Portfolio Selection and Risk Sharing via Risk Budgeting

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Abstract

Risk budgeting is an effective risk management that a decision-maker uses to create a risk portfolio with a pre-determined risk profile. This paper provides a rich discussion about the theory and practice on how to construct risk budgeting portfolios in variety of settings. We revisit the usual portfolio selection setting with and without clustered risk budgeting targets, and we then provide a novel approach on how to extend the usual setting to situations in which a non-hedgeable risk is present or fixed sub-portfolios are aimed by the decision-maker. Another novel approach of this paper is how to include risk budgeting targets in the process of risk sharing, which has not been discussed in the literature. Implementation issues are also discussed, and some bespoke algorithms are provided to identify such risk budgeting portfolios. Numerical experiments are performed for real-life financial and insurance data, and we explain the risk mitigation effect of our proposed portfolio. Specifically, financial risk budgeting portfolios with social responsibility targets are constructed, while insurance risk budgeting portfolios are obtained in an intra-group risk sharing setting.

Keywords:
Risk management, Portfolio selection, Risk budgeting/parity, Risk sharing.

1. Introduction

The idea of risk diversification can be traced back to the origins of probability theory, mainly to Bernoulli's 1954 paper (Bernoulli, 1954). In modern times, diversification has been reconsidered in a portfolio selection set-up by Markowitz in 1952 and it has been ever since the cornerstone of modern finance (Markowitz, 1999). Capital markets and insurance markets originated and evolved somehow differently, but recently, there is an enhanced commonality in the approaches taken to manage risk in two markets (Cummins and Weiss, 2016; Hainaut, 2017; Gatzert et al., 2017). The integration was motivated and facilitated by optimisation techniques applied to the decision making on constructing and managing a portfolio of financial assets or portfolio of insurance liabilities. The focus has shifted from risk optimisation to *risk budgeting/parity* (Roncalli, 2013), since the latter aims to distribute the overall risk in

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pre-defined way across all risks. *Risk parity (RP)*, also known as *Equal Risk Contribution*[1], is a special case of *Risk budgeting (RB)* where all risks have the same risk contribution, and represents the most common RB strategy.

The existing RB literature discusses RB/RP portfolios as a valuable alternative to the well-known portfolio selection methods that focuses on reducing the overall risk portfolio. Some important contributions in this area include the seminal work by Maillard et al. (2010) and some other papers that have provided practical solutions for building RB/RP portfolios when the risk preferences are ordered by a specific risk measure; specifically, *variance* and *standard deviation* risk preferences are discussed in (Roncalli, 2013; Spinu, 2013; Bai et al., 2016), *Conditional-Value-at-Risk* and *expectiles* risk preferences are investigated in Mausser and Romanko (2018) and Bellini et al. (2021), respectively, while a larger class of risk preferences is investigated in Asimit et al. (2023). Such papers provide bespoke numerical methods for real-life implementations of RB/RP portfolios. Besides this strand of research, Roncalli and Weisang (2016) shows the connection between RB portfolios and risk factors, while Kaucic (2019) and Anis and Kwon (2022) consider portfolio construction under some cardinality constraints to achieve lower corresponding portfolio overhead.

Portfolio selection is a risk management exercise that is more specific to financial assets, and it does not take into account any risk transfer from the (portfolio) risk holder to third parties; such risk shifting is known as *Risk sharing (RS)*. Conceptually, the RS theory equally apply to financial and insurance liabilities, though the RS literature tends to focus more on portfolios of insurance liabilities, since RS is an effective risk management exercise for insurance carriers to meet the regulatory requirements and shareholders financial expectations. Moreover, RS can not only improve capital allocation, but also stimulate further financial development (Pagano, 1993; Barattieri et al., 2020). RS problems have been widely studied in the literature (Ludkovski and Young, 2009; Asimit and Boonen, 2018; Asimit et al., 2020, 2021), and this strand of research is much related to intra-group risk transfers, in which an insurance group instructs its separate legal entities, i.e. risk holders, on sharing their liabilities (Asimit et al., 2013, 2016; Weber, 2018; Hamm et al., 2020).

Our contributions to the literature can be described as follows. First, we investigate RB strategies for one risk holder across many assets i) with or without risk contribution constraints on clusters of risks, and ii) with background (or non-hedgeable) risk. Then, we consider the RS problem between two risk holders with risk contribution constraints. We provide theoretical results demonstrating that solutions for such problems exists for a large class of risk preferences, and we provide bespoke algorithms to identify these RB/RS strategies in a practical context.

The paper is organised as follows. Section 2 provides the necessary background, while Section 3 contains the main theoretical RB/RS results. Further, Section 4 provides extensive numerical exemplifications of our new theoretical results, including a data analysis based on a unique database that helps us construct RB portfolios with *Social responsible investment (SRI)* constraints, but also a data analysis that illustrates how to construct RS strategies with RB targets for a real-life insurance portfolio. Section 5 summarises our main conclusions. All proofs

[1] This should not be mistaken for *Equal Weighted (EW)* portfolio that is not a risk-based asset allocation strategy since each risk has the same weight in the portfolio selection process, irrespective of the historical data.

are relegated in Appendix A, while more details about the algorithm and data used in Section 4 are provided in Appendix B and Appendix C, respectively.

2. Problem formulation

In this paper, we study two main RB formulations: i) for one risk holder in Section 2.1, and ii) for two risk holders in Section 2.2. Before providing the mathematical formulation of these two RB strategies, we give a brief introduction on the risk measures that is a key concept to our theoretical results.

Throughout this paper, the economy field is represented by $(\Omega, \mathcal{F}, \mathbb{P})$, an atomless probability space, endowed with $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$, the set of all real-valued random variables on this probability space. Let L^q, $q \in [0, \infty)$, be the set of random variables with finite q^{th} moment, and L^∞ be the set of bounded random variables. A risk measure φ is a function that maps an element of L^0 to a (extended) real number, i.e. $\varphi : L^0 \to \overline{\Re}$. We recall below some properties for a generic risk measure and generic random variable Y that represents the future loss of a financial asset or insurance liability. These properties are well-known in the literature and an extensive introduction on risk measures could be found in Föllmer and Schied (2011).

Convexity: $\varphi(aY_1 + (1-a)Y_2) \leq a\varphi(Y_1) + (1-a)\varphi(Y_2)$ for any $Y_1, Y_2 \in L^0$ and $a \in [0,1]$;

Homogeneous of order $\tau > 0$*:* $\varphi(cY) = c^\tau \varphi(Y)$ for any $Y \in L^0$ and $c \geq 0$;

Shift invariance: $\varphi(Y + c) = \varphi(Y)$ for any $Y \in L^0$ and $c \in \Re$;

Translation invariance: $\varphi(Y + c) = \varphi(Y) + c$ for any $Y \in L^0$ and $c \in \Re$.

Four risk measures are often recalled in this paper: standard deviation, variance, *Value-at-Risk (*VaR*)* and *Conditional-Value-at-Risk (*CVaR*)*. The last two are now formally defined. For any $p \in (0,1)$, VaR at probability level p is $\text{VaR}_p(Y) := \inf_x \left\{\mathbb{P}(Y \leq x) \geq p\right\}$, while CVaR at probability level p is $\text{CVaR}_p(Y) := \min_t \left\{t + \frac{1}{1-p}\mathbb{E}(Y-t)_+\right\}$ with $(\cdot)_+ := \max(\cdot, 0)$ on $\Re$.

2.1. RB for one risk holder

We first define the RB problem of one risk holder (investor) that holds a portfolio of $d \geq 2$ risks, i.e. $\mathbf{X} := (X_1, X_2, \ldots, X_d)^T$ where X_k represents the future loss of the k^{th} risk. The portfolio allocation vector is $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_d)^T$, where α_k represents of proportion of k^{th} risk into the entire portfolio with $k \in \{1, 2, \ldots, d\}$. Therefore, the aggregate position of the risk holder is given as $\boldsymbol{\alpha}^T\mathbf{X}$. Let φ be the risk measure that orders risk holder's risk preferences, and thus, the overall portfolio risk is $\mathcal{R}(\boldsymbol{\alpha}) := \varphi\left(\boldsymbol{\alpha}^T\mathbf{X}\right)$. We assume that $\boldsymbol{\alpha} > \mathbf{0}$, i.e. short sellings are not possible. Further, it is assumed throughout that $\boldsymbol{\alpha} \in \Delta_d := \{\boldsymbol{\alpha} \in \Re^d_{++} : \mathbf{1}^T\boldsymbol{\alpha} = 1\}$, where $\Re^d_{++}$ is the standard polyhedral cone of the positive quadrant of $\Re^d$.

Consider a risk measure φ that is homogeneous of order $\tau > 0$. Then, by Euler's Homogeneous Function Theorem, for any $\boldsymbol{\alpha} \in \Delta_d$,

$$\mathcal{R}(\boldsymbol{\alpha}) = \sum_{k=1}^{d} \mathcal{RC}_k(\boldsymbol{\alpha}) \quad \text{with} \quad \mathcal{RC}_k(\boldsymbol{\alpha}) := \frac{1}{\tau}\alpha_k \frac{\partial \mathcal{R}(\boldsymbol{\alpha})}{\partial \alpha_k} = \frac{1}{\tau}\alpha_k \frac{\partial \varphi\left(\boldsymbol{\alpha}^T\mathbf{X}\right)}{\partial \alpha_k}, \tag{2.1}$$

where $\mathcal{RC}_k(\boldsymbol{\alpha})$ is the risk contribution of the k^{th} individual risk. For each $k \in \{1, 2, \dots, d\}$, $b_k := \mathcal{RC}_k(\boldsymbol{\alpha})/\varphi\left(\boldsymbol{\alpha}^T \mathbf{X}\right)$ is the proportion of the k^{th} individual risk to the overall portfolio risk, which depends on a pre-specified portfolio allocation $\boldsymbol{\alpha} \in \Delta_d$. The RB problem is essentially an inverse problem of the above, and is formalised as Definition 1.

Definition 1. *Let $\boldsymbol{b} := (b_1, b_2, \dots, b_d)^T$ be a given constant vector, such that $\boldsymbol{b} \in \Delta_d$. An allocation strategy $\boldsymbol{\alpha} \in \Delta_d$ is said to be RB, if*

$$\mathcal{RC}_k(\boldsymbol{\alpha}) = b_k \varphi\left(\boldsymbol{\alpha}^T \boldsymbol{X}\right) \text{ for all } k \in \{1, 2, \dots, d\}, \text{ where } \mathcal{RC}_k(\boldsymbol{\alpha}) \text{ is given in (2.1).} \tag{2.2}$$

For any $\mathbf{b} \in \Delta_d$, define $\mathcal{RB}(\mathbf{b}) := \{\boldsymbol{\alpha} \in \Delta_d : \boldsymbol{\alpha} \text{ is RB}\}$ as the set of RB portfolios for a given budgeting allocation vector $\mathbf{b}$ and a general risk measure φ. The constant b_k in (2.2) represents the pre-specified risk contribution proportion of the k^{th} risk to the overall portfolio risk. In particular, if $b_k = \frac{1}{d}$, for all $k \in \{1, 2, \dots, d\}$, the allocation strategy is called *Risk Parity (RP)*.

Table 1 summarises the closed-form risk contributions for the four previously-mentioned risk measures. Note that the RB strategies based on risk preferences order via the standard deviation and variance risk measures are equivalent.

φ	$\mathcal{RC}_k$
Standard deviation	$\frac{\text{Cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T \mathbf{X}\right)}{\sqrt{\text{Var}(\boldsymbol{\alpha}^T \mathbf{X})}}$
Variance	$\text{Cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T \mathbf{X}\right)$
Value-at-Risk at level $p \in (0,1)$	$\mathbb{E}\left[\alpha_k X_k \vert \boldsymbol{\alpha}_j^T \mathbf{X} = \text{VaR}_p\left(\boldsymbol{\alpha}^T \mathbf{X}\right)\right]$
Conditional Value-at-Risk at level $p \in (0,1)$	$\mathbb{E}\left[\alpha_k X_k \vert \boldsymbol{\alpha}^T \mathbf{X} \geq \text{VaR}_p\left(\boldsymbol{\alpha}^T \mathbf{X}\right)\right]$

Table 1: Individual risk contributions for some well-known risk measures.

Various numerical solutions have been found for computing RB portfolios. Spinu (2013) showed that the RB portfolios could be written as an efficient convex optimisation problem, which is a much simpler numerical problem than solving the system of non-linear equations in (2.2), whenever the aggregate risk is measured by the variance (and standard deviance). The CVaR risk measure setting is discussed in Mausser and Romanko (2018), while Bellini et al. (2021) illustrates the expectile risk measure case; both papers provide excellent computationally efficient algorithms that make the RB strategies to be implementable in practice, even for a relatively large number of assets.

2.2. RB for two risk holders

We now define a RS problem between two risk holders with RB constraints. Without loss of generality, it is assumed that each of risk holder has a portfolio of $m \geq 1$ *lines of business (LoB)*, which means that there are in total $d = 2m$ risks in this setting. Let $X_{ik} \in L^0$ be the pre-transfer random loss for the i^{th} risk holder on its k^{th} LoB, with $i \in \{1, 2\}$ and $k \in \{1, 2, \dots, m\}$. The risk holders aim to share their risks for all LoBs. Let α_{ijk} be the proportion of the loss X_{ik} held by the i^{th} risk holder for its k^{th} LoB transferred to the j^{th} risk holder; α_{iik} represents the proportion of the risk retained by the i^{th} risk holder from its k^{th} risk. Therefore, the post-transfer random loss held by the j^{th} risk holder and k^{th} LoB is given by

$\alpha_{1jk}X_{1k} + \alpha_{2jk}X_{2k}$. A pictorial representation of the risk sharing for each LoB is illustrated in Figure 1. If $\boldsymbol{\alpha}_j := (\alpha_{1j1}, \ldots, \alpha_{1jm}, \alpha_{2j1}, \ldots, \alpha_{2jm})^T$ for each $j \in \{1, 2\}$, then $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$ and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [\mathbf{0}, \mathbf{1}]$.

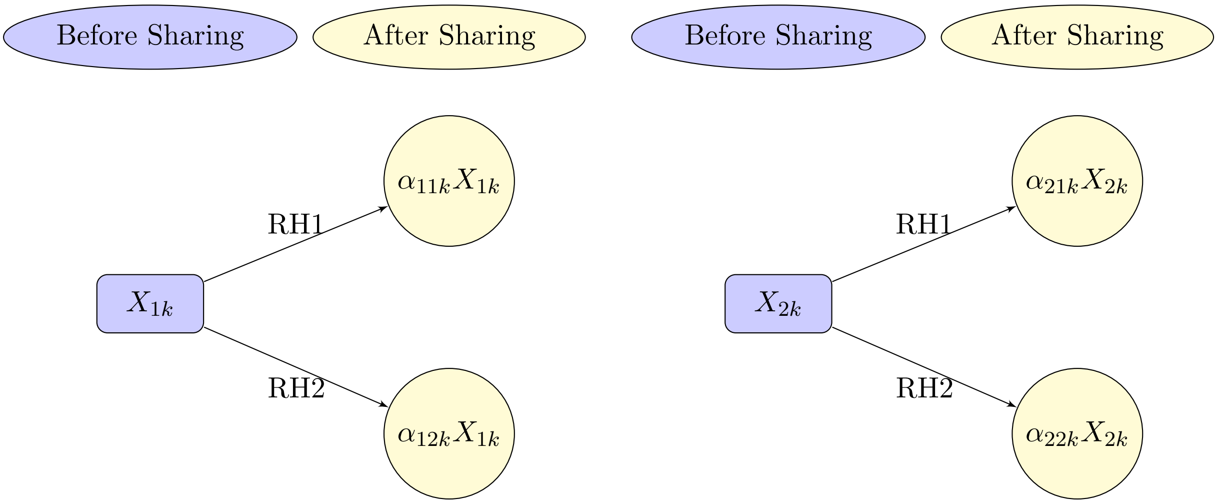

Figure 1: Risk sharing flowchart for the k^{th} LoB, where X_{1k} and X_{2k} are the risk (initially) held by the first (RH1, see left) and the second (RH2, see right) risk holder.

This RB exercise consists in achieving a balance of the risk between the risk holders. Thus, the premium could be decided after the RS is agreed, which is then proportionally allocated amongst the LoBs. Therefore, the aggregate post-transfer risk for the first and the second risk holder is $\boldsymbol{\alpha}_1^T\mathbf{X}$ and $\boldsymbol{\alpha}_2^T\mathbf{X}$, respectively, where $\mathbf{X} := (X_{11}, \ldots, X_{1m}, X_{21}, \ldots, X_{2m})^T$.

Let φ_j be the risk measure that orders the risk preferences of the j^{th} risk holder, where $j \in \{1, 2\}$. Therefore, the post-transfer overall risk position for the first and second risk holder is $\mathcal{R}_1(\boldsymbol{\alpha}_1) := \varphi_1\left(\boldsymbol{\alpha}_1^T\mathbf{X}\right)$ and $\mathcal{R}_2(\boldsymbol{\alpha}_2) := \varphi_2\left(\boldsymbol{\alpha}_2^T\mathbf{X}\right)$, respectively. Assuming further that both risk measures are homogeneous of order $\tau > 0$, the Euler's Homogeneous Function Theorem implies for each $j \in \{1, 2\}$ that

$$\mathcal{R}_j(\boldsymbol{\alpha}_j) = \sum_{i=1}^{2}\sum_{k=1}^{m} \mathcal{RC}_{ijk}(\boldsymbol{\alpha}_j) \quad \text{with} \quad \mathcal{RC}_{ijk}(\boldsymbol{\alpha}_j) := \frac{\alpha_{ijk}}{\tau}\frac{\partial \mathcal{R}_j(\boldsymbol{\alpha}_j)}{\partial \alpha_{ijk}} = \frac{\alpha_{ijk}}{\tau}\frac{\partial \varphi_j\left(\boldsymbol{\alpha}_j^T\mathbf{X}\right)}{\partial \alpha_{ijk}}, \quad (2.3)$$

where $\mathcal{RC}_{ijk}$ is the risk contribution of the k^{th} LoB initially held by the i^{th} risk holder corresponding to the j^{th} risk holder. We are providing the equivalent of Definition 1 for two risk holder setting, which is given as Definition 2.

Definition 2. *Let* $\boldsymbol{b}_j := (b_{1j1}, \ldots, b_{1jm}, b_{2j1}, \ldots, b_{2jm})^T$, $j \in \{1, 2\}$, *be a given constant vector such that* $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \Delta_{2m}$. *A proportional risk sharing* $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ *is said to be RB, if*

$$\mathcal{RC}_{i1k}(\boldsymbol{\alpha}_1) = b_{i1k}\varphi_1\left(\boldsymbol{\alpha}_1^T\boldsymbol{X}\right) \quad \text{and} \quad \mathcal{RC}_{i2k}(\boldsymbol{\alpha}_2) = b_{i2k}\varphi_2\left(\boldsymbol{\alpha}_2^T\boldsymbol{X}\right) \quad (2.4)$$

for all $i \in \{1, 2\}$ *and* $k \in \{1, 2, \ldots, m\}$, *where* $\mathcal{RC}_{ijk}(\boldsymbol{\alpha}_j)$ *is given in* (2.3).

3. Main theoretical results in risk budgeting

This section provides the main results of this paper. The case of one risk holder formulated in Section 2.1 is studied in Section 3.1, and is extended to a clustered formulation in Section 3.2. Further, the RB portfolio allocations with background risk or fixed sub-portfolios for one risk holder are investigated in Section 3.3. Finally, we extend the standard RB with one risk holder developed in Section 3.1 for the case of two risk holders, in Section 3.4. All proofs of our main results from this section are included in Appendix A.

3.1. Standard RB for one risk holder

The main result of this section is Theorem 3, which explains how to identify RB portfolios, i.e. elements of $\mathcal{RB}(\mathbf{b})$, for a given budgeting allocation vector $\mathbf{b}$ and a general risk measure φ. Our Theorem 3 has a different proof than Theorem 1 of Asimit et al. (2023), and extends Theorem 4 of Bellini et al. (2021) that is focused on a specific risk measure, namely the expectiles.

Theorem 3. *Let $\boldsymbol{b} \in \Delta_d$, and φ be a risk measure which is convex, homogeneous of order $\tau \geq 1$, and*

$$\inf_{\boldsymbol{x}\in\Delta_d} \mathcal{R}(\boldsymbol{x}) > 0. \tag{3.1}$$

For any given $\lambda > 0$, the following instance

$$\min_{\boldsymbol{x}\in\Re^d_{++}} \frac{1}{\tau}\mathcal{R}(\boldsymbol{x}) - \lambda \sum_{k=1}^{d} b_k \log x_k, \tag{3.2}$$

admits a unique solution, $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$, that is an interior point of $\Re^d_{++}$. If $\mathcal{R}(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b})$, then $\boldsymbol{\alpha}^*(\boldsymbol{b}) := \boldsymbol{x}^*(\lambda^*, \boldsymbol{b}) = \boldsymbol{x}^*(1, \boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b}) \in \mathcal{RB}(\boldsymbol{b})$, where $\lambda^* = \big(\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b})\big)^{-\tau}$.

While Theorem 3 solves the RB allocation strategy, an approximation of a RB allocation strategy could be achieved by the *Least Squares Estimation (LSE)* formulation, which has been defined in Roncalli (2013) as follows:

$$\min_{\boldsymbol{\alpha}\in\Delta'_d} \sum_{k=1}^{d} \big(\mathcal{RC}_k(\boldsymbol{\alpha}) - b_k\mathcal{R}(\boldsymbol{\alpha})\big)^2, \tag{3.3}$$

where $\Delta'_d := \{\boldsymbol{\alpha} \in \Re^d_+ : \mathbf{1}^T\boldsymbol{\alpha} = 1\}$ is the standard unit d-simplex, where $\Re^d_+$ is the standard polyhedral cone of the non-negative quadrant of $\Re^d$. Note that, if there exists $k_0 \in \{1, 2, \ldots, d\}$ such that $\alpha_{k_0} = 0$, $\mathcal{RC}_{k_0}(\boldsymbol{\alpha}) = 0$, and in turn $b_{k_0} = 0$, which contradicts the fact that $\mathbf{b} \in \Delta_d$. Therefore, (3.3) has the same solutions irrespective of the feasibility set choice, i.e. Δ_d or Δ'_d, but Δ'_d is preferred in numerical optimisation. Bai et al. (2016) has shown that, when φ is taken as the variance, the problem (3.3) could be efficiently solved for approximating RB allocation strategies. We show that the same methodology works for *Clustered Risk Budgeting*, which is defined in the next section.

3.2. Clustered RB for one risk holder

The standard RB allocation assumes a pre-specified risk contribution for each individual risk as explained earlier. A more general formulation is the so-called *Clustered Risk Budgeting (CRB)* allocations that require the pre-specified risk contribution proportions to be applied for clusters of risks, instead of each individual risk. The clustered version of (2.2) requires finding $\boldsymbol{\alpha} \in \Delta_d'$ such that

$$\sum_{i \in \mathcal{I}^{(k)}} \mathcal{RC}_i(\boldsymbol{\alpha}) = b_k \mathcal{R}(\boldsymbol{\alpha}), \quad \text{for all } k \in \{1,2,\dots,l\}, \text{ with } 2 \le l \le d \text{ and } \mathbf{b} \in \Delta_l, \tag{3.4}$$

and $\left\{\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \dots, \mathcal{I}^{(l)}\right\}$ is an l-dimensional partition of $\mathcal{I}_d := \{1,2,\dots,d\}$, i.e.

$$\bigcup_{k=1}^{l} \mathcal{I}^{(k)} = \mathcal{I}_d, \text{ and } \mathcal{I}^{(k_1)} \bigcap \mathcal{I}^{(k_2)} = \varnothing \text{ for all } 1 \le k_1 \ne k_2 \le l.$$

Clearly, the standard (non-clustered) RB allocation from Section 3.1 is achieved when $l = d$. The allocation strategy satisfying (3.4) with $b_k = 1/l$ for all $k \in \{1,2,\dots,l\}$ is known as *Clustered Risk Parity (CRP)* allocation strategy. By definition, the set of all CRB strategies for a given $\mathbf{b} \in \Delta_l$ is denoted by $\mathcal{CRB}(\mathbf{b}) := \{\boldsymbol{\alpha} \in \Delta_d' : \boldsymbol{\alpha} \text{ is CRB}\}$; further, $\mathcal{CRB}\left(\frac{1}{l}\mathbf{1}\right)$ is the set of all CRP strategies for a given l. Note that both sets of all CRB and CRP strategies depend on the choice for the number of clusters of risks l.

It is expected that $\mathcal{CRB}(\mathbf{b})$ has infinitely many portfolios for any given $\mathbf{b} \in \Delta_l$, and each allocation strategy induces a different aggregate risk position. Denote $\boldsymbol{\alpha}^{**}(\mathbf{b})$ as the CRB allocation strategy that minimises the portfolio risk, i.e. $\boldsymbol{\alpha}^{**}(\mathbf{b}) = \arg\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha})$. We next provide a simple example to finding $\boldsymbol{\alpha}^{**}(\mathbf{b})$, which is given as Example 4.

Example 4. *Assume that X_1, X_2, X_3 are three independent risks ($d = 3$), each with a unit variance, and the risk measure φ is given by the variance. The risk holder aims to find CRP strategies with two clusters, namely (X_1, X_2) and X_3; that is, $l = 2$, $\mathcal{I}^{(1)} = \{1,2\}$, $\mathcal{I}^{(2)} = \{3\}$, $b_1 = b_2 = 1/2$. Therefore, by (3.4), a CRP allocation strategy $\boldsymbol{\alpha} \in \Delta_3'$ satisfies $\alpha_1^2 + \alpha_2^2 = \alpha_3^2$, and in turn, the CRP set has a closed-form as follows:*

$$\mathcal{CRB}(1/2,1/2) = \left\{ \boldsymbol{\alpha}(\xi) : \boldsymbol{\alpha}(\xi) = \left(\xi, \frac{\frac{1}{2} - \xi}{1 - \xi}, 1 - \xi - \frac{\frac{1}{2} - \xi}{1 - \xi} \right),\ \xi \in [0, 1/2] \right\}. \tag{3.5}$$

The minimal portfolio variance within the $\mathcal{CRB}\,(1/2,1/2)$ set is obtained when $\xi^* = 1 - \frac{\sqrt{2}}{2}$, i.e.

$$\underset{\xi \in [0,1/2]}{\arg\min} \quad \xi^2 + \left(\frac{\frac{1}{2} - \xi}{1 - \xi} \right)^2 + \left(1 - \xi - \frac{\frac{1}{2} - \xi}{1 - \xi} \right)^2 = 1 - \frac{\sqrt{2}}{2}.$$

Hence, $\boldsymbol{\alpha}^{**}(1/2,1/2) = \boldsymbol{\alpha}(\xi^*)$, where the latter is an element of $\mathcal{CRB}(1/2,1/2)$ in (3.5).

Another way of characterising the set of clustered RB strategies is to identify a parametric set of RB strategies, namely $\mathcal{CRB}(\mathbf{b}) = \big\{\boldsymbol{\alpha}(\mathbf{a}) \in \mathcal{RB}(\mathbf{a}) : \mathbf{a} \in \mathcal{B}(\mathbf{b})\big\}$, where

$$\mathcal{B}(\mathbf{b}) := \left\{\mathbf{a} \in \Delta'_d : \sum_{i \in \mathcal{I}^{(k)}} a_i = b_k \text{ for all } k = 1, 2, \ldots, l\right\}.$$

RB strategies with $\mathbf{a} \in \Delta'_d \backslash \Delta_d$ should be understood as standard RB strategies (see Section 3.1) with a number of individual risks of $d' = d - d_0$, where the d_0 is number of zero-valued elements of $\mathbf{a}$, i.e. the risk set does not include the individual risks with zero budgeting targets.

As mentioned before, $\mathcal{CRB}(\mathbf{b})$ is likely to have infinitely many portfolios for any given $\mathbf{b}$, and a solution of (3.4) may encounter a large aggregate risk position, which is not desirable. Due to the Weierstrass' Theorem, there exists an $\mathbf{a}^*(\mathbf{b}) = \arg\min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}(\mathbf{a}))$, since $\mathcal{B}(\mathbf{b})$ is a compact set and $\mathcal{R}$ is a continuous mapping (as φ is homogeneous). Thus, $\boldsymbol{\alpha}^{**}(\mathbf{b}) = \boldsymbol{\alpha}\big(\mathbf{a}^*(\mathbf{b})\big)$.

Similar to (3.3), CRB allocations could be approximated by an LSE formulation:

$$\min_{\boldsymbol{\alpha} \in \Delta'_d} \sum_{k=1}^{l} \left(\sum_{i \in \mathcal{I}^{(k)}} \mathcal{RC}_i(\boldsymbol{\alpha}) - b_k \mathcal{R}(\boldsymbol{\alpha}) \right)^2. \tag{3.6}$$

Appendix B provides a numerical solution to solve (3.6) if the risk preferences are ordered by the variance or standard deviation risk measure, which is a slight extension of Algorithm 3 in Bai et al. (2016) that focuses only on CRP allocations. Unfortunately, solving (3.6) for other risk measures would require general optimisation algorithms, since so far, we do not have bespoke efficient algorithms for other (than variance or standard deviation) risk measures.

Essentially, finding a CRB portfolio based on (3.6), by either using a bespoke efficient algorithm such as the one detailed in (B.5) (when φ is variance or standard deviation) or global optimisation algorithm (when φ is not variance and standard deviation), consists in finding one element of $\mathcal{CRB}(\mathbf{b})$, i.e. a portfolio $\boldsymbol{\alpha}(\mathbf{a}_0)$ with $\mathbf{a}_0 \in \mathcal{B}(\mathbf{b})$ that is likely to differ from $\boldsymbol{\alpha}^{**}(\mathbf{b})$. A practical remedy for this issue is further explained and described in Algorithm 1. Before providing this algorithm, note that Theorem 1 c) of Asimit et al. (2023), shows that

$$\min_{\mathbf{x} \in \Delta_d} \mathcal{R}(\mathbf{x}) \le \mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \le \mathcal{R}(\mathbf{b}) \text{ and } \mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \le \mathcal{R}\left(\frac{1}{d}\mathbf{1}\right) \text{ for any } \mathbf{b} \in \Delta_d,$$

where $\boldsymbol{\alpha}^*(\mathbf{b}) \in \mathcal{RB}(\mathbf{b})$ is found as in Theorem 3. This suggests that one way to find a CRB portfolio with a low aggregate risk position (instead of directly solving (3.6)) would be to

1) find the budgeting target vector $\mathbf{b}$ that minimises $\mathcal{R}(\mathbf{b})$ by considering the clustering constraints, and then

2) find the RB portfolio based on the budgeting targets found in 1).

In Example 4, Step 1) implies solving $\min_{\mathbf{a} \in \mathcal{B}(1/2, 1/2)} \mathcal{R}(a_1, a_2, 1/2) = a_1^2 + a_2^2 + (1/2)^2$, which is solved by $a_1^* = a_2^* = 1/4$. Step 2) requires finding the RB with $\mathbf{b}_0 = (1/4, 1/4, 1/2)$, i.e. $\boldsymbol{\alpha}^*\big(\mathbf{b}_0\big)$; it is not difficult to show that $\boldsymbol{\alpha}^*\big(\mathbf{b}_0\big) = \boldsymbol{\alpha}(\xi^*)$, and thus, $\boldsymbol{\alpha}^{**}(1/2, 1/2) = \boldsymbol{\alpha}^*\big(\mathbf{b}_0\big)$. In a nutshell, the CRB solution with the lowest aggregate risk portfolio is identified by Algorithm 1 in the setting described in Example 4. This simple illustration – though further examples are in Section 4.1 – shows that the two-step procedure is a good choice for finding CRB portfolios with a low aggregate risk portfolio. For this reason, we propose a *worst-case-scenario*-like strategy and we call this risk portfolio as *WC-CRB*, and its procedure is given as Algorithm 1.

Algorithm 1: CRB algorithm for solving (3.4) with a low aggregate risk portfolio.

Result: Finding the WC-CRB portfolio to approximate $\boldsymbol{\alpha}^{**}(\mathbf{b})$.

Step 1): Find the budgeting target vector, $\mathbf{a}^*(\mathbf{b})$, that minimises the risk portfolio under the worst-case scenario, i.e.

$$\mathbf{a}^*(\mathbf{b}) = \arg\min_{\mathbf{a}\in\mathcal{B}(\mathbf{b})} \mathcal{R}(\mathbf{a}).$$

Step 2): Find $\boldsymbol{\alpha}^*\big(\mathbf{a}^*(\mathbf{b})\big)$, i.e. the RB portfolio based on budgeting targets $\mathbf{a}^*(\mathbf{b})$ via (3.2), (3.3) or any other RB computational procedure.

Note that *WC-CRB* is denoted as *WC-CRP* whenever $\mathbf{b} = \frac{1}{l}\mathbf{1}$. One big advantage of Algorithm 1 is that $\boldsymbol{\alpha}^{**}(\mathbf{b})$ is approximated by a more computationally efficient procedure; that is, one only needs to solve convex instances as in (3.2) instead of solving non-convex LSE formulations as in (3.3). Another advantage is that Algorithm 1 does not require solving clustered variants as in (3.6), which often relies on general optimisation algorithms.

3.3. RB portfolios with background risk or fixed sub-portfolios for one risk holder

This section extends the main result in Section 3.1 in the presence of background risk or fixed sub-portfolios. We first show that the two settings are mathematically equivalent and provide the necessary background.

The background risk setting requires allocating the risk portfolio $\mathbf{X} = (X_1, \dots, X_d)$ for which a non-hedgeable risk (for a financial risk portfolio) or non-insurable risk (for a insurance risk portfolio) Z is present. For example, an investment house focuses on structured finance products covering credit cards, student loans, Small and Medium Enterprise (SME) loans and so on. Each LoB has specific risk that is internally measured, and the investment house funds the purchase of these asset loans by borrowing funds from the market. The market funding risk affects all LoBs and the investment house cannot hedge this risk. Likewise, a maritime insurance portfolio – e.g. a corporate account that includes a variety of insurance sub-portfolios such as hull, cargo and protection & indemnity insurance, breakdown risk, business interruption risk, personal accidents, etc. – such that reputational perils of any kind are not included in the individual coverages. The losses due to such reputational perils are significantly associated with individual losses covered by this maritime insurance portfolio, and the reputational peril represents the (non-insurable) background risk for this bespoke portfolio. The aggregate risk, $\mathcal{R}(\boldsymbol{\alpha}) = \varphi\big(Z + \boldsymbol{\alpha}^T\mathbf{X}\big)$, is spread across all LoBs so that $\boldsymbol{\alpha} \in \Delta_d$ satisfies

$$\mathcal{RC}_k(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha}) \text{ for all } k \in \{1, \dots, d\}, \text{ where } \mathcal{RC}_k(\boldsymbol{\alpha}) := \frac{\alpha_k}{\tau}\frac{\partial\varphi\big(Z + \boldsymbol{\alpha}^T\mathbf{X}\big)}{\partial\alpha_k} \tag{3.7}$$

and budgeting targets $\mathbf{b} = (b_1, \dots, b_d)$ with $\mathbf{b} > \mathbf{0}$ and $\mathbf{1}^T\mathbf{b} < 1$. Note that (3.7) tacitly assumes φ is a homogeneous risk measure of order $\tau \geq 1$.

Now, the fixed sub-portfolio setting assumes a risk portfolio, $(X_1, \dots, X_d, X_{d+1}, \dots, X_{d+d_1})$ with $d_1 \geq 1$, for which the risk holder has a strategy for which the proportions $\tilde{\alpha}_k$ are fixed for all

$d+1 \leq k \leq d+d_1$. The RB strategy is to allocate the aggregate risk, $\mathcal{R}(\boldsymbol{\alpha}) = \varphi\big(\tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}} + \boldsymbol{\alpha}^T\mathbf{X}\big)$, where $\mathbf{X} = (X_1, \ldots, X_d)$ and $\tilde{\mathbf{X}} = (X_{d+1}, \ldots, X_{d+d_1})$. Thus, the aggregate risk is spread across all LoBs so that $(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) \in \Delta_{d+d_1}$ satisfies

$$\mathcal{RC}_k(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha}) \text{ for all } k \in \{1, \ldots, d\}, \text{ where } \mathcal{RC}_k(\boldsymbol{\alpha}) := \frac{\alpha_k}{\tau}\frac{\partial\varphi\big(\tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}} + \boldsymbol{\alpha}^T\mathbf{X}\big)}{\partial\alpha_k} \tag{3.8}$$

and budgeting targets $\mathbf{b} = (b_1, \ldots, b_d)$ with $\mathbf{b} > \mathbf{0}$ and $\mathbf{1}^T\mathbf{b} < 1$. Note that $\tilde{\boldsymbol{\alpha}}$ is the fixed allocation vector for the sub-portfolio $\tilde{\mathbf{X}}$, which means that we only need to solve (3.8) in $\boldsymbol{\alpha}$, since $\tilde{\boldsymbol{\alpha}}$ is set a priori by the risk holder.

Clearly, solving (3.8) is equivalent to solving (3.7) with background risk $Z = \frac{\tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}}}{1-\mathbf{1}^T\tilde{\boldsymbol{\alpha}}}$ and standardised weights $\frac{\boldsymbol{\alpha}}{1-\mathbf{1}^T\tilde{\boldsymbol{\alpha}}}$. This clarifies why the two settings are mathematically equivalent, and from now on, we only focus on the RB portfolios with background risk. The main result of this section explains how to find RB portfolios in the presence of background risk, and is given as Theorem 5.

Theorem 5. *Assume the risk portfolio with background risk defined in (3.7). Further, the aggregate portfolio risk position is given by $\mathcal{R}(\boldsymbol{\alpha}) = \varphi\big(Z + \boldsymbol{\alpha}^T\boldsymbol{X}\big)$, where φ is a convex and homogeneous risk measure of order $\tau \geq 1$. Let $\boldsymbol{b} \in \Re^d_{++}$ with $\boldsymbol{1}^T\boldsymbol{b} < 1$. If $\inf_{\boldsymbol{x}\in\Delta_d}\mathcal{R}(\boldsymbol{x}) > 0$, then for any given $\lambda > 0$, the following instance*

$$\min_{\boldsymbol{x}\in\Re^d_{++}} \frac{1}{\tau}\mathcal{R}(\boldsymbol{x}) - \lambda\sum_{k=1}^{d} b_k \log x_k. \tag{3.9}$$

admits a unique solution, $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$, that is an interior point of $\Re^d_{++}$. If $\mathcal{R}(\boldsymbol{\alpha})$ is differentiable at $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$ for some $\lambda > 0$, then $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$ solves (3.7), where the constraint $\boldsymbol{1}^T\boldsymbol{x}^*(\lambda, \boldsymbol{b}) = 1$ is removed.

The main (technical) difference between Theorem 3 and Theorem 5 is the lack of homogeneity of the aggregate risk $\mathcal{R}$ in (3.9), and therefore, λ acts as a tuning parameter in Theorem 5. That is, we need to find $\lambda > 0$ such that $\mathbf{1}^T\mathbf{x}^*(\lambda, \mathbf{b}) = 1$, and this solution is denoted as $\lambda^*(\mathbf{b})$ if this solution exists as its existence can not be guaranteed. Now, if $\lambda^*(\mathbf{b})$ exists, then $\mathbf{x}^*(\lambda^*(\mathbf{b}), \mathbf{b})$ solves (3.7). In a nutshell, RB portfolios satisfying either (3.7) or (3.8) could be found by iteratively solving (3.9) through the tuning parameter λ.

3.4. RB portfolios for two risk holders

Consider now the setting in Definition 2, where RB portfolios for two risk holders are sought. Recall that m LoBs are in place with $m \geq 1$ where $d = 2m$ such that $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$ and the constraints in (2.4) must hold, i.e. $2m$ and $2(2m-1)$ identity constraints, respectively. That is, the RB portfolios in Definition 2 require $2m + 2(2m-1) = 3d - 2$ identity constraints, though we have $2d$ variables since $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \Re^d_{++}$, which means that the balance condition $3d - 2 \leq 2d$ is achieved if and only if the two risk holders share one LoB, i.e. $m = 1$. We only consider the case in which $m = 1$ in this paper, since the $m > 1$ case would require changing (2.4) into cluster-like constraints, which would involve an extensive analysis that is beyond the reach of this paper. Theorem 6 tells us how to find RB portfolios for two risk holders and general risk preferences ordered by some general risk measures φ_1 and φ_2.

Theorem 6. *Assume that $m = 1$, and let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \Delta_2$. Further, assume that φ_1 and φ_2 be two convex, and homogeneous risk measures of order τ_1 and τ_2 with $\tau_1, \tau_2 \geq 1$, such that $\inf_{\boldsymbol{x}\in\Delta_2} \mathcal{R}_1(\boldsymbol{x}) > 0$ and $\inf_{\boldsymbol{x}\in\Delta_2} \mathcal{R}_2(\boldsymbol{x}) > 0$, where*

$$\mathcal{R}_1(x_{111}, x_{211}) = \varphi_1(x_{111}X_{11} + x_{211}X_{21}) \quad \text{and} \quad \mathcal{R}_2(x_{121}, x_{221}) = \varphi_2(x_{121}X_{11} + x_{221}X_{21}).$$

Then, for any $\lambda_1, \lambda_2 > 0$,

$$\min_{(x_{111},x_{211})\in\Re^2_{++}} \frac{1}{\tau_1}\mathcal{R}_1(x_{111}, x_{211}) - \lambda_1(b_{111}\log x_{111} + b_{211}\log x_{211}) \tag{3.10}$$

and

$$\min_{(x_{121},x_{221})\in\Re^2_{++}} \frac{1}{\tau_2}\mathcal{R}_2(x_{121}, x_{221}) - \lambda_2(b_{121}\log x_{121} + b_{221}\log x_{221}) \tag{3.11}$$

admit a unique solution, $\boldsymbol{x}^*(\lambda_1, \boldsymbol{b}_1; \varphi_1)$ and $\boldsymbol{x}^*(\lambda_2, \boldsymbol{b}_2; \varphi_2)$, respectively, that are interior points of the feasibility set.

i) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b}_1; \varphi_1)$ and $\boldsymbol{x}^*(1, \boldsymbol{b}_2; \varphi_2)$, respectively, then $(\alpha^*_{111}, \alpha^*_{211})$ and $(\alpha^*_{121}, \alpha^*_{221})$ solve (2.4), respectively, where $\alpha^*_{ij1} = t^*_j x^*_i(1, \boldsymbol{b}_j; \varphi_j)$ for all $i, j \in \{1, 2\}$ and

$$\begin{cases} t^*_1 = \frac{x^*_2(1,\boldsymbol{b}_2;\varphi_2)-x^*_1(1,\boldsymbol{b}_2;\varphi_2)}{x^*_1(1,\boldsymbol{b}_1;\varphi_1)x^*_2(1,\boldsymbol{b}_2;\varphi_2)-x^*_2(1,\boldsymbol{b}_1;\varphi_1)x^*_1(1,\boldsymbol{b}_2;\varphi_2)} \\ t^*_2 = \frac{x^*_1(1,\boldsymbol{b}_1;\varphi_1)-x^*_2(1,\boldsymbol{b}_1;\varphi_1)}{x^*_1(1,\boldsymbol{b}_1;\varphi_1)x^*_2(1,\boldsymbol{b}_2;\varphi_2)-x^*_2(1,\boldsymbol{b}_1;\varphi_1)x^*_1(1,\boldsymbol{b}_2;\varphi_2)} \end{cases} \tag{3.12}$$

whenever

$$\big(x^*_1(1, \boldsymbol{b}_1; \varphi_1) - x^*_2(1, \boldsymbol{b}_1; \varphi_1)\big)\big(x^*_1(1, \boldsymbol{b}_2; \varphi_2) - x^*_2(1, \boldsymbol{b}_2; \varphi_2)\big) < 0. \tag{3.13}$$

ii) Assume that $\boldsymbol{b}_1 = \boldsymbol{b}_2$, $\varphi_1 = \varphi_2$, and the fact that $\mathcal{R}_1$ is differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b}_1; \varphi_1)$. If $x^*_1(1, \boldsymbol{b}_1; \varphi_1) = x^*_2(1, \boldsymbol{b}_1; \varphi_1)$, then $(\alpha^*_{111}, \alpha^*_{211}) = (\xi, \xi)$ and $(\alpha^*_{121}, \alpha^*_{221}) = (1 - \xi, 1 - \xi)$ are also solutions of (2.4), respectively, for any $\xi \in (0, 1)$.

iii) Let $\lambda^*_1, \lambda^*_2 > 0$ such that $(\alpha^*_{111}, \alpha^*_{211}) = \boldsymbol{x}^*(\lambda^*_1, \boldsymbol{b}_1; \varphi_1)$, $(\alpha^*_{121}, \alpha^*_{221}) = \boldsymbol{x}^*(\lambda^*_2, \boldsymbol{b}_2; \varphi_2)$ and $\alpha^*_{111} + \alpha^*_{121} = \alpha^*_{211} + \alpha^*_{221} = 1$, then

$$\frac{1}{\tau_1\lambda^*_1}\big(\mathcal{R}_1(\alpha^*_{111}, \alpha^*_{211}) - \mathcal{R}_1(x_{111}, x_{211})\big) + \frac{1}{\tau_2\lambda^*_2}\big(\mathcal{R}_2(\alpha^*_{121}, \alpha^*_{221}) - \mathcal{R}_2(x_{121}, x_{221})\big) \leq 0$$

for any $(x_{111}, x_{121}, x_{211}, x_{221}) \in \Re^4_{++}$ with $x_{111} + x_{121} = x_{211} + x_{221} = 1$.

Theorem 6 i) shows that the risks are fully allocated, i.e. $\alpha^*_{111} + \alpha^*_{121} = \alpha^*_{211} + \alpha^*_{221} = 1$, for any given risk measures and RB under a mild condition stated in (3.13), if the risk profile and risk targets for the two risk holders are quite different. Condition (3.13) requires that the risk appetite for the two risks, (X_{11}, X_{21}), are not the same for the two risk holders; in other words, if $\alpha^*_{111} < \alpha^*_{211}$ then $\alpha^*_{121} > \alpha^*_{221}$, which means that there are incentives for both risk holders to initiate the risk sharing.

Contrary to Theorem 6 i) where there is at most one RB allocation, Theorem 6 ii) suggests that there are infinitely many RB allocation if the risk profile and risk targets for the two risk holders are identical, though a technical condition is required, i.e. $x_1^*(1,\mathbf{b}_1;\varphi_1)=x_2^*(1,\mathbf{b}_1;\varphi_1)$. This setting implies that risk holder 1 retains the same risk proportion in (X_{11},X_{21}), and the second risk holders has the same strategy. Now, $x_1^*(1,\mathbf{b}_1;\varphi_1)=x_2^*(1,\mathbf{b}_1;\varphi_1)$ implies that the RB, i.e. $\mathbf{b}_1=\mathbf{b}_2$, should be chosen such that $x_1^*(1,\mathbf{b}_1;\varphi_1)=x_2^*(1,\mathbf{b}_1;\varphi_1)$; in other words, one should numerically find $\mathbf{c}\in\Delta_2$ such that $y_1^*(\mathbf{c})\approx y_2^*(\mathbf{c})$, where

$$\big(y_1^*(\mathbf{c}),y_2^*(\mathbf{c})\big):=\underset{(y_1,y_2)\Re^2_{++}}{\arg\min}\ \frac{1}{\tau_1}\mathcal{R}_1(y_1,y_2)-\lambda_1(c_1\log y_1+c_2\log y_2).$$

Clearly, we can not guarantee that there exists $\mathbf{c}^*\in\Delta_2$ such that $|y_1^*(\mathbf{c}^*)-y_2^*(\mathbf{c}^*)|\leq\epsilon$ for a sufficiently small $\epsilon>0$, but numerical explorations could answer to this question.

4. Numerical illustrations

This section provides numerical illustrations of how to construct portfolios based on the RB principle. Our numerical implementations disseminate practical implementations on financial and insurance risks for two methods, risk diversification (for one risk holder by applying the RB principle) and RS (for two risk holders by applying the RB principle). Specifically, Section 4.2 focuses on RB portfolios for one risk holder with multiple financial risks where the risk preferences are ordered by the Variance (or Standard Deviation) and CVaR risk measures. Then, Section 4.3 illustrates how RB portfolios for two risk holders that share one LoB could be build whenever the risk preferences are ordered by either Variance (or Standard Deviation) and CVaR risk measures. Before providing these numerical experiments, we provide in Section 4.1 a slight extension of Example 4.

4.1. CRB vs. WC-CRB

As anticipated, we extend Example 4 in Section 3.2 and assume a CRP setting based on variance risk preferences for three independent risks with two clusters such that $\Sigma_{11}=\Sigma_{33}$, i.e. assets 1 and 3 have the same variance. The CRB portfolio is constructed from the solution $\boldsymbol{\alpha}^*_{CRB}$, obtained with the Algorithm 2 in Appendix B. As explained in Section 3.2, the CRB solution is an element of a parametric set of RB solutions, which is obtained by searching for $\boldsymbol{\alpha}\in\Delta_3$ such that $\Sigma_{11}\alpha_1^2+\Sigma_{22}\alpha_2^2=\Sigma_{11}\alpha_3^2$. Denoting $\sigma_{12}=1-\frac{\Sigma_{22}^2}{\Sigma_{11}^2}$, the solution is described by

$$\boldsymbol{\alpha}(\xi):=\left(\frac{\sigma_{12}\xi^2-2\xi+1}{2(1-\xi)},\ \xi,\ 1-\xi-\frac{\sigma_{12}\xi^2-2\xi+1}{2(1-\xi)}\right),$$

for all $0\leq\xi\leq\frac{1-\sqrt{1-\sigma_{12}}}{\sigma_{12}}$ if $\sigma_{12}\in(-\infty,1)\setminus\{0\}$, and $0\leq\xi\leq\frac{1}{2}$ if $\sigma_{12}=0$, since $\sigma_{12}<1$. One may show that minimal variance amongst the $\boldsymbol{\alpha}(\xi)$ portfolios is achieved when $\xi^*=1-\sqrt{\frac{1-\sigma_{12}}{2-\sigma_{12}}}$.

The *WC-CRP* portfolio (defined in Section 3.2) is an element of $\boldsymbol{\alpha}(\xi)$, and it can be found via Algorithm 1 in Section 3.2. For Step 1) we need to solve

$$\underset{\mathbf{a}\in\mathcal{B}(1/2,1/2)}{\arg\min}\quad\Sigma_{11}a_1^2+\Sigma_{22}a_2^2+\Sigma_{33}(1/2)^2:=(a_1^*,a_2^*)=\left(\frac{\Sigma_{22}}{2\big(\Sigma_{11}+\Sigma_{22}\big)},\frac{\Sigma_{11}}{2\big(\Sigma_{11}+\Sigma_{22}\big)}\right).$$

Step 2) requires finding the RB with the budgeting targets $\boldsymbol{\alpha}^*(a_1^*, a_2^*, 1/2)$, which could be identified via a non-clustered version of Algorithm 2 in Appendix B, though a closed-form solution is possible since we only need solving in $\boldsymbol{\alpha} \in \Delta_3'$ the following system of equations

$$\alpha_1^2 = 2a_1^*\alpha_3^2, \quad \Sigma_{22}\alpha_2^2 = 2a_2^*\Sigma_{11}\alpha_3^2 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

The latter is solved by $\boldsymbol{\alpha}^*_{WC-CRB} = (c_1, c_2, c_3)/\mathbf{1}^T\mathbf{c}$, where

$$c_1 := \sqrt{\frac{\Sigma_{22}}{2\Sigma_{11}(\Sigma_{11}+\Sigma_{22})}}, \quad c_2 := \sqrt{\frac{\Sigma_{11}}{2\Sigma_{22}(\Sigma_{11}+\Sigma_{22})}} \quad \text{and} \quad c_3 := \sqrt{\frac{1}{2\Sigma_{11}}}.$$

The following three variance choices are further considered:

a) $\Sigma_{11} = \Sigma_{33} = 1, \Sigma_{22} = \sqrt{0.5}$, i.e. $\sigma_{12} = 0.5$;

b) $\Sigma_1 = \Sigma_2 = \Sigma_3 = 1$, i.e. $\sigma_{12} = 0$;

c) $\Sigma_1 = \Sigma_3 = 1, \Sigma_2 = \sqrt{1.5}$, i.e. $\sigma_{12} = -0.5$.

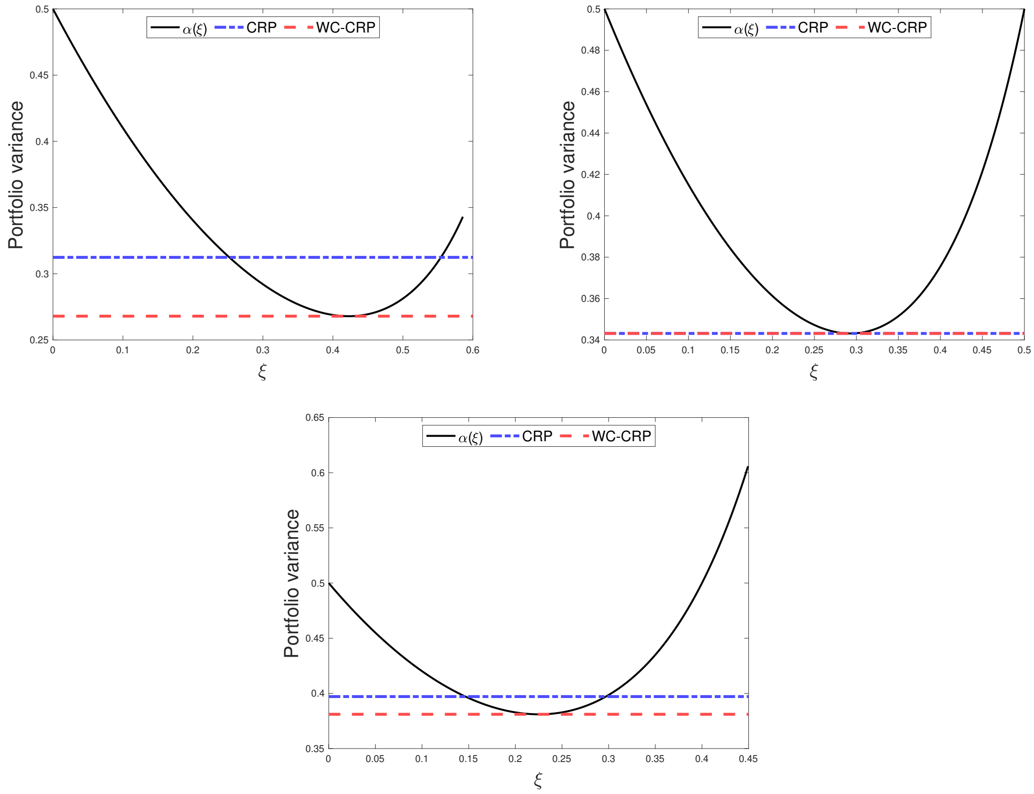

Figure 2: Portfolio variance for the parametric, CRP and WC-CRP portfolios for Case a) (top left), Case b) (top right) and Case c) (bottom).

Figure 2 compares the risk position of the CRP portfolio (computed via Algorithm 2 in Appendix B) and WC-CRP portfolio (computed via Algorithm 1 in Section 3.2) with the risk position of the parametric portfolio with risk allocation $\boldsymbol{\alpha}(\xi)$. The results clearly show the advantage of using the WC-CRP portfolio, besides its obvious computational advantage that was explained in Section 3.2, which reiterates the practical use of Algorithm 1.

4.2. RP for one risk holder

This section provides a data analysis based on our main results in Sections 3.1 and 3.2. That is, we reconsider the investment portfolio in Hallerbach et al. (2004) that was related to portfolio allocation satisfying certain *socially responsible investing (SRI)* characteristics[2].

A global survey of institutional investors on their beliefs about climate risk shows that climate risk is categorised as important, but not as high as financial, legal, and operational risk (Krueger et al., 2020). There are theoretical models constructed on the idea that investors may expect a lower expected return on investments SRI companies (Heinkel et al., 2001). From a theoretical perspective, there are two competing theories trying to explain SRI investment behaviour, the stakeholder value maximisation and the shareholder expense view. There is mixed evidence in general supporting both theories. Walley and Whitehead (1994) show that when firms use their financial resources to improve environmental performance then there is a fall in shareholder value because of higher product prices that translates into a lower profitability. At the same time, investors expect significantly higher returns on stocks that are not passing environmental criteria compared to the stocks of firms not affected by these environmental concerns and furthermore, lenders may also require higher interest rates on the loans issued to stocks of firms in the former category (Chava, 2014). There is evidence from mutual funds industry that socially responsible investors expect to earn lower returns on SRI funds than on standard funds, so investors may be willing to sacrifice financial performance to achieve their social preferences (Riedl and Smeets, 2017). Analyzing the U.S. mutual fund market, Hartzmark and Sussman (2019) discuss evidence that being categorised as low sustainable led to net outflows of more than \$12 billion while being considered high sustainable resulted in net inflows of more than \$24 billion. In addition, SRI investment may be associated with better financial instruments in the market and may provide insurance against event risk (Lins et al., 2017).

Hallerbach et al. (2004) develops a framework for constructing an investment portfolio with the investment opportunities states spanned by a set of attributes that characterises the effects on society, which is known as SRI portfolio. The SRI scores determine the degree of social responsibility embedded in a firm and it helps the investor or portfolio manager to construct the opportunity portfolio (i.e., decide which assets to invest in) by including only those companies that satisfy certain SRI targets before deciding upon the asset allocation (i.e., decide how much to invest in each asset). This two-stage approach provides a 360 degrees approach to construct a SRI portfolio. Our data analysis focuses on the second stage that supports the decision-making process on portfolio composition that is based on RB and CRB allocations. Given the new socio-economic environment that investors and fund managers ought to operate in, there is a growing emphasis on controlling the degree of risk absorbed from different asset classes, from different geographical economic regions or satisfying different ESG, SDG or SRI features. Traditional portfolio management techniques were not designed with these social preferences in mind. In this paper we highlight a new set of techniques that combine risk budgeting with SRI constraints and we show that the portfolio constructed based on these new ideas are not

[2]We would like to thank Aloy Soppe for making the original raw dataset available to us. The original dataset was put together by the Triodos bank, the first European green bank.

underperforming standard benchmarks.

We work with an universe of 374 companies that are grouped in ten *Global Industry Classification Standard (GICS)* sectors[3]. The companies data cover different regions, namely EU, UK, US and REST (firms from countries outside the EU, UK and US). The summary information is reported in Appendix C, see Table C.3.

Together with the SRI information, we have also collected historical stock prices (daily returns) for all firms in our sample from January 2010 to December 2020, from various sources: Datastream, WRDS-CRSP, Compustat, IBES and Yahoo!Finance. The financial performance of those companies is measured by various measures and the summary is tabulated in Table 2. Note that the performance is evaluated for two periods of time, namely before and after the COVID-19 pandemic, but also for the combined period from 2010 until 2020. Table 2 suggests that the financial performance in 2020 alone is significantly different from the performance observed before the COVID-19 pandemic, excepting perhaps the EU. Note that the portfolio performance tabulated in Table 2 assumes that each asset has equal weight in the total portfolio, which is known as the *Equal Weighted (EW)* portfolio, and thus, is considered as a benchmark portfolio that is not easy to outperform in practice, see DeMiguel et al. (2013).

Table 2: Summary of the financial performance per region.

		REGION				
		EU	UK	US	REST	Total
	No. of companies	188	56	96	34	374
	EQUAL WEIGHTED PROTFOLIO (daily returns)					
11 years: 2010 - 2020	Annualised average return	0.0888	0.0724	0.1279	0.1038	0.0996
	Annualised standard deviation	0.1998	0.2039	0.1759	0.1433	0.1651
	Mean	0.0004	0.0004	0.0005	0.0004	0.0004
	Standard deviation	0.0126	0.0128	0.0111	0.0090	0.0104
	Skewness	-0.5116	-0.7272	-0.5146	-0.2725	-0.7435
	Kurtosis	7.6460	14.9598	15.8433	3.3927	11.4002
10 years: 2010 - 2019	Annualised average return	0.0833	0.0869	0.1284	0.0864	0.0974
	Annualised standard deviation	0.1889	0.1823	0.1478	0.1357	0.1492
	Mean	0.0004	0.0004	0.0005	0.0004	0.0004
	Standard deviation	0.0119	0.0115	0.0093	0.0085	0.0094
	Skewness	-0.2089	-0.8002	-0.4618	-0.3067	-0.4145
	Kurtosis	4.7198	13.3674	4.1551	2.5442	5.2876
1 year: 2020	Annualised average return	0.0898	-0.1106	0.0780	0.1502	0.0627
	Annualised standard deviation	0.2879	0.3446	0.3512	0.2091	0.2761
	Mean	0.0005	-0.0002	0.0005	0.0006	0.0004
	Standard deviation	0.0181	0.0217	0.0221	0.0132	0.0174
	Skewness	-1.5274	-0.5124	-0.3906	-0.1649	-1.2751
	Kurtosis	10.7350	7.4429	7.2376	5.1527	10.2675

Our data analysis relies on comparing three RB/CRB portfolios, where the risk preferences are either measured by *standard deviation (SD)* and/or *Conditional-Value-at-Risk (*CVaR*)* risk measures. The first two portfolios, denoted as *SD-RP* and $\text{CVaR}_{95\%}$*-RP*, are standard RB portfolios, as explained in Section 3.1 with $\varphi = SD$ and $\varphi = \text{CVaR}_{95\%}$, respectively. The third

[3]Note that the original investment portfolio in Hallerbach et al. (2004) consists of 590 companies, but 216 firms were delisted or vanished during the 2010-2020 period.

portfolio, denoted as CVaR$_{95\%}$-*SD-CRB*, is a portfolio built on compounding risk measures such that CVaR$_{95\%}$-*SD-CRB* matches the total portfolio risk as measured by SD to the risk measured by the CVaR$_{95\%}$ equivalent portfolio, i.e. the aggregate level of risk measured via SD of the CVaR$_{95\%}$-*SD-CRB* and CVaR$_{95\%}$-*RP* portfolios are equal. This compound measure has the advantage that it meets the regulatory requirements to use the CVaR as the market risk measure, whilst the portfolio allocations are computed via the CRB procedure in Section 3.2 with $\varphi = SD$. Recall that CRB allocations are efficiently computed when $\varphi = SD$, and not when $\varphi = \text{CVaR}_{95\%}$.

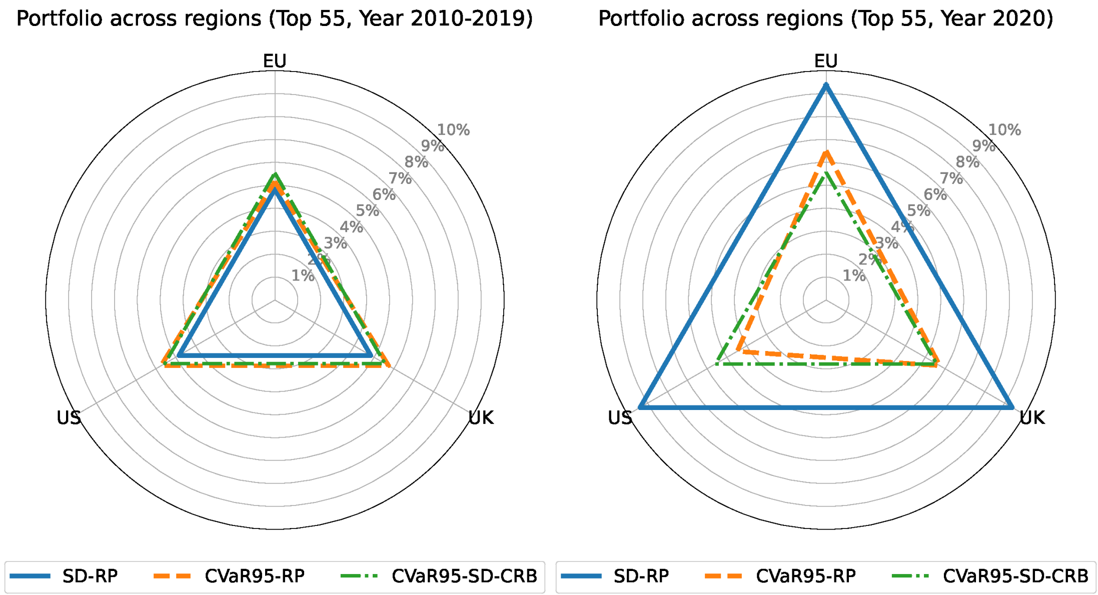

Figure 3: Risk contributions of each region (EU, UK and US) for 2010-2019 (left) and 2020 (right).

Figure 3 compares the (clustered) risk contributions for the three portfolios over the two periods, where the risk contributions are consistently computed with φ being the annualised SD. Each of the three portfolios is composed of $n = 155$ assets by choosing the top 55 SRI ranked companies in each region, namely EU, UK and US.[4] On the left radar chart, when the overall market risk is lower, the three portfolios lead to similar and equal (regional) risk contributions, and in turn, CRP across regions is achieved; a slightly different picture emerges during 2020 when the market risk is significantly higher as seen in Table 2. As anticipated, due to the extremely volatile economic environment of the year 2020, the total SD of *SD-RP* (see graph on the right-side) has a larger overall portfolio's SD than the CVaR$_{95\%}$-*RP* and CVaR$_{95\%}$-*SD-CRB*, which by construction, have the same aggregate level of risk measured via SD. We observe that the US/UK/EU cluster has a higher/similar/lower SD risk contribution for CVaR$_{95\%}$-*SD-CRB* than the SD risk contribution for CVaR$_{95\%}$-*RP*. One possible explanation for this result is the degree of homogeneity or heterogeneity in the companies that are selected in each region. The companies in the EU are subject to more intense regulation and the top companies are expected to have similar SRI scores. This would lead to lower CVaR$_{95\%}$-*SD-CRB*. The US firms are more heterogeneous in behaviour, taking advantage of a more relaxed regulatory regime. This would

[4]The Rest of the World was dropped out because there are only 34 companies in the sample from this region.

make a portfolio constructed with them riskier.

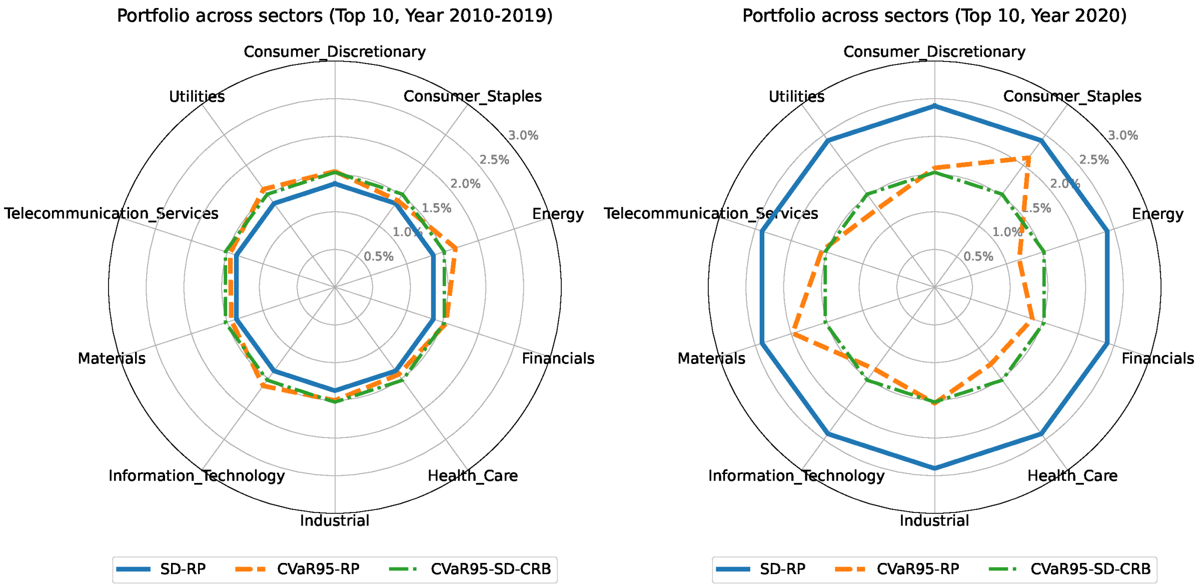

Figure 4: Risk contributions of each of the ten GICS sectors for 2010-2019 (left) and 2020 (right).

We now redo the previous computations by including in each of the three portfolios the top 10 SRI ranked companies in each of the ten GICS sectors listed in Table C.3, and thus, the new three portfolios are composed of $n = 100$ assets. Recall that the new three portfolios have no selection parity imposed at the regional level. The new risk allocations are computed as before, and the results are displayed in Figure 4 that shows a similar pattern to that in Figure 3. The left radar chart in Figure 4 shows the three portfolios are similar during low market risk, and in turn, CRP across sectors is now achieved. The right radar chart in Figure 4 indicates that two sectors, namely *Consumer Staples* and *Materials*, have significantly larger risk allocations for $\text{CVaR}_{95\%}$-*RP* as compared to $\text{CVaR}_{95\%}$-*SD-CRB*, while the individual sectors with high annualised SD, namely *Financial* and *Energy*, have lower risk allocations for $\text{CVaR}_{95\%}$-*RP* as compared to $\text{CVaR}_{95\%}$-*SD-CRB*. This effect can be attributed to the COVID-19 pandemic that has engulfed the major economies and the destabilisation of the world-wide supply chain.

Figure 5 replicates the sector comparison displayed in Figure 4, but only for one specific region, namely the EU. The other two regions (UK and US) are not discussed since the pattern is similar to the EU region. That is, we redo the computations shown in Figure 4 by creating the three portfolios when including only the top 35 and 55 SRI ranked EU companies as displayed on the upper and lower panels, respectively; that is, the upper and the lower panels contain portfolios composed of $n = 35$ and $n = 55$ assets, respectively. For the period 2010-2019, working with a larger pool of companies helps to reduce the risk contributions to each sector, possibly, as a side effect of diversification. The exogenous shock of COVID-19 pandemic in 2020 produces more total risk in all sectors. The general shape in the spider plots is very similar for plots done with the same number of companies, suggesting that the market structure did not change in 2020 but the overall risk levels increased substantially.

Figure 5: Risk contributions of each of the ten GICS sectors for 2010-2019 (left) and 2020 (right). The numbers in each bracket indicate the number of EU companies selected from that particular sector.

4.3. RB for two risk holders

This section provides a data analysis based on our main RB results in Sections 3.4, and our numerical experiments are based on Theorem 6 i). That is, we assume that two insurers holding one LoB, namely a motor insurance portfolio of policies, aim to perform a RS so their resulting risk portfolios are RB balanced. The pre-transfer summary information about monthly aggregated individual claim amount (in euro) is included in the Table C.6 for risk holder 1 and Table C.7 for risk holder 2. Note that the (pre-transfer) aggregate risk for the two risk holders mimics the motor insurance claim experience of two legal entities of an EU well-known insurance group. That is, we use the real-world data to fit and then generate the claim frequency (denoted as F_N) and individual claim amount (denoted as F_X). Based on the *Akaike information criterion (AIC)* information provided in Table C.5, the 'best' *maximum likelihood estimation (MLE)* fitted distributions of the (pre-transfer) risk distributions are: negative binomial distribution (for claim frequency) and log-logistic distribution (for claim amount), though the MLE estimated parameters differ for the two risk holders (for details, see Tables C.6 and C.7). These MLE estimates are then used to generate 10,000 replications of the aggregated (pre-transfer) annual losses for both risk holders, i.e. $S_i = \sum_{k=1}^{12} S_{i,k}, i \in \{1,2\}$, where $S_{i,k}$ is the aggregated monthly loss of risk holder i for the k^{th} month.

The generated data with size $10,000$ become the (pre-transfer) annual losses for the two risk holders. We apply Theorem 6 i) for four cases, where the Risk Holder 1 prefers a RP approach (i.e. $b_{111} = b_{211} = 1/2$) and the target vectors for Risk Holder 2 are varied (i.e., b_{121} is varied within $(0,1)$ such that $b_{111} + b_{211} = 1$ and (3.13) is satisfied). The four cases are as follows: i) *Case 1* with $\varphi_1 = \varphi_2 = SD$, ii) *Case 2* with $\varphi_1 = SD$ and $\varphi_2 = \text{CVaR}_{95\%}$, iii) *Case 3* with $\varphi_1 = \text{CVaR}_{95\%}$ and $\varphi_2 = SD$, and iv) *Case 4* with $\varphi_1 = \varphi_2 = \text{CVaR}_{95\%}$.

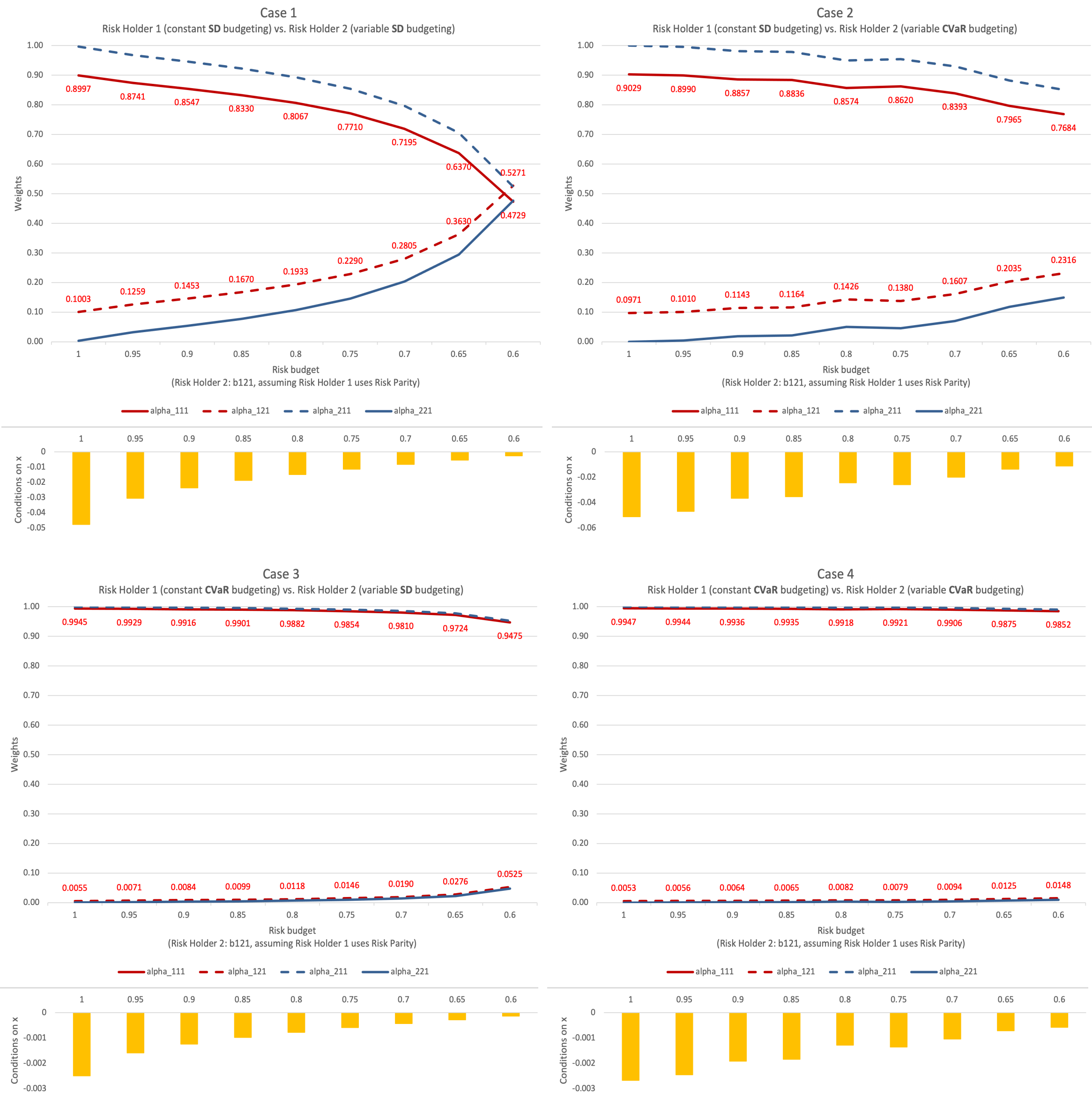

Figure 6: The risk allocations are depicted for the four RS processes as a function of b_{121}, i.e. the budgeting target for the risk transferred from risk holder 1 to risk holder 2. The allocations for the two risk holders are in red and blue for risk holder 1 and risk holder 2, respectively; further, the risk retained by each risk holder appears as solid lines, while the part of risk that is transferred to the other party appears as dashed lines. The orange bars (labelled as 'Conditions on x') represent how negative the value in condition (3.13) is.

The comparative results depicted in Figure 6 show a clear RS process in between the two risk holders in the presence of RB targets, and explain whether there are incentives to the risk holders to adopt this complex RS process. As expected, the conclusions depend on how the risk holders order their risk preferences. RS is appealing for both risk holders in the first two cases, and hardly of interest in the last two RS schemes. This could be explained by the fact that the RP strategy adopted of the first risk holder is quite restrictive for Cases 3 and 4, where the risk holder 1 orders its preferences via a tail risk measure, namely $\text{CVaR}_{95\%}$. Further, the two (pre-transfer) motor insurance portfolios have shown to be quite tail independent, since no common extreme climate event was recorded during the period of observation that contributed to tail insensitive (post-transfer) portfolios.

5. Conclusions

This paper provides an extensive discussion about the theory and practice around constructing RB portfolios in variety of settings. We have started out with revisiting the usual one risk holder setting with and without clustered RB targets, and we then show how those settings could be extended to situations in which a non-hedgeable risk is present or fixed sub-portfolios are aimed by the risk holder. The latter are novel approaches, which widen the application of RB portfolio construction. Another novel approach of this paper is a combination of the concepts of RS and RB, which has not been discussed in the wider risk analysis and risk management literature.

Our theoretical results are accompanied by numerical procedures to identify such RB and RB-RS portfolios. Numerical experiments are provided for pure RB portfolios, where we show how to apply our methods to constructing RB and clustered RB by considering SRI factors. Such SRI factors are becoming now more and more popular given changes in stakeholders' preferences towards societal benefits.

Numerical experiments are also provided for RB-RS portfolios. We could conclude that such RS scheme would raise interest to insurance players if the risk holders face pre-transfer risks that are heavy tailed and tail dependent, and thus, the tail risk preferences govern the risk holders' perception of risk. The other scenario where our novel RS scheme would be of interest is when the the risk holders face pre-transfer risks that are weakly tail dependent and the non-tail risk preferences govern the risk holders' perception of risk.

Funding and interest

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors declare that they have no conflict of interest.

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Appendix A. Proofs

Appendix A.1. Proof of Theorem 3

Note that (3.2) is a strictly convex optimisation problem since $-\lambda\sum_{k=1}^{d} b_k \log x_k$ is a strictly convex function in $\mathbf{x}$ over the convex cone $\Re_{++}^d$. Let $F(\mathbf{x};\lambda)$ be the objective function of (3.2). To show that the solution of (3.2) lies in the interior of $\Re_{++}^d$, it suffices to show that

$$\liminf_{\mathbf{x}\to\mathbf{x}'} F(\mathbf{x};\lambda)=\infty, \quad \text{for any } \mathbf{x}'\in\partial\Re_{++}^d = B_1\cup B_2, \quad \text{where} \tag{A.1}$$

$$B_1 := \bigcup_{\mathcal{I}\subseteq\{1,2,\ldots,d\};\ |\mathcal{I}|\geq 1} \{\mathbf{x} : x_k=\infty \text{ for all } k\in\mathcal{I}, \text{ and } x_k\in[0,\infty)\,, \text{ for all } k\in\{1,2,\ldots,d\}\setminus\mathcal{I}\}\,,$$

$$B_2 := \bigcup_{\mathcal{I}\subseteq\{1,2,\ldots,d\};\ |\mathcal{I}|\geq 1} \{\mathbf{x} : x_k=0 \text{ for all } k\in\mathcal{I}, \text{ and } x_k\in(0,\infty)\,, \text{ for all } k\in\{1,2,\ldots,d\}\setminus\mathcal{I}\}\,.$$

Fix an $\mathbf{x}'\in B_1$, and by the homogeneity of φ, one may get that

$$\begin{aligned} F(\mathbf{x};\lambda) &= \frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau}\varphi\left(\frac{\mathbf{x}^T\mathbf{X}}{\mathbf{1}^T\mathbf{x}}\right) - \lambda\sum_{k=1}^{d} b_k\log\left(\frac{x_k}{\mathbf{1}^T\mathbf{x}}\right) - \lambda\log\left(\mathbf{1}^T\mathbf{x}\right) \\ &\geq \frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau}\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right) - \lambda\sup_{\mathbf{y}\in\Delta_d}\sum_{k=1}^{d} b_k\log y_k - \lambda\log\left(\mathbf{1}^T\mathbf{x}\right) \qquad \text{(A.2)} \\ &= \frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau}\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right) - \lambda\sum_{k=1}^{d} b_k\log b_k - \lambda\log\left(\mathbf{1}^T\mathbf{x}\right)\,, \end{aligned}$$

for any $\mathbf{x}\in\Re_{++}^d$. Therefore,

$$\frac{F(\mathbf{x};\lambda)}{\mathbf{1}^T\mathbf{x}} \geq \frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau-1}\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right) - \frac{\lambda}{\mathbf{1}^T\mathbf{x}}\sum_{k=1}^{d} b_k\log b_k - \lambda\frac{\log\left(\mathbf{1}^T\mathbf{x}\right)}{\mathbf{1}^T\mathbf{x}} \quad \text{for any } \mathbf{x}\in\Re_{++}^d.$$

Clearly, $\sum_{k=1}^{d} b_k\log b_k < 0$ since $\mathbf{b}\in\Delta_d$. Moreover, there exists an $M>0$ such that $\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau-1} > M$ for any $\mathbf{x}$ sufficiently close to $\mathbf{x}'$, since $\tau\geq 1$. Furthermore, for any small $\epsilon>0$, there is a neighbourhood of $\mathbf{x}'$ such that $|\log\left(\mathbf{1}^T\mathbf{x}\right)/\mathbf{1}^T\mathbf{x}|<\epsilon$ since $\log y = o(y)$ as $y\to\infty$ and $\mathbf{x}'\in B_1$. Putting all these together with $\epsilon\downarrow 0$ and keeping (3.1) in mind, one may conclude that

$$\liminf_{\mathbf{x}\to\mathbf{x}'}\frac{F(\mathbf{x};\lambda)}{\mathbf{1}^T\mathbf{x}} > 0, \quad \text{and thus, } \liminf_{\mathbf{x}\to\mathbf{x}'} F(\mathbf{x};\lambda)=\infty \text{ for any } \mathbf{x}'\in B_1.$$

Fix an $\mathbf{x}'\in B_2$; then, there exists an $\mathcal{I}\subseteq\{1,2,\ldots,d\}$ with $|\mathcal{I}|\geq 1$ such that $x'_k=0$ for all $k\in\mathcal{I}$, and $x'_k\in(0,\infty)$ for all $k\in\{1,2,\ldots,d\}\setminus\mathcal{I}$. Similar to (A.2), one may get that

$$F(\mathbf{x};\lambda) \geq \frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau}\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right) - \lambda\sum_{k=1}^{d} b_k\log x_k$$

for any $\mathbf{x}$ sufficiently close to $\mathbf{x}'$. Since $\lambda > 0$ and $\mathbf{b} > \mathbf{0}$, the above equation implies that $\liminf_{\mathbf{x}\to\mathbf{x}'} F(\mathbf{x};\lambda) = \infty$ for any $\mathbf{x}' \in B_2$.

Equation (A.1) implies that there exist an $a > 0$ and an $\epsilon \in (0, a]$ such that

$$\inf_{\mathbf{x}\in\Re^d_{++}} F(\mathbf{x};\lambda) = \inf_{\mathbf{x}\in B_{a,\epsilon}} F(\mathbf{x};\lambda), \quad \text{where } B_{a,\epsilon} := \{\mathbf{x} \in B_a : \min_{1\le k\le d} x_k \ge \epsilon\}$$

with $B_a := \{\mathbf{x} \in \Re^d_{++} : ||\mathbf{x}|| \le a\}$ and $||\cdot||$ being the Euclidean distance. Since $B_{a,\epsilon}$ is a compact set, the global minimum of $F(\cdot;\lambda)$ on $\Re^d_{++}$, i.e. $\mathbf{x}^*(\lambda,\mathbf{b})$, is an interior point of the feasibility set for any given $\lambda > 0$.

It remains to prove that

$$\mathbf{x}^*(\lambda^*,\mathbf{b}) = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b}) \in \mathcal{RB}(\mathbf{b}). \tag{A.3}$$

Firstly, we show that the unique solution of (3.2), i.e. $\mathbf{x}^*(\lambda,\mathbf{b})$, satisfies

$$\mathbf{x}^*(\lambda,\mathbf{b}) = \lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b}) \quad \text{for any } \lambda > 0. \tag{A.4}$$

Assume, on the contrary, that $\lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})$ does not solve (3.2) for a given $\lambda \in \Re_{++} \setminus \{1\}$; that is, there exists $\tilde{\mathbf{x}} \in \Re^d_{++}$ such that

$$\frac{1}{\tau}\mathcal{R}\big(\tilde{\mathbf{x}}\big) - \lambda\sum_{k=1}^d b_k \log \tilde{x}_k < \frac{1}{\tau}\mathcal{R}\big(\lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})\big) - \lambda\sum_{k=1}^d b_k \log\big(\lambda^{1/\tau}x^*_k(1,\mathbf{b})\big).$$

By this inequality and the homogeneity of φ,

$$\begin{aligned}&\frac{\lambda}{\tau}\mathcal{R}\big(\lambda^{-1/\tau}\tilde{\mathbf{x}}\big) - \lambda\sum_{k=1}^d b_k \log\big(\lambda^{-1/\tau}\tilde{x}_k\big) - \lambda\sum_{k=1}^d b_k\frac{\log\lambda}{\tau}\\ &\quad< \frac{\lambda}{\tau}\mathcal{R}\big(\mathbf{x}^*(1,\mathbf{b})\big) - \lambda\sum_{k=1}^d b_k \log\big(x^*_k(1,\mathbf{b})\big) - \lambda\sum_{k=1}^d b_k\frac{\log\lambda}{\tau},\end{aligned}$$

which further implies that

$$\frac{1}{\tau}\mathcal{R}(\lambda^{-1/\tau}\tilde{\mathbf{x}}) - \sum_{k=1}^d b_k \log\big(\lambda^{-1/\tau}\tilde{x}_k\big) < \frac{1}{\tau}\mathcal{R}\big(\mathbf{x}^*(1,\mathbf{b})\big) - \sum_{k=1}^d b_k \log\big(x^*_k(1,\mathbf{b})\big).$$

This contradicts that $\mathbf{x}^*(1,\mathbf{b})$ solves (3.2) with $\lambda = 1$, as $\lambda^{-1/\tau}\tilde{\mathbf{x}} \in \Re^d_{++}$, and concludes (A.4).

Secondly, we show that $\left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b}) \in \Delta_d \cap \mathcal{RB}(\mathbf{b})$. Note that $\mathbf{x}^*(\lambda,\mathbf{b}) \in \Re^d_{++}$, but not guaranteed to be in Δ_d, and thus, is not necessarily in $\mathcal{RB}(\mathbf{b})$. Since $\mathcal{R}(\mathbf{x})$ is differentiable at $\mathbf{x}^*(1,\mathbf{b})$ (and thus at $\mathbf{x}^*(\lambda,\mathbf{b})$ for any $\lambda > 0$ due to (A.4)) and the fact that $\mathcal{R}$ is a homogeneous function, the first-order conditions in (3.2) imply that $\mathcal{RC}_k\big(\mathbf{x}^*(\lambda,\mathbf{b})\big) = b_k\mathcal{R}\big(\mathbf{x}^*(\lambda,\mathbf{b})\big)$ for all $k \in \{1, 2, \ldots, d\}$. However, due to the homogeneity of φ, $\mathcal{RC}_k$ is also homogeneous of the same order as φ, and thus

$$\mathcal{RC}_k\big(t\mathbf{x}^*(\lambda,\mathbf{b})\big) = b_k\mathcal{R}\big(t\mathbf{x}^*(\lambda,\mathbf{b})\big), \quad \text{for all } k \in \{1,2,\ldots,d\} \quad \text{and any } t > 0.$$

In particular, choose $t = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}$ to find that $\left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b}) \in \Delta_d \cap \mathcal{RB}(\mathbf{b})$.

Thirdly, $\mathbf{x}^*(\lambda^*,\mathbf{b}) = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b})$ is true due to (A.4), and in turn $\mathbf{x}^*(\lambda^*,\mathbf{b}) \in \Delta_d$. The latter justifies (A.3), which concludes our proof.

Appendix A.2. Proof of Theorem 5

Let $F(\mathbf{x};\lambda)$ be the objective function in (3.9). One could show that the equivalence of (A.1) holds, and in turn, the global minimum of $F(\cdot;\lambda)$ on $\Re_{++}$, i.e. $\mathbf{x}^*(\lambda,\mathbf{b})$, is an interior point of the feasibility set. As before, the first order conditions imply that $\mathbf{x}^*(\lambda,\mathbf{b})$ solves (3.7). The proof is now complete.

Appendix A.3. Proof of Theorem 6

The proof is similar to the proof of Theorem 3, and thus, we only provide the necessary arguments. We apply the conclusions of (3.2) from Theorem 3 with $\lambda = \{\lambda_1,\lambda_2\}$ in (3.10) and (3.11), and conclude that (3.10) and (3.11) admit unique solutions that are interior points of the feasibility set.

We now show part i). Due to the homogeneity of φ_1 and φ_2, then for any $t_1,t_2 > 0$, $t_1\mathbf{x}^*\left(1,\mathbf{b}_1;\varphi_1\right)$ solves (3.10) with $\lambda_1 = t_1^{-1/\tau_1}$, and $t_2\mathbf{x}^*\left(1,\mathbf{b}_2;\varphi_2\right)$ solves (3.11) with $\lambda_2 = t_2^{-1/\tau_2}$. Thus, we need to find (t_1,t_2) such that the risks are fully allocated within the LoB, i.e. solving

$$t_1x_1^*\left(1,\mathbf{b}_1;\varphi_1\right) + t_2x_1^*\left(1,\mathbf{b}_2;\varphi_2\right) = t_1x_2^*\left(1,\mathbf{b}_2;\varphi_1\right) + t_2x_2^*\left(1,\mathbf{b}_2;\varphi_2\right) = 1, \tag{A.5}$$

which is solved by (3.12). Now, (3.12) leads to a feasible risk allocation if and only if $t_1^*,t_2^* > 0$, which is equivalent to (3.13). The proof of part i) is concluded.

Part ii) could be argued in the same way as part i). Since $x_1^*(1,\mathbf{b}_1;\varphi_1) = x_2^*(1,\mathbf{b}_1;\varphi_1)$, then (A.5) is guaranteed for any $(t_1,t_2) \in \Re^2_{++}$ such that $t_1 + t_2 = 1/x_1^*(1,\mathbf{b}_1;\varphi_1)$, which concludes this part ii).

We now show part iii). Since $(\alpha^*_{111},\alpha^*_{211})$ solves (3.10) with $\lambda = \lambda_1^*$, then

$$\frac{1}{\tau_1\lambda_1^*}\left(\mathcal{R}_1(\alpha^*_{111},\alpha^*_{211}) - \mathcal{R}_1(x_{111},x_{211})\right) \le b_{111}\log\frac{\alpha^*_{111}}{x_{111}} + b_{211}\log\frac{\alpha^*_{211}}{x_{211}}, \tag{A.6}$$

for any $(x_{111},x_{211}) \in \Re^2_{++}$. Similarly, since $(\alpha^*_{121},\alpha^*_{221})$ solves (3.11) with $\lambda = \lambda_2^*$, then

$$\frac{1}{\tau_2\lambda_2^*}\left(\mathcal{R}_2(\alpha^*_{121},\alpha^*_{221}) - \mathcal{R}_2(x_{121},x_{221})\right) \le b_{121}\log\frac{\alpha^*_{121}}{x_{121}} + b_{221}\log\frac{\alpha^*_{221}}{x_{221}}, \tag{A.7}$$

is true for any $(x_{121},x_{221}) \in \Re^2_{++}$. Combining (A.6) and (A.7) imply that

$$\begin{aligned}
&\frac{1}{\tau_1\lambda_1^*}\left(\mathcal{R}_1(\alpha^*_{111},\alpha^*_{211}) - \mathcal{R}_1(x_{111},x_{211})\right) + \frac{1}{\tau_2\lambda_2^*}\left(\mathcal{R}_2(\alpha^*_{121},\alpha^*_{221}) - \mathcal{R}_2(x_{121},x_{221})\right) \\
&\quad\le \min_{\substack{(x_{111},x_{121})\in\Delta_2\\(x_{211},x_{221})\in\Delta_2}} b_{111}\log\frac{\alpha^*_{111}}{x_{111}} + b_{121}\log\frac{\alpha^*_{121}}{x_{121}} + b_{211}\log\frac{\alpha^*_{211}}{x_{211}} + b_{221}\log\frac{\alpha^*_{221}}{x_{221}} \\
&\quad= b_{111}\log\frac{\alpha^*_{111}}{b_{111}} + b_{121}\log\frac{\alpha^*_{121}}{b_{121}} + b_{211}\log\frac{\alpha^*_{211}}{b_{211}} + b_{221}\log\frac{\alpha^*_{221}}{b_{221}} \\
&\qquad + (b_{111}+b_{121})\log(b_{111}+b_{121}) + (b_{211}+b_{221})\log(b_{211}+b_{221})
\end{aligned}$$

$$
\begin{aligned}
&\le \max_{(x_{111},x_{121})\in\Delta_2} b_{111}\log\frac{x_{111}}{b_{111}} + b_{121}\log\frac{x_{121}}{b_{121}} + \max_{(x_{211},x_{221})\in\Delta_2} b_{211}\log\frac{x_{211}}{b_{211}} + b_{221}\log\frac{x_{221}}{b_{221}} \\
&\qquad +(b_{111}+b_{121})\log(b_{111}+b_{121}) + (b_{211}+b_{221})\log(b_{211}+b_{221}) \\
&= (b_{111}+b_{121})\log\frac{1}{b_{111}++b_{121}} + (b_{211}+b_{221})\log\frac{1}{b_{211}++b_{221}} \\
&\qquad +(b_{111}+b_{121})\log(b_{111}+b_{121}) + (b_{211}+b_{221})\log(b_{211}+b_{221}) \\
&= 0,
\end{aligned}
$$

where the second inequality is due to $\alpha^*_{111}+\alpha^*_{121} = \alpha^*_{211}+\alpha^*_{221} = 1$. The proof is now complete.

Appendix B. SD/Variance-based CRB

The SD and variance-based CRB portfolios are the same, and thus, this is true for CRP counterparts. The mathematical formulation of variance-based CRB portfolio is as follows:

$$
\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} = b_k\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha} \quad \text{for all } k\in\{1,\dots,l\}, \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha}=1 \text{ and } \boldsymbol{\alpha}\ge\mathbf{0}. \tag{B.1}
$$

Solving (B.1) is quite challenging, and the only efficient solution is to rely on the equivalent LSE-like formulation in (3.3), which is given as

$$
\min_{\boldsymbol{\alpha}\ge\mathbf{0}}\sum_{k=1}^{l}\left(\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}\right)^2 \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha}=1. \tag{B.2}
$$

The optimisation problem from (B.2) is non-convex and any off-the-shelf general optimisation tools may lead to unstable solutions. Alternatively, a relaxation of (B.2) is suggested in Bai et al. (2016), which could be efficiently solved via the *Alternating Linearisation Method (ALM)*. An appropriation of the ALM approach is provided, and (B.2) is reformulated as

$$
\min_{\boldsymbol{\alpha}\ge\mathbf{0},\theta}\sum_{k=1}^{l}\frac{1}{b_k}\left(\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\theta^2\right)^2 \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha}=1. \tag{B.3}
$$

Algorithm 3 from Bai et al. (2016) precisely solves (B.3) when an equal budget problem (i.e., CRP is sought), and we now adapt the same algorithm for our non-level CRB setting. For ease of notation, we denote $\mathbf{x}^T=(\boldsymbol{\alpha}^T,\theta)\in\Re^{1\times(d+1)}$ and $|\mathcal{I}^{(k)}|=d_k$, where $d_1+d_2+\ldots+d_l=d$, since $\left\{\mathcal{I}^{(1)},\dots,\mathcal{I}^{(l)}\right\}$ is a partition of $\mathcal{I}_d$. Note that

$$
\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\theta^2 = \mathbf{x}^T M_k\mathbf{x}, \quad \text{where} \quad M_k := \begin{bmatrix} \Sigma_{\mathcal{I}^{(k)}}\Gamma_{\mathcal{I}^{(k)}} & \mathbf{0} \\ \mathbf{0}^T & -b_k \end{bmatrix},
$$

and $\Sigma_{\mathcal{I}^{(k)}}\in\Re^{d\times d_k}$ is a submatrix of Σ where the columns of Σ are extracted based only on the indexes of $\mathcal{I}^{(k)}$. Moreover, $\Gamma_{\mathcal{I}^{(k)}}\in\Re^{d_k\times d}$ is a binary matrix such that $\left(\Gamma_{\mathcal{I}^{(k)}}\right)_{st}=\mathbb{1}_{t=\pi^k(s)}$, where $\mathbb{1}_A$ is the indicator function that takes the value 1 if A is true, and 0 otherwise. Further, $\pi^k:\{1,2,\dots,d_k\}\to\mathcal{I}_d$ maps the columns of $\Sigma_{\mathcal{I}^{(k)}}$ of Σ. Therefore, the system of equations in

(B.1) is solved by running a much simpler task:

$$\min_{\mathbf{x}\geq\mathbf{0}} F(\mathbf{x}) := \sum_{k=1}^{l} \frac{1}{b_k}\left(\mathbf{x}^T M_k \mathbf{x}\right)^2 \quad \text{s.t. } \mathbf{c}^T\mathbf{x} = 1, \text{ where } \mathbf{c}^T = (\mathbf{1}^T, 0) \in \Re^{1\times(d+1)}. \tag{B.4}$$

We solve (B.4) by approximating $\mathbf{x}^*$, a local optimum of (B.4). That is, we generate two sequences $\{\mathbf{x}_s : s \geq 0\}$ and $\{\mathbf{y}_s : s \geq 0\}$ such that $\mathbf{x}_s \to \mathbf{x}^*$ and/or $\mathbf{y}_s \to \mathbf{x}^*$. Similar to Algorithm 3 in Bai et al. (2016), a two-block variant of (B.4) is required to solve:

$$\min_{\mathbf{x},\mathbf{y}\in\mathcal{X}} G(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{l} \frac{1}{b_k}\left(\mathbf{x}^T M_k \mathbf{y}\right)^2 \quad \text{s.t. } \mathbf{x} = \mathbf{y}, \tag{B.5}$$

where $\mathcal{X} := \{\mathbf{x} \geq \mathbf{0} : \mathbf{c}^T\mathbf{x} = 1\}$ is the feasible set. Note that (B.5) is a *convex quadratic programming (QP)* instance in $\mathbf{x}$ for any given $\mathbf{y}$ that could be efficiently solved; the same holds if $\mathbf{x}$ and $\mathbf{y}$ are swapped. Further, note that the partial derivatives of G are

$$G_1(\mathbf{x},\mathbf{y}) := \frac{\partial G}{\partial \mathbf{x}} = 2\sum_{k=1}^{l} \frac{\mathbf{x}^T M_k \mathbf{y}}{b_k} M_k \mathbf{y} \quad \text{and} \quad G_2(\mathbf{x},\mathbf{y}) := \frac{\partial G}{\partial \mathbf{y}} = 2\sum_{k=1}^{l} \frac{\mathbf{x}^T M_k \mathbf{y}}{b_k} M_k^T \mathbf{x}.$$

Denote

$$\begin{aligned} H_1(\mathbf{x},\mathbf{y};\mu) &:= G(\mathbf{x},\mathbf{y}) + \langle G_2(\mathbf{y},\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{y}\|_2^2, \\ H_2(\mathbf{x},\mathbf{y};\mu) &:= G(\mathbf{x},\mathbf{y}) + \langle G_1(\mathbf{x},\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{y}\|_2^2, \end{aligned}$$

with $\mu > 0$.The algorithm for solving (B.4), and thus (B.5), is described next as Algorithm 2.

Algorithm 2: CRB algorithm for solving (B.5)

Result: $(\mathbf{x}_{s^*}, \mathbf{y}_{s^*})$ that approximates $\mathbf{x}^*$, a local optimum of (B.4), where s^* is the termination step
$\mu_{1,0} = \mu_{2,0} = \mu_0 > 0$, $\alpha \in (0,1)$, and $\mathbf{x}_0 = \mathbf{y}_0 \in \mathcal{X}$;
for $s \in \{0,1,\ldots\}$ **do**
  $\mathbf{x}_{s+1} := \arg\min_{\mathbf{x}\in\mathcal{X}} H_1(\mathbf{x},\mathbf{y}_s;\mu_{1,s})$;
  if $F(\boldsymbol{x}_{s+1}) \leq H_1(\boldsymbol{x}_{s+1},\boldsymbol{y}_s;\mu_{1,s})$ **then**
    choose $\mu_{1,s+1} \geq \mu_{1,s}$;
  else
    find the lowest $n_{1,s} \geq 1$ such that $F(\mathbf{z}_{1,s}) \leq H_1(\mathbf{z}_{1,s},\mathbf{y}_s;\mu_{1,s}^*)$, where $\mu_{1,s}^* = \mu_{1,s}\alpha^{n_{1,s}}$ and $\mathbf{z}_{1,s} := \arg\min_{\mathbf{x}\in\mathcal{X}} H_1(\mathbf{x},\mathbf{y}_s;\mu_{1,s}^*)$;
    $\mu_{1,s+1} := \mu_{1,s}^*/\alpha$ and $\mathbf{x}_{s+1} := \mathbf{z}_{1,s}$;
  end
  $\mathbf{y}_{s+1} := \arg\min_{\mathbf{y}\in\mathcal{X}} H_2(\mathbf{x}_{s+1},\mathbf{y};\mu_{2,s})$;
  if $F(\boldsymbol{y}_{s+1}) \leq H_2(\boldsymbol{x}_{s+1},\boldsymbol{y}_{s+1};\mu_{2,s})$ **then**
    choose $\mu_{2,s+1} \geq \mu_{2,s}$;
  else
    find the lowest $n_{2,s} \geq 1$ such that $F(\mathbf{z}_{2,s}) \leq H_2(\mathbf{x}_{s+1},\mathbf{z}_{2,s};\mu_{2,s}^*)$, where $\mu_{2,s}^* = \mu_{2,s}\alpha^{n_{2,s}}$ and $\mathbf{z}_{2,s} := \arg\min_{\mathbf{y}\in\mathcal{X}} H_2(\mathbf{x}_{s+1},\mathbf{y};\mu_{2,s}^*)$;
    $\mu_{2,s+1} := \mu_{2,s}^*/\alpha$ and $\mathbf{y}_{s+1} := \mathbf{z}_{2,s}$;
  end
end

Appendix C. Empirical Data

Table C.3: Number of firms for each country within each of the ten GICS sectors and four regions. GICS sectors: Consumer Discretionary (CD), Consumer Staples (CS), Energy (E), Financials (F), Health Care (HC), Industrials (I), Information Technology (IT), Materials (M), Telecommunication Services (TS), Utilities (U).

		GICS SECTOR										
REGION	COUNTRY	**C.D.**	**C.S.**	**E.**	**F.**	**H.C.**	**I.**	**I.T.**	**M.**	**T.S.**	**U.**	**Total**
EU (188)	Austria			1	1				1	1	1	5
	Belgium	1	1		3	1		1	1			8
	Denmark					1			1			2
	Finland						1	1	2		1	5
	France	13	5	1	5	2	7	4		2	1	40
	Germany	7	3		5	2	4	2	4	1	2	30
	Greece				1					1		2
	Ireland				2		1		1			4
	Italy	2		1	6		1	1		1	2	14
	Netherlands	2	3	1	2		2	1	3	1		15
	Norway		1	2			1			1		5
	Portugal				1					1	1	3
	Spain	2		1	3		1			1	3	11
	Sweden	1			4		7	1	2	2		17
	Switzerland	3	1		6	4	6	2	4	1		27
UK (56)	UK	12	7	1	16	3	8	2		2	5	56
US (96)	US	13	12	5	16	13	17	15	1	2	2	96
REST (34)	Australia						2		2			4
	Canada			1			1				2	4
	China	3			2		1				1	7
	Japan	3	1		2	3	1	6		1		17
	Korea							1				1
	Singapore				1							1
	Total	**62**	**34**	**14**	**76**	**29**	**61**	**37**	**22**	**18**	**21**	**374**

Table C.4: Granular financial performance for EW portfolios per GICS sector by including all 374 companies across all four regions in three periods: 2010 - 2020 (top), 2010-2019 (middle) and 2020 only (bottom).

	GICS SECTOR									
	C.D.	C.S.	E.	F.	H.C.	I.	I.T.	M.	T.S.	U.
No. companies	62	34	14	76	29	61	37	22	18	21
EQUAL WEIGHTED PORTFOLIO (daily returns)										
11 years: 2010 - 2020										
Annual. Return	0.097	0.086	-0.006	0.062	0.115	0.114	0.166	0.125	0.042	0.062
Annual. Stdev	0.186	0.135	0.237	0.215	0.131	0.195	0.169	0.190	0.161	0.162
Mean	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
Stdev	0.012	0.009	0.015	0.014	0.008	0.012	0.011	0.012	0.010	0.010
Skewness	-0.663	-0.602	-0.552	-0.497	-0.575	-0.538	-0.714	-0.308	-0.459	-1.082
Kurtosis	12.967	8.552	16.407	12.326	8.069	9.400	8.825	4.902	7.170	15.943
10 years: 2010-2019										
Annual. Return	0.101	0.089	0.019	0.068	0.115	0.107	0.156	0.106	0.049	0.060
Annual. Stdev	0.165	0.123	0.196	0.195	0.116	0.174	0.151	0.185	0.153	0.144
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
Stdev	0.010	0.008	0.012	0.012	0.007	0.011	0.010	0.012	0.010	0.009
Skewness	-0.547	-0.278	-0.112	-0.349	-0.466	-0.292	-0.454	-0.168	-0.156	-0.303
Kurtosis	5.569	2.913	2.180	7.920	2.175	4.552	3.075	3.367	3.895	3.559
1 year: 2020										
Annual. Return	0.043	0.027	-0.207	-0.065	0.104	0.139	0.271	0.237	-0.030	0.074
Annual. Stdev	0.321	0.219	0.475	0.365	0.218	0.326	0.299	0.258	0.251	0.287
Mean	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
Stdev	0.020	0.014	0.030	0.023	0.014	0.021	0.019	0.016	0.016	0.018
Skewness	-0.624	-1.144	-0.643	-0.735	-0.727	-0.817	-1.065	-0.931	-1.532	-2.136
Kurtosis	6.312	10.920	7.968	7.932	7.440	6.917	7.935	10.693	11.804	15.615

Table C.5: Fit the 'best' MLE for F_N (i.e., **nbinom***: Negative Binomial distribution) and F_X (i.e., **llogis***: Log-logistic distribution) via AIC.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Risk holder 1												
F_N: AIC												
pois	264	312	461	522	572	467	508	435	414	424	318	248
nbinom*	135	138	145	146	149	146	148	145	144	145	139	133
F_X: AIC												
lnorm	37430	37619	43096	40138	45983	46294	48116	45040	44074	44935	39497	36671
exp	39297	39540	45648	42456	48423	48512	50671	47202	46159	47659	41502	38386
gamma	39262	39538	45650	42457	48424	48500	50673	47192	46161	47631	41500	38335
weibull	39021	39343	45429	42268	48223	48420	50474	47109	45984	47241	41304	38074
llogis*	37428	37442	42937	39924	45647	46092	47831	44832	43867	44715	39387	36627
invweibull	37855	39677	43907	41851	49184	48276	50383	47115	48679	47066	40396	37826
pareto	38276	38536	44424	41337	47180	47588	49421	46301	45123	46070	40490	37338
Risk holder 2												
F_N: AIC												
pois	261	372	435	498	396	474	461	415	411	432	316	197
nbinom*	135	141	145	145	144	147	147	145	144	145	139	129
F_X: AIC												
lnorm	37363	37613	43289	40346	46220	46447	47976	45474	43493	44761	39560	36492
exp	39248	39487	45673	42388	48426	48771	50283	47717	45587	47133	41349	38232
gamma	39217	39476	45672	42389	48416	48768	50279	47718	45579	47113	41349	38203
weibull	38959	39253	45432	42235	48332	48638	50167	47572	45483	46800	41174	37962
llogis*	37277	37515	43058	40104	45984	46228	47774	45318	43341	44521	39354	36428
invweibull	38346	38644	44703	42805	48296	48125	49706	46704	45047	46669	41991	37619
pareto	38158	38460	44428	41390	47500	47722	49306	46715	44681	45752	40405	37217

Table C.6: Summary information about the individual claim amount (in euro) for Risk Holder 1. The MLE estimates for F_X and F_N are as given in Table C.5.

	Empirical data (individual claim amount in euro)					MLE estimates (amount & freq)			
	Mean	SD	Quantiles			F_X: log-logistic		F_N: neg. binomial	
			q = 50%	q = 75%	q = 90%	shape	scale	size	mu
Jan	1079.13	1894.54	522.91	1068.30	2410.23	1.67	544.76	34.59	528.62
Feb	945.64	1776.45	493.53	930.13	1875.84	1.78	505.24	27.88	553.57
Mar	891.36	1789.33	467.68	885.45	1727.29	1.83	478.88	18.99	650.85
Apr	874.65	1904.59	465.84	855.00	1727.60	1.85	477.30	15.02	608.96
May	845.58	2024.08	454.15	849.35	1668.39	1.85	469.19	15.81	702.13
Jun	859.16	1482.66	477.27	892.81	1728.68	1.84	486.85	19.80	699.85
Jul	893.49	1717.87	478.42	888.14	1750.00	1.84	490.46	17.76	721.95
Aug	872.76	1507.28	475.98	898.01	1791.71	1.84	491.81	20.99	677.55
Sep	938.92	1672.51	495.84	955.62	1873.66	1.78	509.93	20.96	647.64
Oct	986.02	2455.77	490.19	939.29	1974.40	1.78	506.66	20.12	658.68
Nov	955.87	1759.53	494.41	961.48	1949.64	1.76	509.90	27.31	579.09
Dec	1116.23	2062.02	535.59	1128.97	2463.10	1.64	560.96	36.40	511.27

Table C.7: Summary information about the individual claim amount (in euro) for Risk Holder 2. The MLE estimates for F_X and F_N are as given in Table C.5.

	Empirical data (individual claim amount in euro)					MLE estimates (amount & freq)			
	Mean	SD	Quantiles			F_X: log-logistic		F_N: neg. binomial	
			q = 50%	q = 75%	q = 90%	shape	scale	size	mu
Jan	1073.97	2038.80	533.77	1059.76	2204.65	1.69	548.85	35.14	528.70
Feb	941.48	1846.69	482.46	953.32	1925.49	1.72	496.01	21.32	553.58
Mar	892.53	1717.38	460.53	871.87	1765.93	1.80	477.89	20.31	650.95
Apr	869.81	1630.98	464.14	873.45	1725.15	1.82	478.86	16.20	609.13
May	845.77	1472.67	466.59	883.71	1699.71	1.85	478.72	25.98	702.10
Jun	873.41	1577.57	477.67	902.74	1702.56	1.84	488.32	19.06	699.93
Jul	871.92	1485.73	473.98	886.88	1792.83	1.83	485.88	20.96	722.00
Aug	903.43	1548.28	478.29	914.71	1862.66	1.80	494.89	21.14	677.53
Sep	901.52	1685.62	500.49	937.22	1801.89	1.83	508.89	20.57	647.70
Oct	950.89	1993.65	479.82	938.70	1880.82	1.75	495.98	19.19	658.79
Nov	943.98	1690.15	492.61	963.82	1950.41	1.76	508.40	28.40	579.18
Dec	1100.38	2076.28	547.08	1118.78	2308.01	1.68	563.51	56.85	511.34