Knightian Uncertainty Modelling and its Impact on Option Pricing:

Applications of Fuzzy Set Theory, Fuzzy Measure Theory and Fuzzy Differential Calculus

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Dedication

This thesis is dedicated to my father,

Abdul Karim Assi,

my mother,

Zeinab Saleh,

and

my husband,

Ibrahim Moussa.

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Abstract

This research tackles the issue of uncertainty due to lack of information, alternatively known as Knightian Uncertainty, and its impact on option pricing. In the presence of such uncertainty, Probability Theory becomes restrictive and alternative tools are called for. In this research, we consider tools of Fuzzy Theory. We introduce three Option Pricing Models the first of which is a fuzzy binomial model based on the standard CRR binomial model. The model performs option pricing in a fuzzy world characterized by blurred prices. In such a world, it is no longer possible to price by replication. So we introduce a fuzzy pricing approach that employs Sugeno integration and fuzzy measures, and generates bounds on the possible option price. The second model is a fuzzy Black-Scholes model, which prices options in the presence of uncertain or fuzzy volatility. We model such volatility by establishing bounds on the corresponding fuzzy values thereby generating fuzzy bounds on the possible option price. Finally, the third model is an extension on an existing one period fuzzy binomial model that prices options when the underlying price is characterized by opacity. The option price returned by this model is dependent on a market parameter that summarizes its completeness. However, it is possible to defuzzify the last two models to obtain one crisp price that summarizes market information. The last two models outperform their standard counterparts.

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Chapter 1

Introduction

It has long been said that the only thing that is certain is uncertainty. One has to deal with uncertainty on daily basis irrespective of how aware he/she may be. Bertrand Russell has once said that "Everything is vague to a degree you do not realize till you have tried to make it precise." So to capture uncertainty, we try to formulate approximations of the world around us. This approach is also extended to mathematical and scientific disciplines as well as social sciences.

However, approximations are not always easy to come by because we frequently stumble on imperfect or vague information, which gives rise to uncertainty. Uncertainty in such instances is differentiated from risk since, in the latter case, we have enough information to estimate our chances or probabilities while, in the former case, this is not always possible. Sometimes, it is still possible to formulate approximations by resorting to the Principle of Insufficient Reasoning and taking up few assumptions. At other times, this is not enough and we need other means. As problems get more complex, the means of handling them get more sophisticated.

The primary tool of dealing with uncertainty has always been Probability Theory. Probability Theory has come a long way in solving many important problems in various disciplines. But it requires enough information about the issue under examination. As a result, it may become restrictive in the presence of imperfect information, which motivates us to look for other alternatives. This has been recognized as far back as 1921 by Knight in a quotation which says, "If risk were exclusively of the nature of known chance or mathematical probability, there could be no reward for risk-taking; the fact of risk could exert no considerable influence on the distribution of income in any way. For if the actuarial chance of gain or loss in any transaction is ascertainable, either by calculation a priori or by the application of statistical methods to past experience, the burden of bearing the risk can be avoided by the payment of a small fixed cost limited to the administrative expense of providing insurance." ([93], p.46) The purpose of this research is to look at uncertainty arising from imperfect information, otherwise known as Knightian uncertainty, and at means of dealing with it particularly in Option Pricing.

Uncertainty has been dealt with independently in different disciplines. We will look at two such disciplines that are relevant to our research and which are Fuzzy Theory and Economic Theory. The difference between the two theories is that the former provides alternatives to Probability Theory while the latter provides applications to which such tools are applied. A common factor in both theories is that uncertainty is distinguished from risk, even though the decomposition of uncertainty is more detailed in Fuzzy Theory. On the other hand, the means of dealing with uncertainty in both disciplines are different. Therefore, we attempt to incorporate both theories into one coherent framework and transmit the information to dealing with options in an uncertain environment.

Options are actually exposed to such uncertainty from two sources, the options market itself and the financial market corresponding to the underlying asset on which the option is contingent. Uncertainty on the underlying generates uncertainty on the option itself. In particular, we will be considering the uncertainty generated by the underlying. There is a very important distinction between risk and uncertainty that needs to drawn at this point in order to avoid confusion. Options are indeed traded for investment purposes but they are also perceived as insurance against risk that arises from the behaviour of the underlying in the future. In many cases, we have enough information to know our risk and hence employ Probability Theory. However, at other times, the nature of the underlying or the market on which it is traded, as well as many other factors, generate uncertainty characterized by lack of information. In this case, we have to look for alternatives to Probability Theory. Researchers have been using a variety of tools and approaches that serve as alternatives to Probability Theory in such circumstances, as we will see during the course of Chapter 3. The approach we adopt to study the implications of such uncertainty on options is to price them in uncertain environments using Fuzzy Theory tools.

Fuzzy Theory provides us with tools of dealing with the uncertainty and imprecision frequently encountered within the social sciences. Such tools may also serve as alternatives to Probability Theory within the appropriate problem context. In fact, fuzziness in economics and finance has been recognized by fuzzy experts for some time, who have addressed it using Fuzzy Theory tools. However, it is only until recently that finance researchers have recognized the value of such tools. The most sophisticated applications are those in Fuzzy Option Pricing. But the applications of Fuzzy Theory in Economics and Finance are far from being exhausted. We believe that Fuzzy Theory can add significant value to Economics and Finance and has the potential of solving many pending issues. It is our aim in this research to illustrate these points.

We use the term Fuzzy Theory to include both, Fuzzy Set Theory and Fuzzy Measure Theory, even though they are two separate theories that have been developed independently. Fuzzy Set Theory has been developed before Fuzzy Measure Theory and has received wider popularity. Consequently, it is more 'advanced' and it has been applied in a variety of fields more frequently than Fuzzy Measure Theory has. However, both theories are relatively new and some important concepts pertaining to them are still scattered in various publications. Hence, we assume the task of gathering and presenting them in this research.

We start Chapter 2 with the foundations of both theories and associated concepts. We leave the basics to the appendices for readers who are new to such approaches. In this chapter, we discuss important issues related to Fuzzy Set Theory such as the membership function and probability and possibility distributions. We also look at the building blocks of Fuzzy Measure Theory including fuzzy measures and fuzzy integration. We specifically consider elicitation approaches of the fuzzy measure that will be used at a later stage in this research. We consider two different nonlinear integration approaches, which are Sugeno Integral and Choquet Integral.

Chapter 3 brings the uncertainty aspect in Finance Theory and Economic Theory in general, and Option Pricing in particular, together with that in Fuzzy Theory. For this purpose, we present a brief overview of Knightian uncertainty in Fuzzy Theory and Economic Theory, recent developments in Option Pricing and fuzziness in Economics and Finance. Finance and Economics researchers have long been aware of the presence of Knightian uncertainty and of the need for alternatives to Probability Theory in some circumstances. On the other hand, Fuzzy Theory researchers have always recognized the presence of imprecision and uncertainty due to lack of information. In fact, this is one of the basic premises of Fuzzy Theory and the reason why it has been founded in the first place as an alternative to Probability Theory in such environments. However, research in those fields has been developed separately, despite the fact that they can both contribute to each other, except for the relatively few occasions where fuzzy researchers have attempted to apply Fuzzy Theory to Finance and Economics problems that have not been examined thoroughly by Finance researchers further hindering the spread of Fuzzy Theory tools to Finance and Economics. Fortunately and more recently, there have been several successful attempts on the part of Finance researchers to apply Fuzzy Theory to problems in Finance, mainly in Option Pricing. So in this chapter, we will look at how Economic Theory and Fuzzy Theory perceive uncertainty and at the suggested approaches to handle it. We also look at the applications of Fuzzy Theory in Economics and Finance as proposed by both Fuzzy and Finance researchers.

We will also review recent developments in Option Pricing. This is important in giving us an idea as to where our research stands vis-a-vis existing Option Pricing research. Lately, there has been a recognition of the impact of uncertainty on options in the form of uncertain parameters for example. As a result, several approaches have been proposed to handle this uncertainty. The general result has been that, in this environment, we can only obtain no-arbitrage bounds on possible option prices. We will examine the various tools and approaches that existing Option Pricing literature has employed. In this manner, we will have presented a general framework through which we can present our research as well as equip our reader with a comprehensive view of uncertainty and associated tools. In brief, we introduce two Fuzzy Option Pricing models plus another model that extends on an existing one. We apply those models to empirical data for the purpose of studying their applicability and comparability to other crisp models.

So starting with the first model in Chapter 4, which is our major contribution in this research, we present a Fuzzy Binomial Model, which builds on the CRR binomial OPM in a fuzzy world. We consider two aspects of uncertainty in this problem, one that is associated with the Bid/Ask quotations and spread and another that is associated with vagueness concerning future states of the world. We will not dwell into these issues now since we will discuss them in details in Chapter 4.

These types of uncertainties generate fuzziness around the call price, which we attempt to price. Due to the problem definition, it will no longer be possible to price using conventional tools. For a start, Probability Theory cannot be applied because the additivity pre-requisite becomes too restrictive. We can substitute this with Fuzzy Measure Theory. We introduce two new fuzzy measures, a conditional measure that captures the uncertainty associated with Bid/Ask quotations and spread and a regular one that captures the vagueness associated with future states of the world. However, due to fuzziness, we can no longer price the option by replication but we are able to reserve the preference-free characteristic in the fuzzy measures we introduce.

The general methodology involves transforming fair or model prices into observable or fuzzy prices revolving around the fair ones. Then we price the option by backward induction across the binomial. To perform the pricing, we calculate the expectations in a risk-neutral world. However due to the nonadditive nature of the fuzzy measures, we can no longer use linear integration to calculate the expected value. We have to resort to nonlinear integration, of which we try two approaches, the Sugeno Integral and the Choquet Integral. In fact, we concentrate on the Sugeno Integral, whose use in this research represents our most important contribution, since it has not been used before in Finance and Economics. We apply the Choquet integral for comparison purposes only. Similarly to recent models in Option Pricing, we obtain bounds on the possible option prices that envelope and super- or sub-replicate the CRR or the fair model value. So, as we will see, this model is original in the idea it presents as well as the tools and methodology.

In Chapter 5, we apply the model to examples from markets characterized by varying levels of uncertainty. This is a specialized model that applies when uncertainty is high so one has to be careful when to apply it. It is not proposed as an alternative to existing models across all environments. As the empirical analysis shows us, it is only acceptable in markets characterized by high uncertainty. We look at American as well as European Call Options from the NASDAQ, currency and Index Options markets. As expected, the model performs best in the NASDAQ market and worst in the Index (S&P500) market due to the transparency and relatively low level of uncertainty of the latter.

We look at how our bounds behave when expectations are calculated using a Sugeno Integral as to how they behave when using a Choquet Integral. We also analyze the behaviour of the bounds across moneyness and expiration. Since this model builds on the standard CRR OPM, it is useful to look at how the bounds vis-a-vis that value. Finally, we look at the implications of using a historical volatility versus using an implied volatility.

Chapter 6 presents a Fuzzy Black-Scholes model. Fuzziness is generated by the uncertainty of volatility. Volatility is perhaps the only input to an Option Pricing Model that cannot be estimated with precision. Rather than having to have a complete view of the volatility in the market, literature has frequently resorted to getting around this problem by using stochastic volatility models and deterministic volatility surfaces among others. Yet other approaches have encouraged the use of an uncertain volatility which is bounded by two extreme yet possible values. Our work is more in line with such approaches but it differs from them in the way volatility is modelled and in the approach to solving the problem. We model volatility as a fuzzy number and so we obtain a fuzzy original PDE that we solve with the help of Fuzzy Differential Calculus, which is also a new area in Fuzzy Theory that we summarize in this research based on scattered publications. The solution to this fuzzy PDE is a fuzzy Call Price that is bounded between two values and dependent on a parameter that can be perceived as summarizing the level of information in the market, or the level of market completeness or the level of confidence.

Another contribution of the model is its flexibility. The user who is interested in one value for the Call Price can defuzzify the Call Price to get one value summarizing information in the market. Alternatively, an expert may prefer to manipulate the parameter on which the bounds are dependent, thereby manipulating the bounds to suit his view of the market in the presence of varying degrees of uncertainty. It is also possible to use the formula to back out bounds for the implied volatility in the market.

In Chapter 7, we compare this model to the standard Black-Scholes OPM as well as to the Uncertain Volatility Model, which employs a similar idea and has been recently introduced. To compare it to the Black-Scholes model, we have to defuzzify it, establish the bounds on the implied volatility by solving an optimization problem and then forecast option prices in a different sample. Whereas to compare it to the Uncertain Volatility Model, we have to vary the 'information' parameter and analyze where the values given by the latter model fit. It is also possible to substitute the bounds of the implied volatility and compare the bounds given by both models.

In Chapter 8, we draw a comparative analysis between existing Fuzzy Option Pricing approaches. This area is still in its infancy and the corresponding applications are still few; in fact, there are only three published papers. However, they are comprehensive examples after which other applications can follow. As such, they serve to lay the foundations for Fuzzy Option Pricing. We will look at two fuzzy binomial models ([110],[112]) which price options in the presence of uncertainty but within different problem definitions. One of the models is a one period model, so we build on it a mutli-period fuzzy model that serves as a rough extension. Both models use tools of Fuzzy Set Theory and Fuzzy Arithmetic to solve the problem. The third model [30] we look at employs Fuzzy Measure Theory to solve a Fuzzy Black-Scholes equation. We apply the models, whenever possible, to empirical data. Finally, we conclude in Chapter 9.

Chapter 2

Fuzzy Set Theory and Fuzzy Measure Theory

Albert Einstein has once said, "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." There has been an increasing need for modelling imprecision and uncertainty in various fields recently, which has induced researchers to come up with alternatives to or variations of Probability Theory and Ordinary Set Theory. Fuzzy Set Theory and Fuzzy Measure Theory are two prime candidates that can serve as appropriate alternatives to Probability Theory and Set Theory in certain contexts especially those characterized by a high level of Knightian uncertainty. The fact that they have a more developed mathematical foundation than other alternatives, but certainly not more than Probability Theory and Set Theory, has also increased their popularity. Besides, their attempt to model approximations and patterns of human thinking makes them intuitive theories.

Fuzzy Set Theory and Fuzzy Measure Theory are two separate disciplines within Fuzzy Theory. They have also been developed independently and they target different types of problems. Generally, Fuzzy Set Theory deals with the degree to which a certain element is compatible with a set while Fuzzy Measure Theory deals with the degree of evidence as to whether a certain element belongs to a set or not. However, Fuzzy Set Theory has been used more than Fuzzy Measure Theory in various applications including engineering, finance and medicine. It is only recently that Fuzzy Measure Theory has witnessed an increase in popularity.

In this chapter, we will present a brief overview of the foundations of each theory. There are still some vague areas that experts in these theories have not agreed on; consequently, the definitions and concepts may sometimes be loosely used and interpreted in literature. However, we have tried our best to be clear and to draw sharp distinctions whenever possible. Besides, due to the fact that such areas are relatively new, significant amounts of research are increasingly introduced without being documented in reference books even though they discuss elements pertaining to the basic foundations especially in Fuzzy Measure Theory. We have also tried to provide as comprehensive a literature review as possible. The process of writing such a review and bringing the various pieces of literature together has been a tedious process due to the diversity of material. So this chapter also stands as a contribution by itself.

We will confine the literature review in this chapter to those two theories only so that we do interrupt the flow of concepts and their relationships. However, in the next chapter, we will look at other alternatives to Probability Theory. We will also see how literature has been dealing with Knightian uncertainty. Within the course of the chapter, we will also consider various applications of fuzzy theory in the finance and economics literature.

The chapter is divided into two parts, Fuzzy Set Theory and Fuzzy Measure Theory. Starting with Fuzzy Set Theory in the first section, we will look at basic definitions and concepts pertaining to the theory. We specifically look at the concepts of membership functions and possibility distributions and their comparative behaviour to each other and the concept of probability. In section two, we move to Fuzzy Measure Theory. We discuss basic concepts concentrating on the Sugeno measure, which lies at the heart of Fuzzy Measure Theory, and the Fuzzy Integral (or Sugeno Integral). We also look at approaches to constructing the fuzzy measure. Those approaches are scattered in various publications but we have managed to melt them down into five approaches and to summarize them in one section. We will also look at conditional as well as dual fuzzy measures. Finally, we provide a brief review of the concept of Choquet integration in discrete time. The Choquet Integral, which is an alternative to Sugeno Integration, is more popular in the finance literature perhaps because it is more 'conventional,' it is more acknowledged and it converges to the Lebesgue integral in certain cases.

2.1 Fuzzy Set Theory

Fuzzy Set Theory has been introduced around the mid-1960s by Zadeh [149]. The basic idea behind this theory is to model the intrinsic imprecision and uncertainty prevalent in the real world. To accomplish this task, it employs approximations similar to those of human reasoning incorporating graded rather than precise sets with sharp boundaries. Zadeh emphasized that the source of uncertainty in such instances is the lack of "sharply defined criteria of class membership rather than the presence of random variables." ([149], p.339)

Ever since its introduction, Fuzzy Set Theory has known increasing popularity and it has been applied in various areas such as pattern recognition, decision making, control theory and many others. We will not go over the very basics of Fuzzy Set Theory in this section. However, they are summarized in Appendix A so the reader, who is not familiar with the theory, is advised to read Appendix A before reading this chapter.

2.1.1 Grade of Membership, Possibility and Probability

The concept of grade of membership is the building block of Fuzzy Set Theory while the concept of a possibility distribution is the building block of Possibility Theory. Fuzzy Set Theory and Possibility Theory are two conceptually different theories, which necessarily implies that the concept of grade of membership and that of a possibility distribution are conceptually different. However, some literature uses the two terms interchangeably. So, in this chapter, we will attempt to clarify on the differences and similarities between the two as well as compare them to probability theory. We also present a literature review on the elicitation of membership functions and possibility distributions in Appendix A.

Fuzzy Set Theory lays the grounds for Possibility Theory by providing it with its mathematical tools. Zadeh [150] argues that Fuzzy Set Theory to Possibility Theory is just like Measure theory to Probability theory (but we will see later a different point of view with Sugeno [133]). Within this context, a fuzzy restriction can be viewed as a possibility distribution where the membership function serves as the possibility distribution function and a fuzzy variable is linked to a possibility distribution just like a random variable is linked to probability theory [150].

Zadeh [150] has introduced possibility distributions in an attempt to relate the Theory of Possibility to the Theory of Fuzzy Sets. In his paper, he defines a possibility distribution as a fuzzy restriction that acts as an elastic constraint on the values which a fuzzy variable can attain. We only provide a summary of his point of view here. Let $U = \{u\}$ be our universe of discourse, F a fuzzy set in U characterized by its membership function μ_F, and X a fuzzy variable that assumes values in U. Let F act as an elastic (removable) constraint on the possible values of X meaning that it is a fuzzy restriction on X. Then the proposition "X is F" induces a possibility distribution over the values of X that are equal to the values of u in F such that the possibility of those values is equivalent to $\mu_F(u)$. Hence, the possibility distribution function is *numerically* equal to the membership function μ_F. The reason that F is a fuzzy restriction on X is that the statement "X is F" restricts the values that X can assume thereby inducing a possibility distribution $\prod_X$ equivalent to F. So the possibility distribution is analogous to the fuzzy restriction on the variable and the possibility distribution function is similar to the membership function of the fuzzy restriction. The possibility that X takes the value u, $\pi_X(u)$, is equivalent to the grade of membership, $\mu_F(u)$. So, effectively, fuzzy membership functions model possibility distributions.

Of course, the distinction has been developed and clarified more over the years. Bilgic and Turksen [56] argue that having fuzzy set memberships as the basis of gradual possibility does not mean that the concepts of fuzzy sets and possibility distributions are the same thing. They compare equating $\pi_X(u)$ to $\mu_F(u)$ to equating a probability function to a frequency, which does not mean that the two are the same. The distinction they draw between $\pi_X(u)$ and $\mu_F(u)$ centers on the interpretation and underlying assumptions of each concept. So $\pi_X(u)$ estimates the possibility that the fuzzy variable, X, assumes a value u, given the incomplete state of knowledge "X is F." Dubois and Prade [49] interpret the possibility distribution function as the possibility that each element of F be a possible value of X rather than as the possibility of X belonging to F. On the other hand, $\mu_F(u)$ estimates the degree of compatibility of the variable X assuming a value u with the statement "X is F." There is no incomplete information concerning the value that X assumes in the latter case: we are given the precise information $X = u$ and we need to estimate its degree of compatibility with F or the degree to which it satisfies the statement "X is F."

Lai and Hwang [100] also draw a distinction between the two concepts based on their interpretations. They consider membership functions as preference-based in the sense that they indicate a subjective degree of satisfaction, implying compatibility, while they interpret the grade of possibility as the subjective or objective degree of occurrence of the event. Throughout our work, we do recognize that the two terms are conceptually different but numerically equivalent.

2.1.2 Definitions

Since the introduction of Fuzzy Set Theory in the 1960s, the interpretation and measurement of the membership function have been controversial in the sense that no consensus has been reached. However, Fuzzy Set Theory is a very comprehensive and flexible theory that can be adapted to many real world problems involving different forms of uncertainty. Since the membership function is one of the building blocks of this theory, there is no need to have a consensus as to the form of the membership function. Fuzziness arises due to a number of different reasons and so it can assume many interpretations. As a result, the membership function is highly context-dependent in some instances and can assume various interpretations and forms depending on the problem for which it is being formulated and the assumed view of fuzziness. For this specific reason, it is very important to be careful about the shape of the membership function one uses. It is quite easy to find a function that satisfies the conditions of a membership function but it may not be easy to find one that is compatible with the set under consideration. Choosing a function that is not compatible can lead to model misspecification and this is when elicitation approaches step in.

The membership function can assume many functional forms based on the interpretation of the concept of grade of membership. We will first present the formal definition of the membership function and then move on to present its different interpretations. Assume we have a universe, U and a fuzzy set, F, then the membership function $\mu_F(.)$ is defined as a mapping from the universe X to the unit interval $[0, 1]$, that is, $\mu_F : X \rightarrow [0, 1]$. A similar definition holds for the possibility distribution, that is, $\pi : X \rightarrow [0, 1]$ is a possibility distribution.

Bilgic and Turksen [56] present five different views of the membership functions based on the two trends in the interpretations of fuzziness, namely the subjective versus objective fuzziness trend and that stemming from the individual rather than the group trend. In brief, they divide the interpretations of the membership function into five categories, which are the likelihood view, the random set view, the similarity view, the utility view and the measurement view. The first two subscribe to the objective view, the third and fourth subscribe to the subjectivity view and the last one is a combination of both.

We will only present a briefing on each view. The reader who is interested in more information is referred to [56]. The likelihood view subscribes to a probabilistic setting in interpreting the grade of membership whereby the membership function returns the likelihood that the variable in question is assigned a value in the corresponding fuzzy set. The random set view corresponds to the membership function as an interval which sometimes can correspond to the α-cuts of the fuzzy set. The similarity view assumes there is an ideal example that fully belongs to the fuzzy set and the rest belong to the set with a grade that depends on their distances from the ideal example. The utility view examines the membership function within a decision theoretic setting whereby logic and utility theory can be used to arrive at the membership function's final form. Finally, the measurement theoretic view maps an algebraic structure to a numerical one such that both the representation and the meaningfulness of the representation of a phenomenon are represented.

Unfortunately, there is no general consensus concerning the appropriate shape of a membership function or possibility distribution for a particular problem. Rather, there are various ways of eliciting them, which can be highly subjective at times. So far, there is no comprehensive survey that documents all possible alternatives. As a consequence, we have done a general survey of the possible shapes such distributions can assume besides their elicitation methods and included them in Appendix A.

2.1.3 Grade of Membership versus Probability

Given the definition of the grade of membership, one is tempted to compare it to the concept of probability. There is always a tendency to compare Fuzzy Set Theory with Probability Theory because they look similar and also because we are used to Probability Theory modelling uncertainty [151]. Fuzzy Set Theory and Possibility Theory are always inherent in human reasoning; however, Probability Theory is more popular when it comes to applications as it has a more developed and coherent mathematics backing it than the former theories do. But, in real life, we really use fuzzy or possibilistic statements rather than probabilistic ones. For example, if someone asks us if we think that it is going to rain, the answer will have inherent possibility rather than probability because we do not have enough information to specify a precise number; in other words, our reasoning is very fuzzy to allow us to come up with a fuzzy number.

Understandably, this is a very controversial subject that has been started ever since Fuzzy Set Theory has been introduced in the 1960s. However, sometimes the possibility or feasibility of comparing the two concepts has been questioned. As we will see later, Sugeno [133] has not been able to draw a comparison between the two and this is how he has come up with Fuzzy Measure Theory. Zimmerman [151] points out that it is hard to compare the two concepts and he attributes this to the fact that the comparison can be carried out on various levels (linguistic, mathematical...) and to the generality of Fuzzy Set Theory.

The two concepts may look the same but there is a general consensus in the literature that they are conceptually different. Probability deals with the frequency of occurrence of a variable while the grade of membership deals with how close or how compatible a given variable is with the concept of the fuzzy set under consideration. However, in another line of thought, Hisdal [78] questions the validity of this argument (and also presents a reference of literature that agrees with her argument). She actually suggests that there is a connection between the two and introduces the TEE (Threshold, Error, assumption of Equivalence) model for grades of membership that takes into account this connection. This model asserts that a grade of membership, $\mu_X(x)$, is an estimate of the probability that the label X takes the value x. In a comparative work, Giles [71] argues that grades of membership behave like probabilities in many respects but they are not probabilities. Her definition of a grade of membership implies that it is a standardized utility but it does reduce to a probability in some circumstances. In fact, Giles' definition of a grade of membership coincides with that of Hisdal under certain circumstances as well.

2.1.4 Possibility versus Probability

As we can conclude from the previous analysis, possibility and probability are two different concepts. They both deal with the degree of occurrence of a variable but from different angles. So while possibility answers the question "What is possible?" probability answers the question "What is likely to happen?" In other words, probability tells us that an event may occur with a certain degree or chance, possibility tells us the degree of ease with which this event occurs. Possibility really represents a weaker form of information than probability because quantifying the possibility of occurrence of an event is less assertive and requires less information (and, hence, the popularity of possibility distributions with problems suffering from missing data) than quantifying its probability of occurrence. This, in fact, goes in parallel with the fact that possibility can be used when we have missing information but probability cannot. We need more information to formulate probabilities and this is why probabilities represent a stronger form of information.

Yet, a variable can be associated with both theories through the possibility/probability consistency principle. The possibility/probability consistency principle expresses a weak connection between probability and possibility. A high degree of possibility does not necessarily mean a high degree of probability and a low degree of probability does not have to imply a low degree of possibility. But if an event is impossible then it is definitely improbable. This heuristic relation is what is known as the possibility/probability consistency principle. Zadeh [150] emphasizes that this is not a precise law; rather, it is an attempt to formalize the heuristic connection between probability and possibility which implies that a lower possibility implies a lower probability but not vice versa.

Based on this and on the fact that possibility distributions to possibility theory are like probability densities to probability theory, it is possible to draw a possibility distribution from a probability distribution and vice versa. We will confine our analysis to the probability to possibility distribution transformation case. There are two lines of thought in this area: the bijective transformation method ([56],[106],[90]) and the conservation of uncertainty method ([52],[56], [53],[106]).

The bijective transformation approach has been introduced by Dubois and Prade [52] in 1983. The basic idea behind this approach is to preserve as much information as possible. The intuition behind it is that possibility distributions represent a weak form of information whereas a probability one represents a strong form of information. Therefore, moving from a possibility distribution results in more information while moving from a probability to a possibility one results in loss of information.

This approach views a possibility measure as nested focal elements, which are elements corresponding to beliefs of individuals about events, and approximates it by means of a probability measure through interpreting each focal element as a conditional probability. Consider $X = \{x_i \mid i = 1, ...n\}$ as our universe of discourse, E_i as a focal element, $P(. \mid E_i)$ as a conditional probability uniformly distributed over E_i. Let $p_i = P(\{x_i\})$ be the probability of occurrence of x_i, π_i be its possibility of occurrence. Then the atom of probability associated with an element $x \in X$ can be defined as,

$$p(x) = \sum_{i=1}^{n} P(x \mid E_i)m(X) = \sum_{x \in E_i} \frac{m(E_i)}{|E_i|}, \tag{2.1}$$

where $|E_i|$ is the number of elements in E_i. So, effectively, one probability measure, which is bounded by the necessity and possibility measures, has been chosen, namely,

$$N(A) \leq P(A) \leq \Pi(A), \quad \forall A. \tag{2.2}$$

Then those probability atoms can be computed directly from the possibility distribution

$$p_i = \sum_{j=i}^{n} \frac{\pi_j - \pi_{j+1}}{j}, \tag{2.3}$$

where $\pi_1 = 1 \geq \pi_2 \geq ... \geq \pi_n$, and $\pi_{n+1} = 0$ by convention. The inverse of this formula is

$$\pi_i = \sum_{j=1}^{n} \min(p_i, p_j) = ip_i + \sum_{j=i+1}^{n} p_j, \tag{2.4}$$

where the x_i's are arranged in a way such that their probabilities are in descending orders, that is, $p_1 \geq ... \geq p_n$ and $p_{n+1} = 0$. This formula implies that a fuzzy set can be defined from a histogram.

Those equations imply that the shape of the possibility distribution is the same as that of the probability. However, the possibility distribution envelopes the probability one, which, in formal terms can be expressed as $\pi_i \geq \frac{p_i}{p_{\max}}$, where $p_{\max} = \max_i(p_i)$, $\forall i = 1, ..., n$.

The conservation of uncertainty method has been introduced by Klir [90] in 1990. This approach subscribes to a measurement theoretic view of probability/possibility distributions and rests on three main assumptions, the first of which is the principle of uncertainty and information invariance. It requires that the amount of inherent uncertainty must be preserved through the transformation. This implies that the Shannon entropy, which is the measure of uncertainty in Probability Theory, be numerically equal to its probabilistic counterpart leading to the uncertainty invariance or principle of uncertainty conservation equation, namely,

$$H(\mathbf{p}) = N(\boldsymbol{\pi}) + D(\boldsymbol{\pi}), \tag{2.5}$$

where $H(\mathbf{p}) = -\sum_{i=1}^{n} p_i \log_2 p_i$ is the Shannon entropy and $N(\boldsymbol{\pi}) = -\sum_{i=2}^{n} \pi_i \log_2 \left(\frac{i}{i-1}\right)$ is nonspecificity and $D(\boldsymbol{\pi}) = -\sum_{i=1}^{n-1} (\pi_i - \pi_{i+1}) \log_2 \left[1 - i \sum_{j=i+1}^{n} \frac{\pi_j}{j(j-1)}\right]$ is discord, which are the possibilistic counterparts. It is also possible to replace discord and nonspecificity by Higashi and Klir's logarithmic measure index.

A second assumption requires that all numerical values in one transformation be scaled to their counterparts in the other transformation. It forces each value π_i to be a function of p_i/p_1 [56]. The scale can be interval scale, ratio scale, Log-interval scale... Finally, this approach requires that all transformations satisfy an uncertainty conservation principle whereby $\pi(u) \geq p(u)$, $\forall u$, which means everything that is probable must be possible.

Based on this, Klir introduces his log-interval scale transformation
$\pi_i = \left[\frac{p_i}{p_1}\right]^\alpha$,
where $\alpha \in [0, 1]$ is a constant that can be determined by solving the corresponding equation. He argues that this transformation is the one that holds for all distributions and is unique. The drawback with this approach is that the measures of discord and nonspecificity are not comprehensive measures of total uncertainty [106]. Dubois and Prade [56] also question this assumption. Besides, they argue that this approach does not respect the probability/possibility consistency principle.

Dubois and Prade [56] present summary and references on a more recent approach on such transformations based on confidence intervals. The confidence interval is that corresponding to maximal probability and it is usually taken to have a degree of confidence of 95%. So the most specific possibility distribution consistent with the probability distribution is

$$\pi(a_L) = \pi(a_L + L) = 1 - P(I_L), \quad \forall L, \tag{2.6}$$

where L is the length of the confidence interval $I_L = [a_L, a_L + L]$ and $p(a_L) = p(a_L + L)$. Therefore, the α-cut of the most specific possibility distribution π is the $1 - \alpha$ confidence interval of p. Based on this approach, Lasserre has been able to transform a uniform probability distribution into a triangular possibility distribution. Cvinalar and Trussell [35] have also introduced an approach based on the confidence interval (in fact, we have seen their membership function earlier) and derived membership functions for several known pdfs.

These approaches are suitable when randomness is hard to work with so that it is hard to achieve a probabilistic model and uncertainty is better described within a fuzzy framework. They also require a large amount of data to estimate the pdf. Besides, a transformation from a probability distribution, usually represented by a normalized histogram, to a possibility distribution is justified only when a probability distribution is hard to come by.

Within the context of investment decisions, Gupta [76] does not recommend using possibility distributions and suggests transforming them into probability distributions for several reasons. The most important reason is that the information related to future outcomes of investment proposals is probabilistic rather than possibilistic. However, because the future is vague, we have to resort to possibilistic estimates. In line with this is the fact that a decision maker is not interested in what is possible in the future; rather, he/she is interested in what is likely to happen. The second reason falls within the scope of the type of information embedded in a probability or possibility distribution. He argues that, since converting from a probability to a possibility distribution results in loss of information, the final decision will be imprecise casting a higher level of risk on the decision maker. Besides, he believes that Probability Theory has a richer structure to work with and it is more developed than Fuzzy Set Theory.

In conclusion, it is very important to be aware of the models and concepts we have presented above. Membership functions and possibility distributions are highly context dependent at times and so one must be careful about using them. Besides, one must also be careful of the framework within which they are to be used because choosing an inappropriate approach will result in loss of information and wrong results. These concepts will prove to be useful later on when we work on possibilities of occurrences of states in the presence of uncertainty.

2.2 Fuzzy Measure Theory

Sugeno [133] has first introduced Fuzzy Measure Theory in an attempt to compare fuzziness, traditionally represented by fuzzy sets, to randomness, represented by probability. According to him, fuzzy sets are only extensions of ordinary sets and, given that the concept of probability is different from that of sets, a direct comparison will fail. Fuzzy measures are monotonic set functions, which are functions that make sets correspond to numerical values, and, hence, it is possible to compare them to probability measures.

Fuzzy measures are concerned with the degree to which evidence proves that a certain element belongs to a certain set or not. So the idea behind fuzzy measures is not returning the grade of membership of a particular element in a set or several sets under consideration, even though they admit membership functions as a special case; rather it has to do with whether the evidence is enough to prove it. So full evidence indicates full membership in the set and if we are considering more than one set, then the element at hand has full membership in one and only one set. However, the evidence or pieces of information are hardly full or perfect which gives rise to uncertainty. It is important to understand that, within this context, uncertainty arises due to information deficiency and not vagueness whereby sets do not have sharp boundaries.

A fuzzy measure represents this uncertainty by assigning a value which indicates the extent to which the variable under consideration belongs to every possible set. This value stands for the degree of evidence or belief or contribution of a piece of information that such an element belongs to a certain set. It is not necessary to work with fuzzy sets as well since fuzzy measures apply to both, fuzzy as well as crisp sets.

These characteristics make fuzzy measures ideal for application to problems where uncertainty, including Knightian uncertainty, is prevalent. As we will see in the next chapter, Knightian uncertainty is quite prevalent in business and economics, in general, and in the financial markets, in particular, which is the domain of our research. Given that options are contingent on the underlying asset (and we are trying to price options under uncertainty), they are affected by this uncertainty; let alone that they have a lot of such uncertainty themselves arising from the option market. So we will utilize those measures in our research showing their full power in option pricing. But before that, we need to provide a literature review on Sugeno measure and methods of determining it, the fuzzy integral and fuzzy conditional measures, which is the purpose of this chapter.

Fuzzy measures and fuzzy integrals have been applied in a wide variety of areas. The broad categories for applications include: subjective evaluation processes ([133],[134],[80],[115]), decision-making process ([134],[143],[135]) and learning process [134]. The latter two utilize conditional fuzzy measures.

2.2.1 Sugeno Measure

Suppose we observe a transaction price in the market but we do not know whether it belongs to the bid or ask set. It is uncertain or fuzzy for us in which set it is. Sugeno measure, being a special case of fuzzy measures, allows us to assign a degree measuring or indicating the extent to which the observed price belongs to the bid or ask sets. It is within this context that this measure includes the concept of grade of membership as a special case. The measure specifically measures the fuzziness of a set; however, the set under consideration does not have to be fuzzy. One possible interpretation of such fuzziness is the subjectivity involved in guessing. In fact, Sugeno [134] views a fuzzy measure as a mathematical model of subjective evaluation. Other interpretations of the Sugeno measure (and fuzzy measures in general) are grade of importance, degree of similarity, belief, evidence, likelihood, certainty, or plausibility that a specific element belongs to a specific set.

Assume we have a nonempty set X and a nonempty class ζ of subsets of X. Let $g : \zeta \rightarrow [0, \infty]$ be a fuzzy measure defined on (X, ζ). Then, g satisfies the following:
(i) $g(\phi) = 0$.
(ii) Monotonicity: $A, B \in \zeta$ and $A \subset B \Longrightarrow g(A) \leq g(B)$.
(iii) Continuity from below: $A_n \in \zeta, \{A_n\}$ is monotone such that $A_1 \subset A_2 ...$, and $\bigcup\limits_{n=1}^{\infty} A_n \in \zeta \Longrightarrow \lim\limits_{n} g(A_n) = g(\bigcup\limits_{n=1}^{\infty} A_n)$.
(iv) Continuity from above: $A_n \in \zeta, \{A_n\}$ is monotone such that $A_1 \supset A_2 ...$, $\mu(A_1) < \infty$ and $\bigcap\limits_{n=1}^{\infty} A_n \in \zeta \Longrightarrow \lim\limits_{n} g(A_n) = g(\bigcap\limits_{n=1}^{\infty} A_n)$.
g is called regular if $X \in \zeta$ and $g(X) = 1$. The last two conditions are not important if we are considering finite sets. Now, compare those conditions to those of probability measure. Given the probability space $(X, ß, P)$, then P satisfies the following conditions
(i) $0 \leq P(E) \leq 1$ for all $E \in ß$ and $P(E) = 1$.

(ii) If $E_n \in ß$ for $1 \leq n < \infty$ and $E_i \cap E_j = \Phi$ for $i \neq j$, then $P(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

So a probability measure eventually satisfies the monotonicity and continuity properties of the fuzzy measure, which makes it a special fuzzy measure. Alternatively, a fuzzy measure is a set function derived by loosening the properties of probability measures. The value added by Fuzzy Measure Theory is that it relieves individuals from additivity which is a restricting condition imposed by Probability Theory and replaces it with monotonicity.

In fact, Klir et al. [91] draw interesting conclusions about the interaction between the two sets A and B using the sign of the inequality. So when $g(A \cup B) < g(A) + g(B)$, A and B exhibit inhibitory interaction; when $g(A \cup B) > g(A) + g(B)$, A and B exhibit synergetic interaction and when $g(A \cup B) = g(A) + g(B)$, A and B exhibit no interaction, which corresponds to the probability measure case. Of course, the interactions are considered with respect to the property being measured by g.

Going back to the fuzzy measure, suppose that $A, B \in \zeta$, $A \cap B = \phi$ and $A \cup B \in \zeta$, where $g(A \cup B) = g(A) + g(B) + \lambda.g(A).g(B)$ and $\lambda \in (-\frac{1}{\sup g}, \infty) \cup \{0\}$ such that $\sup g = \sup_{A \in \zeta} g(A)$, then we say g satisfies the λ-rule. g satisfies the finite λ-rule (on ζ) iff there exists λ as defined above and

$$g(\bigcup_{i=1}^{n} A_i) = \begin{cases} \frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1 + \lambda.g(A_i)] - 1\right\} & \lambda \neq 0, \\ \sum_{i=1}^{n} g(A_i) & \lambda = 0. \end{cases} \quad (2.7)$$

μ satisfies σ-λ-rule iff there exists λ as defined above and

$$g(\bigcup_{i=1}^{n} A_i) = \begin{cases} \frac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1 + \lambda.g(A_i)] - 1\right\} & \lambda \neq 0, \\ \sum_{i=1}^{\infty} g(A_i) & \lambda = 0. \end{cases} \quad (2.8)$$

If g satisfies σ-λ-rule on ζ and there exists $E \in \zeta$ such that $g(E) < \infty$, then g is called a λ-fuzzy measure and is denoted by $g_\lambda(.)$. When ζ is a σ-algebra and $g_\lambda(X) = 1$, $g_\lambda(.)$ is also called a Sugeno Measure [140]. This is the measure we will be dealing with.

In fact, the behaviour of the measure is dependent on the behaviour of λ.Note that when $\lambda = 0$, g is the classical probability measure, and the λ-rule, finite λ-rule and σ-λ -rule are equivalent to additivity, finite additivity and σ-additivity respectively. When $\lambda \neq 0$, g is subadditive or superadditive depending on the sign of λ. The following properties summarize the statements made so far,

(i) Superadditivity: $\lambda > 0$ when $\sum_{i=1}^{n} g(A_i) < g(X)$,

(ii) Additivity: $\lambda = 0$ when $\sum_{i=1}^{n} g(A_i) = g(X)$,

(iii) Subadditivity: $-\frac{1}{g(X)} < \lambda < 0$ when $\sum_{i=1}^{n} g(A_i) > g(X)$.,

where $X, A, B \in \zeta, A \cap B = \phi$ and $X = A \cup B$.

Those properties also define the basic properties of the Sugeno measure. Now, we move on to show how it is constructed. Assume we are given a monotone sequence $F = F_1 \subset F_2 \subset ... \subset F_n$, then the fuzzy measure of $F' \subset F$ is given by

$$g_\lambda(F') = \frac{1}{\lambda}\left[\prod_{s_i \in F'} (1 + \lambda g^i) - 1\right], \qquad -1 < \lambda < \infty, \tag{2.9}$$

where $g^i = g_\lambda(\{s_i\})$ is the fuzzy density function of the fuzzy measure and $\lambda \in (-1, \infty)$, (λ always has a unique root on $(-1, \infty)$). Note that

$$g_\lambda(\{s_i, s_j\}) = g^i + g^j + \lambda g^i g^j, \qquad i \neq j; \tag{2.10}$$

$$g(F_i) = g^i + g(F_{i-1}) + \lambda g^i g(F_{i-1}). \tag{2.11}$$

Generally,

$$g_\lambda(\{s_1, ..., s_k\}) = \sum_{i=1}^{k} g^i + \lambda \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} g^i g^j + ... + \lambda^{k-1} g^1 g^2 ... g^k. \tag{2.12}$$

Given that $g(F) = 1$ for such measures, we can compute λ using

$$g(F) = 1 = \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g^i) - 1\right]. \tag{2.13}$$

If we define $H(s_i)$ to be the fuzzy distribution function, which is similar to a probability density function for the monotone sequence K, then we can write

$$H(x) = g_\lambda((-\infty, x)), \tag{2.14}$$

$$g_\lambda([a,b]) = \frac{H(b) - H(a)}{1 + \lambda H(a)}. \tag{2.15}$$

So,

$$g^i = \frac{H(s_i) - H(s_{i+1})}{1 + \lambda H(s_{i+1})}, \qquad 1 \leq i \leq n-1, \tag{2.16}$$

$$g^n = H(s_n), \tag{2.17}$$

$$H(s_i) = g_\lambda(F_i), \qquad K_i = \{s_i, s_{i+1}, ...s_n\}. \tag{2.18}$$

Sugeno points out that, in constructing $g_\lambda(.)$, it is not necessary to start with $H(s_i)$ first, rather we can start with g^i such that $0 \leq g^i \leq 1$, for $1 \leq i \leq n$. Note that the degrees of freedom of $H(s)$, g^i and λ is n. Fuzzy measures are very important for defining the fuzzy integral, which we will introduce in the next section.

2.2.2 The Fuzzy Integral

The fuzzy integral, also known as Sugeno integral, has been introduced first in an attempt to model the subjective evaluation of objects embodied by fuzzy measures, but, later on, it has found its way into various other applications. The fuzzy integral is a nonlinear integral or functional, which allows it to represent fuzzy expectations, as opposed to probability expectation. As we will see later, it also serves as an aggregate means.

The fuzzy integral is computed in a different way from the Lebesgue integral. So while the Lebesgue integral is an additive model that utilizes addition, the fuzzy integral is a comparison operation that utilizes comparison operators such as the minimum and the maximum. Besides, we can view the Lebesgue integral as measuring the area under the curve but we cannot do the same for the fuzzy integral; it is viewed in a completely different way.

Wierschon [144] interprets the fuzzy integral as a search for an optimal grade of agreement between two opposite tendencies. Onisawa and Sugeno et al. [115] interpret the fuzzy integral as a model of the subjective evaluation of fuzzy objects whose attributes are measured by a fuzzy measure and their individual characteristic functions are integrated with respect to a fuzzy measure. They interpret the fuzzy measure as a subjective scale for guessing whether an a priori unlocated element belongs to a certain set and also as the grade of subjective importance associated with an attribute. The characteristic function can be given objectively from the physical properties of the attribute or subjectively according to subjective evaluation. These two interpretations are adopted in most literature.

Consider the F-measurable space $(X, 2^X)$ and a function $h : X \longrightarrow [0, 1]$. Let A be a subset of X, then h_A is the membership function of A. The fuzzy integral over A in $(X, 2^X)$ can be defined as

$$\oint_A h(x) \circ g(.) = \sup_{F \in 2^X} [\inf_{x \in F} h(x) \wedge g(A \cap F)], \tag{2.19}$$

where g is a fuzzy measure of $(X, 2^X)$, $\circ$ is the rule of composition (max-min) in fuzzy sets theory and $\wedge$ denotes the minimum. However, it is not necessary to consider only $(X, 2^X)$, that is, to consider 2^X as the domain of g, rather any family of sets that includes a monotone sequence will do and we can consider the fuzzy measure space $(X, \Im, g)$. But due to some technical difficulties beyond the scope of this research [133], the Borel field ß can be used as the domain of g. So we will be working with the fuzzy measure space $(X, ß, g)$. The fuzzy measure of a fuzzy set A is given by

$$g(A) = \oint h_A(x) \circ g(.), \tag{2.20}$$

where the fuzzy integral can take the form

$$\oint_A h(x) \circ g(.) = \oint [h_A(x) \wedge h(x)] \circ g(.). \tag{2.21}$$

Since, A does not have to be fuzzy and so the fuzzy integral can be expressed as

$$\oint_A h(x) \circ g(.) = \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)], \tag{2.22}$$

where $F_\alpha = \{x \mid h(x) \geq \alpha\}$ (α-cut). A is known as the domain of integration when $A = X$, $g(A \cap F_\alpha)$ is written as $g(F_\alpha)$. As α increases, F_α decreases and, given that g is monotone, $g(A \cap F_\alpha)$ decreases as well.

Now let $h(x)$ be a simple function on X such that

$$h(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x), \tag{2.23}$$

where $X = \sum_{i=1}^{n} E_i, E_i \in ß$, and $E_i \cap E_j = \phi$ $(i = j)$. Then we can evaluate the fuzzy integral independent of the value of h as

$$\oint_A h(x) \circ g(.) = \bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)], F_1 = X, \tag{2.24}$$

where $\vee$ denotes the maximum. It is also possible to evaluate a measurable function, which does not have to be bounded by 0 and 1, using the fuzzy integral and a fuzzy measure that is not bounded by ∞ from above (i.e. not a Sugeno measure).

The fuzzy integral satisfies the following properties:

(i) $\oint a \circ g = a, a \in [0, 1]$,

a) $\oint (a \vee h) \circ g(.) = a \vee \oint h \circ g(.)$,

b) $\oint (a \wedge h) \circ g(.) = a \wedge \oint h \circ g(.)$,

c) $\oint (h_1 \vee h_2) \circ g(.) \geq \oint h_1 \circ g(.) \vee \oint h_2 \circ g(.)$,

d) $\oint (h_1 \wedge h_2) \circ g(.) \leq \oint h_1 \circ g(.) \wedge \oint h_2 \circ g(.)$,

e) $\oint_{E \cup F} h \circ g(.) \geq \oint_E h \circ g(.) \vee \oint_F h \circ g(.)$,

f) $\oint_{E \cap F} h \circ g(.) \geq \oint_E h \circ g(.) \wedge \oint_F h \circ g(.)$.

(ii) If $h_1 \leq h_2, \oint h_1(x) \circ g \leq \oint h_2(x) \circ g$.

(iii) If $E \subset F, \oint_E h(x) \circ g \leq \oint_F h(x) \circ g$.

(iv) $\oint [\bigvee_{i=1}^{n} (a_i \wedge h_i)] \circ g(.) = \bigvee_{i=1}^{n} [a_i \wedge \oint h_i \circ g(.)]$, where $\{h_n\}$ is monotonically decreasing sequence of ß-measurable and $\{a_i\}$ is a monotonically increasing sequence of real numbers.

(v) $\lim_{n\to\infty} \oint h_n \circ g(.) = \oint h \circ g(.)$.

Assume now that X is a finite set (note that we need not assume continuity of fuzzy measures in this case) such that

$X = \{x_1, x_2, ..., x_n\}$

and let h be

$h(x_1) \geq h(x_2) \geq ... \geq h(x_n)$.

If h does not follow this order, it has to be re-ordered and the elements of X renumbered. As such, the fuzzy integral in $(X, 2^X, g)$ can be defined as follows

$$\oint_A h(x) \circ g = \bigvee_{i=1}^{n} [h(x_i) \wedge g(A \cap H_i)], \tag{2.25}$$

where $H_i = \{x_1, x_2, ..., x_n\}$ (A and h are still as defined previously). The fuzzy integral can also be written as

$$\oint h(x) \circ g = \bigvee_{i=1}^{n} [h(x_i) \wedge g(H_i)], \tag{2.26}$$

where $H_i = \{x_1, x_2, ..., x_n\}$.

So as we have seen, the fuzzy integral is a rather nonconventional integral. It is connected with monotonicity whereas the Lebesgue integral is connected with additivity. Interestingly enough Sugeno [133] showed that the difference between a Lebesgue integral and a fuzzy integral is not more than $\frac{1}{4}$. Namely, given a probability space $(X, ß, P)$, then

$$\left| \int_X h(x)dP - \oint_X h \circ P(.) \right| \leq \frac{1}{4}. \tag{2.27}$$

2.2.3 Measure Construction

In this section, we present the different approaches used to estimate Sugeno measure, which normally boil down to estimating the fuzzy density functions. It is important to note that, in most literature, the λ-fuzzy measure and Sugeno measure are used interchangeably especially in practical applications. We will classify the approaches by which we can estimate the fuzzy density function g_i into five categories: the subjective approach, probability measure transformation approach, fuzzy distribution derivation approach, membership function approximation approach and statistical data approach.

Subjective Approach

This is the most popular approach in literature. It usually involves some form of an optimization technique. The basic idea here is to minimize an error function through an iteration process so that the fuzzy density function eventually approximates the subjective weights or evaluations provided by experts. Literature has dealt with this problem using different algorithms. When Sugeno [133] first introduced the theory of fuzzy measures and fuzzy integrals, he applied it to the subjective evaluation of female faces and pattern recognition. The corresponding optimization problems involved minimizing the error between human evaluation and the fuzzy integral model input. In pattern recognition, he presents different subjects with a set of patterns and asks for their evaluation of the similarity of the patterns. We will first present the algorithm for the pattern recognition problem.

$\Psi(C) = \int P_C \circ g_\lambda(.)$,

$w(C) = \frac{\Psi(C)-\Psi(B)}{\Psi(A)-\Psi(B)}$,

minimize $J = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(d_i - w_i)^2}$,

where $P = \{A, B\} \cup \{C, D...\}$ such that A and B are standard patterns and C, D... are general patterns. Therefore, $w(C)$ would represent the membership function of patterns that are similar to A but not to B. $g_\lambda(.)$ represents the fuzzy measure and d_i is the subjectively given membership function. The process is iterated until a value of $g_\lambda(.)$ is found that minimizes J.

As for the subjective evaluation problem([133], [134]), Sugeno uses the following algorithm,

$e = \oint h(s) \circ g$,

$w_j = \frac{\overline{d}-\underline{d}}{\overline{e}-\underline{e}} e_j + \frac{\underline{d}\overline{e}-\overline{d}\underline{e}}{\overline{e}-\underline{e}}$, minimize $J = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(d_i - w_i)^2}$,

where d is the subjective overall valuation and $h(s)$ is the subjective partial evaluation.

This is a rather indirect approach to constructing $g_\lambda(.)$. However, it is possible to arrive at it directly ([134]). In the following example, the Sugeno measure is viewed as the grade of importance of the object under evaluation. Suppose a subject is given a set K and is asked to assign a grade of importance for all subsets of K resulting in $d : 2^K \to [0, 1]$. Assume that $d(.)$ satisfies all properties of fuzzy measures so that the problem now is to minimize

$$J = \sqrt{\frac{1}{2^K}\sum_{E \in 2^K}(d(E) - g_\lambda(E))^2},$$

So by minimizing J with respect to the fuzzy densities and the parameter λ, which differs by individuals, it is possible to arrive at a direct approximation for d_i by $g_\lambda(.)$.

Wang et al. [141] use genetic algorithm to determine g^i from data, which also involves solving an optimization problem. Tahani and Keller [135] determine g^i subjectively or from training data and then compute λ using $\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g^i)$. The fuzzy integral $e = \max_{i=1}^{n}[\min(h(x_i), g(A_i))]$, where $h : X \to [0, 1]$, $X = \{x_1, ..., x_n\}$ and $A_i = \{x_1, ..., x_i\}$ is a partition of X, can then be computed given that $g(A_i)$ can be determined recursively by

$$g(A_1) = g(\{x_1\}) = g^1,$$

$$g(A_i) = g^i + g(A_{i-1}) + \lambda g^i g(A_{i-1}).$$

They apply this to an object recognition problem where g^i is determined according to

$$sum = \sum_{i,j} p_{ij},$$

$$b_{ij} = \frac{p_{ij}.dsum}{sum},$$

$$g^i = \sum_{j=1}^{m} b_{ij},$$

where p_{ij} is performance of the classifier and $dsum$ is the desired sum of fuzzy densities, which still has an element of subjectively.

In an attempt to work with general fuzzy measures rather than specifically λ- or Sugeno measures, Ishi and Sugeno [80] introduce the Fuzzy Measure Learning Identification Algorithm to approximate the human evaluation process mathematically. In broad terms, the model involves starting with arbitrary fuzzy measures and gradually adjusting them according to the data set. g^i is regarded as the importance of the corresponding element as an information input. Once again, the process is iterated until the difference between the subjective and the model outputs goes below a certain tolerance level, i.e., $|e|=|z-\overline{z}|< tol$, where $\overline{z} = \oint_K h(s_i) \circ g(.)$, $K = \{s_1, ..., s_n\}$ representing the set of information elements and $h : K \rightarrow [0,1]$ is a function that quantifies information. Note that the index has to be altered for each set of data such that $h(s_i) \geq h(s_i+1)$. The adjustment of $g(.)$ depends on the adaptive gain $\alpha = \gamma \wedge \frac{\sigma_e^2}{\varsigma}$, where γ and ς are constants and $\sigma_e = \sqrt{\sum_{j=1}^{N} \frac{(z-z_j)^2}{N}}$ is the root mean square error. In another paper, Onisawa and Sugeno (et al.) [115] also use FLIA but they use factor analysis to collect the subjective judgements.

Probability Measure Transformation Approach

Sugeno measure can be derived from a classical probability measure by using an appropriate transformation. Let $(X,ß,p)$ be a measurable space where p is a Lebesgue measure. Then, the composition $f \circ p$ produces a Sugeno measure iff

$$f(x) = \frac{1}{\lambda}(c^x - 1)$$

(Theorem 2.1,[145], p.71). This can be used to derive a Sugeno measure from a classical probability measure. Assume that g is a Sugeno measure defined on ß, then

$$p(A) = \frac{\log(1+\lambda g(A))}{\log(1+\lambda)}$$

$$\Rightarrow g(A) = \frac{(1+\lambda)^{p(A)}-1}{\lambda}$$

([96],[97],[144],[145], [142]).

Kruse ([96],[97]) uses this approach and solves an optimization problem to arrive at the fuzzy measure. Wierzchoń [145] also uses this approach where he solves the problem using the Least Squares Method. He starts with minimizing

$$J^2 = \frac{1}{m}\sum_{j=1}^{m}(w_j - G_j)^2$$

where w_j represents the *subjective* weights of $X_j \subset X = \{x_1, ...x_n\}$ and $G_j = \frac{1}{\lambda}\left[\prod_{x_i \in X_j}(1+\lambda g^i) - 1\right]$. Using the transformation, the author replaces this equation with

$$R^2 = \frac{1}{m}\sum_{j=1}^{m}(v_j - P_j)^2$$

where $v_j = \log_{(1+\lambda)}(1+\lambda w_j)$ and $P_j = \sum_{x_i \in X_j} p_i$. Using LS method

$$\frac{\partial R^2}{\partial p_i} = 0, \qquad i = 1, 2, ..., n,$$

with the constraints

$$\sum_{i=1}^{n} p_i = 1, \qquad p_i \geq 0,$$

the author arrives at the following equations

$$\prod_{j=1}^{m} (1 + \lambda w_j)^{k_j} = (\lambda + 1)^{2^{n-2}(n+1)-n}$$

to compute λ, and

$$z_i = \sum_{j=1}^{m} v_j d_{ij} = \log_{(1+\lambda)} \prod_{j=1}^{m} (1 + \lambda w_j d_{ij})$$

$$p_i = 2^{2-n}(z_i + 1) - 1$$

to determine probabilities. Finally, he computes the fuzzy densities using

$$g^i = \frac{1}{\lambda}[(1+\lambda)^{p_i} - 1].$$

So, again, we see that subjectivity plays a main role in the determination λ and, consequently, the fuzzy densities. Finally, Kruse [97] and Wenxiu [142] show that if f is a fuzzy event, then

$$\int_A f dg = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{\int_A \frac{\log(1+\lambda f)}{\log(1+\lambda)} dp}$$

but, once again, λ needs to be determined.

Fuzzy Distribution Derivation Approach

Sugeno [133] defines a λ-fuzzy measure in terms of an F-distribution function. He defines an F-distribution (fuzzy distribution) function, which has the same properties as the distribution function of a probability measure, as follows ([133], Definition 2.7, p.16): a function with the following properties on R^1 is called an F-distribution function.

1. $0 \leq H(x) \leq 1$ for $x \in R^1$.
2. If $x \leq y$, then $H(x) \leq H(y)$.
3. $\lim_{x \to a+0} H(x) = H(a)$.
4. $\lim_{x \to -\infty} H(x) = 0$.
5. $\lim_{x \to \infty} H(x) = 1$.

Defining this fuzzy distribution function for a monotone sequence $K = K_1 \supset K_2 \supset ... \supset K_n$, Sugeno defines a λ-fuzzy measure as follows

$$1 = H(s_1) \geq H(s_2) \geq ... \geq H(s_n) \geq 0,$$

$g^i = \frac{H(s_i)-H(s_{i+1})}{1+\lambda H(s_{i+1})}$,

$g^n = H(s_n)$

$\Rightarrow H(s_i) = g_\lambda(K_i)$ where $K_i = \{s_i, s_{i+1}, ..., s_n\}$,

$g^i = g_\lambda(\{s_i\})$,

so that

$H(s_i) = \frac{1}{\lambda}[\prod_{k=i}^{n}(1+\lambda g^k) - 1] = 1, 1 \leq i \leq n$,

and g^i is called a fuzzy density of g_λ. Particularly, for a half-open interval

$g((a,b]) = \frac{H(b)-H(a)}{1+\lambda H(a)}$,

$g((-\infty, a]) = H(a)$.

No practical applications are provided in the literature. Wang and Klir [140] provide a couple of examples in their book. They refer to the F-distribution function $H(x)$ as a left-continuous probability distribution function or as a distribution function for g_λ. Wierzchoń [143] tackles this point briefly. He suggests determining the fuzzy distribution function by having the subject order the states in some way $x_1, x_2, ..., x_n$ and, then, determine his judgements $H(i) = w(\{x_1, ..., x_n\})$, whose number is only $n_1 = I - 1$ because $H(I) = 1$, where $I = card(X)$, X is the set of possible states of the world, and $w(\{x_i\})$ is the grade of credibility or importance of occurrence of state i. In another paper, Wierzchoń [144] recovers an underlying probability distribution function P from a fuzzy distribution function $H(x) = g((-\infty, x])$,

$g((x_1, x_2]) = \frac{F(x_2)-F(x_1)}{1+\lambda F(x_1)}$,

$P(x) = \log_{(\lambda+1)}(1 + \lambda F(x))$.

Membership Function Approximation Approach

Sugeno [133] has proved that the grade of membership is a special case of the grade of fuzziness. Recall that the λ measure (or Sugeno measure) measures the grade of fuzziness of a set i.e. whether a certain element belongs to the set or not whereas the membership function measures the grade of membership of an element or the degree of compatibility of this element with the set, given that we know that it belongs to the set.

Assume that we pick up an element x_0 from a set E, then

$Gr(x_0 \in E) = \chi_E(x_0)$

where $Gr(x_0 \in E)$ is the grade of membership of x_0 in E, and $\chi_E(x_0)$ is the characteristic function of E. Then if E is a fuzzy set A, we can write

$$g(x_0, A) = \oint_X h_A(x) \circ g(x_0, .)$$
$$= h_A(x_0)$$

where $h_A(x_0)$ is the grade of membership of x_0 in A.

Wenxiu [142] argues that, given a possibility distribution $f(x_i) = \pi_i (i = 1, 2, ..., n)$ such that $f : X \rightarrow [0, 1]$ and $\pi_i \in [0, 1]$, then, for $\lambda \neq 0$, if f satisfies

$$\frac{1}{\lambda}[\prod_{i=1}^{n}(1 + \lambda\pi_i) - 1] = 1,$$

then it can generate a g_λ-measure such that

$$g_\lambda = \frac{1}{\lambda}[\prod_{x_i \in A}(1 + \lambda\pi_i) - 1]$$

Of course, here we are implicitly assuming that the possibility distribution is numerically equivalent to the membership function.

Statistical Data Approach

This approach involves using statistical data to arrive at an approximation of the fuzzy density. Leszczynski et al. [102] introduce a fuzzy clustering algorithm using Sugeno measure. However, they do not rely directly on human evaluation of the measure but they rely on actual data. They solve an optimization problem involving Lagrangian function and Euclidean distance to arrive at the fuzzy measure.

2.2.4 Conditional Fuzzy Measures

Conditional fuzzy measures ([133],[137],[143]) serve as conditional probabilities. A fuzzy measure of Y, $\sigma_Y(. \mid x)$, for any $x \in X$ is called a conditional fuzzy measure of Y with respect to X. A fuzzy measure of Y, $g_Y(.)$, can be constructed utilizing its conditional fuzzy measure [137] so that

$$g_Y(F) = \oint_X \sigma_Y(F \mid x) \circ g_X(.).$$

Integrating over a measurable function, we get

$$\oint_Y h(y) \circ g_Y = \oint_X [\oint_Y h(y) \circ \sigma_Y(. \mid x)] \circ g_X$$
$$= \bigvee_{i=1}^{I} \left\{ h(y) \wedge \left[\bigvee_{i_1=1}^{i} (\sigma_Y(. \mid x_{i_1}) \wedge g(\{x_1, ..., x_{i_1}\}) \right] \right\}.$$

In fact, Sugeno ([137],[133]) was able to derive the equivalent of Bayes' theorem for fuzzy measure theory utilizing this measure, namely,

$$\oint_F \sigma_X(E \mid x) \circ g_Y = \oint_E \sigma_Y(F \mid x) \circ g_X,$$

where, within this context, g_X is called a priori fuzzy measure and $\sigma_X(. \mid y)$ is called a posteriori fuzzy measure. Sugeno[133] also proves the equivalent of Fubini and Radon-Nikodym theorems in fuzzy measure theory, whereby conditional fuzzy measures are the analogue of conditional probability measures.

2.2.5 Fuzzy Dual Measures

Given a regular fuzzy measure μ, then ν, such that,

$$\nu(E) = 1 - \mu(\overline{E}),$$

is also a regular fuzzy measure and is known as a dual fuzzy measure. For a λ-fuzzy measure, g_λ,

$$\nu_{\lambda'}(E) = 1 - g_\lambda(\overline{E}),$$

So for every subadditive measure g, we can construct a superadditive measure ν such that they add up to 1 and $\lambda' = \frac{-\lambda}{1+\lambda}$. Derive ν from the definition of g,

$$g_\lambda(\overline{E}) = \frac{1-g_\lambda(E)}{1+\lambda g_\lambda(E)},$$

we get,

$$\nu_{\lambda'}(E) = \frac{(1+\lambda)g_\lambda(E)}{1+\lambda g_\lambda(E)}.$$

2.2.6 Choquet Integral

The Choquet integral is another form of nonlinear integration which found a lot of popularity in the economics and finance literature in non-additive measure applications. In continuous-time, the Choquet integral assumes the form,

$$(c)\int u(x)d\nu = \int_{-\infty}^{0} [\nu\,(u(x) > t) - \mu(\Omega)]dt + \int_{0}^{\infty} \nu\,(u(x) > t)\,dt, \qquad \mu(\Omega) < \infty,$$

where μ is the measure and Ω is the universe. $\mu(\Omega) = 1$ in our case. When $u(x)$ is a non-negative function, the integral becomes,

$$(c)\int u(x)d\nu = \int_{0}^{\infty} \nu\,(u(x) > t)\,dt.$$

In its discrete form, the Choquet integral can be written as

$$C_g(h) = (c)\int hdg = \sum_{i=1}^{n} [h(x_i) - h(x_{i+1})].g(A_i)$$

where $h(x_1) \geq ... \geq h(x_n)$, $h(x_{n+1}) = 0$, $A_i = \{x_1, ..., x_i\}$ and $g(A_i)$ is a non-additive measure. The Choquet integral converges to the Lebesgue one when the measure is additive.

It can also be written in another form,

$$C_g(h) = \sum_{i=1}^{n} h(x_i).[g(A_i) - g(A_{i-1})]$$

where $A_i = \{x_1, ..., x_i\}$, $0 \leq g(A_1) \leq g(A_2) \leq ... \leq 1$, $g(A_0) = 0, g(A_n) = 1$.

In a similar logic to linear conditional expectations, the conditional Choquet integral in discrete form, can be written as

$$\tilde{C}(S_j) = \int h(S_j)dg(S_j \mid x_{i_1}) = \sum_{i_1=1}^{n} h(S_j).[g(S_j \mid x_{i_1}) - g(S_j \mid x_{i_1-1})].$$

2.3 Conclusion

We have presented a comprehensive yet brief literature review of the foundations of Fuzzy Set Theory and Fuzzy Measure Theory. For this purpose, we have presented an overview of the concepts of membership function and possibility distribution touching on the distinction between them. We have also compared those concepts to those of a probability distribution. Besides, We have tackled the issue of deriving a membership function and a possibility distribution. We have also provided a literature review on fuzzy measures and methods of constructing them as well as on nonlinear integration approaches.

Those theories serve as alternatives to Probability Theory in problems characterized by Knightian uncertainty and vagueness. In the next chapter, we will look closely at the concept of Knightian uncertainty. We will specifically consider finance and economics applications and see how alternatives other than Fuzzy Measure Theory and Fuzzy Set Theory can be utilized to deal with Knightian uncertainty. We will also see applications of Fuzzy Measure Theory and Fuzzy Set Theory in finance and economics.

Chapter 3

Literature Review

The previous chapter has paved the way to understanding the tools utilized in the models we will present during the course of this research. Those tools serve as alternatives to Probability Theory within the Knightian uncertainty frameworks and settings that we will consider. However, they are not the only alternatives to Probability Theory. Researchers have been able to deal successfully with problems characterized with Knightian uncertainty using other tools. In this chapter, we will look at the literature contribution to dealing with Knightian uncertainty in finance and economics. In addition, there is existing literature that uses fuzzy tools to deal with Knightian uncertainty in finance and economics. The most sophisticated treatment is really in Fuzzy Option Pricing, which is exactly the area we target in our research. So we will also provide a literature overview of fuzziness in economics and finance and ways of dealing with it but we will leave Fuzzy Option Pricing to a later chapter (Chapter 8) where we perform a detailed study of those approaches. To avoid redundancy, we will only touch briefly upon this area in this chapter.

Before delving into the literature review, we need to clarify the term 'uncertainty.' We, more often than not, come across uncertainty as a synonym to risk in finance and economics literature. However, the two terms are different, which is the reason why the tools of handling them are different, and consequently the reason why we are attempting to use alternative tools to probability.

Uncertainty has been broken down into several branches in Fuzzy Theory depending on the framework of the problem. As such, it is distinguished from risk. This development has been taking place independently of the Knightian uncertainty distinction in finance and economics. We will attempt to bring together those two separate fields that attempt to discuss the same matter and draw on the tools that researchers have introduced to deal with such uncertainty in either field.

Section one discusses Knightian Uncertainty. We look at uncertainty in Economic Theory and uncertainty in Fuzzy Theory. In section two, we consider recent developments in Option Pricing literature since they have comparative implications for the models we will be introducing later on. In the third section, we will look at fuzziness in economics and finance. The need for alternative tools to Probability Theory in the fuzzy literature has been long recognized by fuzzy researchers. However, they have come into notice by finance researchers only recently. Not surprisingly, the most challenging and sophisticated, and hence promising, applications are those introduced by finance researchers. Therefore, we will be drawing on literature from different areas in an attempt to present various pieces of related research, which has been developed independently, into one coherent framework.

3.1 Knightian Uncertainty

Risk and uncertainty are usually used interchangeably, with more association between risk and unfavourable outcomes. Based on this, probability has always been assumed to be able to model uncertainty. However, as we have mentioned in the introduction, the two terms are quite different and probability theory alone is not enough to model all forms of uncertainty. It is our goal in this section to illustrate the difference between risk and uncertainty by drawing on literature from Economic Theory as well as Fuzzy Theory.

3.1.1 Uncertainty in Economic Theory

Knight [93] has presented a detailed analysis distinguishing between risk and uncertainty as far as 1921. He refers to risk as the measurable uncertainty and to uncertainty, which has later assumed the name Knightian uncertainty, as the unmeasurable one. According to Knight, we have only partial knowledge of the world around us. Knowledge is "variable in degrees" and problems usually relate to those degrees of knowledge rather than to the presence or absence of knowledge. This necessarily dictates that we do not have quantitative probabilities of every possible outcome, and, hence, our knowledge of the future is imperfect.

So risk holds in cases where we have enough information or knowledge to derive distributions of outcomes and formulate our probabilities, which Knight divides into a priori and statistical. A priori probability can be estimated from groups of instances that are generated by calculations based on general principles or repetitive experiments, such as the throwing of a die, while statistical probability can be estimated from groups of instances that are generated empirically, such as the burning of a building. On the other hand, uncertainty refers to those cases where we do not have groups of instances, or any valid basis for classifying them, and so we do not have information to formulate estimates of probability. Knight associates the term "objective probability" with risk and the term "subjective probability" with uncertainty.

He argues that the cases exhibiting uncertainty refer to events that are highly unique, which is quite prevalent, but unfortunately ignored, in business and economics. In fact, he believes that measurable uncertainty or risk do not induce any uncertainty into business and so, in such cases, there can be no reward for risk-taking. It is always possible to insure against such type of uncertainty. According to Knight, uncertainty can be reduced using six approaches. The first approach is the "consolidation" approach and it involves using groups of cases rather than separate instances because the larger the group, the lower the uncertainty. The second approach is the "specialization" approach and it accommodates for different views of individuals regarding it. The third, fourth and fifth approaches are control of the future, increased power of prediction and "diffusion" of the consequences of untoward contingencies, which is closely associated with the consolidation approach. The sixth approach is directing industrial activity towards lines involving a minimal degree of uncertainty.

Before moving on to present other literature, we would like to present an interesting observation which Knight notes and which is a little bit out of line here but relevant to our research later. Knight observes that classes of objects do overlap sometimes. But scientific thinking drives us to eliminate such statements by trying to interpolate from other general facts about the history of the subject crisp statements related to its occurrences or non-occurrences. Yet, it is not always possible to achieve that. However, in our case, we do not have to worry about this because we will allow for overlapping classes of objects using Fuzzy Set Theory.

Keynes [88] recognizes the fact that probabilities cannot always be measurable and numerically estimated due to lack of rational basis or our lack of apprehension of evidence, that is, in our terminology, due to our incomplete or lack of knowledge. He associates with such probabilities risks that are not insurable. He also recognizes the existence of vague knowledge but does not attempt to deal with it due to its complexity. He also considers cases where probability does not exist.

Another important insight that Keynes draws is that additivity does not always hold. In particular, he questions the soundness of the mathematical expectations as measuring our preferences given different courses of action. In other words, he questions whether the undesirability of a particular course of action should rightfully increase in direct proportion to the increase of attaining its object rather than by more than proportionately and, hence, accommodating for the risk assumed. As we will see later, we mend this drawback by using expectations that are means of compromise rather than additive.

Ellsberg [60] analyzes the assertion that all uncertainties reduce to risk and, hence, one can always formulate numerical or qualitative probabilities. He finds that this does not hold. To illustrate his point, Ellsberg provides a betting example that allows one to measure subjective probabilities of the subjects in an experiment. The example involves two urns containing red and black balls. The total number of balls in either urns is known but the ratio of red to black is not known for the first urn; however, this ratio for the second urn is 50 red to 50 black. A gambler is asked to bet on drawing a specific colour using four scenarios.

The first two scenarios involve betting on one colour from one urn, for example, betting on drawing red from the first urn or drawing black from the first urn as well. The third and fourth scenarios involve betting on a specific colour from either urn, for example, betting on drawing red from the first urn or drawing red from the second urn. Basically, the first two scenarios involve a choice of colour but one urn. The third and fourth scenarios involve one colour but choice of urn. The gambler is not allowed to choose an urn or a colour.

Biases aside, the typical response for the first two scenarios is indifference. As for the third and fourth scenarios, the majority prefers betting on the second urn where the probabilities are known while only a minority prefers betting on the first urn where the probabilities are unknown. The implication of this experiment is that people, in general, prefer to act on known probabilities or to avoid ambiguity but it is not possible to infer the additive subjective probabilities from such choices. So probabilities cannot be formulated and, hence, uncertainty does not reduce to risk always.

Interestingly enough, Knight has also presented a similar example ([93], p.218). He considers one urn and assumes that one man does not know the composition of red and black balls in it while another does. So we can assume that the first man assumes equal chances of drawing a red ball or a black ball but we can also assume that the right probability is that corresponding to the knowledge of the second man but the first man does not know it. However, when it comes to taking a decision of conduct, Knight argues that the man will assume the equal chances case. Ellsberg argues that Knight contradicts his uncertainty view in this case because it implies that people can always resort to the Principle of Insufficient Reasoning and so they will not be interested in Knightian uncertainty when it comes to decisions of conduct.

In fact, Ellsberg wrote his paper in an attempt to examine closely Savage axioms (besides Knightian uncertainty). He shows that they do not always hold. However, Raiffa [126] attempts to prove that such axioms always hold and carries an experiment within the classroom setting. We are not really interested with these specific application in our research. We are, in fact, drawing on Raiffa's paper because of a comment he has made, within the context of his experiment, and which says " I found that when relative frequencies or so-called objective probabilities were given in numerical form as data of a decision problem, then these were often used in computing various indices (e.g., expected or actuarial values) which served as a guide to action. But if certain uncertainties in the problem were in cloudy or fuzzy form, then very often there was a shifting of gears and no effort at all was made to think deliberately and reflectively about the problem." ([126], p.691) We believe that the reason that subjects, in the presence of cloudy or fuzzy uncertainties, do not think deliberately and reflectively about the problem is the fact that they are thinking in terms of probabilities. We do believe that had the subjects approached the problem from a different angle, they would have been able to deal better with such uncertainties. Within the course of our research, we will attempt to fix this problem by using possibilities or other measures that are especially tailored for such cases.

We find ourselves more in line with Fellner [65]. He recognizes the existence of nonadditive subjective probabilities, which he calls uncorrected but attempts correcting them to be able to calculate mathematical expectations. Uncertainty, in this case, is measured by the nonadditivity, which he calls distortion, of those uncorrected subjective probabilities. Our research is similar in the sense that we will be working with nonadditive measures but we will not correct them; rather, we use fuzzy expectations instead of mathematical expectations that require additivity.

More recently, Knightian uncertainty and nonadditive probabilities started finding an appeal in the economic literature. Curley and Yates [37] do recognize uncertainty due to insufficient information. They distinguish between two types of uncertainty, which are uncertainty related to the occurrences of an outcome and uncertainty related to the likelihood of the outcomes. Interestingly enough, they distinguish between ambiguity and uncertainty. They believe that uncertainty about a decision's outcome is captured by probabilities, so we can infer that they meant by this uncertainty risk, in our terminology. On the other hand, they also recognize an uncertainty of "ambiguity," which we infer to be the Knightian uncertainty in our terminology. They define ambiguity as uncertainty related to the "processes by which outcomes are determined, and has been characterized as uncertainty about the outcome probabilities themselves." ([37], p.274) Based on this, they deal with ambiguity, or Knightian uncertainty, using "uncertain probabilities," which are characterized by intervals defined by their ranges and centers. The greater the range, the higher the ambiguity. They argue that people tend to exhibit ambiguity avoidance and they find out that the degree of ambiguity avoidance increases with the center.

Schmeidler [127] deals with such uncertainty, which has to do with the occurrences of the state itself, using nonadditive probabilities. He argues that such measures can transmit or record (vague) information that additive probabilities cannot. He combines both subjective and objective probabilities. Objective probability, viewed as a physical concept, is a special case of the subjective or personal one and is prevalent when there is no Knightian uncertainty. This does not impose an additivity assumption over the probabilities. To compute the expectations using such probabilities, Schmeidler uses the Choquet Integral (hence, the Choquet expected utility model) which allows nonadditive probabilities (and converges to the Lebesgue integral in the case of additive probabilities so that the corresponding expectations converge to mathematical expectations). He also introduces the notions of uncertainty aversion, uncertainty attraction and uncertainty neutrality in analogy to their risk counterpart. He argues that additive probabilities correspond to the uncertainty neutrality case.

Dow and Werlang [46] study the impact of uncertainty in an investment decision problem. Such uncertainty arises from the uncertainty of the underlying asset itself. They deal with this uncertainty using subjective nonadditive probabilities and they argue that if those probabilities add up to less than one, then uncertainty aversion prevails. They show that the highest price an agent is willing to buy the asset at is the asset's expected value (using Choquet integral) over the nonadditive probabilities and the lowest price he is willing to sell the asset at is the expected value of selling the asset short. Those two prices embody only beliefs and uncertainty aversion and not attitude to risk. The authors also draw implications for insurance whereby they argue that, based on such logic, there will be a range of prices for which the agent can fully insure.

In another paper, they [47] attempt to interpret the high stock price volatility within a Knightian uncertainty framework given that the stock market is influenced by many factors that induce a high degree of such type of uncertainty. They again utilize nonadditive probabilities that reflect uncertainty aversion. However, in this case, the event and its complement add up to less than one and the difference serves as a measure of the uncertainty of the corresponding event from the agent's point of view. They also use the Choquet integral to compute expectations.

Epstein and Wang [61] study the determination of asset prices under Knightian uncertainty using an extended Bayesian model, namely the multiple-prior model, that allows a distinction between risk and uncertainty. They use an uncertainty adjusted probability kernel that is allowed to be multivalued (to accommodate uncertainty). They also account for uncertainty aversion.

Epstein [62] has recently tried to introduce a definition of uncertainty aversion. However, he does not distinguish between vagueness, ambiguity and Knightian uncertainty; we will differentiate between those terms later on when we study uncertainty in fuzzy set theory. The value added with this new definition is that it is also suitable for applications within a Savage framework (and it assumes eventwise differentiability of utility but this is not relevant here). The model is more general than the multiple-priors and the Choquet expected utility models.

In the literature we have considered so far, no formal definition of ambiguity has been presented. It is only recently that Epstein and Zhang [63] have attempted defining ambiguity formally. The interesting feature about such a theory is that it endogenizes unambiguous events. However, the definition is quite long and out of line with our research that we will not present it in this document but the reader is referred to the corresponding paper for a meaningful definition.

In summary, most of the work done on uncertainty in economics accounts for Knightian uncertainty using non-additive probabilities and some form of adapted utility model such as the Choquet expected utility and the multiple-priors models. In the following section, we will look at different aspects of uncertainty and different ways of dealing with it.

3.1.2 Uncertainty in Fuzzy Theory

As we have seen in the previous section, probability, irrespective of what its proponents (especially Bayesian statisticians) say, is not enough to model uncertainty and it may fail if used within an unsuitable problem definition. There are many ways by which we can deal with this problem and we will explore them in the following chapter. At this point, however, we will analyze the problem from a Fuzzy Theory point of view.

Probability, in fact, models the randomness aspect of uncertainty. But, as we have seen in the previous section, randomness is not really the only aspect of uncertainty. There are uncertainties arising due to factors other than chance and so they are not random and cannot exactly be modelled by probability theory. We will merge those uncertainties under a broad title for now and call it fuzziness until we are well equipped to draw a distinction between the two terms.

Both probability and fuzzy theory deal with uncertainty numerically and they share some structural similarities (such as commutativity, associativity and distributivity among sets). In fact, in the chapters to follow, we will carry a deeper comparison between elements of Probability Theory and elements of Fuzzy Set Theory and Fuzzy Measure Theory but, at the moment, we will suffice by keeping the comparison at a conceptual level.

The root of dissimilarity between the two is the treatment of the event and its negation. Probability theory draws a sharp edge between the two to be able to distinguish between the two concepts. Fuzzy theory, on the other hand, does not distinguish between the two. In other words, in probability theory, it is assumed that we can distinguish between an event and its negation but, in fuzzy theory, we do not have this restriction and the two entities can actually overlap. In the real world, especially within our context the financial markets, we cannot always differentiate between an event and its negation, which has important implications particularly for incomplete contracts or contingent claims [129].

Kosko [95] presents a very interesting paradox within this framework. He measures the fuzziness of a set A by how much the superset $A^c \cup A$, where A^c is the complement of A, is a subset of its own subset $A \cap A^c$. In Probability Theory, this is impossible and $P(A^c \cap A \mid A^c \cup A) = P(\phi \mid X) = 0$, where X is the sample space or sure event and ϕ is the empty set or impossible event. On the other hand, in Fuzzy Theory, we do not classify the world as black or white; we do have grey areas and, perhaps, plenty of them.

Kosko defines fuzziness as event ambiguity, that is, it measures the degree to which an event occurs. Randomness, on the other hand, deals with whether or not an event will occur. As Kosko puts it, "Whether an event occurs is random. To what degree it occurs is fuzzy." ([95], p.213) Of course, the event itself can be ambiguous and we can question its occurrences which will produce the probability of a fuzzy event but this is another story now. To illustrate the distinction further, Kosko gives a very interesting example: suppose that there is a 50% chance that there is an apple in the refrigerator, which can be arrived at using subjective or objective probability tools. Now suppose there is half an apple in the refrigerator. These are two physically distinct events or "states of affair," as Kosko calls them, that have the same numerical uncertainty. The first refers to randomness and the second to fuzziness.

Fuzziness is a type of deterministic uncertainty that does not dissipate with the release of new information like randomness does, that is, fuzziness is inherent to the definition of the event itself. If we use Probability Theory alone in the presence of ambiguity, the sample-space will still have the same ambiguity after randomness is eliminated because information clarifies the degree of occurrences. It is only until we use tools of fuzzy theory, that we can eliminate or accommodate this ambiguity.

So far, we have been using fuzziness, ambiguity and uncertainty interchangeably. In fact, those terms model different entities and it is time to draw a distinction between the three terms. Klir [89] divides uncertainty into vagueness and ambiguity. Vagueness deals with linguistic uncertainty, that is, with uncertainty associated with sets that do not have sharp boundaries. This is particularly prevalent when it is hard to draw distinctions in the real world, for example, between an event and its negation. Vagueness is usually modelled by Fuzzy Set Theory, specifically, grades of membership. Fuzziness is really associated with vagueness.

On the other hand, ambiguity deals with one-to-many relationships, that is, with instances involving several alternatives among which the choice is indeterminate. It is usually modelled by Fuzzy Measure Theory. As we will see in later chapters, this theory represents the uncertainty associated with whether a certain element of the universe of discourse, which is not a priori allocated to any subset of this universe, belongs to the set (which is a subset of the universe of discourse) under consideration.

The difference between vagueness and ambiguity can be further illustrated using Dowlatshahi et al.'s definitions ([48], definition 9.1.1, p.289) keeping in mind that they establish those definitions within the context of linguistics; however, there is no reason to believe that this cannot be applicable to other contexts. Vagueness arises when we have more than one interpretation but there are no grounds on which to decide on the best interpretation. Ambiguity, on the other hand, admits grounds for the decision but they point to more than one alternative. Vagueness is relative to a view or perspective, whereby each view or perspective provides a different interpretation, but ambiguity is not. However, they argue that vagueness is more general than ambiguity but we do not agree with that; we, generally, assume the position that the two entities are two distinctive types of the more general concept of uncertainty.

Ambiguity, in turn, can be further divided into three types. The first type is non-specificity or imprecision of evidence. Specificity has to do with the values a variable can assume in a subset or the alternatives a decision situation can assume. Nonspecificity arises when such alternatives or values or interpretations are left undecided. It involves the size of the subset. So the larger the subset, that is the more alternatives involved, the less specific the characterization is, and, obviously, when the subset is a singleton, that is involving one alternative, the situation or the characterization is fully specific.

The second type of ambiguity is conflict or dissonance in evidence and is associated with the disjoint subsets of the universe of discourse to which the element under consideration may belong. The evidence on one subset could conflict with evidence on the others. The harder for us to distinguish between evidence or alternatives, the greater the dissonance. The third type of ambiguity has to do with those subsets that do not overlap or partially overlap. So we will have partially or totally conflicting evidence on the subsets. This type of ambiguity has the meaning of confusion in evidence. In fact, those last two types of ambiguity are quite similar.

We have seen in the previous section how incompleteness of information gives rise to Knightian uncertainty. But we have not established a link between this uncertainty and the four types of uncertainty within the context of fuzzy theory yet. Dowlatshahi et al. [48] divide incompleteness into two types: internal incompleteness which has to do with the lack of information in a framework and external incompleteness where the framework itself is incomplete. However, they treat incompleteness as a separate entity to vagueness and ambiguity. We, of course, do not agree with this view. We

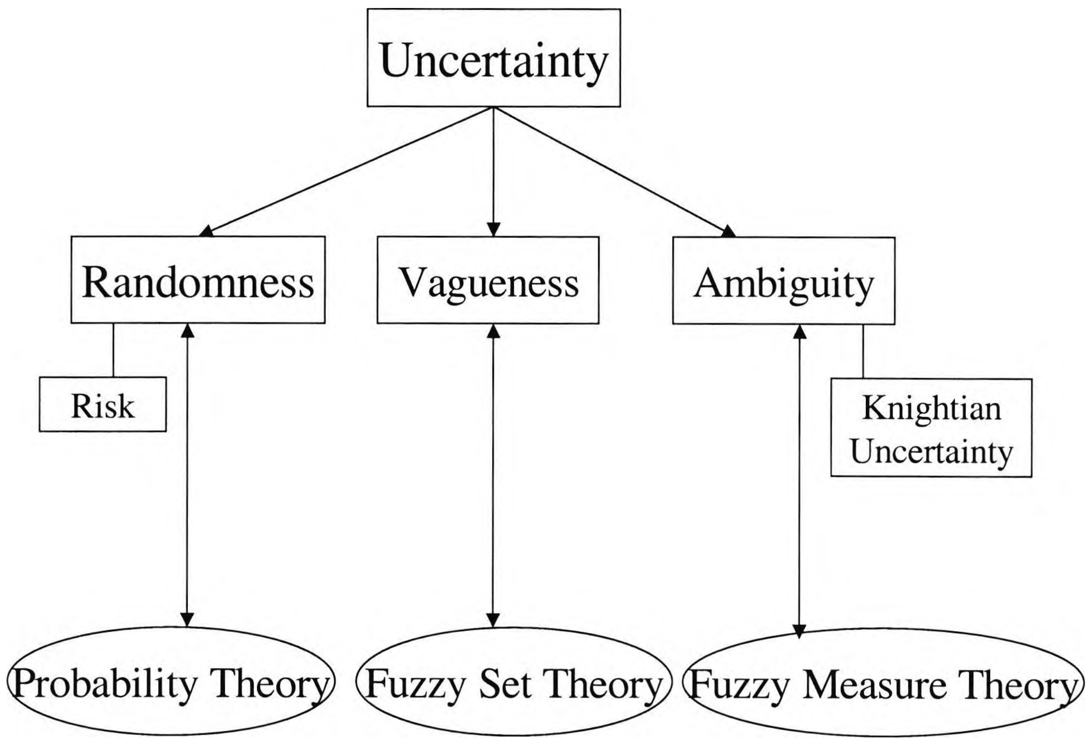

Figure 3-1: General Uncertainty Framework

tend to view incompleteness as one of the causes that give rise to uncertainty, whether in the form of ambiguity or vagueness.

In fact, we tend to agree with Klir's point of view [90] on this issue. Incompleteness is a form of information deficiency or, using a better description, is imperfect information. In such instances, information can be incomplete, imprecise, vague or contradictory and so it gives rise to different types of uncertainties.

Figure 3.1 shows a map of the decomposition of uncertainty and tools that deal with each decomposition. Several approaches have been proposed to measure the amount of uncertainty in a system. We will not talk about them now but we have collected a comprehensive literature review in Appendix B.

3.2 Option Pricing

The models we present in this research build upon the CRR OPM or the Black-Scholes OPM. Those models serve as fair price models, that is, they are assumed to hold in a 'perfect' market. Given the assumptions they impose, one does not expect them to be empirical models. Nevertheless, they are still used by practitioners but this is really an issue of simplicity and self-fulfilling prophecy. In our opinion, it is not 'scientifically' correct to assume that those models are the right ones just because the market uses them. As support to this statement we point to the general agreement that the Black-Scholes OPM generally underprices in-the-money options and overprices out-of-the-money ones.

According to Bhattacharya [11], testing Option Pricing Models usually falls into testing three hypotheses:

1. mathematical structure of the formula,
2. measurement of inputs and outputs such as simultaneity issues, and
3. efficiency of the options market.

In our research, we are more concerned with the third point. Models that give a single no-arbitrage or fair price impose stringent assumptions that cannot be met in actual trading markets. In fact, recent developments in Option Pricing have been more inclined towards establishing an interval of no-arbitrage prices rather than a single price.

In a frictionless market, options can be priced by a 'no-arbitrage' strategy, whereby the option is hedged against the underlying asset such that the portfolio earns the riskless rate of return, and is rebalanced continuously until expiry. However, perfect replication is not possible in an imperfect market such as listed option markets. Literature has tackled various market imperfections and studied their impact on option pricing. The general finding is that those market imperfections impose wide bounds on equilibrium option prices rather than the one price model output given under a perfect market. This also has implications for hedging and for testing Option Pricing Models.

Before exploring this literature, we need to make a clarification on the use of the term 'imperfection.' It is quite frequent to come across this term in Option Pricing literature (such as [67],[57]) as it is the case in Economic Theory. The idea behind using this term has to do with the fact that replication or arbitrage strategy, which is the main approach for pricing options in complete markets and the approach by which the Black-Scholes and CRR OPMs are derived, is only possible in a frictionless market. The fair price models assume a frictionless market and so for those models to hold, we need to have a 'perfect' market in the sense that the corresponding market has to satisfy the assumptions imposed by those models. Deviations from those assumptions will give rise to imperfections such that option replication, and hence the application of fair price models, is no longer possible and other approaches are called for. These 'imperfections' give rise to market incompleteness.

One of the most important market imperfections is uncertain volatility. The Black-Scholes model assumes a constant volatility, which is not observed in actual options markets. So given that volatility is assumed constant by a fair price (or 'perfect') model, it follows that it can be considered a market imperfection if it violates the assumption of this fair price model.

Volatility fluctuates a lot in the financial markets. It is the hardest input to estimate in an Option Pricing Model and there is no general consensus on how it must be estimated. Even historical or realized volatility differs from the actual one depending on the statistical technique used. Volatility is important because it affects the risk in options arbitrage by directly affecting the fair or model price of the option. It also affects the return of such a strategy since an agent who has a better estimate of the market volatility, that is an estimate closer to the true volatility, will be able to earn more than the riskless rate of return.

There have been some successful attempts at modelling volatility as a stochastic volatility or as a deterministic volatility surface while preserving the no-arbitrage principle. However, those models are only appropriate within certain problem frameworks. Besides, stochastic volatility models have been criticized as complicated especially in the presence of a market risk parameter while deterministic volatility surfaces have been accused of mispricing. In either case, we will not really go in details into those approaches because they are beyond the scope of our research. We are really interested in uncertain volatility models whereby the volatility is known to fluctuate within a band and so it is no longer possible to price by replication. The alternative in such instances is to establish bounds on the value a volatility can assume which in turn gives rise to portfolio super- or sub-replication leading to bounds on the option price itself.

Another important imperfection is the existence of transaction costs in actual markets. This imperfection also leads to bounds around the fair price of the option such that arbitrage opportunities do not exist. Other imperfections include portfolio constraints, short selling constraints among others, which we will see in the following literature review.

Figlewski [67] studies the implications of market imperfections such as uncertain volatility, transaction costs, indivisibility and rebalancing at discrete intervals on option pricing and hedging, more precisely on the no-arbitrage strategy. The author argues that, in a frictionless market, arbitrage drives the price of an option to its fair Black-Scholes value. However, the presence of imperfections creates friction in the market, and, consequently, one can only establish bounds on the option price. The option price will be free to fluctuate within those bounds such that no arbitrage is possible since it would be costly to do so. His methodology involves setting up an experiment that is based on simulated data and introduces those imperfections one by one subsequently comparing them to the Black-Scholes OPM value. Jameson and Wilhelm [81] confirm this finding. They argue that uncertain volatility and the inability to rebalance continuously, or the sigma and gamma of an option, generate nondiversifiable risks that account for a significant proportion of the bid-ask quotes entailing important implications for theoretical bounds on option pricing, which is in line with Figlewski's argument.

A more recent stream of literature that attempts arriving at no-arbitrage bounds using various complex mathematical approaches has emerged and it is usually referred to as portfolio sub- or super-replication. It generally targets pricing and hedging in incomplete markets characterized by various imperfections like transaction costs, uncertain volatilities and others. The problem in incomplete markets, other than the fact that replication is not possible, is that there exists more than one martingale measure that is equivalent to the initial probability measure and among which the choice is indeterminate. Therefore, instead of one no-arbitrage price, one would get a range of no-arbitrage prices.

Edirisinghe et al. [57] look at the effect of transaction costs and trading constraints, such as lot size and position limits, on the portfolio replication. Their approach is to minimize the initial cost of the terminal payoff required to super-replicate the desired payoff irrespective of the agent's preferences. The problem is developed as a nonlinear programming problem which is then reformulated as linear programming and two stage recursive models. A binomial tree in discrete space is constructed where the final solution, after switching to the linear programming approach, is path dependent so that the number of constraints grows exponentially as the trading frequency increases and a global optimum is computationally difficult to reach. As a consequence, they work with a reduced binomial tree where the agent's decisions are conditioned on the stock price and time, and not on the path followed. Besides, the tree grows only quadratically with the trading frequency and so a global optimum is easier to reach. The cost of replicating the option increases as trading costs increase such that, at a certain point, there is no trading. The authors argue that the super-replicating strategy in the presence of transaction costs starts off with a larger investment than that with no transaction costs but involves less trading up to expiry and it is no longer optimal to rebalance at each period. However, because of the exponential growth and integer constraints, the authors develop a two-stage dynamic programming model whereby the value function grows only quadratically and not exponentially so it is computationally more efficient. Based on this model, a set of conditions impose policy actions which entails revising the portfolio only if the corresponding marginal gain more than offsets the cost of trading. It involves using in the first stage a backward recursive model to minimize the amount by which the generated cash flow based on the initial wealth is lower than the desired one and then, in the second stage, the minimum initial wealth that makes this difference zero is computed.

Avellaneda et al. [5] consider the scenario whereby volatility is uncertain and they allow it to fluctuate within a band bounded by a maximum and a minimum possible volatility that can be chosen subjectively or statistically. Those bounds lead to a new non-linear (parabolic) PDE, which the authors refer to as the Black-Scholes-Barenblatt BSB equation. The BSB equation is in fact a generalization of the Black-Scholes diffusion equation and converges to it in the crisp case (one volatility value). Solving this equation leads to two consecutive bounds for the option price where the volatility values used in the equation are selected based on the convexity of the value function. The authors argue that the agent can short an option at the upper bound price and risklessly hedge it using the hedge ratio corresponding to this price since it is the minimal initial possible cost. On the other hand, they can long an option at the lower bound price, which serves as the maximal bid value, with a hedge ratio corresponding to the lower bound price. They also solve the equation is discrete time using trinomial trees and finite differences. They show that the equation can be used for hedging using other derivatives.

Jouini and Kallal [82] study portfolio super-replication under short sales constraints and different borrowing and lending rates. The authors argue that a no-arbitrage condition for assets that cannot be sold short is to have supermartingale price processes while for those that can only be sold short is to have a submartingale price process. The super- and submartingale probability (risk-neutral) measures go into the pricing equation to give us the bounds for the set of option prices. The upper bound is the minimal initial cost needed to super-replicate a desired payoff, i.e. an agent will not agree to pay more than this amount for the option, and the lower bound is the maximum amount that can be borrowed against it, that is an agent would not agree to sell the option at less than this amount. The authors argue that those bounds are the tightest possible bounds in such a market that can be obtained irrespective of agents' preferences. For the diffusion case, they derive a PDE, which they solve numerically.

El Karoui and Quenez [58] work with a stochastic dynamic approach to solve the problem of pricing in an incomplete market. They also obtain a range of prices whereby the upper price serves as a selling price and is hedged by a portfolio consisting of one riskless asset and a number of risky assets, and the lower bound corresponds to a buying price.

Cvitanić and Karatzas [38] study the problem of super-replication in the presence of transaction costs using an optimization problem to maximize utility. The authors define shadow state price densities and use them to solve the marginal utility and its dual problem. The optimal portfolio is then the inverse of the marginal utility evaluated at the shadow price-density. The upper bound of the set of option prices is the minimal hedging price, which is quite high, for example, it is equal to the underlying asset's price in the case of European options (buy and hold strategy), which makes the model impractical. The authors also find that under certain conditions, it is optimal not to trade.

Karatzas and Kou [86] consider hedging American options under portfolio constraints in the form of restrictions on borrowing and short-selling as well as restrictions on investments in certain assets using martingale theory and stochastic processes. They introduce an interval for no-arbitrage prices whereby any price lying outside the interval is an arbitrageable price. Those bounds correspond to a supermartingale measure. Under convex constraints, they find that the lower bound that sub-replicates the desired portfolio is equivalent to the payoff and the upper bound that super-replicates it is less than or equal to the underlying asset's initial price. In a market with convex coefficients, they find that the lower hedge bound corresponds to the initial payoff while the upper bound corresponds to the expectations of the terminal payoff.

Cvitanić et al [39] establish a closed-form solution for portfolio super-replication of European options in the presence of portfolio constraints. Again, they find that the minimal super-replicating strategy associated with the upper bound corresponds to a buy and hold strategy, that is buy the underlying and hold it until maturity, which also holds for general path-dependent options. Their work is an extension to that of Cvitanić and Karatzas [38] whereby they reach a closed form solution using PDE, stochastic control and martingale theory tools. They solve the problem in a Markovian continuous time setting whereby the minimum hedge price is given by the supremum of the expectations of the claim over all supermartingale measures.

Frey and Sin [69] compute no-arbitrage bounds for European options under a stochastic volatility constraint. Those bounds correspond to the infimum and the supremum of the range of possible prices. They correspond to the most expensive sub-replication and the cheapest super-replication strategy respectively. For a general stochastic volatility model, they find that the infimum is greater or equal to the expected value of the payoff while the supremum is less than or equal to the underlying price, which means that the cheapest superreplication strategy is to hold the stock. This is, of course, not quite a contribution and other models are called for. So the authors introduce subjective bounds on the volatility. To hedge the option, they introduce a tracking error that measures the deviation between the terminal payoff and the one it is supposed to replicate, which is the based on the Black-Scholes model. So if this error is positive, it means that the agent's terminal value of the hedge portfolio will always cover the option payoff. A similar result has been obtained by El Karoui et al. [59].

Frey [70] studies super-replication in a stochastic volatility model. However, because all superhedging in stochastic volatility models results in a price that is too high such as the buy and hold case, the author introduces bounds on the stochastic volatility that can be subjective or derived from historical data and serve as a confidence interval. The value process is characterized by an optimal stopping problem within the context of the Black-Scholes model. The author finds that the superhedging price of a European option under stochastic volatility is equivalent to the value of the corresponding American option under constant volatility.

Perrakis and Lefoll [120] look at the implications of transactions costs on American options with dividends and study their pricing and hedging across the binomial tree. The underlying asset is assumed to be of physical delivery rather than cash settlement. The presence of transaction costs naturally leads to bounds for prices whereby exercise is sometimes dependent on the holder's preferences. They have been able to establish replication, but with a more complex approach than that of the binomial tree, for both the bid and the ask. But it is conditional for the Bid case and depends on a transactions cost parameter.

In summary, in an imperfect market, we can only get no-arbitrage range of prices, whereby any value beyond the upper and lower bounds can be an arbitrageable price. As we have seen, common approaches utilize tools of probability and martingale theories, stochastic control and optimization models. We will be using Fuzzy Measure Theory and Fuzzy Set Theory to price options in the presence of Knightian uncertainty. Knightian uncertainty fits into the framework by giving rise to market incompleteness due to lack of information. As we have seen in Chapter 2, fuzzy tools are appropriate for modelling such uncertainty and imprecision so it will be worthwhile to examine which problems in finance and economics that have been solved using such tools, which will be presented in the following section.

3.3 Fuzzy Reasoning in Economics and Finance

Fuzzy theory is a logical approach for solving problems in economics and finance. Fuzzy Set Theory and Fuzzy Measure Theory offer researchers flexibility in dealing with economics and finance problems especially that decision makers are sometimes faced with a high level of uncertainty whereby precision is no longer feasible. Using Fuzzy Theory, it is possible to accommodate imprecision and to incorporate natural language if necessary into the problem. The decision maker does not really have to summarize information into one precise number. Another important point is the nonlinearity of data in the sense that changes are not proportional all the time. Data in economics and finance are sometimes nonlinear or non-proportionate, for example, twice the input does not mean twice the output or an increase in stock price does not result in proportional increase in expected return. Sometimes, this can have an impact on the final output. In those circumstances, Fuzzy Theory is an appropriate candidate.

Generally, the approaches that have been used so far utilize tools of Fuzzy Set Theory. However, recently, more applications have been utilizing tools of Fuzzy Measure Theory especially for complex applications involving option pricing and credit risk. Such tools come in handy when it is hard to measure states or preferences and additivity becomes too restrictive. The applications utilizing Fuzzy Set Theory are numerous but we will restrict the review to a collection of papers that provide a comprehensive representation of the models or approaches used generally.

3.3.1 Fuzziness in Economics

Baumont [9] studies the influence of business experience in the evolution of the labor market structure. Since Probability Theory 'fails' when there is lack of information, the author resorts to Possibility Theory since it accounts for human perceptions about the future as well as discounts unpredictable events and, in this particular example, it models job preferences and promotions better than probability theory does. So she studies the possibility of getting a job with given abilities and then the possibility of getting a job with new acquired abilities or work experience. The major finding is that the labor market structure converges in possibility into a stable state.

Ponard [121] studies the contribution of Fuzzy Set Theory to the foundations of economics. To this end, he considers three models: fuzzy economic choice, fuzzy economic calculation and fuzzy general economic equilibrium. In the first problem, he considers imprecise preferences and incorporates them using a fuzzy or max-min utility function. A fuzzy economic calculation framework is introduced to model those imprecise preferences and fuzzy utility. Finally, the fuzzy model of general equilibrium studies the compatibility of fuzzy partial equilibria of consumers and producers. The author reaches the conclusion that fuzzy theory has major contribution to economics via imprecise preferences and fuzzy utility and is, in fact, able to solve problems that classical theory is not able to solve.

Billot [12] provides a comparative review of the use of Fuzzy Set Theory and nonadditive probabilities in economics. He concludes that the nonadditive probability approach has found wider acceptance in the economics circle than Fuzzy Set Theory.

Finally, Buckley ([18],[19]) solves a fuzzy Leontief's open input-output problem. Fuzziness lies in the imprecise coefficients, which are represented by fuzzy numbers. As a result, output is comprised of parametric bounds. In the same papers, he solves a fuzzy demand-supply model by, again, fuzzifying the coefficients in the respective differential equation.

3.3.2 Fuzziness in Finance

Fuzzy reasoning seems to be more popular in financial applications than in economics ones. Besides, the applications tend to be more sophisticated since here, we will actually see applications of Fuzzy Measure Theory and nonlinear integration. However, there is a general tendency to get the final value or function in terms of parametric bounds, which can be quite subjective at times. Most of the fuzzy financial applications tend to be investment problems with few exceptions of Fuzzy Option Pricing.

Tanaka et al. [136] address the problem of decision making at higher level since it is characterized by a higher degree of imprecision than it is at a lower level. The problem, which is particularly an investment problem, is defined by fuzzy actions, states and information. However, they do also introduce probabilities to represent the uncertainty of the occurrence of the fuzzy objects, hence, combining randomness (probability) and fuzziness (meaning of events). They deal with the problem using membership functions, probabilities and fuzzy utility functions from which they are able to derive worth and quantities of information. Their model is a nonparametric model but the membership functions are subjectively determined.

Buckley [16] tackles the problem of ranking investment proposals characterized by fuzzy cash flows, fuzzy project duration and fuzzy interest rates. He represents the fuzziness by fuzzy numbers, that are special types of fuzzy sets. He uses standard fuzzy arithmetic, which is quite different from classical arithmetic. The projects are then ranked based on their fuzzy net present value. The proposals are eventually considered equally best. The author [17] uses a similar approach to solve an elementary compound interest problem. He uses fuzzy numbers to represent fuzzy cash flows, fuzzy interest rates and fuzzy number of periods and gets a fuzzy present value and a fuzzy future value utilizing standard fuzzy arithmetic. In arriving at the final decision, he utilizes the max-min approach. In a later paper, Buckley [18] solves a fuzzy internal rate of return problem where cash flows are uncertain and, hence, represented by fuzzy numbers. The author solves a fuzzy equation to arrive at parametric bounds for the internal rate of return IRR.

Östermark [116] tackles portfolio management using a fuzzy mathematical programming approach. He particularly works with a fuzzy Capital Asset Pricing Model (CAPM) to make it a more realistic model. Fuzziness enters into the framework through the coefficients of the policy constraints since they generally have an element of imprecision in them.

Li Calzi [26] attempts to establish a general setting for the fuzzy mathematics of finance. Unlike Buckley, he does not restrict himself to the compound interest model, rather he deals with the foundations and extends it to models such as the discount model.

Benanchenhou [7] works on another portfolio management problem. She introduces an intelligent (fuzzy) trading system, which includes a fuzzy rule extraction tool that formulates membership functions, extracts rules and then use those rules for trading. So the fuzzy system fuzzifies time series, derives membership functions, extracts rules, executes those rules, evaluates them and then defuzzifies the result. The results are quite dependent on the degree to which the membership functions overlap such that the higher the degree of overlap, the less successful the results.

Kuchta [98] also works on an investment (choice) problem. The author extends the classical capital budgeting problem into a fuzzy one which involves comparing projects dependent on fuzzy cash flows, duration time, and required rate of return. The approach followed involves replacing the fuzzy objects with fuzzy numbers but here standard arithmetic rather than fuzzy arithmetic is used. The value of the project is represented by a parametric bound but no specific approach for comparing fuzzy numbers was specified.

Gupta [76] adopts a rather nonconventional approach. He does acknowledge that investment problems have a high degree of imprecision. However, he does not recommend working with possibility distributions but suggests transforming them to probability distributions instead. He argues that converting from probability to possibility distributions results in loss of information. Probability theory has a well developed mathematics to support it and, most importantly, managers are more interested in what is likely to happen rather than what will possibly happen. So he presents a new possibility/probability consistency principle and models fuzzy cash flows by normal probability distributions using this principle.

Simonelli [129] extends classical common knowledge theory into the fuzzy case. Her model has particular implications for 'incomplete contracts' in the financial markets since Fuzzy Theory does not require an agent to discriminate between an event and its negation while classical Probability Theory does. She introduces a new partition whereby states of the world are described by fuzzy events where their uncertainty of occurrence is described by probability theory. The model is able to establish fuzzy interactive knowledge between two agents concerning their expectations about those probabilities and derives conditions for no-arbitrage opportunities based on the agents' interactive knowledge.

Yao and Lin [148] derive an optimal fuzzy profit from a fuzzy demand function and a linear cost function. The fuzzy demand function can be quadratic or linear with fuzzy coefficients. The results are only similar to the classical case when the fuzzy case converges to the crisp case but they are different when the fuzzy and crisp cases are different. The author recommends treating the problem in the fuzzy sense in the latter case.

Wu [146] prices three complementary products such that the fuzzy profit is maximized in a perfect competition. Fuzziness arises from the fact that the demand function cannot be precisely defined like in the monopolistic case. It is achieved by fuzzifying the coefficients, that is by replacing them with fuzzy numbers. The fuzzy demand functions are maximized for all cases and then combined to yield an optimal solution.

More sophisticated applications of Fuzzy Theory to Finance Theory have really been introduced more recently. Muzzioli and Torricelli ([110], [112]) have been the first to introduce Fuzzy Set Theory to Option Pricing. They apply their technique to binomial option pricing. In their first paper ([110]), they introduce a one period fuzzy binomial model where fuzziness is characterized by an opaque payoff. We extend this model in our research to the mutliperiod case and compare it to the CRR model. In their second paper ([112]), they introduce a multi-period fuzzy binomial model whereby fuzziness arises from opacity of future states of the world. The two papers will be discussed thoroughly in Chapter 8.

On the other hand, Cherubini, and Cherubini and Della Lunga ([30],[31],[32]) have probably been the first (and so far the only ones) to publish work on applications of Fuzzy Measure Theory to option pricing and other advanced financial applications in an attempt to model Knightian uncertainty. He uses fuzzy measures and their duals to establish parametric bounds on the value under consideration. Cherubini interprets the defining parameter as an indicator of uncertainty.

In [32], the authors introduce a fuzzy VaR model that discounts liquidity risk by establishing bounds on the possible values VaR can assume. They look at VaR as the difference between the forward value of a position in a risky asset and a protective put that is deep out of the money. Fuzzy Measure Theory steps into the picture when a probability interval is used instead of a precise value for the probability distribution.

In [31], the authors apply their approach to the valuation of corporate claims. They follow Merton's model of valuing such claims whereby they are valued as a derivative written on the value of the assets of the firm. The challenge in the real world is coming up with a precise estimate of the probability distribution of the underlying, which is close to impossible. So the authors resort to Fuzzy Measure Theory to induce a set of probabilities over an interval. The bounds of this probability set define the upper and lower value for the derivative. Using these bounds, they arrive at bounds to the yield to maturity of corporate bonds, which serve directly in computing the credit spread. Finally, they value a default put option, which is decomposed into default probability and loss given default, implied by corporate debt. Again, the probability distribution cannot be estimated directly so intervals have to be used instead, which leads to intervals in the default put option. In the same manner, he evaluates a corporate bond in [30]. Our interest lies in Fuzzy Option Pricing, which has been witnessing increasing popularity lately. We have not dwelled into those models in details now since we have dedicated one whole chapter (Chapter 8) for this issue with a comparative study.

3.4 Conclusion

In this chapter, we have brought together related research from different areas and presented them within a coherent framework. We have specifically looked at uncertainty and the various approaches of dealing with it in both Fuzzy Theory as well as Economic Theory. This is important for clarifying the general framework of the problems we will be tackling as well as how the solution tools we will use compare with alternative tools. We have also presented a literature review on recent developments in Option Pricing Theory, which is relevant because we propose a similar solution structure but using different tools. Besides, it helps us formulate a picture as to how our models fit within the Option Pricing literature given that we attempt to use fuzzy tools in this area. Finally, we have considered fuzziness in economics and finance because it is also important to know where we stand vis-a-vis existing 'fuzzy literature.' Alongside the previous chapter which has laid the foundations of Fuzzy Theory, this chapter equips the reader with the necessary background to understanding the models that we present in the rest of this research. Starting with the next chapter, we present a Fuzzy Binomial Model which is a variation of the CRR OPM. It targets Option Pricing in a world characterized by high uncertainty. The model incorporates tools of both Fuzzy Set Theory as well as Fuzzy Measure Theory and will provide the most original contribution of this research.

Chapter 4

Fuzzy Binomial OPM

We have presented a literature review in the previous chapters combining research from different areas into one coherent framework, thereby laying the foundations for the models that will be forward. In this chapter and the following ones, we will introduce models that employ fuzzy tools to address new as well as existing problems that have been traditionally addressed by conventional mathematical tools.

In this chapter, we consider the effect of uncertainty, generated by lack of information and vagueness, on option pricing. Conventional probabilistic tools can be restrictive in such circumstances due to the additivity requirement. So we relax this requirement by using a specific class of nonadditive measures, namely, fuzzy measures which we have detailed in chapter two. The general methodology involves using a preference-free fuzzy expectation pricing approach whereby expectations are computed using nonlinear integration. However, due to the prevailing uncertainty, we can only obtain a range of possible option prices instead of one fair value. So the bounds will be derived by performing the fuzzy expectation operations over the fuzzy measures and their duals. To calculate the expectations, we consider two approaches to nonlinear integration, namely, the Sugeno Integral and the Choquet integral. The next chapter presents the empirical applications of the model.

We start by stating the problem definition or the intuition after which we move to describe the technical approach. To this end, we define the fuzzy measures irrespective of preferences and then we present the model using both Sugeno integration as well as Choquet integration to compute fuzzy expectations.

4.1 Intuition

Asset prices in the financial markets sometimes deviate from equilibrium prices depending on the level of opacity and illiquidity associated with the asset such that the more illiquid and opaque the asset is, the more frequent the deviation. One of the important factors that contribute to this deviation is Knightian uncertainty, which, in turn, can arise due to different reasons. We consider two sources of such uncertainty. The first one is associated with the Bid/Ask spread while the second has to do with vagueness associated with possible future states. In this model, we are more concerned with the uncertainty associated with the underlying asset itself and how this uncertainty is transmitted into the option price.

At any single instance of time, there is a multiplicity of Bid/Ask prices, that is, several Bid/Ask quotations from different sources come through at one instance of time. Besides, the observable price at which a trade is done is not known to be a bid or an ask precisely. As Lyons ([103]) says, "... there is no way to determine on the basis of broker data available whether the bid or offer was cleared by any given transaction." (p. 335) This creates uncertainty or fuzziness around transaction prices. This type of uncertainty is due to lack of information.

There is another aspect to the uncertainty generated by the Bid/Ask spread. The actual spread, which is based on transaction data, is usually less than the actual quoted spread, which is based on screen quotations. This spread is usually calculated based on consecutive transaction prices. So when this spread is low, the volatility in prices, as measured by the difference in prices, is expected to be low because the corresponding transaction prices will be somewhere between the close quotations and vice versa. Price volatility, as well as the size of the spread, are positively related to uncertainty as we will see later on. In other words, a high volatility and a wider spread are associated with higher uncertainty. This point is particularly important for the conditional fuzzy measure that we define later on. However, both points are closely related and the measure will be designed to accommodate both of them.

On the other hand, there is vagueness associated with the future state of the asset especially in periods of high uncertainty. This is due to lack of evidence about future states of the universe and has to do with possible occurrences of those states. It is very important to stress that such types of uncertainty are important depending on the level of liquidity and transparency of the asset. Generally, they are more important the more illiquid the asset and the higher the level of opacity.

We propose that such issues have particular implications for option pricing even though similar scenarios have not been considered previously. In such circumstances, it is no longer possible to use equilibrium models such the Black-Scholes model and the CRR binomial model. The fuzziness associated with the underlying is transmitted to the option itself and conventional tools will be either too restrictive or too complex to deal with such problems. Fuzzy tools provide an easier and a more appropriate alternative.

It is best to handle the problem in discrete time. For this specific purpose, the CRR binomial model is the primary candidate to illustrate the problem and tackle it thereafter. The CRR model is built around a set of fair unobservable prices that are supposed to hold in equilibrium. However, within the framework we are considering, we actually observe fuzzy prices that revolve around those unobservable ones and accommodate uncertainty. When uncertainty is high, those deviations are wider and so the fair option price that is based on the fair unobservable underlying prices is no longer 'fair.' As the uncertainty increases, other approaches are needed to handle this issue and at some point it is no longer possible to obtain one precise value for the option; rather, we can only get a range of possible prices. To handle option pricing in such a fuzzy setting, we propose a fuzzy approach, which will be outlined in the following section.

4.2 The General Fuzzy Pricing Approach: The Solution Framework

In this section, we outline the basic fuzzy pricing approach, which will be described in details in the following sections. As we have mentioned earlier, the CRR binomial model is based on a set of hypothetical unobservable (asset) prices that hold in 'equilibrium.' However, depending on the market under consideration and the level of transparency and liquidity, we quite often observe deviations from those prices. We consider the case where we can only observe fuzzy prices that move around those unobservable 'equilibrium' prices. In such instances, those fuzzy prices are the actual prices that determine the option price.

We propose that the reasons why those observed prices deviate from equilibrium prices in this example are the uncertainty aspects that we have talked about in the previous section. Observed prices, unlike equilibrium prices, accommodate uncertainty and so the actual option price, unlike the model option price, reflects uncertainty.

To handle the pricing in such instances, we propose a new fuzzy approach in discrete time, which builds upon the existing CRR model and which is called the Fuzzy Binomial Model (FBM). The underlying information structure is still Markovian but we impose fuzziness on top of it such that the fuzzy prices revolve around the fair prices. Those fuzzy prices serve as a proxy for observable prices that accommodate uncertainty. As a result, we get a fuzzy binomial tree (Figure 4-1) discounting uncertainty. This uncertainty is incorporated and transmitted into the final fuzzy model value via fuzzy measures and expectations.

In the previous section, we have distinguished between two types of uncertainty within the context of our model, the first of which has to do with the Bid/Ask spread and the second has to do with vagueness of future states of the world. To capture such uncertainties, we resort to Fuzzy Measure Theory. Fuzzy Measure Theory is better able to handle such uncertainties than Probability Theory is because the latter is too restrictive in this case due to the fact that it requires additivity and, hence, complete information.

To model the uncertainty generated by the Bid/Ask spread, we introduce a conditional fuzzy measure that attempts to relate the spread to the volatility of prices. The volatility of prices, which is usually a measure of uncertainty, is measured by the absolute change between two consecutive transaction prices. For transparency issues, it is better to estimate the spread from transaction prices even though it is also possible to use the quoted spread as a proxy keeping in mind that the latter is wider than the former. When this spread is wide, the change in prices is expected to be high and vice versa. So, as such, we can say that the spread is generating volatility in prices. Of course, as we will see later, there are several determinants of the Bid/Ask spread but we are interested in the uncertainty aspect and consequently in how this spread generates volatility in prices. In brief, uncertainty leads to a wider Bid/Ask spread and higher volatility, and the conditional measure attempts to capture this.

On the other hand, we capture the vagueness associated with future states of the world by regular fuzzy measures. This is easy to illustrate within the context of the binomial model. For example, when we say the 'up state,' we do not necessarily mean one number capturing this statement; rather, it can be a range of values that satisfy this criteria with varying degrees. So, given the binomial model, those values revolve around the number summarizing the 'up state' and which is given by the fair or equilibrium value. This is particularly useful in periods characterized by high uncertainty and partial information.

In summary, we incorporate uncertainty into our model by introducing a set of fuzzy measures that translates the set of equilibrium or unobservable prices into a new set of prices, which accommodate uncertainty and serve as a proxy to observable prices, using fuzzy conditional measures. This translates in our model into transforming the existing payoff based on fair prices into a fuzzy one. Then, this set of new prices or payoffs is used to arrive at the final value of the option using fuzzy expectations or fuzzy integration over a new set of regular fuzzy measures. This process gives us the upper bound for the set of option prices. To get the lower bound, we introduce the dual measures of the sets of regular and conditional fuzzy measures and use the same basic model. It is important to stress that those measures are independent of investors' beliefs and preferences so the model is also preference-free.

4.2.1 A Note on the Fuzzy Pricing Approach

Pricing of options in complete markets rests on two important concepts, replication and risk-neutrality. In such a world, the Equivalent Martingale Measure is unique. So pricing under such a measure must yield a no-arbitrage price. However, replication actually lies at the heart of arbitrage pricing so that the particular choice of the probability measure is no longer relevant. Problems arise in incomplete markets where replication is no longer possible and the Equivalent Martingale Measure (EMM) is no longer unique as we have seen in Section 3.2. So the concepts of risk-neutrality and EMM are more important in such circumstances.

A risk-neutral economy is a hypothetical economy whereby all discounted price processes are martingales. Of course, this is only a tool that facilitates arbitrage pricing and it is not consistent with the real world. What this concept implies is that the best estimate for the expected price is the currently prevalent one. However, an investor is much better off investing all his money in a risk-free asset rather than undertaking a risky investment that is expected to have the same value it currently has. Yet, this concept seems somehow to be working quite well for pricing derivatives.

Some researchers try to preserve this risk-neutrality argument within an incomplete market setting by choosing a measure that preserves the martingale property. Others have considered alternative measures that are not necessarily martingales ([57],[109]). The corresponding models can be invariant as well as sensitive to the chosen underlying probability measure [109]. In the first case, it is not necessary to specify the probability process describing the distribution of the underlying at maturity. Portfolio super-replication approaches fall under this category as expected. They are usually solved by linear programming, relative entropy minimization, and PDE approach methodologies. In the second case, the pricing depends on the chosen probability measure so subjectivity and investor preferences may enter into the picture. Mean-variance hedging approaches fall into this category. They can be solved by utility maximization or risk minimization among others. However, the expectations are taken over the actual or the subjective (not necessarily martingale) probability measure ([57],[109]).

Our pricing methodology is more consistent with the second category. The basic purpose behind using this approach is to model the fuzziness in the real world so the actual pricing methodology does not assume a risk-neutral economy. Moreover, since this fuzziness generates market incompleteness, replication is no longer possible.

The fuzzy pricing approach we adopt involves calculating the fuzzy payoff and then discounting this value backward at the risk-free rate using fuzzy expectations over the preference-free fuzzy measures, which gives us one bound on the option. To get the other bound, we compute the discounted expectations over the dual measures.

To clarify the fuzzy methodology, we will explain the model from a CRR 'window.' The parameters in the CRR binomial model are calculated by matching the drift and variance in discrete time to those corresponding to the asset's distribution in continuous time. The value of the option is then calculated by discounting back the payoff at maturity at the risk-free rate assuming a risk-neutral world. However, it is also possible to perform the pricing by replication such that, at each node, the following conditions are satisfied,

$$\Delta Su + rB = C$$

$$\Delta Sd + rB = C$$

where Δ and B are the amounts of risky asset and riskless asset that have to be held, S is the value of the underlying asset, r is the risk-free rate, u and d are the proportions by which the asset price goes up or down respectively and C is the value of the option at the corresponding node calculated using expectation pricing where the martingale measure, otherwise known as risk-neutral probability, is equivalent to,

$$p = \frac{r-d}{u-d}.$$

In the real world, this is actually not possible due to uncertainty and investors' expectations. In the fuzzy world we are specifically considering, this is not possible due to the uncertainty scenarios under consideration, which give rise to market incompleteness. We can perform an equivalent fuzzy expectation approach discounted at the risk-free rate and using preference-free fuzzy measures. In the binomial model, the risk-neutral valuation involves discounting the payoff, which is based on S, at the risk-free rate using the risk-neutral probabilities. In the fuzzy binomial model, the risk-neutral valuation involves discounting the fuzzy payoff, based on a fuzzy S, at the risk-free rate using the non-additive preference-free fuzzy measures. However, it is more complex to deal with the replication issue.

A similar analogy is an option on a non-traded event whereby valuation is performed by discounting expectations of future cashflows. However, it is not possible to perfectly replicate the option and one has otherwise to consider an asset that is highly correlated to the corresponding asset. In our model, the option is indeed on a traded asset but the fuzziness makes it impossible to replicate it and we can only resort to the portfolio it is supposed to be replicating, namely, that which prevails in the absence of uncertainty. This portfolio corresponds to the one in the CRR model. To deal with this issue, we can introduce a tracking error that super-replicates this portfolio if it is positive, sub-replicates it if it is negative and replicates it if it is zero. The intuition behind this analysis is that the fuzzy binomial tree is built around the CRR one. In the absence of uncertainty, the CRR model and the corresponding option price prevail, which makes the model the obvious candidate. However, we will leave this matter to future research since it is beyond the scope of this research.

It is quite understandable that some readers maybe worried about performing the expectations over measures that have not been proved to be martingales. However, the reader must not perceive the model from a 'conventional' risk-neutral point of view because the underlying assumptions no longer hold. We are now performing the pricing in a fuzzy world where prices are blurred but still rest on the fair model prices. Perhaps future research will be able to formalize arbitrage in such a fuzzy world and set guidelines for testing it.

In fact, option pricing in incomplete markets has seen several applications whereby expectations are carried over a measure that is determined subjectively or statistically (refer to [57],[109] for examples). In theory, this is not supposed to happen but the problem framework frequently imposes such paths. In our case, we have been able to avoid subjectivity by developing measures using fuzzy tools and existing data, which is similar to a statistical approach, but we have not been able to study their risk-neutrality since, by assumption, risk-neutrality no longer holds.

In the following sections, we will go into the technicalities of the fuzzy pricing approach from defining the measures to carrying out the fuzzy expectations. Determining the fuzzy measures has been a tricky issue. As we have repeatedly mentioned, in the CRR model as well as in most lattice methods, we match the mean and variance across the tree to those corresponding to the distribution in continuous time. However, we can no longer do this in a fuzzy setting because, first, we do not know how the distribution will look like in a fuzzy world and second, we cannot formulate the equations due to non-additivity and nonlinear operators. So the best we can do is define those measures independently using conventional fuzzy approaches whenever possible making sure they are preference-free. Similarly, defining the fuzzy expectations, especially the conditional one, has been a very tedious operation since it has not been done before, neither in Finance Theory nor in Fuzzy Theory. So, in fact, this model extends contributions to both areas. The following sections analyze those matters in detail.

4.3 The Conditional Fuzzy Measure

4.3.1 Definition

This section presents the conditional fuzzy measure, which has two purposes. First, it attempts to capture the uncertainty generated by the spread. Second, it serves as an indication as to what extent an observed transaction price is a bid or an ask. As we have mentioned earlier, one cannot know affirmatively whether an observed transaction price is a bid or an ask, which generates information uncertainty. Literature usually resorts to guessing or inferring from data whether the transaction price is a bid or an ask. Moreover, it is not possible to estimate probabilities for this purpose due to lack of information. Therefore, fuzzy measures serve as a suitable alternative.

The conditional fuzzy measure we introduce captures such uncertainty and provides a tool by which one can guess how much the price under consideration is a bid or an ask based on the magnitude and direction of movement of next period's price. If the price goes up in the next period, then the current price is most likely to be an ask but if it goes down, then the current price is most likely to be a bid.

It is very important, and quite interesting too, to note that the bid and ask sets themselves do not have to be disjoint. They can be overlapping, that is, for a particular state, there are always two sets the bid and the ask but the price can belong to either based on the consecutive price. Besides, guessing that a price is in the bid set does not necessarily mean that it is not also in the ask set with a different degree which does not necessarily "complement", in the traditional meaning of the word, the degree to which this price can be in the bid set due to nonadditivity. It necessarily follows that knowing or approximating the degree to which the price under consideration belongs to the bid (ask) set will not permit us to infer the degree to which it belongs to the ask (bid) set unless the degree of nonadditivity is known.

Section 2.2.3. provides a comprehensive overview of constructing fuzzy measures. We adopt the distribution approach because it does not involve any subjectivity. Thus, the conditional fuzzy measure, which is a distance measure, is of the form

$g = H(\frac{|\Delta P|}{spread})$

where H is a fuzzy distribution just like a probability distribution, ΔP corresponds to the change in transaction prices between period t_0 and the next period t_1 and serves as a measure of the volatility of prices, and *spread* corresponds to period t_0 bid-ask spread. This measure tries to capture the uncertainty or fuzziness generated by the spread on spot prices. We consider the absolute change because we are interested in the magnitude of the change and not its direction of movement. If we do actually discount the direction of movement, the measure will be very low for negative change, which is counter-intuitive because it is known that volatility and uncertainty are higher when there is a negative change and so the measure must give a high value instead of a low one.

Empirical Analysis shows that an appropriate distribution for this measure is the t-distribution due to fat tails. So the final form of g is,

$g = t(\frac{|\Delta Spot|}{spread}, dof = 4)$,

where dof stands for degrees of freedom and t stands for the Student t distribution.

The measure is dependent on the size of the spread and the corresponding volatility of spot prices. It measures the volatility of prices generated by the spread. So when the spread is low (high) with respect to the volatility in prices, the measure is high (low). Generally, a high value of the measure signals that the spread is generating volatility in the spot prices and vice versa. There are three scenarios that need to be considered when analyzing the measure whereby the first two involve one of the variables changing in one direction with respect to the other and the third involves the change of both. In essence, this is quite contingent on the interplay of those variables and outside factors influencing them. To clarify this matter, a brief literature review emphasizing the determinants of the Bid/Ask spread and touching upon its interaction with the volatility of prices follows.

4.3.2 Determinants of the Bid/Ask Spread

There is well documented literature on the Bid/Ask spread (BAS) and its determinants in both equity and foreign exchange markets, which fall within the scope of this research. The spread has been observed to display seasonality, i.e., intra-day and weekly pattern, in both foreign exchange and equity markets ([1], [105],[72],[10],[14]). An interesting observation is that the BAS tends to be smaller in currency markets than in equity markets, which can be due to economies of scale, lower variance and less dealer's expected losses due to asymmetric information [10].

Broadly speaking, the two major schools of thought on determinants of the BAS spread divide those determinants into inventory-based models and information-based models. Specifically, the major factors that are emphasized in the literature are immediacy [41], asymmetric information or the adverse selection problem ([6], [131], [36], [73], [99], [77], [1], [132], [68], [103]), uncertainty ([79],[36]) and inventory control ([3],[77],[132]). Information and uncertainty are the most relevant factors within the context of g and they will be highlighted the most.

There is an inverse relationship between immediacy and the BAS since higher immediacy leads to lower waiting costs. The relationship between asymmetric information and the BAS is not as clear-cut since it is dependent on two other factors. The theory behind asymmetric information that there are informed traders, with whom the dealer cannot win due to their superior private information, and uninformed or liquidity-motivated traders, with whom the trader always wins due to their low information level and sole interest in liquidity. So the trader always sets the BAS as a trade-off between the two whereby setting a wide BAS limits losses to informed traders but may lead to losing potential gains from uninformed ones. Thus, in general, the spread is wider the better the informed trader's private information or the higher the percentage of informed traders relative to liquidity-motivated traders or the higher the elasticities of the expected supply and demand functions of liquidity traders.

Information is very important within the context of g since it affects the measure through the spread. There is wide evidence in the literature ([6],[132],[105]) that the spread widens when there is a sizable information change. Like information, uncertainty generally has a positive relation with the spread in that higher uncertainty results in a wider BAS. Literature concentrates on transactions uncertainty (specifically uncertainty related to the timing of transactions) and information or return uncertainty, which is more relevant to this analysis. It can be measured by risk, volatility and price level interchangeably. There is a general agreement in the literature ([3],[79],[36],[105],[10]) on a positive relation between risk, as measured by stock variance or volatility, and spread. Therefore, a wider spread is usually associated with a higher volatility in prices. Interestingly enough, the share price level and the spread seem to have a direct relation such that as the price increases, the spread widens, which is really the opposite of what g measures.

Uncertainty and risk also play a large role in the foreign exchange literature. They are usually reflected by the variation of the spot rate. The risk is generally exchange rate risk but Overturf [117] includes credit risk as well. Large variability in the spot rate is usually associated with wide BAS ([66],[10],[14],[72]). However, Bollerslev [14] argues that it is possible for the bid and ask to move in the same direction in the presence of more uncertainty such that the size of the spread is not affected.

The positive relation between the spread and uncertainty about the future level of exchange rate holds in all literature ([66],[2],[117],[15],[72],[14]). Dealers do not usually know whether the customer is a buyer or seller so they have to set a quote at which they will be indifferent to this issue. Given that dealers quote spreads based on the last piece of information they have got, they know that as they are quoting, rates can be changing. So they have to quote a spread that reflects what they believe is going on in the market. This reflects uncertainty due to lack of information about the level of exchange rates; in other words, dealers base their quotes at time t on information available at time t-1 (information at time t is missing or insufficient). Besides, rates could change from the time the bank accepts an order to the time it covers its position resulting in a profit or a loss. So dealers protect themselves by quoting a wider BAS especially when uncertainty is accompanied by large fluctuations.

Fieleke [66] uses three measures to test the effect of uncertainty on the spread, which are variations in covered interest differential, change in the exchange rate and actions, and announcements of government officials. The first two variables seem to have a more important effect than the last one. He assumes uncertainty is exogenous. On the other hand, Overturf [117] argues that transaction costs represent uncertainty at times of large fluctuations, which effectively means that uncertainty is endogenous. He measures risk or uncertainty by the standard deviation of the exchange rate.

Glassman [72] divides the spread into a transaction costs component and a risk component. The risk component is considered a measure of exchange rate uncertainty. It is affected by the frequency of transactions and the volatility of prices calculated from exchange rate changes. Interestingly enough, she finds an opposite effect of the frequency of transactions in foreign exchange markets than in securities markets, where there is a negative relation between the spread and the frequency of transactions. But she finds consistent results concerning the positive relation between the spread and price volatility in the foreign exchange markets similar to the securities market. Boothe [15] also measures risk by exchange rate volatility. The spread is expected to widen in response to an increase in the price level. Demsetz [41] asserts a positive relation between the spread and the price level but the relation is not clear as to whether it holds with a short-term, long-term or present price level.

In an interesting survey on dealers' quoting process in Tokyo, Hong Kong and Singapore, Cheung and Wong [33] present some new findings. According to their survey, the majority of dealers base their spread on market convention. The primary reasons for deviating from market convention, generally quoting a wider spread, are liquidity and uncertainty triggered by unexpected news release, increased market volatility and unexpected change in market activity. These two factors are also important for analyzing inventory costs. But they also support the asymmetric information theory. A wider spread always signals new information that the dealer may not be aware of. They also find that speculation plays an important role whereby it increases market volatility.

A relevant determinant of the spread is market liquidity because it dictates the size of the spread, which in turn affects the magnitude of the measure. Market liquidity and the spread are inversely related ([6],[36],[99]). As for the inventory control effect on the BAS, the results in existing literature are contradictory ([79], [77],[10],[103]). Other factors influencing the spread include competition ([41],[105], [131],[36],[66]) and trading activity ([41],[36],[73],[105] [66],[72],[10]), which are both inversely related to spread except for Glassman[72], transaction size and order costs, which have a direct relation with the spread, and foreign exchange restrictions in the form of taxes or impediments to trading, which are only pertinent to the foreign exchange market and have an inverse relation with the spread. However, these factors are beyond the scope of this research and, hence, they will not be given much emphasis.

4.3.3 BAS effect

Therefore, there are many factors that influence the BAS and volatility of prices and it is hard to isolate the effect of each factor alone. So, by relating the volatility in prices to the spread, g is concentrating on the uncertainty aspect, specifically information uncertainty because the volatility of prices, as measured by the absolute change in prices, is a measure of information uncertainty. When the spread is estimated based on transaction prices, this measure captures the volatility generated by the spread. If the market maker sets a wide spread, the change in transaction prices is expected to be wide and vice versa. It is in this sense the spread can generate volatility in prices.

When there is information uncertainty, both volatility and the spread change but when there is transaction uncertainty, then only the spread widens. When the spread is wide but the change in spot prices is low, the measure is low indicating that the spread is not generating volatility in prices.

When both variables change, then the magnitude of the measure depends on the size of change in each variable. When the spread is narrow but the change in spot prices is high, then the measure is high meaning that the spread is indeed generating volatility in the spot prices. At first glance, this may seem counter-intuitive; for example, if the spread is narrow on average but the level of prices can be high such that a small change in the spread leads to high volatility in prices.

Within the context of the fuzzy binomial model, this measure contributes to transforming the crisp payoff into a fuzzy payoff. However, it also has some relevant applications in financial markets that are characterized by opacity and a high level of uncertainty such as NASDAQ and the foreign exchange market, particularly, applications that try to study price volatility and spread behaviour as well as inferring the degree to which a transaction price is a bid or an ask but this is beyond the scope of this research. At the moment, we will be concentrating on its contribution to computing fuzzy expectations. However, to compute fuzzy expectations, we need a measure indicating the possibility of occurrence of states of the world, which we will do in the next section.

4.4 The State Measure

This section introduces the fuzzy measure that captures uncertainty related to future states of the world. This measure is the equivalent of the risk-neutral probabilities in a fuzzy world. It is useful for computing both the fuzzy payoff and fuzzy expectations. It actually measures the possibility of occurrence of a certain state. Since it does not require as much information as a probability measure does, it is very useful in periods characterized by fuzziness and lack of information.

To derive the state measure, we can use one of three approaches. We can either induce a membership function for the up and down states, or induce a possibility distribution from an existing probability distribution or transform a fuzzy measure from a risk-neutral probability measure. In the first two approaches, we do not have a fuzzy per se. However, given that a grade of membership is a special case of the fuzzy measure which also implies that a possibility distribution is also a special case of the fuzzy measure (Chapter 2), we will consider the three concepts numerically equivalent. In the following subsection, we will consider each approach separately.

4.4.1 Approaches to determining the State Measure

Membership Function Approximation

This approach corresponds to the parametrized membership function approach which subscribes to the similarity view of the grade of membership (Appendix A). It relies on comparing the value of an object to a prototype, which is the value that the object is supposed to assume in an ideal setting. The basic idea behind this approach is that the membership function is related to the similarity between the object and the prototype and inversely related to the distance between them. As Appendix A shows, there are many forms which the membership function can assume depending on the context. Let us consider the simple form ([56])

$$\mu(x) = \frac{1}{1+d(x)}$$

where $d(x)$ is the distance between the up or down price and the prototype. Medasani et al. [106] argue that it is possible for $d(x) = 1/x$. However, because it has been proved that the relation between a physical entity and human perception is generally exponential, they suggest using a better distance function which is $1/\exp(-a(x-b))$, where a and b are two parameters defining the slope of the membership function and the inflection point, which is the point reflecting the tendency in the subject's attitude to change from being rather positive to rather negative, respectively.

Pal and Majumdar [118] present a general form for the membership function $\mu(x) = g\{d(X,C)\}$, where $d(X,C)$ is the weighted distance between the object and the prototype. However, they do include $\mu(x) = [1 + d(X,C)]^{-1}$ and $\mu(x) = \exp[-d(X,C)]$ as two simple forms for the membership function. The distance has the form

$$d(X,C) = (X-C)'A(X-C),$$

where $X = [x_1, ..., x_N]'$ is the feature vector, $C = [c_1, ..., c_N]'$ is the prototype vector and A is a symmetric positive definite weight matrix. The values for C and A can be estimated from a set of data, assuming that the form of the membership function is known.

Chaudhuri and Majumdar [28]) work on exactly the same problem. Their estimates of C and A in the presence of statistical information and with respect to a measurement scale are

$$C = E(Y),$$

$$A_{ii} = [\sigma_i(Y)]^{-1},\ A_{ij} = 0,\ i \neq j,$$

where $E(.)$ and $\sigma_i(.)$ denote, respectively, the expectation vector and the standard deviation of the i-th feature, and Y is the feature measurement vector of the prototype. However, according to the authors the disadvantages of this approach are that assuming the protoype and the distance related to the expected value and the inverse of the standard deviation is quite restrictive, and it requires a relatively large number of the prototype, which may not be available at all times. So they introduce another non-statistical approach, which we will not present here because it is irrelevant to the research at this point but the reader is referred to their paper [28]. Nevertheless, this approach is particularly relevant to our problem as we will see later.

Lai and Hwang [100] cite Zimmerman and Zysno's [152] functions,

$d(x) = \exp(-a(x-b))$,

$\mu(x) = \frac{1}{1+\exp(-a(x-b))}$,

for the distance measure and the membership function, where a and b are semantic parameters from a linguistic point of view. They represent the slope and the inflection point respectively.

So far we have worked with protoypes and known mathematical expressions for the membership function. However, this is not the case always but we will not dwell on this subject much because it is not relevant to our problem; the interested reader is referred to ([49], [28], [118]). For the reader interested in scaling, this measure is more meaningful on an interval or ratio scale. The reason for that is that we are interested in the degree of similarity between our object and the prototype meaning that we are dealing with metric distances, which are more informative on an interval or ratio scale ([56],[100]). In fact, Lai and Hwang [100] present an interesting approach to adapt the membership function into subjective preferences involving transformation into interval scale and least squares of deviation. However, we do not have to worry about this now because we have excluded subjectivity from the beginning.

Probability to Possibility Distribution Transformation

"... the proper framework for information analysis is possibilistic rather than probabilistic in nature..." (Zadeh [150]). Since we are dealing with informational uncertainty and its effect on option pricing, we are tempted to induce a possibility distribution from the probability distribution, which can accommodate other alternatives than the up or down prices. This is effectively like saying that in a coin toss experiment, we do not necessarily have a head or tail, rather it is possible that the coin rolled away or landed on its edge or that we have an unfair coin and so we do not have complete randomness [52]. Traditionally, we ignore such cases and confine ourselves to head or tail [64] but here we do not have to do this anymore. The analogy for the binomial tree is that the price overshoots up or down or remains the same. But we cannot quantify this uncertainty of occurrence with probability theory because we do not have enough information so we resort to possibility theory. Besides, transformations are meaningful when we have uncertainty combination with heterogeneous sources [53] of, for example, statistical data, linguistic data..., which is applicable within our context.

Applying this approach to our problem can be quite a tricky issue. First, we do not know how the probability distribution will look like in a fuzzy world. Second, if we do know it, we are much better off using it because then it means we have full information about possible states of the world and transforming the probability distribution to a possibility one means we will be losing valuable information. However, this issue will remain open for future research.

Probability Measure Transformation

Using this approach, we can derive a transformation of risk-neutral probabilities. Generally, we can derive a $\lambda-$fuzzy (or Sugeno) measure from a classical probability measure by using an appropriate transformation (Chapter 2). Let (X,ß,p) be a measurable space where p is a Lebesgue measure. Then, the composition $f \circ p$ produces a Sugeno measure iff

$f(x) = \frac{1}{\lambda}(c^x - 1)$

(Theorem 2.1,[145], p.71). Assume that g is a Sugeno measure defined on ß, then

$p(A) = \frac{\log(1+\lambda g(A))}{\log(1+\lambda)}$

$\Rightarrow \; g(A) = \frac{(1+\lambda)^{p(A)}-1}{\lambda}$

([96],[97],[144],[145], [142]).

Therefore, in our case, we can transform the risk-neutral probabilities into non-additive (Sugeno) measures. Hence, fuzzy measures for the up and down states are, respectively,

$g(up) = \frac{(1+\lambda)^{p(up)}-1}{\lambda},$

$g(down) = \frac{(1+\lambda)^{p(down)}-1}{\lambda}$,

where $p(up)$ and $p(down)$ are the risk neutral probabilities. So we have only one unknown, which is λ.

We can think of λ as the uncertainty aversion factor. However, the problem is how to determine λ such that it is not subjective. So let us assume that we are in a risk-neutral economy. Then, λ will be modelling the uncertainty aversion, which cannot be captured by probabilistic models. What this tells us is that even if an economy is risk-neutral, it can still exhibit degrees of uncertainty aversion. Using pricing in a risk-neutral economy and fuzzy expectations, we have the following equation

$S\exp[(R_d - R_f)\frac{T}{N}] = \{[S_u \wedge g(\{Su\})] \vee [S_d \wedge g(\{Su, Sd\})]\}$,

If we substitute for $g(\{Su\}$ the transformation equation and replace $g(\{Su, Sd\}) = 1$, we will obtain one equation with one unknown, λ. This equation has to be solved numerically. So from the value we get for λ, we can obtain the values for the Sugeno measures for the up and down states at each node and then solve for the call price. This approach involves a high level of unjustified complexity so it is also left for future research. The approach that is most appropriate for our problem is the membership function elicitation approach. Since only a two state space will be considered, a membership function is induced for each of the up and down states.

4.4.2 The Final Form of the State Measure

The state measure is derived using the parametrized membership function approach which subscribes to the similarity view of the grade of membership (Appendix A.5). This approach relies on comparing the value of an object to a prototype, which is the value that the object is supposed to assume in an ideal setting. The basic idea behind this approach is that the membership function is related to the similarity between the object and the prototype and inversely related to the distance between them. There are many forms which the membership function can assume depending on the context. The general form adopted is,

$\mu(x) = \frac{1}{1+f(d(x))}$

where $f(d(x))$ is a function of the distance between the up or down price and the prototype and it can take on many forms. Since we are assuming that the measure is numerically equivalent to the membership function, we will use the standard notation g rather than μ for the state measure from now on.

The up or down state, as it is, is a linguistically vague concept. So its meaning can be formally represented by the membership function. Besides, it is not known for sure which state will occur so even if the state is measured or defined precisely, that is, a precise number is associated with the state, such as Su in the crisp binomial tree, there will still be problems concerning its degree of occurrence.

At any point in time, there is an expected price that is believed to prevail ideally and which will serve as a prototype in the membership function definition. So any other price that occurs will be a divergence from this value and its similarity to the prototype will be measured by its membership function which will depend on the deviation between the sample and the prototype. The degree of importance with which a particular state will occur or the possibility of occurrence of the considered state or the degree to which a certain price is compatible with the notion of an up or down state, depending on how the measure is interpreted, is dependent on the price's distance from the prototype.

The prototype within the fuzzy binomial model's setting corresponds to the expected price in a risk-neutral world, which is, irrespective of subjective preferences and avoiding information context dependency, $E(S) = \exp[(R_d - R_f)\frac{\tau}{N}]S_0$ for foreign exchange and $E(S) = \exp[R\frac{\tau}{N}]S_0$ for equity, where R_d is the domestic risk free rate of return, R_f is the foreign risk free rate of return, R is the risk-free rate, τ is the time to maturity, N is the number of steps in the binomial tree and S_0 is the current price.

The distance function $f(d(x))$ in the membership function is of the form $a(S_n - E(S))^b$. So as S_n approaches $E(S)$, d gets smaller and the grade of membership gets larger meaning that,

1. the up or down price closely resembles the prototype if the membership function assumed to measure how closely the sample resembles the prototype.

2. the up or down price has a high possible degree of occurrence if the membership function is considered numerically equivalent to the possibility distribution.

3. the up or down price has a high degree of belonging to the set of possible states

(not necessarily observable) for next period if the membership function is interpreted as a special case of Sugeno measure.

4. the degree of evidence that a particular state will occur is high or the degree of importance with which a particular state will occur is high.

The empirical form of $f(d(x))$ has been calibrated to $3(S_n - E(S))^2$. So the membership function has the form,

$g(S_n) = \frac{1}{1+3(S_n-E(S))^2}$.

The function is peaked at $E(S)$, with a value that is equivalent to one, and then decreases symmetrically as S_n deviates from $E(S)$.

For the reader interested in scaling, this measure is more meaningful on an interval or ratio scale. The reason for that is that we are interested in the degree of similarity between our object and the prototype meaning that we are dealing with metric distances, which are more informative on an interval or ratio scale.

4.5 The Model

In this section, we put all pieces together and introduce a fuzzy mathematical model. Figure 4-1 shows the structure of the fuzzy binomial tree. The underlying structure is that corresponding to the conventional binomial tree as shown by the solid lines in the figure. Imposing fuzziness on top of the structure, we get the observable prices revolving around the 'equilibrium' prices as the dashed lines show. For every branch in the tree, there exist two conditional fuzzy measures and two state measures dependent on the direction and magnitude of the change in price from one period to the next. Figures 4-1 and 4-2 show an example of a two period model. The tree for the underlying asset's price evolution has to be projected one step further to that corresponding to the Call tree or the valuation period because the conditional fuzzy measures for the second period are dependent on the prices in the third period.

Each branch has two conditional measures, namely,

$g[C_t/S_{t+1}\_up] = t[\frac{(S_{t+1}\_up-S_t)}{spread}, 4]$,

$g[C_t/S_{t+1}\_down] = t[\frac{|(S_{t+1}\_down-S_t)|}{spread}, 4]$,

where C_t is the payoff at period t, S_{t+1}_up (S_{t+1}_down) is the price next period

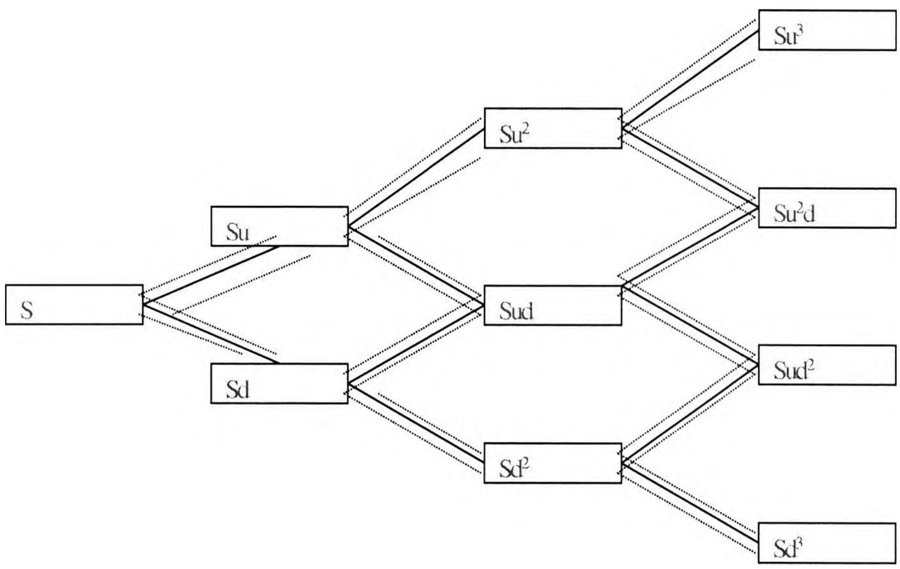

Figure 4-1: Stock Price Evolution in a Fuzzy World

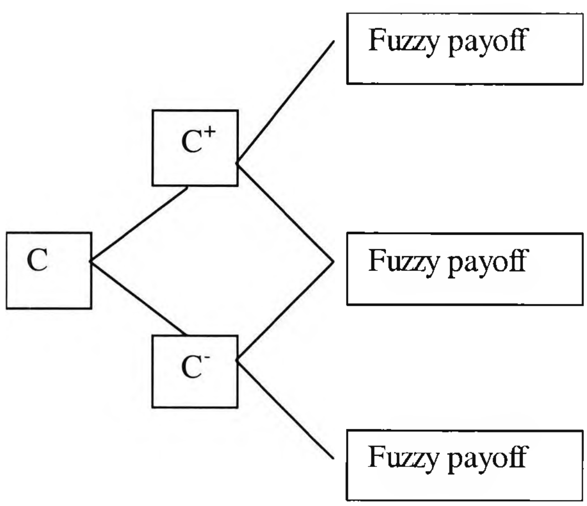

Figure 4-2: Call Valuation in a Fuzzy World

in case an upward (downward) movement in the tree is realized and S_t is the price at period t. The same analysis holds for state measures. Those fuzzy state and conditional measures are combined with the crisp payoff in a fuzzy mathematical formula to get the conditional expectations of each payoff at each point in time. This case is quite unusual because the value of the measure rather than the magnitude of the price is conditional on next period's prices, that is,

$$S_{t+1}(\varpi_{t+1}) = \begin{cases} S_{t+1}\_up & \varpi_{t+1} = \{up\_state\} \\ S_{t+1}\_down & \varpi_{t+1} = \{down\_state\} \end{cases},$$

$$S_t(\varpi_{t+1}) = \begin{cases} S_t & \varpi_{t+1} = \{up\_state\} \\ S_t & \varpi_{t+1} = \{down\_state\} \end{cases},$$

with the above conditional measures. Notice that S_t does not really change values depending on next period's prices, rather its measure does. Because this measure is not additive, this will lead to a price different from the one currently prevalent. This is why spot prices always need to be projected one step further to maturity. This will produce a set of new prices that embody uncertainty. This new set of spot prices induces a fuzzy payoff,

$$C_t(\varpi_{t+1}) = \begin{cases} C_t & \varpi_{t+1} = \{up\_state\} \\ C_t & \varpi_{t+1} = \{down\_state\} \end{cases},$$

which is propagated through the tree by fuzzy expectations. So, in essence, a set of fuzzy measures generates a new set of prices using fuzzy conditional expectations. The following analysis details the process.

4.5.1 Deriving the New Set of Prices

To transform the call payoff based on the conditional fuzzy measure, conditional fuzzy expectations are used, according to ([133],[143]),

$$\tilde{C}(S_j) = \oint \oint h(S_j) \circ g(S_j \mid x) \circ g(x), \tag{4.1}$$

where $\tilde{C}(S_j)$ is the new transformed payoff, S_j (where $j = 1, ..., N$, N is the number of steps in the binomial tree) is the spot price at maturity for the jth state, $h(S_j) = \frac{(S_j - K)^+}{K}$ is the non-transformed payoff, $g(S_j \mid x)$ is the fuzzy measure conditional on next period's prices and $g(x)$ is the degree of occurrence or importance of the price next state. A change of numeraire (so the call value is in terms of asset units rather than currency) in the non-transformed call payoff $h(S_j)$ has been introduced for comparison purposes because both fuzzy measures are bounded between 0 and 1 so we need our function to be bounded between 0 and 1 as well. The non-transformed payoff $h(S_j)$ only depends on current period's price irrespective of next period's prices while the transformed one $\tilde{C}(S_j)$ depends on both.

This equation translates to,

$$\tilde{C}(S_j) = \bigvee_{i=1}^{I} \left\{ h(S_j) \wedge \left[\bigvee_{i_1=1}^{i} (g(S_j \mid x_{i_1}) \wedge g(\{x_1, ..., x_{i_1}\}) \right] \right\}, \tag{4.2}$$

where $\wedge$ and $\vee$ are the comparison operators standing for minimum and maximum respectively, $I = \{1, 2\}$ is the number of possible states (equivalent to up and down) and x_{i_1} is next period's price. $g(S_j \mid x_{i_1})$ must be arranged in decreasing order on $\{x_1, ..., x_{i_1}\}$. If the value of $h(S_j)$ changes depending on the next state, then it must be ordered in decreasing order as well.

Of course, the problem has to be solved numerically; however, the algorithm is not hard to implement. To illustrate the algorithm, a one period model is considered but it will be generalized into a multi-period model later on. Assuming that, from each node, there are two branches or two possible states, there must be two payoffs at maturity, which can be transformed as follows,

$$\tilde{C}(S_u) = \bigvee_{i=1}^{2} \left\{ \frac{S_{u,i} - K}{K} \wedge \left[\bigvee_{i_1=1}^{i} (g(S_{u,i} \mid x_{i_1}) \wedge g(\{x_1, ..., x_{i_1}\}) \right] \right\},$$

$$\tilde{C}(S_d) = \bigvee_{i=1}^{2} \left\{ \frac{S_{d,i} - K}{K} \wedge \left[\bigvee_{i_1=1}^{i} (g(S_{d,i} \mid x_{i_1}) \wedge g(\{x_1, ..., x_{i_1}\}) \right] \right\}.$$

But since the spot price does not really change values, $S_{u,i} = Su$ and $S_{d,i} = Sd$. Let the up state assume the value $i = 1$ and the down state assumes $i = 2$. Then, the transformed payoffs will be

$$\begin{aligned} \tilde{C}(S_u) = {} & [\tfrac{(Su-K)^+}{K} \wedge (g(Su \mid Su^2) \wedge g(\{Su^2\})] \vee \\ & \left\{ [\tfrac{(Su-K)^+}{K} \wedge (g(Su \mid Su^2) \wedge g(\{Su^2\})] \vee [\tfrac{(Su-K)^+}{K} \wedge (g(Su \mid Sud) \wedge g(\{Su^2, Sud\})] \right\}, \\ \tilde{C}(S_d) = {} & [\tfrac{(Sd-K)^+}{K} \wedge (g(Sd \mid Sud) \wedge g(\{Sud\})] \vee \\ & \left\{ [\tfrac{(Sd-K)^+}{K} \wedge (g(Sd \mid Sud) \wedge g(\{Sud\})] \vee [\tfrac{(Sd-K)^+}{K} \wedge (g(Sd \mid Sd^2) \wedge g(\{Sud, Su^2\})] \right\}. \end{aligned}$$

Note that $g(\{Su^2, Sud\}) = g(X) = 1$ because we are assuming that we only have two states of nature and that the conditional measures are indeed arranged in decreasing order, that is, $g(Su \mid Su^2) > g(Su \mid Sud)$. In fact, throughout the tree, the fuzzy measure conditional on the up-state is always greater than that conditional on the down-state. So those equations in the current form, but not necessarily, the same variables hold throughout the tree. To illustrate this idea further, recall the equations for the conditional measures,

$$g(Su \mid Su^2) = t\left(\frac{Su^2 - Sud}{spread}, 4\right),$$
$$g(Su \mid Sud) = t\left(\frac{abs(Sud - Su)}{spread}, 4\right),$$

where the spread is assumed to be constant and *abs* stands for absolute. So the only factor that affects the comparison operation is the numerator, which is the absolute change between prices at time $t+1$ and the price at time t. The change in price between two time periods, t and $t+1$, is always a function of the price at time t and either $(u-1)$ in the case of an up movement or $abs(d-1)$ in the case of a down movement. However, given that we will be working with the binomial tree for equity and forex options valuations, we have $u = \exp(\sigma\sqrt{\frac{T}{N}})$ and $d = \frac{1}{u}$ for equity options and $u = \exp[(R_d - R_f)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}]$ and $d = \exp[(R_d - R_f)\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}]$ for foreign exchange options. So $(u-1) > abs(d-1)$ for equity options but $(u-1) < abs(d-1)$ for currency options always. This means that, for equity options, (S_up - S_j) > *abs*(S_down - S_j) always. So $g(S_j$ |S_up) > $g(S_j$ |S_down) always meaning that the measure conditional on the up-state is always greater than the measure conditional on the down-state, provided that they both branch from the same state. This analysis translates, in the one period model example, to $(Su^2 - Sud) < abs(Sud - Su)$ and so $g(Su \mid Su^2) > g(Su \mid Sud)$. The opposite analysis holds for currency options. The next step would be to calculate the call price using fuzzy expectations throughout the tree.

4.5.2 Valuing the Call Option

Given a two state space, namely, $X = \{x_1, x_2\}$, the fuzzy measures of the up and down states are defined as $g^1 = g_\lambda(\{x_1\}), g^2 = g_\lambda(\{x_2\})$, respectively, and $g(X) = 1$. To compute the Sugeno integral, the function or set under integration has to be ordered in a decreasing sequence, e.g. consider $X = \{x_1, x_2, ..., x_n\}$ and define a function h

such that $h(x_1) \geq h(x_2) \geq ... \geq h(x_n)$. So when X is a finite set, the function has to be ordered according to size and the elements of X have to be renumbered. The solution to the fuzzy integral is

$$\oint h(x) \circ g = \bigvee_{i=1}^{n} [h(x_i) \wedge g(H_i)], \tag{4.3}$$

so, in a multiperiod model, the call price will have the form

$$C_0 = \frac{1}{R} E(\tilde{C}) = \frac{1}{R} \oint \tilde{C}(S_j) \circ g(H_j) = \frac{1}{R} \bigvee_{j=1}^{N} [\tilde{C}(S_j) \wedge g(\{S_1, ..., S_j\})], \tag{4.4}$$

where $E(\tilde{C})$ is the fuzzy expectations, $R = \exp[R_d * \frac{T}{N}]$ is the discount factor, $H_i = \{S_1, S_2, ..., S_i\}$ and $\tilde{C}(S_j)$ is arranged in decreasing order.

Therefore, considering the previous example on the one period model, the expected call price at time 0 has the form,

$C_0 = \frac{1}{R}\{[\tilde{C}(S_u) \wedge g(\{Su\})] \vee [\tilde{C}(S_d) \wedge g(\{Su, Sd\})]\}$,

keeping in mind that $g(\{Su, Sd\}) = g(X) = 1$ since there are two states where $g(X) = 1$ by definition, and assuming that $\tilde{C}(S_u) > \tilde{C}(S_d)$, which does not necessarily have to be the case. We always have to compare those two quantities and arrange the fuzzy or Sugeno integral accordingly. For American options, the exercise condition is evaluated at every node.

The fuzzy densities $g(up\_state)$ and $g(down\_state)$, equivalent to $g(Su)$ and $g(Sd)$, are given by the membership function approximation given above. The notation g will be used instead of μ for the state measures and $g(S \mid .)$ for the conditional fuzzy measures.

Those measures, alongside fuzzy integration, return the upper bound of the set of option prices. The lower bound is computed using the same model but with the dual measures of the conditional fuzzy measures and the state measures, which is introduced in the next section.

4.5.3 The Dual Fuzzy Measures

Section 2.2.5 presents the theory behind dual fuzzy measures. This section presents an application whereby they are utilized to their full power and used alongside the fuzzy integral to get the lower bound of the set of option prices. Given the conditional fuzzy measures,

$g(S\_current/S\_up) = t(\frac{S\_up - S\_current}{spread}, 4)$,

$g(S\_current/S\_down) = t(\frac{S\_down - S\_current}{spread}, 4)$,

$1 = g\_up + g\_down + \lambda * g\_up * g\_down.$

such that g_up and g_down are λ-complements of each other, that is,

$g(S\_current/S\_down) = \frac{1 - g\_up}{1 + \lambda g\_up}$,

$g(S\_current/S\_up) = \frac{1 - g\_down}{1 + \lambda g\_down}$,

then the dual fuzzy measures can be defined as,

$\nu\_up = 1 - g(S\_current/S\_down) = \frac{(1+\lambda)g(S\_current/S\_up)}{1+\lambda g(S\_current/S\_up)}$,

$\nu\_down = 1 - g(S\_current/S\_up) = \frac{(1+\lambda)g(S\_current/S\_down)}{1+\lambda g(S\_current/S\_down)}$.

The same analysis holds for the state measures which have the general form

$g\_up = \frac{1}{1+3(Sup - S\exp(R*\frac{T}{N}))^2}$,

$g\_down = \frac{1}{1+3(Sdown - S\exp(R*\frac{T}{N}))^2}$,

whereby their dual fuzzy measures are defined by,

$\upsilon\_up = 1 - g\_down = \frac{(1+\lambda)g\_up}{1+\lambda g\_up}$,

$\upsilon\_down = 1 - g\_up = \frac{(1+\lambda)g\_down}{1+\lambda g\_down}$.

The next step is just to plug those values in the Fuzzy conditional expectations and the fuzzy expectation formulae in a similar manner to the above approach used to evaluate the upper bound.

4.5.4 Analyzing The Fuzzy Densities

This section analyzes the fuzzy densities of the conditional measures and the state measures and their interaction. The analysis will be carried over the equity options model but the same analysis holds for currency options. The fuzzy densities for the up and down states from any node in the tree are

$g^1 = g(\{x_1\}) = \frac{1}{1+3(Sup - S\exp(R*\frac{T}{N}))^2}$, and

$g^2 = g(\{x_2\}) = \frac{1}{1+3(Sdown - S\exp(R * \frac{\tau}{N}))^2}$,

where S is underlying asset's price at time 0, R is the risk-free rate of return, τ is the time to maturity, N is the number of steps in the tree, and Sup and $Sdown$ are the up and down prices which can assume any values of Su^n, Sd^n and $Su^x d^{n-x}$ depending which node is being considered. Use the normalization condition,

$g^1 + g^2 + \lambda g^1 g^2 = 1$,

which is derived from,

$g(X) = 1 = \frac{1}{\lambda}\left[\prod_{i=1}^{2}(1+\lambda g^i) - 1\right]$.

Substituting for $g^1$and g^2,

$\lambda = 9(Sup - S\exp(R * \frac{\tau}{N}))^2(Sdown - S\exp(R * \frac{\tau}{N}))^2 - 1$,

which has a unique root in the interval $(-1, +\infty)$. So λ can assume any sign depending ultimately on the distance between the price of the node under examination and the expected risk-neutral price, which in turn depends on the number of steps in the tree and on the branch under examination, particularly, whether it is closer to the center or edges. When the number of steps is low or the node is closer to the center, λ is negative meaning that $g_1 + g_2 > g(X)$ and the measure is subadditive while its dual is superadditive. So the two states display inhibitory interaction with respect to the occurrence of the spot price meaning that their joint contribution is smaller than the sum of their individual contributions. As the number of steps increases or the node moves towards the edges of the tree, λ increases until it crosses zero meaning that the measure becomes additive and the two states do not interact. When it increases beyond zero, the measure becomes superadditive and its dual superadditive. The two states in this case exhibit synergetic interaction meaning that their joint contribution is higher than the sum of their individual contributions.

As for the densities of the conditional fuzzy measures, they are defined by,

$g(. \mid Sup) = t\left(\frac{Sup - S}{spread}\right)$,

$g(. \mid Sdown) = t\left(\frac{Sdown - S}{spread}\right)$,

which ultimately depends on $(u-1)$ and $|d-1|$ since S is the current price and Sup and $Sdown$ are next period's prices. The same above analysis holds here but the equation for λ is not as explicit. Generally, when the market is liquid and the spread is narrow, the densities tend to be large and the measure is subadditive leading to a superadditive dual measure. In a market characterized by illiquidity and high uncertainty, the spread is wide, the densities are low and the measure is superadditive meaning that its dual is subadditive. However, given this particular definition of the conditional measure, it cannot be less than 0.5 since the argument is not allowed to be less than zero. So $\lambda \geq 0$ only, meaning that the measure in this particular instance can only be additive or superadditive and the dual measure can be also additive or subadditive.

4.5.5 The Fuzzy Expected Value: A Final Comment

We need to make one final comment about the approach we use above. Kandel [69] argues that averages are not 'natural realities;' rather, they are artificial constructs that allow us to understand reality. So using the fuzzy expected value instead of the average is a more natural way of representing central tendency. This is exactly what the fuzzy binomial approach attempts. For a start, we do not have to restrict ourselves to the exhaustiveness restriction imposed by probability theory whereby every point has to be confined into a well defined set [69]. Besides, extreme observations do not affect the final result and they are eliminated through the max-min composition rule. So what we get is roughly a mirror of conventional probability theory measures of central tendency in fuzzy environments. The max-min composition approach is a particularly powerful approach since it rests on maximizing a minimum value.

Comparing our approach involving fuzzy expectation or a fuzzy expected value to the conventional approach involving probabilistic expectation or roughly an average, we notice that essentially our fuzzy expected value can be used as an alternative to the average. An average is the sum of weighted observations which span the whole space. A fuzzy expected value is, on the other hand, a compromise between those observations as well as their 'weights.' So all we do is get a compromise between those observations, which is roughly what we do with an average. In fact, Kandel [69] shows that the fuzzy expected value is very close to measures of central tendency and it may sometimes be a better approximation to the mean because it does not get affected with extreme values.

In fact, Sugeno has shown that the difference between the fuzzy expected value and Lebesgue integration is only $\frac{1}{4}$. It is also possible to use Choquet integral, in the spirit of economic and fuzzy option pricing approaches that utilize nonadditive measures, instead of the fuzzy integral as the following section shows.

4.6 The Choquet Integral Approach

Section 2.2.6 presents Choquet integration in discrete time. In this section, the Choquet integral replaces the Sugeno integral in the fuzzy binomial model. We are interested in the behaviour of the Choquet integral as compared to that of the Sugeno integral especially that economic and finance applications use this type of integration when it comes to nonlinear integration and non-additive measure. In the following chapter, empirical applications, which compare those two types of nonlinear integration, are presented.

Using the definition of the Choquet integral in discrete time, such that monotonicity is preserved and whereby,

$$C_g(h) = (c)\int h.dg = \sum_{i=1}^{n} h(x_i).[g(A_i) - g(A_{i-1})] \tag{4.5}$$

and $A_i = \{x_1, ..., x_i\}$, $0 \leq g(A_1) \leq g(A_2) \leq ... \leq 1$, $g(A_0) = 0, g(A_n) = 1$, we get,

$$E[N = Su \ / \ Y = Su^2] = (c)\int Ndg = \sum_{i=1}^{2} N.[g(N \ / \ A_i) - g(N \ / \ A_{i-1})]$$
$$= N.[g(N \ / \ A_1) - g(N \ / \ A_0)] + N.[g(N \ / \ A_2) - g(N \ / \ A_1)].$$

But $g(N \ / \ A_0) = 0$ and $g(N \ / \ A_2) = g(N \ / \ A_n) = 1$ by definition. So,

$$E[N = Su \ / \ Y = Su^2] = N.[g(N \ / \ A_1) - 0] + N.[1 - g(N \ / \ A_1)] = N.$$

So the Choquet integral does not seem to be able to capture the fuzzy payoff like the fuzzy or Sugeno integral does. And the fuzzy payoff is the same as the non-fuzzy one, meaning that the spread has no effect on the payoff.

Working backward through the tree and using the Choquet integral in discrete time, we get

$$C_g(h) = \tfrac{1}{R}(c)\int h.dg = \tfrac{1}{R}\sum_{i=1}^{n} h(x_i).[g(A_i) - g(A_{i-1})]$$
$$= \tfrac{1}{R}\left\{(Su - K)^+[g(A_1) - g(A_0)] + (Sd - K)^+[g(A_2) - g(A_1)]\right\}.$$

But $g(A_0) = 0, g(A_n) = g(A_2) = 1$. So

$$C_g(h) = \frac{1}{R}\{[(Su-K)^+ - (Sd-K)^+]g(A_1) + (Sd-K)^+\}$$
$$= \frac{1}{R}\{[(Su-K)^+ - (Sd-K)^+]g(\{Su\}) + (Sd-K)^+\}.$$

Therefore, the only measure that seems to affect the call is the state measure or, more specifically, the deviation of the spot price from the expected one since it is the basis of the state measure. This state measure still serves the role of risk neutral probabilities in a fuzzy, uncertain world. As for the case with Sugeno integration, this gives the upper bound for the set of options. The lower bound is computed by substituting the duals of those fuzzy measures.

Walley [139] shows that the Choquet integral does not define lower expectations properly if lower measures, which are the duals in our case, are not 2-monotone. A measure is 2-monotone if

$$g(A \cup B) + g(A \cap B) \geq g(A) + g(B),$$

which in our case translates to, using the proper notation for the duals,

$$\upsilon(A \cup B) + \upsilon(A \cap B) \geq \upsilon(A) + \upsilon(B),$$

where A represents the 'up' state and B represents the 'down' state. However, because we have assumed that the 'up' and 'down' states span the whole universe

$$\upsilon(A \cup B) = 1,$$

and,

$$\upsilon(A \cap B) = \upsilon(\Phi) = 0,$$

by definition. So the equation boils down to whether,

$$1 \geq \upsilon(A) + \upsilon(B),$$

which is dependent on the level of non-additivity of the duals. In its current form, the equation means that the dual measures must be superadditive. This will vary depending on which node in the tree we are valuing the measures at. If we are close to the center, then the measures themselves are high and so their duals are very low, whereby they sum to less than one (superadditive) and the above equation holds. However, as we move towards the edges, those duals become increasingly subadditive and the above equation will not hold. So, in general, we cannot say that the duals are 2-monotone and the Choquet integral is expected to perform poorly, which will be examined in the next chapter.

4.7 Conclusion

In this chapter, we have presented a fuzzy binomial model. The model deals with fuzzy prices revolving around crisp model prices and their impact on the call value. The first step is to fuzzify the crisp prices by means of conditional fuzzy measures and conditional fuzzy expectations and the second step is to value the call using fuzzy measures and fuzzy expectations. To this end, we have defined preference-free fuzzy measures and nonlinear operators in discrete time.

This chapter extends contributions to both Fuzzy Theory and Finance Theory. In both cases, the exact definition of both fuzzy measures is original. The conditional fuzzy expectations are also original. As for Finance Theory, we are not aware of any previous work that incorporates Sugeno integration or fuzzy expectation in this sense. Moreover, the specific fuzzy pricing approach that has been followed is a new methodology to pricing in a fuzzy world. Finally, the types of uncertainty considered and their impact on option pricing have not been taken into consideration in any previous option pricing models. We only need to consider how this model fares when it is applied to actual financial applications. In the next chapter, we will study empirical applications of the Fuzzy Binomial Model. To this end, we consider markets with different levels of opacity and liquidity and analyze the results.

Chapter 5

The Fuzzy Binomial Model: Empirical Applications

In the previous chapter, we have introduced the Fuzzy Binomial Model within a theoretical setting. The model attempts to price options in the presence of uncertainty, which gives rise to market incompleteness. Therefore, we can only obtain a range of possible option values rather than one value. The model utilizes fuzzy measures and fuzzy or nonlinear integration.

In this chapter, we will study the empirical behavior of the model. The application has to target markets with a high level of uncertainty. We consider two such markets, NASDAQ as well as currency options markets. However, we also consider S&P 500 options, which is traded in a transparent market, for comparative purposes. The model performs best in uncertain markets as expected.

We do not restrict the application to one type of option. Instead, we look at both American and European options and analyze the performance of the model across each type. We find out that the model gives better results for American options for reasons that we will discuss later. We also vary the volatility estimation approach so we study the performance of the model when we use historical versus implied volatility. The results are mixed in the sense that using implied volatility is better in some cases than using the historical one and vice versa. We do not confine the application to Sugeno integration but we also test the model using Choquet integration. We find out that, as expected, the Sugeno integration is better within this model framework than the Choquet one (particularly because, as we have shown in the previous section, the dual fuzzy measures are not 2-monotone).

The model and the underlying problem are new to the literature so there are no other models to which it can be compared. However, since it builds on the 'fair-value' binomial model, we find it appropriate to look at where the binomial model values stand vis-a-vis our model results. The latter tends to envelope the former with some variations depending on the application considered.

We start by laying down the general methodology in section 1, which also incorporates an analysis of the empirical behaviour of the measures as well as the general model. The data and results analysis are introduced and discussed for each market in separate sections. We start with the NASDAQ market, which is presented in section 2, followed by the currency options and Indexed option markets in sections 3 and 4 respectively. We finally summarize and conclude in section 5.

5.1 General Methodology

The model is solved numerically using Visual C++ code. Originally, the program has been written in VBA/Excel macros but, as the system complexity increases, the program slows down and sometimes crashes. The methodology follows the model outline detailed in chapter 4 such that the payoff is first transformed to accommodate fuzziness at the end of the period using fuzzy conditional measures and conditional fuzzy expectations. Then, discounted fuzzy expectations are computed across the tree in a backward manner to arrive at the final call price. However, there are certain minor issues that need to be sorted out empirically. The most important issue is the treatment of the crisp payoff. Due to the comparison operators, we cannot use the crisp payoff as it is since it has to be compared to the conditional measures and the state measures, which are bounded between 0 and 1 by definition. So we transform its value into a comparable one using a change of numeraire such that it is given in terms of shares rather than in terms of currency.

Another issue is the evolution of the BAS across the tree. It makes sense to get the implied spread from the tree, that is, using the spot price evolution across the tree rather than assuming a constant spread based on historical data. The simplest model is, of course, Roll's estimator, which is

$$spread = 2\sqrt{-cov(\Delta p_t}, \Delta p_{t+1}), \tag{5.1}$$

where Δp_t and Δp_{t+1} are the price changes at times t and $t+1$ respectively, and *cov* stands for covariance. However, there are several drawbacks with using this estimator and which are particularly unpleasant for this specific problem. For a start, it assumes an informationally efficient market, which contradicts the basic underlying premises of the fuzzy binomial model and the conditional measure. Second, it assumes that the probability of two consecutive increases or decreases in prices is zero, which does not hold for the outer edges of the tree. Third, positive covariances will create a problem. So, in the last two cases, there will be no approximation to the spread, which can be treated as missing observations but some sort of approximation to the spread will still be needed to be able to apply the fuzzy binomial model. There are other spread measures in the literature, which make up for this problem but are very cumbersome at this point especially that our intention is to illustrate the model rather than study the behaviour of the spread across the tree. So, given that a constant spread across the tree is too restrictive, we will allow the spread to evolve across the tree in a similar manner to that of the underlying asset. The logic behind this approach is that if the underlying asset's price is allowed to evolve in a certain way, then the bid and ask prices, and consequently the spread, must also be allowed to evolve in the same way.

For the purposes of computing the conditional fuzzy measure and the fuzzy payoff, the trees for the evolution of underlying prices and spread always have to be projected one step further to the maturity of the option. The deviation of each of those projected prices from the risk-neutral price serves as the basis to computing the state measures. This risk-neutral price is an ideal or hypothetical price that is not really supposed to occur in reality. If it does occur, then it gets a fuzzy density of one and the rest of the sets are not likely at all and, hence, assume fuzzy densities of zero (Lemma 2.3, [102]). So, for our present purposes, we will assume that such a state does not coexist with the up and down states.

Wide bounds will be established on the set of possible option prices whereby the upper bound is determined by using the conditional fuzzy and state measures, and the lower bound is determined using the corresponding measures' duals. The models are computed using implied as well as historical volatility and a comparison is carried out between the two to study their implications under a Sugeno integral. Bounds are also computed using the Choquet integral approach and the results are compared to those obtained using the Sugeno or fuzzy integral. We also account for the implications of using volatility generated by different methods. We particularly employ historical and implied volatility estimates and compare the results generated by each approach.

The same basic algorithm is used for all options considered with minor variations. The algorithm for American options accommodates early exercise and that for currency options accommodates foreign and domestic risk-free rates and other modifications they impose on the tree.

5.1.1 Empirical Behaviour of the Fuzzy Measures

In this section, the empirical behaviour of the conditional fuzzy measures and the state measures is examined. Looking closely at the conditional measure, we can easily show that the measure corresponding to an upward movement, $g(. \mid Sup)$, is proportional to $(u-1)\frac{S_0}{spread}$ (since the *spread* and *spot* S at each node are multiplied by the same factor), and that corresponding to a downward movement, $g(. \mid Sdown)$, is proportional to $|d-1|\frac{S_0}{spread}$, where Sup refers to the asset's price after an upward movement from any node in the tree, $Sdown$ corresponds to a downward one, S_0 corresponds to the initial asset's price, and *spread* is the initial spread on the underlying (the evolution of the spread and spot cancels out since they are both multiplied by the same factor because the tree is recombinant).

For equity options, $u = \exp(\sigma\sqrt{\frac{T}{N}})$ and $d = \frac{1}{u}$ so the conditional measures corresponding to the two states for equity options are dependent on volatility, time to maturity, number of steps in the tree, initial underlying asset's price and initial spread on that underlying. Due to the relationship between u and d, the conditional measure for the up state diverges from that of the down state as u increases (d decreases) while it converges to that of the down state otherwise. So, as the volatility of the asset increases, holding everything else constant, the conditional measure for the up state increases while that for the down state decreases. A similar behavior is observed for long-term options provided that the number of steps is constant. However, if the number of steps is increased, we notice the opposite behavior.

On the other hand, for forex options, $u = \exp\left[(R_d - R_f)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\right]$ and $d = \exp\left[(R_d - R_f)\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}\right]$ so the conditional measures for currency options are dependent on volatility, time to maturity, number of steps in the tree, initial underlying asset's price, initial spread on that underlying, and domestic and foreign risk free rates. One can observe that the effects of time to maturity and risk-free rates are more pronounced for conditional measures of forex options than for those of equity options. They are also dependant on one more factor, which is the foreign risk-free rate. Generally, the conditional measures corresponding to an upward movement tend to increase with an increase in volatility and time to maturity while those corresponding to a downward movement tend to decrease when volatility and time to maturity increase. As the number of steps increases, both conditional measures decrease. For a very liquid asset (narrow spread, \$/£) or an asset with a high price level (S&P 500 index), the conditional measures are relatively low. Finally, they are higher when domestic interest rates are higher than foreign interest rates.

The effects of the spot and spread on the conditional measures are equivalent for both markets. In absolute terms, as the spot price increases or the spread decreases, holding everything else constant, the conditional measures for both states will increase and vice versa. But, as we have seen in Chapter 2, there is a direct relationship between the spread and the price level so the value of the measure will depend on the interplay between those two factors. When the price level is high, the spread is also expected to be high but the absolute value of the measure really depends on how the two values compare with each other. In liquid and transparent markets, the spread is generally low and so the measure is expected to be high. Alternatively, in illiquid or uncertain markets, the spread is usually wide and so the measure is expected to be low.

The state measures are not really constant; they are quite dependent on the node of the tree they are evaluated at. The form for the state measures is $\frac{1}{1+3(S\_current-E(S))^2}$ whereby $S\_current$ corresponds to the underlying asset's price at the node at which the state measure is being evaluated and $E(S)$ is equivalent to $S_0 \exp(R\frac{T}{N})$ for equity and $S_0 \exp\left[(R_f - R_d)\frac{T}{N}\right]$. Clearly, $S\_current$ is a variable but $E(S)$ is a constant. Therefore, the state measures are dependent on the node price (which ultimately depends on the u^n and d^n factors where $n = \{0, ..., N\}$), the risk free rate(s) and time to maturity. In the short run, that is for a low number of steps, an increase in time to maturity leads to an increase in the state measure but not with direct proportion. But, as the number of steps increases, the effect of $S\_current$ takes over for the outer edges of the tree and the relation between the time to maturity and the state measure becomes inversely proportional and the state measure is quite low. However, around the center, the state measure is high irrespective of the time to maturity (since the effect tends to cancel out when time to maturity in both $S\_current$ and $E(S)$ is considered simultaneously), which is logical because the underlying asset's price is quite close to the prototype or the expected preference-free price.

So, as the above analysis shows, the volatility and the initial price level of the underlying asset, the time to maturity of the option, the risk-free rate and the number of steps in a binomial tree affect both conditional and state measures. However, moneyness has no effect on either of them. A similar analysis holds for the dual measures since they are the other side of the coin so that dual measures corresponding to an upward measure behave in an opposite manner to the fuzzy measures corresponding to the downward move (due to additivity) and, similarly, dual measures corresponding to a downward move behave in an opposite manner to the fuzzy measures corresponding to the upward move.

5.1.2 Empirical Behaviour of the General Model

The set of fuzzy measures influences the behaviour of the general model first through the fuzzy payoff and then through the propagation across the tree. The definition of the fuzzy payoff for equity as well as for forex options depends on the comparative sizes of the fuzzy densities. The value of $g(. \mid Sup)$ is proportional to $(u-1)$ and the value of $g(. \mid Sdown)$ is proportional to $|d-1|$.

In the case of equity options, where $d = \frac{1}{u}$ and $u > 1$, $|d-1| < (u-1)$ always because $|\frac{u-1}{u}| < (u-1)$ while for forex options, $|d-1| > (u-1)$. This means that, for equity options, $g(. \mid Sup) > g(. \mid Sdown)$ and vice versa for forex options. So the final form of the fuzzy payoff at each node for equity options is,

$$\tilde{C}(Su^x d^{n-x}) = \left\{ \begin{array}{l} [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^{x+1} d^{n-x}) \wedge g(\{Su^{x+1} d^{n-x}\})\right] \vee \\ [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^{x+1} d^{n-x}) \wedge g(\{Su^{x+1} d^{n-x}\})\right] \\ \vee [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^x d^{n-x+1}) \wedge g(\{Su^{x+1} d^{n-x}, Su^x d^{n-x+1}\})\right] \end{array} \right\},$$

and that for forex options is,

$$\tilde{C}(Su^x d^{n-x}) = \left\{ \begin{array}{l} [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^x d^{n-x+1}) \wedge g(\{Su^x d^{n-x+1}\})\right] \vee \\ [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^x d^{n-x+1}) \wedge g(\{Su^x d^{n-x+1}\})\right] \\ \vee [\frac{(Su^x d^{n-x}-K)^+}{K} \wedge \left(g(Su^x d^{n-x} \mid Su^{x+1} d^{n-x}) \wedge g(\{Su^{x+1} d^{n-x}, Su^x d^{n-x+1}\})\right] \end{array} \right\},$$

where $Su^x d^{n-x}$ is the price at the current node or, using the above terminology, $S\_current$, $Su^{x+1} d^{n-x}$ is next period's price corresponding to an upward move, $Su^x d^{n-x+1}$ is next period's price corresponding to a downward move and $g(\{Su^{x+1} d^{n-x}, Su^x d^{n-x+1}\} = 1$. It is clearer now how the comparative sizes of the variables influence the fuzzy payoff.

The state measures tend to be the most cumbersome variables. It is hard to know which state measure, that is the one corresponding to an upward move or the one corresponding to a downward move, is greater than the other. Generally, towards the outer edge of the upper half of the tree (when the half is considered with respect to $E(S)$ and not the initial asset's price), the state measure corresponding to a downward move is greater than the measure corresponding to an upward move since the corresponding price is closer to the expected one $E(S)$ and vice versa for the lower half. Around the center, the comparative sizes of the state measures depend on the node under consideration, more specifically, they depend on where the expected price line falls in between the two branches coming out of the node. The relative weights of upward and downward states fluctuate depending on the values of those measures, which has further implications for the empirical behaviour of the model as the following analysis shows.

Given a call option, the payoff will be zero in the half of the tree lying below the initial asset's price line. Hence, irrespective of the values of the fuzzy measures, which are greater than zero, the fuzzy payoff will be zero. The picture is not as clear for the upper half of the tree. There are different scenarios depending on the node under consideration. At nodes that are very close to the center, the crisp payoff plays the major role in determining the fuzzy payoff because it will assume the minimum value among all variables. Towards the upper edge of the tree, it is not really expected to play a major role because it assumes a high value. However, the final result will still depend on the values of the measures and how they compare to the crisp payoff and to each other.

This influences the final call price by evaluating the fuzzy expectations at each node,

$$C(Su^{x-1}d^{n-x}) = \frac{1}{R}\left[\begin{array}{l} \left(\tilde{C}(Su^x d^{n-x}) \wedge g(\{Su^x d^{n-x}\})\right) \\ \vee \left(\tilde{C}(Su^{x-1}d^{n-x+1}) \wedge g(\{Su^x d^{n-x}, Su^{x-1}d^{n-x+1}\})\right) \end{array}\right],$$

for $\tilde{C}(Su^x d^{n-x}) > \tilde{C}(Su^{x-1}d^{n-x+1})$, and

$$C(Su^{x-1}d^{n-x}) = \frac{1}{R}\left[\begin{array}{l} \left(\tilde{C}(Su^{x-1}d^{n-x+1}) \wedge g(\{Su^{x-1}d^{n-x+1}\})\right) \\ \vee \left(\tilde{C}(Su^x d^{n-x}) \wedge g(\{Su^x d^{n-x}, Su^{x-1}d^{n-x+1}\})\right) \end{array}\right],$$

for $\tilde{C}(Su^x d^{n-x}) < \tilde{C}(Su^{x-1}d^{n-x+1})$,and where $g(\{Su^x d^{n-x}, Su^{x-1}d^{n-x+1}\}) = 1$. Therefore, the expected Call price at each node is dependent on the fuzzy payoff and its possibility of occurrence as given by the corresponding state measure. Around the center, the fuzzy payoff is expected to play the major role since the state measures will be high and so they will be ruled out by the minimum operation. According to the above analysis, this means that the crisp payoff is expected to play the major role. Towards the upper edge of the tree, the fuzzy payoff corresponding to a lower state measure is expected to take over since it will be compared with $g(\{Su^x d^{n-x}, Su^{x-1}d^{n-x+1}\})$, which is equal to one, in the second part of the integral and with the state measure corresponding to the complement state in the first part of the integral, which will be very low. So, due to the maximum operator, it will win over. In this setting, that is the one corresponding to the upper edge of the tree, the values of the crisp payoff can be quite high due to the maximum operator and the fact that it will be the minimum in the last part of the conditional fuzzy expectations. But, fortunately and due to the definition of the fuzzy expectations, such high values will be ruled out by the minimum operator across the tree. Ultimately, those values will be influenced by input variables like volatility and time to maturity, which has to be studied empirically as we show in the following sections.

5.2 NASDAQ Options

The NASDAQ options we consider are those corresponding to options written on Dell and Microsoft, which are American-type options. The NASDAQ market has a high level of uncertainty due to opacity unlike physical equity exchanges. So we expect the model application corresponding to those options to outperform that corresponding to the currency and index options we consider in later sections.

We will first describe the data set and then present the results. As we have mentioned earlier, we will establish the bounds using fuzzy measures for one bound, and their duals for the other, as well as fuzzy expectations. We also consider the two approaches to computing the expectations, namely, the Sugeno or fuzzy integration approach and the Choquet integration approach. This comparison illustrates the implications of using different nonlinear integration approaches to the same problem as well as the same input variables. Moreover, we look at the implications of using a historical volatility estimation approach as opposed to using an implied volatility one.

5.2.1 Data Set

The data sets for both Dell and Microsoft Call options have been quoted on the 10th of May 2002 from the Bloomberg database. Each data set is comprised of moneyness, current stock price (S), strike price (X), risk-free rate (r), dividend yield (q), annualized time to maturity (tyr), annualized volatility (implied volatility in this table), spread on the underlying (spread), and bid, ask and last prices for the option. The options expire on the Saturday of the third Friday of every month. For computation purposes, the zeroes in bid, ask and last columns correspond to no quotations. The dividend yields for both asset's are zero. Moneyness is determined by the ratio of the current stock price (S) to the strike (X), namely,

if $\frac{S}{X} > 1.02$, then the option is in-the-money (ITM),

if $\frac{S}{X} < 0.98$, then the option is out–of-the-money (OTM), and if $0.98 < \frac{S}{X} < 1.02$, then the option is at-the-money (ATM).

The data will be categorized according to moneyness and expirations for comparison purposes.

Implied volatilities are those corresponding to the options under consideration. Historical volatilities are quoted from Bloomberg corresponding to short-term volatility (30 day), medium-term volatility (60 day) and long-term volatility (90 day) whereby they are matched with the maturity of the option. Dell options expire in May 2002, June 2002, August 2002, November 2002, January 2003 and January 2004. Microsoft options expire in May 2002, June 2002, July 2002, October 2002, January 2003 and January 2004. Short-term volatility is associated with May 2002 expiration, medium-term volatility is associated with June 2002 and long-term volatility is associated with the longer maturities.

5.2.2 Result Analysis

Appendix C displays the results for Dell and Microsoft options individually. Table 1C shows the results of applying the fuzzy binomial model to Dell options using Sugeno integration. It also shows the results of computing the expectations using Choquet integration. For comparative purposes, the binomial model value as well as the market bid, ask and last quotations are included. The table illustrates the behavior across different moneyness and expirations. The results are computed using implied volatility. Since there are no quotations for the bid or last at times, the implied volatility corresponding to the ask price is used.

The 'Dual Fuzzy' column stands for the lower bound for the range of possible option prices while the 'Fuzzy' column stands for the upper bound. Both bounds are computed using Sugeno expectations. The difference between the two is that the latter employs the conditional fuzzy measures and the state ones while the former uses the corresponding duals. The 'Fuzzy Spread' column represents the spread or difference between the 'Fuzzy' and the 'Dual Fuzzy' values. The 'Binomial' column shows the values given by the binomial model. Finally, the 'DualChoquet' and 'Choquet' columns stand for the lower and upper bounds of the call price respectively using Choquet integration.

We shall compare the results across different maturities and moneyness. To be able to understand the results, we need to understand the general behaviour of the determinants of the bounds across the tree. The payoff increases as we move towards the outer edges since it is directly proportional to the spot price, which increases as we move in that direction. Exactly the opposite behaviour is observed for the state measures since they are inversely related to the spot. The conditional measures are independent of the behaviour of the spot price because the spread is assumed to evolve in the same way as the spot price and so the effect cancels out.

There is a common general pattern for the bounds across different moneyness for all maturities (Figure 5-1). Generally, the dual fuzzy model converges to the fuzzy one for very deep in-the-money options, that is the lower bound converges to the upper bound. As the option becomes less in-the-money, the bounds grow farther apart with a wide difference until the option becomes at-the-money where it reaches a maximum. Then this difference starts decreasing as the option becomes out-of-the-money and it keeps getting smaller until it matches the spread for deep out-of-the-money options. But the problem with out-of-the-money options quoted data is that it is not always reliable due to illiquidity, which is also clear in Table 1C and Figure 5-1.

This behaviour is observable whether we use the implied volatility for the specific moneyness we are considering or whether we use the same at-the-money implied volatility for all moneyness. This implies that the strike is playing the major role for a single maturity especially that the state measures and the conditional measures are not really affected by the moneyness. The strike affects the final model values through its impact on the fuzzy payoff and the early exercise condition for American options. For very deep-in-the-money options, the effect of the strike takes over that of the fuzzy measures and so the two bounds converge. Recall that the strike, and consequently the payoff and the early exercise condition, is the same for the fuzzy model and its dual while the fuzzy measures are different. As we move away from deep in-the-money options, the fuzzy measures seem to be more important where they become the most important for at-the-money options.

The conditional measures are quite low, which makes their duals quite high. The effect of the conditional measures is propagated through the fuzzy payoff or the early exercise condition. The fuzzy measures are ruled out by the maximum operator except when the state measures and the crisp payoff are very low themselves. This tends to take place around the center. This behaviour is specifically observable for at-the-money options where the payoff around the center is quite low. We emphasize the effect of the payoff in this case because the state measures and conditional measures are really the same across different moneyness for a single maturity.

Therefore, for a single maturity, each bound decreases in value as the strike increases indicating that the strike is playing the major role through the fuzzy payoff and the early exercise condition since the state measures and the conditional measures are not affected by moneyness. The bounds converge for deep in-the-money options and start to diverge as we move more towards at-the-money where maximum divergence is reached. Then, they start converging again as we move more towards out-of-the-money. The payoff is the most important factor for deep in-the-money options where the effect of the measures is almost negligible due to the maximum operators. The measures, especially the conditional ones, become more important around the center specifically for at-the-money case. Since the dual measures are the mirror images for the measures, the 'Dual Fuzzy' model exhibits exactly the opposite behaviour to the 'Fuzzy' one.

The spread between those bounds does not really reflect that in the market nor is it expected to do so. Rather those bounds generally envelope the bid/ask quotations quoted in the market, which is clear in the sample options in Figure 5-2. The maximum difference or spread between the two bounds occurs for options with the shortest maturity and then it decreases systematically as the time to maturity increases (Figure 5-4). Generally, those bounds as well as the difference between them grow smaller as the time to maturity increases. This spread pattern is also observed across different maturities (Figure 5-5). So as the time to maturity increases, the spread systematically decreases. Therefore, the spread seems to have patterns across different strikes and maturities but not volatilities.

When the time to maturity is varied, the bounds are consistently lower and closer for options with the same strike. For some deep in-the-money options, values are the same for the same strike irrespective of the different values of volatility, time to

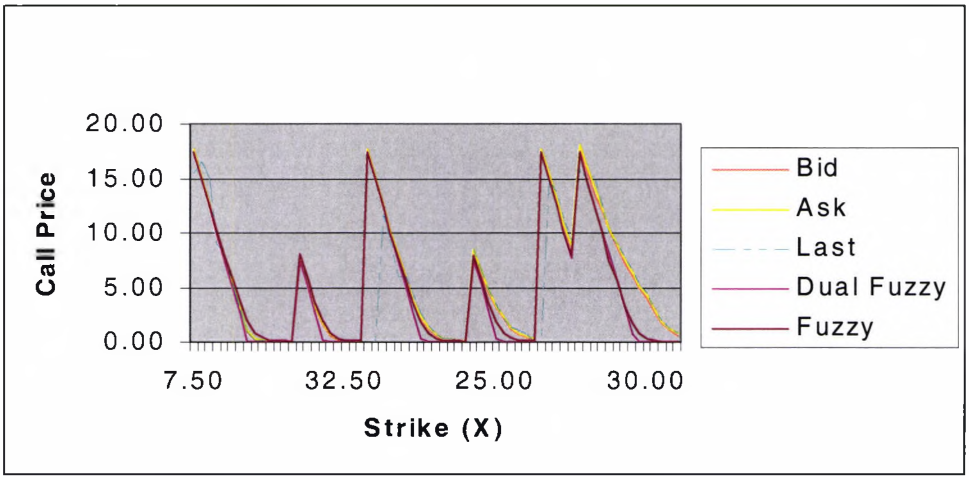

Figure 5-1: Dell Option prices for different maturities. Dual Fuzzy, Fuzzy and Binomial OPM results were computed using implied volatility.

maturity and the risk-free rate across various maturities, except for those with very long maturity, in a similar manner to those values given by the CRR tree. This empahsizes the earlier point we have made about the importance of the strike for deep in-the-money options. The same observation is recorded for deep out-of-the-money options, except that it is not the case for the values given by the CRR model. For other in-the-money and out-of-the-money options, the values are different across different maturities. The results for at and out-of-the-money options are not reliable for very long maturities. As Figure 5-1, they are well below the quoted bid and ask prices.

The value given by the binomial model always falls between those bounds. This serves as an empirical proof to our earlier reasoning that the binomial model gives a price that is based on fair or unobservable prices while, empirically, prices revolving around the fair ones are observed. The final outcome is a set of observable prices bounded by two extremes in between which the option price can assume any value.

The same pattern of behaviour is observed for the Dual Choquet and Choquet OPMs, which use Choquet rather than Sugeno integration. The bounds follow the same behaviour of option prices, that is the prices they give have the same decreasing and increasing patterns as actual option prices. However, they tend to be very wide

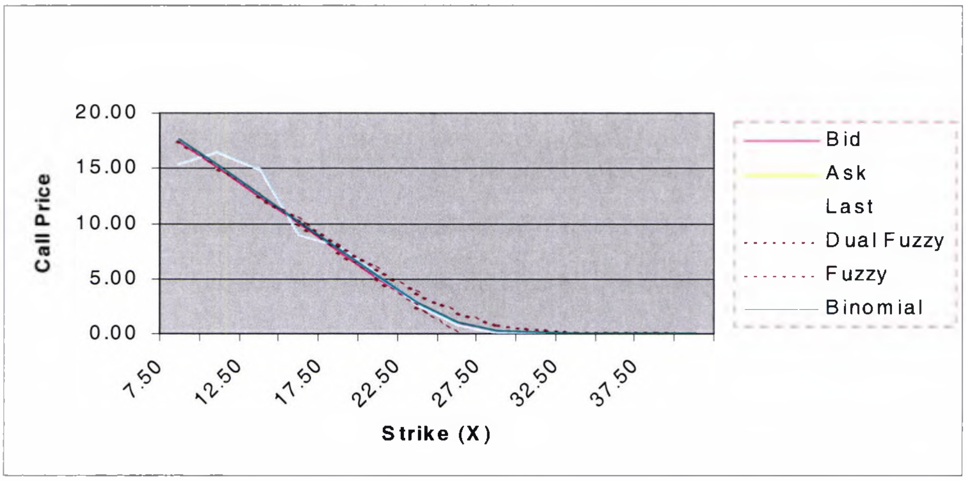

Figure 5-2: Dell Option prices with May maturity. Dual Fuzzy, Fuzzy and Binomial OPM are computed using implied volatility.

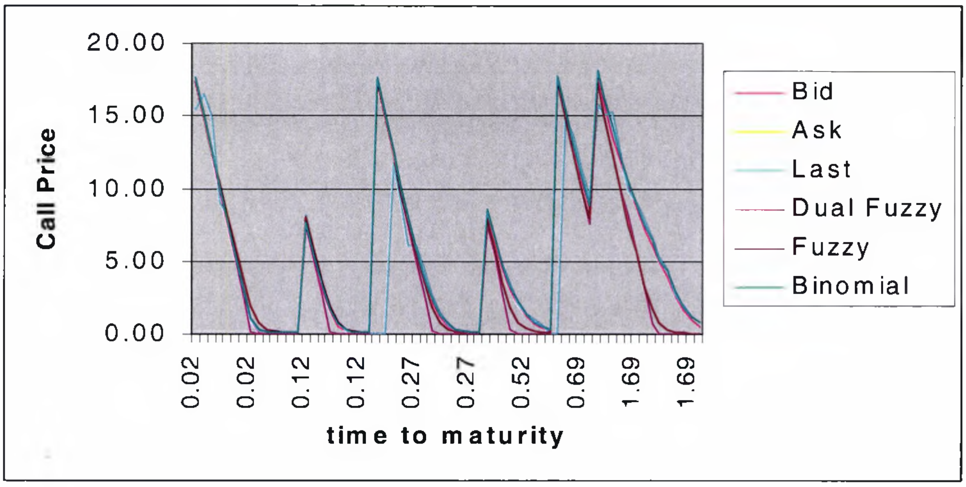

Figure 5-3: Dell option prices for different maturities. Dual Fuzzy, Fuzzy and Binomial are computed using implied volatility.

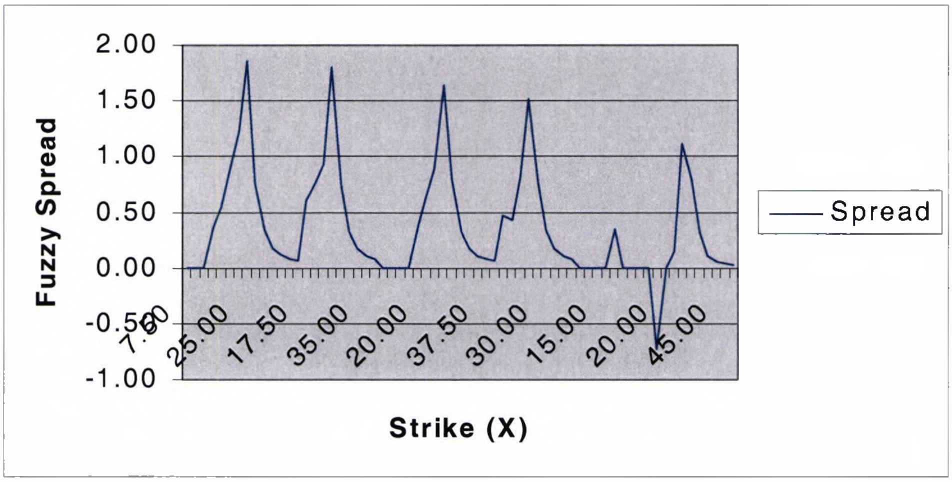

Figure 5-4: Fuzzy spread representing the difference in call prices given by the fuzzy and dual fuzzy OPM.

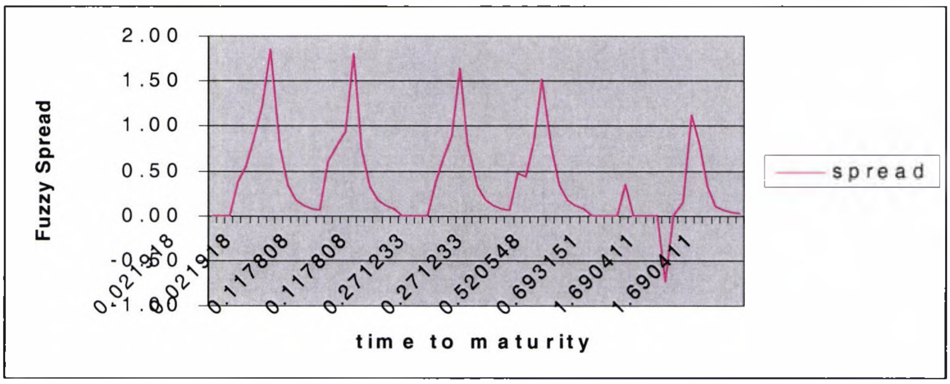

Figure 5-5: Dell option prices across different maturities. Dual Fuzzy, Fuzzy and Binomial OPMs are computed using implied volatility.

due to severe underpricing on the Dual Choquet OPM part rendering the model useless. The Dual Choquet OPM significantly underprices the option and the Choquet overprices it but with a much lower proportion. As a result, the bounds are very wide as Table 1C (Appendix C) shows. However, the Choquet OPM is not really bad and perhaps can be used with a fuzzy volatility similar to the approach used in the following chapter. But when it is combined with the Dual Choquet one, the results are not practical. This has already been expected since, as we have shown in the previous chapter, the dual measures are not 2-monotone and so the Dual Choquet (lower expectation) is expected to perform poorly.

The conditional measure plays no role here. The values are determined by the state measures and their duals. The state measures are quite low towards the outer edges of the tree especially as the number of steps increases. However, this is balanced by the high payoff at those nodes. It is quite high around the center, which balances the low payoff there. Because the conditional measure plays no role, the results are quite high for the 'Choquet' bound. The 'DualChoquet' displays the mirror image of the 'Choquet' behaviour. When the state measures are high, their duals are low. So whenever the payoff is high, the duals are high and whenever the payoff is low, the duals are low. So towards the outer edges, both the duals and the payoff are high. Similarly, both quantities are low towards the center. However, the weight of the nodes towards the outer edge of the tree is low and the effect tends to diminish as we move backward through the tree.

This proves that, within the context of this problem, Choquet integration does not really model uncertainty like Sugeno integration does. The reader at his point is reminded that the Choquet integration approach does not capture the fuzziness of the payoff because the conditional measures cancel out due to the use of linear operators as well as the definition of the Choquet integral. Sugeno integration is better able to capture the fuzzy payoff due to the nonlinear operators. A possible reason for the shortcoming of Choquet integration in this model is that it converges to the Lebesgue integral while the Sugeno one does not. It is possible that this small deviation, arising from nonlinear operators, between the Sugeno and Lebesgue integrals is what counts for the success of the Sugeno integral for this particular problem.

Finally, the fact that the model (using Sugeno integration) is not affected by volatility is emphasized in Table 2C. This table shows the results when we use historical, rather than implied, volatility. The results are very similar to those in Table 1C, whereby implied volatility is used for valuation. However, it is interesting to note that the binomial option value sometimes coincides with the bounds, which are convergent in this case, for deep in-the-money options. In general, we observe the same pattern of behaviour and indeed very similar values when we use historical volatility as we do when we use implied volatility.

The behavior of the model for Miscrosoft options is pretty much the same as it is for Dell options. However, for Microsoft options, the spread tends to be wider and there is a slightly more significant difference between the results reported using implied volatility from those reported using historical volatility. Tables 3C and 4C in Appendix C report the results and the corresponding data set.

Figure 5-6 plots a graph of option prices across different maturities versus the strike price. Almost the same pattern of behaviour observed for Dell options is observed for Microsoft options except that for the case of Microsoft options, the bounds are more well behaved for out-of-the-money options than they are for Dell options. The bounds are convergent for very deep in-the-money options and then they grow wider until they reach their widest closer to at-the-money. They envelope the bid and ask prices for other observations. However, they are not very reliable for very long maturity.

The spread also exhibits the decreasing pattern prevalent for Dell options (Figure 5-7). But volatility seems to affect the results for Microsoft options more than it does for Dell options. Figure 5-8 plots the fuzzy spread obtained using implied volatility with that obtained using historical volatility against the strike price. There is no systematic pattern for the behaviour of either spread. Rather they can vary randomly but there is a general tendency for the bounds to be wider in the case of historical volatility than that in the case of implied volatility.

There is an interesting observation pertaining to the bounds' behaviour. At some point the bounds flip, which is also the case for Dell options, but it is an infrequent observation. However, for Microsoft options, it occurs twice for the set under consideration using an implied volatility but once using historical volatility for an option that is well in-the-money and with shorter maturity. Both observations correspond to the same strike, which is 35 while the spot is 58. This can be due to the behaviour of the measures and their duals and their interplay with the payoff across the tree since the payoff by itself is the same for the two bounds. However, this behaviour is not really expected or justified but it can be that in such instances the duals are generally higher than the corresponding measures. Volatility definitely plays a role in those cases since a different volatility value can make the bounds flip or unflip. It influences the values through every term in the fuzzy payoff as well as the backward induction and it is very hard to isolate the effects of each especially in the presence of nonlinear operators.

The historical volatility in this case is half the implied volatility. Besides, bounds corresponding to historical volatility are not as convergent to each other for deep in-the-money options as those corresponding to implied volatility are. But for out-of-the-money options, especially deep ones, the results are almost the same. The only difference between the two really lies for in-the-money options, especially those close to at-the-money. Table 5C in Appendix C illustrates this analysis. On the other hand, the Choquet model performs better when using historical volatility. It overprices a little bit when it is computed using implied volatility but it gives much better results when computed using historical volatility (Tables 3C and 4C in Appendix C). So for the purposes of using uncertain volatility to price options, it is better to use historical volatility based estimations for the interval of volatility than use the implied one.

5.3 Currency Options

5.3.1 Data Set

The forex option we consider is a European option on \$/£ listed on the Philedelphia Stock Exchange on the 17th of September 2001 as quoted from Bloomberg database. This is an illiquid option written on a liquid asset. Since the underlying price is quoted in lots of 100, we have multiplied the spread by 100 as well for consistency purposes. The domestic and foreign risk-free rates are respectively taken from the US and UK LIBOR. The option has November 2001, December 2001, March 2002 and

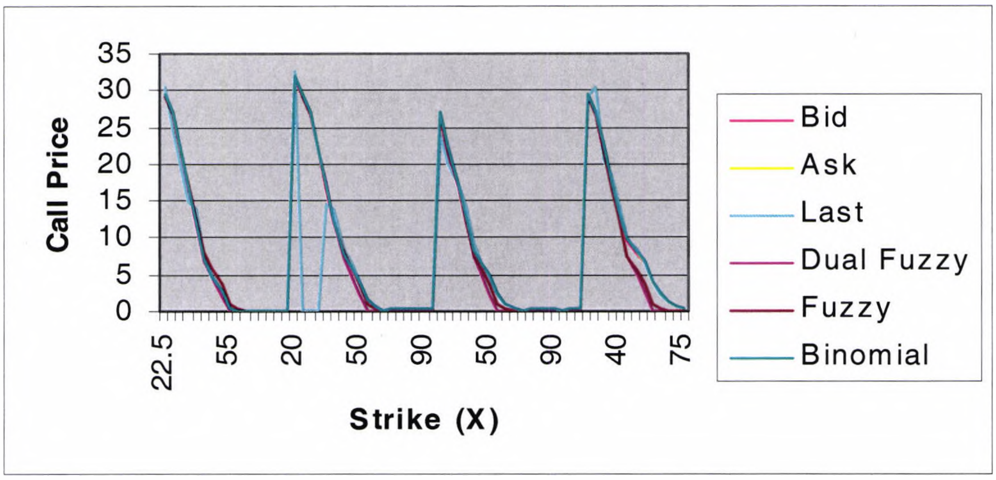

Figure 5-6: Option prices for Microsoft Options for all maturities across the strike price. The Dual Fuzzy, Fuzzy and Binomial OPM results are computed using implied volatility.

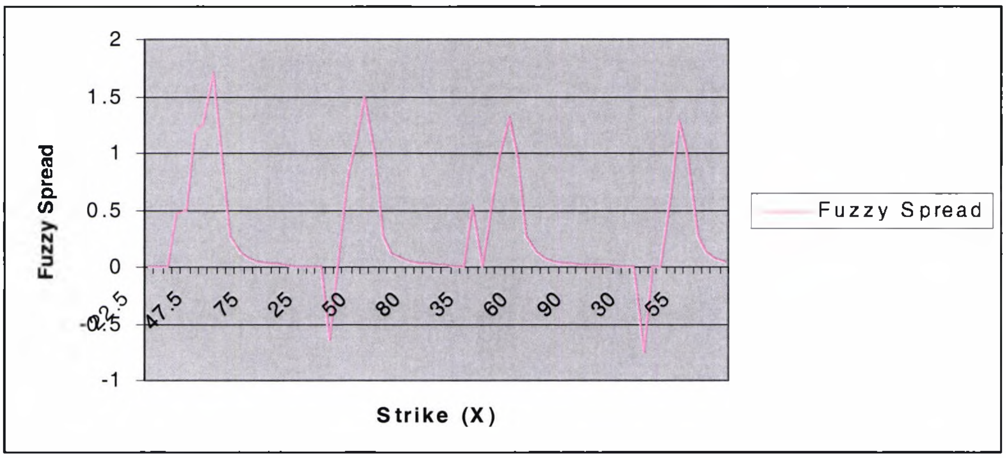

Figure 5-7: Fuzzy Spread corresponding to implied volatility across all maturities versus strike price.

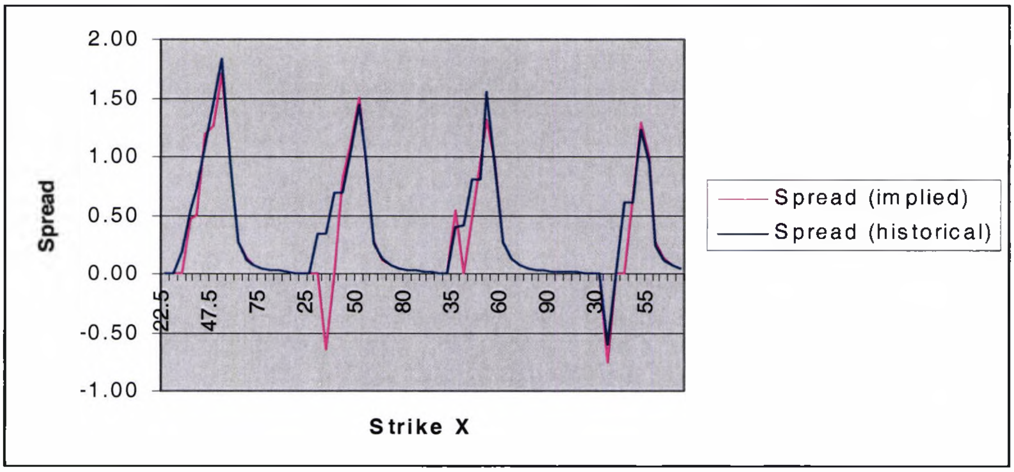

Figure 5-8: Comparison between the fuzzy spread obtained using implied volatility and that obtained using historical volatility.

September 2002 expirations. They expire on the Friday before the third Wednesday of the expiry month.

5.3.2 Result Analysis

Tables 6C and 7C in Appendix C report the results for the given data set. As the tables show, the results for this option using the dual fuzzy and fuzzy models are not as good as they are for the previous ones. One possibility is that, judging by the narrow spread and relatively low implied (and historical) volatility, the underlying asset is not characterized by a high uncertainty like the NASDAQ stocks. Another possibility is that those models perform better for American options than for European options like the one at hand. A third possibility is that the option is illiquid and so it necessarily dictates wider bounds than expected.

Empirically, the conditional measure for this option is higher than that for Dell and Microsoft ones while the payoff, for a comparative level of moneyness, is much lower. The state measures are roughly similar. Besides, this currency option is European and there is no early exercise condition. So the upper ('Fuzzy') bound is relatively high (vis-a-vis the ask quote) because the high values for the measures are not really balanced out by the early exercise condition, which will be low in this case if it did exist, as it is the case for the Dell and Microsoft options. As a result, the high values are propagated through the tree. On the other hand, the lower bound is so low because the payoff is very low while the duals are very high so they get ruled out by the minimum operator.

The bounds do envelope the Bid/Ask quotations but they are very wide for in-the-money and at-the-money options. An interesting observation is that they can be convergent for out-of-the-money options rather than in-the-money options like the NASDAQ options case. For the bounds to converge, they have to agree on a value, which is the payoff in this case. The payoff is expected to play the most important role since it is the only variable that is not different for the two bounds while the measures and their duals are quite different. It is generally lower for out-of-the-money options than it is for options with different moneyness while the measures are the same. So this low value takes over the measures' values through the minimum operation.

Another difference is that the spread or the difference between those bounds tends to be greatest for in-the-money options decreasing gradually until the option becomes out-of-the-money where it becomes acceptable (Figure 5-9). The payoff as well as measures tend to be quite high around the center so the result of the minimum operation will be high anyway. Towards the outer edges of the tree (specifically upper), the payoff is very high but the measures are low so the high value of the payoff is ruled out. However, the latter result does not survive as we compare it to values closer to the center using the maximum operation due to the high values around the center. This leads to a high value for the upper bound for in-the-money options, which decreases as the payoff decreases. On the other hand, the lower bound is very low because, towards the center, the duals in this case will be low but the payoff will be high while, towards the edges, the duals will be high but the payoff will be low. So either way the low value is taking over due to the minimum operation. A closely related issue is that the behaviour of the upper bound captures market behaviour, that is, the bound decreases as the strike increases. However, we do not note the same observation for the lower bound. This again can be due to the payoff as the above analysis shows.

Unlike the NASDAQ options, the results obtained using a historical volatility are much better than those obtained using an implied volatility. In the former case, the bounds are narrower and so the spread is smaller (Table 8C in Appendix C). Historical volatility is lower than implied volatility. So the payoff will be smaller and the measures will be moderate in the sense that they are not as high around the center nor are they as low towards the outer edges and so the corresponding duals will be also moderate. This means that the upper bound will be lower since the values around the center, that ultimately determine its value, are lower now. On the other hand, the lower bound is higher now because the dual measures are generally higher.

The results are shown in Figure 5-10. As can be seen from the plot, the model generally underprices out-of-the-money options but overprices in-the-money ones. The dual fuzzy model is not stable and tends to be nondecreasing in a counter manner to the behaviour of actual option prices. The fuzzy model gives closer results to the ask price than the modified binomial model does in the presence of historical volatility. However, it returns a higher value than the ask for short maturities but a lower one for longer maturities.

The results for the Choquet and Dual Choquet approach, using both historical and implied volatility, give a lower spread but show systematic underpricing. The high payoff is constantly being made smaller through multiplication by the low state measure towards the edges. On the other hand, it is low towards the center while the state measure is high but it is not high enough to overcome the low payoff. So the overall effect is lower values. However, the bounds still exhibit the decreasing pattern associated with increasing strike for Call option prices.

5.4 Index Options

5.4.1 Data Set

The option under consideration here is an S&P500 option (European). The data is quoted on the 24th of July 2002 from Bloomberg database. It includes as well historical volatility and risk-free rate data. Such options have expirations of August 2002, September 2002, October 2002, December 2002, March 2003, June 2003, December 2003 and June 2004. They expire on the Saturday of the third Friday of

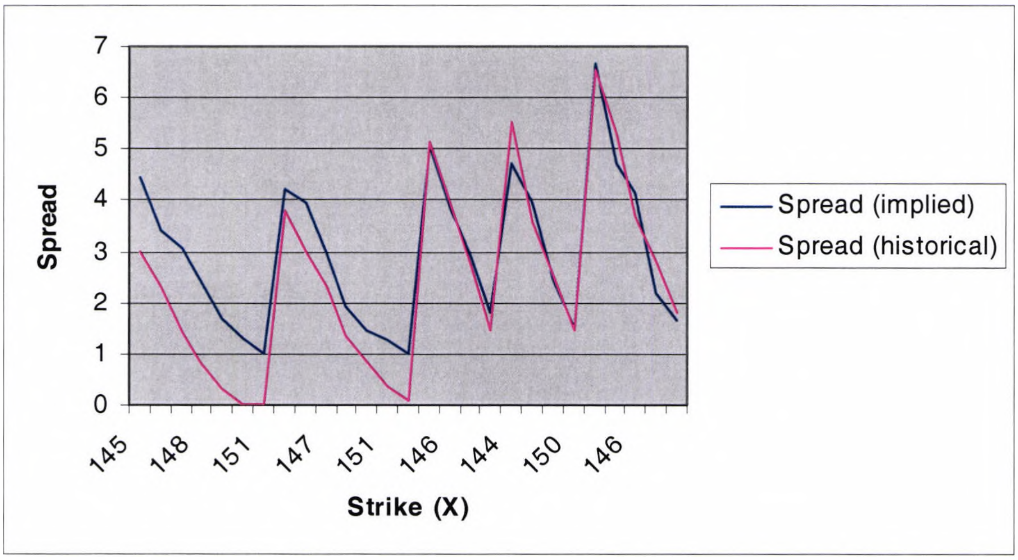

Figure 5-9: Comparison between fuzzy spreads obtained using historical and implied volatilities versus the strike price.

Figure 5-10: Forex option prices for all maturities versus the strike price. The Dual Fuzzy, Fuzzy and Binomial OPM prices were obtained using historical volatility.

the expiration month.

5.4.2 Result Analysis

We see similar scenarios for S&P 500 index options as we have seen for the $/£ currency option that we have just analyzed. Much of the analysis we have carried out in the previous section holds here. Both bounds capture the decreasing behaviour of Call option prices as the strike increases. However, there is considerable underpricing as Tables 9C and 10C in Appendix C show and many times the bounds flip such that the fuzzy model prices are less than the dual ones. So, in this case, the upper bound is given by the dual fuzzy model and the lower one is given by the fuzzy model.

The values are generally more extreme due to the higher level of the index. For example, the measures tend to be very low towards the outer edges of the tree but very high towards its center and so their duals will be behave in an exactly opposite way. This translates to more extreme values in the bounds, which will be very wide. The bounds flip for the same reason. For the 'Fuzzy' bound, the very low payoff towards the center wins over the very high measures through the minimum operation while it loses to the very low measures towards the outer edges where it assumes a very high value again due to the minimum operation. So the values will be very low. On the other hand, the 'Dual Fuzzy' bound is higher than the 'Fuzzy' one because both the payoff and the duals will be high or low at the same time (since the duals are the mirror image of the fuzzy ones).

The historical volatility results, shown in Figure 5-11, are better than the implied volatility ones, which again proves our point about the high values. The historical volatility is lower than the implied volatility and so all the variables will be consistently lower. The dual fuzzy model behaves better in this case than fuzzy one as expected because it has relatively more moderate variables. It is also more stable unlike the previous two options we have considered so far. It actually behaves better than the binomial model in the presence of historical volatility. But for the implied volatility case, the binomial model fares better than both the fuzzy and dual fuzzy ones even though the latter gives very close results to it. The spread is also very wide. It is much wider for implied volatility calculations than it is for historical volatility ones

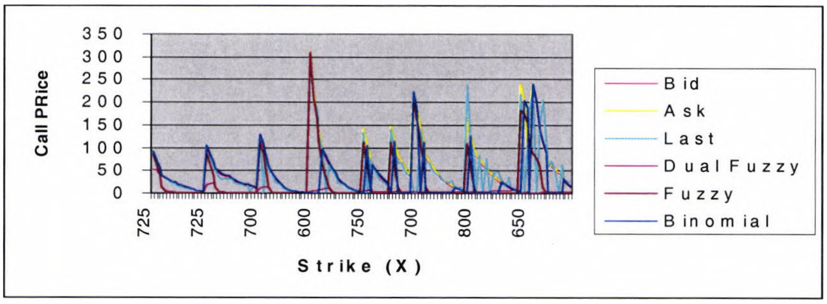

Figure 5-11: S&P 500 option prices across all maturities versus the stock price. The Fuzzy, Dual Fuzzy and Binomial OPMs are computed using historical volatility

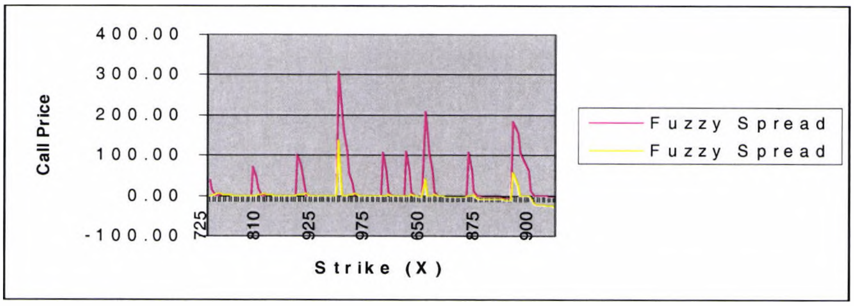

Figure 5-12: Fuzzy Spread for S&P 500 options based on historical and implied volatilities (pink corresponds to a fuzzy spread with implied volatility and the other with historical)

(Figure 5-12).

The Dual Choquet model does not give good results at all as expected. But the Choquet one is not bad actually and it is not affected by changes in volatility like the other models especially the binomial one, that is, it gives similar results for both historical and implied volatilities. It is actually quite close to the 'Dual Fuzzy' one. Generally, the behaviour of those two bounds is very similar to that of the \$/£ option bounds and the same analysis holds here.

5.5 Conclusion

In summary, we have applied the model to sample options from the NASDAQ, currency and Index option markets using Sugeno as well as Choquet integration. We have analyzed the performance of the model in the presence of implied versus historical volatility. We have also looked at how the binomial model is placed within those different approaches.

Within the sample considered, the model holds best for American NASDAQ options. For the European options, it holds better for forex options than for index options. The upper and lower bounds, given by the fuzzy and dual fuzzy values respectively, are acceptable for American options but quite wide sometimes for currency and Index options. Generally, the fuzzy model is more stable than its dual. But the dual fuzzy one is able to detect errors in data better than the fuzzy one does. It tends to behave erratically when the input parameters are wrong, for example, we have had data entry errors in the implied volatility and time to maturity for S&P 500 but, due to the unreasonable values returned by the dual fuzzy model, we have been able to detect them. On the technical side, Sugeno integration performs better than the Choquet integration approach within this framework. Finally, the results corresponding to the different volatility estimates are mixed.

As we have seen in the previous chapter, the model has been built on 'fuzzifying' the binomial model, which gives fair prices, to accommodate market uncertainty. The problem, as well as the specific solution approach (namely the use of Sugeno integration), that have been proposed in our research have never been considered before in the finance literature nor have they been considered in the fuzzy literature. However, the model does have practical applications and is not only a hypothetical model. It is quite interesting to see how fuzzy integration as well as fuzzy measures can provide sophisticated tools to tackle tough issues in finance.

In the chapters to follow, we will consider 'conventional' fuzzy option pricing approaches. In the next chapter, we look at option pricing in the presence of a fuzzy volatility. We use fuzzy set theory to model the latter and then use fuzzy differential calculus to solve the problem. Those tools are used more by fuzzy experts to tackle problems in finance and particularly in options pricing theory. Besides, the problem itself has been tackled several times by finance researchers using stochastic control problems and differential calculus so we will look at the tools and the solution approach fuzzy theory has to offer.

Chapter 6

Fuzzy Black-Scholes Model

In this chapter, we tackle the problem of uncertain volatility. Volatility is the input that is the most uncertain and the hardest to estimate in an Option Pricing Model. This uncertainty leads to uncertainty in determining a fair option value and gives rise to market incompleteness. Therefore, the issue of estimating the volatility that has to go into an Option Pricing Model has motivated researchers over the years to come up with various alternatives to deal with volatility.

This has often raised the question of which volatility estimate is the best to input into an Option Pricing Model. The usual approaches involve using stochastic volatility models, local volatility estimates or deterministic volatility surfaces. Generally, empirical estimates usually generate a single number representing a complete view of the market's volatility at a particular instance of time. More recently, several successful approaches in option pricing literature ([5],[69]) attempting to model volatility as an interval rather than as a precise number have been initiated. The general idea is to restrict the volatility path to a 'band' in which volatility can assume any value. This, in turn, generates no-arbitrage bounds on the value the option price can assume.

In this chapter, we consider the same concept but a different approach. Particularly, we use tools of Fuzzy Set Theory and Fuzzy Differential Calculus. We model uncertain volatility as a fuzzy number and work with its α-cuts to fuzzify the basic PDE and its solution. We do not really introduce a new model; rather, we amend the Black-Scholes OPM such that it accommodates a volatility band.

The fuzzy approach that we utilize here is different from the one we have utilized in the previous chapter. Our work here is more in line with existing Fuzzy Option Pricing literature such as that of Cherubini [30], Della Lunga and Cherubini ([31],[32]), and Muzzioli and Torricelli ([110],[112]). The general structure involves using Fuzzy Set Theory to derive a fuzzy PDE and then employing Fuzzy Differential Calculus to arrive at a fuzzy Black-Scholes equation. This is due to the fact that fuzzy volatility transmits fuzziness into the diffusion equation, which has to be solved now using Fuzzy Differential Calculus tools. Fuzzy Differential Calculus is a relatively new area. Several solution approaches have been suggested in the literature (for a review of those approaches, refer to Appendix D). In our model, we consider two solution approaches; we will first study the existence of the Buckley Feuring Solution (BFS) to the diffusion equation; if it does not exist, we will study the existence of the Seikkala Solution (SS). But, first, a literature overview of solving fuzzy PDEs, which are essential for solving our problem, is presented. The background to solving such equations lies in solving fuzzy equations and fuzzy differential equations, which are summarized in Appendix D as well (in fact, the reader is strongly advised to review those models before reading this chapter). There are different approaches by which a single equation can be solved and a solution does not always exist in the fuzzy case even though it can exist in the crisp case.

This model differs from the previous one in that it is parametric, in the sense that it depends on a market parameter α, and utilizes Fuzzy Set Theory rather than Fuzzy Measure Theory. This market parameter is assumed to summarize the degree of market completeness or incompleteness such that the Black-Scholes OPM can be used within a complete, as well as incomplete, market setting. So, by manipulating this factor, we will be able to allow the model to converge to the standard Black-Scholes model and to generate a fair option value under a complete market setting or to generate no-arbitrage bounds on the option value that envelope the standard Black-Scholes fair option value. This will be analyzed within a worst case/best case scenario framework. We will also be able to generate a defuzzified option value based on the Muzzioli-Torricelli defuzzification approach whereby a constrained optimization problem is performed on the fuzzy option value to deduce the 'implied' volatility bounds from the market.

In this chapter, we present the theoretical model and in the following chapter, we consider empirical applications. To this end, we carry out a comparison between the defuzzified fuzzy Black-Scholes option value and the standard Black-Scholes one. We also carry out a comparison between the uncertain volatility model [5], which is finding increasing popularity in the option pricing literature, and our Fuzzy Black-Scholes model. Usually, in such models, the bounds on volatility can be estimated subjectively, or based on historical behaviour, or deduced from the volatility smile. In our case, we consider two different approaches in establishing bounds on volatility in our empirical application. The first approach involves utilizing the implied volatility bounds deduced from the market by solving the constrained optimization problem. The second approach involves subjective estimates based on the implied Black-Scholes volatility such that it lies in between. The performance of the Fuzzy Black-Scholes model and the Uncertain Volatility one vis-a-vis each other and vis-a-vis the Black-Scholes model is analyzed.

This model extends contributions to both Fuzzy Theory and Option Pricing Theory with more emphasis on the latter. As far as Option Pricing Theory is concerned, we introduce Fuzzy Differential Calculus applications into this theory and provide an extension to existing approaches that model uncertain parameters. We also utilize an approach by which a volatility band can be deduced from the market, that is, we will be able to deduce between what values the market thinks the volatility should be confined by using the Muzzioli-Torricelli defuzzification approach [112]. We also provide a general framework, which can tackle both complete and incomplete market cases. Last but not least, we will be able to provide worst case and best case scenarios which are particularly useful in an uncertain environment. The advantage of a worst case scenario is that it protects an investor against adverse events.

As for the contribution to Fuzzy Theory, we present as comprehensive a documentation on Fuzzy Differential Calculus as possible. As we have mentioned earlier, Fuzzy Differential Calculus is a new area and the majority of research is scattered in various papers. So we try to combine various approaches together. Besides, fuzzy finance is still in its infancy and the majority of models in Fuzzy Theory that tackle applications in Finance Theory are trivial (apart from Fuzzy Option Pricing which has been tackled by 'finance', rather than 'fuzzy', experts). We contribute to extending a more practical approach in line with the increasing literature in fuzzy option pricing.

We start with a brief literature review of the various approaches proposed for solving fuzzy equations, fuzzy differential equations and fuzzy partial differential equations. Appendix D provides more elaboration on fuzzy derivatives. Then we move on to fuzzify the Black-Scholes PDE, starting with the basic PDE, and its solution, which is the Black-Scholes call option value. We consider two solution approaches and study the associated conditions that have to be satisfied. Finally, we establish the final form of the model and carry out a worst case/best case scenario analysis, with empirical applications to follow in the next chapter.

6.1 Solving Fuzzy Partial Differential Equations

6.1.1 Buckley-Feuring Solution

Appendix D presents a brief overview of solving fuzzy equations and fuzzy differential equations, which serve as the background to the solution approaches reviewed in this section. In solving fuzzy differential equations, Buckley and Feuring [21] introduce a new solution approach. In [20], they apply their new solution concept to solve fuzzy partial differential equations. They define the elementary fuzzy partial differential equation (FPDE) as

$\varphi(D_x, D_y)U(x,y) = F(x,y,k)$

$F(x,y,k)$ is a continuous function for $(x,y) \in I_1 \times I_2$ and $k = (k_1, ..., k_n)$ is a vector of constants,

$\varphi(D_x, D_y)$ is an operator which is a polynomial in D_x and D_y, the partials w.r.t. x and y respectively with constant coefficients.

$U(x,y)$ is a continuous function with continuous partials in x and y with $(x,y) \in I_1 \times I_2$,

where $I_1 = [0, M_1]$, $I_2 = [0, M_2]$, and $M_1, M_2 > 0$.

The equation can be subject to boundary conditions of different forms. The solution to the FPDE with boundary conditions is

$$U(x,y) = G(x,y,k,c),$$

where k and c are constants. But they are uncertain. Hence, we can model this uncertainty by substituting triangular fuzzy numbers for them. The solution approach goes as follows

(i) Fuzzify the crisp PDE to obtain the elementary FPDE and compute $\overline{F}$ from F using the extension principle ($\overline{F}(x,y,\overline{K})$). Also, U becomes $\overline{U}$, which maps $I_1 \times I_2$ to fuzzy numbers, i.e., $U(x,y) = \overline{Z}$, which is a fuzzy number. The fuzzified PDE is

$$\varphi(D_x, D_y)\overline{U}(x,y) = \overline{F}(x,y,\overline{K})$$

with fuzzy boundary conditions such as: $\overline{U}(0,y) = \overline{C}_1, \overline{U}(x,0) = \overline{C}_2, ..., \overline{U}(0,y) = \overline{g}_1(y;\overline{C}_4), \overline{U}(x,0) = \overline{f}_1(x;\overline{C}_5)$... We get $\overline{g}_i$ and $\overline{f}_i$ from g_i and f_i using the extension principle.

Let $\overline{Y}(x,y)[\alpha] = [y_1(x,y,\alpha), y_2(x,y,\alpha)]$,

$$\overline{F}(x,y)[\alpha] = [F_1(x,y,\alpha), F_2(x,y,\alpha)],$$

where $y_1(x,y,\alpha) = \min\{G(x,y,k,c) \mid k \in \overline{K}[\alpha], c \in \overline{C}[\alpha]\}$,

$$y_2(x,y,\alpha) = \max\{G(x,y,k,c) \mid k \in \overline{K}[\alpha], c \in \overline{C}[\alpha]\},$$

$$F_1(x,y,\alpha) = \min\{F(x,y,k) \mid k \in \overline{K}[\alpha]\},$$

$$F_2(x,y,\alpha) = \max\{F(x,y,k) \mid k \in \overline{K}[\alpha]\}.$$

(ii) Let $\Gamma(x,y,\alpha) = [\varphi(D_x, D_y)y_1(x,y,\alpha), \varphi(D_x, D_y)y_2(x,y,\alpha)]$

Assuming $y_i(x,y,\alpha)$ have continuous partials so that $\varphi(D_x, D_y)y_i(x,y,\alpha)$ is continuous for all (x,y) in $I_1 \times I_2$, and all $\alpha, i = 1,2$. If $\Gamma(x,y,\alpha)$ defines the α-cuts of a fuzzy number for all (x,y) in $I_1 \times I_2$, and all $\alpha, i = 1,2$, then $\overline{Y}(x,y)$ is differentiable and

$$\varphi(D_x, D_y)\overline{Y}(x,y)[\alpha] = \Gamma(x,y,\alpha)$$

Therefore, for $\overline{Y}(x,y)$ to be a BFS to the FPDE, the following conditions must hold

a) $\overline{Y}(x,y)$ is differentiable.

b) $\overline{Y}(x,y)$ satisfies $\varphi(D_x, D_y)\overline{Y}(x,y) = \overline{F}(x,y,\overline{K})$

c) $\overline{Y}(x,y)$ satisfies the boundary conditions.

(iii) Therefore, $\overline{Y}(x,y)$ is a BFS (without considering boundary conditions) if $\overline{Y}(x,y)$ is differentiable and $\varphi(D_x, D_y)\overline{Y}(x,y) = \overline{F}(x,y,\overline{K})$, in other words,

$$\varphi(D_x, D_y)y_1(x,y,\alpha) = F_1(x,y,\alpha)$$

$\varphi(D_x, D_y)y_2(x,y,\alpha) = F_2(x,y,\alpha)$

Sufficient conditions for the existence of BFS

These are the same as the ones imposed for solving FODEs. However, in this case, the authors ignore boundary conditions because each one needs separate studying and so they omit the constant c_i from the problem. Hence,

$U(x,y) = G(x,y,k)$

$\Longrightarrow \overline{Y}(x,y) = \overline{G}(x,y,\overline{K})$.

6.1.2 Seikkala Solution

If $\overline{Y}(x,y)$ is not a BFS, then we have to look for the Seikkala Solution (SS). Let $\overline{U}(x,y)[\alpha] = [u_1(x,y,\alpha), u_2(x,y,\alpha)]$. We have to solve the following system of FPDEs

$\varphi(D_x, D_y)u_1(x,y,\alpha) = F_1(x,y,\alpha)$

$\varphi(D_x, D_y)u_2(x,y,\alpha) = F_2(x,y,\alpha)$

with boundary conditions, e.g. $\overline{U}(0,y) = \overline{C}_1, \overline{U}(M_1,0) = \overline{C}_2$,

$u_1(0,y,\alpha) = c_{11}(\alpha)$,

$u_2(0,y,\alpha) = c_{12}(\alpha)$,

$u_1(M_1,y,\alpha) = c_{21}(\alpha)$,

$u_2(M_2,y,\alpha) = c_{22}(\alpha)$.

So $u_i(x,y,\alpha)$ is a solution to the system if $[u_1(x,y,\alpha), u_2(x,y,\alpha)]$ define the α-cut of a fuzzy number. Then, $\overline{U}(x,y)$ is the SS.

Therefore, we have also seen various approaches to solving Fuzzy Equations, DEs and PDEs. The next step will be applying these approaches to the heat diffusion equation and, derive a fuzzy Black-Scholes OPM.

6.2 The Crisp Model

The model is the basic Black-Scholes OPM but adapted to accommodate uncertain volatility. So rather than using one precise value for volatility, we allow the use of an interval bounded by an upper and lower volatility value that bound the true volatility value. Those bounds can be subjective or inferred from historical data. They lead to intervals of option prices that are dependent on a market completeness factor. The original PDE is,

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}C(S,t) + \frac{\partial C(S,t)}{\partial t} + (r - D_0)S\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \quad (6.1)$$

and its solution is the Black-Scholes equation,

$$C(S,t) = Se^{-D_0 t}N(d_1) - Ee^{-r\tau}N(d_2) \quad (6.2)$$

where $d_1 = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$, and $d_2 = \frac{\ln(\frac{S}{E})+(r-D_0-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}$.

The first step would be to fuzzify this PDE and then move on to apply fuzzy differential calculus tools.

6.3 Fuzzification of PDE

Assuming volatility is the only variable that is uncertain and can be modelled by fuzzy set theory, we will fuzzify it by substituting fuzzy triangular numbers for it, whereby $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with α-cuts $\overline{\sigma} = [\sigma_1(\alpha), \sigma_2(\alpha)]$, where

$\sigma_1(\alpha) = (\sigma_2 - \sigma_1)\alpha + \sigma_1$,

$\sigma_2(\alpha) = (\sigma_2 - \sigma_3)\alpha + \sigma_3$.

This means that d_1 and d_2 are also fuzzy and can be defined in terms of α-cuts. However, to write down their equations in terms of the α-cuts of the volatility, we need to know the sign of their first derivatives w.r.t. volatility. Hence,

$\frac{\partial d_1}{\partial \sigma} = \sqrt{\tau} - \frac{\ln(\frac{S}{E})+\left(r-D_0+\frac{\sigma^2}{2}\right)\tau}{\sigma^2\sqrt{\tau}}$,

which is positive iff $\frac{\sigma^2\tau}{2} > (r - D_0)\tau + \ln\left(\frac{S}{E}\right)$, meaning that the α-cuts of $d_1 = [d_{11}(\alpha), d_{12}(\alpha)]$ can be written as follows

$d_{11}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_1^2(\alpha)}{2})\tau}{\sigma_1(\alpha)\sqrt{\tau}}$,

$d_{12}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_2^2(\alpha)}{2})\tau}{\sigma_2(\alpha)\sqrt{\tau}}$,

otherwise, i.e. when $\frac{\partial d_1}{\partial \sigma}$ is negative,

$d_{11}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_2^2(\alpha)}{2})\tau}{\sigma_2(\alpha)\sqrt{\tau}}$,

$d_{12}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_1^2(\alpha)}{2})\tau}{\sigma_1(\alpha)\sqrt{\tau}}$.

Similarly,

$\frac{\partial d_2}{\partial \sigma} = -\sqrt{\tau} - \frac{\ln\left(\frac{S}{E}\right)+\left(r-D_0-\frac{\sigma^2}{2}\right)\tau}{\sigma^2\sqrt{\tau}}$,

which is positive iff $\frac{\sigma^2\tau}{2} < -(r - D_0)\tau - \ln\left(\frac{S}{E}\right)$, meaning that the α-cuts of $d_2 = [d_{21}(\alpha), d_{22}(\alpha)]$ can be written as follows

$d_{21}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_1^2(\alpha)}{2})\tau}{\sigma_1(\alpha)\sqrt{\tau}}$,

$d_{22}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_2^2(\alpha)}{2})\tau}{\sigma_2(\alpha)\sqrt{\tau}}$,

otherwise, i.e. when $\frac{\partial d_1}{\partial \sigma}$ is negative,

$d_{21}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_2^2(\alpha)}{2})\tau}{\sigma_2(\alpha)\sqrt{\tau}}$,

$d_{22}(\alpha) = \frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_1^2(\alpha)}{2})\tau}{\sigma_1(\alpha)\sqrt{\tau}}$.

Therefore, the equations for $d_1$and d_2 depend on the moneyness and the maturity of the option. Table 1C in Appendix C shows the bounds for the fuzzy volatility estimated from the implied volatilities of the bid and ask quotations. This is effectively like saying 'The volatility is around the mid value.' So, given that volatility is a triangular fuzzy number by definition, we choose the lower and upper limits to be 2% higher and 2% lower respectively (of course, the 2% is only an example and we can have variations of it). Then, we compute the α-cuts of the fuzzy volatility as shown above and substitute them in $d_1$and d_2. Alternatively, we can work with the fuzzy volatility itself, rather than fuzzy $d_1$and d_2.

To fuzzify the PDE, let us first write it in the following form:

$$\varphi(D_S, D_t)U(S,t) = F(S,t,\sigma), \tag{6.3}$$

we get:

$$\frac{\partial^2}{\partial S^2}C(S,t) = \frac{2}{\sigma^2 S^2}[rC(S,t) - \frac{\partial C(S,t)}{\partial t} - (r - D_0)S\frac{\partial C(S,t)}{\partial S}], \tag{6.4}$$

hence,

$$\varphi(D_S, D_t) = D_S D_t, \tag{6.5}$$

$$U(S,t) = C(S,t). \tag{6.6}$$

Let $S \in I_1 = [0, M_1]$ and $t \in I_2 = [0, M_2]$, where $M_1 > 0$ and $M_2 > 0$. Now, we can fuzzify it. Using the extension principle, compute $\bar{F}(S,t,\bar{\sigma})$ from $F(S,t,\sigma)$. The function $C(S,t)$ becomes $\bar{C}(S,t)$ so that $\bar{C}$ maps $I_1 \times I_2$ to fuzzy numbers, that is, $\bar{C}(S,t) = \bar{Z}$ where $\bar{Z}$ is a fuzzy number. We get,

$$F(S,t,\sigma) = \frac{2}{\sigma^2 S^2}[rC(S,t) - \frac{\partial C(S,t)}{\partial t} - (r - D_0)S\frac{\partial C(S,t)}{\partial S}], \tag{6.7}$$

$$C(S,t) = G(S,t,\sigma) = Se^{-D_0 t}N(d_1) - Ee^{-r\tau}N(d_2). \tag{6.8}$$

We get the following fuzzy PDE,

$$\varphi(D_S, D_t)\bar{C}(S,t) = \bar{F}(S,t,\bar{\sigma}). \tag{6.9}$$

Now, fuzzify G by computing $\bar{Y}(S,t) = \bar{G}(S,t,\bar{\sigma})$ using the extension principle. Let

$\overline{Y}(S,t)[\alpha] = [y_1(S,t,\alpha), y_2(S,t,\alpha)]$,

$\overline{F}(S,t,\overline{\sigma})[\alpha] = [F_1(S,t,\alpha), F_2(S,t,\alpha)]$,

where

$y_1(S,t,\alpha) = \min\{G(S,t,\sigma) \mid \sigma \in \overline{\sigma}[\alpha]\}$,

$y_2(S,t,\alpha) = \max\{G(S,t,\sigma) \mid \sigma \in \overline{\sigma}[\alpha]\}$,

$F_1(S,t,\overline{\sigma},\alpha) = \min\{F(S,t,\sigma) \mid \sigma \in \overline{\sigma}[\alpha]\}$,

$F_2(S,t,\overline{\sigma},\alpha) = \max\{F(S,t,\sigma) \mid \sigma \in \overline{\sigma}[\alpha]\}.$,

for all S, t and α. Define

$$\Gamma(S,t,\alpha) = [\varphi(D_S, D_t)y_1(S,t,\alpha), \varphi(D_S, D_t)y_2(S,t,\alpha)], \tag{6.10}$$

If $\Gamma(S,t,\alpha)$ defines the α-cuts of a fuzzy number, then $\overline{Y}(S,t)$ is differentiable and we can write

$$\varphi(D_S, D_t)\overline{Y}(S,t)[\alpha] = \Gamma(S,t,\alpha). \quad (6.11)$$

6.4 Fuzzy Solutions

6.4.1 Buckley-Feuring Solution

Recall that for $\overline{Y}(S,t)$ to be a BFS (Buckley-Feuring Solution), we should have,

(i) $\overline{Y}(S,t)$ is differentiable,

(ii) $\overline{Y}(S,t)$ satisfies the PDE $\varphi(D_S, D_t)\overline{Y}(S,t)[\alpha] = \overline{F}(S,t,\overline{\sigma},\alpha)$ i.e.

$\varphi(D_S, D_t)y_1(S,t)[\alpha] = F_1(S,t,\overline{\sigma},\alpha)$,

$\varphi(D_S, D_t)y_2(S,t)[\alpha] = F_2(S,t,\overline{\sigma},\alpha)$,

(iii) $\overline{Y}(S,t)$ satisfies the boundary conditions:

$C(0,t) = 0$,

$C(S,t) = \max(S-E, 0)$,

$C(S,t) \approx S$ as $S \to \infty$.

Recall also that the sufficient condition for the existence of a BFS, assuming that $\overline{Y}(S,t)$ is differentiable and ignoring boundary conditions, is

$\frac{\partial G}{\partial \sigma}\frac{\partial F}{\partial \sigma} > 0$,

otherwise, $\overline{Y}(S,t)$ is not a BFS and we have to look for the Seikkala Solution SS. Therefore, we will proceed by checking for the existence of the BFS and, if it does not exist, we move on to SS.

For the solution to be a BFS, it has to satisfy its sufficient condition. So after fuzzifying F and G, we find their first derivatives w.r.t. σ

(i) $\frac{\partial G}{\partial \sigma} = Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial \sigma}$

$= Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r\tau}N'(d_2)\frac{\partial d_1}{\partial \sigma} + Ee^{-r\tau}N'(d_2)\sqrt{\tau}$,

But $Se^{-D_0\tau}N'(d_1) = Ee^{-r\tau}N'(d_2)$ (substituting $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ and the equations for d_1 and d_2), we get

$\frac{\partial G}{\partial \sigma} = Ee^{-r\tau}N'(d_2)\sqrt{\tau} > 0$ always. Hence, for BFS to exist, $\frac{\partial F}{\partial \sigma}$ must be positive.

(ii) $\frac{\partial F}{\partial \sigma} = \frac{-4\sigma S^2}{\sigma^4 S^4}[rC - \frac{\partial C}{\partial t} - (r - D_0)S\frac{\partial C}{\partial S}] + \frac{2}{\sigma^2 S^2}[r\frac{\partial C}{\partial \sigma} - \frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial t}) - (r - D_0)S\frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial S})]$

Calculating each term separately, we find

(a) vega $\frac{\partial C}{\partial \sigma} = Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial \sigma}$

$= Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r\tau}N'(d_2)\frac{\partial d_1}{\partial \sigma} + Ee^{-r\tau}N'(d_2)\sqrt{\tau}$

$= Se^{-D_0\tau}N'(d_1)\sqrt{\tau}$,

(b) delta $\frac{\partial C}{\partial S} = e^{-D_0\tau}N(d_1) + Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial S}$

$= e^{-D_0\tau}N(d_1) + Se^{-D_0\tau}N'(d_1)[\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}]$

But $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$

$\Longrightarrow \frac{\partial C}{\partial S} = e^{-D_0\tau}N(d_1)$,

(c) theta $\frac{\partial C}{\partial t} = D_0Se^{-D_0\tau}N(d_1) + Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial t} - rEe^{-r\tau}N(d_2) - Ee^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial t}$

$= D_0Se^{-D_0\tau}N(d_1) + Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial t} - rEe^{-r\tau}N(d_2) - Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial t} + Se^{-D_0\tau}N'(d_1)(\frac{\sigma}{2\sqrt{\tau}})$

but $-rEe^{-r\tau}N(d_2) = -rSe^{-D_0\tau}N(d_1) + rC$

$\Longrightarrow \frac{\partial C}{\partial t} = rC - Se^{-D_0\tau}N(d_1)[r - D_0] - \frac{S\sigma N'(d_1)e^{-D_0\tau}}{2\sqrt{\tau}}$

$= rC - Se^{-D_0\tau}N(d_1)[r - D_0] - \frac{\sigma}{2\tau}\frac{\partial C}{\partial \sigma}$

$= rC - S(r - D_0)\frac{\partial C}{\partial S} - \frac{\sigma}{2\tau}\frac{\partial C}{\partial \sigma}$,

(d) $\frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial t}) = r\frac{\partial C}{\partial \sigma} - S(r - D_0)\frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial S}) - \frac{1}{2\tau}\frac{\partial C}{\partial \sigma} - \frac{\sigma}{2\tau}\frac{\partial^2 C}{\partial \sigma^2}$

but $\frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial S}) = e^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \sigma}$,

and $\frac{\partial d_1}{\partial \sigma} = \frac{-\sqrt{\tau}\ln(S/E)}{\sigma^2\tau} + \frac{\sigma\tau\sigma\sqrt{\tau} - (r - D_0 + \sigma^2/2)\tau\sqrt{\tau}}{\sigma^2\tau}$

$= \frac{-1}{\sigma}[\frac{\ln(S/E)}{\sigma\sqrt{\tau}} - \frac{(\sigma^2/2 - r + D_0)\sqrt{\tau}}{\sigma}]$

$= \frac{-d_2}{\sigma}$

$\Longrightarrow \frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial t}) = -e^{-D_0\tau}N'(d_1)\frac{d_2}{\sigma}$

$= \frac{-d_2}{\sigma S\sqrt{\tau}}\frac{\partial C}{\partial \sigma}$,

and $\frac{\partial^2 C}{\partial \sigma^2} = \frac{\partial}{\partial \sigma}(Se^{-D_0\tau}N'(d_1)\sqrt{\tau})$

$= Se^{-D_0\tau}\sqrt{\tau}N''(d_1)\frac{\partial d_1}{\partial \sigma}$

$= -Se^{-D_0\tau}\sqrt{\tau}N''(d_1)\frac{d_2}{\sigma}$

but $N''(d_1) = -d_1N'(d_1)$

$\Longrightarrow \frac{\partial^2 C}{\partial \sigma^2} = \frac{d_1 d_2}{\sigma}\frac{\partial C}{\partial \sigma}$

$\therefore \frac{\partial}{\partial \sigma}(\frac{\partial C}{\partial t}) = r\frac{\partial C}{\partial \sigma} + S(r - D_0)\frac{d_2}{\sigma S\sqrt{\tau}}\frac{\partial C}{\partial \sigma} - \frac{1}{2\tau}\frac{\partial C}{\partial \sigma} - \frac{\sigma}{2\tau}\frac{d_1 d_2}{\sigma}\frac{\partial C}{\partial \sigma}$

$= [r + (r - D_0)\frac{d_2}{\sigma\sqrt{\tau}} - \frac{1}{2\tau} - \frac{d_1 d_2}{2\tau}]\frac{\partial C}{\partial \sigma}$

Substituting in $\frac{\partial F}{\partial \sigma}$, we get $\frac{\partial F}{\partial \sigma} = \frac{-4}{\sigma^3 S^2}[rC - rC + S(r - D_0)\frac{\partial C}{\partial S} + \frac{\sigma}{2\tau}\frac{\partial C}{\partial \sigma} - (r - D_0)S\frac{\partial C}{\partial S}] + \frac{2}{\sigma^2 S^2}[r\frac{\partial C}{\partial \sigma} - r\frac{\partial C}{\partial \sigma} - (r - D_0)\frac{d_2}{\sigma\sqrt{\tau}}\frac{\partial C}{\partial \sigma} + \frac{1}{2\tau}\frac{\partial C}{\partial \sigma} + \frac{d_1 d_2}{2\tau}\frac{\partial C}{\partial \sigma} + (r - D_0)S\frac{d_2}{\sigma S\sqrt{\tau}}\frac{\partial C}{\partial \sigma}]$

$= \frac{-4}{\sigma^3 S^2}[\frac{\sigma}{2\tau}\frac{\partial C}{\partial \sigma}] + \frac{2}{\sigma^2 S^2}[\frac{1}{2\tau}\frac{\partial C}{\partial \sigma} + \frac{d_1 d_2}{2\tau}\frac{\partial C}{\partial \sigma}]$

$= \frac{-1}{\sigma^2 S^2 \tau}\frac{\partial C}{\partial \sigma} + \frac{d_1 d_2}{\sigma^2 S^2 \tau}\frac{\partial C}{\partial \sigma}$

$= (d_1 d_2 - 1)(\frac{1}{\sigma^2 S^2 \tau})\frac{\partial C}{\partial \sigma}$

Recall that for BFS to exist, $\frac{\partial F}{\partial \sigma} > 0$ must hold. But $\frac{\partial F}{\partial \sigma} > 0$ iff $d_1 d_2 > 1$. There are three cases to consider:

(i) at-the-money (call) options: $S \approx E \Longrightarrow \ln(S/E) \approx 0$

$\Longrightarrow d_1 = \frac{(r - D_0 + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$,

$d_2 = \frac{(r - D_0 - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$

$\Longrightarrow d_1 d_2 = \frac{(r - D_0 + \frac{\sigma^2}{2})(r - D_0 - \frac{\sigma^2}{2})\tau}{\sigma^2} = \frac{[(r - D_0)^2 - \sigma^4/4]\tau}{\sigma^2}$

$\therefore d_1 d_2 > 1$ iff $\frac{[(r - D_0)^2 - \sigma^4/4]\tau}{\sigma^2} > 1 \Longrightarrow (\frac{r - D_0}{\sigma})^2\tau - \frac{\sigma^2}{2}\tau > 0$

But if $\sigma >>$ ($\sigma \to \infty$), the first term $\to 0$ and the $d_1 d_2 \to -\infty$

$\Longrightarrow \frac{\partial F}{\partial \sigma} < 0$ and BFS does not exist.

However, if $\sigma << \Longrightarrow (\frac{r - D_0}{\sigma})^2 >>$ and $\frac{\sigma^2}{2}\tau <<$, then $d_1 d_2 > 1$

$\Longrightarrow \frac{\partial F}{\partial \sigma} > 1$ and BFS exists. Therefore, we have a change of sign for at the money call options.

(ii) out-of-the-money options: $S < E \Longrightarrow \ln(S/E) < 0$

$d_1 d_2 = d_1(d_1 - \sigma\sqrt{\tau}) = d_1^2 - d_1\sigma\sqrt{\tau} + \frac{\sigma^2\tau}{4} - \frac{\sigma^2\tau}{4}$

$= (d_1 - \frac{\sigma\sqrt{\tau}}{2})^2 - \frac{\sigma^2\tau}{4}$

$= \left[\frac{\ln(S/E) + \left(r - D_0 + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right]^2 - \frac{\sigma^2\tau}{4}$

$= \left[\frac{\ln(S/E) + \left(r - D_0 + \frac{\sigma^2}{2}\right)\tau - \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right]^2 - \frac{\sigma^2\tau}{4}$

$= \left[\frac{\ln(S/E) + r\tau - D_0\tau}{\sigma\sqrt{\tau}}\right]^2 - \frac{\sigma^2\tau}{4}$

$\therefore d_1 d_2 > 1$ iff $\left[\frac{\ln(S/E)+r\tau-D_0\tau}{\sigma\sqrt{\tau}}\right]^2 - \frac{\sigma^2\tau}{4} > 1$

If $\sigma \to \pm\infty$, the first term becomes infinitesimal while the second term becomes very large meaning that $\frac{\partial F}{\partial \sigma} < 0$ and, hence, the BFS does not exist. However, if $\sigma <<$, the first term becomes a positive very large number while the second becomes infinitesimal, which means that $\frac{\partial F}{\partial \sigma} > 0$ and BFS would exist. Moving on to the effect of the stock price and the exercise price, we know that for out-of-the-money calls $S < E \Longrightarrow \frac{S}{E} \in \,]0,1[\, \Longrightarrow \ln(\frac{S}{E}) \in \,]-\infty, 0[$. So for deep out-of-the-money calls, $S << E \Longrightarrow \ln\frac{S}{E} \to -\infty \Longrightarrow \left(\ln\frac{S}{E}\right)^2 >>$ and $d_1 d_2 > 1 \Longrightarrow \frac{\partial F}{\partial \sigma} > 0$ and BFS exists unless $\sigma >>$, where the case at hand will have to be studied individually. Therefore, we again have a change of sign.

As it is the case for in-the-money calls, $S > E \Longrightarrow \frac{S}{E} \in \,]1,\infty[\, \Longrightarrow \ln\frac{S}{E} \in \,]0,\infty[$. Therefore, the same analysis as in the out-of-the-money calls case holds here and, hence, we have a change of sign once again. However, the numerator in the first term (in $d_1 d_2$) for in-the-money calls will always be larger than that for out-of-the-money ones.

When the BFS exists, the solution to the fuzzy PDE is

$$Y_i(S,t,\alpha) = Se^{-D_0\tau}N\left(\frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}\right) - Ee^{-r\tau}N\left(\frac{\ln(\frac{S}{E})+(r-D_0-\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}\right).$$

6.4.2 Seikkala Solution

For the case where BFS does not exist, we have to look for the SS. Let $\overline{C}(S,t)[\alpha] = [c_1(S,t,\alpha), c_2(S,t,\alpha)]$. Hence, we have to solve the following system of partial differential equations,

$$\frac{\partial^2}{\partial S^2}C_1(S,t,\alpha) = \frac{2}{\sigma_1^2(\alpha)S^2}[rC_1(S,t,\alpha) - \frac{\partial C_1(S,t,\alpha)}{\partial t} - (r-D_0)S\frac{\partial C_1(S,t,\alpha)}{\partial S}],$$
$$\frac{\partial^2}{\partial S^2}C_2(S,t,\alpha) = \frac{2}{\sigma_2^2(\alpha)S^2}[rC_2(S,t,\alpha) - \frac{\partial C_2(S,t,\alpha)}{\partial t} - (r-D_0)S\frac{\partial C_2(S,t,\alpha)}{\partial S}],$$

subject to

$C_i(0,t) = 0,$

$C_i(S,t) = \max(S-E, 0),$

$C_i(S,t) \approx S$ as $S \to \infty,$

$i = 1, 2.$

The solution is

$C_i(S,t,\alpha) = Se^{-D_0\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}) - Ee^{-r\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0-\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}),$

$i = 1, 2.$

SS exists if

$[c_1(S,t,\alpha), c_2(S,t,\alpha)]$

defines the α-cuts of a fuzzy number. Since $C_i(S,t,\alpha)$ are continuous and $C_1(S,t,1) = C_2(S,t,1)$, we only have to check if $\frac{\partial C_1}{\partial \alpha} > 0$ and $\frac{\partial C_2}{\partial \alpha} < 0$. Starting with $\frac{\partial C_1}{\partial \alpha}$, we get

$$\frac{\partial C_1}{\partial \alpha} = Se^{-D_0\tau}N'(d_1)\frac{\partial d_1}{\partial \alpha} - Ee^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial \alpha},$$

$$\frac{\partial d_i}{\partial \alpha} = \frac{\partial d_i}{\partial \sigma_i(\alpha)}\frac{\partial \sigma_i(\alpha)}{\partial \alpha}; \qquad i = 1, 2,$$

$$\frac{\partial d_1}{\partial \sigma} = \sqrt{\tau} - \frac{\ln(\frac{S}{E})+\left(r-D_0+\frac{\sigma^2}{2}\right)\tau}{\sigma^2\sqrt{\tau}},$$

$$\frac{\partial d_2}{\partial \sigma} = -\sqrt{\tau} - \frac{\ln(\frac{S}{E})+\left(r-D_0-\frac{\sigma^2}{2}\right)\tau}{\sigma^2\sqrt{\tau}},$$

$$\frac{\partial \sigma_1(\alpha)}{\partial \alpha} = \sigma_2 - \sigma_1 > 0,$$

$$\frac{\partial \sigma_2(\alpha)}{\partial \alpha} = \sigma_2 - \sigma_3 < 0.$$

After some calculation, we get

$$\frac{\partial d_1}{\partial \alpha} = \left[\frac{\frac{\sigma^2}{2}\tau+D_0\tau-\left[\ln(\frac{S}{E})+r\tau\right]}{\sigma^2\sqrt{\tau}}\right](\sigma_2-\sigma_1),$$

$$\frac{\partial d_2}{\partial \alpha} = \left[\frac{-\frac{\sigma^2}{2}\tau+D_0\tau-\ln(\frac{S}{E})-r\tau}{\sigma^2\sqrt{\tau}}\right](\sigma_2-\sigma_1).$$

Substituting in $\frac{\partial C_1}{\partial \alpha}$ and using $Se^{-D_0\tau}N'(d_1) = Ee^{-r\tau}N'(d_2)$, we get

$$\frac{\partial C_1}{\partial \alpha} = Se^{-D_0\tau}N'(d_1)\left[\frac{\partial d_1}{\partial \alpha} - \frac{\partial d_2}{\partial \alpha}\right]$$
$$= Se^{-D_0\tau}N'(d_1)(\sigma_2-\sigma_1)\sqrt{\tau} > 0.$$

Similarly, for $\frac{\partial C_1}{\partial \alpha}$ and working with $\sigma_2 - \sigma_3$, we get

$$\frac{\partial C_1}{\partial \alpha} = Se^{-D_0\tau}N'(d_1)(\sigma_2-\sigma_3)\sqrt{\tau} < 0.$$

Therefore, $C_1(S,t,1) = C_2(S,t,1)$ do define the α-cuts of a fuzzy number and so SS always exists.

6.5 Algorithm

1. Start with BFS.

1.1. Fuzzify the PDE by computing $\bar{F}$ and $\bar{G}$ from F and G using the extension principle.

1.2. Test the sufficient condition for the existence of the BFS: since $\frac{\partial G}{\partial \sigma} > 0$ always, the BFS exists iff $\frac{\partial F}{\partial \sigma} > 0$. But $\frac{\partial F}{\partial \sigma} > 0$, only if $d_1 d_2 > 1$. So if this does not hold, look for the SS. If BFS exists, then the solution has the form

$$\bar{Y}_i(S,t,\alpha) = Se^{-D_0\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0+\frac{\bar{\sigma}^2(\alpha)}{2})\tau}{\bar{\sigma}(\alpha)\sqrt{\tau}}) - Ee^{-r\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0-\frac{\bar{\sigma}^2(\alpha)}{2})\tau}{\bar{\sigma}(\alpha)\sqrt{\tau}}).$$

2. If BFS does not exist, we have to check for the existence of the Seikkala solution.

2.1. Write the solution in terms of α-cuts and derive the system of Partial Differential Equations.

2.2. Check if those α-cuts define the α-cuts of a fuzzy number by testing whether $\frac{\partial C_1}{\partial \alpha} > 0$ and $\frac{\partial C_2}{\partial \alpha} < 0$. If they do, then the SS exists and the solution is defined as

$$C_i(S,t,\alpha) = Se^{-D_0\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0+\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}) - Ee^{-r\tau}N(\frac{\ln(\frac{S}{E})+(r-D_0-\frac{\sigma_i^2(\alpha)}{2})\tau}{\sigma_i(\alpha)\sqrt{\tau}}),$$

$i = 1, 2$.

6.6 Analysis

This model can be used within a complete as well as an incomplete market setting depending on the level of α assumed. It returns no-arbitrage bounds on the call value or a range of the possible no-arbitrage prices a call value can assume. However, it is also possible to defuzzify the call value and get one crisp number, which is independent of α, as we will see in the following chapter.

α can be viewed as summarizing the level of market completeness. Other plausible interpretations of α are confidence level or information level. It can assume any value between 0 and 1. From a fuzziness point of view, the case where $\alpha = 0$ corresponds to complete fuzziness while that where $\alpha = 1$ corresponds to complete 'crispness.' Within the context of our model, $\alpha = 0$ corresponds to 'total' market incompleteness, or highest level of uncertainty, and widest bounds on the uncertain parameter and the corresponding option price while $\alpha = 1$ corresponds to market completeness, or no uncertainty, and the bounds on the uncertain parameter and option price converge to one certain value. As α increases, the 'level' of market completeness increases and the bounds converge or tighten. The case where it assumes a value of 1 is the Black-Scholes case, hence, our interpretations of α as measuring the level of market completeness since the Black-Scholes OPM is assumed to hold within a complete market setting.

Using this framework, it is possible to carry out a historical study of the values

α has assumed whereby the Black-Scholes call value is assumed to be the fair price and observable prices are only deviations of it. However, for the purpose of calculating the no-arbitrage bounds, α can be estimated subjectively or historically. For emerging markets, a plausible value of α should be below 0.5 while for more developed markets, it should be above 0.5 and approaching one as the market approaches 'total completeness.'

The major contribution of this model is that it gives no-arbitrage range for option values in an uncertain environment such that anything inside this range is possible, without having to assume anything about its probability of occurrence, while anything outside it is impossible or arbitrageable. But it is still possible to get one crisp number even though, as we will see in the following section, it is possible to argue that the value-added does not really justify the computational cost involved.

Therefore, in the presence of an uncertain parameter, it is best to deal with a range of possible option prices. So, given that any value inside the range is possible and that we do not have any information about its probability of occurrence, it is plausible to work with a best case and a worst case scenario. Of course, it is not sensible to assume a best case scenario since we can incur huge losses if events turn around negatively. A worst case scenario is reasonable to assume especially in the presence of high uncertainty since anything better will be good and even if such a state does actualize, we would be properly hedged assuming we had properly discounted it. On the downside, we may suffer from an opportunity loss since we do not discount or use all information. The underlying assumption of the previous statement is that a worst case scenario entails or requires the least information. In such a case, our model can be used as a correction by manipulating the level of α as we will see in the following chapter.

Assuming very high uncertainty and from an objective point of view, $\alpha = 0$ corresponds to a worst case scenario since in that case, we have complete fuzziness and the parameter under consideration, such as volatility in this case, is 'maximally' uncertain or fuzziest. The bounds we get on such a parameter and the corresponding bounds generated on the option price itself are widest. On the other hand, $\alpha = 1$ corresponds to a best case scenario since we do not have uncertainty and the uncertain parameter's bounds converge to its most possible value. In this instance, the range of option prices also converge to one value, which is the Black-Scholes market. In the case where uncertainty is not, or is not expected to be, very high, the expert can manipulate α to the level he/she finds appropriate.

Thus, the best way to utilize this model depends on the expert. If the expert desires one value and does not wish to lose information, it is possible to use a defuzzification process with minimal loss of information. If the level of uncertainty is very high, it is best to assume the worst case/best case scenario analysis presented above. On the other hand, the expert can always manipulate α to the level of uncertainty he/she thinks is appropriate.

6.7 Conclusion

Therefore, in this chapter, we have presented a model that prices options in the presence of uncertain volatility, which can also be applied in the presence of other uncertain parameters, such as uncertain dividends or interest rates. Option Pricing Theory has tackled this matter using standard tools including stochastic control methods and Partial Differential Equations, as we have seen in Chapter 3 and will talk more in detail about one such approach, namely the Uncertain Volatility Model [5], in the following chapter. In our model, we have used a different approach, which involves Fuzzy Theory.

We model the uncertain volatility as a fuzzy number. This fuzziness generates fuzziness in the basic Black-Scholes diffusion equation, which has to be solved using Fuzzy Differential Calculus, and its solution, which is the call value. This means that we obtain a range of possible values for the option price, which depends on a parameter α summarizing the level of market completeness. Existing Option Pricing Models that deal with uncertain parameters agree on the fact that an uncertain parameter has to generate bounds on option pricing, in other words, in the presence of uncertainty, we no longer get one no-arbitrage value, rather we get no-arbitrage bounds for the option price in between which any value is possible. In such instances, it is best to work with a worst case/best case scenario analysis.

However, in our case, it is possible to defuzzify the fuzzy price to get a crisp number or one value, that best summarizes information, instead of a range. Besides, our model is quite simpler than other equivalent models in Option Pricing Theory and allows the expert to manipulate the width of the bounds since those bounds tend to be quite wide as most of the results given by the latter models show. Finally, it accommodates more than one uncertain parameter at once easily, which is not offered by other approaches.

This model is also different from the Fuzzy Binomial Model that we have presented in Chapter 4 even though both models tackle uncertainty and utilize some form of Fuzzy Theory. The Fuzzy Binomial Model deals with uncertainty using fuzzy measures of a fuzzy event but this model does not really consider measures; rather, it works within the same framework as the crisp approach but introduces fuzziness into it. So it implicitly assumes working with the probability of a fuzzy event with no explicit definition.

The following chapter will present an empirical application and a comparison between this model and the Black-Scholes OPM as well as the Uncertain Volatility Model [5]. We will also look at the empirical application of the worst case/best case scenario analysis and the behaviour of the bounds as α is varied.

Chapter 7

The Fuzzy Black-Scholes Model: Empirical Applications

In the previous chapter, we have seen a fuzzified form of the Black-Scholes OPM that accommodates uncertain parameters. We have particularly considered uncertain volatility and fuzzified the Black-Scholes model accordingly so that we do not have the restriction of the constant volatility assumption anymore. In this chapter, we consider empirical applications as well as comparisons to existing Option Pricing Models.

In the first part, we defuzzify the fuzzy call option value using the Muzzioli-Torricelli defuzzification approach [112] to get a crisp number independent of α. This approach facilitates analyzing the predictive power of the fuzzy Black-Scholes OPM as well as comparing it to the standard Black-Scholes equation. In the second part, we compare it to the Uncertain Volatility Model [5], which is the most popular approach in modelling uncertain volatility in the existing option pricing literature.

7.1 Fuzzy Black-Scholes OPM vs Black-Scholes OPM

7.1.1 Methodology

In this section, we will be comparing the forecasting ability of the Fuzzy Black-Scholes (FBS) model to that of the Black-Scholes (BS) one. This is a two-fold process. The first stage involves deducing the implied volatilities from these models using a training set. The second stage involves using those implied volatilities in testing sets by matching strikes and expirations to forecast option prices. The results of the two models will be compared on the basis of the RMSE and RMSE as percentage of price.

In the FBS case, we need to get three values $(\sigma_1, \sigma_2, \sigma_3)$ for the volatility (section 6.3) that describe the fuzzy number by training the model on a given data set. Then, we have to input those values into the fuzzy model and get a defuzzified forecasted value for a different out-of-sample test. For the BS model, we get the implied volatility using the same training data set we have used for training FBS and then use this implied volatility to forecast prices using the testing data set. We compare the implied volatilities as well as the root mean square error, RMSE, and RMSE as percent of price of both models.

We follow the algorithm outlined in section 6.5 in the previous chapter to write the coding for FBM. To be able to obtain the implied volatilities, we use the Muzzioli-Torricelli defuzzification approach [112]. The authors solve the following problem,

$$\min_{P_T} D(C, P_T) = \int_0^1 (\underline{C} - P_T)^2 \, d\alpha + \int_0^1 \left(\overline{C} - P_T\right)^2 d\alpha,$$

where D is the metric distance between a crisp number and a fuzzy number, and from the first order condition, it follows that

$$P_T = \frac{1}{2} \int_0^1 \left(\underline{C} + \overline{C}\right) d\alpha.$$

This is substituted into

$$D(C, P_T) = \int_0^1 (\underline{C} - P_T)^2 \, d\alpha + \int_0^1 \left(\overline{C} - P_T\right)^2 d\alpha,$$

which can be written as,

$$D(C, P_T) = \int_0^1 \left(\underline{C}^2 + \overline{C}^2\right) d\alpha - 2P_T^2.$$

The square root of this metric is an index of fuzziness measuring the dispersion of the fuzzy prices around the defuzzified one.

The optimization problem we look at is constrained by,

$\alpha \in [0, 1]$

$\sigma_1 \leq \sigma_2 \leq \sigma_3$.

We solve the non-linear constrained minimization problem using Excel Solver, which implements the Generalized Reduced Gradient (GRG2) nonlinear optimization code. However, if we build the whole model in Excel, it will crash due to technical constraints. So we have built a C++ COM object (dll) that has all the calculation code and then created an object of COM type in VBA code in a dynamic Excel spreadsheet that allows the user to access the functions in the COM object. So effectively, the training data set has been entered in Excel and then a VBA function is called through which the input data is passed and gets processed in C++ COM. The COM object computes the distance between the FBS option value and the market option value (Last) to be minimized. The result is passed back to the VBA function and displayed in the spreadsheet. Another VBA macro has been built that calls the Excel Solver from VBA and performs the minimization technique subject to the given constraints. On the other hand, to compute the implied volatilities for the BS model, we have used a simple Newton-Raphson technique and implemented it in Excel/VBA.

This gives us the implied volatilities for both models, which will be used for forecasting the call option price using a separate testing data set. To forecast the call option prices, we build a C++ console application that takes as input the testing data set with the implied volatilities and returns results for the FBS and BS option values, and the root mean square errors for both models.

7.1.2 Data Set

To test the models, we look at S&P 500 index options because it is the most actively traded European option. We consider three market data sets: a training one and two out-of-sample testing one. The training set has been quoted on the 24th of July 2002 at 16:46 and is comprised of 134 observations. The out-of-sample testing sets are quoted such that they allow us to study the implications of the degree of closeness of the training set to the testing set on the performance of the models. The first testing set corresponds to same day closing prices, namely 24th July 2002, as the training set. The second set has been quoted on the 4th of October 2002.

Each data set consists of quotes for underlying, strike, interest rates, dividend yield and expiration. Interest rates are those rates corresponding to TB of matching maturities with the options. The expiration is used to compute the time to expiry. There are usually March, June, August, September, October and December expirations with the option expiring on the Saturday after the third Friday of each month. We use the first set to calculate the implied volatilities. The implied volatilities corresponding to a particular expiration and strike from the training set are matched to similar expirations and strikes in the testing sets. Those values, alongside the second data set, are used to run the comparative results.

7.1.3 Result Analysis

Tables 1E and 2E in Appendix E show the results for the option values using both models as well as implied volatilities for 24th July and 4th October sets respectively. Also shown are moneyness and expirations. ITM, OTM and ATM in the moneyness column stand for in-the-money, out-of-the-money and at-the-money respectively. The BS implied volatility is given by sigmaBS and the FBS implied volatilities are given by sigma1, sigma2 and sigma3. As Table 2E shows, we do not have forecasting results for the whole set of options quotations for that day because we do not have matching information for the whole set.

As figure 7-1 shows, the FBS implied volatilities sometimes coincide for sigma1 and sigma2, or sigma2 and sigma3 except for some deep-out-of-the-money options. The reason that sigma1 and sigma2 or sigma2 and sigma3 sometimes coincide is due to the way the volatility constraint is defined since we allow for equality between the mid and the extremes so we do get an interval sometimes. The BS implied volatility tends to be enveloped by the FBS implied volatilities except for deep-out-of-the-money options where it exceeds the upper bound but it is never below the lower bound. However, it can be between sigma1 and sigma2 or sigma2 and sigma3. The observations where the BS implied volatility is closer or overshoots the FBS implied volatilities also correspond to the widest spread between the FBS upper and lower volatility bounds. On the other hand, the observations where the BS implied volatility is closer to the lower FBS lower volatility bound correspond to the smallest spread between the two bounds. Sometimes, the bounds coincide for at-the-money options.

As for the valuation, the two models produce very close numbers. However, their performance depends on how close the testing set is to the training set. For the July testing set, the RMSE and RMSE as percent of price for FBS model are 23.4674 and 1.31694 respectively while those for BS model are 24.0083 and 1.55599 respectively. So, generally, the FBS model gives better results than the BS model. For the October

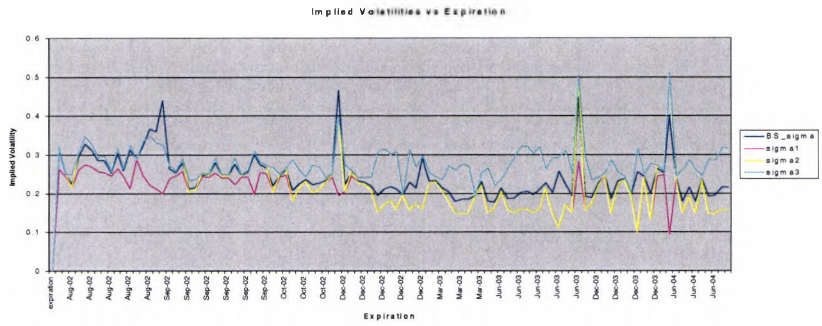

Figure 7-1: Implied volatilities of the BS and FBS models

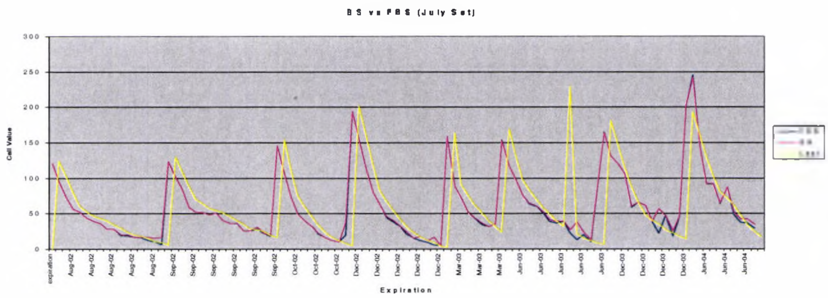

Figure 7-2: FBS and BS call option values for the July forecasting set

testing sets, there is general underpricing for both models, which increases as the option moves in or out-of-the-money. The least underpricing occurs for at-the-money options. The corresponding RMSE and RMSE as percent of price for the FBS and BS models are 20.1728 and 0.41685, and 20.0986 and 0.486109 respectively. So, generally, the BS model performs slightly better than the FBS one. Figures 7-2 and 7-3 show the prediction results for both forecasting sets. There is general overpricing for in-the-money options but underpricing as the option goes out-of-the-money and it is systematic for both models.

Interestingly enough, the two models seem to perform better for the testing set that is farther from the training set, that is, they give better results for the October testing set than they do for the July one. This indicates that the closeness of the training to the testing set is not really very relevant. Rather, what is relevant is the input data going into both models. The October testing set has closer data to the

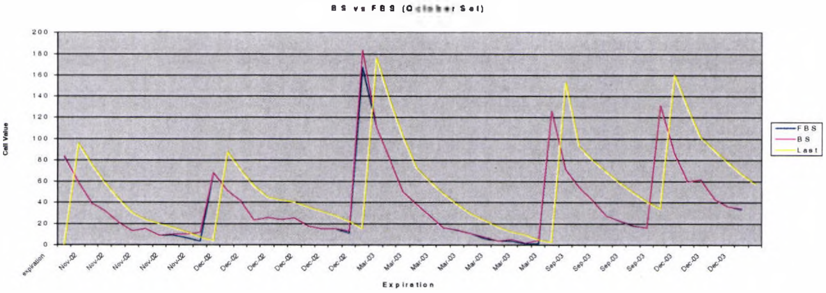

Figure 7-3: FBS and BS call option values for the October forecasting set

training set, for example, the underlying price of this set is closer to the underlying price of the training set than the July testing set price is even though the time to expiries do not match exactly. However, the training set under consideration is not comprehensive and, hence, one has to be careful when training those models to have as comprehensive a data set as possible. When the user is working with a small training data set and the circumstances change or the behaviour of the data changes, the results are expected be offline, irrespective of where the testing set stands vis-a-vis the training one.

This is actually a drawback of the FBS model, which is the also the case for the BS model, because it means that it has to be trained on a new data set every time the settings change, which happens quite often in the financial markets. The alternative is to train the model on a comprehensive data set. On the other hand, if the FBS model is used to look at the volatility of the market, then it can be very helpful because it gives the trader bounds on the values the market views volatilities and, consequently, gives him a margin of error.

Another drawback is that it is computationally cumbersome, which can make it quite slow if the appropriate hardware is not available. A possible solution is to build the whole system in C++, that is, to program the optimization code in C++ because that definitely speeds up the computation and to use a powerful machine. Besides, with a data set of 134 observations, which is not considered a large data set, Excel has crashed several times and the results have been lost.

It is also possible to use market data to determine α, signalling the level of market information or incompleteness of market. The estimation of α can be subjective but it is also possible to study its value and behaviour pattern historically and draw implications or use the result for forecasting. In the following section, we will consider such applications. Besides, we will be comparing the bounds on the call value we get using the FBS model to those we get using the Uncertain Volatility Model.

7.2 Fuzzy Black-Scholes OPM vs Uncertain Volatility Model OPM

In the previous section, we have seen how the FBS model fairs vis-a-vis the BS model by defuzzifying the call price and calculating the implied volatility bounds from market data. In this section, we will study how the FBS model fairs vis-a-vis other option pricing models that address uncertain parameters. As we have mentioned earlier, we will be particularly considering the Uncertain Volatility (UV) model that has been proposed by Avellaneda et al [5]. This model extends an approach for pricing and hedging derivative securities in an uncertain volatility environment where the volatility is bounded between two values. This, in turn, generates bounds on the possible no-arbitrage price an option can assume. So it will be appropriate to look at the FBS bounds and analyze them in the light of the UV bounds. We will start by a brief review of the UV model and then move to present the methodology and result analysis.

7.2.1 The Uncertain Volatility (UV) Model: A Brief Review

Avellaneda et al [5] have introduced a model for pricing and hedging derivative securities in an uncertain environment. They consider uncertain volatility so that rather than having to summarize a complete view of volatility as a single number, one is allowed to bound volatility between two extremes and allow it to fluctuate or assume any value in between those extremes such that $\sigma_{\min} \leq \sigma \leq \sigma_{\max}$. As a result, the option value must lie between two no-arbitrage bounds where the upper bound and the lower bound satisfy the following two equations respectively,

$$W^{+}(S_t,t) = \sup_p E_t^P\left[\sum_{j=1}^{N} e^{-r(t_j-t)}F_j(S_{t_j})\right]$$

and

$$W^{-}(S_t,t) = \inf_p E_t^P\left[\sum_{j=1}^{N} e^{-r(t_j-t)}F_j(S_{t_j})\right]$$

where the expectations are taken over the set of possible probability paths P, and $F_j(S_{t_j})$ represents the stream of cashflows characterizing the derivative security.

The PDE has the following form,

$$\frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2\left(\frac{\partial^2 W}{\partial S^2}\right)S^2\frac{\partial^2 W(S,t)}{\partial S^2} = 0$$

As the PDE shows, the volatility is dependent on the convexity of the option. So the best way of utilizing this model is by considering it within the context of a worst case/best case scenario. In the case of a long position, the worst case scenario dictates assuming the volatility coincides with the lower bound, that is, $\sigma_{\min} = \sigma$ and vice versa for the corresponding best case scenario. In the case of a short position, the worst case scenario dictates assuming the volatility coincides with the upper bound, that is, $\sigma = \sigma_{\max}$ and vice versa for the best case scenario. This analysis can be summarized by the following equations for the bounds of the option price where W^+ stands for the best case case scenario and W^- stands for the worst case scenario,

$$\sigma\left(\frac{\partial^2 W^+}{\partial S^2}\right) = \begin{cases} \sigma_{\max} & \frac{\partial^2 W}{\partial S^2} \geq 0 \\ \sigma_{\min} & \frac{\partial^2 W}{\partial S^2} < 0 \end{cases},$$

and

$$\sigma\left(\frac{\partial^2 W^-}{\partial S^2}\right) = \begin{cases} \sigma_{\max} & \frac{\partial^2 W}{\partial S^2} \leq 0 \\ \sigma_{\min} & \frac{\partial^2 W}{\partial S^2} > 0 \end{cases}.$$

7.2.2 Data and Methodology

As we have mentioned earlier, we have to view the FBS result in terms of bounds rather than a defuzzified value. We also need to calculate the bounds given by the UV model. To this end, we have built an Excel spreadsheet with VBA macros that implement the models. For the FBS model, we have followed the algorithm presented in the previous chapter. We do not go for the defuzzification approach, rather we vary α to get the bounds. For the UV model, we use the finite difference scheme proposed by the authors [5]. They propose the following parameters for the trinomial tree,

$$u = e^{\sigma_{\max}\sqrt{\Delta t} + r\Delta t},$$

$m = e^{r\Delta t}$,

$ud = e^{-\sigma_{\max}\sqrt{\Delta t}+r\Delta t}$,

$p_u = p\left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)$,

$p_m = 1 - 2p$,

and

$p_u = p\left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)$,

where $\frac{\sigma^2_{\min}}{2\sigma^2_{\max}} \leq p \leq \frac{1}{2}$.

We have used the same data sets corresponding to S&P 500 Index options as the previous section. We basically adopt two main approaches. In the first approach, we consider the implied volatility values that we have obtained in the previous section where for the UV model $\sigma_1 = \sigma_{\min}$ and $\sigma_2 = \sigma_{\max}$. We consider three sets for the applications. The first set is the July training set that we have used to deduce bounds for market implied volatility, namely, the July 24th interday snapshot. The second set is the July forecasting set, which corresponds to July 24th close. The third set is the October forecasting set.

The volatility values that are input into the models are the ones calculated in the previous section based on the July training set. So these values will be fed, alongside other inputs, into the models for each set. The intuition behind including the July training set in the applications is that we want to study how the bounds behave if they are fed the exact volatility from the market irrespective of the forecasting power of the model. On the other hand, the July forecasting and October forecasting sets applications include an element of forecasting power in them.

In the second approach, we consider subjective bounds for the volatility. We assume that the most possible volatility, or σ_2, is the one corresponding to the BS implied volatility and we allow σ_1 and σ_3 to vary by -0.1 and $+0.1$ respectively. So we get the same volatility spread for all moneyness and expirations. The data set we have used for this application is the July training set because we want to study how the Call option bounds will vary around the Last price irrespective of the forecasting power of the model under consideration. So σ_2 will be the market implied volatility corresponding to the specific option.

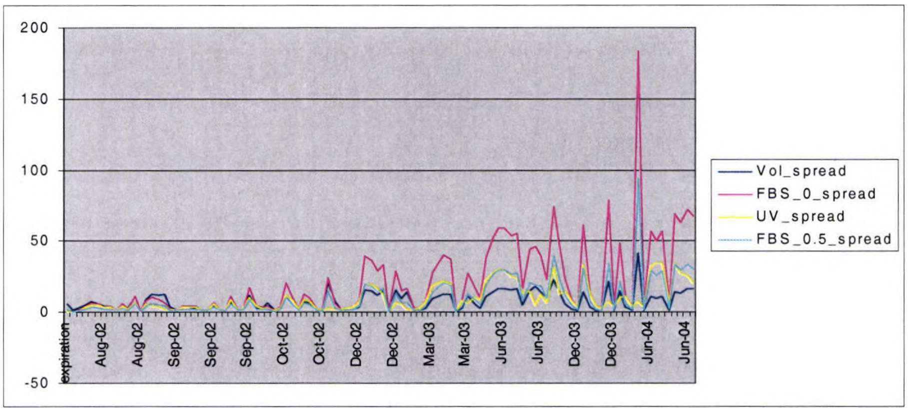

Figure 7-4: Spread of the FBS and UV bounds for the July training set across all maturities.

7.2.3 Result Analysis

The results are shown in tables 3 through 7 in Appendix E. Tables 3E, 4E and 5E present the results for the July training set, the July forecasting set and the October forecasting set respectively. They show the UV bounds and the FBS bounds corresponding to an α of 0, 0.5 and 0.1 across different moneyness and maturities. The tables also show the Last (or settlement) price for each option. Table 6E displays the results for the approach that imposes subjective bounds on the volatility across various moneyness and maturities. It also shows the Last price. Table 7E shows how the bounds of the FBS model change as α changes.

In line with the analysis of the UV bounds, UV1 (in the tables) corresponds to the worst case option price if we are long the option but best case scenario if we are short. On the other hand, UV2 corresponds to best case scenario if we are long the option but worst case scenario if we are short. For the FBS model, the worst case scenario corresponds to an $\alpha = 0$ because it corresponds to maximum fuzziness and uncertainty and, as the tables show, the bounds are widest. The best case scenario corresponds to $\alpha = 1$ because it corresponds to a minimum uncertainty case. As the tables show, the bounds, in this case, converge to one value, that of the BS OPM.

The FBS bounds corresponding to an $\alpha = 0$ seem to be, on average, wider than the UV bounds as measured by the spread in figure 7-4 shows. This figure also shows

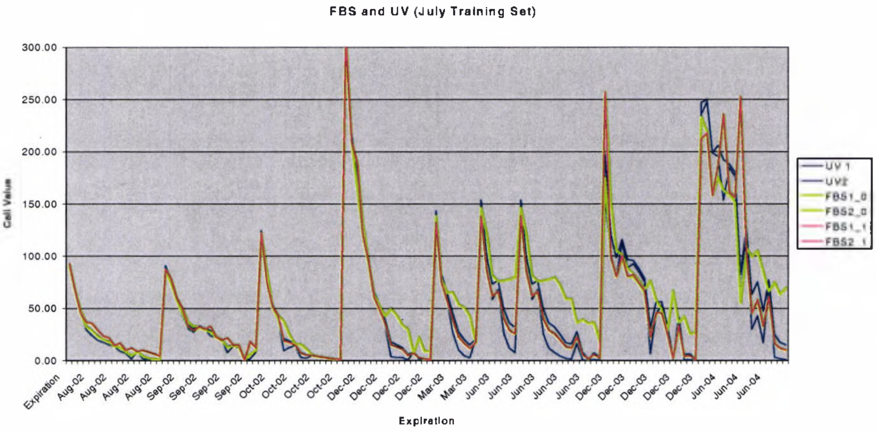

Figure 7-5: FBS and UV bounds across all maturities for the July training set

that the width of the UV bounds seems to be matching the FBS bounds corresponding to an $\alpha = 0.5$. This pattern is quite consistent across all sets. This can be due to the fact that the FBS model, by definition, is more comprehensive and especially tailored for high uncertainty scenarios.

However, both bounds show the same pattern of behaviour. As tables 3E through 7E show and as expected, bounds for both models widen as the bounds on volatility widen. The results show that the highest spread corresponds to at-the-money options, perhaps because they are the ones most sensitive to volatility. Figure 7-5 presents a graph of the actual bounds rather than the spread. The wide bounds correspond to observations around at-the-money options. This pattern is consistent across all maturities and all testing sets. However, it seems to be more emphasized for longer dated options. UV bounds and FBS bounds corresponding to $\alpha = 0.5$ seem to be quite close or similar.

However, as the tables and figures show, one of the FBS bound corresponding to $\alpha = 0$ mostly coincides with the FBS value corresponding to an $\alpha = 1$. This is again due to the equality constraint imposed on the volatility constraint in the optimization problem. When the bounds on the volatility do not coincide, the former bounds envelope the latter one. It is also empirically obvious that the FBS bounds narrow as α increases until they finally converge to the BS value when $\alpha = 1$. In fact,

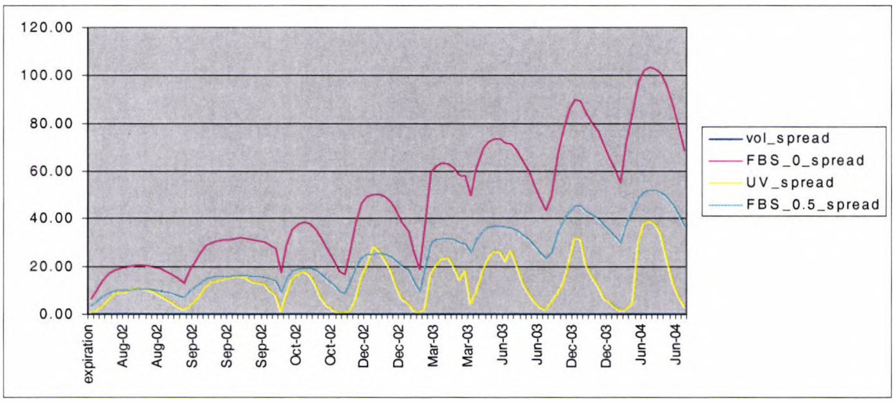

Figure 7-6: FBS and UV spread across all maturities

there are two cases where the FBS bounds converge, holding everything else constant. The first case corresponds to $\alpha = 1$ as tables 3E through 7E show. The second case is when sigma converges to one value, as some observations in tables 3E to 5E show. Table 7E shows this behavior more explicitly as α assumes values of $0, 0.5, 0.75, 0.9$ and 1. It is also clear in figure 7-4 how the spread, as a measure of the width on the bounds, decreases, meaning that the bounds are narrowing down, as α decreases. In general, FBS bounds corresponding to $\alpha = 0$ envelope the Last price but it is not the case for the UV ones.

These patterns are actually emphasized when we introduce subjective bounds. This approach allows us to analyze the models irrespective of their forecasting power. It is actually clearer how the bounds vary across different maturities and moneyness. The volatility has a fixed spread in this case. However, as figure 7-7 shows, the bounds for both UV and FBS, corresponding to all values of $\alpha < 1$, are widest for at-the-money and, not very deep, out-of-the-money options. So the spread is highest for those observations as figure 7-6 shows. So we actually observe the same pattern irrespective of the forecasting power of the model; however, they tend to be more emphasized when we introduce subjective bounds on the volatility.

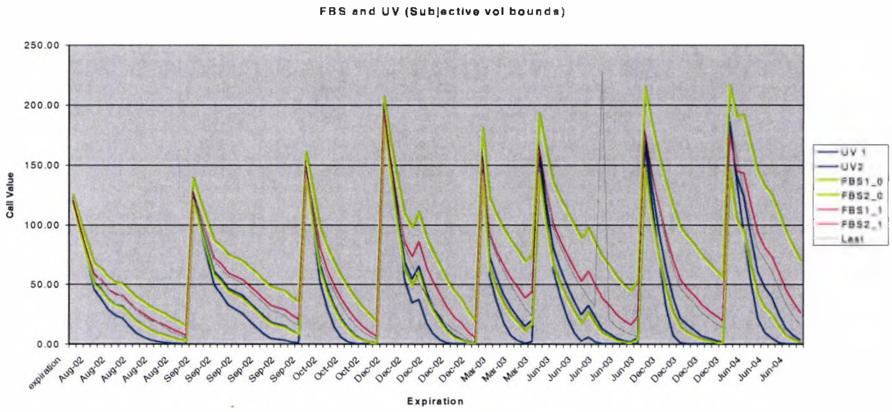

Figure 7-7: Bounds for the UV and FBS models using subjective volatility bounds

7.3 Conclusion

Therefore, we have studied in this chapter empirical and comparative applications of the FBS model. As we have seen, it is possible to get one crisp value from the FBS model summarizing market information. It is also possible to derive market implied volatilities in the form of bounds on the possible values a volatility can assume. Comparing it to the BS model, it seems to have more predictive power than the BS model does.

It is also possible to work with uncertain volatility to derive bounds on the option price in the presence of uncertainty. In such instances, it is safer to work with bounds rather than single values. So we have analyzed the empirical behavior of the FBS model and compared it to that of the UV model. As we have seen, the model's bounds can be manipulated according to the expert's personal opinion or historical analysis. However, in the presence of high uncertainty, it is best to follow the worst case/best case scenario analysis we have carried earlier. In general, the FBS model is more comprehensive and more flexible than the UV model. But both models show the same pattern of behavior, namely, their bounds widen as we move more towards at-the-money and longer-term options, perhaps because there is more uncertainty associated with those options.

This provides potential for applying the model to other uncertain parameter problems. It also means that we do not really have to resort to complex solutions every time we have an uncertain parameter since this model serves as an underlying platform or prototype that can be used for other uncertain parameters such as uncertain dividends, interest rates, etc... It also extends more potential for providing solutions to finance using fuzzy theory, which is finding more applications in option pricing as the next chapter shows.

Chapter 8

Fuzzy Option Pricing: A Comparative Approach

In the previous four chapters, we have presented two Fuzzy Option Pricing models with empirical applications. We have also seen applications of Fuzzy Theory to finance and economics in Chapter 3. The applications have been relatively simplistic and targeting investment decision making in most of the cases. In this chapter, we will explore existing literature on applications of Fuzzy Theory to Option Pricing, which is a more recent stream of literature. But those applications are relatively scarce in comparison to fuzzy applications in finance and economics. The existing trustworthy research that we are aware of boils down into three papers. This maybe due to the fact that finance researchers are not really exposed to the full potential of Fuzzy Theory yet. Most fuzzy finance research is carried out by fuzzy researchers and thus tends to be trivial from a finance and economics point of view. However, given that the fuzzy applications to Option Pricing Theory are being introduced by finance researchers who are more familiar with existing Option Pricing problems, the applications turn out to be the most sophisticated applications of Fuzzy Theory to finance and economics, and may also serve as laying the foundations for Fuzzy Option Pricing.

We will present a literature review of the existing three papers on Fuzzy Option Pricing, two of which utilize tools of Fuzzy Set Theory while the other utilizes tools of Fuzzy Measure Theory. So we will start with the Muzzioli-Torricelli approaches which basically build on the existing CRR OPM within a fuzzy framework and devise a fuzzy solution using Fuzzy Set Theory. They actually introduce two different models whereby the first is a one period model and the second is a mutliperiod one that solve different problems. We extend the one-period model for comparative purposes at a later stage. We then move on to the Cherubini approach, which incorporates Fuzzy Measure Theory and Choquet integration. Cherubini introduces a fuzzy Black-Scholes OPM. We attempt to draw a comparison between those existing models. Unfortunately, this attempt has been hampered either by technical fall backs of the models or lack of explicit definitions.

8.1 The Muzzioli-Torricelli Models

Muzzioli and Torricelli ([110],[112]) combine Fuzzy Set Theory with binomial option pricing. In the first paper, they introduce a one period model for pricing a call option with a fuzzy payoff. In the second paper, they introduce both a one and a multi-period model to price a European call option on an underlying asset that has an opaque price under different states of the world.

8.1.1 The MTM1 Model

In their first paper [110], the authors present a one period model for pricing an option on an asset with a fuzzy price at the end of the period. The opacity of the price can be contributed to many factors that are summarized in the information level, α, of the market. In their pricing methodology, the authors price the option both by replicating the portfolio and by using risk-neutral valuation.

Triangular fuzzy numbers (Appendix A) are used to model the fuzzy price of the asset after one period such that $P_1^R =< a_1/a_2/a_3 >$, with level sets $P_1^R(\alpha) = [a_1(\alpha), a_2(\alpha)]$ and characterized by the following membership function,

$$P_1^R(x) = \begin{cases} 0, & 0 \leq x \leq a_1; \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2; \\ \frac{x-a_3}{a_2-a_3}, & a_2 \leq x \leq a_3; \\ 0, & x \geq a_3; \end{cases}$$

The notation $P_1^R(x)$, in this case, stands for the membership or characteristic function. Then, P_1^R induces a possibility distribution over the value of x equivalent to the membership function. The no-arbitrage principle still holds and is expressed as,

$$a_1(\alpha) \leq P_0^R(1+r) \leq a_3(\alpha),$$

where P_0^R is the initial price of the underlying asset.

Let K denote the exercise price and assume that $a_1(\alpha) \leq K \leq a_3(\alpha)$. So the fuzzy payoff of the call at the end of the period is

$$P_1^C = \max[0, P_1^R - K] = \max[0, a_1 - K, a_2 - K, a_3 - K]. \tag{8.1}$$

For the payoff to satisfy no-arbitrage and $a_1(\alpha) \leq K \leq a_2(\alpha)$, it is defined only for $\alpha < \overline{\alpha}$, where

$\overline{\alpha} = \min[\alpha^*(K), \alpha^*(P_0^R(1+r)), \alpha^{**}(K), \alpha^{**}(P_0^R(1+r))]$

$\alpha^*(i) = \frac{i-a_1}{a_2-a_1}, i = K, P_0^R(1+r);$

$\alpha^{**}(j) = \frac{a_3-K}{a_3-a_2}, j = K, P_0^R(1+r).$

In terms of α-cuts,

$$P_1^C = \max[0, a_3(\alpha) - K], \tag{8.2}$$

which is not a triangular fuzzy number.

The authors first construct a portfolio that replicates the payoff of a call option. Assume that N_i represents the number of units of asset i, where $i = \{M, R\}$, with M representing the money market account and R representing the risky asset. Then,

$P_1^C = N_M P_1^M + N_R P_1^R.$

Substituting α-cuts and employing some mathematics, the authors arrive at

$N_M = -\frac{N_R[a_1+\alpha(a_2-a_1)]}{(1+r)}$,

$N_R = -\frac{a_3(1-\alpha)-K+a_2\alpha}{(1-\alpha)(a_3-a_1)}$.

Finally, $P_0^C = N_M P_0^M + N_R P_0^R$.

Alternatively, the authors use the risk neutral valuation approach, where

$P_0^C = \frac{1}{1+r}\widehat{E}[P_1^C]$

The risk-neutral probabilities are computed according to the standard approach except that α-cuts are used for the up and down jumps denoted by u and d respectively. The triangular fuzzy number representing the movement of the price is $< d/m/u >$, where $u = \frac{a_3}{P_0^R}, m = \frac{a_2}{P_0^R}, d = \frac{a_1}{P_0^R}$, with α-cuts $[d(\alpha), u(\alpha)]$. After some algebra, the authors finally arrive at the following values for the risk neutral probabilities,

$$\begin{aligned} q_1 &= \frac{(1+r)-d(\alpha)}{u(\alpha)-d(\alpha)}, \\ q_2 &= \frac{u(\alpha)-(1+r)}{u(\alpha)-d(\alpha)}. \end{aligned} \tag{8.3}$$

The final value of the fuzzy call is

$$P_0^C = \frac{a_3(1-\alpha)-K+a_2\alpha}{(1-\alpha)(a_3-a_1)} * \left[\frac{P_0^R(1+r)-a_1-\alpha(a_2-a_1)}{(1+r)}\right]. \tag{8.4}$$

As the equation shows, the fuzzy call price does not really have the conventional fuzzy form in the sense that it is not constrained between two numbers, an upper bound and a lower bound, where it can assume any value in between depending on the level of α. Rather, it is one value that is dependent on α or the level of information as the authors call it.

The authors study the first derivative of the call w.r.t. the various variables. In summary, they find out that the fuzzy call has the same properties as the standard call option w.r.t. the strike price, interest rate and the underlying price. The fuzzy call is decreasing in α, the information level, meaning that as information increases, volatility decreases leading to a decrease in the value of the call, which is similar to the standard call case. As for the upper and lower bounds, a_1 and a_3 respectively, of the underlying price, there is an inverse relation between the call price and the lower bound and a direct relation between the call and the upper bound, which makes sense because as the lower bound increases or the upper bound decreases, volatility declines and so does the call price.

Within the context of the binomial model, the price in the downward state, D, corresponds to the lower bound and that in the upward state, U, corresponds to the upper bound, namely, $P_1^R(D) = a_1$ and $P_1^R(U) = a_3$. Using the conventional notations employed in the standard binomial model, this is equivalent to saying that the price of the underlying at maturity is bounded by the 'up' and 'down' prices, that is, $Sd < P_1^R < Su$. However, usually the upper and lower bounds are defined as the least possible so this implies that Su and Sd are least possible.

Comparing the results of their model to those of the standard binomial OPM, the authors find out that the latter is only a special case of the former corresponding to $\alpha = 0$. This is actually a counter-intuitive result in the fuzzy sense since the case where $\alpha = 0$ corresponds to the support of the fuzzy set of the various values the underlying price can take and, consequently, it corresponds to the case of 'extreme fuzziness' where all the values are possible, of course, depending on their membership grade. But the standard binomial result has to refer to the no-fuzziness case, namely, $\alpha = 1$. However, α cannot be one; otherwise, the number of shares and the risk-neutral probabilities will be indeterminate.

Usually in Fuzzy Theory when $\alpha = 0$, the bounds, or α-cuts, are the widest and when $\alpha = 1$, the bounds converge to one value, corresponding to the crisp case. As we have seen, the issue of α is quite tricky and unusual in this problem. In this setting, the bounds on the underlying are widest or fuzziest when bounded by the up and down states as given by the binomial tree and then they get smaller as α increases meaning they get less fuzzy until α reaches one, where the state $E(S)$ actualizes. In this case, the future price or the bounds of the future price converge to the most possible price, $E(S)$, corresponding to complete certainty. However, since by definition of the risk-neutral probabilities that α cannot be one, this state cannot actualize. So the model does not really converge to a crisp case, or more specifically, a crisp case for this model does not exist, and so it holds only under high uncertainty. It does not converge to the standard binomial case because it is not based around the binomial prices; rather, it is bounded by those prices, which are least possible in this setting and will be the first to go out of the picture as α increases.

However, taking a closer look at the model, we can see that it is actually the underlying price that is bounded by two values that are dependent on α. So the previous analysis holds only for the underlying. As for the call price, it is not the case. The call price is not really bounded by α-cuts; rather, it is one value that is dependent on α. So α can no longer be interpreted as summarizing the level of fuzziness in the conventional sense. In fact, it is quite an unconventional interpretation or view of α.

However, the model, as it stands now, is not very practical in the sense that it is not ready yet for empirical applications but it serves as laying the foundations of option pricing in discrete time using Fuzzy Set Theory. For the purpose of comparing it to other Fuzzy Option Pricing Models, we will extend this model to the multi-period case.

The Extended MTM1 Model

Extending this model to a multiperiod model can be quite tricky. We will preserve the same assumptions that the MTM1 assumes but we will add to it one more assumption which is the fact that the most expected underlying price at any period is the preference-free expected price (martingale). To avoid confusion, we will be using the standard binomial model notations, namely, S for the price of the underlying or risky asset rather than P_0^R. There are two possibilities for extension. The first possibility involves binding the future price from below by the lowermost state and from above by the uppermost state, for example, in a three period model, $Sd^3 < P_3^R < Su^3$ if the price is to be modelled by a triangular fuzzy number. But if the price is to be modelled by a different fuzzy number, such as the trapezoidal one, then the choice of the vertices will be an issue. In both cases, the model will get very complicated as the number of steps in the tree increases. The second possibility is within a binomial model setting. It considers the future price at each node in a similar manner to the one-period model. So, for example, if we are at node Sud^2, the future price will be bounded by Sud^3 from below and Su^2d^2 from above; similarly, at the adjacent node Su^2d, the future price will be bounded by Su^2d^2 from below and Su^3d from above. Following this approach, the model can be extended into the multi-period case. This is better than the other possibility and so we will utilize it to extend the model into the multi-period case.

In such a setting, the price is completely vague and fuzzy in the future, which makes the model quite appropriate for pricing options in periods of very high uncertainty. The future price is opaque then and is only known to be bounded between two values. Besides, because of the very high prevalent uncertainty, we can only view the price as opaque on a period by period case. In other words, we do not look at the price as opaque three periods from now into the future; rather, we look at it from this period to the next one, and then in the next period, we look at it to the following period and so on, depending on which node we are at.

Within a general binomial setting, there are no longer up and down states but there is a continuum of possible states bounded from above by the most possible state and from below by the least possible state, for example, in a one period model, the price will be bounded by Su in the up state and Sd in the down state. Those bounds are given by the standard up and down jump factors. So the underlying structure is given by the standard binomial tree and fuzziness is imposed over it such that the bounds on the underlying are given by the binomial prices.

Prices of the underlying are represented by triangular numbers such that $S_i =< a_1/a_2/a_3 >$. Since the underlying will be bounded by the up and down states, then $a_1 = Su^{j-1}d^{n-j+1}$ and $a_3 = Su^jd^{n-j}$ which leaves out a_2, which is by definition the most possible value. A logical approximation to the most possible value is the preference-free expected price at any point, given by $E(S) = S_{i-1} * \exp(R\frac{T}{N})$ or $(1+r) * S_{i-1}$. It is also possible to approximate a_2 by $\frac{Su^{j-1}d^{n-j+1}+Su^jd^{n-j}}{2}$ so that we have a symmetrical triangular number but the first approach is really more intuitive and makes more sense within the context of this problem. Figure 8-1 shows a two-period extended MTM1 model. There is a continuum of prices in every state bounded

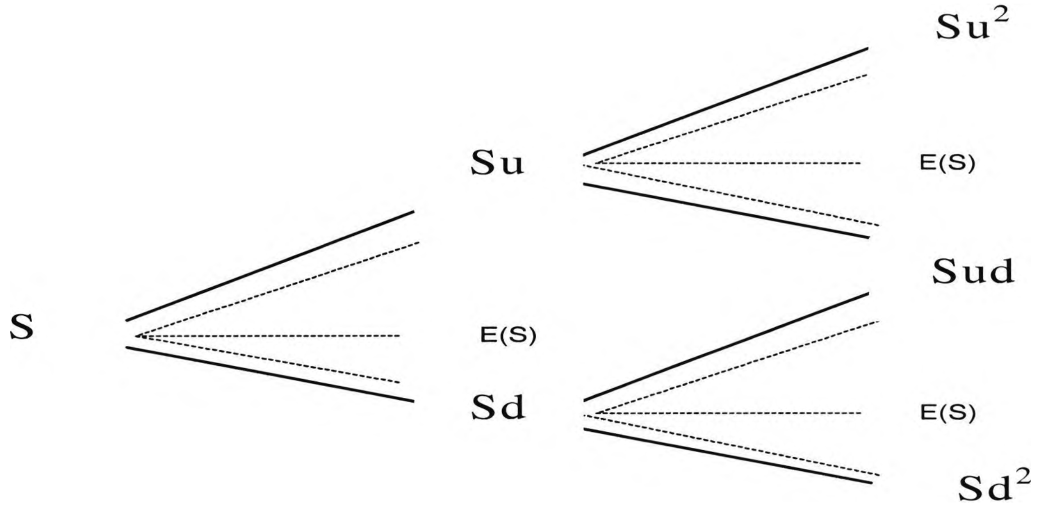

Figure 8-1: The Extended MTM1 in a two-period setting.

by the up and down states and the expected price falls almost in the middle.

The vertices of the triangular number can be used to establish $\alpha-$cuts or, consequently, bounds on the possible range of prices in each state. Those bounds will lead to bounds on the risk-neutral probabilities as given by equation 8.3 such that the value given by the upper bound returns the risk-neutral probability of occurrence of the upper bound of the price and similarly for the lower bounds. Those risk-neutral probability bounds have special implications for the incomplete market case.

As mentioned before, in an incomplete market, there is no one unique no-arbitrage probability measure; rather, there is a set of possible probability measures that satisfy the no-arbitrage principle among which the decision is left indeterminate. The bounds given by the risk-neutral probabilities in the MTM1 model behave as limits on the possible values a probability measure in incomplete markets can assume. So the extended model serves as a model for pricing options in discrete-time in an incomplete market.

In summary, at each node, we have a continuum of possible states or possible prices the stock can assume in the future. This leads to a continuum of possible risk-neutral probability measures. Within the context of incomplete markets, this range represents the range of possible probability measure values a risk-neutral probability measure can assume. Within the context of the standard binomial model, the maximum price a share can assume is the price corresponding to an up state and the minimum it can assume is that corresponding to a down state. The expected price is $E(S)$, which does not by condition actualize but theoretically has the highest possibility (similar to our fuzzy binomial model). The same holds for probability measures. The exact values, that are assumed, are dependent on a parameter α, which can be interpreted as summarizing the level of market information or market incompleteness. So by $\alpha = 1$, it means that the model cannot hold for a complete market scenario. But if we assume that the model is true, then we can infer that the market cannot be complete or no one can have full information, which is quite realistic.

To price the option, we use the risk-neutral valuation approach. It is important to emphasize that this is not a fuzzy version of the binomial tree so we cannot use the standard binomial model logic to price the option. In other words, there are no payoffs corresponding to an up and down state. Instead, each continuum of prices results in one payoff that is bounded from above and below and is not necessarily a triangular fuzzy number. For example, assuming a one-period setting, the continuum of prices is bounded by Su and Sd and the whole range results in a fuzzy payoff that also depends on $E(S)$ and has two bounds. Given that $E(S)$ differs according to the node we are standing at, it will result in a non-recombinant tree for call prices despite the fact that it is recombinant for the share price. The reason for that is that the lower bound of the call price for one state does not coincide with the upper bound of the lower adjacent state.

So we start off with computing the risk-neutral probability bounds which differ across states. We also compute the call payoff bounds dependent on the share price bounds. Then we discount the payoff across the tree using the risk-neutral probabilities and fuzzy arithmetic. We illustrate the method using a two-period model and then the extension to a multi-period scenario is straightforward.

Assuming triangular fuzzy numbers, the general form for any price will be $P =< a_1/a_2/a_3 >$ and for the stock price it will be $S_i =< Sdown/E(S)/Sup >$, which translates into $S_{ni} =< Su^j d^{n-j}/E(S)/Su^{j+1}d^{n-j-1} >$. Figure 8-2 shows the frame-

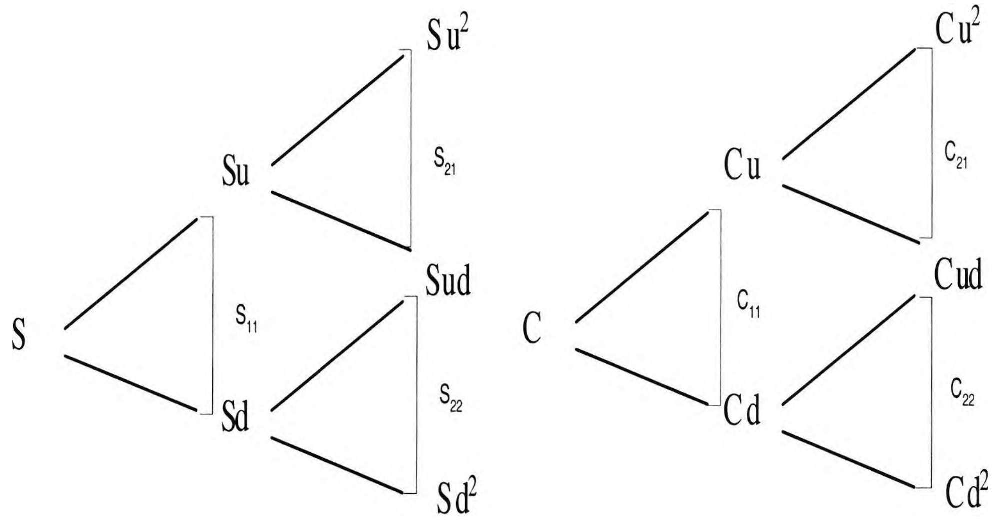

Figure 8-2: The two-period Extended MTM1 share and call trees.

work for the model. It illustrates the difference between the binomial model and the extended MTM1. Note the range of possible values for a given price bounded by the binomial values.

$S_{21} =< Sud/Su\exp(R\frac{T}{N})/Su^2 >$,

so the α-cuts or bounds for S_{21} are,

$S_{21}^1(\alpha) = Sud + \alpha(Su\exp(R\frac{T}{N}) - Sud)$,

$S_{21}^2(\alpha) = Su^2 + \alpha(Su\exp(R\frac{T}{N}) - Su^2)$,

where the superscripts 1 and 2 stand for the lower and upper bounds respectively. Similarly,

$S_{22} =< Sd^2/Sd\exp(R\frac{T}{N})/Sd^2 >$,

$S_{22}^1(\alpha) = Sd^2 + \alpha(Su\exp(R\frac{T}{N}) - Sd^2)$,

$S_{22}^2(\alpha) = Sud + \alpha(Su\exp(R\frac{T}{N}) - Sud)$,

and

$S_{11} =< Sd/S\exp(R\frac{T}{N})/Su >$,

$S_{11}^1(\alpha) = Sd + \alpha(Su\exp(R\frac{T}{N}) - Sd)$,

$S_{11}^2(\alpha) = Su + \alpha(Su\exp(R\frac{T}{N}) - Su)$.

In a similar approach to equations 8.1 and 8.2, the call payoffs will be translated to,

$$C^1_{21}(\alpha) = \max[0, S^1_{21}(\alpha) - K],$$
$$C^2_{21}(\alpha) = \max[0, S^2_{21}(\alpha) - K],$$

and

$$C^1_{22}(\alpha) = \max[0, S^1_{22}(\alpha) - K],$$
$$C^2_{22}(\alpha) = \max[0, S^2_{22}(\alpha) - K].$$

where K is the strike price. Using risk-neutral valuation and the same logic employed in deriving equations 8.3, we get

$$C^1_{11}(\alpha) = \frac{1}{1+R}\left[q^{21}_1 C^1_{21}(\alpha) + q^{21}_2 C^2_{21}(\alpha)\right],$$
$$C^2_{11}(\alpha) = \frac{1}{1+R}\left[q^{22}_1 C^1_{22}(\alpha) + q^{22}_2 C^2_{22}(\alpha)\right].$$

Therefore, the final call price is

$$C_0 = \frac{1}{1+R}\left[q^{11}_1 C^1_{11}(\alpha) + q^{11}_2 C^2_{11}(\alpha)\right].$$

So the call price is not bounded between values that are dependent on α. Instead, it is one price that is dependent on the level of information or degree of market incompleteness. Note that interval arithmetic is employed to arrive at the final value of the call (Appendix A).

Generally, the usefulness of a model lies in its forecasting power. For this reason, we need to get an objective value that is not dependent on a subjective parameter. In other words, we need to change the fuzzy value we get from this equation into a non-fuzzy one or crisp one; in other words, we need to defuzzify it.

The two most frequently used defuzzification or decomposition methods are: composite moments or centroid method and composite maximum or maximum height method. The centroid or center of gravity method calculates the weighted mean of the fuzzy region. It is the most widely used method because defuzzified values move smoothly around the output fuzzy region, it is easy to calculate and it can be applied to both singleton and fuzzy output set geometries.

Concerning the composite maximum method, the point with the highest truth value is chosen. However, if it is ambiguous such as lying on a plateau, any of the following techniques will be chosen: average maximum, center of maximums and simple composite maximum. This method applies to a narrower class of problems than the centroid technique because the expected value here is sensitive to a single rule that dominates the fuzzy region as well as the expected value tends to jump among frames as the fuzzy region changes shape. However, it is more qualified for a wider range of applications that assess the maximum of a fuzzy property. For our purposes, we consider the center of gravity method (COG), whereby,

$$C = \frac{\int_0^1 C(\alpha)d\alpha}{\int_0^1 \alpha d\alpha},$$

where $C(\alpha)$ is the call price dependent on the level of α, and C is the crisp call price. Of course, the denominator evaluates to 1. In section 8.3, we analyze the forecasting power of this model and compare it to the standard CRR model.

8.1.2 The MTM2 Model

In their second paper [112], Muzzioli and Torricelli introduce a multiperiod fuzzy binomial model for pricing European call options using the risk-neutral valuation approach. Fuzziness in this case stems from the vagueness of future possible states of the world and it is transmitted into the model through a fuzzy volatility that leads to fuzziness in the u and d factors in the CRR binomial tree. The risk-neutral probabilities and the stock price are, consequently, fuzzy, that is, they are represented by weighted intervals rather than point estimates. This model is appropriate for situations where the up and down states are fuzzy that is in periods of lower uncertainty than that within the MTM1 context. Figure 8-3 illustrates the basic idea behind MTM2. It is characterized by a fuzzy up and a fuzzy down state rather than a vague future price as in extended MTM1 (Figure 8-1).

Volatility is modelled as a fuzzy triangular number, and, because $u > 1$ and proportional to volatility, it can also be modelled as a triangular fuzzy number defined by $u_1 = e^{\sigma_1 \sqrt{\Delta t}}, u_2 = e^{\sigma_2 \sqrt{\Delta t}}$ and $u_3 = e^{\sigma_3 \sqrt{\Delta t}}$. Similarly, d can be modelled by triangular fuzzy numbers but since it is inversely proportional to volatility and u, it must be defined as $d_1 = \frac{1}{u_3}, d_2 = \frac{1}{u_2}$ and $d_3 = \frac{1}{u_1}$ (using fuzzy arithmetic). This, in turn, induces possibility distributions for the stock price and the risk-neutral probabilities. The standard risk-neutral valuation approach yields

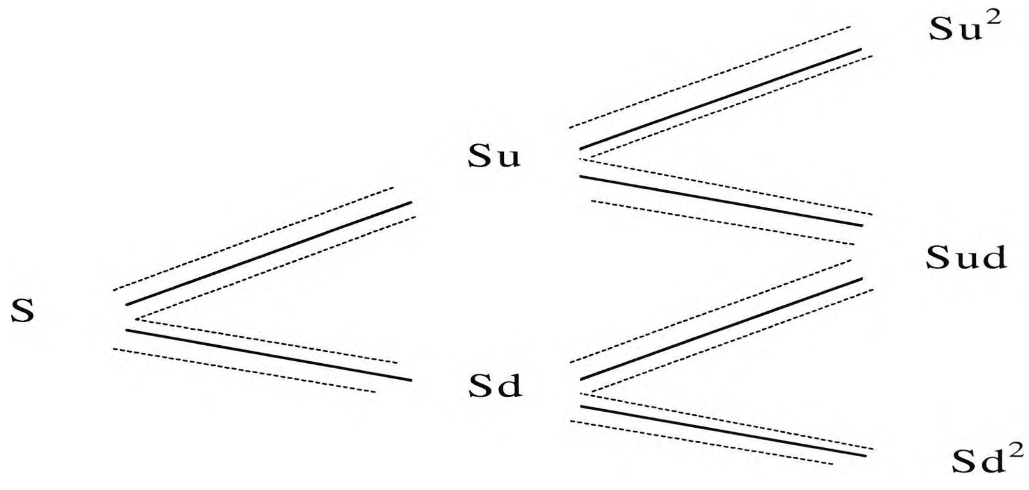

Figure 8-3: A rough sketch of MTM2 in a two period setting.

$$\begin{bmatrix} 1 & 1 \\ \frac{P_0 d}{1+r} & \frac{P_0 u}{1+r} \end{bmatrix} \begin{bmatrix} p_d \\ p_u \end{bmatrix} = \begin{bmatrix} 1 \\ P_0 \end{bmatrix},$$

whereby P_0 is the initial stock price, p_d and p_u are, respectively, the down and up risk-neutral probabilities and r is the risk-free rate. In the presence of fuzziness, u and d can be replaced by their α-cut intervals such that,

$$u_1(\alpha) = u_1 + \alpha(u_2 - u_1),$$

$$u_2(\alpha) = u_3 + \alpha(u_2 - u_3),$$

$$u(\alpha) = [u_1(\alpha), u_2(\alpha)],$$

and similarly for $d(\alpha) = [d_1(\alpha), d_2(\alpha)]$. Solving the fuzzy systems, the authors establish the intervals for the risk-neutral probabilities as

$$p_u = [\underline{p_u}, \overline{p_u}] = \left[\frac{(1+r)-d_3+\alpha(d_3-d_2)}{u_3-d_3-\alpha(u_3-u_2-d_3+d_2)}, \frac{(1+r)-d_1-\alpha(d_2-d_1)}{u_1-d_1+\alpha(u_2-u_1-d_2+d_1)}\right],$$

$$p_d = [\underline{p_d}, \overline{p_d}] = \left[\frac{u_1+\alpha(u_2-u_1)-(1+r)}{u_1-d_1+\alpha(u_2-u_1-d_2+d_1)}, \frac{u_3-\alpha(u_3-u_2)-(1+r)}{u_3-d_3+\alpha(u_3-u_2-d_3+d_2)}\right],$$

where $\overline{p_u}+\underline{p_d} = 1$ and $\underline{p_u}+\overline{p_d} = 1$ so that $\underline{p_i}$ and $\overline{p_i}$ are dual measures. The fuzzy risk-neutral probability distributions are not necessarily triangular fuzzy numbers; rather their shapes depend on the shape of u and d. The case where $\alpha = 1$ corresponds to the crisp case where each interval collapses to a point estimate.

The authors use their model to price a European call option. They start with a one period model and then move on to a two period model from which the extension into a multi-period model is straightforward. The exercise price is assumed to lie between the lowest and highest bounds of the option price such that,

$P_0 d_3^{j+1} u_3^{n-j-1} \leq X \leq P_0 d_1^j u_1^{n-j}$,

where $j = 0...n-1$. The stock price at maturity can assume any of the following values,

$P_0 d^i u^j = \left(P_0 d_1^i u_1^j, P_0 d_2^i u_2^j, P_0 d_3^i u_3^j \right)$,

where $i, j = 0, ..., n$,and $i = n - j$.

In the one-period model,

$P_0 d_3 \leq X \leq P_0 u_1$

The call payoff in the 'down' state is $C(d) = 0$ and that in the 'up' state is $C(u) = (P_0 u - X)$. Therefore, the call payoff at the end of the period is $C(u) = (P_0 u_1 - X, P_0 u_2 - X, P_0 u_3 - X)$. The call price at time t $= 0$, C_0, can be determined using the risk-neutral valuation approach; hence,

$C_0 = \frac{1}{1+r} \widehat{E}[C_1] = \frac{1}{1+r}[p_u * C(u)]$

$$\Longrightarrow C_0 = [\underline{C_0}, \overline{C_0}] = \begin{bmatrix} \frac{P_0 u_1 - X + \alpha P_0 (u_2 - u_1)}{1+r} * \frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)}, \\ \frac{P_0 u_3 - X - \alpha P_0 (u_3 - u_2)}{1+r} * \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 - \alpha(u_2 - u_1 - d_2 + d_1)} \end{bmatrix}.$$

The interval for the call price is fuzziest when $\alpha = 0$ and least fuzzy when $\alpha = 1$. The latter case corresponds to the complete market scenario and the call price in this case corresponds to that given by the standard binomial approach. This case also corresponds to the most possible value case meaning that the interval for the call price is built around the most possible or crisp or complete market value for the call price. As for the shape of the call price, it is increasing in α in the left bound and decreasing in α in the right bound. However, it is concave or convex in either cases depending on the sign $(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)$. It is linear when $(u_2 - d_2)(P_0 u_3 - X) = (u_1 - d_1)(P_0 u_2 - X)$. For the multiperiod case, the call price is not necessarily a triangular fuzzy number due to standard fuzzy arithmetic.

Similarly, the call price for the two period case is

$$C_0 = [\underline{C_0}, \overline{C_0}] = \frac{1}{(1+r)^2} \begin{bmatrix} (P_0 u_1^2 - X + \alpha P_0 (u_2^2 - u_1^2)) * \left(\frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \right)^2, \\ (P_0 u_3^2 - X - \alpha P_0 (u_3^2 - u_2^2)) * \left(\frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 - \alpha(u_2 - u_1 - d_2 + d_1)} \right)^2 \end{bmatrix}.$$

The fuzzy volatility is derived by solving the following non-linear optimization problem,

$$\min_{\sigma_1, \sigma_2, \sigma_3} \sum_{i=1}^{n} (P_T(\sigma_1, \sigma_2, \sigma_3) - P_M)^2$$

such that $\sigma_1 < \sigma_2 < \sigma_3$ and $e^{-\sigma_1 \sqrt{\Delta t}} < e^{r\sqrt{\Delta t}} < e^{\sigma_1 \sqrt{\Delta t}}$, and where P_T is the theoretical price, P_M is the market price and n is the number of observations. In implementing the model, P_T corresponds to the defuzzified price while, in comparing with the binomial model, it is the binomial price. To compute the defuzzified price, the authors use their defuzzification approach, which we have outlined in the previous chapter. They argue that the defuzzified price is different from the binomial one because it discounts market information, which further serves in arbitraging. They implement their model using DAX-index options. Within the sample they consider, the fuzzy model outperforms the binomial one.

This model converges to the binomial model in the complete market case ($\alpha = 1$) where there is perfect information. So it relates the option price to the level of information quite well. However, problems arise when we need to defuzzify the model in a multiperiod setting using the metric defined above. The problem is that the integrand is unbounded, that is, as the number of steps increases, the integrand increases indefinitely particularly because the upper bound for the call price goes to infinity as the number of steps goes to infinity. So the model will not converge in this case. Of course, in a two period binomial model, similar to the one the authors consider, does not suffer of this drawback but there will be no convergence for the model as a whole. To overcome this problem, we have to use a metric that incorporates a weighting that is inversely proportional to the number of steps in the tree. However, the model remains important as laying the foundations for Fuzzy Option Pricing. Finally, this model and the previous one serve as a good example of uncertainty versus risk since the risk-neutral probabilities, which are measures of risk, are dependent on α, which can serve as a measure of fuzziness or uncertainty. So we have a case of coexisting risk and uncertainty.

8.2 The Cherubini Model CM

In [30], Cherubini uses g_λ-fuzzy measures and nonadditive expected utility model for option pricing. The standard utility approach is modelled as,

$U = \int u(x)dH = \int H(x : u(x) \geq \alpha)d\alpha,$

while that of Cherubini, called the g_λ-utility approach, is defined using a Choquet integral as

$U_* = \int g_\lambda(x : u(x) \geq \alpha)d\alpha = \int \left(\frac{1-H(\alpha)}{1+\lambda H(\alpha)}\right) d\alpha,$

where $H(.)$ is a probability distribution whose support is defined on the set of non-negative real numbers. In fact,

$g_\lambda(x : u(x) \geq \alpha) = 1 - g_\lambda(-\infty, \alpha) = \left(\frac{1-H(\alpha)}{1+\lambda H(\alpha)}\right)$

because $g_\lambda(-\infty, \alpha) = H(\alpha)$ and so $g_\lambda(u(x) \geq \alpha)$ serves as the λ-complement to $g_\lambda(-\infty, \alpha)$. So the author is distorting expected utility by λ, which he defines as an indicator of uncertainty to arrive at a parametrization of non-additive expected utility.

Cherubini defines a set of probability functions as a set of probability measures, which he refers to as the core of $g^H_\lambda()$, bounded from below and from above by the g_λ-measures, whereby

$\Gamma(H, \lambda) = \left\{P : g^H_{\lambda^*}(A) \geq P(A) \geq g^H_\lambda(A), \forall A \in \Im, \lambda^* = \frac{-\lambda}{(1+\lambda)}\right\},$

where P defines the probability measures of the set, $\Im$ is a σ-field and $g^H_\lambda()$ represents the g_λ-measures defined using the probability distribution approach $H()$. The core is non-empty if the measure is convex meaning that λ can assume values only in the interval $(0, \infty)$. So the measure is subadditive.

This results in two bounds for the expected utility model, a lower one defined as

$$U_*(H, \lambda) = \int \left(\frac{1 - H(\alpha)}{1 + \lambda H(\alpha)}\right) d\alpha = \min \left\{\int u(x)dP : P \in \Gamma(H, \lambda)\right\}, \qquad (8.5)$$

and an upper one defined by

$$U^*(H,\lambda) = U_*(H,\lambda^*) = \int \left(\frac{1-H(\alpha)}{1+\lambda^* H(\alpha)}\right) d\alpha = \max\left\{\int u(x)dP : P \in \Gamma(H,\lambda)\right\}, \tag{8.6}$$

using lower and upper Choquet integrals respectively. Since λ can only be in the interval $(0, \infty)$, λ^* lies in $(-1, 0)$ so the dual fuzzy measures are superadditive.

Cherubini applies his approach to corporate debt valuation and levered fuzzy replication (or fuzzy Black-Scholes equation). Our interest is, of course, in the second application. In this model, the author establishes bounds for the set of possible option prices, which he refers to as bid-ask bounds. Cherubini looks at the Black-Scholes equation as a levered option replication method whereby the agent buys the stock and borrows against it an amount of money K, which is the strike, maturing at a certain point in the future. This replicated portfolio is equivalent to a call option at maturity and so the value of the call option is the value of the asset plus a debt contract. Since such a portfolio has to be customized in an unofficial market, fuzzy valuation seems more plausible than standard approaches.

The lower and upper bounds for the call option price or the Bid/Ask quotes are defined as

$$c_*(V,t;K) = V(t) - \exp\left[-r(T-t)\right] E^*\left[\min\left(V(T),K\right) \mid V(t)\right],$$
$$c^*(V,t;K) = V(t) - \exp\left[-r(T-t)\right] E_*\left[\min\left(V(T),K\right) \mid V(t)\right],$$

where r is the risk-free rate, $(T-t)$ is the time to maturity, E^* is the non-additive expectation evaluated using Choquet integration in continuous time, $V(T)$ is the value of the underlying asset at maturity and K is the strike price. Choquet integration is evaluated in the same manner as in equations 8.5 and 8.6 where $H(.)$ is the lognormal distribution. In the presence of additivity, the model converges to the Black-Scholes value.

The author applies this approach to a snapshot of S&P500 Index call options, quoted on the 7^{th} of March 1997, and recovers parametric bounds for the option price dependent on λ. To make the Bid/Ask spread plausible (around 1 dollar), λ is calibrated to 0.08. The continuous-time Choquet integration is evaluated using the Romberg method. The consecutive empirical results show that the model's spread increases as the strike increases or as the option gets more out-of-the-money even though the implied volatility tends to decrease. The author interprets this increase in spread as resulting from an increase of uncertainty since as the strike increases, the levered position increases leading to this increase in uncertainty irrespective of the uncertainty on the underlying as modelled by the implied volatility.

We have tried to replicate the model but we came across basic impediments that hindered us from obtaining reasonable results. Of the most important issues are the approximations for the standard deviation and mean employed in the lognormal distribution that is supposed to recover the price. Another problem lies in the limits of the integral or the expectations on the right-hand side of the equation. Cherubini is not explicit about this matter even though it is quite crucial for the result. Finally, the dividends do not seem to play any part in the formula even though they are quite important. We have considered some assumptions of our own but the results have not been very useful so this model is excluded from the comparative study.

8.2.1 Other Models

There are other models proposed in the literature that employ a logic similar to that employed in fuzzy applications. We will mention those models briefly. However, we will not analyze them thoroughly since they are out of the scope of this research. The first model has been proposed by Walley and De Souza [138]. They use imprecise probabilities in analyzing the impact of uncertainty and indeterminacy on energy options. They introduce a probabilistic decision-theoretic approach whereby a decision has to be made as to whether to invest in solar energy or not.

Another model has been proposed recently by Muzzioli and Torricelli [111]. They extend the Derman-Kani implied tree for option pricing to account for illiquidity as well as put-call parity violations. To this end, they deal with intervals of probabilities and stock prices rather than precise values. Consequently, the expectations are computed using Choquet integration over intervals in discrete time. The intervals are established by deriving two implied trees using call and put options separately. Those intervals impose interval bounds on the option price as well which the authors interpret as Bid/Ask prices.

8.3 Empirical Analysis

In this section, we will study empirical applications of the models we have referenced in this chapter. As we mentioned earlier, we have not been able to get reasonable results for the Cherubini model, particularly, because some of the basic assumptions are missing. As for the Muzzioli-Torricelli (MTM2) model, we have not been able to get reliable results either because the integrand is not bounded and so the model does not converge. So it has not been possible to derive implied volatilities. However, with a different defuzzification technique, the model is expected to fair quite well. Therefore, we will only be analyzing the extended Muzzioli-Torricelli (MTM1) model and comparing it to the CRR binomial model.

8.3.1 MTM1 OPM vs CRR Binomial OPM

Methodology

In this section, we analyze the forecasting ability of the extended MTM1 model and compare it to that of the CRR binomial model. The implementation in this case is much simpler than it is for the FBS model. It involves building a C++ model that returns the option values and root mean square errors. As we have seen earlier, extended MTM1 returns an option price, rather than option bounds, that is dependent on α. So, by using the centroid defuzzification method, we can get one crisp option value that summarizes market information.

Data Set

The data set we use is that of S&P 500 index options quoted on the 24th of July. It is made up of the underlying price, strike, interest rates, dividend yields, expirations and volatility. The volatility corresponds to the Black-Scholes implied volatility. We consider all expirations, namely, March, September, November and December. The model does not have to be trained and can be automatically tested on any data set.

Result Analysis

Table 7E in Appendix E shows the results for the option value calculations using MTM1 and CRR binomial models. The RMSE and RMSE as percent of price for both models are 15.7332 and 0.227716 for the former model and 15.9341 and 0.248146 for the latter indicating that MTM1 is a little bit better than the CRR one. This tells us that the system complexity does not justify the value added in such a market characterized by transparency and a low level of Knightian uncertainty. However, in a market characterized by illiquidity and Knightian uncertainty, it is useful to apply the model. Another important issue is the defuzzification technique used, which is the centroid method. This method is similar to a weighted mean. Since the fuzzy call price is a triangular number that is almost symmetric around the most possible price which is close to the crisp price, the result is very close to the CRR value.

If we view MTM1 model as summarizing the market information, we can conclude that the level of information does not make much of a difference probably because of the high level of transparency and liquidity of the option under consideration. Table 7E shows that both models predict prices that are close to the Ask price rather than the Last price.

8.4 Conclusion

In this chapter, we have provided a review of existing literature on Fuzzy Option Pricing. This is a relatively new area in option pricing and those models serve as laying the foundations for Fuzzy Option Pricing. We have looked at models that fuzzify the CRR OPM and the Black-Scholes OPM using Fuzzy Set Theory as well as Fuzzy Measure Theory. We have also extended a one period model introduced by Muzzioli and Torricelli into a multiperiod setting.

We have also attempted to carry an empirical comparative study. Unfortunately, we have not been able to do that for two of the existing models, MTM2 and CM. MTM2 assumes a defuzzification technique using a quadratic metric distance which implies that the model does not converge as the number of steps increase. The Cherubini model lacks explicit definitions for the practical application to be performed. We have been, however, able to carry an empirical application of the extended MTM1 and the results returned have been very much similar to that of the CRR model. The complexity of the model and the computational burden does not really justify using the model when Knightian uncertainty is not prevalent. It becomes useful in a market characterized by a high level of uncertainty.

Chapter 9

Conclusion

In this research, we have tackled the issue of uncertainty in the financial markets with specific emphasis on its implications for Option Pricing Theory. The approach we adopt is a pricing approach whereby we consider the impact of uncertainty on options and then we develop an Option Pricing Model, which accommodates the type of uncertainty under consideration. Such uncertainty is initiated by lack of information; it is also known as Knightian uncertainty. The issue of uncertainty is not a new one. However, it has been approximated by risk for a long while due to several reasons, the most important of which is the lack of sophisticated tools that can handle such problems. As a result, Probability Theory has been employed to handle all types of problems irrespective of whether the problem is characterized by risk or uncertainty. With the increasing complexity of problems, it has become evident that alternatives to Probability Theory that relax the additivity constraint are needed.

Consequently, several alternatives have been introduced and applied successfully to various fields such as engineering, finance and medicine. In our research, we have considered two of such alternatives, Fuzzy Set Theory and Fuzzy Measure Theory motivated by the fact that they have considerable potential in solving problems in finance and economics that has been proved successful by applications to problems having similarities to those in finance and economics as well as by the fact that the existing applications of fuzziness in finance and economics are still in their infancy. The existing applications of Fuzzy Theory in finance and economics tend to be trivial unfortunately despite their capacity to solve complex problems. An exception to those applications is Fuzzy Option Pricing, which has been introduced by finance researchers. Existing literature on Fuzzy Option Pricing, even though it boils down to few papers, serves as laying down the basic foundations for this new area.

We have viewed the issue of uncertainty and tools to handle it from as comprehensive a point of view as possible. Uncertainty in general has been separately developed in Economics Theory as well as in Fuzzy Theory. So we have gathered literature from both fields and presented them within one coherent framework. We have also looked at recent developments in Option Pricing Theory which are important for comparisons with our models. Besides, we have presented a literature review on fuzziness in finance and economics to make clear the placement of our models in Fuzzy Theory. The major contribution of all literature reviews is that they bring different areas of research together.

On the technical side, we have documented important issues in Fuzzy Set Theory from independent research, which is scattered in various forms of literature. We have particularly considered controversial issues such as the comparative behaviour and definitions of the membership function, possibility distributions and probability distributions. We have also presented the basics of Fuzzy Measure Theory. Our major contribution here is the section on the fuzzy measure elicitation approaches, which are also scattered in different applications and publications. Naturally, we have incorporated nonlinear integration, dual measures, conditional fuzzy measures and Choquet integration as well. There are also very important concepts that have been included in the appendices so that we do not get involved in matters beyond the scope of our research.

Being fully equipped with the necessary literature and tools, we have moved to present our Fuzzy OPMs. Those models boil down to three models, two of which are new and one is an extension to an existing Fuzzy OPM. In the first model, we consider pricing in a fuzzy environment whereby fuzziness is associated with the BAS as well as with states of the world. We have utilized Fuzzy Measure Theory and Fuzzy Integration to establish bounds on option prices in this case. The model we have developed extends contributions to both Fuzzy Theory and Option Pricing Theory. The major contribution lies in the use of the conditional expectations and the corresponding mathematics involved. This model exhibits the first Sugeno integral application in the finance and economics literature.

We have also applied the model to empirical data spanning examples from the currency, NASDAQ and Index option markets. The types of options that have been considered are American as well as European. The model has given the best results for American (NASDAQ) stock options, where the NASDAQ market is characterized with the highest uncertainty among the three markets that have been tackled.

However, the bounds are still relatively wide and, in very few observations, flip for long-term expirations (except for SP500 Index options, where those observations are more frequent). So further work in this area needs to be done. Another issue that needs consideration is calculating the implied volatilities from this model. This is quite a tricky issue because of the comparison operators. Finally, the model is still open for hedging techniques. The challenge in this case comes from two sources, the bounds for the call price and fuzziness. One suggestion is to introduce a tracking error function that measures the difference between the theoretical portfolio, which is the one corresponding to the standard CRR binomial model, and the one that is supposed to be hedged. If the tracking error is positive, then the actual portfolio super-replicates the theoretical one. If it is zero, it replicates it and if it is negative it sub-replicates it. Another approach is to replicate the upper bound and lower bound separately using fuzzy values. This should be done for both bounds so, at each node, we have a minimum and a maximum number of stocks and bonds to be held. The problem with this approach is that it tends to be messy over multiple periods. Of course, those approaches serve only as guidelines and still need to be implemented and tested.

In the second model, we have fuzzified the Black-Scholes OPM using Fuzzy Set Theory and Fuzzy Differential Calculus. Fuzzy Differential Calculus is still in its infancy so we have had to go through the difficult task of collecting and refining the existing literature. The source of fuzziness is the uncertain volatility which is prevalent in certain circumstances in the options markets. The fuzzy volatility generates fuzziness in the final call price, which means that again we establish bounds on the value a call option can assume.

We have also provided empirical applications of this model. Our findings show that, using the Muzzioli-Torricelli defuzzification approach [112], this model beats the standard Black-Scholes model perhaps because it summarizes the level of information in the market better. Besides, the volatility bounds that have been suggested by this model bound the Black-Scholes implied volatility, which better reflects the market's behaviour. Those bounds are expected to be wider when market uncertainty increases but this is still open for further research. In fact, since this model converges to the Black-Scholes model in the complete market case, it can be exploited to measure the level of market completeness or to measure the degree by which the Black-Scholes model deviates from the market. We have also compared our model to the Uncertain Volatility Model [5], which has been witnessing increased popularity lately. The bounds of our model tend to envelope those of the Uncertain Volatility one. It is also possible to manipulate α to change the width of the bounds.

Our third model is an extension of the one-period Muzzioli-Torricelli Fuzzy Binomial Model [110] into a multi-period setting, whereby fuzziness is characterized by an opaque future asset price. This model also converges to the standard CRR model so it can serve as a measure of market completeness. Using the center of gravity defuzzification technique, we have been able to get a value for the call option and to show that it does actually beat the CRR model.

Finally, we have drawn a comparative study on existing Fuzzy Option Pricing Models, which provide the foundation for Fuzzy Option Pricing. Three models are examined for this purpose. MTM1, which is a one-period model, has been extended (our third model) to facilitate comparison. The second model is MTM2 but we have not been able to obtain results for this model since it does not converge given the proposed type of defuzzification technique. The third model is the Cherubini model but again replication has not been possible due to lack of explicit definitions. The model that seems to give the best results is the extended Muzzioli-Torricelli multi-period Fuzzy Binomial Model. Generally, those models tend to have a common drawback, which is the fact that they are slow so some pruning and use of sophisticated machines are needed to make them practical in a fast pace market.

Appendix A

Basics of Fuzzy Set Theory

This appendix introduces the basics of Fuzzy Theory. However, our intention is not to analyze this theory in detail but to present the concepts that will be relevant for further understanding of our research. So we will stick to the basics only. The reader who is familiar with these basics may choose to skip this part.

A.1 Fuzzy Sets

Zadeh defines fuzzy sets as "a class with a continuum of grades of membership." ([149], p.339) To illustrate this idea, let X be a space of points, or a universe, with a generic element x so that $X = \{x\}$, and let A be a set in X. A is called a fuzzy set if it is characterized by a membership or characteristic function $\mu_A(x)$ whose value denotes the grade of membership of x in A. $\mu_A(x)$ associates with each element x in X a value in the real interval [0,1], whereby a higher value designates a higher grade of membership. In contrast, if A were an ordinary set, its membership function will take two values only: 0 or 1.

A.1.1 Some Properties of Fuzzy Sets

- The support of A is: supp $A = \{x \in X, \mu_A(x) > 0\}$.

- The height of A is: hgt $A = \sup_{x \in X} \mu_A(x)$.

- The α-cut A_α of A is: $A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$, α–cuts are also called level sets.

- A is said to be normalized iff $\exists x \in X, \mu_A(x) = 1$..

- A is said to be an empty set iff $\forall x \in X, \mu_A(x) = 0$..

- A is said to be convex iff: $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \forall x_1 \in X, x_2 \in X, \alpha \in [0, 1]$;

A is convex iff its α-cuts are convex.

A.1.2 Operations on Fuzzy Sets

The arithmetic operations on fuzzy sets are different from those on ordinary sets. However, there is no agreement as to how a particular operation is to be carried out and there are quite many definitions. The union and intersection of two fuzzy sets are denoted as the maximum and minimum respectively. Particularly interesting is the complement. The complement of a fuzzy set A is defined as $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$. This means that the fuzzy set and its complement can actually overlap. Sugeno [133] has introduced another definition for complement known as λ-complement, such that $\lambda \in (-1, \infty)$,

$\mu_{\bar{A}}(x) = \frac{1-\mu_A(x)}{1+\lambda\mu_A(x)}$.

A.2 Fuzzy Numbers

A fuzzy number is essentially a fuzzy set. There are several definitions of fuzzy numbers in the literature. Dubois and Prade ([49], p.26) define a fuzzy number as a convex normalized fuzzy set A of the real line $\Re$ such that:

(a) $\exists! x_0 \in X$, $\mu_A(x_0) = 1$ (x_0 is called the mean value of A).

(b) μ_A is piecewise continuous.

Goetschel and Voxman ([75], Definition 1.1, p.87) refer to an earlier definition by Dubois and Prade whereby a fuzzy number is a fuzzy set $\mu : R^1 \to I = [0, 1]$ with the properties:

(i) μ is upper continuous,

(ii) $\mu(x) = 0$ outside of some interval $[c, d]$.

(iii) there are real numbers a and b, $c \leq a \leq b \leq d$, such that m is strictly increasing on $[c, a]$, strictly decreasing on $[b, d]$, and $\mu(x) = 1$ for each $x \in [a, b]$.

The authors alter this definition slightly and introduce two other conditions instead of (i) and (iii):

(i') μ is upper semicontinuous (u.s.c.),

(iii') there are real numbers a and b, $c \leq a \leq b \leq d$, such that μ is increasing on $[c, a]$, decreasing on $[b, d]$, and $\mu(x) = 1$ for each $x \in [a, b]$.

These changes are initiated in order to resolve the inconsistency in Dubois and Prade's definition (because it allows $c = d$; therefore, the resulting fuzzy number could be a real number) and to allow defining a metric for the family of fuzzy numbers $\Im$ so as to study the topological properties of fuzzy numbers.

In a later paper, Goetschel and Voxman ([75], p. 31) view fuzzy numbers from a different perspective. They accept the above definition and add to it the following:

Define $C_r(\mu)$, for $0 \leq r \leq 1$, as:

$C_r(\mu) = \{(x, r) \mid \mu(x) \geq r\}$ if $0 < r \leq 11$,

cl(supp μ) if $r = 0$,

where cl(supp μ) is the closure of the support of μ. Then, μ is a fuzzy number iff:

(i) $C_r(\mu)$ is a closed and bounded interval for each r, $0 < r \leq 1$, and

(ii) $C_1(\mu) \neq \phi$.

Thus, a fuzzy number is characterized by the endpoints of the interval C_r. The authors represent the fuzzy number μ by the parameterized triplets:

$\{a(r), b(r), r) \mid 0 \leq r \leq 1\}$,

where $a(r)$ denotes the left endpoint of $C_r(\mu)$ and $b(r)$ denotes the right endpoint. Therefore, the fuzzy number is characterized by the endpoint functions a and b.

The authors introduce a list of sufficient conditions ([75], Theorem 1.1, p. 32) that a: $R^1 \to I = [0, 1]$ and b: $R^1 \to I = [0, 1]$ must satisfy so that $\mu : R^1 \to I = [0, 1]$, defined by $\mu(x) = \sup\{r \mid a(r) \leq x \leq b(r)\}$, is a fuzzy number with the above parameterization. These conditions are:

(i) a is a bounded increasing function,

(ii) b is a bounded decreasing function,

(iii) $a(1) \leq b(1)$,

(iv) for $0 < k \leq 1$, $\lim_{r \to k}{}^{-}a(r) = a(k)$ and $\lim_{r \to k}{}^{-}b(r) = b(k)$,

(v) $\lim_{r \to 0}{}^{+}a(r) = a(0)$ and $\lim_{r \to 0}{}^{+}b(r) = b(0)$.

The converse is also true, i.e. if $\mu : R^1 \to I = [0,1]$ is a fuzzy number, then a and b satisfy the above conditions. Those conditions will prove to be very useful in solving fuzzy differential equations.

There are different forms of fuzzy numbers. However, I will discuss only triangular fuzzy numbers because they are most relevant to our research. Buckley and Qu define triangular fuzzy numbers in [22]. Let $\overline{N}$ be a real fuzzy number with membership function $y = \mu(x \mid \overline{N})$ partially specified by $(n_1/n_2/n_3)$. Then:

(i) $n_1 < n_2 < n_3$;

(ii) $\mu(x \mid \overline{N}) = 0$ outside (n_1, n_3) and it equals one at $x = n_2$..

(iii) $\mu(x \mid \overline{N})$ is continuous and strictly increasing from zero to one on $[n_1, n_2]$; and

(iv) $\mu(x \mid \overline{N})$ is continuous and strictly decreasing from one to zero on $[n_2, n_3]$.

If the graph of $\mu(x \mid \overline{N}) = y$ is a straight line segment on $[n_1, n_2]$ and on $[n_2, n_3]$, then we can write $\overline{N} = (n_1/n_2/n_3)$ and call $\overline{N}$ a real fuzzy number; otherwise, we have to describe

$y = \mu(x \mid \overline{N})$ on (n_1, n_2) and on (n_2, n_3). The α-cut $\overline{N}(\alpha)$ of $\overline{N}$ is a closed interval given by $\overline{N}(\alpha) = [n_1(\alpha), n_2(\alpha)]$ where:

$n_1(\alpha) = (n_2 - n_1)\alpha + n_1$

$n_2(\alpha) = (n_2 - n_3)\alpha + n_3$

And the following holds for any fuzzy number $\overline{N}$:

(i) $\overline{N} \geq 0$ if $n_1 \geq 0$.

(ii) $\overline{N} > 0$ if $n_1 > 0$.

(iii) $\overline{N} \leq 0$ if $n_3 \leq 0$.

(iv) $\overline{N} < 0$ if $n_3 < 0$.

The support of $\overline{N} = (n_1, n_3)$.

Based on Goetschel and Voxman's sufficient conditions for the parameters of a fuzzy number, Buckley ([19], p. 4) presents sufficient conditions, which must be satisfied by the α-cuts $[n_1(\alpha), n_2(\alpha)]$ of a fuzzy number $\overline{N}$, and which are:

(i) $n_1(0) < n_1(1) \leq n_2(1) < n_2(0)$;

(ii) $n_1(\alpha)$ is upper semi-continuous and increasing on $[n_1(0), n_1(1)]$;

(iii) $n_2(\alpha)$ is upper semi-continuous and decreasing on $[n_2(1), n_2(0)]$.

A.3 Fuzzy Vectors

If V is a subset of R^n with membership function $y = \mu(x \mid V)$, then V is a fuzzy vector iff:

(i) $y = \mu(x \mid V)$ is u.s.c.

(ii) $V(\alpha)$ is compact, simply connected, and arcwise connected for $0 \leq \alpha \leq 1$, and

(iii) $V(1)$ is not empty.

In fact, when $n = 1$, V becomes a real fuzzy number so, basically, a fuzzy vector is only a generalization of a fuzzy number, $n \geq 2$. ([19])

A.4 Interval Analysis

As we will see in solving fuzzy equations, interval arithmetic is very important in certain approaches to solving fuzzy equations. In interval analysis ([108]), a number is viewed as an interval and, hence, characterized by the two endpoints: $[a, b] = \{x : a \leq x \leq b\}$. However, what we really need from interval analysis for our purpose is interval arithmetic:

Let $Y = [y_1, y_2]$ and $X = [x_1, x_2]$ where $y_1 \leq y_2$ and $x_1 \leq x_2$

(i) $X + Y = [x_1 + y_1, x_2 + y_2]$

(ii) $-X = [-x_2, -x_1]$

(iii) $X - Y = [x_1 - y_2, x_2 - y_1]$

(iv) $1/X = [1/x_2, 1/x_1]$, provided the interval X does not contain 0. If X contains zero, then it is unbounded and cannot be represented as an interval with endpoints that are real numbers.

(v) $X \cdot Y$:

$x_1 \cdot y_1 = \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$

$x_2 \cdot y_2 = \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$

$\therefore$ we obtain 9 cases based on the signs of the endpoints:

1. $x_1 \geq 0, y_1 \geq 0 : X \cdot Y = [x_1y_1, x_2y_2]$

2. $x_1 < 0 < x_2, y_1 \geq 0 : X \cdot Y = [x_1y_2, x_2y_2]$

3. $x_2 \leq 0, y_1 \geq 0 : X \cdot Y = [x_1y_2, x_2y_1]$

4. $x_1 \geq 0, y_1 < 0 < y_2 : X \cdot Y = [x_2y_1, x_2y_2]$

5. $x_2 \leq 0, y_1 < 0 < y_2 : X \cdot Y = [x_1y_2, x_1y_1]$

6. $x_1 \geq 0, y_2 \leq 0 : X \cdot Y = [x_2y_1, x_1y_2]$

7. $x_1 < 0 < x_2, y_2 \leq 0 : X \cdot Y = [x_2y_1, x_1y_1]$

8. $x_2 \leq 0, y_2 \leq 0 : X \cdot Y = [x_2y_2, x_1y_1]$

9. $x_1 < 0 < x_2, y_1 < 0 < y_2 :$

   $x_1 \cdot y_1 = \min(x_1y_2, x_2y_1)$

   $x_2 \cdot y_2 = \max(x_1y_1, x_2y_2)$

(vi) $X/Y = X \cdot (1/Y)$

Algebraic Properties of interval arithmetic

$X + (Y + Z) = (X + Y) + Z$

$X(YZ) = (XY)Z$

$X + Y = Y + X$

$XY = YX$

$0 + X = X + 0 = X$

$0X = X0 = 0$

$1X = X1 = X$

$x(Y + Z) = xY + xZ$; x is real and Y and are Z intervals

$X(Y + Z) = XY + XZ$; if $YZ > 0$

A.5 Membership Function Elicitation Methods

Generating membership functions (or possibility distributions) is not a trivial task. There are no general guidelines that one can follow to arrive at a final form for the membership function, rather the membership function tends to have many functional forms which can be adapted to the problem at hand. As we have seen above, there are different interpretations for the membership function which result in different lines of thought for the function generation. Based on that, authors tend to have different categories for the functional forms.

Lai and Hwang [100] divide elicitation methods into two approaches, axiomatic and semantic, based on earlier work by Giles [71]. The axiomatic approach is similar to utility theory in that it subscribes to mathematical considerations whereas the semantic approach concentrates on the interpretations of the terms. Chaudhuri and Majumder [28] divide membership function elicitation methods into two approaches according to the availability or non-availability of mathematical expressions for the membership function.

We prefer to divide the elicitation methods into two broad categories, namely, subjective and objective approaches. Subjective approaches are based on preferences or perceptions while objective approaches are based on training data or mathematical concepts rather than subjective preferences. Most of the examples we will see rely on the first approach because most of the work done so far tends towards modelling subjective phenomenon and interpreting linguistic vagueness. However, lately, the literature has been witnessing increased interest in generating the membership function using a set of data.

Subjective approaches map perceptions about vague concepts into numerical structures. Based on that, they utilize tools from measurement theory and scaling (a good reference on measurement and scaling of membership functions is [113]). There are many techniques that try to elicit membership functions based on preferences.

1. Polling ([56],[49],[106]): This approach corresponds to the likelihood view of membership functions. A subject is presented with the object and asked whether this object belongs to a set. The subject is expected to provide "Yes/No" answers.

2. Direct Rating ([56],[106],[113]): This approach can be applied to a single individual or a group. The subject is asked to explicitly give the grade of membership of an object in a set (characteristic...). Such methods require evaluation measurement to be at least on an interval scale.

3. Reverse Rating ([56],[106],[113]): This approach can also be applied to a single individual as well as to a group. However, in this approach, the subject is given the membership function and asked to give the object that would best suit such a grade of membership. It also requires evaluation to be measured on at least a membership scale.

4. Interval Estimation [56]: This approach undertakes the random set-view of the membership function. The subject is asked to give an interval that describes or includes a fuzzy set. It is believed that this approach deals with the uncertainty rather than the vagueness aspect.

5. Membership Exemplification ([56],[49]): This approach relies on partial information to elicit a membership function. A subject is asked to give values for the membership function of a fuzzy set at certain points and then the membership function of the set is elicited from this partial information.

6. Pairwise Comparison ([56],[106]): This approach involve comparing the membership function of one set with that of another. It requires evaluation on a ratio scale.

7. True-Valued Approach [100]: The basic idea behind this approach is that the degree of membership of an element x in set A, $\mu_A(x)$, is numerically equivalent to the truth value of the statement 'x is A,' $v(x \text{ is } A)$.

8. Interpolation [29]: This approach subscribes to the measurement theory view of membership functions. Measurement theory provides an axiomatic framework for interpolating the membership function, which is based on subjective preferences. Chen and Otto [29] use it within the context of design engineering. They collect data for the grade of membership of some points from experts (i.e. subjective data) and try to interpolate the membership function for the rest of the data points. However, since the membership function is constrained (boundedness, and monotonicity and convexity conditions) which hinders the application of usual interpolation schemes, such as least squares or spline methods, they introduce a constrained interpolation method using the Bernstein polynomials and an algorithm for computing shape preserving quadratic splines. In fact, their approach is very close to that of subjective probabilities. The basic idea is to introduce a flexible interpolation method which can accommodate for the discontinuities in the membership function. This can be accomplished by segmenting the membership function into pieces subject to certain end-conditions.

9. Payoff Function ([71],[100]): This approach subscribes to the utility view of the concept of grade of membership. It considers that the assertion 'x is A' instead of the statement itself to be fundamental and to be identified by the payoff function across

different states of the world similar to decision theory.

10. Parametrized Membership functions ([106],[100],[56],[28]) subscribe to the similarity view. They rely on parameters or a distance measure between the observation and an ideal one. The most popular forms of such a membership function are

$\mu(x) = \frac{1}{1+\exp(a-bx)}$,

$\mu(x) = \frac{1}{1+d(x)}$,

$\mu(x) = \frac{1}{1+f(d(x))}$,

$\mu(x) = \frac{1}{\exp(-a(x-b))}$,

where a and b are parameters that can be determined from statistical data.

11. Conjugate Gradient Search Technique [27]: This approach was presented in an attempt to introduce a Fuzzy Delphi Method. It allows fitting continuous mathematically explicit membership functions to discrete membership functions. It provides an estimation of the bounds of the membership function based on interval-valued surveys elicited from experts.

Medasani et al. [106] argue that subjective approaches usually lack a 'general category' like the maximum likelihood for estimating probability densities. But they attribute that to the fact that we do not yet fully understand human perception of vagueness.

Now we move on to present objective approaches to generating a membership function. We will divide those approaches into two categories: those that deal with heuristic methods and those that deal with training data. We will start with the former approach.

Heuristic approaches ([106],[100]) use a predefined shape for membership functions. The functions can be made suitable for the problem at hand; however, it is not easy to tune the parameters especially for complex problems. Heuristic approaches include piecewise linear functions and piecewise monotonic functions. In the following, we present a summary of such membership functions.

Zimmerman's linear function: $\mu(x) = 1 - \frac{x}{a}$

Tanaka, Uejima and Asai's triangular function: $\mu(x) = \begin{cases} 1 - \frac{|a-x|}{\alpha} & \text{if } \alpha - a \leq x \leq \alpha + a \\ 0 & \text{otherwise} \end{cases}$

piecewise linear: $\mu(x) = \begin{cases} 0 & \text{if } x \leq a \\ w_1 \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ w_2 \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } x > d \end{cases}$

piecewise linear: $\mu(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{1}{b-a}x + \frac{a}{a-b} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$

S-function: $S(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ 2\left(\frac{x-a}{c-a}\right)^2 & \text{if } a < x \leq b \\ 1 - 2\left(\frac{x-a}{c-a}\right)^2 & \text{if } b < x \leq c \\ 1 & \text{if } x > c \end{cases}$

where $b = \frac{a+c}{2}$

π-function: $\pi(x; a, b, c) = \begin{cases} S(x; c-b, c-\frac{b}{2}, c) & \text{if } x \leq c \\ 1 = S(x; c, c+\frac{b}{2}, c+b) & \text{if } x > c \end{cases}$

piecewise monotonic function: $\mu(x) = \exp(-b(x-a)^2)$

Svarowski's sin function: $\mu(x) = \frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right), \quad x \in [a, b]$

Zadeh's unimodal functions: $\mu_{young}(x) = \begin{cases} \frac{1}{\left\{1+\left[\frac{(x-25)}{5}\right]^2\right\}} & \text{if } x > 25 \\ 1 & \text{if } x \leq 25 \end{cases}$

$$\mu_{old}(x) = \begin{cases} \frac{1}{\left\{1+\left[\frac{(x-50)}{5}\right]^2\right\}} & \text{if } x \geq 50 \\ 0 & \text{if } x < 50 \end{cases}$$

Dimitru and Luban's power functions: $\mu(x) = \frac{x^2}{a^2} + 1, \quad x \in [0, a]$

$$\mu(x) = -\frac{x^2}{a^2} - 2\frac{x}{a} + 1, \quad x \in [0, a]$$

Lai and Hwang [100] acknowledge some of the functional forms presented above but categorize them differently and add some other forms too. They actually combine functional forms for preference-based membership functions as well as possibility distribution and put them in 4 categories. They categorize Zadeh's unimodal functions, Dimitru and Luban's power functions and Svarowski's sin function under membership functions based on heuristic determination category, which is the first category.

They consider Zimmerman's linear function and Tanaka, Uejima and Asai's symmetric triangular function under membership functions based on reliability concerns with respect to the particular problem, which is the second category. But they also add to this category the following membership functions,

Hannan's piecewise linear function: $\mu(x) = \sum_j \alpha_j |x - a_j| + \beta x + r,\ j = 1, ..., N$

$$\alpha_j = \frac{(t_{j+1} - t_j)}{2},$$
$$\beta = \frac{(t_{N+1} + t_1)}{2},$$
$$r = \frac{(s_{N+1} + s_1)}{2}.$$

where $\mu(x) = t_i x + s_i$, for each segment i, $a_{i-1} \leq x \leq a_i$.

Leberling's hyperbolic function: $\mu(x) = \frac{1}{2} + \left(\frac{1}{2}\right) \tanh(a(x - b)),\ -\infty \leq x \leq \infty$

where a is a parameter.

Sakawa and Yumine's exponential and hyperbolic inverse functions, respectively:

$$\mu(x) = c(1 - \exp\left(\tfrac{b-x}{b-a}\right)),\ x \in [a, b],$$
$$\mu(x) = \tfrac{1}{2} + c\tanh^{-1}(d(x - b)),$$

where c and d are parameters.

Dimitru and Luban's function: $\mu(x) = \frac{1}{1+(x/a)}$,

where a ia parameter.

Dubois and Prade's L-R fuzzy number: $\mu(x) = \begin{cases} L\left(\frac{(a-x)}{\alpha}\right) & \text{if } x < a \\ R\left(\frac{(x-b)}{\beta}\right) & \text{if } x > b \\ 1 & \text{if } a \leq x \leq b \end{cases}$

where $L(.)$ and $R(.)$ are reference functions.

The third category encompases the functional forms for the membership functions based on more theoretical demand. Under this category, they present the following forms,

Civanlar and Trussell's function [35]: $\mu(x) = \begin{cases} \text{ap}(x) & \text{if ap}(x) \leq 1 \\ 0 & \text{otherwise} \end{cases}$

where a $\in [0, 1]$ is a parameter and p(x) is the probability density function

Svarovski's function: $\mu(x) = \begin{cases} 0 & \text{if } x < a \\ K(x-a)^2 & \text{if } a \le x \le b \\ K_2x^2 + K_1x + K_0 & \text{if } b < x \le c \\ 1 & \text{if } x > c \end{cases}$

where K, K_0, K_1, and K_2 are parameters.

The fourth and final category that Lai and Hwang present is that corresponding to membership functions that serve as a model for human concepts. Such membership functions include

Hersh and Caramazza's function: $\mu(x) = \frac{1}{2} + d\frac{r}{10}$,

where $d(x) = 1$ for "yes" answers and $d(x) = -1$ for "no" answers, and r is a confidence value. In fact, this function has been determined empirically in an attempt to model the implications of context on the interpretation of a set of linguistic terms.

Zimmerman and Zysno's function: $\mu(x) = \frac{1}{2} + \left(\frac{1}{d}\right)\left[\frac{1}{1+\exp(-a(x-b))} - c\right]$,

which is, in fact, another function that utilizes distance.

Dombi's function: $\mu(x) = \frac{(1-s)x^2}{[(1-s)x^2+s(1-x)^2]}$,

where s is the characteristic value of the shape. It is the intersection value of $y = \mu(x)$ and $y = x$.

There are other functions, which are generally used for possibility distributions and are quite popular in the literature especially with fuzzy numbers.

1. Arc tangent: $\mu = \frac{tg^{-1}(s(x-m))}{\pi} + 0.5$

   where s is the scalar fact and m is the midpoint

2. Gaussian: $\mu = \exp\left(-0.5\left(\frac{x-m}{s}\right)^2\right)$

   where s is the standard deviation and m is the mean

3. Inverse: $\mu = \frac{1}{(1+a(x-c)^b)}$

   where a is the scalar fact, b is the power and c is the starting value

4. Linear: $\mu = \begin{cases} 1 & b = d = 1, x \le c \\ 1 & d = f = 1, c \le x \\ c + k(x-a) & b \ne d, d \ne f, a \le x \le c, k = \frac{d-b}{c-a} \\ f * k(x-c) & c < x \le d, k = \frac{d-b}{a-c} \end{cases}$

   where a is the starting point, b, d and f are 1|0, c is the midpoint, and e is the last point.

5. Sigmoidal: $\mu = \begin{cases} 1 & x \geq c \\ 0.5\left(\frac{x-a}{b-a}\right)^2 & a < x \leq b \\ 1 - 0.5\left(\frac{x-a}{b-a}\right)^2 & b < x \leq c \\ 0 & x \leq a \end{cases}$

where a is when the curve is at its minimum (close to zero), b is when the curve is at its midheight (close to 0.5), and c is when it is at its maximum.

6. Triangular: $\mu = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ \frac{x-c}{b-c} & b \leq x \leq c \\ 0 & x > c \end{cases}$

where $a < b < c$ are the three vertices of the triangle.

7. Trapezoidal: $\mu = \begin{cases} 0 & 0 \leq x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \frac{x-d}{c-d} & c \leq x \leq d \\ 0 & x \geq d \end{cases}$

where $a < b < c < d$ are the four vertices of the trapezoid.

8. Fuzzy Normal Distribution: $\mu(x) = \exp[-k(x-a)^2]$

where $k > 0$ and $a \in \Re$.

9. Fuzzy Sharp Gamma Distribution: $\mu(x) = \begin{cases} \exp[k(x-a)], & x \leq a \\ \exp[-k(x-a)], & x > a \end{cases}$,

where $k > 0$ and $a \in \Re$.

10. Fuzzy Cauchy Distribution: $\mu(x) = \frac{1}{1+\alpha(x-a)^{\beta}}$,

where $\alpha > 0$ and β is positive even.

Clearly, we do need to estimate the parameters for the membership functions. This can be quite a tricky issue actually after defining the membership function. There are some attempts in the literature to estimate such parameters. Kai-Yuan [25] presents an approach to estimating the parameters of a normal fuzzy number (or membership function). As we will see later, Dishkant [43] introduces a new approach based on many-valued logic to estimate parameters.

It is also possible to elicit a possibility distribution from a probability distribution ([49],[106]). So if we agree that membership functions are numerically equivalent to possibility distributions, it is possible to generate membership functions from histograms, given that normalized histograms can be treated as probability distributions and, in the presence of a large data sample, this can be used to approximate a pdf [106]. We will not dwell into this approach much now because we will elaborate more on it later on.

As for the objective approaches, they generally depend on training data. In what follows, we will present the most popular techniques that are utilized in this area.

1. Fuzzy KNN Algorithm [106]: The fuzzy K-nearest neighbor algorithm is a popular objective approach. It is applied to a set of data whereby it assigns a membership value to a sample vector. This membership function describes what fraction of the vector rests in the defined classes. So rather than assigning the vector to a defined class, it assigns class memberships to the test or sample vector. Consider $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$ a set of labeled samples. Then, the membership function of a vector $\mathbf{x}$ in class i is

$$\mu_i(\mathbf{x}) = \frac{\sum_{j=1}^{K} \mu_{ij} \left(\frac{1}{d_j^2}\right)^{\frac{1}{(m-1)}}}{\sum_{j=1}^{K} \left(\frac{1}{d_j^2}\right)^{\frac{1}{(m-1)}}},$$

where μ_{ij} is the membership of the jth labeled vector in the ith class, d_j is the distance between the test vector $\mathbf{x}$ and the jth labeled vector and m is a real number greater than 1 that represents the strength of the fuzzy distance function.

2. Neural Networks and Neuro-Fuzzy Techniques ([56], [106]): Neural networks can be used to generate membership functions from a set of data. Generally, a multilayer feedforward network is used. The empirical justification for using this approach is that the activation function of the neural network is very similar to some of the functional forms of the membership function, which we have presented earlier. It is also possible to use neural-fuzzy approaches by using a fuzzy neuron.

The advantage of such approaches is that it allows generating complex membership functions due to the nonlinearity of the network. However, their disadvantage is that the shape of the membership function can not be known in regions for which there is no data. But it is possible to make up for this by using artificial data with desired output values [106].

3. Fuzzy Clustering Techniques ([7],[56],[106]): These approaches are recommended for generating membership functions when sufficient data is available. Medasani et al. [106] provide a relatively comprehensive summary of two fuzzy clustering techniques, the Fuzzy C-Means method and the Robust agglomerative mixture decomposition, that can be used. Benachenhou [7] utilizes fuzzy logic in trading. To this end, she needs to use a method to fuzzify time series. So she uses clustering techniques to extract the size of the membership function. She also presents a summary of two clustering techniques that have been used to fuzzify time series.

However, we will only present a general outline here. In general, the output data is partitioned into clusters or classes, which is then projected on the input data to generate clusters and select the variables involved in the input-output relation. The shape of the membership functions have to be determined. They allow a graduated, rather than crisp, assignments of the data to clusters ranging between 0 and 1. Divide the data into three groups for model building and testing. Note that clustering techniques involve a distance measure. Usually the data space used is based on the Euclidean norm so one has to carry further analysis before using membership functions that are based on non-Euclidean norms.

4. Many-Valued Logic Approach ([43],[85]): This approach is based on Lukasiewicz many-valued logic t_w. Dishkant [43] introduces a Limit Theorem, which defines a membership type

$$\rho(z) \triangleq \max\{1 - \tfrac{1}{2}c(z-a)^2, 0\},$$

but the parameters c and a remain unknown. So he suggests an approach to parameter estimation where he interprets unknown parameters as fuzzy variables.

The basic idea is to find an optimal estimate t_0 of a fuzzy parameter A. $t_0 \in T$ is a an optimal estimate of A if

$$\exists\, t\lambda_A(t) \rightarrow \lambda_A(t),$$

is true in t_w, where $\lambda_A(t)$ is the membership function of A such that $\lambda_A : T \rightarrow [0,1]$ and T is an arbitrary set of the parameter values. It is possible to have more than one optimal estimate or none at all. In fact, the above condition implies that λ_A is maximal for t_0 as a value of t which can be translated to

$t_0 = \arg\max \lambda_A(t)$.

Assume that we have $\mu_A(x, t)$ as a truth value of the assertion "If A assumes a value t, then the fuzzy variable S assumes a value x," and that $\mu : X \times T \to [0, 1]$ is a priori known. We can arrive at the optimal estimate t_0 by utilizing empirical data which can be subjectively drawn or experimentally obtained. This empirical data spans the truth values μ_k of statements:"S takes a value x_k," where $k = 1, 2, ..., n$. So empirical situation in t_w can be defined by the condition

$$\bigwedge_{1 \le k \le n} (\mu(x_k, t) \leftrightarrow \mu_k),$$

where $\bigwedge_{1 \le k \le n}$ denotes the multinomial rigid conjunction (returns the minimum of two functions). This condition also serves as the membership function of the parameter A which now becomes a fuzzy variable. Therefore, the optimal estimate t_0 to A becomes

$$t_0 = \arg\min_t \max_{1 \le k \le n} |\mu(x_k, t) - \mu_k| .$$

In fact, Dishkant utilizes this formula to arrive at an equivalent of the maximum likelihood approach in Probability Theory, which is not relevant now but the interested reader is referred to [43]. Effectively, Dishkant has established a formula for the membership function of a fuzzy variable, which is the sum of many fuzzy variables with known membership functions.

There are other examples dealt with in the literature. Lai and Hwang reference other less known approaches to eliciting membership functions such as hard c-means algorithm, fuzzy alternative to regression analysis, magnitude estimation etc... ([100], section 2.3.2.4). Dubois and Prade [49] mention a deformable prototype approach, which is similar to other distance approaches but different in that the prototype is deformed such that a maximal similarity is obtained. They also talk about the relative preference method, which utilizes eigenvalues. The membership values are calculated from sets of data representing relative membership values between different elements in a set. They also present comparison of subsets and filter function approaches. The former approach derives a set of inequalities that describe membership values based on which fuzzy subset better matches the fuzzy set under consideration. The latter approach is a filter function that identifies membership functions of fuzzy sets that model adjectives. It depends on the location of the neutral point and the transition between membership and nonmembership. In another book [56], they provide references on approaches involving mathematical derivation and others deriving continuous membership functions from discrete points. They also present literature that draws comparisons among scaling and elicitation methods.

Appendix B

Uncertainty Measures and Nonadditive Measures

B.1 Uncertainty Measures

In this section, we present an overview of the various measures of uncertainty. Even though this is not directly related to our research but it is important to include it because it is relevant within the framework of uncertainty and we have to clarify the difference between such measures and nonadditive measures, which are at the core of our research. In fact, it is the different theories lying behind those nonadditive measures that give rise to different interpretations of uncertainty.

Measures of uncertainty measure the amount of uncertainty itself in the system while nonadditive measures try to model or capture it. The usual measure of uncertainty in Probability Theory is the Shannon entropy. On the other hand, in Fuzzy Theory, there is more than one measure. Given that we have different types of uncertainty, we have different measures of it which will be presented within the framework of Evidence Theory.

The first type of uncertainty is vagueness. Measures of vagueness are also known as measures of fuzziness or, sometimes, as indices of fuzziness. As the name implies, those measures indicate the degree to which the set is fuzzy, that is, the membership of elements in this set is ambiguous. It reflects the difficulty in trying to assess which elements belong to a fuzzy set and which do not.

There are several measures of fuzziness that have been proposed in the literature. Generally, a measure of vagueness or fuzziness is a function

$f : P(X) \to \Re$

where X is the universe of discourse, $P(X)$ is the power set of X. Any measure of fuzziness has to satisfy three requirements, the first of which is unique while the other two depend on the meaning associated with the degree of fuzziness. The unique requirement entails that the degree of fuzziness must be zero for all crisp subsets of the power set, that is, if a subset A is crisp or, in other words, has sharp distinctive boundaries or not fuzzy, its degree of fuzziness given by $f(A)$ has to be zero. The second requirement deals with measures of fuzziness as comparison of sharpness between subsets so that if A is sharper than B ($A < B$), then $f(A) \leq f(B)$. The third requirement states that the degree of fuzziness of a set should be maximal only for a subset that is perceived as maximally fuzzy.

Having defined the basic requirements, we now move on to present such measures ([89],[90], [55],[83]).

1. De Luca and Termini's measure:

$$f(A) = -\sum_{x \in X} [\mu_A(x) \log_2 \mu_A(x) + (1 - \mu_A(x)) \log_2(1 - \mu_A(x))],$$

where $\mu_A(x)$ is the grade of membership which is $\frac{1}{2}$ for a maximally fuzzy set. So for $A < B$ and for all $x \in X$:

$$\mu_A(x) \leq \mu_B(x) \qquad \text{for } \mu_B(x) \leq \tfrac{1}{2},$$

$$\mu_A(x) \geq \mu_B(x) \qquad \text{for } \mu_B(x) \geq \tfrac{1}{2}.$$

2. Kaufmann's index of fuzziness: This measure is defined in terms of a metric distance so that

$$f(A) = \sum_{x \in X} |\mu_A(x) - \mu_B(x)|,$$

when the Hamming distance is used, and

$$f(A) = \left(\sum_{x \in X} [\mu_A(x) - \mu_B(x)]^2\right)^{\frac{1}{2}},$$

when the Euclidean distance is used; other distances can be used as well such as the Minkowskian class of distances

$$f_w(A) = \left(\sum_{x \in X} [\mu_A(x) - \mu_B(x)]^w\right)^{\frac{1}{w}},$$

where $w \in [1, \infty]$; obviously this last measure can serve as a generalization of the other two.

3. Higashi and Klir's measure:

$$f_{c,w}(A) = D_{c,w}(Z, Z^c) - D_{c,w}(A, A^c),$$

where $D_{c,w}(A, A^c)$ is a distance from the Minkowski class defined as

$$D_{c,w}(A, A^c) = \left[\sum_{x \in X} \delta_{c,A}^{w}(x) \right]^{\frac{1}{w}},$$

and

$$\delta_{c,A}(x) = \left| \mu_A(x) - c\left(\mu_A(x)\right) \right|,$$

Z is a crisp set and c stands for the complement. This is based on Yager's requirement that an index of fuzziness has to reflect the lack of discriminating between an event and its complement.

As for measures of ambiguity, they are broadly described via plausibility measures and belief measures (equivalent to upper and lower probabilities respectively). We will not introduce those measures now since we will be talking since we have talked about them indetail in Chapter 2. However, we like to note that, given that probability measures are perceived as a special type of plausibility measures, they can be used to measure ambiguity. Since there are three distinct types of ambiguity, we have to introduce the measures that govern them and which are embedded in plausibility and belief measures.

For the nonspecificity of evidence, there are four possible measures (([89],[90],[55]).

1. Hartley's measure of information:

$$I(N) = K_0 \log_b N,$$

where N is the total number of alternatives and K_0 is an arbitrary positive constant. It is associated with the ambiguity related to the selection of one element from a set of possible alternatives.

2. U-uncertainty:

$$U(\pi) = \int_0^1 \log_2 \mid c(\pi, \alpha) \mid d\alpha,$$

where π is a normalized possibility distribution and $\mid c(\pi, \alpha) \mid$ is the cardinality of the α-cut. This is a possibilistic measure of uncertainty.

3. Yager's measure:

$$a(\pi) = 1 - \int_0^1 \frac{d\alpha}{|c(\pi,\alpha)|},$$

which is another possibilistic measure but less justified than the U-uncertainty one. It is perceived as a possibilistic counterpart to hyperbolic entropy.

4. Generalized U-uncertainty:

$$U_m(m) = \sum_{A \subseteq X} m(A) \log_2 | A |,$$

where $m(A)$ is a basic probability assignment. This is a U-uncertainty measure generalized to plausibility and belief measures.

As for dissonance or confusion in evidence cases, there exist three measures.

1. Shannon Entropy:

$$H(p) = - \sum_{i=1}^{n} p_i \log p_i,$$

where p is a probability distribution. So it is only meaningful for probability measures.

2. Yager's measure of dissonance:

$$E(m) = \sum_{A \subseteq X} m(A) Con(Bel, Bel_A),$$

$$Con(Bel_1, Bel_2) = -\log(1-k),$$

$$k = \sum_{i,j} m_1(A_i) \cdot m_2(B_j), \;\; A_i \cap B_j = \phi,$$

where $Con(Bel_1, Bel_2)$ is the weight of conflict between the two beliefs and Bel stands for the belief function representing the basic probability assignment m. This measure of dissonance is also written as,

$$E(m) = - \sum_{A \subseteq X} m(A) \ln Pl(A).$$

Maximal ambiguity in this case is obtained when $m(A)$ is scattered over, but equally assigned to, a maximal number of disjoint subsets. Notice that E, the measure of dissonance here, and U, regular or generalized U-uncertainty, serve opposing purposes whereby the former discriminates among probability measures but not fuzzy sets and vice-versa for the latter.

3. Measure of confusion:

$$C(m) = - \sum_{A \subseteq X} m(A) \log_2 Bel(A),$$

which is maximal when m is uniformly distributed among the largest possible number of subsets of X such that none of them is a subset of the other. This measure shows that the greater the number of subsets and the more uniform the distribution, the higher our confusion by evidence will be.

A measure of uncertainty can also be considered a measure of information in the sense that the amount of uncertainty eliminated (due to new information) is the same as the amount of information gained. But the measure of uncertainty as a measure of information does not involve pragmatic or semantic aspects of information, that is, information based on uncertainty does not include any pragmatic or semantic aspect but represents the necessary core that any semantic or pragmatic information must contain. In all cases, this is not directly relevant to our research and so we will limit the discussion to this point. We move on to present nonadditive measures that capture such uncertainties and which form the basis of our research.

B.2 Nonadditive Measures

Probability Theory is only a part of classical Measure Theory. Restricted by the additivity assumption imposed by classical Measure Theory, mathematicians started developing theories that challenge this theory. The first of those is the theory of capacities developed by Choquet. Later on, more practical theories encompassing nonadditive measures have been introduced. In 1967, Dempster introduced a new theory that involves upper and lower probabilities and, hence, admits a range of probabilities within an interval. The theory has been later developed by Shafer and became to what is known now as the Dempster-Shafer or Evidence Theory. It introduces two nonadditive measures where one is superadditive, known as a belief measure, and the other is subadditive, known as a plausibility measure. The two measures induce each other, that is, it is enough to know one to infer the other.

In 1978, Zadeh [150] introduced Possibility Theory. We have talked about this theory in detail during the course of our research. As we have seen, possibility measures are special cases of plausibility measures and, hence, they also arise from Evidence Theory. They are usually associated with necessity measures. More recently, Walley introduced imprecise probabilities.

We will be presenting all of those measures in this appendix. However, our main focus will be on a theory that has been developed in 1974 by Sugeno. Sugeno ([133],[134]) has introduced Fuzzy Measure Theory, which includes the nonadditive measures, known as fuzzy measures, and the nonlinear integral, known as Sugeno or fuzzy integral, in an attempt to compare probabilities to fuzzy sets. We will not dwell much on this topic here since we have devoted an entire chapter (Chapter 2) to analyze it. However, we like to note that, sometimes, the term "fuzzy measure" is used as a general term to all nonadditive measures such that all nonadditive measures we have mentioned so far become special cases of it.

B.2.1 Dempster-Shafer Theory

This theory is based on evidential reasoning. That is it allows us to develop partial beliefs based on pieces of evidence that convey vague, imprecise or incomplete information. The basic element in this theory is a function m, known as a basic probability assignment, where

$m : 2^X \rightarrow [0,1]$,

$m(\phi) = 0$,

$\sum_{A \subseteq X} m(A) = 1$,

where X is a finite set that could represent the set of alternatives or solutions. When $m(A) > 0$, the subsets A are known as focal subsets making up $\Im$. $(\Im, m)$ represents the body of evidence that entails a piece of information. It is within this framework that $m(A)$ represents the confidence level of the subset A as a representative of the piece of information. Another way to look at $m(A)$ is as the degree of belief that a certain element of X belongs to A.

However, it is in fact allocated to ignorance so that as $m(A)$ increases, ignorance increases because A is viewed as an imprecise observation. The focal subsets do not have to be disjoint and they do not have to span the entire set X. Focal subsets represent mutually exclusive possible values of a variable. In such cases, information is known to be disjunctive. But information can also be conjunctive, that is, each focal subset can represent a set of values that the variable can take.

B.2.2 Belief and Plausibility Measures

Those measures are associated with the basic probability assignment such that

$$Bel(A) = \sum_{B \subseteq A} m(B),$$
$$Pl(A) = \sum_{B \cap A \neq \phi} m(B),$$
$$Pl(A) = 1 - Bel(\bar{A}),$$

where Bel and Pl stand for belief and plausibility measures respectively. Belief measures indicate the weight of evidence characterizing subsets of A. On the other hand, plausibility measures indicate the weight of evidence that does not focus on $\bar{A}$.

When all focal subsets are singletons, both measures converge to the probability measures (in fact, a probability measure is a special plausibility measure). m is usually a probability assignment on 2^X but in this case, it is usually a usual probability assignment. When the focal subsets are not singletons, the following inequalities hold,

superadditivity:$Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$,

subadditivity:$Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B)$.

B.2.3 Necessity and Possibility measures

When the focal subsets are nested such that $\Im = \{A_1 \subset A_2 \subset ... \subset A_n\}$, the bodies of evidence is called consonant. This gives rise to two new nonadditive measures which are special cases of belief and plausibility measures. Those measures are possibility, which is a special plausibility measure, and necessity, which is a special belief measure. A possibility measure is defined by taking the supremum of the possibility distribution. It was first introduced by Zadeh [150] in 1978. Let π be a possibility distribution function such that

$$\pi : X \to [0, 1],$$
$$Pos(A) = \Pi(A) = \sup_{x \in A} \pi(x),$$
$$Nec(A) = 1 - \Pi(A),$$

where Pos (or Π) stand for the possibility measure and $Nec(A)$ stands for the necessity measure. Those measures usually satisfy the following inequalities

$$Pos(A \cup B) = \max [Pos(A), Pos(B)],$$
$$Nec(A \cap B) = \min [Nec(A), Nec(B)].$$

The possibility distribution can in fact be related to the basic assignment m (we will talk in detail about possibility distributions in the following chapter). Let $n = |X|$

be the cardinality of X and order x_i such that $\pi(x_i)$ is monotonically increasing for $i = 1, ..., n$. Therefore,

$$\pi(x_i) = \sum_{j=i}^{n} m_j,$$

which can also be written as

$$m_i = \pi(x_i) - \pi(x_{i+1}).$$

Note that $\pi(x_{n+1}) = 0$ by convention and $\pi(x_1) = 1$. Given that a membership function is numerically equivalent to a possibility distribution, we can infer that a fuzzy set is a consonant body of evidence. Dubois and Prade [55] differentiate between those measures and probability measures. The former arise from imprecise but consonant pieces of information (nested subsets) while probability measures stem from elementary focal subsets (singletons).

B.2.4 A Fuzzy Dempster-Shafer Theory

A fuzzy basic assignment can be defined as

$$\tilde{m} : \tilde{P}(X) \to [0, 1],$$

where $\tilde{P}(X)$ is the fuzzy power set or the set of all fuzzy subsets of X. Plausibility and belief measures can now be defined as,

$$\widetilde{Bel}(A) = \sum_{B \in \tilde{\Im}} \tilde{m}(B) \left[1 - \max_{x \in X} \min\left(1 - \mu_A(x), \mu_B(x)\right)\right],$$

$$\widetilde{Pl}(A) = \sum_{B \in \tilde{\Im}} \tilde{m}(B) \left[\max_{x \in X} \min\left(\mu_A(x), \mu_B(x)\right)\right],$$

where $\mu_A(x)$ and $\mu_B(x)$ are the grades of membership of x in fuzzy sets A and B, respectively, and $\tilde{\Im}$ is the set of all fuzzy focal elements associated with $\tilde{m}$.

B.2.5 Fuzzy Measures

We have already discussed fuzzy measures in detail in chapter two so we only include it here for the purpose of being comprehensive. Evidence Theory, Possibility Theory and Probability Theory are all special cases of the more general Fuzzy Measure Theory which necessarily means that the corresponding measures are also special cases of fuzzy measures as well. This is, of course, the general agreement in the literature but there are still some controversies over the truth of such assertions [124].

B.2.6 Imprecise Probabilities

As we have seen, the additivity requirement in classical Probability Theory is quite restrictive since it imposes estimating probability with a precise number and requires a clear distinction between an event and its negation. But imperfect information due to many widespread sources is widely prevalent and to be able to model those situations, we have to relax the precision assumption. This can be done using imprecise probabilities which generally involve an interval defined by a lower and an upper probability value. They represent minimum and maximum acceptable rates respectively. This theory induces decision making to satisfy principles of coherence and avoiding sure loss.

The idea behind the introduction of imprecise probabilities is to introduce a unified theory that can accommodate all types of uncertainty and include all measures as special cases. Imprecise probabilities are supposed to include the measures we have discussed so far as special cases and adds to them other more general measures. Walley [139] presents those measures in order of increasing generality manner as

1. necessity and possibility measures
2. belief and plausibility measures
3. Choquet capacities of order 2
4. coherent upper and lower probabilities
5. coherent upper and lower previsions
6. sets of probability measures
7. sets of desirable gambles
8. partial preference ordering.

Model 9 is supposed to be a special case of models 5-8 so it does not exactly fit in this list

9. partial comparative probability ordering.

Of course, as we have seen in the literature on fuzzy measures, fuzzy measures are supposed to include models 1, 2 and 6. Walley refers to fuzzy measures as capacities or capacities of order 1. It is not in our interest to try to sort out the differences in points of view, if there are any, nor are we interested in a detailed analysis of imprecise probabilities so we will only confine our analysis in to a brief overview of each of these models. We have already seen models 1 and 2 so we only have to present the remaining.

Choquet Capacities of order 2

Let Ω be our sample space or the set of all possible states. In the first four models, it is enough to define only the lower probabilities because the upper one can be derived from it. So if we define $\underline{P}(A)$ as a lower probability, the upper probability can be derived using $\overline{P}(A) = 1-\underline{P}(A^c)$, where $A \in X$, a set of subsets of Ω and c stands for conjugate. It has to satisfy the following requirements

$\underline{P}(\phi) = 0,$

$\underline{P}(\Omega) = 1,$

$0 \leq \underline{P}(A) \leq 1.$

It is said to be a Choquet capacity of order 2 if it satisfies the following

$\underline{P}(A \cup B)+\underline{P}(A \cap B) \geq \underline{P}(A)+\underline{P}(B).$

Coherent Lower Probabilities

Coherent lower probabilities serve as a lower envelope of a set of probability measures. A lower probability is coherent iff there exists a non-empty set of probability measures, M, such that

$\underline{P}(A) = \inf\{P(A) : P \in M\}.$

The shortcomings of such types of probabilities that hinders their generality are their inability to model comparative probability judgements and to determine unique conditional probabilities. However, their most important shortcoming within our context is their inability of determining unique lower or upper expectations.

Coherent Lower Previsions

A gamble is defined as a mapping from Ω to $\Re$. If X is a non-empty set of gambles, then a lower prevision can be defined as

$P : X \rightarrow \Re.$

It is called coherent if it forms an envelope for a set of linear expectation such that, assuming M is a set of probability measures,

$\underline{P}(A) = \inf\{E_P(A) : P \in M\}$,

where $E_P(A)$ stands for the expectations of A with respect to P. The conjugate upper prevision is determined using

$\overline{P}(A) = -\underline{P}(-A)$.

When X is a linear space of gambles, $\underline{P}(A)$ has to satisfy three axioms

1. $\underline{P}(A) \geq \inf\{X(\varpi) : \varpi \in \Omega\}$,
2. $\underline{P}(cA) = c\underline{P}(A), \qquad c > 0$,
3. $\underline{P}(A + B) \geq \underline{P}(A) + \underline{P}(B)$.

Unlike coherent lower probabilities, coherent lower previsions are able to model comparative probabilities, and determine expectations and conditional probabilities. They can also solve the problem of missing information. However, they have two main shortcomings. When $\underline{P}(A) = 0$,they cannot determine conditional lower previsions $P(. \mid A)$, which is important when updating lower previsions after observing A. Besides, they cannot distinguish preference from weak preference.

Sets of Probability Measures

Sets of probability measures reduce to a single measure in the presence of precise probabilities. A closed convex set of probability measures is the set off all such measures that are have lower previsions as their lower envelope and that have expectations that dominate the lower prevision. The convexity assumption is not really necessary and such sets, when not considered in combination with others, do not change behaviour in the absence of convexity. Those measures are more informative than lower previsions but they still have the same drawbacks.

Sets of Desirable Gambles and Partial Preference Orderings

Assume that X is the set of all gambles whereby a gamble assumes the same definition given to it in section 4.6.3. Let D denote the set of desirable gambles, which is a subset of X. It is called coherent if it satisfies the following axioms, assuming that A, $B \in X$

1. if $A \in D$ and $0 > A$ then $A \notin D$,
2. if $A \in D$ and $A > 0$ then $A \in D$,
3. if $A \in D$ and $c \in \Re^+$ then $c\ A \in D$,
4. if $A \in D$ and $B \in D$ then $A \in D$,

where $A > B$ means that $A \geq B$ and $A(\varpi) > B(\varpi)$, for $\varpi \in \Omega$. A coherent set of desirable gambles is a convex cone of gambles allowing only positive gambles and excluding negative ones.

A partial preference ordering $>$ is a partial ordering of the gambles in X. So that $A > B$ implies that A is preferred to B. There is a one-to-one correspondence between sets of desirable gambles and coherent partial preference ordering, which implies that both models are equally general. This correspondence is defined by $A > B$ holds iff $A - B \in D$. However, sets of desirable gambles are simpler mathematically than coherent partial preference ordering. Those models include all information in previous models in imprecise probabilities and add additional information that allows conditioning on 0 and differentiates between preference and weak preference.

APPENDIX C

Table 1C: This table shows the option bounds for Dell stock options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Black-Scholes implied volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	May-02	17.30	17.60	15.40	17.44	17.44	0.00	17.60	2.56	19.28
ITM	May-02	14.90	15.10	16.50	14.94	14.94	0.00	15.10	1.56	16.69
ITM	May-02	12.40	12.60	14.90	12.44	12.44	0.00	12.60	1.03	13.98
ITM	May-02	9.90	10.10	9.00	9.94	10.30	0.36	10.10	0.69	11.37
ITM	May-02	7.40	7.60	8.00	7.44	7.98	0.54	7.60	0.44	8.73
ITM	May-02	5.00	5.20	5.30	4.94	5.80	0.86	5.20	0.26	6.13
ITM	May-02	2.75	2.85	2.80	2.44	3.67	1.23	2.85	0.11	3.51
ATM	May-02	0.95	1.05	0.95	0.06	1.92	1.86	1.05	0.00	0.95
OTM	May-02	0.10	0.25	0.15	0.00	0.75	0.75	0.25	0.00	0.00
OTM	May-02	0.00	0.10	0.10	0.00	0.33	0.33	0.10	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.17	0.17	0.05	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.11	0.11	0.05	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.08	0.08	0.05	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.06	0.06	0.05	0.00	0.00
ITM	Jun-02	7.50	7.70	8.10	7.44	8.04	0.60	7.70	0.44	8.76
ITM	Jun-02	5.20	5.40	5.70	4.94	5.70	0.76	5.40	0.26	6.19
ITM	Jun-02	3.20	3.40	3.50	2.67	3.59	0.93	3.40	0.12	3.65
ATM	Jun-02	1.60	1.70	1.85	0.13	1.93	1.80	1.70	0.01	1.09
OTM	Jun-02	0.50	0.65	0.75	0.00	0.74	0.74	0.65	0.00	0.00
OTM	Jun-02	0.15	0.25	0.20	0.00	0.32	0.32	0.25	0.00	0.00
OTM	Jun-02	0.00	0.10	0.10	0.00	0.17	0.17	0.10	0.00	0.00
OTM	Jun-02	0.00	0.10	0.00	0.00	0.11	0.11	0.10	0.00	0.00
OTM	Jun-02	0.00	0.10	0.00	0.00	0.08	0.08	0.10	0.00	0.00
ITM	Aug-02	17.30	17.60	0.00	17.44	17.44	0.00	17.60	13.07	19.22
ITM	Aug-02	14.90	15.10	0.00	14.94	14.94	0.00	15.10	1.71	16.62
ITM	Aug-02	12.40	12.60	11.70	12.44	12.44	0.00	12.60	1.03	13.92

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	Aug-02	10.10	10.30	9.40	9.94	9.94	0.00	10.30	0.69	11.42
ITM	Aug-02	7.80	8.00	6.00	7.44	7.81	0.37	8.00	0.45	8.87
ITM	Aug-02	5.50	5.80	6.00	4.94	5.58	0.64	5.80	0.26	6.27
ITM	Aug-02	3.70	3.90	4.40	2.73	3.61	0.88	3.90	0.12	3.75
ATM	Aug-02	2.20	2.35	2.55	0.20	1.84	1.63	2.35	0.01	1.22
OTM	Aug-02	1.15	1.25	1.45	0.00	0.79	0.79	1.25	0.00	0.00
OTM	Aug-02	0.50	0.65	0.60	0.00	0.32	0.32	0.65	0.00	0.00
OTM	Aug-02	0.10	0.25	0.30	0.00	0.17	0.17	0.25	0.00	0.00
OTM	Aug-02	0.05	0.20	0.20	0.00	0.11	0.11	0.20	0.00	0.00
OTM	Aug-02	0.00	0.10	0.10	0.00	0.08	0.08	0.10	0.00	0.00
OTM	Aug-02	0.00	0.10	0.05	0.00	0.06	0.06	0.10	0.00	0.00
ITM	Nov-02	8.30	8.50	8.40	7.44	7.91	0.47	8.50	0.48	8.93
ITM	Nov-02	6.30	6.50	6.50	5.36	5.79	0.43	6.50	0.27	6.41
ITM	Nov-02	4.60	4.80	4.80	2.84	3.65	0.81	4.80	0.13	3.91
ATM	Nov-02	3.10	3.30	3.40	0.30	1.82	1.51	3.30	0.01	1.37
OTM	Nov-02	2.05	2.20	2.05	0.00	0.76	0.76	2.20	0.00	0.00
OTM	Nov-02	1.20	1.35	1.35	0.00	0.34	0.34	1.35	0.00	0.00
OTM	Nov-02	0.65	0.80	1.00	0.00	0.17	0.17	0.80	0.00	0.00
OTM	Nov-02	0.35	0.50	0.60	0.00	0.10	0.10	0.50	0.00	0.00
OTM	Nov-02	0.15	0.30	0.00	0.00	0.08	0.08	0.30	0.00	0.00
ITM	Jan-03	17.30	17.70	0.00	17.44	17.44	0.00	17.70	203.63	19.12
ITM	Jan-03	15.10	15.40	14.80	14.94	14.94	0.00	15.40	36.48	16.62
ITM	Jan-03	12.80	13.00	13.60	12.44	12.44	0.00	13.00	2.79	14.10
ITM	Jan-03	10.60	10.80	11.30	9.94	9.94	0.00	10.80	1.14	11.54
ITM	Jan-03	8.60	8.80	9.40	7.61	7.96	0.35	8.80	0.64	8.98
ITM	Jan-04	17.90	18.10	15.80	17.44	17.44	0.00	18.10	42508.85	19.18
ITM	Jan-04	15.80	16.20	15.00	14.94	14.94	0.00	16.20	46949.38	16.73
ITM	Jan-04	13.50	14.30	15.20	12.44	12.44	0.00	14.30	13495.68	14.17
ITM	Jan-04	11.90	12.20	12.30	9.94	9.94	0.00	12.20	1096.32	11.49
ITM	Jan-04	10.20	10.40	9.90	8.18	7.44	-0.74	10.40	313.81	9.00
ITM	Jan-04	8.70	8.90	9.00	5.66	5.66	0.00	8.90	180.32	6.57

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	Jan-04	7.20	7.50	7.80	3.13	3.27	0.14	7.50	91.69	4.14
ATM	Jan-04	6.00	6.40	6.50	0.63	1.74	1.11	6.40	73.01	1.71
OTM	Jan-04	5.00	5.10	5.30	0.00	0.79	0.79	5.10	24.38	0.01
OTM	Jan-04	3.90	4.30	4.40	0.00	0.32	0.32	4.30	21.28	0.00
OTM	Jan-04	2.55	2.85	2.70	0.00	0.10	0.10	2.85	9.81	0.00
OTM	Jan-04	1.65	1.80	1.90	0.00	0.06	0.05	1.80	4.57	0.00
OTM	Jan-04	1.00	1.15	1.15	0.00	0.03	0.03	1.15	2.87	0.00
OTM	Jan-04	0.50	0.75	0.80	0.00	0.03	0.02	0.75	2.16	0.00

Table 2C: This table shows the option bounds for Dell stock options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Historical volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	May-02	17.30	17.60	15.40	17.44	17.44	0.00	17.44	2.33	18.38
ITM	May-02	14.90	15.10	16.50	14.94	14.94	0.00	14.94	1.49	15.88
ITM	May-02	12.40	12.60	14.90	12.44	12.44	0.00	12.44	1.00	13.38
ITM	May-02	9.90	10.10	9.00	9.94	10.31	0.37	9.95	0.66	10.88
ITM	May-02	7.40	7.60	8.00	7.44	8.03	0.59	7.45	0.43	8.38
ITM	May-02	5.00	5.20	5.30	4.94	5.82	0.88	4.95	0.25	5.89
ITM	May-02	2.75	2.85	2.80	2.44	3.73	1.29	2.57	0.11	3.39
ATM	May-02	0.95	1.05	0.95	0.04	1.91	1.87	0.84	0.00	0.89
OTM	May-02	0.10	0.25	0.15	0.00	0.77	0.77	0.15	0.00	0.00
OTM	May-02	0.00	0.10	0.10	0.00	0.33	0.33	0.01	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.18	0.18	0.00	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.11	0.11	0.00	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.08	0.08	0.00	0.00	0.00
OTM	May-02	0.00	0.05	0.05	0.00	0.06	0.06	0.00	0.00	0.00
ITM	Jun-02	7.50	7.70	8.10	7.44	7.89	0.45	7.54	0.44	8.64
ITM	Jun-02	5.20	5.40	5.70	4.94	5.72	0.78	5.28	0.26	6.14
ITM	Jun-02	3.20	3.40	3.50	2.53	3.71	1.18	3.26	0.12	3.62
ATM	Jun-02	1.60	1.70	1.85	0.15	1.87	1.72	1.83	0.01	1.13
OTM	Jun-02	0.50	0.65	0.75	0.00	0.74	0.74	0.93	0.00	0.00
OTM	Jun-02	0.15	0.25	0.20	0.00	0.33	0.33	0.43	0.00	0.00
OTM	Jun-02	0.00	0.10	0.10	0.00	0.18	0.18	0.18	0.00	0.00
OTM	Jun-02	0.00	0.10	0.00	0.00	0.11	0.11	0.07	0.00	0.00
OTM	Jun-02	0.00	0.10	0.00	0.00	0.08	0.08	0.03	0.00	0.00
ITM	Aug-02	17.30	17.60	0.00	17.44	17.44	0.00	17.48	2.36	18.72
ITM	Aug-02	14.90	15.10	0.00	14.94	14.94	0.00	14.99	1.52	16.22
ITM	Aug-02	12.40	12.60	11.70	12.44	12.44	0.00	12.50	1.02	13.73
ITM	Aug-02	10.10	10.30	9.40	9.94	10.23	0.29	10.05	0.68	11.23
ITM	Aug-02	7.80	8.00	6.00	7.44	8.02	0.58	7.72	0.44	8.74

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	Aug-02	5.50	5.80	6.00	4.94	5.82	0.88	5.64	0.26	6.25
ITM	Aug-02	3.70	3.90	4.40	2.73	3.62	0.89	3.91	0.12	3.75
ATM	Aug-02	2.20	2.35	2.55	0.23	1.76	1.53	2.58	0.01	1.26
OTM	Aug-02	1.15	1.25	1.45	0.00	0.78	0.78	1.64	0.00	0.00
OTM	Aug-02	0.50	0.65	0.60	0.00	0.34	0.34	1.00	0.00	0.00
OTM	Aug-02	0.10	0.25	0.30	0.00	0.17	0.17	0.60	0.00	0.00
OTM	Aug-02	0.05	0.20	0.20	0.00	0.11	0.11	0.35	0.00	0.00
OTM	Aug-02	0.00	0.10	0.10	0.00	0.08	0.08	0.20	0.00	0.00
OTM	Aug-02	0.00	0.10	0.05	0.00	0.06	0.06	0.11	0.00	0.00
ITM	Nov-02	8.30	8.50	8.40	7.44	7.84	0.40	8.20	0.45	8.89
ITM	Nov-02	6.30	6.50	6.50	5.34	5.75	0.41	6.37	0.27	6.40
ITM	Nov-02	4.60	4.80	4.80	2.84	3.66	0.82	4.84	0.13	3.91
ATM	Nov-02	3.10	3.30	3.40	0.34	1.80	1.46	3.62	0.02	1.43
OTM	Nov-02	2.05	2.20	2.05	0.00	0.77	0.77	2.66	0.00	0.00
OTM	Nov-02	1.20	1.35	1.35	0.00	0.30	0.30	1.94	0.00	0.00
OTM	Nov-02	0.65	0.80	1.00	0.00	0.18	0.18	1.40	0.00	0.00
OTM	Nov-02	0.35	0.50	0.60	0.00	0.11	0.11	1.01	0.00	0.00
OTM	Nov-02	0.15	0.30	0.00	0.00	0.08	0.08	0.72	0.00	0.00
ITM	Jan-03	17.30	17.70	0.00	17.44	17.44	0.00	17.55	2.49	18.83
ITM	Jan-03	15.10	15.40	14.80	14.94	14.94	0.00	15.11	1.62	16.35
ITM	Jan-03	12.80	13.00	13.60	12.44	12.44	0.00	12.74	1.10	13.87
ITM	Jan-03	10.60	10.80	11.30	9.94	9.94	0.00	10.53	0.75	11.38
ITM	Jan-03	8.60	8.80	9.40	7.44	7.90	0.46	8.54	0.50	8.90
ITM	Jan-04	17.90	18.10	15.80	17.44	17.44	0.00	17.89	603.97	18.84
ITM	Jan-04	15.80	16.20	15.00	14.94	14.94	0.00	15.71	452.74	16.35
ITM	Jan-04	13.50	14.30	15.20	12.44	12.44	0.00	13.71	362.00	13.87
ITM	Jan-04	11.90	12.20	12.30	9.94	9.94	0.00	11.93	301.51	11.43
ITM	Jan-04	10.20	10.40	9.90	8.17	7.44	-0.73	10.35	258.30	9.00
ITM	Jan-04	8.70	8.90	9.00	5.67	5.67	0.00	8.97	225.89	6.57
ITM	Jan-04	7.20	7.50	7.80	3.17	3.17	0.00	7.78	200.69	4.14
ATM	Jan-04	6.00	6.40	6.50	0.67	1.55	0.88	6.74	180.52	1.72

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
OTM	Jan-04	5.00	5.10	5.30	0.00	0.60	0.60	5.86	164.09	0.01
OTM	Jan-04	3.90	4.30	4.40	0.00	0.32	0.32	5.09	150.41	0.00
OTM	Jan-04	2.55	2.85	2.70	0.00	0.11	0.11	3.87	128.93	0.00
OTM	Jan-04	1.65	1.80	1.90	0.00	0.05	0.05	2.96	112.81	0.00
OTM	Jan-04	1.00	1.15	1.15	0.00	0.03	0.03	2.29	100.27	0.00
OTM	Jan-04	0.50	0.75	0.80	0.00	0.02	0.02	1.78	90.25	0.00

Table 3C: This table shows the option bounds for Microsoft stock options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Black-Scholes implied volatility is used in the calculations.

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	May-02	0.01	29.20	29.40	30.30	29.30	29.30	0.00	29.40	1.34	31.24
ITM	May-02	0.01	26.70	27.00	24.50	26.80	26.80	0.00	27.00	1.11	28.74
ITM	May-02	0.01	21.70	22.00	19.50	21.80	21.80	0.00	22.00	0.75	23.61
ITM	May-02	0.01	16.70	16.90	14.60	16.80	17.27	0.47	16.90	0.49	18.33
ITM	May-02	0.01	11.70	12.00	13.70	12.08	12.57	0.49	12.00	0.30	13.26
ITM	May-02	0.01	6.90	7.00	7.40	6.80	7.99	1.19	7.00	0.16	8.02
ITM	May-02	0.01	4.50	4.70	4.90	4.51	5.76	1.26	4.70	0.09	5.48
ITM	May-02	0.01	2.45	2.70	2.60	1.99	3.70	1.71	2.70	0.04	2.93
OTM	May-02	0.01	0.30	0.40	0.45	0.00	0.97	0.97	0.40	0.00	0.00
OTM	May-02	0.01	0.00	0.10	0.05	0.00	0.27	0.27	0.10	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.12	0.12	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.07	0.07	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.10	0.00	0.05	0.05	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.00	0.00	0.03	0.03	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.03	0.03	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.00	0.00	0.02	0.02	0.05	0.00	0.00
ITM	Jun-02	0.01	31.60	31.90	32.60	31.80	31.80	0.00	31.90	1.71	33.79
ITM	Jun-02	0.01	29.10	29.40	0.00	29.30	29.30	0.00	29.40	1.36	31.15
ITM	Jun-02	0.01	26.70	26.90	0.00	26.80	26.80	0.00	26.90	1.11	28.59
ITM	Jun-02	0.01	21.70	21.90	0.00	21.80	21.80	0.00	21.90	0.75	23.45
ITM	Jun-02	0.01	16.50	17.30	14.40	17.45	16.80	-0.65	17.30	0.50	18.56
ITM	Jun-02	0.01	12.00	12.20	13.70	12.24	12.24	0.00	12.20	0.31	13.29
ITM	Jun-02	0.01	7.60	7.80	8.30	7.19	7.99	0.80	7.80	0.16	8.27
ITM	Jun-02	0.01	5.60	5.80	6.30	4.67	5.78	1.12	5.80	0.10	5.73
ITM	Jun-02	0.01	3.90	4.00	3.90	2.14	3.63	1.49	4.00	0.04	3.17
OTM	Jun-02	0.01	1.45	1.55	1.50	0.00	0.97	0.97	1.55	0.00	0.00
OTM	Jun-02	0.01	0.40	0.50	0.45	0.00	0.27	0.27	0.50	0.00	0.00
OTM	Jun-02	0.01	0.05	0.15	0.10	0.00	0.12	0.12	0.15	0.00	0.00

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
OTM	Jun-02	0.01	0.00	0.20	0.05	0.00	0.07	0.07	0.20	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.05	0.00	0.05	0.05	0.20	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.03	0.03	0.20	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.02	0.02	0.20	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.02	0.02	0.20	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.02	0.02	0.20	0.00	0.00
ITM	Jul-02	0.01	26.50	26.90	24.00	26.80	26.80	0.00	26.90	1.10	28.54
ITM	Jul-02	0.01	21.30	22.30	19.90	21.80	21.80	0.00	22.30	0.78	23.62
ITM	Jul-02	0.01	16.80	17.10	17.10	16.80	17.34	0.54	17.10	0.50	18.40
ITM	Jul-02	0.01	12.10	12.90	14.00	12.41	12.41	0.00	12.90	0.31	13.52
ITM	Jul-02	0.01	8.10	8.30	9.20	7.27	7.74	0.47	8.30	0.16	8.31
ITM	Jul-02	0.01	6.10	6.60	6.60	4.77	5.72	0.95	6.60	0.10	5.82
ITM	Jul-02	0.01	4.60	4.80	4.70	2.23	3.54	1.31	4.80	0.04	3.30
OTM	Jul-02	0.01	2.10	2.35	2.30	0.00	0.95	0.95	2.35	0.00	0.00
OTM	Jul-02	0.01	0.85	0.95	0.90	0.00	0.27	0.27	0.95	0.00	0.00
OTM	Jul-02	0.01	0.30	0.35	0.35	0.00	0.12	0.12	0.35	0.00	0.00
OTM	Jul-02	0.01	0.05	0.15	0.15	0.00	0.07	0.07	0.15	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.10	0.00	0.05	0.05	0.20	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.03	0.03	0.20	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.02	0.02	0.20	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.02	0.02	0.20	0.00	0.00
OTM	Jul-02	0.01	0.00	0.05	0.10	0.00	0.02	0.02	0.05	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.01	0.01	0.20	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.01	0.01	0.20	0.01	0.00
ITM	Oct-02	0.01	29.30	29.50	29.20	29.30	29.30	0.00	29.50	1.41	30.97
ITM	Oct-02	0.01	26.70	27.10	30.50	26.80	26.80	0.00	27.10	1.28	28.52
ITM	Oct-02	0.01	21.60	22.60	20.60	21.80	21.80	0.00	22.60	1.10	23.64
ITM	Oct-02	0.01	17.70	17.80	18.50	17.56	16.80	-0.76	17.80	0.55	18.53
ITM	Oct-02	0.01	13.40	13.60	14.30	12.52	12.52	0.00	13.60	0.34	13.55
ITM	Oct-02	0.01	9.60	9.80	9.80	7.47	7.47	0.00	9.80	0.18	8.56
ITM	Oct-02	0.01	7.70	8.10	8.40	4.94	5.60	0.65	8.10	0.11	6.05

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	Oct-02	0.01	6.30	6.60	6.50	2.43	3.70	1.28	6.60	0.06	3.55
OTM	Oct-02	0.01	3.90	4.10	4.00	0.00	0.98	0.98	4.10	0.01	0.00
OTM	Oct-02	0.01	2.20	2.40	2.30	0.00	0.27	0.27	2.40	0.00	0.00
OTM	Oct-02	0.01	1.20	1.25	1.20	0.00	0.12	0.12	1.25	0.00	0.00
OTM	Oct-02	0.01	0.55	0.70	0.65	0.00	0.07	0.07	0.70	0.00	0.00
OTM	Oct-02	0.01	0.25	0.35	0.40	0.00	0.04	0.04	0.35	0.00	0.00

Table 4C: This table shows the option bounds for Microsoft stock options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Historical volatility is used in the calculations.

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	May-02	0.01	29.20	29.40	30.30	29.30	29.30	0.00	29.31	1.31	30.42
ITM	May-02	0.01	26.70	27.00	24.50	26.80	26.80	0.00	26.81	1.08	27.92
ITM	May-02	0.01	21.70	22.00	19.50	21.80	21.98	0.18	21.81	0.73	22.92
ITM	May-02	0.01	16.70	16.90	14.60	16.80	17.34	0.54	16.81	0.48	17.92
ITM	May-02	0.01	11.70	12.00	13.70	11.80	12.52	0.72	11.82	0.30	12.92
ITM	May-02	0.01	6.90	7.00	7.40	6.80	7.89	1.09	6.87	0.15	7.92
ITM	May-02	0.01	4.50	4.70	4.90	4.30	5.76	1.46	4.57	0.09	5.42
ITM	May-02	0.01	2.45	2.70	2.60	1.98	3.81	1.83	2.65	0.04	2.92
OTM	May-02	0.01	0.30	0.40	0.45	0.00	0.98	0.98	0.52	0.00	0.00
OTM	May-02	0.01	0.00	0.10	0.05	0.00	0.27	0.27	0.05	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.12	0.12	0.00	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.07	0.07	0.00	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.10	0.00	0.05	0.05	0.00	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.00	0.00	0.03	0.03	0.00	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.05	0.00	0.03	0.03	0.00	0.00	0.00
OTM	May-02	0.01	0.00	0.05	0.00	0.00	0.02	0.02	0.00	0.00	0.00
ITM	Jun-02	0.01	31.60	31.90	32.60	31.80	31.80	0.00	31.84	1.61	33.14
ITM	Jun-02	0.01	29.10	29.40	0.00	29.30	29.30	0.00	29.35	1.32	30.65
ITM	Jun-02	0.01	26.70	26.90	0.00	26.80	26.80	0.00	26.85	1.09	28.15
ITM	Jun-02	0.01	21.70	21.90	0.00	21.80	22.14	0.34	21.86	0.74	23.15
ITM	Jun-02	0.01	16.50	17.30	14.40	16.80	17.14	0.34	16.88	0.49	18.16
ITM	Jun-02	0.01	12.00	12.20	13.70	11.80	12.48	0.68	11.98	0.30	13.16
ITM	Jun-02	0.01	7.60	7.80	8.30	7.14	7.83	0.69	7.51	0.16	8.17
ITM	Jun-02	0.01	5.60	5.80	6.30	4.64	5.68	1.04	5.61	0.10	5.67
ITM	Jun-02	0.01	3.90	4.00	3.90	2.14	3.58	1.44	4.02	0.04	3.17
OTM	Jun-02	0.01	1.45	1.55	1.50	0.00	1.00	1.00	1.82	0.00	0.00
OTM	Jun-02	0.01	0.40	0.50	0.45	0.00	0.26	0.26	0.70	0.00	0.00
OTM	Jun-02	0.01	0.05	0.15	0.10	0.00	0.12	0.12	0.23	0.00	0.00

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
OTM	Jun-02	0.01	0.00	0.20	0.05	0.00	0.07	0.07	0.07	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.05	0.00	0.05	0.05	0.02	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.03	0.03	0.00	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.03	0.03	0.00	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.02	0.02	0.00	0.00	0.00
OTM	Jun-02	0.01	0.00	0.20	0.00	0.00	0.02	0.02	0.00	0.00	0.00
ITM	Jul-02	0.01	26.50	26.90	24.00	26.80	26.80	0.00	26.89	1.09	28.23
ITM	Jul-02	0.01	21.30	22.30	19.90	21.80	21.80	0.00	21.90	0.74	23.24
ITM	Jul-02	0.01	16.80	17.10	17.10	16.80	17.20	0.40	16.95	0.49	18.25
ITM	Jul-02	0.01	12.10	12.90	14.00	12.20	12.60	0.40	12.15	0.30	13.26
ITM	Jul-02	0.01	8.10	8.30	9.20	7.20	8.01	0.81	7.88	0.16	8.27
ITM	Jul-02	0.01	6.10	6.60	6.60	4.70	5.51	0.81	6.08	0.10	5.77
ITM	Jul-02	0.01	4.60	4.80	4.70	2.20	3.75	1.55	4.55	0.04	3.28
OTM	Jul-02	0.01	2.10	2.35	2.30	0.00	0.93	0.93	2.34	0.00	0.00
OTM	Jul-02	0.01	0.85	0.95	0.90	0.00	0.27	0.27	1.08	0.00	0.00
OTM	Jul-02	0.01	0.30	0.35	0.35	0.00	0.12	0.12	0.45	0.00	0.00
OTM	Jul-02	0.01	0.05	0.15	0.15	0.00	0.07	0.07	0.17	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.10	0.00	0.04	0.04	0.06	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.03	0.03	0.02	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.03	0.03	0.01	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.02	0.02	0.00	0.00	0.00
OTM	Jul-02	0.01	0.00	0.05	0.10	0.00	0.02	0.02	0.00	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.01	0.01	0.00	0.00	0.00
OTM	Jul-02	0.01	0.00	0.20	0.05	0.00	0.01	0.01	0.00	0.00	0.00
ITM	Oct-02	0.01	29.30	29.50	29.20	29.30	29.30	0.00	29.49	1.34	30.93
ITM	Oct-02	0.01	26.70	27.10	30.50	26.80	26.80	0.00	27.01	1.10	28.44
ITM	Oct-02	0.01	21.60	22.60	20.60	21.80	21.80	0.00	22.10	0.75	23.46
ITM	Oct-02	0.01	17.70	17.80	18.50	17.40	16.80	-0.60	17.38	0.50	18.48
ITM	Oct-02	0.01	13.40	13.60	14.30	12.40	12.40	0.00	13.05	0.32	13.49
ITM	Oct-02	0.01	9.60	9.80	9.80	7.40	8.02	0.61	9.35	0.17	8.51
ITM	Oct-02	0.01	7.70	8.10	8.40	4.90	5.52	0.61	7.78	0.11	6.02

Moneyness	Expiration	spread	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Binomial	DualChoquet	Choquet
ITM	Oct-02	0.01	6.30	6.60	6.50	2.40	3.63	1.23	6.41	0.05	3.52
OTM	Oct-02	0.01	3.90	4.10	4.00	0.00	0.94	0.94	4.22	0.01	0.00
OTM	Oct-02	0.01	2.20	2.40	2.30	0.00	0.24	0.24	2.69	0.01	0.00
OTM	Oct-02	0.01	1.20	1.25	1.20	0.00	0.12	0.12	1.66	0.01	0.00
OTM	Oct-02	0.01	0.55	0.70	0.65	0.00	0.06	0.06	1.00	0.00	0.00
OTM	Oct-02	0.01	0.25	0.35	0.40	0.00	0.05	0.05	0.59	0.00	0.00

Table 5C: This table shows a comparison between the behaviour of the bounds and corresponding spread for Dual Fuzzy and Fuzzy binomial OPMs using implied versus historical volatility (for Microsoft stock options).

								implied volatility			historical volatility		
Moneyness	Expiration	sigma (hist)	spread	dof	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Dual Fuzzy	Fuzzy	Fuzzy Spread
ITM	May-02	0.5231	0.01	4.00	29.20	29.4	30.3	29.30	29.30	0.00	29.30	29.30	0.00
ITM	May-02	0.5231	0.01	4.00	26.70	27	24.5	26.80	26.80	0.00	26.80	26.80	0.00
ITM	May-02	0.5231	0.01	4.00	21.70	22	19.5	21.80	21.80	0.00	21.80	21.98	0.18
ITM	May-02	0.5231	0.01	4.00	16.70	16.9	14.6	16.80	17.27	0.47	16.80	17.34	0.54
ITM	May-02	0.5231	0.01	4.00	11.70	12	13.7	12.08	12.57	0.49	11.80	12.52	0.72
ITM	May-02	0.5231	0.01	4.00	6.90	7	7.4	6.80	7.99	1.19	6.80	7.89	1.09
ITM	May-02	0.5231	0.01	4.00	4.50	4.7	4.9	4.51	5.76	1.26	4.30	5.76	1.46
ITM	May-02	0.5231	0.01	4.00	2.45	2.7	2.6	1.99	3.70	1.71	1.98	3.81	1.83
OTM	May-02	0.5231	0.01	4.00	0.30	0.4	0.45	0.00	0.97	0.97	0.00	0.98	0.98
OTM	May-02	0.5231	0.01	4.00	0.00	0.1	0.05	0.00	0.27	0.27	0.00	0.27	0.27
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0.05	0.00	0.12	0.12	0.00	0.12	0.12
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0.05	0.00	0.07	0.07	0.00	0.07	0.07
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0.1	0.00	0.05	0.05	0.00	0.05	0.05
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0	0.00	0.03	0.03	0.00	0.03	0.03
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0.05	0.00	0.03	0.03	0.00	0.03	0.03
OTM	May-02	0.5231	0.01	4.00	0.00	0.05	0	0.00	0.02	0.02	0.00	0.02	0.02
ITM	Jun-02	0.4273	0.01	4.00	31.60	31.9	32.6	31.80	31.80	0.00	31.80	31.80	0.00
ITM	Jun-02	0.4273	0.01	4.00	29.10	29.4	0	29.30	29.30	0.00	29.30	29.30	0.00
ITM	Jun-02	0.4273	0.01	4.00	26.70	26.9	0	26.80	26.80	0.00	26.80	26.80	0.00
ITM	Jun-02	0.4273	0.01	4.00	21.70	21.9	0	21.80	21.80	0.00	21.80	22.14	0.34
ITM	Jun-02	0.4273	0.01	4.00	16.50	17.3	14.4	17.45	16.80	-0.65	16.80	17.14	0.34
ITM	Jun-02	0.4273	0.01	4.00	12.00	12.2	13.7	12.24	12.24	0.00	11.80	12.48	0.68
ITM	Jun-02	0.4273	0.01	4.00	7.60	7.8	8.3	7.19	7.99	0.80	7.14	7.83	0.69
ITM	Jun-02	0.4273	0.01	4.00	5.60	5.8	6.3	4.67	5.78	1.12	4.64	5.68	1.04
ITM	Jun-02	0.4273	0.01	4.00	3.90	4	3.9	2.14	3.63	1.49	2.14	3.58	1.44
OTM	Jun-02	0.4273	0.01	4.00	1.45	1.55	1.5	0.00	0.97	0.97	0.00	1.00	1.00
OTM	Jun-02	0.4273	0.01	4.00	0.40	0.5	0.45	0.00	0.27	0.27	0.00	0.26	0.26

								implied volatility			historical volatility		
Moneyness	Expiration	sigma (hist)	spread	dof	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Dual Fuzzy	Fuzzy	Fuzzy Spread
OTM	Jun-02	0.4273	0.01	4.00	0.05	0.15	0.1	0.00	0.12	0.12	0.00	0.12	0.12
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0.05	0.00	0.07	0.07	0.00	0.07	0.07
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0.05	0.00	0.05	0.05	0.00	0.05	0.05
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0	0.00	0.03	0.03	0.00	0.03	0.03
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0	0.00	0.02	0.02	0.00	0.03	0.03
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0	0.00	0.02	0.02	0.00	0.02	0.02
OTM	Jun-02	0.4273	0.01	4.00	0.00	0.2	0	0.00	0.02	0.02	0.00	0.02	0.02
ITM	Jul-02	0.3893	0.01	4.00	26.50	26.9	24	26.80	26.80	0.00	26.80	26.80	0.00
ITM	Jul-02	0.3893	0.01	4.00	21.30	22.3	19.9	21.80	21.80	0.00	21.80	21.80	0.00
ITM	Jul-02	0.3893	0.01	4.00	16.80	17.1	17.1	16.80	17.34	0.54	16.80	17.20	0.40
ITM	Jul-02	0.3893	0.01	4.00	12.10	12.9	14	12.41	12.41	0.00	12.20	12.60	0.40
ITM	Jul-02	0.3893	0.01	4.00	8.10	8.3	9.2	7.27	7.74	0.47	7.20	8.01	0.81
ITM	Jul-02	0.3893	0.01	4.00	6.10	6.6	6.6	4.77	5.72	0.95	4.70	5.51	0.81
ITM	Jul-02	0.3893	0.01	4.00	4.60	4.8	4.7	2.23	3.54	1.31	2.20	3.75	1.55
OTM	Jul-02	0.3893	0.01	4.00	2.10	2.35	2.3	0.00	0.95	0.95	0.00	0.93	0.93
OTM	Jul-02	0.3893	0.01	4.00	0.85	0.95	0.9	0.00	0.27	0.27	0.00	0.27	0.27
OTM	Jul-02	0.3893	0.01	4.00	0.30	0.35	0.35	0.00	0.12	0.12	0.00	0.12	0.12
OTM	Jul-02	0.3893	0.01	4.00	0.05	0.15	0.15	0.00	0.07	0.07	0.00	0.07	0.07
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.1	0.00	0.05	0.05	0.00	0.04	0.04
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.05	0.00	0.03	0.03	0.00	0.03	0.03
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.05	0.00	0.02	0.02	0.00	0.03	0.03
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.05	0.00	0.02	0.02	0.00	0.02	0.02
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.05	0.1	0.00	0.02	0.02	0.00	0.02	0.02
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.05	0.00	0.01	0.01	0.00	0.01	0.01
OTM	Jul-02	0.3893	0.01	4.00	0.00	0.2	0.05	0.00	0.01	0.01	0.00	0.01	0.01
ITM	Oct-02	0.3893	0.01	4.00	29.30	29.5	29.2	29.30	29.30	0.00	29.30	29.30	0.00
ITM	Oct-02	0.3893	0.01	4.00	26.70	27.1	30.5	26.80	26.80	0.00	26.80	26.80	0.00
ITM	Oct-02	0.3893	0.01	4.00	21.60	22.6	20.6	21.80	21.80	0.00	21.80	21.80	0.00
ITM	Oct-02	0.3893	0.01	4.00	17.70	17.8	18.5	17.56	16.80	-0.76	17.40	16.80	-0.60
ITM	Oct-02	0.3893	0.01	4.00	13.40	13.6	14.3	12.52	12.52	0.00	12.40	12.40	0.00

								implied volatility			historical volatility		
Moneyness	Expiration	sigma (hist)	spread	dof	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Dual Fuzzy	Fuzzy	Fuzzy Spread
ITM	Oct-02	0.3893	0.01	4.00	9.60	9.8	9.8	7.47	7.47	0.00	7.40	8.02	0.61
ITM	Oct-02	0.3893	0.01	4.00	7.70	8.1	8.4	4.94	5.60	0.65	4.90	5.52	0.61
ITM	Oct-02	0.3893	0.01	4.00	6.30	6.6	6.5	2.43	3.70	1.28	2.40	3.63	1.23
OTM	Oct-02	0.3893	0.01	4.00	3.90	4.1	4	0.00	0.98	0.98	0.00	0.94	0.94
OTM	Oct-02	0.3893	0.01	4.00	2.20	2.4	2.3	0.00	0.27	0.27	0.00	0.24	0.24
OTM	Oct-02	0.3893	0.01	4.00	1.20	1.25	1.2	0.00	0.12	0.12	0.00	0.12	0.12
OTM	Oct-02	0.3893	0.01	4.00	0.55	0.7	0.65	0.00	0.07	0.07	0.00	0.06	0.06
OTM	Oct-02	0.3893	0.01	4.00	0.25	0.35	0.4	0.00	0.04	0.04	0.00	0.05	0.05

Table 6C: This table shows the option bounds for $/£ currency options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Black-Scholes implied volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DualChoquet	Choquet
ATM	Nov-01	3.47	3.72	0	0.4023	4.8310	3.7213	0.0646	2.1539
ATM	Nov-01	2.92	3.17	0	0.4075	3.8305	3.1700	0.0566	1.2216
ATM	Nov-01	2.51	2.67	0	0.4150	3.4867	2.6711	0.0503	0.3126
ATM	Nov-01	2.04	2.19	0	0.4416	2.8345	2.1907	0.0427	0.0031
ATM	Nov-01	1.66	1.81	0	0.4432	2.1263	1.8104	0.0424	0.0000
OTM	Nov-01	1.33	1.48	1.45	0.4453	1.7598	1.4808	0.0420	0.0000
OTM	Nov-01	1.05	1.2	0	0.4457	1.4351	1.2000	0.0420	0.0000
ATM	Dec-01	3.82	4.07	0	0.2287	4.4491	4.0705	0.2168	1.7576
ATM	Dec-01	3.3	3.55	0	0.2290	4.1876	3.5517	0.2109	0.9787
ATM	Dec-01	2.82	3.07	0	0.2309	3.1929	3.0701	0.2037	0.2171
ATM	Dec-01	2.05	2.2	0	0.2480	2.1785	2.2015	0.1748	0.0000
OTM	Dec-01	1.71	1.86	0	0.2489	1.6935	1.8604	0.1734	0.0000
OTM	Dec-01	1.42	1.57	0	0.2471	1.4983	1.5691	0.1763	0.0000
OTM	Dec-01	1.16	1.31	0	0.2477	1.2472	1.3109	0.1754	0.0000
ITM	Mar-02	5.2	5.45	0	0.0695	5.1118	5.4515	1.7574	0.7125
ATM	Mar-02	4.18	4.43	0	0.0713	3.8963	4.4310	1.6921	0.2541
ATM	Mar-02	3.3	3.55	0	0.0729	2.9328	3.5501	1.6404	0.0008
OTM	Mar-02	2.56	2.81	0	0.0738	1.8678	2.8086	1.6098	0.0000
ITM	Jun-02	5.58	5.83	0	0.0388	4.7486	5.8275	3.6489	0.1061
ATM	Jun-02	4.62	4.87	0	0.0395	3.9964	4.8721	3.5690	0.0365
ATM	Jun-02	3.78	4.03	0	0.0400	2.5046	4.0318	3.5062	0.0003
OTM	Jun-02	3.05	3.3	0	0.0406	1.5402	3.2986	3.4439	0.0000
ITM	Sep-02	7.09	7.34	0	0.0210	6.6602	7.3394	6.7970	0.0115
ITM	Sep-02	6.07	6.32	0	0.0214	4.7103	6.3229	6.6498	0.0069
ATM	Sep-02	5.15	5.4	0	0.0219	4.1579	5.3971	6.5087	0.0023
ATM	Sep-02	4.33	4.58	0	0.0223	2.2082	4.5827	6.3799	0.0000
OTM	Sep-02	3.61	3.86	0	0.0226	1.6727	3.8606	6.2769	0.0000

Table 7C: This table shows the option bounds for $/£ currency options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Historical volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DualChoquet	Choquet
ATM	Nov-01	3.47	3.72	0	1.8972	4.8791	2.7044	0.0111	2.3818
ATM	Nov-01	2.92	3.17	0	1.9103	4.1958	2.1233	0.0042	1.3876
ATM	Nov-01	2.51	2.67	0	1.9234	3.3473	1.6259	0.0003	0.3987
ATM	Nov-01	2.04	2.19	0	1.9365	2.7382	1.2157	0.0003	0.0025
ATM	Nov-01	1.66	1.81	0	1.9496	2.2741	0.8837	0.0003	0.0000
OTM	Nov-01	1.33	1.48	1.45	1.9225	1.9225	0.6256	0.0003	0.0000
OTM	Nov-01	1.05	1.2	0	1.4559	1.4559	0.4307	0.0003	0.0000
ATM	Dec-01	3.82	4.07	0	1.0420	4.8421	2.9299	0.0142	2.3310
ATM	Dec-01	3.3	3.55	0	1.0492	4.0456	2.3798	0.0073	1.3445
ATM	Dec-01	2.82	3.07	0	1.0564	3.3621	1.9010	0.0032	0.3814
ATM	Dec-01	2.05	2.2	0	1.0707	2.3930	1.1504	0.0030	0.0000
OTM	Dec-01	1.71	1.86	0	1.0779	1.9111	0.8705	0.0030	0.0000
OTM	Dec-01	1.42	1.57	0	1.0851	1.4309	0.6463	0.0029	0.0000
OTM	Dec-01	1.16	1.31	0	1.0923	1.1812	0.4705	0.0029	0.0000
ITM	Mar-02	5.2	5.45	0	0.2573	5.3939	4.1471	0.1785	2.6941
ATM	Mar-02	4.18	4.43	0	0.2608	4.1653	3.1102	0.1623	1.0480
ATM	Mar-02	3.3	3.55	0	0.2644	2.9405	2.2615	0.1540	0.0014
OTM	Mar-02	2.56	2.81	0	0.2680	1.7195	1.5972	0.1500	0.0000
ITM	Jun-02	5.58	5.83	0	0.1193	5.6361	4.5178	0.7663	1.4926
ATM	Jun-02	4.62	4.87	0	0.1210	3.6789	3.5482	0.7419	0.5582
ATM	Jun-02	3.78	4.03	0	0.1226	2.6300	2.7336	0.7226	0.0011
OTM	Jun-02	3.05	3.3	0	0.1243	1.5868	2.0661	0.7066	0.0000
ITM	Sep-02	7.09	7.34	0	0.0676	6.6079	5.8197	1.8520	0.9686
ITM	Sep-02	6.07	6.32	0	0.0685	5.3568	4.7707	1.8122	0.5792
ATM	Sep-02	5.15	5.4	0	0.0695	3.7584	3.8558	1.7734	0.2010
ATM	Sep-02	4.33	4.58	0	0.0704	2.8594	3.0714	1.7377	0.0009
OTM	Sep-02	3.61	3.86	0	0.0714	1.8799	2.4103	1.7043	0.0000

Table 8C: This table shows a comparison between the behaviour of the bounds and corresponding spread for Dual Fuzzy and Fuzzy binomial OPMs using implied versus historical volatility (for $/£ currency options).

					implied volatility			historical volatility		
Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Fuzzy Spread	Dual Fuzzy	Fuzzy	Fuzzy Spread
ATM	Nov-01	3.47	3.72	0	0.4023	4.8310	4.4287	1.8972	4.8791	2.9819
ATM	Nov-01	2.92	3.17	0	0.4075	3.8305	3.4230	1.9103	4.1958	2.2855
ATM	Nov-01	2.51	2.67	0	0.4150	3.4867	3.0717	1.9234	3.3473	1.4239
ATM	Nov-01	2.04	2.19	0	0.4416	2.8345	2.3929	1.9365	2.7382	0.8017
ATM	Nov-01	1.66	1.81	0	0.4432	2.1263	1.6831	1.9496	2.2741	0.3245
OTM	Nov-01	1.33	1.48	1.45	0.4453	1.7598	1.3145	1.9225	1.9225	0.0000
OTM	Nov-01	1.05	1.2	0	0.4457	1.4351	0.9894	1.4559	1.4559	0.0000
ATM	Dec-01	3.82	4.07	0	0.2287	4.4491	4.2204	1.0420	4.8421	3.8001
ATM	Dec-01	3.3	3.55	0	0.2290	4.1876	3.9587	1.0492	4.0456	2.9964
ATM	Dec-01	2.82	3.07	0	0.2309	3.1929	2.9620	1.0564	3.3621	2.3057
ATM	Dec-01	2.05	2.2	0	0.2480	2.1785	1.9305	1.0707	2.3930	1.3223
OTM	Dec-01	1.71	1.86	0	0.2489	1.6935	1.4446	1.0779	1.9111	0.8332
OTM	Dec-01	1.42	1.57	0	0.2471	1.4983	1.2512	1.0851	1.4309	0.3458
OTM	Dec-01	1.16	1.31	0	0.2477	1.2472	0.9995	1.0923	1.1812	0.0889
ITM	Mar-02	5.2	5.45	0	0.0695	5.1118	5.0423	0.2573	5.3939	5.1366
ATM	Mar-02	4.18	4.43	0	0.0713	3.8963	3.8250	0.2608	4.1653	3.9045
ATM	Mar-02	3.3	3.55	0	0.0729	2.9328	2.8600	0.2644	2.9405	2.6761
OTM	Mar-02	2.56	2.81	0	0.0738	1.8678	1.7940	0.2680	1.7195	1.4515
ITM	Jun-02	5.58	5.83	0	0.0388	4.7486	4.7098	0.1193	5.6361	5.5168
ATM	Jun-02	4.62	4.87	0	0.0395	3.9964	3.9569	0.1210	3.6789	3.5579
ATM	Jun-02	3.78	4.03	0	0.0400	2.5046	2.4645	0.1226	2.6300	2.5074
OTM	Jun-02	3.05	3.3	0	0.0406	1.5402	1.4996	0.1243	1.5868	1.4625
ITM	Sep-02	7.09	7.34	0	0.0210	6.6602	6.6392	0.0676	6.6079	6.5403
ITM	Sep-02	6.07	6.32	0	0.0214	4.7103	4.6889	0.0685	5.3568	5.2883
ATM	Sep-02	5.15	5.4	0	0.0219	4.1579	4.1361	0.0695	3.7584	3.6889
ATM	Sep-02	4.33	4.58	0	0.0223	2.2082	2.1860	0.0704	2.8594	2.7889
OTM	Sep-02	3.61	3.86	0	0.0226	1.6727	1.6501	0.0714	1.8799	1.8085

Table 9C: This table shows the option bounds for S&P 500 index options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Black-Scholes implied volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ITM	Aug-02	89.50	92.50	91.00	50.1661	87.5705	92.5035	0.1340	87.2191
ITM	Aug-02	69.60	72.60	66.50	49.8387	62.6074	72.6110	0.0968	62.3541
ITM	Aug-02	51.60	54.60	45.00	37.6443	37.6443	54.5964	0.0612	37.4915
ATM	Aug-02	36.00	38.00	35.00	12.6812	12.6812	38.0049	0.0268	12.6305
ATM	Aug-02	30.70	33.70	33.00	2.6960	4.6252	34.9381	0.0158	2.6859
ATM	Aug-02	25.80	28.80	26.80	0.0142	1.2730	28.7819	0.0111	0.0000
ATM	Aug-02	23.90	25.00	22.00	0.0143	1.4240	25.0043	0.0099	0.0000
ATM	Aug-02	21.50	24.50	20.00	0.0144	0.5718	24.5169	0.0109	0.0000
ATM	Aug-02	17.90	20.30	13.70	0.0145	0.3333	20.3017	0.0103	0.0000
ATM	Aug-02	14.60	16.00	15.00	0.0147	0.1587	16.0124	0.0094	0.0000
ATM	Aug-02	12.00	13.60	8.80	0.0149	0.1136	13.6106	0.0095	0.0000
ATM	Aug-02	10.10	11.00	9.50	0.0151	0.0694	10.9905	0.0092	0.0000
ATM	Aug-02	8.20	9.00	7.90	0.0152	0.0611	8.9929	0.0084	0.0000
ATM	Aug-02	7.00	7.80	7.80	0.0153	0.0477	7.7991	0.0087	0.0000
ATM	Aug-02	5.00	6.00	6.00	0.0156	0.0386	5.9927	0.0090	0.0000
ATM	Aug-02	4.40	4.50	4.50	0.0157	0.0313	4.5006	0.0079	0.0000
ATM	Aug-02	2.60	3.20	2.60	0.0160	0.0222	3.2010	0.0084	0.0000
ITM	Sep-02	100.40	103.40	85.00	16.6947	87.3819	103.3946	0.3599	85.2005
ITM	Sep-02	81.80	84.80	77.00	18.6054	62.4726	84.7994	0.2975	60.9081
ITM	Sep-02	64.80	67.80	59.00	20.8223	37.5632	67.8266	0.2394	36.6198
ATM	Sep-02	49.70	52.70	48.60	12.6539	12.6539	52.7052	0.1863	12.3353
ATM	Sep-02	44.20	47.20	33.00	2.6902	2.6902	48.5482	0.1797	2.6226
ATM	Sep-02	39.10	42.10	29.00	0.0858	2.5298	42.1144	0.1559	0.0001
ATM	Sep-02	36.80	39.80	31.90	0.0863	0.6328	39.8093	0.1536	0.0000
ATM	Sep-02	34.70	37.70	29.50	0.0869	0.6370	37.7174	0.1524	0.0000
ATM	Sep-02	30.30	33.30	30.70	0.0879	0.2891	33.3209	0.1461	0.0000
ATM	Sep-02	26.40	29.40	22.00	0.0890	0.1648	29.4222	0.1415	0.0000
ATM	Sep-02	22.90	24.50	19.00	0.0900	0.1142	24.5081	0.1256	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Sep-02	18.40	20.80	18.80	0.0916	0.0560	20.7860	0.1282	0.0000
ATM	Sep-02	17.00	19.40	14.00	0.0921	0.0566	19.4155	0.1265	0.0000
ATM	Sep-02	15.80	18.20	14.10	0.0926	0.0430	18.1888	0.1260	0.0000
ATM	Sep-02	14.60	17.00	0.00	0.0931	0.0433	17.0126	0.1248	0.0000
ATM	Sep-02	11.90	14.30	14.30	0.0942	0.0354	14.3092	0.1180	0.0000
OTM	Sep-02	10.10	12.50	11.40	0.0952	0.0285	12.5099	0.1173	0.0000
ITM	Oct-02	104.80	107.80	0.00	11.7294	87.2313	107.7986	0.8715	82.5381
ITM	Oct-02	86.70	89.70	77.00	13.1733	62.3649	89.7080	0.7471	58.9999
ITM	Oct-02	54.50	57.50	51.00	12.6321	12.6321	57.4831	0.5137	11.9471
ATM	Oct-02	41.50	44.50	42.00	0.1877	0.5136	44.4884	0.4437	0.0000
ATM	Oct-02	30.70	33.70	22.90	0.1934	0.1346	33.7080	0.4017	0.0000
ATM	Oct-02	22.20	25.20	18.40	0.1991	0.0615	25.2122	0.3743	0.0000
ATM	Oct-02	15.10	17.50	14.50	0.2048	0.0373	17.5074	0.3275	0.0000
ATM	Oct-02	10.10	12.50	8.60	0.2105	0.0243	12.4762	0.3105	0.0000
ATM	Oct-02	6.70	8.20	6.00	0.2162	0.0150	8.2027	0.2790	0.0000
ATM	Oct-02	4.30	5.50	4.50	0.2218	0.0115	5.5085	0.2641	0.0000
ATM	Oct-02	2.65	3.80	3.50	0.2287	0.0086	3.8042	0.2707	0.0000
ATM	Oct-02	1.95	2.70	0.00	0.2332	0.0069	2.6964	0.2596	0.0000
ATM	Oct-02	1.15	1.90	0.00	0.2389	0.0062	1.8919	0.2596	0.0000
ATM	Oct-02	0.60	1.35	0.00	0.2446	0.0050	1.3480	0.2608	0.0000
ATM	Oct-02	0.50	1.05	0.00	0.2503	0.0043	1.0456	0.2729	0.0000
ATM	Dec-02	306.70	310.70	296.00	2.0732	309.8278	310.6966	18.1500	263.6000
ITM	Dec-02	214.00	217.00	205.00	3.8563	210.7463	217.0064	8.2750	179.0000
ITM	Dec-02	170.80	173.80	181.00	4.8044	161.2056	173.7718	6.3430	136.8000
ITM	Dec-02	130.60	133.60	123.00	6.1343	111.6648	133.6158	4.7260	94.7100
ITM	Dec-02	94.40	97.40	94.00	7.9434	62.1241	97.4297	3.4707	52.6501
ITM	Dec-02	78.40	81.40	0.00	8.9393	37.3537	81.4188	3.0129	31.6431
ITM	Dec-02	63.80	66.80	61.00	10.0764	12.5833	66.7811	2.6091	10.6548
ATM	Dec-02	50.80	53.80	48.70	0.5564	1.3970	53.7871	2.2813	0.0000
ATM	Dec-02	39.90	42.90	36.50	0.5732	0.1668	42.9066	2.0698	0.0000
ATM	Dec-02	30.70	33.70	22.90	0.5901	0.0618	33.7173	1.8977	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Dec-02	22.10	25.10	20.00	0.6070	0.0337	25.1304	1.6438	0.0000
ATM	Dec-02	16.10	18.50	16.00	0.6238	0.0214	18.5114	1.4717	0.0000
ATM	Dec-02	12.20	13.00	10.00	0.6407	0.0152	13.0085	1.2931	0.0000
ATM	Dec-02	8.10	9.60	6.50	0.6575	0.0110	9.6010	1.2399	0.0000
ATM	Dec-02	6.00	7.50	6.00	0.6710	0.0095	7.5021	1.2068	0.0000
ATM	Dec-02	2.85	4.00	2.50	0.7081	0.0055	3.9952	1.1908	0.0000
ATM	Dec-02	1.15	1.90	1.65	0.7418	0.0042	1.9001	1.0733	0.0000
ATM	Mar-03	136.60	139.60	133.00	3.8490	111.0809	139.6175	19.7190	73.9700
ATM	Mar-03	101.80	104.80	0.00	4.9871	61.7992	104.7931	14.2056	41.0769
ITM	Mar-03	71.70	74.70	75.00	6.4286	12.5175	74.6853	10.3787	8.3046
ITM	Mar-03	58.70	61.70	58.00	1.3670	1.0948	61.7012	8.9763	0.0000
ATM	Mar-03	47.60	50.60	44.90	1.4084	0.1255	50.6232	8.0350	0.0000
ATM	Mar-03	37.50	40.50	29.50	1.4498	0.0472	40.4668	7.0423	0.0000
ATM	Mar-03	29.80	32.80	24.00	1.4912	0.0337	32.8218	6.6174	0.0000
ATM	Mar-03	22.80	25.80	18.50	1.5326	0.0195	25.8084	6.0218	0.0000
ATM	Mar-03	17.30	19.70	16.80	1.5741	0.0163	19.6925	5.3835	0.0000
ATM	Jun-03	143.20	146.20	140.00	2.4594	110.5545	146.1605	62.7560	53.8500
ATM	Jun-03	109.70	112.70	93.50	3.1668	61.5064	112.7239	44.3236	29.8693
ITM	Jun-03	79.70	82.70	66.00	4.1862	12.4582	82.6927	30.4574	6.0331
ITM	Jun-03	67.10	70.10	68.00	2.4086	0.8371	70.1139	26.5514	0.0000
ATM	Jun-03	55.30	58.30	48.40	2.4815	0.0970	58.2774	22.6310	0.0000
ATM	Jun-03	45.40	48.40	40.50	2.5545	0.0571	48.4405	20.1994	0.0000
ATM	Jun-03	37.20	40.20	38.10	2.6275	0.0377	40.1911	18.6756	0.0000
ATM	Jun-03	219.10	222.10	0.00	1.5859	208.6926	222.0942	113.5070	103.8000
ATM	Jun-03	178.60	181.60	0.00	2.0835	159.6346	181.6092	76.6486	79.1960
ITM	Jun-03	142.10	145.10	140.00	2.5847	110.5767	145.0693	57.0960	54.6800
ITM	Jun-03	108.80	111.80	93.50	3.2861	61.5187	111.8312	41.2208	30.3306
ITM	Jun-03	78.90	81.90	66.00	4.3248	12.4607	81.8847	28.5773	6.1263
ITM	Jun-03	66.30	69.30	68.00	2.3587	0.8530	69.3374	24.9674	0.0000
ATM	Jun-03	54.70	57.70	48.40	2.4302	0.0984	57.6901	21.5128	0.0000
ATM	Jun-03	44.80	47.80	40.50	2.5017	0.0581	47.8313	19.1607	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Jun-03	36.70	39.70	38.10	2.5732	0.0382	39.6672	17.7917	0.0000
ATM	Jun-03	29.40	32.40	32.00	2.6446	0.0198	32.4287	16.2500	0.0000
ATM	Jun-03	23.10	26.10	23.50	2.7161	0.0158	26.0976	14.8196	0.0000
ATM	Jun-03	18.40	20.80	22.50	2.7876	0.0103	20.8055	13.5816	0.0000
ATM	Jun-03	14.80	17.20	25.50	2.8448	0.0087	17.2255	12.7096	0.0000
ATM	Jun-03	10.60	13.00	12.50	2.9305	0.0063	12.9755	11.8022	0.0000
ATM	Jun-03	8.30	9.80	7.50	3.0020	0.0057	9.8125	10.6473	0.0000
ATM	Jun-03	6.10	7.60	12.00	3.0735	0.0051	7.5851	10.0782	0.0000
ATM	Jun-03	4.50	5.70	5.30	3.1450	0.0040	5.7011	9.3392	0.0000
ATM	Dec-03	123.30	126.30	112.00	1.7139	60.5191	126.2852	312.7810	5.2200
ATM	Dec-03	95.40	98.40	86.00	2.1399	12.2583	98.3384	223.4292	1.0499
ITM	Dec-03	81.20	84.20	0.00	2.5368	0.5854	84.1966	171.7934	0.0000
ITM	Dec-03	71.60	74.60	82.00	2.6092	0.1464	74.6088	167.1807	0.0000
ATM	Dec-03	61.50	64.50	0.00	2.8376	0.0674	64.5077	148.5073	0.0000
ATM	Dec-03	51.90	54.90	74.00	3.1455	0.0248	54.8904	128.0466	0.0000
ATM	Dec-03	44.00	47.00	0.00	3.3726	0.0177	47.0217	116.5040	0.0000
ATM	Dec-03	36.30	39.30	35.00	3.7270	0.0136	39.3149	11.1762	0.0000
ATM	Dec-03	30.40	33.40	48.00	3.9301	0.0106	33.4257	94.5283	0.0000
ATM	Dec-03	25.90	28.90	46.00	4.1669	0.0085	28.9162	87.3523	0.0000
ATM	Dec-03	20.10	24.10	25.50	4.2691	0.0069	24.1198	85.4810	0.0000
ATM	Dec-03	16.70	18.70	15.00	4.9224	0.0051	18.6962	69.8130	0.0000
ATM	Dec-03	13.20	15.20	34.00	5.2724	0.0045	15.2020	63.7658	0.0000
ATM	Dec-03	10.50	12.50	11.50	5.5192	0.0039	12.5151	60.2510	0.0000
ATM	Dec-03	6.90	7.90	8.50	6.3023	0.0032	7.8983	50.6713	0.0000
ATM	Dec-03	4.30	5.10	5.00	6.8773	0.0026	5.0775	45.5126	0.0000
ATM	Jun-04	236.30	239.30	217.00	0.3187	182.1240	239.3420	16658.0000	0.0000
ATM	Jun-04	200.70	203.70	0.00	1.1967	172.6728	203.7274	894.6380	0.3800
ATM	Jun-04	86.70	89.70	105.00	1.6250	153.7705	185.0888	525.1003	0.3390
ITM	Jun-04	167.40	170.40	184.00	0.8310	106.5146	170.4644	2059.2000	0.0000
ITM	Jun-04	236.30	239.30	217.00	0.0949	87.6123	239.0570	1006738.0000	0.0095
ITM	Jun-04	200.70	203.70	0.00	0.1980	78.1611	203.7049	86673.3800	39512.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ITM	Jun-04	137.30	140.30	154.50	1.0275	59.2588	140.3463	1426.9600	74804.0000
ITM	Jun-04	110.50	113.50	203.50	1.2539	12.0030	113.4771	1020.8400	60122.0000
ITM	Jun-04	86.70	89.70	105.00	1.5254	0.1069	89.7417	738.3695	3.0000
ITM	Jun-04	66.60	69.60	58.30	1.8207	0.0285	69.6550	556.3046	0.0000
ATM	Jun-04	58.10	61.10	69.00	1.9621	0.0187	61.1309	495.8467	0.0000
ATM	Jun-04	50.00	53.00	45.00	2.1377	0.0134	53.0578	433.4462	0.0000
ATM	Jun-04	43.00	46.00	0.00	2.2988	0.0101	46.0357	388.2157	0.0000
ATM	Jun-04	38.10	41.10	62.00	2.4132	0.0078	41.1246	361.6630	0.0000
ATM	Jun-04	26.20	30.20	29.50	2.6807	0.0052	30.2289	312.9018	1.0000
ATM	Jun-04	19.10	21.10	23.00	3.1795	0.0038	21.1143	240.7829	6.0000
ATM	Jun-04	12.90	14.90	23.00	3.5771	0.0029	14.9234	203.3274	2.0000

Table 10C: This table shows the option bounds for S&P 500 index options using fuzzy as well as Choquet integration. The binomial option value is also shown for comparative purposes. Historical volatility is used in the calculations.

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ITM	Aug-02	89.50	92.50	91.00	87.5705	87.5705	87.3297	0.1218	87.4226
ITM	Aug-02	69.60	72.60	66.50	62.6074	62.6074	62.3794	0.0844	62.5095
ITM	Aug-02	51.60	54.60	45.00	37.6443	39.6698	37.8119	0.0495	37.5995
ATM	Aug-02	36.00	38.00	35.00	12.6812	15.8368	16.5149	0.0167	12.6926
ATM	Aug-02	30.70	33.70	33.00	2.6960	7.1853	10.2906	0.0042	2.7306
ATM	Aug-02	25.80	28.80	26.80	0.0142	2.6197	5.7941	0.0009	0.0000
ATM	Aug-02	23.90	25.00	22.00	0.0143	1.3400	4.1640	0.0009	0.0000
ATM	Aug-02	21.50	24.50	20.00	0.0144	0.8119	2.9160	0.0009	0.0000
ATM	Aug-02	17.90	20.30	13.70	0.0145	0.3355	1.3001	0.0008	0.0000
ATM	Aug-02	14.60	16.00	15.00	0.0147	0.1821	0.5095	0.0008	0.0000
ATM	Aug-02	12.00	13.60	8.80	0.0149	0.1244	0.1768	0.0007	0.0000
ATM	Aug-02	10.10	11.00	9.50	0.0151	0.0839	0.0535	0.0007	0.0000
ATM	Aug-02	8.20	9.00	7.90	0.0152	0.0731	0.0281	0.0007	0.0000
ATM	Aug-02	7.00	7.80	7.80	0.0153	0.0537	0.0069	0.0007	0.0000
ATM	Aug-02	5.00	6.00	6.00	0.0156	0.0392	0.0007	0.0006	0.0000
ATM	Aug-02	4.40	4.50	4.50	0.0157	0.0339	0.0003	0.0006	0.0000
ATM	Aug-02	2.60	3.20	2.60	0.0160	0.0243	0.0000	0.0006	0.0000
ITM	Sep-02	100.40	103.40	85.00	87.3819	87.3819	86.7914	0.1275	86.2124
ITM	Sep-02	81.80	84.80	77.00	62.4726	62.4726	61.9054	0.0900	61.6233
ITM	Sep-02	64.80	67.80	59.00	37.5632	39.6955	37.5227	0.0550	37.0530
ATM	Sep-02	49.70	52.70	48.60	12.6539	14.7862	16.6506	0.0221	12.5011
ATM	Sep-02	44.20	47.20	33.00	2.6902	6.9604	10.5678	0.0095	2.6853
ATM	Sep-02	39.10	42.10	29.00	0.0858	2.3495	6.1202	0.0061	0.0000
ATM	Sep-02	36.80	39.80	31.90	0.0863	1.2010	4.4862	0.0059	0.0000
ATM	Sep-02	34.70	37.70	29.50	0.0869	0.7274	3.1987	0.0058	0.0000
ATM	Sep-02	30.30	33.30	30.70	0.0879	0.3492	1.5084	0.0056	0.0000
ATM	Sep-02	26.40	29.40	22.00	0.0890	0.1819	0.6324	0.0053	0.0000
ATM	Sep-02	22.90	24.50	19.00	0.0900	0.1219	0.2380	0.0051	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Sep-02	18.40	20.80	18.80	0.0916	0.0701	0.0436	0.0048	0.0000
ATM	Sep-02	17.00	19.40	14.00	0.0921	0.0613	0.0235	0.0047	0.0000
ATM	Sep-02	15.80	18.20	14.10	0.0926	0.0541	0.0123	0.0046	0.0000
ATM	Sep-02	14.60	17.00	0.00	0.0931	0.0482	0.0062	0.0045	0.0000
ATM	Sep-02	11.90	14.30	14.30	0.0942	0.0388	0.0015	0.0043	0.0000
ATM	Sep-02	10.10	12.50	11.40	0.0952	0.0304	0.0003	0.0041	0.0000
ITM	Oct-02	124.10	127.10	121.20	112.0977	113.3502	111.2282	0.1634	639.0000
ITM	Oct-02	104.80	107.80	0.00	87.2313	88.4838	86.3618	0.1235	86.3355
ITM	Oct-02	86.70	89.70	77.00	62.3649	63.6174	61.4955	0.0863	61.7233
ATM	Oct-02	54.50	57.50	51.00	12.6321	16.3954	13.2943	0.0187	12.5851
ATM	Oct-02	41.50	44.50	42.00	0.1877	1.4031	1.3399	0.0028	0.0000
ATM	Oct-02	30.70	33.70	22.90	0.1934	0.1888	0.0173	0.0023	0.0000
ATM	Oct-02	22.20	25.20	18.40	0.1991	0.0738	0.0000	0.0019	0.0000
ATM	Oct-02	15.10	17.50	14.50	0.2048	0.0390	0.0000	0.0016	0.0000
ATM	Oct-02	10.10	12.50	8.60	0.2105	0.0241	0.0000	0.0013	0.0000
ATM	Oct-02	6.70	8.20	6.00	0.2162	0.0166	0.0000	0.0010	0.0000
ATM	Oct-02	4.30	5.50	4.50	0.2218	0.0122	0.0000	0.0008	0.0000
ATM	Oct-02	2.65	3.80	3.50	0.2287	0.0089	0.0000	0.0006	0.0000
ATM	Oct-02	1.95	2.70	0.00	0.2332	0.0075	0.0000	0.0004	0.0000
ATM	Oct-02	1.15	1.90	0.00	0.2389	0.0061	0.0000	0.0003	0.0000
ATM	Oct-02	0.60	1.35	0.00	0.2446	0.0052	0.0000	0.0002	0.0000
ATM	Dec-02	0.50	1.05	0.00	0.2503	0.0044	0.0000	0.0001	0.0000
ITM	Dec-02	306.70	310.70	296.00	173.6136	309.8278	308.3260	0.6530	649.0000
ITM	Dec-02	214.00	217.00	205.00	208.3363	210.7463	209.2445	0.3790	850.0000
ITM	Dec-02	170.80	173.80	181.00	161.2056	161.2056	159.7038	0.2737	81.0000
ITM	Dec-02	130.60	133.60	123.00	111.6648	111.6648	110.1630	0.1833	850.0000
ITM	Dec-02	94.40	97.40	94.00	62.1241	63.7628	60.6235	0.1051	58.2993
ITM	Dec-02	78.40	81.40	0.00	37.3537	38.9924	36.0001	0.0698	35.0021
ATM	Dec-02	63.80	66.80	61.00	12.5833	15.8641	14.0890	0.0366	11.8082
ATM	Dec-02	50.80	53.80	48.70	0.5564	1.2105	2.4479	0.0193	0.0000
ATM	Dec-02	39.90	42.90	36.50	0.5732	0.1856	0.1356	0.0169	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Dec-02	30.70	33.70	22.90	0.5901	0.0717	0.0021	0.0147	0.0000
ATM	Dec-02	22.10	25.10	20.00	0.6070	0.0377	0.0000	0.0128	0.0000
ATM	Dec-02	16.10	18.50	16.00	0.6238	0.0240	0.0000	0.0110	0.0000
ATM	Dec-02	12.20	13.00	10.00	0.6407	0.0165	0.0000	0.0094	0.0000
ATM	Dec-02	8.10	9.60	6.50	0.6575	0.0121	0.0000	0.0080	0.0000
ATM	Dec-02	6.00	7.50	6.00	0.6710	0.0098	0.0000	0.0070	0.0000
ATM	Dec-02	2.85	4.00	2.50	0.7081	0.0061	0.0000	0.0047	0.0000
ATM	Mar-03	1.15	1.90	1.65	0.7418	0.0044	0.0000	0.0031	0.0000
ITM	Mar-03	136.60	139.60	133.00	111.0809	111.0809	108.3987	0.2599	518.0000
ITM	Mar-03	101.80	104.80	0.00	61.7992	61.7992	59.1426	0.1769	49.5880
ATM	Mar-03	71.70	74.70	75.00	12.5175	14.7916	14.8663	0.1043	9.9521
ATM	Mar-03	58.70	61.70	58.00	1.3670	1.2408	3.6801	0.0829	0.0000
ATM	Mar-03	47.60	50.60	44.90	1.4084	0.1878	0.4591	0.0746	0.0000
ATM	Mar-03	37.50	40.50	29.50	1.4498	0.0723	0.0269	0.0671	0.0000
ATM	Mar-03	29.80	32.80	24.00	1.4912	0.0380	0.0007	0.0602	0.0000
ATM	Mar-03	22.80	25.80	18.50	1.5326	0.0233	0.0000	0.0539	0.0000
ATM	Jun-03	17.30	19.70	16.80	1.5741	0.0163	0.0000	0.0481	0.0000
ITM	Jun-03	143.20	146.20	140.00	110.5545	110.5545	106.3130	0.3929	117.0000
ITM	Jun-03	109.70	112.70	93.50	61.5064	61.5064	57.3847	0.3013	39.0120
ATM	Jun-03	79.70	82.70	66.00	12.4582	14.8730	15.2513	0.2212	7.7553
ATM	Jun-03	67.10	70.10	68.00	2.4086	1.2213	4.5504	0.1938	0.0000
ATM	Jun-03	55.30	58.30	48.40	2.4815	0.1956	0.8333	0.1775	0.0000
ATM	Jun-03	45.40	48.40	40.50	2.5545	0.0703	0.0901	0.1624	0.0000
ATM	Jun-03	37.20	40.20	38.10	2.6275	0.0380	0.0057	0.1484	0.0000
ITM	Jun-03	219.10	222.10	0.00	168.4801	208.6926	204.2912	0.6171	86.0000
ITM	Jun-03	178.60	181.60	0.00	159.6346	159.6346	155.2333	0.4942	49.0000
ITM	Jun-03	142.10	145.10	140.00	110.5767	110.5767	106.1753	0.3887	846.0000
ITM	Jun-03	108.80	111.80	93.50	61.5187	61.5187	57.2392	0.2975	39.3895
ATM	Jun-03	78.90	81.90	66.00	12.4607	14.8760	15.1548	0.2175	7.8337
ATM	Jun-03	66.30	69.30	68.00	2.3587	1.2215	4.5071	0.1904	0.0000
ATM	Jun-03	54.70	57.70	48.40	2.4302	0.1957	0.8223	0.1743	0.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ATM	Jun-03	44.80	47.80	40.50	2.5017	0.0703	0.0885	0.1595	0.0000
ATM	Jun-03	36.70	39.70	38.10	2.5732	0.0380	0.0055	0.1458	0.0000
ATM	Jun-03	29.40	32.40	32.00	2.6446	0.0238	0.0002	0.1331	0.0000
ATM	Jun-03	23.10	26.10	23.50	2.7161	0.0162	0.0000	0.1214	0.0000
ATM	Jun-03	18.40	20.80	22.50	2.7876	0.0118	0.0000	0.1106	0.0000
ATM	Jun-03	14.80	17.20	25.50	2.8448	0.0098	0.0000	0.1025	0.0000
ATM	Jun-03	10.60	13.00	12.50	2.9305	0.0074	0.0000	0.0913	0.0000
ATM	Jun-03	8.30	9.80	7.50	3.0020	0.0061	0.0000	0.0828	0.0000
ATM	Jun-03	6.10	7.60	12.00	3.0735	0.0050	0.0000	0.0748	0.0000
ATM	Dec-03	4.50	5.70	5.30	3.1450	0.0043	0.0000	0.0675	0.0000
ITM	Dec-03	154.70	157.70	236.00	108.7800	108.7800	106.7335	1.0061	68.0000
ITM	Dec-03	123.30	126.30	112.00	60.5191	60.5191	58.8723	0.8747	10.0503
ATM	Dec-03	95.40	98.40	86.00	12.2583	15.2159	18.7824	0.7597	1.9103
ATM	Dec-03	81.20	84.20	0.00	8.1139	1.1206	7.4293	0.7097	0.0000
ATM	Dec-03	71.60	74.60	82.00	8.3597	0.1946	2.1820	0.6650	0.0000
ATM	Dec-03	61.50	64.50	0.00	8.6056	0.0699	0.4668	0.6232	0.0000
ATM	Dec-03	51.90	54.90	74.00	8.8515	0.0355	0.0717	0.5840	0.0000
ATM	Dec-03	44.00	47.00	0.00	9.0974	0.0228	0.0081	0.5471	0.0000
ATM	Dec-03	36.30	39.30	35.00	9.3432	0.0158	0.0007	0.5125	0.0000
ATM	Dec-03	30.40	33.40	48.00	9.5891	0.0116	0.0000	0.4799	0.0000
ATM	Dec-03	25.90	28.90	46.00	9.7858	0.0096	0.0000	0.4553	0.0000
ATM	Dec-03	20.10	24.10	25.50	10.0809	0.0072	0.0000	0.4204	0.0000
ATM	Dec-03	16.70	18.70	15.00	10.3267	0.0058	0.0000	0.3931	0.0000
ATM	Dec-03	13.20	15.20	34.00	10.5726	0.0049	0.0000	0.3674	0.0000
ATM	Dec-03	10.50	12.50	11.50	10.8185	0.0042	0.0000	0.3431	0.0000
ATM	Jun-04	6.90	7.90	8.50	11.3102	0.0032	0.0000	0.2985	0.0000
ATM	Jun-04	4.30	5.10	5.00	11.8020	0.0025	0.0000	0.2587	0.0000
ITM	Jun-04	236.30	239.30	217.00	124.2657	182.1240	185.9503	2.1058	374.0000
ITM	Jun-04	200.70	203.70	0.00	126.2699	172.6728	176.4991	2.0574	95.0000
ITM	Jun-04	86.70	89.70	105.00	130.2785	153.7705	157.5968	1.9650	1.6615
ITM	Jun-04	167.40	170.40	184.00	106.5146	106.5146	110.3486	1.7571	569.0000

Moneyness	Expiration	Bid	Ask	Last	Dual Fuzzy	Fuzzy	Binomial	DuallChoquet	Choquet
ITM	Jun-04	236.30	239.30	217.00	87.6123	87.6123	91.4945	1.6821	0.7343
ITM	Jun-04	200.70	203.70	0.00	78.1611	78.1611	82.1205	1.6461	0.6335
ITM	Jun-04	137.30	140.30	154.50	59.2588	59.2588	63.6832	1.5770	0.4501
ATM	Jun-04	110.50	113.50	203.50	19.7029	15.3739	24.1580	1.4194	0.0783
ATM	Jun-04	86.70	89.70	105.00	20.9344	0.1672	4.5125	1.2806	0.0000
ATM	Jun-04	66.60	69.60	58.30	22.1658	0.0350	0.3518	1.1574	0.0000
ATM	Jun-04	58.10	61.10	69.00	22.7815	0.0224	0.0702	1.1008	0.0000
ATM	Jun-04	50.00	53.00	45.00	23.3972	0.0156	0.0112	1.0473	0.0000
ATM	Jun-04	43.00	46.00	0.00	24.0129	0.0114	0.0015	0.9965	0.0000
ATM	Jun-04	38.10	41.10	62.00	24.5055	0.0091	0.0002	0.9578	0.0000
ATM	Jun-04	26.20	30.20	29.50	25.8601	0.0058	0.0000	0.8590	0.0000
ATM	Jun-04	19.10	21.10	23.00	27.0915	0.0041	0.0000	0.7780	0.0000
ATM	Jun-04	12.90	14.90	23.00	28.3230	0.0031	0.0000	0.7043	0.0000

Appendix D

Fuzzy Differential Calculus

D.1 Fuzzy Derivatives

D.1.1 Dubois-Prade derivative

Dubois and Prade ([50],[125]) deal with differentiation of ordinary functions at a fuzzy point and differentiation of fuzzy functions at a non-fuzzy point. In general, both approaches yield a fuzzy value.

In the first case, there is uncertainty about the value of the derivative at a fuzzy point due to the uncertainty about the precise location of this point. Let f be a differentiable mapping from $[a, b] \subset \Re \times \Re$, with derivative f'. Let X_0 be the fuzzy point with support $S(X_0) = x \in \Re \mid \mu_{X_0}(x) > 0\} \subseteq [a, b]$ at which we want to differentiate f. Then, $f'(X_0)$ is the fuzzy set of the possible values of f' at a point whose possible positions are restricted by X_0. is defined by its membership function using the extension principle:

$$\mu_{f'(X_0)}(y) = \sup_{x \in f'^{-1}(y)} \mu_{X_0}(x)$$

Of course, if $f'(x)$ is constant on $S(X_0)$, then $f'(X_0)$ is non-fuzzy.

In the second case, assume that $\tilde{f}$ is a non-fuzzy mapping from $[a, b]$ to the set of fuzzy sets of $\Re$, where $\tilde{f}(x)$ is normalized, continuous, support-bounded and strictly convex $\Longrightarrow \tilde{f}(x)$ is a fuzzy number. $\tilde{f}(x)$ can be viewed as the possibility distribution of a priori possible values at x, where x is an ordinary point belonging to $[a, b]$, of an imprecise mapping roughly specified by $\tilde{f}$.

Let f_α^+ and f_α^- be the upper and lower α-curves of $\tilde{f}$ respectively. If f_α^+ and f_α^- are differentiable on [a,b], then the fuzzy value of $\tilde{f}'(x_0)$ of the derivative of $\tilde{f}$ at x_0 can be defined by its membership function:

$$\mu_{f'(x_0)}(y) = \sup_{h:y=h'(x_0)} \mu_{\varpi(\tilde{f})}(h)$$

where $S[\varpi(\tilde{f})] = \{ f_\alpha^-, f_\alpha^+ \mid \alpha \in]0,1[\} \bigcup \{f_1\}$ and $\mu_{\varpi(\tilde{f})}(f_\alpha^-) = \mu_{\varpi(\tilde{f})}(f_\alpha^+) = \alpha, \mu_{\varpi(\tilde{f})}(f_1) = 1$. In other words,

$$\mu_{f'(x_0)}(y) = \sup\{\alpha \mid y = (f_\alpha^\varepsilon)'(x_0), \varepsilon \in \{+,-\}$$

where $\forall\ x,\ f_\alpha(x) \triangleq \{y \mid \mu_{F(x)}(y) \geq \alpha\}$ is closed,

$$f_\alpha^+ = \sup f_\alpha(x) \text{ and } f_\alpha^- = \inf f_\alpha(x),$$

and $f'(x)$ can be found by differentiating f_α^+ and f_α^-:

$$f_\alpha'(x_0) = [f_\alpha^{-'}(x_0)\ , f_\alpha^{+'}(x_0)],$$

which holds if it satisfies the following conditions:

(i) $f_\alpha^{-'}(x_0) \leq f_\alpha^{+'}(x_0), \forall\ \alpha$,

(ii) $\alpha \leq \beta \Longrightarrow [f_\alpha^{-'}(x_0)\ , f_\alpha^{+'}(x_0)] \supseteq [f_\beta^{-'}(x_0)\ , f_\beta^{+'}(x_0)]$,

(iii) $\lim_{\beta \uparrow \alpha} f_\beta^{-'}(x_0) = f_\alpha^{-'}(x_0), \lim_{\beta \uparrow \alpha} f_\beta^{+'}(x_0) = f_\alpha^{+'}(x_0)$.

However, this derivative is not always a fuzzy number so Buckley and Feuring suggested defining $f'(x_0) = 1, \forall\ x$ satisfying $x_2'(t,1) < x < x_2'(t,1)$ whenever $x_2'(t,1) < x_2'(t,1)$. Then DPD$\overline{X}(t)$ will be a fuzzy number.

D.1.2 Puri-Ralescu Derivative

Puri and Ralescu ([125],[21],[54]) use the Hausdorff metric for this derivative whereby $D(\overline{X}(t), \overline{Z}(t)) = \sup H(\overline{X}(t)[\alpha], \overline{Z}(t)[\alpha])$, where $\overline{X}(t)$ and $\overline{Z}(t)$ are fuzzy numbers for $t \in I$. They generalize Rădstorm Embedding theorem to define the differential of a fuzzy function. To do so, they propose a theorem ([125], Theorem 2.2, p. 555) whereby they prove that: $\exists$ a normed space χ such that $F_0(X)$ can be embedded isometrically into χ, where $F_0(X)$ is a subset of $F(X)$ the set of fuzzy subsets of X. Denoting this embedding by $j : F_0(X) \rightarrow \chi_0$, then the fuzzy function $F : U \rightarrow F_0(X)$ is called differentiable at $x_0 \in U$ if the map $\hat{F} = j \circ F$ is differentiable at x_0 ([125], Definition 3.1, p. 556), i.e. F is differentiable at $x_0 \in U$ if $\exists$ a linear bounded operator: $\tilde{F}'(x_0) : X \rightarrow Y$ such that: $\lim_{x \rightarrow x_0} [\| \tilde{F}(x) - \tilde{F}(x_0) - \tilde{F}'(x_0)(x -$

$x_0) \|\|\|\| x - x_0 \|\|] = 0$

They then extend the concept of Hukuhara difference to define their differential. So they define a fuzzy function F: $U \longrightarrow F_0(\Re^n)$ to be H-differentiable at $x_0 \in U$ if $\exists\ D\text{F}(x_0) \in F_0(\Re^n)$, such that:

$\lim_{h \to o^+} [(\text{F}(x_0 + h)) - \text{F}(x_0))/h] = D\text{F}(x_0)$, and

$\lim_{h \to o^+} [(\text{F}(x_0)) - \text{F}(x_0 - h))/h] = D\text{F}(x_0)$,

and both limits exist. Non-standard fuzzy subtraction, namely Hukuhara difference, is employed here ([125], Definition 3.2, p. 557). In their discussion of this derivative, Dubois and Prade [54], and Buckley and Feuring [21] note that this derivative $\text{F}^{'}_{\alpha}(x_0)$ $= [f_{\alpha}^{-'}(x_0)\ , f_{\alpha}^{+'}(x_0)] = D\text{F}(x_0)$ is always a fuzzy number. However, Dubois and Prade [50] show that it does not always exist.

D.1.3 Goetschel-Voxman derivative

Goetschel and Voxman [75] define the derivative $f^{'}(x_0)$ of $f : \Re^1 \longrightarrow F$, where F is a subset of the topological vector space $V = \{(a(r), b(r), r) \mid 0 \leq r \leq 1; a : I \longrightarrow \Re^1$and $b : I \longrightarrow \Re^1$ are bounded functions$\}$. A fuzzy number here is represented by the parametrized triplets $\{(a(r), b(r), r) \mid 0 \leq r \leq 1\}$, where a and b are the endpoint functions. The derivative is defined as:

$f'(x) = \lim_{h \to o} [(f(x + h)) - f(x))/h]$,

$f'(x)(\alpha) = [f_1'(x, \alpha), f_2'(x, \alpha)]$,

with the limit being taken with respect to the metric D:

$D(\{(a(r), b(r), r) \mid 0 \leq r \leq 1\}, \{(c(r), d(r), r) \mid 0 \leq r \leq 1\}) = \sup\{\max\{|a(r) - b(r)|, |c(r) - d(r)|\}$ $0 \leq r \leq 1\}\}$

Note that D and V together form a topological vector space. The authors prove that $f'(x)$ exists for each $x \in \Re^1$ if a_x and b_x are continuos, where a_x and $b_x$are the partial derivatives of a and b and w.r.t. x ([75], Theorem 2.3, p. 34). They also prove that in a neighborhood N of a point $x_0 \in \Re^1$, if the families $\{a_x(r, x) \mid r \in I\}$ and $\{b_x(r, x) \mid r \in I\}$ exist and are equicontinuous w.r.t. x at x_0, then $f'(x_0)$ exists ([75], Theorem 2.4, p. 35).

Goetschel and Voxman also present the Fundamental Theorem of Fuzzy Calculus, which says: Suppose $f : \Re^1 \to \Im$ is a continuos fuzzy function and let $F(x) = \int_c^x f(t)dt$. Then $F'(x)$ exists and $F'(x) = f(x)$ ([75], Theorem 3.5, p. 40).

Buckley and Feuring ([2]) note that the subtraction employed in the definition of the GVD is not standard fuzzy subtraction. Using standard fuzzy subtraction, $[f(x+h) - f(x)][\alpha] = [f_1(x+h,\alpha) - f_2(x,\alpha), f_2(x+h,\alpha) - f_1(x,\alpha)]$,but using the subtraction employed in GVD, $[f(x+h) - f(x)][\alpha] = [f_1(x+h,\alpha) - f_1(x,\alpha), f_2(x+h,\alpha) - f_2(x,\alpha)]$. Note also that if $\overline{N}$ is a fuzzy number, $-\overline{N}$ is not. So GVD $f'(x)$ may not be a fuzzy number, then $-f'(x)$ is a fuzzy number.

D.1.4 Seikkala derivative

We have already talked in details about this derivative, so we will only define it briefly. Denote a mapping $x : I \to E$, where I is a real interval and E is the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals. Then, x is a fuzzy process with α-level sets:

$$[x(t)]_\alpha = [x_1^\alpha(t), x_2^\alpha(t)],\ t \in I,\ 0 < \alpha \leq 1,$$

with derivative $\text{SD}\overline{X}(t) = [x'(t)]_\alpha = [(x_1^\alpha)'(t), (x_2^\alpha)'(t)]$, which is a fuzzy derivative for all $t \in I$.

D.1.5 Kandel-Friedman-Ming derivative

Buckley and Feuring [21] present a summary of this derivative. In this case, fuzzy numbers do not necessarily have to have compact support. The metric used for this derivative is:

$$D_P(\overline{X}(t), \overline{Z}(t)) = \max\{[\int_0^1 |x_1(t,\alpha) - z_1(t,\alpha)|^P d\alpha]^{1/P}, [\int_0^1 |x_2(t,\alpha) - z_2(t,\alpha)|^P d\alpha]^{1/P}\},$$

where $\overline{X}(t)(\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$ and $\overline{Z}(t)(\alpha) = [z_1(t,\alpha), z_2(t,\alpha)]$ define fuzzy numbers.

The derivative $f'(x)$ has the same definition as GVD. If it exists, then:

$$f'(x)(\alpha) = [f_1'(x,\alpha), f_2'(x,\alpha)],$$

which is always a fuzzy number $\forall t \in I$. However, $f(x)$ is differentiable at $t_0 \in I$ if there is a fuzzy number $\underline{f(x_0)}$ such that:

$$\lim_{h\to 0} D_P[\frac{\overline{X}(t_0+h) - \overline{X}(t_0)}{h}, f(x_0)] = 0$$

But the subtraction in this case is not standard as well. They use the same approach as that used to define GVD.

D.2 Relationships between Derivatives and between Solutions

Theorem 1: ([21], Theorem 3.1, p. 46)

1. If the Goetschel-Voxman derivative (GVD$\overline{X}(t)$) exists and is a fuzzy number for each $t \in I$, then Seikkala Derivative (SD$\overline{X}(t)$) exists and GVD$\overline{X}(t)$ = SD$\overline{X}(t)$.

2. If the Puri-Ralescu derivative (DP$\overline{X}(t)$) exists, then SD$\overline{X}(t)$ exists and PRD$\overline{X}(t)$ = SD$\overline{X}(t)$.

3. If the Kandel-Friedman-Ming derivative (KFM$\overline{X}(t)$) exists, then SD$\overline{X}(t)$ exists and KFMD$\overline{X}(t)$ = SD$\overline{X}(t)$.

4. If SD$\overline{X}(t)$ exists and if $x_1^{'}(t, \alpha)$ and $x_2^{'}(t, \alpha)$ are both continuous in α for each $t \in I$, then SD$\overline{X}(t)$ = DPD$\overline{X}(t)$.

Theorem 2: ([21], Theorem 3.2, p. 46) Assume the continuity condition holds. If SD$\overline{X}(t)$ exists, then SD$\overline{X}(t)$ = DPD$\overline{X}(t)$ = GVD$\overline{X}(t)$ = PRD$\overline{X}(t)$ = KFMD$\overline{X}(t)$.

Theorem 3: (Theorem 3.3, p.47) Assume the continuity condition holds. If one of the derivatives SD, GVD and it is a fuzzy number, PRD, or KFMD exist, then so do the others and they are all equal.

Theorem 4:([21], Theorem 4.2, p.49)

1. If BFS = $\overline{Y}(t)$, then SS = $\overline{Y}(t)$.

2. If PRS = $\overline{X}(t)$, then SS = $\overline{X}(t)$.

3. If KFMS = $\overline{X}(t)$, then SS = $\overline{X}(t)$.

where BFS, SS, PRS and KFMS stand for the Buckley-Feuring solution, Seikkala solution, Puri-Ralescu solution and Kandel-Friedman-Ming solution respectively.

Theorem 5: ([21], Theorem 4.3, p. 49))

1. If BFS = $\overline{Y}(t)$ and PRD$\overline{Y}(t)$ exists, then PRS = $\overline{Y}(t)$.

2. If BFS = $\overline{Y}(t)$ and KFMD$\overline{Y}(t)$ exists, then KFMS = $\overline{Y}(t)$.

Theorem 6: ([21], Theorem 4.4, p. 49)

Assume the continuity condition holds and $I = [0, M]$, where $M > 0$.

1. If SD$\overline{X}(t)$ exists, then SS = PRS = KFMS.

2. If BFS = $\overline{Y}(t)$ and the continuity holds for $\overline{Y}(t)$, then BFS = SS = PRS = KFMS.

Theorem 7: ([21], Theorem 4.5, p. 49)

1. If PRS = $\overline{X}(t)$ and the derivative condition (refer to Solving Fuzzy Differential Equations, item II) holds, then BFS = $\overline{X}(t)$.

2. If SS = $\overline{X}(t)$ and the derivative condition holds, then BFS = $\overline{X}(t)$.

3. If KFMS = $\overline{X}(t)$ and the derivative condition holds, then BFS = $\overline{X}(t)$.

D.3 Solving Fuzzy Equations

Several methods have been proposed for solving fuzzy equations. We will consider the extension principle approach, and the α-cuts and interval arithmetic approach.

D.3.1 Extension Principle Approach

Considering an equation of the form $y = f(x_1, ..., x_n)(f : R^n \to R)$, this approach goes as follows [23],

1. Substitute triangular fuzzy numbers $\overline{X}_i$ for x_i.

2. Obtain $\overline{Y}$ using the extension principle:

$\mu(y \mid \overline{Y}) = \sup\{\min\{\mu(x_i \mid \overline{X}_i)\} \mid f(x_1, ..., x_n) = y\}, 1 \leq i \leq n$

3. Get the united extension of f

$\Omega(\alpha) = \{y \mid y = f(x), x \in \overline{X}(\alpha)\}$

where $\overline{X}(\alpha) = \overline{X}_1(\alpha) \times ... \times \overline{X}_n(\alpha)$, and $\overline{X}_i(\alpha) = [x_{i1}(\alpha), x_{i2}(\alpha)]$ is a closed interval because $y = \mu(x_i \mid \overline{X}_i)$ is continuous.

4. Define $\overline{W}$, a fuzzy subset of the real numbers, by:

$\mu(y \mid \overline{W}) = \sup\{\alpha \mid y \in \Omega(\alpha)\}$

where the supremum of the empty set, i.e. when there are no values for x_i, is zero. The authors prove that $\overline{W} = \overline{Y}$, and, if f is continuous, $\overline{W}(\alpha) = \overline{Y}(\alpha) = \Omega(\alpha)$.

To evaluate the fuzzy equation, Buckley [19] outlines the extension principle approach for solving a fuzzy quadratic equation with only one variable of the form $\overline{A}\ \overline{X}^2 + \overline{B}\ \overline{X} = \overline{C}$:

Denote the set of all values of $(ax^2 + bx)$, where $x \in \overline{X}(\alpha)$, $a \in \overline{A}(\alpha)$, $b \in \overline{B}(\alpha)$, by $\overline{S}_1(\alpha) = [\theta_1(\alpha),\, \theta_2(\alpha)]$. Given that $\overline{X}(\alpha) = [x_1(\alpha),\, x_2(\alpha)]$, then to solve the equation for $\overline{X}$, we have to have $\overline{S}_1(\alpha) = \overline{C}(\alpha)$ where $\overline{C}(\alpha) = [c_1(\alpha),\, c_2(\alpha)]$. So we set $\theta_i(\alpha) = c_i(\alpha)$, $i = 1, 2$, and solve the two equations for $x_1(\alpha)$ and $x_2(\alpha)$.

In an earlier paper, Buckley and Qu [22] use the extension principle approach to solve various linear and quadratic fuzzy equations. However, they only tackle existence and computation, but not uniqueness, of the solution. They also consider real as well as complex solutions. We will confine the overview to real fuzzy numbers only. Note that all parameters and variables are considered triangular fuzzy numbers.

Linear Equations

1. $\overline{A} + \overline{X} = \overline{C}$

Existence of Solution

A solution $\overline{X}$ to the above equation exists iff:

$c_1 - a_1 < c_2 - a_2 < c_3 - a_3$.

Computation of Solution

$\overline{X}$ is a triangular fuzzy number, whereby

$\overline{X} = (c_1 - a_1 < c_2 - a_2 < c_3 - a_3)$,

([22], Theorem 1, p. 45).

2. $\overline{A}\,\overline{X} = \overline{C}$

Assume zero does not belong to the support of A.

Existence of Solution (assuming 0 does not belong to the support of A) A solution exists iff

a) Assuming zero does not belong to the support of C:

$a_1c_2 > c_1a_2$ and $a_3c_2 < c_3a_2$ when $\overline{A} > 0$, $\overline{C} \geq 0$;

$a_1c_2 < c_1a_2$ and $a_3c_2 > c_3a_2$ when $\overline{A} < 0$, $\overline{C} \leq 0$;

$a_3c_2 > c_1a_2$ and $a_1c_2 < c_3a_2$ when $\overline{A} > 0$, $\overline{C} \leq 0$;

$a_3c_2 < c_1a_2$ and $a_1c_2 > c_3a_2$ when $\overline{A} < 0$, $\overline{C} \geq 0$;

b) Assuming zero belongs to the support of $C (c_2 = 0)$:

Zero belongs to the support of $\overline{X}$.

Computation of Solution

a) Zero does not belong to the support of C:

$x_1(\alpha) = c_1(\alpha)\ /\ a_1(\alpha)$, $x_2(\alpha) = c_2(\alpha)\ /\ a_2(\alpha)$ when $\overline{A} > 0$, $\overline{C} \geq 0$;

$x_1(\alpha) = c_2(\alpha)\ /\ a_2(\alpha)$, $x_2(\alpha) = c_1(\alpha)\ /\ a_1(\alpha)$ when $\overline{A} < 0$, $\overline{C} \leq 0$;

$x_1(\alpha) = c_1(\alpha)\ /\ a_2(\alpha)$, $x_2(\alpha) = c_2(\alpha)\ /\ a_1(\alpha)$ when $\overline{A} > 0$, $\overline{C} \leq 0$;

$x_1(\alpha) = c_2(\alpha)\ /\ a_1(\alpha)$, $x_2(\alpha) = c_1(\alpha)\ /\ a_2(\alpha)$ when $\overline{A} < 0$, $\overline{C} \geq 0$;

b) Zero belongs to the support of $C :\Longrightarrow x_2 = 0$

$x_1(\alpha) = c_1(\alpha)\ /\ a_2(\alpha)$, $x_2(\alpha) = c_2(\alpha)\ /\ a_2(\alpha)$ when $\overline{A} > 0$;

$x_1(\alpha) = c_2(\alpha)\ /\ a_1(\alpha)$, $x_2(\alpha) = c_1(\alpha)\ /\ a_1(\alpha)$ when $\overline{A} < 0$;

([22], Theorem 3, p.46)

3. $\overline{A}\,\overline{X} + \overline{B} = \overline{C}$

This equation boils down to equation 2 above. So substitute $c_1(\alpha) - b_1(\alpha)$ and $c_2(\alpha) - b_2(\alpha)$ for $c_1(\alpha)$ and $c_2(\alpha)$ respectively, and $c_1 - b_1$, $c_2 - b_2$ and $c_3 - b_3$ for c_1, c_2 and c_3 respectively in case 2.([22], Theorem 6, p.48)

Quadratic Equations

1. $\overline{A}\,\overline{X}^2 = \overline{C}$

Assume that zero does not belong to the support of A. But since $\overline{X}^2 \geq 0$, we must have $\overline{C} \geq 0$ and $\overline{C} \leq 0$.

Existence of Solution

Solutions $\overline{X}_1 \geq 0$ and $\overline{X}_2 = -\overline{X}_1$ exist iff

a) $a_1c_2 > c_1a_2$ and $a_3c_2 < c_3a_2$ when $\overline{A} > 0$, $\overline{C} \geq 0$;

b) $a_1c_2 < c_1a_2$ and $a_3c_2 > c_3a_2$ when $\overline{A} < 0$, $\overline{C} \leq 0$.

Computation of Solution

Let $\overline{U} = \overline{X}^2 \Longrightarrow$ you can solve the equation as $\overline{A}\,\overline{U} = \overline{C}$. Then, compute $\overline{X}_1$ according to $\overline{X}_1 = \overline{U}^{0.5}$

2. $\overline{A}\,\overline{X}^2 + \overline{B} = \overline{C}$

Let $c_i^* = c_i - b_i$, $i = 1, 2, 3$.

Existence of Solution

Solutions $\overline{X}_1 \geq 0$ and $\overline{X}_2 = -\overline{X}_1$ exist iff:

c) $c_i^* \geq 0$ for all i, $a_1 c_2^* > c_1^* a_2$ and $a_3 c_2^* < c_3^* a_2$ when $\overline{A} > 0$;

d) $c_i^* \leq 0$ for all i, $a_1 c_2^* < c_1^* a_2$ and $a_3 c_2^* > c_3^* a_2$ when $\overline{A} < 0$;

Computations of Solution

Let $\overline{U} = \overline{X}^2 > 0 \Longrightarrow$ you can solve the equation as $\overline{A}\,\overline{U} = \overline{C}^*$. Then, compute $\overline{X}_1$ according to $\overline{X}_1 = \overline{U}^{0.5}$. ([22], Theorem 9, p.48)

3. $\overline{A}\,\overline{X}^2 + \overline{B}\,\overline{X} = \overline{C}$

To solve those equations, we need to take the α-cuts of the parameters and variables. Therefore, we obtain two simultaneous equations. The authors obtain eight cases for those simultaneous equations based on the sign of $\overline{A}$, $\overline{B}$ and $\overline{X}$ (see Table 1, Appendix D, [22]).

Existence of Solution

a) Case $P(1,1)$: $\overline{A} > 0$, $\overline{B} \geq 0$ and $\overline{X} \geq 0$, a solution exists in this case iff $0 \leq x_1 < x_2 < x_3$ and:

$$c_2 - c_1 > (a_2 - a_1)x_2^2 + (b_2 - b_1)x_2$$

$$c_3 - c_2 > (a_3 - a_2)x_3^2 + (b_3 - b_2)x_3$$

b) Case $P(2,2)$: $\overline{A} < 0$, $\overline{B} \leq 0$ and $\overline{X} \geq 0$, a solution exists in this case iff $0 \leq x_1 < x_2 < x_3$ and:

$$c_2 - c_1 > (a_2 - a_1)x_3^2 + (b_2 - b_1)x_3$$

$$c_3 - c_2 > (a_3 - a_2)x_2^2 + (b_3 - b_2)x_2$$

c) Case $N(1,2)$: $\overline{A} > 0, \overline{B} \leq 0$ and $\overline{X} \leq 0$, a solution exists in this case iff $x_1 < x_2 < x_3 \leq 0$ and:

$$c_2 - c_1 > (a_2 - a_1)x_2^2 + (b_3 - b_2)x_2$$

$$c_3 - c_2 > (a_3 - a_2)x_1^2 + (b_1 - b_2)x_1$$

d) Case $N(2,1)$: $\overline{A} < 0, \overline{B} \geq 0$ and $\overline{X} \leq 0$, a solution exists in this case iff $x_1 < x_2 < x_3 \leq 0$ and:

$$c_2 - c_1 > (a_2 - a_1)x_1^2 + (b_3 - b_2)x_1$$

$$c_3 - c_2 > (a_3 - a_2)x_2^2 + (b_1 - b_2)x_2$$

The authors do not present sufficient conditions for the existence of (real) solutions to the remaining four cases because they are complicated and too restrictive. To solve them, the authors wrote a computer program and inputted many values for the fuzzy parameters but never really found a combination that could solve the equations (i.e.

give a real fuzzy number as a solution). ([22], Theorem 13, p. 49)

Therefore, the extension principle approach is very restrictive and many fuzzy equations do not have a solution, not even a complex one. So the authors introduce a new solution concept, as we will see later. In a later paper, Buckley and Qu ([23]) consider equations of the form $y = f(x_1, ..., x_n)$. They particularly consider two cases 1. One variable appears more than once in $f(x_1, ..., x_n)$. Then, we compute $\overline{Y}(\alpha) = \Omega(\alpha)$ by substituting all a_i in $\overline{X}_i(\alpha)$ for x_i, for $1 \leq i \leq n$, in $f(x_1, ..., x_n)$. But the same a_1 has to be substituted for all the x_1 in $f(x_1, ..., x_n)$.

The problems with this approach are that it is not always easy to determine $\overline{S}_1(\alpha)$, and it is not possible to get a solution all the time even for simple fuzzy equations.

D.3.2 α-Cuts and Interval Arithmetic Approach

Consider the equation of the form $y = f(x_1, ..., x_n)$. Let $\overline{V}(\alpha) = f(\overline{X}_1(\alpha), ..., \overline{X}_n(\alpha))$, where $\overline{V}(\alpha)$ is computed using interval arithmetic. The following arithmetic operations will be needed to evaluate the fuzzy equation:

$(\overline{M}\ \overline{N})(\alpha) = \overline{M}(\alpha)\ \overline{N}(\alpha)$

$(\overline{M} \pm \overline{N})(\alpha) = \overline{M}(\alpha) \pm \overline{N}(\alpha)$

$(\overline{M}\ /\ \overline{N})(\alpha) = \overline{M}(\alpha)\ \overline{N}^{-1}(\alpha)$

where the right hand sides of the equalities are calculated using interval arithmetic. However, one must not mistakenly conclude that, as a result, $(\overline{A} + \overline{B}\ \overline{X}^2)(\alpha) = \overline{A}(\alpha) + \overline{B}(\alpha)(\overline{X}(\alpha))^2$ holds because not all fuzzy numbers $\overline{A}$, $\overline{B}$, and $\overline{X}$ generate an equality.

Before moving on to the computation of the solution, we will present the comparison Buckley draws between the extension principle solution, denoted by $\overline{Y}(\alpha)$, and this solution, denoted by $\overline{V}(\alpha)$. We would expect $\overline{Y}(\alpha) \subset \overline{V}(\alpha)$ (see inclusion monotonicity above). However, we will not always get $\overline{Y}(\alpha) = \overline{V}(\alpha)$, meaning that, in general, we can not use α-cuts to evaluate fuzzy equations. In ([19], p. 3), Buckley argues that the a-cuts approach produces a fuzzier solution than the extension principle approach does.

To illustrate this point, he considers a quadratic equation of the form $y = ax^2 + bx$. Therefore, we find the α-cuts of $\overline{Y}(\alpha)$ by computing the intervals: $\Omega(\alpha) = \{ax^2 + bx \mid$

$x \in \overline{X}(\alpha)$, $a \in \overline{A}(\alpha)$, $b \in \overline{B}(\alpha)\}$, while we find $\overline{V}(\alpha)$ by using interval arithmetic: $\overline{V}(\alpha) = \{ax_1x_2 + bx_3 \mid x_i \in \overline{X}(\alpha),\ i = 1,2,3,\ a \in \overline{A}(\alpha),\ b \in \overline{B}(\alpha)\}$. Therefore, we use one value of x, $x_1 = x_2 = x_3 = x$, in computing $\overline{Y}(\alpha)$ but three different values of x to compute $\overline{V}(\alpha)$. Hence, $\overline{Y}(\alpha)$ is a proper subset of $\overline{V}(\alpha)$, which is fuzzier.

D.4 Solving Fuzzy Differential Equations

D.4.1 Seikkala Solution

Seikkala [128] considers the initial value problem:

$x'(t) = f(t, x(t)), x(0) = x_0$

He presents two solutions: one using the extension principle approach (and which is the solution which Buckley and Feuring refer to as Seikkala Solution later on), and another using the extremal solution approach. Before proceeding, we should present the two properties, which Seikkala defines for α-level sets to represent a fuzzy number (and vice versa). They are:

1. Inclusion Property: $[a_1^\alpha, a_2^\alpha] \supset [a_1^\beta, a_2^\beta], 0 \prec \alpha \preceq \beta$, i.e., $a_1^\alpha \preceq a_1^\beta \preceq a_2^\beta \preceq a_2^\alpha$
2. Continuity Property: $[\lim_{k\to\infty} a_1^{\alpha_k}, \lim_{k\to\infty} a_2^{\alpha_k}] = [a_1^\alpha, a_2^\alpha]$,whenever (α_k) is a nondecreasing sequence converging to $\alpha_k \in (0, 1]$

The initial value problem has a deterministic function f, which is a continuous mapping: $\Re_+ \times \Re \to \Re$ and a fuzzy initial value x_0, where $x_0$is a fuzzy number in E (E is the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals) with α-level intervals: $[x_0]_\alpha = [x_{01}^\alpha, x_{02}^\alpha], 0 \prec \alpha \leq 1$. Now, we move to present the two solutions.

Extension Principle Approach

Using the extension principle of Zadeh, we can define $f(t, x)$, where x is a fuzzy number, as:

$f(t, x)(s) = \sup\{x(\tau) : s = f(t, \tau)\}, s \in \Re$

$\Longrightarrow [f(t, x)]_\alpha = [\min\{f(t, u) : u \in [x_1^\alpha, x_2^\alpha]\}, \max\{f(t, u) : u \in [x_1^\alpha, x_2^\alpha]\}]$

where $x \in E$ with $[x]_\alpha = [x_1^\alpha, x_2^\alpha], 0 \prec \alpha \leq 1$

But since the fuzzy derivative of a fuzzy process x is:

$[x'(t)]_\alpha = [(x_1^\alpha)'(t), (x_2^\alpha)'(t)], 0 \prec \alpha \leq 1$

$\Longrightarrow x : \Re_+ \rightarrow E$ is a fuzzy solution to the initial value problem on interval $I = [0, T)$ if:

$(x_1^\alpha)'(t) = \min\{f(t, u) : u \in [x_1^\alpha(t), x_2^\alpha(t)]\}, (x_1^\alpha)(0) = x_{01}^\alpha$

$(x_2^\alpha)'(t) = \max\{f(t, u) : u \in [x_1^\alpha(t), x_2^\alpha(t)]\}, (x_2^\alpha)(0) = x_{02}^\alpha$

where $t \in I$ and $0 \prec \alpha \leq 1$. Thus, if we can solve the equation uniquely, all we have to do is prove that $[x_1^\alpha(t), x_2^\alpha(t)], 0 \prec \alpha \leq 1$, define a fuzzy number $x(t)$ in E. For the equation to have a unique solution, f has to satisfy a generalized Lipschitz condition: $\mid f(t,x) - f(t,\overline{x}) \mid \leq g(t, \mid x - \overline{x} \mid), t \geq 0$, and $x, \overline{x} \in \Re^2$, where $g : \Re_+ \times \Re \rightarrow \Re$ is a continuous mapping such that $r \rightarrow g(t, r)$ is nondecreasing.

Extremal Solution Approach

Using this approach, we have to solve the crisp initial value problems (separately)

$x'(t) = f(t, x(t)),\ x(0) = x_{01}^\alpha,$

& $x'(t) = f(t, x(t)),\ x(0) = x_{02}^\alpha.$

Then, we study the dependence of the solution on α. The following equations should define a fuzzy process

$x : [0, t_\beta] \rightarrow E$, for $\beta \in [0, 1], t_\beta > 0,$

$[x(t)]_\alpha = [x_1^\alpha(t), x_2^\alpha(t)],\ \beta \leq \alpha \leq 1,$

& $[x(t)]_\alpha = [x_1^\beta(t), x_2^\beta(t)],\ 0 < \alpha \leq \beta,$

where $x_1^\alpha(t)$ & $x_2^\alpha(t)$ are the minimal and maximal solutions of the two initial value problems.

D.4.2 Buckley-Feuring Solution

Buckley and Feuring [21] also consider the initial value problem

$y'(t) = f(t, y, k),\ y(0) = c,$

where $y = g(t, k, c)$ defines the solution to this equation, and the parameters k_i and the initial condition c are uncertain. This uncertainty is modeled by respectively substituting triangular numbers $\overline{K}_i$ (hence, $\overline{K} = (\overline{K}_1, ..., \overline{K}_n)$ becomes a fuzzy vector) and $\overline{C}$ for them. Hence, y becomes a fuzzy function $\overline{Y}(t)$, where

$$\frac{d\overline{Y}}{dt} = f(t, \overline{Y}, \overline{K}),\ \overline{Y}(0) = \overline{C}$$

The sufficient conditions for the uniqueness of the solution (p.43) are

1. (0,c) is in $\Re$.
2. f is continuous in $\Re$ (k is held fixed).
3. $\dfrac{\partial f}{\partial y}$ is continuous in $\Re$.

Continuity is always assumed. The continuity condition (p.46) holds when $x_i'(t, \alpha)$ is continuous on $I \times [0, 1],\ i = 1, 2$.

Several solutions (see Appendix D) are analyzed in this paper besides the authors' new solution. However, the authors finally consider two solutions: Seikkala Solution (denoted by SS) and Buckley-Feuring Solution (denoted by BFS) due to a set of relationships between the various derivatives (see Appendix E). Besides, some solutions are discarded because they are not always equal to a fuzzy number.

The BFS solution approach goes as follows

$$\frac{d\overline{Y}}{dt} = f(t, \overline{Y}, \overline{K}),\ \overline{Y}(0) = \overline{C}.$$

Let $\overline{K}(\alpha) = \overline{K}_1[\alpha] \times ... \times \overline{K}_n[\alpha]$,

$$\Phi(\alpha) = K(\alpha) \times \overline{C}[\alpha],\ 0 \le \alpha \le 1$$

Assume $\Phi(0) \subset K \times C$ so that g will be continuous on $I \times \Phi(\alpha)$ for all α, then

Step I: Fuzzify the crisp solution $y = g(t, k, c)$ to obtain $\overline{Y} = g(t, \overline{K}, \overline{C})$ using either one of

- the extension principle approach,
- the α-cuts approach, where $\overline{Y}(t)[\alpha] = [y_1(t, \alpha), y_2(t, \alpha)]$

  $y_1(t, \alpha) = \min\{g(t, k, c) \mid k \in \overline{K}[\alpha],\ c \in \overline{C}[\alpha]\}$,

  $y_2(t, \alpha) = \max\{g(t, k, c) \mid k \in \overline{K}[\alpha],\ c \in \overline{C}[\alpha]\}$,

  $t \in I,\ \alpha \in [0, 1]$, I is a closed bounded interval which includes zero

- the united extension approach

  define $\Omega(\alpha) = \{g(t, k, c) \mid (k,\ c) \in \Phi(\alpha)\}$,

$\overline{Y}(t)(x) = \sup\{\alpha \mid x \in \Omega(\alpha)\}$,

$t \in I, \ \alpha \in [0,1]$,

where $\overline{Y}(t)(x)$ is the membership function of $\overline{Y}(t)$.

$\Longrightarrow \overline{Y}(t)[\alpha] = \Omega(\alpha)$ and $\overline{Y}(t)$ is a fuzzy number for all $t \in I$. (Theorem 2.1, p. 44)

Step II: Assume that $y_i(t, \alpha)$ is differentiable for all $t \in I, \ \alpha \in [0,1]$. Denote the partial of $y_i(t, \alpha)$ with respect to t by $y_i'(t, \alpha), \ i = 1, 2$. Let

$\Gamma(t, \alpha) = [y_1'(t, \alpha), y_2'(t, \alpha)]$

If $\Gamma(\alpha)$ defines the α-cuts of a fuzzy number, for all $t \in I$, then $\overline{Y}(t)$ is differentiable and

$\dfrac{d\overline{Y}(t)}{dt}[\alpha] = \Gamma(t, \alpha) = [y_1'(t, \alpha), y_2'(t, \alpha)]$, for all $t \in I, \ \alpha \in [0,1]$.

Note that this equation is the derivative of $\overline{Y}(t)[\alpha] = [y_1(t, \alpha), y_2(t, \alpha)]$ and so we can write it as $\dfrac{d}{dt}(\overline{Y}(t)[\alpha])$.

Step III: In order for $\overline{Y}(t)$ to be a solution, the fuzzy initial value problem (FIVP) must hold and $\dfrac{d\overline{Y}(t)}{dt}$ must exist. To check the FIVP, we need to compute the α-cuts of $f(t, \overline{Y}, \overline{K})$, where

$f(t, \overline{Y}, \overline{K})[\alpha] = [f_1(t, \alpha), f_2(t, \alpha)]$,

$f_1(t, \alpha) = \min\{f(t, y, k) \mid y \in \overline{Y}(t)[\alpha], \ k \in \overline{K}[\alpha]\}$,

$f_2(t, \alpha) = \max\{f(t, y, k) \mid y \in \overline{Y}(t)[\alpha], \ k \in \overline{K}[\alpha]\}$,

$t \in I, \ \alpha \in [0,1]$.

Hence, $\overline{Y}(t)$ must satisfy the following equations

$y_1'(t, \alpha) = f_1(t, \alpha)$,

$y_2'(t, \alpha) = f_2(t, \alpha)$,

$y_1(0, \alpha) = c_1(\alpha)$,

$y_2(0, \alpha) = c_2(\alpha)$,

where $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$.

This solution does not solve all fuzzy equations as we will see later. The conditions (*derivative condition*) for the existence of BFS i.e. BFS $= \overline{Y}(t)$, provided that the Seikkala derivative (SD$\overline{Y}(t)$) exists for $t \in I$, are ([21], Theorem 4.1, p. 48)

$\dfrac{\partial f}{\partial y} > 0, \quad \dfrac{\partial g}{\partial c} > 0,$

& $\left(\frac{\partial g}{\partial k_i}\right)\left(\frac{\partial f}{\partial k_i}\right) > 0$,
$i = 1...n$. If any of these conditions is violated, then $\overline{Y}(t)$ does not solve the equation. In this case, we would have to look for the Seikkala solution (SS). Note that
when $\frac{\partial g}{\partial k} < 0$ and $\frac{\partial g}{\partial c} > 0$, $y_1(t,\alpha) = g(t, k_2(\alpha), c_1(\alpha))$ and $y_2(t,\alpha) = g(t, k_1(\alpha), c_2(\alpha))$,
when $\frac{\partial g}{\partial c} > 0$ and $\frac{\partial g}{\partial k} > 0$, $y_1(t,\alpha) = g(t, k_1(\alpha), c_1(\alpha))$ and $y_2(t,\alpha) = g(t, k_2(\alpha), c_2(\alpha))$,
when $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial f}{\partial k} < 0$, $f_1(t,\alpha) = f(t, y_1(t,\alpha), k_2(\alpha))$ and $f_2(t,\alpha) = f(t, y_2(t,\alpha), k_1(\alpha))$,
when $\frac{\partial f}{\partial y} < 0$ and $\frac{\partial f}{\partial k} > 0$, $f_1(t,\alpha) = f(t, y_2(t,\alpha), k_1(\alpha))$ and $f_2(t,\alpha) = f(t, y_1(t,\alpha), k_2(\alpha))$.

We have already talked about the Seikkala solution. However, the approach Buckley and Feuring refer to is the extension principle approach. Namely, SS is a solution to the FIVP, such that SS $= \overline{X}(t)$, if the Seikkala derivative (SD$\overline{X}(t)$) exists and

$$SD\overline{X}(t) = f(t, \overline{X}(t), \overline{K}),$$
$$\overline{X}(0) = \overline{C},$$

which is equivalent to

$$x_1'(t,\alpha) = f_1(t,\alpha),$$
$$x_2'(t,\alpha) = f_2(t,\alpha),$$
$$x_1(0,\alpha) = c_1(\alpha),$$
$$x_2(0,\alpha) = c_2(\alpha),$$

where $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$, and f is as defined above. If $[x_1(t,\alpha), x_2(t,\alpha)]$ define the α-cuts of a fuzzy number $\overline{X}(t)$, then SS $= \overline{X}(t)$. Recall that sufficient conditions for $[x_1(t,\alpha), x_2(t,\alpha)]$ to define the α-cuts of a fuzzy number are ([21], p.44)

- $x_1(t,\alpha)$ and $x_2(t,\alpha)$ are continuous on $I \times [0,1]$.
- $x_1(t,\alpha)$ is an increasing function of α for each $t \in I$.
- $x_2(t,\alpha)$ is a decreasing function of α for each $t \in I$.
- $x_1(t,1) \leq x_2(t,1)$ all $t \in I$.

In fact, SS is the most general solution to the FIVP (refer to Appendix A for a summary of relationship between various solutions). So in solving equations, as we will see in the following applications, we will look first for BFS and, if does not exist, look for SS.

In an earlier paper, Buckley and Feuring [20] have used this approach in solving elementary fuzzy partial differential equations. However, the FPDEs are elementary in the sense that their solutions are elementary (no Bessel functions, Fourier series...).

APPENDIX E

Table 1E: This table shows the comparative option values for FBS and BS OPM for S&P 500 Index Options as of close of 24th July 2002

Moneyness	Expiration	Last	sigma BS	sigma1	sigma2	sigma3	FBS Value	BS Value
ITM	Aug-02	123.20	0.31353	0.26164	0.31968	0.31968	119.52079	119.69317
ITM	Aug-02	100.70	0.24645	0.24164	0.24796	0.24796	94.45801	94.45941
ITM	Aug-02	79.30	0.22368	0.21556	0.21556	0.24729	70.80056	70.79295
ITM	Aug-02	59.80	0.29346	0.25988	0.30454	0.30454	55.40911	55.40513
ITM	Aug-02	52.60	0.32668	0.27392	0.34427	0.34427	51.58117	51.56812
ITM	Aug-02	45.80	0.31100	0.26761	0.32541	0.32541	43.91728	43.91604
ITM	Aug-02	42.60	0.28352	0.25600	0.29262	0.29262	38.32030	38.32435
ATM	Aug-02	39.50	0.28348	0.25294	0.29347	0.29347	35.45985	35.47315
ATM	Aug-02	33.70	0.25342	0.24443	0.25631	0.25631	27.16164	27.16918
ATM	Aug-02	28.50	0.30305	0.26361	0.31422	0.31422	27.30145	27.45095
ATM	Aug-02	23.70	0.25884	0.24296	0.26368	0.26368	18.82390	18.85635
OTM	Aug-02	19.90	0.31120	0.21200	0.32192	0.32192	18.72341	20.31454
OTM	Aug-02	17.95	0.28744	0.28743	0.28743	0.28743	16.37026	16.37117
OTM	Aug-02	14.55	0.32253	0.24421	0.33261	0.33261	15.51851	16.57639
OTM	Aug-02	9.80	0.36471	0.22037	0.34723	0.34723	12.17049	16.23749
OTM	Aug-02	8.60	0.35885	0.20927	0.32840	0.32840	9.75587	14.49281
OTM	Aug-02	5.20	0.43847	0.19793	0.32443	0.32443	6.07091	16.54793
ITM	Sep-02	129.40	0.26301	0.23583	0.27100	0.27100	122.62985	122.65956
ITM	Sep-02	108.50	0.25326	0.24423	0.25623	0.25623	100.72160	100.72178
ITM	Sep-02	88.60	0.27998	0.25528	0.28820	0.28820	84.50662	84.49812
ITM	Sep-02	70.20	0.21005	0.20291	0.20291	0.23146	58.05915	58.05190
ITM	Sep-02	63.40	0.21428	0.20796	0.20796	0.23320	52.08019	52.07716
ITM	Sep-02	57.10	0.24752	0.24211	0.24932	0.24932	51.00409	51.00417
ITM	Sep-02	54.20	0.24526	0.24115	0.24588	0.24813	47.86832	47.86849
ATM	Sep-02	51.00	0.27904	0.25057	0.28836	0.28836	50.40671	50.42640
ATM	Sep-02	44.90	0.24349	0.23972	0.24376	0.24671	39.82694	39.82788
ATM	Sep-02	39.90	0.24390	0.24002	0.24427	0.24701	35.22603	35.22759
ATM	Sep-02	35.00	0.27511	0.22114	0.28928	0.28928	35.42438	35.87123
OTM	Sep-02	28.40	0.24479	0.24052	0.24551	0.24752	25.43166	25.43497
OTM	Sep-02	26.50	0.25475	0.24098	0.25901	0.25901	25.19724	25.23247
OTM	Sep-02	24.70	0.30043	0.19589	0.30874	0.30874	27.51484	30.47870
OTM	Sep-02	18.70	0.27369	0.25216	0.27973	0.27973	21.74188	21.85864
OTM	Sep-02	15.90	0.26615	0.24927	0.27093	0.27093	18.06445	18.14648
ITM	Oct-02	153.80	0.21868	0.20250	0.20250	0.26474	144.79864	144.71978
ITM	Oct-02	111.90	0.24510	0.24057	0.24614	0.24757	104.72523	104.72448
ITM	Oct-02	74.90	0.25965	0.24733	0.26375	0.26375	72.50419	72.50352
ITM	Oct-02	58.90	0.20682	0.17964	0.17964	0.28550	48.40863	48.52965
ATM	Oct-02	44.90	0.22283	0.21074	0.21074	0.25792	39.18762	39.24360
OTM	Oct-02	33.20	0.23437	0.23198	0.23198	0.24147	31.38255	31.38581
OTM	Oct-02	23.30	0.22233	0.20273	0.20273	0.27242	20.87824	21.20338
OTM	Oct-02	16.10	0.22532	0.20846	0.20846	0.26722	15.28266	15.55496
OTM	Oct-02	10.75	0.23175	0.22908	0.22908	0.23947	11.76008	11.76760
OTM	Oct-02	7.15	0.24478	0.24031	0.24552	0.24732	9.71029	9.72297
OTM	Oct-02	4.20	0.46472	0.19362	0.39732	0.39732	19.38530	36.92764

Moneyness	Expiration	Last	sigma BS	sigma1	sigma2	sigma3	FBS Value	BS Value
ITM	Dec-02	201.40	0.22225	0.20374	0.20374	0.27187	193.57236	193.50090
ITM	Dec-02	158.20	0.25377	0.24396	0.25699	0.25699	153.58484	153.58566
ITM	Dec-02	118.10	0.23105	0.22805	0.22805	0.24004	111.89493	111.89229
ITM	Dec-02	82.40	0.22748	0.22372	0.22372	0.23876	78.79241	78.79149
ITM	Dec-02	67.00	0.21707	0.20904	0.20904	0.24106	62.63373	62.64053
ATM	Dec-02	53.40	0.19374	0.15175	0.15175	0.30597	44.02739	44.89017
OTM	Dec-02	41.60	0.21002	0.16945	0.16945	0.31381	37.79027	38.83298
OTM	Dec-02	31.40	0.21552	0.18178	0.18178	0.29949	30.79692	31.69614
OTM	Dec-02	23.20	0.21002	0.15759	0.15759	0.30619	20.62927	23.48697
OTM	Dec-02	16.80	0.19520	0.19520	0.19520	0.19520	15.05346	15.05346
OTM	Dec-02	11.90	0.22761	0.15686	0.15686	0.31026	11.91571	16.69783
OTM	Dec-02	9.05	0.21212	0.17133	0.17133	0.26427	8.79614	11.02967
OTM	Dec-02	3.60	0.28900	0.15951	0.15951	0.29818	4.86140	16.34617
OTM	Dec-02	1.90	0.23017	0.22319	0.22319	0.25064	4.40008	4.37563
ITM	Mar-03	163.20	0.23228	0.22941	0.22941	0.24090	158.27127	158.26716
ITM	Mar-03	90.50	0.21285	0.20612	0.20612	0.23304	87.75242	87.74966
ITM	Mar-03	75.50	0.20176	0.17920	0.17920	0.26868	71.25739	71.31530
ATM	Mar-03	62.00	0.17750	0.14944	0.14944	0.25673	51.67245	52.06559
OTM	Mar-03	50.10	0.18258	0.14791	0.14791	0.27374	42.63334	43.58256
OTM	Mar-03	39.90	0.18312	0.14795	0.14795	0.26871	33.86269	35.19409
OTM	Mar-03	31.20	0.19234	0.19234	0.19234	0.19234	30.67144	30.67144
OTM	Mar-03	23.90	0.22897	0.22187	0.22187	0.24995	34.71167	34.72171
ITM	Jun-03	168.20	0.17870	0.14941	0.14941	0.26302	152.79749	152.37038
ITM	Jun-03	131.20	0.17595	0.16168	0.16168	0.21880	115.51752	115.42899
ITM	Jun-03	97.90	0.21280	0.20595	0.20595	0.23333	97.70943	97.70698
ITM	Jun-03	83.30	0.18470	0.15814	0.15814	0.26332	75.00499	75.10531
ATM	Jun-03	69.60	0.18649	0.14960	0.14960	0.29146	63.80649	64.35025
OTM	Jun-03	57.80	0.20164	0.15874	0.15874	0.31900	58.85336	59.85780
OTM	Jun-03	48.50	0.20339	0.15817	0.15817	0.31991	50.06410	51.62340
OTM	Jun-03	39.50	0.19452	0.14867	0.14867	0.30092	38.41942	40.73859
OTM	Jun-03	31.70	0.20823	0.15837	0.15837	0.31838	35.94511	38.69469
OTM	Jun-03	228.80	0.22584	0.21327	0.21327	0.26120	38.34687	38.48707
OTM	Jun-03	20.70	0.20105	0.14753	0.14753	0.28978	22.10089	26.39608
OTM	Jun-03	14.80	0.25568	0.11277	0.11277	0.29083	12.84651	37.41728
OTM	Jun-03	11.30	0.22193	0.17047	0.17047	0.30882	19.62206	23.05899
OTM	Jun-03	8.45	0.19262	0.15038	0.15038	0.24947	9.92258	12.66481
OTM	Jun-03	6.45	0.44827	0.28152	0.50128	0.50128	86.78869	86.89326
ITM	Dec-03	180.00	0.19066	0.15587	0.15587	0.29244	163.99257	163.54600
ITM	Dec-03	145.60	0.18761	0.17303	0.17303	0.23144	130.40934	130.34442
ITM	Dec-03	114.80	0.22526	0.22069	0.22069	0.23897	118.99131	118.99130
ATM	Dec-03	88.00	0.24537	0.24262	0.24466	0.24954	105.76084	105.76165
OTM	Dec-03	65.60	0.18529	0.14749	0.14749	0.28329	58.53673	60.13548
OTM	Dec-03	47.40	0.23048	0.22277	0.22277	0.25319	64.81684	64.85160
OTM	Dec-03	39.90	0.23628	0.23404	0.23404	0.24297	60.61155	60.61461
OTM	Dec-03	34.70	0.19510	0.19357	0.19357	0.19962	38.61550	38.61863
OTM	Dec-03	26.80	0.25445	0.09982	0.09982	0.31545	21.73239	56.37200
OTM	Dec-03	22.10	0.24244	0.23996	0.24138	0.24695	46.22470	46.22907
OTM	Dec-03	17.60	0.19706	0.13131	0.13131	0.27519	17.57136	25.09672
OTM	Dec-03	14.20	0.26350	0.24176	0.27060	0.27060	45.30200	45.31283

Moneyness	Expiration	Last	sigma BS	sigma1	sigma2	sigma3	FBS Value	BS Value
ITM	Jun-04	192.90	0.25331	0.24646	0.25559	0.25559	202.37002	202.36891
ITM	Jun-04	159.70	0.39889	0.09365	0.50866	0.50866	244.24368	241.74708
ITM	Jun-04	129.40	0.23725	0.23540	0.23540	0.24281	143.62891	143.62874
ATM	Jun-04	103.75	0.17714	0.14910	0.14910	0.25885	91.16495	91.51110
OTM	Jun-04	80.40	0.21359	0.18918	0.18918	0.28530	91.14159	91.34841
OTM	Jun-04	70.60	0.17777	0.14862	0.14862	0.25850	63.83310	64.59017
OTM	Jun-04	61.50	0.23546	0.23282	0.23282	0.24334	86.62404	86.62813
OTM	Jun-04	47.00	0.19008	0.14747	0.14747	0.28621	48.38961	51.62174
OTM	Jun-04	33.80	0.19290	0.14829	0.14829	0.28259	37.22546	41.18129
OTM	Jun-04	24.20	0.21369	0.15658	0.15658	0.31549	35.53719	41.47445
OTM	Jun-04	16.70	0.21369	0.15658	0.15658	0.31549	28.90000	33.78092

Table 2E: This table shows the comparative option values for FBS and BS OPM for S&P 500 Index Options as of close of 4th October 2002

Moneyness	Expiration	Last	sigmaBS	sigma1	sigma2	sigma3	FBS Value	BS Value
ITM	Nov-02	94.90	0.31353	0.26164	0.31968	0.31968	82.42211	82.93652
ITM	Nov-02	75.70	0.24645	0.24164	0.24796	0.24796	58.38240	58.38758
ITM	Nov-02	58.20	0.22368	0.21556	0.21556	0.24729	38.57830	38.59264
ATM	Nov-02	42.80	0.29346	0.25988	0.30454	0.30454	31.89697	31.90591
OTM	Nov-02	29.90	0.28352	0.25600	0.29262	0.29262	20.35740	20.36088
OTM	Nov-02	23.40	0.25342	0.24443	0.25631	0.25631	12.60884	12.61505
OTM	Nov-02	19.60	0.30305	0.26361	0.31422	0.31422	14.49264	14.61665
OTM	Nov-02	16.30	0.25884	0.24296	0.26368	0.26368	8.43823	8.46026
OTM	Nov-02	12.10	0.31120	0.21200	0.32192	0.32192	8.63127	9.78783
OTM	Nov-02	6.75	0.36471	0.22037	0.34723	0.34723	6.48013	9.61524
OTM	Nov-02	3.85	0.43847	0.19793	0.32443	0.32443	3.00049	11.18055
ITM	Dec-02	87.10	0.26301	0.23583	0.27100	0.27100	67.36550	67.45358
ITM	Dec-02	70.40	0.25326	0.24423	0.25623	0.25623	50.34091	50.34480
ATM	Dec-02	55.50	0.27998	0.25528	0.28820	0.28820	40.98672	40.98893
OTM	Dec-02	44.80	0.21428	0.20796	0.20796	0.23320	22.80353	22.80250
OTM	Dec-02	42.40	0.24752	0.24211	0.24932	0.24932	25.67379	25.67384
OTM	Dec-02	40.00	0.24526	0.24115	0.24588	0.24813	23.54861	23.54869
OTM	Dec-02	35.50	0.27904	0.25057	0.28836	0.28836	24.87593	24.88852
OTM	Dec-02	31.30	0.24349	0.23972	0.24376	0.24671	17.06853	17.06899
OTM	Dec-02	27.40	0.24390	0.24002	0.24427	0.24701	14.54075	14.54154
OTM	Dec-02	22.30	0.27511	0.22114	0.28928	0.28928	14.57612	14.85763
OTM	Dec-02	15.30	0.30043	0.19589	0.30874	0.30874	10.90920	12.77520
ITM	Mar-03	175.30	0.46472	0.19362	0.39732	0.39732	166.49400	182.76564
ITM	Mar-03	136.70	0.22225	0.20374	0.20374	0.27187	110.25709	110.34840
ITM	Mar-03	102.00	0.25377	0.24396	0.25699	0.25699	80.86917	80.87571
ATM	Mar-03	72.10	0.23105	0.22805	0.22805	0.24004	49.32066	49.32069
OTM	Mar-03	59.20	0.22748	0.22372	0.22372	0.23876	37.93444	37.93410
OTM	Mar-03	47.80	0.21707	0.20904	0.20904	0.24106	27.06333	27.06135
OTM	Mar-03	37.80	0.19374	0.15175	0.15175	0.30597	15.79794	15.91874
OTM	Mar-03	29.40	0.21002	0.16945	0.16945	0.31381	13.43478	13.61621
OTM	Mar-03	22.85	0.21552	0.18178	0.18178	0.29949	10.26862	10.42519
OTM	Mar-03	16.80	0.21002	0.15759	0.15759	0.30619	6.03452	6.74743
OTM	Mar-03	12.30	0.19520	0.19520	0.19520	0.19520	3.40220	3.40220
OTM	Mar-03	9.50	0.22761	0.15686	0.15686	0.31026	3.38083	4.81466
OTM	Mar-03	4.55	0.21212	0.17133	0.17133	0.26427	1.14572	1.46376
OTM	Mar-03	2.43	0.28900	0.15951	0.15951	0.29818	0.76521	3.95455
ITM	Sep-03	152.90	0.23017	0.22319	0.22319	0.25064	125.53529	125.55091
ATM	Sep-03	92.50	0.23228	0.22941	0.22941	0.24090	70.48842	70.48841
OTM	Sep-03	80.00	0.21285	0.20612	0.20612	0.23304	54.08103	54.08017
OTM	Sep-03	68.50	0.20176	0.17920	0.17920	0.26868	41.64318	41.64935
OTM	Sep-03	58.10	0.17750	0.14944	0.14944	0.25673	26.97159	27.08586
OTM	Sep-03	48.90	0.18258	0.14791	0.14791	0.27374	22.03929	22.39438
OTM	Sep-03	40.70	0.18312	0.14795	0.14795	0.26871	17.00776	17.55244
OTM	Sep-03	33.60	0.19234	0.19234	0.19234	0.19234	15.51551	15.51551
ITM	Dec-03	159.90	0.22865	0.22137	0.22137	0.25012	131.08524	131.09774
ITM	Dec-03	128.70	0.17841	0.14944	0.14944	0.26184	85.51420	85.69536

Moneyness	Expiration	Last	sigmaBS	sigma1	sigma2	sigma3	FBS Value	BS Value
ATM	Dec-03	101.10	0.17573	0.16144	0.16144	0.21865	59.26932	59.26886
OTM	Dec-03	88.70	0.21260	0.20572	0.20572	0.23327	61.47959	61.47853
OTM	Dec-03	77.30	0.18452	0.15811	0.15811	0.26271	42.85646	42.86574
OTM	Dec-03	66.80	0.18633	0.14956	0.14956	0.29096	35.59806	35.74857
OTM	Dec-03	57.40	0.20150	0.15857	0.15857	0.31893	33.52371	33.87934

Table 3E: This table shows the comparative option values for the Call option bounds of FBS and UV OPM for S&P 500 Index Options corresponding to July training set

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Aug-02	91	89.45	90.67	89.32	91.60	90.37	91.60	91.60	91.60
ITM	Aug-02	66.5	67.19	67.47	66.25	66.58	66.42	66.58	66.58	66.58
ITM	Aug-02	45	43.63	45.57	44.41	46.80	44.41	45.59	44.41	44.41
ATM	Aug-02	35	29.14	32.66	31.82	36.06	33.94	36.06	36.06	36.06
ATM	Aug-02	33	23.29	28.72	27.88	34.71	31.29	34.71	34.71	34.71
ATM	Aug-02	26.8	18.73	23.37	22.59	28.21	25.39	28.21	28.21	28.21
ATM	Aug-02	22	16.89	20.03	19.35	22.88	21.12	22.88	22.88	22.88
OTM	Aug-02	20	14.47	17.75	17.11	20.96	19.03	20.96	20.96	20.96
OTM	Aug-02	13.7	12.44	13.44	12.90	13.96	13.43	13.96	13.96	13.96
OTM	Aug-02	15	8.51	12.18	11.66	16.06	13.83	16.06	16.06	16.06
OTM	Aug-02	8.8	6.55	7.98	7.63	9.20	8.40	9.20	9.20	9.20
OTM	Aug-02	9.5	0.85	4.11	3.88	11.49	7.39	11.49	11.49	11.49
OTM	Aug-02	7.9	8.26	8.26	7.90	7.90	7.90	7.90	7.90	7.90
OTM	Aug-02	7.8	1.18	3.97	3.76	9.24	6.29	9.24	9.24	9.24
OTM	Aug-02	6	0.10	1.60	1.50	7.73	4.10	7.73	7.73	7.73
OTM	Aug-02	4.5	0.04	1.01	0.96	5.90	2.94	5.90	5.90	5.90
OTM	Aug-02	2.6	0.00	0.29	0.27	3.62	1.40	3.62	3.62	3.62
ITM	Sep-02	85	90.33	90.33	86.79	86.79	86.79	86.79	86.79	86.79
ITM	Sep-02	77	73.45	76.77	73.92	78.04	75.95	78.04	78.04	78.04
ITM	Sep-02	59	58.76	60.29	57.76	59.41	58.59	59.41	59.41	59.41
ATM	Sep-02	48.6	42.64	47.05	44.90	49.83	47.37	49.83	49.83	49.83
ATM	Sep-02	33	30.13	33.94	31.92	36.25	31.92	34.08	31.92	31.92
ATM	Sep-02	29	26.42	29.87	28.04	31.88	28.04	29.96	28.04	28.04
ATM	Sep-02	31.9	31.84	32.91	31.08	32.17	31.63	32.17	32.17	32.17
OTM	Sep-02	29.5	29.53	30.58	28.88	29.93	29.24	29.76	29.59	29.59
OTM	Sep-02	30.7	23.15	28.02	26.47	32.11	29.29	32.11	32.11	32.11
OTM	Sep-02	22	21.76	22.78	21.46	22.46	21.75	22.25	22.04	22.04
OTM	Sep-02	19	18.81	19.77	18.47	19.43	18.76	19.24	19.05	19.05
OTM	Sep-02	18.8	7.02	13.14	12.20	21.05	16.51	21.05	21.05	21.05
OTM	Sep-02	14	13.62	14.50	13.47	14.34	13.78	14.22	14.09	14.09
OTM	Sep-02	14.1	11.29	13.41	12.46	14.65	13.54	14.65	14.65	14.65
OTM	Sep-02	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
OTM	Sep-02	14.3	0.96	5.84	5.27	17.54	10.88	17.54	17.54	17.54
OTM	Sep-02	11.4	7.31	9.93	9.21	12.14	10.64	12.14	12.14	12.14
ITM	Oct-02	121.2	122.19	124.08	119.52	121.77	120.62	121.77	121.77	121.77
ITM	Oct-02	77	72.79	78.74	74.73	83.95	74.73	79.23	74.73	74.73
ATM	Oct-02	51	52.06	53.30	50.18	51.45	50.68	51.32	51.19	51.19
ATM	Oct-02	42	39.60	42.46	39.73	42.76	41.24	42.76	42.76	42.76
OTM	Oct-02	22.9	8.85	20.21	18.26	37.00	18.26	27.50	18.26	18.26
OTM	Oct-02	18.4	11.94	18.16	16.49	24.20	16.49	20.29	16.49	16.49
OTM	Oct-02	14.5	14.04	15.47	14.16	15.53	14.16	14.84	14.16	14.16
OTM	Oct-02	8.6	2.50	7.36	6.55	15.14	6.55	10.55	6.55	6.55
OTM	Oct-02	6	1.77	5.17	4.59	10.57	4.59	7.33	4.59	4.59
OTM	Oct-02	4.5	3.99	4.71	4.30	5.12	4.30	4.70	4.30	4.30
OTM	Oct-02	3.5	3.11	3.50	3.22	3.67	3.39	3.61	3.55	3.55
OTM	Oct-02	0	2.15	2.53	2.30	2.66	2.43	2.61	2.56	2.56

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
OTM	Oct-02	0	1.36	1.65	1.50	1.77	1.60	1.73	1.69	1.69
OTM	Oct-02	0	0.85	1.05	0.97	1.16	1.03	1.13	1.10	1.10
OTM	Oct-02	0	0.52	0.66	0.61	0.75	0.66	0.73	0.71	0.71
ITM	Dec-02	296	317.29	317.29	308.33	308.33	308.33	308.33	308.33	308.33
ITM	Dec-02	205	218.21	218.21	209.24	209.24	209.24	209.24	209.24	209.24
ITM	Dec-02	181	168.68	171.17	162.63	188.17	173.07	188.17	188.17	188.17
ITM	Dec-02	123	122.44	128.10	120.40	131.17	120.40	125.52	120.40	120.40
ITM	Dec-02	94	95.89	98.46	91.93	94.69	93.31	94.69	94.69	94.69
ATM	Dec-02	61	62.90	65.63	60.29	63.14	60.29	61.71	60.29	60.29
ATM	Dec-02	48.7	49.02	52.45	47.79	51.42	47.79	49.61	47.79	47.79
OTM	Dec-02	36.5	31.87	38.44	34.61	42.19	34.61	38.39	34.61	34.61
OTM	Dec-02	22.9	2.95	17.21	14.47	49.02	14.47	31.13	14.47	14.47
OTM	Dec-02	20	2.33	14.67	12.62	43.21	12.62	27.12	12.62	12.62
OTM	Dec-02	16	2.35	12.17	10.53	33.43	10.53	21.22	10.53	10.53
OTM	Dec-02	10	0.15	5.47	4.38	29.22	4.38	15.05	4.38	4.38
OTM	Dec-02	6.5	7.60	7.60	6.50	6.50	6.50	6.50	6.50	6.50
OTM	Dec-02	6	0.01	2.33	1.78	21.75	1.78	9.45	1.78	1.78
OTM	Dec-02	2.5	0.03	1.24	0.99	8.29	0.99	3.69	0.99	0.99
OTM	Dec-02	1.65	0.00	0.27	0.20	8.41	0.20	2.48	0.20	0.20
ITM	Mar-03	133	137.95	142.97	131.45	137.70	131.45	134.54	131.45	131.45
ATM	Mar-03	75	79.40	82.71	74.14	77.58	74.14	75.86	74.14	74.14
ATM	Mar-03	58	56.04	63.37	55.96	64.13	55.96	60.04	55.96	55.96
OTM	Mar-03	44.9	26.13	44.71	38.18	65.14	38.18	51.60	38.18	38.18
OTM	Mar-03	29.5	9.68	26.89	21.87	52.83	21.87	37.02	21.87	21.87
OTM	Mar-03	24	4.08	19.43	15.62	50.14	15.62	32.14	15.62	15.62
OTM	Mar-03	18.5	2.09	14.35	11.19	41.85	11.19	25.47	11.19	11.19
OTM	Mar-03	16.8	19.97	19.97	16.80	16.80	16.80	16.80	16.80	16.80
ITM	Jun-03	140	146.85	153.37	137.99	146.08	137.99	142.00	137.99	137.99
ITM	Jun-03	93.5	82.66	98.83	84.59	120.56	84.59	102.33	84.59	84.59
ATM	Jun-03	66	57.82	72.63	60.99	81.01	60.99	71.01	60.99	60.99
ATM	Jun-03	68	67.35	76.08	65.56	75.31	65.56	70.44	65.56	65.56
OTM	Jun-03	48.4	25.42	48.33	39.22	76.03	39.22	57.55	39.22	39.22
OTM	Jun-03	40.5	11.44	35.48	28.43	77.24	28.43	52.44	28.43	28.43
OTM	Jun-03	38.1	7.09	31.27	24.79	79.03	24.79	51.17	24.79	24.79
ITM	Jun-03	140	147.01	153.41	138.03	145.94	138.03	141.95	138.03	138.03
ITM	Jun-03	93.5	82.45	98.74	84.49	120.86	84.49	102.43	84.49	84.49
ATM	Jun-03	66	57.81	72.63	61.00	80.99	61.00	71.00	61.00	61.00
ATM	Jun-03	68	67.40	76.09	65.58	75.27	65.58	70.42	65.58	65.58
OTM	Jun-03	48.4	25.29	48.27	39.17	76.19	39.17	57.60	39.17	39.17
OTM	Jun-03	40.5	11.36	35.43	28.39	77.35	28.39	52.48	28.39	28.39
OTM	Jun-03	38.1	7.10	31.28	24.81	79.00	24.81	51.16	24.81	24.81
OTM	Jun-03	32	3.79	24.75	19.24	71.90	19.24	44.38	19.24	19.24
OTM	Jun-03	23.5	1.47	16.63	12.56	58.67	12.56	33.89	12.56	12.56
OTM	Jun-03	22.5	0.95	15.31	11.47	58.42	11.47	32.86	11.47	11.47
OTM	Jun-03	25.5	16.12	27.08	22.14	35.85	22.14	28.79	22.14	22.14
OTM	Jun-03	12.5	0.14	7.07	4.95	38.85	4.95	19.15	4.95	4.95
OTM	Jun-03	7.5	0.00	1.32	0.79	34.85	0.79	12.41	0.79	0.79
OTM	Jun-03	12	0.22	6.86	5.16	36.02	5.16	17.98	5.16	5.16

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
OTM	Jun-03	5.3	0.07	3.03	2.05	17.48	2.05	7.94	2.05	2.05
ITM	Dec-03	236	155.47	196.85	174.39	256.55	215.45	256.55	256.55	256.55
ITM	Dec-03	112	96.07	120.19	98.44	152.94	98.44	125.51	98.44	98.44
ATM	Dec-03	86	79.44	97.95	79.76	104.70	79.76	92.25	79.76	79.76
ATM	Dec-03	0	112.53	115.53	97.95	100.97	99.03	100.54	100.11	100.11
OTM	Dec-03	82	88.52	96.08	80.01	87.96	80.01	83.99	80.01	80.01
OTM	Dec-03	0	92.10	95.18	79.67	82.70	80.75	82.27	81.84	81.84
OTM	Dec-03	74	84.03	87.02	72.81	75.80	73.25	74.75	73.70	73.70
OTM	Dec-03	0	74.87	77.94	64.42	67.39	65.48	66.97	66.54	66.54
OTM	Dec-03	35	5.97	30.59	21.87	76.03	21.87	47.74	21.87	21.87
OTM	Dec-03	48	44.58	55.88	44.96	57.20	44.96	51.02	44.96	44.96
OTM	Dec-03	46	52.02	55.76	45.13	48.63	45.13	46.87	45.13	45.13
OTM	Dec-03	25.5	30.34	32.68	25.00	27.02	25.00	26.00	25.00	25.00
OTM	Dec-03	15	0.00	3.33	1.51	66.30	1.51	25.93	1.51	1.51
OTM	Dec-03	34	38.48	41.24	33.15	35.58	33.39	34.61	33.64	33.64
OTM	Dec-03	11.5	0.03	5.46	3.27	41.79	3.27	18.35	3.27	3.27
OTM	Dec-03	8.5	0.30	5.80	3.85	25.20	3.85	12.45	3.85	3.85
OTM	Dec-03	5	0.00	0.73	0.33	26.42	0.33	7.76	0.33	0.33
ITM	Jun-04	217	235.27	246.65	211.90	232.73	211.90	221.99	211.90	211.90
ITM	Jun-04	0	247.41	249.51	215.88	218.30	216.74	217.95	217.61	217.61
ITM	Jun-04	105	198.37	198.37	157.60	157.60	157.60	157.60	157.60	157.60
ITM	Jun-04	184	194.93	205.51	174.88	187.05	180.95	187.05	187.05	187.05
ITM	Jun-04	217	153.11	192.04	162.32	235.22	198.79	235.22	235.22	235.22
ITM	Jun-04	0	184.51	187.41	158.30	161.41	159.41	160.97	160.52	160.52
ITM	Jun-04	154.5	175.36	179.44	151.36	155.55	153.45	155.55	155.55	155.55
ATM	Jun-04	203.5	56.71	81.58	54.56	252.31	155.34	252.31	252.31	252.31
OTM	Jun-04	105	123.50	127.18	104.08	107.76	104.08	105.92	104.08	104.08
OTM	Jun-04	58.3	29.33	62.70	44.85	98.98	44.85	71.67	44.85	44.85
OTM	Jun-04	69	42.01	74.75	57.20	104.64	57.20	80.75	57.20	57.20
OTM	Jun-04	45	16.36	46.41	32.21	84.32	32.21	57.57	32.21	32.21
OTM	Jun-04	62	71.80	76.81	60.75	65.75	60.75	63.24	60.75	60.75
OTM	Jun-04	29.5	2.52	24.29	15.72	74.17	15.72	42.48	15.72	15.72
OTM	Jun-04	23	1.16	17.12	11.09	62.82	11.09	33.91	11.09	11.09
OTM	Jun-04	23	0.44	14.62	9.48	69.20	9.48	35.15	9.48	9.48

Table 4E: This table shows the comparative option values for the Call option bounds of FBS and UV OPM for S&P500 Options corresponding to July forecasting set

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Aug-02	123.2	119.62	120.07	118.63	119.86	119.15	119.86	119.86	119.86
ITM	Aug-02	100.7	95.57	95.73	94.33	94.50	94.41	94.50	94.50	94.50
ITM	Aug-02	79.3	70.81	71.80	70.44	71.92	70.44	71.14	70.44	70.44
ITM	Aug-02	59.8	51.03	53.87	52.70	56.32	54.48	56.32	56.32	56.32
ITM	Aug-02	52.6	43.20	47.93	46.84	53.18	49.97	53.18	53.18	53.18
ITM	Aug-02	45.8	36.38	40.87	39.81	45.29	42.54	45.29	45.29	45.29
ITM	Aug-02	42.6	33.65	36.71	35.67	39.20	37.43	39.20	39.20	39.20
ATM	Aug-02	39.5	30.03	33.37	32.47	36.46	34.46	36.46	36.46	36.46
ATM	Aug-02	33.7	26.02	27.17	26.26	27.46	26.86	27.46	27.46	27.46
ATM	Aug-02	28.5	19.98	24.29	23.47	28.58	26.02	28.58	28.58	28.58
ATM	Aug-02	23.7	16.10	17.97	17.29	19.34	18.31	19.34	19.34	19.34
OTM	Aug-02	19.9	5.15	11.49	10.94	21.36	16.06	21.36	21.36	21.36
OTM	Aug-02	17.95	16.84	16.84	16.37	16.37	16.37	16.37	16.37	16.37
OTM	Aug-02	14.55	5.03	10.18	9.70	17.50	13.51	17.50	17.50	17.50
OTM	Aug-02	9.8	1.05	5.24	4.96	14.72	9.51	14.72	14.72	14.72
OTM	Aug-02	8.6	0.58	3.80	3.59	11.97	7.42	11.97	11.97	11.97
OTM	Aug-02	5.2	0.06	1.42	1.36	7.89	4.06	7.89	7.89	7.89
ITM	Sep-02	129.4	122.80	124.29	120.91	123.23	122.01	123.23	123.23	123.23
ITM	Sep-02	108.5	102.11	103.09	99.90	101.00	100.44	101.00	101.00	101.00
ITM	Sep-02	88.6	81.04	84.45	81.50	85.52	83.49	85.52	85.52	85.52
ITM	Sep-02	70.2	56.65	59.81	57.10	60.96	57.10	59.01	57.10	57.10
ITM	Sep-02	63.4	50.65	53.73	51.18	54.81	51.18	52.98	51.18	51.18
ITM	Sep-02	57.1	51.57	52.62	50.19	51.28	50.73	51.28	51.28	51.28
ITM	Sep-02	54.2	48.53	49.55	47.24	48.31	47.60	48.14	47.96	47.96
ATM	Sep-02	51	43.08	48.23	46.00	51.88	48.94	51.88	51.88	51.88
ATM	Sep-02	44.9	40.29	41.37	39.23	40.33	39.55	40.10	39.87	39.87
ATM	Sep-02	39.9	35.46	36.54	34.61	35.72	34.95	35.50	35.29	35.29
ATM	Sep-02	35	20.88	29.18	27.40	38.10	32.74	38.10	38.10	38.10
OTM	Sep-02	28.4	25.34	26.40	24.79	25.85	25.16	25.70	25.55	25.55
OTM	Sep-02	26.5	22.05	24.66	23.16	25.88	24.52	25.88	25.88	25.88
OTM	Sep-02	24.7	6.59	16.31	15.04	31.75	23.23	31.75	31.75	31.75
OTM	Sep-02	18.7	16.51	20.10	18.81	22.73	20.75	22.73	22.73	22.73
OTM	Sep-02	15.9	14.33	16.92	15.88	18.80	17.33	18.80	18.80	18.80
ITM	Oct-02	153.8	147.54	148.96	143.85	147.98	143.85	145.66	143.85	143.85
ITM	Oct-02	111.9	107.78	108.65	104.15	105.04	104.50	104.95	104.86	104.86
ITM	Oct-02	74.9	71.48	74.05	70.35	73.22	71.78	73.22	73.22	73.22
ITM	Oct-02	58.9	34.91	46.68	43.48	63.24	43.48	53.33	43.48	43.48
ATM	Oct-02	44.9	32.25	39.66	36.93	45.97	36.93	41.45	36.93	36.93
OTM	Oct-02	33.2	31.63	33.35	30.94	32.72	30.94	31.83	30.94	30.94
OTM	Oct-02	23.3	10.95	19.51	17.86	30.07	17.86	23.86	17.86	17.86
OTM	Oct-02	16.1	7.67	14.29	13.01	22.29	13.01	17.51	13.01	13.01
OTM	Oct-02	10.75	11.18	12.59	11.41	12.82	11.41	12.11	11.41	11.41
OTM	Oct-02	7.15	9.30	10.13	9.19	10.03	9.50	9.92	9.81	9.81

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
OTM	Oct-02	4.2	0.01	2.72	2.42	26.10	11.90	26.10	26.10	26.10
ITM	Dec-02	201.4	199.86	201.34	192.40	197.60	192.40	194.61	192.40	192.40
ITM	Dec-02	158.2	158.73	160.35	152.22	154.05	153.12	154.05	154.05	154.05
ITM	Dec-02	118.1	116.58	118.67	111.33	113.60	111.33	112.46	111.33	111.33
ITM	Dec-02	82.4	80.87	84.13	77.92	81.42	77.92	79.67	77.92	77.92
ITM	Dec-02	67	59.51	66.20	60.67	68.52	60.67	64.59	60.67	60.67
ATM	Dec-02	53.4	19.66	38.73	34.38	72.95	34.38	53.68	34.38	34.38
OTM	Dec-02	41.6	13.61	32.95	28.88	64.68	28.88	46.67	28.88	28.88
OTM	Dec-02	31.4	10.87	27.22	23.81	52.06	23.81	37.70	23.81	23.81
OTM	Dec-02	23.2	2.17	14.63	12.63	45.75	12.63	28.36	12.63	12.63
OTM	Dec-02	16.8	16.98	16.98	15.05	15.05	15.05	15.05	15.05	15.05
OTM	Dec-02	11.9	0.23	6.60	5.45	33.65	5.45	17.82	5.45	5.45
OTM	Dec-02	9.05	0.93	6.49	5.38	20.16	5.38	11.93	5.38	5.38
OTM	Dec-02	3.6	0.01	1.80	1.45	17.89	1.45	7.58	1.45	1.45
OTM	Dec-02	1.9	2.49	4.41	3.79	6.33	3.79	4.98	3.79	3.79
ITM	Mar-03	163.2	168.09	170.20	157.69	160.03	157.69	158.85	157.69	157.69
ITM	Mar-03	90.5	88.68	95.71	85.76	93.75	85.76	89.75	85.76	85.76
ITM	Mar-03	75.5	55.23	73.32	64.35	91.99	64.35	78.17	64.35	64.35
ATM	Mar-03	62	31.23	50.99	43.22	77.00	43.22	60.13	43.22	43.22
OTM	Mar-03	50.1	17.81	39.60	32.81	72.22	32.81	52.42	32.81	32.81
OTM	Mar-03	39.9	10.56	29.91	24.77	61.56	24.77	42.86	24.77	24.77
OTM	Mar-03	31.2	35.90	35.90	30.67	30.67	30.67	30.67	30.67	30.67
OTM	Mar-03	23.9	30.54	37.91	32.70	40.78	32.70	36.71	32.70	32.70
ITM	Jun-03	168.2	158.60	164.70	146.44	173.51	146.44	158.75	146.44	146.44
ITM	Jun-03	131.2	117.41	127.30	111.32	128.34	111.32	119.65	111.32	111.32
ITM	Jun-03	97.9	100.54	108.88	95.32	104.87	95.32	100.09	95.32	95.32
ITM	Jun-03	83.3	55.90	77.88	65.53	103.41	65.53	84.49	65.53	65.53
ATM	Jun-03	69.6	34.89	61.73	50.80	102.75	50.80	76.83	50.80	50.80
OTM	Jun-03	57.8	24.01	53.80	44.17	102.95	44.17	73.53	44.17	44.17
OTM	Jun-03	48.5	15.04	44.00	35.47	94.21	35.47	64.57	35.47	35.47
OTM	Jun-03	39.5	7.77	31.92	25.23	78.91	25.23	51.40	25.23	25.23
OTM	Jun-03	31.7	5.11	28.84	22.51	77.60	22.51	49.06	22.51	22.51
OTM	Jun-03	228.8	27.85	41.24	34.37	50.44	34.37	42.28	34.37	34.37
OTM	Jun-03	20.7	1.38	15.98	11.83	55.34	11.83	31.80	11.83	11.83
OTM	Jun-03	14.8	0.00	4.42	2.86	48.84	2.86	21.40	2.86	2.86
OTM	Jun-03	11.3	1.18	13.63	10.55	49.56	10.55	28.04	10.55	10.55
OTM	Jun-03	8.45	0.48	7.03	5.01	26.90	5.01	14.28	5.01	5.01
OTM	Jun-03	6.45	6.45	37.69	32.21	105.81	67.17	105.81	105.81	105.81
ITM	Dec-03	180	167.24	179.27	152.85	199.06	152.85	174.74	152.85	152.85
ITM	Dec-03	145.6	133.83	148.48	124.77	147.50	124.77	136.01	124.77	124.77
ITM	Dec-03	114.8	130.46	137.66	117.03	124.89	117.03	120.96	117.03	117.03
ATM	Dec-03	88	119.95	123.02	104.53	107.62	104.99	106.53	105.44	105.44
OTM	Dec-03	65.6	24.41	57.57	43.39	104.24	43.39	73.63	43.39	43.39

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
OTM	Dec-03	47.4	62.87	75.25	61.47	74.89	61.47	68.16	61.47	61.47
OTM	Dec-03	39.9	68.57	72.63	59.65	63.51	59.65	61.58	59.65	59.65
OTM	Dec-03	34.7	46.23	48.84	38.01	40.44	38.01	39.22	38.01	38.01
OTM	Dec-03	26.8	0.00	7.15	4.20	82.60	4.20	37.41	4.20	4.20
OTM	Dec-03	22.1	52.60	55.53	45.24	48.04	45.53	46.92	45.81	45.81
OTM	Dec-03	17.6	0.24	10.31	6.80	54.55	6.80	27.21	6.80	6.80
OTM	Dec-03	14.2	35.78	46.20	37.09	48.08	42.50	48.08	48.08	48.08
ITM	Jun-04	192.9	229.10	232.46	199.56	203.31	201.43	203.31	203.31	203.31
ITM	Jun-04	159.7	136.72	143.69	107.49	290.91	197.21	290.91	290.91	290.91
ITM	Jun-04	129.4	167.64	171.15	142.73	146.34	142.73	144.53	142.73	142.73
ATM	Jun-04	103.75	69.90	102.29	77.12	133.17	77.12	105.25	77.12	77.12
OTM	Jun-04	80.4	66.04	100.39	78.71	128.40	78.71	103.59	78.71	78.71
OTM	Jun-04	70.6	34.62	69.16	49.76	106.27	49.76	77.85	49.76	49.76
OTM	Jun-04	61.5	100.51	106.06	85.27	90.69	85.27	87.98	85.27	85.27
OTM	Jun-04	47	11.96	45.73	31.63	100.10	31.63	64.82	31.63	31.63
OTM	Jun-04	33.8	5.74	32.79	22.26	84.56	22.26	51.62	22.26	22.26
OTM	Jun-04	24.2	2.60	27.66	18.62	90.09	18.62	51.55	18.62	18.62
OTM	Jun-04	16.7	1.11	20.80	13.46	80.15	13.46	43.16	13.46	13.46

Table 5E: This table shows the comparative option values for the Call option bounds of FBS and UV OPM for S&P500 Options corresponding to October forecasting set

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Nov-02	94.9	79.12	81.31	79.76	83.35	81.46	83.35	83.35	83.35
ITM	Nov-02	75.7	59.05	59.50	58.02	58.50	58.26	58.50	58.50	58.50
ITM	Nov-02	58.2	36.52	39.10	37.81	40.90	37.81	39.35	37.81	37.81
ATM	Nov-02	42.8	25.15	29.27	28.28	33.10	30.69	33.10	33.10	33.10
OTM	Nov-02	29.9	14.94	18.30	17.50	21.31	19.40	21.31	21.31	21.31
OTM	Nov-02	23.4	11.23	12.32	11.77	12.89	12.33	12.89	12.89	12.89
OTM	Nov-02	19.6	7.86	11.45	11.01	15.67	13.30	15.67	15.67	15.67
OTM	Nov-02	16.3	6.17	7.61	7.22	8.85	8.02	8.85	8.85	8.85
OTM	Nov-02	12.1	0.52	3.34	3.11	10.64	6.49	10.64	10.64	10.64
OTM	Nov-02	6.75	0.10	1.71	1.58	8.37	4.40	8.37	8.37	8.37
OTM	Nov-02	3.85	0.00	0.34	0.32	4.16	1.63	4.16	4.16	4.16
ITM	Dec-02	87.1	63.30	66.62	64.19	68.43	66.29	68.43	68.43	68.43
ITM	Dec-02	70.4	49.74	51.32	49.10	50.76	49.93	50.76	50.76	50.76
ATM	Dec-02	55.5	34.97	39.22	37.39	42.19	39.79	42.19	42.19	42.19
OTM	Dec-02	44.8	20.10	23.29	21.90	25.52	21.90	23.71	21.90	21.90
OTM	Dec-02	42.4	25.39	26.40	24.90	25.93	25.42	25.93	25.93	25.93
OTM	Dec-02	40	23.45	24.42	22.97	23.95	23.30	23.79	23.64	23.64
OTM	Dec-02	35.5	17.66	22.22	20.96	26.19	23.56	26.19	26.19	26.19
OTM	Dec-02	31.3	16.80	17.72	16.58	17.49	16.84	17.30	17.10	17.10
OTM	Dec-02	27.4	14.23	15.06	14.07	14.92	14.33	14.76	14.59	14.59
OTM	Dec-02	22.3	4.50	9.58	8.82	16.56	12.54	16.56	16.56	16.56
OTM	Dec-02	15.3	0.41	3.80	3.42	13.68	7.94	13.68	13.68	13.68
ITM	Mar-03	175.3	155.56	157.59	150.67	172.73	159.54	172.73	172.73	172.73
ITM	Mar-03	136.7	108.96	114.07	107.92	117.58	107.92	112.51	107.92	107.92
ITM	Mar-03	102	81.77	84.07	78.98	81.50	80.24	81.50	81.50	81.50
ATM	Mar-03	72.1	50.28	52.72	48.68	51.25	48.68	49.96	48.68	48.68
OTM	Mar-03	59.2	37.50	40.58	37.13	40.34	37.13	38.73	37.13	37.13
OTM	Mar-03	47.8	22.36	28.21	25.44	31.96	25.44	28.68	25.44	25.44
OTM	Mar-03	37.8	1.03	10.40	8.95	37.56	8.95	22.36	8.95	8.95
OTM	Mar-03	29.4	0.82	8.85	7.60	32.33	7.60	18.94	7.60	7.60
OTM	Mar-03	22.85	0.83	7.16	6.11	23.96	6.11	14.13	6.11	6.11
OTM	Mar-03	16.8	0.02	2.58	2.08	20.39	2.08	9.38	2.08	2.08
OTM	Mar-03	12.3	3.84	3.84	3.40	3.40	3.40	3.40	3.40	3.40
OTM	Mar-03	9.5	0.00	0.91	0.69	14.33	0.69	5.35	0.69	0.69
OTM	Mar-03	4.55	0.00	0.40	0.34	4.54	0.34	1.71	0.34	0.34
OTM	Mar-03	2.425	0.00	0.07	0.05	4.59	0.05	1.05	0.05	0.05
ITM	Sep-03	152.9	130.07	135.55	123.88	130.54	123.88	127.18	123.88	123.88
ATM	Sep-03	92.5	74.83	78.20	69.61	73.12	69.61	71.37	69.61	69.61
OTM	Sep-03	80	52.16	59.61	52.02	60.27	52.02	56.14	52.02	52.02
OTM	Sep-03	68.5	23.07	40.99	34.93	61.92	34.93	48.33	34.93	34.93
OTM	Sep-03	58.1	7.78	24.29	19.46	50.09	19.46	34.34	19.46	19.46
OTM	Sep-03	48.9	3.08	17.24	13.85	47.82	13.85	29.94	13.85	13.85
OTM	Sep-03	40.7	1.55	12.85	9.91	39.92	9.91	23.72	9.91	9.91
OTM	Sep-03	33.6	18.85	18.85	15.52	15.52	15.52	15.52	15.52	15.52
ITM	Dec-03	159.9	136.93	143.29	129.08	137.13	129.08	133.08	129.08	129.08
ITM	Dec-03	128.7	72.79	89.97	76.67	112.29	76.67	94.31	76.67	76.67

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ATM	Dec-03	101.1	50.46	65.03	54.39	73.91	54.39	64.16	54.39	54.39
OTM	Dec-03	88.7	60.27	68.78	59.12	68.56	59.12	63.84	59.12	59.12
OTM	Dec-03	77.3	20.39	41.97	34.08	69.39	34.08	51.59	34.08	34.08
OTM	Dec-03	66.8	8.30	30.43	24.16	70.64	24.16	46.86	24.16	24.16
OTM	Dec-03	57.4	4.71	26.76	20.98	72.43	20.98	45.77	20.98	20.98

Table 6E: This table shows the comparative option values for the Call option bounds of FBS and UV OPM for S&P500 Options using volatility subjective bounds

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Aug-02	123.20	119.48	119.82	118.34	124.45	119.11	122.21	120.41	120.41
ITM	Aug-02	100.70	94.53	95.83	94.42	104.20	96.10	101.10	98.37	98.37
ITM	Aug-02	79.30	69.74	73.04	71.66	85.18	74.44	81.33	77.71	77.71
ITM	Aug-02	59.80	45.44	51.28	50.05	66.84	53.89	62.38	58.04	58.04
ITM	Aug-02	52.60	37.26	45.59	44.46	62.72	48.84	58.02	53.38	53.38
ITM	Aug-02	45.80	28.43	37.47	36.46	55.53	41.12	50.69	45.87	45.87
ITM	Aug-02	42.60	23.48	32.50	31.72	51.11	36.49	46.21	41.33	41.33
ATM	Aug-02	39.50	21.09	31.19	30.26	50.02	35.17	45.06	40.11	40.11
ATM	Aug-02	33.70	14.56	25.14	24.33	44.46	29.36	39.43	34.40	34.40
ATM	Aug-02	28.50	8.78	19.15	18.43	38.62	23.47	33.57	28.52	28.52
ATM	Aug-02	23.70	5.58	15.44	14.86	34.77	19.77	29.75	24.74	24.74
OTM	Aug-02	19.90	3.05	11.80	11.28	30.59	15.94	25.64	20.75	20.75
OTM	Aug-02	17.95	1.63	9.23	8.73	27.49	13.17	22.62	17.83	17.83
OTM	Aug-02	14.55	0.95	7.31	7.08	24.94	11.18	20.20	15.59	15.59
OTM	Aug-02	9.80	0.39	5.14	4.85	21.07	8.36	16.60	12.32	12.32
OTM	Aug-02	8.60	0.12	3.37	3.22	18.25	6.27	13.95	9.92	9.92
OTM	Aug-02	5.20	0.03	1.98	1.93	14.60	4.24	10.73	7.23	7.23
ITM	Sep-02	129.40	121.14	124.16	120.80	138.78	124.21	133.41	128.49	128.49
ITM	Sep-02	108.50	96.41	101.56	98.29	120.05	102.85	113.92	108.14	108.14
ITM	Sep-02	88.60	72.12	80.17	77.07	102.49	82.82	95.68	89.09	89.09
ITM	Sep-02	70.20	48.93	60.46	57.68	86.27	64.54	78.91	71.65	71.65
ITM	Sep-02	63.40	41.07	54.30	51.81	81.49	59.07	73.96	66.47	66.47
ITM	Sep-02	57.10	32.25	46.03	43.90	74.35	51.43	66.68	59.03	59.03
ITM	Sep-02	54.20	28.68	43.14	40.87	71.67	48.52	63.94	56.22	56.22
ATM	Sep-02	51.00	25.17	40.37	38.07	69.16	45.82	61.37	53.59	53.59
ATM	Sep-02	44.90	18.90	34.35	32.39	63.87	40.27	56.01	48.14	48.14
ATM	Sep-02	39.90	13.55	29.01	27.39	58.99	35.29	51.10	43.19	43.19
ATM	Sep-02	35.00	8.11	23.16	21.45	52.90	29.26	45.00	37.11	37.11
OTM	Sep-02	28.40	4.51	17.78	16.82	47.59	24.31	39.76	31.98	31.98
OTM	Sep-02	26.50	3.64	16.31	15.18	45.60	22.52	37.81	30.10	30.10
OTM	Sep-02	24.70	2.84	15.01	13.80	43.83	20.97	36.08	28.44	28.44
OTM	Sep-02	18.70	1.25	10.96	10.11	38.66	16.66	31.10	23.73	23.73
OTM	Sep-02	15.90	0.59	8.34	7.63	34.83	13.64	27.45	20.34	20.34
ITM	Oct-02	153.80	147.20	148.29	143.02	160.32	145.61	154.55	149.56	149.56
ITM	Oct-02	111.90	98.06	105.86	101.28	130.04	107.60	122.20	114.66	114.66
ITM	Oct-02	74.90	51.66	66.39	62.47	97.83	71.06	88.81	79.88	79.88
ITM	Oct-02	58.90	30.31	47.06	43.69	81.11	53.00	71.73	62.36	62.36
ATM	Oct-02	44.90	14.43	32.06	29.74	68.03	39.31	58.46	48.89	48.89
OTM	Oct-02	33.20	5.17	20.94	19.00	56.57	28.18	47.04	37.56	37.56
OTM	Oct-02	23.30	1.37	13.14	11.55	46.92	19.74	37.63	28.52	28.52
OTM	Oct-02	16.10	0.16	6.83	5.85	37.49	12.46	28.66	20.22	20.22
OTM	Oct-02	10.75	0.02	3.48	3.04	30.57	8.08	22.32	14.71	14.71
OTM	Oct-02	7.15	0.00	1.54	1.24	23.94	4.61	16.47	9.88	9.88
OTM	Oct-02	4.20	0.00	0.61	0.44	18.25	2.44	11.72	6.31	6.31
ITM	Dec-02	201.40	199.60	199.87	190.59	206.70	192.18	200.50	195.54	195.54
ITM	Dec-02	158.20	150.08	151.40	142.46	168.74	146.57	160.28	152.73	152.73

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Dec-02	118.10	100.62	106.56	98.31	136.07	106.27	125.60	115.56	115.56
ITM	Dec-02	82.40	53.03	68.45	61.84	108.37	73.13	96.52	84.75	84.75
ITM	Dec-02	67.00	33.84	54.29	48.51	97.51	60.74	85.25	72.99	72.99
ATM	Dec-02	53.40	36.95	65.02	60.59	110.40	73.09	98.00	85.56	85.56
OTM	Dec-02	41.60	18.03	44.18	39.77	89.70	52.19	77.19	64.67	64.67
OTM	Dec-02	31.40	8.68	31.14	28.27	77.27	40.21	64.81	52.43	52.43
OTM	Dec-02	23.20	3.15	21.18	18.32	65.16	29.27	52.92	40.91	40.91
OTM	Dec-02	16.80	0.67	12.20	10.46	53.75	19.87	41.92	30.53	30.53
OTM	Dec-02	11.90	0.07	6.51	5.27	43.81	12.72	32.59	22.08	22.08
OTM	Dec-02	9.05	0.01	3.85	3.02	37.63	8.98	26.97	17.24	17.24
OTM	Dec-02	3.60	0.00	1.13	0.78	26.26	3.87	17.18	9.53	9.53
OTM	Dec-02	1.90	0.00	0.31	0.18	18.58	1.64	11.07	5.28	5.28
ITM	Mar-03	163.20	154.19	156.58	142.34	180.65	149.02	168.99	158.25	158.25
ITM	Mar-03	90.50	56.94	73.06	62.67	121.72	77.06	106.75	91.83	91.83
ITM	Mar-03	75.50	34.80	55.31	46.01	107.71	61.41	92.30	76.86	76.86
ATM	Mar-03	62.00	17.67	40.82	33.57	96.46	49.32	80.79	65.07	65.07
OTM	Mar-03	50.10	6.76	30.12	23.82	86.33	39.19	70.57	54.84	54.84
OTM	Mar-03	39.90	2.53	21.80	17.33	78.18	31.77	62.50	46.97	46.97
OTM	Mar-03	31.20	0.54	14.69	10.96	68.77	23.84	53.32	38.23	38.23
OTM	Mar-03	23.90	1.63	19.90	15.97	73.74	29.05	58.37	43.38	43.38
ITM	Jun-03	168.20	158.23	162.07	143.36	192.84	152.92	178.54	165.03	165.03
ITM	Jun-03	131.20	109.37	118.30	101.51	162.20	115.15	146.06	130.26	130.26
ITM	Jun-03	97.90	61.60	79.95	66.04	135.37	83.07	117.89	100.43	100.43
ITM	Jun-03	83.30	39.56	62.74	49.99	121.95	68.00	104.02	86.03	86.03
ATM	Jun-03	69.60	21.74	47.95	37.43	110.66	55.80	92.44	74.15	74.15
OTM	Jun-03	57.80	9.12	35.13	26.46	99.50	44.48	81.14	62.78	62.78
OTM	Jun-03	48.50	2.37	24.38	17.43	88.74	34.36	70.40	52.20	52.20
OTM	Jun-03	39.50	5.51	31.87	25.82	97.07	42.83	78.75	60.61	60.61
OTM	Jun-03	31.70	1.53	21.97	16.39	84.62	31.85	66.53	48.82	48.82
OTM	Jun-03	228.80	0.17	12.36	8.85	72.33	21.99	54.66	37.68	37.68
OTM	Jun-03	20.70	0.03	8.46	5.75	65.36	17.13	48.08	31.75	31.75
OTM	Jun-03	14.80	0.00	4.68	2.83	56.13	11.55	39.59	24.40	24.40
OTM	Jun-03	11.30	0.00	2.50	1.48	49.49	8.20	33.65	19.51	19.51
OTM	Jun-03	8.45	0.00	1.60	0.79	43.92	5.89	28.82	15.74	15.74
OTM	Jun-03	6.45	0.01	4.98	3.32	52.39	11.15	36.75	22.72	22.72
ITM	Dec-03	180.00	166.27	175.04	147.18	214.84	161.79	196.42	178.57	178.57
ITM	Dec-03	145.60	118.04	131.19	105.97	183.50	124.11	163.39	143.49	143.49
ITM	Dec-03	114.80	71.11	93.35	72.50	158.13	93.75	136.71	115.22	115.22
ATM	Dec-03	88.00	30.55	62.52	45.34	134.98	67.88	112.74	90.37	90.37
OTM	Dec-03	65.60	7.21	38.56	25.99	114.92	47.57	92.38	69.86	69.86
OTM	Dec-03	47.40	0.84	21.38	13.91	98.03	32.67	75.65	53.69	53.69
OTM	Dec-03	39.90	0.18	15.86	9.30	89.56	26.03	67.45	46.02	46.02
OTM	Dec-03	34.70	0.04	11.38	6.36	82.87	21.22	61.06	40.19	40.19
OTM	Dec-03	26.80	0.00	6.65	3.58	74.20	15.71	52.93	33.03	33.03
OTM	Dec-03	22.10	0.00	4.66	2.13	67.60	12.08	46.89	27.90	27.90
OTM	Dec-03	17.60	0.00	2.81	1.20	61.51	9.16	41.43	23.44	23.44
OTM	Dec-03	14.20	0.00	1.47	0.57	55.20	6.52	35.89	19.08	19.08
ITM	Jun-04	192.90	183.83	185.70	145.47	216.41	158.43	195.74	176.05	176.05
ITM	Jun-04	159.70	136.72	141.51	104.51	189.70	123.19	166.95	144.58	144.58

Money	Expiry	Last	UV 1	UV2	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS1_1	FBS2_1
ITM	Jun-04	129.40	93.18	123.07	94.39	191.44	118.61	167.28	142.97	142.97
ATM	Jun-04	103.75	52.61	90.54	65.28	167.02	90.98	141.89	116.53	116.53
OTM	Jun-04	80.40	20.92	59.91	41.29	144.38	66.93	118.66	92.80	92.80
OTM	Jun-04	70.60	9.42	46.86	29.75	131.93	54.56	106.07	80.20	80.20
OTM	Jun-04	61.50	4.69	38.44	23.68	124.26	47.42	98.37	72.63	72.63
OTM	Jun-04	47.00	0.83	24.08	14.13	109.79	35.03	84.10	58.94	58.94
OTM	Jun-04	33.80	0.04	13.01	6.69	93.74	23.20	68.65	44.74	44.74
OTM	Jun-04	24.20	0.00	7.16	3.23	81.50	15.81	57.27	34.84	34.84
OTM	Jun-04	16.70	0.00	3.24	1.28	69.83	10.02	46.73	26.16	26.16

Table 7E: This table shows the comparative option values for the Call option bounds of FBS and UV OPM for S&P500 Options showing behaviour of FBS bounds (wrt α)

Money	Expiry	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS2_0.75	FBS2_0.75	FBS1_1	FBS2_1
ITM	Aug-02	118.34	124.45	119.11	122.21	119.70	121.25	120.41	120.41
ITM	Aug-02	94.42	104.20	96.10	101.10	97.17	99.69	98.37	98.37
ITM	Aug-02	71.66	85.18	74.44	81.33	76.03	79.48	77.71	77.71
ITM	Aug-02	50.05	66.84	53.89	62.38	55.94	60.19	58.04	58.04
ITM	Aug-02	44.46	62.72	48.84	58.02	51.10	55.69	53.38	53.38
ITM	Aug-02	36.46	55.53	41.12	50.69	43.49	48.27	45.87	45.87
ITM	Aug-02	31.72	51.11	36.49	46.21	38.90	43.76	41.33	41.33
ATM	Aug-02	30.26	50.02	35.17	45.06	37.63	42.58	40.11	40.11
ATM	Aug-02	24.33	44.46	29.36	39.43	31.88	36.91	34.40	34.40
ATM	Aug-02	18.43	38.62	23.47	33.57	25.99	31.04	28.52	28.52
ATM	Aug-02	14.86	34.77	19.77	29.75	22.25	27.24	24.74	24.74
OTM	Aug-02	11.28	30.59	15.94	25.64	18.33	23.19	20.75	20.75
OTM	Aug-02	8.73	27.49	13.17	22.62	15.48	20.21	17.83	17.83
OTM	Aug-02	7.08	24.94	11.18	20.20	13.35	17.88	15.59	15.59
OTM	Aug-02	4.85	21.07	8.36	16.60	10.30	14.43	12.32	12.32
OTM	Aug-02	3.22	18.25	6.27	13.95	8.03	11.89	9.92	9.92
OTM	Aug-02	1.93	14.60	4.24	10.73	5.66	8.93	7.23	7.23
ITM	Sep-02	120.80	138.78	124.21	133.41	126.26	130.88	128.49	128.49
ITM	Sep-02	98.29	120.05	102.85	113.92	105.42	110.98	108.14	108.14
ITM	Sep-02	77.07	102.49	82.82	95.68	85.91	92.36	89.09	89.09
ITM	Sep-02	57.68	86.27	64.54	78.91	68.07	75.27	71.65	71.65
ITM	Sep-02	51.81	81.49	59.07	73.96	62.76	70.21	66.47	66.47
ITM	Sep-02	43.90	74.35	51.43	66.68	55.22	62.85	59.03	59.03
ITM	Sep-02	40.87	71.67	48.52	63.94	52.36	60.07	56.22	56.22
ATM	Sep-02	38.07	69.16	45.82	61.37	49.70	57.48	53.59	53.59
ATM	Sep-02	32.39	63.87	40.27	56.01	44.21	52.07	48.14	48.14
ATM	Sep-02	27.39	58.99	35.29	51.10	39.24	47.14	43.19	43.19
ATM	Sep-02	21.45	52.90	29.26	45.00	33.18	41.05	37.11	37.11
OTM	Sep-02	16.82	47.59	24.31	39.76	28.13	35.86	31.98	31.98
OTM	Sep-02	15.18	45.60	22.52	37.81	26.29	33.94	30.10	30.10
OTM	Sep-02	13.80	43.83	20.97	36.08	24.68	32.24	28.44	28.44
OTM	Sep-02	10.11	38.66	16.66	31.10	20.15	27.39	23.73	23.73
OTM	Sep-02	7.63	34.83	13.64	27.45	16.92	23.86	20.34	20.34
ITM	Oct-02	143.02	160.32	145.61	154.55	147.43	151.94	149.56	149.56
ITM	Oct-02	101.28	130.04	107.60	122.20	111.06	118.38	114.66	114.66
ITM	Oct-02	62.47	97.83	71.06	88.81	75.45	84.33	79.88	79.88
ITM	Oct-02	43.69	81.11	53.00	71.73	57.67	67.04	62.36	62.36
ATM	Oct-02	29.74	68.03	39.31	58.46	44.10	53.68	48.89	48.89
OTM	Oct-02	19.00	56.57	28.18	47.04	32.85	42.29	37.56	37.56
OTM	Oct-02	11.55	46.92	19.74	37.63	24.08	33.04	28.52	28.52
OTM	Oct-02	5.85	37.49	12.46	28.66	16.23	24.38	20.22	20.22
OTM	Oct-02	3.04	30.57	8.08	22.32	11.24	18.42	14.71	14.71
OTM	Oct-02	1.24	23.94	4.61	16.47	7.05	13.04	9.88	9.88
OTM	Oct-02	0.44	18.25	2.44	11.72	4.16	8.85	6.31	6.31
ITM	Dec-02	190.59	206.70	192.18	200.50	193.64	197.84	195.54	195.54
ITM	Dec-02	142.46	168.74	146.57	160.28	149.44	156.37	152.73	152.73

Money	Expiry	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS2_0.75	FBS2_0.75	FBS1_1	FBS2_1
ITM	Dec-02	98.31	136.07	106.27	125.60	110.80	120.51	115.56	115.56
ITM	Dec-02	61.84	108.37	73.13	96.52	78.91	90.62	84.75	84.75
ITM	Dec-02	48.51	97.51	60.74	85.25	66.86	79.12	72.99	72.99
ATM	Dec-02	60.59	110.40	73.09	98.00	79.32	91.78	85.56	85.56
OTM	Dec-02	39.77	89.70	52.19	77.19	58.43	70.93	64.67	64.67
OTM	Dec-02	28.27	77.27	40.21	64.81	46.29	58.61	52.43	52.43
OTM	Dec-02	18.32	65.16	29.27	52.92	35.03	46.88	40.91	40.91
OTM	Dec-02	10.46	53.75	19.87	41.92	25.09	36.16	30.53	30.53
OTM	Dec-02	5.27	43.81	12.72	32.59	17.22	27.22	22.08	22.08
OTM	Dec-02	3.02	37.63	8.98	26.97	12.88	21.96	17.24	17.24
OTM	Dec-02	0.78	26.26	3.87	17.18	6.41	13.15	9.53	9.53
OTM	Dec-02	0.18	18.58	1.64	11.07	3.17	7.93	5.28	5.28
ITM	Mar-03	142.34	180.65	149.02	168.99	153.40	163.48	158.25	158.25
ITM	Mar-03	62.67	121.72	77.06	106.75	84.42	99.28	91.83	91.83
ITM	Mar-03	46.01	107.71	61.41	92.30	69.13	84.58	76.86	76.86
ATM	Mar-03	33.57	96.46	49.32	80.79	57.20	72.94	65.07	65.07
OTM	Mar-03	23.82	86.33	39.19	70.57	47.00	62.70	54.84	54.84
OTM	Mar-03	17.33	78.18	31.77	62.50	39.31	54.71	46.97	46.97
OTM	Mar-03	10.96	68.77	23.84	53.32	30.92	45.72	38.23	38.23
OTM	Mar-03	15.97	73.74	29.05	58.37	36.10	50.81	43.38	43.38
ITM	Jun-03	143.36	192.84	152.92	178.54	158.75	171.65	165.03	165.03
ITM	Jun-03	101.51	162.20	115.15	146.06	122.59	138.10	130.26	130.26
ITM	Jun-03	66.04	135.37	83.07	117.89	91.73	109.15	100.43	100.43
ITM	Jun-03	49.99	121.95	68.00	104.02	77.02	95.03	86.03	86.03
ATM	Jun-03	37.43	110.66	55.80	92.44	64.98	83.30	74.15	74.15
OTM	Jun-03	26.46	99.50	44.48	81.14	53.61	71.96	62.78	62.78
OTM	Jun-03	17.43	88.74	34.36	70.40	43.22	61.28	52.20	52.20
OTM	Jun-03	25.82	97.07	42.83	78.75	51.65	69.65	60.61	60.61
OTM	Jun-03	16.39	84.62	31.85	66.53	40.21	57.61	48.82	48.82
OTM	Jun-03	8.85	72.33	21.99	54.66	29.62	46.06	37.68	37.68
OTM	Jun-03	5.75	65.36	17.13	48.08	24.15	39.76	31.75	31.75
OTM	Jun-03	2.83	56.13	11.55	39.59	17.59	31.77	24.40	24.40
OTM	Jun-03	1.48	49.49	8.20	33.65	13.40	26.31	19.51	19.51
OTM	Jun-03	0.79	43.92	5.89	28.82	10.32	21.97	15.74	15.74
OTM	Jun-03	3.32	52.39	11.15	36.75	16.56	29.49	22.72	22.72
ITM	Dec-03	147.18	214.84	161.79	196.42	170.00	187.40	178.57	178.57
ITM	Dec-03	105.97	183.50	124.11	163.39	133.71	153.40	143.49	143.49
ITM	Dec-03	72.50	158.13	93.75	136.71	104.48	125.97	115.22	115.22
ATM	Dec-03	45.34	134.98	67.88	112.74	79.14	101.57	90.37	90.37
OTM	Dec-03	25.99	114.92	47.57	92.38	58.67	81.11	69.86	69.86
OTM	Dec-03	13.91	98.03	32.67	75.65	43.01	64.60	53.69	53.69
OTM	Dec-03	9.30	89.56	26.03	67.45	35.77	56.62	46.02	46.02
OTM	Dec-03	6.36	82.87	21.22	61.06	30.37	50.46	40.19	40.19
OTM	Dec-03	3.58	74.20	15.71	52.93	23.93	42.75	33.03	33.03
OTM	Dec-03	2.13	67.60	12.08	46.89	19.46	37.12	27.90	27.90
OTM	Dec-03	1.20	61.51	9.16	41.43	15.70	32.10	23.44	23.44
OTM	Dec-03	0.57	55.20	6.52	35.89	12.12	27.08	19.08	19.08
ITM	Jun-04	145.47	216.41	158.43	195.74	166.88	185.73	176.05	176.05
ITM	Jun-04	104.51	189.70	123.19	166.95	133.70	155.69	144.58	144.58

Money	Expiry	FBS1_0	FBS2_0	FBS1_0.5	FBS2_0.5	FBS2_0.75	FBS2_0.75	FBS1_1	FBS2_1
ITM	Jun-04	94.39	191.44	118.61	167.28	130.79	155.14	142.97	142.97
ATM	Jun-04	65.28	167.02	90.98	141.89	103.78	129.24	116.53	116.53
OTM	Jun-04	41.29	144.38	66.93	118.66	79.86	105.74	92.80	92.80
OTM	Jun-04	29.75	131.93	54.56	106.07	67.32	93.12	80.20	80.20
OTM	Jun-04	23.68	124.26	47.42	98.37	59.92	85.46	72.63	72.63
OTM	Jun-04	14.13	109.79	35.03	84.10	46.76	71.42	58.94	58.94
OTM	Jun-04	6.69	93.74	23.20	68.65	33.56	56.50	44.74	44.74
OTM	Jun-04	3.23	81.50	15.81	57.27	24.75	45.75	34.84	34.84
OTM	Jun-04	1.28	69.83	10.02	46.73	17.37	36.04	26.16	26.16

Table 8E: This table shows the comparative option values for MTM1 and CRR OPM for S&P 500 Index Options quoted on 24th July 2002

Moneyness	Expiration	Bid	Ask	Last	MTM1 Value	CRR Value
ITM	Aug-02	89.50	92.50	91.00	92.2009	92.6213
ITM	Aug-02	69.60	72.60	66.50	72.2259	72.7426
ITM	Aug-02	51.60	54.60	45.00	54.2085	54.2891
ATM	Aug-02	36.00	38.00	35.00	37.6079	38.3207
ATM	Aug-02	30.70	33.70	33.00	34.5058	34.7258
ATM	Aug-02	25.80	28.80	26.80	28.3928	28.9063
ATM	Aug-02	23.90	25.00	22.00	24.6101	25.3082
OTM	Aug-02	21.50	24.50	20.00	24.0934	24.8932
OTM	Aug-02	17.90	20.30	13.70	19.8928	20.4789
OTM	Aug-02	14.60	16.00	15.00	15.6353	15.7846
OTM	Aug-02	12.00	13.60	8.80	13.2324	13.7935
OTM	Aug-02	10.10	11.00	9.50	10.6343	11.2465
OTM	Aug-02	8.20	9.00	7.90	8.6523	9.0282
OTM	Aug-02	7.00	7.80	7.80	7.4714	7.5234
OTM	Aug-02	5.00	6.00	6.00	5.6867	6.0995
OTM	Aug-02	4.40	4.50	4.50	4.2102	4.5920
OTM	Aug-02	2.60	3.20	2.60	2.9459	2.9973
ITM	Sep-02	100.40	103.40	85.00	102.8372	103.7549
ITM	Sep-02	81.80	84.80	77.00	84.2479	84.7875
ITM	Sep-02	64.80	67.80	59.00	67.2074	68.1630
ATM	Sep-02	49.70	52.70	48.60	52.1201	52.9597
ATM	Sep-02	44.20	47.20	33.00	47.9467	48.1709
ATM	Sep-02	39.10	42.10	29.00	41.5268	42.0765
ATM	Sep-02	36.80	39.80	31.90	39.2288	40.0513
OTM	Sep-02	34.70	37.70	29.50	37.1201	38.1316
OTM	Sep-02	30.30	33.30	30.70	32.7253	33.8498
OTM	Sep-02	26.40	29.40	22.00	28.8298	29.7314
OTM	Sep-02	22.90	24.50	19.00	23.9590	24.1766
OTM	Sep-02	18.40	20.80	18.80	20.2711	20.8951
OTM	Sep-02	17.00	19.40	14.00	18.8759	19.6665
OTM	Sep-02	15.80	18.20	14.10	17.6782	18.5651
OTM	Sep-02	14.60	17.00	0.00	16.4691	17.3871
OTM	Sep-02	11.90	14.30	14.30	13.7761	14.5156
OTM	Sep-02	10.10	12.50	11.40	11.9911	12.3367
ITM	Oct-02	124.10	127.10	121.20	126.4795	127.0783
ITM	Oct-02	104.80	107.80	0.00	107.1699	108.2427
ITM	Oct-02	86.70	89.70	77.00	89.0664	89.2304
ATM	Oct-02	54.50	57.50	51.00	56.8527	57.7166
ATM	Oct-02	41.50	44.50	42.00	43.8645	44.7100
OTM	Oct-02	30.70	33.70	22.90	33.0627	34.1793
OTM	Oct-02	22.20	25.20	18.40	24.6020	24.9439
OTM	Oct-02	15.10	17.50	14.50	16.9362	17.9107
OTM	Oct-02	10.10	12.50	8.60	11.9702	11.9779
OTM	Oct-02	6.70	8.20	6.00	7.7345	8.3879
OTM	Oct-02	4.30	5.50	4.50	5.0791	5.2893
OTM	Oct-02	2.65	3.80	3.50	3.4551	3.7772

Moneyness	Expiration	Bid	Ask	Last	MTM1 Value	CRR Value
OTM	Oct-02	1.95	2.70	0.00	2.3761	2.7062
OTM	Oct-02	1.15	1.90	0.00	1.6285	1.6346
OTM	Oct-02	0.60	1.35	0.00	1.1320	1.2740
OTM	Oct-02	0.50	1.05	0.00	0.8479	1.0156
ITM	Dec-02	306.70	310.70	296.00	310.3583	310.3980
ITM	Dec-02	214.00	217.00	205.00	216.4134	216.8023
ITM	Dec-02	170.80	173.80	181.00	173.1552	173.7260
ITM	Dec-02	130.60	133.60	123.00	132.8530	134.1365
ITM	Dec-02	94.40	97.40	94.00	96.6239	97.2543
ITM	Dec-02	78.40	81.40	0.00	80.6111	82.1297
ATM	Dec-02	63.80	66.80	61.00	66.0348	66.9328
ATM	Dec-02	50.80	53.80	48.70	53.0403	53.9185
OTM	Dec-02	39.90	42.90	36.50	42.1359	43.5982
OTM	Dec-02	30.70	33.70	22.90	32.9595	33.4203
OTM	Dec-02	22.10	25.10	20.00	24.4317	25.4377
OTM	Dec-02	16.10	18.50	16.00	17.8277	18.8072
OTM	Dec-02	12.20	13.00	10.00	12.4274	12.7303
OTM	Dec-02	8.10	9.60	6.50	9.0532	9.8187
OTM	Dec-02	6.00	7.50	6.00	6.9737	7.5467
OTM	Dec-02	2.85	4.00	2.50	3.6027	4.0283
OTM	Dec-02	1.15	1.90	1.65	1.6216	1.7021
ITM	Mar-03	136.60	139.60	133.00	138.7629	140.1048
ITM	Mar-03	101.80	104.80	0.00	103.8815	105.1383
ATM	Mar-03	1.70	74.70	75.00	73.8117	74.7287
ATM	Mar-03	58.70	61.70	58.00	60.8291	61.7253
OTM	Mar-03	47.60	50.60	44.90	49.7326	51.4103
OTM	Mar-03	37.50	40.50	29.50	39.6577	40.6919
OTM	Mar-03	29.80	32.80	24.00	32.0101	32.5951
OTM	Mar-03	22.80	25.80	18.50	25.0229	26.3636
OTM	Mar-03	17.30	19.70	16.80	18.9529	19.8506
ITM	Jun-03	143.20	146.20	140.00	145.2266	146.2500
ITM	Jun-03	109.70	112.70	93.50	111.6482	113.4101
ATM	Jun-03	79.70	82.70	66.00	81.6938	82.6244
ATM	Jun-03	67.10	70.10	68.00	69.1127	70.0224
OTM	Jun-03	55.30	58.30	48.40	57.3164	59.1602
OTM	Jun-03	45.40	48.40	40.50	47.4597	49.0001
OTM	Jun-03	37.20	40.20	38.10	39.2995	39.3879
ITM	Jun-03	219.10	222.10	0.00	221.2839	222.0826
ITM	Jun-03	178.60	181.60	0.00	180.6176	181.9137
ITM	Jun-03	142.10	145.10	140.00	144.1522	145.2286
ITM	Jun-03	108.80	111.80	93.50	110.7638	112.4880
ATM	Jun-03	78.90	81.90	66.00	80.8959	81.8253
ATM	Jun-03	66.30	69.30	68.00	68.3458	69.2543
OTM	Jun-03	54.70	57.70	48.40	56.7327	58.5667
OTM	Jun-03	44.80	47.80	40.50	46.8596	48.3709
OTM	Jun-03	36.70	39.70	38.10	38.7926	38.8310
OTM	Jun-03	29.40	32.40	32.00	31.5340	32.7615
OTM	Jun-03	23.10	26.10	23.50	25.2390	26.7140
OTM	Jun-03	18.40	20.80	22.50	19.9637	20.7094

Moneyness	Expiration	Bid	Ask	Last	MTM1 Value	CRR Value
OTM	Jun-03	14.80	17.20	25.50	16.4603	16.7643
OTM	Jun-03	10.60	13.00	12.50	12.2902	13.2611
OTM	Jun-03	8.30	9.80	7.50	9.1302	9.9302
OTM	Jun-03	6.10	7.60	12.00	6.9985	7.0507
OTM	Jun-03	4.50	5.70	5.30	5.1925	5.6156
ITM	Dec-03	154.70	157.70	236.00	156.5681	156.6882
ITM	Dec-03	123.30	126.30	112.00	125.0816	127.3556
ATM	Dec-03	95.40	98.40	86.00	97.1412	98.0799
ATM	Dec-03	81.20	84.20	0.00	83.0201	83.9366
OTM	Dec-03	71.60	74.60	82.00	73.4150	75.4734
OTM	Dec-03	61.50	64.50	0.00	63.3654	65.5587
OTM	Dec-03	51.90	54.90	74.00	53.7775	55.1150
OTM	Dec-03	44.00	47.00	0.00	45.9258	46.2613
OTM	Dec-03	36.30	39.30	35.00	38.2320	39.7404
OTM	Dec-03	30.40	33.40	48.00	32.3447	34.1679
OTM	Dec-03	25.90	28.90	46.00	27.8457	29.3556
OTM	Dec-03	20.10	24.10	25.50	23.1095	23.4479
OTM	Dec-03	16.70	18.70	15.00	17.8296	18.6872
OTM	Dec-03	13.20	15.20	34.00	14.3406	15.5413
OTM	Dec-03	10.50	12.50	11.50	11.6676	12.7026
OTM	Dec-03	6.90	7.90	8.50	7.2484	7.6004
OTM	Dec-03	4.30	5.10	5.00	4.5193	5.1440
ITM	Jun-04	236.30	239.30	217.00	237.8941	239.9074
ITM	Jun-04	200.70	203.70	0.00	202.6886	204.0861
ITM	Jun-04	86.70	89.70	105.00	184.1261	185.6660
ITM	Jun-04	167.40	170.40	184.00	169.1172	170.1196
ITM	Jun-04	236.30	239.30	217.00	236.7908	241.0986
ITM	Jun-04	200.70	203.70	0.00	201.7212	205.4582
ITM	Jun-04	137.30	140.30	154.50	138.9681	141.5432
ATM	Jun-04	110.50	113.50	203.50	112.1193	113.0539
OTM	Jun-04	86.70	89.70	105.00	88.3661	90.5324
OTM	Jun-04	66.60	69.60	58.30	68.2920	70.4139
OTM	Jun-04	58.10	61.10	69.00	59.7884	60.6644
OTM	Jun-04	50.00	53.00	45.00	51.7847	52.6108
OTM	Jun-04	43.00	46.00	0.00	44.7740	46.5894
OTM	Jun-04	38.10	41.10	62.00	39.8643	41.9719
OTM	Jun-04	26.20	30.20	29.50	28.9928	29.8492
OTM	Jun-04	19.10	21.10	23.00	20.0620	21.2808
OTM	Jun-04	12.90	14.90	23.00	13.9063	15.0931

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