Scientific African 21 (2023) e01765

Contents lists available at ScienceDirect

Scientific African

journal homepage: www.elsevier.com/locate/sciaf

Modelling Asymmetric Dependence in Stochastic Volatility and Option Pricing: A Conditional Copula Approach

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A R T I C L E I N F O

Article history:
Received 25 February 2023
Revised 2 June 2023
Accepted 14 June 2023

Editor: DR B Gyampoh

Keywords:
Stochastic volatility
Asymmetric dependence
Copula h-function
Dynamic conditional copula
Option pricing

A B S T R A C T

In this paper, stochastic volatility models with asymmetric dependence were presented and applied to pricing options. A dynamic conditional copula approach was proposed to capture this dependence asymmetry. This approach offered simplicity and flexibility, and yielded closed-form solutions for option pricing under different model constructions for the stochastic volatility based on a mean-reverting Gaussian, a square-root and a lognormal process. Empirical experimentation based on S&P 500 options showed that the developed dynamic option pricing models under asymmetric stochastic volatility significantly and consistently outperformed the basic Heston model across option maturities, strike prices and various copula function specifications. The square-root model combined with a Joe copula was the best ranked, having achieved 32.33% overall performance improvement. This superior empirical performance in option pricing, the unique flexibility to various dependence asymmetry considerations, and the analytical tractability added to the benefits of the proposed models framework.

Introduction

The Black and Scholes [1] model in its assumptions results in a flat implied volatility surface profile. However, it is empirically shown that implied volatilities exhibit a dependence on maturity time and strike price in a given cross-section and time-series. This empirical implied volatility surface has been noted to exhibit stylized 'smile', 'skew' or 'smirk' features which change over time as a result of option market dynamics. Empirical studies on the behavior of implied volatilities have pointed out some commonality across markets, particularly on the smile patterns, the term structure and the time-series patterns which exhibit high autocorrelation and mean-reversion. Furthermore, the asset market returns present non-constant volatility corroborated by volatility clustering, conditional heavy tails, and correlation asymmetry under stressed market conditions, such as leverage effects of negative correlation between asset returns and volatility, and a gain-loss asymmetry where down-size movements are disproportionately larger than upward movements, in addition to jumps in returns and volatility. Consequently, the assumption of asset returns' normality is invalidated in practice given these features affecting the tails and skewness of their distribution ([2–10]).

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https://doi.org/10.1016/j.sciaf.2023.e01765

In practice however, even though the Black–Scholes model assumes explicitly a constant volatility and interest rates, it is implemented as if volatility and interest rates are time-varying and are extracted by model inversion from daily option prices and the yield curve (2). This treatment further demonstrates the weakness of its static assumptions, since this treatment inherently implies that the randomness of volatility and interest rates exists. Several approaches have been proposed that attempt to explicitly define and capture these characteristics in option pricing. Models that capture the instantaneous implied volatility profile have been proposed, such as local volatility models, non-linear diffusions, jump diffusions, GARCH models, and stochastic volatility models (2,3).

The literature is rich with traditional models of stochastic volatility, such as, for example, Hull and White [11], Johnson and Shanno [12], Scott [13], Wiggins [14], Stein and Stein [15], Heston [16], Ball and Roma [17], Bates [18], Bates [9], Scott [19], Bakshi et al. [20], Sabanis [21], Lee [22] and Cao et al. [2]. These models generally address the option pricing problem using Fourier inversion techniques resulting in semi-analytical formulae for option pricing (e.g., see 23,22). Hull and White [11] instead derive an expected Black–Scholes-style price representation conditional on the integrated squared volatility and solve the problem approximately by making use of its moments and a Taylor series expansion. Fourier transform techniques, although in principle are considered very accurate, their computational efficiency can suffer in practical applications, such as when calibrating to observed market prices, due to the effect of the estimation iterations on the parameter values and, consequently, on the behaviour of the associated characteristic function. Similarly, accuracy concerns arise when applied to hedging strategies combined with scenario generation. Nevertheless, they remain the method of choice so far, being also generally more easily adaptable to different model dynamics than, for example, numerical schemes for partial differential equations ([24–26]).

Despite the various earlier contributions, a model that is sufficiently flexible and analytically tractable to capture the stochastic volatility dynamics, the associated heavy tails and skewness in the distribution of the underlying asset returns still needs further consideration. In the aforementioned traditional model approaches, the dependence between the return and volatility shocks is modelled directly by their linear correlation. However, the use of linear correlations is an unsatisfactory measure of dependence (27). Therefore, traditional return distribution modelling needs to be refocused to accommodate these non-Gaussian features and a potentially nonlinear return-volatility dependence structure, in addition to time variation and persistence in dependence that has been identified from empirical data (28,29). Asymmetric and dynamic dependence structures exist between returns and volatility shocks and need to be incorporated into the modelling. For example, Veraart and Veraart [30] and Lu et al. [31] considered a stochastic correlation parameter; Huang and Huang [29] examined via Monte Carlo simulation a static and a dynamic copula with autoregressive persistence in the dependence structure between returns and volatility shocks. They found, in their numerical simulation of a discretized generalized stochastic volatility model, that option prices are mainly affected by different nonlinear dependence structures and they showed that a dynamic copula yielded the highest option prices.

In this paper, stochastic volatility models with asymmetric dependence are presented and applied to option pricing. A dynamic conditional copula h-function is proposed to capture the existing nonlinear dependence structures between the stock price and stochastic volatility processes. This h-function also captures the conditional probability distribution of the underlying stock price process. To highlight the simplicity and flexibility of the proposed conditional copula approach, we construct different volatility models based on a mean-reverting Gaussian process; a square-root diffusion; and a lognormal process. Closed-form price formulae for European options under these different model specifications are presented. The particular pricing approach is termed dynamic, since it allows several alternative dependence structures between the driving processes to be captured by a malleable conditional copula h-function whose formulation is flexible to varying configurations of the stochastic volatility process. This approach represents an important development over the techniques currently used in the relevant literature as it exhibits much more accurate option price modelling performance, a closed-form solution, in addition to being unique in its flexibility, extensibility, and analytical tractability for various stochastic volatility models.

Each stochastic volatility configuration developed in this paper is implemented and evaluated in terms of its performance in the pricing of in-the-money S&P 500 Index European call options across time to maturity and strike price levels, and then compared to the basic Heston model. Aggregate performance improvements over the Heston model of 30.64%, 27.79% and 11.60% are respectively recorded for the Gaussian mean-reverting, square-root and lognormal models. Copula-specific performance variation is observed across all three models. The square-root model under the Joe copula is found to achieve the highest performance improvement of 32.33%.

The remainder of this paper is structured as follows. In Section 2, we present the development of the proposed stochastic volatility model framework with asymmetric dependence and its application in option pricing. In Section 3, we investigate and empirically evaluate our dynamic pricing models. Section 4 concludes the paper.

Option price modelling with asymmetric stochastic volatility

In this section, a general stochastic volatility model with asymmetric dependence is presented and extended to option pricing by employing a dynamic conditional copula. The asymmetric stochastic volatility model allows a nonlinear dependence relationship to be modelled between the asset price and the variance processes, which has traditionally been modelled by a linear correlation. When applied to option pricing in the subsequent section, a conditional copula h-function is used to capture this nonlinear dependence relationship. A dynamic option pricing model that can accommodate a variety of dependence structures and volatility processes specifications is thereby developed.

Stochastic volatility models with asymmetric dependence

Let $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\})$ be a filtered probability space where the filtration satisfies the usual conditions with $\mathcal{F}_0$ trivial. This filtered probability space supports all processes that follow and Q denotes the risk neutral probability measure.

We consider a stochastic model $(S_t, \upsilon_t)_{t\geq 0}$ where S_t and υ_t denote, respectively, the price of a non-dividend paying stock and the instantaneous variance process. The model is generally given by

$$\begin{cases} dS_t = rS_t dt + \sqrt{\upsilon_t} S_t \Big(\rho dW_{2,t} + \sqrt{1-\rho^2} dW_{1,t}\Big) \\ d\upsilon_t = \eta_t dt + \delta_t dW_{2,t} \end{cases}, \tag{1}$$

where W_1 and W_2 are independent standard Brownian motions on the filtered probability space, r is the continuously compounded risk-free interest rate and $\rho \in [-1, 1]$ is the instantaneous correlation between the two processes.

The variance dynamics υ_t can take several different forms. For the purposes of this study, we consider three cases where:

$$(\eta_t, \delta_t) = \begin{cases} (k(\theta - \upsilon_t), \sigma), & \text{Gaussian process} \\ (\alpha(\beta - \upsilon_t), \gamma\sqrt{\upsilon_t}), & \text{Square-root diffusion} \\ (\mu\upsilon_t, \omega\upsilon_t), & \text{Lognormal process} \end{cases}$$

under the measure Q. In the first case, the parameter θ denotes the long-run mean, k the speed of mean-reversion, and σ controls the volatility of the variance process. The role is similar for parameters β, α and γ in the second case. Nevertheless, while the Gaussian mean-reverting process may, in principle, take negative values, in the case of the square-root diffusion, if the so-called Feller condition, $2\alpha\beta \geq \gamma^2$, is satisfied, then the zero boundary is unattainable; otherwise, this is attracting and attainable. At the zero boundary though, the process is immediately reflected into the positive domain. Even for the Gaussian mean-reverting process, it is possible with actual volatility data to fit it so that it is restricted in the positive domain (see 15). In the last case, the variance is assumed to evolve according to a geometric Brownian motion with parameters $\mu, \omega \geq 0$.

Therefore, by solving (1) under the risk-neutral measure Q, we have that

$$\ln\frac{S_t}{S_u} = r(t-u) - \frac{1}{2}V_{u,t}(t-u) + Y_{u,t} + \int_u^t \sqrt{\Big(1-\rho^2\Big)\upsilon_s}\, dW_{1,s},$$

where

$$Y_{u,t} = \int_u^t \rho\sqrt{\upsilon_s}\, dW_{2,s} \text{ and } V_{u,t} = \frac{1}{t-u}\int_u^t \upsilon_s ds.$$

Further, we note at this point that the process υ_s is independent of $W_{1,s}$ under the measure Q, and that conditionally on $V_{u,t}$, then $\ln\frac{S_t}{S_u e^{Y_{u,t}}}$ follows a conditional normal distribution (see 32) as

$$\left(\ln\frac{S_t}{S_u e^{Y_{u,t}}} \;\middle|\; V_{u,t}\right) \sim N\Big(m_{u,t}(V_{u,t}),\; s^2_{u,t}(V_{u,t})\Big),$$

where

$$m_{u,t}(V_{u,t}) = r(t-u) - \frac{1}{2}V_{u,t}(t-u),$$

$$s^2_{u,t}(V_{u,t}) = \Big(1-\rho^2\Big)V_{u,t}(t-u).$$

By Girsanov's theorem, the process

$$\tilde{W}_{1,t} = W_{1,t} - \int_0^t \sqrt{\upsilon_s} ds \text{ and } \tilde{W}_{2,t} = W_{2,t} - \int_0^t \lambda(\upsilon_s) ds \tag{2}$$

is a two-dimensional Brownian motion under the equivalent probability measure $\tilde{Q}$ resulting in

$$\tilde{\eta}_t = \eta_t + \delta_t\lambda(\upsilon_t)$$

whereby for the respective stochastic volatility processes under $\tilde{Q}$ we have that

$$\lambda(\upsilon_t) = \begin{cases} \sigma, & \text{Gaussian process} \\ \gamma\sqrt{\upsilon_t}, & \text{Square-root diffusion} \\ \sqrt{\omega}, & \text{Lognormal process} \end{cases}$$

and

$$\tilde{\eta}_t = \begin{cases} k\Big(\tilde{\theta} - \upsilon_t\Big), & \tilde{\theta} = \theta + \sigma^2/k \\ \tilde{\alpha}\Big(\tilde{\beta} - \upsilon_t\Big), & \tilde{\alpha} = \alpha - \gamma^2, \tilde{\beta} = \alpha\beta/\Big(\alpha - \gamma^2\Big) \\ \tilde{\mu}\upsilon_t, & \tilde{\mu} = \mu + \omega^2 \end{cases}.$$

Thus, the model dynamics under measure $\tilde{Q}$ are given by

$$\ln \frac{S_t}{S_u e^{\tilde{Y}_{u,t}}} = r(t-u) + \frac{1}{2}V_{u,t}(t-u) + \int_u^t \sqrt{(1-\rho^2)\nu_s}\,d\tilde{W}_{1,s},$$

$$d\nu_t = \tilde{\eta}_t dt + \delta_t d\tilde{W}_{2,t}.$$

Under measure $\tilde{Q}$ and conditionally on $V_{u,t}$, $\ln \frac{S_t}{S_u e^{\tilde{Y}_{u,t}}}$ similarly follows a conditional normal distribution:

$$\left(\ln \frac{S_t}{S_u e^{\tilde{Y}_{u,t}}} \,\middle|\, V_{u,t} \right) \sim N\left(\tilde{m}_{u,t}(V_{u,t}),\ s^2_{u,t}(V_{u,t})\right),$$

where

$$\tilde{m}_{u,t}(V_{u,t}) = r(t-u) + \frac{1}{2}V_{u,t}(t-u),$$

$$s^2_{u,t}(V_{u,t}) = \left(1-\rho^2\right)V_{u,t}(t-u).$$

Option pricing by conditional copula

Conditional on the information of the volatility path generated up until expiry time T, the price of a European plain vanilla call option $f(S,u)$ with strike price K by risk-neutral valuation is given by

$$\begin{aligned} f(S,u) &= \mathbb{E}^{Q}[e^{-r(T-u)}(S_T-K)\cdot \mathbf{1}_{\{S_T>K\}}|\mathcal{F}^{\nu}_T] \\ &= \mathbb{E}^{Q}[S_T e^{-r(T-u)}|\mathcal{F}^{\nu}_T]\mathbb{E}^{\tilde{Q}}[\mathbf{1}_{\{S_T>K\}}|\mathcal{F}^{\nu}_T] - Ke^{-r(T-u)}\mathbb{E}^{Q}[\mathbf{1}_{\{S_T>K\}}|\mathcal{F}^{\nu}_T] \\ &= \mathbb{E}^{Q}[S_T e^{-r(T-u)}|\mathcal{F}^{\nu}_T]P^{\tilde{Q}}(S_T>K|\mathcal{F}^{\nu}_T) - Ke^{-r(T-u)}P^{Q}(S_T>K|\mathcal{F}^{\nu}_T). \end{aligned} \tag{3}$$

In order to obtain the option's payoff probabilities $P^{\tilde{Q},Q}(S_T>K|\mathcal{F}^{\nu}_T)$ of expiring in the money under the measures $\tilde{Q}$ and Q, we make use of the conditional copula h-function probability defined next.

Definition 1 (Conditional copula (see Patton [33])**.** Consider the random vector (X,Y,Z). Let $F_{\cdot|Z}(\cdot|z)$ denote the marginal conditional cumulative distribution function given $Z=z$. Then, based on Sklar's theorem, the conditional joint cumulative distribution function of $(X,Y|Z)$ is

$$F_{X,Y|Z}(x,y|z) = C(F_{X|Z}(x|z),\ F_{Y|Z}(y|z)|z),$$

where $C(\cdot,\cdot|z)$ is the conditional copula function for all z.

Proposition 1 (Conditional copula h-function)**.** *Given the continuous marginals $\phi = F_{X|Z=z}(\cdot|z)$ and $\psi = F_{Y|Z=z}(\cdot|z)$, there exists a unique conditional copula h-function $h(\phi,\psi|z)$ with the dependence parameter φ that is obtained as the partial derivative of the conditional copula distribution $C(\phi,\psi|z)$ with respect to ψ such that*

$$P(\{X\le x|Y=y\}|Z=z) = \frac{\partial C(\phi,\psi|z)}{\partial\psi} = h(\phi,\psi|z).$$

Proof. We have that

$$\begin{aligned} h(\phi,\psi|z) &= \lim_{\Delta y\to 0} \mathrm{P}(\{X\le x|X\le xy\le Y\le y+\Delta yy\le Y\le y+\Delta y\}|Z=z) \\ &= \lim_{\Delta y\to 0} \frac{F_{X,Y|Z}(x,\ y+\Delta y|z) - F_{X,Y|Z}(x,\ y|z)}{F_{Y|Z}\ (y+\Delta y|z) - F_{Y|Z}\ (y|z)} \\ &= \lim_{\Delta y\to 0} \frac{C(F_{X|Z}(x|z),\ F_{Y|Z}\ (y+\Delta y|z)|z) - C(F_{X|Z}(x|z), F_{Y|Z}\ (y|z)|z)}{F_{Y|Z}\ (y+\Delta y|z) - F_{Y|Z}\ (y|z)} \\ &= \frac{\partial C\left(F_{X|Z}(x|z),\ F_{Y|Z}(y|z)|z\right)/\partial y}{\partial F_{Y|Z}(y|z)/\partial y} = \frac{\partial C(\phi,\psi|z)}{\partial\psi}. \end{aligned}$$

Therefore, the option's conditional payoff probability $P^{\tilde{Q},Q}(S_T>K|\mathcal{F}^{\nu}_T)$ of expiring in the money at maturity time T is conditionally dependent on the integrated variance to maturity $V_{u,T}$ such that by application of the conditional copula formulation (see Proposition 1), this payoff probability is obtained under both measures $\tilde{Q}$ and Q as

$$P^{\tilde{Q},Q}\left(S_T>K|\mathcal{F}^{\nu}_T\right) = P^{\tilde{Q},Q}(\{S_T>K|V_{u,T}\}|\nu_u)$$

$$= \frac{\partial C^{\tilde{Q},Q}\left(F_S\left(S_T > K \,|\, \{V_{u,T} = \hat{v}\big|\nu_u\}\right), F_V\left(V_{u,T} \le \hat{v}\big|\nu_u\right)|\nu_u\right)}{\partial F_V\left(V_{u,T} \le \hat{v}\big|\nu_u\right)} = \frac{\partial C^{\tilde{Q},Q}(\phi_S, \psi_V|\nu_u)}{\partial \psi_V}$$
$$= h^{\tilde{Q},Q}(\phi_S, \psi_V),$$

where $C^{\tilde{Q},Q}$ denotes the copula function that captures the asymmetric dependence between the stock price process and the integrated variance process, and $h^{\tilde{Q},Q}(\phi_S, \psi_V)$ is the conditional copula h-function with respect to ψ_V. The continuous marginals ϕ_S and ψ_V represent the conditional probability F_S of the stock price process which follows a conditional lognormal distribution and the conditional probability F_V of the integrated variance process respectively.

The price of a European plain vanilla call option $f(S,u)$ under asymmetric stochastic volatility by the conditional copula approach is therefore obtained as

$$f(S,u) = S_u \Sigma_{u,T}\left(0,\hat{v}\right) h\left(\Phi\left(d_1\left(0,\hat{v}\right)\right), \psi_V^1\left(\hat{v}\right)\right) - Ke^{-r(T-u)} h\left(\Phi\left(d_2\left(0,\hat{v}\right)\right), \psi_V^2\left(\hat{v}\right)\right), \tag{4}$$

where

and $d_2(Y_{u,T}, V_{u,T}) = \frac{\ln\frac{S_u}{K} + Y_{u,T} + m_{u,T}(V_{u,T})}{\sqrt{s_{u,T}^2(V_{u,T})}}$,

$$\Sigma_{u,T}(Y_{u,T}, V_{u,T}) = e^{m_{u,T}(V_{u,T}) + \frac{1}{2}s_{u,T}^2(V_{u,T}) - r(T-u)}\mathbb{E}^Q\left(e^{Y_{u,T}}\right),$$

S_u is the current stock price, r is the risk-free interest rate, K is the strike price, $\hat{v}$ is the estimate of the aggregate variance from time u until expiry of the option at time T, h is the conditional copula h-function with respect to the second argument with dependence parameter φ and Φ is the standard normal cumulative distribution function. $\psi_V^{1,2}$ are the cumulative distribution functions under the respective measures $\tilde{Q}$ and Q of the integrated variance. Under the three stochastic volatility models specifications – Gaussian, square-root and lognormal – these functions are estimated by a Gaussian, Gamma (see Prayoga and Privault [34]) and lognormal distribution (see Levy [35], Turnbull and Wakeman [36]) respectively. More specifically, the cumulative distribution functions under the three specifications are as follows:

a. Under the Gaussian mean-reverting stochastic volatility, $\psi_V^{1,2}$ are obtained by the standard normal cumulative distribution function as $\Phi\left(g_1^v\right)$ and $\Phi\left(g_2^v\right)$ respectively, where $g_1^v = \frac{\hat{v} - \tilde{m}_g}{\sqrt{s_g^2}}$ and $g_2^v = \frac{\hat{v} - m_g}{\sqrt{s_g^2}}$ with

$$m_g = \theta + \frac{(\nu_u - \theta)\left(1 - e^{-k(T-u)}\right)}{k(T-u)},$$

$$\tilde{m}_g = \tilde{\theta} + \frac{\left(\nu_u - \tilde{\theta}\right)\left(1 - e^{-k(T-u)}\right)}{k(T-u)} \text{ with } \tilde{\theta} = \theta + \frac{\sigma^2}{k},$$

$$s_g^2 = \frac{\sigma^2}{k^2(T-u)^2}\left((T-u) - \frac{2\left(1 - e^{-k(T-u)}\right)}{k} + \frac{1}{2k} - \frac{e^{-2k(T-u)}}{2k}\right).$$

b. Under the square-root mean-reverting stochastic volatility, $\psi_V^{1,2}$ are obtained by the cumulative gamma distribution functions $F_1\left(\hat{v}; a_1, b_1\right)$ and $F_2\left(\hat{v}; a_2, b_2\right)$ respectively where the shape and scale parameters are $a_1 = \frac{\tilde{m}_f}{b_1}$ and $b_1 = \frac{s_f^2}{\tilde{m}_f}$, $a_2 = \frac{m_f}{b_2}$ and $b_2 = \frac{s_f^2}{m_f}$ with

$$m_f = \nu_u\left[\frac{1 - e^{-\alpha(T-u)}}{\alpha(T-u)}\right] + \beta\left[\frac{e^{-\alpha(T-u)} + \alpha(T-u) - 1}{\alpha(T-u)}\right],$$

$$\tilde{m}_f = \nu_u\left[\frac{1 - e^{-\tilde{\alpha}(T-u)}}{\tilde{\alpha}(T-u)}\right] + \tilde{\beta}\left[\frac{e^{-\tilde{\alpha}(T-u)} + \tilde{\alpha}(T-u) - 1}{\tilde{\alpha}(T-u)}\right] \text{ with } \tilde{\alpha} = \alpha - \gamma^2 \text{ and } \tilde{\beta} = \frac{\alpha\beta}{\alpha - \gamma^2},$$

$$s_f^2 = \gamma^2\nu_u\left[\frac{1 - 2\alpha(T-u)e^{-\alpha(T-u)} - e^{-2\alpha(T-u)}}{\alpha^3(T-u)^2}\right] + \gamma^2\beta\left[\frac{e^{-2\alpha(T-u)} + 2\alpha(T-u) + 4(\alpha(T-u)+1)e^{-\alpha(T-u)} - 5}{2\alpha^3(T-u)^2}\right].$$

c. Under the lognormal stochastic volatility model, $\psi_V^{1,2}$ are obtained by the standard normal cumulative distribution functions $\Phi\left(l_1^v\right)$ and $\Phi\left(l_2^v\right)$ respectively where $l_1^v = \frac{\ln\hat{v} - \tilde{m}_l}{\sqrt{\tilde{s}_l^2}}$ and $l_2^v = \frac{\ln\hat{v} - m_l}{\sqrt{s_l^2}}$ such that

$m_l = \ln\nu_u\left[\frac{e^{\mu(T-u)} - 1}{\mu(T-u)}\right] - 0.5\ \varepsilon^2$ and where $\tilde{\mu} = \mu + \omega^2$, and $\tilde{s}_l^2 = \ln \frac{\tilde{\varepsilon}^2}{v_u^2\left[\frac{(e^{\tilde{\mu}(T-u)}-1)^2}{\tilde{\mu}^2(T-u)^2}\right]}$,

$$\varepsilon^2 = 2v_u^2\left[\frac{\mu e^{(\omega^2+2\mu)(T-u)} - (\omega^2+2\mu)e^{\mu(T-u)} + (\omega^2+\mu)}{(\omega^2+\mu)(\omega^2+2\mu)\mu(T-u)^2}\right],$$

$$\tilde{\varepsilon}^2 = 2v_u^2\left[\frac{\tilde{\mu} e^{(\omega^2+2\tilde{\mu})(T-u)} - (\omega^2+2\tilde{\mu})e^{\tilde{\mu}(T-u)} + (\omega^2+\tilde{\mu})}{(\omega^2+\tilde{\mu})(\omega^2+2\tilde{\mu})\tilde{\mu}(T-u)^2}\right].$$

□

Empirical evaluation

This section examines the relative empirical performance of the dynamic option pricing models under asymmetric stochastic volatility that were introduced in the previous section. The data, obtained from Eikon, include daily prices for 3-year in-the-money S&P 500 Index call options expiring on 18 December 2019 and 25 strike price levels ranging from 1800 to 3000 USD. The underlying prices of the S&P 500 Index are obtained for the 5-year period from 19 August 2015 to 19 August 2020. In the performance analysis of the models, the following copula h-functions are fitted: the empirically estimated Joe copula, Gaussian, Student *t*, Clayton, Gumbel and Frank copulae.

Table 1 reports the summary statistics of the data used in the analysis, while in Table 2 we present summaries of the estimated parameters of the Gaussian mean-reverting, square-root, and lognormal stochastic volatility models. Table 3 reports the dependence parameter estimates of the fitted copula h-functions. Figs. 1 and 2 exhibit plots of S&P 500 Index daily prices and S&P 500 Index stochastic volatility from 19 August 2015 to 19 August 2020. The empirical copula dependence plot is given in Fig. 3.

In Table 4, the summaries of the pricing errors by root mean square error (RMSE) are presented. The RMSE is computed from the estimated daily option prices under the Gaussian mean-reverting, the square-root and the lognormal models and compared with the actual daily option prices for the various strike prices. The RMSEs obtained are then aggregated and the means, standard deviations, minimums and maximums are presented for the different models. The performance evaluation is presented for the various copula h-functions specifications under the different models. The findings reveal that the aggre-

Table 1
Summary Statistics of S&P 500 Index prices, log-returns, stochastic variance and integrated variance from 19-August-2015 to 19-August-2020.

	Min	Median	Mean	Max	Std. Dev	Skewness	Kurtosis
S&P 500 prices	1829	2636	2575	3390	388.63	-0.0044	2.0040
S&P 500 log-returns	-0.1277	0.0004	0.0004	0.0897	0.0121	-1.0909	25.1340
Stochastic variance	0.0000	0.0155	0.0236	0.1092	0.0234	1.1193	3.4949
Integrated variance	0.0036	0.0066	0.0089	0.0228	0.0055	1.0622	2.9067

Table 2
Estimation outcome of the different stochastic volatility processes.

Gaussian mean-rev.		Square-root diffusion		Lognormal process	
k	2.3799	α	1.2089	μ	0.0595
θ	0.0245	β	0.0347	ω	0.3446
σ	0.0538	γ	0.4013		

Table 3
Estimated empirical Joe copula parameter φ as well as other fitted copula parameters with standard errors in parentheses.

Estimated empirical copula parameter	
Joe	4.8851 (0.17)
Other estimated copula parameters	
Gaussian	0.6546 (0.02)
Student *t*	0.7850 (0.02), 2.1011 (0.19)
Clayton	0.5813 (0.06)
Gumbel	2.5570 (0.08)
Frank	6.9302 (0.32)

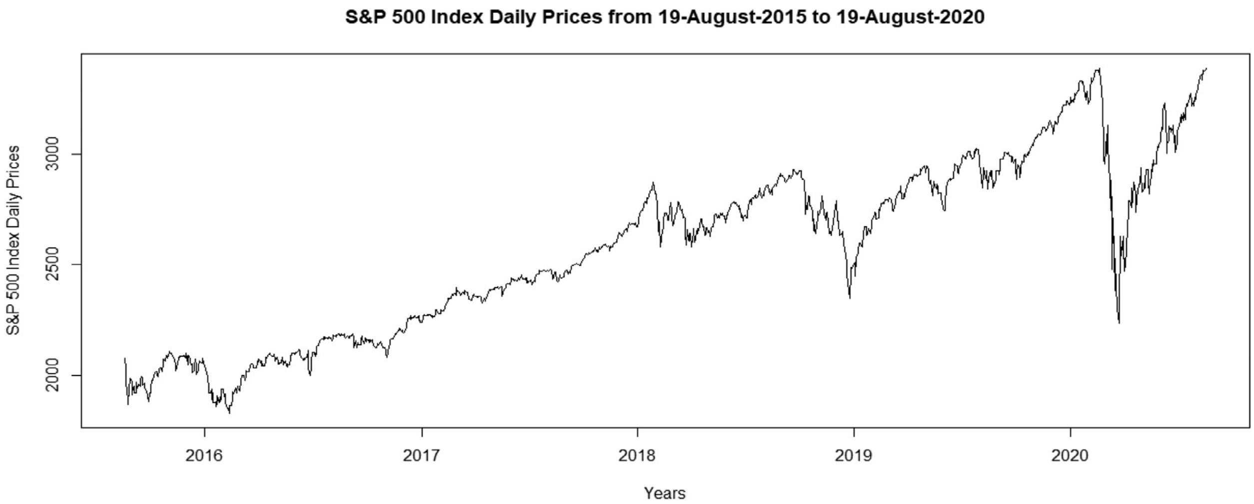

Fig. 1. Plot of S&P Index daily prices from 19 August 2015 to 19 August 2020.

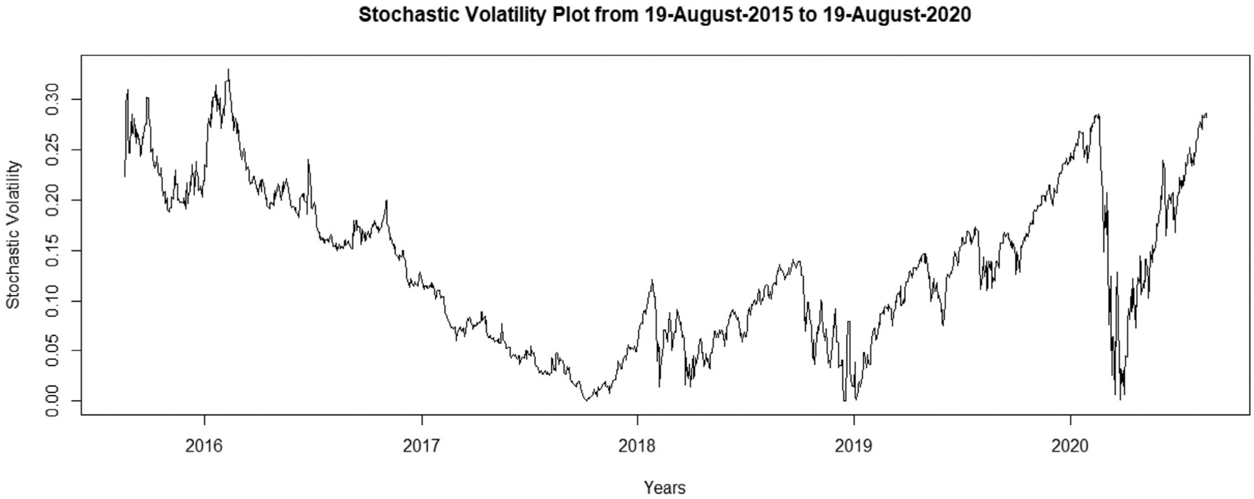

Fig. 2. S&P Index stochastic volatility plot from 19 August 2015 to 19 August 2020.

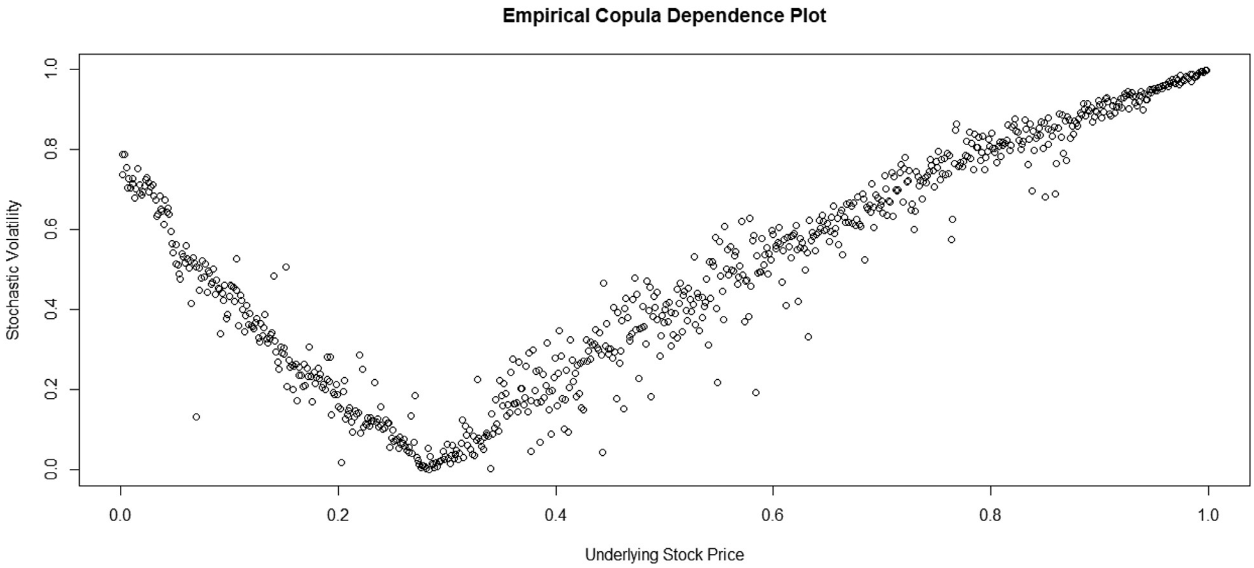

Fig. 3. Empirical Copula Dependence Plot of Stochastic Volatility and Underlying Stock Prices.

Table 4
Pricing evaluation by RMSE of dynamic option pricing models under asymmetric stochastic volatility for various copula configurations for in-the-money S&P 500 Index European call options in comparison with the Heston model. A 3-year evaluation with 700 daily option prices estimates across 25 strike price levels are used. The daily RMSEs are then aggregated across the different strikes and the mean, minimum, maximum and standard deviation are reported.

Option pricing performance evaluation under Gaussian mean-reverting process							
RMSE	Heston	Joe	Gaussian	Student *t*	Clayton	Gumbel	Frank
Mean	64.1333	44.2738	45.1845	43.5226	46.4559	43.6266	43.8352
Std. dev.	17.3127	13.7439	13.3196	14.6077	11.7463	14.4757	14.3233
Min	35.9580	25.3154	25.4379	23.5761	30.3753	24.1904	23.7262
Max	88.8123	65.3999	65.3999	65.3999	65.3999	65.3999	65.3999
Option pricing performance evaluation under square-root diffusion							
RMSE	Heston	Joe	Gaussian	Student *t*	Clayton	Gumbel	Frank
Mean	64.133	43.399	48.542	45.311	47.149	45.254	48.206
Std. dev.	17.313	14.024	13.872	15.109	9.823	15.627	15.477
Min.	35.958	24.993	30.202	24.365	33.392	23.611	29.154
Max.	88.812	83.958	91.906	95.612	65.634	98.509	104.889
Option pricing performance evaluation under lognormal process							
RMSE	Heston	Joe	Gaussian	Student *t*	Clayton	Gumbel	Frank
Mean	64.133	56.883	56.907	56.462	56.907	56.542	56.448
Std. dev.	17.313	18.798	18.818	19.440	18.800	19.316	19.472
Min	35.958	28.013	26.212	25.227	27.339	25.830	24.610
Max	88.812	82.260	82.260	82.260	82.260	82.260	82.260

Table 5
Relative percentage performance improvement in RMSE of dynamic option pricing models under various copula configurations against the Heston model across time to maturity and 55 strike price levels. The relative percentage performance improvements are obtained from comparing the RMSE values for the different models in Table 4.

Model	Joe	Gaussian	Student *t*	Clayton	Gumbel	Frank	Aggregate	Std. Dev
Gaussian mean-rev.	30.97%	29.55%	32.14%	27.56%	31.98%	31.65%	30.64%	1.62%
Square-root diffusion	32.33%	24.31%	29.35%	26.48%	29.44%	24.83%	27.79%	2.84%
Lognormal process	11.31%	11.27%	11.96%	11.27%	11.84%	11.98%	11.60%	0.33%

gate pricing errors are lower for all the dynamic option pricing models as compared to the Heston model as summarized in Table 4. The Heston model is computed using the same parameters as those obtained for the square-root diffusion model.

The standard deviation of the RMSE captures the performance variation across the various strike prices. The Heston model has the highest RMSE with the highest deviation of 17.31 which is higher than the aggregate deviations of the Gaussian mean-reverting and square-root diffusion models, respectively 13.98 and 13.70, across the copulae h-function specifications. The lognormal dynamic pricing model has a higher RMSE deviation across strike price levels at an aggregate of 19.11. The lowest variation in performance was recorded by the square-root model under the Clayton copula at 9.82. Taking into account the asymmetric dependence in stochastic volatility, a significant improvement in performance is witnessed across copula specifications in the three different models. The Heston model resulted in an aggregate RMSE across strike price levels of 64.13, whereas the lognormal, square-root and Gaussian models yielded aggregate RMSEs across strike levels and copula specifications of 56.69, 46.31 and 44.48, respectively. The relative performance improvements are compared to the Heston model and are summarized in Table 5.

Table 5 presents the relative percentage performance improvements in RMSE by comparing the dynamic option pricing models under the various copula configurations to the Heston model. A positive performance improvement indicates that the dynamic option pricing models had lower RMSE than the Heston model. The dynamic option pricing model under asymmetric Gaussian mean-reversion improves the performance in pricing of S&P 500 in-the-money European call options by an aggregate of 30.64% across copulae compared to the Heston model. Under the asymmetric square-root model, the pricing performance improvement is at an aggregate level of 27.79%, while for the asymmetric lognormal model this stands at 11.60%. The square-root model under the Joe copula achieved the highest improvement of 32.33%, whereas the lowest of 11.27% was given by the lognormal model under the Gaussian and Clayton copulae specifications.

Furthermore, copula-specific performance variation was witnessed for the three models. More specifically, the lognormal model yielded the lowest variation in performance across copulae. The square-root model had the highest performance variation from a low of 24.31% for the Gaussian copula to a high of 32.33% by the Joe copula. The square-root model, across the copula h-function specifications, was observed to have the highest performance variation at 2.84% performance variation. The pricing errors were uniform and lowest under the lognormal model across copula specifications and, as such, the lowest performance variation was observed at 0.33% across copula h-function specifications. The performance variation for the Gaussian mean-reverting model across copula specifications was recorded at 1.62%. With our dynamic conditional copula approach to pricing options under asymmetric stochastic volatility, we find that, compared to the Heston model, the performance of the square-root model under the Gaussian copula specification exhibits a significant performance improvement of 24.31%. Here similar parameters for the Heston and the square-root diffusion model were used. This demonstrates the superior tractability and performance accuracy of the proposed dynamic conditional copula approach to pricing options under asymmetric stochastic volatility over previous approaches.

Conclusion

The aim of this paper is to develop stochastic volatility models with asymmetric dependence and apply them to option pricing. Closed-form solutions for European options are presented based on the application of a dynamic conditional copula approach to capture asymmetric dependence. Evaluation of the empirical performance for a mean-reverting, square-root, and lognormal models is performed under various copula specifications.

The derived pricing RMSEs for 3-year S&P 500 Index European call options based on our constructed models suggest that these perform significantly better and consistently across maturity levels, strike price levels, and different specifications of the copula h-functions than the basic Heston model. Overall performance improvements of 30.64%, 27.79% and 11.60% were achieved across all copulae under the mean-reverting Gaussian, square-root and lognormal models, respectively. The square-root model under the Joe copula specification was found to be the best with a performance improvement of 32.33%. Across maturity and strike price levels, the square-root stochastic volatility model under the Clayton copula specification yielded the lowest variation of 9.82 compared to that of the Heston model of 17.31. The performance of the lognormal stochastic volatility model was found to be rigid to the choice of the copula function, whereas the square-root stochastic volatility and Gaussian stochastic volatility models were quite affected by the choice of the copula function.

The selection herein of the copula h-function, the stochastic volatility process and its cumulative distribution function were seen to impact the general performance of the dynamic option pricing models. The dynamism of this conditional copula approach for pricing options under asymmetric stochastic volatility allows freedom in the construction of the driving stochastic volatility model. It also allows flexibility in the choice of a copula h-function without imposing any restrictive modelling assumption on the nature of dependence between the asset price and stochastic volatility processes, thus offering a wealth of model specifications of this kind. This dynamism is a modelling panacea for option pricing under stochastic volatility since it allows liberty for a variety of stochastic volatility models and various copula specifications and is practically extensible to allow jumps in the stock price and volatility processes and also allow stochastic interest rates.

The proposed closed-form dynamic option pricing approach is practical, fully analytical, simple, and accurate, hence represents a suitable alternative to classical models. It provides a suitable alternative for option pricing under stochastic volatility and a substantive framework for application of the dynamic copula approach for other option classes under varied stochastic dynamics.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Brian Wesley Muganda: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Validation, Visualization, Writing – original draft, Writing – review & editing. **Ioannis Kyriakou:** Supervision, Writing – original draft, Writing – review & editing. **Bernard Shibwabo Kasamani:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Methodology, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Acknowledgements

The first author gratefully acknowledges the financial support from the German Academic Exchange Service (DAAD) in the In-Country Doctoral Scholarship Programme. This support has immensely facilitated the production of this work.

References

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ. 81 (1973) 637–654.
[2] C. Cao, S. Bakshi, Z. Chen, Option Pricing and Hedging Performance Under Stochastic Volatility and Stochastic Interest Rates, Handbook of Quantitative Finance, C. F. Lee, J. Lee, & A. C. Lee (Eds.), Springer, 2010.
[3] R. Cont, Empirical properties of asset returns: sytlized facts and statistical issues, Q. Finance 1 (2) (2001) 223–236.
[4] R. Cont, J. Da Fonseca, V. Durrleman, Stochastic models of implied volatility surfaces, Econ. Notes 31 (2) (2002) 361–377.
[5] B. Dumas, J. Fleming, R. Whaley, Implied volatility function: empirical tests, J. Finance 53 (6) (1998) 2059–2106.
[6] C. Alexander, Principles of the skew, Risk 14 (1) (2001) 529–532.
[7] M. Wallmeier, R. Hafner, Dynamics of dax implied volatilities, University of Augsburg Working Paper (2000).
[8] P. Fouque, G. Papanicolaou, K.R. Sircar, Financial modelling in a fast mean-reverting stochastic volatility environment, Asia-Pacific Financ. Market. 6 (1999) 37–48.

[9] D.S. Bates, Empirical option pricing: a retrospection, University of Iowa and the National Bureau of Economic Research. (2001).
[10] M.S. Johannes, B. Eraker, N.G. Polson, The impact of jumps in volatility and returns, Chicago Booth RPS: Econ. Stat. (Topic) (2000).
[11] J. Hull, A. White, The pricing of options on assets with stochastic volatilities, J. Finance 42 (1987) 282–300.
[12] H. Johnson, D. Shanno, Option pricing when the variance is changing, J. Financ. Q. Anal. (1987) 143–151.
[13] L.O. Scott, Option pricing when the variance changes randomly: theory, estimation, and application, J. Financ. Quant. Anal. 22 (1987) 419–438.
[14] J.B. Wiggins, Option values under stochastic volatilities, J. Financ. Econ. 19 (1987) 394–419.
[15] E. Stein, J. Stein, Stock price distribution with stochastic volatility: an analytic approach, Rev. Financ. Stud. 4 (1991) 727–752.
[16] S.L. Heston, A closed-form solution for option with stochastic volatility with application to bond and currency options, Rev. Financ. Stud. 6 (2) (1993) 327–343.
[17] C.A. Ball, A. Roma, Stochastic volatility option pricing, J. Financ. Quant. Anal. 29 (4) (1994) 589–607.
[18] D.S. Bates, Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark option, Rev. Financ. Stud. 9 (1) (1996) 69–107.
[19] L.O. Scott, Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: applications of fourier inversion methods, Math. Finance 7 (4) (1997) 413–426.
[20] G.S. Bakshi, C. Cao, Z. Chen, Empirical performance of alternative option pricing models, J. Finance 52 (5) (1997) 2003–2049.
[21] S. Sabanis, Stochastic volatility and the mean reverting process, J. Futures Market. 23 (1) (2003) 33–47.
[22] R.W. Lee, Option pricing by transform methods: extensions, unification and error control, J. Comput. Finance 7 (3) (2004) 51–86.
[23] P. Carr, D.B. Madan, Option valuation using the fast fourier transform, J. Computat. Finance 2 (4) (1999) 61–73.
[24] S. Fadugba, C. Nwozo, Valuation of european call options via the fast fourier transform and the improved mellin transform, J. Math. Finance 06 (2016) 338–359.
[25] A. van Haastrecht, R. Lord, A. Pelsser, D. Schrager, Pricing long-dated insurance contracts with stochastic interest rates and stochastic volatility, Insurance Math. Econ. 45 (2009) 436–448.
[26] R.K. Thulasiram, P. Thulasiraman, Performance evaluation of a multithreaded fast fourier transform algorithm for derivative pricing, J. Supercomput. 26 (2003) 43–58.
[27] A.J. McNeil, R. Frey, P. Embrechts, Quantitative Risk Management: Concepts, Techniques and Tools, Businesspro Collection, Princeton University Press, 2005.
[28] C. Ning, D. Xu, T. Wirjanto, Modeling the leverage effect with copulas and realized volatility, Finance Res. Lett. 5 (4) (2008) 221–227.
[29] L. Huang, S. Huang, Option pricing under copula-based asymmetric dynamic leverage effects, J. Inf. Optim. Sci. 31 (5) (2010) 1041–1059.
[30] A. Veraart, L.A.M. Veraart, Stochastic volatility and stochastic leverage, Annal. Finance 8 (2012) 205–233.
[31] X. Lu, G. Meissner, H. Sherwin, A unified stochastic volatilitystochastic correlation model, J. Math. Finance 10 (2020) 679–696.
[32] M. Romano, N. Touzi, Contingent claims and market completeness in a stochastic volatility model, Math. Finance 7 (1997) 399–412.
[33] A. Patton, Estimation of Copula Models for Time Series of Possibly Different Lengths, Discussion Papers (University of California, San Diego), volume 17, University of California, San Diego, San Diego, 2001.
[34] A. Prayoga, N. Privault, Pricing cir yield options by conditional moment matching, Asia Pacific Financ. Market. 24 (2017) 19–38.
[35] E. Levy, Pricing european average rate currency options, J. Int. Money Finance 14 (1992) 474–491.
[36] S. Turnbull, L. Wakeman, A quick algorithm for pricing european average options, J. Financ. Quant. Anal. 26 (1992) 377–389.