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RESEARCH

Hochschild cohomology of symmetric groups and generating functions, II

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Abstract

We relate the generating functions of the dimensions of the Hochschild cohomology in any fixed degree of the symmetric groups with those of blocks of the symmetric groups. We show that the first Hochschild cohomology of a positive defect block of a symmetric group is nonzero, answering in the affirmative a question of the third author. To do this, we prove a formula expressing the dimension of degree one Hochschild cohomology as a sum of dimensions of centres of blocks of smaller symmetric groups. This in turn is a consequence of a general formula that makes more precise a theorem of our previous paper describing the generating functions for the dimensions of Hochschild cohomology of symmetric groups.

Keywords: Hochschild cohomology, Symmetric groups, Partition identities, Blocks

Mathematics Subject Classification: 20C20

1 Introduction

Let p be a prime number and k a field of characteristic p. As a consequence of results in [4], using the classification of finite simple groups, if G is a finite group of order divisible by p, then $HH^1(kG)$ is nonzero. It is an open question [5, Question 7.4] whether for G a finite group and B a block of kG, if the defect groups of B are non-trivial, then $HH^1(B)$ is nonzero. This question has been shown to have a positive answer in some cases in [8] and [6], for instance. We prove that this question has an affirmative answer if G is a symmetric group $\mathfrak{S}_n$ on n letters.

Theorem 1.1 *Let B a block of $k\mathfrak{S}_n$ with non-trivial defect groups. Then $HH^1(B) \neq 0$.*

In fact, we give a precise formula for the dimension of the first Hochschild cohomology of a block of a symmetric group as a sum of dimensions of the centres of blocks of smaller symmetric groups (Theorem 1.2), and this easily implies Theorem 1.1.

In order to state the formula, let us recall that to each block B of $k\mathfrak{S}_n$ is associated a non-negative integer $w \leqslant \lfloor n/p \rfloor$ called the weight of B, with the property that the Sylow p-subgroups of $\mathfrak{S}_{pw}$ are defect groups of B under the natural inclusion $\mathfrak{S}_{pw} \leqslant \mathfrak{S}_n$. In particular, B has non-trivial defect groups if and only if $w > 0$. Moreover, by Theorem 7.2 of Chuang and Rouquier [3] if B and B' are blocks of possibly different symmetric groups,

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with the same weight, then B and B' are derived equivalent algebras, and consequently, for any $r \geqslant 0$, we have $\dim_k HH^r(B) = \dim_k HH^r(B')$.

For $w \geqslant 0$, denote by B_{pw} the principal block of $k\mathfrak{S}_{pw}$. Then B_{pw} has weight w and by the above, $\dim_k HH^r(B) = \dim_k HH^r(B_{pw})$ for any weight w block B of a symmetric group algebra. Thus Theorem 1.1 is a consequence of the following result. For $w \geqslant 0$, let $\rho(pw, \varnothing)$ equal the number of partitions of pw with empty p-core.

Theorem 1.2 *Let B be a weight w-block of a symmetric group algebra over k. If $p = 2$, then*

$$\dim_k HH^1(B) = 2\sum_{j=0}^{w-1} \dim_k Z(B_{pj}) = 2\sum_{j=0}^{w-1} \rho(pj, \varnothing).$$

If $p \geqslant 3$, then

$$\dim_k HH^1(B) = \sum_{j=0}^{w-1} \dim_k Z(B_{pj}) = \sum_{j=0}^{w-1} \rho(pj, \varnothing).$$

The proof of the above formula goes through the following theorem relating generating functions of dimensions of Hochschild cohomology of blocks of symmetric groups with those of the entire group algebra and then invoking the results of our previous paper [1]. Denote by $p(n)$ the number of partitions of n, and by $P(t)$ the generating function $\sum_{n=0}^{\infty} p(n)t^n$. Note that $\dim_k HH^0(k\mathfrak{S}_n) = \dim_k Z(k\mathfrak{S}_n) = p(n)$.

Theorem 1.3 *Set $Z(t) = \sum_{n=0}^{\infty} \dim_k Z(B_{pn})\, t^n$. For any $r \geqslant 1$, there exists a rational function $\phi(t)$ (depending on p and r) with $\phi(0)$ nonzero, such that*

$$\sum_{n=0}^{\infty} \dim_k HH^r(B_{pn})\, t^n = t\phi(t)Z(t)$$

and

$$\sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_n)\, t^n = t^p\phi(t^p)P(t).$$

Remark 1.4 In [1], we proved that

$$\sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_{pn})\, t^n = R_{p,r}(t)P(t)$$

where $R_{p,r}(t)$ is a rational function of t. Ken Ono asked us whether $R_{p,r}(t)$ is a rational function of t^p, and Theorem 1.3 proves that this is the case, with $R_{p,r}(t) = t^p\phi(t^p)$.

Remark 1.5 The constant coefficient of $\phi(t)$ in Theorem 1.3 is equal to $y_1 = \dim_k(HH^r(B_p))$. Since B_p is derived equivalent to $k(C_p \rtimes C_{p-1})$, we have $y_1 = \dim_k HH^r(k(C_p \rtimes C_{p-1}))$. An easy calculation, using the centraliser decomposition, shows that for $r \geqslant 1$ we have $y_1 = 2$ if $r \equiv 0$ or -1 modulo $2(p-1)$ (which is in particular the case if $p = 2$) and $y_1 = 1$ otherwise. If we drop the condition $\phi(0) \neq 0$, then the two formulas in Theorem 1.3 hold trivially for $r = 0$ with $\phi(t) = \frac{1}{t}$.

Remark 1.6 We note that the above results do not depend on the choice of the field k. If k' is an extension field of k and B a block of $k\mathfrak{S}_n$ for some positive integer n, then $B' = k' \otimes_k B$ is a block of $k'\mathfrak{S}_n$ having the same defect groups as B, and for any finite-dimensional k-algebra A we have a graded k-algebra isomorphism $HH^*(k' \otimes_k A) \cong k' \otimes_k HH^*(A)$.

2 Proofs.

We begin with an elementary lemma.

Lemma 2.1 *Let R be an integral domain and m a positive integer. If $h(t) \in R[[t]]$ is such that $h(t^m) \in R((t))$ is a rational function, then $h(t)$ is also a rational function.*

Proof Let $h(t) = \sum_{n=0}^{\infty} h_n t^n$, $h_n \in R$ and suppose that $a(t), b(t) \in R[t]$ are such that

$$\sum_{n=0}^{\infty} h_n t^{mn} = h(t^m) = \frac{a(t)}{b(t)}.$$

If $h(t) = 0$, then there is nothing to prove. So, we assume that $h(t) \neq 0$. Write

$$a(t) = \sum_{s=0}^{m-1} a_s(t^m)\, t^s, \qquad b(t) = \sum_{s=0}^{m-1} b_s(t^m)\, t^s,$$

for $a_s(t), b_s(t) \in R[t]$, $0 \leqslant s \leqslant m-1$. Comparing coefficients of powers of t, the equality

$$a(t) = b(t) \sum_{n=0}^{\infty} h_n\, t^{mn}$$

implies the equality

$$b_s(t^m) = a_s(t^m) \sum_{n=0}^{\infty} h_n\, t^{mn}$$

for each s, $0 \leqslant s \leqslant m-1$. Choose s such that $b_s(t) \neq 0$. Then $h(t) = a_s(t)/b_s(t)$ is a rational function of t. □

Proof of Theorem 1.3 Let $r \geqslant 1$. For $n \geqslant 0$, let $c(n)$ denote the number of p-core partitions of n and for each s, $0 \leqslant s \leqslant p-1$, set $C_s(t) = \sum_{n=0}^{\infty} c(np+s)\, t^n$. For $w \geqslant 0$, set $z_{pw} = \dim_k Z(B_{pw})$ and $y_{pw} = \dim_k HH^r(B_{pw})$. We use the notation

$$Z(t) = \sum_{n=0}^{\infty} z_{pn}\, t^n$$

from the statement of Theorem 1.3 and we set

$$Y(t) = \sum_{n=0}^{\infty} \dim_k HH^r(B_{pn})\, t^n = \sum_{n=0}^{\infty} y_{pn}\, t^n.$$

Note that by [3, Theorem 7.2], y_{pw} is the dimension of the degree r Hochschild cohomology of any weight w block of a symmetric group algebra. Also, recall that the ordinary irreducible characters of $\mathfrak{S}_n$ are labelled by partitions of n. By the Nakayama conjecture, proved by Brauer [2] and Robinson [7], two ordinary irreducible characters of $\mathfrak{S}_n$ belong to the same p-block of $\mathfrak{S}_n$ if and only if the partitions labelling them have the same p-core, and consequently the set of blocks of $k\mathfrak{S}_n$ is in bijection with the set of p-cores of partitions of n. Further, a block of $k\mathfrak{S}_n$ has weight w if and only if the corresponding p-core is a partition of size $n-pw$. Thus, the number of weight w blocks of $k\mathfrak{S}_n$ equals $c(n-pw)$. Since the

Hochschild cohomology of a finite dimensional algebra is the direct sum of the Hochschild cohomologies of its blocks, we have that for any s with $0 \leqslant s \leqslant p-1$, and any $n \geqslant 0$,

$$\dim_k Z(k\mathfrak{S}_{pn+s}) = \sum_{w=0}^{n} z_{pw} c(p(n-w)+s),$$

and

$$\dim_k HH^r(k\mathfrak{S}_{pn+s}) = \sum_{w=0}^{n} y_{pw} c(p(n-w)+s).$$

In other words, for any s, $0 \leqslant s \leqslant p-1$,

$$\sum_{n=0}^{\infty} \dim_k Z(k\mathfrak{S}_{pn+s})\, t^{pn+s} = t^s Z(t^p) C_s(t^p) \tag{2.2}$$

and

$$\sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_{pn+s})\, t^{pn+s} = t^s Y(t^p) C_s(t^p). \tag{2.3}$$

Since $B_0 \cong k$, we have $z_0 = \dim_k Z(B_0) = 1$ and from this it follows that $Z(t)$ is invertible in $\mathbb{Z}[[t]]$. On the other hand, we have $y_0 = \dim_k HH^r(B_0) = 0$ and $y_1 = \dim_k HH^r(B_p) \neq 0$ (cf. Remark 1.5). In particular, $Y(t)$ is divisible by t but not by t^2 in $\mathbb{Z}[[t]]$. So we may define a power series in t, with nonzero constant term,

$$\phi(t) = (t^{-1} Y(t)) Z(t)^{-1} \in \mathbb{Z}[[t]],$$

and then we have

$$Y(t) = t\phi(t) Z(t). \tag{2.4}$$

Now,

$$\begin{aligned}\sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_n)\, t^n &= \sum_{s=0}^{p-1} \sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_{pn+s})\, t^{pn+s} = Y(t^p) \sum_{s=0}^{p-1} C_s(t^p)\, t^s \\ &= t^p \phi(t^p) \sum_{s=0}^{p-1} Z(t^p) C_s(t^p)\, t^s\end{aligned}$$

where we have used (2.3) for the second equality and (2.4) for the third equality. On the other hand, by (2.2),

$$P(t) = \sum_{s=0}^{p-1} \sum_{n=0}^{\infty} \dim_k Z(k\mathfrak{S}_{pn+s})\, t^{pn+s} = \sum_{s=0}^{p-1} Z(t^p) C_s(t^p)\, t^s.$$

So,

$$\sum_{n=0}^{\infty} \dim_k HH^r(k\mathfrak{S}_n)\, t^n = t^p \phi(t^p) P(t).$$

Now by Theorem 1.3 of [1] we have that $t^p\phi(t^p)$ is a rational function of t. It then follows from Lemma 2.1 that $\phi(t)$ is a rational function of t, and by the above, $\phi(0)$ is nonzero. □

Proof of Theorem 1.2 By Theorem 1.2 of [1],

$$\sum_{n=0}^{\infty} \dim_k HH^1(k\mathfrak{S}_n)\, t^n = \begin{cases} \dfrac{2t^2}{1-t^2}\, P(t) & p=2, \\ \dfrac{t^p}{1-t^p}\, P(t) & p \geqslant 3. \end{cases}$$

Note that in [1] the base field is taken to be $\mathbb{F}_p$, but the above result clearly holds as well for any field of characteristic p (cf. Remark 1.6). By Theorem 1.3 and its notation, applied with $r = 1$,

$$\sum_{n=0}^{\infty} \dim_k HH^1(B_{pn})\, t^n = \begin{cases} \dfrac{2t}{1-t}\, Z(t) & p=2, \\ \dfrac{t}{1-t}\, Z(t) & p \geqslant 3. \end{cases}$$

Here, note that $h(t) \to h(t^m)$ is an injective map on $\mathbb{Z}[[t]]$. This, along with [3, Theorem7.2] proves the equations

$$\dim_k HH^1(B_{pw}) = \begin{cases} 2\displaystyle\sum_{j=0}^{w-1} \dim_k Z(B_{pj}) & p=2, \\ \displaystyle\sum_{j=0}^{w-1} \dim_k Z(B_{pj}) & p \geqslant 3. \end{cases}$$

of Theorem 1.2. The remaining equations follow from the Nakayama Conjecture, proved in [2,7], which states that characters of a symmetric group belong to the same p-block if and only if the corresponding partitions have the same p-core, implying in particular that $\dim_k Z(B_{pj}) = \rho(pj, \varnothing)$. □

Theorem 1.1 is an immediate consequence of Theorem 1.2.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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