\nabla^2 g = 0$, $\underset{\sim}{p} \neq \underset{\sim}{q}$. Accordingly, V is a harmonic function everywhere except at δB.

(iii) $$V(\underset{\sim}{p}) = |\underset{\sim}{p}|^{-1} \int_{\partial B} \sigma(\underset{\sim}{q})dq + |\underset{\sim}{p}|^{-3} \int_{\partial B} (\underset{\sim}{p}.\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + 0(|\underset{\sim}{p}|^3), \tag{5}$$

as $|\underset{\sim}{p}| \to \infty$

Therefore V is a regular harmonic function in B_e.

(iv) $V(\underset{\sim}{p})$ exists at every $\underset{\sim}{p} \subset \partial B$, and it is continuous at $\underset{\sim}{p}$ with respect to its neighbouring values in B_i, B_e, i.e.

$$\begin{aligned} V(\underset{\sim}{p}_i) &\to V(\underset{\sim}{p}) \text{ as } \underset{\sim}{p}_i \to \underset{\sim}{p}, \; \underset{\sim}{p}_i \subset B_i \\ V(\underset{\sim}{p}_e) &\to V(\underset{\sim}{p}) \text{ as } \underset{\sim}{p}_e \to \underset{\sim}{p}, \; \underset{\sim}{p}_e \subset B_e \end{aligned} \tag{6}$$

(v) $V(\underset{\sim}{p})$ is continuous and differentiable on ∂B. Also, in line with (3)

$$\frac{\partial V(\underset{\sim}{p})}{\partial t} = \int_{\partial B} \frac{\partial}{\partial t} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \; \underset{\sim}{p} \subset \partial B, \tag{7}$$

where $\frac{\partial}{\partial t}$ denotes differentiation along any tangential direction to ∂B at $\underset{\sim}{p}$.

(vi) $V(\underset{\sim}{p})$ has two formally distinct normal derivatives V'_e, V'_i at $\underset{\sim}{p} \subset \partial B$ pointing into B_e, B_i respectively. These may be constructed by writing

$$V'_e(\underset{\sim}{p}) = \int_{\partial B} g'_e(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}) \; ; \; \underset{\sim}{p} \subset \partial B. \tag{8}$$

$$V'_i(\underset{\sim}{p}) = \int_{\partial B} g'_i(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}) \; ; \; \underset{\sim}{p} \subset \partial B, \tag{9}$$

where $g'_e(\underset{\sim}{p},\underset{\sim}{q})$ denotes the exterior normal derivative of g at $\underset{\sim}{p}$ keeping $\underset{\sim}{q}$ fixed, and similarly for g'_i. Since

$$g'_i(\underset{\sim}{p},\underset{\sim}{q}) + g'_e(\underset{\sim}{p},\underset{\sim}{q}) = \underset{\sim}{0} \tag{10}$$

it follows that

$$V'_i(\underset{\sim}{p}) + V'_e(\underset{\sim}{p}) = -4\pi\sigma(\underset{\sim}{p}) \; . \tag{11}$$

According to the interior Dirichlet existence-uniqueness theorem, there exists a unique harmonic function ϕ in B_i, which assumes prescribed continuous boundary values on a closed Liapunov surface ∂B. To construct ϕ in B_i, we write

$$\phi(\underset{\sim}{p}) = \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \; \underset{\sim}{p} \subset B_i \tag{12}$$

where σ appears as a hypothetical Hölder-continuous source density to be determined. An effective way forward is to note that both sides remain continuous at ∂B, so yielding the boundary relation

$$\int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = \phi(\underset{\sim}{p}) \; ; \; \underset{\sim}{p} \subset \partial B \; . \tag{13}$$

This may be viewed as a Fredholm integral equation of the first kind for σ in terms of ϕ on δB, with a unique solution which enables us to generate ϕ throughout B_i from (12).

Similarly, according to the exterior Dirichlet existence uniqueness theorem, there exists a unique regular harmonic function ϕ in B_e which assumes prescribed continuous boundary values on ∂B. Clearly ϕ may be constructed by solving equation (13) as before and utilising σ to generate the simple-layer potential

$$\phi(\underset{\sim}{p}) = \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \; \underset{\sim}{p} \subset B_e \qquad (14)$$

Of course

$$\sigma(\underset{\sim}{p}) = -\frac{1}{4\pi}\left[\phi_i'(\underset{\sim}{p}) + \phi_e'(\underset{\sim}{p})\right] , \qquad (15)$$

in line with (11). Accordingly, if ϕ is available both in B_i and B_e, then σ is immediately known from (15) so yielding the identities:

$$\phi(\underset{\sim}{p}) \equiv -\frac{1}{4\pi}\int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})[\phi_i'(\underset{\sim}{q}) + \phi_e'(\underset{\sim}{q})]dq, \; p \subset B_i \qquad (16)$$

$$-\frac{1}{4\pi}\int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})[\phi_i'(\underset{\sim}{q}) + \phi_e'(\underset{\sim}{q})]dq \equiv \phi(\underset{\sim}{p}) \; ; \; \underset{\sim}{p} \subset \partial B \qquad (17)$$

$$\phi(p) \equiv -\frac{1}{4\pi}\int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})[\phi_i'(\underset{\sim}{q}) + \phi_e'(\underset{\sim}{q})]dq \; ; \; \underset{\sim}{p} \subset B_e \qquad (18)$$

In place of (12), (13), (14) respectively.

VECTOR POTENTIAL THEORY

Classical linear elastostatics may be formulated by a vector potential theory which closely parallels scalar potential theory. It would, indeed be advantageous to employ the same symbolism in each theory, its interpretation depending on the context. Thus the scalar potential ϕ becomes the elastostatic displacement vector $\underset{\sim}{\phi}$. The normal derivative ϕ' becomes the traction vector $\underset{\sim}{\phi}^*$ associated with $\underset{\sim}{\phi}$. The Newtonian unit-source potential $g(\underset{\sim}{p},\underset{\sim}{q})$ becomes the fundamental displacement dyadic of the medium. More precisely, in this context we mean that

$$\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}) = \begin{bmatrix} g(\underset{\sim}{p}_1,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_1,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_1,\underset{\sim}{q}_3) \\ g(\underset{\sim}{p}_2,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_2,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_2,\underset{\sim}{q}_3) \\ g(\underset{\sim}{p}_3,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_3,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_3,\underset{\sim}{q}_3) \end{bmatrix} \qquad (19)$$

where $g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\beta)$ signifies the displacement component in the α-direction at $\underset{\sim}{p}$ generated by a unit point-force in the β-direction at $\underset{\sim}{q}$. Clearly column 1 defines the displacement vector at $\underset{\sim}{p}$ generated by a unit force acting in the 1-direction at $\underset{\sim}{q}$, etc. By virtue of $g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\beta) = g(\underset{\sim}{q}_\beta,\underset{\sim}{p}_\alpha)$, we see that row 1 defines the displacements vector at $\underset{\sim}{q}$ generated by a unit point-force acting in the 1-direction at $\underset{\sim}{p}$, etc. In the isotropic continuum,

$$\begin{aligned} g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\beta) &= \frac{(1-\kappa)}{\mu R}\delta_{\alpha\beta} + \frac{\kappa}{\mu R}\frac{\partial R}{\partial p_\alpha}\frac{\partial R}{\partial p_\beta} \\ &= \frac{(1-\kappa)}{\mu R}\delta_{\alpha\beta} + \frac{\kappa}{\mu}\frac{(p_\alpha - q_\alpha)(p_\beta - q_\beta)}{R^3} \qquad (20) \\ &= \frac{1}{\mu R}\delta_{\alpha\beta} - \frac{\kappa}{\mu}\frac{\partial^2 R}{\partial p_\alpha \partial p_\beta} \quad ; \; \alpha, \beta = 1, 2, 3 \end{aligned}$$

where $R = |\underset{\sim}{p}-\underset{\sim}{q}|$. This is Kelvin's solution expressed in tensor notation, Love (1927).

Corresponding with $g'(\underset{\sim}{p},\underset{\sim}{q})$ we construct the fundamental traction dyadic of the medium, i.e.

$$\underset{\approx}{g}^*(\underset{\sim}{p},\underset{\sim}{q}) = \begin{bmatrix} g^*(\underset{\sim}{p}_1,\underset{\sim}{q}_1) & g^*(\underset{\sim}{p}_1,\underset{\sim}{q}_2) & g^*(\underset{\sim}{p}_1,\underset{\sim}{q}_3) \\ g^*(\underset{\sim}{p}_2,\underset{\sim}{q}_1) & g^*(\underset{\sim}{p}_2,\underset{\sim}{q}_2) & g^*(\underset{\sim}{p}_2,\underset{\sim}{q}_3) \\ g^*(\underset{\sim}{p}_3,\underset{\sim}{q}_1) & g^*(\underset{\sim}{p}_3,\underset{\sim}{q}_2) & g^*(\underset{\sim}{p}_3,\underset{\sim}{q}_3) \end{bmatrix} \qquad (21)$$

where $g^*(p_\alpha,q_\beta)$ signifies the traction component in the α-direction at $\underset{\sim}{p}$ generated by a unit point-force acting in the β-direction at $\underset{\sim}{q}$. Clearly column 1 defines the traction vector at $\underset{\sim}{p}$ generated by a unit point-force acting in the 1-direction at $\underset{\sim}{q}$, etc.

Finally, corresponding with $g(\underset{\sim}{p},\underset{\sim}{q})'$ we construct the traction dyadic

$$\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q})^* = \begin{bmatrix} g(\underset{\sim}{p}_1,\underset{\sim}{q}_1)^* & g(\underset{\sim}{p}_1,\underset{\sim}{q}_2)^* & g(\underset{\sim}{p}_1,\underset{\sim}{q}_3)^* \\ g(\underset{\sim}{p}_2,\underset{\sim}{q}_1)^* & g(\underset{\sim}{p}_2,\underset{\sim}{q}_2)^* & g(\underset{\sim}{p}_2,\underset{\sim}{q}_3)^* \\ g(\underset{\sim}{p}_3,\underset{\sim}{q}_1)^* & g(\underset{\sim}{p}_3,\underset{\sim}{q}_2)^* & g(\underset{\sim}{p}_3,\underset{\sim}{q}_3)^* \end{bmatrix} \qquad (22)$$

where row 1 defines the traction vector at $\underset{\sim}{q}$ generated by a unit point-force acting in the 1-direction at $\underset{\sim}{p}$, etc.

The simple-source density σ now becomes a vector simple-source density $\underset{\sim}{\sigma} = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$. This enables us to construct a vector simple-layer potential corresponding to (2), viz

$$\underset{\sim}{V}(\underset{\sim}{p}) = \int_{\partial B} \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}).\underset{\sim}{\sigma}(\underset{\sim}{q})dq = \int_{\partial B} \underset{\sim}{\sigma}(\underset{\sim}{q}).\underset{\approx}{g}(\underset{\sim}{q},\underset{\sim}{p})dq \, , \tag{23}$$

with components

$$V_\alpha(\underset{\sim}{p}) = \int_{\partial B} g(p_\alpha, q_\beta)\sigma_\beta(\underset{\sim}{q})dq \; ; \; \alpha,\beta = 1, 2, 3 \; . \tag{24}$$

This has properties at ∂B entirely analogous to those of the scalar simple-source potential, e.g. formulae (8), (9) may be read as traction formula, and it defines an elastostatic displacement field for any choice of $\underset{\sim}{p}$. These properties have been proved by Kupradze (1965) for the linear isotropic elastic continuum, but we may conjecture that they also hold for the general linear anisotropic elastic continuum.

SPHERE PROBLEMS

We now construct two distinct elastostatic fields external to a spherical boundary, and we show how these may be represented by vector potentials. First, we seek a field $\underset{\sim}{\phi}$ with the properties:-

(i) $\underset{\sim}{\phi} = 0\,(r^{-1})$ as $r \to \infty$,

(ii) $\underset{\sim}{\phi} = \langle 0, 0, t_3 \rangle$, i.e. a rigid-body translation of magnitude t_3 on the boundary $r = a$ of a spherical cavity within an infinite isotropic linear elastic continuum.

Utilising the Papkovitch-Neuber representation (Jaswon and Symm, 1977), the required field is

$$\underset{\sim}{\phi} = \langle 0, 0, \frac{\alpha}{r} \rangle - \kappa\nabla\left(\frac{\alpha x_3}{r} + \beta\frac{\partial r^{-1}}{\partial x_3}\right) \; ; \; r \geqslant a \tag{25}$$

where $\kappa = \dfrac{1}{4(1-\nu)}$; ν = Poisson's ratio ($0 < \nu \leqslant \frac{1}{2}$)

$$\text{and} \quad \alpha = \frac{3at_3}{3-2\kappa} \, , \; \beta = \frac{a^2}{3}\alpha = \frac{a^3 t_3}{3-2\kappa} \; . \tag{26}$$

In terms of components:-

$$\left.\begin{aligned} \phi_1 &= \gamma\left(\frac{x_1x_3}{r^3} - \frac{a^2x_1x_3}{r^5}\right) \\ \phi_2 &= \gamma\left(\frac{x_2x_3}{r^3} - \frac{a^2x_2x_3}{r^5}\right) \\ \phi_3 &= \gamma\left(\frac{1-\kappa}{\kappa c} + \frac{x_3^{\,2}}{r^3} + \frac{a^2}{3r^3} - \frac{a^2x_3^{\,2}}{r^5}\right) \end{aligned}\right\} \tag{27}$$

where $\gamma = \dfrac{3a\kappa}{3-2\kappa}t_3$. This provides a traction vector

$$\underset{\sim}{\phi}^*_e = \langle 0, 0, -\frac{\mu\gamma}{\kappa a^2} \rangle \quad \text{over } r = a \tag{28}$$

yielding a resultant force

$$\int_{\partial B} \underset{\sim}{\phi}^*_e dq = \langle 0, 0, \frac{-4\pi\mu\gamma}{\kappa} \rangle \tag{29}$$

and a resultant moment

$$\int_{\partial B} \underset{\sim}{q}\wedge\underset{\sim}{\phi}^*_e dq = \int_{\partial B} \frac{\mu\gamma}{\kappa a^2}\langle -x_2, x_1, 0 \rangle dq = \langle 0, 0, 0 \rangle \, , \tag{30}$$

acting on the boundary.

We also seek a field $\underset{\sim}{\phi}$ with the properties:-

(i) $\underset{\sim}{\phi} = 0(r^{-1})$ as $r \to \infty$

(ii) $\underset{\sim}{\phi} = \langle 0, 0, w_3 \rangle \wedge \langle x_1, x_2, x_3 \rangle = \langle -w_3x_2, w_3x_1, 0 \rangle$, i.e. rigid-body rotation of magnitude w_3 on $r = a$.

The required field is readily seen to be

$$\underset{\sim}{\phi} = \langle \frac{-a^3}{r^3}w_3x_2 \, , \frac{a^3}{r^3}w_3x_1, 0 \rangle = \frac{a^3}{r^3}\langle -w_3x_2, w_3x_1, 0 \rangle \; ; \; r \geqslant a \tag{31}$$

which provides a traction vector

$$\underset{\sim}{\phi}^*_e = \frac{3\mu w_3}{a}\langle x_2, -x_1, 0 \rangle \text{ over } r = a \, , \tag{32}$$

yielding a resultant force

$$\int_{\partial B} \underset{\sim}{\phi}_e^*(\underset{\sim}{q})dq = \langle 0, 0, 0 \rangle \tag{33}$$

and a resultant couple

$$\int_{\partial B} \underset{\sim}{q}\wedge\underset{\sim}{\phi}_e^*(q)dq = \int_{\partial B} \frac{3\mu w_3}{a} \langle x_1x_3, x_2x_3, -x_1^2-x_2^2 \rangle dq$$

$$= \langle 0, 0, -8\pi a^3\mu w_3 \rangle \tag{34}$$

acting on the boundary.

The above boundary displacements are particular cases of the general rigid-body displacement

$$\underset{\sim}{\phi} = \underset{\sim}{t} + \underset{\sim}{w} \wedge \underset{\sim}{r} \tag{35}$$

where $\underset{\sim}{t}$, $\underset{\sim}{w}$ are constant vectors.

It is convenient to break this down into the six independent vectors:-

$$\underset{\sim}{\mu}_1 = \langle 1, 0, 0 \rangle,\ \underset{\sim}{\mu}_2 = \langle 0, 1, 0 \rangle,\ \underset{\sim}{\mu}_3 = \langle 0, 0, 1 \rangle$$

$$\underset{\sim}{\mu}_4 = \langle 1, 0, 0 \rangle \wedge x,\ \underset{\sim}{\mu}_5 = \langle 0, 1, 0 \rangle \wedge \underset{\sim}{r},\ \underset{\sim}{\mu} = \langle 0, 0, 1 \rangle \wedge \underset{\sim}{r} \tag{36}$$

The fields corresponding with $\underset{\sim}{\mu}_3$, $\underset{\sim}{\mu}_6$ have been determined and therefore by symmetry we may immediately write down the fields corresponding with $\underset{\sim}{\mu}_s$; s = 1, 2, 3, 4, 5 .

INTEGRAL REPRESENTATIONS: TRANSLATION PROBLEM

Within the interior domain $r \leqslant a$ there exists mathematically a field $\underset{\sim}{\phi} = \langle 0, 0, t_3 \rangle$ which assumes the same boundary values as the exterior field (27), i.e. $\langle 0, 0, t_3 \rangle$ on $r = a$. This interior field yields the traction vector $\underset{\sim}{\phi}_i^* = 0$. Accordingly, from (15) both the interior and exterior fields may be generated from the vector source density.

$$\underset{\sim}{\sigma} = -\frac{1}{4\pi}(\underset{\sim}{\phi}_e^* + \underset{\sim}{\phi}_i^*) = -\frac{1}{4\pi}\underset{\sim}{\phi}_e^* = \langle 0, 0, \frac{\mu\gamma}{4\pi\kappa a^2} \rangle \tag{37}$$

on bearing in mind (28). Substituting for $\underset{\sim}{\sigma}$ into the integral

$$\underset{\sim}{V}(p) = \int_{\partial B} \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}).\underset{\sim}{\sigma}(q)dq . \tag{38}$$

we should identically obtain $\underset{\sim}{\phi}$ for any choice of $\underset{\sim}{p}$. In practice it is very difficult to evaluate the integral exactly, but its asymptotic behaviour can be examined as follows. Note that

$$\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}) \to \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0}) \text{ as } \underset{\sim}{p} \to \infty \tag{39}$$

so that

$$\int_{\partial B} \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}).\underset{\sim}{\sigma}(\underset{\sim}{q})dq \to \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0}).\int_{\partial B} \underset{\sim}{\sigma}(q)dq \text{ as } p \to \infty. \tag{40}$$

Now from (37)

$$\int_{\partial B} \underset{\sim}{\sigma}(\underset{\sim}{q})dq = \langle \int_{\partial B}\sigma_1(\underset{\sim}{q})dq, \int_{\partial B}\sigma_2(\underset{\sim}{q})dq, \int_{\partial B}\sigma_3(\underset{\sim}{q})dq \rangle = \langle 0, 0, \frac{\mu\gamma}{\kappa} \rangle, \tag{41}$$

so that (40) has the components

$$\left.\begin{aligned}
V_1(\underset{\sim}{p}) &= \sum_{j=1}^{3} g(\underset{\sim}{p}_1,\underset{\sim}{0}_j)\int_{\partial B}\sigma_j(\underset{\sim}{q})dq = g(\underset{\sim}{p}_1,\underset{\sim}{0}_3)\frac{\mu\gamma}{\kappa} = \frac{\gamma x_1x_2}{r^3}\\
V_2(\underset{\sim}{p}) &= \sum_{j=1}^{3} g(\underset{\sim}{p}_2,\underset{\sim}{0}_j)\int_{\partial B}\sigma_j(\underset{\sim}{q})dq = g(\underset{\sim}{p}_2,\underset{\sim}{0}_3)\frac{\mu\gamma}{\kappa} = \frac{\gamma x_2x_3}{r^3}\\
V_3(\underset{\sim}{p}) &= \sum_{j=1}^{3} g(\underset{\sim}{p}_3,\underset{\sim}{0}_j)\int_{\partial B}\sigma_j(\underset{\sim}{q})dq = g(\underset{\sim}{p}_3,\underset{\sim}{0}_3)\frac{\mu\gamma}{\kappa} = \frac{\gamma(1-\kappa)}{\kappa r} + \frac{\gamma x_3^2}{r^3}
\end{aligned}\right\} \tag{42}$$

which agree exactly with the asymptotic components of $\underset{\sim}{\phi}$ as given by (27). We remark that the integral (38) can be evaluated exactly at the centre of the sphere, i.e. putting $\underset{\sim}{p} = 0$ in the expression (20), and we find $\underset{\sim}{V} = \langle 0, 0, t_3 \rangle$ as expected.

Within $r \leqslant a$ there exists a field $\underset{\sim}{\phi} = < -w_3x_2, w_3x_1, 0 >$ which becomes identical with the exterior field (31) on $r = a$. Accordingly from (15), both fields may be generated by the vector simple-source density

$$\underset{\sim}{\sigma} = -\frac{1}{4\pi}(\underset{\sim}{\phi}^*_e + \underset{\sim}{\phi}^*_i) = \frac{-1}{4\Pi}\underset{\sim}{\phi}^*_e = \frac{-3\mu w_3}{4\Pi a} < x_2, -x_1, 0 > ;\ x_1, x_2 \subset \partial B \tag{43}$$

on bearing in mind (32) and $\underset{\sim}{\phi}^*_i = 0$. If so, by symmetry,

$$\int_{\partial B} \underset{\sim}{\sigma}(\underset{\sim}{q})dq = < 0, 0, 0 > \tag{44}$$

showing that the first-order asymptotic approximation to $\underset{\sim}{g}(\underset{\sim}{p},\underset{\sim}{q})$ gives a null result. In the second approximation we have, Jaswon and Symm (1977), Jaswon (1984),

$$\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}) = \underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0}) + \underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q})_{\underset{\sim}{q}=o} + O(|\underset{\sim}{p}|^{-3}) , \tag{45}$$

so that

$$\underset{\sim}{\phi}(\underset{\sim}{p}) \to \int_{\partial B}[\underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0})].\underset{\sim}{\sigma}(\underset{\sim}{q})dq \quad \text{as } \underset{\sim}{p} \to \infty \quad . \tag{46}$$

To evaluate the components of $[\underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0})]$:

$$[\underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0})]_{\alpha\beta} = \underset{\sim}{q}.\nabla g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\beta)_{\underset{\sim}{q}=o} = \sum_{j=1}^{3} q_j \frac{\partial}{\partial q_j} g(\underset{\sim}{p}_\alpha,\underset{\sim}{g}_\beta)_{\underset{\sim}{q}=o}$$

$$= \sum_{\alpha=1}^{3} g_j \frac{\partial}{\partial q_j}\left(\frac{1}{\mu R}\delta_{\alpha\beta} - \frac{\kappa}{\mu}\frac{\partial^2 R}{\partial p_\alpha \partial p_\beta}\right)_{q=o} ;\ R = |\underset{\sim}{p}-\underset{\sim}{q}| \ . \tag{47}$$

Accordingly

$$\left.\begin{aligned}
\phi_1(\underset{\sim}{p}) &= \int_{\partial B}\sum_{\beta=1}^{3}[\underset{\sim}{q}.\nabla\underset{=}{g}(p,0)]_{1\beta}\sigma_\beta(\underset{\sim}{q})dq = \frac{-a^3 w_3 p_2}{R^3} \\
\phi_2(\underset{\sim}{p}) &= \int_{\partial B}\sum_{\beta=1}^{3}[\underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{0})]_{2\beta}\sigma_\beta(\underset{\sim}{q})dq = \frac{a^3 w_3 p_1}{R^3}
\end{aligned}\right\} \tag{48}$$

equation continued

$$\left.\phi_3(\underset{\sim}{p}) = \int_{\beta}\sum_{\beta=1}^{3}[\underset{\sim}{q}.\nabla\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{o})]_{3\beta}\sigma_\beta(q)dq = 0 \right\} \tag{48}$$

This asymptotic field agrees exactly with ϕ as given by (31) allowing for a slight adaptation of symbols. We remark that the integral (38) can also be evaluated exactly at the centre of the sphere i.e. at $\underset{\sim}{p} = 0$:

$$\underset{\sim}{\phi}(0) = \int_{\partial B}\underset{\approx}{g}(\underset{\sim}{0},\underset{\sim}{q}).\underset{\sim}{\sigma}(\underset{\sim}{q})dq = < 0, 0, Q > \tag{49}$$

$$= < -w_3x_2, w_3x_1, 0 > \text{ at } \underset{\sim}{p} = < 0, 0, 0 > \text{ as expected.}$$

THE RIGID-BODY DISPLACEMENT FIELD

This plays an analogous role in vector potential theory (elastostatics) to that of the constant harmonic function in scalar potential theory. Thus $\underset{\sim}{\phi} = \underset{\sim}{\mu}_s$; $s = 1, 2, \ldots 6$ on ∂B implies $\underset{\sim}{\phi} = \underset{\sim}{\mu}_s$ in $B_i + \partial B$ and $\underset{\sim}{\phi}^*_i = 0$ on ∂B, in line with well known corresponding properties of the harmonic function $\phi = k$ (a constant) on ∂B. Also, given an arbitrary source-free displacement field $\underset{\sim}{\psi}$ on $B_i + \partial B$, it satisfies the boundary conditions

$$\int_{\partial B}\underset{\sim}{\psi}^*_i.\underset{\sim}{\mu}_s dq = 0 ;\quad s = 1, 2, \ldots 6 \tag{50}$$

These express the fact that the tractions associated with $\underset{\sim}{\psi}$ produce no resultant force ($s = 1, 2, 3$) and no resultant moment ($s = 4, 5, 6$) acting on ∂B, in line with the Gauss condition for the flux of a harmonic function over ∂B.

Given $\underset{\sim}{\mu}_s$ on ∂B, we may generate this by a vector source-density $\underset{\sim}{\lambda}_s$ which satisfies the vector integral equation of the first kind

$$\int_{\partial B}\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}).\underset{\sim}{\lambda}_s(\underset{\sim}{q})dq = \underset{\sim}{\mu}_s(\underset{\sim}{p}) ;\ \underset{\sim}{p} \subset \partial B \ . \tag{51}$$

Analytical solutions of (51) have been found for a spherical boundary; i.e. (28), (32) bearing in mind (15). For other boundaries, solutions can only be achieved by numerical methods.

If λ_s is available for ∂B, regarded as the internal boundary of an infinite external domain, then we may exploit the following theorem:

> given arbitrary continuous displacements ψ on ∂B, the tractions associated with ψ produce resultant forces and moments which satisfy the relations

$$-\frac{1}{4\pi}\int_{\partial B} \psi_e^* \cdot \mu_s \, dq = \int_{\partial B} \psi \cdot \lambda_s \, dq \ ; \quad s = 1, 2, \dots 6 \tag{52}$$

These may be proved by introducing a vector source density σ which satisfies the equation

$$\int_{\partial B} g(p,q) \cdot \sigma(q) \, dq = \psi(p) \ ; \quad p \subset \partial B \tag{53}$$

i.e. a generalisation of (51). Operating upon both sides of equation (53) by $\cdot\int_{\partial B} \dots \lambda_s(p)dp$, we find

$$\int_{\partial} \psi(p) \cdot \lambda_s(p) dp = \int_{\partial B} \lambda_s(p) dp \cdot \int_{\partial B} g(p,q) \cdot \sigma(q) dq$$

$$= \int_{\partial B} \lambda_s(p) \cdot g(p,q) dp \cdot \int_{\partial B} \sigma(q) dq = \int_{\partial B} \mu_s(q) \cdot \sigma(q) dq \tag{54}$$

on interchanging the order of integration (Fubini's theorem) and using (51) with p,q interchanged. The right-hand side of (54) may be written

$$-\frac{1}{4\pi}\int_{\partial B} \mu_s \cdot [\psi_i^* + \psi_e^*] dq = -\frac{1}{4\pi}\int_{\partial B} \mu_s \cdot \psi_e^* dq \tag{55}$$

by virtue of (52), so defining the force- and moment-resultants required. This proves the theorem.

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The Representation of Elastostatic Fields by Vector Potentials
(Invited contribution)

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ABSTRACT

Fields in the infinite region exterior to a spherical cavity have been represented by vector simple-layer potentials (BETECH 86). We now attempt to represent these fields by vector double-layer potentials. This poses a far more difficult problem, both in defining the sources and in ensuring acceptable behaviour at infinity. These issues do not appear to have been previously explored.

INTRODUCTION

At Betech 86[1] we introduced linear elastostatic fields on the infinite region exterior to a spherical cavity, and we represented these fields by simple-layer vector potentials. In this paper we represent fields by double-layer potentials. However the problem is now more difficult for two reasons:

1. The double-layer potential has asymptotic behaviour $O(r^{-2})$ as $r \to \infty$, whereas the regular elastostatic field generally has $O(r^{-1})$, behaviour as $r \to \infty$. Accordingly either suitable terms must be superposed upon the potential or appropriate terms must be removed from the field, before such a potential can be constructed.

2. The determination of double-layer source densities proves to be considerably more complicated than that of simple-layer source densities since, as will be explained below, the former essentially involves solving a boundary-value problem whereas the latter only involves the straightforward computation of tractions.

The plan of our paper is as follows. First we briefly summarise the Betech 86 paper. Next we introduce double-layer potentials. Then we reduce the given field to $O(r^{-3})$ behaviour as $r \to \infty$ and calculate the relevant source-density distribution. This enables us to construct the required double-layer potential and to demonstrate its exact or at least asymptotic equivalence with the reduced field.

Single-potential representations of elastostatic fields, first proposed by Kupradse[2], provide interesting theoretical alternatives to Somigliana's formula, which involves a superposition of potentials. A clear advantage of the formula is that it involves directly the boundary displacements and tractions, i.e. the data of immediate engineering significance. However it may not necessarily yield numerical solutions of greater accuracy, for the same cost, as those which might be achieved by the Kupradse boundary formulations. A useful testing ground for syptematic numerical comparisons is available though the exact solutions presented in this paper.

SUMMARY OF BETECH 86 PAPER

Within the infinite region exterior to a spherical cavity there exists a linear elastostatic displacement field

$$\underset{\sim}{\phi}_e(\underset{\sim}{x}) = \gamma < \frac{x_1 x_3}{r^3} - \frac{a^2 x_1 x_3}{r^5},$$

$$\frac{x_2 x_3}{r^3} - \frac{a^2 x_2 x_3}{r^5}, \frac{1-\kappa}{\kappa r} + \frac{x_3^2}{r^3} + \frac{a^2}{3r^3} - \frac{a^2 x_3^2}{r^5} >; \quad r \geqslant a \qquad (1)$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$ and

$$\gamma = \frac{3a\kappa}{3-2\kappa} t_3, \quad \kappa = \frac{1}{4(1-\nu)}; \quad \nu = \text{Poisson's ratio } (0<\nu\leqslant 1/2), \quad t_3 = \text{a constant}. \qquad (2)$$

This field is characterised by the behaviour:

(1) $\underset{\sim}{\phi}_e = O(r^{-1})$ as $r \to \infty$;

(2) $\underset{\sim}{\phi}_e = \langle 0, 0, t_3 \rangle$ on $r = a$, i.e. a rigid-body translation of the boundary in the x_3-direction.

If so there exists an accompanying interior field

$$\underset{\sim}{\phi}_i = \langle 0, 0, t_3 \rangle; \quad r \leqslant a. \qquad (3)$$

Both $\underset{\sim}{\phi}_e, \underset{\sim}{\phi}_i$ may be represented by the simple-layer vector potential

$$\underset{\sim}{V}(\underset{\sim}{x}) = \int_{\partial B} \underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})\, \underset{\sim}{\sigma}(\underset{\sim}{y})\, dy; \quad \underset{\sim}{y} \subset \partial B\ (r=a), \quad \underset{\sim}{x} \subset B_e\ (r \geqslant a) \text{ or } \underset{\sim}{x} \subset B_i\ (r \leqslant a) \qquad (4)$$

Where dy denotes the element of surface area at $\underset{\sim}{y} \subset \partial B$, $\underset{\sim}{\sigma}$ is the simple-layer section source density at $\underset{\sim}{y}$, and $\underset{\approx}{g}$ is the fundamental displacement dyadic of the medium. Since $\underset{\sim}{\phi}_e, \underset{\sim}{\phi}_i$ are known we may immediately write (* denotes the traction operation)

$$\underset{\sim}{\sigma} = -\frac{1}{4\pi}(\underset{\sim}{\phi}_e{}^* + \underset{\sim}{\phi}_i{}^*) = -\frac{1}{4\pi}\underset{\sim}{\phi}_e{}^*$$

$$= -\frac{1}{4\pi} \langle 0, 0, \frac{-\mu\gamma}{\kappa a^2} \rangle, \qquad (5)$$

since of course $\underset{\sim}{\phi}_i^* = 0$. Also

$$\underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y}) = \begin{bmatrix} g(x_1,y_1) & g(x_1,y_2) & g(x_1,y_3) \\ g(x_2,y_1) & g(x_2,y_2) & g(x_2,y_3) \\ g(x_3,y_1) & g(x_3,y_2) & g(x_3,y_3) \end{bmatrix} \qquad (6)$$

where

$$g(x_\alpha, y_\beta) = \frac{1-\kappa}{\rho}\delta_{\alpha\beta} + \frac{\kappa}{\rho}\frac{\partial\rho}{\partial x_\alpha}\frac{\partial\rho}{\partial x_\beta}; \quad \begin{matrix}\alpha = 1, 2, 3\\ \beta = 1, 2, 3\end{matrix} \tag{7}$$

$$\rho = |x - y|,$$

This dyadic element signifies the α-component of displacement at x generated by a unit point-force acting in the β-direction at y; alternatively, for an isotropic medium, it could also signify the β-component of displacement at y generated by a unit point-force acting the α-direction at x. Expressed in component form, (4) appears as

$$V_\alpha(x) = \int_{\partial B} g(x_\alpha, y_\beta)\,\sigma_\beta(y)dy\ ; \quad \begin{matrix}\alpha = 1, 2, 3\\ \beta = 1, 2, 3\end{matrix} \tag{8}$$

where σ_β ; $\beta = 1, 2, 3$ is defined in (5).

It appears not possible to evaluate the integral (4) exactly, but its asymptotic behaviour can be examined as follows. Note that

$$g(x, y) \to g(x, 0) \quad \text{as} \quad r \to \infty, \tag{9}$$

so that

$$\int_{\partial B} g(x,y)\sigma(y)dy \to g(x,0)\int_{\partial B}\sigma(y)dy \quad \text{as} \quad r \to \infty, \tag{10}$$

which gives the asymptotic results

$$V(x) = \gamma < \frac{x_1x_3}{r^3}, \ \frac{x_2x_3}{r^3}, \ \frac{1-\nu}{\kappa r} + \frac{x_3^2}{r^3} >, \tag{11}$$

agreeing exactly with the asymptotic components of ϕ_e extracted from (1). No such procedure is possible for ϕ_i. We may point out that V can be evaluated exactly at the centre of the cavity, i.e. putting $x = 0$ in (4), yielding the expected result

An exterior field of different character is

$$\phi_e = \frac{a^3}{r^3} < -\omega_3x_2,\ \omega_3x_1,\ 0 >; \quad r \geqslant a;\ \omega_3 = \text{const.} \tag{13}$$

This has the behaviour:

(1) $\phi_e \to O(r^{-2})$ as $r \to \infty$

(2) $\phi_e = < -\omega_3x_2,\ \omega_3x_1\ 0 > \equiv < -\omega_3y_2,\ \omega_2y_1\ 0 >$ (14)

on $r=a$.

If so there exists an interior field

$$\phi_i = < -\omega_3x_2,\ \omega_3x_1,\ 0 > ; \quad r \leqslant a. \tag{15}$$

As before $\phi_i^* = 0$ and we find

$$\sigma = -\frac{1}{4\pi}\phi_e^* = \frac{3\mu\omega_3}{4\pi a} < -y_2,\ y_1,\ 0 >. \tag{16}$$

Now, however $\int_{\partial B}\sigma(y)dy = 0$,

showing that the first-order asymptotic approximation gives a null result by virtue of (10). Expanding to the second approximation (Jaswon & Symm[3]):

$$g(x,y) = g(x,0) + y.\nabla g(0,x) + O(r^{-3}) \quad \text{as} \quad r \to \infty, \tag{18}$$

so that

$$V(x) = \int_{\partial B}[y.\nabla g(0, x)]\,\sigma(y)dy + O(r^{-3}) \quad \text{as} \quad r \to \infty. \tag{19}$$

On computing the components of the dyadic $[\mathbf{y}.\ \mathbf{g}(\mathbf{x},0)]$ and evaluating the integral in (19), we recover precisely the field (13). This implies that the asymptotic expansion (18) provides a route for the exact integration of $\mathbf{V}$ everywhere in B_e. As before, $\mathbf{V}$ can be evaluated exactly at $\mathbf{x} = 0$, yielding the expected result

$$\int_{\partial B} \mathbf{g}(0,\mathbf{y})\sigma(\mathbf{y})d\mathbf{y} = \langle 0, 0, 0 \rangle$$

$$= \langle -\omega_3 x_2,\ \omega_3 x_1,\ 0 \rangle \quad \text{at} \quad \mathbf{x} = 0. \qquad (20)$$

REPRESENTATION BY DOUBLE-LAYER VECTOR POTENTIALS

Corresponding with $\mathbf{g}(\mathbf{x},\mathbf{y})$ we introduce the traction dyadic

$$\mathbf{g}(\mathbf{x},\mathbf{y})^* = \begin{bmatrix} g(\mathbf{x}_1,\mathbf{y}_1)^* & g(\mathbf{x}_1,\mathbf{y}_2)^* & g(\mathbf{x}_1,\mathbf{y}_3)^* \\ g(\mathbf{x}_2,\mathbf{y}_1)^* & g(\mathbf{x}_2,\mathbf{y}_2)^* & g(\mathbf{x}_2,\mathbf{y}_3)^* \\ g(\mathbf{x}_3,\mathbf{y}_1)^* & g(\mathbf{x}_3,\mathbf{y}_2)^* & g(\mathbf{x}_3,\mathbf{y}_3)^* \end{bmatrix} \qquad (21)$$

where

$$g(\mathbf{x}_\alpha,\mathbf{y}_\beta)^* = \frac{2\nu-1}{2(1-\nu)}\ \frac{1}{\rho^2}\Big[\frac{\partial\rho}{\partial y_\beta}n_\alpha - \frac{\partial\rho}{\partial y_\alpha}n_\beta + \frac{\partial\rho}{\partial n}$$

$$\Big(\delta_{\alpha\beta} + \frac{3}{1-2\nu}\ \frac{\partial\rho}{\partial y_\alpha}\ \frac{\partial\rho}{\partial y_\beta}\Big)\Big]$$

$$\rho = |\mathbf{x} - \mathbf{y}| . \qquad (22)$$

This dyadic element has two distinct interpretations: it is either the β-component of traction at $\mathbf{y}$ generated by a unit point force acting in the α-direction at $\mathbf{x}$, or it is the α-component of displacement at $\mathbf{x}$ generated by a unit traction source oriented in the β-direction at $\mathbf{y}$. Only the latter interpretation applies here. Utilising (21) we construct the double-layer vector potential

$$\mathbf{W}(\mathbf{x}) = \int_{\partial B} \mathbf{g}(\mathbf{x},\mathbf{y})^*\ \boldsymbol{\mu}(\mathbf{y})d\mathbf{y}\ ; \qquad \mathbf{y} \subset \partial B$$

$$\mathbf{x} \subset B_i \text{ or } B_e , \qquad (23)$$

where $\boldsymbol{\mu}$ signifies the double-layer source density at $\mathbf{y}$. Expressed in terms of components, (23) appears as

$$W_\alpha(\mathbf{x}) = \int_{\partial B} g(\mathbf{x}_\alpha,\mathbf{y}_\beta)^*\ \mu_\beta(\mathbf{y})d\mathbf{y}\ ; \quad \alpha = 1, 2, 3 \qquad (24)$$

$$\beta = 1, 2, 3 .$$

An important feature of $\mathbf{W}$ is that

$$\mathbf{W} = O(r^{-2}) \quad \text{as} \quad r \to \infty , \qquad (25)$$

so that it could not represent the field (1) as it stands. We therefore remove the $O(r^{-1})$ terms to obtain a reduced field

$$\phi_e^{(2)} = \phi_e - \frac{\gamma}{\kappa}\ \mathbf{g}(\mathbf{x},0) \qquad (26)$$

$$= -\gamma a^2 \left\langle \frac{x_1x_3}{r^5},\ \frac{x_2x_3}{r^5},\ -\frac{1}{3r^3} + \frac{x_3^2}{r^5} \right\rangle \qquad (27)$$

$$= O(r^{-3}) \quad \text{as} \quad r \to \infty .$$

The tractions $[\boldsymbol{\phi}_e^{(2)}]^*$ associated with an $O(r^{-3})$ field produce neither a resultant force nor a resultant moment. Accordingly if the $O(r^{-3})$ condition is met, then $\boldsymbol{\phi}_e^{(2)}$ could be represented by $\mathbf{W}$ for a suitable choice of $\boldsymbol{\mu}$.

It has been shown (Jaswon & Symm[3]) that

$$\underset{\sim}{\mu} = \frac{1}{4\pi}\left[\underset{\sim}{\phi}_e{}^{(2)} - \underset{\sim}{\phi}_i{}^{(2)}\right]_{r=a} \tag{28}$$

where $\underset{\sim}{\phi}_i{}^{(2)}$ signifies the interior field ($r \leqslant a$) defined by

$$[\underset{\sim}{\phi}_e{}^{(2)}]^* + [\underset{\sim}{\phi}_i{}^{(2)}]^* = 0. \tag{29}$$

Since the interior tractions $[\underset{\sim}{\phi}_i{}^{(2)}]^*$ must constitute a self-equilibrated system of forces, the same applies to $[\underset{\sim}{\phi}_e{}^{(2)}]^*$ so explaining the null resultant moment condition. Now

$$[\underset{\sim}{\phi}_e{}^{(2)}]^* = 2\mu\gamma \left< \frac{3a^2y_1y_3}{r^6}, \frac{3a^2y_2y_3}{r^6}, -\frac{(y_1^2+y_2^2)}{r^4} + \frac{2a^2y_3^2}{r^6} \right>; \quad r = a \tag{30}$$

from which follows the resultant

$$\int_{\partial B} \underset{\sim}{y} \wedge [\underset{\sim}{\phi}_e{}^{(2)}]^*\, dy = \langle 0, 0, 0 \rangle \tag{31}$$

By virtue of (30)

$$[\underset{\sim}{\phi}_i{}^{(2)}]^* = -\frac{2\mu\gamma}{a^4} \langle 3y_1y_3, 3y_2y_3, -y_1^2 - y_2^2 + 2y_3^2 \rangle \tag{32}$$

which yields through fairly straightforward calculations (El-Damanawi[4])

$$\underset{\sim}{\phi}_i{}^{(2)} = \frac{-2\gamma}{a^4(3\lambda+2\mu)} \langle (\lambda+4\mu)y_1y_3, (\lambda+4\mu)y_2y_3, -(2\lambda+3\mu)(y_1^2+y_2^2) - (\lambda-\mu)y_3^2 \rangle. \tag{33}$$

To this may be added an arbitrary rigid body displacement

$$\underset{\sim}{d} + \underset{\sim}{b} \wedge \underset{\sim}{x} \tag{34}$$

where d, b are constant vectors. Substituting (27), (33), (34) into (28) gives

$$\underset{\sim}{\mu} = \frac{5\gamma}{16\pi a^2(1-2\kappa)(1+4\kappa)} \left< y_1y_3, y_2y_3, -\frac{5+8\kappa}{3}a^2 + y_3^2 \right> + \underset{\sim}{d} + \underset{\sim}{b} \wedge \underset{\sim}{x} \tag{35}$$

which is the vector source density required.

The integral (23) can be evaluated exactly by means of the asymptotic expansion

$$g(\underset{\sim}{x},\underset{\sim}{y})^* =$$

$$\begin{bmatrix} \frac{(1-2\kappa)\underset{\sim}{x}.\underset{\sim}{y}}{a\rho^3} + \frac{6\kappa x_1^2 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{-(1-2\kappa)(x_1y_2-x_2y_1)}{a\rho^3} + \frac{6_1 x_1x_2 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{-(1-2\kappa)(x_1y_3-x_3y_1)}{a\rho^3} + \frac{6\kappa x_1x_3 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} \\ \frac{-(1-2\kappa)(x_2y_1-x_1y_2)}{a\rho^3} + \frac{6\kappa x_1x_2 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{(1-2\kappa)\underset{\sim}{x}.\underset{\sim}{y}}{a\rho^3} + \frac{6\kappa x_2^2 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{-(1-2\kappa)(x_2y_3-x_2y_3)}{a\rho^3} + \frac{6\kappa x_2x_3 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} \\ \frac{(1-2\kappa)(x_1y_3-x_3y_1)}{a\rho^3} + \frac{6\kappa x_1x_3 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{(1-2\kappa)(x_2y_3-x_3y_2)}{a\rho^3} + \frac{6\kappa x_2x_3 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} & \frac{(1-2\kappa)\underset{\sim}{x}.\underset{\sim}{y}}{a\rho^3} + \frac{6\kappa x_3^2 \underset{\sim}{x}.\underset{\sim}{y}}{a\rho^5} \end{bmatrix} \tag{a}$$

$$
+\begin{bmatrix}
\dfrac{2(1-2\kappa)x_1x_3y_1y_3}{a\rho^5} + \dfrac{12\kappa x_1^3x_3y_1y_3}{a\rho^7} & \dfrac{12\kappa x_1x_3x_2^2y_2y_3}{a\rho^7} & \dfrac{6\kappa x_1x_3(x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2)}{a\rho^7} \\
\dfrac{12\kappa x_1^2x_2x_3y_1y_3}{a\rho^7} & \dfrac{2(1-2\kappa)x_2x_3y_2y_3}{a\rho^5} + \dfrac{12\kappa x_2^3x_3y_1y_3}{a\rho^7} & \dfrac{6\kappa x_2x_3(x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2)}{a\rho^7} \\
\dfrac{12\kappa x_1^2x_3^2y_1y_3}{a\rho^7} & \dfrac{12\kappa x_2^2x_3^2y_1y_3}{a\rho^7} & \left(\dfrac{(1-2\kappa)}{a\rho^3} + \dfrac{6\kappa x_3^2}{a\rho^7}\right)(x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2)
\end{bmatrix} \quad \text{(b)}
$$

$$+ \; O(r^{-4}) \quad \text{as} \quad r \to \infty , \qquad (36)$$

i.e. dyadic (a) integrates to zero and dyadic (b) gives

$\underset{\sim}{W} = \underset{\sim}{\phi}_e^{(2)}$ everywhere provided that

$$\text{(i)} \quad \underset{\sim}{b} = 0, \quad \underset{\sim}{d} = \frac{\gamma(-\lambda+6\mu)}{4\pi a^3(3\lambda+2\mu)} < 0,\ 0,\ \left(\frac{5\lambda}{\lambda-6\mu} - \frac{5+8\kappa}{3}\right)a^2 >, \qquad (37)$$

and

(ii) $\underset{\sim}{\mu}$ in (35) is multiplied by the factor

$$\frac{5(3\lambda+2\mu)}{4(-\lambda+6\mu)(1-2\kappa)(1+4\kappa)} ,$$

i.e. provided that

$$\mu_{\pm} = \frac{5\gamma}{16\pi a^2(1-2\kappa)(1+4\kappa)} < y_1y_3,\ y_2y_3,\ -\frac{5+8\kappa}{3}a^2+y_3^2 >. \qquad (38)$$

The rotation field (13) has $O(r^{-2})$ behaviour as $r \to \infty$, and the associated tractions produce a resultant moment

$$\int_{\partial B} \underset{\sim}{y} \wedge \underset{\sim}{\phi}_e^* \, dy = \frac{3\mu\omega_3}{a} \int_{\partial B} < y_1,\ y_2,\ y_3 > \wedge < y_2,\ -y_1,\ 0 > dy$$

$$= \frac{3\mu\omega_3}{a} \int_{\partial B} (y_1y_3,\ y_2y_3,\ -y_1^2-y_2^2)dy \qquad (39)$$

$$= < 0,\ 0,\ -8\pi a^3\mu\omega_3 >. \qquad (40)$$

This moment generates the entire field, leaving no provision for a contribution by $\underset{\sim}{W}$.

CONCLUSION

Owing to the jump in $\underset{\sim}{W}$ at ∂B, μ as defined in (38) satisfies the vector boundary integral equation

$$\int_{\partial B} \underset{\approx}{g}(\underset{\sim}{x},\underset{\sim}{y})^* \underset{\sim}{\mu}(\underset{\sim}{y})dy + 2\pi\underset{\sim}{\mu}(\underset{\sim}{x}) =$$

$$-\gamma a^2 < \frac{x_1x_3}{r^5},\ \frac{x_2x_3}{r^5},\ -\frac{1}{3r^3} + \frac{x_3^2}{r^5} >_{r = a}$$

$$= -\frac{\gamma}{a^3} < x_1x_3,\ x_2x_3,\ -\frac{a^2}{3} + x_3^2 > \qquad (41)$$

- see equation (27).

Also σ as defined in (5) satisfies the vector boundary integral equation

$$\int_{\partial B} g(x,y)\sigma(y)dy = \langle 0, 0, t_3 \rangle , \qquad (42)$$

and σ as defined in (16) satisfies the vector boundary integral equation

$$\int_{\partial B} g(x,y)\sigma(y)dy = \langle -\omega_3 y_2 , \omega_3 y_1 , 0 \rangle . \qquad (43)$$

These provide exact analytical solutions against which numerical solutions could be usefully calibrated.

Instead of reducing Φ_e we may superpose suitable contributions upon W to accommodate the effects of a resultant force and resultant moment, according to the general theory put forward by Jaswon[5] .

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WHAT IS A DISLOCATION ?
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WHAT IS A DISLOCATION?

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ABSTRACT

A Volterra dislocation is the elastostatic analogue of a uniform magnetic shell or vortex-equivalent sheet. Just as these may be regarded mathematically as uniform dipole sheets,so dislocations may be regarded as specialised traction sheets. This model is briefly explained and connected up with the theory of Taylor dislocations in a crystal.

1. INTRODUCTION

A Volterra (1907) dislocation is a sheet within the linear elastic continuum, across which the displacement field jumps by a rigid-body translation or rotation without impairing the continuity of the strain and stress components. Some simple dislocation models may be constructed with a hollow cut cylinder as exhibited in fig.1

We may regard the sheet as a specialised distribution of traction sources, and these generate a vector double-layer potential which may be identified as the elastostatic field of the dislocation. The bounding contour of the sheet, i.e. the dislocation line, plays an important physical role in the theory of crystal dislocations Taylor (1934); Nabarro (1967).

Clearly a dislocation sheet is the vector analogue of a uniform magnetic shell or vortex-equivalent sheet, which are particular examples of a uniform dipole sheet. This generates a scalar double-layer potential, eg. a magnetostatic potential or velocity potential, which jumps by a uniform amount on crossing the sheet without impairing the continuity of the magnetostatic field or fluid velocity. Just as in the dislocation case, the bounding contour of the sheet plays an important physical role, being identified as an electric current or vortex line as the case may be.

In his original treatment, Volterra utilised Somigliana's formula, which is the fundamental formula of vector potential theory, Jaswon & Symm (1977).

However, since this involves the superposition of vector simple-layer and double-layer potentials, it obscures the useful dipole analogy. We exploit the analogy to calculate the field of a dislocation having the form of a circular disc.

2. Scalar double-layer potentials

A continuous distribution of dipoles over a sheet S contained by a contour ∂S generates the potential

$$W(\underline{x}) = \int_S g(\underline{x},\underline{y})' \mu(\underline{y}) dy; \quad \underline{y} \subset S, \quad \underline{x} \not\subset S, \tag{1}$$

where $\mu(\underline{y})$ signifies the dipole source density at $\underline{y}$ and dy signifies the element of area at $\underline{y}$. Also

$$g(\underline{x},\underline{y}) = |\underline{x}-\underline{y}|^{-1}, \tag{2}$$

and

$$g(\underline{x},\underline{y})' = g'(\underline{y},\underline{x}) = \frac{d}{dn_y} g(\underline{y},\underline{x}) \tag{3}$$

i.e. $g(\underline{x},\underline{y})'$ is the normal derivative of $g(\underline{x},\underline{y})$ at $\underline{y}$ keeping $\underline{x}$ fixed. Physically expressed, $g(\underline{x},\underline{y})$ signifies the potential at $\underline{x}$ generated by a unit dipole source at $\underline{y}$. It is well established, Kellogg (1929) that W has the following general properties:

(i) W is continuous and differentiable at least to the second order, and satisfies $\nabla^2 W = 0$, everywhere except at S, i.e. W defines a harmonic function of $\underline{x}$ everywhere except at S.

(ii) $W = 0(r^{-2})$ as $|\underline{x}| \longrightarrow \infty$.

(iii) $[W] = 4\pi\mu(\underline{x})$ at $\underline{x} \subset S$, where [ ] signifies the jump in a quantity on crossing S.

(iv) $[\frac{\partial W}{\partial n}] = 0$, i.e. the normal derivative (but not necessarily the tangential derivatives) of W remains continuous on crossing S.

If $\mu(=m)$ is uniform over S, then W has the following additional properties:

(v) $[\nabla W] = 0$, i.e. both the normal and tangential derivatives of W remain continuous on crossing S.

(vi) $[W] = \int_{\gamma} \nabla W . d\underline{\gamma} = 4\pi m$

for any circuit γ which loops ∂S (fig.2).

These two properties characterise a uniform magnetic shell or vortex-equivalent sheet, focusing attention upon ∂S as the physically significant entity i.e. seat of an electric current or of fluid vorticity as the case may be.

To fix the ideas we choose S to be a circular disc of radius c in the y_1, y_2 plane with centre at $y_1 = 0$, $y_2 = 0$. If so

$$\underline{y} = \langle y_1, y_2, y_3 \rangle = \langle r\cos\theta , r\sin\theta , h \rangle_{h=0} \quad (4)$$

$$dy = r dr d\theta$$

Also , for ease of integration we consider only

$$\underline{x} = \langle x_1, x_2, x_3 \rangle = \langle 0, 0, z \rangle \quad ; z > 0 \quad (5)$$

$$g(\underline{x}, \underline{y}) = [r^2 + (z-h)^2]^{-1/2}_{h=0} ,$$

$$g(\underline{x}, \underline{y})' = \frac{d}{dh} [r^2 + (z-h)^2]^{-1/2}_{h=0}$$

$$= \frac{z}{(r^2+z^2)^{3/2}} \quad ; \quad z > 0. \quad (6)$$

Inserting (6), with $\mu(\underline{y}) = m$, into the integral (1) gives:

$$W = W(z) = 2\pi m \int_{r=0}^{r=c} \frac{z r dr}{(r^2+z^2)^{3/2}}$$

$$= 2\pi m \left(1 - \frac{z}{(z^2+c^2)^{1/2}}\right) \quad ; z > 0 \quad (8)$$

$$= 2\pi m (1 - \cos\alpha) \quad \alpha = \cos^{-1} \frac{z}{(z^2+c^2)^{1/2}}. \quad (9)$$

This is of course a well known classical result usually obtained by the method of solid angles.

Note that:

(i) $W = O(z^{-2})$ as $z \longrightarrow \infty$ as follows from (8).

(ii) $W \longrightarrow 2\pi m$ as $z \longrightarrow 0$ as also follows from (8).

(iii) $W = 0$ for $z = 0$ as follows from (7) and also directly from the fact that $g(\underline{x},\underline{y})' = 0$ for $\underline{x} \subset S$.

Referring to the integral(1) these last two results appear respectively as

$$\text{(ii)} \quad \lim_{z \to 0} \int_S g(\underline{x},\underline{y})' m dy = 2\pi m \,, \qquad (10)$$

$$\text{(iii)} \quad \int_S \lim_{z \to 0} g(\underline{x},\underline{y})' m dy = 0 \,, \qquad (11)$$

with a jump which arises from the non-uniform convergence of the function:

$$U_z(r) = \frac{zr}{(z^2+r^2)^{3/2}} \quad , \text{as } z \longrightarrow 0. \qquad (12)$$

Since $U_z(r)$ is anti-symmetric with respect to z, W is also anti-symmetric with respect to,z i.e.

$$W(z) = -2\pi m\left(1 - \frac{|z|}{(z^2+r^2)^{1/2}}\right) \; ; \; z < 0 \qquad (13)$$

so yielding:

$$[W] = 4\pi m \; ; \; \left[\frac{dW}{dz}\right] = 0 \,, \qquad (14)$$

in line with general theory. A graph of W(z) appears in fig.3 (m=1).

3. Vector double-layer potential

Corresponding with W, we introduce the vector double-layer potential:

$$\underline{W}(\underline{x}) = \int_S \underline{\underline{g}}(\underline{x},\underline{y})^* \cdot \underline{\mu}(\underline{y}) dy \quad ; \quad \underline{y} \subset S \quad \underline{x} \not\subset S \quad , \qquad (15)$$

where $\underline{\underline{g}}(\underline{x},\underline{y})^*$ signifies the fundamental traction dyadic of the medium and $\underline{\mu}(\underline{y})$ signifies a vector source-density. In terms of components:

$$\underline{\underline{g}}(\underline{x},\underline{y})^* = \begin{bmatrix} g(\underline{x}_1,\underline{y}_1)^* & g(\underline{x}_1,\underline{y}_2)^* & g(\underline{x}_1,\underline{y}_3)^* \\ g(\underline{x}_2,\underline{y}_1)^* & g(\underline{x}_2,\underline{y}_2)^* & g(\underline{x}_2,\underline{y}_3)^* \\ g(\underline{x}_3,\underline{y}_1)^* & g(\underline{x}_3,\underline{y}_2)^* & g(\underline{x}_3,\underline{y}_3)^* \end{bmatrix} \qquad (16)$$

where $g(\underline{x}_\alpha,\underline{y}_\beta)^*$ provides the β-component of traction at $\underline{y}$ generated by a unit point-force acting along the α- direction at $\underline{x}$. Clearly row 1 of (16) defines the traction vector at $\underline{y}$ generated by a unit point-force acting along the 1-direction at $\underline{x}$,etc. Also column 1 of (16) defines an elastostatic displacement field , i.e. that generated by a unit traction-source acting along the 1-direction at $\underline{y}$, etc. This means that $\underline{\underline{g}}(\underline{x}\cdot\underline{y})^*$ plays the role of a vector dipole potential corresponding with the scalar dipole potential $g(\underline{x},\underline{y})'$. Writing $\underline{\mu} = \langle \mu_1,\mu_2,\mu_3 \rangle$, (15) appears in component form as:

$$W_\alpha(\underline{x}) = \int_S g(\underline{x}_\alpha, \underline{y}_\beta)^* \mu_\beta(\underline{y}) dy \quad ; \quad \alpha, \beta = 1,2,3 \qquad (17)$$

assuming the summation convention for dummy subscripts.

To evaluate (16) we must first compute the fundamental displacement dyadic of the medium:

$$\underline{g}(\underline{x},\underline{y}) = \begin{bmatrix} g(\underline{x}_1,\underline{y}_1) & g(\underline{x}_1,\underline{y}_2) & g(\underline{x}_1,\underline{y}_3) \\ g(\underline{x}_2,\underline{y}_1) & g(\underline{x}_2,\underline{y}_2) & g(\underline{x}_2,\underline{y}_3) \\ g(\underline{x}_3,\underline{y}_1) & g(\underline{x}_3-\underline{y}_2) & g(\underline{x}_3,\underline{y}_3) \end{bmatrix} \qquad (18)$$

where $g(\underline{x}_\alpha,\underline{y}_\beta)$ provides the β-component of displacement at $\underline{y}$ generated by a unit point-force acting along the α-direction at $\underline{x}$. Alternatively since $g(\underline{x}_\alpha,\underline{y}_\beta) = g(\underline{y}_\beta,\underline{x}_\alpha)$, it also provides the α-component of displacement at $\underline{x}$ generated by a unit point-force acting along the β-direction at y. Clearly both row 1 and column 1 of (18) define elastostatic displacement vectors, etc. For an infinite linear isotropic elastic continuum, the dyadic components are nothing more than Kelvin's point-force solution, Love (1927) written systematically in subscript notation

It has been shown by Kupradze (1965) that $\underline{W}$ has the following properties in a linear isotropic elastic continuum:

(i) $\underline{W}$ is continuous and differentiable at least to the second order, and satisfies the Cauchy-Navier equation, everywhere except at S, i.e. $\underline{W}$ defines an elastostatic displacement field everywhere except at S.

(ii) $\underline{W} = O(r^{-2})$ as $|\underline{x}| = r \longrightarrow \infty$.

(iii) $[\underline{W}] = 4\pi\underline{\mu}(\underline{x})$ at $\underline{x} \subset S$.

If $\underline{\mu}(\underline{y}) = \underline{b} + \underline{\omega} \wedge \underline{y}$, where $\underline{b}$, $\underline{\omega}$ are constant vectors, i.e. $\underline{\mu}$ varies as a rigid-body displacement over S, then $\underline{W}$ has the following additional property analogous to $[\nabla W] = 0$ in the scalar case:

(iv) $$\left[\frac{\partial W_\alpha}{\partial x_\beta} + \frac{\partial W_\beta}{\partial x_\alpha} \right] = 0 \quad ; \quad \alpha, \beta = 1,2,3,$$

i.e. the strains associated with $\underline{W}$ remain continuous on crossing S. This means that the stresses and therefore the tractions remain continuous on crossing S so identifying the sheet as a Volterra dislocation.

If $\underline{\omega} = 0$, i.e. no rotational jump, then $\underline{W}$ has the following additional property which replaces (iii) above:

(v) $$[\underline{W}] = \oint_\ell \nabla\underline{W}.d\underline{\ell} = 4\pi\underline{b} \quad ; \quad 4\pi\underline{b} = \text{Burger's vector},$$

for any circuit which loops the dislocation line ∂S. Here $4\pi\underline{b}$ is the Burger's vector of the dislocation line as defined in the theory of crystal dislocations (see section 6).

4 Circular dislocations

Choosing a circular sheet of radius c as before, and again writing

$$\underline{x} = \langle o,o,z \rangle \ , \quad \underline{y} = \langle y_1, y_2, h \rangle_{h=0} \tag{18}$$

we compute the components of $\underline{\underline{g}}(\underline{x},\underline{y})^*$ from the known components of $\underline{\underline{g}}(\underline{x},\underline{y})$. Details are given by El-Damanawi (1989), he obtained:

$$\underline{\underline{g}}(\underline{x},\underline{y}) = \frac{1-\kappa}{\mu\rho}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\kappa}{\mu\rho^3}\begin{bmatrix} y_1^2 & y_1 y_2 & -y_3 z \\ y_1 y_2 & y_2^2 & -y_2 z \\ -y_1 z & -y_2 z & z^2 \end{bmatrix}, \tag{19}$$

$$\underline{\underline{g}}(\underline{x},\underline{y})^* = \frac{(1-2\kappa)z}{\mu\rho^3}\begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & -y_2 \\ y_1 & y_2 & 1 \end{bmatrix} + \frac{6\kappa}{\mu\rho^5}\begin{bmatrix} y_1^2 & y_1 y_2 & y_1 z \\ y_1 y_2 & y_2^2 & y_2 z \\ y_1 & y_2 & -z^2 \end{bmatrix} \tag{20}$$

The integral (17) may then be evaluated exactly for the six independent rigid-body displacements:

$$b_1 \langle 1 \ , \ 0 \ , \ 0 \rangle \ , \ b_2 \langle \ 0, \ 1 \ , \ 0 \ \rangle \ , \ b_3 \langle 0 \ , \ 0 \ , \ 1 \rangle$$

$$\omega_1 \langle 0,0,y_2 \rangle , \ \omega_2 \langle 0,0,-y_1 \rangle , \ \omega_3 \langle -y_2, y_1, 0 \rangle$$

5. Two-dimensional continuum dislocations

Problems of two-dimensional linear isotropic elastostatics, in the absence of body forces, may be conveniently formulated through a stress function χ which satisfies the biharmonic equation:

$$\nabla^4\chi = \nabla^2 \ (\nabla^2\chi) \ = 0 \ ; \ \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{21}$$

The displacement components associated with χ are given by the formulae:

$$2\mu W_1 = (1-\nu)H - \frac{\partial\chi}{\partial x_1} \quad , \quad 2\mu W_2 = (1-\nu)H^* - \frac{\partial\chi}{\partial x_2} \tag{22}$$

where H, H^* are conjugate harmonic functions defined by

$$\frac{\partial H}{\partial x_1} = \frac{\partial H^*}{\partial x_2} = \nabla^2\chi \tag{23}$$

and μ, ν denote the shear modulus and Poisson's ratio respectively. These formulae simplify somewhat by introducing the Almansi (1897) representation for χ:

$$\chi = x_1\phi + \psi (\text{or} \ \ x_2\phi + \psi) \tag{24}$$

where ϕ, ψ are harmonic functions, since then

$$\nabla^4\chi = 2\frac{\partial\phi}{\partial x_1}, \quad H = 2\phi, \quad H^* = 2\phi^* \tag{25}$$

so enabling us to replace (22) by

$$2\mu W_1 = 2(1-\nu)\phi - \frac{\partial\chi}{\partial x_1}, \quad 2\mu W_2 = 2(1-\nu)\phi^* - \frac{\partial\psi}{\partial x_2}. \tag{26}$$

Note that the functional equation $x_1\phi + \psi = 0$ has the two independent non-trivial solutions:

$$\phi = 1, \ \psi = -x_1 \ ; \ \phi = x_2, \ \psi = -x_1x_2 \tag{27}$$

showing that an arbitrary rigid-body displacement may be superposed upon W_1, W_2 keeping χ invariant.

Formulae (27) point to the dislocation solution (omitting dimensional coefficients)

$$\chi = x_1 \log r \ ; \ \phi = \log r, \ \phi^* = \theta, \ \psi = 0 \tag{28}$$

yielding the translation jumps:

$$[W_1] = 0, \ [W_2] = \frac{2\pi}{\mu}(1-\nu) \tag{29}$$

for any complete circuit about the origin. They also point to a second, independent, solution:

$$\chi = x_2 \log r \ ; \ \phi = -\theta, \ \phi^* = \log r, \ \psi = (x_2 \log r + x_1\theta) \tag{30}$$

yielding the translation jumps:

$$[W_1] = -\frac{2\pi}{\mu}(1-\nu), \ [W_2] = 0. \tag{31}$$

Here the dislocation line coincides with the x_3-axis as exhibited in fig.4 , so identifying the dislocation sheet as the half-plane $x_2 = 0$, $x_1 < 0$. These are purely mathematical models. Physical models could only be constructed by making the body multiply-connected, i.e. replacing the dislocation line by a hollow tube or core which in general has the form of a torus enclosing ∂S. We then cut through the material so as to intersect the core, rigidly translate one side of the cut relative to the other, and weld the sides together again in the new configuration. Six independent dislocations can be constructed across the cut, of which two examples have appeared in fig.1

6. Crystal dislocations

The atomistic structure of an edge dislocation is modelled in fig.5 , which depicts a section of the crystal at right angles to the dislocation line. This provides a crystalline version of the continuum dislocation modelled in fig.4 . Here the straight lines numbered 1,2,...,6 mark the traces of crystal planes at right angles to the slip direction, i.e. that of the translation jump (31). Fig. 5(a) refers to the perfect crystal . Figure 5(b) shows the crystal severed into two halves across the slip plane, fig.5(c) shows an extra half-plane, denoted p, inserted symmetrically between the upper half-planes 3 and 4 . In fig. 5 (d) the two halves of the crystal have been stitched together by re-introducing the atomic forces, matching as far as possible half- planes of the same number, and thereby leaving the central half-plane without a partner. This operation requires the upper half-plane to be compressed and the lower half-plane to be extended. The lower edge of p, identified as the x_3-axis of fig. 4 marks the edge dislocation line.

The dislocation lies at the centre of a small region of misfit bounded by the almost perfectly matched half-planes 1 and 6, beyond which the crystal is perfect. Since the misfit also falls off vertically the region is preferably pictured as a cylindrical domain, sometimes termed the dislocation core, and indicated by the circled area in fg. 5(d). In 5(e) the dislocation has effectively jumped forward by one inter-atomic spacing to the right compared with 5 (d), as the central spot now falls between the lower half-planes 4,5 instead of between 3,4. This jump does not imply any movement of matter: p still remains the neighbour of the upper half-plane 3 , but its lower part deviates slightly to the right, thereby becoming aligned with the lower half-plane 4. The upper half-plane 4 is left without a partner, to assume the role formerly held by p. The dislocation thus propagates very much like a travelling wave or disturbance, instantaneously separating the slipped from the unslipped regions of prefect crystal. It eventually becomes blocked at some particular point, or passes right out of the crystal as shown in fig. 5(f). Since the configurations 5 (d) and 5 (e) have the same energy, the dislocation, to a first approximation, moves under a vanishingly small stress. This provides the essential mechanism of plastic deformation.

The locked-up stress field generated by the continuum dislocation provides a very good approximation to that of a crystal dislocation outside the region of the dislocation core. Within the latter region, the strains are so large that classical elasticity can no longer be applied and a direct calculation of atomic displacements becomes necessary.

Since dislocations are singularities in stress fields, they interact with other dislocations, and more generally, with other geometrical imperfections. For instance two edge dislocation in the same slip plane repel or attract each other, according to whether their signs are like or unlike. If they are on different slip planes the situation becomes more complex, but the general possibility arises of dislocations blocking or locking each other by virtue of their mutual interactions, an effect which provides the essential mechanism in all theories of work hardening.

7. Concluding remarks

Vector potentials play a key role in the formulation of elastostatic boundary-value problems by boundary integral equations. In these problems the potential is generated from simple-laye or double-layer vector sources on a closed surface. However vector potentials may also be generated from sources on an open surface (sheet). By analogy with the uniform magnetic shell or vortex - equivalent sheet, which involve a uniform distribution of scalar dipoles over the sheet, we can introduce a specialised distribution of vector dipoles over the sheet so constructing a Volterra dislocation as described in section 4. This paper accordingly demonstrates the essential mathematical unity between the foundations of B.E.M. and the foundations of dislocation theory. In consequence the computational methods developed with B.E.M. could also be applied to the computation of dislocation fields.

The elastic continuum is a smooth approximation to the underlying crystal medium. By the same token Volterra dislocations are smooth versions of crystalline dislocations. These have proved to be extremely effective in general, but fail in one important respect: Volterra dislocations react elastically to applied stress, whilst crystalline dislocations become mobile so providing familiar metallurgical effects beyond the scope of continuum theory. A simplified model of a crystalline dislocation is briefly described in section 6, corresponding with the two - dimensional Volterra

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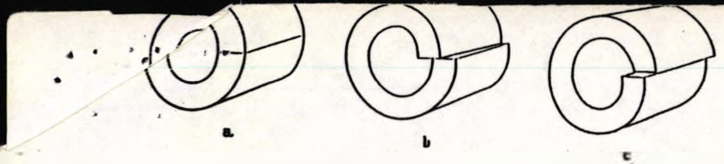

figure 1

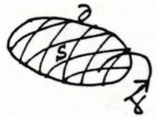

figure 2

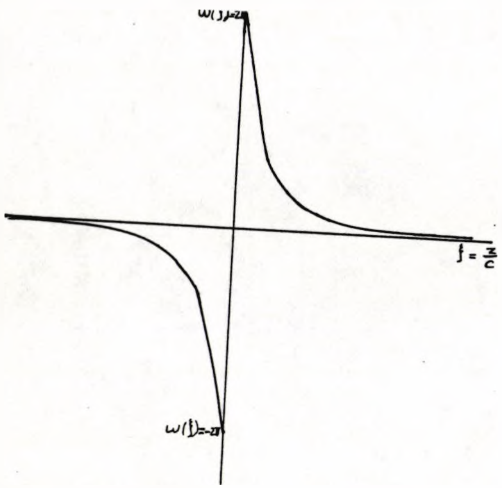

figure 3

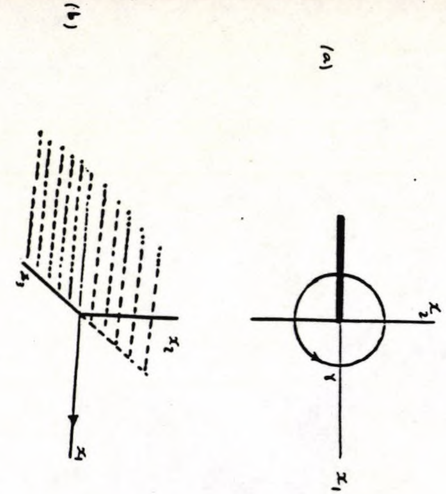

figure 4

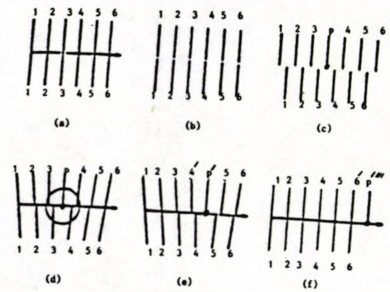

figure 5