Copulae and Correlation Products

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Abbreviations

ARCH	=	Autoregressive Conditional Heteroskedasticity
CDO	=	Collateralised Debt Obligation
CDS	=	Credit Default Swap
d.f.	=	distribution function
d.o.f.	=	degrees of freedom
EDS	=	Equity Default Swap
EGARCH	=	Exponential GARCH
EMS	=	Empirical Martingale Simulation
GARCH	=	Generalised ARCH
i.i.d.	=	independent identically distributed
NGARCH	=	nonlinear GARCH
NGARCH-M	=	nonlinear GARCH in mean
OECD	=	Organisation of Economic and Cooperation Development
PDE	=	Partial Differential Equation
p.d.f.	=	probability distribution function
R.T.	=	Rating Transition
r.v.	=	random variable
s.d.e.	=	stochastic differential equation

Symbols

$1_{\{E\}}$ = indicator function of the event E

τ = *r.v.* time of default

$\mathbb{Q}$ = risk-neutral probability measure

$\mathbb{P}$ = actual probability measure

$\mathrm{E}^{\mathbb{Q}}(\cdot)$ = expectation under $\mathbb{Q}$

$\mathrm{E}^{\mathbb{P}}(\cdot)$ = expectation under $\mathbb{P}$

$\mathbb{Q}(\tau \leq t)$ = risk-neutral probability of the event $\tau \leq t$

$\mathbb{P}(\tau \leq t)$ = actual probability of the event $\tau \leq t$

$(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{G}, \mathbb{P})$ = reference filtered probability spaces

$\mathcal{H}$ = filtration that carries full information about default events

$\mathcal{F}$ = filtration that carries all information regarding other financial and economic processes

$F(\cdot)$ = distribution function

$S(\cdot)$ = survival function

$F_i(\cdot)$ = distribution function of credit (or obligor) i

$f_i(\cdot)$ = probability distribution function of credit (or obligor) i

$F^{-1}(\cdot)$ = quantile function, pseudo-inverse or generalised inverse of $F(\cdot)$

$F_h(\cdot)$ = distribution function of a credit (or obligor) in the rating class h

$\mathbb{N}$ = the set of natural numbers

$\Re$ = $(-\infty,+\infty)$ the real line

$\Re^n$ = $\underbrace{(-\infty,+\infty)\times(-\infty,+\infty)\times\cdots\times(-\infty,+\infty)}_{n \quad times}$

$\mathrm{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{n,t})^T$ n-dimensional (column) vector

C = copula

c = copula-density

C^- = lower Fréchet bound

C^+ = upper Fréchet bound (copula for $n = 2$)

$C^\perp$ = product copula

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Declaration

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Abstract

This thesis studies copula applications to correlation products. There are six self-contained but related projects in this research, with the following objectives: 1) to review reduced-form approaches to model the default process of a single-name obligor and their extension to model the joint distribution of defaults in a portfolio of obligors; 2) to set up the CDO market; 3) to introduce copulae and to provide a justification of copulae as modelling tools; 4) to provide with our view regarding the most suitable copula when modelling complex correlation products; 5) to prepare a time-inhomogeneous intensity model for valuing cash-flow CDOs, which explicitly incorporates the credit rating of the firms in the collateral portfolio as the indicator of the likelihood of default; 6) to prepare a pricing model for CDOs of EDSs.

We found strong evidence that the Clayton copula is a suitable tool when modelling correlation products with Li's Survival model. The Clayton copula had some important consequences: we noticed a redistribution of losses from the Junior note to the Mezzanine and Senior notes; in addition, it picked up some extra risk in the Senior note, and finally, when compared with the Normal copula, it overestimated the fair compensation of the Senior and the Mezzanine notes and underestimated the fair compensation of the Junior note. We also found the Clayton copula was very adaptable into the dynamic copula framework of Schönbucher and Schubert.

Modelling the notes of cash-flow CDOs with copulae and time-inhomogeneous transition matrices has not been an easy task. This is because the computation of the transition matrices for arbitrary periods of time was based on an annual transition matrix. In addition, this matrix, as most of the empirical annual transition matrices, was not compatible with a continuous Markov process since it did not admit a valid generator. Therefore, we computed a modified version of a true generator. Following this, we successfully applied one method, originally advanced by JLT (1997), to calibrate the *adjusted* matrix to the S&P's probabilities of default. Finally, we described how to simulate the credit rating migration of one single credit, and how to join n -credit rating migrations *via* the Normal copula. Modelling the collateral credit risk in this way is very powerful, since it allowed us to take into account quality trigger linked to the rating-performance of the collateral and to keep the model of the joint credit rating migrations, totally separate with copulae. For example, when there are performance triggers linked to the collateral average rating, our Rating Transition Copula model perfectly captures the diversion of cash from the interest waterfall to the principal waterfall for the benefit of the Senior and Mezzanine notes.

To price single-name Equity Default Swaps and CDOs of Equity Default Swaps, we extended the GARCH option framework of Duan (1995). Volatility of the underlying equity price is the critical factor affecting option prices, and in our EDS model, the variance of the equity return followed a nonlinear GARCH in mean. When pricing single-name EDS, we proposed two nonlinear GARCH in mean (NGARCH-M): normal and t-Student NGARCH-M model. As a benchmark, we assumed that the equity returns moved accordingly to the standard homoskedastic lognormal process of Black & Scholes and priced the single-name EDS with the Rubinstein and Reiner model for binary barrier options. The problem we found with this approach was that the implied volatilities for very deep out-of-the-money put options were not available. When the volatility was modelled as a GARCH process, it was not possible to derive the future distribution of the underlying equity. Therefore, our model relied on Monte Carlo simulations. To ensure that the simulated option price did not violate rational option pricing bounds, we used the empirical martingale simulation originally advanced by Duan and Simonato, coupled with the standard variance reduction technique. To address the issue of how to price a basket of EDSs, we resorted to the concept of copula. With copulae, we were able to decouple the pricing problem: keeping the aspect of modelling the marginal distribution of the equity returns *via* NGARCH-M, totally separate from addressing the dependence problem.

Chapter 1: Credit risk literature review

1.1 Background of the study

1.1.1 Before copulae

In recent years, banks and other financial institutions have completely changed the way they see credit risk. Traditionally, a loan would remain on the bank balance sheet until it matured or it defaulted. There was little scope to efficiently price a loan, as there was no efficient way to transfer the credit risk to another party *via* capital markets. Whether or not a loan was mis-priced was not possible to determine with certainty, as there was not a transparent market where credit risk could be traded.

With the introduction of credit derivatives, all this has changed. Credit derivatives allow banks to *synthetically* lay-off loans from their balance sheets, while keeping the business relationship with the obligor. This process is called *securitisation.*

Using credit derivatives, banks were initially allowed to exploit loopholes in the regulatory discipline of Basel I. The new Basel II tries to correct and remove some of the arbitrages between banking businesses that Basel I brought. It is still too soon to foresee the effect that, the new regulation will have on the banking businesses. However, as *advanced* banks will be allowed to use their internal credit risk models to measure their economic capital, we can reasonably expect a boost towards an active credit risk management.

More importantly, credit derivatives allow banks and other financial institutions to buy and sell credit risk. Schönbucher (2003) says *credit is now a traded asset.* However, the trading credit risk market is still far from becoming an efficient location to price credit risk in all its forms. We believe, Schönbucher's observation is correct for some credit products, such as credit default swaps (CDSs), but incorrect for correlation products, such as CDOs, as they do not enjoy the same level of liquidity.

Although intensity based models are now widespread and commonly used to price contingent claims dependent on the default process of one single obligor, such as CDSs, literature is still evolving with regard to correlation products. The most obvious way of adding correlation is by imposing correlation on the default intensities of the obligors. However, extensive research proves that, if the default dynamics are modelled as a pure diffusion process, the order of the default correlation that can be reached is just too small (Schönbucher (2003), Jouanin *et al.* (2001) and Rogge and Schönbucher (2003)).

There have been attempts to correct the low-correlation problem: *joint jumps* in the default intensities and *joint defaults*. The possibility of joint jumps in the default intensities allows reaching any degree of default correlation. A good example is Duffie and Garleanu (1999) who modelled the obligors default intensities as affine jump-diffusion processes. The issue with this model is the poor analytical tractability, particularly when the model needs to be calibrated to actual CDS spreads. Moreover, it is also difficult to analyse the joint loss distribution implied to this model.

Alternatively, joint defaults models cause joint defaults of several obligors at the same time. Examples of this approach are Duffie and Singleton (1998), Davis and Lo (2000) and Jarrow and Yu (2001). The former model becomes very impractical as the number of obligors in the reference portfolio grows. The basic idea behind the two other models is the phenomenon called *default contagion*, that is, the default of one obligor causes a jump in the default intensities of other obligors. This phenomenon is frequently observed in the market: for example, the default of Enron and the financial crises in the Asian markets in the late 1990's. The problem of these models is how to achieve a calibration. With Jarrow and Yu model, we also face the problem of selecting those *primary* obligors whose default status triggers the default of *secondary* obligors.

To date, when adapting structural models to capture default dependence, financial literature has relied on a natural extension of, either Merton type models, or Black and Cox type models. Two examples are KMV (1998) and CreditMetrics (Gupton, Finger and Bhatia (1997)). Yet, these models cannot be adapted to price modern financial products such as correlation products. This is because they are essentially static models and cannot capture timing risk of default (Rogge and Schönbucher (2003)), which is very important when managing CDOs. They also cannot be calibrated to CDS spreads and thus, cannot be used for hedging purposes. Finally, they assume that the joint loss distribution of a portfolio of obligors is Gaussian, and therefore, fail to capture key modern features of risk management, such as tail dependence, which is very important in pricing senior notes.

1.1.2 Correlation products: the CDOs market

CDOs are typically diversified portfolios of debt collateral, which include corporate debts, commercial and residential mortgages, equities and other asset-backed securities. They are offered in response to banks, insurance companies and fund managers who wish to efficiently redistribute and transfer or speculate on credit risk.

In a typical CDO structure, the originator, also called the sponsor, sells a portfolio of debt instruments, or transfers their credit risk, to a special purpose vehicle (SPV) and takes an equity interest. The SPV funds the purchase of the portfolio through issuing prioritised notes: senior, mezzanine and junior notes. Senior notes are usually purchased by relatively risk-averse investors and are repaid before the mezzanine and the junior notes. Any remaining cash is paid to the equity holders. CDOs are usually distinguished as cash-flow CDOs and synthetic CDOs (or synthetics).

In general, cash-flow CDOs are difficult to compare, as they provide large flexibility in arranging deals, ranging from subordination, quality of the collateral, trigger tests, diversity, spread trapping mechanisms and turbo repayment features. This is why simple models cannot cope with the huge complexity of CDOs.

Synthetic CDOs on the other hand, just as n^{th} to default baskets, do not contain the same degree of structural protection available to the note holders. For these deals, the main element to consider is how the loss materialises while the deal ages. Hence, correctly modelling dependence within the reference portfolio is paramount, so that mis-pricing a deal through unrealistic assumptions on the dependence structure can be avoided.

In practice, a CDO can be seen as a portfolio of credit default swaps (CDSs) and to manage these products, an accurate understanding of the market price dynamics of CDSs is needed, as well as the ability to capture their default dependency in a realistic manner.

More recently, financial markets have witnessed the development of standardised synthetic CDOs, such as CDX.NA.IG (125) and CDX.NA.HY (100). They are essentially buying and selling notes of synthetic corporate CDOs, where each note has a different sensitivity to correlation within the reference corporate portfolio. A way of looking at this new CDO is as options on portfolio losses (McGinty *et al.* (2004)) with a payoff dependent on default time correlation. In particular, when we look at the equity piece, its premium is insensitive to default time correlation. This is because, when increasing default time correlation, the defaults cluster together, but only few defaults are needed to trigger a principal loss. On the contrary, senior notes are in essence deep-out-the money options, as their payoff depends upon the risk that losses will cluster in the future, which is seen by many as a very unlikely event.

The effect of this latest innovation in the CDO market is that correlation is now a tradable asset, and hence it is market-priced.

The complex nature of CDOs means that rating agencies play a key role as gatekeepers in this market (Perraudin (2004)).

1.1.3 What are copulae?

Copulae describe the interdependence between various random variables. They are a powerful and flexible way to describe dependence between risk components. Using copulae enables to keep the model of the marginal distribution of variables, independent of the model of the dependence between the same variables.

Copulae also allow dropping the assumption of joint normality and highlighting the pitfalls behind the use of linear correlation. They have become an unavoidable modelling tool, for valuing securitised products, such as CDOs and Asset Backed Securities in general, where modelling co-movements between non-normal variables may be very relevant.

The concept of copula was first introduced by Sklar (1959) and studied by many other authors such as Nelsen (1999) and Joe (1997). They have been first applied to multidimensional problems related to credit risk. They are currently finding their way as modelling tool in the equity options area, to confirm their flexibility. In this thesis, we will use copulae to price baskets of CDSs and of equity default swaps as well.

1.1.4 Modelling correlation products with copulae

A classical assumption used for modelling credit risk in portfolios of obligors, is that the joint distribution between the given individual risks or marginal distributions, can be represented with the Normal distribution. When a copula joins the marginal distributions *via* the Normal distribution, we talk of Normal copula. However, not all individual risks, when pooled together to form a portfolio, are likely to behave in a Normal distribution fashion. Moreover, when the joint Normal distribution is assumed, we fail to catch important joint behaviours in the tails of the distribution, with some relevant consequences on the prices of correlation products.

With copulae, we are given plenty of their families. As well as the Normal copula, there are also the *t*-Student copula, the Clayton copula, the Gumbel copula, and many others. Consequently, copulae are flexible tools to produce richer dependences between marginal distributions, than the one produced by the Normal copula.

The Normal copula is often considered the natural assumption, especially if we consider that the credit market lacks of data on the dependence, to motivate the use of an alternative copulae. Yet the reason in practise financial institutions select the Normal copula is motivated by its easy tractability, especially under Monte Carlo simulations. For this reason, the Normal copula has become the standard tool to price standardised synthetic CDOs and baskets of CDSs today. It is too often neglected that by assuming a different copula, the price of these products would inevitably change.

It is standard approach to use the Normal copula to price correlation products in a fashion, which is consistent with a set of initial term structures of intensity rates. It is further assumed that those intensities do not change at future dates. To appreciate all consequences that the choice of a copula has on future survival rates, it is necessary to analyse the copula dynamics. For example, what happens to the credit spreads and to the premiums of CDO notes when few credits in the *reference* portfolio default. Hence, addressing the correct copula dynamics is clearly very important for hedging.

1.1.5 Copulae, rating transition matrices and cash-flow CDOs

Cash-flow CDOs are characterised by several waterfall triggers linked to different performance statistics of the collateral. When they are triggered, they divert cash due to pay the interests of the junior notes to accelerate the amortisation of the senior notes. One of the most frequent waterfall trigger is the one linked to the credit ratings of the firms in the collateral portfolio.

Credit rating transitions have been modelled in the past as a finite Markov chain, which assumes that, the credit rating changes from one rating to another at given time intervals with a certain probability. Due to the Markov property, the probabilities of future credit rating only depend on the current rating. Jarrow, Lando and Turnbull (1997) were the first to develop a Rating Transition (RT) model, where the credit rating classes evolve according to time-homogeneous and time-inhomogeneous Markov process.

Transition matrices can be obtained by the three main rating agencies, Moody's, Fitch and S&P. However, transition matrices published by rating agencies are not suited for valuing default-risky financial instruments, since they are typically available only for annual frequencies, with the shortest period being one year. Many financial instruments have maturities shorter than a year, and thus require transition matrices over arbitrary time horizons.

The approach that obtains transition matrices of any arbitrary time horizons involves embedding the discrete-time Markov chain into a continuous-time Markov process (Kingman (1962)). Solving the embedding problem essentially means to find a generator matrix (Kreinin and Sidelnikova (2001)). The extra complexity in the valuation of a default-risky financial instrument with a rating agency transition matrix is the fact that the rating agency transition matrix is not compatible with a continuous-time Markov process.

Copulae are the perfect tool to address the issue of modelling migration dependence. To date, copula research has been focused on modelling joint defaults and it has lacked of the same interest towards modelling joint rating migrations, with some few exceptions such as Hamilton, James and

Webber (2001). However, because under the new banking regulation (BIS (2004)), capital requirements are driven in part by rating migrations, we foresee an increase of interest in combining copulae with the rating migration models.

There is also lack of copula application on modelling the actual legal structure of cash-flow CDOs. The large part of current research is on synthetic CDOs, where calculating the joint loss distribution of the reference portfolio is sufficient to infer the loss distribution of the CDO notes. Cash-flow CDOs represent a substantial portion of the CDO universe, hence addressing the correct modelling approach is clearly very important.

1.1.6 Copulae, NGARCH-M and equity default swaps

Equity default swaps (EDSs) are similar to CDSs as the protection buyer makes regular payments and receives a payment from the protection seller should a trigger event happen. The difference lies in how the occurrence of the trigger event is determined. In the CDS the trigger event occurs when the reference entity defaults, whereas, in the EDS the trigger event is defined as the drop in the equity price of the reference equity below a specified percentage of the equity price, called the trigger value, at the beginning of the trade.

While any combination of the trigger value and the recovery rate can be considered, the current EDS market seems to be in favour of a trigger value of 30% and a recovery rate of 50% of the notional amount. In this respect, EDSs are similar to path-dependent deep-out-of-the money equity digital options. Unlike digital options where the premium is paid up front, the premium of the EDS is spread until it matures.

EDSs have found their way onto the CDO market. They are structured as single-tranche CDOs and are privately rated by Moody's, S&P or Fitch. They are different from baskets of deep-out-of-the money equity digital options because the others have never been rated by a rating agency. The rating is the prerequisite to bring new CDOs of EDSs to the credit investors and in our view is the key factor for a much larger market for EDSs. Moody's (2004), S&P (2004) and Fitch (2004) consider the depth of historic data available on the equity markets as the main advantage for the rating process, and look at the past volatility of the equity returns as the main variable for the rating.

It seems natural that the large amount of work researchers have put into modelling the volatility of equity and index returns should also be very relevant for modelling the prices of EDSs, since the default leg and the premium leg of an EDS are linked to the performance of the reference equity or index. It is well documented in the literature that the equity returns possess properties such as fat-tails, volatility clustering and time-varying variances. Following the path-breaking paper by Engle (1982), a new literature has focused on autoregressive conditional heteroskedasticity (ARCH) models. Bollerslev (1986) and Taylor (1986) proposed the generalised ARCH (GARCH), Nelson

(1991) the exponential GARCH (EGARCH), Engle and Ng (1993) the nonlinear GARCH (NGARCH).

With reference to the application of GARCH in the option pricing area, Duan (1995) developed a risk-neutral model within the GARCH framework. He characterised the transition between the actual and the risk-neutral probability distributions if the dynamic of the underlying equity price is given by a GARCH process, and thus established the foundation for the valuation of options under GARCH. His model is also called nonlinear GARCH in mean (NGARCH-M).

Stochastic volatility models are alternative methods to GARCH to model the time-varying nature of the equity return volatility. It is in the estimation exercise that GARCH models have a distinct advantage over stochastic volatility models. Because the volatility of the equity prices is not easily observable, rather it has to be *implied* from current option prices, the implementation of a stochastic volatility model is a very difficult task. In contrast, GARCH models have the advantage that the volatility is observable from the history of equity prices, without requiring any information on option prices. Heston and Nandi (2000) emphasised that with GARCH models only a finite number of parameters need to be estimated regardless of the length of the time series. Option pricing applications under GARCH also benefit from the fact that they are relatively easy to re-estimate. Whereas, for stochastic volatility models, it may not be realistic to repeat the estimation procedure on a rolling basis, as new option prices become available.

Copulae are the natural tools to model dependence within a CDO of EDSs. Brownian motion frameworks have been used to model multivariate option prices for a very long time. In the past, the dependence among equity returns has been represented by a multivariate normal distribution, where correlation has been the measure of dependence. More recently, Cherubini and Luciano (2002) addressed the issues of non-normal returns, and the dependence in the multivariate contingent pricing problem, was addressed using copulae. Van den Goorbergh and Genest (2004) used a dynamic copula model to price multivariate options where the dependence structure between two asset prices is time-varying over time and expressed with copulae. None of these authors linked copulae to a NGARCH-M pricing model.

1.1.3 The problem statement

This thesis investigates copula applications on correlation products. In order to present the results in a meaningful and manageable manner, six self-contained but interrelated chapters are included in this thesis. In this section, we will state the objectives for each of them separately.

1.2.1 Objective of chapter 1

Chapter 1, entitled "Credit risk literature review", is a review of current structural and reduced-form approaches to model the default process in the case of a single obligor, and of literature efforts to extend reduced-form approaches to model the joint distribution of default times in a portfolio of obligors. The chapter is heavily weighted towards reduced-form models, as they are the most popular class of credit derivative pricing models, they are easily calibrated to market prices and they provide realistic dynamics of default-risky bonds. We also investigate how to implement them under a variety of different approaches: Cox *et al.* (1985), Heath *et al.* (1992), numerical trees and PDEs based. Before closing the chapter, we will cover attempts to extend them to the multi-obligors case. Our objective is to help navigate in the reduced-form modelling literature, before moving to copula-base models, where reduced-form models have found new applications.

This chapter also establishes the background of the research, highlights the problems, explains the significance and outlines the structure of the thesis.

1.2.2 Objectives of chapter 2

Chapter 2, entitled "A survey of CDOs and their use in bank balance sheet management", aims to describe the nature of typical CDOs in detail, and to explain different categories of transactions and features in the CDO market. In particular, we show why it is important to distinguish between cash-flow and synthetic CDOs. The distinction will motivate using two different pricing models for the two categories and are prepared in chapter 4 and 5.

The second objective is to show the rationale behind different transactions. Thirdly, this chapter helps understanding the universe of buyers in the CDO market: who should consider investing in CDOs, and who should not do so. Fourthly, as rating agencies have acquired, in the structured products market, the important role of ensuring that the transaction is constantly monitored, we review one of the rating agencies cash-flow CDO model with which the CDO notes are rated.

1.2.3 Objectives of chapter 3

Chapter 3, entitled "Copulae", has as objective to introduce copulae and to provide a justification of copulae as modelling tools. By the end of this chapter, it will be become apparent why linear default correlation is not a canonical measure of dependence between random variables, whereas copulae, along with Kendall's tau, Spearman' correlation and tail dependence, provide a much more flexible way to represent market co-movements and dependence.

1.2.4 Objectives of chapter 4

The objective of this chapter, with title "Modelling correlation products with copulae", is to provide with our view regarding the most suitable copula when modelling complex correlation products, such as n^{th} to default. To do so, we review the solutions proposed in the financial literature to the problem of extending intensity-based models to the multivariate case *via* several different copulae. By modelling different real-life transactions, we want to draw conclusions regarding how the prices of senior, mezzanine and junior notes would change when different copulae are used in place of the Normal copula. Furthermore, we want to measure how their prices would change when modelling with dynamic copulae.

1.2.5 Objectives of chapter 5

The objective of chapter 5, entitled "Structuring and rating cash-flow CDOs with rating transition matrices" is to prepare a time-inhomogeneous intensity model for valuing cash-flow CDOs, which explicitly incorporates the credit rating of the firms in the collateral portfolio as the indicator of the likelihood of default. Our model can prove very useful for the pricing, structuring, rating and risk management of CDO notes, whenever the legal structure of the transaction, includes waterfall triggers linked to the credit ratings of the firms in the collateral portfolio. If the waterfall triggers are breached, they divert cash due to pay the interests of the junior notes to accelerate the amortisation of the more senior notes. For this reason, we believe that in order to measure the risk and value of the CDO notes, it is necessary to combine a credit risk model with the exact cash-flow waterfall model of the given structure.

To reach our objective, we firstly need to model the rating transition process of each single obligor in the portfolio according to a set of time-inhomogeneous transition matrices, calibrated to historical default probabilities. Only then, to address the issue of modelling migration dependence, we rely on the concept of copula.

The algorithms developed by Lando (1998a) and Israel, Rosenthal and Wei (2001) will be very useful for preparing the transition matrices to feed our cash-flow CDO model.

1.2.6 Objectives of chapter 6

Chapter 6, entitled "Pricing and rating CDOs of equity default swaps with NGARCH-M copulae" has the objective of preparing a pricing model for CDOs of EDSs. The pricing model that we want to develop must be capable of replicating the observed equity return properties such as fat-tails, volatility clustering and time-varying variances.

To model the single-name EDS we will use a nonlinear GARCH in mean (NGARCH-M). The dependence within a CDO of EDSs is modelled *via* copulae.

1.3 The significance of the study

The first three chapters are introductory studies to develop the framework within which the research of chapter four to six will fall. In what follows, we explain the research significance of chapters 4, 5 and 6.

1.3.1 The significance of chapter 4

The first application of copulae for pricing baskets of credits is attributed to Li (1999 and 2000). He proposed the Survival copula to model the joint default dependency in the collateral portfolio of credits. Finger (2000) was the first to compare the results of the Li's Survival model with the Merton model. In his work, he joined the times of default of homogeneous credits with the Normal copula. He found that the Survival model tends to underestimate the expected loss of the junior loss note and overestimate the expected loss of the mezzanine and senior notes.

More recently, Meneguzzo and Vecchiato (2002), and Mashal, Naldi and Zeevi (2003) expanded the work of Finger and used the Li's Survival model with the Clayton copula and the *t*-Student copula respectively, to real-life credit portfolios. Meneguzzo and Vecchiato found that the Clayton copula is not useful in modelling the tail dependence of times of default. Mashal, Naldi and Zeevi concluded that the Normal copula would generally overestimate the 1st to default and underestimate the 3rd and the 2nd to default when compared with the *t*-Student copula with 12 d.o.f..

The findings of Mashal, Naldi and Zevi, and of Meneguzzo and Vecchiato puzzle us. Mashal, Naldi and Zevi empirically studied the application of the *t*-Student copula, with 12 degrees of freedom (d.o.f.) on a 1st, 2nd and 3rd to default on a five-name basket with a maturity of five years. Our view is that it is not possible to draw general conclusions regarding redistribution of losses from the 1st to default into the 2nd and the 3rd to default, only by modelling a reference portfolio of five-names. Therefore, we will expand their work and look for general rules, as the reference portfolio grows.

More importantly, Mashal, Naldi and Zevi empirical results would also suggest a redistribution of losses when Normal, *t*-Student and Clayton copulae are used with default time correlation equal to zero. In our view, when default time correlation is zero, the three copulae ought to calculate the same values since the times of default are not really joint together *via* copulae. Hence, we will investigate the role of correlation and the wider concept of dependence, when they are modelled with different copulae.

Meneguzzo and Vecchiato found that the Clayton copula does not appear useful. We have a positive expectation with regard to the Clayton copula, and want to use this chapter to explore the benefits of the Clayton copula when it used to price real-life transactions.

Once we have established the significance of the Clayton copula when modelling correlation products, we want to prove its power by showing how easy it is to extend into a dynamic copula set-up.

1.3.2 The significance of chapter 5

The significance of this chapter, is to measure the impact of modelling credit rating transitions as trigger events which, when breached, divert the distribution of cash from the junior to senior notes, so accelerating the repayment of senior notes. No previous research has so far linked rating transition matrices and copulae to the actual legal structure of a cash-flow CDO. To our knowledge, this is the first model of cash-flow CDOs, which uses a rating agency transition matrix, publicly available that we successfully calibrate using historic cumulative default rates.

A further innovative feature of our research is the application of the rating transition copula model to determine the rating of CDO notes. The rating methodology that we want to develop is based on the expected losses that the holder of a rated note would suffer when investing in cash-flow CDO notes. Based on both the marginal and the joint probability distribution of the credit rating migrations, we calculate the amount of the rated debt by comparing the expected cumulative losses of the note with the expected cumulative losses associated with that rating category.

The rating transition copula model will be numerically compared with Li's Survival model (1999 and 2000). The advantage of the Survival model is clearly the saving in the computational time required to perform the simulations. This is because the final time of default and not the full migration path until the credits mature or default is simulated. However, as we will see, with the Survival model, we will not be able to model correctly the interest and the principal waterfalls.

1.3.3 The significance of chapter 6

To date, there has been no known literature, which incorporates NGARCH-M, and more in general, GARCH, into the pricing of EDS. To illustrate our original methodology, we use two nonlinear GARCH-M processes: the nonlinear normal-GARCH-M (1,1) and the nonlinear *t*-GARCH-M (1,1). We will also borrow from Duan (1995) to explain the transition between the actual and the risk-neutral probability distributions.

Because the multi-period distribution of the nonlinear GARCH-M process in unknown, we need to recur to some numerical procedures. Heston and Naldi (2000) closed-form solutions for

European options under GARCH, are not applicable in our exercise, since the EDS default payment can be triggered before the final maturity. Ritchken and Trevor (1999) trinomial trees to price American options under GARCH, could be useful only for single-name EDSs. Since our research deals with CDO of EDSs, the numerical scheme of Ritchken and Trevor will soon become impractical as the number of EDSs in the CDO grows. Hence, our model will rely on Monte Carlo simulations.

With reference to Monte Carlo simulations Duan and Simonato (1995) observed that the simulated option price violates rational option pricing bounds and hence it is not a sensible price estimate. The simulated path of the equity price fails to possess the martingale property even though the theoretical model does. As a solution, Duan and Simonato proposed a correction to the standard Monte Carlo technique which ensured that the simulated paths of the equity price are "empirical" martingales. This correction was called the empirical martingale simulation (EMS). Our EDS model uses Duan and Simonato EMS-Monte Carlo model, coupled with the standard variance reduction technique.

In our model, the dependence within a CDO of EDSs is modelled through copulae. Our contribution is on measuring the impact of different copula functions on the price of the same CDO of EDSs. We propose to use the following copulae with the EMS-NGARCH-M model: Normal, *t*-Student and Clayton.

A further innovative feature of our research is to use our model to calculate the rating of three CDO of EDS notes. The rating methodology that we want to develop is based on the expected losses that the holder of a rated note would suffer when investing in a product whose reference portfolio is made of EDSs. In a single-name EDS, the probability of triggering the seller payment is driven mainly by the time-varying volatility of the underlying equity price, exactly as for a deep-out-the money equity digital option the probability of expiring in the money depends on the same time-varying volatility. In a portfolio of EDSs, where the seller payment is triggered by any of the prices of the equities in the reference portfolio hitting the trigger value, the dependence between pairs of equity prices is another important component to analyse. Based on both the marginal and the joint probability distribution of the equity prices, we propose to calculate the amount of the rated debt as it is generally done for CDO notes, and compare the expected cumulative losses of the note with the expected cumulative losses associated with that rating category.

1.4 The organisation of the study

The structure of this thesis is as follows. Chapter 1 contains the literature review of credit risk. Chapter 2 investigates the market of CDOs. Chapter 3 introduces the concept of copulae. Chapter 4 examines the application of different copulae to value real-life correlation products. Chapter 5 develops a rating-transition-copula model for cash-flow CDOs. Chapter 6 explores nonlinear

GARCH in mean and copulae to price EDS and CDOs of EDSs. Chapter 7 summarises and discusses the results, and suggests direction for future research.

Throughout this study, we will first present models and then apply them to real-life transactions. All the outputs of the numerical exercises will be given with tables and figures. The complete list of these tables and figures is in the Appendices.

1.5 Literature review

In the literature of modelling the default event, a variety of alternative approaches has been explored to characterise the pricing equation of a default-risky bond. Using the same nomenclature adopted in all financial literature on default, two distinct approaches have evolved: the *structural* approach and the *reduced-form* approach. While in the structural approach the value of the firm is used to model the time of default, in the reduced-form approach the intensity of default is modelled as a jump-process, and the time of default is expressed as the first jump of the intensity.

Structural models try to provide the link between the credit quality of a firm and its financial strength. Hence, the time of default is endogenously generated within the model. Alternatively, in reduced-form models, the time of default is the first jump of an exogenously given jump process.

Another important difference between the two approaches is on the treatment of the recovery, when default happens. Reduced-form models exogenously specify recovery rates, whereas in structural models, the recovery value depends upon the firm's asset and liability values, at the time of default.

One of the new areas in financial research is modelling the behaviour of *default dependency*, that is, the manner in which the joint likelihood of default of various obligors changes over time. An understanding of this phenomenon is obviously of great importance for those who want to trade or invest in correlation products, such as CDOs and n^{th} to defaults.

Both structural and reduced-form approaches can be adapted to incorporate dependency among several obligors.

In the structural approach, default dependency between several obligors is easily introduced through asset correlation. This is the approach taken by KMV (1998) and CreditMetrics (Gupton, Finger and Bhatia (1997)). However, since defaults remain predictable, jumps in credit spreads cannot appear at all, hence these models cannot be used for pricing purposes. Moreover, they are essentially static models, as they model only default risk over a fixed time horizon, and do not capture timing risk of default. Last critique to KMV and CreditMetrics is that the joint defaults are only modelled in a Gaussian framework, hence it is not possible to model extreme events. In spite of these limitations, KMV and CreditMetrics provide important insights into the relationship between the firm's fundamentals and correlated default events.

The first and most obvious way of adding correlation with reduced-form models, is to introduce correlation between the default intensities of the obligors.[1] If this is done using only diffusion-based dynamics for the default intensities, the default correlation that can be reached is of the same order of magnitude as of the default probabilities.[2] Thus, for highly good credit quality

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[1] In this thesis, we will use the terms of credit and obligor as synonymous.

[2] In Schönbucher (2003) p. 317.

portfolios, the degree of default correlation is just too low.[3] This would not be acceptable for modelling dependence in a default-risky portfolio of highly dependent obligors, for example in the same industry, and with very low probabilities of default.

There are essentially two ways out of low default correlation with reduced-form models: *joint jumps* in the default intensities and *joint defaults*.

The possibility of *joint jumps* in the default intensities allows reaching high default correlation, in principle even perfect one, by letting all intensities jump to infinity at the same time. A good example is Duffie and Garleanu (1999). To impose a higher degree of default correlation, but still not enough, they let intensity experience correlated jumps and assumed an *affine* dependence on a set of state variables. The issue with this model is the poor analytical tractability and difficult calibration to market prices.

One of the first examples of *joint default* models is Duffie and Singleton (1998), where defaults are triggered by idiosyncratic as well as other regional, industry, or wide-economic shocks. The typical common shock, which may trigger joint defaults, is the natural catastrophe of an earthquake, or the liquidity systemic crisis in the banking sector. The arrival time of the default event of each single obligor is a homogeneous Poisson process with constant intensity. Thus, the single obligor time of default is exponential distributed. Likewise, the arrival times of default caused by common shocks are exponential distributed. In this way, if the industry shock arrives before the idiosyncratic shock, all obligors in the same industry would default at the same time. Other examples of *joint default* models are Kijima (2000), and Kijima and Muromachi (2000). A first critique that is moved to these models is how to achieve calibration. For a small portfolio of ten obligors, we already have 1024 possible joint default events to calibrate. A second critique relates to the non-realistic cluster of defaults that these models generate: joint defaults do not all happen at the same time. Furthermore, it should be noted that this approach find an easy and direct application into the Marshall and Olkin copula.

A second way of looking at *joint default* is *via* the credit phenomenon of *contagion*,[4] which is the phenomenon of joint sudden and large jumps in credit spreads across several obligors (Davis and Lo (2001), Jarrow and Yu (2001) and Giesecke and Weber (2002 and 2003)). For example, after the default of Enron, several utility companies in the US experienced large upward changes in their credit spreads. Giesecke (2002) refers to the contagion phenomenon as "the market is *re-assessing* the default probabilities of the companies" that experience sudden change in their credit spread. The idea behind these models is that the default intensity of an obligor would jump upwards if another related obligor defaults. Essentially, every obligor's default intensity depends on every other obligor's survival, and thus also on the other obligors default intensities, which in turn depend on the

[3] For a proof, see Jouanin, Rapuch, Riboulet and Roncalli (2001) p. 5.
[4] So defined by Allen and Gale (2000).

first obligor's survival.[5] This is a sort of loop dependency very difficult to model. Jarrow and Yu (2001) simplified the loop dependency, by only modelling one-way dependency: the intensity rate of a secondary firm jumps at the default event of another firm, called primary firm.

A third way to include dependence, and not only correlation, among obligors is through copulae. We will review copulae, as modelling tools, and copula-based approaches, in the following chapters of this thesis.

Our objective in the rest of this chapter is to review some of the most important single-name reduced-form models and summarise the key features of reduced-form models when default dependency is added. Hence, the structure of this chapter continues as follows: in section seven we very briefly review structural models, in section eight and nine we introduce the reduced-form approach and review some single-name reduced-form models. Finally, in section ten we review two default dependency reduced-form models: Duffie and Garleanu (1999) and Jarrow and Yu (2001). Before reviewing these models, we will introduce some probabilistic notations.

1.6 Probability framework

The economic uncertainty is modelled via specifying a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of possible states of nature, $\mathcal{F} = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$, and $\mathbb{P}$ is the actual probability measure.[6]

The model for the default-free term structure of interest rate is given by a non-negative, bounded and $\mathcal{F}_t$-adapted default-free short-rate process $r(t)$.[7]

We also assume there are no arbitrage opportunities and there exists a risk-neutral probability measure $\mathbb{Q}$, equivalent to the actual probability measure $\mathbb{P}$, under which the price of a contingent claim, which pays no coupons or dividends, becomes a $\mathcal{F}$-martingale under $\mathbb{Q}$, when it is discounted using the saving account $\beta(t)$ given by[8]

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right) \tag{1.1}$$

The price of a default-free zero-coupon bond is given by

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\frac{\beta(t)}{\beta(T)}\middle|\mathcal{F}_t\right) \tag{1.2}$$

[5] In Rogge and Schönbucher (2003) p. 5.
[6] In section 1.8, we will expand the probability space to deal with reduced-form models.
[7] In Elizalde (2003) p. 3.
[8] In Harrison and Kreps (1979), Harrison and Pliska (1981), and Bielecki, Jeanblanc and Rutkowski (2003).

We denote the time of default as τ, the indicator function of the default event as

$$1_{\{\tau \le t\}} = \begin{cases} 1 & if \quad \tau \le t \\ 0 & if \quad else \end{cases} \tag{1.3}$$

Finally, under the risk-neutral probability measure $\mathbb{Q}$, we indicate the distribution function of the time of default τ as

$$\mathrm{E}^{\mathbb{Q}}\left(1_{\{\tau \le t\}}\right) = \mathbb{Q}(\tau \le t) \qquad \text{with } t > 0 \tag{1.4}$$

The use of the actual or risk-neutral probability measure depends upon the final objective of the implementation. Generally for pricing, we will prefer the probability measure $\mathbb{Q}$, whereas, for risk management purposes, the probability measure $\mathbb{P}$ will be our choice. Under the actual probability measure $\mathbb{P}$, we indicate the distribution function of the time of default τ as

$$\mathrm{E}^{\mathbb{P}}\left(1_{\{\tau \le t\}}\right) = \mathbb{P}(\tau \le t) \qquad \text{with } t > 0 \tag{1.5}$$

1.7 Structural models

The structural approach tries to establish a direct link between some economic fundamentals, such as the capital structure of a firm, and the probability of downgrade or default.

It is the oldest of the pricing models of default-risky securities. It was first proposed by Black and Scholes (1973) and subsequently expanded by Merton (1974) and Black and Cox (1976).

This type of modelling is in theory solid, with some important advantages.

(1) The link to an economic context that underlies the event of default and the clear definition of the default itself.

(2) An easy option price framework.

(3) A stochastic process of default linked to the firm's asset value.

There are also some important issues.

(1) The assumption that the firm's asset value can be observed is not realistic, since in practise, continuous-time observations of the firm's asset value are not available.

(2) The assumption that the firm's asset is a tradable security is also incorrect.

(3) It is difficult to estimate the volatility of the firm's asset value.

(4) Since the firm's asset value is a quantity difficult to estimate, remains unclear how to infer the model parameters from market data.

(5) The Merton's original expression is easy to implement, however, it leads to unrealistic credit spreads that are not observed in the credit markets. Credit spreads for maturities going to zero are zero which means that investors do not ask for a premia, when they invest in default-risky bonds with very short maturity.

Most recent contributions have added complexity to the original proposition. Zhou (1997) allowed jumps in the process of the firm's asset value, which solved the problem of unrealistic low short-term credit spreads that are typical in all structural models.

Other models were developed to take into account the term structure of default-free interest rates and to model the recovery value in the event of default as exogenous and independent from the firm's asset value.

The capital structure of a firm is sensitive to interest rates and it changes accordingly, amongst other things. There is reason to include some correlation between the credit default and the default-free interest rates. One of the first models that tried to find a solution to the problem of credit risk in presence of stochastic interest rates was Shimko, Tejima and Van Deventer (1993). As Merton (1974) before them, they assumed that the firm can only default at the debt maturity and the default can only occur when the firm's assets are exhausted. This assumption was removed in Longstaff and Schwartz (1995), and in Saa-Requejo and Santa Clara (1997). Longstaff and Schwartz (1995) reached closed-form solutions for default-risky bond prices where the default can be triggered before the maturity of the debt (as in Black and Cox (1976)) and the firm's asset value is correlated with stochastic default-free interest rates. Saa-Requejo and Santa Clara (1997) based their proposal on both Black and Cox (1976) and Longstaff and Schwartz (1995) models, and changed the constant default barrier into a stochastic barrier specifically assimilated in the value of the liabilities of the firm. Longstaff and Schwartz (1995) and Saa-Requejo and Santa Clara (1997) are special cases of Briys and de Varenne (1997), which is the most generic of this type of models. Briys and de Varenne (1997) allowed a degree of correlation between the firm's asset value and the default-free interest rates, and in the event of default, both the probability of default and the recovery value are related to the level of the firm's asset value.

Most recently, Duffie and Lando (2001) generated realistic credit spreads by allowing imperfect information. In a market place where only incomplete information is available, investors are not able to assess how close the firm's asset value is to the default threshold. The default is now a total unpredictable event as in the reduced-form models.

The basic elements in all structural models are the firm's asset process and the barrier process. We indicate with $A(t)$ the firm's asset value and with $H(t)$ the barrier value. The time of default is expressed in terms of the time of hitting a barrier.

Definition 1.1: *The time of default τ is typically defined in terms of $A(t)$ and $H(t)$ as follows*

$$\tau = \inf\{t > 0 : t \in [0,T], A(t) \le H(t)\} \tag{1.6}$$

Typically, τ is an $\mathcal{F}$-stopping time, and since the underlying filtration $\mathcal{F}$ in most structural models, is generated by a standard Brownian motion, τ is a $\mathcal{F}$-predictable stopping time. This means that with a structural model, there is a sequence of stopping times, which announces the time of default. In this sense, the default time is predictable.

In what follows we will only review Merton and Black and Cox model. For a review of Longstaff and Schwartz, Saa-Requejo and Santa Clara, Briys and de Varenne, Zhou models and more in general on structural models, we refer to Cossin and Pirotte (2001).

1.7.1 Merton model

To value bonds subject to default risk, Merton (1974) considered a firm, which is financed by equity and by zero-coupon bond. The firm's asset value $A(t)$, under the actual probability measure $\mathbb{P}$, follows a $(\mathcal{F}_t)$-diffusion process given by

$$dA(t) = A(t)\mu dt + A(t)\sigma dW(t) \tag{1.7}$$

where μ is the drift, σ is the volatility, and $W(t)$ is a standard Brownian motion.

The firm has a single debt with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T and face value $L > 0$. The firm cannot issue new senior debt and cannot pay dividends. There is a constant default-free interest rate *r*, uncorrelated with the firm's asset value. Default may happen only at the maturity date T of the default-risky zero-coupon bond, and the default event corresponds to $\{A(T) < L\}$.

In the Merton model, the random time of default τ and the default indicator function $1_{\{\tau=T\}}$ are defined as

$$\tau = \begin{cases} T & if \quad A(T) < L \\ \infty & if \quad else \end{cases} \tag{1.8}$$

$$1_{\{\tau=T\}} = \begin{cases} 1 & if \quad \tau = T \Leftrightarrow A(T) < L \\ 0 & if \quad else \end{cases} \tag{1.9}$$

The default-risky zero-coupon bond at maturity is evaluated as

$$\begin{aligned} D(T,T) &= \min(A(T), L) \\ &= A(T) - \max(A(T) - L, 0) = A(T) - V(T) \end{aligned} \tag{1.10}$$

which is the firm's asset value minus equity, where the equity is $V(T) = \max(A(T) - L, 0)$.

The terminal debt payoff is equivalent to owning the asset and being short a call option on the same asset with an exercise price equal to the face value of the debt. Thus, the equity holder is in the same position as the holder of a call option on the same asset, with a strike value equal to face value, also called book value, of the zero-coupon debt. The equity value $V(t)$ and the debt value $D(t,T)$ can be found by using one of the two approaches for European derivatives. The first approach consists in building a hedging portfolio, which returns the default-free rate. The second approach is based on the Feynman-Kac theorem, which states that the conditional expectation of a stochastic process obeys a partial differential equation.[9] With either approach, the equity and debt values are given as solutions of two PDEs with some given boundary conditions. It is possible to obtain closed-form solutions for the equity, the default-risky zero-coupon bond and the risk-neutral default probability.

The Merton model remains a milestone in the derivative modelling: it was the first to value the firm's equity as a call option on the firm value. However, the following problems have limited its application.

It is unrealistic to assume that the default can occur only at maturity of the debt when the firm's asset value is not sufficient to cover the debt obligations. This is because safety covenants provide the bondholders with the right to force the firm into bankruptcy or reorganization if the firm is poorly performing.

The classes of bondholders are not homogeneous and senior debt should not be priced as a junior debt.

Empirical studies like Jones *et al.* (1984) showed that the Merton model systematically produces too low credit spreads that do not match the credit spreads observed in the market.

The Merton model reaches closed-form solutions with debt expressed as zero-coupon bond, and once coupon bonds are added to the equation, Merton achieved closed-form solution only with infinite maturity.

1.7.2 Black and Cox model

Black and Cox (1976) introduced a default boundary level $H(t)$, with $0 \leq H_0 < A_0$, at which the bondholders will liquidate the firm. Whenever the asset value drops below the level $H(t)$, the firm defaults, whether or not at the maturity of the debt.

Black and Cox chose a default boundary level that evolves deterministically according to

$$H(t) = Ke^{-Y(T-t)} \tag{1.11}$$

for some discount rate Y, and where K, $0 < K \leq L$, is a constant.

For a given boundary level $H(t)$ the early time of default τ' is defined as

$$\tau' = \inf\{t \in [0,T) : A(t) \leq H(t)\} \tag{1.12}$$

and the Merton's time of default τ at maturity is equal to

$$\tau = T1_{\{A(T) \leq L\}} + \infty 1_{\{A(T) > L\}} \tag{1.13}$$

[9] In Tavella and Randall (2000) p. 24.

The pricing PDE that $D(t,T)$ follows is the same as in Merton model, but with different boundary conditions.[10]

1.8 Reduced-form models

Reduced-form or intensity models do not use the company capital structure to trigger default but directly model the time of default or downgrade as the time of the first jump of a Poisson process. More specifically, the risk of default is reflected either by a deterministic default intensity function, or by a stochastic intensity. The advantage of these models is that they create a pricing framework very similar to default-free interest rate theory. The main result is of discounting promised payments with a default-adjusted rate, instead of discounting them with the default-free interest rate. The adjustment is exactly the default intensity (or the default intensity times the loss rate in case it is assumed a fractional recovery at default).

They were originally initiated by Lando (1994 and 1998), Jarrow and Turnbull (1995) and Duffie and Singleton (1998). Jarrow and Turnbull considered the simple case of the time of default modelled as a homogeneous-time Poisson process with known payoff at default. Duffie and Singleton used the same Poisson process, but with a constant recovery, proportional to the value of the default-risky bond just before default. Lando (1994) was the first to apply the Cox process (and the iterated conditional expectations) to model default. Lando (1998) also extended his previous work to include a Markov chain model of credit rating migrations. Further references are found in Bielecky and Rutkowski (2002) and Bielecky, Jeanblanc and Rutkowski (2003).

1.8.1 Hazard function

We assume that we are given a reference filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where it is convenient to model the filtration $\mathcal{G}$ as $\mathcal{G} = \mathcal{F} \vee \mathcal{H}$, where $\mathcal{H}$ is the filtration that carries full information about default events, whereas the filtration $\mathcal{F}$ carries all information regarding other financial and economic processes, such as default-free interest rates, but it does not carry full information about the default event. We begin by defining the following objects.[11]

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[10] The closed-form expressions of the equity and the default-risky zero-coupon bond, in Merton and Black and Cox models, are found in Cossin and Pirotte (2001).
[11] The following definitions are in Rutkowski (2000) pp. 2-3.

Definition 1.2: *We let* τ *be a finite, non-negative random variable on a probability space* $(\Omega, \mathcal{G}, \mathbb{P})$, *referred as the random time of default. We also assume that* $\mathbb{P}(\tau = 0)$, *and* $\mathbb{P}(\tau > t) > 0$ *so* τ *is unbounded.*

Definition 1.3: *We let* F *be the right-continuous distribution function of* τ $F(t) = \mathbb{P}(\tau \leq t)$, *and* $S(t)$ *be the survival function of* τ $S(t) = \mathbb{P}(\tau > t)$.

Definition 1.4: *The increasing right-continuous function* Γ, *given by the formula*

$$\Gamma(t) = -\ln(1 - F(t)) \tag{1.14}$$

is called the hazard function of a random time of default τ.

Definition 1.5: *If the distribution function* $F(t)$, *is an absolutely continuous function, that is if*

$$F(t) = \int_0^t f(u)du \tag{1.15}$$

then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - \exp\left(-\int_0^t h(u)du\right) \tag{1.16}$$

where we set

$$h(t) = \frac{f(t)}{1 - F(t)} \tag{1.17}$$

The function $h(t)$ *is called the hazard rate of* τ.

We now introduce the right-continuous jump process $H(t) = 1_{\{\tau \leq t\}}$ and $\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}$ is the filtration generated by the process $H(t)$.

Definition 1.6:[12] *For any* $t \leq s$, *the conditional probabilities of the objects in definition* 1.3 *are*

$$\mathbb{P}(\tau > s | \mathcal{H}_t) = 1_{\{\tau > t\}} \mathbb{P}(\tau > s | \tau > t) = 1_{\{\tau > t\}} \exp\left(-\int_t^s h(u)du\right) \tag{1.18}$$

$$\mathbb{P}(t < \tau < s | \mathcal{H}_t) = 1_{\{\tau > t\}} \mathbb{P}(t < \tau < s | \tau > t) = 1_{\{\tau > t\}} \left(1 - \exp\left(-\int_t^s h(u)du\right)\right) \tag{1.19}$$

[12] For a proof, see Bielecki *et al.* (2003) p. 23.

which are very useful to analyse the distribution of next jump times, at later points in time $t > 0$.

1.8.2 Poisson and Cox processes

Poisson process provides a convenient way of modelling the arrival of default in intensity models. We first let $N(t)$ be a counting process associated with an increasing sequence of stopping time $\{\tau_i, i \in \mathbb{N}\} = \{\tau_1, \tau_2, ...\}$, defined by

$$N(t) = \sum_i 1_{\{\tau_i \le t\}} \qquad \text{with } [\tau_i < \tau_{i+1}] \tag{1.20}$$

Definition 1.7: *The time of default* τ *is the time of the first jump of* $N(t)$

$$\tau = \inf\{t \ge 0; N(t) > 0\} \tag{1.21}$$

Definition 1.8: *The sequence* (τ_i) *is an inhomogeneous Poisson process with deterministic intensity function* $\lambda(t)$, *if the increments* $N(T) - N(t)$ *are independent on the* σ*-field* $\mathcal{G}_t$ *and for* $t < T$, *the Poisson process is given by*

$$\mathbb{P}\big(N(T) - N(t) = k\big) = \frac{1}{k!}\left(\int_t^T \lambda(u)du\right)^k \exp\left(-\int_t^T \lambda(u)du\right) \tag{1.22}$$

Definition 1.9: *Alternatively,* $\lambda(t)$ *is the intensity of the counting process* $N(t)$ *if and only if*

$$\Lambda(t) = \int_0^t \lambda(s)ds \tag{1.23}$$

is the predictable compensator of $N(t)$, *i.e.* $\mathrm{M}(t) = N(t) - \Lambda(t)$ *is a local martingale.*

With the inhomogeneous Poisson process, we set the time of default equal to the first jump time of the Poisson process $N(t)$. The survival probability can be calculated from (1.22) where there are no jumps until time t

$$S(t) = \mathbb{P}(\tau > t|\mathcal{G}_0) = \mathbb{P}(N(t) = 0) = \exp\left(-\int_0^t \lambda(u)du\right) \tag{1.24}$$

and hence $\lambda(t) = h(t)$.

The density of the time of the first jump, given that no jump has occurred until t is

$$\mathrm{P}\left(\tau_{n+1}-\tau_n \in (t,t+dt)\right)=\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right) \tag{1.25}$$

We can further simplify the intensity in (1.22) and assume it is a constant λ.

Definition 1.10: *The sequence of the arrival times* (τ_i) *is now called a homogeneous process with intensity* λ *if the inter arrival times* $\tau_{i+1}-\tau_i$ *are independent and exponentially distributed with parameter* λ. *The probability of having* k *jumps is given by*

$$\mathrm{P}\left(N(T)-N(t)=k\right)=\frac{1}{k!}\lambda^k (T-t)^k e^{-\lambda(T-t)} \tag{1.26}$$

and the survival probability simplifies to

$$S(t)=e^{-\lambda t} \tag{1.27}$$

As originally specified by Lando (1998),[13] we now assume that the time of default τ, is generated by a *Cox* process with stochastic intensity $\lambda(t)$.

Definition 1.11:[14] *A Cox process*[15] $N(t)$ *with intensity* $\lambda(t)$, *is a Poisson process with stochastic intensity, with the restriction that conditional on the realisation of* $\lambda(t)$, $N(t)$ *is a time-inhomogeneous Poisson process with intensity* $\lambda(t)$.

The stochastic intensity $\lambda(t)$ can be presented as $\lambda(t)=\lambda(X(t))$, where the process X represents the dynamics of the state variables driving the intensity $\lambda(t)$. The state variables may include the default-free interest rate $r(t)$, stock prices, credit ratings and any other relevant variables.

We denote with $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by X and we assume that the default-free process $r(t)$ and the stochastic intensity λ are $(\mathcal{F}_t)_{t\geq 0}$-adapted. Moreover, we let $(\mathcal{H}_t)_{t\geq 0}$ be the filtration generated by the counting process $N(t)$, which, for every $t \geq 0$, reveals the information about defaults up to time t. The full filtration $(\mathcal{G}_t)_{t\geq 0}$ is then defined as the enlarged σ-field, $(\mathcal{G}_t)_{t\geq 0}=(\mathcal{F}_t)_{t\geq 0} \vee (\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{F}_t=\sigma\{X(s):0\leq s\leq t\}$ and

[13] Other contributors of the Cox process applied to reduced-form models are: Duffie and Singleton (1998), Madan and Unal (1998) and Schönbucher (1998 and 1999).
[14] In Giesecke (2003).
[15] Also known as *doubly stochastic Poisson process*.

$\mathcal{H}_t = \sigma\{1_{\{\tau \le s\}} : 0 \le s \le t\}$, and hence represents the information set, on the path of the state variable X, and on defaults occurred, available at time t.

The law of iterated expectations[16] then leads to the jump probabilities

$$\begin{aligned} \mathbb{P}(N(T) - N(t) = k) &= \mathbb{E}\left(1_{\{N(T)-N(t)=k\}}\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(1_{\{N(T)-N(t)=k\}} \middle| \lambda\right)\right) \\ &= \mathbb{E}\left(\frac{1}{k!}\left(\int_t^T \lambda(u)du\right)^k \exp\left(-\int_t^T \lambda(u)du\right)\right) \end{aligned} \tag{1.28}$$

Hence, the unconditional survival probability, the conditional survival probability, and the density are respectively given by

$$\mathbb{P}(\tau > t) = \mathbb{E}\left(\exp\left(-\int_0^t \lambda(u)du\right)\right) \tag{1.29}$$

$$\mathbb{P}(\tau > s | \lambda(t)) = \exp\left(-\int_t^s \lambda(u)du\right) \tag{1.30}$$

$$\mathbb{P}(\tau_{n+1} - \tau_n \in (t, t+dt)) = \mathbb{E}\left(\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)\right) \tag{1.31}$$

A way of simulating the first jump τ of a Cox process N, is to let Z be a unit exponential random variable, assumed independent from $(\mathcal{F}_t)_{t\ge 0}$, and hence from the state variable X, and define the time of default as

$$\tau = \inf\left\{t : \int_0^T \lambda(X(s))ds \ge Z\right\} \tag{1.32}$$

The compensated process $\mathrm{M}(t)$ reads the same as the compensated process of the inhomogeneous Poisson process, with the only difference that $\lambda(t)$ is now stochastic. Also important is the expected increment of the Cox process, which is

$$\mathbb{E}(N(t)|\mathcal{F}_t) = \lambda(t)dt \tag{1.33}$$

In essence, there are two different ways of modelling the default event, first through the point of view of stochastic processes and predictable compensator $\Lambda(t)$, hence including intensities, and second by analysing the distribution of the next jump time, thus using hazard rates.

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16 In Schönbucher (2003) p. 125, Duffie and Singleton (2003) p. 33, Giesecke (2003) and Lando (1998) pp. 99-120.

1.8.3 Pricing blocks

As an application of the Cox process, we recall the well known result of for the pricing of default-risky claims of Lando (1998). A default-risky claim of maturity T is a *r.v.* X measurable with respect to $\mathcal{G}_T$. We assume that there is a martingale measure $\mathbb{Q}$, and the pre-default arbitrage-free price at time t of the claim X is

$$D(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds \right) X \middle| \mathcal{G}_t \right) \tag{1.34}$$

We further assume that the claim pays Y, if no default occurs until maturity, or a recovery φ if there is a default. We focus here on the case where recovery is zero if there is a default, that is $X = 1_{\{\tau > T\}} Y$. The case of positive recovery is treated in the following sections. We also suppose that Y is itself $\mathcal{F}_T$-measurable.[17]

Proposition 1.1:[18] *For a positive or bounded $\mathcal{F}_T$-measurable r.v. Y, we have on $\{\tau > t\}$*

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds \right) 1_{\{\tau > T\}} Y \middle| \mathcal{G}_t \right) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T \big(r(s) + \lambda(s)\big)ds \right) Y \middle| \mathcal{F}_t \right) \tag{1.35}$$

With the inhomogeneous Poisson process and assuming that the default-free interest rate process $r(t)$ and the default event are independent, and both non-stochastic, the price simplifies into[19]

$$D(t,T) = \exp\left(-\int_t^T \big(r(s) + \lambda(s)\big)ds \right) \tag{1.36}$$

1.8.4 Example: a credit option with the intensity approach

We now proceed, using both an inhomogeneous Poisson process and a Cox process, to price a credit option, which pays £1 at time T_{k+1}, if and only if the default of a reference default-risky bond happens in the period $[T_k, T_{k+1}]$. The example was originally proposed in Schönbucher (2003).[20]

[17] See Rutkowski (2000) for details.
[18] For a proof, see Jouanin *et al.* (2001) p. 4.
[19] In Jarrow and Turnbull (1995) pp. 53-85.
[20] In Schönbucher (2003) p. 118.

We indicate with $e(0,T_k,T_{k+1})$ the credit option price at time $t=0$, with $B(0,T_k,T_{k+1})$ the forward price at time $t=0$ of a default-free zero-coupon bond which matures at time T_{k+1} for the period $[T_k,T_{k+1}]$, and with $D(0,T_k,T_{k+1})$ the forward price at time $t=0$ of a default-risky zero-coupon bond which matures at time T_{k+1} for the period $[T_k,T_{k+1}]$. The credit option price is written as

$$e(0,T_k,T_{k+1}) = \mathrm{E}^{\mathrm{Q}}\left(\exp\left(-\int_0^{T_{k+1}} r(s)ds\right)\left(1_{\{N(T_k)=0\}} - 1_{\{N(T_{k+1})=0\}}\right)\right) \tag{1.37}$$

where the inner parenthesis contains the probability of default in the interval $[T_k,T_{k+1}]$.

Calling upon the assumption of independency between the default event and the default-free interest rate, the price in (1.37) becomes

$$\begin{aligned} e(0,T_k,T_{k+1}) &= \mathrm{E}^{\mathrm{Q}}\left(\exp\left(-\int_0^{T_{k+1}} r(s)ds\right)\right)\left(\mathrm{E}^{\mathrm{Q}}\left(1_{\{N(T_k)=0\}}\right) - \mathrm{E}^{\mathrm{Q}}\left(1_{\{N(T_{k+1})=0\}}\right)\right) \\ &= B(0,T_{k+1})\exp\left(-\int_0^{T_k}\lambda(s)ds\right)\left(1-\exp\left(-\int_{T_k}^{T_{k+1}}\lambda(s)ds\right)\right) \\ &= D(0,T_k)B(0,T_k,T_{k+1})\left(1-\exp\left(-\int_{T_k}^{T_{k+1}}\lambda(s)ds\right)\right) \\ &= D(0,T_k)B(0,T_k,T_{k+1}) - D(0,T_k,T_{k+1}) \end{aligned} \tag{1.38}$$

The final step in (1.38) has an arbitrage interpretation: the credit option in (1.37) is equivalent to a default-risky zero-coupon maturing at time T_k, whose face value is reinvested in a forward default-free zero-coupon bond for the period $[T_k,T_{k+1}]$, and being short on a forward default-risky bond maturing at time T_{k+1}.

As $T_{k+1}\to T_k$, i.e. $\Delta t\to 0$, we can write (1.38) in continuous time as

$$e(0,T) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}e(0,T,T+\Delta t) = D(0,T)\lambda(T) \tag{1.39}$$

The credit option has a different expression under the Cox process. To see this, we write the payoff of £1 discounted from the time of default τ as

$$\exp\left(-\int_0^{\tau} r(s)ds\right)1_{\{\tau<T\}} = \int_0^T \exp\left(-\int_0^t r(s)ds\right)1_{\{\tau\geq t\}}\,dN(t) \tag{1.40}$$

which is the sum of the discounted values until default occurs.

We take the expectation of (1.40) and use the expression of the predictable compensator of $N(t)$ in (1.33), and write the expected payoff as

$$\mathrm{E}^{Q}\left(\exp\left(-\int_0^{\tau} r(s)ds\right)1_{\{\tau<T\}}\right)=\mathrm{E}^{Q}\left(\int_0^{T}\exp\left(-\int_0^{t} r(s)ds\right)1_{\{\tau\geq t\}}\lambda(t)dt\right)$$

$$=\int_0^{T}\mathrm{E}^{Q}\left(\exp\left(-\int_0^{t} r(s)ds\right)1_{\{\tau\geq t\}}\lambda(t)\right)dt \qquad (1.41)$$

We are now able to write the credit option price in (1.40) as

$$e(0,T)=\mathrm{E}^{Q}\left(\exp\left(-\int_0^{T}\left(r(s)+\lambda(s)\right)ds\right)\lambda(T)\right) \qquad (1.42)$$

1.8.5 Recovery schemes

For modelling purposes, we can consider three recovery conventions that may be useful in practical situations:[21]

(1) recovery of treasury (RT),

(2) recovery of par value (RPV) and

(3) recovery of market value (RMV).

The recovery of treasury (RT) assumes that when there is a default, investors receive a constant fraction $\varphi \in [0,1]$ of an equivalent default-free zero-coupon bond, which is held until maturity. Hence, the price of a default-risky bond becomes

$$D(t,T)=\mathrm{E}_t^{Q}\left(\exp\left(-\int_t^{T} r(s)ds\right)1_{\{\tau>T\}}+\exp\left(-\int_t^{T} r(s)ds\right)\varphi 1_{\{\tau\leq T\}}\right) \qquad (1.43)$$

The recovery of par value (RPV) assumes that the firm is liquidated at the time of default and the bond investors receive a fraction payment of the face amount at the time of default, and the price of a default-risky bond becomes

$$D(t,T)=\mathrm{E}_t^{Q}\left(\exp\left(-\int_t^{T} r(s)ds\right)1_{\{\tau>T\}}+\exp\left(-\int_t^{\tau} r(s)ds\right)\varphi 1_{\{\tau\leq T\}}\right)$$

$$=\mathrm{E}_t^{Q}\left(\exp\left(-\int_t^{T} r(s)ds\right)1_{\{\tau>T\}}+\varphi\int_t^{T}\pi(t,u)\exp\left(-\int_t^{u} r(s)ds\right)du\right) \qquad (1.44)$$

where $\pi(t,u)$ is the density of the time of default, which under a Cox process, is in (1.31), and for the inhomogeneous Poisson process, is in (1.25).

[21] We also note that the face value of this default-risky bond is of £.1.

If the bond investors, upon default, receive a fraction of the pre-default bond market value, the scheme takes the name of the recovery of market value (RMV), and the price of a default-risky bond can be written as

$$D(t,T) = \mathrm{E}_t^{\mathrm{Q}}\left(\exp\left(-\int_t^T r(s)ds \right) 1_{\{\tau > T\}} + \exp\left(-\int_t^\tau r(s)ds \right) (1 - l(\tau)) D(\tau^-, T) 1_{\{\tau \le T\}} \right) \quad (1.45)$$

where $1 - l(\tau)$ is the recovery of a pre-default market value $D(\tau^-, T)$.

Duffie and Singleton (1998) wrote the price of a default-risky bond, under the RMV as follows

$$D(t,T) = \mathrm{E}_t^{\mathrm{Q}}\left(\exp\left(-\int_t^T (r(s) + l\lambda(s)) ds \right) \right) \quad (1.46)$$

where $r(t)$ and $\lambda(t)$ are independent.

The price in (1.46) is the consequence of the following pricing rule,[22]

$$\mathrm{E}_\tau^{\mathrm{Q}}\left(\frac{dD(\tau,T)}{D(\tau^-,T)} \middle| \mathcal{F}_\tau \right) = (r(\tau) + l\lambda(\tau)) dt \quad (1.47)$$

that is, holding the default-risky bond should return the default adjusted short rate $r + l\lambda$.

The expectation in (1.47) shows that it is not possible to simultaneously extract both the recovery and the intensity rate from market prices, because what we observe is the product $l\lambda$. One possibility is to use the recovery rates as published by Moody's, S&P or Fitch for different rating classes and location of the issuer, and then to proceed to derive the term structure of the intensity rates. It should be noted that this is correct so long as, after deriving the term structure of the default rates, the default model used to price default-risky bonds maintains the assumption of independency between default-free interest rate and the default event.

1.8.6 Random recoveries

To model random recoveries, we recur to the compound Poisson process.

Definition 1.12: *Let $\{N(t) : t \ge 0\}$ be a Poisson process with intensity $\lambda(t)$, and suppose that the time of each event τ_i is associated to a realization of a r.v. Φ_i, where $\{\Phi_n : n > 0\}$ is a family of i.i.d. random variables sharing some distribution function $F(\varphi) = \mathbb{P}(\Phi_i \le \varphi)$. We also let $\{\Phi_n : n > 0\}$ be independent of $\{N(t) : t \ge 0\}$. Then the stochastic process*

[22] For a proof, see Schönbucher (2003) p. 139.

$$Z(t) = \sum_{i=1}^{N(t)} \Phi_i \tag{1.48}$$

is said to be a compound Poisson process.

In a compound Poisson process at each time of jump τ_i of a Poisson process, a *r.v.* Φ_i is drawn from a distribution $F(\varphi)$. Φ_i is called the marker to the point of jump τ_i, and the whole set $\{(\tau_i, \Phi_i), i \in \mathbb{N}\}$ of points in time and markers, is called a marked Poisson process.[23] The marked Poisson process is a generalisation of the Poisson process $N(t)$.

The marker can be any type of *r.v.*, we only assume that it is drawn from a measurable probability space (E, ε), where E is the unit interval $[0,1]$, and the $\sigma-$algebra ε is the Borel sets on E.

To represent the information contained in a marked Poisson process, we recur to the jump measure.

Definition 1.13:[24] *The jump measure of a marked Poisson process* $\{(\tau_i, \Phi_i), i \in \mathbb{N}\}$ *is a random measure on* $E \times \Re_+$ *and on a time interval* $[0,t]$, *such that for all* $E \in \varepsilon$

$$\mu(\omega, E, [0,t]) = \int_0^t \int_E \mu(\omega, e, s) de ds = \sum_{i=0}^{\infty} 1_{\{\tau_i(\omega) \le t\}} 1_{\{\Phi_i(\omega) \in E\}} \quad \text{with } \omega \in \Omega \tag{1.49}$$

which counts the number of events (τ_i, Φ_i), with a marker from E during the time interval $[0,t]$.

With the jump measure in (1.49), we can describe the realisations of the marked Poisson process. To specify the probabilities of the realisations of the marked Poisson process, we define a *r.v.* $\nu(de, dt)$ as the predictable compensator measure of $\mu(de, dt)$.

Definition 1.14:[25] *The compensator measure of* $\mu(de, dt)$ *is the unique predictable r.v.* $\nu(\omega, de, dt)$ *with the following property.*

Let $f(\omega, e, s)$ *be a predictable stochastic function.*[26] *Let* $M(\omega, t)$ *be defined as follows:*

[23] In Schönbucher (2003) and Taylor and Karlin (1984).
[24] In Schönbucher (2003) p. 92.
[25] In Schönbucher (2003) p. 94.
[26] The predictable stochastic function can be the function, which describes the recovery payoff. See also sections 1.9.4 and 1.9.5.

$$M(\omega,t)=\int_0^t\int_E f(\omega,e,s)\mu(\omega,e,s)deds-\int_0^t\int_E f(\omega,e,s)\nu(\omega,e,s)deds \qquad (1.50)$$

then $M(t)$ is a local martingale.

We now write $\nu(de,dt)$ as the product of the density of the recovery rate Φ at time t, $f(t,e)$, and the probability of having a jump in the next very small time step dt,

$$\nu(de,dt)=f(e,t)\lambda(t)dedt \qquad (1.51)$$

with $$\int_E f(e)de=1$$

As an example,[27] we price the recovery of par value (RPV) of a default-risky bond. Using the jump measure of the marked point process in (1.49), the payoff at default can be written as

$$\int_0^T\int_E B(0,s)e\mu(e,s)deds=\begin{cases}B(0,t)\varphi & if \quad \tau<T\\ 0 & otherwise\end{cases} \qquad (1.52)$$

Taking the expectation of the integral in (1.52), we can write the recovery price at time $t=0$,

$$\mathrm{E}^{Q}\int_0^T\int_E B(0,s)e\mu(e,s)deds \qquad (1.53)$$

Combining the results in (1.50), (1.51) and (1.52), we can write (1.53) as

$$=\mathrm{E}^{Q}\int_0^T\int_E B(0,s)ef(e,s)de\lambda(t)dt$$

$$=\mathrm{E}^{Q}\int_0^T B(0,s)\left(\int_E ef(e)de\right)\lambda(t)dt$$

$$=\mathrm{E}^{Q}\int_0^T B(0,s)g(t)\lambda(t)dt \qquad (1.54)$$

This is a very useful result as it shows that the pricing problem with a random recovery payoff Φ, can be solved as a pricing problem with a predictable process $g(t)$ as recovery payoff.[28]

1.8.7 Calibration

Credit default swaps (CDSs) are the most liquid products in the credit derivative market. Hence, they are used to extract risk-neutral probabilities.

[27] Originally in Schönbucher (2003) p. 144.
[28] In Schönbucher (2003) p. 144.

Typically, there are five main assumptions behind the calibration of risk-neutral probabilities on CDS premiums.

(1) The default-free interest rate process and the default process are independent.

(2) The default process is modelled as a time-inhomogeneous Poisson process.

(3) Defaults can only happen at a set of finite and discrete dates.

(4) Default payments are settled immediately upon default.

(5) Recoveries are assumed constant.

In a CDS the protection buyer makes regular payments and receives a payment from the protection seller should a default event happen. If the default event occurs, a percentage $1-\varphi$ of M is paid to the protection buyer, where φ is the recovery rate and M is the notional amount of the contract. The amount $M(1-\varphi)$ is called the default payment.

At the payment dates t_j, with $j \in \{1,2,..,m\}$ and $t_m = T$, the protection buyer pays the regular payment fee of

$$s\delta_j 1_{\{t_j \leq \tau\}} \qquad \text{at} \qquad t_j \tag{1.55}$$

to the protection seller. Here s denotes the CDS premium, δ_j is the day count fraction for the interval $[t_{j-1}, t_j]$ and $1_{\{t_j \leq \tau\}}$ is the indicator function that the default event has not occurred before the payment date t_j.

Furthermore, at the time of default τ, the protection buyer makes a final payment covering the time between the last payment date and the time τ of the trigger event. Let $j^* = \max\{j | t_j \leq \tau\}$ be the last payment date before the time τ. Then, the protection buyer pays the extra fee of

$$s\delta_{j^*} 1_{\{t_{j^*} \leq \tau < t_j\}} \qquad \text{at} \qquad \tau \tag{1.56}$$

to the protection seller. Here δ_{j^*} is the day count fraction for the interval $[t_{j^*}, \tau]$. If we denote $V_P(t)$ as the value at time t of receiving 1bp of fee payments and no fees after the interval $[t_{j-1} \leq \tau \leq t_j]$, then we can write

$$V_P(t) = M\mathbb{E}^{\mathbb{Q}}\left(\sum_{j=1}^{m} B(0,t_j)\delta_j 1_{\{t_j \leq \tau\}} + B(0,\tau)\delta_{j^*} 1_{\{t_{j^*} \leq \tau < t_j\}} \middle| \mathcal{G}_t \right) \tag{1.57}$$

where $B(0,t)$ is the discount factor of the default-free interest rate.

If the default occurs before the final payment date $t_m = T$, i.e. if $\tau \leq t_m = T$, then the protection seller pays to the protection buyer the default amount $M(1-\varphi)$. Thus, the payment is

$$M(1-\varphi)1_{\{\tau<T\}} \qquad \text{at} \qquad \tau \tag{1.58}$$

We denote the value of the default leg at a time t as

$$V_D(t) = M\mathbb{E}^{\mathbb{Q}}\left(B(0,\tau)(1-\varphi)1_{\{\tau<T\}}\middle|\mathcal{G}_t\right) \tag{1.59}$$

Given (1.57) and (1.59), the fair CDS premium rate at time t, is that rate at which the premium leg has the same value as the default leg

$$s = \frac{V_D(t)}{V_P(t)} \tag{1.60}$$

Under the assumptions one to five, and that $B(0,t)$ is a deterministic function, the default and the premium legs simplify into

$$V_P(t) = M\left(\sum_{j=1}^{m} B(0,t_j)\delta_j\left(1-F(t_j)\right)+\delta_{j}^{\cdot}\left(F(t_j)-F(t_{j-1})\right)B(0,t_j)\right) \tag{1.61}$$

and

$$V_D(t) = M(1-\varphi)\left(\sum_{j=1}^{m}\left(F(t_j)-F(t_{j-1})\right)B(0,t_j)\right) \tag{1.62}$$

where $\delta_{j}^{\cdot}$ is assumed to be half a coupon period.

To calibrate the hazard function, we now follow Arvanitis and Gregory (2001), and Duffie (1999) and assume that the hazard function in (1.17) is a deterministic piece-wise constant, i.e. it is constant between maturity dates of the CDS $\{T_1, T_2, .., T_n\}$, to which we are calibrating

$$h(u) = \alpha_i \quad \text{for } i \in [t_{j-1}, t_j] \tag{1.63}$$

which means that the risk-neutral probabilities of default are given

$$\mathbb{Q}\left(\tau \le t_j\right) = 1-\exp\left(-\sum_{j=1}^{m}\alpha_j\left(t_j - t_{j-1}\right)\right) \tag{1.64}$$

The CDS with the shortest maturity T_1 is used to calibrate α_1. Once we know α_1, we can proceed to calibrate α_2 on the CDS with maturity T_2, and so on.

1.9 Implementation of stochastic intensity models

There are two main aspects when implementing stochastic intensity models. The first is the choice of the model specification and the second is the numerical implementation. In terms of the correct choice of the model, the reduced-form approach literature has hugely relied on the analogy between survival probabilities and forward credit spreads with the term structure models of short-term interest rates and applied many of the short-term interest rate models to derive reduced-form models.[29] The dynamics of the credit spreads either, have been specified in such a way that the solutions of survival probabilities and default-risky bond prices have been derived analytically, or whenever that has not been possible, numerical implementations such as numerical trees, numerical methods for solving PDEs, and Monte Carlo simulations have been applied.

In the rest of this section, we will review the extension of some of the most important default-free interest rate models to value default-risky bonds with stochastic intensity rates as in the spirit of Lando (1998). We will start with a model that has closed-form solutions: one-factor Cox *et al.* intensity model without and with jumps. After that, we will review the application, to default-risky bonds, of default-free interest rate models which require a numerical implementation: two-factor Hull and White numerical tree, HJM forward rate model, and PDE-based model with stochastic recoveries.

1.9.1 Cox *et al.* intensity model

Duffie and Garleanu (1999) showed how to apply the single factor Cox *et al.* (1985) to model the intensity rate process. The dynamics of the intensity rate process are given by

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t) \tag{1.65}$$

where

$W(t)$ is a standard Brownian motion,

θ is the long-term mean of $\lambda(t)$,

κ is the mean rate of reversion to the long-term mean, and

σ is the volatility coefficient.

The survival probability at time t, from t to T, is given analytically by

$$\mathrm{Q}\left(\tau > T \middle| \mathcal{F}_t\right) = \mathrm{E}_t^{\mathrm{Q}}\left(\exp\left(-\int_t^T \lambda(u)du\right)\right) = \exp\left(\alpha(T-t) + \beta(T-t)\lambda(t)\right) \tag{1.66}$$

where α and β are time-dependent coefficients and can be found in Duffie and Garleanu (1999).

By changing each of the parameters in (1.65), they also draw reach term structures of the intensity rates, which we summarise below. The volatility, expressed as $\sigma / \sqrt{\lambda(t)}$, causes a decline in the probability of default, given a survival until a certain time. This is because the change on the volatility does not affect the mean of the intensity rate process, but increases the dispersion around the mean, that is the variance. This, in turn, increases the survival probability and therefore reduces

[29] For a review of default-free interest rate models, see James and Webber (2000).

the default probability. The mean-reversion κ adds a further complexity to the intensity rate. For example, as $\kappa \to 0$, the variance increases and the probability of default drops. A high rate on mean-reversion keeps the intensity rates close to its initial level, and reduces the effect of the volatility on the shape of the intensity rates. A high initial $\lambda(0)$ with respect to the mean reversion level indicates that a firm has a poor credit quality, but it will improve with the time, conditioned on survival. How fast it improves depends on the mean-reversion. The conclusion is of a trade-off between the volatility and the mean-reversion parameters.

Duffie and Garleanu also extended the CIR to include jumps. They adopted a pre-intensity process λ, solving a stochastic differential equation of the form

$$d\lambda(t) = \kappa(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t) + JdN(t) \tag{1.67}$$

where $N(t)$ is independent of $W(t)$, the jumps sizes J are independent and exponential distributed with mean μ_J, the jump times are those of an independent Poisson process with intensity rate c. All remaining parameters are the same as already specified in (1.65).

The survival probability at time t, from t to T, is given as in (1.66) but with different α and β coefficients which can be found in Duffie and Garleanu (1999).

For comparing calibration issues of Cox *et al.* intensity model, with and without jumps, we revert to Duffie and Singleton (2003).

1.9.2 Tree-based models: Schönbucher (1999)

Schönbucher model is based upon the two-factor Hull and White (1994) model for default-free interest rates. It can be fitted to any given term structure of default-free and default-risky bond prices. The initial set up is a Gaussian specification, also known as the extended Vasicek (1977) model. As consequence of that choice, intensities and interest rates can become negative at some node of the tree. A remedy to this problem is also given and it is to specify a tree only for the logarithm of interest rate and intensity. For example, the dynamics of the interest rate and default intensity can be specified with the Black and Karazinsky (1991). The disadvantage of this remedy is that the calibration can only be done numerically.

As a tree-based model, it shares all the advantages of this type of models: its mechanics are intuitive, the hedge ratios are found automatically with the prices, and for pricing credit derivatives with an early exercise of the option it dominates Monte Carlo methods. However, since this model can be viewed as an explicit finite difference scheme to solve a partial differential equation (as all tree-based models), it shares all the weaknesses of the explicit finite difference schemes when compared with the implicit finite difference schemes and the Crank-Nicolson method.[30]

Schönbucher also modelled the effect of correlation in the dynamics of interest rates and intensities.

The model builds the dynamic of the interest rate process in a recombining trinomial tree structure with steps up, middle and down. For each time step of length Δt the short rate will evolve as $r(t+\Delta t) = r(t) + k\Delta t$ where $k \in [-1,0,1]$ is a negative or positive integer and Δr is given by

$$\Delta r = \sigma\sqrt{3\Delta t} \tag{1.68}$$

The branching probabilities at the nodes are evaluated by the use of three constraints: firstly, the first two moments of the process, secondly, the probabilities must add up to 1, and thirdly, the tree may not grow infinitely otherwise probabilities might become negative (this last requirement has an effect on the geometry of the tree).

After the branching probabilities and the temporary tree for the interest rate process are built, the calibration procedure starts. This is done iteratively using *forward induction*,[31] and it will lead to the fitted tree of the interest rate process, that is the tree with which the default-free bond prices $B(0,t)$ are recovered without arbitrage.

Once the tree of the interest rate process is completed, the model moves to build the tree of the default intensity, and then to combine the two trees into one three-dimensional tree.

Where the dynamics of the default-free interest rates and of the default intensity are correlated, the nodes of the three-dimensional tree are refitted. This is done by redistributing the probability weights among the nodes of the tree of the same time step. One problem remains unsolved. When correlation is not zero, the default-risky bond prices cannot exactly be recovered from the three-dimensional tree because there is no closed-form solution.[32] Schönbucher, only in a later research,[33] suggested a method to find the k of the default intensity parametrically.

Schönbucher extracted some important results when he moved to numerical applications. For example, when the implied default probabilities are extracted from his model, they increase as the correlation increases. He explained this as intuitive. When interest rates and credit spreads are positively correlated, defaults are more likely in the economic environment with high interest rates (in the model, this corresponds to the nodes with high interest rates). In an economic environment with higher interest rates, the prices of default-risky bonds are further compressed by discounting with high interest rate than with low interest rates. To reach a given price for a default-risky bond,

[30] For a comparison of explicit, implicit and Crank-Nicolson methods in finance, see James and Webber (2000), or in more general, Thomas (1995).

[31] Jamshidian (1991) introduced the method of forward induction and placed the tree methods on a sound theoretical ground, also enabling them to be calibrated to market prices.

[32] When correlation is $\rho \neq 0$ the following is true $D(t,T) \neq B(t,T)\mathrm{E}_t^{\mathbb{Q}}\left(\exp\left(-\int_t^T \lambda(s)ds\right)\right)$.

[33] In Schönbucher (2003) pp. 187 - 200.

the absolute default likelihood must therefore be higher. The argument runs conversely for negative correlation. These results are only in part supported by recent empirical research: Christiansen (2000) found that the credit spreads and the level of the term structure of interest rates are negatively correlated. Longstaff and Schwartz (1995) reported that changes in credit spreads are negatively related to changes in Treasury prices.

Two points we think Schönbucher needed to clarify: the first is the default-risky bond prices used in the calibration, and secondly, the parametric methodology used to fit the *three-dimensional* tree.

1.9.3 An intensity model in a HJM framework

An alternative approach assumes that the default intensities are determined by the forward default-risky rate as in the spirit of the term structure modelling approach of Heath, Jarrow and Morton (1992). In this section, we will show how Duffie and Singleton (1998), and Schönbucher (1998) extended the HJM model to incorporate default risk.[34]

In the HJM model, an entire curve of forward rates evolves simultaneously, according to a set of volatility curves.[35]

First we write the dynamics of the continuously compounded default-free and default-risky forward rate as

$$df(t) = \mu(t)dt + \sum_{i=1}^{n} \sigma_i(t)dW_i(t) \tag{1.69}$$

and

$$df^D(t) = \mu_{f^D}(t)dt + \sum_{i=1}^{n} \sigma_{f^D_i}(t)dW_{f^D_i}(t) \tag{1.70}$$

where W and W_{f^D} are two n-dimensional vector Brownian motions with independent components.

From (1.69) and (1.70) we can write the default-free zero-coupon bond $B(t,T)$ and the default-risky zero-coupon bond $D(t,T)$ as

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right) \tag{1.71}$$

$$D(t,T) = (1 - N(t))\exp\left(-\int_t^T f^D(t,s)ds\right) \tag{1.72}$$

and the default-free interest rates $r(t)$ and the default-risky interest rate $r_D(t)$ as

[34] As references of the HJM model, we recommend Baxter and Rennie (1996), Bjork (1998), Rebonato (1998) and James and Webber (2000).

[35] James and Webber (2001) show how Ho and Lee (1986) can be embedded into the HJM model and subsequently use this model to derive the price of a European interest rate call option.

$$r(t) = \tfrac{1}{T-t} \int_t^T f(t,s)ds \tag{1.73}$$

and
$$r_D(t) = \tfrac{1}{T-t} \int_t^T f^D(t,s)ds \tag{1.74}$$

The factor $1 - N(t)$ in (1.72) ensures that the default-risky zero-coupon bond $D(t,T)$ price drops to zero at the time of default τ.[36]

By Ito's lemma, the dynamics of the default-free zero-coupon bonds $B(t,T)$ and of the default-free interest rates $r(t)$ are given, respectively as

$$\frac{dB(t,T)}{B(t,T)} = \left[-a(t,T) + r(t) + \frac{1}{2}\sum_{i=1}^{n} b_i^2(t,T) \right] dt + \sum_{i=1}^{n} b_i(t,T)dW_i(t) \tag{1.75}$$

and
$$r(t) = f(0,t) + a(t,T) + \sum_{i=1}^{n} b_i(t,T)dW_i(t) \tag{1.76}$$

where
$$b(t,T) = -\int_t^T \sigma_i(t,v)dv \tag{1.77}$$

$$a(t,T) = \int_t^T \mu(t,v)dv \tag{1.78}$$

By Ito's lemma, the dynamics of the default-risky zero coupon bonds $D(t,T)$ and of the default-risky interest rates $r_D(t)$ are given, respectively as

$$dD(t,T) = D(t,T)\left[-a_D(t,T) + r_D(t) + \frac{1}{2}\sum_{i=1}^{n} b_{D,i}^2(t,T) \right] dt +$$

$$D(t,T)\sum_{i=1}^{n} b_{D,i}(t,T)dW_{D,i}(t) - D(t,T)dN(t) \tag{1.79}$$

and
$$r_D(t) = f_D(0,t) + a_D(t,T) + \sum_{i=1}^{n} b_{D,i}(t,T)dW_{D,i}(t) \tag{1.80}$$

where

$$b_D(t,T) = -\int_t^T \sigma_{D,i}(t,v)dv \tag{1.81}$$

$$a_D(t,T) = \int_t^T \mu_D(t,v)dv \tag{1.82}$$

To avoid arbitrage opportunities, the dynamics of the forward rates in (1.69) and (1.70) must

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[36] In Schönbucher (1998) p. 207.

satisfy the following two no-arbitrage conditions. A drift restriction, which is an extended version of the HJM restriction for the default-free bond, and a drift restriction for the default-risky bond, which ensures that the default-risky interest rates are always greater than the default-free interest rates. Under the martingale measure $\mathbb{Q}$, all bond price processes must have a drift r, hence

$$\mathrm{E}^{\mathbb{Q}}[dB(t,T)] = r(t)B(t,T)dt \tag{1.83}$$

and
$$\mathrm{E}^{\mathbb{Q}}[dD(t,T)] = r(t)D(t,T)dt \tag{1.84}$$

Taking the expectation of (1.75) and setting this equal to (1.83) we are able to write

$$a(t,T) = \frac{1}{2}\sum_{i=1}^{n} b_i^2(t,T) \tag{1.85}$$

and using (1.77) and (1.78)

$$\int_t^T \mu(t,v)dv = \frac{1}{2}\sum_{i=1}^{n}\left(\int_t^T \sigma_i(t,v)dv\right)^2 \tag{1.86}$$

Removing the integral by taking the first derivative with respect to T, (1.86) leads to

$$\begin{aligned}\mu(t,v) &= \frac{1}{2}\sum_{i=1}^{n}\sigma_i(t,v)\int_t^T \sigma_i(t,v)dv \\ &= \sum_{i=1}^{n} -\sigma_i(t,v)b_i(t,v)\end{aligned} \tag{1.87}$$

where $\mu(t,T)$ and $\sigma_i(t,T)$ are the coefficient functions in the forward rate process of (1.69).

Schönbucher defined the drift condition of (1.86) as the *Forward Rate Drift Condition* which corresponds to the drift of the original HJM model, applied to default-free interest rates.

To write the *Forward Rate Drift Condition* of the default-risky forward rate in (1.70), we take the expectation of (1.79) and setting this equal to (1.84), we have

$$a_D(t,T) = r_D(t) - r(t) - \lambda(t) + \frac{1}{2}\sum_{i=1}^{n} b_{D,i}^2(t,T) \tag{1.88}$$

where we have used the fact that $\mathrm{E}^{\mathbb{Q}}[dN(t)] = \lambda(t)dt$.

Using (1.81) and (1.82), (1.88) becomes

$$\int_t^T \mu_D(t,v)dv = r_D(t) - r(t) - \lambda(t) + \frac{1}{2}\sum_{i=1}^{n}\left(\int_t^T \sigma_{D,i}(t,v)dv\right)^2 \tag{1.89}$$

where $\mu_D(t,T)$ and $\sigma_{D,i}(t,T)$ are the coefficient functions of the forward rate process in (1.70). We can notice that as $T \to t$, (1.89) simplifies into[37]

[37] Schönbucher (2003) writes the second condition $\lambda(t) = r_D(t) - r(t)$ as $\lambda(t) = f_D(t,t) - f(t,t)$, where as $\mathrm{T} \to t$, the two expressions are equivalent.

$$\lambda(t) = r_D(t) - r(t) \tag{1.90}$$

that is, the difference between the default-free interest rate $r(t)$ and the default-risky interest rate $r_D(t)$ is indeed the intensity rate $\lambda(t)$.

Using the result in (1.90), and removing the integral in (1.89) by taking the first derivative with respect to T, (1.89) leads to

$$\mu_D(t,v) = \frac{1}{2}\sum_{i=1}^{n} \sigma_{D,i}(t,v)\int_t^T \sigma_{D,i}(t,v)dv$$

$$= \sum_{i=1}^{n} - \sigma_{D,i}(t,v) b_{D,i}(t,v). \tag{1.91}$$

The equation in (1.87), (1.90) and (1.91) are the three conditions that ensure the absence of arbitrage conditions. Schönbucher suggested to modelling the spread between the default-free forward rate and the default-risky forward rate. For example, the square root model of Cox *et al.* could be used to model the dynamics of the spread $s(t)$, because it would generate positive spread rates. In this case, it is necessary to ensure the following drift condition on the dynamics of the spread,[38]

$$\mu_s(t,v)dv = \frac{1}{2}\sum_{i=1}^{n} \sigma_{s,i}(t,v)\int_t^T \sigma_{s,i}(t,v)dv \tag{1.92}$$

where the dynamics of the spread $s(t)$ are expressed as

$$ds(t) = \mu_s(t)dt + \sum_{i=1}^{n} \sigma_{s,i}(t)dW_{s,i}(t). \tag{1.93}$$

1.9.4 PDE with random recovery

Here we show how to derive the partial differential equation and how a numerical scheme could be implemented, to price the default-risky zero-coupon bond where, when default happens, the recovery $\varphi(\tau)$ is no longer constant, but it is a *r.v.*. Later on, a PDE of the same type is implemented to value the credit put option to hedge the default risk on the same default-risky bond. The problem was analysed by Wilmott (1998), Tavella and Randall (2000) and Schönbucher (2003).

We assume that the value of a default-risky zero-coupon bond depends on the default-free interest rate $r(t)$, default intensity $\lambda(t)$, and time t. To describe the dynamics of the model, we also assume the following stochastic process of the default-free interest rate $r(t)$, and of the default intensity $\lambda(t)$

[38] In Schönbucher (2000).

$$dr(t) = \mu(t)dt + \sigma(t)dW(t) \quad (1.94)$$

and
$$d\lambda(t) = \mu_\lambda(t)dt + \sigma_\lambda(t)dW_\lambda(t) \quad (1.95)$$

with
$$dW(t)dW_\lambda(t) = \rho dt \quad (1.96)$$

where

$W(t)$ and $W_\lambda(t)$ are the standard Brownian motions of the default-free interest rate and default intensity,

$\mu(t)$ and $\mu_\lambda(t)$ are their drifts, and

$\sigma(t)$ and $\sigma_\lambda(t)$ are their volatilities.

We also assume that the random recovery $\varphi(\tau)$ is described by the marked Poisson process, defined in section 1.8.6.

With the default-free interest rate, the intensity rate and the marked Poisson processes so defined, we move to price a default-risky zero-coupon bond with the following payoff structure.

(1) The final payoff at time T, is $D(T)$, which is a constant face value.

(2) No payment is received by the default-risky zero-coupon holder before maturity or default.

(3) At the time of default τ, the payoff is a function $g(r(t), \lambda(t), \varphi(\tau), \tau)$ of the interest rate process $r(t)$, the default intensity $\lambda(t)$, and of the random recovery rate $\varphi(\tau)$.

(4) In the time between t and $\tau \wedge T$, the default-risky zero-coupon bond is represented as function of $D(r(t), \lambda(t), t)$ for $t \leq \tau \wedge T$.

By Ito's lemma, the partial differential equation satisfied by D is as follows,[39, 40]

$$dD = \\ = \left(\frac{\partial D}{\partial t} + \mu\frac{\partial D}{\partial r} + \mu_\lambda\frac{\partial D}{\partial \lambda} + \frac{1}{2}\sigma_\lambda^2\frac{\partial^2 D}{\partial \lambda^2} + \frac{1}{2}\sigma^2\frac{\partial^2 D}{\partial r^2} + \rho_{r,\lambda}\sigma\sigma_\lambda\frac{\partial^2 D}{\partial r \partial \lambda}\right)dt \\ + \int_{[0,1]}\left(g(\lambda, r, t, \varphi) - D(\lambda, r, t)\right)\mu(t, e, \varphi)de dt + \sigma\frac{\partial D}{\partial r}dW(t) + \sigma_\lambda\frac{\partial D}{\partial \lambda}dW_\lambda(t) \quad (1.97)$$

Since the default-risky zero-coupon bond defaults when N jumps, the integral represents the decrement in value of the default-risky zero-coupon bond at default, which will take D to the payoff $g(\cdot)$. The payoff at default $g(\cdot)$ is contained in the dynamics of D, while in most options the payoff is on the boundary conditions only.

[39] From Schöbucher (2003) p. 202, Willmott (1998) p. 569, and Tavella and Randall (2000) p. 36.
[40] For the Ito's lemma in presence of jumps, see Jacod and Shiryaev (1988).

Under the fundamental pricing rule that the expected rate of return of holding any security under the martingale measure $\mathbb{Q}$ is the default-free interest rate $r(t)$, the risk-neutral expectation of dD, $\mathbb{E}^{\mathbb{Q}}(dD)$, must be equal to $rDdt$, and (1.97) becomes

$$\begin{aligned} rDdt &= \\ &= \left(\frac{\partial D}{\partial t} + \mu \frac{\partial D}{\partial r} + \mu_\lambda \frac{\partial D}{\partial \lambda} + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2 D}{\partial \lambda^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial r^2} + \rho_{r,\lambda}\sigma\sigma_\lambda \frac{\partial^2 D}{\partial r \partial \lambda} \right) dt \\ &+ \int_{[0,1]} \big(g(\lambda, r, t, \varphi) - D(\lambda, r, t)\big) f(e) de \lambda dt \end{aligned} \tag{1.98}$$

where the expectation of the jump measure $\mathbb{E}^{\mathbb{Q}}(\mu(t,e,\varphi))$ is replaced with its predictable compensator measure $f(e)de\lambda dt$ in (1.51).

Removing from (1.98) dt, and remembering that $\int_{[0,1]} f(e)de = 1$, the pricing PDE is

$$\begin{aligned} &\frac{\partial D}{\partial t} + \mu \frac{\partial D}{\partial r} + \mu_\lambda \frac{\partial D}{\partial \lambda} + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2 D}{\partial \lambda^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial r^2} + \rho_{r,\lambda}\sigma\sigma_\lambda \frac{\partial^2 D}{\partial r \partial \lambda} \\ &- (\lambda + r)D + \lambda \int_{[0,1]} \big(g(\lambda, r, t, \varphi)\big) f(e) de = 0 \end{aligned} \tag{1.99}$$

Schönbucher (2003) called $\int_{[0,1]} \big(g(\lambda, r, t, \varphi)\big) f(e) de$ the *locally expected default payoff*.

The PDE in (1.99) is a partial integro-differential equation PIDE. To simplify the solution procedure Wilmott (1998) and Tavella and Randal (2000) assumed a known recovery value

$$\int_{[0,1]} \big(g(\lambda, r, t, \varphi)\big) f(e) de = z \tag{1.100}$$

For example, if there is a default, the value of the default-risky zero-coupon bond drops to $zD(\tau^-)$, where $D(\tau^-)$ is the value of the default-risky zero-coupon bond just before the time of default τ.

The assumption simplifies the solution of the partial integro-differential equation into a partial differential equation, and (1.99) becomes

$$\begin{aligned} &\frac{\partial D}{\partial t} + \mu \frac{\partial D}{\partial r} + \mu_\lambda \frac{\partial D}{\partial \lambda} + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2 D}{\partial \lambda^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial r^2} + \rho_{r,\lambda}\sigma\sigma_\lambda \frac{\partial^2 D}{\partial r \partial \lambda} \\ &- (\lambda + r)D + \lambda z D = 0 \end{aligned} \tag{1.101}$$

The change in value can also be written as known loss fraction lD, and we have

$$rD + \lambda D - \lambda z D = rD + \lambda l D \quad \text{with } l = 1 - z. \tag{1.102}$$

1.9.5 A default protection option with known recovery

The investor of the default-risky zero-coupon bond in (1.101) faces two types of risk: firstly, in case of default, he receives at the time of default a random recovery of the bond face value, secondly, in case the perceived risk of default rises, the credit spread will widen, and the default-risky zero-coupon bond price drops.

Credit spread options are designed to give protection in case of spread widening. The put version gives the holder the right to sell the default-risky zero-coupon bond for a pre-specified yield spread H (the strike of the put option) in case the yield spread between the default-risky zero-coupon bond and the default-free zero-coupon bond becomes higher than H before the option expires.

To hedge the default and the credit spread risk of the default-risky zero-coupon bond, the investor needs to buy a default protection option (put option) with the following variables and payoff structure.

(1) φ is the bond recovery in case of default, assumed here to be equal to a constant z.

(2) $B(t,T_B)$ is the default free zero coupon bond that matures at time T_B.

(3) $D(t,T_D)$ is the default-risky zero-coupon bond that matures at time T_D, with $T_D = T_B$.

(4) H is the strike (a constant).

(5) When there is no default until the option maturity, T_O, with $T_O < T_D = T_B$, the option pays

$$\max[0, HB(T_O,T_B) - D(T_O,T_D)] = [HB(T_O,T_B) - D(T_O,T_D)]^+$$

(6) If default occurs before the option maturity, the option has the following payoff

$$HB(\tau,T_B) - \varphi D(\tau,T_D)$$

and with default the option comes to an end.

(7) No payment is received by the option holder before maturity or default.

Such payoff protects against a drop in the value of the default-risky zero-coupon bond as well. In fact, when the perceived risk of default rises, the credit spread will widen, and at maturity, the option will be in the money.

In the time between t and earliest of default time τ and maturity of the option T_O, the option price is represented as function of $C(r(t),\lambda(t),t)$ for $t \le \tau \wedge T_O$.

The PDE of this option, is very similar to the one found in (1.99) for the default-risky zero-coupon bond. We have only to change D with C, and substitute the known default payoff of the default-risky zero-coupon bond zD with the option payoff in case of default, $[HB - zD]^+$,

$$\frac{\partial C}{\partial t}+\mu\frac{\partial C}{\partial r}+\mu_{\lambda}\frac{\partial C}{\partial \lambda}+\frac{1}{2}\sigma_{\lambda}^{2}\frac{\partial^{2} C}{\partial \lambda^{2}}+\frac{1}{2}\sigma^{2}\frac{\partial^{2} C}{\partial r^{2}}+\rho_{r.h}\sigma\sigma_{\lambda}\frac{\partial^{2} C}{\partial r\partial \lambda} \\ -(r+\lambda)C+\lambda[HB-zD]^{+}=0 \tag{1.103}$$

The PDEs in (1.101) and (1.103) can be solved with the numerical method of Crank-Nicolson. To correctly build the numerical scheme of the option in (1.103), we first need to build the numerical scheme of the default-risky $D(t,T)$ and default-free zero-coupon $B(t,T)$ bonds, to know their values on the grid, before moving to build the numerical scheme of the option in (1.103).

The numerical scheme is broken down in three steps.

(1) Solve the PDE of the default-free zero-coupon bond $B(t,T)$, which is a well known equation[41]

$$\frac{\partial B}{\partial t}+\mu_{r}\frac{\partial B}{\partial r}+\frac{1}{2}\sigma_{r}^{2}\frac{\partial^{2} B}{\partial r^{2}}-rB=0 \tag{1.104}$$

(2) Solve the PDE of the default-risky zero-coupon bond $D(t,T)$ in (1.101), and

(3) Solve the PDE of the default protection option $C(t,T)$ in (1.103).

1.10 Default correlation

In the previous pages, we saw how structural and reduced-form approaches distinctively model the credit event of one obligor or credit. This section reviews two models which incorporate the default dependence between credits within a reduced-form approach: Duffie and Garleanu (1999) and Jarrow and Yu (2001). They are essentially two distinct efforts to resolve the issue of low default correlation, typical of reduced-form models. The first is an example of *joint jumps* in the default intensities and the second of *joint defaults*. Before moving on, we remind the definition of times of default in a portfolio of credits.

1.10.1 Defining the times of default

We consider *n* credits with associated random times of default $\tau_1,\ldots,\tau_n$ defined on a given reference complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$. As in the spirit of Lando (1998), to model the credit

[41] From the derivation of the PDE, see Wilmott (1999).

event of default, we associate to each credit i, an $(\mathcal{F}_t)$-adapted,[42] non-negative and continuous process $\lambda_i(t)$ and set

$$\Lambda_i(t) = \int_0^t \lambda_i(s)ds \qquad \text{with a.s. } \int_0^t \lambda_i(s)ds < \infty . \tag{1.105}$$

Definition 1.15: *In a reduced-form set-up, the time of default of credit i is defined as*

$$\tau_i = \inf\left\{ t : \int_0^t \lambda_i(s)ds \geq Z_i \right\} \tag{1.106}$$

where $\lambda_i(t)$ is the intensity rate of credit i, and Z_i is an exponential random variables of parameter 1, which can be seen as a stochastic barrier.

There is an alternative and equivalent representation of the time of default in (1.106).

Definition 1.16: *Given a Cox process $N_i(t)$ with intensity $\lambda_i(t)$, we set*

$$\tau_i = \inf\{t \geq 0; N_i(t) > 0\} \tag{1.107}$$

and τ_i is a stopping time with respect to the filtration generated by the Cox process $N_i(t)$. The two definitions in (1.106) and (1.107) are equivalent since we have chosen $\lambda_i(t)$ as intensity of the Cox process $N_i(t)$.

In general, the τ_i s are not stopping times with respect to $(\mathcal{F}_t)$. For this reason, we consider the filtration generated by the survival process $\mathcal{H}_{i,t} = \sigma\{1_{\{\tau_i \leq s\}} : 0 \leq s \leq t\}$, which contains information on the credit event of the credit i, and we introduce the enlarged σ-field, $(\mathcal{G}_t)_{t\geq 0} = \mathcal{F}_t \vee \mathcal{H}_{i,t} \vee \mathcal{H}_{2,t} .... \vee \mathcal{H}_{n,t}$, which represents all information available to an agent at time t.

1.10.2 Duffie and Garleanu model: *joint jumps*

For baskets of credits, the correlation is an important issue and Duffie and Garleanu (1999) proposed to solve it by allowing correlation through the state variables.

[42] We denote by $(\mathcal{F}_t)$, the filtration generated by all state variables, economic, interest rates, currencies, etc., including the intensity process $\lambda_i(t)$.

To study the implication of changing correlation in the times of default of a portfolio of credits, Duffie and Garleanu assumed an *affine*[43] dependence on a set of state variables, which have an *affine* specification. They used the fact that a basic affine model can be written as the sum of independent basic affine models, provided that, the mean rate of reversion, the volatility coefficient and the mean of the jump sizes are common to the underlying independent basic affine processes.[44] As they wrote, this is to ensure a parsimonious and a tractable 1-factor model[45] to model the probability of default of each credit.

They assumed a collateral portfolio of n credits, whose times of default $\tau_1, \tau_2, .., \tau_n$ have basic affine intensity processes $\lambda_1, \lambda_2, .., \lambda_n$. The correlation is introduced in the following way

$$\lambda_i = X_i + X_C \tag{1.108}$$

where X_i and X_C are independent basic affine processes, meaning nonnegative affine process $X_i(t)$ and $X_C(t)$, which satisfy the following

$$dX_i(t) = \kappa(\theta_i - X_i(t))dt + \sigma\sqrt{X_i(t)}dW_i(t) + J_i dN_i(t) \tag{1.109}$$

and

$$dX_C(t) = \kappa(\theta_C - X_C(t))dt + \sigma\sqrt{X_C(t)}dW_C(t) + J_C dN_C(t) \tag{1.110}$$

where the jumps sizes J_i and J_C are independent and exponential distributed with common jump mean $\mu_J = \mu_i$, the jump times are those of independent Poisson processes with different intensity rates ℓ_i and ℓ_C, and the parameters κ and σ are common to the underlying pair of independent affine processes. All remaining parameters are the same as already specified in (1.65).

They explained X_i as the state variable, which governs the idiosyncratic default of credit i, whereas X_C is the state variable common to all $\lambda_i(t)$, which explains common aspects of economic performances such as, industry, sector and geographic location of the credits. The $n+1$ underlying state variables $X_1, X_2, .., X_n, X_C$ are assumed independent. Because $\lambda_i(t)$ is the sum of X_i and X_C, it is itself a basic affine process with parameters $(\kappa, \theta_i + \theta_C, \sigma, \mu, \ell_i + \ell_C)$.

In this model, the risk-neutral survival probability of credit i has an analytic expression equal to

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[43] For a review of affine processes, see Duffie (2002).
[44] In Duffie and Singleton (2003) p. 256. For a proof, see Duffie and Garleanu (2001).
[45] Duffie and Garleanu also extend the one-factor to handle multifactor risk.

$$\mathbb{Q}(\tau_i > T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_0^T \lambda_i(s, X(s))ds\right)\right)$$
$$= \exp(\alpha_i(T) + \alpha_C(T) + \beta_i(T)X_i(0) + \beta_C(T)X_C(0)) \quad (1.111)$$

where the expressions of α_i, α_C, β_i and β_C are found in their paper.

If we move from the expression of the survival probability of one single credit in (1.111), to the expression of the joint survival distribution, the issue becomes more complex. In general, the valuation of a basket of credits will depend on the order in which credits default. In a basket of fifty credits, there are more than 10^{16} different ways in which, the first 10 defaults can happen.[46] To reduce the number of permutations, one would use the Moody's *diversity score*, which substitutes the original collateral of n credits with a hypothetic collateral of credits in which all credits are independent and identical.[47] Unfortunately, in this way the correlation is lost.

To see the approach of Duffie and Garleanu, we indicate the default probability of credit j as $p_j = \mathbb{Q}(\tau_j \leq T)$, the complementary survival probability as $s_j = \mathbb{Q}(\tau_j > T)$, the probability that at least one of the first j credits defaults is $d_j = \mathbb{Q}(p_1 \cup p_2 \cup ... \cup p_j)$, the number of defaults as M, and the number of credits in the collateral portfolio as n.

Duffie and Garleanu introduced the following important assumptions in their model.

(1) The probability that more than one default event occurs at the same time is zero, $\mathbb{Q}(\tau_j = \tau_i) = 0$, for $i \neq j$.

(2) All correlated credits have the same intensity $\lambda_j(t) = \lambda_i(t)$ and the same notional $A_j = A_i$, for $i \neq j$.

(3) $X_1, X_2, ..., X_n, X_C$ are $n+1$ independent basic affine processes.

Under the first assumption, the intensity associated with the first default, $\tau_{(1)} = \min(\tau_1, \tau_2, ..., \tau_j)$ is

$$\lambda_{(1)} = \min(\tau_1, \tau_2, ..., \tau_n) = \lambda_1 + \lambda_2 + ... + \lambda_n \quad (1.112)$$

and the probability that at least one of the first j credits defaults d_j, is equal to

$$d_j = 1 - \mathbb{Q}(\tau > T) = 1 - \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_0^T \sum_{i=1}^{j} \lambda_i(t)dt\right)\right) \quad (1.113)$$

---

[46] The number of unique ways (given that position is important) in which 50 objects (credits) can be placed in 10 positions is called permutations, and it is given by 50!/(50-10)!=50x49x...x41.

[47] See Cifuentes and O'Connor (1996), and Gluck and Remeza (2000). We will also describe in more detail the Moody's diversity score in chapter two.

With the second assumption, they managed to find an analytical expression for the unconditional joint default distribution, which is the probability of M defaults out of n credits,[48] equal to

$$Q(M = k) = \binom{n}{k} Q(p_1 \cap \ldots \cap p_n \cap s_{k+1} \cap \ldots \cap s_n) \tag{1.114}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{1.115}$$

and
$$Q(p_1 \cap \ldots \cap p_n \cap s_{k+1} \cap \ldots \cap s_n) = \sum_{j=1}^{n} (-1)^{(j+k+n+1)} \binom{k}{n-j} d_j \tag{1.116}$$

Under the third assumption, Duffie and Garleanu were able to find an analytic expression for the expectation operator in (1.113) as

$$d_j = 1 - \exp(\alpha_C(T) + \beta_C(T)X_C(0) + j\alpha_i(T) + j\beta_i(T)X_i(0)) \tag{1.117}$$

where α_C, β_C, α_i and β_i, are given explicitly and can be read in their paper.

Critically, they recognised that the case in (1.113) to (1.117) is a special one, and the way to prepare more realistic analytic expressions, is to form sub-groups of credits, so as to reduce the number of permutations of credits to be considered.

This is exactly the approach taken by Huge (2001). He initially decomposed the *m*-to-defaults[49] in a portfolio of *n* credits, into a portfolio of 1st-to-default contracts, each of which is priced using affine intensities. In this way, the order in which, the credits of the original collateral default, is no longer important, and the pricing approach is based on a purely arbitrage argument. Huge achieved a remarkable reduction of the necessary computational time, but still insufficient for large portfolio. For example, if there are 50 credits and the first *m*-to-defaults is 10, then the number of 1st-to-default to be calculated is[50] $\sum_{j=0}^{m} \binom{n}{j} \cong 10^{10}$, against the case of 10^{16} if we want to look at all unique ways in which 50 credits can be placed in the first *10*-to defaults.

Huge found another way to simplify the computation. Instead of making the whole collateral as homogenous, as Moody's *diversity score* does, he divided the collateral in heterogeneous subsets

48 This extends the traditional binomial-expansion formula to include correlation.

49 He also extends his methodology to CDOs by treating the *m*-to-default as the first *m* defaults in a CDO.

50 The number of unique ways (given that position is not longer important) in which 50 objects (credits) can be placed in 10 positions is called combinations, and it is given by:
$50!/[10!(50-10)!] = \binom{50}{10}$.

of homogeneous credits. In this way, he achieved considerable speed-ups in pricings, without making the heterogeneous subsets independent.[51]

One point we think Huge did not clarified is how the joint distribution of the heterogeneous subsets compares with the joint distribution of the original collateral.

As an alternative to analytical expressions, Duffie and Garleanu explored the route of Monte Carlo simulations. Their computation consists of simulating the paths of $X_1, X_2,.., X_n, X_C$ and $\lambda_1, \lambda_2,.., \lambda_n$ for time steps of one week. In this way, they were able to simulate the compensator Λ_i in (1.105). At the same time, they simulated n standard unit-mean exponential distributed variables $Z_1, Z_2,.., Z_n$. The time of default τ_i was set as in (1.106).

1.10.3 Jarrow and Yu model: *joint default*

To model dependent defaults, Jarrow and Yu (2001) proposed to make a distinction between primary firms and secondary firms. In the former group, they included those firms whose probabilities of default are only influenced by macroeconomic conditions, and not by the default of other firms. In the second group, they included those firms whose default probabilities depend in part or entirely on the credit status of primary firms. The construction of their model is based on the assumption of asymmetric information in the market place.

For the primary firms, they assumed the existence of a family of $(\mathcal{F}_t)$-adapted intensity processes $\lambda_1, \lambda_2,.., \lambda_k$ that produces a collection of $(\mathcal{F}_t)$-conditionally independent random times of default $\tau_1, \tau_2,..., \tau_k$ with the following method

$$\tau_i = \inf\left\{ t > 0 : \int_0^t \lambda_i(u)du \geq -\ln \xi_i \right\} \tag{1.118}$$

where ξ_i, $i = 1,..., k$ are mutually independent identically uniform distributed random variables, under the martingale measure $\mathbb{Q}$, and with $-\ln \xi_i = Z_i$, where Z_i is a unit exponential random variable, so to link (1.118) with (1.106).

For the secondary firms, they assumed that the probability space is large enough to support a set of ξ_i, $i = k+1,..., n$ of mutually independent identically uniform distributed random variables. These variables are independent not only from the filtration $(\mathcal{F}_t)$, but also from the already constructed times of default of the primary firms.

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[51] For the pricing expressions for the n^{th} to default and CDO, see Huge (2001).

The times of default for the secondary firms, $\tau_{k+1}, \tau_{k+2}, ..., \tau_{k+p}$, are defined by means of the same method as in (1.118)

$$\tau_{k+j} = \inf\left\{ t > 0 : \int_0^t \lambda_{k+j}(u)du \geq -\ln \xi_{k+j} \right\} \tag{1.119}$$

where the intensities $\lambda_{k+1}, \lambda_{k+2}, .., \lambda_{k+p}$ are now given by

$$\lambda_{k+j}(t) = \mu_{k+j}(t) + \sum_{l=1}^{k} \nu_{k+j,l}(t) 1_{\{\tau_l \leq t\}} \tag{1.120}$$

In (1.119), the ξ_{k+j}, $j = 1,..., p$, are mutually independent identically uniform distributed random variables, and μ_{k+j} and $\nu_{k+j,l}$ are $(\mathcal{F}_t)$ -adapted stochastic processes. In case the default of the l^{th} primary firm does not affect the default of the j^{th} secondary firm, we set $\nu_{k+j,l}$ to zero.

There are some important issues with this model. To investigate them, we let $(\mathcal{G}_t)_{t\geq 0} = \mathcal{F}_t \vee \mathcal{H}_{i,t} \vee \mathcal{H}_{2,t} .... \vee \mathcal{H}_{k,t}$ be the enlarged filtration, and we let $(\mathcal{F}_t^{\sim})_{t\geq 0} = \mathcal{F}_t \vee \mathcal{H}_{k+1,t} \vee \mathcal{H}_{k+2,t} .... \vee \mathcal{H}_{n,t}$ be the filtration generated by the reference filtration $(\mathcal{F}_t)$ and the observations on defaults of secondary firms. Then, the times of default of primary firms are no longer conditionally independent when the filtration $(\mathcal{F}_t)$ is replaced by $(\mathcal{F}_t^{\sim})$, as the defaults of primary firms become dependent on defaults of secondary firms. Schönbucher (2003) pointed out that, "the resulting joint process of the default indicators $(N_1(t),.., N_n(t))$, is not a Cox process any more, as from the joint intensities, we can draw conclusions about the time of default".

We also observe that it will not be easy to establish the criteria, which differentiates primary from secondary firms.

A third issue is how to calibrate $\mu(t)$ and $\nu(t)$, in (1.120), to historical observations of default frequencies or CDS premiums.

Jarrow and Yu initially considered the very simple case of two firms, A and B. A is the primary firm, and B is the secondary firm. Then, they moved to analyse the multiple case, where A and B are two primary firms and C is the secondary firm dependent simultaneously on A and B. In their examples, the default process is assumed independent of the default-free interest rate. In what follows, we will only analyse the case of two firms.

The default intensity of firm A is assumed constant λ_A, so that (1.118) can be written as

$$\tau_A = \inf\{t > 0 : \int_0^t \lambda_A(u)du = \lambda_A t \geq -\ln \xi_A\} \tag{1.121}$$

For the secondary firm B, the default intensity is simplified and satisfies the following

$$\lambda_B(t) = b_1 1_{\{\tau_A > t\}} + b_2 1_{\{\tau_A \le t\}} \quad (1.122)$$

where $b_1 > 0$ and $b_2 > 0$ are two constants and must be chosen ensuring the positivity of the default intensity in (1.122). Thus, the time of default of B is defined as

$$\tau_B = \inf\{t > 0 : \int_0^t \lambda_B(u) du \ge -\ln \xi_B\} \quad (1.123)$$

where ξ_A and ξ_B are mutually independent.

Jarrow and Yu prepared the prices of two default-risky bonds, respectively issued by A and by B. The default-risky bond issued by A is equal to

$$D_A(t,T) = B(t,T)(\varphi_A + (1-\varphi_A)e^{-\lambda_A(T-t)} 1_{\{\tau_A > t\}}) \quad (1.124)$$

For the default-risky bond issued by B, with the default intensity in (1.122), in the case where A has not defaulted, we have

$$D_B(t,T) = B(t,T)(\varphi_B + (1-\varphi_B)K(\lambda_A, \lambda_B, a)(T-t)1_{\{\tau_B > t\}}) \quad (1.125)$$

and where A has defaulted,

$$D_B(t,T) = B(t,T)(\varphi_B + (1-\varphi_B)e^{-a(T-t)} 1_{\{\tau_B > t\}}) \quad (1.126)$$

where, $K(\lambda_A, \lambda_B, a) = (1 + a(T-t))e^{-(a+b_1)(T-t)}$ if $b_2 = a$

and $K(\lambda_A, \lambda_B, a) = \dfrac{b_2 e^{-(a+b_1)(T-t)} - a e^{-(b_1+b_2)(T-t)}}{b_2 - a}$ if $b_2 \ne a$.

We can summarise the following properties in the two-firm set-up.[52]

(1) $\lambda_A(t)$ is the intensity of τ_A, with respect to the filtration $(\mathcal{F}_t)$,

(2) $\lambda_B(t)$ is the intensity of τ_B, with respect to the filtration $(\mathcal{F}_t) \vee (\mathcal{H}_{1,t})$ and

(3) $\lambda_A(t)$ is not the intensity of τ_A, with respect to the filtration $(\mathcal{F}_t) \vee (\mathcal{H}_{2,t})$.

[52] In Bielecki *et al.* (2003) p. 69.

Chapter 2: A survey of CDOs and their use in bank balance sheet management

2.1 Introduction

This chapter explores the market of collateralised debt obligations (CDOs) and synthetic CDOs and their use in bank balance sheet management. We first review different types of CDOs used in capital markets and their economic rationales and then discuss the growth in synthetic CDOs under structural and balance sheet management perspectives. Following this, we analyse the CDO equity piece and how it can be used in portfolio management. The last section is dedicated to Moody's table of idealised cumulative expected losses.

2.2 The CDO structure

A CDO is a special purpose company or vehicle (SPV), complete with assets, liabilities and a manager. Typically, the CDO's assets consist of a diversified portfolio of illiquid and default-risky assets such as high yield bonds (CBO) or bank leverage loans (CLO).[1]

We have set up a typical CDO structure in Figure 2.1. The assets, also called the collateral portfolio, are transferred to the SPV that funds the purchase, from cash proceeds of the notes it has issued.

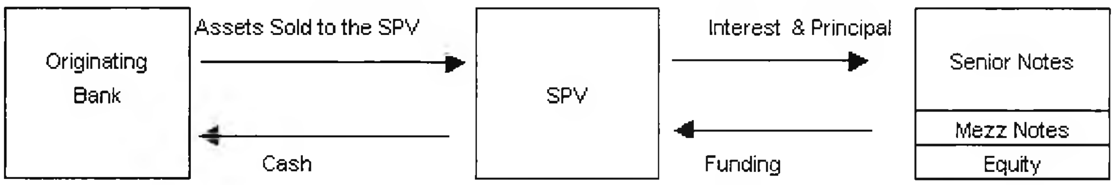


Figure 2.1: CDO diagram.

The CDO structure allocates interest income and principal repayment from a pool of different debt instruments to a prioritised collection of securities notes called tranches. Senior notes are paid before mezzanine and lower rated notes. Any residual cash flow is paid to the equity investors. This makes the senior notes significantly less risky than the collateral.

[1] Most recently, CDO technology has been extended to emerging market debts, structured finance securities, commercial real estate-linked debt, distressed assets, and last to arrive, private equity funds.

On every payment date, the equity investors receive cash distributions after the scheduled payments and other costs have been paid off. The equity is also called the "first-loss" position in the collateral portfolio. This is because it is exposed to the risk of the first pound loss in the portfolio.

The CDO rating is based on its ability to service the notes, with the cash flows generated by the collateral portfolio. The service of the notes depends on the diversification and quality of the collateral, subordination and structural protection.

As we move down the CDO's capital structure, the level of risk increases. The equity investors bear the highest risk, and their gain lies in the residual cash, also called excess spread between the interest income from the assets sold to the SPV, and the interests due to the notes plus any non-interest costs and losses.

The equity investors also have the option to call the CDO notes after the end of the non-call period, which in most cases lasts three to five years. In this case, the assets are liquidated and the proceeds are used to pay down the notes and interests due.

The typical CDO has a ramp-up period, a reinvestment period and an amortising period. During the ramp-up period the collateral portfolio is formed. In the reinvestment period, all proceeds received from the assets matured, are promptly reinvested in new assets. The amortising period is the terminal phase, when the notes are repaid in order of seniority using the proceeds from the collateral portfolio.

In the ramp-up period, there is the risk of negative carry, which is the condition in which the cost of borrowing money exceeds the return obtained from it. This is because, before the CDO is fully ramped-up, the proceeds received from issuing the notes, are invested in highly rated and secured bonds, such as US Government Bonds and Pfandbriefe,[2] with very little return.
The above problem is rectified in synthetic CDOs,[3] since the notes issued represent only a part of the collateral portfolio.

Figure 2.2 displays an example capital structure, where the high yield bonds collateralise CDO liabilities.

[2] For the list of these securities see Deacon (2003) and updates from press release of Bank of International Settlements, URL: http//www.bis.org.
[3] In a Synthetic CDO the credit risk is synthetically transferred from the originating bank to capital markets, via Credit default swaps. See section 2.8 for synthetic CDOs.

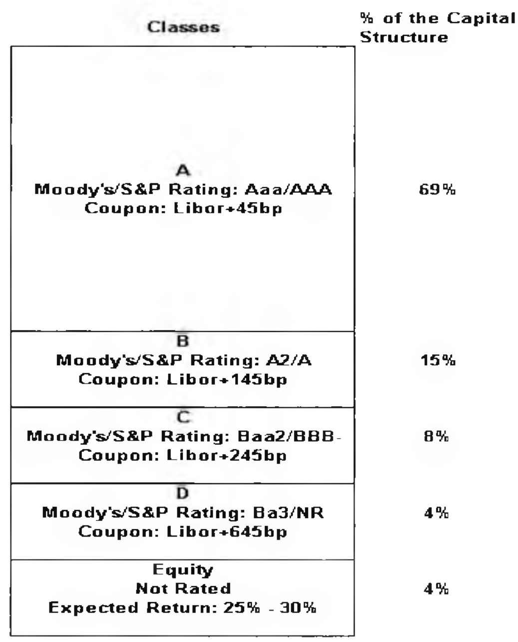
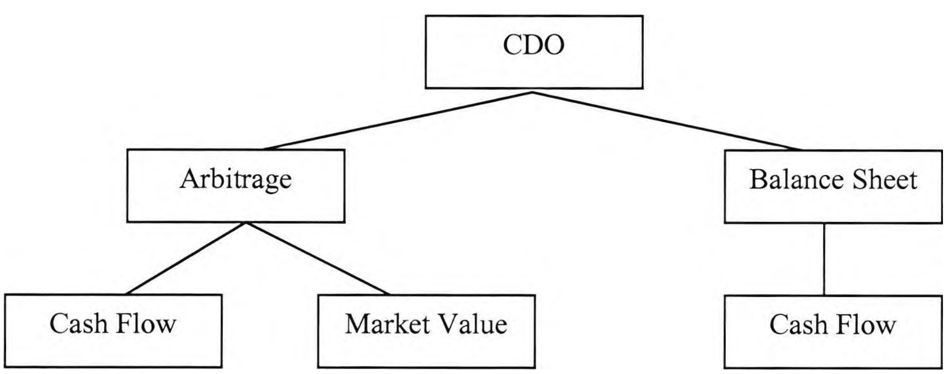


Figure 2.2: CDO capital structure.

2.3 Arbitrage and Balance Sheet CDOs

Most CDOs can be placed into one of two main groups: arbitrage and balance sheet transactions. Figure 2.3 shows the conceptual breakdown between the two structures.



Source: Morgan Stanley (2001)[4]

Figure 2.3: CDO structure.

A cash flow CDO is one where the collateral portfolio is not actively traded by the CDO manager.[5] The CDO performance depends on whether the notes are fully repaid and there are no

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[4] Schorin and Weinreich (2001).

[5] We will look at the role of the CDO manager in section 2.6.

interest payment delays. Losses are the main source of risk, and not the volatility of the market value of the collateral portfolio.

A market value CDO is one where the performance of the CDO notes is primarily a mark-to-market performance, i.e. all securities in the collateral are marked to market with high frequency. In the case where the collateral market value drops below a certain value, called the market value trigger, the CDO manager is obliged to pay the CDO notes by selling all the assets of the collateral portfolio. The market value trigger is set to create a reserve between the market value of the collateral portfolio and the par values of the CDO notes.

Because the collateral of market value CDOs is constantly traded, the CDO performance depends upon the CDO manager, who will buy and sell the collateral assets. As part of normal due diligence, a potential market value CDO investor needs to evaluate the ability of the manager, the institutional structure around him, and the suitability of the management style to a leveraged investment vehicle.

Balance-sheet cash flow CDOs are structures for the purpose of regulatory capital relief, where the assets of the collateral portfolio are lower yield debt instruments. The regulatory capital relief reduces funding costs or increases return on equity, by selling to the SPV the assets that take too much regulatory capital, and thus removing them from the originator's balance sheet.

These transactions rely on the quality of the collateral that is made of guaranteed bank loans with a very high recovery rate.[6] In this case the CDO takes the name of collateralised loan obligations (CLOs). The relatively low coupon of these assets results in a smaller excess spread[7] than the arbitrage CDOs. Given their relative superior credit quality when compared to assets used in arbitrage CDOs, they require less subordination.[8] The size of a typical balance sheet CLO is in general larger than an arbitrage CDO, as the transaction must have an impact on the return on regulatory capital (RORC) of the originating bank looking for capital relief. Because the collateral is represented by bank loans with an opaque market value, CLOs are only in the form of balance sheet-cash flow.

More recently, active credit portfolio management has been cited as another reason behind CLOs. Through active credit portfolio management, banks are able to measure the risk-adjusted efficiency of their credit portfolio.[9] The CLO is now the tool for a bank to redeploy freed-up resources in higher yield and better diversified instruments.

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[6] Statistics regarding the recovery rates for CDO have been published in, S&P (20 Feb 2004), Recovery Rates for credits in global synthetic CDOs Q4 2003, whereas the statistics of corporate defaults have been published in, S&P (Feb 2004), Corporate Defaults in 2003 Recede from Recent Highs. Both reports can be found on URL: http//www.standardandpoors.com.

[7] Excess spread is the difference between the interests earned from the collateral assets and the expenses and the interests paid to the notes. We will look at the excess spread in section 2.4.

[8] Subordination is the prioritisation of investor claims, and it defined in section 2.4.

[9] See Jobst (2002) on how to use active credit portfolio management to remove credit risk from the bank balance sheet.

Market value-arbitrage CDOs go through a very extensive trading by the CDO manager, necessary to exploit perceived price appreciations. This type of CDO relies on the market value of the collateral portfolio, which is monitored on a daily basis. Every security traded in capital markets, with a liquid market price, can be included in this type of CDO. The important aspect is the collateral manager's capacity to generate a high total rate of return. In these CDOs, the CDO manager has a great deal of flexibility in terms of the assets included in the deal. He can also increase or decrease the funding amount, which changes the leverage of the structure, during the reinvestment period.

In cash flow-arbitrage CDOs, the collateral assets have been purchased at market price and are negotiable instruments, primarily bonds. However, syndicated loans, usually tradable, have been included in past transactions. In the majority of the cases, the collateral assets have already been used in a previous CDO, and by merging the collateral with a more diversified pool, or by simply re-tranching the original notes, it is possible to extract an arbitrage benefit. Unlike market value-arbitrage CDOs, the collateral assets are not traded very frequently.

The aim of arbitrage CDOs is to capture the arbitrage opportunity that exists in the credit-spread differential, between the high yield collateral and the highly rated notes.[10] Most arbitrage CDOs are private deals, where size is not large and the number of assets included in the collateral portfolio are very limited compared to the cash flow type.

2.4 Credit enhancement

Credit enhancement is a way to protect senior note and mezzanine note investors from quality deterioration of the collateral portfolio. It is ensured through prioritisation of investor claims, also known as subordination, over-collateralisation and excess spread.

2.4.1 Subordination

The principal repayment and the interest income of the collateral portfolio are generally distributed sequentially to the note investors in order of their seniority, *via* the principal waterfall and the interest waterfall.

The interest waterfall specifies the order in which interest income is paid in the CDO. The fees, the hedge costs such as interest rate swap payments are paid before the interest due to the senior notes and the interest due to the mezzanine notes. Any additional income is called extra spread and is paid to the equity investors.

[10] For a detail analysis on how to create an arbitrage CDO, see Goodman (2001).

The interest waterfall is subject to changes where certain triggers[11] are breached. For example, a trigger can be defined in such a way that no more than a certain percentage of the par value of the collateral portfolio is rated below BBB. In these instances, the interest proceeds that would normally go to pay the interest due to the mezzanine notes are used to pay down (turbo or accelerate) the senior notes. This form of early amortisation ensures that the investors of the senior notes are insured against further reduction of the value of the collateral portfolio due to rising defaults or downgrades.

The principal waterfall is the distribution of the principal repayment. At the top, are the senior notes. The mezzanine notes are repaid as soon as the senior notes are fully redeemed. In this way, the performance of the mezzanine notes is subordinated to the good performance of senior notes.

2.4.2 Over-collateralisation

Over-collateralisation (OC) provides a further protection to the senior notes by imposing two quality tests on the collateral portfolio: the par value test and the interest coverage test.

The par value test requires that the collateral portfolio A, is sufficient to repay the various notes at any future date and until the notes are redeemed. For example, a par value test may be applied on the senior note L_S, so that they are never greater than a percentage β, of the collateral portfolio minus defaults. In this case, it is defined as

$$[A_t - defaults_t]\beta - L_{S,t} > 0 \quad \text{with} \quad 0 < \beta \leq 1 \tag{2.1}$$

The par value test is also applicable to the mezzanine note L_M. In this case, the percentage is selected at a lower rate, and it is defined as

$$[A_t - defaults_t - L_{S,t}]\alpha - L_{M,t} > 0 \text{ with } \quad 0 < \alpha < \beta \leq 1 \tag{2.2}$$

The interest coverage test guarantees that all expenses and interests due on the liability side are fully covered by the interest proceeds from the collateral portfolio.[12] We indicate as I_A the interests earned on the collateral assets, as Ex the expenses and as I_L the interests on the notes, and we write the test as

$$I_A - Ex - I_L \geq 0 \tag{2.3}$$

If either test is breached, the interest waterfall is changed, and the repayment of the senior notes (or mezzanine notes) is accelerated.

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[11] The triggers are all defined in the Offering Circular of the CDO transaction.

[12] See Moody's (22 March 2002), Collateralised Debt Obligations Indices: January 2002, for a report on par value tests on current CDOs in US and Europe. For example for arbitrage cash flow CDOs, the senior note average par value OC test was in January 2002 equal to 126.33%, whereas the senior note average interest rate coverage was, during the same period, 226.15%.

2.4.3 Excess spread

The excess spread is what is left in the interest waterfall after paying all expenses and the interests due to the notes. Before the excess spread is distributed to the equity investors, it is used to cover losses in the collateral portfolio. In this way, it represents the first layer of credit protection of the note investors.

In a real life CDO, it is very difficult to estimate the excess spread that can be used to protect the senior notes and then returned to the equity investors. For example, if the CDO suffers a large number of defaults in the last stage of the transaction and the par value test is breached so that the interest waterfall is changed to accelerate the repayment of the senior notes with the extra spread, the equity investors have had plenty of time to collect excess spread and achieve a remarkable return. In contrast, if the CDO suffers a large number of defaults in the early days of the transaction, the equity investors would bear the first loss and would not collect any excess spread, because it would be used to accelerate the senior notes.

2.5 Credit enhancement in market value CDOs

Market value CDOs have three further forms of credit enhancement: advance rates, market value over-collateralisation test, and minimum net worth test.

The advance rate is the maximum percentage of the collateral portfolio market value that can be used to issue senior and mezzanine notes. Rating agencies assign the collateral advance rate according to the historic volatility of the collateral return and on the liquidity of its market price. Assets with a higher return volatility and lower liquidity are given lower advance rates. Table 2.1 shows the advance rates that Fitch would apply to different types of assets as collateral. For example, when a portfolio of Certificate of Deposits (CD) or Commercial Paper (CP) is used as collateral of a note rated AA, Fitch allows to uses only 95% of the CD and CP market value. In this instance, 5% of the market value of the CD and CP is the equity that the originating bank retains. The advance rate of CD and CP would rise to 100% where the note is rated BB.

Asset Category	AA	A	BBB	BB	B
Cash and Equivalents	100%	100%	100%	100%	100%
CD and CP	95%	95%	95%	100%	100%
Senior Secured Bank Loans	85%	90%	91%	93%	96%
BB-High Yield Debt	71%	80%	87%	90%	92%
<BB-High Yield Debt	69%	75%	85%	87%	89%
Convertible Bonds	64%	70%	81%	85%	87%
Convertible Preferred Stock	59%	65%	77%	83%	86%
Mezzanine Debt, Distressed, Emerging Market	55%	60%	73%	80%	85%
Equity, Illiquid Debt	40%	50%	73%	80%	85%

Source: Fitch

Table 2.1: Fitch's advance rates.

In market value CDOs, the par value OC test is called the market value OC test and is applied on the market value of the collateral portfolio MVA_t, as follows

$$[MVA_t - defaults_t]\beta - L_{S,t} > 0 \qquad \text{with } 0 < \beta \leq 1 \qquad (2.4)$$

To illustrate how the advance rate and the market value OC tests work in practise, we introduce a simple example with the collateral portfolio and the liability structure of Table 2.2.

Collateral			Notes			
Assets	**Market Value (euro)**	**%**	**Tranche**	**Rating**	**Face Value (euro)**	**%**
CD and CP	£230	46%	Senior Notes	AA	£375	75%
BB-High Yield Debt	£225	45%	Mezz. Notes	BBB	£50	10%
Convertible Bonds	£10	2%	Junior Notes	B	£25	5%
Mezz. Debt	£25	5%	Equity	NR	£50	10%
Equity	£10	2%				
Total	£500	100%	**Total**		£500	100%

Table 2.2: Market value CDO.

In Table 2.3, we apply the AA advance rates of Table 2.1 to each asset of the collateral portfolio of Table 2.2. In column 5, we have calculated the senior advance amounts of each asset, by multiplying the AA advance rates (from Table 2.1) by the asset market values of column 2 of Table 2.2. Their sum gives the Total Advance Amount of £402.

To prepare the AA OC test, we take the difference between the Total Advance Amount of £402 (in Table 2.3) and the Senior Note face value of £375 (in Table 2.2). The difference of £27m is the Borrowing Amount Surplus and is the maximum change in the collateral market value that the CDO can sustain before breaching the AA market value OC test.

	Collateral			
Assets	**Market Value (euro)**	**%**	**AA Advance Rates**	**Sr. Advance Amounts**
CD and CP	£230	46%	95%	£219
BB-High Yield Debt	£225	45%	71%	£160
Convertible Bonds	£10	2%	64%	£6
Mezz. Debt	£25	5%	55%	£14
Equity	£10	2%	40%	£4
Total	£500	100%	Total Advance Amount	£402
			Senior Notes	£375
			Borrowing Amount Surplus	£27

Table 2.3: Market value OC test.

A collateral manager must ensure that the market value tests are not violated. A breach of the market value OC test is quite serious, and when it happens, the collateral manager must remedy it within a cure period that is usually between two to ten business days.

There are usually two options:

(1) to sell asset(s) with lower advance rate and buy one or more with higher advance rate,

(2) or to sell asset(s) with lower advance rate and repay some or a portion of the notes, starting from the most senior notes, until the OC test is in compliance.

The first action is preferred when the OC test is slightly out of compliance. The second is a drastic cure.

The minimum net worth test is designed to offer a further credit protection to the senior notes. This is achieved by imposing that the collateral market value MVA_t, less the par value of all notes, is never lower than the equity face value multiplied by a percentage δ, indicated by the rating agency[13]

$$[MVA_t - L_{S,t} - L_{M,t} - L_{J,t}]\delta - V \geq 0 \qquad \text{with } 0 < \delta \leq 1 \qquad (2.5)$$

In cases where the test is breached, the manager has a cure period to bring the CDO into compliance, by either

(1) redeeming part or all of the senior notes, or

(2) by generating enough capital gains by selling some assets.

The latter is preferable for the equity investors since the manager would not de-leverage the deal.

If the collateral manager cannot comply with the market value OC test or the minimum net worth test, the notes investors have the legal power to take control of the CDO and liquidate the collateral portfolio.[14]

2.6 The CDO manager

The manager of the CDO is responsible for the credit performance of the collateral portfolio and for ensuring that the transaction meets the diversification, quality and structural guidelines specified by the rating agencies. In return for managing the collateral portfolio, the manager receives a fee, typically divided into base and incentive components. During the reinvestment period, the CDO manager continuously evaluates the state of the collateral portfolio and of the overall market. He trades out positions at risk for credit deterioration, and takes advantage of appreciation opportunities.

The key to a successful market value CDO is the manager's ability to generate high returns through research, market knowledge and trading ability. Different managers stress different strategies. For example, an insurance company may depend on its portfolio risk management system, a mutual fund group may use its size and market knowledge, and a private equity sponsor may rely on its knowledge of leveraged companies. The market value CDO is typically only accessible to

[13] See Yvonne and Gluck (1998) p. 12.

[14] For an exhaustive explanation of the SPV laws, regulations, tax and accounting, see Deacon (2003).

managers who have established track records and who have demonstrated a high level of organisational commitment to the CDO business.

Today, successful CDO management franchises are found in a variety of asset management organisations, including mutual fund groups, insurance companies, banks, private equity firms and hedge funds.

In recent CDOs, the collateral manager has the opportunity to use CDSs to hedge positions or lock profits, which can then be used to provide a further credit enhancement to the transaction. Where the manager purchases protection without having exposure, these transactions are called *naked short bucket transactions*. This new facility combines elements of a cash flow-arbitrage CDO with elements of a synthetic CDO and further protects the portfolio against credit deterioration. In more general terms, the collateral manager can enter into short CDSs to buy protection and hedge his credit exposure to the obligors under cash obligations, total return swaps and other CDSs. The only condition to enter into short CDSs is to have sufficient excess spread available to pay the upfront premium. If, in subsequent periods, there is no longer a sufficient excess spread, the manager will close out the trade by entering into offsetting CDSs. CDS gives extra flexibility to a manager: if a sector is perceived as too risky, rather than simply selling assets, he can close out positions by buying credit protection.

2.7 The motivations for CDOs

The main motivations for CDOs are market price inefficiency, funding costs and regulatory capital relief.

2.7.1 Market price inefficiency

CDOs make most economic sense for collateral portfolios in markets where there is limited information (inefficient) with the possibility of high risk-adjusted returns through active management. Default-risky assets, such as the debt of leveraged corporations, are often difficult to analyse and value, thus limiting their potential investor base and creating a gap in the economy between the demand and supply of risky finance. As a result, corporate debts are relatively illiquid in the secondary market. The CDO structure addresses this market inefficiency by bringing a specialised manager to the transaction and allocating much of the credit risk and of the liquidity risk in the equity class.

The CDO acts as a cushion and hedges the collateral portfolio from defaults (in a cash flow CDO) and from the direct impact of mark-to-market changes in the value of the collateral (in a market value CDO).

2.7.2 Funding cost

If the main motivation is funding cost, which is the interest paid on the notes, the originating bank would try to reduce the funding costs by minimizing the size of the subordinated notes (mezzanine notes and the equity piece).

Funding can be an issue for banks whose rating has declined to a level where funding from other sources is expensive. The advantage of funding through a CDO is that the size of the subordinated notes, and so the funding cost, depends on the good quality of the collateral portfolio. To improve it, the originating bank would also fund the Cash Collateral Account (CCA), which is a cash deposit that reinforces the credit protection, by creating another layer of protection before the equity piece, usually in the range of 1% of the collateral portfolio.

2.7.3 Regulatory capital

Regulatory capital relief is another important motivation for issuing CDOs. Under the BIS rules, loans require regulatory capital RC, of the size of 8% of the risk weighted assets RW_{Loan}, of 100%.[15]

In Figure 2.4, we have prepared an example to outline the originating bank regulatory capital before and after the CDO: in the left box we have a portfolio of loans before they are transferred to the CDO and in the right box after they are purchased by the CDO. We assume that the loan collateral A, is of £100, all loans are of the same amount of £2 and pay the default-free rate $r(t)$, assumed the libor rate of 5%, plus a spread s_{Loan}, of 100 bps.

Ignoring any loan eligibility rule for a RW_{Loan} reduction, the regulatory capital charge of the loan portfolio RCC_{Loan}, before the CDO is

$$RCC = RW_{Loan} * £100 * RC$$

$$RCC = £8 \tag{2.6}$$

[15] At the time we are writing this chapter, the new Basel regulation on capital charges before and after securitisation, is under negotiation. An update can be found in, Basel Committee on Banking Supervision, Charges to the Securitisation Framework (30/1/04), URL: http://www.bis.org.

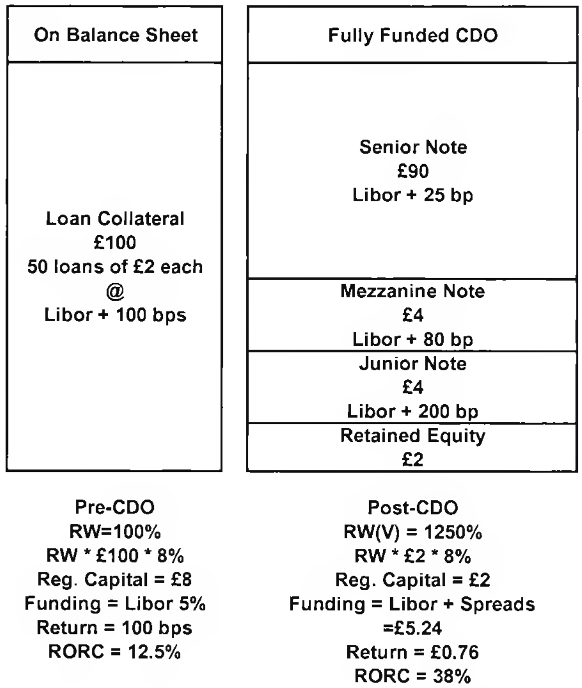

Figure 2.4: Regulatory capital relief and RORC.

We also assume that the funding cost of the portfolio is libor (with no spread) and we derive the $RORC$,

$$RORC = \frac{s_{Loan}}{RC_{Loan}}$$

$$RORC = \frac{100bps}{8\%} = 12.5\% \tag{2.7}$$

With the CDO in the right box of Figure 2.4, the originating bank sells, through senior, mezzanine and junior notes, 98% of the original loan portfolio and retains on its balance sheet only 2%. Under the BIS rules, the only capital charge is on the equity piece V, which receives full regulatory capital deduction,[16] that is, $RW_V = 1250\%$. With (2.6) we can calculate the new regulatory capital of the originating bank, which drops from £8 to £2,

$$RCC = RW_V * £2 * RC$$

$$RCC = £2 \tag{2.8}$$

To calculate the $RORC$, we first compute the funding cost, FC, of the three notes, where L_S, L_M and L_J, are respectively the notionals of the three notes, and s_S, s_M and s_J, are their spreads over the libor $r(t)$,

[16] For the current rules on regulatory capital charges before and after securitisaton see, Basel Committee on Banking Supervision, Charges to the Securitisation Framework (30/1/04).

$$FC = L_S(r(t)+s_S) + L_M(r(t)+s_M) + L_J(r(t)+s_J)$$
$$FC = £5.24 \tag{2.9}$$

After this, we subtract from the interest on the asset, I_{Loan}, the funding cost in (2.9)

$$I_{Loan} - FC = £6 - £5.24$$
$$I_{Loan} - FC = £0.76 \tag{2.10}$$

where $I_{Loan} = (r(t) + s_{Loan})A$, and divide (2.10) by (2.8)

$$RORC = \frac{£0.76}{£2}/100\% = 38\% \tag{2.11}$$

If this transaction had a bigger volume it would hugely affect the RORC of the overall bank.[17]

2.8 Synthetic CDOs

In the CDOs we have seen so far, assets are actually transferred into the SPV. The process of transferring loans to the SPV requires significant up front work. A loan-by-loan analysis is necessary to check it complies with the securitisation programme and to verify that there are no special clauses attached to any loan limiting its transfer. We would like to call these CDOs *conventional*, to distinguish them from a new generation of CDOs called *synthetics*.

The term synthetic appeared when the credit risk was transferred into the SPV through a credit default swap,[18] and the underlying credit ownership of the underlying pool remained in the originating bank's balance sheet. In this instance, the term *synthetic* was used, since the risk was synthetically transferred out of the originating bank's balance sheet. With synthetic CDOs, the big advantage is that sensitive client relationship issues arising from loan transfer notification, assignment provisions and other restrictions can be avoided. Client confidentiality is maintained. It also takes less time to complete the transaction.

In the following two sections, we will review how CDSs can be used to transfer the credit risk from the originating bank to capital markets, with both fully funded and partially funded CDOs.

2.8.1 Fully funded synthetic CDOs

In Figure 2.5, we illustrate a simplified example of how a fully funded synthetic CDO is structured.

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[17] This is an exemplified opportunity cost for regulatory capital relief. A full calculation would normally include: rating agency costs, lawyer structuring fees, underwriting costs, and would take into account the amortising profile of the notes.

[18] Swiss Bank brought Glacier Finance 1997-1 and 1997-2 to market in late 1997. Swiss Bank transferred the credit risk to the SPV, via a portfolio of credit linked notes.

The loan collateral A, is £100. The SPV issues one note L_{AAA}, rated as AAA, of £98 that pays the default free rate $r(t)$, assumed the libor rate, plus the spread s_{AAA}, and the originator bank O, retains the equity V, of £2. The proceeds of the note are invested in high quality securities S, which have 0% risk weight assets, and return the default-free rate $r(t)$.

In order to transfer the credit risk of the loan collateral, the originating bank enters into a CDS with an OECD bank, referred to the loan collateral A, for a notional amount of £100. With the CDS the originating bank buys credit protection on the loan collateral A, in return for the premium, γ_O. Then, the SPV sells a CDS with reference to the same loan collateral A, to the same OECD bank in return for the premium γ_{SPV}, where we assume $\gamma_{SPV} = \gamma_O$. At this point, the originating bank has removed the credit risk from its balance sheet, and the SPV has acquired the credit risk, both entities *via* the CDS market. The CDS premium γ_{SPV}, that the SPV receives, plus the interests on the 0% risk weight assets $r(t)$, should be large enough to pay the interests on the notes $r(t)$, plus the spread s_{AAA}. This is the main condition for the originating bank to go ahead with a fully funded synthetic CDO. We write this condition

$$\gamma_{SPV} A + Sr(t) > L_{AAA}(r(t) + s_{AAA})$$

$$\gamma_{SPV} A + L_{AAA} s_{AAA} > 0 \qquad \text{with} \quad A > L_{AAA} = S$$

$$\gamma_{SPV} > s_{AAA} \frac{L_{AAA}}{A} \tag{2.12}$$

Both γ_{SPV} and s_{AAA} are what we like to call, *market inputs*: s_{AAA} is the spread over the default-free rate that the AAA investor is willing to earn to hold the AAA note, γ_{SPV} is the credit premium that an OECD bank is willing to earn to take over the credit risk of the loan collateral. As long as the loan collateral A has a greater credit risk than the note L_{AAA}, and $\gamma_{SPV} > s_{AAA}$, the SPV generates guarantee excess spread that flows back to the originator bank.

We now look at the regulatory capital charge of the originating bank after the fully funded synthetic CDO.

Under the BIS rules, the regulatory capital charge, $RCC_{\%,CDS}$ on the CDS is 1.6%, calculated as the risk weight asset $RW_{CDS,OECD}$, of 20%, times the regulatory capital RC, of 8%

$$RCC_{\%,CDS} = RW_{CDS,OECD} * RC$$

$$RCC_{\%,CDS} = 1.6\% \tag{2.13}$$

The 1.6% is then multiplied by the notional amount of the CDS of £100, to give the regulatory capital charge $RCC_{£,CDS}$, of £1.6

$$RCC_{£,CDS} = RCC_{\%,CDS} * CDS$$

$$RCC_{£,CDS} = £1.6 \qquad (2.14)$$

The originating bank has retained the equity of £2. The regulatory capital charge of the equity is the risk weight asset RW_V, of 1250%, times the regulatory capital RC, of 8%, times the equity of £2, which is £2,

$$RCC_{£,V} = RW_V * RC * V$$

$$RCC_{£,V} = £2 \qquad (2.15)$$

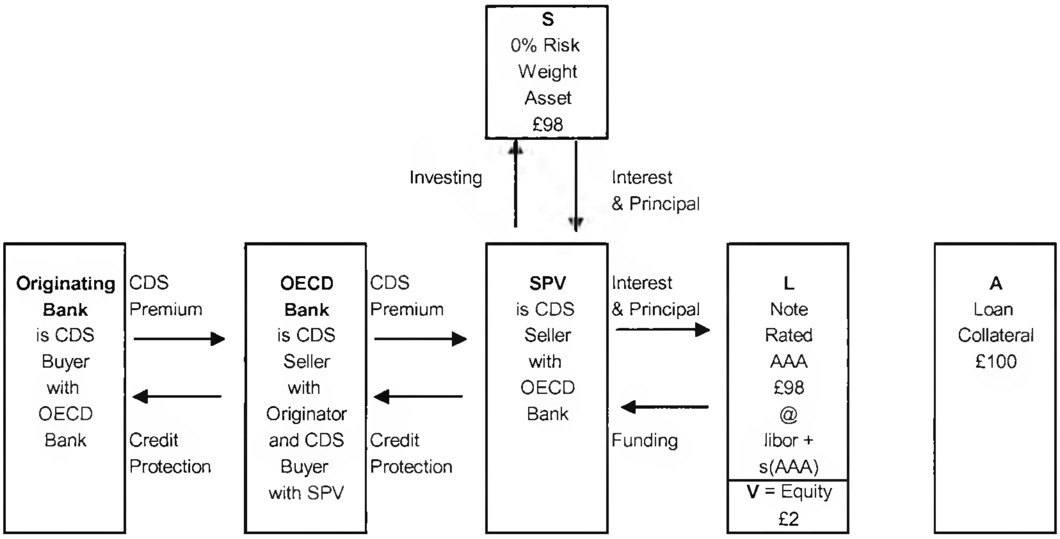
As result, the total capital charge of this transaction is £3.6,

$$RCC_{£,V} + RCC_{£,CDS} = £3.6 \qquad (2.16)$$

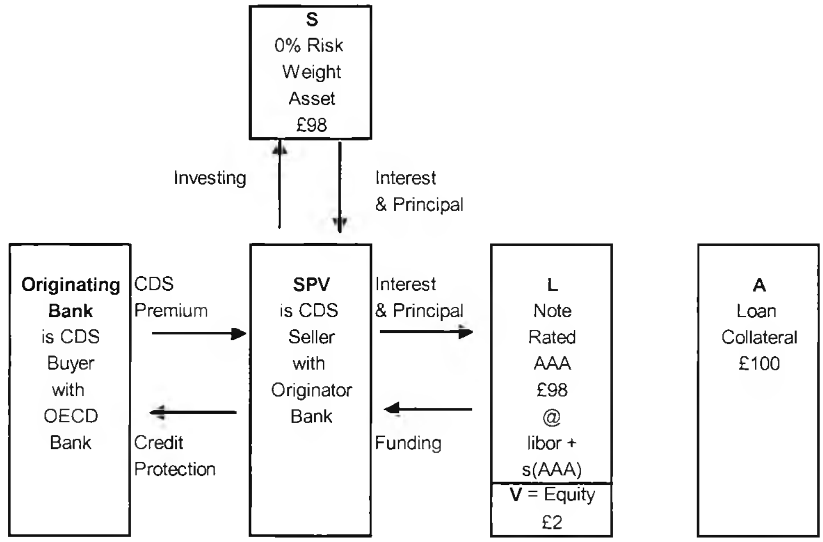
With the fully funded synthetic CDO in Figure 2.5, the originating bank does not achieve a further reduction of regulatory capital, when moving from a conventional CDO. Because the CDS is with an OECD bank, there is actually an additional regulatory capital of £1.6.

In Figure 2.6, we show an alternative structure to the one in Figure 2.5. All variables are the same, except that the CDS is directly with the SPV. Since the proceeds from the note L_{AAA}, are invested in high quality securities that have a 0% risk weight assets, the CDS does not bring any regulatory capital charge and $RW_{CDS,SPV} = 0$. Therefore the only regulatory capital remains the £2 of the equity.

As in Figure 2.5, with the structure in Figure 2.6, the originating bank has synthetically transferred the credit risk to capital markets, and preserved confidentiality: it does not notify its clients that the loan is transferred to the SPV.



Source: Investing in Collateralized Debt Obligations, Frank J. Fabozzi, Laurie S. Goodman.
Figure 2.5: Fully funded synthetic CDO with CDS with an OECD bank.



Source: Investing in Collateralized Debt Obligations, Frank J. Fabozzi, Laurie S. Goodman

Figure 2.6: Fully funded synthetic CDO with CDS with an SPV.

2.8.2 Partially funded structures

In the fully funded synthetic CDO in Figure 2.5, the SPV has issued a note equal to 98% of the loan collateral. This is a very expensive funding programme and thus, the originating bank is far from achieving an efficient capital use. The alternative is a partially funded synthetic CDO transaction where the originating bank still buys credit protection, either directly from an SPV (Figure 2.7) or from an OECD bank (Figure 2.8). The difference is that in this instance, the SPV issues a lower amount of notes.

In Figures 2.7 and 2.8, the SPV issues one note L_A, of £18, rated A, that pays the default-free rate $r(t)$ plus the spread s_A, whose proceeds, are invested in high quality securities S, that have 0% risk weight assets, and return the default-free rate $r(t)$. The originating bank retains the equity V, of £2. To transfer to capital markets the credit risk of the same collateral portfolio A, of £100, the originating bank enters in two CDSs: one called super senior CDS, CDS^S, also called *un-funded piece*, and a second called junior CDS, CDS^J, also called *funded piece*. The notional of the junior CDS, CDS^J is £20

$$CDS^J = L_A + V$$

$$CDS^J = £20 \tag{2.17}$$

What really characterises this structure is the super senior CDS, CDS^S, whose notional is £80

$$CDS^S = A - CDS^J$$

$$CDS^S = £80 \quad (2.18)$$

To see how the super senior CDS influences the funding benefit between a fully funded and a partially funded synthetic CDO, we write as k^S_{SPV} and as v^J_{SPV}, respectively the super senior CDS premium and the junior CDS premium and we assume that market forces preserve the following relationship between the premiums of this example and the premium of the previous one,

$$v^J_{SPV} > \gamma_{SPV} > k^S_{SPV} \quad (2.19)$$

Then, we rewrite the condition of (2.12) as

$$v^J_{SPV} CDS^J + Sr(t) > L(r(t) + s_A)$$

$$v^J_{SPV} CDS^J > L_A s_A \qquad \text{with } CDS^J > L_A = S \quad (2.20)$$

We can notice two relationships in (2.20). Firstly, from (2.17) and (2.18), the larger the notional of CDS^S, the smaller the size of the note L_A, and the smaller the funding cost, $L_A(r(t) + s_A)$. Secondly, v^J_{SPV} and k^S_{SPV} are the premiums that two OECD banks are willing to earn to take over the credit risk of the loan collateral. As long as the funded piece CDS^J, has a greater credit risk than the note L_A, the SPV generates guarantee excess spread that flows back to the originating bank.

The size, the rating and the spread of the note L_A are what drive the funding benefit of this structure when compared with the fully funded CDO, which we can write as follows

$$L_{AAA} s_{AAA} - L_A s_A > 0 \quad (2.21)$$

We now look at the regulatory capital charge, of the originating bank, after the partially funded synthetic CDO.

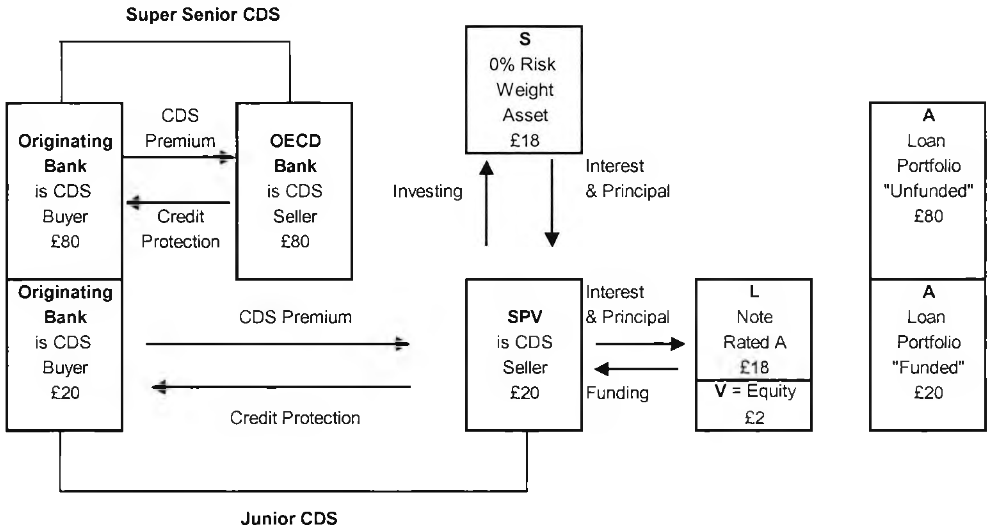
Under the BIS rules, if the super senior CDS, CDS^S, is with another OECD bank, the risk weight asset, $RW_{CDS^S,OECD}$ is 20%, and returns the regulatory capital charge, $RCC_{£,CDS^S}$, is £1.6,

$$RCC_{£,CDS^S} = RW_{CDS^S/OECD} * RC * CDS^S$$

$$RCC_{£,CDS^S} = £1.6 \quad (2.22)$$

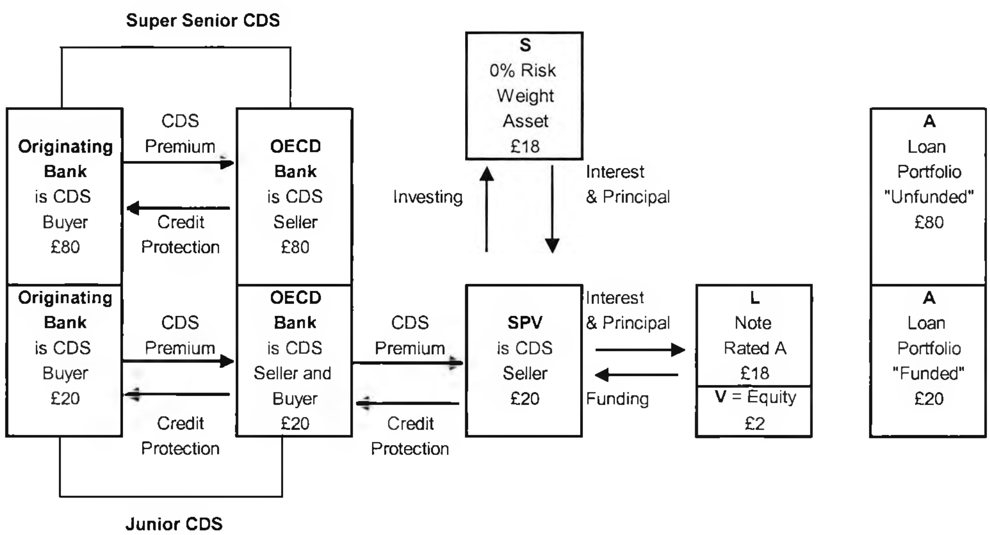
The regulatory capital rules on the equity piece and on the junior CDS, are the same as those applied on the fully funded synthetic CDOs in 2.8.1, and we have to distinguish between the case where junior CDS is directly with the SPV or with another OECD bank.

Figures 2.7 and 2.8 illustrate the mechanics of these types of CDOs, whereas Table 2.4 shows the regulatory capital results of the structures of Figures 2.5 to 2.8, and compares their funding costs, where we have assumed a libor rate $r(t)$, of 5%, and spreads over the libor, s_{AAA} and s_A of 10 bps and 50 bps, respectively of the notes rated AAA and A.



Source: Investing in Collateralized Debt Obligations, Frank J. Fabozzi, Laurie S. Goodman.

Figure 2.7: Partially funded synthetic CDO with CDS with an SPV.



Source: Investing in Collateralized Debt Obligations, Frank J. Fabozzi, Laurie S. Goodman.

Figure 2.8: Partially funded synthetic CDO with CDS with an OECD bank.

	Notional	Figure 5	Figure 6	Figure 7	Figure 8
		Regulatory Capital			
CDS	£100.00	£1.60	£0.00	£0.00	£0.00
Senior CDS	£80.00	£0.00	£0.00	£1.60	£1.60
Junior CDS	£20.00	£0.00	£0.00	£0.00	£1.60
Equity	£2.00	£2.00	£2.00	£2.00	£2.00
Total		£3.60	£2.00	£3.60	£5.20
		Funding Cost			
Note Notional		£98.00	£98.00	£18.00	£18.00
Note Rating		AAA	AAA	A	A
Libor		5%	5%	5%	5%
spread (bps)		10	10	50	50
		£5.00	£5.00	£0.99	£0.99

Table 2.4: Regulatory capitals and funding costs of Figures 2.5 to 2.7.

2.9 Balance sheet management with CDSs

Figure 2.9 contains a new partially funded CDO structure, where the loan collateral is the same as in the previous examples.

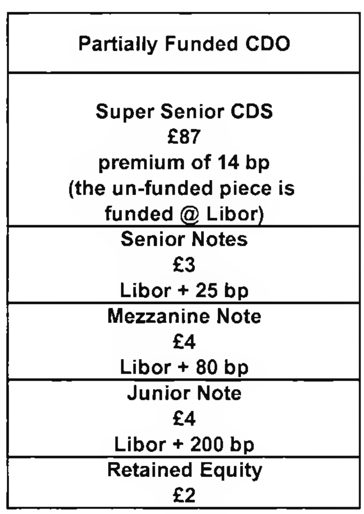


Figure 2.9: Partially funded synthetic CDO.

This structure is also free of any interest rate risk: it pays libor and receives libor. The spread, s_{Loan} of 100 bps over libor compensates the bank for taking the credit risk. The spreads on the three notes, ranging from 25 to 200 bps, compensate the notes investors for taking different credit risks.

We can remove the libor, leave the spreads on the note side and write the loan collateral as 50 loans paying the spread s_{Loan}, of 100 bps over libor, as prepared in Table 2.5.

		Partially Funded Synthetic CDO			
Assets	**%**	**Spreads**	**Liabilities**	**%**	**Spreads**
Loan 1	2%	100 bps	Super Senior CDS	87%	14 bps
Loan 2	2%	100 bps	Senior Notes	3%	25 bps
Loan 3	2%	100 bps	Mezzanine Notes	4%	80 bps
....	...	100 bps	Junior Notes	4%	200 bps
Loan 50	2%	100 bps	Retained Equity	2%	Dividend

Table 2.5: CDO structure with hedged interest rate risk.

A CDS is designed to mimic the credit behaviour of a floating rate note, such as the loans in Table 2.5. The loan spread, that is constant until the loan matures, is equivalent to the fixed leg of a CDS. In fact, the CDS seller, who seeks credit exposure, receives X basis points, i.e. a spread, per year until the credit reference matures or defaults. The constant spread is the fixed leg of the CDS. As a consequence, we can remove the loans and add the CDSs on the asset side.

		Partially Funded Synthetic CDO			
Assets	**%**	**Spreads**	**Liabilities**	**%**	**Spreads**
CDS 1	2%	100 bps	Super Senior CDS	87%	14 bps
CDS 2	2%	100 bps	Senior Notes	3%	25 bps
CDS 3	2%	100 bps	Mezzanine Notes	4%	80 bps
....	...	100 bps	Junior Notes	4%	200 bps
CDS 50	2%	100 bps	Retained Equity	2%	Dividend

Table 2.6: CDO structure with hedged interest rate risk and with CDS in place of loans.

With the CDO in Table 2.6 the originating bank is now exposed to the credit risk of 50 synthetic assets. To hedge its position, the bank borrows via three different notes and a CDS. Retaining the equity, gives it the right to a possible dividend.

Viewed from this angle, a CDO is a hedged portfolio. The assets are a portfolio of CDSs, the liabilities are the notes with different seniority. By hedging its balance sheet from credit risk, the bank is trying to achieve a higher return than investing in default-free treasury bonds. By partially funding the CDO structure, the bank has also achieved a leverage position, with potentially huge returns. The problem is how to correctly price the portfolio of CDSs.

2.10 CDO equity piece

The CDO equity piece is a truly hybrid security. It exhibits the features of a coupon bond, a corporate equity, a call option on the collateral and a managed fund.

As a coupon bond, CDO equity is issued at or near par[19] and has a final maturity date. Like convertible bonds, cash payments are not specified, although the range of expected distributions is established at the time of issuance. In a similar way to a call option, the value of CDO equity increases with the price and volatility of the underlying assets, the collateral portfolio. As with any actively managed investment, the contribution of the manager is a crucial determinant of CDO equity performance.

2.10.1 The CDO equity piece performance

The equity of a CDO represents a leveraged investment in the underlying asset class and in the asset management skills of the CDO manager. The leverage is achieved by issuing investment and sub-investment-grade debt as term[20] asset-backed securities.

Credit losses are the obvious drivers of the CDO equity piece performance and can affect investors in two ways. First, as collateral shrinks because of defaults (in cash flow CDOs) or realised price deterioration (in market value CDOs), the amount of collateral portfolio reduces and with it the size of received interest payments. Second, if the par size of the collateral drops below the par value test or where the interest coverage test is breached, the excess spread that is normally paid to the equity investors is redirected to pay down the senior liabilities, thereby de-leveraging the CDO.

Equity payments resume only after the tests are restored above the trigger levels. Redirection of equity distributions can also be triggered by a drop in the interest income relative to the interest cost of the transaction.

According to Moody's (2002),[21] the average excess spreads in cash flow-arbitrage CDOs, in January 2002, was of 3.14%. In the same month the average life, in cash flow-arbitrage CDOs, was of 5.95 years. That means that, on average, in cash flow arbitrage CDOs, the return on the CDO equity was of 18.7% (3.14% times 5.95 years).

2.10.2 The CDO embedded call option

Depending on the collateral asset type and the timing of the transaction, the call option embedded in CDO equity may be quite valuable. Figure 2.10 shows the historical spread over Libor on the Goldman Sachs Single B Bond Index and the estimated cost of funding of CDO liabilities.[22] The

[19] This means that if an investor would like to purchase the full equity piece, he will have to pay an amount equal to the equity piece face value (£2 in Figure 2.6).
[20] The "term" is used to differentiate the term *securitisation* where the SPV liabilities are bonds, from *conduit* where the SPV liabilites are commercial papers.
[21] See Moody's (22 March 2002), Collateralised Debt Obligations Indices: January 2002, for a report on excess spreads, average lives and other statistics on CDOs in US and Europe.
[22] From Olberg *et al.* (2001).

wider the gap between the income from the assets and the cost of the liabilities, the greater the investment incentive for CDO equity. Is it also greater the capital appreciation for the equity investors as a consequence of calling the CDO.[23]

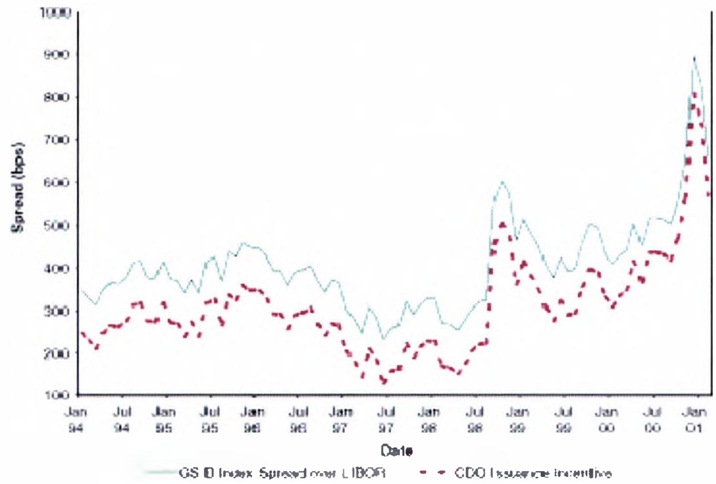


Figure 2.10: GS Single B Bond Index over Libor and the bond-backed CDO issuance incentive.[24]

The upside from calling the transaction depends hugely on the type of collateral, making the distinction between CBOs and CLOs imperative. Those CDOs where the collateral is represented by bonds purchased at low prices (when interest rates were high) and where the structure is financed through cheap term notes (current low interest rates) offer the most likely possibility of significant capital appreciation. Floating-rate collaterals such as leveraged loans can easily be refinanced. The underlying borrowers can prepay outstanding loans and refinance at a lower funding rate. For this reason, a manager of a CLO will be in a very difficult position if he wishes to generate outsized capital appreciation. In other words, they do not offer as much potential for significant appreciation.

As the remaining expected returns fall, the equity holders are likely to exercise their option during the repayment period, either to take advantage of potential appreciation in CBOs, or to minimise the impact of a difficult credit environment with CLOs.

2.10.3 Investing in CDO equity

For long-horizon investors such as pension plans, endowments and insurance companies, portfolio diversification is an important investment consideration. In principle, diversification across asset classes lowers portfolio volatility without altering expected returns.

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[23] Here we refer to the right of the equity investors to liquidate the collateral portfolio and use the proceeds to pay down the notes and interests due.
[24] From Olberg *et al.* (2001).

Traditionally, investments in properties and foreign securities have been seen as effective diversification strategies. More recently, as volatilities in financial markets have increased, asset investors have also turned to more illiquid asset types such as private equity, hedge fund investments, commodities, insurance risk securities and ultimately CDO equities.

CDO equities are perceived to have a lower correlation if compared with the traditional asset classes which they are made. This is not a surprise since the CDO cash flow structure hedges the equity investment against short-term liquidity or technical fluctuations in the value of the collateral. Indeed, the combination of non-generic collateral and active management should result in a low correlation between CDO equity returns and returns on benchmark asset classes, such as public equity, investment-grade corporate liabilities and government debt. Low long-term horizon return correlations, along with high expected returns, should lead asset investors such as insurance companies, pension plans, endowments and foundations, to consider investment in CDO equity as an effective diversification strategy and "alternative investment" bucket in the portfolios of long-term horizon investments.

The problem is that historical data on CDO equity returns are unavailable because the market is relatively new and remains a very private one.

Orberg *et al.* (2001) looked at how to measure the correlation of CDO equities. Their route was to look at the underlying collateral markets as a starting point for thinking about correlations between returns on CDO equities and returns in other asset classes. In their analysis they took the Merrill Lynch Single B Index as the proxy of the CDO equities to measure correlations with other market indices.

Historical (12/89 - 12/00) asset class return statistics

	ML single B	LB Govt	LB Credit	S&P 500	Russell 2000
Average (%)	9.3	7.62	8.21	15.48	12.85
Standard deviation (%)	21.48	14.17	16.26	47.98	63.64

Correlations

	ML single B	LB Govt	LB Credit	S&P 500	Russell 2000
ML single B	1.00	0.31	0.47	0.49	0.57
LB Govt	0.31	1.00	0.95	0.34	0.15
LB Credit	0.47	0.95	1.00	0.43	0.26
S&P 500	0.49	0.34	0.43	1.00	0.69
Russell 2000	0.57	0.15	0.26	0.69	1.00

Table 2.7: Historical (12/89 – 12/00) asset class return statistics and correlations (from Orberg *et al.*).

Table 2.7 shows the historical annualised monthly return averages, standard deviations and correlations for the Merrill Lynch Single B Index, the Lehman Brothers Government Index, the Lehman Brothers Credit Index, the S&P 500 Index and the Russell 2000 Index. As they expected, the high-yield Merrill Lynch Single B Index returns display the highest correlation with the small-cap Russell 2000 Index and lowest correlation with the Lehman Brothers Government Index.
Table 2.7 is useful for fund managers who would like to invest in Government bonds and does not want to lose the appreciation given by investing in equities. They are interested in seeing how the CDO equity would diversify their portfolio.

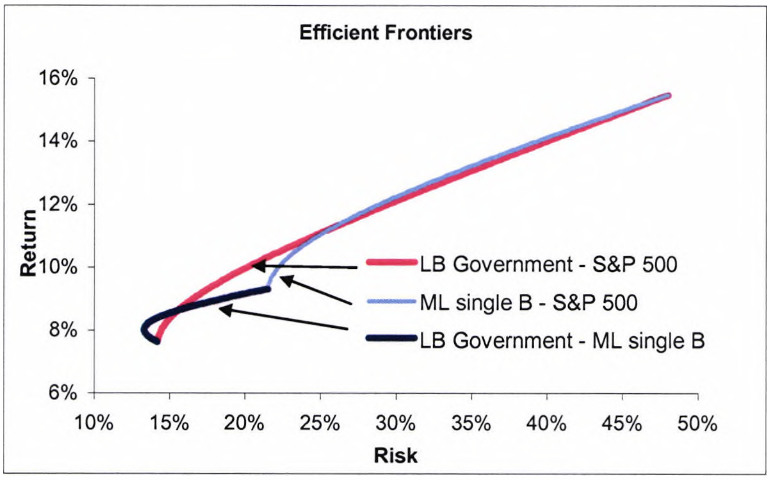

Figure 2.11: Three efficient frontiers: ML single B - LB Governments, ML single B -S&P and S&P - LB Governments.

Using the data in Table 2.7, we have formed three different portfolios: LB Governments and ML single B, LB Governments and S&P, and ML single B and S&P. In Figure 2.11, we have prepared the efficient frontiers of these portfolios. The first name in each of the three portfolios has the initial weight of 100%. Subsequently, the portfolio weights change, the first name weight moves from 100% to 0%, whereas the second name weight moves from 0% to 100%.

We can see that below a risk (standard deviation) of 15.8%,[25] the fund manager should diversify by combining Government bonds with CDO equities (ML single B) rather than combining Government bonds with equities (S&P 500). From Figure 2.11, we can make a further point. It is never good to combine equities with CDO equities.

[25] At 15.8%, the split between Government bonds and ML single B is 40% to 60%.

2.11 The Moody's diversity score and the table of idealised cumulative expected losses

The ratings are important features of the CDO market. They summarise the view that, one, two or all three main international rating agencies, Moody's, S&P and Fitch, not only have on the credit quality of the collateral portfolio, but also on the quality of the CDO structure (principal waterfall, interest waterfall, par value tests, interest coverage, excess spread diversion etc).

Moody's (Cifuentes and O'Connor (1996), Cifuentes and Wilcox (1998) and Tabe (2002)), S&P (2002) and Fitch (2003) have their own methodologies to assess the credit risk in a collateral portfolio. For a comparison of the three rating methodologies, we recommend Zhu *et al.* (2003). In what follows, we will review three aspects of the Moody's methodology: the table of idealised cumulative expected losses, the binomial expansion technique (BET) and the diversity score.

Moody's methodology is based upon the concept of expected losses, which reflects the amount that the CDO notes investors may lose until the CDO note matures. The approach Moody's takes is to calculate, at a given rating category, the level of subordination below a certain note, so that the cumulative expected losses of the note, are the same as the benchmark of cumulative expected losses associated with that rating category. The benchmarks of expected cumulative losses are also known as the Table of idealised cumulative expected losses and are shown in Table 2.8. These are cumulative losses based on the historical performances of rated bonds. The data includes actual credit default experiences of corporate bonds and asset-backed securities. The cumulative losses are calculated as frequencies of issuer losses over the total number of rated issuers. The issuers are then characterised by rating classes and maturities. For example, a loss probability of 2.035% means that if we were to hold the totality of bonds rated as Baa3, we would make a loss of 2.035% over a period of 6 years. The consequence is that for each level of expected cumulative losses and maturity, there is an associated rating category, or otherwise said, the bond rating depends on its maturity and on its expected loss.

	Year									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Aaa	0.000%	0.000%	0.000%	0.001%	0.002%	0.002%	0.003%	0.004%	0.005%	0.006%
Aa1	0.000%	0.002%	0.006%	0.012%	0.017%	0.023%	0.030%	0.037%	0.045%	0.055%
Aa2	0.001%	0.004%	0.014%	0.026%	0.037%	0.049%	0.061%	0.074%	0.090%	0.110%
Aa3	0.002%	0.010%	0.032%	0.056%	0.078%	0.101%	0.125%	0.150%	0.180%	0.220%
A1	0.003%	0.020%	0.064%	0.104%	0.144%	0.182%	0.223%	0.264%	0.315%	0.385%
A2	0.006%	0.039%	0.122%	0.190%	0.257%	0.321%	0.391%	0.456%	0.540%	0.660%
A3	0.021%	0.083%	0.198%	0.297%	0.402%	0.501%	0.611%	0.715%	0.836%	0.990%
Baa1	0.050%	0.154%	0.308%	0.457%	0.605%	0.754%	0.919%	1.085%	1.249%	1.430%
Baa2	0.094%	0.259%	0.457%	0.660%	0.869%	1.084%	1.326%	1.568%	1.782%	1.980%
Baa3	0.231%	0.578%	0.941%	1.309%	1.678%	2.035%	2.382%	2.734%	3.064%	3.355%
Ba1	0.488%	1.111%	1.722%	2.310%	2.904%	3.438%	3.883%	4.340%	4.780%	5.170%
Ba2	0.858%	1.909%	2.849%	3.740%	4.626%	5.374%	5.885%	6.413%	6.958%	7.425%
Ba3	1.546%	3.030%	4.329%	5.385%	6.523%	7.419%	8.041%	8.641%	9.191%	9.713%
B1	2.574%	4.609%	6.369%	7.618%	8.866%	9.840%	10.522%	11.127%	11.682%	12.210%
B2	3.938%	6.419%	8.553%	9.972%	11.391%	12.458%	13.206%	13.833%	14.421%	14.960%
B3	6.391%	9.136%	11.567%	13.222%	14.878%	16.060%	17.050%	17.909%	18.579%	19.195%
Caa	14.300%	17.875%	21.450%	24.134%	26.813%	28.600%	30.388%	32.174%	33.963%	35.750%

Table 2.8: Moody's idealised cumulative expected losses.[26]

Moody's sometimes refers to the table of idealised cumulative expected default rates,[27] which is prepared from Table 2.8, after adjusting the losses for a constant recovery rate of 45%.

To aggregate the single creditor loss into the losses of the collateral portfolio, Moody's uses the binomial expansion technique (BET). The BET is based on the following assumptions: a) the probability that one particular credit of the collateral portfolio defaults is independent of the probability that any other credit in the same portfolio defaults, b) all credits have the same size, maturity, probability of default p, and loss l, in case of default.

The consequence of such assumptions is the probability of a loss of $X = nl$, with $n \leq D$ can be calculated with the Binomial Distribution

$$\mathrm{P}(X = nl) = \binom{D}{n} p^n (1-p)^{D-n} \tag{2.23}$$

where D is the diversity score of the collateral portfolio, explained below.

To be able to use the BET, any portfolio of credits (bonds, loans etc) must be conveniently approximated with a number D of *homogeneous* credits. The diversity score reduces the actual portfolio of collateral credits with correlated default probabilities, to a *homogenous* portfolio of D credits with uncorrelated default probabilities. Moody's defines the diversity score of a given collateral portfolio as the number D of bonds in an idealised comparison portfolio formed by matching the first two moments of the loss distribution of the original and comparison portfolios. A detailed explanation of alternative diversity score method calculations, is given in Cifuentes *et al.* (1999), whereas applications are discussed in Schorin and Weinreich (1998) and Duffie and Garleanu (1999).

[26] From Cifuentes and O'Connor (1996).
[27] Tabe (2002).

These attempts have succeeded in bridging the gap of knowledge amongst practioners on one side and Moody's on the other. However, as Schönbucher (2003) highlighted, the Moody's BET is not a formal portfolio default model, it is inaccurate and is unsuitable for pricing. Besides, Moody's has never been generous in disclosing how the diversity score is calculated and updated, and more importantly, how it has approximated the default behaviour of real portfolios of credits, especially during the years 2002 and 2003 of very high defaults. In spite of these observations, the diversity score has become a market standard for communicating the degree of diversification of a portfolio.

An alternative approach is taken by S&P (2002) and Fitch (2003),[28] who use default probabilities to find the CDO note ratings. Perraudin and Peretyatkin (2003) discussed the differences between expected loss and default probability based ratings for all asset-backed securities (to which CDOs belong). They showed that thick notes (such as senior notes rated AAA) tend to have superior ratings if the expected loss approach is taken. They found this to be consistent with the view that Moody's enjoys greater market share for rating senior notes, while S&P and Fitch obtain a larger share for rating mezzanine notes.

We also note that the S&P and Fitch models do not arrive at an *idealised* construction of defaults, but rather build a distribution of aggregate default rates for the collateral portfolio. From the distribution, they apply different confidence levels for different ratings, and calculate the credit enhancements, firstly without and ultimately with recoveries.

In Table 2.9 we have extracted the rating confidence levels that S&P and Fitch use for rating CDO notes, together with the S&P stressing factors.

	AAA	AA	A	BBB	BB	B
Fitch Confidence Levels	99.62%	98.98%	97.86%	94.52%	84.04%	68.70%
S&P Confidence Levels	99.95%	99.55%	97.87%	90.94%	47.54%	33.09%
S&P Stressing Factors	120%	111%	102%	93%	81%	72%
S&P Pre Stressing Factor Confidence Levels	99.05%	99.05%	97.87%	95.25%	83.95%	74.71%

Table 2.9: Fitch and S&P confidence levels for rating CDO notes (AAA to B).

[28] S&P uses the CDO Evaluator to simulate correlated default times, with a one-step period Monte Carlo simulation. Fitch uses the Vector CDO model, a Merton-based model, where the aggregated distribution of correlated default times is calculated with a multi step Monte Carlo simulation.

Chapter 3: Copulae

3.1 Introduction

In this chapter, we introduce the notion of copulae, which allows us to combine a general framework of default dependency with the calibration of the marginal distribution of probability of default either on market CDS premiums or historical default information. We will do so, by separating the individual term structure of credit spreads (the marginal distributions) from the default dependency using copulae. We will also examine survival copula, and the use of copulae in setting probability bounds for sums of random variables.

3.2 Copula definitions

We denote by $f(\mathrm{x})$ and $F(\mathrm{x})$, respectively the joint probability distribution function (p.d.f.) and the joint distribution function (d.f.) of an *n*-dimensional random vector $\mathrm{X}_t = (X_{1,t}, X_{2,t},..., X_{n,t})^T$ at point $\mathrm{x}_t = (x_{1,t}, x_{2,t},..., x_{n,t})^T$. The univariate margins[1] p.d.f. and d.f. of each element of X_t at points x_t are denoted by $f_i(x_i)$ and $F_i(x_i)$ respectively, with $i = 1,2,..., n$.

Before defining the copula, we recall the following proposition of probability-integral and quantile function that we will use repeatedly in this chapter.

Proposition 3.1:[2] *Let X be a random variable with distribution function F, with notation $X \sim F$, and let $F^{-1}(t)$ be the quantile function or the generalised inverse of F, defined as*

$$F^{-1}(t) = \inf\{x \in \Re : F(x) \geq t\} \tag{3.1}$$

for all $t \in [0,1]$. Then,

(1) *for any uniformly distributed $U \sim U(0,1)$, we have $F^{-1}(U) \sim F$;*

(2) *if F is continuous, then the random variable X is uniformly distributed, i.e. $F(X) \sim U(0,1)$.*

(1) provides a simple way of simulating random variables with the distribution function F, by drawing uniformly distributed numbers.

[1] In this thesis, we will use the terms of margins and marginals as synonymous.

[2] In Embrechts, McNeil and Straumann (1999) p. 4. The proof is in Wang (1997).

Definition 3.1:[3] *An n-dimensional copula is a function* C *defined on* $[0,1]^n$ *with range on* $[0,1]$ *with the following properties*:

(1) (grounded) *for every* u *in* $[0,1]^n$, $C(u)=0$ *if at least one coordinate* $u_j = 0$ *with* $j=1,2,..,n$.

(2) (reflectiviness) *if all coordinates of* u *are* 1 *except* u_j *then,* $C(u)=u_j$ *with* $j=1,2,..,n$.

(3) (n-increasing) *for every point* u_1 *and* u_2 in $[0,1]^n$ *the* C*-volume* $V_C([u_1,u_2])$ *is non-negative,*

$$\sum_{j_1=1}^{2}\cdots\sum_{j_n=1}^{2}(-1)^{j_1+j_2+\ldots+j_n}C(u_{1,j_1},u_{2,j_2},\ldots,u_{n,j_n})\geq 0 \tag{3.2}$$

for all $(u_{1,1},u_{2,1},\ldots,u_{n,1})$ *and* $(u_{1,2},u_{2,2},\ldots,u_{n,2})$ *in* $[0,1]^n$ *with* $u_{j,1}\leq u_{j,2}$ *and* $j=1,2,..,n$.[4]

From its definition, a copula represents the joint d.f. of n-standard uniform random variables, $U_1,U_2,..,U_n$, that is, a multivariate distribution with uniform margins

$$C(u_1,u_2,..,u_n)=\mathbb{P}(U_1\leq u_1,U_2\leq u_2,..,U_n\leq u_n) \tag{3.3}$$

A copula may be used to represents joint d.f. of any general random variables. As a consequence of probability-integral and quantile function we have

$$\begin{aligned}&F(x_1,x_2,..,x_n)=\\ &\mathbb{P}(F_1(X_1)\leq F_1(x_1),F_2(X_2)\leq F_2(x_2),..,F_n(X_n)\leq F_n(x_n))\\ &=C(F_1(x_1),F_2(x_2),..,F_n(x_n))\end{aligned} \tag{3.4}$$

that is, any continuous multivariate d.f. $F(x_1,x_2,..,x_n)$ of random variables $X_1,X_2,..,X_n$ can be decomposed into a composition of the individual marginal d.f. $F_i(x_i)$ and the copula $C(\cdot)$. This is formally stated in the following theorem.

Theorem 3.1 (Sklar's 1959):[5] *Let* F *be an n-dimensional distribution function with margins* $F_1(x_1),F_2(x_2),..,F_n(x_n)$. *Then there exists an n-copula* C *such that for all* x *in* $\Re^n$,

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[3] In Nelsen (1999) p. 39, Embrechts *et al.* (1999) p. 4, and Meneguzzo and Vecchiato (2002) pp. 5-6. The original definition was in Sklar (1959) in french.

[4] From Embrechts *et al.* (1999) p. 4. Property 2 follows from the fact that the marginals are uniform [0,1]; property 3 is true because the sum in (3.2) can be interpreted as $=\mathbb{P}(u_{1,1}\leq U_1\leq u_{1,2},..,u_{n,1}\leq U_n\leq u_{n,2})$ which is non-negative.

[5] For a proof, see Sklar (1996).

$$F(x_1, x_2,.., x_n) = C(F_1(x_1), F_2(x_2),.., F_n(x_n)) \quad (3.5)$$

This theorem explains why copula reveals the link between multivariate distribution function and its individual margins. An important corollary is the following.

Corollary 3.1:[6] *Let* F *be an n-dimensional distribution function with continuous margins* $F_1(x_1), F_2(x_2),.., F_n(x_n)$ *and copula* C *(where* C *satisfies* (3.5)). *Then for any* u *in* $[0,1]^n$

$$C(u_1, u_2,.., u_2) = F(F_1^{-1}(u_1),.., F_n^{-1}(u_n)) \quad (3.6)$$

In this way, a copula is a joint distribution function of uniform variates U_i, $i = 1,2,...,n$, each of which has a standard uniform distribution.

Remark 3.1: *From the previous definition of copula, in terms of density, we have the following representation of the copula-density* c[7]

$$f(x_1, x_2,.., x_n) = c(F_1(x_1), F_2(x_2),..., F_n(x_n)) \cdot \prod_{i=1}^{n} f_i(x_i) \quad (3.7)$$

where
$$c(F_1(x_1), F_2(x_2),..., F_n(x_n)) = \frac{\partial^n C(F_1(x_1), F_2(x_2),..., F_n(x_n))}{\partial F_1(x_1) \partial F_2(x_2)...\partial F_n(x_n)} \quad (3.8)$$

and
$$f_i(x_i) = \frac{\partial F_i(x_i)}{\partial x_i} \quad (3.9)$$

There is also a copula K, the survival copula, that separates an *n*-dimensional survival function S, from its univariate survival margins $S_1(x_1), S_2(x_2).., S_n(x_n)$. In what follows, we start with the relationship between the survival copula and the copula in the bi-dimensional case.

Before defining a survival copula, we remind an elementary relationship of survival analysis in the bi-dimensional case.[8] We denote by $S(x_1, x_2)$ the joint survival function, which can also be written as

$$S(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2) \quad (3.10)$$

Similarly, the survival function $C_S(u_1, u_2) = \mathbb{P}(U_1 > u_1, U_2 > u_2)$ of two $U(0,1)$ random variables, with copula $C(u_1, u_2)$ is

[6] In Embrechts, Lindskog and McNeil (2001) p. 4.
[7] For a proof for the bivariate case, see Meneguzzo and Vecchiato (2002) p. 11.
[8] In Bowers *et al.* (1997).

$$C_S(u_1,u_2) = 1 - u_1 - u_2 + C(u_1,u_2) \tag{3.11}$$

Since $C(F_1(x_1), F_2(x_2)) = F(x_1,x_2)$, we have $C_S(F_1(x_1), F_2(x_2)) = S(x_1,x_2)$. (3.12)

Definition 3.2: *For a copula C, we define the survival copula K as*

$$K(u_1,u_2) = C_S(1-u_1, 1-u_2)$$
$$= u_1 + u_2 - 1 + C(1-u_1, 1-u_2) \tag{3.13}$$

where we note that K is a copula, whereas C_S in (3.11) *is not.*[9]

Sklar's theorem can be easily restated in terms of a survival copula K.

Theorem 3.2: *Let S be an 2-dimensional joint survival function with margins $S_1(x_1)$ and $S_2(x_2)$. Then there exists a 2-copula K such that for every (x_1,x_2) in $\Re^2$,*

$$S(x_1,x_2) = K(S_1(x_1), S_2(x_2)) \tag{3.14}$$

For the multidimensional one, the n-survival copula K is

$$S(x_1,..,x_n) = K(S_1(x_1),..,S_n(x_2)) \tag{3.15}$$

where the relationship between the multidimensional survival copula K and copula C, cannot be easily expressed as for the bi-dimensional case in (3.13). For this relationship, we refer to Georges *et al.* (2001).

3.3 Special copulae

We let $C^-(u), C^\perp(u)$ and C^+ be three functions defined on $[0,1]^n$ as follows

$$C^-(u) = \max\left(\sum_{i=1}^{n} u_i - n + 1, 0\right) \tag{3.16}$$

$$C^\perp(u) = u_1 u_2 \cdots u_n \tag{3.17}$$

$$C^+(u) = \min(u_1, u_2, ..., u_n) \tag{3.18}$$

with $C^\perp(u)$ the product copula.

Theorem 3.3 (Fréchet 1951):[10] *If C is a n-copula, then for every u in $[0,1]^n$,*

[9] For a proof see Cherubini and Vecchiato (2004) p. 75.
[10] For more details and a proof, see Nelsen (1999).

$$C^{-}(u) \le C(u) \le C^{+}(u) \tag{3.19}$$

With Frechet's theorem, it is possible to fix some *bounds* to joint distribution functions: $C^{-}(u)$ is the lower bound and $C^{+}(u)$ is the upper bound. The same theorem suggests some sort of *ordering* in the values of a copula. For the cases of $n = 2$, the bounds are themselves copulae, since every copula is the distribution function of a random vector $(U_1, U_2)^T$, and we can write

$$C^{-}(u_1, u_2) = \mathrm{P}(U \le u_1, 1 - U \le u_2) \tag{3.20}$$

$$C^{+}(u_1, u_2) = \mathrm{P}(U \le u_1, U \le u_2) \tag{3.21}$$

We say that $C^{+}(u_1, u_2)$ describes perfect positive dependence and $C^{-}(u_1, u_2)$ describes perfect negative dependence between two random variables. Whereas, $C^{\perp}(u_1, u_2)$ describes the case of independence. Yaari (1987) expressed the relationship between X_1 and X_2, if they have the copula C^{+} as *co-monotonic* and if they have the copula C^{-} as *counter-monotonic*.

To understand the role played by *co-monotonicity* in baskets of credits, consider the following CDO structure[11] where the investor 1, called originator, retains the first portion of the loss L, called equity piece Eq, and the investor 2 suffers any loss above the equity Eq. The loss of the originator, X_1, and the loss of the investor, X_2, can be represented as follows

$$X_1 = \begin{cases} L & if \;\; L \le Eq \\ Eq & if \;\; L > Eq \end{cases} \qquad X_2 = \begin{cases} 0 & if \;\; L \le Eq \\ L - Eq & if \;\; L > Eq \end{cases} \tag{3.22}$$

Note that $X_1 = g_1(L)$ and $X_2 = g_2(L)$ are not linearly correlated since we cannot write one as a linear function of the other. However, since they are always non-decreasing functions of the loss L, they are co-monotonic.[12]

3.4 Measure of dependence

The Pearson's linear correlation coefficient, which is by far the most used measure to test dependence in the financial community, is not a measure of general, but only *linear* dependence. If two or more random variables are well represented by a multivariate Elliptical distribution, their dependence structure is linear. Hence, the linear correlation coefficient is a meaningful measure of dependence. Outside the Elliptical distribution, their dependence structure is not linear, and the use of linear correlation coefficient as a measure of dependence may lead to incorrect conclusions.

[11] For a review of the CDO market, see Picone (2003).

[12] This example was originally proposed by Wang (1997) to aggregate losses in a portfolio of correlated insurance claims.

Therefore, Copula methods for capturing wider forms of dependence should be considered. Copulae are more general tools to describe dependence between random variables than the Pearson's linear correlation. In this section we review the inadequacies of linear correlation as measure to describe non linear dependency between random variables, together with the properties of copulae.

3.4.1 Pearson's Linear Correlation

Definition 3.3: *Let* $(X_1, X_2)^T$ *be a vector of two random variables with nonzero variances. The Pearson's linear correlation is*

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} \tag{3.23}$$

where $Cov(X_1, X_2)$ is the covariance of $(X_1, X_2)^T$ and $Var(X_1)$ and $Var(X_2)$ are the variances of X_1 and X_2. An alternative representation of (3.23) is the following

$$\rho(X_1, X_2) = \beta \frac{\sqrt{Var(X_2)}}{\sqrt{Var(X_1)}} \tag{3.24}$$

where β is the slope of the following equation $X_1 = \alpha + \beta X_2$, for a constant $\alpha > 0$. This explains why (3.23) describes a linear relationship between two random variables.

The popularity of the Pearson's linear correlation can be attributed to three important reasons. Firstly, Pearson's linear correlation is straightforward to calculate. Secondly, Pearson's linear correlation and covariances are easy to manipulate under strictly linear operations, and this property is extensively used to calculate the inputs to derive the efficient frontier and portfolio analysis in the Markowitz portfolio theory.[13] A third reason is due to the quality of the Pearson's linear correlation of being a natural measure of dependence in multivariate normal distributions, and more in general, of multivariate spherical and elliptical distributions.[14]

Pearson's linear correlation is not a copula based measure of dependence, and very often leads to misleading results. The following shortcomings should prevent everybody to look at the Pearson's linear correlation as the canonical measure of dependence.

The first problem is that the Pearson's linear correlation is defined only where the variances are finite. Where the two random variables, X_1 and X_2, are elliptically joint with the *t*-Student

[13] For more on how to manipulate linear correlation under linear operations, see Elton and Gruber (1991) p. 98.

[14] For more on Elliptical and Spherical distributions, see Lindskog (2000) p. 11 and Embrechts *et al.* (1999) pp. 7-13.

distribution with degree of freedoms $\upsilon \leq 2$, the Pearson's linear correlation is not defined since the second moments are infinite.[15]

The second problem is that the Pearson's linear correlation is not an invariant measure under non-linear increasing transformations. For example, if the two variables are log transformed, we have the following

$$\rho(X_1, X_2) \neq \rho[\log(X_1), \log(X_2)]. \tag{3.25}$$

The reason is that the Pearson's linear correlation depends on the distribution of the margins, thus the measure is affected by non-linear change in the scale of the margins, such as the log transformation.[16] Bouye and Salmon (2000) explained the above problem writing the Pearson's linear correlation at copula level as

$$\rho(X_1, X_2) = \frac{1}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) dF_1^{-1}(u_1) dF_2^{-1}(u_2) \tag{3.26}$$

From (3.26) we see that the Pearson's linear correlation depends on the copula and on the margins, F_1 and F_2.

A common misunderstanding around the Pearson's linear correlation is to confuse zero linear correlation with independence. Independence of two random variables implies a Pearson's linear correlation equal to zero, but the reverse is not in general true. It is true only when the marginals are normal distributed and the joint distribution is normal as well.[17]

One last limitation is that the Pearson's linear correlation is not a measure to capture extreme co-movements in two or more random variables. As we shall see in the following section, *tail dependence* is a property of the underlying copula.

3.4.2 Dependence with copulae

Perhaps, one of the most important properties of copulae is the one captured in the following theorem, which explains why copulae are better tools to describe dependence between random variables than the Pearson's linear correlation.

Theorem 3.4 (Invariance):[18] *Let* $(X_1, ..., X_n)^T$ *be the random vector with copula* C. *If* $(g_1(X_1), .., g_n(X_n))^T$ *are strictly increasing on the range of* $X_1, ..., X_n$, *then they also have* C

[15] See Embrechts *et al.* (2001) pp. 6-13.
[16] For a proof see Schweizer and Wolff (1981).
[17] For more about marginals normally distributed and non-normal joint distributions see Embrechts *et al.* (1999).
[18] For a proof see Embrechts, McNeil and Straumann (1999) p. 6.

as their copula.

This theorem shows that the new variables $g_i(X_i)$ have the same copula as the original variables X_i. Therefore, the way variables move together, or are dependent, is captured by the copula, regardless of the scale in which the same variables are expressed.[19] This statement holds for Pearson's linear correlation only under linear transformations.

There are indeed two measures of dependence that do not share the same limitation as the Pearson's linear correlation and are very useful when it is required to compare any two copulae. These are the Spearman's rank correlation $\rho_S(X_1, X_2)$ and the Kendall's rank correlation $\rho_\tau(X_1, X_2)$.

Definition 3.4:[20] *Kendall's rank correlation of a random vector* (X_1, X_2) *is defined as*

$$\rho_\tau(X_1, X_2) = \mathrm{P}\big((X_1 - X_1')(X_2 - X_2') > 0\big) - \mathrm{P}\big((X_1 - X_1')(X_2 - X_2') < 0\big) \tag{3.27}$$

where (X_1', X_2') is an independent realisation of the joint distribution of (X_1, X_2). The Kendall's rank correlation in (3.27) looks like the difference of two probability measures, the first is the probability of concordance and the second is the probability of discordance.

Definition 3.5:[21] *Spearman's rank correlation of a random vector* (X_1, X_2) *is defined as*

$$\rho_S(X_1, X_2) = 3\Big(\mathrm{P}\big((X_1 - X_1')(X_2 - X_2'') > 0\big) - \mathrm{P}\big((X_1 - X_1')(X_2 - X_2'') < 0\big)\Big) \tag{3.28}$$

where (X_1', X_2') and (X_1'', X_2'') are independent realisations of the joint distribution of (X_1, X_2).

Both Spearman's ρ_S and the Kendall's ρ_τ can be written at copula level as follows[22]

$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 C(u_1, u_2)\,du_1 du_2 - 3 \tag{3.29}$$

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)\,dC(u_1, u_2) - 1 \tag{3.30}$$

With (3.29) and (3.30), both Spearman's ρ_S and Kendall's ρ_τ depend only on the copula, whereas, we showed in (3.26) that the Pearson's linear correlation depends not only on the copula, but on the marginal distributions as well.

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[19] Schweizer and Wolff (1981) showed that the copula accounts for all dependence between two *r.v.*s.
[20] In Embrechts *et al.* (1999) p. 16.
[21] In Embrechts *et al.* (2001) p. 13.
[22] In Embrechts *et al.* (1999), theorem 3. p. 16.

Spearman's ρ_S and Kendall's ρ_τ are measures of the degree of *monotonic* dependence between two random variables, whereas the Pearson's linear correlation is only the measure of linear dependence. For example, if $\rho_S(X_1,X_2)=\rho_\tau(X_1,X_2)=1$ the random variables X_1,X_2, have joint distribution C^+. In case $\rho_S(X_1,X_2)=\rho_\tau(X_1,X_2)=-1$ the random variables X_1,X_2, have joint distribution C^-.

For last, they are also invariant under monotonic transformations. This is because both can be expressed in terms of copula, which is invariant under strictly increasing transformations of the marginal distributions (theorem 3.4).

Rather than looking at global measures of dependence such as Spearman's ρ_S and Kendall's ρ_τ, further insight into the association between two random variables can be gained by considering local measures of dependence. Upper and lower tail dependence are local dependence measures of bi-variate extremes, which cannot be measured by the Pearson's linear correlation.

The concept of tail dependence relates to the dependence that arises in extreme observations, that is, those located in the upper right and lower left quadrant tail of the bivariate distribution. It is defined as follows.

Definition 3.6:[23] *Let* (X_1,X_2) *be a vector of continuous random variables with marginal functions* F_1 *and* F_2. *The coefficient of upper and lower tail dependence of* (X_1,X_2) *are*

$$\lambda_U=\lim_{u\to 1}\mathrm{P}\left(X_2>F_2^{-1}(u_2)\mid X_1>F_1^{-1}(u_1)\right) \tag{3.31}$$

$$\lambda_L=\lim_{u\to 1}\mathrm{P}\left(X_2<F_2^{-1}(u_2)\mid X_1<F_1^{-1}(u_1)\right) \tag{3.32}$$

provided that the limits λ_U and $\lambda_L \in [0,1]$ exists.

It is evident that (3.31) (and (3.32)) is the probability that one component is extremely large (small) given that the other component is extremely large (small).

An alternative and equivalent definition, from which it can be seen that the concepts of upper and lower tail dependence are indeed copula properties is the one proposed by Joe (1997)

$$\lambda_U=\lim_{u\to 1}\frac{1-2u+C(u,u)}{1-u} \tag{3.33}$$

and

$$\lambda_L=\lim_{u\to 0}\frac{C(u,v)}{u}. \tag{3.34}$$

[23] In Embrechts *et al.* (2001) pp. 15-16 and Georges, Lamy, Quibel and Roncalli (2001). For a proof see Joe (1997) p. 33.

3.5 Examples of copula

There are many choices of copulae available in the literature that would permit various dependence structures. We begin with Marshall and Olkin exponential copula. Following this, we introduce two Elliptical copulae: Normal and *t*-Student, and two Archimedean copulae: Clayton and Gumbel.

3.5.1 Marshall and Olkin exponential copula

The Marshall and Olkin exponential copula (1967) is used to model the credit events as triggered by either idiosyncratic shocks or a by a shock common to a number of obligors, which is in the spirit of the approach of Duffie and Singleton (1998). It has been used for years in the actuarial field, but just recently applied to finance. Among the first and major contributors are Embrechts, Lindskog and Straumann (1999), Embrechts, Lindskog and McNeil (2001) and Giesecke (2002).

We begin with bivariate Marshall and Olkin exponential copula and present analytic expressions for the rank correlation and tail dependence. Then, we move to the multivariate case.

For simplicity we begin with an economy characterised by only two credits. In this simplified world, we let the defaults of the two credits be driven by firm-specific factors, as well economic-wide factors.

We let t_1 and t_2 denote the maturity of the two credits. Furthermore, assume there are three independent time-homogeneous Poisson processes N_1, N_2 and N with intensity rate λ_1, λ_2 and λ_{12} that we can interpret as the idiosyncratic shocks of credit 1 and 2 (λ_1, λ_2), whereas we think of λ_{12}, as the systematic shock that affects both credits simultaneously. Also, since there are only two credits in the economy, the economic-wide factor, λ, coincides with systematic shock of credit 1 and 2, λ_{12}, and we can write $\lambda_{12} = \lambda$.

Definition 3.7: *We define the random time of default τ_i of credit i as*

$$\tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\} \tag{3.35}$$

which means that default takes place unexpectedly if either the idiosyncratic or a systematic shock strikes the credit i. Thus, credit i defaults with an intensity equal to $\lambda_i + \lambda$. The random times τ_1, τ_2 and τ, of occurrence of the three shocks are independent exponential random variables with intensity rate λ_1, λ_2 and λ respectively. Hence $\mathrm{P}(\tau_1 = \tau_2) > 0$.

The univariate survival margin of credit i is

$$S_i(t_i) = \mathbb{P}(\tau_i > t_i) = \exp(-(h_i + h)t) \ (i = 1, 2 \text{ and } t \geq 0), \quad (3.36)$$

and the joint survival of credit 1 and 2 is

$$S(t_1, t_2) = \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \mathbb{P}(\tau_1 > t_1)\mathbb{P}(\tau_2 > t_2)\mathbb{P}(\tau > \max(t_1, t_2)) \quad (3.37)$$

From the fact that $\max(t_1, t_2) = t_1 + t_2 - \min(t_1, t_2)$, we can re-write (3.37) as

$$\begin{aligned} S(t_1, t_2) &= \exp(-(\lambda_1 + \lambda)t_1 - (\lambda_2 + \lambda)t_2 + \lambda \min(t_1, t_2)) \\ &= S_1(t_1)S_2(t_2)\min(\exp(\lambda t_1), \exp(\lambda t_2)) \end{aligned} \quad (3.38)$$

We let $S_i(t_i) = u_i$ and θ_i be the ratio of the joint intensity rate λ and the intensity rate of credit i λ_i, $\theta_i = \lambda / (\lambda_i + \lambda)$. Then, $\exp(\lambda t_1) = u^{\theta_1}$, and $\exp(\lambda t_2) = u^{\theta_2}$. Hence, the survival copula K is given by

$$\begin{aligned} K(u_1, u_2) &= S(S_1^{-1}(u_1), S_2^{-1}(u_2)) \\ &= u_1 u_2 \min(u_1^{-\theta_1}, u_2^{-\theta_2}) = \min(u_1^{1-\theta_1} u_2, u_1 u_2^{1-\theta_2}) \end{aligned} \quad (3.39)$$

where the vector (θ_1, θ_2) controls, as Giesecke (2002) pointed out, the degree of dependency between the two times of default. The copula in (3.39) is known as the Marshall and Olkin copula.

When defaults are independent, $\lambda = 0$, then $\theta_1 = \theta_2 = 0$, and we are able to write the product copula in (3.17) as

$$K(u_1, u_2) = u_1 u_2 \quad (3.40)$$

When defaults are perfectly positively correlated (we prefer to speak of co-monotone defaults), i.e. credits default simultaneously, then $\lambda_1 = \lambda_2 = 0$ and $\theta_1 = \theta_2 = 1$. Hence, we write the Fréchet upper bound copula in (3.18)

$$K(u_1, u_2) = \min(u_1, u_2) \quad (3.41)$$

Otherwise, we write $u_1 u_2 \leq K(u_1, u_2) \leq \min(u_1, u_2)$, meaning that with the Marshall and Olkin copula, defaults can only be positively related.[24]

Embrechts *et al.* (2001)[25] showed that the Spearman's rho and Kendall's tau (and Pearson's correlation)[26] of the times of default τ_1 and τ_2 are easily evaluated for this copula as

$$\rho_S(\tau_1, \tau_2) = \frac{3\theta_1\theta_2}{2\theta_1 + 2\theta_2 - \theta_1\theta_2} = \frac{3\lambda}{3\lambda + 2\lambda_1 + 2\lambda_2} \quad (3.42)$$

and

$$\rho_\tau(\tau_1, \tau_2) = \rho(\tau_1, \tau_2) = \frac{\theta_1\theta_2}{\theta_1 + \theta_2 - \theta_1\theta_2} = \frac{\lambda}{\lambda_1 + \lambda_2 + \lambda} \quad (3.43)$$

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[24] In Giesecke (2002) p. 5.
[25] In Embrechts *et al.* (2001) pp. 18-19.
[26] In Lindskog (2000) p. 23.

In Figure 3.1 we plot the two rank default correlation measures as a function of the joint default intensity λ, and with the idiosyncratic shocks λ_1 and λ_2 equal to 0.1.

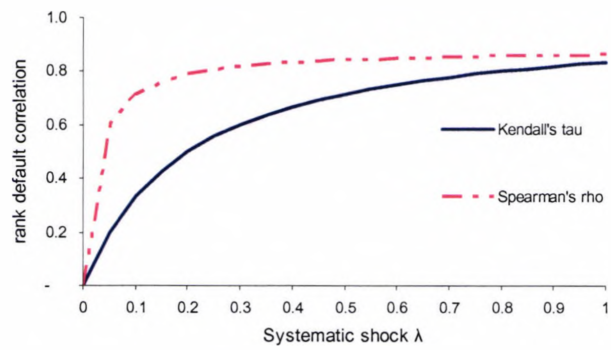

Figure 3.1: Spearman's rho and Kendall's tau measures with idiosyncratic shocks $\lambda_1 = \lambda_2 = 0.1$.

When $\lambda = 0$, the rank default correlation measures are equal to zero and the credits default independently. When $\lambda_1 = \lambda_2 = 0$, the joint shock dominates the idiosyncratic component and the firms will default simultaneously. As the joint default intensity λ increases, Spearman's rho dominates the Kendall's tau. However, this is not always true. In Figure 3.2 we show the case with the idiosyncratic shocks λ_1 and λ_2 equal to 0.5, and for λ greater than 0.5 the Spearman's measure underestimates the Kendall tau measure.

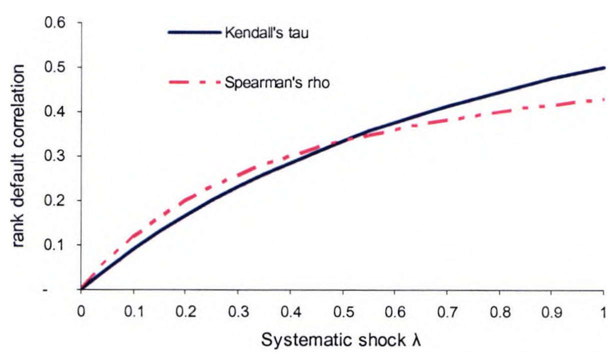

Figure 3.2: Spearman's rho and Kendall's tau measures with idiosyncratic shocks $\lambda_1 = \lambda_2 = 0.5$.

In Figure 3.3 we plot the Spearman's rho and Kendall's tau measures with constant joint default intensity λ of 0.5, and with λ_1 and λ_2 changing from 0 to 1. Intuitively, the rank default correlation measures decrease as the two idiosyncratic rates increase, and the idiosyncratic component dominates the joint shock component of default.

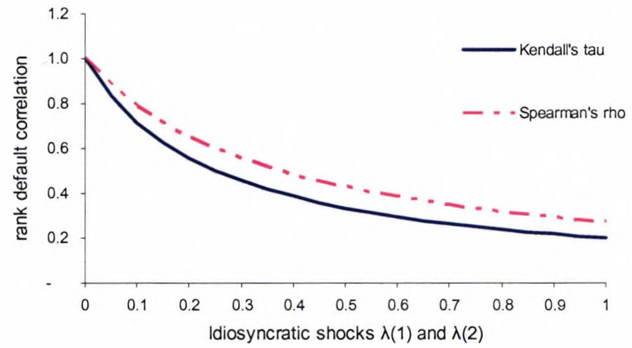

Figure 3.3: Spearman's rho and Kendall's tau measures with systematic shock $\lambda = 0.5$.

The upper tail dependence has a closed form solution[27] equal to

$$\lambda_u = \min(\theta_1, \theta_2) \tag{3.44}$$

Thus, the ratio of the joint intensity rate, λ, and the intensity rate of credit 1, λ_1, (or the intensity rate of credit 2, λ_2), controls the asymptotic measure of tail dependence between the two credits.

Before we move to the multivariate extension of the bivariate Marshall and Olkin exponential copula, we set up the three-variate case. We assume a three-credit economy, with $n = 3$, where the default of an individual firm is driven by firm-specific factors, some sector, geographic or country factors common to pairs of credits, as well an economic-wide factor common to all three credits. The factors are independent Poisson processes, N_1, N_2, N_3, N_{12}, N_{13}, N_{23} and N_{123} with respective intensities λ_1, λ_2, λ_3, λ_{12}, λ_{13}, λ_{23} and λ_{123}. The maturities of the three credits are denoted t_1, t_2 and t_3.

To indicate whether a non-firm specific factor leads to the firm default we introduce a matrix[28] with elements $(a_{ij})_{nXm}$, where the value 1 indicates that the factor leads to the default and the value 0 does not:

$$\beta_j = \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_{12} \quad \lambda_{13} \quad \lambda_{23} \quad \lambda_{123}$$

$$(a_{i,j}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{3.45}$$

with $j = 1,2,..,7$ and $i = 1,2,3$.

[27] The derivation of that result can be found in Embrechts *et al.* (2001) p. 19.
[28] This matrix representation was taken from Giesecke (2002).

Definition 3.8: *We define the random time of default* τ_1 *of credit* 1 *as*[29]

$$\tau_1 = \inf\{t \geq 0 : a_{11}N_1(t) + a_{14}N_{12}(t) + a_{15}N_{13}(t) + a_{17}N_{123}(t) > 0\} \tag{3.46}$$

The survival probability of credit 1 can be written as

$$S_1(t_1) = \mathbb{P}(\tau_1 > t_1) = \exp\left(-\sum_{j=1}^{7} a_{1j}\beta_j t_1\right) \tag{3.47}$$

and the joint survival probability of the three credits is

$$\begin{aligned} S(t_1,t_2,t_3) &= \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3) \\ &= \exp\left[-\sum_{i=1}^{3} \beta_i t_i - \sum_{i<j} a_{ij}\beta_i \max(t_i,t_j) - \beta_7 \max(t_1,t_2,t_3)\right] \end{aligned} \tag{3.48}$$

with β_i as defined in (3.45).

Similar arguments yield to the following n-dimensional exponential survival function

$$S(t_1,...,t_n) = \mathbb{P}(\tau_1 > t_1,...,\tau_n > t_n) =$$

$$\exp\left[-\sum_{i=1}^{n} \lambda_i t_i - \sum_{i<j} a_{ij}\lambda_{ij} \max(t_i t_j) - \sum_{i<j<k} a_{ijk}\lambda_{ijk} \max(t_i t_j t_k) - ... - a_{12..n}\lambda_{12..n} \max(t_1,t_2,..,t_n)\right]$$

(3,49)

Using the same arguments used for the two-dimensional copula, once i and $j \in \{1,2,..,n\}$ are fixed, with $i \neq j$, the two-dimensional survival copula of credit i and j is

$$\begin{aligned} K(u_i,u_j) &= K(1,..,1,u_i,1,..,u_j,1,..,1) \\ &= \min(u_j u_i^{1-\theta_i}, u_i u_j^{1-\theta_j}) \end{aligned} \tag{3.50}$$

where θ_i and θ_j are the ratios of the joint default intensity of credit i and j to the default of credit i (or j for the latter) equal to

$$\theta_i = \frac{\sum_{k=1}^{7} a_{ik}a_{jk}\beta_k}{\sum_{k=1}^{7} a_{ik}\beta_k} \qquad \theta_j = \frac{\sum_{k=1}^{7} a_{ik}a_{jk}\beta_k}{\sum_{k=1}^{7} a_{jk}\beta_k} \tag{3.51}$$

In analogy to the bivariate case, the $(i, j)th$ entry of the Spearman's and Kendall's rank default time matrix are respectively given by

$$\rho_S(\tau_i,\tau_j) = \frac{3\theta_i\theta_j}{2\theta_i + 2\theta_j - \theta_i\theta_j} \tag{3.52}$$

[29] Similarly, the random time of default of credit 2 and 3 can be defined.

and
$$\rho_\tau(\tau_i,\tau_j) = \frac{\theta_i\theta_j}{\theta_i + \theta_j - \theta_i\theta_j} \tag{3.53}$$

The critique we can move to the multivariate Marshall and Olkin exponential copula is the large number of parameters to calibrate. Given the lack of available market data, this seems an impossible task. Embrechts *et al.* (2001) suggested setting the shock intensities for subgroups with more then two elements, $\{\lambda_{ijk}\},\ldots,\{\lambda_{ijk\ldots n}\}$, to zero. As they underlined, in this case the copula would have only a bivariate dependence.

This is also the route taken by Wong (2000). By setting $a_{ijk}\lambda_{ijk}$, and all higher orders in (3.49) to zero, he derived a multivariate extension of the exponential distribution calibrated to the parameters specified in the KMV model. With Wong's assumptions, (3.49) becomes

$$S(t_1,\ldots,t_n) = \mathrm{P}(\tau_1 > t_1,\ldots,\tau_n > t_n) = \exp\left[-\sum_{i=1}^{n}\lambda_i t_i - \sum_{i<j} a_{ij}\lambda_{ij}\max(t_i t_j)\right] \tag{3.54}$$

The simplest model, which yet accounts for default dependency in a credible way, is to set the shock intensities for subgroups with more then one element, $\{\lambda_{ij}\},\{\lambda_{ijk}\},\ldots$ to zero, with the exception of the subgroup with the highest number of elements, $\{\lambda_{ijk\ldots n}\}$. In this way, the times of default are still dependent upon each other through the common shock with intensity $\{\lambda_{ijk\ldots n}\}$.

3.5.2 Normal copula

Definition 3.9: *Let* Φ *denote the standard univariate normal distribution and* Φ_Σ^n *denote the standard multivariate normal distribution with correlation matrix* Σ. *Then the multivariate Normal copula is defined as follows*

$$C_\Sigma^N(u_1,\ldots,u_n) = \Phi_\Sigma^n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)) \tag{3.55}$$

where Φ^{-1} is the inverse of the standard normal cumulative distribution function.

In the bivariate case the normal copula can be written as[30]

$$C_\Sigma^N(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2(X_1,X_2)}}\exp\left\{-\frac{x_1^2 - 2\rho(X_1,X_2)x_1x_2 + x_2^2}{2\left(1-\rho^2(X_1,X_2)\right)}\right\}dx_1dx_2 \tag{3.56}$$

where $\rho(X_1,X_2)$ is Pearson's linear correlation of the two random variables.

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[30] In Lindskog, Mc Neil and Schmock (2001) p. 25.

The Pearson linear correlation $\rho(X_1, X_2)$ for a normal copula can be expressed in terms of both Kendall's $\rho_\tau(X_1, X_2)$ and Spearman's $\rho_S(X_1, X_2)$ [31] in the following way

$$\rho(X_1,X_2) = 2\sin\left(\frac{\pi}{6}\rho_S(X_1,X_2)\right) = \sin\left(\frac{\pi}{2}\rho_\tau(X_1,X_2)\right) \tag{3.57}$$

This copula fails to incorporate tail dependence and correlates the random variables near the mean and not in the tails. In spite of its limitations, it has been extensively applied in finance, given its nice properties.

3.5.3 *t*-Student copula

Definition 3.10: *Let t_υ denote the univariate t-Student distribution and $t^n_{\upsilon,\Sigma}$ denote the multivariate t-Student distribution with correlation matrix Σ and υ degrees of freedom. Then the multivariate t-Student copula is defined as follows*

$$C^t_{\upsilon,\Sigma}(u_1,u_2,...u_n) = t^n_{\upsilon,\Sigma}\left(t^{-1}_\upsilon(u_1), t^{-1}_\upsilon(u_2),.., t^{-1}_\upsilon(u_n)\right) \tag{3.58}$$

In the bivariate case the copula expression can be written as[32]

$$C^t_{\upsilon,\Sigma}(u_1,u_2) = \int_{-\infty}^{t^{-1}_\upsilon(u_1)} \int_{-\infty}^{t^{-1}_\upsilon(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2(X_1,X_2)}} \left\{1 + \frac{x_1^2 - 2\rho(X_1,X_2)x_1x_2 + x_2^2}{\upsilon\left(1-\rho^2(X_1,X_2)\right)}\right\}^{-(\upsilon+2)/2} dx_1 dx_2 \tag{3.59}$$

where $\rho(X_1, X_2)$ is Pearson's linear correlation and t_υ is defined only for $\upsilon > 2$.

It can be shown that the *t*-Student copula has upper (and lower) tail dependence equal to[33]

$$\lambda_U = \lambda_L = 2t_{\upsilon+1}\left(\sqrt{\upsilon+1}\sqrt{1-\rho(X_1,X_2)} / \sqrt{1+\rho(X_1,X_2)}\right) \tag{3.60}$$

where $t_{\upsilon+1}$ denotes the tail of a univariate *t*-Student distribution.

Table 3.1 shows some tail dependence coefficients of a bivariate *t*-Student copula for different degrees of freedom and linear correlation values. For example, with a bivariate normal distribution, the probability associated with joint tail events, given that a marginal tail event has occurred is zero (last row of Table 3.1 with infinity as degrees of freedom). However, if we use the *t*-Student copula to describe the same phenomena, the probability of these events would be different from zero. With 3 degrees of freedom and a correlation value of 0.5, the probability is 31.25%; with 10 degrees of freedom and a correlation value of 0.9, the probability is 46.27%, and so on.

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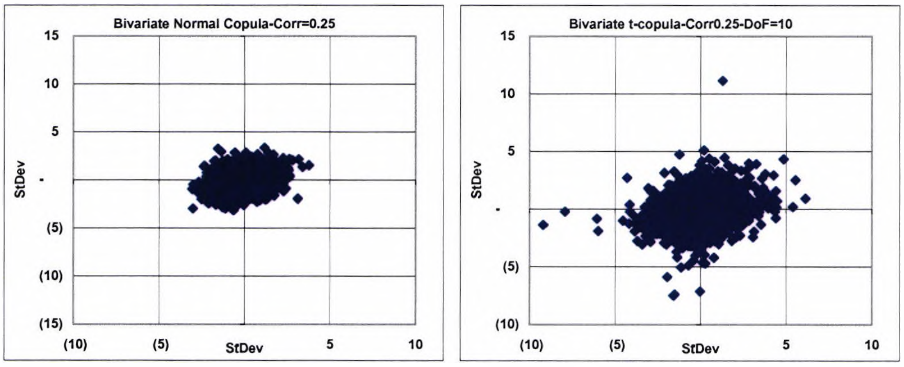
[31] In Lindskog *et al.* (2001). This result will prove to be very important for comparing samples drawn from Elliptical copulae with those drawn from Archimedean copulae.
[32] In Lindskog *et al.* (2001) p. 26.
[33] In Embrechts *et al.* (1999) p. 19.

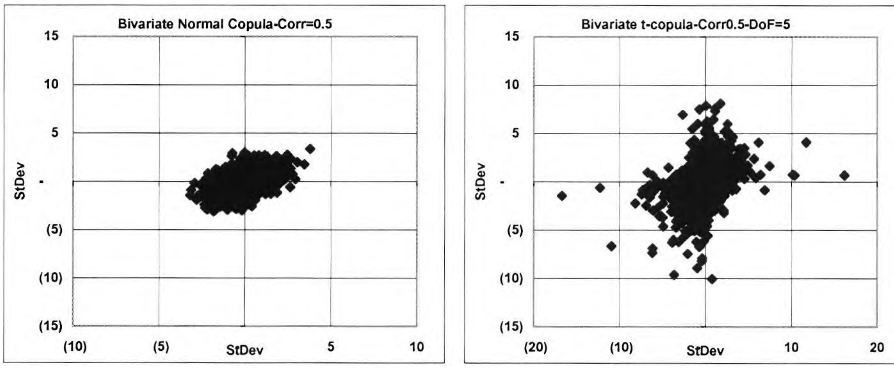
		Correlation					
		-0.5	0	0.3	0.5	0.9	1
degrees of freedom	3	0.0257	0.1161	0.2161	0.3125	0.6701	1
	10	0.0001	0.0068	0.0331	0.0819	0.4627	1
	15	3.00E-06	0.0010	0.0097	0.0346	0.3724	1
	20	9.00E-08	0.0002	0.0029	0.0151	0.3051	1
	infinity	0	0	0	0	0	0

Table 3.1: Bivariate *t*-Student copula tail dependence coefficients.

Another illustration of the different weights the *t*- Student copula gives to extreme events is given in Figures 3.4a, 3.4b, 3.5a, and 3.5b. Figure 3.4a and 3.5a show two charts with 5,000 runs from a Normal-copula with linear correlation $\rho(X_1, X_2)$ of 0.25 and 0.5, respectively. Figure 3.4b shows a chart of 5,000 runs from a *t*-Student copula with 10 degrees of freedom and linear correlation $\rho(X_1, X_2)$ of 0.25, whereas Figure 3.5b shows a chart of 5,000 runs from the *t*-Student copula with 5 degrees of freedom and linear correlation $\rho(X_1, X_2)$ of 0.5. Clearly, the *t*-Student copula generates more joint extreme events.



Figures 3.4a (left) and 3.4b (right): The left chart shows 5,000 runs from a Normal copula with linear correlation $\rho(X_1, X_2)$ of 0.25. The right chart shows 5,000 runs from a *t*-Student copula with linear correlation $\rho(X_1, X_2)$ of 0.25 and 10 degrees of freedom.



Figures 3.5a (left) and 3.5b (right): The left chart shows 5,000 runs from a Normal copula with linear correlation $\rho(X_1, X_2)$ of 0.5. The right chart shows 5,000 runs from a *t*-Student copula with linear correlation $\rho(X_1, X_2)$ of 0.5 and 5 degrees of freedom.

Heavy tails phenomena concern the potential for *extreme co-movements* between financial assets. Recent empirical studies suggest that extreme joint movements are intrinsically present in the joint behaviour of many financial assets. Karolyi and Stultz (1996), Straetmans (2000) and Longin and Solnik (2001) found strong evidence of this behaviour across international equity markets. More recently, Mashal, Naldi and Zeevi (2003) found that the VAR of portfolios of options on the G5 equity market indices, currencies and metals, calculated on the assumption of a normal dependence structure, are typically between 10% to 50% smaller than those calculated with a dependence structure, which supports extreme co-movements, such as the *t*-Student distribution. In particular, they found a non-trivial degree of dependence in the tails. They also rejected, with negligible error probability, the consistency of normal dependence structure and the statistical significance of their results became stronger as the number of underlying assets increases.

In baskets and CDOs of credits, extreme co-movements correspond to joint defaults. Thus, large joint movements that are not captured via the multivariate normal distribution can lead to large discrepancies in the calculated prices of these instruments.

The Normal copula places little mass on the tails relative to the *t*-Student copula. Unlike the Normal copula where dependence is captured only via linear correlation, the *t*-Student copula retains the use of linear correlation while introducing an additional parameter that controls the heaviness of the joint tails, the degree of freedom. Another important property of the *t*-Student copula is the so-called *tail dependence*. This means that even in very extreme events, asset returns remain dependent while within the Normal copula they become asymptotically independent. Both properties lead to a higher probability of occurrence of joint defaults and therefore greater risk in baskets or CDOs.

It is easy to understand the relationship of the *t*-Student copula with the Normal copula: the multivariate *t*-Student distribution is a generalisation of the multivariate normal distribution, in the sense that as the degrees of freedom go to infinity, the *t*-Student distribution approaches the normal distribution. An approach commonly taken to generate draws from a *t*-Student distribution is to define $Y = X\sqrt{\upsilon / S}$, where $X \sim N(0,\Sigma)$ is multivariate standard normal distributed, $S \sim \chi^2_\upsilon$ is chi-squared distributed with υ degrees of freedom, and X and S are independent. Then, Y will be distributed as $t_{\upsilon,\Sigma}$.

3.5.4 Archimedean copulae

Elliptical copulae are easy to simulate, but have two main disadvantages: they do not have closed-form expressions and therefore are expensive in computational terms, and cannot generate those asymmetries necessary to model dependence in financial markets, i.e., stronger dependence between large losses than large returns. In contrast, Archimedean copulae have closed-form expressions and can generate a wide range of dependence structures.

Definiton 3.11:[34] *Let us consider a function* $\varphi(u)$: $[0,1] \rightarrow [0,\infty]$ *which is continuous, strictly decreasing* $\varphi'(u) < 0$ *for all* $u \in [0,1]$, *convex* $\varphi''(u) > 0$ *for all* $u \in [0,1]$. $\varphi(u)$ *is called the generator of the copula and uniquely determines an Archimedean copula.*

Definiton 3.12:[35] *We define the pseudo-inverse of* $\varphi(u)$ *as follows*

$$\varphi^{-1}(u) = \begin{cases} \varphi^{-1}(u) & if \quad 0 < u \leq \varphi(0) \\ 0 & if \quad \varphi(0) \leq u < \infty \end{cases} \tag{3.61}$$

If $\varphi(0) = \infty$, the pseudo-inverse collapses into an ordinary inverse function. When this happens, we say that $\varphi(u)$ is a strict generator, and the copula is called a strict Archimedean copula.

Note that $\varphi^{-1}(u)$: $[0,\infty] \rightarrow [0,1]$ is continuous, strictly decreasing. Furthermore,

$\varphi^{-1}(\varphi(u)) = u$ on $[0,1]$, and

$$\varphi(\varphi^{-1}(u)) = \begin{cases} u & if \quad 0 < u \leq \varphi(0) \\ \varphi^{-1}(0) & if \quad \varphi(0) \leq u < \infty \end{cases} \tag{3.62}$$

[34] In Nelsen (1999) p. 90.
[35] In Nelsen (1999) p. 90.

Theorem 3.5 (Kimberling 1974): *Let* C *be the function from* $[0,1]^n$ *to* $[0,1]$ *given by*

$$C(u_1,u_2,..,u_n) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + .. + \varphi(u_n)) \tag{3.63}$$

Then C *is a copula if and only if* φ^{-1} *is convex.*[36]

Gumbel and Clayton are classical Archimedean copulae. Below we report their expression, in the bivariate form, dependent on one parameter a, together with the Kandall's rank correlation ρ_τ and the tail dependence.[37]

Definition 3.13 (Clayton copula 1978): *Let* $\varphi(u) = u^{-a} - 1$ *with* $a \in [-1,\infty) \setminus \{0\}$ *then using theorem* 3.5 *we have*

$$C(u_1,u_2) = \max[(u_1^{-a} + u_2^{-a} - 1)^{-1/a}, 0] \tag{3.64}$$

with
$$\rho_\tau = \frac{a}{a+2} \tag{3.65}$$

and
$$\lambda_L = 2^{-1/a} \tag{3.66}$$

Definition 3.14 (Gumbel copula 1960): *Let* $\varphi(u) = (-\ln(u))^a$ *with* $a \in [1,\infty)$ *then using theorem* 3.5 *we have*

$$C(u_1,u_2) = \exp(((-\ln(u_1))^a + (-\ln(u_1))^a)^{-1/a}) \tag{3.67}$$

with
$$\rho_\tau = 1 - a^{-1} \tag{3.68}$$

and
$$\lambda_U = 2 - 2^{1/a} \tag{3.69}$$

3.5.5 Drawing from multivariate Archimedean copulae

Drawing from Archimedean copulae is more complex than drawing from Elliptical copulae. A method to simulate draws from a chosen copula is the *conditional sampling*.

Genest (1987) and Genest and Rivest (1993) introduced the idea of simulating the full distribution of random variables $(X_1, X_2,.., X_k,.., X_n)$ by recursively simulating the conditional distribution of X_k given $(X_1, X_2,.., X_{k-1})$. The task is to generate pairs (u,v) $\in [0,1]$ of uniformly distributed random variables U and V whose joint distribution is the copula C. Assume

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[36] For a proof see Nelsen (1999) p. 91.

[37] For more on definitions, generators and dependence measures of Archimedean copulae, see Joe (1997), Nelsen (1999) and Embrechts *et al.* (2001).

that we have already drawn from the uniformly random variable U, and that we want to draw from another random variable V joint to U through the function C, i.e. the copula. The idea is to use the conditional distribution of V, given the draw value of U, as defined below

$$c_u(v) = \mathbb{P}(V \leq v \mid U = u) = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial C}{\partial u} = C_u(v) \qquad (3.70)$$

The application of the conditional sampling to the Archimedean copulae leads to the following sampling algorithm.[38]

(1) Simulate k random variates $(u_1, u_2, ..., u_k)$ from $U(0,1)$.

(2) Set $v_1 = u_1$ (the first draw).

(3) Set the random variate $u_2 = C_2(v_2 \mid v_1)$ from which $v_2 = C_2^{-1}(u_2 \mid v_1)$.

(4) ....

(5) Set the random variate $u_k = C_k(v_k \mid v_1, v_2, v_3, .., v_{k-1})$

from which $v_k = C_k^{-1}(u_k \mid v_1, v_2, v_3, .., v_{k-1})$.

In the following two sections, we will apply the sampling algorithm to generate multivariate Clayton and Gumbel copulae.

3.5.6 Clayton *n*-copula

The Clayton *n*-copula is given by

$$C(v_1, v_2, .., v_n) = \left[\sum_{i=1}^{n} v_i^{-a} - n + 1 \right]^{-1/a} \qquad (3.71)$$

From the definition of the Clayton copula, its generator is given by $\varphi(v) = v^{-a} - 1$ and the inverse is $\varphi^{-1}(t) = (t+1)^{-1/a}$, with $t = \varphi^{-1}(v)$.

The derivatives of the inverse $\varphi^{-1}(t)$, are as follows

$$\varphi^{-1(1)}(t) = -\frac{1}{a}(t+1)^{-1/a-1}$$

$$\varphi^{-1(2)}(t) = \frac{a+1}{a^2}(t+1)^{-1/a-2}$$

[38] For a detailed description of the application of conditional sampling to general Archimedean copulae and derivation of the sampling algorithms, see Genest (1987), Genest and Rivest (1993), Lee (1993), Frees and Valdez (1998), Marshall and Olkin (1988), Embrechts *et al.* (2001) and Meneguzzo and Vecchiato (2002).

$$\varphi^{-1(k)}(t) = (-k)^k \frac{(a+1)(a+2)\ldots(a+k-1)}{a^k}(t+1)^{-1/a-k} \tag{3.72}$$

With the knowledge of the derivatives, draws from a Clayton *n*-copula can be simulated with the following algorithm.

(1) Simulate *k* random variates $(u_1, u_2, \ldots, u_k)$ from $U(0,1)$.

(2) Set $v_1 = u_1$ (the first draw).

(3) Set $u_2 = C_2(v_2 \mid v_1)$, with $u_2 = \dfrac{\varphi^{-1(1)}(c_2)}{\varphi^{-1(1)}(c_1)}$, $c_1 = \varphi(v_1) = v_1^{-a} - 1$

and $c_2 = \varphi(v_1) + \varphi(v_2) = v_1^{-a} + v_2^{-a} - 2$.

It follows that $u_2 = \left(\dfrac{v_1^{-a} + v_2^{-a} - 1}{v_1^{-a}}\right)^{-\frac{1}{a}-1}$ and $v_2 = \left(v_1^{-a}\left(u_2^{-\frac{a}{a+1}} - 1\right) + 1\right)^{-\frac{1}{a}}$. (3.73)

(4) Set $u_3 = C_3(v_3 \mid v_2, v_1)$, with $u_3 = \dfrac{\varphi^{-1(2)}(c_3)}{\varphi^{-1(2)}(c_2)}$ and $c_3 = \varphi(v_1) + \varphi(v_2) + \varphi(v_3)$.

It follows that $$u_3 = \left(\frac{v_1^{-a} + v_2^{-a} + v_3^{-a} - 2}{v_1^{-a} + v_2^{-a} - 1}\right)^{-\frac{1}{a}-2}$$

and $$v_3 = \left(u_3^{-1/\left(\frac{1}{a}+2\right)}\left(v_1^{-a} + v_2^{-a} - 1\right) - \left(v_1^{-a} + v_2^{-a} - 2\right)\right)^{-1/a}. \tag{3.74}$$

(5) …

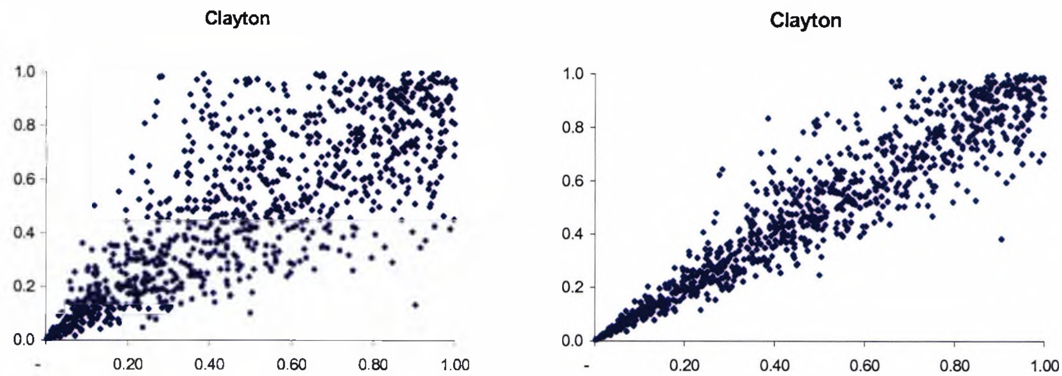
(6) Set $u_k = C_k(v_k \mid v_{k-1,\ldots,} v_2, v_1)$, with $u_k = \dfrac{\varphi^{-1(2)}(c_k)}{\varphi^{-1(2)}(c_{k-1})}$,

and $c_k = \varphi(v_1) + \varphi(v_2) + \ldots + \varphi(v_k)$.

It follows that v_k is equal to

$$v_k = \left(\left(v_1^{-a} + v_2^{-a} + \ldots + v_{k-1}^{-a} - k + 2\right)\left(u_k^{\frac{a}{a(1-k)-1}} - 1\right) + 1\right)^{-1/a}. \tag{3.75}$$

An illustration of the different weights the Clayton copula gives to extreme events is given in Figures 3.6a and 3.6b. Figure 3.6a shows a chart with 1,000 runs with lower tail dependence λ_L of 0.794 and Kendall ρ_τ of 0.6. In Figure 3.6b we have achieved a stronger dependence in the lower tail by moving the tail dependence λ_L to 0.917 and Kendall ρ_τ to 0.8. Note the concentration at the lower-left quadrant.



Figures 3.6a (left) and 3.6b (right). 1,000 runs from Clayton copula, left with $a = 3$, $\lambda_L = 0.794$ and $\rho_\tau = 0.6$; right with $a = 8$, $\lambda_L = 0.917$ and $\rho_\tau = 0.8$.

3.5.7 Gumbel *n*-copula

The Gumbel *n*-copula is given by

$$C(v_1, v_2, .., v_n) = \exp\left\{-\left[\sum_{i=1}^{n}(-\ln v)^{1/a}\right]\right\} \text{ with } a > 1 \tag{3.76}$$

From the definition of the Gumbel copula, its generator is given by $\varphi(v) = (-\ln(v)^a)$ and the inverse is $\varphi^{-1}(t) = \exp\left(-t^{1/a}\right)$, with $t = \varphi^{-1}(v)$.

The derivatives of the inverse $\varphi^{-1}(t)$, are as follows

$$\varphi^{-1(1)}(t) = -\frac{\exp\left(-t^{1/a}\right)t^{\left(\frac{1}{a}-1\right)}}{a}$$

$$\varphi^{-1(2)}(t) = \frac{\exp\left(-t^{1/a}\right)t^{\left(\frac{1}{a}-2\right)}(t^{1/a}+1-a)}{a^2} \tag{3.77}$$

Knowing the derivatives, draws from a Gumbel *n*-copula, can be simulated with the following algorithm.

(1) Simulate *k* random variates $(u_1, u_2, ..., u_k)$ from $U(0,1)$.

(2) Set $v_1 = u_1$ (the first draw).

(3) Set $u_2 = C_2(v_2 \mid v_1)$ with $u_2 = \dfrac{\varphi^{-1(1)}(c_2)}{\varphi^{-1(1)}(c_1)}$, $c_1 = \varphi(v_1) = (-\ln v_1)^a$ and

$c_2 = \varphi(v_1) + \varphi(v_2) = (-\ln(v_1))^a + (-\ln(v_2))^a$

and solve for v_2 so that $u_2 = \dfrac{\varphi^{-1(1)}(c_2)}{\varphi^{-1(1)}(c_1)}$ is satisfied.

(4) Set $u_3 = C_3(v_3 \mid v_2, v_1)$ with, $u_3 = \dfrac{\varphi^{-1(2)}(c_3)}{\varphi^{-1(2)}(c_2)}$, and solve for v_3,

(5) and so until v_k is solved.

An illustration of the different weights the Gumbel Copula gives to extreme events is given in Figures 3.7a and 3.7b. Figure 3.7a shows a chart with 1,000 runs with lower tail dependence λ_U of 0.794 and Kendall ρ_τ of 0.6. In Figure 3.7b we have achieved a stronger dependence in the upper tail by changing the tail dependence λ_L to 0.917 and Kendall ρ_τ to 0.8. Note the concentration at the upper-right quadrant.

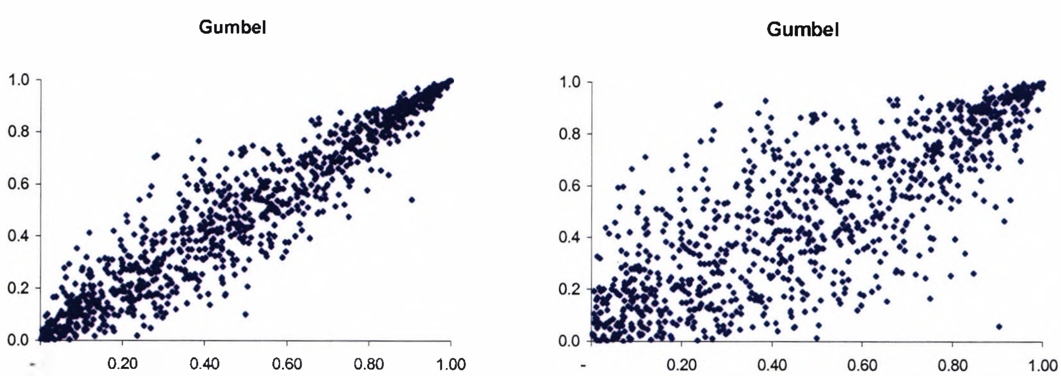


Figure 3.7a and 3.7b: 1,000 samples from a Gumbel copula, left with $a = 2.5$ and $\tau = 0.6$; right with $a = 5$ and $\tau = 0.8$.

It is not always possible to calculate the inverse function analytically, and this is the case of multivariate Gumbel copula. In this case a numerical algorithm must be used. Marshall and Olkin (1988) found an alternative route to a numerical algorithm, and suggested generating multivariate outcomes from *compound* constructions of copulae and using the inverse Laplace transform for the generator. We refer to Marshall and Olkin (1988) for this method.

Chapter 4: Modelling correlation products with copulae

4.1 Introduction

The goal of this chapter is to compare copulae and copula models when applied to real-life transactions. To do so, we review the solutions proposed in the financial literature to the problem of extending intensity-based models to the multivariate case by modelling the dependence structure among the defaults of different obligors or credits through the copula. We will also provide our view regarding the most suitable copula when modelling correlation products.

The first applications of copulae were in the actuarial field. Frees and Valdez (1997) explored copula properties and applied different copula functions for joint life mortality and multi-decrement applications. They also explained Monte Carlo algorithms and covered the issue of the calibration of the copula. Wang (1997) applied copulae for modelling insurance correlated risk and proposed Monte Carlo simulations and fast Fourier transform, as efficient methods of aggregating losses.

These pioneer copula applications would not have been possible without the work of Marshall and Olkin (1988) who prepared most of the numerical algorithms to simulate multivariate random variables for several copula functions.

In finance, the first applications of copulae were in the Value at Risk area, and are attributed to Embrechts, McNeil and Straumann (1999).

The first author to use the copula approach for pricing baskets of credits was Li (1999 and 2000). He proposed the Survival copula to model the joint default dependency in the collateral portfolio of credits. In his model, the specification of the stochastic dependence between each pair of credits is completely independent of their marginals. This model is called the Survival model. Li achieved speed and flexibility. Moreover, calibration is also relatively easy, if historical default time data is available.

Based on the same principle, Schönbucher and Schubert (2001) extended the Survival model to incorporate the dynamics of the default intensities and named this the Default Trigger model. In their model, the default occurs when the intensity process of a credit reaches a pre-specified threshold, the trigger, which is an exponential random variable. The originality is in linking the different thresholds with a dynamic copula.

Both Li's and Schönbucher and Schubert's models are computationally intense since they rely on Monte Carlo simulations. Laurent and Gregory (2002) proposed a simplified version of the Schönbucher and Schubert's model and obtained closed-form formulae for certain conditional intensities of default, by making specific assumptions regarding the copula. In this way, they were able to write fast computational expressions for the stochastic intensities of default times, credit spreads, and prices of large baskets of credits. Their approach is known as Factor copula.

Finger (2000) was the first to compare the results of the Li's Survival model with the Merton model. In his work, he joined the default times of homogeneous credits with the Normal copula. He found that the Survival model tends to underestimate the expected loss of the junior loss note and overestimate the expected loss of the mezzanine and senior notes, when compared with the Merton model. The differences increases as default time correlation increases. Jouanin, Rapuch, Riboulet and Roncalli (2001) compared the Survival model of Li (1999) with the Default Trigger model of Schönbucher and Schubert (2001) using the Normal copula, and concluded that the two models give very different simulated default times, even if the same margins are used.

More recently, Meneguzzo and Vecchiato (2002), and Mashal, Naldi and Zeevi (2003) expanded the work of Finger and used the Survival model of Li (1999) with the Clayton copula and the *t*-Student copula respectively, to real-life credit portfolios. Meneguzzo and Vecchiato found that the Clayton copula is not useful in modelling the tail dependence of default times. Mashal, Naldi and Zeevi concluded that the Normal copula, when compared with the *t*-Student copula, tends to overestimate the expected loss of the junior loss note and underestimate the expected loss of the mezzanine and senior notes.

We argue that the Normal copula, which is the current industry standard in the financial industry, cannot generate realistic default dependency, whereas the Clayton copula can. Therefore, we will use this chapter as an opportunity to research the advantages of the Clayton copula over other copulae and disprove the conclusions of Meneguzzo and Vecchiato. We also find difficult to read the research of Mashal, Naldi and Zeevi, where they concluded that with zero correlation, the Normal copula tends to overestimate the expected loss of the junior loss note and underestimate the expected loss of the mezzanine and senior notes, when compared with the *t*-Student copula. We argue that with zero correlation, all copulae would calculate the same expected loss. Hence, we will study how the change in default time correlation, effects the prices of real-life CDOs, under different copula assumptions.

With the Li's Survival model, we cannot specify the dynamics of the default intensities. Even though, as we aim to show by the end of this chapter, we have a method that generates default scenarios that are consistent with an initial set of default intensities, we cannot control the future evolution of the credit spreads. Schönbucher and Schubert provided with a dynamic framework, where, if the copula function that is used to generate dependent defaults is the Clayton copula, all default intensities jointly jump upwards by a discrete amount at a default of one obligor. This is a very realistic behaviour of credit spreads. Furthermore, with the Clayton copula, we are able to work with a joint distribution function in closed-form, which also provide an efficient algorithm to generate dependent defaults.

Both Li's Survival and Schönbucher and Schubert models can be easily calibrated to individual term structures of default intensities. Hence, we can start with the same set of time-inhomogeneous default intensities, model the same real-life CDO, and analyse the results.

This chapter develops through the following sections. Firstly, we describe correlation products, which will be used throughout this chapter for comparing copula models. Section three deals with modelling dependence with the Marshall and Olkin copula, whereas in section four we review Li's Survival Normal copula and extend it to include the *t*-Student copula and Clayton copula. Section seven gives a short overview over the Schönbucher and Schubert dynamic copula. In this chapter, we also present four numerical exercises: one with the Marshall and Olkin copula, in section 3, two with Li' Survival model, in sections 5 and 6, and one with Schönbucher and Schubert dynamic copula in section 8. Their goal is to highlight the differences in the rating, in the price and in the expected shortfall of baskets and CDOs, that inevitably are found when a copula is preferred to another, and when a dynamic copula replaces a static copula.

4.2 Correlation products

The copula models we review in this chapter are for pricing, hedging and managing the credit risk of correlation products which can be grouped into two categories: n^{th} to default baskets and CDOs. As we will see in the next two sections, where those products are formally introduced, in order to correctly evaluate the effect of diversification on the credit risk generated by the underlying portfolio of credits, the key is how to handle the dependence between the n random times of default τ_i s. The pricing problem is not elementary, whenever the hypothesis of independence of perfect linear correlation between the underlying credits is dropped. It becomes even more complex when the questionable assumption of joint normality is also abandoned. Furthermore, no closed-form solution is available. Thus, we will pay particular care to Monte Carlo simulation. In what follows we will explore the application of the copulae explained in the previous chapter as pricing tools for correlation products. Before doing so, we formally introduce the two main categories of correlation products.

4.2.1 n^{th} to default baskets

We consider thereafter the case of m out of n defaults, where there is a default payment when the m^{th} default occurs, and n is the number of credits. The payout depends on the temporal ranking of the defaults. We associate with the collection of the times of default of n credits $\tau_1,..,\tau_m,..,\tau_n$, the ordered sequence of the same times $\tau_{(1)},..,\tau_{(m)},..,\tau_{(n)}$. By definition $\tau_{(1)} = \min(\tau_{(1)},..\tau_{(m)},..,\tau_{(n)})$

is the first-to-default, and $\tau_{(n)} = \max(\tau_{(1)},..,\tau_{(m)},..,\tau_{(n)})$ is the last-to-default. We indicate with M_i the notional or current balance of the credit i and with φ_i its recovery rate, then the default payment of the m out of n credits can be written as

$$\sum_{i=1}^{n} M_i(1-\varphi_i)1_{\{\tau_i=\tau_{(m)}\}}1_{\{\tau_{(m)}<T\}} \qquad \text{at} \qquad \tau_{(m)} \tag{4.1}$$

We denote the value of the default leg of the m out of n defaults at a time $t < t_{j-1}$ as

$$V_{D,(m)}(t) = \mathrm{E}^{\mathrm{Q}}\left(B(0,\tau_{(m)})\sum_{i=1}^{n} M_i(1-\varphi_i)1_{\{\tau_i=\tau_{(m)}\}}1_{\{\tau_{(m)}<T\}} \middle| \mathcal{G}_t \right) \tag{4.2}$$

If we denote $V_{P,(m)}(t)$ as the payment leg, i.e. the value at time t of receiving 1bp of fee payments and no fees after the interval $[t_{j-1} \le \tau_{(m)} \le t_j]$, then we can write

$$V_{P,(m)}(t) = \mathrm{E}^{\mathrm{Q}}\left(\sum_{j=1}^{m} B(0,t_j)\delta_j \sum_{i=1}^{n} M_i 1_{\{i=(m)\}}1_{\{t_j\le\tau_{(m)}\}} + B(0,\tau_{(m)})\delta_{j^*} \sum_{i=1}^{n} M_i 1_{\{\tau_i=\tau_{(m)}\}}1_{\{t_{j^*}\le\tau_{(m)}<t_j\}} \middle| \mathcal{G}_t \right) \tag{4.3}$$

where $1_{\{i=(m)\}}$ means that the m^{th} defaulting credit is the i^{th} credit. In (4.3), we have also taken into account for the final payment that the protection buyer makes to cover the time between the last payment date and the time $\tau_{(m)}$ of the m^{th} default event. The fair premium rate at time t of the m out of n defaults can be written as

$$s_{(m)} = \frac{V_{D,(m)}(t)}{V_{P,(m)}(t)} \tag{4.4}$$

4.2.2 CDOs

CDOs are characterised by a payout dependent on percentiles of the loss distribution caused by default events. They involve the tranching of the reference portfolio of credits into different classes of notes:

(1) $Sen(t)$, the senior note,

(2) $Mez(t)$, the mezzanine note,

(3) $Jun(t)$, the junior note.

The proceeds from the issuance of the notes are invested in default-free bonds, which guarantee the repayment of the principals of the notes, if no default has occurred. The difference between the reference portfolio of credits, $\sum_{i=1}^{n} M_i$, and the sum of the notes, is the equity piece, $Eq(t) = \sum M_i(t) - Sen(t) - Mez(t) - Jun(t)$.

Definition 4.1: *The cumulative loss of a portfolio of credits at time t is indicated as the pure jump process*

$$L(t) = \sum M_i(1-\varphi_i)N_i(t) \tag{4.5}$$

where $N_i(t) = 1_{\{\tau_i \le t\}}$ is the counting process which jumps from 0 to 1 at the time of default *i*.

The cumulative loss $CL(t)$ of any given note can be written as follows

$$CL(t) = \begin{cases} 0 & if \quad L(t) \le D(t) \\ L(t) - D(t) & if \quad D(t) < L(t) \le C(t) \\ L(t) - C(t) & if \quad C(t) < L(t) \le B(t) \\ L(t) - B(t) & if \quad B(t) < L(t) \le \sum_{i=1}^{n} M_i \end{cases} \tag{4.6}$$

where $D(t) = Eq(t)$, $C(t) = Eq(t) + Jun(t)$ and $B(t) = C(t) + Mez(t)$.

$CL(t)$ is a pure jump process as $L(t)$. Every time there is jump in $CL(t)$, then a default payment occurs, equal to $CL(\tau_i) - CL(\tau_i^-)$, where τ_i^- is the time just before the jump. Laurent and Gregory (2002) and Meneguzzo and Vecchiato (2002) noticed that, since $CL(t)$ is an increasing process, it can be defined the Stieltjes integrals with the respect to $CL(t)$. Besides, because $CL(t)$ is constant between jump times, any Stieltjes integral with the respect to $CL(t)$, $\int g(t)dCL(t)$, for some function $g(t)$, is written as a discrete sum with respect to every jump time, i.e. $\sum_i g(\tau_i)[CL(\tau_i) - CL(\tau_i^-)]$.

As a direct consequence of the notation of Laurent and Gregory, and Meneguzzo and Vecchiato, we can write the default payments of the CDO notes as a discounted payoff

$$\int_0^T B(0,t)dCL(t) = \sum_i B(0,\tau_i)[CL(\tau_i) - CL(\tau_i^-)] \tag{4.7}$$

Thus, the default payment leg of any given CDO note, under the martingale measure $\mathbb{Q}$, is as follows

$$\mathrm{E}^{\mathbb{Q}}\left(\int_0^T B(0,t)dCL(t)\right) \tag{4.8}$$

Using integration by part and Fubini's theorem[1] we have

$$\mathrm{E}^{\mathrm{Q}}\left(\int_0^T B(0,T)dCL(t)\right) = B(0,T)\mathrm{E}^{\mathrm{Q}}\big(CL(T)\big) + \int_0^T f(0,t)B(0,t)\mathrm{E}^{\mathrm{Q}}\big(CL(t)\big)dt \quad (4.9)$$

with $f(0,t)$ as the instantaneous forward rate.

For any given note, the premium payment leg is the payment made, provided that the cumulative loss distribution $CL(t)$ is not greater than the note itself,

$$\begin{aligned}\mathrm{E}^{\mathrm{Q}}\Bigg(&\sum_{j=1}^{m}\delta_{t_j}s_{Eq}B(0,t_j)Eq(t_j)1_{\{L(t_j)\le D(t_j)\}} + \\ &\sum_{j=1}^{m}\delta_{t_j}s_{Jun}B(0,t_j)Jun(t_j)1_{\{D(t_j)<L(t_j)\le C(t_j)\}} + \\ &\sum_{j=1}^{m}\delta_{t_j}s_{Mez}B(0,t_j)Mez(t_j)1_{\{C(t_j)<L(t_j)\le B(t_j)\}} + \\ &\sum_{j=1}^{m}\delta_{t_j}s_{Sen}B(0,t_j)Sen(t_j)1_{\{B(t_j)<L(t_j)\le\sum M_j\}}\Bigg)\end{aligned} \quad (4.10)$$

where m is the number of payment dates.

At launch, the premiums s_{Eq}, s_{Jun}, s_{Mez} and s_{Sen} are found by putting into equivalence the default leg with the premium leg.

4.3 Modelling with Marshal and Olkin exponential copula

In this section, we apply the Marshal and Olkin copula to price a n^{th} to default basket. With this copula, the calculation of the aggregated loss distribution becomes an easy and tractable intensity-based model in the spirit of Duffie and Singleton (1998). However, as we saw in the previous chapter, when we assume that defaults can be triggered by idiosyncratic as well as other regional, industry, or economic-wide shocks, the probability of multiple defaults at the same time is not zero. If we keep this assumption and the claim depends upon the temporal ranking of the credit events, the Marshal and Olkin copula can be used with Monte Carlo simulations. If we assume that the probability of multiple defaults at the same time is zero, we move into an exponential set-up where a closed-form pricing formula exists for the 1[st] to default. For us, this is the natural benchmark to compare the Marshal and Olkin copula with Monte Carlo simulations.

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[1] In Meneguzzo and Vecchiato (2002) p. 24.

4.3.1 1st to default with an exponential model

To price a 1st to default with an exponential model, we assume that the valuation is carried on under the risk-neutral measure $\mathbb{Q}$, and hence, in the credit default swap market, the quotes of the credit spreads are available, from which we extract the risk-neutral time-homogeneous intensity rates $\lambda_i^{\mathbb{Q}}$.[2]

Definition 4.2: *We define the risk-neutral intensity rate of credit i as*

$$\lambda_i^{\mathbb{Q}} = \lambda_i + \lambda_{1,2,..,i,..,n} \tag{4.11}$$

where $\lambda_{1,2,..,i,..,n}$ is the joint intensity, i.e. the intensity of the systematic shock that affects all credits simultaneously.

A further assumption is that there is zero probability that more than one credit defaults at the same time $\mathbb{Q}(\tau_i = \tau_j, i \neq j) = 0$.

Proposition 4.1:[3] *Suppose that* $\mathbb{Q}(\tau_i = \tau_j, i \neq j) = 0$. *Then* $\tau_{(1)} = \min(\tau_1, \tau_2, ..., \tau_n)$, *i.e. the first time of default, occurs at an intensity rate*

$$\lambda_{(1)} = \lambda_1 + \lambda_2 + ... + \lambda_n \tag{4.12}$$

that is, the joint distribution of the first time of default is still exponential distributed, with a intensity rate $\lambda_{(1)}$ equal to the sum of the individual intensity rates.

Expressing (4.12) in terms of risk-neutral intensity rates, we obtain

$$\lambda_{(1)}^{\mathbb{Q}} = \lambda_1^{\mathbb{Q}} + \lambda_2^{\mathbb{Q}} + .. + \lambda_n^{\mathbb{Q}} - (n-1)\lambda_{1,2,..,i,..,n} \tag{4.13}$$

To value the two legs of the 1st to default, we also assume that the default-free interest rate process $r(t)$, is constant, hence, correlation between the default-free interest rate and intensity rate do not materialise. We further assume that the credits in the reference portfolio have the same maturity T_i, the same notional M_i, and when there is a default the net loss is the same, i.e. $L = L_i = (1-\varphi_i)M_i$ for every $i = 1,2,..., n$.

The default payment leg of the 1st to default in (4.2) further simplifies into[4]

[2] See chapter 1 on how to calibrate the intensity rate to the market quotes of CDSs.
[3] In Duffie (1998) p. 3. For a proof, see pp. 3-4.
[4] In Li (1999) p. 45.

$$V_{D,(1)}(t) = L\int_t^T \lambda_{(1)}^{Q} e^{-(r+\lambda_{(1)}^{Q})s} ds$$

$$= L\frac{\lambda_{(1)}^{Q}}{r+\lambda_{(1)}^{Q}}\left(1 - e^{-(T-t)(r+\lambda_{(1)}^{Q})}\right) \quad (4.14)$$

In (4.14) the default payment leg of the 1st to default is effectively valued by discounting its real cash flows with the credit risk adjusted discount factor. When the intensity and default-free interest rate are two independent processes, the credit risk adjusted discount factor is the product of the credit risk free discount factor and the pure credit risk discount factor. Therefore, in (4.14) we can interpret e^{-rT} as the default free discount factor, $e^{-\lambda_{(1)}^{Q}T}$ as the pure default discount factor and $L\lambda_{(1)}^{Q}$ as the real financial instrument cash flows.

Figure 4.1 shows what happens to the value of the default payment leg of the 1st to default in (4.14), of a basket of five credits where we assume different values of the joint intensity of the systematic shock, $\lambda_{1,2,3,4,5}$. The five credits have an identical nominal value of £1, the same marginal intensity rate λ_i^{Q} at 10%, and a maturity greater or equal to 2 years, which is also the maturity of the 1st to default. According to (4.13), when the joint intensity $\lambda_{1,2,...,n}$, rises, $\lambda_{(1)}^{Q}$ falls.

When the five credits are independent, $\lambda_{1,2,3,4,5} = 0$, the reference portfolio 1st to default risk-neutral intensity rate $\lambda_{(1)}^{Q}$ in (4.13) reverts to (4.12), and the value of the default payment leg is £0.5864.

When $\lambda_{1,2,3,4,5} = 1$, we expect the value of the default payment leg to be the same as the value of the default payment leg default on only one of the five credits. This is because when one credit defaults, the other four credits default too and we do not benefit from any diversification of risk.

In all other cases, the reference portfolio intensity rate $\lambda_{(1)}^{Q}$ is calculated with (4.13), and using the relationship in (4.14), we calculate the values shown in Figure 4.1. The values follow the expected pattern: as the joint intensity $\lambda_{1,2,3,4,5}$ rises, the value falls. In addition, when $\lambda_{1,2,3,4,5} = 1$, the value is 0.1638, the same as the value of the default payment leg default on only one of the five credits.

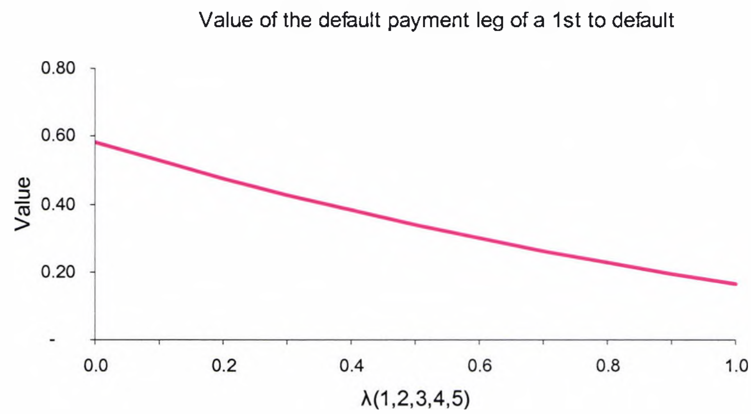

Figure 4.1: Value of default payment leg default of a 1[st] to default as $\lambda_{1,2,3,4,5}$ changes from 0 to 1.

4.3.2 Simulating times of default with Marshall and Olkin copula

The Marshal and Olkin exponential copula can easily be used to price n^{th} to defaults and not only a 1[st] to default, and without assuming that $\mathbb{Q}(\tau_i = \tau_j, i \neq j) = 0$. This is done with Monte Carlo simulations.

To simulate default times we rely on the quantile transforms in (3.1). With this method, first a model is used to calculate the survival probability $S_i(t_i)$. Following, a uniformly distributed *r.v.* U_i, is drawn, and the time of default τ_i, is found as follows

$$S_i(t_i) \sim U_i$$

$$t_i = S_i^{-1}(U_i) \tag{4.15}$$

In the exponential model, to simulate the time of default τ_i, of credit i, with a risk-neutral time-homogeneous intensity rate $\lambda_i^{\mathbb{Q}}$, (4.15) becomes

$$\exp(-\lambda_i^{\mathbb{Q}}\tau_i) = u_i$$

$$\tau_i = -\frac{1}{\lambda_i^{\mathbb{Q}}}\ln(u_i) \tag{4.16}$$

This is still incomplete, since we want to draw dependent default times. We can do this in the following way.

In (4.16) we substitute $\lambda_i^{\mathbb{Q}}$ with the joint intensity $\lambda_{1,2,\ldots,n}$. Then, we simulate two times of default: the time τ when the joint intensity $\lambda_{1,2,\ldots,n}$ strikes credit i, and the time τ_i when the intensity λ_i^Q strikes credit i. If $\tau < \tau_i$, credit i defaults at time τ. Otherwise it defaults at time τ_i (when the idiosyncratic shock λ_i^Q strikes).

The simulation algorithm can be written as follows (as adapted from Giesecke (2002)).

(1) Simulate $m+1$ independent exponential times of default with given intensities $\lambda_1^Q, \lambda_2^Q, ..., \lambda_m^Q, \lambda_{12..m}$, by drawing $m+1$ uniform random variates $u_1, u_2, ..., u_m, u_{12..m}$.

(2) The credit i $(i = 1,2,...,m)$ defaults at time

$$\tau_i = -\frac{\ln(u_i)}{\lambda_1^Q - \lambda_{12..m}}$$

and the time the joint intensity strikes credit i is

$$\tau_{m+1} = -\frac{\ln(u_{m+1})}{\lambda_{12..m}}.$$

(3) To simulate the exponential times of default for the credit i, take the minimum of the time of default τ_i and the joint default time τ_{m+1}

$$\tau_i = \min(\tau_i, \tau_{m+1}).$$

(4) Generate the joint uniform random variate v_i, with

$$v_i = \exp(-\tau_i \lambda_i^Q).$$

(5) To arrive at the credit i dependent default time, set

$$z_i = -\frac{\ln(v_i)}{\lambda_i^Q}.$$

(6) and so on for the other credits.

This algorithm is very flexible and correctly distinguishes between the intensity rate λ_i, and risk-neutral intensity rate λ_i^Q. When we want to add other joint intensity rates we can do so by calculating their times of default as in point 1 and 2, and then using the minimum condition in 3 where applicable.

Also, we can use an exponential model where the default times are simulated by drawing correlated standard normal random variables $X_1, X_2, .., X_m, X_{12...m}$. If we set $u_i = \Phi(X_i)$, then we have effectively joint uniform random variates u_i, through the Normal copula, where $\Phi(\cdot)$ is the standard normal distribution.

4.3.3 Numerical example

In this section, we use the simulation algorithm to calculate the value of the default payment leg of the 1st, 2nd and 3rd to default of a basket with the number of credits in the reference portfolio going from 1 to 50. We assume that the intensity rate is flat at 10% and it is common to all credits. The term structure of interest rates is flat at 10% and the default time linear correlations $\rho(\tau_i, \tau_j)$, are 0.25 and 0.5. Using the relationship in (3.43), default time linear correlation of 0.25 and 0.50 are equivalent to joint intensity rates $\lambda_{1,\dots,n}$, respectively of 4% and 6.67%.

The contract matures in two years. At default, the protection seller pays when the default occurs. We disregard the effect of the accrued premium in (4.3). The number of runs in the Monte Carlo is 10,000 simulations. Table 4.1 summarises the inputs of the exponential model.

Parameters	
Maturity	2 Years
Interest Rate	10%
Intensity Rate (the same for all credits)	10%
Number of Assets	1 to 50
Case 1	
Joint Intensity Rate	4.0%
Default Time Linear Correlation	0.25
Case 2	
Joint Intensity Rate	6.67%
Default Time Linear Correlation	0.5

Table 4.1: n^{th} to default parameters.

In Table 4.2 (in the Appendix 1) we have computed the reference portfolio 1st to default intensity rate using (4.13), with different default time linear correlation assumptions.

Tables 4.3 to 4.7 show the Monte Carlo results: values, standard errors and the time necessary to complete the calculation. The time always refers to the total time (in seconds) necessary to value the 1st, 2nd and 3rd to default together.

To benchmark the Monte Carlo algorithm, we use the valuation formula in (4.14) and call the value it calculates, the analytic value. To perform our benchmarking, we see whether the analytic value is contained in the range of the value $\pm$ the standard error as calculated with the simulation algorithm. From Tables 4.3 and 4.4 we can see that the value calculated with the Monte Carlo algorithm is out of that range only once, and this occurs in Table 4.3 when the number of credits is two. However, the same analytic value is contained within the wider range of the value $\pm$ 1.5 x the standard error.

Tables 4.5 and 4.6 contain the values and standard errors of the 2nd and 3rd to default, with default time linear correlation of 0.25 and 0.5. The computational time is the same as in Tables 4.3 and 4.4, respectively.

In Figures 4.5, 4.6 and 4.7 we show the accuracy of the Monte Carlo results as the number of sample paths goes from 5,000 to 300,000. All values refer to the 1^{st}, 2^{nd} and 3^{rd} with two-year maturity, on a reference portfolio of 10 credits. All credits have the intensity rate at 10%, the term structure of interest rate is flat at 10%, and the default time linear correlation $\rho(\tau_i, \tau_j)$ is 0.50. The computational times are in Table 4.7.

Figure 4.5 compares the Monte Carlo values, computed with different sample paths, with the analytic values. The analytic value is within the standard error bars of the Monte Carlo values, with the exception of the value computed with 5,000 sample paths. We can therefore say that even with only 10,000 sample paths, the results are very accurate for the 1^{st} to default.

For the 2^{nd} and 3^{rd} to default, we do not have the analytic value to use as a benchmark, nevertheless, Figures 4.6 and 4.7 point to a good quality already with 10,000 sample paths.

Clearly, accuracy improves as the number of sample paths rises, but with that so does the time to obtain the results. The computational times under different sample paths are in Table 4.7: with 51 seconds, it is possible to value a basket of 10 credits. However, when trying with 300,000 sample paths, the computational time of 1,556 sec. becomes prohibitive.

The simulation algorithm we have used is the crude Monte Carlo, where the random numbers are generated with the Bays-Durham algorithm.[5] The programming code is in VBA. We can reasonably expect a further reduction in the computational time, where a faster Monte Carlo algorithm, such as the stratified sampling or Sobol with variance reduction, together with C++, is used.

4.4 Li's Survival copula and Merton model

This section shows how Li's Survival (1999) model uses copulae to add dependence in baskets of credits. This model differs from the Schönbucher and Schubert (2001) model (which we will treat in section 4.7) as the copula is directly put on the survival times. For this reason, it is called Survival copula. Alternatively, in the Schönbucher and Schubert (2001) model, the copula is put on the exponential random variables, which behave like random thresholds or triggers. This is why this model is sometimes referred as, Default Threshold copula.

We will also review the Merton model, which we will use as a benchmark to compare the results produced with Li's Survival model.

[5] The algorithm is from Press, Flannery, Teukolesky and Vetterling (1997).

4.4.1 Dependent times of default with Merton model

The Merton model assumes that the firm's asset value $A(t)$, is a stochastic quantity that triggers the default when it hits a certain threshold value, a_t, with $0 < a_0 < A_0$.

Definition 4.3: *The time of default τ_i of credit i is given by*

$$\tau_i = \inf\{t > 0 : A_i(t) \leq a_{it}\} \tag{4.17}$$

We can write the processes of all firm's asset values $A_i(t)$, as

$$\begin{aligned} dA_1(t) &= A_1(t)\mu_1 dt + A_1(t)\sigma_1 dW_1(t) \\ dA_2(t) &= A_2(t)\mu_2 dt + A_2(t)\sigma_2 dW_2(t) \\ &\dots \\ dA_n(t) &= A_n(t)\mu_n dt + A_n(t)\sigma_n dW_n(t) \end{aligned} \tag{4.18}$$

where μ_i and σ_i are the return and volatility of the i^{th} firm value and $dW_i s$ are correlated normal variates. We also assume that the correlation is constant with respect to time.

The threshold value that triggers the i^{th} firm's default can be calculated from the fact that $W_i(t)$ is normally distributed. From the s.d.e. of the i^{th} firm in (4.18), and after it is discretised, we can extract $W_i(t)$ with the following

$$W_i(t) = \frac{\ln\left(\dfrac{A_i(t)}{A_i(0)}\right) - \mu_i t}{\sigma_i \sqrt{t}} \tag{4.19}$$

which is the normalised return of the i^{th} firm over time *t*, i.e. $\sim N(0,1)$.

We want to calculate the probability that the i^{th} firm's asset value will fall at or cross the value, a_{it}, that triggers the default, during the time *t*, which from (4.19), can be written as

$$\begin{aligned} \mathrm{P}(A_i(t) \leq a_{it}) &= \mathrm{P}\left(W_i(t) \leq \frac{\ln\left(\dfrac{a_{it}}{A_i(0)}\right) - \mu_i t}{\sigma_i \sqrt{t}} \right) \\ &= \mathrm{P}(W_i(t) \leq w_{it}) \\ &= \Phi(\alpha_{it}) \end{aligned} \tag{4.20}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution.

From (4.20), we can relate the scaled threshold value, w_{it}, to the normalised return, α_{it}, as

$$\alpha_{it} = \Phi^{-1}\left(\mathbb{P}\left(W_i(t) \le w_{it}\right)\right) \tag{4.21}$$

where $\Phi^{-1}(\cdot)$ is the inverse of the cumulative standard normal distribution.

From (4.21) we can draw two important conclusions: we can recover α_{it} from the required cumulative probability; as we have substituted the threshold value a_{it}, in (4.20), with the scaled threshold value w_{it}, we can work with the normalised return α_{it}. In this particular case, α_{it} can be recovered, either from the cumulative probabilities of default over time t implied in the CDS premiums of firm *i*, or from the actual cumulative probabilities of default over time t estimated for example by rating agencies.[6] We indicate as $q_{i;0,t}$ the cumulative probabilities of default over time t, and α_{it} is recovered with the following

$$\alpha_{it} = \Phi^{-1}\left(q_{i;0,t}\right) \tag{4.22}$$

This guarantees that the probability that the i^{th} firm's value hits the threshold, during the first period, is equal to the probability of default, $q_{i;0,1}$,

$$\mathbb{P}\left(A_i(1) \le a_{i1}\right) = q_{i;0,1} \tag{4.23}$$

To calculate the threshold for the next period, we have to condition upon the fact that the i^{th} firm has not defaulted at the end of period 1. We rely on the following condition[7]

$$\mathbb{P}\left(A_i(1) > a_{i1} \cap A_i(2) \le a_{i2}\right) = \mathbb{E}\left(1_{W_i(1)>\alpha_{i1}} 1_{W_i(2)\le\alpha_{i2}}\right) = q_{i;1,2} \tag{4.24}$$

which can also be written as

$$q_{i;1,2} = \frac{q_{i;0,2} - q_{i;0,1}}{1 - q_{i;0,1}} \tag{4.25}$$

where $q_{i;1,2}$ is the i^{th} firm's probability of default between the period 1 and the period 2, condition upon being survived until period 1, and the term is defined in (4.23).

Finally, the correlation among the n-normal variates W_i, for $i = 1,2,..,n$ is imposed with the *Cholesky* decomposition as follows: from the correlation matrix Σ we construct a lower triangular matrix

[6] For example, the actual probabilities of default are those prepared by Moody's and given in Table 2.8 in chapter two, after been adjusted by the recovery rate of 45%.

[7] In Arvanitis and Gregory (2001) p. 161.

$$\mathrm{B} = \begin{pmatrix} b_{11} & 0 & . & . & . & 0 \\ b_{21} & b_{22} & . & . & . & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ b_{n1} & b_{n2} & . & . & . & b_{nn} \end{pmatrix} \tag{4.26}$$

such that $\Sigma = \mathrm{BB}^T$, where B^T is the transpose of B. The elements of the matrix can be calculated from the *Cholesky* algorithm as proposed by Burden and Faires (1989), Johnson (1987) and Wang (1997): [8]

$$b_{ij} = \frac{\rho_{ij} - \sum_{s=1}^{j-1} b_{is} b_{js}}{\sqrt{1 - \sum_{s=1}^{j-1} b_{js}^2}}, \qquad 1 \le j \le i \le n \tag{4.27}$$

with the convention that $\sum_{s=1}^{0}(\cdot) = 0$, and ρ_{ij} an element of the correlation matrix Σ.

To simulate the time when the i^{th} firm's asset value hits the threshold value and the firm defaults, we proceed with the following numerical algorithm.

(1) Collect the cumulative default probabilities of all the n credits, $q_{i:0,1}, q_{i:0,2}, \ldots, q_{i:0,T}$ for $i = 1,2,\ldots,n$, where T is the number of periods until the n^{th} to default matures.

(2) In the first period, we set the threshold values with (4.22), i.e. $\alpha_{i1} = \Phi^{-1}(q_{i;0,1})$.

(3) For the following periods, we set the threshold values, α_{ik}, equal to the conditional default probability of the k^{th} period as,

$$\alpha_{ik} = \Phi^{-1}\left(\frac{q_{i;0,k} - q_{i;0,k-1}}{1 - q_{i;0,k-1}}\right)$$

(4) In each k^{th} period, we draw correlated standard normal *r.v.*s X_{ik} by *Cholesky* decomposition and compared them to the threshold values of the same period, α_{ik}, and when $X_{ik} \le \alpha_{ik}$, the firm defaults.

The main advantage of this model is that it is easy to draw correlated random variables using the *Cholesky* decomposition, and it can be used in waterfall CDOs, where it is important to know when the credit defaults.

This model has two main drawbacks: by construction, it does not allow any correlation of defaults through time. For example, if a high rate of default is realised in one period, in the next

[8] Wang (1997) reminds us to calculate the element of B from top to bottom and from left to right.

period the simulation starts without any memory of what has happened in the previous one. This is not how the financial credit market works. If the quality of credit *i* deteriorates in one period, market participants will tend to overweight risk of default in the next period. In order to capture this behaviour, Finger (2000) simulated default times with random variables correlated in consecutive periods. This creates a sort of migration to higher or lower rating classes. He called this approach the *single diffusion process*. Even with Finger's correction, the Merton's model maintains its second drawback: it requires a stepwise approach, where the defaults of the portfolio are simulated at very small time steps. At each time step, the defaulted credits are dropped and the simulation continues on the remaining credits for the following time step and so on. Clearly this is very time consuming.

4.4.2 Li's Survival model

We start by defining, in a inhomogeneous Poisson set up, the survival function $S_i(t_i)$, the joint survival function $S(t_1,t_2,..,t_n)$, the default function $F_i(t_i)$ and the joint default function $F(t_1,t_2,..,t_n)$ of the random vector $\tau=(\tau_1,\tau_2,...,\tau_n)$ of the times of default as follows

$$S_i(t_i)=\mathbb{P}(\tau_i>t_i)=\exp\left(-\int_0^{t_i}\lambda_i(s)ds\right) \tag{4.28}$$

$$S(t_1,t_2,..,t_n)=\mathbb{P}(\tau_1>t_1,\tau_2>t_2,..,\tau_n>t_n) \tag{4.29}$$

$$F_i(t_i)=\mathbb{P}(\tau_i\le t_i)=1-\exp\left(-\int_0^{t_i}\lambda_i(s)ds\right) \tag{4.30}$$

$$F(t_1,t_2,..,t_n)=\mathbb{P}(\tau_1\le t_1,\tau_2\le t_2,..,\tau_n\le t_n) \tag{4.31}$$

with the time of default τ_i, as in definition 1.15.

If the intensity processes are independent, we obtain

$$S(t_1,t_2,..,t_n)=\prod_{i=1}^{n}S_i(t_i) \tag{4.32}$$

Otherwise, using the Sklar's theorem, Li proposed to model the joint survival function directly with the Survival copula as follows

$$S(t_1,t_2,..,t_n)=\mathbb{P}(\tau_1>t_1,\tau_2>t_2,..,\tau_n>t_n)=K(S_1(t_1),S_2(t_2),..,S_n(t_n)) \tag{4.33}$$

The Survival approach has one apparent drawback: it is essentially a static model, since it does not account for changes in the intensity rates, i.e. it does not condition on the new flow of information which has become available in the market place.

To calibrate the Li's model, we assume that $(q_{i:0,1},q_{i:0,2},...,q_{i:0,T})$ is a vector of a given set of cumulative probabilities of default of credit *i*, for example extracted from market prices of CDSs or from the Moody's idealised cumulative expected losses in Table 2.8 in chapter 2. Then, the calibration is achieved by linking a vector of *n* correlated standard normal *r.v.*s $(X_1, X_2,.., X_n)$, *via* the normal copula as follows

$$C_\Sigma^N(u_1,u_2,..,u_n) = \mathbb{P}\left(q_{1;0,T} < \Phi^{-1}(X_1),.., q_{n;0,T} < \Phi^{-1}(X_n)\right) \quad (4.34)$$

This leads to the following algorithm to simulate correlated times of default.

(1) Draw a correlated standard normal variable and set the time of default as

$$\tau_i = \Phi^{-1}(X_i)$$

(2) If $\tau_i > q_{i:0,T}$, then credit *i* does not default before the maturity of the n^{th} to default.

(3) Otherwise, we locate the time of default in the k^{th} period with $q_{i;0,k-1} < \tau_i \le q_{i;0,k}$.

The correlated X_i numbers are drawn with *Cholesky* factorisation.

The Li Normal copula model can be adjusted to include the *t*-Student copula. We consider a vector of correlated normal distributed *r.v.*s $(X_1, X_2,.., X_n)$, and a vector of χ_υ^2 distributed *r.v.*s $(S_1, S_2,.., S_n)$ with υ degrees of freedom. We also assume S is independent of X, then $Y_i = X_i\sqrt{\upsilon / S_i}$ is *t*-Student distributed.

The calibration is achieved by linking the vector $(Y_1, Y_2,.., Y_n)$, via the *t*-Student copula as follows

$$C_{\upsilon,\Sigma}^t(u_1,u_2,..,u_n) = \mathbb{P}\left(q_{1;0,T} < t_\upsilon^{-1}(Y_1),.., q_{n;0,T} < t_\upsilon^{-1}(Y_n)\right) \quad (4.35)$$

The Survival model with the *t*-Student copula can be used with the following simulation algorithm.

(1) Simulate *n* correlated standard normal *r.v.*s X_i with one extra *r.v.* S_i distributed as a χ_υ^2 with υ degrees of freedom.

(2) Create *t*-Student distributed *r.v.*s by taking $Y_i = X_i\sqrt{\upsilon / S_{(\chi^2,\upsilon)}}$ and set the time of default as

$$\tau_i = t_{\upsilon,\Sigma}^{-1}(Y_i)$$

(3) If $\tau_i > q_{i:0,T}$, then credit *i* does not default before the maturity of the n^{th} to default.

(4) Otherwise, we locate the default time in the k^{th} period with $q_{i:0,k-1} < \tau_i \le q_{i:0,k}$.

Alternatively, if we consider a vector of random variables $(v_1, v_2,.., v_n)$ joined with the Clayton copula, we are able to write the following

$$C(v_1, v_2, .., v_n) = \mathbb{P}(v_1 \le q_{1:0,T}, .., v_n \le q_{n;0,T}) \quad (4.36)$$

With (4.36), we draw n times of default, joined with the Clayton copula. If $v_i \le q_{i;0,T}$, then credit i defaults before time T, and we locate the time of default in the k^{th} period with $q_{i;0,k-1} < v_i \le q_{i;0,k}$. Otherwise, it survives.

4.5 Numerical exercise: pricing 1st, 2nd and 3rd to default with Li's Survival copula

In this section we use a numerical example to illustrate the differences in the values of the default payment leg of a 1st, 2nd and 3rd to default, when the random times of default are joined with different copulae via the Survival model. We also use the Merton model as a sort of benchmark. With the Merton model, the assets paths are only joined with the Normal copula.

We will change the default time correlation[9] and the number of credits in the reference portfolio to emphasize the differences in prices. When the Clayton copula is used, we will rely upon the only available parameter a to capture both the lower tail dependence λ_L and Kendall's ρ_τ. In this way, the prices calculated with the Clayton copula will be comparable with those calculated with the Elliptical copulae.

For simplicity, the contract is the same as in section 4.3.3 and we assume the same flat intensity rate at 10% common to all credits.

In the Monte Carlo set-up, the following parameters are used.

(1) We apply three default time Pearson's linear correlation $\rho(\tau_i, \tau_j)$, to analyse their impact on the valuation of the 1st, 2nd and 3rd to default: 0, 0.25 and 0.5. We do not expect to notice any difference in the values calculated with the Merton Normal copula, and the Survival model with the Normal and t-Student copulae when $\rho(\tau_i, \tau_j) = 0$.

(2) The degrees of freedom to generate t-Student distributed random variates are 100, 12, 8 and 4. We believe this choice will allow us to see the effect of joint default events in the final values.

(3) We use nine Clayton copulae. To calibrate them we find iteratively that a in (3.66) which allows us to calculate the lower tail dependence λ_L in (3.66) which is equal to the upper and lower tail dependence of the t-Student copula in (3.60).

(4) The maturity of the 1st, 2nd and 3rd to default is 2 years, and the default time step is a quarter.

(5) With the Monte Carlo algorithm, we simulate the quarter when the default occurs and not the exact time of default. To accommodate for the fact that the contract pays when default occurs,

[9] Note that with the Merton model the default time correlation is the asset correlation.

we adjust the default payment leg by assuming that the default happens midway between quarters.

(6) The interest rate used to discount the losses is assumed to be flat at 10%.

(7) The number of sample paths is 10,000, and it was chosen since it gave a good trade-off between accuracy and time in section 4.3.3.

The results produced by the models are contained in the Appendix 2. We distinguish between three sets of Tables. In Tables 4.8 to 4.10 we show premiums and standard errors for the Survival model with Normal copula. This is to allow monitoring the quality of the Monte Carlo. In Tables 4.11 to 4.19 we compare the differences between the Merton and the Survival model where the Elliptical copulae are used. Finally, the differences in the Clayton copulae are reported in Tables 4.20 to 4.22.

4.5.1 Survival model with Normal copula and analytic formula, with default time correlation equal to 0

In Table 4.8, we can read the Monte Carlo values, standard errors and computational time of the 1st, 2nd and 3rd to default, calculated with Survival model with Normal copula when default time correlation $\rho(\tau_i, \tau_j)$ is zero.

As we did in section 4.3.3, we want to check whether the analytic values in Table 4.2 are in the range of the simulation values $\pm$ their standard errors. When we built those ranges with the values and standard errors from Table 4.8, most of the analytic values lie outside. We believe this is so, since the analytic formula models the exact time of default, whereas in this numerical example, we model the quarter when default occurs. This translates in lower values of default payment leg.

4.5.2 Survival model with Normal and Marshall-Olkin copula

To compare the Survival model with the Marshall-Olkin copula of section 4.3.3, we look at the values in Tables 4.9 and 4.10, and in Tables 4.3 to 4.6. The values of the 1st, 2nd and 3rd to default are higher when we use the Marshall-Olkin copula. This is true for the three contracts. In part this can be explained with the same argument used earlier with the analytic model, that is, with the Marshall-Olkin copula we simulated the time of default and not the quarter when default occurs. A secondary reason depends on how default time correlation is modelled: the Marshall-Olkin copula models default time correlation through a joint intensity rate (the systematic factor), whereas the Survival approach models correlated default times through correlated random variates.

An important advantage of the Normal copula over the Marshall and Olkin copula is the much lower time necessary to simulate the times of default. For example, in a portfolio of fifty credits we need more than 250 secs. with the Marshall-Olkin copula versus only 11 seconds. when we use the Normal copula.

4.5.3 Survival and Merton model

Tables 4.11 to 4.19 show the values of the 1st, 2nd and 3rd to default calculated with the two models. When the default time correlation $\rho(\tau_i, \tau_j)$ is zero there is virtually no difference between the values of the 1st, 2nd and 3rd to default, calculated with the two models.[10]

With the default time correlation $\rho(\tau_i, \tau_j)$ equal to 0.25 and 0.50, the Merton model overvalues the 1st to default and the 2nd to default in all cases. It undervalues the 3rd to default until the number of credits is 9. From 10 to 50 credits, the 3rd to default is overvalued as well. In summary, the losses are distributed differently between the 1st, the 2nd and the 3rd to default as long as the number of credits in the reference portfolio is not greater than 9. With much larger collateral, the Merton model hugely overvalues the losses.

4.5.4 Normal and *t*-Student copulae with the Survival model

Tables 4.11 to 4.19 show that accounting for joint extreme events with the *t*-Student copula has some important implications for the values of the 1st, 2nd and 3rd to default, if compared with the values calculated with the Normal copula.

When the default time correlation $\rho(\tau_i, \tau_j)$ is 0, we cannot find any meaningful difference in the values of the 1st, 2nd and 3rd to default calculated by the *t*-Student copula with 100, 12, 8 and 4 degrees of freedom. This confirms what was anticipated earlier.

With the default time correlation $\rho(\tau_i, \tau_j) = 0.25$ and $\rho(\tau_i, \tau_j) = 0.5$, the *t*-Student copula increases the values of the 1st, 2nd and 3rd to default as the degrees of freedom drop.

Our findings contradict the results and conclusions of Mashal, Naldi and Zevi (2003) on modelling the default times with the *t*-Student copula for valuing n^{th} to defaults. They empirically studied the application of the *t*-Student copula, with 12 d.o.f. on a 1st, 2nd and 3rd to default on a five name basket with a maturity of five years. In their work, they used a constant intensity rate at 1% per year together with three different levels of default time correlation, very similar to ours, of 0, 0.20

[10] We remind that with the Merton model the default time correlation is the asset correlation.

and 0.50. Their results suggest the Normal copula will generally underestimate the 3rd and the 2nd to default and overestimate the 1st to default when compared with the *t*-Student copula with 12 d.o.f..

Our results do not show the same redistribution of losses from the 1st to default into the 2nd and the 3rd to default. One reason for the empirical differences between their findings and ours, and we want to emphasise, only where the default time correlation $\rho(\tau_i, \tau_j)$ is 0.25 and 0.50, can be attributed to the longer maturity they used, five years against two years in our work, and to the lower intensity rate they applied, 1% against 10% in our work. We cannot explain the differences between our results and theirs where the default time correlation $\rho(\tau_i, \tau_j)$ is 0. In our view, the Normal, the *t*-Student and the Clayton copulae ought to calculate the same values since the times of default are not really joint together *via* copulae if $\rho(\tau_i, \tau_j) = 0$.

From a computational point of view, the *t*-Student copula is relatively more expensive than the Normal copula (see Table 4.23).

4.5.5 Survival model with Clayton copula

The results calculated with the Clayton copula are in Tables 4.20 to 4.22. Reading these results is more complex since the default time dependency is captured through the common parameter a, as calculated in (3.66), rather than through the default time correlation used for the *t*-Student copula and Normal copula. To make the comparisons with the *t*-Student copula easier, we have preferred to calibrate a on the lower tail dependence λ_L in (3.66). For example, the results in Table 4.20, second column, are obtained with a=5.6*E*-08, which is calibrated on the *t*-Student copula with $\lambda_L = 0$, $\rho_\tau(\tau_i, \tau_j) = 0$ and $\upsilon = 12$. For completeness, we report below all the values of a that we used in Tables 4.20 to 4.22, together with the parameters of the *t*-Student copula.

t-Student copula				Clayton copula		
$\rho(\tau_i, \tau_j)$	λ_L	$\rho_\tau(\tau_i, \tau_j)$	υ	λ_L	$\rho_\tau(\tau_i, \tau_j)$	a
0.25	0.02	0.16	12	0.02	0.082	0.178
0.50	0.06	0.33	12	0.06	0.110	0.246
0	0.01	0	8	0.01	0.07	0.151
0.25	0.05	0.16	8	0.05	0.104	0.231
0.50	0.12	0.33	8	0.12	0.14	0.327
0	0.08	0	4	0.08	0.121	0.274
0.25	0.14	0.16	4	0.14	0.15	0.353
0.50	0.25	0.33	4	0.25	0.2	0.500

The results in Tables 4.20 to 4.22 show that the Clayton copula is a very powerful tool to use to value the 1st, 2nd and 3rd to default. With the choice of the parameter a, we can model the default time dependency in the upper left quadrant of the distribution, and generate values either strongly or weakly dependent on joint extreme events.

As a increases, the effect on the values depends on the number of credits. With 50 credits, the values of the 1st, 2nd and 3rd to default drop. With 5 credits, this is still true for the 1st to default, whereas the values of the 2nd and 3rd to default rise as a increases. This shows the significant impact that the Clayton copula has on the distribution of losses across the 1st, 2nd and 3rd to default: with small reference portfolios, expected losses are redistributed from the 1st to default to the 2nd and 3rd to default. This is in line with what happens when we used the Normal or *t*-Student copulae and increased the default time correlation. This point will be further explained in the following section.

The sensitivity to a change in a is more pronounced when the 1st, 2nd and 3rd to default are computed with a larger number of credits.

With $a = 5.6E\text{-}08$, i.e. when the default time dependency between each pair of credits is zero, the Clayton copula calculates the same values of the 1st, 2nd and 3rd to default as the Normal and *t*-Student copulae when the default time correlation $\rho(\tau_i, \tau_j)$ is 0. This is exactly what we expected.

From a computational point of view, the Clayton copula is the fastest of all approaches so far developed. In only one occasion (Table 4.23), the computational time exceeds 1 sec.

4.5.6 Default time correlation with Normal and *t*-Student copulae

Figures 4.15, 4.16 and 4.17 show what happens to the default payment leg of two reference portfolios of 5 and 20 credits, as default time correlation changes. In Figure 4.15 the default payment leg of 1st to default is always monotonically decreasing in default time correlation. In Figure 4.16 the default payment leg of 2nd to default is monotonically decreasing in default time correlation with 20 credits. With 5 credits, the price is almost flat and has a turning point at a default time correlation of 0.6. The turning point depends on the initial parameters, but above all on the number of credits in the basket. In Figure 4.17 we can draw the same conclusion for 3rd to default for the case of 20 credits. With 5 credits, the default payment leg increases as default time correlation increases. This has a huge effect on trading strategies. If an investor holds a 3rd to default note, and expects default time correlation to move up, the change in value of his investment will depend on the number of credits that makes the reference portfolio.

The relationships in Figure 4.15 to 4.17 can be generalised with saying that the 1st to default is always long in default time correlation. Otherwise said, when default time correlation goes up, the value of the 1st to default goes up as well. As we move up in the time ranking of the credit defaults

(2nd, 3rd, 4th and so on) the relationship becomes more complex and depends primarily on the number of credits.

Another important component of the relationship between value and default time correlation is the complexity of the n^{th} to default structure. In many practical cases, the first k defaults are pulled together to create the First Loss note; the second group of defaults, $k + l$, in the same n^{th} to default, are pulled together to create the Second Loss note, and so on. In all these cases we can generalise and say that only the First Loss is always long in default time correlation. For the other n^{th} Loss notes this tends to be untrue. We dedicate section 4.6 to further research the relationship between default time correlation and values, and we will do so analysing a typical real-life synthetic CDO.

4.5.7 Number of credits and maturity

We expect that as the number of credits rises, the default payment leg of 1st, 2nd and 3rd to default rise as well: the chance of one or more defaults in the next two years increases. All Tables confirm this. Figure 4.18 shows that the default payment leg of 1st, 2nd and 3rd to default also grows, as maturity rises. Clearly the probability of default rises with time and this rule is reflected in the three values.

4.6 Rating synthetic CDOs

As a final comparison of the Survival copula models used in the previous sections, we consider a numerical example where we calculate the ratings of the notes of a real-life synthetic CDO.

4.6.1 Moody's rating methodology

Moody's bases the rating of correlation products on the concept of expected loss (EL) and standard error (SE).[11] It uses a proprietary Monte Carlo model to calculate the timing of default and the name of the defaulted credits. Different recovery assumptions are applied to the defaulted credits to arrive at the CDO notes expected loss and standard errors. The EL and SE are used to determine the final rating of the notes as follows: Moody's takes the EL and the SE of the note and maps their sum to the Moody's idealised cumulative expected losses of Table 2.8 in chapter 2. For example, a note with a contractual maturity of 10 years should bear an expected loss no greater than 0.006% to be rated as Aaa.

[11] Tabe (2002).

To follow the Moody's methodology, in each Monte Carlo run, we produce the time and amount of losses and discount them back to the present time, using a constant Libor of 5%. In addition, to calculate the losses that hit the First Loss, the Second Loss and finally the Third Loss, we apply the allocation of losses as written in the contract description that will follow. The expected loss and standard errors are calculated as present value in Sterling and in percentage.

4.6.2 Rating arbitrage

We define rating arbitrage as the different rating that Moody's would grant to the note, had a different default time model or correlation assumption been used. There is the risk of calculating more beneficial ratings for the Mezzanine and Senior notes, by using generous assumptions for the default time correlation. In the numerical example that follows, we will show the effect of the default time correlation on the final rating. We will also show how the rating could be different if the Clayton or the *t*-Student copula is used.

4.6.3 Expected shortfall

A disadvantage of using the EL for rating synthetic CDO notes is that it is not sensitive to the likelihoods of losses in excess of certain confidence level ℓ. For example two different models may calculate the same EL, and therefore calculate the same rating, but one may show significantly higher likelihood of large losses.

In addition to EL one could measure the unexpected loss UL, which is defined through the following[12]

$$\mathbb{P}(\mathrm{UL}_\ell > \mathrm{L}_\ell) = (1-\ell) \tag{4.37}$$

where L_ℓ is the loss at the confidence level ℓ.

The UL is the maximum loss that will incur at a confidence level ℓ until the note matures. However, it provides no information on the distribution of the $1-\ell$ of losses above this value. With this in mind, Arvanitis and Gregory (2001) and Duffie and Singleton (2003) considered a similar quantity called expected shortfall defined as

$$\mathbb{E}(\mathrm{L}_\ell | \mathrm{UL}_\ell > \mathrm{L}_\ell) \tag{4.38}$$

which is the conditional expected loss, given that the loss is at least as large as the loss level L_ℓ. The unexpected loss is the actual worst-case loss at a given confidence level, whereas the expected shortfall is the average of all losses above this worst-case loss.

[12] In Arvanitis and Gregory (2001) p. 100.

It is still unclear to us how to use the expected shortfall as a substitute of the EL to calculate the rating of the synthetic CDO notes. Nevertheless, we believe this is a very important tool to compare the quality and performance of the models analysed so far, when used for pricing and rating CDO notes.

4.6.4 Contract description

The CDO structure consists of three notes: First Loss Protection, Second Loss Protection and the Third Loss Protection.

(1) First Loss Protection or Junior note has a notional of £40,000.

(2) Second Loss Protection or Mezzanine note has a notional of £50,000.

(3) Third Loss Protection or Senior note suffers a loss if the Junior and the Mezzanine notes are fully utilised. Its notional is £910,000.

The reference portfolio consists of fifty credits, and it is shown in Table 4.24. For each credit, we show the maturity, the par value (or current balance), the rating, the industry classification and the CDS premium as implied in the Moody's idealised cumulative expected losses of Table 2.8 in chapter 2 (and using a constant recovery rate of 45%).
The CDO contractual maturity dates are 4 years and the recovery rate is assumed to be 45% for all credits. All notes mature at the earliest of their contractual maturity and the date when their notionals are exhausted.

4.6.5 Calculating the premium

The First Loss, Second Loss and Third Loss pay Libor plus a premium. To calculate the premiums we rely on the formulae derived in section 4.2. Hence, the premium is calculated by putting into equivalence the premium leg in (4.10) with the default leg in (4.9).

As we simulate the quarter when default occurs, we assume that the simulated default happens midway between quarters. Furthermore, since the default probabilities are calibrated on the Moody's idealised cumulative expected losses of Table 2.8 in chapter 2, rather than on the CDS market premiums, our model will not produce risk-neutral prices.[13]

[13] We envisage a natural extension of this exercise: calibrating two sets of default probabilities. A first set, on Moody's idealised cumulative losses of Table 2.8 in chapter 2 to calculate the rating, and a second set is calibrated on the CDS market premiums to calculate risk-neutral premiums.

4.6.6 Rating results

In Tables 4.25 to 4.33, the numerical results are presented. For all three notes, we calculate the premium, the EL (in percentage and in value), the standard errors, the Moody's rating, and the expected shortfall with three confidence levels of 97%, 99% and 99.5%. We show the results in three tables, one with a default time correlation of 0, a second with a default time correlation of 0.25 and a third with a default time correlation of 0.50. In each table, the numerical results are calculated using the Merton model and the Survival model. For the Survival model we produce the results with three copulae: Normal, *t*-Student and Clayton, whereas the Merton model is used with the Normal copula only. With the *t*-Student copula we use two different degrees of freedom υ, of 12 and 4. The Clayton copula is calibrated on the parameters of the *t*-Student copula with 12, 8 and 6 degrees of freedom υ.

4.6.7 Survival and Merton model with the Normal copula

With a default time correlation equal to 0, Tables 4.25 and 4.26 show that the Merton model and the Survival model calculate the same expected loss and rating for the three notes. However, when we look at the expected shortfalls, the Merton model shows a Mezzanine note much riskier than the Survival model.

With a default time correlation equal to 0.25 and 0.50, the Merton model overvalues the expected loss and the expected shortfalls of the Junior, and undervalues the expected loss and the expected shortfalls of the Mezzanine and Senior notes when compared with the Survival model (Tables 4.28 to 4.30). We also find their differences increase as the default time correlation moves to 0.50. This is in line with the numerical results illustrated in section 4.5, and with the results reported by Finger (2000).

4.6.8 Default time correlation

- Junior note-First Loss

Tables 4.25, 4.28 and 4.31 show that as default time correlation increases, the premium of the Junior note decreases. The relationship between the premium and the default time correlation is reversed when looking at the Mezzanine note and the Senior note.

- Mezzanine note-Second Loss

Tables 4.26, 4.29 and 4.32 show the premium of the Mezzanine note increases. The relative change (in %) in the premium is greater when the Survival model with the *t*-Student copula with 4 d.o.f. is used: it moves from 16.36 bps when default time correlation is zero (Table 4.26, 5th column), to 101.36 bps when default time correlation is 0.50 (Table 4.32, 5th column).

- Senior note-Third Loss

Tables 4.27, 4.30 and 4.33 show what happens to the Senior note. When default time correlation moves from zero to 0.5, the relative change in the premium is greater when the Survival model is used together with the *t*-Student copula and with 4 d.o.f.: the premium moves from 0 bps (Table 4.27, 5th column) to 2.63 bps (Table 4.33, 5th column).

In summary, high default time correlation has the effect of increasing the probability of joint defaults, however, the effect on the premiums is different and depends on the ranking of the note in terms of loss allocation. As correlation rises, the protection that the Junior note offers to the Mezzanine and Senior notes diminishes. Therefore, a larger premium is required for the Mezzanine and Senior notes to offset a greater risk.

With the current structure, the Junior note is long in default time correlation.[14] The reverse is true for the Senior note. In this structure, the Mezzanine note behaves more as a Senior note, when the default time correlation changes.

4.6.9 Normal and *t*-Student copulae

- EL and Rating

Table 4.25 shows that the *t*-Student copula has no or little impact on the EL and rating of the Junior note. On the other hand, the rating of the Mezzanine note drops one notch (Table 4.26), signalling the collateral portfolio is much riskier, but still insufficient to raise the expected loss of the Senior note that remains equal to zero.

With the default time correlation of 0.25, the *t*-Student copula only influences the ratings of the Junior and the Mezzanine notes (Tables 4.28 and 4.29). When moving the default time correlation to 0.5, the rating of Senior notes drops two notches and becomes A1 from Aa2.

- Expected shortfall

The expected shortfalls show that the *t*-Student copula changes the risk of the Mezzanine and the Senior notes. When we move from the Normal copula to the *t*-Student copula with 12 d.o.f., the expected shortfalls, at all three confidence levels, rise, and signal the drop of the credit quality of the collateral portfolio. The credit quality further drops when the *t*-Student copula with 4 d.o.f. is used.

[14] This means that the value of the Junior note increases as correlation increases.

The exception is the Junior note: Tables 4.25, 4.28 and 4.31 show the expected shortfalls do not change when the *t*-Student copula is used in place of the Normal copula and when the default time correlation changes from 0 to 0.5.

6.6.10 Clayton copula

The results of the Clayton copula are presented in Tables 4.25 to 4.33 under the title *No (linear) default time correlation*. To model the dependence, we used nine values for a.

- Clayton copula and Normal and *t*-Student copulae with zero default time correlation

Table 4.25 shows that, as a moves from 5.6E-08 to 0.198, the EL of the Junior note diminishes and the rating improves, from Caa to B3. The effect on expected shortfall, at all three confidence levels, is very small.

The effect on the Mezzanine note (Table 4.26) is reversed when a is increased: the EL rises and as a consequence, the rating drops.

With regard to the Senior note, when the Elliptical copulae are used, in no circumstance is the EL different from 0. However, the Clayton copula with a equal to 0.151 and 0.198 (in Table 4.27), would suggest a much riskier note, and with a rating of Aa2 or Aa3.

In summary, the drop of the EL of the Junior note is compensated for an equal increase of the EL of the Mezzanine and Senior notes.

- Clayton copula and Normal and *t*-Student copulae with default time correlation equal to 0.25 and 0.5

To compare Clayton copula with the Elliptical copulae when default time correlation equal to 0.25 and 0.5, we have set a at 0.177, 0.231 and 0.274, in Table 4.28 to 4.30, and at 0.246, 0.327 and 0.391 in Tables 4.31 to 4.33.

This has had no impact on the ratings and on expected shortfalls of the Junior note, which are not materially different from those calculated with the Elliptical copulae. However, the effect on the EL is significant: it drops as a rises.

The results on the Mezzanine note and the Senior note are drastically different: the tables indicate both notes are much riskier when modelled with the Clayton copula. The rating of the Mezzanine note drops one notch down, from Ba1 to Ba2, when a is set at 0.391, whereas, the rating of the Senior note drops to Aa3 (in Table 4.30), and further down to A1 (in Table 4.33). The expected shortfalls at 99% and 99.5% confidence levels in Tables 4.30 and 4.33, are also much larger than the expected shortfalls calculated with the Elliptical copulae.

Looking at the EL, we can notice that as a rises, the EL of the Junior note drops, and this is compensated for an equal increase of the EL of the Mezzanine and Senior notes, (see Tables 4.31, 4.32 and 4.33). This implies that the Normal copula underestimates the fair compensation of the Senior and the Mezzanine notes and overestimates the fair compensation of the Junior note.

These results are very important and contradict the work of Meneguzzo and Vecchiato (2002) on modelling default times with the Clayton copula in CDO and n^{th} to defaults. They found that the Clayton copula does not appear useful, because it is not able to capture the upper tail dependence. In their model, they first simulated the survival function and then calculated the exact default time by taking the inverse of the survival function. This approach simulated too many survival times and fewer default times.

Our approach is to simulate the joint times of default *via* simulating the quarters when default occurs. In this way, we are able to use the Clayton copula and capture the lower tail dependence.

4.6.11 Clayton copula and *t*-Student copula: our conclusion

When the *t*-Student copula is used, by reducing the degrees of freedom, we are able to calculate a Senior note much riskier. However, the consequence on Mezzanine and Junior notes is contained.

When we increase the level of dependence in the lower left quadrant of the distribution through the parameter a, the Clayton copula on one hand reduces the risk of the Junior note, on the other hand, it makes the Mezzanine and the Senior notes much riskier.

More important, the Clayton copula picks up some extra risk in the Senior note. This is confirmed by the expected shortfalls that capture the fatter tails of the loss distribution. For example, in Tables 4.27 and 4.30, the expected shortfalls calculated with the *t*-Student copula are much lower compared with the values calculated with the Clayton copula. Only when the default time correlation is 0.5, do the expected shortfalls become consistent between the two copulae.

These are important results and point towards the Clayton copula as a flexible and important tool for structuring and rating purposes, with very useful properties when compared to *t*-Student copula for modelling extreme events.

4.7 Schönbucher and Schubert dynamic copula model

This section gives a short overview over the Schönbucher and Schubert (2001) copula modelling framework, together with an application to price a real-life synthetic CDO structure. More details and proofs can be found in their original article.

Schönbucher and Schubert proposed to use copulae to join default threshold exponential *r.v.*s for a default countdown process. More precisely, a default occurs when the intensity process reaches a default threshold which is an exponential *r.v.* with parameter 1, independent of the intensity process. To induce dependency between the times of default, the default thresholds are joint with copulae.

Definition 4.4: *Let* $\lambda_i(t)$ *be an* $(\mathcal{F}_t)$ *-adapted stochastic process and define the default countdown process for credit i as*

$$y_i(t) = \exp\left(- \int_0^t \lambda_i(s) ds \right) \tag{4.39}$$

Definition 4.5: *As in the spirit of Lando* (1998), *the time of default of credit* i *is defined as*

$$\tau_i = \inf\{t : y_i(t) \leq U_i\} \tag{4.40}$$

and the default indicator function is indicated as $N_i(t) = 1_{\{\tau \leq t\}}$.

Assumption 4.1: *The copula* C *is twice differentiable and its partial derivatives are written as*

$$\frac{\partial}{\partial u_i} C(\cdot) = C_{u_i}(\cdot) \tag{4.41}$$

Assumption 4.2: U_i *s are uniformly distributed on* $[0,1]$ *and independent of* $(\mathcal{F}_\infty)$. *Then if we take* $Z_i = -\ln U_i$, Z_i *is a unit exponential r.v. and it follows that* (4.40) *is equivalent to* (1.106).

Definition 4.6: *The filtration* $\mathcal{H}_{i,t}$ *is the augmented filtration that is generated by* $N_i(t)$. *It is convenient to denote with*

$$\mathcal{H}_t = (\mathcal{H}_{1,t} \vee \mathcal{H}_{2,t} \vee \ldots \vee \mathcal{H}_{n,t}) \tag{4.42}$$

all information about the defaults of all obligors until time t.

At this point, we have a very flexible model. Firstly, it joins the default thresholds in such a way that the single intensities can be calibrated to the term structure of CDS premiums or can be made stochastic, while the dependence of the default thresholds can be separately modelled with any copula. Secondly, it provides us with a very simple algorithm to simulate the time of default τ_i in (4.40), as it follows.

(1) Simulate $(u_1, u_2, .., u_n)$ from the selected copula.

(2) For each credit i, simulate the process $\lambda_i(t)$, and calculate $y_i(t)$.

(3) Stop when $y_i(t) \leq u_i$ and take $\tau_i = t_i$ as the time of default.

Schönbucher and Schubert observed that in this way, the full information available to an agent is not completely used. To see what they meant by that, we indicate as $h_i(t)$ the intensity of credit i for an agent who has access to the complete information set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ available at time t, where $\mathcal{F}_t$ represents the information regarding the general state of the economy and $\mathcal{H}_t$ gives the information on the default states of all credits. The complete information set is taken into account to distinguish the case before default from the case after default. Whenever a default occurs a partial derivative of the copula is taken with the respect to the defaulted credit and the value of the countdown process in (4.39) is fixed at $y_i(\tau_i)$. In this way, the survival functions are updated with the respect to the full information that arrives in the form of default or survival.

Proposition 4.2 (Survival probabilities before default of credit j):[15]

(1) *Under* $(\mathcal{G}_{i,t})_{t\in[0,T]}$, *which is the case where an agent has no access to the information regarding the state of default or survival of all credits, on the set* $\tau_i > t$, *the individual survival probability of credit* i *is*

$$Q(\tau_i > T|\mathcal{G}_{i,t}) = E^Q\left(\frac{y_i(T)}{y_i(t)}\middle|\mathcal{G}_{i,t}\right) = E^Q\left(\exp\left(-\int_t^T \lambda_i(s)ds\right)\middle|\mathcal{G}_{i,t}\right) \quad (4.43)$$

(2) *The intensity of* $N_i(t)$ *under* $\mathcal{G}_{i,t}$ *is*

$$g_i(t) = 1_{\{\tau_i > t\}}\lambda_i(t) \quad (4.44)$$

(3) *Under* $(\mathcal{G}_t)_{t\in[0,T]}$, *which is the case where an agent can observe all other credits and so has full access to the information regarding the state of default or survival of the other obligors, and no default has occurred, the individual survival probability differs from* (4.43) *and is*

$$Q(\tau_i > T|\mathcal{G}_t) = E^Q\left(\frac{C(y_1(t),\ldots,y_i(T),\ldots,y_n(t))}{C(y_1(t),\ldots,y_n(t))}\middle|\mathcal{G}_t\right) \quad (4.45)$$

(4) *The intensity of* $N_i(t)$ *differs under* $(\mathcal{G}_t)_{t\in[0,T]}$ *and it is*

$$\begin{aligned} h_i(t) &= \lambda_i(t)y_i(t)\frac{\frac{\partial}{\partial u_i}C(y_1(t),\ldots,y_n(t))}{C(y_1(t),\ldots,y_n(t))} \\ &= \lambda_i(t)y_i(t)\frac{\partial}{\partial u_i}\ln C(y_1(t),\ldots,y_n(t)) \end{aligned} \quad (4.46)$$

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[15] For a proof, see Schönbucher and Schubert (2001).

with $t \leq T$ on the set $\{\tau_1 > t,...,\tau_n > t\}$.

The process $\lambda_i(t)$, which controls the dynamic of the default countdown process in (4.39), coincides with the default intensity of credit i $h_i(t)$, only in the independence case, $C = C^{\perp}$, or when the information is restricted to $(\mathcal{F}_t) \vee (\mathcal{H}_{i,t})$. Hence, the information regarding the general state of the economy and only the default state of credit i. For this reason, the process $\lambda_i(t)$ takes the name of *pseudo-default* intensity of credit i.

Proposition 4.3 (Joint survival probability, survival probability and default intensity of credit i, after k out of n credits have defaulted):[16]

(1) *The conditional joint survival probability is*

$$\mathbb{Q}\left(\tau > \mathrm{T} \middle| \mathcal{G}_t \wedge \{\tau_j = t_j \ \ for \ 1 \leq j \leq k\} \wedge \{\tau_j > t \ \ for \ k < j \leq n\}\right) = $$

$$= \frac{\mathrm{E}^{\mathbb{Q}}\left(\dfrac{\partial^k}{\partial u_1 ... \partial u_n} C(y_1(t_1),..., y_k(t_k), y_{k+1}(T_{k+1}).., y_n(T_n)) \middle| \mathcal{G}_t \right)}{\dfrac{\partial^k}{\partial u_1 ... \partial u_n} C(y_1(t_1),.., y_k(t_k), y_{k+1}(t),.., y_n(t))} \tag{4.47}$$

where $T_j > \tau_j$ for $j > k$.

(2) *The survival probability* $\mathbb{Q}_i^{-j}(\cdot)$ *of credit* i, *when* j *has defaulted, is*

$$\mathbb{Q}_i^{-j}(\tau_i > T_i | \mathcal{G}_t) = \mathrm{E}^{\mathbb{Q}}\left(\frac{\dfrac{\partial}{\partial u_j} C(y_1(t),..., y_i(T),.., y_n(t))}{\dfrac{\partial}{\partial u_j} C(y_1(t),.., y_n(t))} \middle| \mathcal{G}_t \right) \tag{4.48}$$

(3) *The default intensity* $h_i^{-j}(\cdot)$ *of credit* i, *when* j *has defaulted, is*

$$h_i^{-j}(t) = \lambda_i(t) y_i(t) \frac{\dfrac{\partial^2}{\partial u_j \partial u_i} C(y_1(t),.., y_n(t))}{\dfrac{\partial}{\partial u_j} C(y_1(t),.., y_n(t))} \tag{4.49}$$

Proposition 4.4 (Dynamics):[17]

(1) *The dynamic of the default intensity of credit* i, $h_i(t)$, *is as follows*

[16] For a proof see Schönbucher and Schubert (2001).
[17] For a proof see Schönbucher and Schubert (2001).

$$\frac{dh_i}{h_i} = \frac{d\lambda_i}{\lambda_i} + \left(h_i \left(1 - \frac{C_{u_i u_j} C}{C_{u_i}^2} \right) - \lambda_i \right) dt - dN_i$$

$$+ \sum_{j=1, i \neq i}^{n} \left(\frac{C_{u_i u_j} C}{C_{u_i} C_{u_j}} - 1 \right) (dN_j - h_j dt) \tag{4.50}$$

where $C_{u_i u_j} = \frac{\partial^2}{\partial u_j \partial u_i} C(y_1(t),.., y_n(t))$ and $C_{u_i}^2 = \frac{\partial^2}{\partial u_i^2} C(y_1(t),.., y_n(t))$.

The influence of the default of other credits is contained in the summation term in the second line of (4.50), which mainly depends on the copula. If credit j does not default, then $dN_j = 0$ and there is a drift influence of

$$- \sum_{j=1, i \neq i}^{n} \left(\frac{C_{u_i u_j} C}{C_{u_i} C_{u_j}} - 1 \right) h_j dt \tag{4.51}$$

which reduces the default intensity of credit i, if a positive dependence is assumed between the credits i and j.[18] The s.d.e. in (4.50) gives an alternative specification to simulate the time of default based on the countdown process $y_i(t)$. When (4.50) is compared with the countdown process where only $\lambda_i(t)$ is used, the same joint distribution of default time is reached, however only (4.50) gives the correct dynamics of the default intensity of credit i, $h_i(t)$ under $\mathcal{G}_t$.

Schönbucher and Schubert showed that, when the dynamics of the default intensities are joint with the Clayton copula, the expressions in (4.48), (4.49) and (4.50) nicely simplify in

$$h_i(t) = \left(\frac{C(y_1(t), y_2(t),.., y_n(t))}{y_i(t)} \right)^a \lambda_i(t) \tag{4.52}$$

$$h_i^{-j}(t) = h_i(t)(1 + a) \tag{4.53}$$

$$\frac{dh_i}{h_i} = \frac{d\lambda_i}{\lambda_i} + \lambda_i dt - (1 + a) dN_i + \sum_{j=1}^{n} (1 + a)(dN_j - h_j dt) \tag{4.54}$$

With (4.54), at default of credit j, the credit spreads of the surviving credits jump by a constant factor $(1 + a)$. The drift correction $(1 + a) h_j dt$, tends to continuously reduce the default intensity as long as no default happens. In this way, the return to the pre-default intensity $h_i(t)$ is

[18] We refer to their original paper for a full interpretation of the terms is (4.50).

not modelled through an exponential distributed time-decay as in Davis and Lo (1999), but flows directly from the copula set up.[19]

4.8 Numerical exercise with Schönbucher and Schubert dynamic copula model

As anticipated earlier, the Li's Survival model is a static copula model, where no insight on copula dynamics is given. In what follows, we present a numerical exercise which will allow us to highlight the different results that Li's Survival model and Schönbucher and Schubert (2001) dynamic copula model would calculate.

For simplicity, the collateral portfolio and the structure are the same as in section 4.6. We assume the same alfa parameter of the Clayton copula as in Tables 4.28 to 4.33 and the same interest rate of 5%. We also suppose the same time-inhomogeneous *pseudo-default* intensity rates $\lambda_i(t)$, for all underlying credits. They are extracted from the Moody's table of idealised cumulative expected losses (Table 2.8), assuming they remain constant through the quarterly payment dates and using a recovery rate of 45% equal for all credits. In this way, credits with the same rating are given the same *pseudo-default* intensity rate, but their dynamics $h_i(t)$ are different.

All results are in Tables 4.34 to 4.36. We are not surprised to note that the results for Junior note are not dissimilar from those in Tables 4.28 and 4.31. However, when we look at the Mezzanine and at the Senior notes, their premiums, ELs, Ratings and expected shortfalls are all much worse than when they were calculated with the Li's Survival model (in Tables 4.29-4.30 and 4.32-4.33). For example, with $a = 0.391$, the premium of the Senior note in Table 4.36 is 26.38 bps, whereas it was 2.88 bps in Table 4.33. The highest rating of the Senior note is A1, and it calculated with $a = 0.177$, whereas with the Li's Survival model we calculated a rating of Aa2.

Therefore, even with the same initial intensity rates $\lambda_i(t)$, the default mechanisms in Li's Survival model and Schönbucher and Schubert (2001) dynamic copula are very different.

4.9 Conclusion

In this chapter, we studied three types of intensity copula models: Marshall and Olkin exponential copula, the Li's Survival copula model and the Schönbucher and Schubert dynamic copula.

The first of these models provides an easy way to exponentially join the times of default where the random arrival times are modelled as time-homogeneous Poisson processes. We also

[19] In Schönbucher and Schubert (2001) p. 24.

tested a Monte Carlo algorithm to price 1st, 2nd and 3rd to default, which proved to be an efficient algorithm with 10,000 simulations. However, when it was compared with the Li's Survival model, we found that the latter provided with a faster Monte Carlo algorithm, which could be efficiently used to price large collateral portfolio as well.

When modelling correlation products with the Li's Survival model, we found that their prices depend upon the following components: 1) the size of the reference portfolio, 2) the default time correlation, 3) the complexity of the structure of the correlation product and 4) the copula that is used to join the times of default.

The relationship between premiums and correlation can be generalised with saying that the 1st to default is always long in default time correlation. As we moved up in the time ranking of the credit defaults (2nd, 3rd, 4th and so on) the relationship become more complex and depended on the number of credits. Hence, the size of the reference portfolio is one of the key elements when analysing the sensitivity of the premiums to changes in the default time correlation.

Another important component of the relationship between premium and default time correlation is the complexity of the structure of the correlation product. High default time correlation increases the probability of joint defaults. In addition, the sensitivity on the premiums depends on the ranking of the note in terms of loss allocation. In principle, as correlation rises, the protection that the Junior note offers to the Senior notes diminishes. Therefore, a larger premium is required for the Senior notes to offset a greater risk. However, it is not always easy to say how the Mezzanine note premium would react to a change of correlation. With the structure analysed in section 4.6, the Junior note was long in default time correlation, whereas the Senior note was short. The Mezzanine note behaved more as a Senior note.

Our numerical results of section 4.5 contradicted the results and conclusions of Mashal, Naldi and Zevi when modelling with the *t*-Student copula. We did not find the same redistribution of losses from the 1st to default into the 2nd and the 3rd to default, when the Normal copula is changed with the *t*-Student copula. Our results showed a general increase of their premiums. We also found that the Normal, *t*-Student and Clayton copulae calculated the same premiums when the times of default were not correlated or dependent, which was not the case in Mashal, Naldi and Zevi.

In section 4.6, we found strong evidence that the Clayton copula is a suitable tool when pricing CDO notes. By changing the only available parameter of dependence a, we noticed a redistribution of losses from the Junior note to the Mezzanine note and the Senior note. We also compared the Clayton copula with the Normal copula and the *t*-Student copula, and we found that the Clayton copula overestimated the fair compensation of the Senior and the Mezzanine notes and underestimated the fair compensation of the Junior note. More importantly, the Clayton copula picked up some extra risk in the Senior note. This was confirmed by an increase of the expected shortfalls of the Senior note.

Our findings contradicted the work of Meneguzzo and Vecchiato on modelling default times with the Clayton copula. They used the Clayton copula to simulate the exact default time. In this way, they simulated too many survival times and fewer default times. Alternatively, we simulated the quarters when default occurred. In this way, we were able to use the Clayton copula and capture the lower tail dependence.

Another important advantage of the Clayton copula over Elliptical copulae is that it simplifies into nice closed-forms. Therefore, we found easy to apply the Clayton copula into the Schönbucher and Schubert dynamic copula framework.

In order to show that the default mechanism implied with the Li's Survival model is different from the Schönbucher and Schubert dynamic copula, we compared these models with a Clayton copula, *via* pricing a real-life CDO. We assumed the same time-inhomogeneous intensity rates for the underlying credits and found much larger losses with the dynamic Clayton copula.

Chapter 5: Structuring and rating cash-flow CDOs with rating transition matrices

5.1 Introduction

This chapter presents a time-inhomogeneous intensity model for valuing cash-flow CDOs that explicitly incorporates the credit rating of the firms in the collateral portfolio as the indicator of the likelihood of default. This model will prove useful for the pricing, structuring, rating and risk management of CDO notes in presence of waterfall triggers linked to the credit ratings of the firms in the collateral portfolio, that if breached, divert cash due to pay the interests of the more junior notes to accelerate the amortisation of the more senior notes.

Moody's (Cifuentes and O'Connor (1996)), Fitch (2003) and S&P (2002) have developed their own in-house models for rating cash-flow CDOs. With the support of these analytical tools, they determine the amount of credit risk present in the collateral portfolio. Once the credit risk is calculated, it is compared to the credit protection offered by the structure to determine the correct rating of the CDO notes. The critique we can move to the rating agency approaches is that the credit rating transitions are not fully captured within their models.

Because the rating agency models do not specifically use credit rating information, they cannot be used to model CDO notes whose performance directly depends on credit rating changes. In our view, the probabilities of downgrade and upgrade ought to be properly modelled, since we are not only interested in the risk of default when the CDO is launched, but also how the CDO structure performs when the risk profile of the collateral deteriorates over time. For example, quality tests would trigger if the average credit quality of the collateral were to drop below a certain rating, imposing the CDO manager to sell some of the credits and therefore reduce the internal rate of return of the CDO equity investors. In order to measure the risk and value of the CDO notes, we believe it is necessary to combine a credit risk model with the exact cash-flow waterfall model of the given structure.

Credit rating transitions have been modelled in the past as a finite Markov chain,[1] which assumes that, the credit rating changes from one rating to another at given time intervals with a certain probability. The probabilities of the credit rating form a transition matrix, due to the Markov property, the probabilities of future credit rating only depend on the current rating. Jarrow, Lando and Turnbull (JLT[2] 1997) were the first to develop a Rating Transition (RT) model, where the credit rating classes evolve according to time-homogeneous and time-inhomogeneous Markov process. In the time-inhomogeneous version, the rating transition process was modelled by calibrating the

[1] For a review on Markov chain, see Häggström (2002) and Norris (1998).
[2] JLT in the rest of the chapter.

intensities of the generators over discrete times, with a set of observed default-risky instrument prices. Further evolution was Lando (1998a). In a doubly stochastic framework, he allowed for stochastic transition matrices to model the effect of the economic environment on the default risk.

Transition matrices can be obtained by the three main rating agencies, Moody's, Fitch and S&P. However, transition matrices published by rating agencies are not suited for valuing default-risky financial instruments. They are typically available only for annual frequencies, with the shortest period being one year. Many financial instruments have maturities shorter than a year, and thus require transition matrices over arbitrary time horizons.[3] The approach that obtains transition matrices of any arbitrary time horizons involves embedding the discrete-time Markov chain into a continuous-time Markov process (Kingman (1962)). Because, for any continuous-time Markov process, any transition matrix can be expressed as the exponential of a generator (or intensity) matrix, solving the embedding problem essentially means to find a generator matrix (Kreinin and Sidelnikova (2001)). To further complicate the valuation of a default-risky financial instrument with a rating agency transition matrix, is the fact that its true generator does not exist (Israel, Rosenthal and Wei (IRW[4] 2001)), otherwise said, the rating agency transition matrix is not compatible with a continuous-time Markov process. IRW identified a set of conditions under which a valid Markov generator exists and presented algorithms for calculating an approximate generator when the true one does not exist. We will rely on the algorithms developed by Lando (1998a) and IRW to prepare the transition matrices to feed our cash-flow CDO model.

Another issue is to match a transition matrix with historic cumulative default probabilities used by rating agencies for rating cash-flow CDO notes. These default probabilities are based on cohorts of historically observed default frequencies. Moreover, when they are transformed in cumulative multi-year default probabilities, they have a tendency to look like saturated at longer maturities.[5] Hence, assuming time-homogeneity in the rating transitions (which implies that the transition probabilities do not change with time and are constant over the entire time horizon), would inevitably leads to large differences between the cumulative multi-year default probabilities as calculated with the rating transitions, and the historic cumulative frequencies as reported by rating agencies. Besides, given sufficient time, all firms will eventually default in a time-homogeneous system (Jafry and Schuermann (2004)), even though the average time of default may be very large for some initial rating classes. This long-term pattern is a mathematical artefact caused by the simplistic time-homogeneous assumption embedded in the one-year rating agency transition matrix. Lando (1998a) further expanded the work of JLT and showed three methods to calibrate the generator of the one-year transition matrix to the market prices. In this way, he obtained transition

[3] See Jarrow, Lando and Turnbull (1997) for a discussion on pricing and hedging credit derivatives within a Markovian model of rating transition.

[4] IRW in the rest of the chapter.

[5] See Bluhm (2003) p. 13.

matrices for shorter and longer maturities. This makes his research a very valuable one. We will borrow from Lando (1998a) to calibrate the transition matrix to historic default probabilities used by one of main rating agencies. The agency time-homogeneous transition matrix, after being calibrated, will result in a time-inhomogeneous one, and ready to be applied for rating. The introduction of time-inhomogeneous system is a very relevant time-evolution feature of our cash-flow CDO model, necessary to analyse how the credit risk ages.[6] In this way, the transition intensities change as the collateral portfolio ages.

The transition matrices used in our model are calibrated to historical default probabilities. Because modern financial theory requires that hedging and pricing of option type products takes place under a risk-neutral valuation framework, our model will not produce risk-neutral prices. We foresee one easy extension of our model: calibrating the default probabilities to those probabilities implied in the CDS premiums (as in the spirit of JLT), so to be capable of calculating the rating and the price of the CDO notes at the same time. We leave this to another research.

Alternatively, to avoid all the problems embedded with a rating agency transition matrix, the preferred approach is to estimate the generator or generators directly, using the original data set, which must include the exact time transitions occurred. Then, transition matrices for the required period of time can be easily constructed. Such approach is known as the continuous-time approach and was taken by Christensen, Hansen and Lando (2002). There is also evidence for non-Markovian behaviours of the rating process. Nickell, Perraudin and Varotto (2000), Lando and Skødeberg (2002), Christensen, Hansen and Lando (2002), Duffie and Wang (2003) and Hamilton and Cantor (2004) showed the presence of non-Markovian behaviours, such as rating drifts,[7] industry heterogeneity and time variation due in particular to the business cycle. For example, conditioning the future rating on the duration of the past rating implies that, two firms, with the same current rating, may not be assigned the same distribution of future rating. Instead, our approach does not provide a way of computing two different distributions of future rating for firms in the same current rating. We do not deny the value of an empirical exercise, which, at the same time, could be capable of estimating generators, and testing non-Markovian effects. However, we fear our work would be in the area of risk management, with little immediate practical implementation in the area of rating structure finance bonds, where we would like this work to belong. For this reason, we will leave this study to another time.

To address the issue of modelling migration dependence we rely on the concept of copula. A copula is a multivariate distribution function with uniform marginals on the unit intervals. From a practical point of view, the copula gives the advantage of selecting first the marginal distributions of

[6] See D'Amico, Janssen and Manca (2004) for a comparison between the credit risk problem and the more general problem of the reliability of a stochastic system.

[7] Meaning that, a firm recently downgraded, for example from AA to B, is more likely to be further downgraded than an obligor who was always in B.

the rating class of the credit in the CDO, and then linked them through the most suited copula to represent the dependence among the same components. The use of copulae is certainly not new to model portfolios of default-risky instruments.[8] Since initiated by Li (1999 and 2000) and further developed by Schönbucher and Schubert (2001), many copulae have been proposed as alternative to the market standard Normal copula, such as *t*-Student (Mashal and Nandi (2001) and Frey and McNeil (2003)) and Clayton (Schönbucher and Schubert (2001)). Factor copulae (Laurent and Gregory (2002)) are also available, and provide faster and alternative algorithms to copula Monte Carlo approaches.

We found it surprising that all copula research is concentrated around modelling joint defaults and it lacks of the same interest towards modelling joint rating migrations. However, because under the new banking regulation (BIS (2004)), capital requirements are driven in part by rating migrations, we foresee an increase of interest in combining copulae with the rating migration models. Hamilton, James and Webber (2001) were the first to investigate with copulae, dependency relationships occurring between rating events in different rating classes. In their model, the credit rating process is an observable marked point process, determined by an underlying state variable. They calibrated the single-name and the multi-name models by looking at the distribution of rating changes through time and using the Moody's Corporate Bond Default database.

Furthermore, no previous research has so far linked Rating Transition and copula model to the actual legal structure of a cash-flow CDO. All past research is in the area of synthetic CDOs, where calculating the joint loss distribution of the reference portfolio is sufficient to infer the loss distribution of the CDO notes. Our contribution is, when modelling cash-flow CDOs, on measuring the impact of modelling credit rating transitions as trigger events which, when breached, divert the distribution of cash from the junior to senior notes, so accelerating the repayment of senior notes. In our model, the rating process does not dependent on underlying state variables, like in Hamilton *et al* (2001), but it is directly modelled by using actual time-inhomogeneous transition probabilities over discrete times. To induce dependence within the collateral portfolio, we propose to use the Normal copula function. However, our approach can easily be extended to include other copula functions.

A further innovative feature of our research is the application of the RT-Copula model to determine the rating of CDO notes. The rating methodology that we want to develop is based on the expected losses that the holder of a rated note would suffer when investing in a cash-flow CDO note. Based on both the marginal and the joint probability distribution of the credit rating migrations, we propose to calculate the amount of the rated debt by comparing the cumulative expected losses of the note with the cumulative expected losses associated with that rating category.

To our knowledge, this is the first model of cash-flow CDOs, which uses a rating agency transition matrix, publicly available that we successfully calibrate using historic cumulative default

[8] Copulae were originally applied in risk management by Embrechts, Lindskog and Straumann (1999).

rates. Our RT-Copula model is an extension of the time-inhomogeneous JLT model, applied to structuring and rating cash-flow CDOs. The model we suggest involves five steps:

(1) For each credit, we model the rating migrating process via time-inhomogeneous Markov chains.

(2) In this chapter, we aim to calculate the rating of the CDO notes with the RT-copula model. Because of this, we calibrate the S&P's one-year rating transition matrix on the S&P's cumulative default rates.

(3) Join the n -credit rating migration processes *via* the Normal copula.

(4) Implement Monte Carlo simulations, whereby correlated simulated paths work out the credits that migrate and default.

(5) Losses and recoveries are then allocated to the waterfall, generating notes cash flows.

The RT-Copula model will be numerically compared with Li's Survival model (1999 and 2000). The advantage of the Survival model is clearly the saving in the computational time required to perform the simulations. This is because the final time of default and not the full migration path until maturity or default of the credits is simulated. However, as we will see, we will not be able to correctly model the interest and the principal waterfalls.

This chapter develops through the following sections. In section two and three we review the JLT model; in section four and five we show an original way to simulate the single-name rating migration within a time-inhomogeneous Markov process, the multi-name rating migration processes are then joined *via* the Normal copula; section six shows the way rating agencies estimate transition matrices versus alternative and more efficient continuous-time estimators; in section seven we review the conditions for a true generator as in IRW; section eight deals with three ways to calibrate a transition matrix as in JLT and Lando (1998a); sections nine to eleven report the S&P one-year rating transition matrix of which we firstly find an approximated generator, and subsequently, we calibrate on the S&P historic default rates; in section twelve we review two benchmark models, the Survival and the Merton, that will help us to measure the quality of the RT-Copula model. Section thirteen explains the mechanics of interest and principal waterfalls. The correct CDO price strongly depends on the correct model of these waterfalls, as much as on the correct model of the default dependence.

In the last section, we will provide a numerical example and compare the results of the RT-Copula model, based on the probability distribution of joint rating migration, with the Survival model and the Merton model, which only model the joint probabilities of default. In this way, it will become apparent that the correct model is the RT-Copula when the interest and the principal waterfalls of the CDO structure contain trigger events on the credit ratings. Hence, it is required to model the full migration path until maturity or default of the credits.

5.2 Rating transitions

In the rating transition approach, the transition probabilities and defaults are modelled by using an adapted stochastic process $\eta(\omega,t)$, on a finite state space $S=\{s_1,s_2,..,s_K\}$, where $\eta:\Omega\times[0,T]\to S$ and S denotes the set of the K rating classes. The underlying uncertainty is represented by a filtered probability space $(\Omega,\mathbb{P},\mathcal{F}_t)$, where $\mathbb{P}$ is the actual probability measure, whereas with $\mathbb{Q}$, we will indicate the risk-neutral probability measure. The dependence on $\omega\in\Omega$ from now on is suppressed.

Definition 5.1: *The $(K\times K)$ transition matrix for the period $[t,T]$, under the actual probability measure $\mathbb{P}$, is written as*

$$P(t,T)=\begin{pmatrix} p_{1,1}(t,T) & p_{1,2}(t,T) & \dots & p_{1,K}(t,T) \\ p_{2,1}(t,T) & p_{2,2}(t,T) & \dots & p_{2,K}(t,T) \\ \dots & \dots & \dots & \dots \\ p_{K-1,1}(t,T) & p_{K-1,2}(t,T) & \dots & p_{K-1,K}(t,T) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

where $p_{h,j}(t,T)\geq 0$ for all $h,j,h\neq j$ and $p_{h,h}(t,T)=1-\sum_{j=1,j\neq h}^{K}p_{h,j}(t,T)$.

In (5.1) the rating classes are ordered by their credit risk: the rating class 1 represents the best credit rating, the class $K-1$ is the worst credit rating, and the class K is the state of default.

Definition 5.2: *For all h, the time of default τ_h is defined as*

$$\tau_h=\inf\{s\geq t:\eta(s)=K\}, \tag{5.2}$$

which is the first time the firm credit rating hits the state of default K, assumed an absorbing state.

Definition 5.3: *For all $h,j,h\neq j$, $p_{h,j}(t,T)$ is the actual probability of going to state j from state h in the period $[t,T]$*

$$p_{h,j}(t,T)=\mathbb{P}\big(\eta(T)=j\big|\eta(t)=h\big) \quad \forall\ h,j\in S,\ t\leq T \tag{5.3}$$

The probabilities in (5.1) are observed at time $t_0 \leq t$, and they may change over time. Thus, we add a third argument t_0 to the probability in (5.3) to reflect the time during which the information to express those probabilities, becomes available.

Definition 5.4: *The probabilities conditional on the information revealed at time t_0 are*

$$p_{h,j}(t_0;t,T) = \mathrm{P}\big(\eta(T) = j \big| \eta(t) = h \wedge \mathcal{F}(t_0)\big) \;\forall\; h,j \in S,\; t_0 \leq t \leq T \qquad (5.4)$$

where the probabilities are now conditional on the information revealed at time t_0.

Definition 5.5: *For all h, $F_{\eta(t)=h}(t,T)$ is the distribution function of $\eta(t)$ under P in the period $[t,T]$*

$$F_{\eta(t)=h}(t,T) = \sum_{j \in S} p_{h,j}(t,T) \qquad t \leq T \qquad (5.5)$$

5.3 Jarrow, Lando and Turnbull model

JLT pioneered the RT modelling and provided with the modelling framework in discrete and continuous-time. In their approach, the rating transition process $\eta(t)$ has the following properties:

(1) Markov property: at any time t, the transition probability to another state until time $T > t$, depends only on the current state $\eta(t)$ of the rating process

$$\mathrm{P}\big(\eta(T) = j \big| \mathcal{F}(t)\big) = \mathrm{P}\big(\eta(T) = j \big| \eta(t)\big) \qquad \forall\; j \in S \qquad (5.6)$$

This is also equivalent to $P(t_0;t,T) = P(t,T)$, that is, conditioning on further information besides the current rating state $\eta(t)$ at time t, does not improve the risk assessment. This assumption precludes stochastic changes in the transition probabilities driven by other factors, such as the point in the economic cycle, during which the transition probabilities are estimated.

(2) Time homogeneity: the transition probabilities depend only on the time interval over which the transitions take place, otherwise said, the same matrix is used for larger intervals of time

$$P(t,T) = P(T-t) \qquad \forall\; t \leq T \qquad (5.7)$$

In the next two sub-sections, we will first focus on the discrete-time and subsequently on the continuous-time model.

5.3.1 Discrete-time

Assuming that the rating transitions only take place at discrete times, $0 = t_0 < t_1 < ... < t_N$, there is a set of small time intervals, $[t_n, t_{n+1}]$, for which the rating transition matrices can be specified. The intervals are non-overlapping and cover the whole time frame.

Keeping the assumption of time-homogeneity, under the actual probability measure $\mathbb{P}$, all one-period transition matrices must coincide to

$$P(t_n, t_{n+1}) = P \tag{5.8}$$

The transition matrix for larger time intervals is then given by taking P to the power of the length of the time interval

$$P(t_n, t_m) = P^{m-n} \qquad \forall \; n \le m \le N \tag{5.9}$$

Definition 5.6: *Under the martingale measure $\mathbb{Q}$, the one-period transition matrix is written as*

$$Q(t,t+1) = \begin{pmatrix} q_{1,1}(t,t+1) & q_{1,2}(t,t+1) & \dots & q_{1,K}(t,t+1) \\ q_{2,1}(t,t+1) & q_{2,2}(t,t+1) & \dots & q_{2,K}(t,t+1) \\ \dots & \dots & \dots & \dots \\ q_{K-1,1}(t,t+1) & q_{K-1,2}(t,t+1) & \dots & q_{K-1,K}(t,t+1) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.10}$$

where $q_{h,j}(t,t+1) \ge 0$ for all $h, j, h \ne j$, $q_{h,h}(t,t+1) = 1 - \sum_{j=1, j\ne h}^{K} q_{h,j}(t,t+1)$, and $q_{h,j}(t,t+1)$ is the martingale probability of going to state j from state h in the period $[t,t+1]$.

JLT assumed the following relationship between the actual one-period transition probabilities in (5.8) and the martingale transition probabilities in (5.10)

$$q_{h,j}(t,t+1) = \pi_h(t) p_{h,j} \tag{5.11}$$

for all $h, j, h \ne j$, where $\pi_h(t)$ is a deterministic function of time.

With (5.11) the martingale transition probabilities are proportional to the actual transition probabilities and to the proportionality factor $\pi_h(t)$, which they called the risk premium. Furthermore, it is also ensured that the martingale probabilities do not depend on the entire history up to the current time t. In this way, the Markov property is satisfied.

In matrix form, (5.11) becomes

$$Q(t,t+1) - I = \Pi(t)[P - I] \tag{5.12}$$

where I is a $(K \times K)$ identity matrix and $\Pi(t) = diag(\pi_1(t), \pi_2(t), ..., \pi_{K-1}(t), 1)$ is a $(K \times K)$ diagonal matrix.

Given the time dependence of the risk premiums in (5.11), under the martingale measure $\mathbb{Q}$, the time-homogeneous Markov chains are replaced with time-inhomogeneous Markov chains.

5.3.2 Continuous-time intensities

Definition 5.7: *A continuous time-homogeneous Markov chain* $\{\eta(t): 0 \le t \le \tau\}$ *is specified in terms of its* $(K \times K)$ *constant generator (or intensity) matrix*

$$\Lambda^{P} = \begin{pmatrix} \lambda_1 & \lambda_{1,2} & \dots & \lambda_{1,K} \\ \lambda_{2,1} & \lambda_2 & \dots & \lambda_{2,K} \\ \dots & \dots & \dots & \dots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \dots & \lambda_{K-1,K} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.13}$$

where $\lambda_{h,j} \ge 0$ for all $h, j, h \ne j$, and $\lambda_h = -\sum_{j=1, j\ne h}^{K} \lambda_{h,j}$ for $h = 1,2,.., K$.

The $(K \times K)$ transition matrix for η, of the $T-t$ period, under the actual probability measure, is given as

$$P(t,T) = \exp\left((T-t)\Lambda^{P}\right) = \sum_{n=0}^{\infty} \left((T-t)\Lambda^{P}\right)^{n} / n! \tag{5.14}$$

In (5.13) the off-diagonal elements are the constant intensities of jumping to rating j from rating h, whereas the diagonal elements are the constant intensities of moving away from rating h. The default state K is again an absorbing state. With the constant generator matrix, the probability of remaining in the same rating h continually until some time $s > t$ is $\approx \exp\left((s-t)\lambda_h\right)$, analogous to modelling the survival probability with a constant default intensity.[9] If a transition occurs, we can interpret $\approx -\lambda_{h,j} / \lambda_h$ as the probability of a move into the rating class j.

Since the intensities of the generator matrix Λ^{P} are constant, the commutative property applies,[10] and the exponential matrix in (5.14) is calculated using the expression

$$P(t,T) = B \exp\left(D(T-t)\right) B^{-1} \tag{5.15}$$

where $D = diag(d_1, d_2, ..., d_{K-1}, d_K)$ is a $(K \times K)$ diagonal matrix, whose entries are the eigenvalues of Λ^{P}, $\exp\left(D(T-t)\right)$ is simply $= diag(\exp(d_1(T-t)), ..., \exp(d_K(T-t)))$ and the columns of B are the eigenvectors of Λ^{P}.

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[9] Otherwise said, the random times between rating transitions are exponential distributed.
[10] See Duffie and Singleton (2003) p. 96.

Similarly to the discrete-time case, JLT transformed the actual probability-generator matrix to a risk neutral-generator matrix multiplying the actual probabilities by the risk premium matrix $U(t)$

$$\Lambda^{Q}(t) = U(t)\Lambda^{P} \tag{5.16}$$

where $U(t) = diag(u_1(t), u_2(t), ..., u_{K-1}(t), 1)$ is a $(K \times K)$ diagonal matrix, whose first $K-1$ entries are strictly positive deterministic function of time t.

In the same way as with the discrete-time case, given the time dependence of the risk premiums in (5.16), under the martingale measure $\mathbb{Q}$, the Markov chains are now time-inhomogeneous.

The $(K \times K)$ transition matrix for η, from time t to T, under the risk-neutral probability measure $\mathbb{Q}$, is given as solution to the Kolmogorov ordinary differential equations (ODE)[11]

$$\frac{\partial Q(t,T)}{\partial t} = -\Lambda^{Q}(t)Q(t,T) \tag{5.17}$$

and

$$\frac{\partial Q(t,T)}{\partial T} = Q(t,T)\Lambda^{Q}(T) \tag{5.18}$$

with the initial condition $Q(t,t) = I$. If the generator matrices $\Lambda^{Q}(t)$ and $\Lambda^{Q}(s)$ commute for all $t \neq s$,[12] then the solution is

$$Q(t,T) = \exp\left(\int_t^T \Lambda^{Q}(u)du\right) \tag{5.19}$$

Under the commutative property, Lando (1998b) further simplified the solution in (5.19), as follows

$$Q(t,T) = B\exp\left(\int_t^T \mu(s)ds\right)B^{-1} \tag{5.20}$$

where $\mu(s)$ are the eigenvalues of Λ^{P} in (5.13), modified to account for the risk premiums $U(t)$, which are assumed strictly positive deterministic function of time.

5.4 Simulating the single-name rating migration

In this section and in the following one, we prepare the Monte Carlo simulation model for rating and structuring cash-flow CDOs. We start with developing the simulation model of the single credit, where its rating migration is modelled as a continuous time-inhomogeneous Markov chain $\eta(t)$.

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[11] This is discussed in JLT (1997) p. 496.
[12] Schönbucher (2003) p. 231.

We assume in the rest of the chapter that $\eta(t)$ is independent of the level of the default-free risk interest rate $r(t)$. We later, take $r(t)$ to be constant, so in our illustrations correlation effects do not materialise.

We begin with defining the credit rating thresholds, which correspond either to the event of migrating to another rating class or to the event of staying in the same rating class as at the end of the period $[t,t+1]$. To do so, we partition the unit interval $[0,1]$ into K sub-intervals

$$0 \le b_K = p_{h,K}(t,t+1) < b_{K-1} = \sum_{j=0}^{1} p_{h,K-j}(t,t+1) < \ldots < b_1 = \sum_{j=0}^{K-1} p_{h,K-j}(t,t+1) = 1 \qquad (5.21)$$

Definition 5.8: *For all* h*, we call the set*

$$\left\{ \left[b_{j+1}, b_j\right)_{j=2,\ldots,K-1} \cup \left[b_2, b_1\right] \cup \left[0, b_K\right) \right\} \qquad (5.22)$$

the credit rating thresholds, at time t*, for* $\eta(t)$.

We now discuss how to simulate the single-name credit rating $\eta(t)$, which, over the period $[t,t+1]$, is distributed according to the distribution function $F_{\eta(t)=h}(t,t+1)$, on a state space S.

We let $\eta(t) = \psi(U)$, where U is a uniform random variable $[0,1]$, and the function $\psi:[0,1] \to S$ is given by

$$\psi(u) = \begin{cases} s_1 & \text{for } u \in [b_2, b_1] \\ s_2 & \text{for } u \in [b_3, b_2) \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ s_j & \text{for } u \in [b_{j+1}, b_j) \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \cdot \quad \cdot \\ s_K & \text{for } u \in [0, b_K) \end{cases} \qquad (5.23)$$

When $u \in [0, b_K)$, then $\eta(t) = K$, the credit defaults and the simulation stops. Otherwise, the credit rating at the end of the period $[t,t+1]$ is $\psi(\cdot)$, and it is used to simulate the credit rating for the following period.

5.5 Multi-name rating migration: copula model

The multi-name rating migration follows directly from linking the copula function to the single-name rating migration process.

Any continuous distribution function $F(x_1, x_2,.., x_n)$ of random variables $\eta_1(t), \eta_2(t),.., \eta_n(t)$ can be decomposed into a composition of the individual marginal distribution function $F_{\eta_i(t)}(x_i)$ and the copula function $C(\cdot)$.

Theorem 5.1 (Sklar's 1959): *Let F be an n-dimensional distribution function with margins $F_{\eta_1(t)}(x_1), F_{\eta_2(t)}(x_2),.., F_{\eta_n(t)}(x_n)$. Then there exists an n-copula C such that for all* x *in* $\Re^n$

$$F(x_1, x_2,.., x_n) = C(F_{\eta_1}(x_1), F_{\eta_2}(x_2),.., F_{\eta_n}(x_2)) \tag{5.24}$$

This theorem explains the reason why copula reveals the link between multivariate distribution function and its individual margins.

Corollary 5.1:[13] *Let F be an n-dimensional distribution function with continuous margins $F_{\eta_1(t)}(x_1), F_{\eta_2(t)}(x_1),.., F_{\eta_n(t)}(x_n)$ and copula C (where C satisfies* (5.24)*). Then for any* u *in* $[0,1]^n$

$$\begin{aligned} C(u_1, u_2,.., u_2) &= F(F^{-1}_{\eta_1(t)}(u_1),.., F^{-1}_{\eta_n(t)}(u_n)) \\ &= \mathrm{P}(U_1 \le u_1,.., U_n \le u_n) \end{aligned} \tag{5.25}$$

where $F^{-1}_{\eta_i(t)}(\cdot)$ denotes the quantile function of $F_{\eta_i(t)}(\cdot)$.[14]

In this way, the copula is a joint distribution function of uniform variates U_i, $i = 1,2,...,n$, each of which has a standard uniform distribution.

There are many choices of copulae available in the literature that would permit various dependence structures among the *n*-credit rating migrations. In our cash-flow CDO model, we will consider just the Normal copula. This copula fails to incorporate tail dependence and correlates the random variables near the mean and not in the tails. In spite of its limitations, we still prefer this copula as we believe it will make the comparison of the results obtained using the RT-Copula model

[13] In Embrechts, Lindskog and McNeil (2001) p. 4.
[14] See also chapter 3.

and the Survival model easier. The Normal copula was introduced in chapter 3. In what follows, we only report its definition.

Definition 5.9: *Let Φ denote the standard univariate normal distribution and Φ_{Σ}^{n} denote the standard multivariate normal distribution with correlation matrix Σ. Then the multivariate Normal copula of the joint credit rating migrations is defined as follows*

$$C_{\Sigma}^{N}(u_1,..,u_n) = \Phi_{\Sigma}^{n}(\Phi^{-1}(u_1),..,\Phi^{-1}(u_n)) \tag{5.26}$$

where Φ^{-1} is the inverse of the standard normal cumulative distribution function.

The vector $(u_1, u_2,..,u_n)$ is used with the function $\psi(\cdot)$ to map the credit rating. When $u_i \in [0, b_K)$, then $\eta_i(t) = K$, the credit i defaults and the simulation stops. Otherwise, the rating of credit i at the end of the period $[t, t+1]$ is $\psi(.)$, and it is used to simulate the credit rating for the following period. To generate correlated uniform numbers $(u_1, u_2,..,u_n)$ we rely upon the Cholesky decomposition.

5.6 Estimation of transition intensities: rating agency model and continuous-time model

The most common method for the estimation of a discrete-time transition matrix of actual probabilities is based upon empirical observations of the rating behaviour of groups of firms with the same initial rating. For example, for each period of interest, a cohort of existing firms is first defined and then tracked through the end of the period.[15]

We denote by $C_h(t)$ the total number of firms in class h at time t, and by $C_{h,j}(t, t+1)$ the total number of firms in class h at the end of the year, that migrated to the class j by the end of the year. Then, the estimate of the one-year transition probability is

$$\bar{p}_{h,j}(t, t+1) = \frac{C_{h,j}(t, t+1)}{C_h(t)} \qquad \text{with } h \neq j \tag{5.27}$$

Assuming time-homogeneous Markov chain dynamics in the rating transitions, the transition event from the rating class h, can be seen as the outcome of $C_h(t)$ independent multinomial

[15] Coleman (2002) p. 2.

trials.[16] Hence, we can collect all the observations over different years and prepare the maximum likelihood estimator of $p_{h,j}$ as[17]

$$\bar{p}_{h,j} = \frac{\sum_{t=0}^{T} C_{h,j}(t,t+1)}{\sum_{t=0}^{T} C_h(t)} \qquad \text{with } h \neq j \qquad (5.28)$$

This is the method that rating agencies use to prepare their one-year transition matrix,[18] which we call the *rating agency* approach.[19] Of course, rating agencies have access to continuous-time data of rating transitions (the exact time that rating transition occurs). However, to our knowledge, they do not publish or declare to use continuous-time data when estimating entries in their transition matrices.

The rating agency approach has three serious drawbacks.

(1) No model can be solely based on one-year step and more flexibility is required to price financial instruments with payoffs occurring at arbitrary points in time.

(2) Rare transitions, which are the migrations from high credit quality rating classes to classes of very poor credit quality, are not actually observed in the rating agencies data set, thus, they cannot provide meaningful estimate probabilities of those rare transition events (and thus, the problems that we will see in sections 5.7.2 and 5.7.3). For example, if the number of firms rated AAA at the beginning of a period and defaulted by the end of the same period is zero, the estimator in (5.27) would be 0. Furthermore, if there are transitions from AAA to a lower rating class and from this class there are defaults by the end of the same period, then the estimator of the default probability of AAA would still be 0. However, the continuous-time estimator would take this information into account and estimate a positive default probability.

(3) Any rating migration activity that occurs within the period is very unfortunately ignored.

Alternative to the cohort method are continuous-time estimation methods, which follow the firm over time as it moves from one rating to another. They are also known as hazard/intensity rate approaches, as they extract a hazard rate to model the random time between transitions. Lando and Skødeberg (2000) and Christensen, Hansen and Lando (2002) proposed a continuous-time method to estimate, either the time-homogeneous generator matrix Λ^{P}, or the time-inhomogeneous generator matrix $\Lambda^{P}(t)$. In order to work, their method, in addition to the cumulative rating transition data, also requires the exact point in time at which the rating transition takes place. Following Lando and Skødeberg (2000), the estimator under the assumption of time-homogeneous transition intensities is

[16] Lando and Skødeberg (2000) and Trück and Özturkmen (2003).

[17] Christensen, Hansen and Lando (2002) p. 9.

[18] An example of a one year credit transition matrix is the S&P matrix in section 5.9.

[19] This is also known as the *cohort* method. In the rest of the chapter, we will use *cohort* method and rating agency method as synonymous.

$$\bar{\lambda}_{h,j} = \frac{M_{h,j}(t)}{\int_0^t Y_h(s)ds} \tag{5.29}$$

where $Y_h(s)$ is the number of firms in rating class h at time s and $M_{h,j}(t)$ is the total number of transitions from h to j, where $h \neq j$, over the period t, where $M_{h,j}(t) \geq C_{h,j}(t)$ since $M_{h,j}(t)$ also includes the transitions of firms that entered class h over the period t. Now, the denominator counts the number of periods that a firm spends in the rating state and it is continuously updated as firms move in and out the rating state. Thus, for a horizon of one year, even if a firm rated AA, transits to the rating class A, before ending in the rating class BBB at the end of the year, the portion of year rated A, will contribute to estimate the probability $p_{AA,A}$.[20] Having prepared the generator $\bar{\Lambda}^{\mathrm{P}}$, then the transition matrix $\bar{P}$ is computed with (5.14).

Where the time-inhomogeneus transition intensities are to be estimated, they proposed to use the Aalen-Johansen estimator[21] (a non-parametric method which imposes fewer assumptions on the data generating process and accounts for all movements within the estimation horizon)

$$\bar{\lambda}_{h,j}(t) = \sum_{\{T_{h,j}(k) \leq t\}} \frac{1}{Y_h(T_{h,j}(k))} \tag{5.30}$$

where $T_{h,j}(1) < T_{h,j}(2) < \ldots$ are the observed times of transitions from h to j, and $Y_h(T_{h,j}(k))$ counts the number of firms rated h, evaluated just before time t.

Having estimated the time-inhomogeneous intensities, the transition matrix is then computed as the following product limit

$$\bar{P}(t,s) = \prod_{k=1}^{m} (I + \Delta \bar{\Lambda}^{\mathrm{P}}(T(k))) \tag{5.31}$$

where I is the identity matrix, $T(k)$ is the time of the jump in the interval $]t,s]$, m is the number of sub-periods in the same interval and

[20] Jafry and Schuermann (2004) p. 20.

[21] For details, see Aalen and Johansen (1978) and Lando and Skødeberg (2000).

$$\Delta \Lambda^{P}(T(k)) = \begin{pmatrix} -\frac{\Delta M_1(T(k))}{Y_1(T(k))} & \frac{\Delta M_{1,2}(T(k))}{Y_1(T(k))} & \cdots & \frac{\Delta M_{1,K}(T(k))}{Y_1(T(k))} \\ \frac{\Delta M_{2,1}(T(k))}{Y_2(T(k))} & -\frac{\Delta M_2(T_h)}{Y_2(T(k))} & \cdots & \frac{\Delta M_{2,K}(T(k))}{Y_2(T(k))} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\Delta M_{K-1,1}(T(k))}{Y_{K-1}(T(k))} & \frac{\Delta M_{K-1,2}(T(k))}{Y_{K-1}(T(k))} & \cdots & \frac{\Delta M_{K-1,K}(T(k))}{Y_{K-1}(T(k))} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.32)$$

The off-diagonal elements count the number of migrations to the rating class j from class h divided by the number of firms at the beginning of the period, whereas, the diagonal elements count the total number of migration away from class h, again divided by the number of firms at the beginning of the period. The Aalen-Johansen estimator is essentially the cohort method applied to very short time intervals.

Lando and Skødeberg (2000) and Yafry and Schuermann (2004) provided examples to analyse the differences between time-homogeneous and time-inhomogeneous transition matrices estimated with the continuous-time method and with the rating agency approach. They concluded that the differences between the entries of time-homogeneous and time-inhomogeneous transition matrices are not dramatically different for larger data set. However, when comparing time-inhomogeneous transition matrices with rating agency transition matrices, they found much larger differences.

Lando and Skødeberg, and Christensen, Hansen and Lando,[22] also summarised the key advantages of using the continuous-time method over the rating agency one, which we report below.

(1) It is possible to calculate non-zero estimates of the probabilities of rare transitions that the rating agency method estimates as zero.

(2) When estimating the generator of a time-continuous Markov chain, transition matrices for arbitrary time horizons can be obtained without incurring in the embedding problem.[23]

(3) It is possible to generate confidence intervals of all transition probabilities.

(4) The continuous-time method allows to model and test non-Markovian behaviours such as, rating drift (i.e. dependence on previous rating) and seasoning effects.

(5) The one-year transition matrix, estimated with the rating agency method, strongly depends on the starting point in time the rating transitions are observed, whereas, with the continuous-time model, the results are no longer dependent on the starting period.

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[22] Christensen, Hansen and Lando (2002) also developed bootstrap methods to estimate confidence sets for transition probabilities.

[23] Embedding is defined as determining whether the empirical transition matrix is compatible with a true generator, see also next section.

(6) The dependence on covariates[24] can be tested and business cycles effects quantified.[25]

5.7 Finding the generator matrix

Having decided not to estimate a credit transition matrix from a credit rating transition history data set, we are left with the rating agency transition matrices. Moving from a discrete-time transition model to a continuous time-inhomogeneous transition model, such as the one we are developing, has the main advantage of using transition and default probabilities at arbitrary points in time. However, since our starting matrix will be a rating agency public available transition matrix, there are some problems, which require particular care to be dealt with: *embedding*, *identification*, *approximation* and *uniqueness* of the generator of the rating agency transition matrix.

Singer and Spilerman (1976) defined the *embedding* as, determining if the empirical transition matrix is compatible with a true generator, otherwise said, whether the generator exists. They also defined the *identification* as, seeking the true generator once its existence has been established. *Approximation* addresses those situations where it has been established that the true generator does not exist, however, it is still possible to obtain an approximation. In essence, these concepts relate to how to choose the correct generator that is most compatible with the bond rating behaviour.[26] There is also the problem of *uniqueness*, as an empirical transition matrix may as well have more than one valid generator.

Authors such as Arvanitis, Gregory and Laurent (1999), and Lando (1998a and 1998b) simply assumed the existence of a generator. The first authors who re-discovered in finance the conditions under which the generator does or does not exist were Israel, Rosenthal and Wei (2001). In the four sub-sections that follow, we will borrow from their original article.

5.7.1 Finding a candidate generator matrix: *embedding*

Let $P(0,1)$ be the one-period $(K \times K)$ transition matrix of a time-homogeneous Markov chain. From now on, we remove the time dependence.

We are interested in finding a generator Λ with the properties stated below.

(1) the matrix Λ is the matrix logarithm of P

$$P = e^{\Lambda} \tag{5.33}$$

(2) the off-diagonal elements of Λ are non-negative

[24] Covariates are explanatory variables which describe the duration in the current rating or whether the previous rating change was an upgrade or a downgrade.

[25] See for example, Nickell, Perraudin and Varotto (2000) and Kavvathas (2001).

[26] From Israel, Rosenthal and Wei (2001) p. 246.

$$\lambda_{h,j} \geq 0 \qquad \forall\ h,j \in S,\ h \neq j \tag{5.34}$$

(3) the diagonal elements of Λ satisfy

$$\lambda_{h,h} = -\sum_{h \neq j} \lambda_{h,j} \qquad \forall\ j \in S \tag{5.35}$$

Under the first property, we are looking for a generator Λ such that

$$P = e^{\Lambda} = \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!} = I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + .. \tag{5.36}$$

In dealing with (5.36), the first task is to calculate V with

$$V = \max\{(a-1)^2 + b^2;\ a+bi \text{ is an eigenvalue of } P,\ a,b \in \Re\}$$

where all the (possible complex) eigenvalues of P, of the form $a+bi$, are examined by computing the absolute square of the eigenvalues minus 1 and taking the maximum of these.

Theorem 5.2:[27] *Let P be a one-period $(K \times K)$ transition matrix, and suppose that the condition $V < 1$ holds. Then the series*

$$\Lambda^{*} = (P-I) - \frac{(P-I)^2}{2} + \frac{(P-I)^3}{3} - \frac{(P-I)^4}{4} + .. \tag{5.37}$$

converges geometrically quickly and it is a $(K \times K)$ matrix, which satisfies the property one and the property three exactly.[28]

Theorem 5.3:[29] *The convergence of the power series in (5.37) is guaranteed if the original matrix P, is strictly diagonally dominated,*[30] *i.e. $p_{h,h} > 0.5$ for all $h \in S$. Then $V < 1$.*

IRW noted that, even if some of the diagonal entries were less than 0.5, the series may as well converge and may as well have $V < 1$. Thus, convergence does not seem to be a serious problem, since most empirical credit transition matrices are strictly diagonally dominated. IRW pointed out that even if the series Λ^{*} does not converge, P may still have a true generator. They provided with an algorithm to search for a valid generator when (5.37) fails to converge. Thus, we can refer to the condition of convergence, as a sufficient but not necessary one. We refer to their original article for more about their algorithm.

[27] From Israel, Rosenthal and Wei (2001) p. 247.
[28] For a proof see Israel, Rosenthal and Wei (2001), Zahl (1955) and Singer and Spilerman (1976).
[29] From Israel, Rosenthal and Wei (2001) p. 247.
[30] For a proof see Horn and Johnson (1985) p. 302 and Israel, Rosenthal and Wei (2001) p. 247.

There is a simpler calculation than the power series in (5.37), which can be used if the matrix P, or Λ^*, can be diagonalised, with all eigenvalues real and positive.

Proposition 5.1:[31] *If there exists a non-singular transformation matrix B and a diagonal matrix D_P such that*

$$P = BD_P B^{-1} \tag{5.38}$$

then, the matrix Λ^*, with $P = e^{\Lambda^*}$, can also be diagonalised with the same transformation matrix B, and it is given by

$$\Lambda^* = BD_\Lambda B^{-1} \tag{5.39}$$

where the diagonal matrices are

$$D_P = diag\{\exp(d_1),..,\exp(d_{K-1}),1\} \qquad and \quad D_\Lambda = diag\{d_1,..,d_{K-1},0\} \tag{5.40}$$

The expressions in (5.38) and (5.40) are very useful to construct a transition matrix P applicable for any small time interval as follows

$$P(s-t) = BD_P^{(s-t)}B^{-1} \qquad \text{with } s > t \tag{5.41}$$

where the diagonal matrix is

$$D_P^{(s-t)} = diag\{(\exp(d_1))^{(s-t)},..,(\exp(d_{K-1}))^{(s-t)},1\} \tag{5.42}$$

5.7.2 The non-negativity condition: *approximation*

Often, there remains unsatisfied the property two in (5.34), that is, it is possible that some off-diagonal elements of Λ^* are negative. In this case Λ^* converges, but it is not a generator matrix. Hence, the matrix P is not a proper Markov transition matrix, because some of its entries will also be negative probabilities.

Where this happens, IRW proposed an algorithm to search for all possible generators for P, when P has distinct eigenvalues.[32] However, if any negative off-diagonal entries of Λ^* are quite small, they suggested to correct the problem by simply replacing these negative entries with 0, and adding the negative values, either back to the diagonal entries, or into all other entries (which have the correct sign) on the same row, proportionally to their absolute values. In this way, they replaced

[31] From Schönbucher (2003) p. 234.
[32] Israel, Rosenthal and Wei (2001) p. 259. The authors claimed that all one-year rating transition matrices are very likely to have distinct eigenvalues. However, where this was not true, they referred to Singer and Spilerman (1976) for guidance.

Λ^* with $\tilde{\Lambda}$, which no longer exactly satisfies $P \neq e^{\tilde{\Lambda}} = \tilde{P}$.[33] A measure of the approximation is L^1 – norm of $P - \exp(\tilde{\Lambda})$.[34]

JLT only addressed the issue of the approximation of the generator and obtained an approximated generator by assuming that the probability of a credit to make more than one rating transition during one year is negligible and thus, could be overlooked. JLT method leads to the following algorithm for approximating a generator

$$\tilde{\lambda}_{h,h} = \ln(p_{h,h}); \qquad \tilde{\lambda}_{h,j} = p_{h,j} \ln(p_{h,h}) / (p_{h,h} - 1) \qquad (h \neq j) \tag{5.43}$$

5.7.3 Other conditions for the non-existence of a valid generator

IRW reported three further conditions, which prevent the existence of a valid generator[35]

Theorem 5.4: *Let P be a one-period $(K \times K)$ transition matrix, and suppose that*

(1) $\det(P) \leq 0$, *or* (5.44)

(2) $\det(P) > \prod_h p_{h,h}$, *or* (5.45)

(3) *there are states h and j such that j is accessible from h, but $p_{h,j} = 0$.* (5.46)

Then, there does not exist an exact generator for P.

It can be proven that if $p_{h,h} > 0.5$ for all h, then necessarily $\det(P) > 0$, so (5.44) never applies.[36]

The third condition, in (5.46) is constantly violated in the historical credit transition matrices. For example, the one-year default probability of a firm in the rating class of the best credit quality is usually zero, even though the default can be reached from lower credit quality classes if future downgrades are taken into account. In reality, the historical transition matrices are observations and not probabilities, and thus, even for very high credit quality classes the default probability cannot be zero, and we should not confuse their historical frequencies with their probabilities of default.

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[33] Kreinin and Sidelnikova (2001) presented an alternative algorithm, called quasi-optimisation, which was then compared with the simple corrections advanced by IRW. They concluded it produces better approximation. They also referred to the approximation issue as *Regularisation*.

[34] L^1-norm only provides a relative comparison between two matrices. For more on formal techniques to compare matrices see Jafry and Schuermann (2004) pp. 4-18.

[35] They also referred to the paternity of the three parts of the theorem, respectively to 1) Kingman (1962); 2) Goodman (1970); 3) Chung (1967).

[36] For a proof, see IRW p. 249.

Kreinin and Sidelnikova (2001) examined 32 empirical transition matrices and found the first and second conditions, (5.44) and (5.45), were never satisfied, while the third condition, in (5.46), was satisfied in the majority of the cases. They concluded that in the majority of the cases, a valid generator does not exist.

5.7.4 *Uniqueness* of generator

It is possible that a transition matrix P has multiple valid generators Λ. This statement is very important, since different generators lead to different values of $P(0,t)=\exp(t\Lambda)$, and potentially, to arbitrage when pricing default-risky securities.

Cuthbert (1972 and 1973) proved the uniqueness of generators for P, with the following theorem.

Theorem 5.5: *Let P be a one-period $(K \times K)$ transition matrix.*

(1) *If $\det(P) > 0.5$, then P has at most one generator.* (5.47)

(2) *If $\det(P) > 0.5$ and $\|P - I\| < 0.5$, then the only possible generator for P is $\ln(P)$.* (5.48)

(3) *If P has distinct eigenvalues and $\det(P) > e^{\pi}$, then the only possible generator for P is $\ln(P)$.* (5.49)

In addition, Singer and Spilerman (1976) observed the following.[37]

Theorem 5.6: *Let P be a one-period $(K \times K)$ transition matrix.*

(1) *If all eigenvalues of P are positive, then is the only real matrix Λ^*, such that $P = \exp(\Lambda^*)$.* (5.50)

(2) *If P has any negative eigenvalues, then there is no real matrix Λ^*, such that $P = \exp(\Lambda^*)$.* (5.51)

IRW combined the results of theorems 5.5 and 5.6, and expressed the conditions for excluding multiple valid generators for the transition matrix P as follows.

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[37] Singer and Spilerman (1976) pp. 29-30.

Corollary 5.2: *Let P be a one-period $(K \times K)$ transition matrix such that one of the following condition holds:*

(1) $\det(P) > 0.5$ *and* $\|P - I\| < 0.5$, *or* (5.52)

(2) P *has distinct eigenvalues and* $\det(P) > e^{\pi}$, *or* (5.53)

(3) P *has distinct real eigenvalues.* (5.54)

If an empirical transition matrix P has indeed multiple valid generators Λ, then there is no well defined concept to point to that generator that represents the empirical transition matrix P the best. Again, IRW observed that, based on observing many empirical transition matrices, it is generally unlikely for a rating to migrate to a remote rating in a short period of time. Based on this empirical observation, they proposed to choose, amongst many valid generators, the one with the smallest value of

$$J = \sum_{h,j} |j - h| |\lambda_{h,j}| \tag{5.55}$$

This ensures that the probability of jumping too far in a very short period of time is minimised.

5.8 Methods to modify and calibrate the transition matrix

The last issue we want to review before moving to the empirical analysis, is how to adjust, modify or change the empirical transition matrix due to economic reasons or if we need to calibrate the empirical transition matrix to market price of default-risky bonds or CDS premiums.

Time-homogeneous transition matrices are not realistic. The changes in the economic cycle, due in first place to recession and expansion are well documented and must be incorporated into the transition matrices.[38] A second issue was described in JLT and concerns how to match the transition matrices with the default probabilities implied in the bond prices. A third issue is specifically related to our main objective in this chapter, which is to compare the RT model with the Survival model for rating purposes. The Survival model, as input to simulate the time of default, uses the S&P's cumulative default rates. Hence, we need to calibrate the transition matrix to the cumulative default rates so to make the two models comparable. The effect of calibrating the transition matrices is to move to a time-inhomogeneous framework.

To deal with this issue we describe three numerical methods, all originally proposed by JLT and Lando (1998a), which can be applied to modify time-homogeneous transition matrices.

[38] See Jongwoo (1999).

We assume as given a set of default probabilities for all rating class h and maturity t

$$G_h(t) = \mathbb{P}(t \leq \tau_h),\ h = \{1,2,..,K-1\} \text{ and } t = \{1,2,..,T\} \tag{5.56}$$

The aim is to create a family of time-inhomogeneous transition matrices $(\tilde{P}(0,t))_{t\geq 1}$ so that the default probabilities in (5.56) for each maturity, match the entries in the last column of $\tilde{P}(0,t)$. We repeat that the last column of $\tilde{P}(0,t)$ is the default column, and it is indicated as $\tilde{P}(0,t)_{h,K}$.

We also assume a given one-year transition matrix P and a generator matrix Λ, with $P = \exp(\Lambda)$, and proceed as follows

(1) Let $\tilde{P}(0,0) = I$,

(2) Given $\tilde{P}(0,t)$, choose $\tilde{P}(0,t+1)$, such that

$\tilde{P}(0,t)\,\tilde{P}(t,t+1) = \tilde{P}(0,t+1)$, where $\tilde{P}(0,t+1)_{h,K} = G(t+1)_h$, for $h = 1,2,..,K-1$

(3) Go to step 2.

In each time step, the generator Λ is modified such that

$$\tilde{P}(0,1) = \exp(\tilde{\Lambda}(0)) \tag{5.57}$$

$$\tilde{P}(1,2) = \exp(\tilde{\Lambda}(1)) \tag{5.58}$$

$$\tilde{P}(2,3) = \exp(\tilde{\Lambda}(2)) \tag{5.59}$$

....

satisfying the following

$$\tilde{P}(0,1)\,\tilde{P}(1,2) = \tilde{P}(0,2) \tag{5.60}$$

$$\tilde{P}(0,1)\,\tilde{P}(1,2)\,\tilde{P}(2,3) = \tilde{P}(0,3) \tag{5.61}$$

....

where $\tilde{\Lambda}(0)$ is a modification of Λ, which depends on the factors $\Pi_1 = (\pi_{1,1}, \pi_{2,1}, .., \pi_{K,1})$, and $\tilde{\Lambda}(1)$ is a modification of Λ, depending on the factors $\Pi_2 = (\pi_{1,2}, \pi_{2,2}, .., \pi_{K,2})$. Both Π_1 and Π_2 are chosen, satisfying the following conditions

$$\tilde{P}(0,1)_{h,K} = G_h(1) \tag{5.62}$$

$$\tilde{P}(0,2)_{h,K} = G_h(2) \tag{5.63}$$

....

JLT and Lando (1998a) proposed three different numerical methods, which differentiate on how the step 2 is performed.

(1) Modifying the default intensities (Lando (1998a)),

(2) Modifying the rows of the generator matrix (JLT) and

(3) Modifying the eigenvalues of the transition matrix P (Lando (1998a)).

5.8.1 Modifying the default intensities

With this method, the default column and the diagonal elements of the generator are simultaneously modified by letting

$$\tilde{\lambda}_{1,K} = \pi_{1,1}\lambda_{1,K} \quad \text{and} \quad \tilde{\lambda}_{1,1} = \lambda_{1,1} - (\pi_{1,1} - 1)\lambda_{1,K}$$

$$\tilde{\lambda}_{2,K} = \pi_{2,1}\lambda_{2,K} \quad \text{and} \quad \tilde{\lambda}_{2,2} = \lambda_{2,2} - (\pi_{2,1} - 1)\lambda_{2,K} \tag{5.64}$$

....

such that, with the new transition matrix $\tilde{P}(0,1)$

$$\tilde{P}(0,1) = \exp(\tilde{\Lambda}(0)) = \sum_{n=0}^{\infty} \frac{\tilde{\Lambda}(0)^n}{n!} \tag{5.65}$$

the condition in (5.62) is met, for $h = 1,2,..,K-1$. In this way, it is guaranteed that the new generator is indeed a generator with rows summing to 0. At each time step t, for $t = \{1,2,..,T\}$, the elements in (5.64), $(\pi_{1,t}, \pi_{2,t},.., \pi_{K-1,t}, \pi_{K,t})$, are simultaneously found for the new time step.

With this method, the changes in the generator only take place in the last column (the default probabilities) and in the diagonal elements, and most of the probability mass is shifted from the default probabilities to the diagonal elements.

5.8.2 Modifying the rows of the generator matrix

The idea is to include in the numerical solution of the previous sub-section, all the columns of the generator matrix. The advantage is that the numerical solution again guarantees that the adjusted generator matrix is indeed a generator. With this method we let $\tilde{\Lambda}(t)$ as

$$\tilde{\Lambda}(t)=\begin{pmatrix} \pi_{1,t}\lambda_1 & \pi_{1,t}\lambda_{1,2} & \cdots & \pi_{1,t}\lambda_{1,K} \\ \pi_{2,t}\lambda_{2,1} & \pi_{2,t}\lambda_2 & \cdots & \pi_{2,t}\lambda_{2,K} \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{K-1,t}\lambda_{K-1,1} & \pi_{K-1,t}\lambda_{K-1,2} & \cdots & \pi_{K-1,t}\lambda_{K-1,K} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.66}$$

We apply numerical solutions and solve for the transition matrix $\tilde{P}(0,1)$, subject to the conditions in (5.62) and (5.65), for $h=1,2,..,K-1$. The numerical procedures are carried out for all time steps, with $t=\{1,2,..,T\}$. Unlike the previous method, in this case the probability mass is shifted from the default columns to all other columns.

5.8.3 Modifying the eigenvalues of the transition matrix *P*

The last method is a case studied in Lando (1998a), where the adjustments are carried out by modifying the eigenvalues of the transition probability matrix P. The assumption is that P and Λ are diagonalisable.

We let B be a matrix of the eigenvectors of P and Λ, and let D_P be a diagonal matrix of the eigenvalues of P. Then, the generator is numerically changed by modifying the eigenvalues as follows

$$\tilde{P}(0,1)=B\Pi(0)D_P B^{-1} \tag{5.67}$$

where B and D_P are as defined in (5.38), $\Pi(0)$ is a diagonal matrix with elements $(\pi_{1,1},\pi_{2,1},..,\pi_{K-1,1},0)$ such that, for $h=1,2,..,K-1$, the conditions in (5.62) and in (5.65) are met.

To see better this method, we let b_{jK}^{-1} denote the $(j,K)^{th}$ entry of B^{-1}, and check that $b_{hK}b_{KK}^{-1}=1$. Then, we define $\beta_{h,j}=b_{h,j}b_{j,K}^{-1}$ and write the probability of survival until the end of period 1 as

$$1-G_h(1)=\sum_{j=1}^{K-1}\beta_{h,j}\exp(d_j(1)) \tag{5.68}$$

for $h=\{1,2,..,K-1\}$ and where $d_h(1)=d_h\pi_{h,1}$. Since $\beta_{h,j}$ is known for all h and j, this system of equations determines $d_h(1)$, and finally $\pi_{h,1}$ from the fact that $\pi_{h,1}=d_h(1)/d_h$ for all h.

For the second period $[1,2]$, we start with extracting the survival probabilities between period 1 and 2, $(1-p(1,2)_h)$, for $h=\{1,2,..,K-1\}$, as

$$1-G_h(2)=\sum_{j=1}^{K-1}\tilde{P}(0,1)_{h,j}(1-p(1,2)_j) \tag{5.69}$$

where $\tilde{P}(0,1)_{h,j}$ is the time-inhomogeneous transition matrix for the period $[0,1]$ calibrated with (5.67) and (5.68).

Following this, we proceed as already illustrated for the previous period and use (5.68), where $G_h(1)$ is replaced with $p(1,2)_h$

$$1-p(1,2)_h=\sum_{j=1}^{K-1}\beta_{h,j}\exp(d_j(2)) \tag{5.70}$$

This system of equation determines $d_h(2)$ and $\pi_{h,2}$ as $\pi_{h,2}=d_h(2)/d_h$ for all h. The procedure is then repeated until the last period $t=T$. At each time step, it must be checked that the generator is indeed a generator.

5.9 S&P's one-year time-homogeneous transition matrix: the *starting matrix*

Having explained the RT-Copula model and the problems we have to solve when using a rating agency transition matrix, we now move to analysing the rating agency transition matrix that will be used in the numerical exercise. Table 5.1 shows the historical average one-year rating transition frequencies of S&P 1981-2003. The eight columns represent the credit ratings $S=\{AAA,AA,A,BBB,BB,B,CCC,Default\}$. The last column WR indicates the withdrawn rating category, which includes the cases where S&P has withdrawn all of an issuer's rating. Each row indicates the rating group at the beginning of a one-period.

The upper left-hand corner indicates, for example, that on average over the period 1981-2003, 88.31% of AAA's have remained in the same rating class over one-year period. Apparently, no issuer rated AAA, has ever defaulted over one-year period, 5.83% of AAA's have been downgraded to AA, and 0.01% of AA's have defaulted. However, we cannot read the percentage of AAA's that once downgraded to AA has defaulted.

Moving down the credit spectrum by three notches, at BBB, on average over the period 1981-2003, 84.17% of BBB's have remained in the same rating class over one-year period, 4.59% have been downgraded to BB, 0.32% have defaulted, and 0.04% have been upgraded to AAA.

The likelihood of a rating withdrawal generally increases as the credit quality decreases. Typically, with the cohort method, firms withdrawn either are removed from the sample, or are given their own rating category. However, with a continuous-time method, the firms withdrawn would contribute to estimate the intensities for the portion of time, which they spent in the rating class i.

The largest values are along the main diagonal, which indicate that the most recurring event is the one of remaining in the same rating at the end of one-year period.

	To Rating								
From Rating	AAA	AA	A	BBB	BB	B	CCC	D	WR
AAA	88.310%	5.830%	0.700%	0.080%	0.080%	0.000%	0.000%	0.000%	5.000%
AA	0.590%	87.270%	7.220%	0.640%	0.070%	0.130%	0.030%	0.010%	4.040%
A	0.070%	2.030%	87.390%	5.390%	0.500%	0.180%	0.040%	0.060%	4.340%
BBB	0.040%	0.210%	4.250%	84.170%	4.590%	0.810%	0.170%	0.320%	5.440%
BB	0.040%	0.090%	0.420%	5.460%	75.830%	7.980%	0.820%	1.310%	8.030%
B	0.000%	0.080%	0.240%	0.310%	4.830%	74.420%	4.390%	5.900%	9.840%
CCC	0.100%	0.000%	0.310%	0.620%	1.550%	8.670%	47.470%	29.410%	11.870%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	1.000%	0.000%

Table 5.1: S&P's Historical Average One-Year Rating Transition Matrix - 1981-2003.[39]

Before using this transition matrix, we want to remove the WR category. Our assumption is that this category is not informative for the rating process.[40] This is done by proportionally distributing the WR weights to the other frequencies in the same row. We obtain the one-year rating transition matrix as shown in Table 5.2.

$P =$

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.958%	6.137%	0.737%	0.084%	0.084%	0.000%	0.000%	0.000%
AA	0.615%	90.944%	7.524%	0.667%	0.073%	0.135%	0.031%	0.010%
A	0.073%	2.122%	91.355%	5.635%	0.523%	0.188%	0.042%	0.063%
BBB	0.042%	0.222%	4.495%	89.012%	4.854%	0.857%	0.180%	0.338%
BB	0.044%	0.098%	0.457%	5.938%	82.469%	8.679%	0.892%	1.425%
B	0.000%	0.089%	0.266%	0.344%	5.357%	82.533%	4.869%	6.543%
CCC	0.057%	0.057%	0.352%	0.704%	1.759%	9.838%	53.864%	33.371%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.2: S&P's Historical Average One-Year Rating Transition Freq.1981-2003 - without WR.

[39] From S&P's, Special Report "Corporate Defaults in 2003 Recede form Recent Highs", February 2004, Table 19 – Region: US, p. 39, URL: http://www.standardandpoors.com.
[40] The same assumption was used in Christensen, Hansen and Lando (2002).

5.10 Finding the generator of the S&P's transition matrix: the *adjusted matrix*

By the condition in (5.46), the transition matrix P does not have a valid generator. For example, $p_{AAA,B} = 0\%$, but $p_{AAA,AA} = 6.137\%$ and $p_{AA,B} = 0.135\%$. This means that the rating class $h = AAA, j = B$ with $\mathrm{P}(j = B|h = AAA) = 0$, can be reached *via* first transiting to the rating class $h = AAA, j = AA$ with $\mathrm{P}(j = AA|h = AAA) = 6.137\%$, and then migrating to $h = AA, j = B$ with $\mathrm{P}(j = B|h = AA) = 0.135\%$.

The convergence of the series Λ^* is guaranteed by the fact that the matrix in Table 5.2 is diagonally dominated. Retaining five digits, we obtain Λ^* with (5.37), which we show in Table 5.3. Furthermore, the condition in (5.35) is not violated.

$\Lambda^* =$

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.07326	0.06674	0.00522	0.00046	0.00088	-0.00012	-0.00002	0.00010
AA	0.00666	-0.09615	0.08249	0.00481	0.00040	0.00139	0.00035	0.00004
A	0.00070	0.02322	-0.09298	0.06237	0.00418	0.00163	0.00041	0.00047
BBB	0.00042	0.00184	0.04973	-0.11992	0.05642	0.00693	0.00193	0.00265
BB	0.00047	0.00096	0.00333	0.06919	-0.19829	0.10469	0.00916	0.01049
B	-0.00005	0.00092	0.00270	0.00146	0.06444	-0.20000	0.07243	0.05810
CCC	0.00079	0.00065	0.00443	0.00887	0.02088	0.14577	-0.62499	0.44358
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.3: the generator Λ^*.

By looking at Table 5.3, the condition (5.34) is violated: there are three nonnegative off-diagonal entries, and we conclude that P does not have a true generator. As a consequence of this, we have to find an approximation of $\tilde{\Lambda}$, such that $\exp(\tilde{\Lambda})$ is close to P. To do so, we apply the first solution advanced by IRW also explained in section 5.7.2, and replace Λ^* with $\tilde{\Lambda}$ (in Table 5.4). Retaining five digits, $\tilde{\Lambda}$ has only one nonnegative off-diagonal entry and still all rows sum up to 0. We decide to neglect the only nonnegative off-diagonal entry, since very small. However, with $\tilde{\Lambda}$ we can only compute an approximation of P, since $P \neq \tilde{P}$ (in Table 5.5), and thus, we need to calculate the L^1 – norm of $P - \exp(\tilde{\Lambda})$, to measure the approximation induced with this method,

$$\mathrm{norm}[P - \exp(\tilde{\Lambda})] = 0.00161.$$

$\tilde{\Lambda} =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.07333	0.06672	0.00525	0.00040	0.00084	-0.00002	0.00000	0.00015
AA	0.00666	-0.09615	0.08250	0.00484	0.00039	0.00136	0.00034	0.00005
A	0.00070	0.02322	-0.09298	0.06237	0.00419	0.00165	0.00041	0.00043
BBB	0.00042	0.00185	0.04972	-0.11991	0.05640	0.00692	0.00192	0.00267
BB	0.00048	0.00094	0.00333	0.06921	-0.19827	0.10469	0.00916	0.01046
B	0.00000	0.00093	0.00272	0.00145	0.06441	-0.20003	0.07243	0.05809
CCC	0.00080	0.00067	0.00446	0.00885	0.02084	0.14574	-0.62504	0.44367
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.4: the approximated generator $\tilde{\Lambda}$.

$\tilde{P} =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.950%	6.138%	0.737%	0.074%	0.078%	0.007%	0.001%	0.016%
AA	0.615%	90.941%	7.526%	0.670%	0.070%	0.130%	0.030%	0.019%
A	0.073%	2.122%	91.351%	5.634%	0.523%	0.190%	0.041%	0.066%
BBB	0.042%	0.223%	4.494%	89.011%	4.853%	0.856%	0.178%	0.343%
BB	0.044%	0.098%	0.457%	5.938%	82.468%	8.677%	0.890%	1.428%
B	0.004%	0.088%	0.267%	0.345%	5.355%	82.530%	4.867%	6.545%
CCC	0.058%	0.060%	0.354%	0.698%	1.752%	9.832%	53.861%	33.383%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.5: the approximated one-year transition matrix $\tilde{P}$.

Alternatively, with the JLT's algorithm in (5.43), we compute the following generator $\tilde{\Lambda}_{JLT}$, which has the conditions in (5.34) and (5.35) satisfied.

$\tilde{\Lambda}_{JLT} =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.07064	0.06162	0.00718	0.00080	0.00083	0.00008	0.00001	0.00011
AA	0.00617	-0.09080	0.07560	0.00649	0.00071	0.00135	0.00031	0.00018
A	0.00073	0.02132	-0.08680	0.05667	0.00509	0.00188	0.00041	0.00070
BBB	0.00042	0.00218	0.04521	-0.11047	0.04912	0.00831	0.00181	0.00341
BB	0.00044	0.00099	0.00443	0.06011	-0.17699	0.08827	0.00877	0.01399
B	0.00003	0.00088	0.00266	0.00318	0.05446	-0.17694	0.05199	0.06374
CCC	0.00060	0.00060	0.00364	0.00725	0.01780	0.10494	-0.48281	0.34797
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.6: the approximated generator $\tilde{\Lambda}_{JLT}$.

We now compute

$$\exp(\tilde{\Lambda}_{JLT}) = \tilde{P}_{JLT}$$

From Rating	To Rating: AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.936%	6.162%	0.718%	0.080%	0.083%	0.008%	0.001%	0.011%
AA	0.617%	90.920%	7.560%	0.649%	0.071%	0.135%	0.031%	0.018%
A	0.073%	2.132%	91.320%	5.667%	0.509%	0.188%	0.041%	0.070%
BBB	0.042%	0.218%	4.521%	88.953%	4.912%	0.831%	0.181%	0.341%
BB	0.044%	0.099%	0.443%	6.011%	82.301%	8.827%	0.877%	1.399%
B	0.003%	0.088%	0.266%	0.318%	5.446%	82.306%	5.199%	6.374%
CCC	0.060%	0.060%	0.364%	0.725%	1.780%	10.494%	51.719%	34.797%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.7: the approximated one-year transition matrix $\tilde{P}_{JLT} = \exp(\tilde{\Lambda}_{JLT})$.

Again, with $\tilde{\Lambda}_{JLT}$ we can only compute an approximation of P. To measure the approximation induced with this method and compare it with the previous one, we calculate the L^1 – norm of $P - \exp(\tilde{\Lambda}_{JLT})$,

$$\text{norm}[P - \exp(\tilde{\Lambda}_{JLT})] = 0.06037,$$

and we conclude that the approximation in Table 5.5 is a better result.

5.11 Calibrating the generator and the time-inhomogeneous transition matrix

The aim of this section is to calibrate the approximated one-year transition matrix $\tilde{P}$ to the S&P cumulative default rates,[41] reported in Table 5.8. S&P reports the default rates for years only. To extract the quarterly cumulative default rates, we assume that the default time is exponential distributed between times, and the intensity rates are constant through the quarters.

$G_h(t) =$

	quarter 1	quarter 2	quarter 3	quarter 4	quarter 5	quarter 6	quarter 7	quarter 8
AAA	0.006%	0.011%	0.017%	**0.023%**	0.033%	0.043%	0.052%	**0.062%**
AA	0.028%	0.056%	0.084%	**0.111%**	0.144%	0.177%	0.210%	**0.242%**
A	0.034%	0.068%	0.102%	**0.136%**	0.182%	0.227%	0.272%	**0.317%**
BBB	0.056%	0.112%	0.169%	**0.225%**	0.328%	0.432%	0.535%	**0.638%**
BB	0.700%	1.396%	2.086%	**2.772%**	3.401%	4.026%	4.647%	**5.265%**
B	2.221%	4.393%	6.517%	**8.594%**	10.111%	11.603%	13.071%	**14.514%**
CCC	5.374%	10.459%	15.270%	**19.824%**	22.547%	25.179%	27.721%	**30.176%**

	quarter 9	quarter 10	quarter 11	quarter 12	quarter 13	quarter 14	quarter 15	quarter 16
AAA	0.077%	0.091%	0.105%	**0.119%**	0.137%	0.156%	0.174%	**0.193%**
AA	0.280%	0.318%	0.356%	**0.394%**	0.437%	0.480%	0.522%	**0.565%**
A	0.373%	0.429%	0.485%	**0.542%**	0.608%	0.675%	0.741%	**0.808%**
BBB	0.774%	0.911%	1.047%	**1.182%**	1.341%	1.499%	1.657%	**1.814%**
BB	5.828%	6.388%	6.945%	**7.498%**	8.000%	8.499%	8.995%	**9.489%**
B	15.553%	16.579%	17.593%	**18.594%**	19.316%	20.033%	20.742%	**21.446%**
CCC	31.634%	33.062%	34.460%	**35.829%**	36.659%	37.478%	38.287%	**39.086%**

Table 5.8: S&P's default rates, in bold original data, otherwise extracted by the author.

[41] The S&P's cumulative default rates are from the CDO Evaluator, version 2.2.

In section 5.8, we reported three methods to achieve the calibration task. Here, we will use them to test which gives the best results. The generator that we aim to calibrate is the one in Table 5.4, where the entry $\lambda_{1,6}$ is corrected to zero.

5.11.1 Calibrating by modifying the default intensities (DI)

This method failed to produce the quarterly transition matrices with the required properties. The generators $\tilde{\Lambda}_{DI}(0)$, of the first quarter, and $\tilde{\Lambda}_{DI}(1)$ of the second, are in Tables 5.9 and 5.10 (in the Appendix 5). They are valid generators since both the conditions (5.34) and (5.35) are satisfied. $\tilde{P}_{DI}(0,1)$ (Table 5.11 in the Appendix) also has the default column matched with the entries in the first quarter of Table 5.8, and $\tilde{P}_{DI}(1,2)$ (in Table 5.12 in the Appendix 5), has no negative entry. Unfortunately, with $\tilde{P}_{DI}(0,2) = \tilde{P}_{DI}(0,1)\tilde{P}_{DI}(1,2)$, we failed to match the default column $\tilde{P}_{DI}(0,2)_K$ with all entries in the second quarter of Table 5.8. We proceeded with calculating $\tilde{\Lambda}_{DI}(2)$ of the third quarter. Again, this is valid generator, but one more time, with $\tilde{P}_{DI}(0,3)$, we failed to match its default column with all entries in the third quarter of Table 5.8.

5.11.2 Calibrating by modifying the rows (R)

With this method, we did not encounter any problem. All generators we computed, $\tilde{\Lambda}_R(t)$, with $t \in \{1,2,..,16\}$, are all valid generators with the conditions (5.34) and (5.35) satisfied. At each quarterly step, we were also able to built the time-inhomogeneous quarterly transition matrices $\tilde{P}_R(t,t+1)$, and in no circumstances, there are negative entries. Besides, all calibrated matrices $\tilde{P}_R(0,t)$, with $t \in \{1,2,..,16\}$, also have their default column matched with the entries of Table 5.8.

Tables 5.13 to 5.44 (in the Appendix 6) show the generators and the time-inhomogeneous quarterly transition matrices.

5.11.3 Calibration by modifying the eigenvalues (E)

The generators $\tilde{\Lambda}_E(0)$ and $\tilde{\Lambda}_E(1)$ are in Tables 5.45 and 5.46 (in the Appendix 7). Unfortunately, only the first generator satisfies both conditions (5.34) and (5.35), whereas, the second fails condition (5.34). When we checked the condition (5.34) for the remaining 14 generators $\tilde{\Lambda}_E(t)$, with $t = \{3,..,16\}$, 13 off-diagonal entries have negative values: they are concentrated in the first seven quarters and are very small, the largest value is -0.00024.

Tables 5.47 and 5.48 (in the Appendix) show the calibrated transition matrices for the first two quarters, $\tilde{P}_E(0,1)$ and $\tilde{P}_E(1,2)$: two entries in $\tilde{P}(1,2)$ are negative, 0.008% and 0.005%. We also calculated the matrices covering the periods up to the end of quarter 16, unfortunately, some of them contain negative entries.

Based on these numerical results on the data available, there is evidence that only method two is capable of calibrating the generator of Table 5.4 to the cumulative default rates of Table 5.8. Therefore, we will only use the time-inhomogeneous transition matrices computed with method two in the rest of this chapter.

5.12 Benchmark models: Survival and Merton models

The alternative models to the RT-Copula, are Li's Survival model and Merton model.

5.12.1 Survival model

We let $G_{i;h}(t)$ be the default function of credit i in the rating class h, as published by S&P's and shown in Table 5.8. We can calibrate the Li's Survival model on a vector of correlated standard normal random variables $(X_1, X_2,.., X_n)$, where the Normal copula of the default times is written as

$$C_\Sigma^N(u_1,u_2,...,u_n) = \mathbb{P}\left(G_{1;h}(T) < \Phi^{-1}(X_1),.., G_{n;h}(T) < \Phi^{-1}(X_n)\right) \quad (5.71)$$

The Li's Survival model is used with the following simulation algorithm.

(1) Collect the default function of all *n* credits.

(2) Draw a correlated standard normal variable and set the default time as $\tau_i = \Phi^{-1}(X_i)$.

(3) If $\tau_i > G_{i;h}(T)$, the credit *i*, in the rating class *h*, does not default before the time T. Otherwise, we locate the default time in the k^{th} period with $G_{i;h}(k-1) < \tau_i \leq G_{i;h}(k)$.

The correlated X_i numbers are drawn with *Cholesky* factorisation.

5.12.2 Merton model

The Merton approach assumes that the firm value is a stochastic quantity that triggers the default when the firm's asset value $A(t)$, hits a certain threshold value a_t, with $0 < a_0 < A_0$.[42]

To simulate the time when the i^{th} firm's asset value hits the threshold value and the firm defaults, we proceed with the following numerical algorithm:

(1) Collect the cumulative default probabilities (of Table 5.8) as done for the Survival model.

(2) In the first period, we set the threshold values with $\alpha_{i.1} = \Phi^{-1}(G_{i;h}(1))$.

(3) For the subsequent periods, we set the threshold values, $\alpha_{i,k}$, equal to the conditional default probability of the k^{th} period, i.e.,

$$\alpha_{ik} = \Phi^{-1}\left(\frac{G_{i;h}(k) - G_{i;h}(k-1)}{1 - G_{i;h}(k-1)}\right)$$

(4) In each k^{th} period, we draw correlated standard normal random variables X_{ik} by *Cholesky* decomposition and compared them to the threshold values of the same period $\alpha_{i,k}$, and when $X_{i,k} \leq \alpha_{i,k}$, the firm defaults.

5.13 The financial product: the cash-flow CDOs

Before moving to the numerical exercise, we will describe the waterfall mechanics of a typical cash-flow CDO. The loss distribution and the pricing were discussed in chapter 4. Here, we remind that, since the transition matrices used in our model are calibrated to historical default probabilities, our model will not produce risk-neutral prices.

The expressions in (4.9) and (4.10) say all we need to do to calculate the default and the premium payment legs of any given note, is to compute the first moment of the distribution of losses of the given note, for example *via* Monte Carlo simulations. This can be calculated with simulating the default times *via* the chosen copula function. However, this is not fully correct, because to price

[42] For more on the Merton model applied to CDO see Arvanitis and Gregory (2001).

the CDO notes, we need to take into account the interest and principal waterfalls and finally, how the defaults and the recoveries are allocated to the notes.

There are many ways CDO waterfalls work in practise, and they primarily differentiate on the basis of synthetic and cash-flow CDOs. In synthetic CDOs, the losses hit straight the notes, whereas in cash-flow CDOs there are several waterfall diversion elements, such as the use of the excess spread to cover the losses before flowing back to the originator, which reduces the impact of losses to the CDO note holders.

In this section, we prepare the interest and principal waterfalls of a cash-flow CDO, which we intend to use to compare the Survival model with the RT-copula model.

At the interest period t, the collateral portfolio pays $U(t)$, which is used in the interest waterfall as follows.

(1) pari-passu to the Trustee and Administrative Fees, and to the Senior Management Fees, which we assume to be equal to zero,

(2) Senior Note interests, $I_{SN}(t)$

$$I_{SN}(t) = \min[U(t), I_{SN}(t)]$$

(3) Mezzanine Note interests, $I_{MN}(t)$

$$I_{MN}(t) = \min[U(t) - I_{SN}(t), I_{MN}(t)]$$

(4) Junior Note interests, $I_{JN}(t)$

$$I_{JN}(t) = \min[U(t) - I_{SN}(t) - I_{MN}(t), I_{JN}(t)]$$

(5) and the remaining to the Equity (which is not a contractual interest).

In the same fashion, with the principal waterfall, we describe how the cash received from the amortisation of the collateral portfolio $CP(t)$, is used to pay down the CDO notes at the interest period t.

(1) Senior Note

$$PR_{SN}(t) = \min[CP(t), SN(t)]$$

(2) Mezzanine Note

$$PR_{MN}(t) = \min[CP(t) - PR_{SN}(t), MN(t)]$$

(3) Junior Note

$$PR_{JN}(t) = \min[CP(t) - PR_{SN}(t) - PR_{MN}(t), JN(t)]$$

(4) Equity piece

$$PR_{Eq}(t) = CP(t) - PR_{SN}(t) - PR_{MN}(t) - PR_{JN}(t).$$

Typically, in a cash-flow CDO, the interest and principal waterfalls are often linked to quality tests, which operate on the collateral portfolio, such as the following tests.

(1) Collateral Test A: when the weighted proportion of the collateral portfolio rated below BB, where the weights are the credit current balances, is equal to or lower 20%, the Equity holders will not receive any excess spread from point 5 of the interest waterfall until all notes are fully redeemed. The excess spread will be used to accelerate the amortisation of the Senior note, then of the Mezzanine note, and at last, of the Junior note.

(2) Collateral Test B: when the weighted proportion of the collateral portfolio rated below BBB is equal to or lower 30%, the Junior notes will not receive any interest from point 4 of the interest waterfall until the Senior and Mezzanine notes are fully redeemed. In this way, the interest of the Junior note will be used to accelerate the amortisation of the Senior note and then of the Mezzanine note.

Clearly, in a cash-flow CDO, if we do not model the rating migration, we would not be able to measure the effect of breaching quality tests on the CDO equity investor return.

5.14 Numerical exercise

In section 5.11, we managed to create a set of time-inhomogeneous transition matrices, calibrated on the S&P's cumulative default rates of Table 5.8. These default rates are the cumulative default rates that S&P uses for rating cash-flow CDOs. Therefore, the effect of calibrating the transition matrix on the S&P's cumulative default rates is to preparing a set of transition matrices, which we can use to calculate the rating of the CDO notes.

In section 5.4, we set up a Monte Carlo model where the rating migration process of a single credit is simulated through time-inhomogeneous transition matrices. In section 5.5 we also extended the single-name process to a multi-name one, where n -credit migration processes are joint together with the Normal copula. The waterfall mechanics were also illustrated in section 5.13. The RT-Copula model is now ready to price and to calculate the rating of a real-life cash-flow CDO.

The underlying collateral consists of fifty floating-rate credits for a total of £1bn, and it shown in Table 5.49 (in the Appendix 8). For each credit, the Table shows the maturity, the current balance, the spread over the Libor rate, the S&P rating, the industry classification and the geographic location. The collateral is used to issue three notes.

(1) Senior note of £.840m, which pays libor plus the premium γ_{SN},

(2) Mezzanine note of £.40m, which pays libor plus the premium γ_{MN},

(3) Junior note of £.60m, which pays libor plus the premium γ_{JN}.

(4) Then, the Equity piece is 60m.

To calculate the premiums γ_{SN}, γ_{MN} and γ_{JN} that put into equivalence the default leg with the premium payment leg for each of the three notes, we use the RT-Copula model and simulate n-correlated rating migration processes where at each time step, the model works out the credits that default or migrate. To analyse the effect of the different default time correlation, we use three correlation values of: 0, 0.25 and 0.5. The number of simulations is 10,000. The libor rate is assumed constant at 5%. Throughout, we will use a constant recovery rate of 45%. The time step is the quarter. The performance of the Equity piece will be measured with its internal rate of return (IRR).

We also compute the final rating of the notes as follows: after calculating their expected loss, we map this value to the Moody's idealised cumulative expected loss (in Table 2.8 in chapter 2). So for example, a note with an average life (AL) of 4 years should bear an expected default rate no greater than 0.001% to be rated Aaa.

The waterfall mechanics, in section 5.13, include two collateral tests: if either is breached, the cash payments to the equity investors are delayed until the three notes are fully redeemed. One collateral test also monitors the cash paid to the Junior note holders: if it is breached, it delays interest payments to the Junior note holder until the Senior and Mezzanine notes are fully redeemed. The prices and ratings of the three notes will be calculated with and without the collateral tests. In this way, the comparison of the RT-Copula model with the Survival approach is more transparent. The role of the collateral tests could be easily changed by amending the interest waterfall. Clearly, this can be done with the RT-Copula model, but it is impossible with the Survival model.

The inputs of the Survival model are the cumulative default rates in Table 5.8, and the default times are joint with the Normal copula as explained in section 5.12.1.

The last model we want to use to benchmark the RT-Copula model is the Merton model.

5.14.1 Numerical outputs

We show our numerical results in Tables 5.50-5.56, which are all in the Appendix 8. For the Junior, the Mezzanine and the Senior notes, the Tables 5.50-5.52 report their premiums, expected losses, standard errors, Moody's ratings, average lives and expected shortfalls with two confidence levels of 99.5% and 99.9%. We also include the interest loss, which, where it is greater than zero, indicates that the income from the collateral is insufficient to pay the interest to the note holders. We remind that in this structure the cash received from the maturity of a credit, is not used to support shortfalls of interest. Hence, the only way to cover the interest loss is to reduce the size of the note that is hit and increase the Equity.

The expected loss and the expected shortfalls of the collateral are in Table 5.53, whereas, the expected Equity IRR without the collateral tests is in Table 5.54.

In each table, the numerical information is split by the correlation assumption and by the model used: RT-Copula model, Survival model and Merton model. Initially, the comparison between the results produced with the three models, is done without the collateral tests of section 5.13. In Tables 5.55 and 5.56, we show the results once the collateral tests are used. The only model that can incorporate them is the RT-Copula model.

5.14.2 Results with the RT-Copula model

As correlation increases, the effect on the notes is twofold: the premiums and the expected losses go up, whereas the ratings drop. The expected shortfalls unequivocally show that the notes become riskier. This is not totally reflected on the expected Equity IRR, which remains little affected by a change in correlation. Furthermore, there are shortfalls of interest as correlation rises: the interest losses are all greater than zero.

The correlation assumption is the key factor when structuring a deal like this. In this particular example, we computed the size of the three notes with a correlation of 0, and calculated their ratings as Ba1 (Junior note), Aa1 (Mezzanine note) and Aaa (Senior note). However, with a correlation of 0.50, two points can be underlined: firstly, the correct ratings ought to be B1, Ba1 and Aa3, secondly, the sizes of the notes ought to be reduced since they suffer interest losses. The latter point will also condition the Equity IRR, which would drop.

As correlation rises, all premiums increase. This seems to contradict what we found in the numerical example of section 4.6, where the premiums of the Senior and the Mezzanine notes increased, whereas the premium of the Junior note dropped. To clarify this point, we look at the collateral expected loss and shortfalls, at the Equity and at the Junior note.

(1) The collateral expected loss drops from 3.34%, when correlation is 0, to 3.28%, when correlation is 0.25, and to 3.15%, when correlation is 0.5. The collateral expected shortfall, at 99.5% confidence level, moves from 8.8% to 17.3% when correlation moves from 0 to 0.5. This is the same change in the distribution of losses that we observed in the numerical example in section 4.6.

(2) In the structure analysed in section 4.6 there was no Equity, and the Junior note of 4%, worked as an Equity piece. In the current structure, the Equity is 6% and nearly twice the collateral expected loss. We also mentioned earlier, that the Equity IRR holds well when correlation rises to 0.5. However, it is difficult to differentiate whether the benefit comes from the drop of the collateral expected loss, or rather at expense of the Junior note, which suffers large interest losses.

5.14.3 RT-Copula model Vs Survival and Merton models

When the correlation is 0, we cannot notice any meaningful difference in the premiums, expected losses, standard errors, ratings, average lives and expected shortfalls calculated with the RT-Copula model and with the Merton model. The results of the Survival model, are very close to those of the RT-Copula model, with the only exception of the rating of the Mezzanine note, which is Aaa with the Survival model, whereas it drops one notch down to Aa1 with the RT-Copula model. We also notice there is virtually no difference between the collateral expected loss and shortfall (Table 5.53), and the Equity IRR (Table 5.54), as calculated with the three models.

When the correlation moves to 0.25, there is no meaningful difference between the values of the Senior note as calculated with the three models. For the Mezzanine and the Junior notes, the differences are in the ratings. The rating of the Mezzanine note is Baa1 with the Survival and the Merton model and drops one notch down to Baa2 with the RT-Copula model. The rating of the Junior note is Ba3 with the RT-Copula and the Merton model and drops one notch down to B1 with the Survival model. It is important to note that there remain just small differences between the collateral expected loss and shortfalls (Table 5.53), and the Equity IRR (Table 5.54), as calculated by the three models.

When the correlation rises to 0.5, there still remain some rating arbitrage opportunities with the Mezzanine note.

We would like to stress the remarkable Equity IRR which is achieved with this structure, which remains between 27.5% and 28.8% even when correlation is as high as 0.5.

Overall, we could only detect some minor differences among the outcomes of the three models used.

5.14.4 RT-Copula model and collateral tests

The advantage of the RT-Copula model over the Survival and the Merton models becomes much clearer when we move to analyse the effect of modelling the two CDO collateral tests. From Table 5.55, we can see that the collateral tests create an additional layer of protection in favour of the Junior note. The main effect is to reduce its premium as the expected loss drops, and to accelerate the repayment of its principal, as its AL shortens. The protection comes at the expense of the Equity piece, whose IRR drops. The results of the Junior note depends on the correlation assumptions: with a correlation of 0.25, the Junior note premium drops to 228 bps (from 239 bps), whereas with correlation of 0.50, the premium drops further, from 436 bps down to 370 bps.

The protection that the CDO collateral tests provide to the Mezzanine and Senior notes is minor and it translates into an acceleration of the repayment of their principal of few months (see their ALs).

The consequences on the ratings are also important in two instances. The Mezzanine note rating, when correlation is 0.25, drops one notch down, from Baa2 to Baa3, and when correlation is 0.5, it moves one notch up, from Ba1 to Baa3.

The last point is the large drop in the IRR (Table 5.56) suffered by the Equity investor, when the collateral tests are introduced to the CDO and properly modelled with the RT-Copula model.

5.15 Conclusion

This chapter presented a new CDO model, RT-Copula, based on time-inhomogeneous transition matrices. The model is flexible enough to allow any type of cash-flow waterfall.

As CDO pricing models have become a mainstream research area in finance, we have identified two key building blocks to value the CDO notes: the copula function and the rating transition matrix.

Since the computation of the transition matrices for arbitrary time periods is based on an annual transition matrix, this has not been an easy task. Most of the empirical annual transition matrices are not compatible with a continuous Markov process since they do not admit a valid generator. Therefore, we computed a modified version of a true generator. The modified generator was extracted using the IRW algorithm. By doing this, we calculated the error that occurred when substituting the *starting* matrix, with an *adjusted* matrix.

Following this, we described and applied three methods, JLT (1997) and Lando (1998a), to calibrate the adjusted matrix to the S&P's probabilities of default. Only with one method, we managed to obtain sixteen calibrated transition matrices covering a period of sixteen quarters. The others methods were unsuccessful.

Finally, we described how to simulate the credit rating migration of one single credit, and how to join n -credit rating migrations *via* the Normal copula.

Modelling the collateral credit risk in this way is very powerful, since it allows us to take into account any quality trigger linked to the performance of the collateral: for example no interest is paid to the Equity if the average rating of the collateral drops below a certain rating.

The numerical section, at the end, provided an example of a CDO, which was priced and analysed with the RT-Copula model. The RT-Copula results are also comparable with those calculated using the Survival and Merton models, and we concluded that, where there is no performance trigger linked to the collateral average rating, there is no pricing or rating arbitrage between the three models. When there are performance trigger linked to the collateral average rating, such as the Collateral Tests A and B, the RT-Copula model perfectly captures the diversion of cash from the interest waterfall to the principal waterfall for the benefit of the Senior and Mezzanine notes. Thus, there is evidence to conclude that the RT-Copula model is the correct approach to model cash-flow CDOs.

Chapter 6: Pricing and rating CDOs of equity default swaps with NGARCH-M copulae

6.1 Introduction

Equity default swaps (EDSs) are similar to CDSs as the protection buyer makes regular payments and receives a payment from the protection seller should a trigger event happen. The difference lies in how the occurrence of the trigger event is determined. In the CDS the trigger event occurs when the reference entity defaults, whereas, in the EDS the trigger event is defined as the drop in the equity price of the reference equity below a specified percentage of the equity price, called the trigger value, at the beginning of the trade. A further difference is how the settlement is determined. In the CDS, it is calculated as the notional amount less the recovery amount, usually calculated at a fixed date from the default time, by a way of market bidding process. Thus, the recovery rate is an important variable in the pricing model. In the EDS, when the trigger event occurs, the protection payment is typically set at a specified percentage of the notional, called the recovery rate. In this way, the uncertainty surrounding the recovery value is removed.

While any combination of the trigger value and the recovery rate can be considered, the current EDS market seems to be in favour of a trigger value of 30% and a recovery rate of 50% of the notional amount. In this respect, EDSs are similar to path-dependent deep-out-of-the money equity digital options. Unlike digital options where the premium is paid up front, the premium of the EDS is spread until it matures.

More recently, EDSs have found their way onto the CDO market. Odysseus, rated Aa2 by Moody's (2004), was the first example of a private rated single-tranche CDO, with a reference portfolio made up of 90% CDSs and 10% EDSs.[1] Odysseus is different from a basket of deep-out-of-the money equity digital options because the others have never been rated by a rating agency. The rating was the prerequisite to bring Odysseus to the credit investors and in our view is the key factor for a much larger market for EDSs. Moody's (2004), S&P (2004) and Fitch (2004) are developing rating methodologies for CDOs using EDSs in the reference portfolio. The depth of historic data available on the equity markets is seen as the main advantage for the rating process, whereas large equity volatilities are the main obstacles for rating CDOs where the reference portfolio exclusively includes EDSs. The three rating agencies also agree in denying any cause and effect relationship between the 70% drop in share price (1 minus the trigger value of 30%), and the event of default on the corporate debt of the issuer.

[1] Others CDOs of EDSs are: Daiwa Securities SMBC US$ 291m (Zest Investments V), Chrome Funding ACEO €70m, Fortis Investment Management and SG CIB €30m (GOYA).

It seems natural that the large amount of work researchers have put into modelling the volatility of equity and index returns should also be very relevant for modelling the prices of EDSs, since the default leg and the premium leg of an EDS are linked to the performance of the reference equity or index. It is well documented in the literature that the equity returns possess properties such as fat-tails, volatility clustering and time-varying variances. Engle (1982) was the first to develop an econometric model where those properties were properly modelled and introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model. Extensions to this model followed: Bollerslev (1986) and Taylor (1986) proposed the generalised ARCH (GARCH), Nelson (1991) the exponential GARCH (EGARCH), Engle and Ng (1993) the nonlinear GARCH (NGARCH).

With reference to the application of GARCH in the option pricing area, Duan (1995) was the first to develop a risk-neutral model within the GARCH framework. His model is also called nonlinear GARCH in mean (NGARCH-M). He characterised the transition between the actual and the risk-neutral probability distributions if the dynamic of the underlying equity price is given by a GARCH process, and thus established the foundation for the valuation of options under GARCH. In his model, the risk-neutral option price depends upon a leverage risk premium parameter, and the variance is negatively correlated with the past equity returns. Such a negative correlation gives rise to a negative skewness in the risk-neutral distribution, which Bates (1996) found to be a key feature in the empirical data. The leverage risk premium parameter is estimated directly from the empirical data since it is part of the conditional expected return of the underlying equity. Hence, Duan's model generates option prices that are consistent with the observed volatility skew.[2] Another feature of the Duan's model (and all GARCH models) is that it is non-Markovian in nature. In particular, the option price depends on the information set generated by the past and current prices of the underlying equity and its past, current and one-step-ahead values of the conditional variance process.[3] In a subsequent study, Chaudhury and Wei (1996) compared Duan's GARCH option price model with the Black & Scholes model. They found that Duan's model is most useful for pricing short maturity out-the-money options. The problems with hedging in the GARCH option pricing model were further discussed by Garcia and Renault (1998). Other works included Ritchken and Trevor (1999) who constructed trinomial trees to price American options under GARCH, Duan, Ritchken and Sun (2004) who extended the standard Duan's option valuation model to include jumps, and Heston and Naldi (2000) who developed an analytic formula to price European options under a more general GARCH. When comparing Duan model and Heston and Naldi model, Hsieh and Ritchken (2000) found that Duan model outperforms Heston and Naldi model, especially in its ability to price deep out-the-money options.

More recently, the importance of GARCH option pricing models has expanded due to their connection with stochastic volatility models, which are alternative methods to GARCH to model the

[2] See Heston and Nandi (2000).
[3] See Duan (1995), corollary 2.3.

time-varying nature of the equity return volatility. Indeed, the GARCH diffusion limits contain many well known stochastic volatility models.[4] It is in the estimation exercise that GARCH models have a distinct advantage over stochastic volatility models. Because the volatility of the equity prices is not easily observable, rather it has to be *implied* from current option prices, the implementation of a stochastic volatility model is a very difficult task. This approach requires calculating numerous implied volatilities from all option records and from all trading days of the time series of the option records.[5] The technique adopted by Bakshi, Cao and Chen (1997), is to collect a time series of cross section of option data, and to estimate the parameters of the daily volatilities (together with the other parameters of the model). Clearly, the computational effort becomes soon severe and prohibitive, as the time series grows. In contrast, GARCH models have the advantage that the volatility is observable from the history of equity prices, without requiring any information on option prices. In addition, Heston and Nandi (2000) emphasised that with GARCH models only a finite number of parameters need to be estimated regardless of the length of the time series. Option pricing applications under GARCH also benefit from the fact that they are relatively easy to re-estimate. Whereas, for stochastic volatility models, it may not be realistic to repeat the estimation procedure on a rolling basis, as new option prices become available. The initial popularity of the stochastic volatility models over the GARCH models could in part be explained by the existence of closed-form solutions[6] under the stochastic volatility, which made the calculation of option prices much easier than under the GARCH models. Until recently, Monte Carlo simulations were the only possible route to implement GARCH models. As we saw before, there is now the analytic formula of Heston and Naldi (2000) to price European options under GARCH. Furthermore, numerical trees by Ritchken and Trevor (1999) are now available and can be used to price American options as well. Duan, Gauthier and Simonato (1999) prepared an analytical approximation of the Duan model to price European options. On the other hand, the parameter estimation for stochastic volatility option pricing models remains typically demanding and problematic. For all these reasons, we focus on the GARCH representation of the time-varying volatility feature of equity returns to price CDOs of EDSs.

The pricing model for EDS and CDOs of EDSs that we want to develop, must be capable of replicating the observed equity return properties mentioned above. To date, there has been no known literature which incorporates nonlinear GARCH in mean (NGARCH-M), and more in general GARCH, into the pricing of EDSs. To illustrate our original methodology, we use two nonlinear GARCH-M processes: the nonlinear normal-GARCH-M (1,1) and the nonlinear *t*-GARCH-M (1,1). We will also borrow from Duan (1995) to explain the transition between the actual and the risk-

[4] Duan (1996, 1997), Corradi (2000) and Nelson (1990), provided detailed relationships between GARCH models and stochastic volatility models.
[5] This approach was taken by Bates (1996 and1999) and Nandi (1998).
[6] See Heston (1993).

neutral probability distributions. The Duan's pricing formulae were originally applied to price European and Asian type equity options. This makes his research a very valuable one.

Because the multi-period distribution of the nonlinear GARCH-M process in unknown, we need to recur to some numerical procedures. Heston and Naldi (2000) closed-form solutions for European options under GARCH, are not applicable in our exercise, since the EDS default payment can be triggered before the final maturity.[7] Ritchken and Trevor (1999) trinomial trees to price American options under GARCH, could be useful only for single-name EDS. Since our research deals with CDO of EDSs, the numerical scheme of Ritchken and Trevor will soon become impractical as the number of EDSs in the CDO grows. Hence, our model will rely on Monte Carlo simulations.

In general, Monte Carlo simulations tend to be a rather numerically intensive method if a high degree of accuracy is desired. This is due to the well-known fact that the standard error of a Monte Carlo estimate is inversely proportional to the square root of the number of the simulated paths. Duan and Simonato (1995) highlighted a less known problem and it is that the simulated option price violates rational option pricing bounds and hence it is not a sensible price estimate. In other words, the simulated path of the equity price fails to possess the martingale property even though the theoretical model does. Duan and Simonato observed that since the equity price dynamics are modelled as an exponential semi-martingale, when the martingale property fails, a sort of multiplicative propagation effect on the errors is generated, and a very large number of simulations are required to reduce simulation errors. As a solution, Duan and Simonato proposed a correction to the standard Monte Carlo technique, which ensured that the simulated paths of the equity price are *empirical* martingales. This correction was called the empirical martingale simulation (EMS). Our EDS model uses Duan and Simonato EMS-Monte Carlo model, coupled with the standard variance reduction technique.

In our model, the dependence within a CDO of EDSs is modelled through copulae. A copula is a multivariate distribution function with uniform marginals on the unit intervals. From a practical point of view, the copula gives the advantage of selecting first the marginal distributions of the components of the basket, and then linked them through the most suited copula to represent the dependence among the same components. The use of copulae is certainly not new to price multivariate options. Brownian motion frameworks have been used to model multivariate option prices for a very long time. However, in the past the dependence among equity returns has been represented by a multivariate normal distribution, where correlation has been the measure of dependence. More recently, Cherubini and Luciano (2002) addressed the issues of non-normal returns, and the dependence in the multivariate contingent pricing problem, was addressed using copulae. Van den Goorbergh and Genest (2004) used a dynamic copula model to price multivariate

[7] See Hsieh and Ritchken (2000) as a research on the performance of the closed-form solutions of Heston and Naldi (2000) and the Duan's GARCH model with Monte Carlo simulations.

options where the dependence structure between two equity prices is time-varying over time and expressed with copulae. None of these authors linked copulae to a NGARCH-M pricing model.

Our contribution is on measuring the impact of different copula functions on the price of the same CDO of EDSs. We propose to use the following copula functions, Normal, *t*-Student and Clayton, within the EMS-NGARCH-M model as follows. To induce dependency within a CDO of EDSs, random variates are drawn from the three copulae, and then transformed in return innovations using the inverse of the normal and the *t*-Student distribution. Following this, three notes of CDO of EDSs are priced with the EMS-NGARCH-M model, *via* Monte Carlo simulations.

A further innovative feature of our research is the application of the EMS-NGARCH-M copula model to determine the rating of three notes of CDO of EDSs. The rating methodology that we want to develop is based on the expected losses that the holder of a rated note would suffer when investing in a product whose reference portfolio is made of EDSs. In a single-name EDS, the probability of triggering the seller payment is driven mainly by the time-varying volatility of the underlying equity price, exactly as for a deep-out-the money equity digital option the probability of expiring in the money depends on the same time-varying volatility. In a portfolio of EDSs, where the seller payment is triggered by any of the prices of the equities in the reference portfolio dropping below the trigger value, the dependence between pairs of equity prices is another important component to analyse. Based on both the marginal and the joint probability distribution of the equity prices, we propose to calculate the amount of the rated debt as it is done for more general CDO tranches, and compare the cumulative expected losses of the note with the cumulative expected losses associated with that rating category.

This chapter develops through the following sections: in section two we introduce the *t*-NGARCH-M pricing model, and also review its main properties. In section three and four, the EDS product is first described in its single-name version and then as a basket or CDO of EDSs. In section five, we show how to link a copula to a NGARCH-M process to simulate dependent trigger events in a CDO of EDSs. In the last section, we will provide an empirical and numerical example of a real-life CDO of EDSs.

6.2 Modelling equity volatilities with NGARCH-M

We begin by preparing the GARCH pricing model, reviewing its properties and introducing the empirical martingale simulation.

6.2.1 GARCH pricing model

We use the now classical mathematical setting of Harrison and Pliska (1981). Our general model, for a market consisting of one equity and one default-free bond, is the following:

Prerequisites:[8] *We let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* T *a positive real number (the terminal time),* $(W_t^*)_{0\le t\le T}$ *a standard Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{P})$, *and* $(\mathcal{F}_t)_{0\le t\le T}$ *the* $\mathbb{P}$ *-completion of the filtration generated by* $(W_t^*)_{0\le t\le T}$. *We assume* $\mathcal{F} = \mathcal{F}_T$. *The filtration* $(\mathcal{F}_t)_{0\le t\le T}$ *fulfils the usual conditions, which are:* $\mathcal{F}_0$ *contains all null sets of* $\mathbb{P}$ *and* $(\mathcal{F}_t)_{0\le t\le T}$ *is right continuous.*

We also assume that the equity does not pay any dividend, we denote its price with S_t, and its instantaneous standard deviation as σ_t. The one-period continuously compounded return on the default-free bond is indicated with r_t. For the rest of this chapter we assume that S_t and r_t are independent.

In the nonlinear GARCH-M (NGARCH-M) option model,[9] the equity return has a form analogous to the framework of Duan (1995), with the difference that Duan used the random variable $\varepsilon_t^* \sim N(0,1)$, whereas we prefer the random variable $\varepsilon_t^* \sim t(0,1;\upsilon)$, where $t(\cdot)$ is the *t*-Student distribution and υ is the degrees of freedom.

Assumption 6.1: *The equity return and the conditional variance dynamics are modelled with a t-*NGARCH-M (1,1) *process,* (leverage GARCH). *Under the physical probability measure* $\mathbb{P}$, *they are*

$$\ln\frac{S_t}{S_{t-1}} \equiv R_t = r_t + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t^* \tag{6.1}$$

and
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2(\varepsilon_{t-1}^* - \theta)^2 \tag{6.2}$$

with
$$\varepsilon_t^* \big| \mathcal{F}_{t-1} \sim t(0,1;\upsilon)$$

where λ is the constant equity risk premium (per unit of conditional standard deviation), θ is a non negative parameter that captures the negative correlation between the return R_t and the volatility

[8] Protter (1990), Theorem I.4.31.

[9] For properties, estimation and tests of GARCH and ARCH models, see Bera and Higgins (1993) and Rossi (2000).

innovations σ_t^2, ε_t^* s form an i.i.d. sequence of standard *t*-Student random variables under the actual probability measure $\mathbb{P}$, ω, β and α are the parameters of the *t*-NGARCH-M (1,1) specification. ω, β and α must remain positive to ensure that the conditional variance stays positive. Furthermore, the long run variance implied by the GARCH parameters $\sigma^{2,\mathrm{P}}$, is constant and is given by

$$\sigma^{2,\mathrm{P}} = \frac{\omega}{1-\beta-\alpha(1+\theta^2)} \tag{6.3}$$

For last, to ensure the long run variance $\sigma^{2,\mathrm{P}}$ is bounded, the following should be satisfied

$$\beta + \alpha(1+\theta^2) < 1 \tag{6.4}$$

The motivation of the nonlinear GARCH is to analyse the asymmetric impact of good news and bad news on the volatility σ_t^2, also called the *leverage effect*. Changes in equity prices tend to be negatively correlated with changes in volatility, thus implying that volatility is higher after negative shocks than after positive shocks of the same magnitude. In other words, a negative ε_t^* has a higher impact on σ_t^2 than a positive value, producing the asymmetric impact on the volatility. The parameter θ, called the *leverage*, models the negative correlation between equity returns and volatility, and leads to skew the distribution of returns.

Using the leverage GARCH model, the presence of a risk premium and the negative return-volatility correlation can be analysed in a time-varying fashion.[10]

In (6.2), the conditional variance appears in the mean as a return premium. This allows the average equity return to depend on the level of risk. The functional form of the risk premium $\lambda\sigma_{t+1}$ prevents arbitrage by ensuring that the equity earns the default-free rate r_{t-1}, when the variance is zero. From (6.1), the conditional expectation of gross returns is

$$\mathrm{E}^{\mathrm{P}}\left(\exp(R_t)|\mathcal{F}_{t-1}\right) = \exp(r_{t-1} + \lambda\sigma_t) \tag{6.5}$$

and the expectation exceeds the default-free rate r_{t-1}, by an amount proportional to the square root of the conditional variance. Hence, at this point we cannot value any option since we do not know yet the risk-neutral distribution of the equity price. Duan (1995) provided sufficient conditions to apply a risk-neutral valuation methodology: *it is sufficient that a representative agent has constant relative risk aversion and that returns and aggregate growth rates in consumption have conditional normal distribution*. This is summarised in the following theorem.[11]

[10] The leverage effect was first discovered by Black (1976) and later substantiated by Christie (1982).
[11] In Duan (1995).

Theorem 6.1: *Under the local risk-neutral probability measure* $\mathbb{Q}$ *, the equity price dynamics are*

$$\ln\frac{S_t}{S_{t-1}} \equiv R_t = r_{t-1} - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t \tag{6.6}$$

and
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2(\varepsilon_{t-1} - \lambda - \theta)^2 \tag{6.7}$$

with
$$\varepsilon_t \,|\mathcal{F}_{t-1} \sim t(0,1;\upsilon)$$

where ε_t's are under the risk-neutral probability measure $\mathbb{Q}$. The proof of this theorem can be found in Duan (1995) and in Heston and Naldi (2000).

The long run variance, under the risk-neutral probability measure $\mathbb{Q}$ is now given by

$$\sigma^{2,\mathbb{Q}} = \frac{\omega}{1-\beta-\alpha(1+(\lambda+\theta)^2)} \tag{6.8}$$

which is higher than the long run variance, under the probability measure $\mathbb{P}$. Furthermore, the risk-neutral conditional variance in (6.7) will be larger than the corresponding conditional variance under P in (6.2), if λ and θ share the same sign.[12]

The *leverage* GARCH (1,1) process defined in (6.1), (6.2), (6.6) and (6.7) reduces to the *standard* GARCH (1,1) if $\lambda = 0$ and $\theta = 0$. Besides, with $\beta = 0$ and $\alpha = 0$, the GARCH process reduces to the standard homoskedastic lognormal process of the Black & Scholes model. In other words, the Black & Scholes model is obtained as a special case.

Different specifications of the volatility dynamic σ_t^2, were introduced by Ding, Granger and Engle (1993), Hentschel (1995) and compared by Christoffersen and Jacobs (2004). To characterise different GARCH models, we re-write the volatility in (6.2) as

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2 f(z_{t-1}) \tag{6.9}$$

where $z_{t-1} \sim N(0,1)$ and $f(z_{t-1})$ is the *news function*. For example, borrowing from Christoffersen and Jacobs (2004), the following specification of f can be considered:

Standard : $$f(z_{t-1}) = \varepsilon_{t-1}^{*2} \tag{6.10}$$

Leverage : $$f(z_{t-1}) = \left(\varepsilon_{t-1}^* - \theta\right)^2 \tag{6.11}$$

News : $$f(z_{t-1}) = \left\{\left|\varepsilon_{t-1}^* - \theta\right| - \kappa\left(\varepsilon_{t-1}^* - \theta\right)\right\}^2 \tag{6.12}$$

Power : $$f(z_{t-1}) = \left(\varepsilon_{t-1}^* - \theta\right)^{2\gamma} \tag{6.13}$$

[12] Empirical evidence tells us that $\lambda > 0$ and $\theta > 0$ for equities and indices.

In this way, the extra parameter or parameters give different shapes to the *news function* f: with the *Leverage* parameter $\theta > 0$, f would shift to the right, with the *News* parameter κ, f would rotate and with the *Power* parameter γ, f would flatten. We revert to Christoffersen and Jacobs (2004) for more on these results.[13]

6.2.2 Additional kurtosis

Another important property of the nonlinear GARCH-M model (and more in general of all GARCH models), is the additional kurtosis, *leptokurtosis*, of the unconditional distribution induced by the changing conditional variance.

For any $\varepsilon_t^* = i.i.d.(0,\mu_2)$ in (6.1), for which $\mu_2 \equiv \mathbb{E}(\varepsilon_t^{*2})$ and $\mu_4 \equiv \mathbb{E}(\varepsilon_t^{*4}) < \infty$, the kurtosis of $\zeta_t^* = \varepsilon_t^* \sigma_t$ is given by[14]

$$\kappa_{\zeta^*} = \kappa_{\varepsilon^*} \frac{1-\gamma_1^2}{1-\gamma_2} \tag{6.14}$$

where κ_{ε^*} is the kurtosis of the process ε_t^*, $\gamma_1 \equiv \mathbb{E}\left(\beta + \alpha(1+\theta^2)\varepsilon_t^{*2}\right)$ and $\gamma_2 \equiv \mathbb{E}\left(\left(\beta + \alpha(1+\theta^2)\varepsilon_t^{*2}\right)^2\right)$. If we assume that $\varepsilon_t^* \sim N(0,1)$, then the term ζ_t^* will be conditionally normally distributed, $\zeta_t^* \sim N(0,\sigma_t^2)$. Consequently, (6.14) becomes

$$\kappa_{\zeta^*} = 3\frac{1-\left(\beta + \alpha(1+\theta^2)\right)^2}{1-\left(3\alpha^2(1+\theta^2)^2 + 2\beta\alpha(1+\theta^2) + \beta^2\right)} \tag{6.15}$$

This result has three important implications: firstly, the kurtosis of ζ_t^* exists if $1-(3\alpha^2(1+\theta^2)^2 + 2\beta\alpha(1+\theta^2) + \beta^2) > 0$, secondly, if $\left(\beta + \alpha(1+\theta^2)\right) = 1$, then the NGARCH-M model does not have heavy-tail behaviour, and thirdly, if $\left(\beta + \alpha(1+\theta^2)\right)^2 < 3\alpha^2(1+\theta^2)^2 + 2\beta\alpha(1+\theta^2) + \beta^2$, then the distribution of ζ_t^*, and consequently of R_t in (6.1), is more heavy-tailed than the normal distribution.

Under the risk-neutral measure $\mathbb{Q}$, the kurtosis of ζ_t becomes

$$\kappa_{\zeta} = \kappa_{\varepsilon} \frac{1-u_1^2}{1-u_2^2} \tag{6.16}$$

[13] Kasch-Haroutounian (2004) presents several random time series with the same unconditional variance, where the GARCH effect on returns is illustrated and compared with normal distributed returns with no-GARCH effect. The author also shows the news function impact on the conditional variance for several GARCH models.

[14] See Teräsvirta (1996) and Duan (2000).

where κ_ε is the kurtosis of the process ε_t, $u_1 \equiv \mathrm{E}\left(\beta_1 + \beta_2\left(1+(\theta+\lambda)^2\right)\varepsilon_t^2\right)$ and $\gamma_2 \equiv \mathrm{E}\left(\left(\beta_1 + \beta_2(1+(\theta+\lambda)^2)\varepsilon_t^2\right)^2\right)$.

Our model assumes that $\varepsilon_t^* \sim t(0,1;\upsilon)$, then the term ζ_t^* will be conditionally t-Student distributed, $\zeta_t^* \sim t(0,\sigma_t^2;\upsilon)$. Expanding on the results of Teräsvirta (1996), the kurtosis of ζ_t, under the data generating probability measure $\mathbb{P}$, is

$$\kappa_{\zeta^*} = \kappa_{\varepsilon^*}\frac{1-\left(\beta+\alpha(1+\theta^2)\right)^2}{1-\left(\beta^2(1+\theta^2)^2\kappa_{\varepsilon^*}+2\beta\alpha(1+\theta^2)+\beta^2\right)} \tag{6.17}$$

where

$$\kappa_{\varepsilon^*} \equiv \mu_4 = 3\frac{(\upsilon-2)}{\upsilon-4},\ \upsilon > 4 \tag{6.18}$$

and with γ_2 now equal to

$$\gamma_2 = \beta^2 + 2\beta\alpha(1+\theta^2) + \alpha^2(1+\theta^2)^2\kappa_{\varepsilon^*} < 1 \tag{6.19}$$

Hence, in our model, the unconditional return R_t, is leptokurtic under the data generating (or actual) probability measure $\mathbb{P}$, and remains leptokurtic under the risk-neutral measure $\mathbb{Q}$. R_t also has a negative skewness if the sum of λ and θ is positive. Duan and Wei (1999) observed that leptokurtosis, when there is no skewness, tends to make an out-the-money option worth more than when the same option is priced with a constant-variance model. Negative skewness tends to have the same effect. Therefore, an out-the-money option is worth more than what a constant-variance model would imply.[15] The factors λ and θ together determine the properties of the EDS pricing model, given that the EDS is similar to a deep out-the-money put option.

6.2.3 Empirical estimation

The GARCH models are estimated using a maximum likelihood approach. The logic is to interpret the density as a function of the parameter set, $\phi = \{\omega,\beta,\alpha,\theta,\lambda,\sigma_0\}$, in (6.1) and (6.2), conditional on a set of sample outcomes.

If we assume that ε_t^* follows a normal distribution the log-likelihood function[16] for a sample of T observations is given by

$$\ln L = -\frac{1}{2}\sum_{i=1}^{T}\left(\ln 2\pi + \ln\sigma_t^2 + \varepsilon_t^{*2}\right) \tag{6.20}$$

[15] To separate the skewness and leptokurtosis effect, we refer to Duan and Wei (1999).
[16] In Lildholdt (2002) p. 7, Christoffersen and Jacobs (2004) p. 61 and Peters (2001) p. 6.

where $\varepsilon_t^* = \frac{1}{\sigma_t}\left[R_t - \left(r_{t-1} + \lambda\sigma_t - \frac{1}{2}\sigma_t^2\right)\right]$.

Our model assumes $\varepsilon_t^* | \mathcal{F}_{t-1} \sim t(0,1;\upsilon)$, then the log-likelihood function[17] is given by

$$\ln L = \ln\left[\Gamma\left(\frac{\upsilon+1}{2}\right)\right] - \ln\left[\Gamma\left(\frac{\upsilon}{2}\right)\right] - \frac{1}{2}\ln[\pi(\upsilon-2)]$$
$$-\frac{1}{2}\sum_{i=1}^{T}\left(\ln\sigma_t^2 + (1+\upsilon)\ln\left(1+\frac{\varepsilon_t^{*2}}{(\upsilon-2)}\right)\right) \qquad (6.21)$$

where $\Gamma(\cdot)$ is the gamma function. When $\upsilon \to \infty$, we have the Normal distribution, so that the lower the fatter the tails.[18]

6.2.4 Empirical Martingale Simulation

From equation (6.6), it follows that the price of the underlying equity at future date T, with $T > t$, is

$$S_T = S_t \exp\left(r(T-t) - \frac{1}{2}\sum_{i=t+1}^{T}\sigma_i^2 + \sum_{i=t+1}^{T}\sigma_i\varepsilon_i\right) \qquad (6.22)$$

where we have assumed $r_t = r$.

The conditional distribution of S_T given today's price S_t can be analytically approximated, as in Duan, Gauthier and Simonato (1999). This is useful only to price a European option, where the underlying equity follows the process in (6.1) and (6.2). However, the same analytic solution is not available for path-dependent options, where it is required to know the probability of S_s being above the trigger value K, anytime before the maturity T. Hence, we have to resort to Monte Carlo simulations.

Duan and Simonato (1995), Boyle *et al* (1995) and Barraquand (1994) pointed out that in a standard Monte Carlo simulation, the following martingale property almost always fails in the simulated sample.

Theorem 6.2:[19] *The discounted equity price* S_t *is a* $\mathbb{Q}$*-martingale, that is for any* $0 \le s \le t$, *is*

$$\mathrm{E}^{\mathbb{Q}}[e^{-rt}S_t | \mathcal{F}_s] = e^{-rs}S_s = S_0 \qquad (6.23)$$

[17] In Peters (2001) p. 7 and Kan and Zhou (2003) p. 10.
[18] For a review of estimation procedures for GARCH we refer to Rossi (2000), Lildholdt (2002), Christoffersen and Jacobs (2004), and Bera and Higging (1993).
[19] Duan and Simonato (1995) p. 3.

which means that the discounted average of S_t will be in almost all cases different from S_0. The difference between the sample value and the theoretical value will depend on the number of simulations, and it is higher for deep-in and deep-out-of-the money options. Even small differences have important effects on the final option price.

Duan and Simonato started from the theoretical expression

$$C_0(t) = e^{-rt}\mathbb{E}^{Q}[\max(S_t - K, 0)|\mathcal{F}_0] \tag{6.24}$$

and subsequently derived the rational option pricing bound

$$C_0(t) > \max(S_0 - Ke^{-rt}, 0) \tag{6.25}$$

where K is the exercise price and $C_0(t)$ is the current price of the European call option at time 0, maturing at time t. To show the violation of the rational option pricing bound in (6.25), we proceed as in Duan and Simonato and define the discounted average of the simulated equity price and the simulated option price as follows.

Definition 6.1:[20] *The discounted average of the simulated equity prices at time* 0*, simulated up to time* t*, with* n *sample paths is*

$$S_0^*(t,n) = \frac{1}{n}e^{-rt}\sum_{i=1}^{n} S_i^*(t) \tag{6.26}$$

where $S_i^*(t)$ is the *i*th simulated equity price at time t, for $i = 1,\ldots,n$.

Definition 6.2: *The Monte Carlo simulated call option price at time* 0 *is denoted by* $C_0^*(t,n)$*, and it is approximated by*

$$C_0^*(t,n) = \frac{1}{n}e^{-rt}\sum_{i=1}^{n}\max[S_i^*(t) - K, 0] \tag{6.27}$$

$$\approx S_0^*(t,n) - Ke^{-rt} \qquad \text{if } K/S_0 \text{ is small.}$$

Then, if $S_0^*(t,n)$ is smaller than S_0, it is possible that the simulated call price $C_0^*(t,n)$ will be in violation of the rational option pricing bound and $C_0^*(t,n) \le \max(S_0 - Ke^{-rt}, 0)$.

The correction advanced by Duan and Simonato imposes the martingale property on the collection of the simulated sample paths at a sequence of time points, $t_1, t_2, \ldots, t_m$, and it is known as empirical martingale simulation (EMS).

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[20] Definitions 6.1 to 6.4 are originally from Duan and Simonato (1995) pp. 3-4.

Definition 6.3: *The EMS dynamic of the equity price for the i^{th} path at time t_j, is*

$$S_i^{**}(t_j,n) = S_0 \frac{Z_i(t_j,n)}{Z_0(t_j,n)} \tag{6.28}$$

where $Z_i(t_j,n)$ is a temporary equity price at time t_j, calculated as the ratio of the ith simulated equity prices $S_i^*(t_j,n)$ and $S_i^*(t_{j-1},n)$ at time t_j and t_{j-1}, before the EMS adjustment, as follows

$$Z_i(t_j,n) = S_i^{**}(t_{j-1},n)\frac{S_i^*(t_j,n)}{S_i^*(t_{j-1},n)} \tag{6.29}$$

and $Z_0(t_j,n)$ is the discounted sample average, updated at each ith simulation

$$Z_0(t_j,n) = \frac{1}{n}e^{-rt_j}\sum_{i=1}^{n} Z_i(t_j,n) \tag{6.30}$$

With (6.28), (6.29) and (6.30), the estimated discounted EMS equity price at time 0 is

$$\begin{aligned} S_0^{**}(t,n) &= \frac{1}{n}e^{-rt}\sum_{i=1}^{n} S_i^{**}(t,n) \\ &= S_0 \end{aligned} \tag{6.31}$$

for any n and $t \in \{t_1, t_2, \ldots, t_m\}$.

Definition 6.4: *The EMS call option price is denoted by $C_0^{**}(t,n)$, and it is approximated by*

$$C_0^{**}(t,n) = \frac{1}{n}e^{-rt}\sum_{i=1}^{n} \max[S_i^{**}(t,n) - K, 0] \tag{6.32}$$

and follows that $C_0^{**}(t,n) > \max[S_0 - Ke^{-rt}, 0] = \max[S_0^{**}(t,n) - Ke^{-rt}, 0]$ is guaranteed.

In summary, with the steps (6.28), (6.29) and (6.30), first the temporary equity price $Z_i(t_j,n)$ is created. Following this, the discounted average value $Z_0(t_j,n)$ is calculated and the equity price $S_i^{**}(t_j,n)$, for the next time step, is computed. The rational behind these steps is to apply a multiplicative adjustor to correct the first moment of $S_i^{**}(t_j,n)$, during each simulated time step. In this way, there will be no domain violation.

Duan and Simonato applied the EMS method to price Asian options under the Black-Scholes and GARCH frameworks and found substantial error reductions, particularly pronounced for in and at-the-money options. For out-the-money options, they concluded that the efficiency gain of the EMS method increases as the maturity increases. Clearly, this is very comforting, in part because as observed earlier, the EDS resembles a deep-out-the-money option, and secondly because when EDSs are used as part of the reference portfolio in synthetic CDOs, we expect to see them with average maturity between four to five years.[21]

6.3 Single-name EDS pricing model

EDSs are similar to CDSs as the protection buyer makes regular payments and receives a payment from the protection seller should a trigger event happen.

Definition 6.5: *We define the random time of default in a EDS as the time τ when the trigger event occurs*

$$\tau = \inf\{t > 0 : t \in [0,T], S_t \leq K\} \tag{6.33}$$

where S_t is an $\mathcal{F}_t$ −adapted equity price process defined in (6.6), K is some deterministic function, or percentage of S_0, called the trigger value. The trigger value K does not have the meaning of a safety covenant on the firm's debt, which in a structural credit risk model works as a protection mechanism for the bondholders against unsatisfactory corporate governance. We also share the same view as Fitch (2004), Moody's (2004) and S&P (2004), and disagree to look at the trigger event of 30% as the equity price drop that would also take the issuer to default its corporate bond obligations.

If the trigger event occurs, a fixed percentage $1-\varphi$ of $M = NS_0$ is paid to the protection buyer at the time τ, where φ is the recovery rate, N is the number of equities and M is the notional amount of the contract. The amount $M(1-\varphi)$ is called the default payment. As a matter of fact, the default payment may also be distributed over time, or at the maturity date T. However, for modelling purposes it suffices to consider recovery payment only at the default time τ, as other possibilities can be reduced to the above by means of forward or backward discounting.

We also indicate the risk-neutral distribution function of the random time τ, as

$$\mathbb{Q}(\tau \leq t) = F(t) \tag{6.34}$$

and denote of the occurrence of the trigger event with the indicator function

$$N(t) = 1_{\{\tau \leq t\}} \tag{6.35}$$

In this respect, the EDS valuation involves the knowledge of the distribution of τ, hence, it is not dissimilar to the valuation of the credit risk with the first passage model of Black and Cox (1976), with the important difference that S_t is now an observable variable. If it is not possible to

[21] Four to five years is the most liquid maturity for the CDS market.

infer the probability in (6.34) through an analytical expression, we would have to recur to Monte Carlo simulations.

At the payment dates t_j, with $j \in \{1,2,..,m\}$ and $t_m = T$, the protection buyer pays the regular payment fee of

$$s\delta_j 1_{\{t_j \le \tau\}} \qquad \text{at} \qquad t_j \tag{6.36}$$

to the protection seller. Here s denotes the EDS rate, δ_j is the day count fraction for the interval $[t_{j-1}, t_j]$ and $1_{\{t_j \le \tau\}}$ is the indicator function that the trigger event has not occurred before the payment date t_j.

Furthermore, at the time τ of the trigger event the protection buyer makes a final payment covering the time between the last payment date and the time τ of the trigger event. Let $j^* = \max\{j | t_j \le \tau\}$ be the last payment date before the time τ. Then the protection buyer pays the extra fee of

$$s\delta_{j^*} 1_{\{t_{j^*} \le \tau < t_j\}} \qquad \text{at} \qquad \tau \tag{6.37}$$

to the protection seller. Here δ_{j^*} is the day count fraction for the interval $[t_{j^*}, \tau]$. If we denote $V_P(t)$ as the value at time t of receiving 1bp of fee payments and no fees after the interval $[t_{j-1} \le \tau \le t_j]$, then we can write

$$V_P(t) = M\mathbb{E}^{\mathbb{Q}}\left(\sum_{j=1}^{m} B(0,t_j)\delta_j 1_{\{t_j \le \tau\}} + B(0,\tau)\delta_{j^*} 1_{\{t_{j^*} \le \tau < t_j\}} \middle| \mathcal{F}_t \right) \tag{6.38}$$

where $\mathbb{E}^{\mathbb{Q}}(\cdot)$ denotes the expectation operator under the risk-neutral measure $\mathbb{Q}$ and where $B(0,t)$ is the discount factor of the default-free interest rate, and it assumed a deterministic function.

If the trigger event occurs before the final payment date $t_m = T$, i.e. if $\tau \le t_m = T$, then the protection seller pays to the protection buyer the default amount $1-\varphi$. Thus, the payment is

$$(1-\varphi)1_{\{\tau < T\}} \qquad \text{at} \qquad \tau \tag{6.39}$$

We denote the value of the default leg at a time $t < t_{j-1}$ as

$$V_D(t) = M\mathbb{E}^{\mathbb{Q}}\left(B(0,\tau)(1-\varphi)1_{\{\tau < T\}} \middle| \mathcal{F}_t \right) \tag{6.40}$$

Given (6.38) and (6.40), the fair EDS premium rate at time t, is that rate at which the premium leg has the same value as the default leg

$$s = \frac{V_D(t)}{V_P(t)} \tag{6.41}$$

In section 6.2.4, we pointed out that, when S_t follows the risk-neutral GARCH process in (6.6), the probability of S_t being lower than the trigger value K, anytime before the EDS maturity T, for $T > t$, cannot be analytically calculated. As a consequence, we cannot analytically express the risk-neutral distribution function of the random time of default τ, $F(T)$. Hence, to calculate the fair EDS premium in (6.41), we have to resort to the EMS. More precisely, the EMS dynamics of S_t are in (6.28) where S_t follows the process in (6.6) and (6.7). When S_t strikes K, the default payment is triggered.

6.4 Baskets and CDOs of EDSs

Baskets and CDOs, where the collateral is represented by credits, were formally introduced in chapter four. When the collateral changes from corporate bonds or CDSs to EDSs, the payoff mechanics do not change.

In an *m* out of *n* EDS default, there is a default payment when the m^{th} EDS triggers, and the payout depends on the temporal ranking of the EDS triggers. In CDOs of EDSs, the payout is dependent on the percentiles of the loss distribution caused by EDS triggers. Hence, we do not see any substantial difference in the payout mechanics of baskets and CDOs, when the EDSs are added to their collateral. For this reason, we refer to chapter four for the formal introduction of these products.

However, we need to formally define the i trigger events. We consider n equity names, with associated random times of the trigger events $\tau_1, \tau_2, ..., \tau_n$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $N_i(T) = 1_{\{\tau_i \leq T\}}$ for $i = 1,..,n$ the indicator functions denoting of the occurrence of the trigger events.

Definition 6.6: *We define the i^{th} random time of default in a basket of EDSs as the time τ_i when the i^{th} trigger event occurs*

$$\tau_i = \inf\{t > 0 : t \in [0, T] : S_{i,t} \leq K_i\} \tag{6.42}$$

where $S_{i,t}$s are $\mathcal{F}_t$ – adapted equity price processes defined in (6.6), and K_i is the i^{th} trigger value.[22] The risk-neutral distribution function of the random time τ_i, is indicated as

$$\mathbb{Q}(\tau_i \leq t) = F_i(t) \tag{6.43}$$

The key determinant in the valuation of a basket or a CDO of EDSs is how to handle the dependence between the n random times of default τ_i, which, through (6.42), is the dependence between the n underlying equities. The pricing problem is not elementary, whenever the hypothesis of independence of perfect linear correlation between the underlying equities is dropped. It becomes more complex when the questionable assumption of joint normality is also abandoned. Copulae are the perfect tools to handle multivariate pricing cases, when joint normality is dropped.

6.5 CDOs of EDSs pricing model: copula model

In this section, we discuss an algorithm for multi event simulation. Simulation of successive $m \leq n$ dependent times of default τ_i is possible by generating a vector $(\varepsilon_1,..,\varepsilon_n)$ of n-random numbers from one of the copulae specified in chapter three. The EMS dynamics for $(S_{1,t},..,S_{n,t})$ are then simulated and when one of the equity prices strikes its trigger value, the default payment is triggered.

In our model, we want to generate two vectors $(\varepsilon_1,..,\varepsilon_n)$ of dependent random numbers, one where dependent random numbers are t-Student distributed, $\varepsilon_{i,t}|\mathcal{F}_{t-1} \sim t(0,1;\upsilon)$, and one where they are normal distributed, $\varepsilon_{i,t}|\mathcal{F}_{t-1} \sim N(0,1)$. To do so, we rely upon the following algorithm.

(1) Initialize a vector $(u_1,u_2,..,u_n)$ containing a set of independent standard uniform random numbers $U \sim (0,1)$, where n is the number of EDSs in the CDO.

(2) To induce dependency we distinguish whether Elliptical copulae or Clayton copula are used.

(3) With Elliptical copulae, we distinguish the Normal copula from the t-Student copula.

(3.1) If we use the Normal copula, we generate a vector $H = \left(\Phi^{-1}(u_1),..,\Phi^{-1}(u_n)\right)$ of independent standard normal distributed numbers.

(3.2) If we use the t-Student copula, we generate a vector of $H = \left(t_\upsilon^{-1}(u_1),..,t_\upsilon^{-1}(u_n)\right)$ of independent t-Student distributed numbers, with υ degrees of freedom.

(3.3) On the elements of the vector H, we apply the Cholesky factorisation and obtain a vector G of correlated elements.

[22] The valuation of baskets and CDOs of EDSs is done with respect to the reference filtration $\mathcal{F}_t$.

(3.4) A vector $(v, v_2,.., v_n)$ of copula dependent numbers is calculated as either $C_{\Sigma}^{N}(v_1,..,v_n) = (\Phi(g_1),..,\Phi(g_n))$, or as $C_{\upsilon,\Sigma}^{t}(v_1,..,v_n) = (t_{\upsilon}(g_1),..,t_{\upsilon}(g_n))$.

(4) With the Clayton copula, the vector $(u_1, u_2,.., u_n)$ is used, as explained in chapter three, section 3.5.6, to generate a vector $(v, v_2,.., v_n)$ of copula dependent numbers.

(5) The vector $(\varepsilon_1,..,\varepsilon_n)$ is calculated as $(t_{\upsilon}^{-1}(v_1),...,t_{\upsilon}^{-1}(v_n))$ if we assume $\varepsilon_{i,t}|\mathcal{F}_{t-1} \sim t(0,1;\upsilon)$, or as $(\Phi^{-1}(v_1),...,\Phi^{-1}(v_n))$ if we assume $\varepsilon_{i,t}|\mathcal{F}_{t-1} \sim N(0,1)$.

(6) and so on for the next time step and sample path.

6.6 Numerical exercise

As a comparison of the copula models introduced in the previous sections, we consider a numerical example where we price and calculate the ratings of a real-life CDO of EDSs.

6.6.1 Contract description

The reference portfolio consists of five EDSs, referenced to five equities. In Table 6.1 we show the equity names, prices, and the EDS notionals. The CDO liability consists of three notes, with a contractual maturity of 2 years, and with the following notionals.

(1) Junior note notional is of £3.534(mil.), which provides the first loss protection.

(2) Mezzanine note notional of £5.5(mil.), which suffers a principal loss if the junior note is fully utilised.

(3) Senior note notional of £97.5(mil.), which suffers a principal loss if the junior and mezzanine notes are fully utilised.[23]

6.6.2 Ratings

We also aim to compute the final rating of the notes as follows: after calculating their expected loss (EL), we map this value to the Moody's idealised cumulative expected losses in Table 2.8. For example, a note with an average life (AL) of 4 years should have an EL rate no greater than 0.1 bps to be rated Aaa. To calculate the EL, in each Monte Carlo run we produce the time and amount of losses and discount them back to the present time, using a constant Libor of 5%.

[23] To simplify the numerical exercise, we assume no equity piece.

A disadvantage of using the EL methodology, is that it is not sensitive to the likelihoods of losses in excess of certain confidence level ℓ. For this reason, we calculate the expected shortfalls, as explained in chapter four. It is still unclear to us how to use the expected shortfall as a substitute of the EL to calculate the rating. Nevertheless, we believe this to be a very important tool to compare the quality of the models we will analyse, when used for rating CDOs.

There is always the risk to calculate ratings that are more beneficial by using generous assumptions for the correlation or a different copula model. In the numerical example that follows, we will show the effect of correlation, and the application of different copulae on the final price and rating.[24] All Tables with the numerical exercise results are in the Appendix 9.

6.6.3 Empirical results

Our data set consists of the daily equity prices of British Airways, Royal Bank of Scotland (or Rbs), Shell, Tesco and Barclays and the 1 month Libor rate.[25] The estimation data spans from 4/1/00 to 14/1/05, with 1,274 observations. Table 6.2 presents key statistics of the raw data of daily returns. Skewness and kurtosis are clearly observed.

Table 6.3 shows the parameter estimates obtained by maximising the log-likelihood function in (6.20) for the normal-NGARCH-M, whereas Table 6.4 reports the parameter estimates obtained with the log-likelihood function in (6.21) for the *t*-NGARCH-M. All parameters are statistically significant, with the exception of λ of the equity returns of British Airways and Shell. As a consequence, in both British Airways' and Shell's GARCH models, λ is set to zero.

The correlation parameters θ_i are substantially positive indicating that shocks to returns and volatility are indeed strongly negatively correlated.

Looking at the standardised residuals of the normal-NGARCH-M model in Tables 6.3, we observe that the residual distribution is leptokurtic (see $Kurt(\varepsilon^*)$), in addition to skewed (see $Skew(\varepsilon^*)$). This is evidence of a misspecification of the normal-NGARCH-M model. The *t*-NGARCH-M model in Table 6.4, with the extra parameter of the degrees of freedom υ, explains better the standardised residuals, which remain leptokurtic.

The normal-NGARCH-M and *t*-NGARCH-M models have heavy-tail behaviour under the data generating probability measure $\mathbb{P}$, and remains heavy-tail under the risk-neutral measure $\mathbb{Q}$: with the exception of the equity returns of Rbs, all $\kappa(\zeta^*)$ and $\kappa(\zeta)$ are larger than 3.

[24] When calculating the price and rating, we will not take into account the extra fee that the protection buyer would pay at the time of default indicated in (6.38).

[25] With the following Bloomberg Tickers: BAY LN Equity, RBS LN Equity, SHEL LN Equity, TSCO LN Equity, BARC LN Equity and BP0001M Index.

Consequently, the four equity returns, R_t in (6.1), will be more heavy-tailed than the normal distribution.

6.6.4 Our benchmark: Rubinstein and Reiner model

Instead of assuming that the equity returns behave accordingly to a NGARCH-M process, we assume they follow the standard homoskedastic lognormal process of the Black & Scholes model. Under this assumption, we suggest to price EDSs with the analytic solution for path-dependent binary barrier options of Rubinstein and Reiner (1991). In particular, we propose to treat the EDS default leg in (6.40) as a down-and-out put option, with the strike equal to zero, the option barrier equal to the EDS trigger value K, and the rebate[26] equal to the EDS recovery rate $1-\varphi$. Next, we assume that the EDS default occurs at a constant intensity rate and from this, we recover the EDS payment leg in (6.38) and the EDS premium. In other words, we assume that the premium of the path-dependent binary barrier option is paid over the EDS life as an annuity, rather than at start in one instalment.

For the Rubinstein and Reiner model, the only input that is unobservable is the future volatility of the underlying equity. The way the market determines this volatility is to select the implied volatility of a standard put option. The problem we faced with this approach was that the implied volatilities for deep out-of-the-money put options were not available for all five equities. Table 6.5 shows what we were able to collect from Bloomberg. Only Shell shows a put option with maturity close to two years. Besides, none of the put options has a strike close to the EDS trigger of 30%. Last important point is that all implied volatilities of Tables 6.5 are substantially lower than the annualised long run volatilities σ^{Q} and σ^{P} of Tables 6.3 and 6.4. Hence, we expect large differences between the EDS premiums calculated with the Rubinstein and Reiner model and the same premiums calculated with the NGARCH-M models.

6.6.5 Model inputs

In the numerical exercise, we simulate daily returns from the two NGARCH-M models. Simulations are performed with 1,000 sample paths and with daily time steps, under the assumption of 256 trading days per year. The interest rate is assumed constant at 5%.

We test how sensitive the single-name EDS premiums are to the level of kurtosis in the return process model by modelling with two NGARCH-M processes: Normal and *t*-Student distributed.

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[26] We also assume that the rebate is paid when the barrier is hit.

For all three notes, we calculate the premium, the EL, the standard errors of the EL, the rating and the expected shortfalls with a confidence level of 99%. The random numbers ε, in (6.2) and (6.7), are joint with the Normal, *t*-Student and Clayton copulae.

With the Elliptical copulae, we use three values for correlation: 0, 0.25 and 0.50. With the *t*-Student copula we use two values for the degrees of freedom υ: 6 and 10. The Clayton copula has the property of generating strong dependency in the lower tail of the distribution, hence generating more dependent negative news. With the Clayton copula, we use two values for a: 1.5 and 3. These values will translate into strong global and lower tail dependence.

In no circumstances we expect the three copulae to compute different results when correlation is 0 and when the Clayton copula parameter $a = 0.001$. For this reason, we only use correlation of 0 with the Normal copula.

6.6.6 Single-name EDS premiums

Tables 6.6 and 6.7 shows the EDS premiums as calculated with the three models. The Rubinstein and Reiner model is by far the less conservative and estimates a total expected loss (EL) of £13k for the five EDSs. Only one EDS has a premium substantially greater than zero, British Airways.

The differences between the EDS premiums, as calculated with the Rubinstein and Reiner model and with the *t*-NGARCH-M model, are indeed very large. A zero premium clearly indicates that there is zero probability of trigger a default payment, but the *t*-NGARCH-M model strongly rejects this conclusion. In fact, the *t*-NGARCH-M model calculates premiums all greater than zero. We raised earlier the possibility of finding those large differences. We also explained this with the lack of implied volatility data for the very deep out-of-the-money put options, under investigation.

To expand the last point, we have compared the three models in a pure equity option environment, and priced five path-dependent binary barrier put options written on the five equities of Table 6.5. To do so, we have set the strike equal to zero, the option barrier equal to the EDS trigger value K, the rebate[27] equal to the EDS recovery rate $1-\varphi$, and a maturity of two years. Table 6.6 shows these results with 10,000 simulation paths. In Table 6.7, we have calculated the implied volatility that the Rubinstein and Reiner model requires as inputs, to calculate the same option premiums as the *t*-NGARCH-M model. They range from 31% of Barclays to 46% of Rbs. In our experience, those implied volatilities look like the natural extrapolation of the smile to price very deep out-of-the-money put options and path-dependent binary barrier put options.

We can also notice differences between the ELs of £915k and 113k in Table 6.6 and the option premiums in of £1,096k and £122k in Table 6.7. In the EDS models of Table 6.6, we do not take into account the extra fee that the protection buyer would pay at the time of default indicated in

[27] We also assume that the rebate is paid when the barrier is hit.

(6.38), and assume that defaults pay at the end of quarters. These simplifications have opposite effect on the EDS premium, but since the latter is stronger, overall the EDS premium results reduced. In the models of Table 6.7, we assume that the rebate is paid on the day the barrier is hit. In summary, we predicted that the ELs of Table 6.6 would be lower than the option premiums of Table 6.7, and this is proven to be the case.

A further point is that Rbs has the largest EDS premium when calculated with the *t*-NGARCH-M model, the largest annualised long run volatilities σ^{Q} of 51.31% (Table 6.4), and surprisingly, the lowest implied volatility in Table 6.5 (17.25%). The market seems to have forgotten that this equity once was a very volatile one.

Last point to mention is that, Table 6.6 can be seen as a basket of EDSs, where the equity returns are assumed zero correlated. Under this assumption, we can draw important information regarding the marginal contribution of each single EDS to the basket premium. For example, the EL of £915k (or 0.86% of £106mil.) is very close to the EL calculated as a weighted average of the three notes ELs of 11.121% (Table 6.10), 5.583% (Table 6.11) and 0.198% (Table 6.12), where the weights are the notes notionals.[28]

6.6.7 *t*-NGARCH-M and normal-NGARCH-M

In Tables 6.8 to 6.13 the copula results are presented. Tables 6.8 to 6.10 show the results of the normal-NGARCH-M model $\left(\varepsilon|\mathcal{F} \sim N(0,1)\right)$, whereas Tables 6.11 to 6.13 contain the numerical results of the *t*-NGARCH-M model, $\left(\varepsilon|\mathcal{F} \sim t(0,1;\upsilon)\right)$, where the degrees of freedom υ are as estimated in Tables 6.4. When the degree of freedom is equal to infinity, the residuals ε are normally distributed. In this way, we can measure the effect of moving away from the general assumption of normally distributed residuals ε.

The main conclusion we can draw is that the normal-NGARCH-M, when compared with the *t*-NGARCH-M, hugely underestimates the fair compensation and the risk of the senior, mezzanine and junior notes. The senior note premiums and ELs calculated with *t*-NGARCH-M, are up to ten times larger than those calculated with normal-NGARCH-M. The rating of the mezzanine note, with the *t*-NGARCH-M, is never above B1, whereas with the normal-NGARCH-M, it never drops below Ba1. The same pattern can also be observed for the junior notes.

In section 6.6.3, we also pointed that *t*-NGARCH-M model fits better the data set of the five equities. For this reason, in what follows, we will only comment the results obtained with the *t*-NGARCH-M model.

[28] The difference is due to the different results that each Monte Carlo run returns. In Table 6.7 we used 10,000 simulated paths, whereas in Table 6.6 the simulated paths were 1,000.

6.6.8 Correlation

As correlation increases and we model with the Normal copula and the *t*-Student copula with 10 d.o.f., the premium and the EL of the three notes drop. We also notice an increase in the expected shortfall of the senior note and this signals a much riskier note. However, this does not translate into a change of its rating as the EL did not rise.

When we use the *t*-Student copula with 6 d.o.f., we can finally observe the redistribution of the premium and of the EL from the junior and the mezzanine notes to the senior note.

In summary, as correlation increases, we cannot always observe the redistribution of the premium and of the EL from the junior note to the senior note. A high correlation has the effect of increasing the probability of joint defaults and this is reflected through an increase of the expected shortfalls. The way correlation is felt through the notes premiums, rather depends on the ranking of the note in terms of loss allocation, and whether the *t*-Student copula is used and with what d.o.f. value.

All this has important consequences for a trader who is pricing this deal with different correlation assumptions. If he mistakenly only uses the Normal copula, he would find beneficial to increase the correlation.

The consequences for a structurer are as much as important. If he mistakenly only uses the Normal copula and the EL methodology to calculate the ratings of the CDO notes, he would observe a drop in premiums and thus a potential drop in the note rating. However, he would see a different deal if he were to model the deal with the *t*-Student copula with 6 d.o.f. and analyse the rating with the expected shortfall. This clearly points out that the EL methodology cannot be the correct approach for rating CDO of EDSs.

6.6.9 Normal copula and *t*-Student copula

With the *t*-Student copula, we have the possibility of using the extra parameter d.o.f. to generate higher defaults. As d.o.f. drop, the protection that the junior note offers to the mezzanine and senior notes diminishes. Therefore, a larger premium is required for the mezzanine and senior notes to offset a larger risk. As expected, the results of the Normal copula and the *t*-Student copula further diverge as correlation rises. The larger differences are with the premium and the EL of the junior and the mezzanine notes.

6.6.10 Clayton copula

With the Clayton copula, we rely upon the only available parameter a, to capture both the lower tail dependence λ_L and the Kendall's ρ_τ global dependence.

To make understanding the results of the Clayton copula easier, we have added, in all Tables, the lower tail dependence λ_L and Kendall's ρ_τ implied in the parameter a. For example, where $a = 1.5$, we can reach a Kendall's $\rho_\tau = 0.43$ and tail dependence $\lambda_L = 0.63$, hence, modelling with much stronger dependency than used in the *t*-Student copula with 6 d.o.f.

The Clayton copula has brilliantly responded and met our expectation. In fact, the Tables show that as a moves from 1.5 to 3, the premium and of the EL are redistributed from the junior and the mezzanine notes to the senior note. It is also important mentioning there is a much larger increase of the notes expected shortfalls. The senior note expected shortfall with $a = 1.5$, is 20.24%, which is twice as much as the expected shortfall calculated with any Elliptical copula.

In terms of how the premiums of Clayton copula compare with those of the Elliptical copulae, the Clayton overprices the senior note and underprices the junior and the mezzanine notes.

It is also worth mentioning that we found the algorithm to generate Clayton copula dependent random numbers as the fastest of those used in our numerical exercise.

6.7 Conclusion

In this chapter, we extended the GARCH option framework of Duan (1995) to price single-name EDSs, and to price and calculate the rating of CDOs of EDSs.

Volatility of the underlying equity price is the critical factor affecting option prices. Because EDS resembles deep-out-of-the money equity digital options, the correct modelling of volatility is vital to price EDSs. In our EDS model, the variance of the equity return follows a nonlinear GARCH in mean. Stochastic volatility models are alternative methods to GARCH to model the time-varying nature of the equity return volatility. It is in the estimation exercise that GARCH models have a distinct advantage over stochastic volatility models. The volatility of the equity prices is not easily observable, and it has to be *implied* from current option prices. This causes the parameter estimation for stochastic volatility option pricing models to be demanding and problematic. Option pricing applications under GARCH benefit from the fact that they are relatively easy to re-estimate. Furthermore, they only need a finite number of parameters to be estimated regardless of the length of the time series. For all these reasons, we chose the GARCH representation of the time-varying volatility feature of equity returns to price EDSs and CDOs of EDSs.

When the volatility is modelled as a GARCH process, it is generally not possible to derive the future distribution of the underlying equity. Thus, no analytical formula exists, and alternative numerical methods, such as Monte Carlo simulation or numerical trees, must be used. Trinomial trees could be useful only for the single-name EDS. Since our research deals with CDO of EDSs, the numerical trees will soon become impractical as the number of EDSs in the CDO grows. Hence, our model relied on Monte Carlo simulations.

Duan and Simonato (1995) highlighted a less known problem of Monte Carlo methods and it is that the simulated option price violates rational option pricing bounds and hence it is not a sensible price estimate. As a solution, we used the empirical martingale simulation (EMS) originally advanced by Duan and Simonato. Monte Carlo methods are typically computer intensive. For this reason, we coupled the EMS with the standard variance reduction technique.

When pricing single-name EDS, we proposed two nonlinear GARCH in mean (NGARCH-M): normal and *t*-Student NGARCH-M model. Our set-up can accommodate most observed empirical regularities of the equity prices such as, fat tails, volatility clustering and time-varying variances. It also allows correlation between the conditional variance and the equity returns. This is especially useful considering the existence of a negative correlation with many equity returns (see Black (1976) and Christie (1982)).

As a benchmark for the NGARCH-M models, we assumed that the equity returns move accordingly to the standard homoskedastic lognormal process of Black & Scholes and priced the single-name EDS with the Rubinstein and Reiner model for binary barrier options. For Black & Scholes models, the way the market determines the future volatility is to select the implied volatility of the option. The problem we found with this approach was that the implied volatilities for very deep out-of-the-money put options were not available.

To address the issue of how to price a basket of EDSs, we resorted to the concept of copula. With the copula, we were able to decouple the pricing problem: keeping the aspect of modelling the marginal distribution of the equity returns *via* NGARCH-M, totally separate from addressing the dependence problem.

A new feature of our model is that, contrary to earlier works on pricing multivariate options, the dependence structure is not necessarily multivariate normal, rather it has the flexibility of being expressed as Normal, *t*-Student and Clayton copulae.

As an example of the power of our model, we presented a real-life CDO of EDSs: five EDSs were used as a reference portfolio to issue three CDO notes. The CDO notes were priced and rated using three copula functions: Normal, *t*-Student and Clayton copulae. The marginal distributions of the five EDSs were also empirically presented and we found evidence of non-normality in the equity returns.

Within the same numerical exercise, we showed the limitation of the Black & Scholes framework. The option market lacks implied volatility data, for very deep out-of-the-money put options, with a maturity of 2 years or longer. Hence, we used implied volatilities of put options with different strikes and shorter maturity than we had originally searched for. We also compared these implied volatilities with the annualised long run variances implied in the NGARCH-M processes and found large differences. Then, we priced EDSs with the Rubinstein and Reiner model, and found all premiums substantially lower than those calculated with the NGARCH-M models. As final point, we found evidence that the EL methodology is not the correct approach for rating CDO of EDSs.

Chapter 7: Conclusions

In this research, we have examined many numerical issues relating to the modelling of correlation products with copulae. The results are summarised in this chapter.

7.1 Results of chapter 4

Throughout chapter 4, we searched for answers to the following questions. What happens to the premiums of correlation products when the default time correlation changes? What happens to the premiums when the times of default are joint with the *t*-Student copula or when we replace the *t*-Student copula with the Clayton copula? What happens to the premiums when we include the dynamics of the Clayton copula.

The size of the reference portfolio is one of the key elements when analysing the sensitivity of the premiums to changes in the default time correlation. We found that this relationship could be generalised with saying that the 1st to default is always long in default time correlation. As we moved up in the time ranking of the credit defaults (2nd, 3rd, 4th and so on) the relationship become more complex and depended on the number of credits.

Our numerical results of section 4.5 contradicted the results and conclusions of Mashal, Naldi and Zevi when modelling 1st to default into the 2nd and the 3rd to default with the *t*-Student copula. We did not find the same redistribution of losses from the 1st to default into the 2nd and the 3rd to default, when the Normal copula is changed with the *t*-Student copula. Our results showed a general increase of their premiums. We also found that the Normal, *t*-Student and Clayton copulae calculated the same premiums when the times of default were not correlated or dependent, which was not the case in Mashal, Naldi and Zevi.

Another important component of the relationship between premium and default time correlation is the complexity of the structure of the correlation product. High default time correlation has the effect of increasing the probability of joint defaults. The effect on the premiums depends on the ranking of the note in terms of loss allocation. In principle, as correlation rises, the protection that the Junior note offers to the Mezzanine and Senior notes diminishes. Therefore, a larger premium is required for the Mezzanine and Senior notes to offset a greater risk. With the structure analysed in section 4.6, the Junior note was long in default time correlation, whereas the Senior note was short. The Mezzanine note behaved more as a Senior note, when the default time correlation changed.

The Clayton copula had important consequences on all three CDO notes in section 4.6. As a rose, we noticed a redistribution of losses from the Junior note to the Mezzanine note and the Senior note. When compared with the Clayton copula, we found that the Normal copula underestimated the fair compensation of the Senior and the Mezzanine notes and overestimated the fair compensation of the Junior note. More importantly, the Clayton copula picked up some extra risk in the Senior note. This was confirmed by an increase of the expected shortfalls of the Senior note.

These findings contradicted the work of Meneguzzo and Vecchiato on modelling default times with the Clayton copula. They used the Clayton copula to simulate the exact default time. In this way, they simulated too many survival times and fewer default times. We preferred simulating the quarters when default occurred. In this way, we were able to use the Clayton copula and capture the lower tail dependence.

In order to show that the default mechanism implied with the Li's Survival model is different from the Schönbucher and Schubert dynamic copula, we compared these models with a Clayton copula, *via* pricing the same CDO notes. We assumed the same time-inhomogeneous intensity rates for the underlying credits and found much larger losses with the dynamic Clayton copula. This raised a red alarm, as the current approach in the financial industry is to rely on static copula models.

7.2 Results of chapter 5

This chapter presented a new model to value the notes of cash-flow CDOs. The model uses both copulae and time-inhomogeneous transition matrices. It is also flexible enough to allow any type of cash-flow waterfall. We called it Rating Transition Copula.

Since the computation of the transition matrices for arbitrary periods of time is based on an annual transition matrix, this has not been an easy task. Most of the empirical annual transition matrices are not compatible with a continuous Markov process since they do not admit a valid generator. Therefore, we computed a modified version of a true generator. The modified generator was extracted using the IRW algorithm. By doing this, we calculated the error, which occurred when substituting the *starting* matrix, with an *adjusted* matrix.

Following this, we described and applied three methods, JLT (1997) and Lando (1998a), to calibrate the adjusted matrix to the S&P's probabilities of default. Only with one method we managed to obtain sixteen calibrated transition matrices covering a period of sixteen quarters. The others methods were unsuccessful.

Finally, we described how to simulate the credit rating migration of one single credit, and how to join n -credit rating migrations *via* the Normal copula.

Modelling the collateral credit risk in this way is very powerful, since it allows us to take into account any quality trigger linked to the performance of the collateral: for example no interest is paid to the Equity if the average rating of the collateral drops below a certain rating.

The numerical section provided an example of a cash-flow CDO that was priced and analysed with the RT-Copula model. The RT-Copula results are also comparable with those calculated using the Survival and Merton models, and we concluded that, where there is no performance trigger linked to the collateral average rating, there is no pricing or rating arbitrage between the three models. When there are performance trigger linked to the collateral average rating, such as the Collateral Tests A and B, the RT-Copula model perfectly captures the diversion of cash from the interest waterfall to the principal waterfall for the benefit of the Senior and Mezzanine notes. Thus, there is evidence to conclude that the RT-Copula model is the correct approach to model cash-flow CDOs.

7.3 Results of chapter 6

7.3.1 Summary

In chapter 6, we extended the GARCH option framework of Duan (1995) to price single-name equity default swaps, and to price and calculate the rating of CDOs of equity default swaps.

Volatility of the underlying equity price is the critical factor affecting option prices and in our EDS model, the variance of the equity return followed a nonlinear GARCH in mean. When the volatility was modelled as a GARCH process, it was not possible to derive the future distribution of the underlying equity. Therefore, our model relied on Monte Carlo simulations. To ensure that the simulated option price did not violate rational option pricing bounds, we used the empirical martingale simulation originally advanced by Duan and Simonato, coupled with the standard variance reduction technique.

When pricing single-name EDS, we proposed two nonlinear GARCH in mean (NGARCH-M): normal and *t*-Student NGARCH-M model. As a benchmark, we assumed that the equity returns moved accordingly to the standard homoskedastic lognormal process of Black & Scholes and priced the single-name EDS with the Rubinstein and Reiner model for binary barrier options. The problem we found with this approach was that the implied volatilities for very deep out-of-the-money put options were not available.

To address the issue of how to price a basket of EDSs, we resorted to the concept of copula. With copulae, we were able to decouple the pricing problem: keeping the aspect of modelling the marginal distribution of the equity returns *via* NGARCH-M, totally separate from addressing the dependence problem.

A new feature of our model is that, contrary to earlier works on pricing multivariate options, the dependence structure is not necessarily multivariate normal, rather it has the flexibility of being expressed as Normal, *t*-Student and Clayton copulae.

The numerical findings of chapter 6 can be further broken down in three main areas: empirical estimation results, single-name EDS premiums, and copula results.

7.3.2 Empirical estimation

The empirical estimation carried on by maximising the log-likelihood functions of the normal-NGARCH-M and the *t*-NGARCH-M, suggested that the *t*-NGARCH-M with the extra parameter of the degrees of freedom υ, fits better the data set. Its parameters are statistically significant, with the exception of the parameter of the risk premium of the equity returns of British Airways and Shell. The correlation parameters are also substantially positive indicating that shocks to returns and volatility are indeed strongly negatively correlated.

7.3.3 Single-name EDS premium

After completing the empirical estimations, we moved to calculating the single-name EDS premiums with the three models made available: Rubinstein and Reiner model, normal-NGARCH-M model and the *t*-NGARCH-M model.

We were not surprised to find out that the Rubinstein and Reiner model was by far the less conservative of the three models: only British Airways EDS premium was substantially greater than zero. Besides, the differences between the EDS premiums as calculated with the Rubinstein and Reiner model and with the *t*-NGARCH-M model were indeed very large. The possibility of finding these large differences was expected and explained with the lack of implied volatility data for the very deep out-of-the-money put options, under investigation.

We also calculated the implied volatility that the Rubinstein and Reiner model required as inputs, to calculate the same option premiums as the *t*-NGARCH-M model. They ranged from 31% of Barclays to 46% of Rbs. Besides, they looked like the natural extrapolation of the smile to price very deep out-of-the-money put options and path-dependent binary barrier put options.

As last point, we analysed the marginal contribution of each single-name EDS to the basket premium. This was done by assuming a basket of EDSs, where the equity returns were uncorrelated.

7.3.4 Copula results

When we moved to analysing the CDO, we found that the normal-NGARCH-M, when compared with the *t*-NGARCH-M, hugely underestimated the fair compensation and the risk of the senior, mezzanine and junior notes.

As correlation increased, we could not always observe the redistribution of the premium and of the EL from the junior note to the senior note. All this has important consequences for a trader who is pricing a deal with different correlation assumptions. If he mistakenly only uses the Normal copula, he would find beneficial to increase the correlation. The consequences for a structurer are as much as important. If he mistakenly only uses the Normal copula and the EL methodology to calculate the ratings of the CDO notes, he would observe a drop in premiums and thus a potential drop in the note rating. However, he would see a different deal if he were to model it with the *t*-Student copula with 6 d.o.f. and analyse the rating with the expected shortfall. This clearly points out that the EL methodology cannot be the correct approach for rating CDO of EDSs.

With the *t*-Student copula, we were able to generate more defaults by reducing the d.o.f.. When this happened, the protection that the junior note offered to the mezzanine and senior notes diminished, therefore a larger premium was required for the mezzanine and senior notes to offset a larger risk. The results of the Normal copula and the *t*-Student copula further diverged as correlation rose.

The Clayton copula brilliantly responded and met our expectation. In fact, all tables showed that, as a increased, the premium and the EL were redistributed from the junior and the mezzanine notes to the senior note. In terms of how the premiums of Clayton copula compared with those of the Elliptical copulae, the Clayton overpriced the senior note and underpriced the junior and the mezzanine notes. As last point, we found the algorithm to generate Clayton copula dependent random numbers, the fastest of those used in our numerical exercise.

7.4 Recommendations for further research

This thesis has provided with new insights into modelling correlation products with copulae, but it has also raised many new questions. The following areas are recommended for additional research.

Firstly, in our research we have not covered statistical inference for copulae, which deals with application to market data, such as calibration to actual risk factors co-movements and VAR measurements. We have established the flexibility of the Clayton copula for modelling correlation products. However, we do not know how the Clayton copula would fit to an actual sample of credit data. A potential problem is the lack of data on which to base this exercise. In any empirical exercise, a choice will have to be made with regard to which data series produces the best estimate for default probability: historical defaults, equity returns, asset returns, and credit spreads. The market consensus is that equity information is the most practical approach. Nevertheless, increasing liquidity in the CDS market may provide an interesting alternative and let us use credit spreads. We also observe that CDSs are used to delta hedge positions in correlation products. In turn, correlation products are priced using equity correlations and hedged using CDSs. Therefore, there may be some mismatch in delta hedging correlation products over their life, which a copula empirical exercise, based on both equity prices and CDS premiums, may find true and possible to exploit for arbitrage.

Secondly, we have modelled the same real-life CDO with Li's Survival Clayton copula and Schönbucher and Schubert dynamic Clayton copula. We found large differences in the prices of the notes when calculated with the two models. Hence, we have reason to believe that once copula dynamics are included, correlation products become riskier and as compensation, would require larger premiums. We see two further extensions of our research. Firstly, we welcome an empirical study to calibrate dynamic copulae to market credit spreads. Secondly, we would compare different dynamic copulae *via* pricing the same real-life correlation products. Elliptical copulae do not simplify into nice closed-forms, hence the challenge is how to efficiently evaluate high-dimensional cumulative distribution functions (Normal and *t*-Student).

Thirdly, it would be of interest to extend the work on transition matrices and cash-flow CDOs of chapter 5, and to prepare a risk-neutral framework. The transition matrices used in our model were calibrated to historical default probabilities. Because modern financial theory requires that hedging and pricing of option type products takes place under a risk-neutral valuation framework, our model did not produce risk-neutral prices. We foresee one easy extension of our model: calibrating the default probabilities to those probabilities implied in the CDS premiums, so to be capable of calculating the rating and the price of the CDO notes at the same time.

Fourthly, in chapter 5 we restricted our investigation on using a publicly available rating agency transition matrix. We motivated our choice, saying that we wanted to locate our work in the area of rating structure finance bonds, and to raise some concern regarding the current approach of rating agencies for rating cash-flow CDOs. One important extension of chapter 5 is to estimate the generator or generators directly, using the original data set. Such approach is known as continuous-time approach and was taken by Christensen, Hansen and Lando (2002). In this way, firstly, we would find evidence for non-Markovian behaviours of the rating process, and secondly, we would test how non-Markovian behaviours influence the price and rating of cash-flow CDOs.

Next, we have proposed a new model to price baskets and CDOs of EDSs. It is still too soon to see whether modelling past volatility as a NGARCH-M for EDSs is indeed correct, as this product is still in its infancy. Therefore, an empirical investigation of modelling EDSs must be postponed until the market develops and more data becomes available.

Lastly, we are very sceptical that the EL methodology is the correct approach when rating CDOs of EDSs notes and welcome any future work, which compares VAR measurements with EL when rating CDOs of EDSs notes.

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Appendices

Appendix 1: Chapter 4 - Numerical exercise in section 4.3

In this appendix, we illustrate the results of the Marshall-Olkin exponential copula applied to the numerical example in section 4.3.

	Analytic			
	$\lambda^{Q}(t)$			Value of 1st TD
No. Credits	Corr = 0	Corr = 0.25	Corr = 0.5	Corr = 0
1	10%	10%	10%	0.1648
2	20%	16%	13%	0.3008
3	30%	22%	17%	0.4130
4	40%	28%	20%	0.5057
5	50%	34%	23%	0.5823
6	60%	40%	27%	0.6458
7	70%	46%	30%	0.6983
8	80%	52%	33%	0.7420
9	90%	58%	37%	0.7782
10	100%	64%	40%	0.8084
15	150%	94%	57%	0.8993
20	200%	124%	73%	0.9381
25	250%	154%	90%	0.9562
30	300%	184%	107%	0.9658
35	350%	214%	123%	0.9715
40	400%	244%	140%	0.9753
45	450%	274%	157%	0.9782
50	500%	304%	173%	0.9804

Table 4.2: Portfolio Intensity rates of the 1st to default $\lambda^{Q}(t)$ with default time correlation 0 - 0.25 and 0.5, with number of assets from 1 to 50; and analytic value of the 1st to default with default time correlation of 0.

	1st to Default Corr = 0.25			
	Analytic	Marshall-Olkin copula		
No. Credits	Value	Value	SE	Time (sec.)
1	0.1648	0.1652	0.000036	8
2	0.2495	0.2455	0.000041	13
3	0.3250	0.3275	0.000044	18
4	0.3922	0.3867	0.000071	22
5	0.4522	0.4522	0.000093	27
6	0.5057	0.4998	0.000113	32
7	0.5534	0.5481	0.000132	37
8	0.5960	0.5876	0.000150	42
9	0.6340	0.6306	0.000166	47
10	0.6680	0.6695	0.000183	52
15	0.7909	0.7912	0.000198	77
20	0.8619	0.8561	0.000214	102
25	0.9037	0.8992	0.000232	127
30	0.9289	0.9303	0.000249	151
35	0.9445	0.9455	0.000266	177
40	0.9547	0.9557	0.000283	203
45	0.9615	0.9613	0.000298	229
50	0.9663	0.9665	0.000314	256

Table 4.3: 1st to default Analytic values and Marshall-Olkin values, with standard errors (SE) and computational time, with default time correlation equal to 0.25 and 10,000 sample paths.

	1st to Default Correlation = 0.50			
	Analytic	Marshall-Olkin copula		
No. Credits	Value	Value	SE	Time (sec.)
1	0.1648	0.1672	0.000036	7
2	0.2131	0.2134	0.000039	12
3	0.2583	0.2550	0.000041	17
4	0.3008	0.2991	0.000065	22
5	0.3406	0.3431	0.000084	27
6	0.3780	0.3784	0.000101	32
7	0.4130	0.4179	0.000117	37
8	0.4459	0.4469	0.000133	42
9	0.4767	0.4739	0.000147	47
10	0.5057	0.5087	0.000162	51
15	0.6259	0.6253	0.000175	77
20	0.7138	0.7124	0.000190	101
25	0.7782	0.7741	0.000206	127
30	0.8256	0.8228	0.000222	151
35	0.8607	0.8698	0.000238	176
40	0.8869	0.8866	0.000255	201
45	0.9065	0.9066	0.000270	227
50	0.9213	0.9200	0.000286	252

Table 4.4: 1st to default Analytic values and Marshall-Olkin values, with standard errors (SE) and computational time, with default time correlation equal to 0.5 and 10,000 sample paths.

	Marshall-Olkin copula with corr. = 0.25			
	2nd to Default		3rd to Default	
No. Credits	Value	SE	Value	SE
1	-	-	-	-
2	0.0778	0.000025	-	-
3	0.0988	0.000028	0.0710	0.000024
4	0.1198	0.000043	0.0730	0.000035
5	0.1525	0.000056	0.0811	0.000044
6	0.1832	0.000068	0.0882	0.000053
7	0.2175	0.000080	0.0975	0.000060
8	0.2544	0.000092	0.1146	0.000068
9	0.2912	0.000105	0.1292	0.000076
10	0.3237	0.000117	0.1435	0.000083
15	0.5007	0.000129	0.2618	0.000094
20	0.6303	0.000145	0.3907	0.000107
25	0.7296	0.000162	0.5181	0.000122
30	0.8065	0.000180	0.6237	0.000139
35	0.8568	0.000199	0.7137	0.000157
40	0.8887	0.000218	0.7807	0.000175
45	0.9080	0.000236	0.8205	0.000193
50	0.9244	0.000253	0.8560	0.000211

Table 4.5: 2nd and 3rd to default Marshall-Olkin values, with standard errors (SE) and computational time, with default time correlation equal to 0.25, and 10,000 sample paths.

	Marshall-Olkin copula with corr. = 0.50			
	2nd to Default		3rd to Default	
No. Credits	Value	SE	Value	SE
1	-	-	-	-
2	0.1132	0.000030	-	-
3	0.1190	0.000031	0.1094	0.000030
4	0.1278	0.000046	0.1120	0.000043
5	0.1433	0.000058	0.1189	0.000054
6	0.1557	0.000069	0.1201	0.000064
7	0.1771	0.000079	0.1237	0.000072
8	0.1848	0.000089	0.1248	0.000079
9	0.1963	0.000098	0.1233	0.000086
10	0.2129	0.000107	0.1325	0.000093
15	0.3106	0.000117	0.1648	0.000100
20	0.4031	0.000129	0.2180	0.000108
25	0.4905	0.000143	0.2712	0.000118
30	0.5722	0.000157	0.3453	0.000128
35	0.6474	0.000172	0.4224	0.000140
40	0.6999	0.000188	0.4830	0.000153
45	0.7524	0.000203	0.5509	0.000166
50	0.7853	0.000219	0.6025	0.000180

Table 4.6: 2nd and 3rd to default Marshall-Olkin values, with standard errors (SE) and computational time, with default time correlation equal to 0.5 and 10,000 sample paths.

Computational Time									
Time	25	51	256	385	858	764	1,028	1,276	1,556
Sample Paths	5,000	10,000	50,000	75,000	100,000	150,000	200,000	250,000	300,000

Table 4.7: Computational Time (in sec.) to value with the Marshall-Olkin copula 1st, 2nd and 3rd to default with sample paths from 5,000 to 300,000.

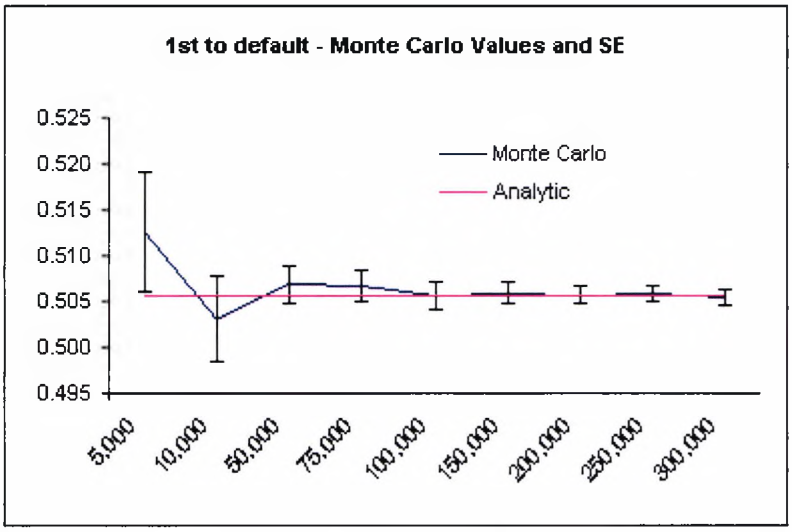

Figure 4.5: 1st to default Monte Carlo values and SE with sample path (5,000 to 300,000) calculated with the Marshall-Olkin copula and with the Analytic expression of section 4.3.

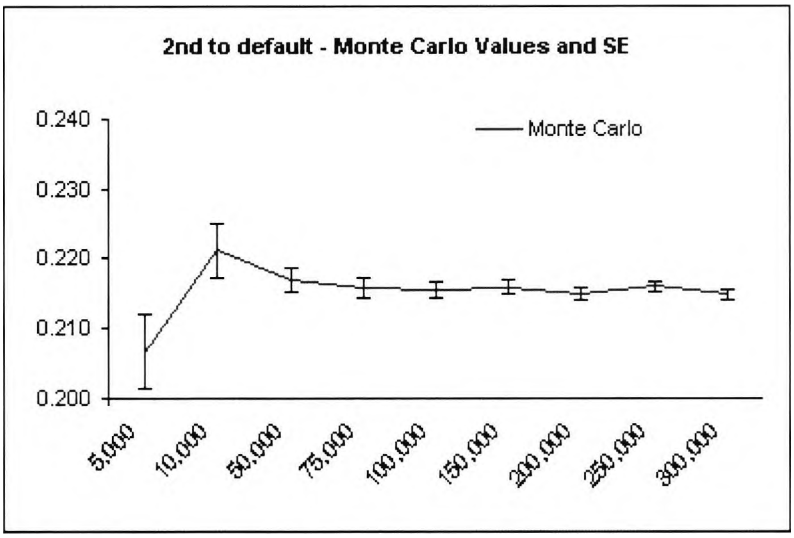

Figure 4.6: 2nd to default Monte Carlo values and SE with sample path (5,000 to 300,000) calculated with the Marshall-Olkin copula.

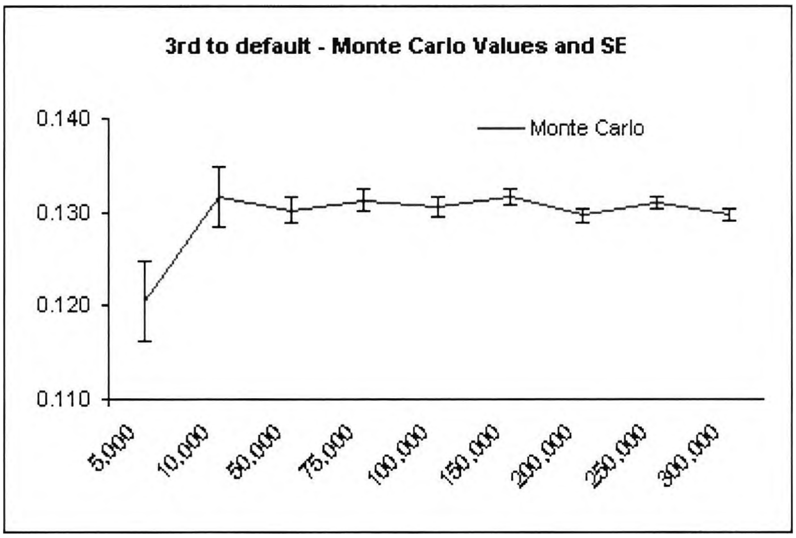

Figure 4.7: 3rd to default Monte Carlo values and SE with sample path (5,000 to 300,000) calculated with the Marshall-Olkin copula.

Appendix 2: Chapter 4 - Numerical exercise in section 4.5

In this appendix, we illustrate the results of the Merton Normal copula model and the Survival model with Normal, *t*-Student and Clayton copulae of the numerical example in section 4.5.

	MCS with Survival model, Normal copula with default time correlation of 0						
	1st to Default		2nd to Default		3rd to Default		
No. Credits	Value	SE	Value	SE	Value	SE	Time (Sec.)
1	0.16826	0.00354	0.00000	0.00000	0.00000	0.00000	-
2	0.30044	0.00430	0.03246	0.00166	0.00000	0.00000	-
3	0.42452	0.00459	0.08410	0.00260	0.00638	0.00074	-
4	0.51229	0.00458	0.14205	0.00325	0.02026	0.00131	-
5	0.57926	0.00448	0.20474	0.00375	0.04034	0.00183	-
6	0.65169	0.00425	0.27183	0.00410	0.07245	0.00241	-
7	0.69297	0.00406	0.32848	0.00431	0.10440	0.00284	-
8	0.74013	0.00379	0.39437	0.00444	0.14440	0.00324	-
9	0.78311	0.00346	0.46356	0.00449	0.19029	0.00361	1
10	0.80849	0.00323	0.50718	0.00447	0.23103	0.00386	1
15	0.90398	0.00201	0.71770	0.00373	0.47069	0.00443	1
20	0.93814	0.00135	0.82690	0.00283	0.65552	0.00398	1
25	0.95645	0.00083	0.89000	0.00195	0.77794	0.00316	2
30	0.96468	0.00059	0.92295	0.00130	0.85265	0.00232	3
35	0.97069	0.00037	0.93917	0.00093	0.89147	0.00172	4
40	0.97393	0.00025	0.94977	0.00061	0.91583	0.00121	6
45	0.97657	0.00022	0.95645	0.00049	0.93041	0.00089	7
50	0.97859	0.00017	0.96157	0.00032	0.93988	0.00062	9

Table 4.8: 1st 2nd and 3rd to default MC values (with standard errors SE and computational time) calculated with Survival model, Normal copula with default time correlation equal to 0.

	MCS with Survival model, Normal copula with default time correlation of 0.25						
	1st to Default		2nd to Default		3rd to Default		
No. Credits	Value	SE	Value	SE	Value	SE	Time (Sec.)
1	0.16624	0.00352	0.00000	0.00000	0.00000	0.00000	-
2	0.28750	0.00425	0.04386	0.00192	0.00000	0.00000	-
3	0.37622	0.00452	0.10709	0.00290	0.01747	0.00122	-
4	0.44499	0.00461	0.16074	0.00344	0.04289	0.00190	-
5	0.49758	0.00462	0.21564	0.00384	0.07865	0.00252	-
6	0.55841	0.00454	0.27827	0.00417	0.12093	0.00305	-
7	0.58173	0.00450	0.32050	0.00433	0.15723	0.00340	-
8	0.61706	0.00441	0.35970	0.00444	0.19691	0.00370	-
9	0.65242	0.00428	0.39995	0.00452	0.22899	0.00390	1
10	0.67151	0.00421	0.43108	0.00454	0.25839	0.00405	1
15	0.75845	0.00373	0.56328	0.00447	0.40312	0.00449	1
20	0.80856	0.00334	0.64156	0.00426	0.50093	0.00453	1
25	0.84276	0.00300	0.69478	0.00404	0.56805	0.00444	2
30	0.86579	0.00275	0.74067	0.00378	0.62979	0.00427	3
35	0.88165	0.00256	0.77573	0.00354	0.67739	0.00409	4
40	0.89447	0.00239	0.79749	0.00337	0.70527	0.00396	6
45	0.90375	0.00225	0.81486	0.00323	0.72828	0.00383	7
50	0.91181	0.00212	0.83389	0.00304	0.75555	0.00366	9

Table 4.9: 1st 2nd and 3rd to default MC values (with standard errors SE and computational time) calculated with Survival model, Normal copula with default time correlation equal to 0.25.

	MCS with Survival model, Normal copula with default time correlation of 0.5						
	1st to Default		2nd to Default		3rd to Default		
No. Credits	Value	SE	Value	SE	Value	SE	Time (Sec.)
1	0.16397	0.00350	0.00000	0.00000	0.00000	0.00000	-
2	0.26415	0.00415	0.07038	0.00241	0.00000	0.00000	-
3	0.32928	0.00442	0.12838	0.00315	0.03717	0.00178	-
4	0.38932	0.00456	0.18630	0.00366	0.08343	0.00260	-
5	0.42512	0.00461	0.22689	0.00393	0.12006	0.00305	-
6	0.45452	0.00462	0.25304	0.00407	0.14614	0.00332	-
7	0.47707	0.00464	0.28984	0.00424	0.17494	0.00356	-
8	0.50313	0.00463	0.31501	0.00434	0.20802	0.00381	-
9	0.52734	0.00461	0.34533	0.00443	0.23271	0.00395	1
10	0.54681	0.00458	0.36387	0.00448	0.25603	0.00408	1
15	0.61231	0.00446	0.45507	0.00460	0.34679	0.00442	1
20	0.65275	0.00432	0.49785	0.00461	0.40203	0.00455	1
25	0.68514	0.00419	0.54594	0.00457	0.45046	0.00460	2
30	0.71116	0.00407	0.57889	0.00451	0.48826	0.00461	3
35	0.72686	0.00399	0.60672	0.00445	0.52105	0.00459	4
40	0.74588	0.00387	0.62850	0.00439	0.55179	0.00455	6
45	0.75592	0.00381	0.64696	0.00432	0.56679	0.00452	7
50	0.76689	0.00373	0.66209	0.00426	0.58738	0.00449	9

Table 4.10: 1st 2nd and 3rd to default MC values (with standard errors SE and computational time) calculated with Survival model, Normal copula with default time correlation equal to 0.5.

	Value of 1st to default with MCS and default time correlation of 0					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1	0.16346	0.16826	0.16155	0.16124	0.16523	0.16469
2	0.30478	0.30044	0.29430	0.29912	0.29902	0.29983
3	0.42135	0.42452	0.41838	0.41094	0.41980	0.42247
4	0.50402	0.51229	0.50246	0.50792	0.50294	0.50826
5	0.58578	0.57926	0.58421	0.58200	0.58551	0.58524
6	0.65399	0.65169	0.63869	0.64697	0.64846	0.64557
7	0.69813	0.69297	0.69959	0.69985	0.70054	0.70016
8	0.74570	0.74013	0.74807	0.74111	0.73951	0.73683
9	0.77553	0.78311	0.77864	0.77717	0.78154	0.76828
10	0.81044	0.80849	0.80611	0.80743	0.81575	0.80771
15	0.90073	0.90398	0.90235	0.89709	0.89904	0.89887
20	0.93639	0.93814	0.93847	0.93901	0.93776	0.93717
25	0.95583	0.95645	0.95557	0.95518	0.95512	0.95421
30	0.96358	0.96468	0.96424	0.96436	0.96361	0.96368
35	0.96967	0.97069	0.97010	0.96947	0.97015	0.96992
40	0.97371	0.97393	0.97315	0.97365	0.97334	0.97393
45	0.97587	0.97657	0.97600	0.97601	0.97606	0.97600
50	0.97757	0.97859	0.97772	0.97804	0.97820	0.97777

Table 4.11: 1st to default MC values with default time correlation of 0. Merton and Survival model with Elliptical copulae.

	Value of 1st to default with MCS and default time correlation of 0.25					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1	0.15763	0.16624	0.16722	0.16586	0.16348	0.16973
2	0.29662	0.28750	0.28306	0.27902	0.27562	0.27998
3	0.39126	0.37622	0.36681	0.37996	0.37143	0.37313
4	0.48237	0.44499	0.43961	0.44429	0.44498	0.44168
5	0.54319	0.49758	0.49586	0.50299	0.50334	0.50801
6	0.60820	0.55841	0.54241	0.54872	0.55778	0.54738
7	0.65376	0.58173	0.57567	0.58802	0.59620	0.60092
8	0.68499	0.61706	0.61650	0.61445	0.62881	0.62866
9	0.72252	0.65242	0.64585	0.64275	0.65134	0.65902
10	0.74976	0.67151	0.67509	0.67319	0.68235	0.68409
15	0.83523	0.75845	0.75683	0.76271	0.76385	0.76902
20	0.88451	0.80856	0.80621	0.81188	0.81770	0.82377
25	0.91405	0.84276	0.83889	0.84722	0.85075	0.86047
30	0.92808	0.86579	0.86417	0.87122	0.87135	0.88543
35	0.94106	0.88165	0.87989	0.88713	0.89321	0.90280
40	0.94744	0.89447	0.89468	0.90246	0.90847	0.91656
45	0.95193	0.90375	0.90570	0.91056	0.91627	0.92765
50	0.95813	0.91181	0.91181	0.92242	0.92939	0.93399

Table 4.12: 1st to default MC values with default time correlation of 0.25. Merton and Survival model with Elliptical copulae.

	Value of 1st to default with MCS and default time correlation of 0.5					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1	0.16460	0.16397	0.16155	0.16124	0.16523	0.16469
2	0.27208	0.26415	0.25607	0.26037	0.26300	0.26277
3	0.37358	0.32928	0.33390	0.33686	0.34020	0.34882
4	0.42892	0.38932	0.37757	0.39269	0.38713	0.39383
5	0.48800	0.42512	0.42195	0.42507	0.43509	0.43895
6	0.53167	0.45452	0.44797	0.46209	0.46793	0.47792
7	0.56488	0.47707	0.48061	0.48909	0.48917	0.51196
8	0.59709	0.50313	0.50948	0.51737	0.51740	0.52840
9	0.63320	0.52734	0.52634	0.53764	0.54319	0.54853
10	0.65781	0.54681	0.54433	0.54717	0.56826	0.57526
15	0.74003	0.61231	0.62038	0.62338	0.62608	0.65727
20	0.79219	0.65275	0.64386	0.66618	0.68278	0.70475
25	0.82157	0.68514	0.68349	0.69832	0.71210	0.74237
30	0.84688	0.71116	0.70914	0.73185	0.74529	0.76896
35	0.85896	0.72686	0.73034	0.75285	0.77239	0.79329
40	0.88332	0.74588	0.74425	0.77514	0.77889	0.81761
45	0.89026	0.75592	0.76297	0.78638	0.79096	0.82989
50	0.89767	0.76689	0.76558	0.79680	0.81327	0.84053

Table 4.13: 1st to default MC values with default time correlation of 0.5. Merton and Survival model with Elliptical copulae.

	Value of 2nd to default with MCS and default time correlation of 0					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2	0.03045	0.03246	0.02764	0.03010	0.02821	0.02774
3	0.07839	0.08410	0.07873	0.07665	0.08120	0.07771
4	0.13546	0.14205	0.13768	0.13746	0.13594	0.13642
5	0.19858	0.20474	0.20328	0.19735	0.20686	0.20577
6	0.26466	0.27183	0.26685	0.25984	0.26536	0.26636
7	0.32750	0.32848	0.33147	0.32701	0.32472	0.33065
8	0.39666	0.39437	0.39774	0.38935	0.38839	0.38763
9	0.44473	0.46356	0.45584	0.44844	0.45441	0.44518
10	0.50515	0.50718	0.49773	0.50483	0.51318	0.50353
15	0.71574	0.71770	0.71896	0.70921	0.70998	0.71252
20	0.82655	0.82690	0.83422	0.82640	0.83430	0.82622
25	0.88793	0.89000	0.88790	0.88819	0.88869	0.88922
30	0.91802	0.92295	0.92171	0.91965	0.92075	0.91970
35	0.93656	0.93917	0.93802	0.93701	0.93749	0.93839
40	0.94952	0.94977	0.94849	0.94903	0.94795	0.94949
45	0.95471	0.95645	0.95611	0.95574	0.95582	0.95602
50	0.95990	0.96157	0.95978	0.96034	0.96066	0.95998

Table 4.14: 2^{nd} to default MC values with default time correlation of 0. Merton and Survival model with Elliptical copulae.

	Value of 2nd to default with MCS and default time correlation of 0.25					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2	0.03662	0.04386	0.04479	0.04613	0.04522	0.04979
3	0.08737	0.10709	0.10267	0.10401	0.10943	0.11094
4	0.14711	0.16074	0.16093	0.16744	0.16342	0.16541
5	0.20457	0.21564	0.21373	0.21892	0.22168	0.22204
6	0.27505	0.27827	0.26411	0.26867	0.27508	0.27438
7	0.32393	0.32050	0.30633	0.31690	0.32092	0.33907
8	0.37965	0.35970	0.35523	0.35528	0.36101	0.36276
9	0.43463	0.39995	0.38396	0.38499	0.39526	0.40971
10	0.47427	0.43108	0.43178	0.43082	0.43713	0.44256
15	0.63344	0.56328	0.55496	0.55898	0.56330	0.56371
20	0.73716	0.64156	0.63814	0.64544	0.65496	0.65183
25	0.80378	0.69478	0.69082	0.70749	0.70362	0.71534
30	0.84456	0.74067	0.74060	0.74799	0.74672	0.76158
35	0.87414	0.77573	0.76845	0.78064	0.78252	0.79676
40	0.89577	0.79749	0.79409	0.80876	0.81184	0.82238
45	0.91017	0.81486	0.81707	0.82618	0.83061	0.84318
50	0.92208	0.83389	0.82859	0.84487	0.85005	0.86246

Table 4.15: 2^{nd} to default MC values with default time correlation of 0.25. Merton and Survival model with Elliptical copulae.

	Value of 2nd to default with MCS and default time correlation of 0.5					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2	0.11188	0.12838	0.13377	0.13049	0.13606	0.14338
3	0.16139	0.18630	0.18036	0.18622	0.18008	0.18627
4	0.21663	0.22689	0.22655	0.22215	0.23120	0.23780
5	0.26078	0.25304	0.25841	0.25919	0.26423	0.27280
6	0.30705	0.28984	0.28643	0.29127	0.28626	0.30387
7	0.34578	0.31501	0.31809	0.32746	0.31945	0.32942
8	0.38056	0.34533	0.34414	0.34616	0.34978	0.35489
9	0.42029	0.36387	0.36000	0.36493	0.37674	0.37728
10	0.53520	0.45507	0.45552	0.45128	0.45415	0.47034
15	0.61854	0.49785	0.49686	0.51140	0.52128	0.53607
20	0.68172	0.54594	0.53973	0.54880	0.55866	0.58575
25	0.72368	0.57889	0.57710	0.59153	0.60168	0.62268
30	0.74310	0.60672	0.60468	0.62037	0.63580	0.65071
35	0.78226	0.62850	0.62761	0.65350	0.65239	0.68392
40	0.80045	0.64696	0.65649	0.66378	0.66971	0.70609
45	0.81387	0.66209	0.65593	0.67973	0.70186	0.72598
50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4.16: 2nd to default MC values with default time correlation of 0.5. Merton and Survival model with Elliptical copulae.

	Value of 3rd to default with MCS and default time correlation of 0					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2						
3	0.00535	0.00638	0.00562	0.00481	0.00533	0.00521
4	0.01569	0.02026	0.01647	0.01862	0.01970	0.01772
5	0.04022	0.04034	0.03766	0.03728	0.03786	0.03902
6	0.06684	0.07245	0.06640	0.06757	0.06521	0.06663
7	0.10155	0.10440	0.10166	0.10247	0.10327	0.10356
8	0.14400	0.14440	0.14307	0.13925	0.14450	0.14572
9	0.18455	0.19029	0.19076	0.18856	0.18864	0.18996
10	0.23527	0.23103	0.22579	0.22947	0.23189	0.23309
15	0.46634	0.47069	0.47405	0.46531	0.46422	0.46335
20	0.65155	0.65552	0.65828	0.64976	0.66251	0.65183
25	0.77305	0.77794	0.77155	0.77892	0.77630	0.77656
30	0.83996	0.85265	0.84636	0.84412	0.84808	0.84479
35	0.88919	0.89147	0.88616	0.88726	0.88854	0.89140
40	0.91560	0.91583	0.91294	0.91253	0.91283	0.91632
45	0.92890	0.93041	0.93050	0.92844	0.93048	0.92925
50	0.93818	0.93988	0.93809	0.93795	0.93877	0.93830

Table 4.17: 3rd to default MC values with default time correlation of 0. Merton and Survival model with Elliptical copulae.

	Value of 3rd to default with MCS and default time correlation of 0.25					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2						
3	0.00885	0.01747	0.01623	0.01622	0.01947	0.02209
4	0.03146	0.04289	0.04949	0.05156	0.04697	0.05247
5	0.05403	0.07865	0.08226	0.08477	0.08416	0.08843
6	0.09363	0.12093	0.11315	0.12093	0.12505	0.12656
7	0.13211	0.15723	0.14302	0.15555	0.15970	0.17360
8	0.16519	0.19691	0.18798	0.19089	0.19752	0.20050
9	0.21334	0.22899	0.21628	0.22117	0.22924	0.23730
10	0.25403	0.25839	0.26329	0.25856	0.26380	0.27110
15	0.42837	0.40312	0.39771	0.39715	0.40596	0.40286
20	0.57004	0.50093	0.49397	0.50371	0.50733	0.50648
25	0.67290	0.56805	0.56164	0.57804	0.57731	0.58408
30	0.73809	0.62979	0.63023	0.63252	0.63282	0.64146
35	0.78711	0.67739	0.66615	0.67548	0.67496	0.69124
40	0.82784	0.70527	0.70134	0.70921	0.71477	0.73156
45	0.85026	0.72828	0.73469	0.74346	0.74103	0.75378
50	0.87343	0.75555	0.75456	0.76536	0.77014	0.78382

Table 4.18: 3^{rd} to default MC values with default time correlation of 0.25. Merton and Survival model with Elliptical copulae.

	Value of 3rd to default with MCS and default time correlation of 0.5					
No. Credits	Merton Normal copula	Survival model, Normal copula	Survival model, t-copula, dof = 100	Survival model, t-copula, dof = 12	Survival model, t-copula, dof = 8	Survival model, t-copula, dof = 4
1						
2						
3	0.01970	0.03717	0.04145	0.03911	0.04431	0.04569
4	0.05110	0.08343	0.07517	0.08226	0.08292	0.09081
5	0.08399	0.12006	0.11878	0.11746	0.12649	0.13121
6	0.12196	0.14614	0.14339	0.15176	0.15596	0.16483
7	0.15645	0.17494	0.17517	0.18307	0.18171	0.19654
8	0.19288	0.20802	0.20644	0.21346	0.21135	0.21575
9	0.21945	0.23271	0.23225	0.23991	0.24137	0.24834
10	0.25934	0.25603	0.24864	0.25794	0.26349	0.26462
15	0.38532	0.34679	0.34837	0.34057	0.34855	0.36081
20	0.48018	0.40203	0.39902	0.40772	0.41506	0.42600
25	0.55426	0.45046	0.44736	0.44960	0.45874	0.48218
30	0.61000	0.48826	0.48776	0.49550	0.50125	0.51816
35	0.63592	0.52105	0.52062	0.52688	0.54242	0.55027
40	0.68772	0.55179	0.54602	0.56334	0.55968	0.58378
45	0.71607	0.56679	0.57686	0.58058	0.58013	0.60851
50	0.73224	0.58738	0.58163	0.60011	0.61647	0.63485

Table 4.19: 3^{rd} to default MC values with default time correlation of 0.5. Merton and Survival model with Elliptical copulae.

	Value of 1st to default with Clayton - Copula with different a								
No. Credits	a = 5.6E-08	a = 0.151	a = 0.178	a = 0.231	a = 0.246	a = 0.274	a = 0.327	a = 0.353	a = 0.5
1	0.16409	0.16535	0.16862	0.16415	0.16603	0.16352	0.16409	0.16475	0.16346
2	0.30025	0.29489	0.28351	0.28581	0.27796	0.28301	0.27338	0.27219	0.26263
3	0.41411	0.38162	0.38409	0.36240	0.36112	0.36152	0.34865	0.35080	0.33954
4	0.50259	0.46386	0.45220	0.43088	0.43299	0.42717	0.41062	0.41588	0.37939
5	0.58011	0.52216	0.50756	0.49783	0.48602	0.47315	0.46477	0.45900	0.41470
6	0.65173	0.57329	0.56197	0.53761	0.53416	0.51693	0.50783	0.49469	0.45505
7	0.68975	0.60895	0.59475	0.58633	0.57252	0.55313	0.54610	0.52096	0.47503
8	0.74489	0.64274	0.62003	0.60306	0.59308	0.58794	0.56749	0.55964	0.50696
9	0.77981	0.67457	0.65636	0.63567	0.62416	0.61083	0.59228	0.57833	0.52544
10	0.80974	0.70110	0.68447	0.65025	0.65139	0.63107	0.61378	0.59229	0.53832
15	0.89903	0.78753	0.77129	0.73897	0.73421	0.71985	0.68832	0.66848	0.60574
20	0.93764	0.83549	0.82434	0.79005	0.77770	0.76395	0.73049	0.72591	0.65055
25	0.95545	0.87443	0.85398	0.81876	0.81642	0.79486	0.76149	0.74920	0.68479
30	0.96339	0.89272	0.87119	0.84608	0.83500	0.81653	0.78890	0.77116	0.70457
35	0.96984	0.90577	0.89145	0.85774	0.85122	0.83454	0.80841	0.79743	0.72198
40	0.97333	0.91381	0.90164	0.87405	0.86391	0.85290	0.82206	0.80836	0.73226
45	0.97538	0.92451	0.90964	0.88171	0.87322	0.86185	0.83223	0.82349	0.75310
50	0.97801	0.93051	0.92022	0.89130	0.88374	0.87368	0.84545	0.82703	0.75937

Table 4.20: 1st to default MC values calculated with Survival model and Clayton copula.

	Value of 2nd to default with Clayton - Copula with different a								
No. Credits	a = 5.6E-08	a = 0.151	a = 0.178	a = 0.231	a = 0.246	a = 0.274	a = 0.327	a = 0.353	a = 0.5
1									
2	0.02904	0.04374	0.04402	0.05004	0.04956	0.05200	0.05460	0.05540	0.06482
3	0.07863	0.09346	0.10131	0.10387	0.10364	0.10628	0.11725	0.11861	0.13005
4	0.13295	0.15744	0.15395	0.16240	0.16789	0.16467	0.17104	0.17599	0.18008
5	0.19971	0.21858	0.21901	0.22402	0.22034	0.21761	0.22151	0.22180	0.21540
6	0.26878	0.27161	0.27164	0.27067	0.26836	0.26939	0.26709	0.26829	0.25937
7	0.31909	0.31906	0.31878	0.32147	0.31035	0.30717	0.30861	0.30098	0.28403
8	0.39427	0.35992	0.35475	0.35458	0.35794	0.35090	0.34473	0.34040	0.32222
9	0.44745	0.40759	0.39839	0.39315	0.38595	0.37712	0.37281	0.36357	0.34941
10	0.51232	0.44601	0.44188	0.41389	0.42150	0.40120	0.40457	0.38190	0.36204
15	0.71159	0.58385	0.57255	0.54475	0.54027	0.52661	0.50741	0.49472	0.44178
20	0.82757	0.68350	0.66584	0.62218	0.61014	0.60178	0.56912	0.57100	0.50336
25	0.88905	0.74287	0.72286	0.68133	0.66885	0.65125	0.61976	0.60442	0.54721
30	0.92029	0.78307	0.75878	0.72452	0.70878	0.69269	0.65406	0.64600	0.58163
35	0.93818	0.81586	0.79076	0.74946	0.74043	0.72251	0.69056	0.68101	0.60468
40	0.94744	0.83360	0.81338	0.77741	0.76019	0.75494	0.71939	0.70393	0.62530
45	0.95558	0.85511	0.83395	0.79369	0.78184	0.76784	0.72880	0.71944	0.64540
50	0.96049	0.86481	0.84821	0.81091	0.79612	0.78582	0.74940	0.73219	0.65348

Table 4.21: 2^{nd} to default MC values calculated with Survival model and Clayton copula.

	Value of 3rd to default with Clayton - Copula with different a								
No. Credits	a = 5.6E-08	a = 0.151	a = 0.178	a = 0.231	a = 0.246	a = 0.274	a = 0.327	a = 0.353	a = 0.5
1									
2									
3	0.00494	0.01317	0.01309	0.01682	0.01828	0.01927	0.02406	0.02496	0.03943
4	0.01714	0.03890	0.04119	0.04713	0.04951	0.05468	0.06270	0.06136	0.07504
5	0.03931	0.07251	0.07546	0.08281	0.08508	0.08674	0.09579	0.09779	0.10906
6	0.06693	0.10564	0.11540	0.12285	0.12160	0.12885	0.13058	0.13897	0.14959
7	0.09845	0.14515	0.14428	0.16680	0.15720	0.15813	0.17036	0.16706	0.17671
8	0.13948	0.18109	0.18250	0.20269	0.19769	0.19919	0.20102	0.20149	0.21030
9	0.18461	0.21962	0.21990	0.22608	0.22527	0.23301	0.23270	0.23477	0.23348
10	0.23989	0.25992	0.26550	0.25900	0.26329	0.25566	0.26125	0.25461	0.25070
15	0.46187	0.40956	0.41010	0.38898	0.39285	0.37877	0.37361	0.37083	0.33668
20	0.65700	0.52981	0.51855	0.48899	0.47669	0.47008	0.45273	0.45569	0.40496
25	0.77320	0.61266	0.59149	0.56058	0.55181	0.53524	0.50959	0.49652	0.45953
30	0.84960	0.67571	0.64694	0.61155	0.59835	0.58270	0.55119	0.54667	0.49604
35	0.88797	0.71626	0.69593	0.64729	0.64146	0.62279	0.60111	0.58778	0.52525
40	0.91271	0.74720	0.72852	0.68569	0.67477	0.66567	0.63320	0.61344	0.55285
45	0.92905	0.77671	0.75535	0.71167	0.69861	0.68487	0.64816	0.63705	0.57085
50	0.93951	0.79738	0.77925	0.73647	0.71411	0.70894	0.67442	0.65609	0.58277

Table 4.22: 3^{rd} to default MC values calculated with Survival model and Clayton copula.

Time (in seconds)				
No. Credits	Merton	Survival Normal copula	Survival t-Student copula	Clayton
1	0	-	0	-
2	0	-	1	0
3	0	-	1	0
4	0	-	1	0
5	0	-	2	0
6	0	-	2	0
7	0	-	2	0
8	1	-	2	0
9	1	1	3	0
10	1	1	3	0
15	1	1	5	0
20	2	1	6	0
25	3	2	8	1
30	4	3	9	1
35	6	4	11	1
40	7	6	13	1
45	9	7	14	1
50	11	9	16	1

Table 4.23: Computational times.

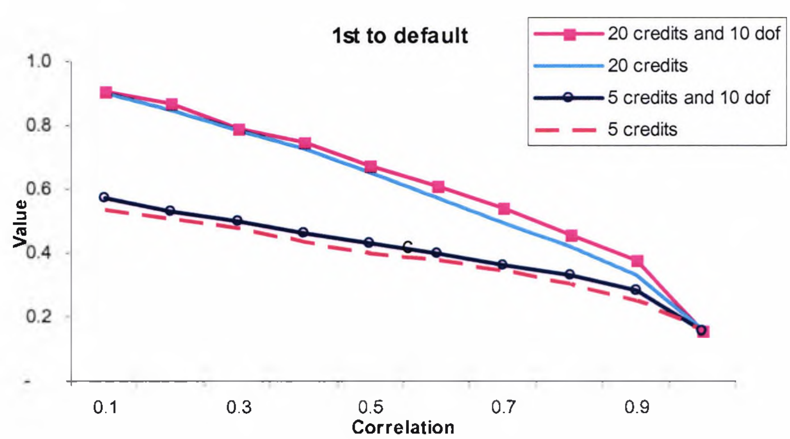

Figure 4.15: 1st to default values calculated with Survival model with Normal copula and *t*-Student copula (10 d.o.f.) with 5 and 20 credits.

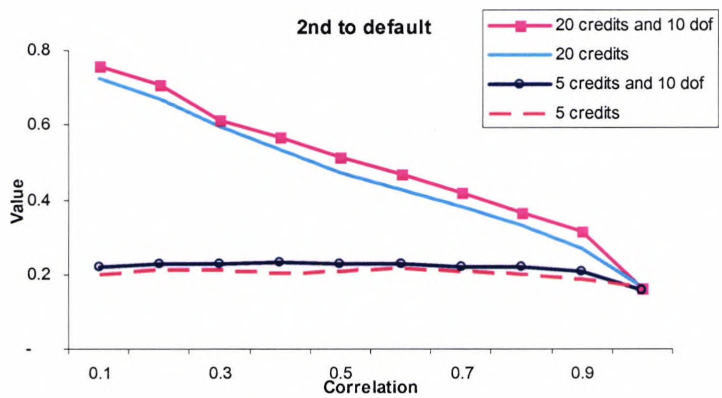

Figure 4.16: 2nd to default values calculated with Survival model and Normal copula and *t*-Student copula (10 d.o.f.) with 5 and 20 credits.

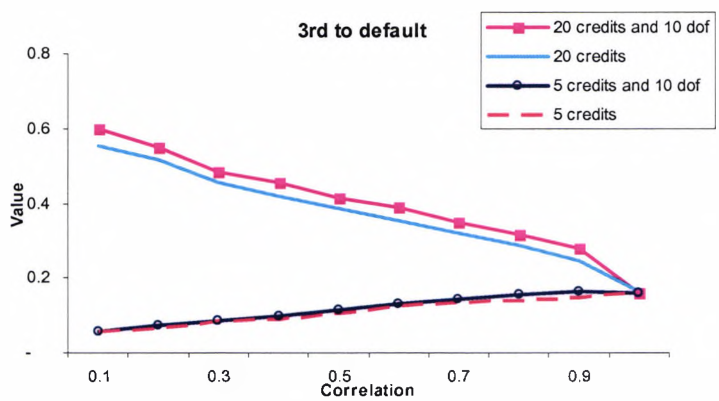

Figure 4.17: 3rd to default values calculated with Survival model with Normal copula and *t*-Student copula (10 d.o.f.) with 5 and 20 credits.

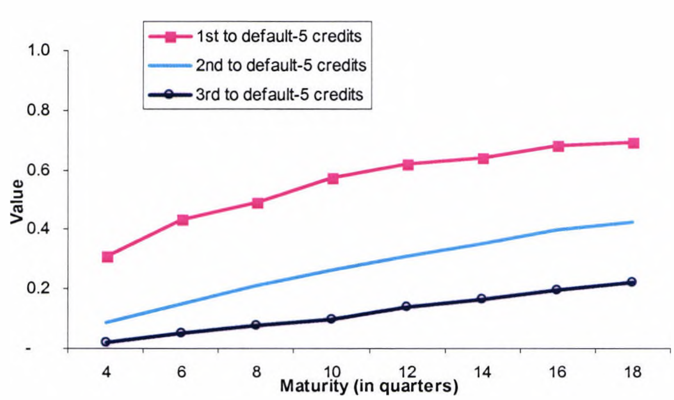

Figure 4.18: 1st – 2nd and 3rd to default values calculated with Survival model with Normal copula and *t*-Student copula, correlation of 0.25 and with 5 credits.

Appendix 3: Chapter 4 – Numerical exercise in section 4.6

In this appendix, we illustrate the results of the Elliptical and Archimedean copulae applied to the numerical example of section 4.6.

Obligor	Maturity	Par	Rating	Industry	CDS Premium Implied in Rating Prob.	Obligor	Maturity	Par	Rating	Industry
1	4 years	30,000	Aaa	Finance	0.01	26	4 years	30,000	A2	Oil and Gas
2	4 years	20,000	Ba1	Beverage, Food and Tobacco	14.65	27	4 years	10,000	Ba1	Banking
3	4 years	20,000	Ba1	Banking	14.65	28	4 years	30,000	Ba1	Buildings and Real Estate
4	4 years	40,000	A1	Finance	0.01	29	4 years	30,000	Baa1	Beverage, Food and Tobacco
5	4 years	10,000	A1	Buildings and Real Estate	0.63	30	4 years	40,000	Baa1	Personal, Food and Misc Services
6	4 years	10,000	Aaa	Buildings and Real Estate	0.01	31	4 years	20,000	Aa3	Insurance
7	4 years	20,000	Ba1	Machinery	14.65	32	4 years	30,000	A3	Utilities
8	4 years	10,000	Baa1	Printing, Publishing and Broadcasting	0.01	33	4 years	10,000	Aaa	Buildings and Real Estate
9	4 years	10,000	Aaa	Printing, Publishing and Broadcasting	2.80	34	4 years	10,000	Baa1	Retail Stores
10	4 years	20,000	Baa1	Utilities	2.80	35	4 years	50,000	Baa1	Finance
11	4 years	10,000	Baa2	Leisure, Amusement, Motion Pictures, Entertainr	4.08	36	4 years	20,000	Ba1	Beverage, Food and Tobacco
12	4 years	20,000	A3	Beverage, Food and Tobacco	0.01	37	4 years	10,000	Ba1	Telecommunications
13	4 years	40,000	Baa1	Chemicals, Plastics and Rubber	0.01	38	4 years	20,000	Ba1	Buildings and Real Estate
14	4 years	10,000	B1	Beverage, Food and Tobacco	1.15	39	4 years	10,000	Aa2	Telecommunications
15	4 years	30,000	B1	Beverage, Food and Tobacco	52.34	40	4 years	20,000	Aa2	Furnishings, Houseware Durable Consumer Prodı
16	4 years	20,000	Baa1	Retail Stores	2.80	41	4 years	10,000	Ba1	Telecommunications
17	4 years	20,000	Baa1	Leisure, Amusement, Motion Pictures, Entertainr	1.15	42	4 years	10,000	Ba2	Banking
18	4 years	10,000	Baa2	Oil and Gas	4.08	43	4 years	20,000	Ba1	Chemicals, Plastics and Rubber
19	4 years	50,000	A3	Buildings and Real Estate	0.01	44	4 years	60,000	A3	Electronics
20	4 years	10,000	Ba1	Containers, Packaging and Glass	0.01	45	4 years	10,000	Ba1	Finance
21	4 years	10,000	Ba1	Printing, Publishing and Broadcasting	1.15	46	4 years	20,000	Aa2	Oil and Gas
22	4 years	30,000	A3	Banking	0.07	47	4 years	10,000	A3	Oil and Gas
23	4 years	10,000	Ba1	Buildings and Real Estate	0.01	48	4 years	10,000	Ba1	Oil and Gas
24	4 years	10,000	B2	Mining, Steel, Iron and Non-precious Metals	71.25	49	4 years	20,000	A2	Telecommunications
25	4 years	10,000	Baa1	Insurance	2.80	50	4 years	10,000	Ba1	Utilities

Table 4.24: Collateral summary information: 50 obligors, with maturity, par value (current balance), rating, industry classification, and CDS premium as implied in the rating probabilities of Moody's idealised cumulative expected losses of Table 2.8, assuming a recovery rate of 45%.

Junior Note							
	(Linear) default time correlation of 0				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	5.6E-08	0.151	0.198
Premium	743.610 bps	758.389 bps	734.162 bps	718.048 bps	745.518 bps	630.188 bps	594.300 bps
EL (£)	10,210.186	10,382.553	10,099.441	9,909.557	10,232.498	8,851.398	8,407.862
SE (£)	107.775	100.718	104.773	105.129	106.248	119.100	121.293
EL (%)	25.5255%	25.9564%	25.2486%	24.7739%	25.5812%	22.1285%	21.0197%
EL+SE (%)	25.7949%	26.2082%	25.5105%	25.0367%	25.8469%	22.4262%	21.3229%
Rating	Caa	Caa	Caa	Caa	Caa	B3	B3
EL|L=97% (£)	37,047.474	36,874.530	37,053.206	37,086.049	37,166.271	38,630.190	38,633.165
EL|L=99% (£)	37,895.033	37,746.769	37,847.423	38,010.503	38,055.192	39,180.680	39,190.245
EL|L=99.5% (£)	38,320.154	38,354.510	38,356.863	38,525.031	38,506.717	39,375.965	39,416.471

Table 4.25: Junior Note with default time correlation of 0 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Mezzanine Note							
	(Linear) default time correlation of 0				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	5.6E-08	0.151	0.198
Premium	14.764 bps	12.312 bps	15.987 bps	16.357 bps	16.685 bps	67.482 bps	82.585 bps
EL (£)	294.358	245.596	318.661	325.998	332.517	1,330.476	1,623.062
SE (£)	19.721	16.597	21.209	23.047	22.114	61.989	69.876
EL (%)	0.5887%	0.4912%	0.6373%	0.6520%	0.6650%	2.6610%	3.2461%
EL+SE (%)	0.6282%	0.5244%	0.6797%	0.6981%	0.7093%	2.7849%	3.3859%
Rating	Baa1	A3	Baa1	Baa1	Baa1	Ba1	Ba1
EL|L=97% (£)	9,920.869	8,948.683	11,383.125	11,388.724	11,770.978	36,381.275	40,343.227
EL|L=99% (£)	16,751.168	13,437.048	18,061.386	19,668.812	18,617.327	44,943.069	46,491.089
EL|L=99.5% (£)	20,991.176	16,954.762	22,620.673	25,022.596	23,586.275	46,663.725	47,637.255

Table 4.26: Mezzanine Note with default time correlation of 0 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Senior Note							
	(Linear) default time correlation of 0				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	5.6E-08	0.151	0.198
Premium	0.000 bps	0.000 bps	0.000 bps	0.000 bps	0.001 bps	0.481 bps	1.139 bps
EL (£)	0.000	0.000	0.000	0.000	0.290	175.184	414.632
SE (£)	0.000	0.000	0.000	0.000	0.290	30.548	51.193
EL (%)	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0193%	0.0456%
EL+SE (%)	0.0000%	0.0000%	0.0000%	0.0000%	0.0001%	0.0226%	0.0512%
Rating	Aaa	Aaa	Aaa	Aaa	Aaa	Aa2	Aa3
EL|L=97% (£)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
EL|L=99% (£)	0.000	0.000	0.000	0.000	0.000	0.000	36,807.047
EL|L=99.5% (£)	0.000	0.000	0.000	0.000	0.000	28,032.648	61,833.396

Table 4.27: Senior Note with default time correlation of 0 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Junior Note							
	(Linear) default time correlation of 0.25				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.177	0.231	0.274
Premium	668.572 bps	610.096 bps	651.570 bps	690.575 bps	603.902 bps	564.658 bps	552.150 bps
EL (£)	9,318.461	8,603.906	9,112.501	9,582.839	8,527.187	8,036.474	7,878.367
SE (£)	109.538	110.694	118.196	117.405	119.757	119.822	121.142
EL (%)	23.2962%	21.5098%	22.7813%	23.9571%	21.3180%	20.0912%	19.6959%
EL+SE (%)	23.5700%	21.7865%	23.0767%	24.2506%	21.6174%	20.3907%	19.9988%
Rating	B3	B3	B3	Caa	B3	B3	B3
EL|L=97% (£)	37,232.394	37,332.685	38,128.156	37,837.359	38,563.832	38,594.275	38,754.624
EL|L=99% (£)	38,153.067	38,121.980	38,731.895	38,475.210	39,097.822	39,187.157	39,253.205
EL|L=99.5% (£)	38,659.036	38,501.176	39,042.436	38,771.273	39,337.939	39,434.465	39,457.255

Table 4.28: Junior Note with default time correlation of 0.25 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Mezzanine Note							
	(Linear) default time correlation of 0.25				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.177	0.231	0.274
Premium	24.126 bps	34.173 bps	57.805 bps	55.290 bps	75.668 bps	82.598 bps	98.451 bps
EL (£)	480.055	678.518	1,142.028	1,092.920	1,489.291	1,623.327	1,928.435
SE (£)	29.549	39.640	55.743	54.066	66.051	70.597	78.915
EL (%)	0.9601%	1.3570%	2.2841%	2.1858%	2.9786%	3.2467%	3.8569%
EL+SE (%)	1.0192%	1.4363%	2.3955%	2.2940%	3.1107%	3.3878%	4.0147%
Rating	Baa2	Baa3	Ba1	Baa3	Ba1	Ba1	Ba1
EL|L=97% (£)	16,065.556	22,590.637	32,359.990	31,205.463	38,328.884	40,848.008	44,160.956
EL|L=99% (£)	24,324.127	33,184.158	42,749.505	42,739.604	45,754.455	46,565.347	46,964.887
EL|L=99.5% (£)	30,887.255	39,110.784	45,483.333	45,314.706	47,056.863	47,641.176	47,910.784

Table 4.29: Mezzanine Note with default time correlation of 0.25 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Senior Note							
	(Linear) default time correlation of 0.25				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.177	0.231	0.274
Premium	0.010 bps	0.155 bps	0.378 bps	0.420 bps	0.740 bps	1.138 bps	1.754 bps
EL (£)	3.591	56.337	137.745	152.861	269.444	413.969	638.103
SE (£)	1.684	13.633	24.457	26.408	37.336	52.995	63.748
EL (%)	0.0004%	0.0062%	0.0151%	0.0168%	0.0296%	0.0455%	0.0701%
EL+SE (%)	0.0006%	0.0077%	0.0178%	0.0197%	0.0337%	0.0513%	0.0771%
Rating	Aaa	Aa1	Aa1	Aa1	Aa2	Aa3	Aa3
EL|L=97% (£)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
EL|L=99% (£)	0.000	0.000	0.000	0.000	26,210.167	36,934.083	53,218.286
EL|L=99.5% (£)	0.000	0.000	27,121.569	25,176.045	43,140.417	58,890.570	75,503.750

Table 4.30: Senior Note with default time correlation of 0.25 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Junior Note							
	(Linear) default time correlation of 0.50				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.246	0.327	0.391
Premium	589.416 bps	453.668 bps	526.120 bps	583.258 bps	560.322 bps	529.721 bps	506.923 bps
EL (£)	8,346.987	6,604.084	7,546.662	8,270.052	7,981.747	7,592.756	7,299.696
SE (£)	114.235	113.134	120.852	121.155	120.885	120.730	120.714
EL (%)	20.8675%	16.5102%	18.8667%	20.6751%	19.9544%	18.9819%	18.2492%
EL+SE (%)	21.1531%	16.7930%	19.1688%	20.9780%	20.2566%	19.2837%	18.5510%
Rating	B3	B2	B3	B3	B3	B3	B3
EL|L=97% (£)	37,932.590	38,173.684	38,485.755	38,305.657	38,716.365	38,650.961	38,777.565
EL|L=99% (£)	38,777.431	38,812.277	38,932.960	38,842.376	39,239.837	39,218.614	39,290.476
EL|L=99.5% (£)	39,331.765	39,158.846	39,180.154	39,110.303	39,410.597	39,467.308	39,483.304

Table 4.31: Junior Note with default time correlation of 0.5 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Mezzanine Note							
	(Linear) default time correlation of 0.50				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.246	0.327	0.391
Premium	52.144 bps	72.439 bps	98.007 bps	101.361 bps	91.041 bps	95.921 bps	108.114 bps
EL (£)	1,031.421	1,426.717	1,919.916	1,984.208	1,786.061	1,879.881	2,113.398
SE (£)	53.092	66.160	78.644	80.823	75.016	77.155	84.147
EL (%)	2.0628%	2.8534%	3.8398%	3.9684%	3.5721%	3.7598%	4.2268%
EL+SE (%)	2.1690%	2.9858%	3.9971%	4.1301%	3.7222%	3.9141%	4.3951%
Rating	Baa3	Ba1	Ba1	Ba1	Ba1	Ba1	Ba2
EL|L=97% (£)	30,940.703	38,997.012	43,886.210	44,363.745	42,305.976	43,743.028	45,572.718
EL|L=99% (£)	41,891.707	45,603.676	46,326.336	46,311.881	46,718.317	47,333.168	47,632.108
EL|L=99.5% (£)	45,119.471	46,697.917	47,236.275	47,202.546	47,729.412	48,277.451	48,399.359

Table 4.32: Mezzanine Note with default time correlation of 0.5 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Senior Note							
	(Linear) default time correlation of 0.50				No (linear) default time correlation		
	Merton model	Survival model					
	Normal copula	Normal copula	*t*-Student copula with dof		Clayton copula with alfa		
			12	4	0.246	0.327	0.391
Premium	0.420 bps	0.856 bps	1.873 bps	2.626 bps	1.335 bps	2.085 bps	2.873 bps
EL (£)	152.697	311.691	681.330	955.295	485.661	758.642	1,045.242
SE (£)	25.642	39.647	69.904	98.777	57.279	77.553	102.976
EL (%)	0.0168%	0.0343%	0.0749%	0.1050%	0.0534%	0.0834%	0.1149%
EL+SE (%)	0.0196%	0.0386%	0.0826%	0.1158%	0.0597%	0.0919%	0.1262%
Rating	Aa1	Aa2	Aa3	A1	Aa3	Aa3	A1
EL|L=97% (£)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
EL|L=99% (£)	0.000	28,412.448	57,308.333	75,469.965	41,648.606	66,271.642	79,463.035
EL|L=99.5% (£)	24,341.484	49,196.000	81,711.875	111,693.750	67,554.118	93,651.250	116,818.333

Table 4.33: Senior Note with default time correlation of 0.5 and with no linear default time correlation - EL, SE, Premium, Rating and Expected Shortfall.

Appendix 4: Chapter 4 – Numerical exercise in section 4.8

In this appendix, we illustrate the results of the Schönbucher and Schuber copula model on the numerical example in section 4.8.

Junior Note						
	Schönbucher and Schubert model					
	Clayton Copula with alfa of					
	0.177	0.231	0.274	0.246	0.327	0.391
Premium	621.14	568.43	565.78	570.77	520.10	493.62
EL (£)	8,740.24	8,084.01	8,050.55	8,113.41	7,469.49	7,127.41
SE (£)	124.10	124.89	127.07	125.18	125.91	126.06
EL (%)	21.8506%	20.2100%	20.1264%	20.2835%	18.6737%	17.8185%
EL+SE (%)	22.1608%	20.5223%	20.4441%	20.5965%	18.9885%	18.1337%
Rating	B3	B3	B3	B3	B3	B3
EL|L=97% (£)	38,749	38,854	39,040	38,972	39,030	39,015
EL|L=99% (£)	39,226	39,260	39,363	39,338	39,379	39,362
EL|L=99.5% (£)	39,403	39,396	39,468	39,484	39,503	39,468

Table 4.34: Junior Note with Schönbucher and Schuber Clayton copula model.

Mezzanine Note						
	Schönbucher and Schubert model					
	Clayton Copula with alfa of					
	0.177	0.231	0.274	0.246	0.327	0.391
Premium	109.99	134.68	158.59	141.42	165.34	184.95
EL (£)	2,149.29	2,618.06	3,067.39	2,745.15	3,193.39	3,557.60
SE (£)	83.84	94.94	104.74	98.23	108.38	115.00
EL (%)	4.2986%	5.2361%	6.1348%	5.4903%	6.3868%	7.1152%
EL+SE (%)	4.4663%	5.4260%	6.3443%	5.6868%	6.6035%	7.3452%
Rating	Ba2	Ba2	Ba2	Ba2	Ba2	Ba3
EL|L=97% (£)	45,065	45,998	46,793	46,423	47,243	47,330
EL|L=99% (£)	46,884	47,483	48,083	47,847	48,364	48,267
EL|L=99.5% (£)	47,651	48,198	48,600	48,462	48,861	48,728

Table 4.35: Mezzanine Note with Schönbucher and Schuber Clayton copula model.

Senior Note						
	Schönbucher and Schubert model					
	Clayton Copula with alfa of					
	0.177	0.231	0.274	0.246	0.327	0.391
Premium	2.78	6.21	11.63	8.59	18.26	26.38
EL (£)	1,010.04	2,256.79	4,223.14	3,122.35	6,619.81	9,548.55
SE (£)	95.22	181.22	271.58	219.95	378.29	475.92
EL (%)	0.1110%	0.2480%	0.4641%	0.3431%	0.7275%	1.0493%
EL+SE (%)	0.1215%	0.2679%	0.4939%	0.3673%	0.7690%	1.1016%
Rating	A1	A2	A3	A3	Baa1	Baa2
EL|L=97% (£)	0	86,066	151,955	116,774	224,759	281,759
EL|L=99% (£)	77,346	160,800	236,825	194,073	302,685	337,727
EL|L=99.5% (£)	109,177	212,708	280,481	239,146	336,952	366,552

Table 4.36: Senior Note with Schönbucher and Schuber Clayton copula model.

Appendix 5: Chapter 5 - generators and transition matrices calculated with modifying the default intensities

$\tilde{\Lambda}_{DI}(0) =$

From Rating	To Rating: AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.0732	0.0667	0.0053	0.0004	0.0008	0.0000	0.0000	0.0000
AA	0.0067	-0.0963	0.0825	0.0048	0.0004	0.0014	0.0003	0.0002
A	0.0007	0.0232	-0.0928	0.0624	0.0042	0.0017	0.0004	0.0003
BBB	0.0004	0.0019	0.0497	-0.1176	0.0564	0.0069	0.0019	0.0003
BB	0.0005	0.0009	0.0033	0.0692	-0.1940	0.1047	0.0092	0.0062
B	0.0000	0.0009	0.0027	0.0014	0.0644	-0.1637	0.0724	0.0218
CCC	0.0008	0.0007	0.0045	0.0089	0.0208	0.1457	-0.2397	0.0583
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.9: generator for the period $[0,1]$, with method 1.

$\tilde{\Lambda}_{DI}(1) =$

From Rating	To Rating: AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.0732	0.0667	0.0053	0.0004	0.0008	0.0000	0.0000	0.0000
AA	0.0067	-0.0963	0.0825	0.0048	0.0004	0.0014	0.0003	0.0002
A	0.0007	0.0232	-0.0929	0.0624	0.0042	0.0017	0.0004	0.0003
BBB	0.0004	0.0019	0.0497	-0.1172	0.0564	0.0069	0.0019	0.0000
BB	0.0005	0.0009	0.0033	0.0692	-0.1924	0.1047	0.0092	0.0046
B	0.0000	0.0009	0.0027	0.0014	0.0644	-0.1622	0.0724	0.0203
CCC	0.0008	0.0007	0.0045	0.0089	0.0208	0.1457	-0.2481	0.0668
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Tables 5.10: generator for the period $[1,2]$, with method 1.

$\tilde{P}_{DI}(0,1) =$

From Rating	To Rating: AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.952%	6.160%	0.720%	0.075%	0.079%	0.008%	0.002%	0.006%
AA	0.617%	90.907%	7.561%	0.653%	0.070%	0.134%	0.036%	0.022%
A	0.073%	2.131%	91.333%	5.674%	0.512%	0.194%	0.049%	0.034%
BBB	0.042%	0.219%	4.526%	89.163%	4.927%	0.848%	0.211%	0.064%
BB	0.044%	0.098%	0.444%	6.031%	82.671%	8.993%	1.028%	0.690%
B	0.004%	0.090%	0.275%	0.329%	5.550%	85.465%	6.076%	2.211%
CCC	0.071%	0.071%	0.424%	0.840%	2.070%	12.263%	78.887%	5.374%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Tables 5.11: time-inhomogeneous transition matrices for the period $[0,1]$, with method 1.

$\tilde{P}_{DI}(1,2) =$

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.952%	6.160%	0.720%	0.075%	0.079%	0.008%	0.002%	0.006%
AA	0.617%	90.907%	7.561%	0.653%	0.070%	0.134%	0.036%	0.022%
A	0.073%	2.131%	91.331%	5.674%	0.512%	0.194%	0.049%	0.035%
BBB	0.042%	0.220%	4.526%	89.194%	4.931%	0.849%	0.210%	0.028%
BB	0.044%	0.098%	0.445%	6.036%	82.806%	9.005%	1.025%	0.541%
B	0.004%	0.090%	0.275%	0.329%	5.557%	85.597%	6.059%	2.088%
CCC	0.071%	0.071%	0.423%	0.837%	2.065%	12.229%	78.204%	6.101%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Tables 5.12: time-inhomogeneous transition matrices for the period $[1,2]$, with method 1.

Appendix 6: Chapter 5 - generators and transition matrices calculated with modifying the rows

$\tilde{P}_{R}(0,1) =$

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	97.194%	2.237%	0.477%	0.045%	0.033%	0.006%	0.001%	0.006%
AA	1.913%	72.308%	23.225%	1.871%	0.151%	0.399%	0.105%	0.027%
A	0.062%	1.389%	93.802%	4.262%	0.295%	0.125%	0.031%	0.034%
BBB	0.009%	0.037%	0.908%	97.795%	0.999%	0.156%	0.040%	0.056%
BB	0.028%	0.052%	0.218%	4.050%	88.383%	5.970%	0.598%	0.700%
B	0.001%	0.031%	0.105%	0.103%	2.246%	92.711%	2.581%	2.221%
CCC	0.010%	0.008%	0.055%	0.113%	0.257%	1.704%	92.481%	5.374%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.13: time-inhomogeneous transition matrices for the period $[0,1]$, with method 2.

$\tilde{P}_{R}(1,2) =$

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	98.109%	1.647%	0.198%	0.020%	0.021%	0.002%	0.000%	0.004%
AA	0.658%	90.528%	7.919%	0.660%	0.053%	0.137%	0.036%	0.009%
A	0.057%	1.642%	93.193%	4.593%	0.315%	0.130%	0.033%	0.036%
BBB	0.006%	0.032%	0.701%	98.286%	0.783%	0.118%	0.030%	0.043%
BB	0.024%	0.048%	0.177%	3.429%	90.184%	5.046%	0.503%	0.590%
B	0.001%	0.033%	0.099%	0.094%	2.212%	92.877%	2.516%	2.168%
CCC	0.010%	0.009%	0.055%	0.114%	0.263%	1.729%	92.367%	5.454%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.14: time-inhomogeneous transition matrices for the period $[1,2]$, with method 2.

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	97.459%	2.313%	0.180%	0.016%	0.028%	0.001%	0.000%	0.005%
AA	0.000%	100.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
A	0.020%	0.663%	97.351%	1.768%	0.122%	0.049%	0.012%	0.014%
BBB	0.006%	0.028%	0.666%	98.391%	0.734%	0.108%	0.028%	0.040%
BB	0.021%	0.044%	0.158%	3.041%	91.290%	4.478%	0.446%	0.523%
B	0.000%	0.034%	0.098%	0.087%	2.198%	92.956%	2.484%	2.142%
CCC	0.010%	0.009%	0.056%	0.115%	0.268%	1.760%	92.232%	5.550%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.15: time-inhomogeneous transition matrices for the period [2, 3].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	97.615%	2.166%	0.172%	0.016%	0.027%	0.001%	0.000%	0.005%
AA	0.033%	99.523%	0.404%	0.029%	0.002%	0.007%	0.002%	0.000%
A	0.027%	0.897%	96.410%	2.400%	0.164%	0.066%	0.017%	0.018%
BBB	0.005%	0.023%	0.534%	98.704%	0.596%	0.085%	0.022%	0.032%
BB	0.017%	0.036%	0.129%	2.536%	92.748%	3.730%	0.370%	0.435%
B	0.000%	0.033%	0.095%	0.080%	2.161%	93.116%	2.423%	2.091%
CCC	0.010%	0.009%	0.057%	0.116%	0.275%	1.791%	92.095%	5.647%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.16: time-inhomogeneous transition matrices for the period [3, 4].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	96.070%	3.361%	0.469%	0.042%	0.045%	0.004%	0.001%	0.008%
AA	0.908%	86.753%	11.174%	0.840%	0.072%	0.191%	0.050%	0.012%
A	0.042%	1.148%	95.183%	3.245%	0.238%	0.094%	0.024%	0.025%
BBB	0.012%	0.057%	1.357%	96.701%	1.516%	0.220%	0.057%	0.081%
BB	0.019%	0.037%	0.150%	2.690%	92.232%	4.029%	0.391%	0.453%
B	0.000%	0.024%	0.074%	0.063%	1.659%	94.695%	1.893%	1.592%
CCC	0.006%	0.006%	0.037%	0.074%	0.173%	1.156%	94.936%	3.612%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.17: time-inhomogeneous transition matrices for the period [4, 5].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.613%	3.941%	0.354%	0.032%	0.049%	0.002%	0.000%	0.009%
AA	0.190%	97.224%	2.349%	0.170%	0.015%	0.040%	0.010%	0.002%
A	0.031%	1.002%	95.964%	2.690%	0.196%	0.077%	0.019%	0.021%
BBB	0.010%	0.051%	1.197%	97.091%	1.339%	0.190%	0.050%	0.071%
BB	0.017%	0.035%	0.131%	2.409%	93.053%	3.602%	0.349%	0.405%
B	0.000%	0.025%	0.073%	0.060%	1.651%	94.741%	1.874%	1.577%
CCC	0.006%	0.006%	0.037%	0.075%	0.175%	1.169%	94.879%	3.653%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.18: time-inhomogeneous transition matrices for the period [5, 6].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	96.058%	3.542%	0.319%	0.028%	0.044%	0.002%	0.000%	0.008%
AA	0.189%	97.238%	2.341%	0.166%	0.014%	0.039%	0.010%	0.002%
A	0.028%	0.885%	96.434%	2.379%	0.172%	0.067%	0.017%	0.018%
BBB	0.010%	0.047%	1.109%	97.307%	1.242%	0.174%	0.046%	0.065%
BB	0.015%	0.031%	0.117%	2.167%	93.755%	3.237%	0.313%	0.363%
B	0.000%	0.025%	0.072%	0.057%	1.645%	94.774%	1.860%	1.566%
CCC	0.007%	0.006%	0.037%	0.075%	0.178%	1.183%	94.819%	3.695%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.19: time-inhomogeneous transition matrices for the period [6, 7].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.996%	3.630%	0.295%	0.026%	0.045%	0.001%	0.000%	0.008%
AA	0.056%	99.181%	0.695%	0.048%	0.004%	0.012%	0.003%	0.001%
A	0.023%	0.776%	96.896%	2.068%	0.148%	0.058%	0.015%	0.016%
BBB	0.008%	0.040%	0.953%	97.689%	1.068%	0.147%	0.039%	0.055%
BB	0.013%	0.028%	0.104%	1.949%	94.392%	2.907%	0.281%	0.326%
B	0.000%	0.025%	0.071%	0.055%	1.639%	94.809%	1.846%	1.555%
CCC	0.007%	0.006%	0.038%	0.076%	0.181%	1.197%	94.755%	3.741%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.20: time-inhomogeneous transition matrices for the period [7, 8].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.255%	4.139%	0.496%	0.042%	0.054%	0.004%	0.001%	0.010%
AA	0.617%	90.956%	7.668%	0.541%	0.047%	0.130%	0.034%	0.007%
A	0.030%	0.873%	96.402%	2.406%	0.181%	0.071%	0.018%	0.019%
BBB	0.014%	0.067%	1.642%	96.022%	1.826%	0.264%	0.069%	0.096%
BB	0.018%	0.036%	0.148%	2.575%	92.533%	3.895%	0.370%	0.425%
B	0.000%	0.018%	0.056%	0.046%	1.248%	96.012%	1.434%	1.186%
CCC	0.004%	0.004%	0.023%	0.047%	0.108%	0.738%	96.785%	2.291%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.21: time-inhomogeneous transition matrices for the period [8, 9].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	94.848%	4.514%	0.521%	0.045%	0.059%	0.004%	0.001%	0.011%
AA	0.555%	91.843%	6.907%	0.497%	0.043%	0.117%	0.031%	0.007%
A	0.032%	0.959%	96.057%	2.639%	0.196%	0.077%	0.019%	0.020%
BBB	0.013%	0.060%	1.448%	96.485%	1.617%	0.232%	0.061%	0.084%
BB	0.017%	0.034%	0.135%	2.411%	93.022%	3.639%	0.346%	0.397%
B	0.000%	0.018%	0.055%	0.045%	1.246%	96.025%	1.429%	1.182%
CCC	0.004%	0.004%	0.023%	0.047%	0.109%	0.743%	96.762%	2.307%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.22: time-inhomogeneous transition matrices for the period [9, 10].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	94.182%	5.277%	0.427%	0.036%	0.065%	0.002%	0.000%	0.012%
AA	0.048%	99.297%	0.599%	0.040%	0.004%	0.010%	0.003%	0.001%
A	0.020%	0.662%	97.357%	1.755%	0.130%	0.050%	0.013%	0.014%
BBB	0.012%	0.057%	1.374%	96.675%	1.529%	0.217%	0.057%	0.079%
BB	0.016%	0.033%	0.127%	2.280%	93.404%	3.439%	0.327%	0.375%
B	0.000%	0.019%	0.055%	0.044%	1.248%	96.026%	1.428%	1.181%
CCC	0.004%	0.004%	0.024%	0.047%	0.110%	0.748%	96.742%	2.321%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.23: time-inhomogeneous transition matrices for the period [10, 11].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	94.290%	5.178%	0.420%	0.036%	0.064%	0.001%	0.000%	0.012%
AA	0.051%	99.242%	0.645%	0.044%	0.004%	0.011%	0.003%	0.001%
A	0.021%	0.711%	97.161%	1.888%	0.138%	0.054%	0.014%	0.014%
BBB	0.010%	0.050%	1.213%	97.061%	1.355%	0.190%	0.050%	0.070%
BB	0.014%	0.031%	0.116%	2.119%	93.881%	3.189%	0.303%	0.348%
B	0.000%	0.019%	0.055%	0.043%	1.249%	96.031%	1.425%	1.179%
CCC	0.004%	0.004%	0.024%	0.047%	0.111%	0.753%	96.719%	2.338%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.24: time-inhomogeneous transition matrices for the period [11, 12].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.825%	6.228%	0.780%	0.065%	0.082%	0.007%	0.001%	0.015%
AA	0.682%	89.878%	8.602%	0.595%	0.052%	0.146%	0.038%	0.008%
A	0.027%	0.782%	96.765%	2.165%	0.163%	0.065%	0.016%	0.017%
BBB	0.015%	0.069%	1.721%	95.842%	1.898%	0.283%	0.072%	0.100%
BB	0.020%	0.041%	0.169%	2.928%	91.493%	4.457%	0.416%	0.475%
B	0.000%	0.014%	0.042%	0.036%	0.930%	97.012%	1.082%	0.885%
CCC	0.003%	0.002%	0.015%	0.029%	0.066%	0.461%	98.001%	1.423%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.25: time-inhomogeneous transition matrices for the period [12, 13].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	93.523%	5.626%	0.701%	0.058%	0.074%	0.006%	0.001%	0.014%
AA	0.673%	90.035%	8.468%	0.585%	0.050%	0.143%	0.037%	0.008%
A	0.026%	0.763%	96.844%	2.116%	0.157%	0.063%	0.016%	0.016%
BBB	0.013%	0.060%	1.497%	96.383%	1.653%	0.245%	0.063%	0.087%
BB	0.020%	0.040%	0.162%	2.858%	91.715%	4.337%	0.405%	0.463%
B	0.000%	0.014%	0.042%	0.036%	0.934%	97.001%	1.085%	0.888%
CCC	0.003%	0.002%	0.015%	0.029%	0.067%	0.463%	97.995%	1.427%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.26: time-inhomogeneous transition matrices for the period [13, 14].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	93.523%	5.626%	0.701%	0.058%	0.074%	0.006%	0.001%	0.014%
AA	0.673%	90.035%	8.468%	0.585%	0.050%	0.143%	0.037%	0.008%
A	0.026%	0.763%	96.844%	2.116%	0.157%	0.063%	0.016%	0.016%
BBB	0.013%	0.060%	1.497%	96.383%	1.653%	0.245%	0.063%	0.087%
BB	0.020%	0.040%	0.162%	2.858%	91.715%	4.337%	0.405%	0.463%
B	0.000%	0.014%	0.042%	0.036%	0.934%	97.001%	1.085%	0.888%
CCC	0.003%	0.002%	0.015%	0.029%	0.067%	0.463%	97.995%	1.427%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.27: time-inhomogeneous transition matrices for the period [14, 15].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	96.087%	3.463%	0.368%	0.029%	0.044%	0.002%	0.000%	0.008%
AA	0.405%	94.058%	5.079%	0.320%	0.027%	0.084%	0.022%	0.004%
A	0.011%	0.345%	98.594%	0.941%	0.068%	0.027%	0.007%	0.007%
BBB	0.010%	0.046%	1.186%	97.148%	1.301%	0.191%	0.049%	0.068%
BB	0.019%	0.038%	0.150%	2.726%	92.122%	4.121%	0.385%	0.439%
B	0.000%	0.014%	0.042%	0.035%	0.933%	97.010%	1.082%	0.885%
CCC	0.003%	0.002%	0.015%	0.029%	0.067%	0.467%	97.974%	1.442%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.28: time-inhomogeneous transition matrices for the period [15, 16].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.07333	0.06672	0.00525	0.00040	0.00084	0.00000	0.00000	0.00015
AA	0.00666	-0.09615	0.08250	0.00484	0.00039	0.00136	0.00034	0.00005
A	0.00070	0.02322	-0.09298	0.06237	0.00419	0.00165	0.00041	0.00043
BBB	0.00042	0.00185	0.04972	-0.11991	0.05640	0.00692	0.00192	0.00267
BB	0.00048	0.00094	0.00333	0.06921	-0.19827	0.10469	0.00916	0.01046
B	0.00000	0.00093	0.00272	0.00145	0.06441	-0.20003	0.07243	0.05809
CCC	0.00080	0.00067	0.00446	0.00885	0.02084	0.14574	-0.62504	0.44367
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.29: generator for the period [0].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.01915	0.01742	0.00137	0.00010	0.00022	0.00000	0.00000	0.00004
AA	0.00694	-0.10011	0.08590	0.00504	0.00040	0.00141	0.00036	0.00006
A	0.00054	0.01781	-0.07131	0.04784	0.00322	0.00127	0.00032	0.00033
BBB	0.00006	0.00027	0.00730	-0.01760	0.00828	0.00102	0.00028	0.00039
BB	0.00025	0.00049	0.00175	0.03627	-0.10391	0.05487	0.00480	0.00548
B	0.00000	0.00035	0.00102	0.00054	0.02404	-0.07467	0.02704	0.02168
CCC	0.00010	0.00009	0.00057	0.00113	0.00265	0.01855	-0.07956	0.05647
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.30: generator for the period [1].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.02574	0.02342	0.00184	0.00014	0.00029	0.00000	0.00000	0.00005
AA	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
A	0.00020	0.00672	-0.02690	0.01805	0.00121	0.00048	0.00012	0.00012
BBB	0.00006	0.00025	0.00680	-0.01640	0.00771	0.00095	0.00026	0.00036
BB	0.00022	0.00043	0.00154	0.03200	-0.09166	0.04840	0.00423	0.00484
B	0.00000	0.00034	0.00100	0.00053	0.02375	-0.07375	0.02671	0.02142
CCC	0.00010	0.00009	0.00058	0.00115	0.00270	0.01889	-0.08102	0.05751
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.31: generator for the period [2].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.02414	0.02197	0.00173	0.00013	0.00028	0.00000	0.00000	0.00005
AA	0.00033	-0.00481	0.00413	0.00024	0.00002	0.00007	0.00002	0.00000
A	0.00028	0.00915	-0.03663	0.02457	0.00165	0.00065	0.00016	0.00017
BBB	0.00005	0.00020	0.00547	-0.01319	0.00620	0.00076	0.00021	0.00029
BB	0.00018	0.00036	0.00127	0.02644	-0.07574	0.03999	0.00350	0.00400
B	0.00000	0.00033	0.00098	0.00052	0.02316	-0.07193	0.02605	0.02089
CCC	0.00011	0.00009	0.00059	0.00117	0.00275	0.01924	-0.08250	0.05856
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.32: generator for the period [3].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.04025	0.03662	0.00288	0.00022	0.00046	0.00000	0.00000	0.00008
AA	0.00988	-0.14253	0.12230	0.00718	0.00058	0.00201	0.00051	0.00008
A	0.00038	0.01255	-0.05027	0.03372	0.00227	0.00089	0.00022	0.00023
BBB	0.00012	0.00053	0.01409	-0.03399	0.01599	0.00196	0.00055	0.00076
BB	0.00020	0.00039	0.00137	0.02840	-0.08136	0.04296	0.00376	0.00429
B	0.00000	0.00025	0.00075	0.00040	0.01770	-0.05496	0.01990	0.01596
CCC	0.00007	0.00006	0.00037	0.00074	0.00174	0.01214	-0.05206	0.03695
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.33: generator for the period [4].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.04488	0.04084	0.00321	0.00024	0.00051	0.00000	0.00000	0.00009
AA	0.00196	-0.02831	0.02430	0.00143	0.00011	0.00040	0.00010	0.00002
A	0.00031	0.01036	-0.04148	0.02782	0.00187	0.00074	0.00018	0.00019
BBB	0.00011	0.00046	0.01238	-0.02986	0.01404	0.00172	0.00048	0.00066
BB	0.00017	0.00034	0.00122	0.02528	-0.07243	0.03825	0.00335	0.00382
B	0.00000	0.00025	0.00074	0.00039	0.01753	-0.05443	0.01971	0.01581
CCC	0.00007	0.00006	0.00038	0.00075	0.00176	0.01228	-0.05266	0.03738
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.34: generator for the period [5].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.04024	0.03661	0.00288	0.00022	0.00046	0.00000	0.00000	0.00008
AA	0.00195	-0.02815	0.02415	0.00142	0.00011	0.00040	0.00010	0.00002
A	0.00028	0.00913	-0.03655	0.02452	0.00165	0.00065	0.00016	0.00017
BBB	0.00010	0.00043	0.01143	-0.02757	0.01297	0.00159	0.00044	0.00061
BB	0.00016	0.00031	0.00109	0.02264	-0.06487	0.03425	0.00300	0.00342
B	0.00000	0.00025	0.00074	0.00039	0.01741	-0.05405	0.01957	0.01570
CCC	0.00007	0.00006	0.00038	0.00075	0.00178	0.01243	-0.05330	0.03783
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.35: generator for the period [6].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.04086	0.03718	0.00293	0.00022	0.00047	0.00000	0.00000	0.00008
AA	0.00057	-0.00826	0.00709	0.00042	0.00003	0.00012	0.00003	0.00000
A	0.00024	0.00790	-0.03165	0.02123	0.00143	0.00056	0.00014	0.00014
BBB	0.00008	0.00036	0.00978	-0.02359	0.01110	0.00136	0.00038	0.00052
BB	0.00014	0.00028	0.00097	0.02026	-0.05805	0.03065	0.00268	0.00306
B	0.00000	0.00025	0.00073	0.00039	0.01728	-0.05365	0.01943	0.01558
CCC	0.00007	0.00006	0.00038	0.00076	0.00180	0.01258	-0.05397	0.03831
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.36: generator for the period [7].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.04873	0.04433	0.00349	0.00026	0.00056	0.00000	0.00000	0.00010
AA	0.00660	-0.09516	0.08165	0.00479	0.00038	0.00134	0.00034	0.00005
A	0.00028	0.00929	-0.03719	0.02495	0.00168	0.00066	0.00017	0.00017
BBB	0.00014	0.00063	0.01702	-0.04104	0.01930	0.00237	0.00066	0.00091
BB	0.00019	0.00037	0.00131	0.02724	-0.07804	0.04121	0.00360	0.00412
B	0.00000	0.00019	0.00056	0.00030	0.01320	-0.04100	0.01485	0.01191
CCC	0.00004	0.00004	0.00023	0.00046	0.00109	0.00763	-0.03273	0.02323
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.37: generator for the period [8].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.05301	0.04823	0.00380	0.00029	0.00061	0.00000	0.00000	0.00011
AA	0.00592	-0.08548	0.07334	0.00430	0.00035	0.00121	0.00031	0.00005
A	0.00031	0.01018	-0.04077	0.02735	0.00184	0.00073	0.00018	0.00019
BBB	0.00013	0.00056	0.01500	-0.03618	0.01702	0.00209	0.00058	0.00080
BB	0.00017	0.00034	0.00122	0.02539	-0.07273	0.03840	0.00336	0.00384
B	0.00000	0.00019	0.00056	0.00030	0.01315	-0.04085	0.01479	0.01186
CCC	0.00004	0.00004	0.00024	0.00047	0.00110	0.00769	-0.03296	0.02340
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.38: generator for the period [9].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.05992	0.05451	0.00429	0.00032	0.00069	0.00000	0.00000	0.00012
AA	0.00049	-0.00709	0.00608	0.00036	0.00003	0.00010	0.00003	0.00000
A	0.00020	0.00672	-0.02693	0.01807	0.00121	0.00048	0.00012	0.00012
BBB	0.00012	0.00053	0.01415	-0.03411	0.01605	0.00197	0.00055	0.00076
BB	0.00016	0.00033	0.00115	0.02395	-0.06860	0.03622	0.00317	0.00362
B	0.00000	0.00019	0.00056	0.00030	0.01314	-0.04082	0.01478	0.01185
CCC	0.00004	0.00004	0.00024	0.00047	0.00111	0.00773	-0.03317	0.02354
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.39: generator for the period [10].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.05878	0.05348	0.00421	0.00032	0.00067	0.00000	0.00000	0.00012
AA	0.00053	-0.00765	0.00656	0.00039	0.00003	0.00011	0.00003	0.00000
A	0.00022	0.00723	-0.02894	0.01941	0.00131	0.00052	0.00013	0.00013
BBB	0.00011	0.00047	0.01248	-0.03009	0.01416	0.00174	0.00048	0.00067
BB	0.00015	0.00030	0.00107	0.02215	-0.06346	0.03351	0.00293	0.00335
B	0.00000	0.00019	0.00055	0.00029	0.01312	-0.04075	0.01476	0.01183
CCC	0.00004	0.00004	0.00024	0.00047	0.00111	0.00779	-0.03341	0.02372
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.40: generator for the period [11].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.07461	0.06789	0.00534	0.00040	0.00085	0.00000	0.00000	0.00015
AA	0.00742	-0.10712	0.09191	0.00539	0.00043	0.00151	0.00038	0.00006
A	0.00025	0.00835	-0.03343	0.02242	0.00151	0.00059	0.00015	0.00015
BBB	0.00015	0.00066	0.01781	-0.04295	0.02020	0.00248	0.00069	0.00095
BB	0.00021	0.00042	0.00150	0.03118	-0.08932	0.04716	0.00412	0.00471
B	0.00000	0.00014	0.00042	0.00022	0.00984	-0.03057	0.01107	0.00888
CCC	0.00003	0.00002	0.00014	0.00029	0.00067	0.00471	-0.02022	0.01435
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.41: generator for the period [12].

	To Rating							
From Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.06712	0.06106	0.00481	0.00036	0.00077	0.00000	0.00000	0.00014
AA	0.00730	-0.10534	0.09039	0.00530	0.00043	0.00149	0.00038	0.00006
A	0.00025	0.00813	-0.03257	0.02185	0.00147	0.00058	0.00015	0.00015
BBB	0.00013	0.00058	0.01544	-0.03724	0.01752	0.00215	0.00060	0.00083
BB	0.00021	0.00041	0.00146	0.03032	-0.08685	0.04586	0.00401	0.00458
B	0.00000	0.00014	0.00042	0.00022	0.00988	-0.03068	0.01111	0.00891
CCC	0.00003	0.00002	0.00014	0.00029	0.00068	0.00473	-0.02028	0.01440
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.42: generator for the period [13].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.06712	0.06106	0.00481	0.00036	0.00077	0.00000	0.00000	0.00014
AA	0.00730	-0.10534	0.09039	0.00530	0.00043	0.00149	0.00038	0.00006
A	0.00025	0.00813	-0.03257	0.02185	0.00147	0.00058	0.00015	0.00015
BBB	0.00013	0.00058	0.01544	-0.03724	0.01752	0.00215	0.00060	0.00083
BB	0.00021	0.00041	0.00146	0.03032	-0.08685	0.04586	0.00401	0.00458
B	0.00000	0.00014	0.00042	0.00022	0.00988	-0.03068	0.01111	0.00891
CCC	0.00003	0.00002	0.00014	0.00029	0.00068	0.00473	-0.02028	0.01440
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.43: generator for the period [14].

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.03998	0.03638	0.00286	0.00022	0.00046	0.00000	0.00000	0.00008
AA	0.00425	-0.06139	0.05267	0.00309	0.00025	0.00087	0.00022	0.00003
A	0.00011	0.00357	-0.01431	0.00960	0.00065	0.00025	0.00006	0.00007
BBB	0.00010	0.00045	0.01210	-0.02917	0.01372	0.00168	0.00047	0.00065
BB	0.00020	0.00039	0.00138	0.02875	-0.08236	0.04349	0.00380	0.00435
B	0.00000	0.00014	0.00042	0.00022	0.00985	-0.03058	0.01107	0.00888
CCC	0.00003	0.00002	0.00015	0.00029	0.00068	0.00478	-0.02049	0.01455
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5.44: generator for the period [15].

Appendix 7: Chapter 5 - generators calculated with modifying the eigenvalues

$\Lambda_E(0) =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.0200	0.0171	0.0026	0.0000	0.0002	0.0002	0.0000	0.0001
AA	0.0017	-0.0254	0.0205	0.0026	0.0000	0.0002	0.0001	0.0003
A	0.0002	0.0058	-0.0239	0.0149	0.0023	0.0003	0.0001	0.0003
BBB	0.0001	0.0008	0.0118	-0.0292	0.0107	0.0045	0.0007	0.0005
BB	0.0001	0.0002	0.0023	0.0130	-0.0412	0.0166	0.0022	0.0069
B	0.0000	0.0002	0.0005	0.0029	0.0102	-0.0415	0.0051	0.0225
CCC	0.0001	0.0002	0.0006	0.0017	0.0034	0.0105	-0.0720	0.0556
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.45: the generator for the period $[0,1]$, with method 3.

$\Lambda_E(1) =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0.0199	0.0173	0.0023	-0.0001	0.0002	0.0002	0.0000	0.0000
AA	0.0017	-0.0255	0.0209	0.0024	-0.0001	0.0002	0.0001	0.0003
A	0.0002	0.0059	-0.0240	0.0151	0.0023	0.0002	0.0000	0.0003
BBB	0.0001	0.0007	0.0120	-0.0294	0.0110	0.0046	0.0007	0.0003
BB	0.0001	0.0002	0.0022	0.0132	-0.0417	0.0170	0.0022	0.0066
B	0.0000	0.0002	0.0005	0.0029	0.0105	-0.0419	0.0052	0.0226
CCC	0.0001	0.0002	0.0006	0.0017	0.0035	0.0107	-0.0730	0.0563
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.46: the generator for the period $[1,2]$, with method 3.

$\tilde{P}_E(0,1) =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	98.018%	1.668%	0.268%	0.000%	0.022%	0.016%	0.003%	0.006%
AA	0.168%	97.496%	2.004%	0.270%	0.003%	0.025%	0.006%	0.028%
A	0.022%	0.566%	97.657%	1.451%	0.233%	0.031%	0.006%	0.034%
BBB	0.012%	0.080%	1.155%	97.141%	1.041%	0.446%	0.069%	0.056%
BB	0.011%	0.022%	0.228%	1.255%	95.979%	1.595%	0.209%	0.700%
B	0.003%	0.023%	0.052%	0.285%	0.984%	95.947%	0.484%	2.221%
CCC	0.008%	0.015%	0.059%	0.163%	0.331%	0.996%	93.054%	5.374%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.47: time-inhomogeneous transition matrices for the period $[0,1]$, with method 3.

$\tilde{P}_E(1,2) =$

From Rating	To Rating AAA	AA	A	BBB	BB	B	CCC	D
AAA	98.029%	1.691%	0.243%	-0.008%	0.022%	0.016%	0.002%	0.005%
AA	0.170%	97.490%	2.036%	0.253%	-0.005%	0.024%	0.006%	0.027%
A	0.021%	0.574%	97.641%	1.475%	0.229%	0.022%	0.005%	0.032%
BBB	0.011%	0.076%	1.174%	97.115%	1.063%	0.453%	0.070%	0.036%
BB	0.011%	0.020%	0.224%	1.282%	95.933%	1.639%	0.216%	0.675%
B	0.003%	0.023%	0.047%	0.289%	1.011%	95.905%	0.492%	2.229%
CCC	0.008%	0.015%	0.058%	0.166%	0.340%	1.013%	92.962%	5.438%
D	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	100.000%

Table 5.48: time-inhomogeneous transition matrices for the period $[1,2]$, with method 3.

Appendix 8: Chapter 5 - Numerical exercise of section 5.14

Credit ID	S&P Credit Rating	Maturity	Current Balance	Current Spread over Libor rate	Country Name	S&P Industry
1	AAA	31-Jul-06	10,000	2.50%	USA	Aerospace & Defence
2	AA	31-Aug-06	3,000	2.50%	USA	Air transport
3	A	30-Sep-06	7,000	2.50%	USA	Automotive
4	BBB	31-Oct-06	20,000	2.50%	USA	Radio & Television
5	BB	30-Nov-06	6,000	2.50%	USA	Radio & Television
6	B	31-Dec-06	14,000	2.50%	USA	Chemicals & plastics
7	CCC	31-Jan-07	30,000	2.50%	USA	Financial intermediaries
8	AAA	28-Feb-07	9,000	2.50%	USA	Insurance
9	AA	31-Mar-07	21,000	2.50%	USA	Oil & gas
10	A	30-Apr-07	40,000	2.50%	USA	Food products
11	BBB	31-May-07	12,000	2.50%	USA	Rail industries
12	BB	30-Jun-07	28,000	2.50%	USA	Steel
13	B	31-Jul-07	50,000	2.50%	USA	Aerospace & Defence
14	CCC	31-Aug-07	15,000	2.50%	USA	Air transport
15	AAA	30-Sep-07	35,000	2.50%	USA	Automotive
16	AA	31-Oct-07	10,000	2.50%	USA	Radio & Television
17	A	30-Nov-07	3,000	2.50%	USA	Radio & Television
18	BBB	31-Dec-07	7,000	2.50%	USA	Chemicals & plastics
19	BB	31-Jul-06	20,000	2.50%	USA	Financial intermediaries
20	B	31-Aug-06	6,000	2.50%	USA	Insurance
21	CCC	30-Sep-06	14,000	2.50%	USA	Oil & gas
22	AAA	31-Oct-06	30,000	2.50%	USA	Publishing
23	AA	30-Nov-06	9,000	2.50%	USA	Rail industries
24	A	31-Dec-06	21,000	2.50%	USA	Steel
25	BBB	31-Jan-07	40,000	2.50%	USA	Aerospace & Defence
26	BB	28-Feb-07	12,000	2.50%	USA	Air transport
27	AAA	31-Mar-07	28,000	2.50%	USA	Automotive
28	A	30-Apr-07	50,000	2.50%	USA	Beverage & Tobacco
29	AAA	31-May-07	15,000	2.50%	USA	Radio & Television
30	AA	30-Jun-07	35,000	2.50%	USA	Chemicals & plastics
31	A	31-Jul-07	10,000	2.50%	USA	Financial intermediaries
32	BBB	31-Aug-07	3,000	2.50%	USA	Insurance
33	AAA	30-Sep-07	7,000	2.50%	USA	Oil & gas
34	B	31-Oct-07	20,000	2.50%	USA	Publishing
35	CCC	30-Nov-07	6,000	2.50%	USA	Rail industries
36	AAA	31-Dec-07	14,000	2.50%	USA	Steel
37	AA	31-Jul-06	30,000	2.50%	USA	Aerospace & Defence
38	A	31-Aug-06	9,000	2.50%	USA	Air transport
39	BBB	30-Sep-06	21,000	2.50%	USA	Automotive
40	BB	31-Oct-06	40,000	2.50%	USA	Cable & satellite television
41	B	30-Nov-06	12,000	2.50%	USA	Radio & Television
42	AAA	31-Dec-06	28,000	2.50%	USA	Chemicals & plastics
43	AAA	31-Jan-07	50,000	2.50%	USA	Financial intermediaries
44	AAA	28-Feb-07	15,000	2.50%	USA	Insurance
45	AAA	31-Dec-07	35,000	2.50%	USA	Oil & gas
46	BBB	31-Dec-07	40,000	2.50%	USA	Industrial equipment
47	BB	31-Dec-07	12,000	2.50%	USA	Rail industries
48	B	31-Dec-07	28,000	2.50%	USA	Steel
49	AA	31-Dec-07	10,000	2.50%	USA	Financial intermediaries
50	AA	31-Dec-07	10,000	2.50%	USA	Financial intermediaries

Table 5.49: Collateral portfolio, with a start date as 1/1/2004 (i.e. 16 quarters until maturity).

Junior Note									
	Rating Transition Model with Correlation			Survival Model with Correlation			Merton Model with Correlation		
	0	0.25	0.5	0	0.25	0.5	0	0.25	0.5
Premium	60.376 bps	239.613 bps	435.924 bps	52.273 bps	235.009 bps	531.447 bps	60.196 bps	210.124 bps	388.849 bps
Interest Loss	0.004%	0.576%	1.321%	0.004%	0.302%	1.854%	0.005%	0.397%	1.235%
AL	3.97 Yrs	3.91 Yrs	3.85 Yrs	3.97 Yrs	3.89 Yrs	3.80 Yrs	3.97 Yrs	3.91 Yrs	3.85 Yrs
EL (%)	1.894%	4.469%	6.726%	1.659%	5.567%	9.480%	1.890%	4.425%	6.390%
SE (%)	0.073%	0.151%	0.196%	0.066%	0.163%	0.235%	0.075%	0.145%	0.189%
EL+SE (%)	1.967%	4.620%	6.921%	1.724%	5.730%	9.715%	1.965%	4.571%	6.579%
Rating	Ba1	Ba3	B1	Ba1	B1	B2	Ba1	Ba3	B1
EL|L=99.5%	61.463%	96.025%	97.785%	56.054%	94.074%	96.986%	63.288%	94.536%	97.486%
EL|L=99.9%	79.136%	98.193%	98.889%	71.673%	96.391%	98.228%	80.064%	97.721%	98.942%

Table 5.50: Junior Note results.

Mezzanine Note									
	Rating Transition Model with Correlation			Survival Model with Correlation			Merton Model with Correlation		
	0	0.25	0.5	0	0.25	0.5	0	0.25	8
Premium	0.088 bps	31.744 bps	78.875 bps	0.000 bps	14.263 bps	95.684 bps	0.086 bps	31.521 bps	78.265 bps
Interest Loss	0.000%	0.064%	0.283%	0.000%	0.029%	0.310%	0.000%	0.094%	0.279%
AL	4.00 Yrs	3.99 Yrs	3.97 Yrs	4.00 Yrs	3.99 Yrs	3.96 Yrs	4.00 Yrs	3.99 Yrs	3.97 Yrs
EL (%)	0.003%	0.525%	1.246%	0.000%	0.315%	1.951%	0.003%	0.400%	1.127%
SE (%)	0.002%	0.060%	0.094%	0.000%	0.043%	0.118%	0.003%	0.053%	0.090%
EL+SE (%)	0.005%	0.585%	1.340%	0.000%	0.358%	2.069%	0.005%	0.453%	1.217%
Rating	Aa1	Baa2	Ba1	Aaa	Baa1	Ba1	Aa1	Baa1	Baa3
EL|L=99.5%	0.000%	78.561%	92.504%	0.000%	54.190%	93.057%	0.000%	68.927%	92.004%
EL|L=99.9%	0.000%	95.064%	97.154%	0.000%	90.055%	96.173%	0.000%	95.245%	97.786%

Table 5.51: Mezzanine Note results.

Senior Note									
	Rating Transition Model with Correlation			Survival Model with Correlation			Merton Model with Correlation		
	0	0.25	0.5	0	0.25	0.5	0	0.25	8
Premium	0.000 bps	0.185 bps	1.107 bps	0.000 bps	0.061 bps	1.175 bps	0.000 bps	0.278 bps	1.219 bps
Interest Loss	0.000%	0.000%	0.005%	0.000%	0.000%	0.004%	0.000%	0.002%	0.008%
AL	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs	3.24 Yrs
EL (%)	0.000%	0.005%	0.029%	0.000%	0.002%	0.031%	0.000%	0.007%	0.032%
SE (%)	0.000%	0.002%	0.005%	0.000%	0.001%	0.004%	0.000%	0.003%	0.006%
EL+SE (%)	0.000%	0.006%	0.034%	0.000%	0.002%	0.035%	0.000%	0.010%	0.037%
Rating	Aaa	Aa1	Aa3	Aaa	Aa1	Aa3	Aaa	Aa1	Aa3
EL|L=99.5%	0.000%	1.873%	5.379%	0.000%	0.000%	5.005%	0.000%	0.000%	6.063%
EL|L=99.9%	0.000%	3.764%	10.341%	0.000%	0.000%	9.464%	0.000%	6.173%	13.971%

Table 5.52: Senior Note results.

Collateral									
	Rating Transition Model with Correlation			Survival Model with Correlation			Merton Model with Correlation		
	0	0.25	0.5	0	0.25	0.5	0	0.25	0.5
EL (%)	3.342%	3.277%	3.149%	3.295%	3.182%	3.099%	3.303%	3.299%	3.199%
EL|L=99.5%	8.840%	13.867%	17.339%	8.569%	12.714%	17.880%	8.941%	13.408%	16.994%
EL|L=99.9%	9.367%	15.441%	19.584%	9.059%	13.876%	19.607%	9.542%	15.220%	19.876%

Table 5.53: Expected loss and shortfalls of the Collateral.

Equity IRR									
	Rating Transition Model with Correlation			Survival Model with Correlation			Merton Model with Correlation		
	0	0.25	0.5	0	0.25	0.5	0	0.25	0.5
IRR (%)	30.254%	29.315%	28.842%	30.502%	28.882%	27.523%	30.377%	29.176%	28.832%

Table 5.54: Equity expected internal rate of return (IRR).

	Rating Transition Model with Collateral Tests								
	Junior Note			Mezzanine Note			Senior Note		
	0	0.25	0.5	0	0.25	0.5	0	0.25	0.5
Premium	57.625 bps	228.028 bps	370.094 bps	0.104 bps	31.317 bps	62.490 bps	0.000 bps	0.313 bps	1.079 bps
Interest Loss	0.002%	0.933%	1.541%	0.000%	0.081%	0.261%	0.000%	0.000%	0.005%
AL	3.96 Yrs	3.89 Yrs	3.84 Yrs	3.93 Yrs	3.91 Yrs	3.89 Yrs	3.10 Yrs	3.10 Yrs	3.09 Yrs
EL (%)	1.808%	5.001%	6.452%	0.003%	0.685%	1.041%	0.000%	0.008%	0.027%
SE (%)	0.071%	0.165%	0.191%	0.003%	0.069%	0.085%	0.000%	0.002%	0.005%
EL+SE (%)	1.879%	5.166%	6.644%	0.007%	0.754%	1.126%	0.000%	0.010%	0.032%
Rating	Ba1	Ba3	B1	Aa1	Baa3	Baa3	Aaa	Aa1	Aa3
EL|L=99.5%	58.965%	97.288%	99.508%	0.000%	86.494%	92.278%	0.000%	0.000%	5.237%
EL|L=99.9%	76.591%	99.898%	99.899%	0.000%	93.836%	96.654%	0.000%	4.282%	10.696%

Table 5.55: Results with Collateral Tests.

	Equity IRR		
	Rating Transition Model with Correlation		
	0	0.25	0.5
IRR (%)	21.834%	21.205%	21.199%

Table 5.56: Equity expected internal rate of return (IRR) with Collateral Tests.

Appendix 9: Chapter 6 – Numerical exercise in section 6.6

ID EDSs	EDS Notionals (000)	Equity	Equity Closing Price as at 11/2/05	Quantity
1	19,964	Tesco	315.50	63,276
2	20,211	Shell	473.75	42,661
3	8,592	Rbs	1,830.00	4,695
4	37,144	British Airways	279.00	133,134
5	20,624	Barclays	606.50	34,005
	106,534			

Table 6.1: EDSs portfolio.

	British Airways	Rbs	Shell	Tesco	Barclays
no. of obser.	1274	1274	1274	1274	1274
mean	-0.0004	0.0004	-0.0001	0.0004	0.0003
uncond. variance	0.0010	0.0005	0.0003	0.0003	0.0005
skew	-0.0826	-0.2026	-0.2388	0.1650	0.1332
kurt	6.8162	7.5852	5.3739	5.4902	4.8774

Table 6.2: Data key statistics of daily returns.

	British Airways	Rbs	Shell	Tesco	Barclays
ML	2,681.80	3,206.88	3,401.20	3,413.28	3,158.04
d.o.f.	∞	∞	∞	∞	∞
λ	0.0000000	0.0057500	0.0000000	0.0499996	0.0521247
s.e.		(0.0007969)		(0.0008008)	(0.0007913)
θ	0.5084075	0.8053928	0.4759412	0.7998863	1.3439123
s.e.	(0.0051381)	(0.0056753)	(0.0073503)	(0.0111737)	(0.0217188)
ω	0.0000209	0.0000052	0.0000073	0.0000199	0.0000138
s.e.	(0.0000014)	(0.0000003)	(0.0000004)	(0.0000008)	(0.0000008)
α	0.0830184	0.0774882	0.0855389	0.0499976	0.0349075
s.e.	(0.0007306)	(0.0007119)	(0.0008799)	(0.0005719)	(0.0004883)
β	0.8773169	0.8689995	0.8737772	0.8478198	0.8652952
s.e.	(0.0023322)	(0.0013999)	(0.0021559)	(0.0031171)	(0.0023978)
σ^{*2} Daily	0.001147	0.001606	0.000340	0.000283	0.000377
σ^{*P} Annualiz.	0.5419	0.6412	0.2952	0.2693	0.3105
$E(\varepsilon^*)$	(0.0079)	0.0002	(0.0163)	(0.0322)	(0.0496)
$Var(\varepsilon^*)$	1.0000	1.0000	1.0040	0.9741	1.0000
$Skew(\varepsilon^*)$	(0.0204)	(0.1582)	(0.4024)	0.1087	0.0032
$Kurt(\varepsilon^*)$	5.6498	6.3228	4.3850	4.4547	3.9428
$Kurt(\zeta^*)$	8.2901	2.3140	4.5395	4.4911	3.9356
$Kurt(\zeta)$	8.3051	2.4958	4.5395	4.4973	3.9337

Table 6.3: Parameter estimates obtained by maximising the log-likelihood function in 6.20. σ^{*P} *Annualiz* is calculated under the assumption of 256 trading days per year.

	British Airways	Rbs	Shell	Tesco	Barclays
ML	3,751.98	4,290.65	4,460.46	4,501.94	4,235.66
d.o.f.	13.73	13.01	14.00	13.62	13.97
λ	0.0000000	0.0139784	0.0040561	0.0186036	0.0039259
s.e.		(0.0008103)	(0.0007854)	(0.0007966)	(0.0007862)
θ	0.5428737	0.8015425	0.4515041	0.2214606	0.9544907
s.e.	(0.0108436)	(0.0108912)	(0.0110862)	(0.0082023)	(0.0168560)
ω	0.0000107	0.0000051	0.0000036	0.0000043	0.0000079
s.e.	(0.0000017)	(0.0000005)	(0.0000005)	(0.0000004)	(0.0000007)
α	0.0579185	0.0740165	0.0709117	0.0776921	0.0550352
s.e.	(0.0008454)	(0.0009599)	(0.0009624)	(0.0008301)	(0.0007530)
β	0.9128292	0.8734285	0.9046500	0.9068993	0.8777104
s.e.	(0.0028510)	(0.0021578)	(0.0025166)	(0.0024977)	(0.0025607)
σ^{*2} Daily	0.000876	0.001029	0.000359	0.000370	0.000462
σ^{*P} Annualiz.	0.4736	0.5131	0.3030	0.3077	0.3440
$E(\varepsilon^*)$	(0.0100)	(0.0079)	(0.0193)	0.0090	(0.0035)
$Var(\varepsilon^*)$	1.0466	1.0000	1.0117	1.0000	1.0090
$Skew(\varepsilon^*)$	(0.1308)	(0.1613)	(0.4477)	0.2421	0.0020
$Kurt(\varepsilon^*)$	6.3742	6.3253	4.8040	4.7833	4.1000
$Kurt(\zeta^*)$	16.0086	1.6395	5.6450	5.3029	4.1320
$Kurt(\zeta)$	16.0236	2.2292	5.6907	5.3565	4.1331

Table 6.4: Parameter estimates obtained by maximising the log-likelihood function in 6.21. σ^{*P} *Annualiz* is calculated under the assumption of 256 trading days per year.

Equity Name	Option Ticker	Strike	EDS Trigger	Maturity	Impl. Vol.	Put Option Mid Price
Tesco	TSCO LN 9 P260 Equity	260.00	94.65	16/09/2005	19.662	1.75
Shell	SHEL LN 12/06 P330 Equity	330.00	142.13	15/12/2006	20.756	4.25
Rbs	RBS LN 9 P1600 Equity	1600.00	549.00	16/09/2005	17.253	19.00
British Airways	BAY LN 9 P260 Equity	260.00	83.70	16/09/2005	26.078	11.50
Barclays	BARC LN 9 P550 Equity	550.00	181.95	16/09/2005	19.959	16.00

Table 6.5: Implied volatilities, source Bloomberg except EDS Triggers, which are calculated as 30% of the equity prices of Table 6.1. All data as at 11/02/2005.

Single Name EDS							
		t-NGARCH-M		normal-NGARCH-M		Rubinstein & Reiner	
Equity Name	EDS Nominal (000)	Premium	EL (£) (000)	Premium	EL (£) (000)	Premium	EL (£) (000)
Tesco	19,964	42.495 bps	159.92	0.000 bps	0.00	0.012 bps	0.05
Shell	20,211	37.263 bps	142.08	0.000 bps	0.00	0.038 bps	0.15
Rbs	8,592	187.305 bps	299.86	38.241 bps	61.88	0.000 bps	0.00
British Airways	37,144	35.029 bps	245.39	7.330 bps	51.48	1.746 bps	12.97
Barclays	20,624	17.376 bps	67.70	0.000 bps	0.00	0.016 bps	0.07
	106,534		914.95		113.36		13.23

Table 6.6: Single-name EDS premiums and ELs calculated with *t*-NGARCH-M, normal-NGARCH-M and Rubinstein and Reiner models.

Path-Dependent Binary Barrier Put Option									
		t-NGARCH-M		normal-NGARCH-M		Rubinstein & Reiner			
Equity Name	No.of Options	Option Premium (000)	EDS Premium	Option Premium (000)	EDS Premium	Option Premium (000)	EDS Premium	Implied Vols t-NGARCH-M
Tesco	63,276	220.70	55.623 bps	0.00	0.000 bps	0.05	0.012 bps	0.3704
Shell	42,661	208.10	51.783 bps	1.87	0.462 bps	0.15	0.038 bps	0.3672
Rbs	4,695	323.00	192.053 bps	68.00	39.750 bps	0.00	0.000 bps	0.4570
British Airways	133,134	293.53	39.689 bps	52.11	7.019 bps	12.97	1.746 bps	0.3537
Barclays	34,005	51.05	12.393 bps	0.00	0.000 bps	0.07	0.016 bps	0.3093
	277,771	1,096.38		121.97		13.23		

Table 6.7: Put option and EDS premiums with Implied Volatilities in t-NGARCH-M. The EDS premiums are calculated assuming constant default intensity. To calculate the binary barrier put option premium of one equity, divide the option premium by the No. of options.

Junior note with normal-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	91.658 bps	1.717%	0.401%	2.119%	Ba3	96.893%
	0.25	0.1609	0.0000	81.430 bps	1.527%	0.379%	1.906%	Ba2	96.896%
	0.50	0.3333	0.0000	86.103 bps	1.615%	0.389%	2.004%	Ba3	96.694%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	91.667 bps	1.718%	0.402%	2.119%	Ba3	97.373%
	0.50	0.3333	0.0819	91.221 bps	1.710%	0.400%	2.110%	Ba3	97.155%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	96.570 bps	1.809%	0.411%	2.221%	Ba3	97.373%
	0.50	0.3333	0.1705	106.905 bps	2.001%	0.432%	2.433%	Ba3	97.135%
	a-parameter					0.000%			
Clayton copula	1.50	0.4286	0.6300	85.592 bps	1.607%	0.387%	1.994%	Ba3	96.500%
	3.00	0.6000	0.7937	81.646 bps	1.531%	0.380%	1.911%	Ba3	96.569%

Table 6.8: junior note results with a normal NGARCH-M (1,1).

Mezzanine note with normal-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	25.241 bps	0.477%	0.168%	0.644%	Ba1	29.238%
	0.25	0.1609	0.0000	19.660 bps	0.371%	0.140%	0.511%	Baa3	27.909%
	0.50	0.3333	0.0000	24.322 bps	0.459%	0.165%	0.625%	Ba1	34.709%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	25.296 bps	0.478%	0.168%	0.646%	Ba1	31.794%
	0.50	0.3333	0.0819	25.131 bps	0.475%	0.167%	0.641%	Ba1	34.918%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	25.915 bps	0.489%	0.168%	0.657%	Ba1	31.709%
	0.50	0.3333	0.1705	27.377 bps	0.517%	0.170%	0.686%	Ba1	33.343%
	a-parameter								
Clayton copula	1.50	0.4286	0.6300	28.386 bps	0.536%	0.188%	0.724%	Ba1	39.360%
	3.00	0.6000	0.7937	23.761 bps	0.449%	0.165%	0.614%	Ba1	33.034%

Table 6.9: Mezzanine note results with a normal-NGARCH-M (1,1).

Senior note with normal-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	1.434 bps	0.027%	0.016%	0.043%	A3	0.000%
	0.25	0.1609	0.0000	0.960 bps	0.018%	0.013%	0.031%	A2	0.000%
	0.50	0.3333	0.0000	1.416 bps	0.027%	0.015%	0.042%	A3	0.000%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	1.440 bps	0.027%	0.016%	0.043%	A3	0.000%
	0.50	0.3333	0.0819	1.428 bps	0.027%	0.016%	0.043%	A3	0.000%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	1.434 bps	0.027%	0.016%	0.043%	A3	0.000%
	0.50	0.3333	0.1705	1.656 bps	0.031%	0.018%	0.050%	A3	0.000%
	a-parameter								
Clayton copula	1.50	0.4286	0.6300	1.884 bps	0.036%	0.018%	0.053%	A3	0.000%
	3.00	0.6000	0.7937	1.633 bps	0.031%	0.018%	0.049%	A3	0.000%

Table 6.10: Senior note results with a normal-NGARCH-M (1,1).

Junior note with t-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	616.748 bps	11.121%	0.962%	12.083%	Caa	97.671%
	0.25	0.1609	0.0000	594.188 bps	10.738%	0.947%	11.686%	Caa	97.597%
	0.50	0.3333	0.0000	537.139 bps	9.726%	0.908%	10.634%	Caa	97.596%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	615.343 bps	11.110%	0.961%	12.071%	Caa	97.596%
	0.50	0.3333	0.0819	568.753 bps	10.283%	0.930%	11.213%	Caa	97.596%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	679.860 bps	12.229%	1.001%	13.230%	Caa	97.596%
	0.50	0.3333	0.1705	674.086 bps	12.133%	0.998%	13.131%	Caa	97.597%
Clayton copula	a-parameter					0.000%			
	1.50	0.4286	0.6300	436.856 bps	7.997%	0.830%	8.827%	B3	97.594%
	3.00	0.6000	0.7937	409.128 bps	7.514%	0.806%	8.321%	B3	97.591%

Table 6.11: Junior note results with a *t*-NGARCH-M (1,1).

Mezzanine note with t-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	301.755 bps	5.583%	0.650%	6.233%	B2	96.500%
	0.25	0.1609	0.0000	282.047 bps	5.219%	0.627%	5.846%	B2	96.493%
	0.50	0.3333	0.0000	248.093 bps	4.599%	0.589%	5.187%	B2	96.409%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	302.317 bps	5.590%	0.651%	6.240%	B2	96.480%
	0.50	0.3333	0.0819	269.780 bps	4.995%	0.614%	5.610%	B2	96.401%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	328.530 bps	6.065%	0.674%	6.739%	B3	96.480%
	0.50	0.3333	0.1705	323.376 bps	5.972%	0.668%	6.640%	B3	96.402%
Clayton copula	a-parameter								
	1.50	0.4286	0.6300	229.143 bps	4.262%	0.578%	4.841%	B2	96.509%
	3.00	0.6000	0.7937	191.128 bps	3.561%	0.522%	4.083%	B1	96.618%

Table 6.12: Mezzanine note results with a *t*-NGARCH-M (1,1).

Senior note with t-NGARCH-M									
	Linear Corr.	Kendall tau	Tail Dep.	Premium	EL (%)	SE (%)	EL+SE (%)	Rating	EL|L=99%
Normal copula	0.00	0.0000	0.0000	10.464 bps	0.198%	0.040%	0.238%	Baa2	10.418%
	0.25	0.1609	0.0000	9.482 bps	0.179%	0.039%	0.218%	Baa2	10.321%
	0.50	0.3333	0.0000	9.656 bps	0.183%	0.043%	0.225%	Baa2	11.764%
t-Student copula, d.o.f. 10	0.25	0.1609	0.0261	11.396 bps	0.215%	0.044%	0.259%	Baa3	10.921%
	0.50	0.3333	0.0819	10.200 bps	0.193%	0.043%	0.236%	Baa2	11.573%
t-Student copula, d.o.f. 6	0.25	0.1609	0.0796	11.550 bps	0.218%	0.042%	0.260%	Baa3	10.445%
	0.50	0.3333	0.1705	14.859 bps	0.281%	0.054%	0.335%	Baa3	12.950%
Clayton copula	a-parameter								
	1.50	0.4286	0.6300	18.532 bps	0.350%	0.081%	0.431%	Baa3	20.236%
	3.00	0.6000	0.7937	20.048 bps	0.379%	0.089%	0.468%	Baa3	24.345%

Table 6.13: Senior note results with a *t*-NGARCH-M (1,1).