VALUATION OF SINGLE-FACTOR INTEREST RATE DERIVATIVES

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Submitted for the degree of
Doctor of Philosophy

City University, London

The research was conducted at:

City University Business School
Centre for Mathematical Trading and Finance

February 2000

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Acknowledgements

It gives me great pleasure to formally acknowledge my debt to Professor Elias Dinenis and Professor Giovanni Barone-Adesi for their constant advice, suggestions and encouragement during the preparation of this Thesis. I am grateful for our many hours of conversation on the subject of this Thesis. I am also grateful to Professor Elias Dinenis for his financial assistance during the period of study.

A paper based upon Chapter 2 has been published in The Journal of Financial Engineering, Volume 6, Number 1, March 1997. I would also like to thank Professor Walter Allegretto for introducing me to the Box Method in Chapter 3, and the free boundary tracking approach in Chapter 4. I would like to thank Dr K.B. Nowman for the parameters used in Chapter 5.

Finally I would like to thank all my friends, colleagues and teachers throughout all my studies. In particular I would like to thank Professor J. Tennyson, Professor I. Allan, and Mr. P. Booth. I would also like to thank my Ph.D. friends and colleagues George, Manoj, Ash, Dennis, Paul, Jeremy, Sotiris, Ane, Alvaro, Illidio, Annabela, Mohammed, Arreeya, Ayo, Simona, John, Mara, Max, Aslihan, Alex, Christina, Shelina, Peter, Ron.

To my parents

Declaration

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Abstract

The seminal papers of Black-Scholes and Merton stimulated growth not only of equity commodity derivatives but also of term structure interest rate models and the valuation of bonds and contingent claims based on these term structure interest rate models. Today research into term structure models is important both to academics and practitioners alike. Unfortunately bond prices and interest rate contingent claim prices based on these term structure models, with few exceptions cannot be valued analytically. To date a number of numerical methods have been developed to solve this problem. The objective of this thesis is to test the existing numerical methods as well as introducing a new method within the context of the single factor interest CKLS model – the CKLS model encloses the earlier single factor term structure of interest rate models.

CHAPTER 1.

LITERATURE REVIEW, OBJECTIVES AND OUTLINE OF THE THESIS

1.1. Introduction

The seminal paper of Black and Scholes (1973) and Merton (1973) resulted in a rapid growth of the financial derivatives market such that today it has become an important and dynamic component of the world financial markets, and an area of active research in academia. Since the publication of the seminal papers academic researchers have focused on the theoretical valuation of both equity and interest rate contingent claims with more emphasis on equity contingent claims. However, recently more attention has been focused the valuation of claims whose values depend on the term structure of interest rates and its evolution over time. This change in research orientation in academia is due to the expansion in recent years of fixed income derivatives as interest rate risk management tools.

Interest rate risk comprises of market risk and yield curve risk. The market risk is due to the changes in the level of interest rate. Yield curve risk arises due to shape risk and volatility risk. The shape risk is due to the changes in the shape of the yield curve, which in turn is, due to changes in the interest. The volatility risk is due to changes in interest rate volatility. In the financial markets many fixed income products are specifically designed to hedge against the above types of risks. For example, interest rate futures, forwards, floating rate notes are used for hedging against market risk. Swaps are used to hedge against shape risk as their returns depend on changes in the shape of the yield curve. Options are used to hedge against volatility risk.

There are two aspects to the modeling of interest rate term structure models and interest rate contingent claims. The first is the specification of alternative interest rate processes leading to arbitrage-free pricing models for bonds and contingent claims. The second is the numerical implementation of these models, where an analytical solution is often not available. Numerical implementation allows incorporation of characteristics not possible with analytical implementation, such as the early exercise feature associated with American contingent claims.

In this Chapter we discuss the different term structure models which have been proposed, as well as the numerical methods used for both stock and interest rate contingent claims. In Section 2 we discuss the interest rate models. Section 3 discusses the numerical methods. In Section 4, we state the objectives of the thesis. Section 5 contains an outline of the thesis.

1.2. Interest Rate Models

The valuation of fixed income instruments is more challenging than the valuation of equity instruments as those two categories of assets exhibit different set of characteristics. For example, one of the main differences between equity and a coupon-paying bond is the certainty at some valuation date of the amounts and corresponding dates of the different coupons and face value. This has the implication that near the final maturity date of the bond; the probability of an increase in value of a par bond is much small than it is at some other valuation date. This is not so for equity. Yet, another result of this price effect is that the corresponding volatility of possible price movements decreases as the maturity date of the bond decreases. This leads to a decrease in the range of possible bond price as the maturity date increases.

One of the basic assumptions in the classical equity option valuation problem is that the interest rate remains constant. Clearly such as an assumption for fixed income instruments is theoretically inconsistent. Another feature distinguishing interest rate models from equity models is the need for interest rate models to exhibit mean reversion and for the volatility to be dependent on the interest rate. Thus the relationship between bond values and the term structure of interest rates implied by future payments leads to stochastic formulation of the yield curve over time.

To date two separate approaches that take the above-mentioned characteristics of fixed income instruments have been proposed. The first approach has been to propose a plausible model for the short-term interest rate, which depends on the market price of risk explicitly. Over the years a number of such short term interest rate models have been proposed including the most general Chan, Karolyi, Longstaff and Sanders (CKLS, 1992). The CKLS model encloses earlier interest rate models proposed by Vasicek (1977), Brennan and Schwartz (1979), and Cox-Ingersloll-Ross (CIR, 1985). The second approach pioneered by Ho-Lee (1986) and HJM (1992) does not take into account the market price of risk explicitly. This approach involves taking the current market term structure of interest rates to develop a no-arbitrage yield curve, which depends on the initial forward rate curve. For subsequent discussions we shall refer to the models based on the first approach as, Equilibrium approach and models based on the second approach as, "Arbitrage Free Models".

1.2.1. Equilibrium Models

In this section, we derive the mathematical structure of single-factor term structure models based on Vasicek (1977). Further, we discuss the major two factor interest rate models that have also been proposed.

We make the following assumptions with regard to single-factor term structure models:

1. The bond market is frictionless: no (distorting) taxes, no transaction costs, no short sale, and all bonds are infinitely divisible.
2. Investors always prefer more wealth to less.
3. All bond prices $P(t,T)$ for all $P > t$ depend only on a single state factor: the short rate r (in addition to t and T). The changes in the yield curve, therefore, at different maturities are perfectly correlated.

Let $P(t,s)$ denote the price at time t of a discount bond maturing at time s, $s \leq t$ with unit maturity value.

$$P(s,s) = 1$$

The yield to maturity $R(t,T)$ on a bond with maturity date $s = t + T$ is:

$$R(t,T) = -\frac{1}{T}\ln P(t,t+T)$$

The instantaneous spot rate at time t is given by:

$$r(t) = R(t,0) = \lim_{T \to 0} R(t,T)$$

Assume that the spot rate $r(t)$ follows a continuous Markov process and is defined by the following stochastic differential equation

$$dr(t) = f(r,t)dt + \rho(r,t)dz \tag{1.1}$$

where $z(t)$ is a Wiener process. $f(r,t)$, $\rho^2(r,t)$ are the instantaneous drift and variance respectively of the process $r(t)$.

Application of Ito's differential rule, leads to the following stochastic differential equation for bond price.

$$dP(t,s,r) = P(t,s,r)\mu(t,s,r)dt - P(t,s,r)\sigma(t,s,r)dz \tag{1.2}$$

where:

$$\mu(t,s,r) = \frac{1}{P(t,s,r)}\left[\frac{\partial}{\partial t} + f(r,t) + \frac{1}{2}\rho^2(r,t)\frac{\partial^2}{\partial r^2}\right]P(t,s,r) \tag{1.3}$$

$$\sigma(t,s,r) = -\frac{\rho(t,s,r)}{P(t,s,r)}\frac{\partial P(r,s,r)}{\partial r} \tag{1.4}$$

Suppose we have an investor who at time t issues an amount W_1 of a bond with maturity date s_1, and simultaneously buys an amount W_2 of bond maturity at time s_2. The total value of this portfolio is $W = W_2 - W_1$. The value of this portfolio changes according to Merton's accumulation equation

$$dW = [W_2\mu(t,s_2,r) - W_1\mu(t,s_1,r)]dt - [W_2\sigma(t,s_2,r) - W_1\sigma(t,s_1,r)]dz \tag{1.5}$$

We now choose W_1 and W_2 so as to make the evolution of the portfolio riskless. We find that the necessary expressions for W_1 , W_2 and dW are:

$$W_1 = \frac{\sigma(t,s_2,r)W}{\sigma(t,s_1,r)-\sigma(t,s_2,r)} \tag{1.6}$$

$$W_2 = \frac{\sigma(t,s_1,r)}{\sigma(t,s_1,r)-\sigma(t,s_2,r)} \tag{1.7}$$

$$dW = \frac{W[\mu(t,s_2,r)\sigma(t,s_1,r)-\mu(t,s_1,r)\sigma(t,s_2,r)]}{\sigma(t,s_1,r)-\sigma(t,s_2,r)} \tag{1.8}$$

Further, we let a riskless loan W accumulate at spot rate r(t) such that:

$$dW = Wr(t)dt \tag{1.9}$$

Equating the above two equations after algebraic manipulation gives:

$$\frac{\mu(t,s_1,r)-r(t)}{\sigma(t,s_1,r)} = \frac{\mu(t,s_2,r)}{\sigma(t,s_2,r)} \tag{1.10}$$

The above expression holds for arbitrary maturity dates s_1 and s_2. Thus the following ratio is independent of s.

$$\frac{\mu(t,s,r)-r(t)}{\sigma(t,s,r)} \tag{1.11}$$

We let $\lambda(r)$ denote the common value of such a ratio for a bond of any maturity date. $\lambda(r)$ may be interpreted as the market price of risk, as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk.

Thus for an arbitrary maturity date s

$$\lambda(r)\sigma(t,s,r)=\mu(t,s,r)-r \qquad (1.12)$$

Substitution into our original stochastic partial differential equation yields.

$$\frac{\partial P}{\partial t}+(f+\rho\lambda)\frac{\partial P}{\partial r}+\frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}-rP=0 \qquad (1.13)$$

The short-term interest rate, which is the variable driving the above partial differential equation is one of the most fundamental and important prices determined in the financial markets. Different researchers have used alternative specifications of the short-term interest rate process. Chan, Karolyi, Longstaff and Sanders (CKLS) (1992) suggested a general formulation, which encloses the common single-factor term structure models. Expressing their general model using our notation:

$$dr_t = k(\theta-r)dt+\sigma r^{\gamma}dz_t \qquad (1.14)$$

(1)	Merton	$dr_t = k\theta dt+\sigma dz_t$
(2)	Vasicek	$dr_t = k(\theta-r)dt+\sigma dz_t$
(3)	CIR SR	$dr_t = k(\theta-r)dt+\sigma\sqrt{r}dz_t$
(4)	Dothan	$dr_t = \sigma r dz_t$
(5)	GBM	$dr_t = -krdt+\sigma dz_t$
(6)	Brennan-Schwartz	$dr_t = k(\theta-r)dt+\sigma r dz_t$
(7)	CIR VR	$dr_t = \sigma r^{\frac{3}{2}}dz_t$

(8) CEV $dr_t = -krdt + \sigma r^{\gamma} dz_t$

Merton (Model 1) (1973) used the simple Brownian motion with drift to model the short-term interest rate process. He derived analytical option prices based on this model. Vasicek (Model 2)(1977) used the Ornstein-Uhlenbeck process to derive an equilibrium model of bond prices. Jamshidian (1989) and Gibson and Schwartz (1990) have further applied this Gaussian model for the interest rate. The square root (SR) (Model 3) process by Cox-Ingersoll-Ross (CIR)(1985) has been extensively applied to value interest-rate contingent claims. For, example Dunn and McConnell (1981) used the SR to value mortgage-backed securities, CIR (1985) to value discount bond and contingent claims, futures and futures option pricing models by Ramaswamy and Sundaresan (1986), the swap pricing model by Sundaresan (1989), and the yield option valuation model by Longstaff (1990). Model 4 is used by Dothan (1978) to value discount bonds and has been further used by Brennan and Schwartz (1977) in developing numerical models of saving retractable, and callable bonds. Model 5 is the Geometric Brownian Motion applied to interest rates. Model 6 is the log-normal interest rate process used by Brennan and Schwartz (1980) in deriving convertible bond prices, and further used by Courtadon (1982) to develop the finite difference numerical method to value bonds and interest rate contingent claims. Model 7 is used by CIR (1980) in the study of variable-rate (VR) securities. Constantinides and Ingersoll (1984) also use a similar model to value bonds in the presence of taxations. Model 8 is used by Marsh and Rosenfeld (1983) to value equilibrium bond prices.

The resulting partial differential equation for the bond and contingent claims subject to the appropriate boundary conditions based on the CKLS model is:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2}[k(\theta - r) - \lambda(r)\sigma(r)]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0 \qquad (1.15)$$

Researchers have given different functional relationships to $\lambda(r)\sigma(r)$. For example Vasicek (1977) uses $\lambda\sigma$, CIR (1985) uses. CKLS take $\lambda = 0$, thus equation (1.15) becomes:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2} + k(\theta - r)\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0 \qquad (1.16)$$

The main advantage of one-factor models is their simplicity as the entire yield curve is a function of single state variable. The single state variable is not directly observable in the market. Proxies are therefore used for this unobservable variable, Chapman, Long and Pearson (1999), hereafter, (CLP). Different researchers have used different proxies, for example Anderson and Stanton (1997) uses the yield on a three-month Treasury bill, CKLS (1992) use one-month Treasury bill yield. A more comprehensive survey of alternative proxies for the short rate are to be found in (CLP,1999). There are, however, several problems associated with single-factor models. First, single-factor models assume that changes in the yield curve, and hence bond returns, are perfectly correlated across maturities. This assumption is contradicted by the empirical evidence available. Furthermore, the assumption of perfect correlation is highly problematic for several practical purposes, for example, Value-at-Risk calculations, and pricing derivatives on interest rate spreads as discussed by Canbarro (1995). Second, the shape of the yield curve is severely restricted. Specifically, the Vasicek and CIR models can only accommodate yield curve that is monotonic increasing or decreasing and humped. An inversely humped yield curve cannot be generated with these models. Finally, with time-invariant parameters one-factor models tend to provide a very poor fit to the actual yield curves observed in the market. To overcome the limitations of single-factor term structure models researchers have put forward a number of two-factor term structure models.

Brennan and Schwartz (1979) proposed a two factor model based on a mean reverting short-term interest rate and a long term interest rate. The long-term interest rate is taken to be the yield on a consol bond. However, this specification of the two-factor model does not lead to analytic bond or contingent claims prices. Schaefer and Schwartz (1984) developed an analytical bond price based on two-factor term structure model. Their two-factor model is very similar to the two-factor model proposed by Brennan and Schwartz, except with one crucial difference. Where as Brennan and Schwartz used a short-term rate and a long-term interest rate, Schaefer and Schwartz used the long term interest rate and the spread, i.e., the difference between the short term interest rate and the long term interest rate. Schaefer and Schwartz (1987) further proposed a two-factor term structure model based on the short-term interest rate and the duration of the bond.

Cox-Ingersoll-Ross (1985) also proposed a two-factor term structure model based on the short-term interest rate and the inflation rate. They develop an analytical solution for the real value of a nominal bond. Longstaff and Schwartz (1992) propose a two factor general equilibrium model using the CIR (1985) framework. The two factors in the Longstaff and Schwartz model are the short-term interest rate and the instantaneous variance of changes in the short-term interest rate. Thus contingent claims based on the Longstaff and Schwartz model will be dependent on both the current level of interest rate and the current level of interest rate volatility. They derive both analytical bond prices and analytical European call option prices based on their model.

Das and Foresi (1997) have put forward a two-factor term structure model that allows for interest rate jumps. They propose that the short-term interest rate follows the process put forward by Vasicek (1977) superimposed with jumps. They proceed to consider two types of jump models. In the first model, the jumps are infrequent events, which change interest rates by discrete amounts but do not change what they call the central tendency. In the second jump model, the jumps change the central tendency. Further they derive analytical solution for bonds and derive numerical scheme for contingent claims.

1.2.2. Arbitrage Free Models

The wide spread popularity of one-factor equilibrium models, such as the Vasicek model, stems from their simplicity. At each date, today and in the future, the entire yield curve is a function of a single state variable, the short rate. However, equilibrium models do not fit the current yield curve exactly, and this tends to limit their effectiveness for pricing fixed income derivatives. By taking the current market term structure of interest rate as the starting point we can overcome this weakness of the equilibrium approach. Below we discuss the major Arbitrage Free Models, which have been proposed over the years.

In its basic form the Ho-Lee model can be stated as a specific case of the Vasicek model.

$$dr_t = \theta(t)dt + \sigma(t)dz_t \qquad (1.17)$$

The Ho-Lee method involves fitting a binomial lattice for discount bond prices, with the restriction that the bond price is pulled to par at maturity. The lattice is constructed such that there is no arbitrage allowed between the pricing along the lattice and current market interest rates. This means that the lattice is constructed such that the market price of risk does not have to be specified. The lattice is analogous to the one suggested by Cox-Ross-Rubinstein (1979) except with three differences. First the lattice is in terms of forward prices rather than spot prices. Second, the up- and down- movements are time dependent. Third the whole term structure is shifted up or down, rather than a single asset price. Other researchers, including Black-Derman-Toy (1990) have extended the Ho-Lee approach, Hull and White (1990a) and Heath-Jarrow-Morton.

The Black-Derman-Toy (BDT) model is based on the assumption that the short-term interest rate is a lognormal process. It is a single factor model in which negative interest rates are prevented because of the log-normality of the short-term interest rate process. The BDT mode is usually constructed using a binomial tree to price exactly any set of bonds and hence contingent claims without requiring any investor risk preference. As such it is an arbitrage free model. The continuous-time equivalent of the BDT interest rate process is:

$$r(t) = u(t)\exp[\sigma(t)z_t] \tag{1.18}$$

With $u(t)$ as the median of the short-term interest rate distribution at time t, $\sigma(t)$ is the volatility of the short-term interest rate process. By making $\sigma(t)$ time dependent, BDT can be used to recover the prices of a wide range instrument.

Hull and White (1990a) generalize the CKLS model by allowing for time dependent mean reversion $\theta'(t)$ and for time dependence in the mean reversion speed $k(t)$ and volatility $\sigma(t)$

$$dr_t = k(t)(\theta'(t) - r)dt + \sigma(t)r^{\gamma}dz_t \tag{1.19}$$

The model corresponds to $\gamma = 0$ be referred to as the Extended Vasicek (EXV). Further at $\gamma = 0$, the Hull and White model can be interpreted as the Ho-Lee model if we express the Hull and White as:

$$dr_t = \theta''(t)dt + \sigma(t)dz_t \tag{1.20}$$

with $\theta''(t) = k(t)(\theta'(t) - r)$

Finally $\gamma = \frac{1}{2}$ leads to the Extended CIR (EXCIR) and $\gamma = 1$ yields the Black-Derman-Toy model. Hull and White (1994b) have extended their approach to two factors. They have achieved this, by incorporating a new stochastic function in the drift of the interest rate for the Extended Vasicek model.

The Heath-Jarrow-Morton (HJM) is based on the martingale approach introduced by Harison and Kreps (1979) and Harison and Pliska (1981). The HJM model is a complete model of the term structure specified in an arbitrage free framework According to Subrahmanyam (1996), the basic set up of the HJM model is similar in spirit to the Vasicek model with one crucial difference. In the case of the HJM model the forward rate is used rather the short rate . The stochastic differential equation for the forward rate is:

$$df(t,T) = a(t,T)dt + b(t,T)dz_t \tag{1.21}$$

Where $a(t,T)$ and $b(t,T)$ are the drift and diffusion terms of the forward rate process, t is the current date, T is the maturity date, and z_t is a Brownian motion. Further $f(t,T)$ is the instantaneous forward interest rate at time t for delivery at date T. The above stochastic differential in its general form is non-Markovian which leads to non-combining lattices when bond prices or contingent claim prices are evaluated. Ritchken and Sankarasubramanian (1995) have proposed a specific classes of volatility structures such that the diffusion process for the forward rate is Markovian.

Below we summarise the main differences between the equilibrium and the arbitrage free approach to bond pricing.

Equilibrium Models	**Arbitrage Free Models**
Main building blocks: stochastic process for the short rate, and assumptions about investor preferences – market price of risk	The prices of these securities are often independent of investor preferences.
The yield curve is determined endogenously in the model – it is not constrained to match the actual market yield curve.	Per construction, arbitrage free term structure models fit the initial yield curve (i.e. today's curve) exactly
Model parameters are constant over time (internal consistency), and typically there are at least two factors.	The models are not stable – the time dependent parameters must be re-calibrated over time (inconsistency).
Models include Vasicek, CIR, BS etc.	Models include HJM, Ho-Lee, as well as equilibrium style models with time dependent parameters such as the BDT and HW extended Vasicek model.
Used mainly for trading bonds (yield curve strategies), less useful for fixed-income derivatives.	In most cases, a single-factor model is used.
Used for risk management purposes.	Used for pricing fixed-income derivatives (not bonds).
Implementation issues: statistical estimation using historical data on the term structure.	Implementation issue: calibration to initial yield curve, and assumptions about the volatility parameter.

1.3. Numerical Methods

Black and Scholes using no arbitrage argument developed an analytical expression for European type contingent claim. However, within the Black-Scholes framework an equivalent analytical expression for an American type contingent claim is not possible. American type contingent claims are distinguished from the European type on the basis that American contingent claims can be exercised anytime prior to the expiry of the option. It is this feature of possible early exercise of the American contingent claim prior to expiry that results in no analytical expression being available.

The key to the valuation of American contingent claims is the location of the early exercise boundary or the free boundary in the terminology of partial differential equations. The early exercise boundary is determined by comparing the intrinsic value with the actual contingent claims price itself. The methods developed for the evaluation of American contingent claims are the Lattice approaches, Analytic approaches, Finite Difference Method, Method of Lines and Monte Carlo Simulation. Below we discuss each of the above mentioned approaches first with respect to equity or commodity contingent claims and then secondly where applicable with respect to interest rate contingent claims.

1.3.1. Lattice Approaches

Based on the earlier work of Sharpe (1978), Cox-Ross-Rubinstein (CRR) (1979) developed the binomial lattice approach for the valuation of contingent claims. Their key assumptions include:

- The expected return from all traded securities is the risk-free interest rate.

- Future cash flows can be valued by discounting their expected values at the risk-free interest rate.
- The probabilities sum to one.
- The mean of the discrete distribution is equal to the mean of the continuous distribution.
- The variance of the discrete distribution is equal to the variance of the continuous distribution.

Based on the above assumptions CRR proved that European option's value in the binomial model converges to the value give by Black-Scholes formula. CRR (1985) further developed their binomial model to value American options on dividend paying stocks. Further they demonstrated the use of the Binomial model, when some of the Black-Scholes assumptions are relaxed. Boyle (1986) further developed the CRR binomial lattice to trinomial lattices. In this case the stock price can jump up to a higher value, jump down to a lower value or stay the same value after a time step. We can generalize the lattice of CRR and Boyle, if we consider a derivative security whose price depends on l underlying variables. The life of the security T is divided into n subintervals of length Δt. At time $i\Delta t$, there exists m_i possible states which we denote by $S_{ij}, (1 \leq j \leq m_i)$. Transition probabilities p_{ijk} are defined as follows:

p_{ijk} - probability of moving from state S_{ij} to state $S_{i+1,j}$ at time $(i+1)\Delta t$.

Further p_{ijk}'s must sum to one and be between zero and one, i.e.:

$\sum_k p_{ijk} = 1$ for all i's and j's.

$0 \leq p_{ijk} \leq 1$ for all i,j, and k.

Once the lattice has been set up, the dynamic programming method can be used. The value of the contingent claim at time T is known for all m_n states at that time. The value of all m_i states at time $i\Delta t$ can be calculated using risk neutral valuation if the value is know for all m_{i+1} states at time $(i+1)\Delta t$. By moving backwards through the tree, the value at time 0 can be obtained.

The lattice approach has been extended to value path dependent options such as Asian options by Hull and White (1993), Lookback options by Cheuk and Vorst (1993). Further schemes to improve the efficiency of lattices have also been developed. These schemes include the control variate method by Hull and White (1988), Richardson extrapolation by Breen (1991).

One of the most important applications of the lattice approach has been for the valuation of bonds and interest rate contingent claims. Rendleman and Barter (RB) (1980) were the first to apply the binomial lattice to value interest rate contingent claims. They assumed that the short term interest rate followed geometric Brownian motion. RB valued interest rate contingent claims as a three-step process. The first step involves generating a lattice of interest rates. The second step involves deriving a lattice of bond prices. The final step involves developing a lattice of interest rate contingent claims based on the lattice of bond prices. The main weakness of the RB lattice is that it is based on the assumption that the short-term interest rate follows a process similar to that of stock prices. Thus the RB lattice cannot be used if the short term interest rate models incorporate both mean reversion and interest rate dependent volatility - a feature of widely used interest rate models. Nelson and Ramaswamy

(NR) (1990) developed a lattice approach that could incorporate both these features. The NR lattice is different from the RB lattice in two aspects. Whereas with the RB lattice, the probability value is fixed throughout the lattice, with the NR lattice, probability value varies from node to node. Further to ensure that the probability values lie between zero and one, multiple jumps are allowed within the NR lattice. The inclusion of multiple jumps in the NR lattice results in it being considerably slower than the RB lattice. Hull and White (HW) (1990) developed a trinomial lattice that incorporated both mean reversion and interest rate dependent volatility. HW lattice ensured that probabilities lied between zero and one by incorporating alternative jump processes. The HW lattice is therefore considerably faster than the NR lattice. Tian (1992) further simplified HW trinomial lattice to a binomial lattice (SB). Tian (1994) tested the NR lattice, HW lattice and SB lattice for bonds and interest rate contingent claims based on the CIR model. He found that for certain combination of parameters both the HW and the SB lattice did not converge to the corresponding analytical bond price and hence interest rate contingent claims. The NR, lattice however, did yield bond and interest rate contingent claim prices which converged for all combination of parameters - albeit at greater computational cost.

1.3.2. Analytic Methods

To avoid the use of numerical schemes for the valuation of American options a number of analytical schemes have been suggested. Johnson (1983) suggested an approximation for an American put option. Blomeyer (1986) further developed Johnson's approximation to value put options that have a dividend date occurring on the underlying asset prior to expiration. The schemes suggested by Johnson and

Blomeyer do not necessarily satisfy the hedging partial differential equation. To avoid this difficulty MacMillan (1986) suggested a numerical scheme based on the decomposition of the American put option as a sum of the value of a European put option plus the early exercise premium. The early exercise premium is assumed to be a function of time and asset price. Barone-Adesi and Whaley (1987) extended MacMillans put approximation to value both American call and put options based on dividend paying stocks and American commodity and futures options with a constant rate of dividend. Their solution is based on the similarity transformation with the solution satisfying the fundamental partial differential equation. The resulting partial differential equation based on the similarity transformation is then converted to an ordinary differential equation by a suitable approximation. This ordinary differential equation is then solved iteratively to determine the critical asset prices and the options prices.

The integral equation method suggested by Kim (1990) again separates the American option into two components. Kim assumes that the American option with time to maturity τ can be expressed as the sum of the value of a European option at time t and the early exercise premium. It is possible to exercise the option at any point in time ν where $t < \nu < \tau$. The early exercise premium is then valued by integrating over the relevant time interval. At each intermediate point of time ν, the critical asset price is determined and thus the decision whether it is optimal to exercise or not is taken. The early exercise premium comprises of two integrals. The first for the probabilistic weighting of not exercising and the second for exercising. The resulting integral equation for the American option is solved using numerical integration. However, this integral equation requires the computation of many early exercise points, Huang, Subrahmanyam and Yu (1996) implement a four-point Richardson extrapolation scheme. As the integral representation method involves only the univariate cumulative normal method, their method is fast, but not very accurate, especially for long expiry options. Ju (1998) proposes an approximation which overcomes this difficulty by approximating the early exercise boundary as a multi-piece exponential function.

The compound option approach for the valuation of American put options is based on the papers of Geske (1977,1979). Since at every instant there is a positive probability of premature exercise, the American option can be interpreted as being equivalent to an infinite sequence of options on options or compound options. Geske and Johnson (1984) develop a solution for the American put. They use four point Richardson extrapolation on a sequence of hypothetical puts, where each put has a finite number of exercise points located at equally spaced time intervals. Evaluating the puts requires calculation of quadrivariate normal integrals. Bunch and Johnson (1992) improve the above scheme. They demonstrate that it is possible to obtain accurate American put prices using two point Richardson extrapolation that involves the valuation of bivrate normal integrals. Ho, Stapleton and Subrahamanyam (1994) suggest a further improvement on Bunch and Johnson's two point Richardson extrapolation procedures. Their improvement is based on an observed approximately exponentially relationship between the value of an American option and the number of exercise points allowed up to the expiry date.

1.3.3. Finite Difference Method

With the finite difference approach, we transform the partial differential equation into a set of finite difference equations. This set is then solved numerically to obtain the value of the contingent claims. Their exists basically two different finite difference schemes. The explicit and the implicit finite difference schemes. Although there are other finite difference schemes, they are essentially a combination of the two. With the explicit finite difference scheme, we can solve the finite difference equations individually. With the implicit finite difference scheme, we need to solve the whole set of finite difference equations simultaneously.

Brennan and Schwartz (1977) used the finite difference approach to solve the free boundary problem. They calculated the value of an American put option for a dividend paying stock and derived the critical asset prices using the implicit finite difference with coefficients depending on the increments of the stock. Schwartz (1977) further expanded this approach to value warrants. Later Brennan and Schwartz (1978) gave intuitive interpretation to the explicit finite difference scheme as a three-jump process. That is, the explicit finite difference scheme can be interpreted as a trinomial lattice. Finally, they interpreted the implicit finite difference scheme as a generalized jump process with infinitely many asset prices.

Courtadon (1982b) further improved the finite difference schemes put forward by Brennan and Schwartz. He used and average of the explicit and the implicit finite difference-schemes - known as the Crank-Nicholson method.

Geske and Shastri (1985) compared the explicit, implicit, and log-transformed explicit and implicit finite difference schemes. They also considered several binomial methods. Their main conclusion was that the explicit finite difference scheme was overall the fastest.

Courtadon (1982a) applied the finite difference method for the valuation, of default-free bonds and interest rate contingent claims. He stated the boundary conditions necessary for valuing default free bonds, European call and put options as well as American call and put option. Using the single factor term structure model proposed by Brennan and Schwartz (1979), he set up the partial differential equation for both default free bonds and contingent claims. Using the implicit finite difference scheme similar to that of Brennan and Schwartz (1977), he set up a system of finite difference equations. By solving this system of equations he obtained the bond prices and contingent claims prices.

Hull and White (1990b) further developed the explicit finite difference scheme to value default free bonds and contingent claims. They noted the conclusion of earlier researchers including Brennan and Schwartz (1978), Geske and Shastri (1985) and others that a suitable transformation of the underlying asset increases the efficiency of the finite difference scheme. Generalizing from this, they introduced a new state variable that had constant instantaneous standard deviation to their finite difference scheme. They modeled their new variable in the same way as the underlying asset. They set up an explicit finite difference scheme in terms of the new state variable and interpreted the coefficients as probabilities of a trinomial lattice introduced by Boyle (1986). Hull and White discussed the conditions under which their proposed explicit finite difference scheme would converge to yield true bond prices and contingent claims prices. To ensure convergence they recommended that the probabilities, i.e. the coefficients should remain positive. This is achieved by using different branching procedures, rather than the usual, up, down and constant branch. Hull and White applied their explicit finite different branching scheme to value bonds and contingent claims based on the short-term interest rate model proposed by Cox-Ingersol-Ross (1985).

1.3.4. Method of Lines

The Method of Line involves converting the second order partial differential equation into a system of first order equations. These first order equations are then discretized and solved iteratively to obtain the value of the contingent claims. To date the Method of Lines has only been used to value put options based on equity by Meyer and Van der Hoek (1994).

1.3.5. Monte Carlo Simulation

The Monte Carlo simulation method for contingent claims valuation was first introduced by Boyle (1977). Until, recently, its main use has been to value path-dependent European type contingent claims. However, in recent years a number of researchers have put forward different Monte Carlo schemes for the valuation of American type contingent claims. The basis of Monte Carlo simulation lies in the insight of Cox and Ross (1976); that if a riskless hedge can be formed the option value can be expressed as the discounted expectation of the payoff it would produce in a risk neutral world. Monte Carlo simulation consists of the following three steps.

- Simulating sample paths of the underlying state variable such as the underlying asset prices over the time increment.
- Evaluating the discounted cash flows of a security on each sample.
- Average the discounted cash flows over sample path.

Boyle (1977) used Monte Carlo simulation to value European call options on discrete dividend paying stocks. Hull and White (1987) used the approach to value options on assets with stochastic volatilities. They found that the Black-Scholes frequently overprices options and that the degree of overpricing increases with the time to maturity. Kemna and Vorst (1990) used Monte Carlo simulation as a valuation method for arithmetic Asian options, Clelow and Caverhill (1994) valued call and look-back call options using Monte Carlo simulation. Caverhill and Pang (1995) evaluated bond prices and call option within Heath-Jarrow-Morton (HJM) framework using Monte Carlo simulation.

One of the main disadvantages of Monte Carlo simulation is that a large number of simulation runs may be required to obtain precise results. Thus variance reduction techniques is required. Boyle (1977) discussed two such variance reduction techniques; the control variate approach and the antithetic variate approach. Kemna and Vorst (1990) used the control variate method in their valuation of Asian options. As a control variate they used the analytical formula for the geometric average option. Recently other variance reduction methods have been introduced. These include moment's matching by Barraquand and Martineau (1995); martingale variance reduction method by Clelow and Caverhill (1994); low discrepancy deterministic sequences by Joy, Boyle and Tan (1996). Low discrepancy sequences have the property that the sequence of points remain evenly dispersed. Deterministic series thus far used include Faure and Sobol.

Tilley (1993) expanded the use of Monte Carlo simulation to value American type options. Till that date, widespread belief existed that Monte Carlo simulation could not be used to value American type options. The basic problem in using Monte Carlo simulation to price American type options is how to incorporate the early exercise feature associated with American options. Tilley dealt with this problem by storing the paths followed by the asset prices, ranking them and further re-ranking them at each possible early exercise date. Tilley uses the valuation of an equity American put option as an example. By grouping the ranked asset prices at each date, he is able to estimate for that group at that date. Barraquand and Martineau (1995) proposed an alternative Monte Carlo scheme for the valuation of American options. Their proposal involved an approach that tracks the conditional probabilities of path specific outcomes in a Monte Carlo simulation. They use their scheme to value put options based on multiple assets. Raymer and Zwecher (1997) extend the Barranquand and Martineau approach to two factor representation of stock prices. Broadie and Glasserman (1997) propose a scheme based on generating two estimates of the asset prices taken from random samples of future state trajectories. One estimate is biased high and one is biased low; both estimates are asymptotically unbiased and converge to the security price. The two estimates are then combined to determine a confidence interval for the security price. Recently Grant, Voran and Weeks (1998) have proposed another Monte Carlo scheme for the valuation of American options. They incorporate the early exercise feature in the Monte Carlo method by linking forward moving simulation and the backward moving recursion through an iterative search process.

1.4. Objectives of the thesis

In the previous sections, we have discussed alternative specifications of possible interest rate models. Further we discussed that there was the Equilibrium approach and the Arbitrage-Free approach to interest rate modeling. For the remainder of the thesis we concentrate on the Equilibrium approach.

Ideally for risk management purposes, analytical prices both for bond and interest rate contingent claim prices is highly desirable. However, except for specific models such as the Vasicek $(\gamma = 0)$, CIR $\left(\gamma = \frac{1}{2}\right)$ analytical solutions are not available. Further CKLS (1990) state that γ is the most important feature differentiating different interest rate models. CKLS, also show that interest rate models, which allow for $\gamma \geq 1$ capture the dynamics of the short-term better than those do, which require $\gamma \leq 1$. Finally CKLS show that these interest rate models differ significantly in their implication for valuing default-free bonds and interest rate contingent claims.

Rebanato (1995) states that 85%+ of variance across rates of different maturity could be satisfactorily explained by using a single factor model. More, specifically he finds in the case of the UK that 92.170% of the variance is explained by a single factor model and 6.93% of the variance (or 99.1% of the total variance) is explained by a two factor model. Thus clearly, a two-factor model is desirable for risk management purposes. However, a two-factor model requires considerably more effort to implement. In addition, with multi-factor models the CPU memory required increases by the power of the factor. As an example, if we declare an array of size N with a single factor model we need to declare an array of size $N_1 \times N_2$ with a two-factor model, or an array of size $N_1 \times N_2 \times ...... \times N_m$ with an m-factor model. Further modeling interest rate derivatives is more demanding than the modeling of equity derivatives. As a result both practitioners and academics have focused their research activities on single-factor term structure models. By focusing on single-factor models researchers are able to gain insights which can be applied in a multi-factor setting.

Our examination of the numerical approaches literature indicates that not all the numerical approaches suggested so far are suitable for general interest rate contingent claim valuation. As discussed in the previous section, different Monte Carlo simulation schemes have been put forward. However, no single approach has been accepted as the standard, unlike the lattice approach as an example. The analytic approaches are not suitable because their starting point is an expression for the European option - an expression generally not available for interest rate contingent claims. This leaves us with the Lattice approach, Finite Difference Method, and the Method of Lines.

The objectives of this thesis is as follows:

1. To test the convergence properties of the simplified binomial lattice of Tian (1994) by varying the γ parameter.
2. To introduce a new numerical scheme in finance from engineering for the evaluation of default-free bonds and interest rate contingent claims based on the CKLS model.
3. To test the convergence and stability of the new method with existing numerical methods.

4. To test the stability of the new numerical scheme by tracking its free boundary for American interest rate put options.
5. To value default-free bonds and interest rate contingent claims for different markets using the new numerical method.

1.5. Outline of the thesis

In Chapter 2 we apply the Simplified Binomial (SB) lattice of Tian to value both default-free bonds and interest rate contingent claims, based on the CKLS model. We test the SB lattice both for stability and convergence.

In Chapter 3 we use the partial differential equation approach to value default-free bonds and interest rate contingent claims. We consider the Finite Difference Method. We develop the Method of Lines approach which has thus far been only used to value equity options to value default-free bonds and interest rate contingent claims. Finally we introduce a new numerical scheme - the Box Method in finance from engineering. As in Chapter 2, we test all three numerical schemes with one another with respect to convergence and stability.

In Chapter 4 we use the Box Method as the starting point to develop a new method to track the free boundary of American interest rate put options. We attempt to track the free boundary of both short dated and long dated options based on widely used interest rate models.

In Chapter 5 we use the Box Method to value default-free bonds and interest rate contingent claims for different markets. In particular we consider Australia, Canada, Japan, Hong Kong, U.K., and U.S.A. We calculate values of default-free bonds across a range of maturity dates and short-term interest rates. We compare the numerical default-free bond values and interest rate contingent claim values with analytical values where available.

Chapter 6 summarizes the results of our research and suggests directions for future research.

CHAPTER 2.

BINOMIAL LATTICE APPROXIMATION TO DIFFUSION PROCESSES

2.1. Introduction

The lattice approach to value contingent claims was first developed by Cox, Ross, and Rubenstein (CRR; 1979). They used a recombining binomial lattice to value equity contingent claims and proved that in the limit $\Delta t \to 0$ contingent claim prices calculated using the binomial lattice approached the contingent claim prices calculated using the Black-Scholes formula. Boyle (1986) further extended the CRR binomial lattice to a trinomial lattice and showed that the trinomial lattice was faster than the binomial lattice. Neither the binomial lattice of CRR or the trinomial lattice of Boyle are directly applicable to widely used interest rate models.

Interest rate stochastic processes are more complex than similar stochastic processes for equities. For example, interest rate processes need to take mean reversion and interest rate dependent volatility into account. This means that when we try to value interest rate dependent contingent claims using the above mentioned lattice approaches recombining of the nodes is no longer guaranteed. Further it may not be possible in some instances to achieve convergence from the discrete to the continuous in the limit $\Delta t \to 0$.

Over the years researchers including Nelson and Ramaswamy (NR; 1990), Hull and White (1990b) and Tian (1992) have attempted to use the lattice approach to value the underlying instruments, i.e. the discounted bond and the contingent claims based on such bonds. The NR binomial lattice method produced both accurate discount bond prices and the contingent claim price based on such bonds. However, this was achieved at the expense of computational speed. HW trinomial lattice method although faster than the NR method suffers from convergence difficulties for certain combination of parameters. HW trinomial lattice was further simplified by Tian (1992) to a simplified binomial lattice (SB). Although the SB lattice is considerably faster and easier to implement than the HW lattice, it nonetheless suffers from the same convergence difficulties as the HW lattice.

Both HW and Tian applied their respective lattices to the Cox, Ingersoll, and Ross (CIR; 1985b) interest rate model and found convergence and stability difficulties with certain combination of parameters. The purpose of this chapter is to further explore the convergence and stability issues that arise when the SB lattice is used to value discount bonds for interest rte processes, that enclosed the CIR as a special case.

The main contribution of this Chapter is to generalise the work of Tian (1994) to the CKLS (1992) model. In Section 2 we discuss the construction of the SB lattice as in Tian for a general one factor stochastic process. In Section 3 we show how the work of Tian (1994) is expanded to the CKLS (1992) model). In Section 4 we discuss results obtained for the CKLS interest rate model. Section 5 concludes this chapter.

2.2. Simplified Binomial Interest Rate Lattice

Consider a general one state variable short term interest rate process:

$$dr = \mu(r,t)dt + \sigma(r,t)dz_t \qquad (2.2.1)$$

where:

$\mu(r,t)$: instantaneous drift of the interest rate process.

$\sigma(r,t)$: volatility of the interest rate process.

dz_t: Standard Wiener process.

In a risk-neutral world, drift rate is adjusted by the market price of risk $\lambda(r,t)$ so that the short term interest rate process becomes:

$$dr_t = [\mu(r,t) - \lambda(r,t)]dt + \sigma(r,t)dz_t \qquad (2.2.2)$$

Taking the discrete time version of the Wiener process as $\Delta z = \varepsilon_k \sqrt{\Delta t}$ the discretized verison of the above equation is:

$$r_{n+1} = r_n + [\{\mu(r_n, t_n) - \lambda(r_n, t_n)\}\sqrt{\Delta t} + \sigma(r_n, t_n)\varepsilon_k(t_n)]\sqrt{\Delta t} \qquad (2.2.3)$$

ε_k has two and three possible outcomes for a binomial and trinomial lattice respectively and a mean of zero and variance of one.

The major problems with the above discretization is that the resulting lattices are non-combining because the volatility is interest rate dependent. This means that the number of nodes increase exponentially as we move forward through the lattice. Such a lattice is said to be path dependent. An alternative lattice where the nodes combine is known as path independent or a simple lattice in the terminology of Nelson-Ramaswamy (1990). The major strength of simple lattices over path dependent lattices is that with simple lattices the number of nodes increase quadratically as we move forward through the lattice. Clearly from the computational viewpoint simple lattices are desirable.

With above researchers in all cases the starting point is to transform equation (2.2.2) to a form that has constant volatility i..e. where the volatility is not dependent on the short term interest rate. This is achieved by letting $\phi = g(r,t)$ such that $r = g^{-1}(\phi,t)$ be the relevant transformation such that process described by equation (2.2.2) becomes.

$$d\phi = q(r,t)dt + \nu dz_t \qquad (2.2.4)$$

where:

$$q(r,t) = \frac{\partial\phi}{\partial t} + (\mu(r,t) - \lambda(r,t))\frac{\partial\phi}{\partial r} + \frac{1}{2}\sigma(r,t)^2\frac{\partial^2\phi}{\partial r^2}$$

$\frac{\partial\phi}{\partial r} = \nu$ - positive constant.

Thus it is possible to construct lattice either in (r,t) or in (ϕ,t) space. The former approach is pursued by Nelson and Ramaswamy and the latter approach is pursued by Hull and White and Tian.

The Simplified Binomial model (SB) is the binomial equivalent of the trinomial Hull and White model. To derive the SB lattice we partition the interval $[t_0, T]$ (where t_0 is the current date and T is the maturity date of the bond or the exercise date of the option) into N subintervals of length Δt such that:

$$\Delta t = \frac{T - t_0}{N}$$

$$t_n = t_0 + n\Delta t$$

for $n = 0,1,2,\ldots\ldots.,N$

Further we assume for the period $(t_n, t_{n+1}]$ r behaves in the following way. Initially its value at time t_n is r_n. For the period $(t_n, t_{n+1}]$ its value still remains at r_n. However, at time t_{n+1}, its value either jumps up to $r_n + u$ with a probability p or jumps down to $r_n - d$ with a probability $(1 - p)$. In order to derive expressions for u, d and p, we equate the mean and variance in discrete and continuous time as follows:

$$pu - (1 - p)d = q\Delta t \qquad (2.2.5)$$

$$pu^2 + (1 - p)d^2 = v^2\Delta t \qquad (2.2.6)$$

$$\text{prob}(\phi_{i+1} = \phi_i + \Delta\phi) = p$$

$$\text{prob}(\phi_{i+1} = \phi_i - \Delta\phi) = 1 - p$$

Thus based on the above two equations, we derive the following expressions for u, d and p.

$$u = d = \Delta\phi = v\sqrt{\Delta t}$$

$$p = \frac{1}{2} + \frac{1}{2}\frac{q\sqrt{\Delta t}}{v}$$

The above expression for p can either be less than zero or greater than one. This leads to the following expression for p.

$$p = \max\left\{0, \min\left\{1, \frac{1}{2} + \frac{q\sqrt{\Delta t}}{2v}\right\}\right\}$$

In order to value the discounted bond prices, the first step is to generate the interest rate lattice by moving forward through time. The second step involves moving backwards through the lattice by calculating the discounted bond price at each node on the lattice. At maturity we take the value of the discounted as 1. Prior to maturity we use the following recursive formula to value the discounted bond price B_{nj} at node j, time n.

$$B_{nj} = \frac{p_{nj}B_{n+1,j+1} + \left(1 - p_{nj}\right)B_{n+1,j}}{1 + r_{nj}\Delta t} \qquad (2.2.7)$$

Once we have calculated the lattice of bond prices, we proceed to calculate the contingent claims based on the bonds. As with bonds we move backwards through the lattice but in this case by calculating the discounted options prices at each node through the lattice prior to the expiry of the option. At maturity we take the value of the call option $\max\{B_{Nj} - E, 0\}$ and put option as $\max\{E - B_{Nj}, 0\}$. E is the exercise price in both cases. At each intermediate step for European type call or put options, value at each node is given by:

$$P_{nj} = \frac{p_{nj} P_{n+1,j+1} + (1 - p_{nj}) P_{n+1,j}}{1 + r_{nj} \Delta t} \tag{2.2.8}$$

where P_{nj} may be call or a put option. However, if the options are American, then value at each node is $\max\{P_{nj}, B_{nj} - E\}$ for call option and $\max\{P_{nj}, E - B_{nj}\}$ for put options.

2.3. CKLS Model

We consider the following CKLS model in a risk neutral world where the short term interest rate is pulled toward a long term value θ at a speed of adjustment k. In an equilibrium model, the market price of risk is incorporated explicitly depending on the model used. For example in the Vasicek model market price of risk is $\lambda\sigma$. The CKLS model is used for the short-term riskless rate and as such the market price of risk is taken to be zero.

$$dr_t = [k\theta - rk]dt + \sigma r^{\gamma} dz_t \qquad (2.3.1)$$

γ: unrestricted parameter

We note that substituting specific values of γ into the above equation leads to specific interest rate models. For example:

$\gamma = 0 \rightarrow$ Vasicek model

$\gamma = \frac{1}{2} \rightarrow$ Cox-Ingersoll-Ross (CIR) model

$\gamma = 1 \rightarrow$ Brennan-Schwartz model

In order to transform equation (2.3.1) so that the volatility is independent of the interest rate,we use the general transformation ϕ for the CKLS interest rate process:

$$\phi = \frac{\nu}{\sigma} \int r^{-\gamma} dr \qquad (2.3.2)$$

where ν can be chosen equal to σ with no loss of generality. Taking the market price of risk as zero, for simplicity, the drift of the process ϕ, q is given by Ito's lemma as:

$$q = k(\theta - r)\frac{\partial \phi}{\partial r} + \frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 \phi}{\partial r^2} \qquad (2.3.3)$$

From equation (2.3.2), we note that there is a singularity at $\gamma = 1$. We therefore integrate equation (2.3.2) for $\gamma = 1$ and $0 < \gamma < 1$ separately.

Thus for $\gamma = 1$, we have:

$$\phi = \frac{\nu}{\sigma}\int \frac{dr}{r} = \frac{\nu}{\sigma}\ln r$$

If we let $\nu = \sigma$

$$\phi = \ln r \qquad (2.3.4)$$

Differentiating the above expression for ϕ with respect to r, once and twice, we have:

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}$$

$$\frac{\partial^2 \phi}{\partial r^2} = -\frac{1}{r^2}$$

Substituting the above expressions for q into equation (2.3.3) and simplifying gives:

$$q = \frac{k\theta}{r} - \frac{1}{2}\left(2k + \sigma^2\right) \qquad (2.3.5)$$

$$q = \frac{a_1}{e^{\phi}} + a_2 \qquad (2.3.6)$$

where:

$$r = e^{\phi}$$

$$a_1 = k\theta$$

$$a_2 = -\frac{1}{2}\left(2k + \sigma^2\right)$$

For $0 < \gamma < 1$, use the following transformation.

$$\phi = \frac{\nu}{\sigma}\int r^{-\gamma}dr = \frac{\nu}{\sigma(1-\gamma)}r^{1-\gamma}$$

Let $\nu = \sigma(1-\gamma)$

$$\phi = r^{1-\gamma} \qquad (2.3.7)$$

Differentiating the above expression for ϕ with respect to r, once and twice, we have:

$$\frac{\partial\phi}{\partial r} = (1-\gamma)r^{-\gamma}$$

$$\frac{\partial^2\phi}{\partial r^2} = -\gamma(1-\gamma)r^{-\gamma-1}$$

Substituting the above expressions for q into equation (2.3.5) and simplifying gives:

$$q = \frac{a_1}{\phi^{\frac{\gamma}{1-\gamma}}} + a_2\phi + \frac{a_3}{\phi} \qquad (2.3.8)$$

where:

$$r = \phi^{\frac{\gamma}{1-\gamma}}$$

$$a_1 = (1-\gamma)k\theta$$

$$a_2 = -(1-\gamma)k$$

$$a_3 = -\frac{1}{2}\gamma(1-\gamma)\sigma^2$$

A necessary condition for convergence of the ϕ process to the r process is that q should be bounded. From equations (2.3.8) we see that q is always bounded if $\phi > 0$. However, from equation (2.3.8) we see that q becomes unbounded if $\phi = 0$. By careful choice of parameters we can ensure that $\phi = 0$ is inaccessible and convergence is always ensured.

The general transformation of r to ϕ ensures that the variance of ϕ is constant and further $r = 0$ is inaccessible for $k\theta > 0$ in equation (2.3.8). $k\theta > 0$ ensures $q > 0$. The positive values of the long-term centrality parameter and the speed of mean reversion of the CKLS interest rate process ensures that these conditions are always met. Hence the interest rate process always converges for $\gamma = 1$.

From equation (2.3.8) we see that for $\gamma \neq 1$, the leading term when ϕ approaches zero is a_1 for $\gamma > \frac{1}{2}$ and a_3 for $\gamma < \frac{1}{2}$. For $\gamma > \frac{1}{2}$ bond prices converge because the term a_1 dominates. Similarly there is no convergence of bond prices for $\gamma < \frac{1}{2}$ because the term a_3 dominates.

2.4. Numerical Experimentation

In this section we perform numerical experiments to determine zero coupon bond prices when the underlying short term interest rate process follows the CKLS process. In particular we examine the rate of convergence and stability of the bond prices in depth.

Tables 2.1 to 2.16 all have the same format. The first two columns give the term to maturity of the bond and the instantaneous short-term interest rate. The third column contains analytical prices calculated using the Cox-Ingersoll-Ross model i.e. for $\gamma = \frac{1}{2}$. The remaining columns contain zero coupon bond prices for different number of annual time steps calculated using Tian's simplified binomial price. These prices will be referred to as SB henceforth. As in Tian (1994) we attempt to value bond prices in two distinctly different circumstances. In the first case we value bonds when the mean reversion rate is high and the volatility of the interest rate is low and in the second case when the mean reversion rate is low and the interest rate volatility is high. We further distinguish these two situations by introducing a variable α_1 where :

$$\alpha_1 = \frac{4k\theta - \sigma^2}{8}$$

$\alpha_1 > 0$ corresponds to low volatility and high mean reversion rate. For $\alpha_1 < 0$ the converse conditions hold.

Tables 2.17 and 2.18 both have the same format, the first column contains the exercise prices. The second column indicates whether the prices are calculated analytically (only occurs when $\gamma = \frac{1}{2}$ i.e. CIR) or using the Simplified Binomial Method. The third, fourth and fifth columns contain the values of α_1, γ, and the bond prices at maturity respectively. The remaining columns contain call or put prices for different terms to expiry.

We calculate prices of zero coupon bonds for different values of γ. Further we examine the rate of convergence and stability by considering prices for different number of annual time steps n. The maturities of the bonds range from 1-25 years. The face value of the zero coupon bond is \$100. Short-term interest rates of 5% and 11% are considered. A difference of 6% between the interest rate scenarios ensures that the approach will remain stable under realistic interest rates. Further, for:

$\alpha_1 = 0.01875 > 0$, $k = 0.5$, $\sigma = 0.1$, $\theta = 0.08$

$\alpha_1 = -0.02725 < 0$, $k = 0.1$, $\sigma = 0.5$, $\theta = 0.08$

α_1, α_2 represent the extreme bounds for the parameters θ and σ. In reality, the parameters will not be as extreme. If a numerical approach yields correct prices under these two extreme conditions, then it will yield correct prices under regular market conditions.

Tables 2.1 and 2.2 show the prices of discount bonds for $\gamma = 1$- Brennan-Schwartz (1980) model for $\alpha_1 > 0$ and $\alpha_1 < 0$ respectively. Both Tables show that the zero coupon bond prices are extremely stable with respect to the annual number of time steps. For example from Table 2.1 consider a 10-year bond, at short-term initial interest rate of 11%. The price of zero coupon bond at $n = 50$ is 42.3708 and the corresponding price at $n = 250$ is 42.3781. Thus an increase in the annual number of time steps by a factor of five has lead to less than one percent change in the zero coupon bond price. Tables 2.1 and 2.2 show that for $\gamma = 1$ zero coupon bond prices are always lower than the correspond analytical CIR price. This difference in bond prices can be explained by noting that bond prices are dependent on the average volatility of the interest rate; which in turn is dependent on the value of γ. A higher value of γ leads to a higher average volatility which in turn leads to a lower bond price. Further this feature between the Brennan and Schwartz model and the CIR model is more pronounced for $\alpha_1 < 0$ and for long maturity bonds.

Tables 2.3 and 2.4 repeat the same calculations but only for $\gamma = \frac{1}{2}$. Note in this case the analytical CIR prices are directly comparable with the SB prices. Table 2.3 shows that for $\alpha_1 > 0$ the SB prices are firstly very stable with respect to the annual number of time steps n and secondly are in excellent agreement with the analytical CIR prices. However for $\alpha_1 < 0$ the situation is totally different as can be seen from Table 2.4. Examination of Table 2.4 shows that SB prices are always lower than the corresponding analytical CIR prices and the difference between the two sets of prices increases with an increase in the term to maturity. Further the zero coupon bond prices are unstable and the level of instability i.e. the range over which the prices fluctuate, increases with an increase of term to maturity of the zero coupon bond.

The sharp difference in the behaviour of bond prices in Tables 2.3 and 2.4 has been explained by Tian (1994). According to Tian for the CIR model i.e. when $\gamma = \frac{1}{2}$, the sign of α_1 will determine convergence of bond prices. In particular if $\alpha_1 < 0$ bond prices will not converge and if $\alpha_1 > 0$ the bond prices will converge.

For $\gamma < \frac{1}{2}$ bond prices do not converge regardless of whether α_1 is positive or negative. Tables 2.5 and 2.6 demonstrate this feature for $\gamma = 0.25$. Again we see that the fluctuations are greater when $\alpha_1 < 0$. Indeed the fluctuations are even more erratic than when $\gamma = \frac{1}{2}$ and further this instability increases as before with $\gamma = \frac{1}{2}$ with term to maturity of the bond. One final feature which will be noticed by examining Table 2.6 is that for $\alpha_1 < 0$ and long maturities the bond prices although unstable are extremely low compared with the corresponding CIR price and that the prices actually seem to be approaching zero as the term to maturity of the bond becomes longer. For example for a 25 year bond at initial interest rate of 11% has bond prices varying between 10.7133 and 0.1759. Contrast this with a 5 year bond at the same short term interest rate where the bond price fluctuates between 59.1822 and 39.9553.

Tables 2.7 and 2.8 indicate that bond prices converge at $\gamma = 0.75$. For all combination of parameters bond prices are stable and close to analytical CIR prices. However, as before the discrepancy between the two sets of prices sensibly increases with an increase of term to maturity. This discrepancy is more stark when $\alpha_1 < 0$.

In tables 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16 we explore the behaviour of bond prices for different values of γ ranging from 0.45 to 0.70 when $\alpha_1 < 0$. In theory, convergence appears at $\gamma > \frac{1}{2}$ if the annual number of time steps n is increased to infinity i.e. with $\gamma < \frac{1}{2}$ we would expect the bond prices to be unstable. This feature is demonstrated in table 2.9 where $\gamma = 0.45$, we see that the bond prices are erratic, with large fluctuations for 15 maturity bonds and apparent stability at very long maturities. This feature of stability at long maturity is deceptive. It can be best appreciated by observing the very high 5 year forward rates implied by the prices of the longer maturity bonds. As we have argued earlier, $\gamma > \frac{1}{2}$ is theoretically sufficient to ensure convergence for the range of γ values, maturities and annual number of time steps selected in our tables convergence is immediately achieved at short maturities, but only for $\gamma = 0.70$ at 25 year maturity.

From Table 2.17, we see that for $\alpha_1 > 0, \gamma = \frac{1}{2}$ SB call prices are in excellent agreement with analytical call prices. However, for $\alpha_1 < 0, \gamma = \frac{1}{2}$ SB call prices are significantly lower than the analytical call prices. This difference is explained by examining the bond price. For $\alpha_1 < 0, \gamma = 0.25$, we find that all the call prices are zero indicating that for the exercise prices chosen, the call options are deep out of the money. The main reason for these values is the collapsed bond price of 12.7424

Table 2.18 contains put prices. As there are no analytical put prices available, direct comparison is not possible. For $\alpha_1 > 0$, we find that the put prices are reasonable given the exercise prices. However, for $\alpha_1 < 0$, we find that the put prices are too expensive due to the low bond prices.

2.5. Conclusion

The development in Section 3 and the results of numerical experimentation in Section 4 indicate that the value of γ is critical for the stability of the lattice. $\gamma > \frac{1}{2}$ ensures that the constant variance binomial tree converges to the underlying interest rate process. Theoretically we could achieve convergence when $\gamma > \frac{1}{2}$, however, in such an instance we need a ridiculously large number of time steps. From a practical viewpoint convergence is achieved around $\gamma = 0.7$.

In this chapter, we have applied the lattice approach and have discovered that it has severe limitations. In the next chapter we use the partial differential equation approach to value discounted bonds and contingent claim prices based on the CKLS model.

Table 2.1 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \Delta r = 0.5\%,\ \gamma = 1.0$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	94.5228	94.5612	94.5258	94.5215	94.5200	94.5193	94.5189
1	11	90.1690	90.1167	90.1508	90.1544	90.1563	90.1570	90.1574
5	5	71.0379	71.9618	70.8604	70.8509	70.8477	70.8462	70.8452
5	11	63.7161	63.3543	63.4452	63.4561	63.4597	63.4615	63.4626
10	5	48.1647	49.1540	47.8332	47.7264	47.7270	47.7271	47.7272
10	11	42.8455	16.5766	42.3708	42.3753	42.3768	42.3776	42.3781
15	5	32.5442	33.9295	32.2294	32.0422	32.0224	32.0229	32.0233
15	11	28.9322	4.7666	28.4065	28.4125	28.4137	28.4143	28.4146
20	5	21.9840	22.2968	21.7123	21.5257	21.4846	21.4792	21.4796
20	11	19.5432	1.1013	17.8643	19.0560	19.0572	19.0577	19.0581
25	5	14.8502	10.3306	14.6238	14.4680	14.4226	14.4088	14.4070
25	11	13.2014	0.2034	12.7746	12.7803	12.7818	12.7824	12.7823

Table 2.2: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.1$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0666	94.9973	94.9962	94.9958	94.9956	94.9955
1	11	90.0672	89.7402	89.7537	89.7554	89.7560	89.7563	89.7564
5	5	83.4832	76.4138	76.3574	76.3511	76.3491	76.3480	76.6474
5	11	72.5572	61.8080	62.0374	62.1065	62.1176	62.1231	62.1264
10	5	75.3333	58.2066	58.2132	58.2172	58.2187	58.2195	58.2200
10	11	65.0224	43.9308	44.3897	44.4505	44.4708	44.4810	44.4872
15	5	68.2741	44.6144	44.6760	44.7017	44.7106	44.7151	44.7178
15	11	58.9177	33.0517	33.5045	33.5824	33.6086	33.6218	33.6297
20	5	61.8442	34.3531	34.3486	34.3977	34.4155	34.4245	34.4299
20	11	53.4022	25.0213	25.6357	25.7297	25.7613	25.7774	25.7870
25	5	56.0925	26.7268	26.3981	26.4750	26.5014	26.5146	26.5225
25	11	48.4052	19.3547	19.6675	19.7748	19.8118	19.8303	19.8415

Table 2.3 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \Delta r = 0.5\%,\ \gamma = 0.5$$

Maturity (years)	**r(%)**	**CIR**	Annual number of time steps (n) **10**	**50**	**100**	**150**	**200**	**250**
1	5	94.5228	94.5662	94.5313	94.5270	94.5256	94.5249	94.5245
1	11	90.1690	90.1256	90.1605	90.1648	90.1662	90.1699	90.1673
5	5	71.0379	71.1473	71.0549	71.0464	71.0436	71.0422	71.0413
5	11	63.7161	63.5822	63.6897	63.7029	63.7073	63.7095	63.7108
10	5	48.1647	48.0527	48.1503	48.1575	48.1599	48.1611	48.1619
10	11	42.8455	42.3731	42.8170	42.8311	42.8359	42.8383	42.8397
15	5	32.5442	30.9070	32.5122	32.5263	32.5323	32.5352	32.5370
15	11	28.9322	20.9780	28.8929	28.9114	28.9184	28.9218	28.9239
20	5	21.9840	9.2427	21.9113	21.9597	21.9677	21.9718	21.9742
20	11	19.5432	15.5713	19.4927	19.5174	19.5260	19.5303	19.5328
25	5	14.8502	6.8207	14.8025	14.8236	14.8319	14.8364	14.8392
25	11	13.2014	11.7957	13.1460	13.1745	13.1828	13.1874	13.1902

Table 2.4: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.5$$

Maturity (years)	**r(%)**	**CIR**	Annual number of time steps (n) **10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0224	94.8532	95.0089	95.0153	95.1216	94.9830
1	11	90.0672	89.8954	89.9402	89.9766	89.9877	90.0419	90.0203
5	5	83.4832	66.0205	78.0677	76.4452	78.0432	81.5666	74.8002
5	11	72.5572	65.7017	64.5628	67.9157	65.7878	65.4420	65.8205
10	5	75.3333	51.0533	54.3538	62.3036	83.4698	58.9077	82.6129
10	11	65.0224	27.8252	48.7119	45.7342	48.2313	53.0214	61.7257
15	5	68.2741	44.3687	59.7569	31.2846	49.7941	36.4932	60.6625
15	11	58.9177	19.3936	20.6762	25.2328	32.5130	43.5086	30.4504
20	5	61.8442	41.3507	11.9593	29.1225	82.3489	39.7977	28.2039
20	11	53.4022	14.4455	18.7966	26.7914	45.8062	23.1155	42.4286
25	5	56.0925	40.3146	8.5513	30.9268	16.2125	60.8095	31.8081
25	11	48.4052	11.3360	19.4318	36.5701	13.0887	31.0861	16.4342

Table 2.5 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \Delta r = 0.5\%,\ \gamma = 0.25$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	94.5228	94.5827	94.5409	94.5355	94.5442	94.5348	94.5414
1	11	90.1690	90.1455	90.1824	90.1871	90.1888	90.1892	90.1899
5	5	71.0379	71.6040	71.6306	71.7011	71.3977	71.3204	71.3013
5	11	63.7161	64.1310	64.1316	64.2805	64.1651	65.3028	64.1602
10	5	48.1647	47.7803	50.8847	49.3457	49.2719	49.5016	48.7296
10	11	42.8455	41.7615	43.8430	43.5653	43.6672	43.9519	43.7139
15	5	32.5442	24.6014	34.0075	33.7530	33.9134	35.7332	33.4930
15	11	28.9322	19.2185	30.1608	30.1282	29.7007	30.4036	30.2003
20	5	21.9840	18.9993	24.2652	25.0777	22.8881	23.3038	22.7247
20	11	19.5432	11.6217	20.3518	20.6131	20.3271	20.2524	20.2861
25	5	14.8502	15.3918	15.7323	15.7820	16.1144	15.5258	16.0132
25	11	13.2014	8.1477	14.2327	18.5408	14.7660	14.9029	13.8099

Table 2.6: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.25$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	89.8455	89.3463	89.8419	90.9203	89.4957	90.7504
1	11	90.0672	87.7584	87.3943	86.4093	88.8362	89.3431	86.5593
5	5	83.4832	57.3250	37.8758	60.7451	44.4512	39.8413	37.3211
5	11	72.5572	39.9553	41.5183	53.9920	37.6150	59.1822	40.0932
10	5	75.3333	45.0088	16.7096	11.6988	9.7896	34.2173	20.9337
10	11	65.0224	21.5378	7.6741	15.4979	10.4151	29.0210	15.8628
15	5	68.2741	41.9875	9.0337	4.9070	3.6089	2.9686	2.5709
15	11	58.9177	14.0994	2.8563	15.9817	5.7811	3.6501	2.7072
20	5	61.8442	42.4194	5.6083	2.3347	1.4857	1.1133	0.9067
20	11	53.4022	10.2821	1.2002	0.4831	5.1569	2.1565	1.3105
25	5	56.0925	44.5442	3.8530	1.2263	0.6701	0.4538	0.3434
25	11	48.4052	8.0393	0.5587	0.1759	10.7133	1.8111	0.8044

Table 2.7 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \Delta r = 0.5\%,\ \gamma = 0.75$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	94.5228	94.5620	94.5268	94.5225	94.5211	94.5204	94.5199
1	11	90.1690	90.1189	90.1532	90.1573	90.1587	90.1594	90.1598
5	5	71.0379	70.9829	70.9004	70.8913	70.8883	70.8868	70.8859
5	11	63.7161	63.4054	63.5040	63.5154	63.5192	63.5210	63.5222
10	5	48.1647	47.9732	47.8177	47.8202	47.8210	47.8214	47.8216
10	11	42.8455	39.5618	42.4755	42.4822	42.4845	42.4856	42.4863
15	5	32.5442	31.8280	32.1254	32.1324	32.1347	32.1359	32.1366
15	11	28.9322	14.4298	28.5207	28.5279	28.5304	28.5317	28.5324
20	5	21.9840	15.1024	21.5670	21.5840	21.5869	21.5884	21.5893
20	11	19.5432	0.1490	19.1496	19.1618	19.1647	19.1661	19.1670
25	5	14.8502	12.4947	14.4907	14.4976	14.5004	14.5020	14.5030
25	11	13.2014	0.0103	12.8473	12.8702	12.8732	12.8748	12.8757

Table 2.8: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.75$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0404	95.0324	95.0314	95.0311	95.0309	95.0308
1	11	90.0672	89.8184	89.8391	89.8417	89.8425	89.8430	89.8432
5	5	83.4832	78.8420	78.8091	78.8130	78.8141	78.8149	78.8152
5	11	72.5572	65.5755	65.9817	66.0355	66.0512	66.0602	66.0656
10	5	75.3333	65.7624	65.5869	65.5154	65.5274	65.5333	65.5379
10	11	65.0224	52.4328	53.3328	53.4099	53.4320	53.4490	53.4582
15	5	68.2741	56.6282	55.3500	55.2521	55.1837	55.2235	55.2216
15	11	58.9177	45.0746	44.7639	44.8633	44.8743	44.8863	44.8862
20	5	61.8442	50.1408	46.9773	46.6991	46.5821	46.5519	46.5757
20	11	53.4032	35.9286	37.4520	37.8161	37.7887	37.8606	37.8371
25	5	56.0925	43.3192	39.5991	39.4030	39.2116	39.2704	39.2767
25	11	48.4052	31.8192	32.2106	32.0108	31.8287	31.9528	31.9573

Table 2.9 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%\ , \gamma = 0.45$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	94.2032	94.6292	94.6325	94.9888	94.7274	94.6481
1	11	90.0672	90.0984	90.0093	89.9506	89.8414	89.8438	89.8945
5	5	83.4832	64.9493	64.8480	67.8887	72.8856	85.0486	71.6882
5	11	72.5572	69.6823	74.7523	64.7510	63.9928	65.2313	68.1789
10	5	75.3333	50.6511	63.6275	38.9979	55.8186	43.6784	69.5384
10	11	65.0224	27.3929	69.3814	69.7575	37.3005	45.9197	72.3476
15	5	68.2741	45.1193	16.2982	35.1366	23.4260	53.0360	35.3029
15	11	58.9177	18.7694	23.0070	32.0187	65.6749	28.3763	67.3119
20	5	61.8442	15.1024	21.5670	21.5840	21.5869	21.5884	21.5893
20	11	53.4032	0.1490	19.1496	19.1618	19.1647	19.1661	19.1670
25	5	56.0925	12.4947	14.4907	14.4976	14.5004	14.5020	14.5030
25	11	48.4052	0.0103	12.8473	12.8702	12.8732	12.8748	12.8757

Table 2.10 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%\ , \gamma = 0.58$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0821	95.1216	95.1189	95.1157	95.1033	95.1151
1	11	90.0672	89.9368	89.9763	89.9759	89.9757	89.9788	89.9784
5	5	83.4832	81.0251	79.8113	81.3975	84.2648	83.3692	83.2332
5	11	72.5572	70.6560	70.0046	71.9769	72.2217	70.4286	71.8480
10	5	75.3333	51.2450	69.1049	66.6272	68.1837	70.9188	74.4992
10	11	65.0224	57.3810	55.9126	60.6087	57.6310	67.7080	67.3645
15	5	68.2741	43.1700	82.7782	79.0011	55.8353	63.7153	74.8207
15	11	58.9177	19.2516	38.7958	54.8187	53.3486	55.1422	58.8431
20	5	61.8442	38.4504	42.8720	51.0803	64.4076	46.7985	61.1943
20	11	53.4032	14.4091	42.2462	38.2465	41.3580	46.9917	55.4109
25	5	56.0925	35.6886	44.9245	59.6391	35.0906	53.4783	39.2244
25	11	48.4052	11.3542	52.4821	46.9518	54.8370	31.2805	41.3935

Table 2.11: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%, \gamma = 0.6$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.1075	95.0999	95.0943	95.0944	95.0920	95.0942
1	11	90.0672	89.9209	89.9518	89.9553	89.9555	89.9571	
5	5	83.4832	79.9592	82.4179	82.3216	81.9253	82.2970	82.1428
5	11	72.5572	69.1971	70.6865	70.1965	70.2367	69.8697	70.2125
10	5	75.3333	51.1666	80.5591	74.4498	73.5735	74.1379	74.8970
10	11	65.0224	66.3192	64.3793	64.3383	60.8978	62.1600	62.7312
15	5	68.2741	42.7809	69.1879	65.6838	67.9129	71.1071	62.3920
15	11	58.9177	58.1810	57.3468	63.9424	60.1880	59.2244	59.2381
20	5	61.8442	37.8085	39.8357	76.4532	57.0481	61.9830	67.5776
20	11	53.4032	14.2278	37.1297	56.5233	54.6404	56.2418	56.2984
25	5	56.0925	34.6973	39.6600	47.9778	60.0109	67.9750	56.5725
25	11	48.4052	11.1764	41.4310	54.6560	40.2539	45.8681	56.6289

Table 2.12: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%, \gamma = 0.62$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0907	95.0845	95.0825	95.0813	95.0813	95.0812
1	11	90.0672	89.9028	89.9321	89.9355	89.9367	89.9374	89.9377
5	5	83.4832	83.0287	81.2949	81.4915	81.2571	81.3433	81.3597
5	11	72.5572	69.1851	69.1929	69.3288	69.2732	69.2627	69.3822
10	5	75.3333	76.8822	74.0661	71.3649	72.4741	72.0064	70.9809
10	11	65.0224	53.0647	59.8233	59.6606	60.9370	59.9992	59.4970
15	5	68.2741	42.3474	62.7260	65.7622	63.2071	63.2079	64.8810
15	11	58.9177	53.3729	51.3007	55.1865	52.5794	52.0288	52.3890
20	5	61.8442	37.1130	66.0067	61.7501	61.7892	56.0105	57.9742
20	11	53.4032	58.7963	55.8751	46.8756	45.5179	46.3959	47.9464
25	5	56.0925	33.7299	69.2213	59.5894	48.9912	54.9683	49.6347
25	11	48.4052	10.9529	47.9763	50.6301	48.2974	45.6670	40.9423

Table 2.13: Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%\ , \gamma = 0.625$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0893	95.0810	95.0797	95.0792	95.0788	95.0787
1	11	90.0672	89.8987	89.9274	89.9310	89.9322	89.9327	89.9331
5	5	83.4832	82.6775	81.4818	81.4145	81.0173	81.2496	81.1598
5	11	72.5572	69.1790	69.0692	69.0528	69.2647	69.2184	69.1407
10	5	75.3333	75.8871	73.0745	72.1282	72.2245	71.8849	71.4673
10	11	65.0224	64.6356	59.0138	59.4726	60.2016	59.2187	59.9499
15	5	68.2741	42.2356	61.5582	61.1200	64.5019	61.9363	63.6556
15	11	58.9177	52.3878	52.5953	53.8121	52.3797	53.4889	53.0590
20	5	61.8442	36.9324	63.6161	59.7124	60.0518	57.7183	56.2875
20	11	53.4032	56.7941	53.5859	45.3150	47.5881	46.1934	46.3374
25	5	56.0925	36.9259	65.5933	58.2714	47.2381	52.7861	48.4052
25	11	48.4052	56.7941	53.5859	45.3150	47.5881	46.1934	46.3374

Table 2.14 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%\ , \gamma = 0.63$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0868	95.0774	95.0765	95.0762	95.0761	95.0760
1	11	90.0672	89.8948	89.9231	89.9265	89.9277	89.9283	89.9286
5	5	83.4832	82.3482	81.4263	81.2418	81.0972	81.0664	80.9329
5	11	72.5572	69.0433	69.1201	69.0911	69.1059	69.0889	69.0547
10	5	75.3333	74.9883	72.2028	72.0862	71.7256	71.4393	71.2799
10	11	65.0224	63.6289	59.2278	59.9959	59.4862	59.6592	59.7580
15	5	68.2741	78.7284	60.5024	65.6778	64.5029	62.6350	62.5240
15	11	58.9177	51.4696	54.3961	52.5946	53.3915	53.5515	53.2485
20	5	61.8442	36.7489	61.5757	57.9624	58.4656	57.7570	54.8019
20	11	53.4032	54.9879	51.6430	46.0033	48.4824	47.5884	45.5517
25	5	56.0925	49.2998	62.5993	56.6698	48.7730	50.8983	51.7188
25	11	48.4052	10.8299	49.7247	46.1582	44.5299	44.2157	42.4201

Table 2.15 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\gamma = 0.65$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0761	95.0679	95.0670	95.0666	95.0664	95.0663
1	11	90.0672	89.8795	89.9062	89.9095	89.9106	89.9112	89.9115
5	5	83.4832	81.3443	80.7477	80.6034	80.5476	80.6047	80.5525
5	11	72.5572	68.3094	68.4811	68.4212	68.5007	68.4915	68.4936
10	5	75.3333	72.0496	69.4732	70.2820	69.5347	69.7904	69.7406
10	11	65.0224	60.5684	59.1816	58.5803	58.5832	58.1211	58.3922
15	5	68.2741	71.5003	63.6979	62.5255	61.8201	61.7378	61.6035
15	11	58.9177	48.3205	53.2996	51.9185	51.4531	50.7654	50.5735
20	5	61.8442	70.2152	55.5803	52.7183	53.4240	54.4168	54.4395
20	11	53.4032	49.2143	45.9072	46.8541	45.7353	45.2751	45.2517
25	5	56.0925	57.8436	54.2841	50.7452	49.6647	46.6671	47.9024
25	11	48.4052	51.9728	44.9968	39.9222	38.7574	39.1568	39.8437

Table 2.16 : Bond Prices calculated analytically (CIR) and the Simplified Binomial Tree for different value of gamma.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\gamma = 0.7$$

			Annual number of time steps (n)					
Maturity (years)	**r(%)**	**CIR**	**10**	**50**	**100**	**150**	**200**	**250**
1	5	95.1632	95.0555	95.0478	95.0468	95.0465	95.0463	94.0462
1	11	90.0672	89.8460	89.8694	89.8723	89.8733	89.8738	89.8741
5	5	83.4832	79.9265	79.6521	79.6343	79.6367	79.6356	79.6335
5	11	72.5572	66.6972	67.1223	67.1777	67.1974	67.2050	67.2115
10	5	75.3333	67.2904	67.8403	67.6072	67.6129	67.5418	67.5425
10	11	65.0224	55.7910	55.8168	55.7263	55.7823	55.7684	55.7886
15	5	68.2741	62.0019	58.5438	58.3626	58.0638	57.9160	57.9140
15	11	58.9177	48.0893	47.6136	47.9384	47.7342	47.8535	47.7471
20	5	61.8442	56.9407	50.7900	50.4015	49.9533	49.9203	49.9642
20	11	53.4032	40.2597	41.9755	41.3674	41.1191	41.0002	41.0875
25	5	56.0925	49.7495	43.4164	43.1277	43.0066	42.6791	42.9227
25	11	48.4052	38.9234	35.3059	35.9617	35.1032	35.2004	35.2243

Table 2.17 : Call Prices calculated analytically (CIR) or using the Simplified Binomial Method (SB)

$\Delta t = 0.05, , r_0 = 8\%$

Exercise Price	**Model**	α_1	γ	**Bond Price**	Expiry (years)				
					5	**4**	**3**	**2**	**1**
35	SB	0.01875	0.25	46.1479	22.4634	20.5693	18.5139	16.2761	13.8294
40	SB	0.01875	0.25		19.0799	16.9153	14.5669	12.0130	9.2211
45	SB	0.01875	0.25		15.6966	13.2635	10.6320	7.8102	4.7823
50	SB	0.01875	0.25		12.3161	9.6355	6.7962	3.9743	1.3899
55	SB	0.01875	0.25		8.9644	6.1383	3.4020	1.2193	0.0908
35	SB	0.01875	0.5	45.4228	21.8763	19.9468	17.8543	15.5820	13.1108
	CIR			45.4273	21.8802	19.9509	17.8585	15.5863	13.1552
40	SB	0.01875	0.5		18.5125	16.3074	13.9160	11.3191	8.4949
	CIR				18.5163	16.3114	13.9201	11.3233	8.4993
45	SB	0.01875	0.5		15.1487	12.6680	9.9778	7.0597	3.9087
	CIR				15.1524	12.6719	9.9819	7.0636	3.9137
50	SB	0.01875	0.5		11.7850	9.0291	6.0521	2.9514	0.4631
	CIR				11.7866	9.0330	6.0560	2.9514	0.4535
55	SB	0.01875	0.5		8.4221	5.4127	2.3833	0.3213	0.0000
	CIR				8.4257	5.4156	2.3804	0.3118	0.0001
35	SB	0.01875	0.75	45.0746	21.5889	19.6420	17.5322	15.2451	12.7647
40	SB	0.01875	0.75		18.2338	16.0087	13.5976	10.9837	8.1490
45	SB	0.01875	0.75		14.8787	12.3755	9.6630	6.7224	3.5339
50	SB	0.01875	0.75		11.5236	8.7423	5.7284	2.4816	0.0678
55	SB	0.01875	0.75		8.1685	5.1093	1.8581	0.0234	0.0000
60	SB	-0.02725	0.25	12.7424	0.0000	0.0000	0.0000	0.0000	0.0000
65	SB	-0.02725	0.25		0.0000	0.0000	0.0000	0.0000	0.0000
70	SB	-0.02725	0.25		0.0000	0.0000	0.0000	0.0000	0.0000
75	SB	-0.02725	0.25		0.0000	0.0000	0.0000	0.0000	0.0000
80	SB	-0.02725	0.25		0.0000	0.0000	0.0000	0.0000	0.0000
60	SB	-0.02725	0.5	53.9393	12.5149	10.4169	8.1685	5.7130	3.2314
				69.9882	23.9008	22.8564	20.2596	19.8902	16.9798
65	SB	-0.02725	0.5		9.4324	7.1913	4.8941	2.6339	0.5631
					20.1770	19.0843	17.7967	16.0922	13.2470
70	SB	-0.02725	0.5		6.3705	4.1718	1.9197	0.0362	0.0000
					16.4887	15.3565	14.0532	12.3971	9.7260
75	SB	-0.02725	0.5		3.5799	1.3652	0.0000	0.0000	0.0000
					12.8444	11.6829	10.3819	8.8038	6.4487
80	SB	-0.02725	0.5		1.0166	0.0000	0.0000	0.0000	0.0000
					9.2570	8.0789	6.8019	5.3528	3.4558
60	SB	-0.02725	0.75	59.0654	16.7839	14.9919	12.9336	10.4183	6.9780
65	SB	-0.02725	0.75		13.4974	11.7328	9.6556	7.2323	4.1805
70	SB	-0.02725	0.75		10.3559	8.5809	6.6552	4.3879	1.9818
75	SB	-0.02725	0.75		7.3550	5.6459	3.9382	2.1082	0.5712
80	SB	-0.02725	0.75		4.5486	3.0461	1.6753	0.5415	0.0193

1) The call option is written on a 10 - year zero coupon bond with a face value of $100.

Table 2.18 : Put Prices calculated analytically (CIR) or using the Simplified Binomial Method (SB)

$\Delta t = 0.05,\ ,r_0 = 8\%$

					Expiry (years)				
Exercise Price	**Model**	α_1	γ	**Bond Price**	**5**	**4**	**3**	**2**	**1**
45	SB	0.01875	0.25	46.1479	0.5438	0.5433	0.5267	0.5267	0.4605
50	SB	0.01875	0.25		3.8522	3.8522	3.8522	3.8522	3.8522
55	SB	0.01875	0.25		8.8522	8.8522	8.8522	8.8522	8.8522
60	SB	0.01875	0.25		13.8521	13.8521	13.8521	13.8521	13.8521
65	SB	0.01875	0.25		18.8521	18.8521	18.8521	18.8521	18.8521
45	SB	0.01875	0.5	45.4228	0.1730	0.1730	0.1729	0.1725	0.1658
50	SB	0.01875	0.5	45.4273	4.5773	4.5773	4.5773	4.5773	4.5773
55	SB	0.01875	0.5		9.5773	9.5773	9.5773	9.5773	9.5773
60	SB	0.01875	0.5		14.5772	14.5772	14.5772	14.5772	14.5772
65	SB	0.01875	0.5		19.5772	19.5772	19.5772	19.5772	19.5772
45	SB	0.01875	0.75	45.0746	0.0472	0.0472	0.0472	0.0472	0.0471
50	SB	0.01875	0.75		4.9524	4.9524	4.9524	4.9524	4.9524
55	SB	0.01875	0.75		9.9254	9.9254	9.9254	9.9254	9.9254
60	SB	0.01875	0.75		14.9254	14.9254	14.9254	14.9254	14.9254
65	SB	0.01875	0.75		19.9254	19.9254	19.9254	19.9254	19.9254
60	SB	-0.02725	0.25	12.7424	47.2576	47.2576	47.2576	47.2576	47.2576
65	SB	-0.02725	0.25		52.2576	52.2576	52.2576	52.2576	52.2576
70	SB	-0.02725	0.25		57.2576	57.2576	57.2576	57.2576	57.2576
75	SB	-0.02725	0.25		62.2576	62.2576	62.2576	62.2576	62.2576
80	SB	-0.02725	0.25		67.2576	67.2576	67.2576	67.2576	67.2576
60	SB	-0.02725	0.5	53.9393	9.1824	8.7773	8.2470	7.6334	7.0000
65	SB	-0.02725	0.5	69.9882	12.0956	11.6961	11.3650	11.0607	11.0607
70	SB	-0.02725	0.5		16.0607	16.0607	16.0607	16.0607	16.0607
75	SB	-0.02725	0.5		21.0607	21.0607	21.0607	21.0607	21.0607
80	SB	-0.02725	0.5		26.0607	26.0607	26.0607	26.0607	26.0607
60	SB	-0.02725	0.75	59.0654	6.0500	5.8748	5.6368	5.2374	4.4324
65	SB	-0.02725	0.75		8.3108	8.1552	7.9227	7.5812	7.0509
70	SB	-0.02725	0.75		11.4158	11.3300	11.2218	11.062	10.9346
75	SB	-0.02725	0.75		15.9346	15.9346	15.9346	15.9346	15.9346
80	SB	-0.02725	0.75		20.9346	20.9346	20.9346	20.9346	20.9346

1) The put option is written on a 10 - year zero coupon bond with a face value of $100.

CHAPTER 3.

PARTIAL DIFFERENTIAL EQUATION APPROACH FOR THE EVALUATION OF DEFAULT-FREE BONDS AND INTEREST RATE CONTINGENT CLAIMS.

3.1. Introduction

The objective of this chapter is to value default-free bonds and interest rate contingent claims based on the CKLS model using the following numerical methods:

a) Crank-Nicholson finite difference approach.

b) Box Method. The Box-Method is wholly new in finance literature

c) Method of Lines. Thus far the Method of Lines approach has only been applied to the valuation of contingent claims based on equity.

The contribution of this chapter is as follows:

a) Crank-Nicholson scheme is generalised to incorporate all possible values of γ.

b) Box Method is applied to finance for the first time

c) Method of lines is extended to fixed income from equities.

We test each of the three numerical methods for their convergence characteristics. In section 2 we derive the numerical schemes for each of the above mentioned numerical methods. In section 3 we investigate each of the numerical methods with each other or when analytical prices are available with analytical prices. Section 4 concludes this chapter. However, before continuing to Section 2, we repeat the CKLS model for the instantaneous short term interest rate.

$$dr_t = k(\theta - r)dt + \sigma r^{\gamma} dz_t \qquad (3.1.1)$$

The resulting partial differential equation based on the above stochastic equation is:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 u}{\partial r^2} + k(\theta - r)\frac{\partial u}{\partial r} - ru + \frac{\partial u}{\partial t} = 0 \qquad (3.1.2)$$

σ, r, k, θ represent the same variables as defined earlier. In equation (3.1.2) [1] $u(r_t, t)$ may represent either $B(r_t, t, T^*)$ or $P(r_t, t, T^*, T)$.

$B(r, t, T^*)$: price of a discount bond at time t, which matures at time T^* with the generated spot rate r_t .

$P(t, T^*, T)$: price of a contingent claim at time t, which expires at time T based on a discount bond which matures at time T^*.

In equation (3.1.2) $u(r_t, t)$ may represent either $B(r_t, t, T^*)$ or $P(r_t, t, T^*, T)$.

$B(r_t, t, T^*)$ is subject to the following boundary conditions:

$$B(0, t, T^*) = 1 \tag{B1}$$

$$B(\infty, t, T^*) = 0 \tag{B2}$$

With $P(r_t, t, T^*, T)$ representing an American call option it is subject to the following boundary conditions:

$$P(r_t, T, T^*, T) = \max[B(r_t, T, T^*) - E, 0] \tag{B3}$$

$$P(\infty, t, T^*, T) = 0 \tag{B4}$$

$$P(r_t, t, T^*, T) = \max[B(r_t, t, T^*) - E, P(r_t, t, T^*, T)] \tag{B5}$$

Finally with $P(r_t, t, T^*, T)$ representing American put options it is subject to the following boundary conditions:

$$P(r_t, T, T^*, T) = \max[E - B(r_t, T, T^*), 0] \tag{B6}$$

---

[1] With the Crank-Nicolson finite difference approach we use the variable $s = \dfrac{cr}{1 + cr}$. Same boundary conditions as with r_t apply except when stated otherwise.

$$P(\infty, t, T^*, T) = E \quad \text{(B7)}$$

$$P(r_t, t, T^*, T) = \max\left[E - B(r_t, t, T^*), P(r_t, t, T^*, T)\right] \quad \text{(B8)}$$

We now transform equation (3.1.2) such that either the bond or the contingent claim evolves from the options expiration date or the bonds maturity date to the present, i.e. we transform the time variable:

$$\tau = T - t \quad (3.1.3)$$

Thus equation (3.1.2) now becomes:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 u}{\partial r^2} + k(\theta - r)\frac{\partial u}{\partial r} - ru = \frac{\partial u}{\partial \tau} \quad (3.1.4)$$

3.2. Numerical Methods

In this section we develop in depth the three numerical methods stated in Section 1 of this chapter to solve the partial differential equation for default free bonds and interest rate contingent claims. A uniform grid of size $M \times N$ is constructed for values of u_n^m - the value of u at time increment t_m and interest rate increment r_n, for each method, where:

[2] $u_n^m = u(n\Delta r, m\Delta t)$

$t_m = t_0 + m\Delta t \qquad\qquad m = 0,1,\ldots,M$

$r_n = r_0 + n\Delta t \qquad\qquad n = 0,1,\ldots,N$

The values of u_n^m are computed column by column from the left column to the right column. And within each column, we solve from bottom to the top. To truncate the grid, we discretize the boundary conditions (B2), (B4) and (B7) respectively as:

$$B\left(j\Delta r, t, T^*\right) = 0 \qquad \text{(B9)}$$

$$P\left(j\Delta r, t, T^*, T\right) = 0 \qquad \text{(B10)}$$

$$P\left(j\Delta r, t, T^*, T\right) = E \qquad \text{(B11)}$$

for $j \geq N+1$

For all susbsequent numerical development we assume that we are at point $(n\Delta r, m\Delta t)$ or (n,m) for short on the grid. For the time derivative in equation (3.1.4), we use the Euler backward difference approximation

$$\frac{\partial u}{\partial \tau} \approx \frac{u_n^m - u_n^{m-1}}{\Delta t} = \frac{u - u_0}{\Delta t} \qquad (3.2.1)$$

---

[2] Same notation is used for $j\Delta s$

Thus equation (3.1.4) now becomes:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 u}{\partial r^2}+k(\theta-r)\frac{\partial u}{\partial r}-ru=\frac{u-u_0}{\Delta t} \qquad (3.2.2)$$

3.2.1. Crank-Nicholson Method

We start firstly by transforming the interest rate grid, using the following transformations:

$$s=\frac{cr}{1+cr} \qquad (3.2.3)$$

where c is a constant

Secondly we transform the variables in equation (3.1.2) as follows:

$$W(s,t)=u(s,t) \qquad (3.2.4)$$

Based on the above transformations, the partial derivatives of equation (3.1.2) becomes:

$$\frac{\partial u}{\partial r}=\frac{\partial W}{\partial s}\frac{ds}{dr}$$

$$\frac{\partial^2 u}{\partial r^2} = \left(\frac{d^2 s}{dr^2}\right)\left(\frac{\partial W}{\partial s}\right) + \left(\frac{ds}{dr}\right)^2\left(\frac{\partial^2 W}{\partial s^2}\right)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial W}{\partial \tau}$$

Substituting the above three transformations for $u, \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}$ into equation (3.1.2) gives:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\left(\frac{ds}{dr}\right)^2\left(\frac{\partial^2 W}{\partial s^2}\right) + \left\{\frac{1}{2}\sigma^2 r^{2\gamma}\left(\frac{d^2 s}{dr^2}\right) + k(\theta - r)\frac{ds}{dr}\right\}\left(\frac{\partial W}{\partial s}\right) - rW = \frac{\partial W}{\partial \tau} \qquad (3.2.5)$$

Furthermore:

$$r = \frac{s}{c(1-s)}$$

$$\frac{ds}{dr} = \frac{c}{(1+cr)^2} = c(1-s)^2$$

$$\left(\frac{ds}{dr}\right)^2 = c^2(1-s)^4$$

$$\frac{d^2s}{dr^2} = -2c^2(1-s)^3$$

Substituting the above expressions into equation (3.2.5) gives:

$$\left\{\frac{1}{2}\sigma^2\left[\frac{s}{c(1-s)}\right]^{2\gamma} c^2(1-s)^4\right\}\frac{\partial^2 W}{\partial s^2}$$

$$+\left\{-\sigma^2\left[\frac{s}{c(1-s)}\right]^{2\gamma} c^2(1-s)^3 + \left[k\theta - \frac{s}{c(1-s)}k\right]c(1-s)^2\right\}\frac{\partial W}{\partial s} \qquad (3.2.6)$$

$$-\frac{s}{c(1-s)}W = \frac{\partial W}{\partial \tau}$$

We discretize the above equation using the following Crank-Nicholson and Euler Backward difference approximations:

$$W = \frac{1}{2}W_n^m + \frac{1}{2}W_n^{m-1}$$

$$\frac{\partial W}{\partial s} = \frac{W_{n+1}^m - W_{n-1}^m}{4\Delta s} + \frac{W_{n+1}^{m-1} - W_{n-1}^{m-1}}{4\Delta s}$$

$$\frac{\partial^2 V}{\partial s^2} = \frac{W_{n+1}^m - 2W_n^m + W_{n-1}^m}{2(\Delta s)^2} + \frac{W_{n+1}^{m-1} - 2W_n^{m-1} + W_{n-1}^{m-1}}{2(\Delta s)^2}$$

$$\frac{\partial W}{\partial \tau} = \frac{W_n^m - W_n^{m-1}}{\Delta t}$$

Substituting the above discretizations leads to the following discrete equation:

$$\frac{\sigma^2\Delta t}{2}\left[\frac{n\Delta s}{c(1-n\Delta s)}\right]^{2\gamma}\frac{c^2(1-n\Delta s)^4}{2(\Delta s)^2}$$

$$\times\left\{W_{n+1}^{m}-2W_{n}^{m}+W_{n-1}^{m}+W_{n+1}^{m-1}-2W_{n}^{m-1}+W_{n-1}^{m-1}\right\}$$

$$+\frac{\Delta t}{4\Delta s}\left\{\begin{matrix}-\sigma^2\left[\dfrac{n\Delta s}{c(1-n\Delta s)}\right]^{2\gamma}c^2(1-n\Delta s)^3\\ \\ +\left[k\theta-\dfrac{n\Delta s}{c(1-n\Delta s)}k\right]c(1-n\Delta s)^2\end{matrix}\right\}$$

$$\times\left\{W_{n+1}^{m}-W_{n-1}^{m}+W_{n+1}^{m-1}-W_{n-1}^{m-1}\right\}$$

$$-\frac{n\Delta s\Delta t}{2c(1-n\Delta s)}W_{n}^{m}-\frac{n\Delta s\Delta t}{2c(1-n\Delta s)}W_{n}^{m-1}=W_{n}^{m}-W_{n}^{m-1} \qquad (3.2.7)$$

We can further simplify the above equation as:

$$A_n\left[W_{n+1}^{m}-2W_{n}^{m}+W_{n-1}^{m}\right]+A_n\left[W_{n+1}^{m-1}-2W_{n}^{m-1}+W_{n-1}^{m-1}\right]$$

$$+B_n\left[W_{n+1}^{m}-W_{n-1}^{m}\right]+B_n\left[W_{n+1}^{m-1}-W_{n-1}^{m-1}\right] \qquad (3.2.8)$$

$$+C_nW_{n}^{m}+C_nW_{n}^{m-1}=W_{n}^{m}-W_{n}^{m-1}$$

where:

$$A_n = \frac{\sigma^2 \Delta t}{2}\left[\frac{n\Delta s}{c(1-n\Delta s)}\right]^{2\gamma} \frac{c^2(1-n\Delta s)^4}{2(\Delta s)^2}$$

$$B_n = \frac{\Delta t}{4\Delta s}\left\{-\sigma^2\left[\frac{n\Delta s}{c(1-n\Delta s)}\right]^{2\gamma} c^2(1-n\Delta s)^3 + \left[k\theta - \frac{n\Delta s}{c(1-n\Delta s)}(k+\lambda)\right]c(1-n\Delta s)^2\right\}$$

$$C_n = -\frac{n\Delta s\Delta t}{2c(1-n\Delta s)}$$

Further rearrangement leads to:

$$\alpha_n = \chi_n W_{n-1}^m + \eta_n W_n^m + \beta_n W_{n+1}^m \tag{3.2.9}$$

where:

$$\alpha_n = -A_n\left[W_{n+1}^{m-1} - 2W_n^{m-1} + W_{n-1}^{m-1}\right] - B_n\left[W_{n+1}^{m-1} - W_{n-1}^{m-1}\right] - \left[1 + C_n\right]W_n^{m-1}$$

$$\chi_n = A_n - B_n$$

$$\eta_n = C_n - 2A_n - 1$$

$$\beta_n = A_n + B_n$$

The matrix equation linking bond prices or contingent claim prices between successive time steps m and m-1 is:

$$\begin{pmatrix} \alpha_1 - \chi_1 W_0^m \\ \alpha_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{N-1} - \beta_{N-1} W_N^m \end{pmatrix} = \begin{pmatrix} \eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \chi_{N-3} & \eta_{N-3} & \beta_{N-3} & 0 \\ \vdots & \ddots & \ddots & 0 & \chi_{N-2} & \eta_{N-2} & \beta_{N-2} \\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1} \end{pmatrix} \begin{pmatrix} W_1^m \\ W_1^m \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ W_{N-1}^m \end{pmatrix} \quad (3.2.10)$$

The above matrix equation applies for $n = 1$ onwards. However to start the above iteration process, we need the option prices at $n = 0$ i.e. at zero interest rate for $m \geq 1$. We start by approximating equation (3.2.5) as $s \rightarrow 0$. This yields the following equation near and at $s = 0$

$$k\theta \frac{ds}{dr} \frac{\partial W}{\partial s} = \frac{\partial W}{\partial \tau} \quad (3.2.11)$$

Noting that as $r \rightarrow 0, s \rightarrow 0$, the above equation simplifies to:

$$k\theta c \frac{\partial W}{\partial s} = \frac{\partial W}{\partial \tau} \quad (3.2.12)$$

To approximate the above first order derivatives, we assume that we are at point (m-1,n) on the grid. Using the forward Euler difference for $\frac{\partial W}{\partial s}$, and $\frac{\partial W}{\partial \tau}$ gives.

$$\frac{\partial W}{\partial s} \approx \frac{W_{n+1}^{m-1} - W_n^{m-1}}{\Delta s} \quad (3.2.13)$$

$$\frac{\partial W}{\partial \tau} \approx \frac{W_n^m - W_n^{m-1}}{\Delta t} \qquad (3.2.14)$$

Substitution of the above two approximations into equation (3.2.12) gives.

$$W_n^m = W_n^{m-1} + kc\theta \frac{\Delta t}{\Delta s}\left(W_{n+1}^{m-1} - W_n^{m-1}\right)$$

(3.2.15)

At $n = 0$ i.e. at zero interest rate, the above expression simplifies to.

$$W_0^m = W_0^{m-1} + kc\theta \frac{\Delta t}{\Delta s}\left(W_1^{m-1} - W_0^{m-1}\right) \qquad (3.2.16)$$

Note that the above approximation applies to both bonds and contingent claims subject to appropriate boundary conditions.

3.2.2. Box Method

The Box Method has been widely applied in engineering. However; to date this method has not been applied in finance. Below we apply the Box Method[3] to partial differential equation based on the CKLS model.

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[3] An introduction to the Box Method can be found in Richard S. Varga's book, Matrix Iterative Analysis (1962).

To derive the algorithm for the Box Method we start by dividing equation (3.1.4) by $\frac{\sigma^2 r^{2\gamma}}{2}$ and we further let:

$$a = \frac{2k\theta}{\sigma^2}$$

$$b = \frac{2k}{\sigma^2}$$

$$c = \frac{2}{\sigma^2}$$

$$d = \frac{2}{\sigma^2}$$

Then the resulting equation is:

$$\frac{\partial^2 u}{\partial r^2} + \left[ar^{-2\gamma} - br^{1-2\gamma}\right]\frac{\partial u}{\partial r} - cr^{1-2\gamma}u = dr^{-2\gamma}\frac{\partial u}{\partial \tau} \qquad (3.2.17)$$

We combine the first term and the second term on the left hand side of the above equation by choosing a function $\Psi(a, b, r, \gamma)$ or $\Psi(r)$ abbreviated such that

$$\frac{1}{\Psi(r)}\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \left[ar^{-2\gamma} - br^{1-2\gamma}\right]\frac{\partial u}{\partial r} \qquad (3.2.18)$$

Expansion and simplification of the above formula leads to the following expression.

$$\frac{1}{\Psi(r)}\frac{\partial \Psi}{\partial r} = ar^{-2\gamma} - br^{1-2\gamma} \qquad (3.2.19)$$

Integrating the previous equation gives:

$$\Psi(r) = \exp\left[\frac{ar^{1-2\gamma}}{1-2\gamma} - \frac{br^{2-2\gamma}}{2-2\gamma}\right] \qquad (3.2.20)$$

Note that with the above expression for $\Psi(r)$ there is singularity at $\gamma = \frac{1}{2}$ and $\gamma = 1$. Thus the above expression for $\Psi(r)$ is not valid at these two specific points. Further if $\gamma \neq 1$ or $\gamma \neq \frac{1}{2}$ but γ is very close to $\gamma = 1$ or $\gamma = \frac{1}{2}$, then the value of $\Psi(r)$ may be excessively because of the nature of the denominators in equation (3.2.20). In such cases we need to use a more complex approach or simply switch to the expression for $\Psi(r)$ when $\gamma = 1$ or $\gamma = \frac{1}{2}$. To derive expression for $\Psi(r)$ when $\gamma = 1$ or $\gamma = \frac{1}{2}$, we substitute, these two values of γ directly into equation (3.2.19) and integrate to give

$$\Psi(r) = \exp\left(\frac{-a}{r}\right) r^{-b} \qquad \text{for } \gamma = 1$$

$$\Psi(r) = \exp(-br) r^{a} \qquad \text{for } \gamma = \frac{1}{2}$$

With this choice of $\Psi(r)$, our original equation becomes

$$\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)-\Psi(r)r^{1-2\gamma}cu=d\Psi(r)r^{-2\gamma}\frac{\partial u}{\partial \tau} \qquad (3.2.21)$$

For $\frac{\partial u}{\partial \tau}$ we use the backward Euler approximation as before, however, for convenience we let $u=u_n^m$ and $u_0=u_n^{m-1}$. Thus equation (3.2.21) becomes:

$$\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)-\Psi(r)r^{1-2\gamma}cu=d\Psi(r)r^{-2\gamma}\left(\frac{u-u_0}{\Delta t}\right) \qquad (3.2.22)$$

Further rearrangement leads to the expression:

$$-\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)+\Psi(r)r^{1-2\gamma}cu+\frac{d\Psi(r)r^{-2\gamma}u}{\Delta t}=\frac{d\Psi(r)r^{-2\gamma}u_0}{\Delta t} \qquad (3.2.23)$$

We integrate the above equation over C_i

$$-\int_{C_i}\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)+\int_{C_i}\left(c\Psi(r)r^{1-2\gamma}+\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}\right)u=\int_{C_i}\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}u_0$$

Approximating each of the integrals, we have for the first integral:

$$-\int_{C_i}\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)=-\Psi(r_b)\left(\frac{u_{n+1}^m-u_n^m}{\Delta r}\right)+\Psi(r_a)\left(\frac{u_n^m-u_{n-1}^m}{\Delta r}\right)$$

For the second integral:

$$\int_{C_i}\left(c\Psi(r)r^{1-2\gamma}+\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}\right)u$$

$$=\Psi(r_n)\left[\frac{cr_b^{2-2\gamma}}{2-2\gamma}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)+\frac{dr_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)\right]u_n^m \text{ if}\gamma\neq\frac{1}{2}\text{or}\gamma\neq 1$$

$$=\Psi(r_n)\left[c(r_b-r_a)+\frac{d}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)\right]u_n^m \quad \text{for } \gamma=\frac{1}{2}$$

$$=\Psi(r_n)\left[-c\ln\left(\frac{r_a}{r_b}\right)+\frac{d}{\Delta t}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)\right]u_n^m \text{ for } \gamma=1$$

For the third integral:

$$\int_{C_i}\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}u_0$$

$$=\Psi(r_n)\left[\frac{dr_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)\right]u_n^{m-1} \qquad \text{if } \gamma\neq\frac{1}{2} \text{ or } \gamma\neq 1$$

$$=\Psi(r_n)\left[-\frac{d}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)\right]u_n^{m-1} \qquad \text{for } \gamma=\frac{1}{2}$$

$$= \Psi(r_n)\left[\frac{d}{\Delta t}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)\right]u_n^{m-} \qquad \text{for } \gamma = 1$$

Substituting the above approximations into the original equation yields

$$\alpha_n = \chi_n u_{n-1}^m + \eta_n u_n^m + \beta_n u_{n+1}^m \qquad (3.2.24)$$

where taking $r_a = \dfrac{r_n + r_{n-1}}{2}$ and $r_b = \dfrac{r_{n+1} + r_n}{2}$:

$$\alpha_n = \frac{d r_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1 - \left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)u_n^{m-1} \quad \text{if } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1$$

$$= -\frac{d}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)u_n^{m-1} \qquad \text{for } \gamma = \frac{1}{2}$$

$$= \frac{d}{\Delta t}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)u_n^{m-1} \qquad \text{for } \gamma = 1$$

$$\chi_n = -\frac{1}{\Delta r}\frac{\Psi(r_a)}{\Psi(r_n)}$$

$$\beta_n = -\frac{1}{\Delta r}\frac{\Psi(r_b)}{\Psi(r_n)}$$

$$\eta_n = \frac{1}{\Delta r}\left(\frac{\Psi(r_b)}{\Psi(r_n)} + \frac{\Psi(r_a)}{\Psi(r_n)}\right) + X$$

where:

$$X = \frac{cr_b^{2-2\gamma}}{2-2\gamma}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right) + \frac{dr_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right) \text{provided } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1$$

$$= c(r_b - r_a) + \frac{d}{\Delta t}\ln\left(\frac{r_a}{r_b}\right) \qquad \text{for } \gamma = \frac{1}{2}$$

$$= -c\ln\left(\frac{r_a}{r_b}\right) + \frac{d}{\Delta t}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \quad \text{for } \gamma = 1$$

As with the Generalised Crank-Nicholson Method we find that the basic matrix equation linking all bond prices or contingent claims prices between two successive time steps m and m-1 as:

$$\begin{pmatrix} \alpha_1 - \chi_1 u_0^m \\ \alpha_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{N-1} - \beta_{N-1} u_N^m \end{pmatrix} = \begin{pmatrix} \eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \chi_{N-3} & \eta_{N-3} & \beta_{N-3} & 0 \\ \vdots & \ddots & \ddots & 0 & \chi_{N-2} & \eta_{N-2} & \beta_{N-2} \\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1} \end{pmatrix} \begin{pmatrix} u_1^m \\ u_1^m \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_{N-1}^m \end{pmatrix} \quad (3.2.25)$$

As with the Crank-Nicholson method, the above matrix starts at n = 1. We, however, need both the bond and option prices at n = 0. We approximate equation (3.1.4) as $r \to 0$:

$$k\theta \frac{\partial u}{\partial r} = \frac{\partial u}{\partial \tau} \tag{3.2.26}$$

Again, we use the forward Euler differences to discretize the above derivatives to yield at n = 0:

$$u_0^m = u_0^{m-1} + k\theta \frac{\Delta t}{\Delta r}\left(u_1^{m-1} - u_0^{m-1}\right) \tag{3.2.27}$$

3.2.3. Solution of Matrix Equation

Both equation (3.2.10) and equation (3.2.25) are general matrix equations both of which, may be more conveniently written as:

$$M\underline{x} = \underline{y} \tag{3.2.28}$$

where M is the general matrix and both $\underline{x}$ and $\underline{y}$ are price vectors, which assuming M is nonsingular leads to the direction solution $\underline{x}$ of prices where:

$$\underline{x} = M^{-1}\underline{y} \tag{3.2.29}$$

Given that the matrix M may comprise of hundreds or thousands of individual elements the above approach from a practical viewpoint is going to be very slow. We thus need to consider alternative approaches of calculating the prices. In fact two separate category of approaches to solve the above equation more efficiently is available. The elimination approach, and the iterative approach. An example of the former is the Gaussian approach. An example of the latter is the Successive Over Relaxation (SOR), approach. We discuss each of the approaches in depth below. For illustrative purposes we concentrate on equation (3.2.24), although the same analysis would hold for equation (3.2.9)

With the Gaussian elimination approach, we initially let:

$$\eta_0 = 1$$

$$\beta_0 = 0$$

$$\chi_0 = 1$$

$$a_0 = \frac{\alpha_0}{\eta_0}$$

$$b_0 = \frac{\beta_0}{\eta_0}$$

We now consider equation (3.2.24) at various points on the grid:

$$\alpha_n = \chi_n u^m_{n-1} + \eta_n u^m_n + \beta_n u^m_{n+1}$$

At n = 0:

$$\alpha_0 = \eta_0 u_0^m + \beta_0 u_1^m \tag{3.2.30}$$

Rearranging the above expression gives:

$$u_0^m + b_0 u_1^m = a_0 \tag{3.2.31}$$

Thus generalising the above expression we have:

$$u_{n-1}^m + b_{n-1} u_n^m = a_{n-1} \tag{3.2.32}$$

Substituting the above expression into discrete equation (3.2.22) and rearranging gives:

$$u_n^m + b_n u_{n+1}^m = a_n \tag{3.2.33}$$

where:

$$b_n = \frac{\beta_n}{\eta_n - b_{n-1}\chi_n}$$

$$a_n = \frac{\alpha_n - \chi_n a_{n-1}}{\eta_n - b_{n-1}\chi_n}$$

Thus once we have the value of u_0^m from the boundary condition we can use equation (3.2.31) to calculate u_1^m and then u_2^m etc until we reach N – 1.

To solve equation (3.2.22) using SOR our starting point is the general matrix is:

$$u_n^m = \left[\frac{\omega}{m_{nn}} \left\{ q_n - \sum_{j=1}^{n-1} m_{nj} u_j^m - \sum_{j=n+1}^{N-1} m_{nj} u_j^m \right\} + (1-\omega) u_n^{m-1} \right] \tag{3.2.34}$$

Further m_{nj} represents individual element of matrix M. Simplification of the above equation leads to equation (3.2.36). Thus the first step of the SOR process involves forming an intermediate quantity z_n^m. Based on this intermediate quantity, a trial solution u_n^m is formed. This trial solution is iterated until, a certain accuracy is achieved between successive iterations. Having achieved this accuracy we move onto n + 1 point on the grid at a particular time step.

$$z_n^m = \frac{1}{\eta_n} \left(\alpha_n - \chi_n u_{n-1}^m - \beta_n u_{n+1}^{m-1} \right) \tag{3.2.35}$$

$$u_n^m = \omega z_n^m + (1-\omega) B_n^{m-1} \tag{3.2.36}$$

3.2.4. Method of Lines

We convert equation (3.2.2) into a system of two first order differential equations.

$$\frac{\partial u}{\partial r} = V(r,\tau) \qquad (3.2.37)$$

$$\frac{\partial V}{\partial r} = c(r,\tau)u(r,\tau) + d(r,\tau)V(r,\tau) + g(r,\tau) \qquad (3.2.38)$$

Substituting equation (3.2.37) and equation (3.2.38) into equation (3.2.2) and comparing coefficients we have:

$$c(r,\tau) = \frac{2}{\sigma^2 r^{2\gamma}}\left(r + \frac{1}{\Delta t}\right)$$

$$d(r,\tau) = -\frac{2}{\sigma^2 r^{2\gamma}}(k\theta - rk)$$

$$g(r,\tau) = -\frac{2}{\sigma^2 r^{2\gamma}\Delta t}u_n^{m-1}$$

Equation (3.2.37) and equation (3.2.38) is related through the Riccati transformation

$$u(r,\tau) = R(r,\tau)V(r,\tau) + w(r,\tau) \qquad (3.2.39)$$

where $R(r,\tau)$ and $w(r,\tau)$ are the solutions of the initial value problems

$$\frac{dR}{dr} = 1 - d(r,\tau)R(r,\tau) - c(r,\tau)R(r,\tau)^2 \qquad (3.2.40)$$

$$\frac{dw}{dr} = -c(r,\tau)R(r,\tau)w(r,\tau) - g(r,\tau)R(r,\tau) \tag{3.2.41}$$

The first step of the discretization process is to numerically integrate equation (3.2.40) and equation (3.2.41) to obtain values for $R(r,\tau)$ and $w(r,\tau)$ at each point on the grid. On the grid we let:

$$c_n = c(n\Delta r, \tau)$$

$$d_n = d(n\Delta r, \tau)$$

$$g_n = g(n\Delta r, \tau)$$

$$R_n^m = R(n\Delta r, m\Delta t)$$

$$w_n^m = w(n\Delta r, m\Delta t)$$

With equation (3.2.40), applying the implicit trapezoidal rule gives:

$$R_{n+1}^m - R_n^m = \frac{\Delta r}{2}\left[1 - d_{n+1}R_{n+1}^m - c_{n+1}\left(R_{n+1}^m\right)^2\right]$$
$$+\frac{\Delta r}{2}\left[1 - d_n R_n^m - c_n\left(R_n^m\right)^2\right] \tag{3.2.42}$$

Rearrangement of the above equation gives the following quadratic equation.

$$c_{n+1}\left(R_{n+1}^m\right)^2 + \left[d_{n+1} + \frac{2}{\Delta r}\right]R_{n+1}^m + \left[d_n - \frac{2}{\Delta r}\right]R_n^m + c_n\left(R_n^m\right)^2 = 0 \tag{3.2.43}$$

Thus the analytical expression for R^m_{n+1} is:

$$R^m_{n+1} = \frac{-\Phi_{n+1} + \sqrt{\Phi^2_{n+1} - 4\left[\sqrt{c_n}\Gamma_n R^m_n + c_n\left(R^m_n\right)^2 - 2\right]}}{\sqrt{2c_{n+1}}} \tag{3.2.44}$$

where:

$$\Phi_{n+1} = \frac{d_{n+1} + \dfrac{2}{\Delta r}}{\sqrt{c_{n+1}}}$$

$$\Gamma_n = \frac{d_n - \dfrac{2}{\Delta r}}{\sqrt{c_n}}$$

Similarly applying the trapezoidal rule to equation (3.2.41) gives:

$$w^m_{n+1} - w^m_n = -\frac{\Delta r}{2}\left[c_{n+1}R^m_{n+1}w^m_{n+1} - g^m_{n+1}R^m_{n+1}\right] -\frac{\Delta r}{2}\left[c_n R^m_n w^m_n - g^m_n R^m_n\right] \tag{3.2.45}$$

Rearrangement of the above equation gives:

$$w^m_{n+1} = \frac{-\Lambda^m_n + \Omega^m_n w^m_n}{\Theta^m_n} \tag{3.2.46}$$

where:

$$\Lambda_n^m = -R_{n+1}^m g_{n+1}^m - R_n^m g_n^m$$

$$\Omega_n^m = \frac{2}{\Delta r} - c_n R_n^m$$

$$\Theta_n^m = \frac{2}{\Delta r} + c_{n+1} R_{n+1}^m$$

Equation (3.2.43) and equation (3.2.45) are subject to the boundary conditions $R_0^m = 0$ and $w_0^m = 0$ respectively.

The next step is to determine the critical exercise price for the contingent claims by iteratively calculating zero for the

following function.

$$\phi_n^m = R_n^m \frac{dP}{dB} - w_n^m + E - B_n^m \qquad (3.2.47)$$

At the critical exercise price let:

$$\frac{dP}{dB} = \varsigma \qquad (3.2.48)$$

$\varsigma = 1$ for a call option and $\varsigma = -1$ for a put option.

Our original partial differential equation is in terms of the derivatives of P and r or B and r, not P and B. We therefore use the following expression to get round this difficulty.

$$\frac{dP}{dB} = \frac{dP}{dr}\frac{dr}{dB} = \varsigma\frac{dr}{dB}$$

$$= \frac{\varsigma}{\frac{dB}{dr}} \tag{3.2.49}$$

We approximate $\frac{dB}{dr}$ using the forward central difference.

$$\frac{dB}{dr} \approx \frac{B_{n+1}^{m} - B_{n-1}^{m}}{2\Delta r} \tag{3.2.50}$$

Thus the final form of equation (3.2.47) is:

$$\phi_n^m = R_n^m \varsigma\left(\frac{2\Delta r}{B_{n+1}^m - B_{n-1}^m}\right) - w_n^m + E - B_n^m \tag{3.2.51}$$

The root of the above equation at this time level is found by using Newton-Raphson iteration. Once the critical exercise price has been determined, u_n^m is calculated by numerically integrating equation (3.2.38) as below and then substituting the result into equation (3.2.39) to obtain u_n^m at time level t_n.

$$\frac{\partial V_n^m}{\partial r} = c_n\left[R_n^m V_n^m + w_n^m\right] + d_n V_n^m + g_n^m \qquad (3.2.52)$$

Again employing the trapezoidal rule we have:

$$V_{n+1}^m - V_n^m = \frac{\Delta r}{2}\left\{c_{n+1}\left[R_{n+1}^m V_{n+1}^m + w_{n+1}^m\right] + d_{n+1} V_{n+1}^m + g_{n+1}^m\right\}$$
$$+\frac{\Delta r}{2}\left\{c_n\left[R_n^m V_n^m + w_n^m\right] + d_n V_n^m + g_n^m\right\} \qquad (3.2.53)$$

Rearrangement of the above equation yields:

$$V_n^m = \frac{\Pi_n^m V_{n+1}^m - H_n^m}{Y_n^m} \qquad (3.2.53)$$

where:

$$\Pi_n^m = 1 - \frac{\Delta r}{2}\left[c_{n+1} R_{n+1}^m + d_{n+1}\right]$$

$$H_n^m = \frac{\Delta r}{2}\left[c_{n+1} w_{n+1}^m + c_n w_n^m + g_{n+1}^m + g_n^m\right]$$

$$Y_n^m = 1 + \frac{\Delta r}{2}\left[c_n R_n^m - d_n\right]$$

With bond prices, the free boundary doesn't exist, thus at each time level, we start the numerical integration from the lowest point on the grid. At this point, we approximate V as:

$$\frac{\partial B}{\partial r} \approx \frac{B_1^m - B_0^m}{\Delta r} \tag{3.2.54}$$

Substituting n as 0 and 1 in equation (3.2.18) gives us the following two equations. Noting that we are interested in approximate bond prices only.

$$B_0^m = B_0^{m-1} + k\theta\xi\left(B_1^{m-1} - B_0^{m-1}\right) \tag{3.2.55}$$

$$B_1^m = B_1^{m-1} + k\theta\xi\left(B_2^{m-1} - B_1^{m-1}\right) \tag{3.2.56}$$

$$V_0^m = \frac{k\theta\xi}{\Delta r} B_2^{m-1} - \left(\frac{2k\theta\xi}{\Delta r} - 1\right) B_1^{m-1} + \left(\frac{k\theta\xi}{\Delta r} - 1\right) B_0^{m-1} \tag{3.3.57}$$

The remaining part of the process is the same as for contingent claims..

3.3. Analysis of Results

In this section, we investigate each of the three numerical methods. Each method is implemented to value bond prices. Due to convergence difficulties with the Method of Lines only the Box method and Crank-Nicholson method could be implemented to value interest contingent claims. Note that as the underlying instrument is a zero coupon bond, the value of the American call option is the same as European call option. We exploit this feature to check the accuracy of our numerical CIR price[4]

As in Tian (1994), we define a quantity $\alpha_1 = (4k\theta - \sigma^2)/8$. $\alpha_1 > 0$,corresponds to low volatility and high mean reversion rate. For $\alpha_1 < 0$ the converse condition holds. We consider the CKLS model for γ taking values of 0.25, 0.50, 0.75. The maturities of the bonds are 5 and 15 years. The face value of the zero coupon bond is \$100. Short -term interest rates of 5% and 11% are considered. For $\alpha_1 > 0, k = 0.5, \sigma = 0.1, \theta = 0.08$, and for $\alpha_1 < 0, k = 0.1, \sigma = 0.5, \theta = 0.08$. Table 3.1 – Table 3.6 contain the bond prices calculated using each of the suggested numerical methods for different

[4] We attempted to use the Vasicek model for $\gamma = 0$ zero as an extra check. However, we found that the analytical formula was unstable and lead to bond prices which were not meaningful. For example for $\alpha_1 < 0$ we found that the bond price was considerably greater than its par value – something not possible for zero coupon bonds. Table 3.13 contains a summary of bond prices for $\alpha_1 < 0$ valued using the numerical methods considered in this chapter.

combinations of α_1 and γ. For the sake of brevity, following notation will be used in all of the tables:

BMS: prices calculated using the Box method, which uses Successive-Over-Relaxation.

BMG: prices calculated using the Box method, which uses Gaussian elimination.

CNS: prices calculated using the Crank Nicholson method, which uses Successive-Over-Relaxation.

CNG: prices calculated using the Crank Nicholson method, which uses Gaussian elimination.

ML: prices calculated using the Method of Lines

Table 3.1 – Table 3.12 contain the bond or call prices calculated using each of the suggested numerical methods for different combinations of α_1 and γ. Table 3.1 – Table 3.6 contains bond prices. Table 3.7 – Table 3.12 contains the call option prices.

Tables 3.1 – Table 3.6 all have the same format and comprise of zero coupon bond prices. In each of these tables, we alter the annual number of time steps from 20 to 1000. This variation serves as a check as to the stability of each of the numerical schemes. Examination of Tables 3.1 – Table 3.6 leads to the following observations:

For $\gamma = 0.25$, gaussian elimination does not lead to sensible bond prices, irrespective of whether $\alpha_1 < 0$ or $\alpha_1 > 0$. Furthermore, for $\alpha_1 < 0$, gaussian elimination does not lead to sensible bond prices irrespective of the value of γ. Also for $\gamma = 0.25$, we find BMS prices are higher the ML prices but lower than CNS prices irrespective of whether $\alpha_1 < 0$ or $\alpha_1 > 0$. For example, from Table 3.3, we see that when the interest rate is 11%, maturity of the bond is 5 years and the annual number of time steps is 1000, BMS price is 64.3104, CNS price is 64.8932 and ML price is 64.2355. Finally all five combinations (i.e. BMS, BMG, CNS, CNG, ML) lead to sensible bond prices for $\gamma = 0.75$.

When all four combinations lead to sensible prices, we find that SOR and gaussian elimination yield almost identical prices with each of the two methods. For example, from Table 3.1 consider, a 5 year bond at 5% interest rate and 50 annual time steps. We find that the Box prices using both SOR and gaussian elimination is identical at $71.0754. Whilst the Crank Nicholson prices are $71.6853 and $71.6958, using SOR and gaussian elimination respectively.

Box bond prices are always lower than Crank Nicholson bond prices. Further, where analytical prices are available, the Box prices are closer to the analytical prices than Crank Nicholson prices. For example, from Table 3.2, we see that a 5 year bond at 5% interest rate and 20 annual time steps is priced at $83.4832 analytically. Whereas, the same bond is price at $84.4832 using the Box method and $84.3837 using the Crank Nicholson method.

Box bond prices are closer to the Method of Lines (ML) bond prices than Crank-Nicholson prices, where the ML prices converge fast enough. We see an example of the former case from Table 3.2 in the case of a 15 year bond at 11% interest rate with 1000 annual time steps, the BMS price is 58.9913, ML price is 58.4592 and CNS price is 59.6010. An example of the latter is found in Table 3.5; for the same maturity bond, at the same interest rate and annual number of time steps, the BMS price is 68.1061, ML price is 66.0925 and CNS price is 69.0801

Tables 3.7 – Table 3.12 all have the same format and comprise of call options based on zero coupon bond prices for various expiry dates and exercise prices. In Tables 3.7 – Table 3.12 the first column indicates the range of exercise prices and the first row indicates the different expiry dates of the option ranging from 1 year to 5 years. All the call options are based on a 10 year zero coupon bond, the call options are during the last 5 years of the bond's maturity date. Further the third column entitled, "Bond Price", indicates the price of a 10 year zero coupon bond based on each of the possible combinations. For example, turning to Table 3.7's, third column, we find that the price of a 10 year zero coupon bond calculated using the Box method is $46.5992, whereas the same bond is priced at $47.0246 using the Crank Nicholson method. Examination of Tables 3.7 – Table 3.12 leads to the following observations:

Where analytical prices are available the Box prices are closer to the analytical prices than Crank Nicholson call prices. For example, from Table 3.8, consider a 5 year call option, exercise at $35. The analytical call price is $21.8802; Box pricing using SOR is $21.9445 and the Crank Nicholson price again using SOR is $22.1132.

As with bonds, Box prices are always lower than the corresponding call prices calculated using Crank Nicholson. However, unlike bonds, the differences are significant in certain cases. In fact these significant differences can be observed in Tables 3.7, 3.10, 3.11, 3.12. To illustrate the differences in call prices between the Box and the Crank Nicholson; consider an example from Table 3.11. In particular, consider a 5 year option, exercise at \$60, the analytical call price is \$23.9008, the Box price is \$23.9476, and the Crank Nicholson price is \$32.2997. In Table 3.8 and Table 3.9, where $\alpha_1 > 0$ and $\gamma \geq 0.5$, both the Box and the Crank Nicholson yield call prices which are close to each other, and close to the analytical price where available (Table 3.8).

Again, as with bonds, when all four combinations yield sensible prices, we again find that SOR and Gaussian elimination lead to almost identical call prices. For example, from Table 3.8, consider a 4 year call option exercised at \$35, we find that the Box price using SOR or Gaussian elimination is identical at \$20.0181. Whilst the Crank Nicholson prices are \$20.1846 using SOR and Gaussian respectively.

3.4. Conclusion

Over the years a number of researchers including HW (1990b) and Tian (1994) have noted convergence and stability difficulties with the evaluation of bond and contingent claims prices, based on the CKLS model for particular combination of parameters.

The findings in this chapter suggests that the convergence difficulties are not restricted to lattice methods alone. We find there are convergence problems both with the Crank-Nicholson Method and the Method of Lines. With the Method of Lines we need to increase the annual number of time steps to a ridiculously high value when $\alpha_1 < 0$ to obtain accurate bond prices. As the free boundary of a call option does not exist, our attention was focused on the put option. However, we were unable to locate the free boundary because the Newton-Raphson iteration scheme diverged. So in summary we were unable to locate the free boundary associated with the option and hence calculate any option price using the Method of Lines. With the Crank-Nicholson Method the bond prices show too much discrepancy with analytical prices, where available when $\alpha_1 < 0$. Of the three numerical methods studied in this chapter only the Box Method converges to produce accurate bond and contingent claim prices for all combination of parameters.

In the next chapter we use the Box Method as the basis to develop a checking procedure to check the free boundary associated with American put options.

Table 3.1: Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \Delta r = 0.5\%,\ \gamma = 0.5$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	CIR	5	71.0379	71.0379	71.0379	71.0379	71.0379	71.0379
	BMS		71.1006	71.0754	71.0670	71.0614	71.0603	71.0595
	BMG		71.1006	71.0754	71.0670	71.0614	71.0603	71.0595
	CNS		71.6853	71.6853	71.6858	71.6914	71.6853	71.6854
	CNG		71.6937	71.6958	71.6966	71.6971	71.6973	71.6973
	ML		70.8065	70.9445	70.9908	71.0218	71.0280	71.0327
5	CIR	11	63.7161	63.7161	63.7161	63.7161	63.7161	63.7161
	BMS		63.7850	63.7475	63.7349	63.7266	63.7249	63.7237
	BMG		63.7850	63.7475	63.7349	63.7266	63.7249	63.7237
	CNS		64.3129	64.3130	64.3134	64.3188	64.3130	64.3131
	CNG		64.3143	64.3147	64.3148	64.3149	64.3150	64.3150
	ML		63.5207	63.6379	63.6772	63.7034	63.7086	63.7126
15	CIR	5	32.5442	32.5442	32.5442	32.5442	32.5442	32.5442
	BMS		32.6428	32.5979	32.5829	32.5728	32.5711	32.5689
	BMG		32.6428	32.5979	32.5289	32.5729	32.5709	32.5694
	CNS		32.8647	32.8648	32.8648	32.8648	32.8646	32.8657
	CNG		32.8745	32.8770	32.8779	32.8785	32.8786	32.8787
	ML		32.4893	32.5209	32.5314	32.5385	32.5399	32.5410
15	CIR	11	28.9322	28.9322	28.9322	28.9322	28.9322	28.9322
	BMS		29.0135	28.9735	28.9601	28.9511	28.9496	28.9476
	BMG		29.0135	28.9735	28.9601	28.9512	28.9494	28.9481
	CNS		29.2251	29.2251	29.2251	29.2251	29.2250	29.2259
	CNG		29.2304	29.2317	29.2322	29.2326	29.2326	29.2327
	ML		28.8842	28.9133	28.9218	28.9281	28.9293	28.9303

Table 3.2: Bond Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.5$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	CIR	5	83.4832	83.4832	83.4832	83.4832	83.4832	83.4832
	BMS		83.6040	83.5409	83.5244	83.5145	83.5115	83.5098
	CNS		84.3837	84.3614	84.3554	84.3538	84.3516	84.3503
	ML		80.7707	82.9406	83.2049	83.3294	83.3509	83.3668
5	CIR	11	72.5572	72.5572	72.5572	72.5572	72.5572	72.5572
	BMS		72.6956	72.6166	72.5961	72.5836	72.5802	72.5781
	CNS		73.2609	73.2389	73.2338	73.2319	73.2305	73.2293
	ML		70.4523	72.1399	72.3481	72.4454	72.4620	72.4741
15	CIR	5	68.2741	68.2741	68.2741	68.2741	68.2741	68.2741
	BMS		68.4127	68.3836	68.3730	68.3668	68.3657	68.3631
	CNS		69.0981	69.0851	69.0807	69.0802	69.0801	69.0801
	ML		63.7759	67.0278	67.4300	67.6422	67.6846	67.7195
15	CIR	11	58.9177	58.9177	58.9177	58.9177	58.9177	58.9177
	BMS		59.0348	59.0095	59.0002	58.9947	58.9940	58.9913
	CNS		59.6168	59.6054	59.6016	59.6011	59.6010	59.6010
	ML		55.1183	57.8758	58.2157	58.3944	58.4300	58.4592

Table 3.3: Bond Prices calculated analytically (CIR), using the Box the Crank Nicholson methods.

$$\alpha_1 = (4k\theta - \sigma^2)/8 > 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.25$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	BMS	5	71.6737	71.6449	71.6352	71.6288	71.6275	71.6266
	CNS		72.2420	72.2396	72.2388	72.2290	72.2382	72.2382
	ML		70.9923	71.2242	71.3480	71.4258	71.4424	71.4551
5	BMS	11	64.3748	64.3354	64.3222	64.3134	64.3116	64.3104
	CNS		64.8956	64.8941	64.8936	64.8843	64.8932	64.8932
	ML		63.8741	64.0713	64.1520	64.2126	64.2255	64.2355
15	BMS	11	34.0294	33.9717	33.9753	33.9477	33.9461	33.9446
	CNS		34.4917	34.2163	34.2163	34.2163	34.2163	34.2162
	ML		33.5352	33.5772	33.5923	33.6028	33.6050	33.6067
15	BMS	11	30.3066	30.2546	30.2416	30.2330	30.2314	30.2301
	CNS		30.4917	30.4914	30.4914	30.4913	30.4913	30.4912
	ML		29.7040	29.7466	29.7607	29.7702	29.7721	29.7735

Table 3.4: Bond Prices calculated using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.75$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	BMS	5	70.9160	70.8907	70.8823	70.8764	70.8756	70.8734
	BMG		70.9160	70.8907	70.8823	70.8766	70.8755	70.8746
	CNS		71.5332	71.5332	71.5332	71.5327	71.5332	71.5332
	CNG		71.5246	71.5221	71.5212	71.5205	71.5204	71.5204
	ML		70.6525	70.7906	70.8367	70.8675	70.8737	70.8783
5	BMS	11	63.6009	63.5630	63.5503	63.5416	63.5403	63.5377
	BMG		63.6009	63.5630	63.5503	63.5419	63.5402	63.5389
	CNS		64.1256	64.1257	64.1257	64.1252	64.1257	64.1257
	CNG		64.1255	64.1256	64.1256	64.1256	64.1256	64.1256
	ML		63.3308	63.4486	63.4879	63.5142	63.5194	63.5234
15	BMS	5	32.2248	32.1793	32.1641	32.1552	32.1527	32.1540
	BMG		32.2248	32.1793	32.1641	32.1539	32.1519	32.1504
	CNS		32.4596	32.4597	32.4597	32.4597	32.4600	32.4603
	CNG		32.4545	32.4532	32.4526	32.4522	32.4521	32.4521
	ML		32.0883	32.1192	32.1295	32.1364	32.1378	32.1388
15	BMS	11	28.6205	28.5799	28.5664	28.5585	28.5562	28.5574
	BMG		28.6205	28.5799	28.5664	28.5573	28.5555	28.5542
	CNS		28.8271	28.8272	28.8272	28.8271	28.8275	28.8277
	CNG		28.8265	28.8264	28.8263	28.8263	28.8262	28.8262
	ML		28.4902	28.5175	28.5267	28.5328	28.5340	28.5349

Table 3.5: Bond Prices calculated using the Box and the Crank Nicholson methods

$$\alpha_1 = (4k\theta - \sigma^2)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5, \Delta r = 0.5\%,\ \gamma = 0.25$$

Maturity (years)	**Model**	**r(%)**	Annual number of time steps (n) **20**	**50**	**100**	**300**	**500**	**1000**
5	BMS	5	87.3004	87.2614	87.2484	87.2398	87.2380	87.2367
	CNS		87.8581	87.8399	87.8339	87.8297	87.8267	87.8284
	ML		60.6888	67.9952	75.0482	82.6325	84.3927	85.7177
5	BMS	11	78.2832	78.2392	78.2246	78.2151	78.2126	78.2115
	CNS		78.7147	78.6982	78.6927	78.6889	78.6807	78.6877
	ML		38.7165	63.9147	66.4312	70.3983	71.0192	71.4567
15	BM	5	76.1355	76.1082	76.0991	76.0930	76.0920	76.0909
	CNS		76.5944	76.5807	76.5761	76.5731	76.5722	76.5718
	ML		12.9002	59.0578	68.3499	72.5920	73.2919	73.8053
15	BM	11	68.1461	68.1216	68.1134	68.1073	68.1069	68.1061
	CNS		69.0981	69.0851	69.0807	69.0802	69.0801	69.0801
	ML		11.6761	52.9119	61.2177	65.0108	65.6352	66.0925

To ensure Method of Line converges $\Delta r = 0.01\%$ is used.

Table 3.6: Bond Prices calculated using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5,\ \Delta r = 0.5\%,\ \gamma = 0.75$$

Maturity (years)	Model	r(%)	Annual number of time steps (n) 20	50	100	300	500	1000
5	BMS	5	79.0790	79.0486	79.0383	79.0320	79.0306	79.0292
	CNS		79.9763	79.9793	79.4685	79.9734	79.9693	79.9685
	ML		78.7005	78.7006	78.7938	78.8054	78.8124	78.8147
5	BMS	11	66.2976	66.2402	66.2209	66.2085	66.2059	66.2037
	CNS		66.9567	66.9551	66.9510	66.6986	66.6960	66.7001
	ML		65.9998	66.0521	66.0694	66.0780	66.0831	66.0849
15	BMS	5	56.2443	56.2246	56.2181	56.2138	56.2133	56.2138
	CNS		56.2805	56.2850	56.2916	56.2716	56.2694	56.2805
	ML		55.1885	55.2279	55.2406	55.2469	55.2505	55.2517
15	BMS	11	45.6853	45.6682	45.6626	45.6588	45.6584	45.6588
	CNS		45.7309	45.7345	45.7383	45.7239	45.7220	45.7303
	ML		44.9083	44.9398	44.9499	44.9549	44.9579	44.9588

To ensure Method of Line converges $\Delta r = 0.01\%$ is used.

Table 3.7: Call Prices calculate using the Box Method.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$\Delta t = 0.05\,, \Delta r = 0.5\%\,, r_0 = 8\%\,, \gamma = 0.25$$

Exercise Price	**Model**	**Bond Price**	**5**	**4**	**3**	**2**	**1**
35	BMS	46.5992	22.9185	21.0493	19.0180	16.8006	14.3676
	CNS	47.0246	31.8368	30.2920	28.4940	26.4229	24.0841
40	BMS		19.5219	17.3859	15.0650	12.5334	9.7567
	CNS		28.3083	26.5429	24.4885	22.1223	19.4490
45	BMS		16.1257	13.7243	11.1067	8.3148	5.2756
	CNS		24.7800	22.7953	20.4900	17.8410	14.8360
50	BMS		12.7325	10.0833	7.2759	4.4155	1.7362
	CNS		21.2544	19.0634	16.5458	13.6856	10.4105
55	BMS		9.3647	6.5630	3.8278	1.5210	0.1853
	CNS		17.7504	15.4169	12.8214	9.9468	6.6336

Table 3.8: Call Prices calculated analytically (CIR), using the Box and the Crank Nicholson methods.

$$\alpha_1 = (4k\theta - \sigma^2)/8 > 0$$

$$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$$

Exercise Price	**Model**	**Bond Price**	Maturity (years) **5**	**4**	**3**	**2**	**1**
	CIR	45.1561[2]	21.8802	19.9509	17.8585	15.5863	13.1552
35	BMS	45.5000[1]	21.9445	20.0181	17.9293	15.6615	13.1957
	BMG	45.5140	21.9445	20.0181	17.9293	15.6615	13.1957
	CNS	45.8809	22.1132	20.1790	18.0921	15.8450	13.4362
	CNG	45.8866	22.1177	20.1846	18.0987	15.8524	13.4438
	CIR		18.5163	16.3114	13.9201	11.3233	8.4993
40	BMS		18.5836	16.3605	13.9887	11.3968	8.5789
	BMG		18.5774	16.3759	13.9887	11.3968	8.5789
	CNS		18.7181	16.5076	14.0513	11.5545	8.8015
	CNG		18.7226	16.5132	14.1291	11.5618	8.8092
	CIR		15.1524	12.6719	9.9819	7.0636	3.9137
45	BMS		15.2104	12.7336	10.0482	7.1352	3.9896
	BMG		15.2104	12.7336	10.0482	7.1351	3.9896
	CNS		15.3230	12.8362	10.1531	7.2662	4.1834
	CNG		15.3275	12.8418	10.1597	7.2735	4.1910
	CIR		11.7886	9.0330	6.0560	2.9514	0.4535
50	BMS		11.8433	9.0919	6.1191	3.0126	0.4788
	BMG		11.8433	9.0919	6.1191	3.0126	0.4788
	CNS		11.9820	9.1653	6.1943	3.1020	0.5267
	CNG		11.9324	9.1709	6.2008	3.1090	0.5317
	CIR		8.4257	5.4156	2.3804	0.3118	0.0001
55	BMS		8.4772	5.4705	2.4305	0.3307	0.0001
	BMG		8.4772	5.4705	2.4305	0.3308	0.0001
	CNS		8.5338	5.5143	2.4679	0.3443	0.0000
	CNG		8.5382	5.5200	2.4746	0.3486	0.0001

Table 3.9: Call Prices calculated using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$\Delta t = 0.05\,, \Delta r = 0.5\%\,, r_0 = 8\%\,, \gamma = 0.75$$

Exercise Price	**Model**	**Bond Price**	Maturity (years) **5**	**4**	**3**	**2**	**1**
35	BMS	45.2609	21.6569	19.7132	17.6073	15.3248	12.8499
	BMG	45.1662	21.6569	19.7132	17.6073	15.3248	12.8499
	CNS	45.5343	21.8253	19.8736	17.7695	15.5079	13.0901
	CNG	45.5299	21.8248	19.8729	17.7687	15.5670	13.0891
40	BMS		18.2985	16.0771	13.6703	11.0617	8.2333
	BMG		18.2985	16.0771	13.6703	11.0617	8.2333
	CNS		18.4388	16.2083	13.8036	11.2189	8.4557
	CNG		18.4382	16.2076	13.8028	11.2180	8.4548
45	BMS		14.9400	12.4409	9.7333	6.7987	3.6173
	BMG		14.9400	12.4409	9.7333	6.7987	3.6173
	CNS		15.0522	12.5430	9.8377	6.9299	3.8208
	CNG		15.0517	12.5423	9.8369	6.9290	3.8119
50	BMS		11.5815	8.8048	5.7964	2.5526	0.0822
	BMG		11.5815	8.8048	5.7964	2.5526	0.0822
	CNS		11.6657	8.8777	5.8718	2.6515	0.0956
	CNG		11.6652	8.8770	5.8710	2.6506	0.0952
55	BMS		8.2231	5.1689	1.9171	0.0258	0.0000
	BMG		8.2231	5.1689	1.9171	0.0252	0.0000
	CNS		8.2792	5.2126	1.9572	0.0265	0.0000
	CNG		8.2787	5.2119	1.9563	0.0262	0.0000

Table 3.10: Call Prices calculated using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$\Delta t = 0.05\,, \Delta r = 0.5\%\,, r_0 = 8\%\,, \gamma = 0.25$$

Exercise Price	Model	Bond Price	Maturity (years) 5	4	3	2	1
60	BMS	77.0820	27.9176	27.2150	26.4227	25.3939	23.5864
	CNS	77.6467	34.7973	34.7470	34.6029	34.1796	32.7911
65	BMS		23.8804	23.1287	22.2966	21.2595	19.4973
	CNS		30.4461	30.3994	30.2653	29.8699	28.5481
70	BMS		19.8563	19.0590	18.1907	17.1465	15.4766
	CNS		26.0955	26.0533	25.9320	25.5731	24.3526
75	BMS		15.8493	15.0081	14.1069	13.0643	11.5241
	CNS		21.7453	21.7085	21.6026	21.2886	20.2041
80	BMS		11.8619	10.9787	10.0480	9.0150	7.6385
	CNS		17.3954	17.3649	17.2776	17.0164	16.0912

Table 3.11: Call Prices calculated analytically (CIR), using the Box Method and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$$

			Maturity (years)				
Exercise Price	**Model**	**Bond Price**	**5**	**4**	**3**	**2**	**1**
	CIR	63.4557	23.9008	22.8564	20.2596	19.8902	16.9798
60	BMS	69.9969	23.9476	22.9006	21.6375	19.9112	16.9769
	CNS	70.8166	32.2997	32.0170	31.4356	30.1946	27.3805
	CIR		20.1770	19.0843	17.7967	16.0922	13.2470
65	BM		20.2200	19.1255	17.8313	16.1109	13.2320
	CNS		28.2373	27.9676	27.4063	26.1936	23.3519
	CIR		16.4887	15.3565	14.0532	12.3971	9.7260
70	BMS		16.5281	15.3950	14.0865	12.4102	9.7061
	CNS		24.1833	23.9313	23.4043	22.2519	19.4636
	CIR		12.8444	11.6829	10.3819	8.8038	6.4487
75	BM		12.8803	11.7194	10.4151	8.8246	6.4317
	CNS		20.1358	19.4299	19.4299	18.3732	15.7371
	CIR		9.2570	8.0789	6.8019	5.3528	3.4558
80	BM		9.2895	8.1135	6.8352	5.3787	3.4527
	CNS		16.0962	15.8990	15.4794	14.5314	12.0601

Table 3.12: Call Prices calculated using the Box and the Crank Nicholson methods

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.75$$

Exercise Price	Model	Bond Price	Maturity (years) 5	4	3	2	1
60	BMS	59.1193	17.1706	15.4195	13.3555	10.7989	7.3479
	CNS	60.1029	21.5125	19.9334	17.9317	15.3181	11.6426
65	BMS		13.8877	12.1152	10.0705	7.6101	4.4391
	CNS		18.1360	16.5416	14.5341	11.9361	8.3080
70	BMS		10.7096	8.9662	7.0142	4.7882	2.1955
	CNS		14.8528	13.2855	11.3311	8.8436	5.4647
75	BMS		7.6751	6.0203	4.2662	2.4419	0.7181
	CNS		11.6925	10.2053	8.3787	6.1095	3.2008
80	BMS		4.8453	3.3704	1.9456	0.7388	0.0655
	CNS		8.6972	7.3556	5.7475	3.8193	1.5839

Table 3.13: Bond Prices calculated using the Box and the Crank Nicholson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$k = 0.1,\ \theta = 0.08,\ \sigma = 0.5,\ \Delta r = 0.5\%,\ \gamma = 0$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	BMS	5	89.4124	89.3811	89.3706	89.3639	89.3624	89.3614
	CNS		89.8061	89.7897	89.7806	89.0273	89.7798	89.7792
	ML		19.0774	23.4223	29.6778	46.4546	56.0525	68.0901
5	BMS	11	81.7602	81.7277	81.7161	81.7095	81.7080	81.7067
	CNS		82.0508	82.0358	82.0276	82.0273	82.0266	82.0261
	ML		18.2252	22.1063	27.7056	42.7791	51.4412	62.3461
15	BMS	5	80.1593	80.1347	80.1265	80.1216	80.1201	80.1195
	CNS		80.4337	80.4205	80.4161	80.4131	80.4124	80.4119
	ML		8.7021	14.5900	22.5353	41.3618	50.8450	61.7635
15	BMS	11	73.2510	73.2293	73.2214	73.2172	73.2160	73.2152
	CNS		73.4741	73.4620	73.4580	73.4552	73.4546	73.4541
	ML		7.9481	13.3216	20.5753	37.7738	46.4424	56.4277

Gaussian elimination did not produce any meaningful prices both with the Box Method and the Crank-Nicholson. Analytical bond prices were unmeaningful. For example 5 year bond at 5% interest is valued at $2873.86 when its value is restricted to be equal to or less than $100.

CHAPTER 4.

A NEW APPROACH TO CHECK THE FREE BOUNDARY OF SINGLE FACTOR INTEREST RATE PUT OPTION

4.1. Introduction

In options pricing literature the location of the free boundary is used to determine the option price. The value of the free boundary at a particular time step before the expiry of the option is the underlying asset value at which an American option ceases to exist. The basis of the analytical option pricing methodology is the location of the free boundary. Thus in traditional option pricing literature the free boundary is assumed to have been correctly identified and the option price calculated.

An alternative scheme is to assume that the option price has been calculated, and use this option price as the basis to locate the free boundary. This approach serves two purposes. First it indicates whether the numerical scheme is stable; secondly it tells us the nature and shape of the free boundary. To date only Courtadon (1982b) has used option prices as the basis to locate the free boundary. Courtadon's approach was, however, very simple in that he used linear interpolation to track the free boundary. In this Chapter we use Green's theorem in conjunction with the Box Method to locate the free boundary. This Chapter represents the first attempt in Finance to track the free boundary in this manner. Section 2, 3 and 4 contain original work.

In Section 2 we set up the American pricing problem as an obstacle. In Section 3 we derive the integral equation in terms of the free boundary at successive time steps. In Section 4 we discretize the integral equation. Section 5 compares the free boundaries of American put options based on the Vasicek model ($\gamma = 0$), CIR model ($\gamma = 0.5$) and Brennan and Schwartz model ($\gamma = 1$). Section 6 contains a summary and conclusion.

4.2. An American Put Option As An Obstacle Problem

The basic starting equation is:

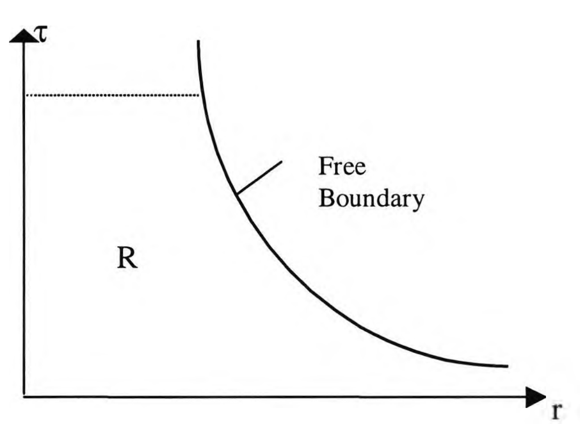
$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 \varepsilon}{\partial r^2} + k(\theta - r)\frac{\partial \varepsilon}{\partial r} - r\varepsilon = \frac{\partial \varepsilon}{\partial \tau} \qquad (4.1.1)$$

where $\varepsilon = P(r_t, t, T^*, T) + B(r_t, t, T^*) = \text{Put on bond} + \text{Bond Price}$.

Further at the free boundary the two following boundary conditions hold:

$$\frac{\partial \varepsilon(s(\tau), \tau)}{\partial r} = 0 \qquad (B1)$$

$$\varepsilon(s(\tau), \tau) = E \text{ (Exercise Price)} \qquad (B2)$$



In the diagram above the curve $r = s(\tau)$ is the free boundary. We integrate equation (4.1.1) in the region R bounded by the free boundary curve $r = s(\tau)$. In particular along the time axis we integrate from $0 \to \tau_m$ at time increment $m\Delta t$ and $0 \to s(\tau)$ along the interest rate axis.

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau + \iint_R k\theta \frac{\partial \varepsilon}{\partial r} dr d\tau - \iint_R rk \frac{\partial \varepsilon}{\partial r} dr d\tau$$

$$- \iint_R r\varepsilon dr d\tau = \iint_R \frac{\partial \varepsilon}{\partial \tau} dr d\tau \qquad (4.1.2)$$

We now integrate and simplify each component of the above equation, starting with the first component $\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$. In particular with the first component, we consider four distinct cases, first $\gamma = 0$, second $\gamma = \frac{1}{2}$, third $\gamma = 1$, and for γ between 0 and 1 excluding the previous values of γ.

First consider the case for $\gamma = 0$

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \iint_R \frac{\sigma^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Now integrating by parts and incorporating boundary condition B1 gives:

$$\int_0^{s(\tau)} \frac{\partial^2 \varepsilon}{\partial r^2} dr = -\frac{\partial \varepsilon(0,\tau)}{\partial r}$$

Further integration of the above expression with respect to time gives:

$$\int_0^{\tau_m}\int_0^{s(t)} \frac{\sigma^2}{2}\frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\frac{\sigma^2}{2}\int_0^{\tau_m} \frac{\partial \varepsilon(0,\tau)}{\partial r} d\tau$$

Second consider the integral for $\gamma = \frac{1}{2}$

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \iint_R \frac{\sigma^2 r}{2}\frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m}\int_0^{s(\tau)} \frac{\sigma^2 r}{2}\frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Integrating the integral $\int_0^{s(\tau)} \frac{\sigma^2 r}{2}\frac{\partial^2 \varepsilon}{\partial r^2} dr$ by parts and inserting the second boundary condition (B2) gives:

$$\int_0^{s(\tau)} \frac{\sigma^2}{2} r \frac{\partial^2 \varepsilon}{\partial r^2} dr = \frac{\sigma^2}{2}\left[\varepsilon(0,\tau) - E\right]$$

Further integrating the above expression with respect to time gives us:

$$\int_0^{\tau_m}\int_0^{s(\tau)} \frac{\sigma^2 r}{2}\frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\frac{\sigma^2}{2} E(m\Delta t) + \frac{\sigma^2}{2}\int_0^{\tau_m} \varepsilon(0,\tau) d\tau$$

Thirdly consider the integral for $\gamma = 1$

$$\iint_R \frac{\sigma^2}{2} r^2 \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Integrating the component $\int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr$ by parts and incorporating the boundary condition B2 gives:

$$\int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr = -\sigma^2 \int_0^{s(\tau)} r \frac{\partial \varepsilon}{\partial r} dr = \sigma^2 s(\tau) E - \sigma^2 \int_0^{s(\tau)} \varepsilon(r, \tau) dr$$

Once again further integrating the above expression with respect to time gives us:

$$\int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\sigma^2 E \int_0^{\tau_m} s(\tau) d\tau + \sigma^2 \int_0^{\tau_m} \int_0^{s(\tau)} \varepsilon(r, \tau) dr d\tau$$

Now for the general case of γ between 0 and 1 and excluding the particular values mentioned above, we have by integrating by parts and by incorporating the boundary condition B1:

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r^{2\gamma}}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\gamma\sigma^2 \int_0^{\tau_m} \int_0^{s(\tau)} r^{2\gamma - 1} \frac{\partial \varepsilon}{\partial r} dr d\tau$$

We now further integrate the component $\int_0^{s(\tau)} r^{2\gamma-1}\frac{\partial\varepsilon}{\partial r}dr$ by parts and insert boundary condition (B1) to give:

$$\int_0^{\tau} r^{2\gamma-1}\frac{\partial\varepsilon}{\partial r}dr = Es(\tau)^{2\gamma-1} - (2\gamma-1)\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r,\tau)dr$$

Thus in the expanded form, the double integral for the general case is:

$$\int_0^{\tau_m}\int_0^{s(\tau)} \frac{\sigma^2 r^{2\gamma}}{2}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = \sigma^2\gamma(2\gamma-1)\int_0^{\tau_m}\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r,\tau)drd\tau - \sigma^2\gamma E\int_0^{\tau_m} s(\tau)^{2\gamma-1}d\tau$$

Note that the above expression also holds for $\gamma = 1$. Thus summarizing all the possible expressions:

$$LHS_0 = \iint_R \frac{\sigma^2}{2}r^{2\gamma}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = \begin{cases} -\frac{\sigma^2}{2}\int_0^{\tau_m}\frac{\partial\varepsilon(0,\tau)dt}{\partial r}, \gamma = 0 \\ -\frac{\sigma^2}{2}E(m\Delta t) + \frac{\sigma^2}{2}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau, \gamma = \frac{1}{2} \\ \sigma^2\gamma(2\gamma-1)\int_0^{\tau_m}\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r,\tau)drd\tau - \sigma^2\gamma E\int_0^{\tau_m} s(\tau)^{2\gamma-1}d\tau, \gamma \neq 0, \gamma \neq \frac{1}{2} \end{cases}$$

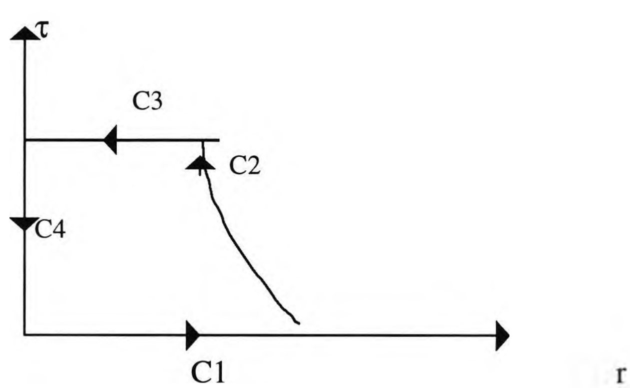
Now integrating the second component of equation (4.1.2) and inserting boundary condition B2 gives:

$$k\theta\iint_R \frac{\partial\varepsilon}{\partial r}drd\tau = k\theta\int_0^{\tau_m}\int_0^{s(\tau)}\frac{\partial\varepsilon}{\partial r}drd\tau = k\theta E(m\Delta t) - k\theta\int_0^{\tau_m}\varepsilon(0,\tau)d\tau$$

Finally integrating the third component of equation (4.1.2) by parts and inserting boundary condition B2 gives us:

$$-k\iint_R r\frac{\partial\varepsilon}{\partial r}drd\tau = -Ek\int_0^{\tau_m}s(\tau)d\tau + k\int_0^{\tau_m}\int_0^{s(\tau)}\varepsilon(r,\tau)drd\tau$$

We now consider the term on the right hand side of the original equation. The figure below indicates the path of integration followed.



Applying Green's theorem gives us:

$$\iint_R \frac{\partial\varepsilon(r,\tau)}{\partial\tau} = -\oint_{C1+C2+C3+C4}\varepsilon(r,\tau)dr$$

We now evaluate each of the components of the above integral separately:

$\int_{C4} \varepsilon(r,\tau)dr = 0$ as we are moving along the time axis only where the interest rate is constant.

$$\int_{C1} \varepsilon(r,\tau)dr = \int_0^{s(0)} \varepsilon(r,0)dr$$

We note that C2 is the free boundary ans such from boundary condition B2 along C2 $\varepsilon(r,\tau) = E$. Hence:

$$\int_{C2} \varepsilon(r,\tau)dr = \int_{\tau=0}^{\tau_m} E\frac{dr}{d\tau}d\tau = E[s(\tau_m) - s(0)]$$

$$\int_{C3} \varepsilon(r,\tau)dr = -\int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

Collecting all the terms on the right hand side gives us:

$$\iint_R \frac{\partial \varepsilon}{\partial \tau} dr d\tau = \int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr + E[s(0) - s(\tau_m)] - \int_0^{s(0)} \varepsilon(r,0)dr$$

Collecting and rearranging the terms both on the left-hand side and the right hand side of equation (4.1.2) gives us:

$$LHS_0 + LHS_1 + LHS_2 + LHS_3 + LHS_4 + LHS_5 + LHS_6 = RHS_0 + RHS_1 \quad (4.1.3)$$

where:

$$LHS_1 = k\theta E(m\Delta t)$$

$$LHS_2 = -k\theta \int_0^{\tau_m} \varepsilon(0,\tau)d\tau$$

$$LHS_3 = \int_0^{\tau_m} \int_0^{s(t)} (k - r)\varepsilon(r,\tau)drd\tau$$

$$LHS_4 = -kE \int_0^{s(\tau)} s(\tau)d\tau$$

$$LHS_5 = -Es(0)$$

$$LHS_6 = \int_0^{s(0)} \varepsilon(r,0)dr$$

$$RHS_0 = \int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

$$RHS_1 = -Es(\tau_m)$$

Observing equation (4.1.3) we see that at any general time step τ_m there is no analytical solution for $s(\tau_m)$. In the next section we use numerical integration to solve equation (4.1.3)

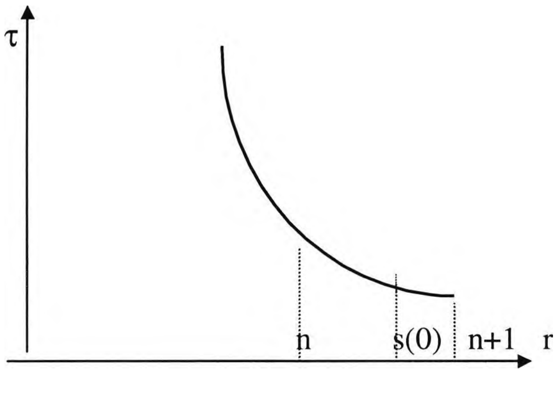
4.3. Discretization of the Integral Equation

Each of the single integral is discretized using the implicit trapezium rule. We start by discretizing the simplest integrals first:

$$\int_0^{\tau_m} \varepsilon(0,t)d\tau \approx \Delta t\left[\frac{1}{2}\varepsilon(0,0)+\varepsilon(0,\Delta t)+\varepsilon(0,2\Delta t)+....+\varepsilon(0,(m-1)\Delta t)+\frac{1}{2}\varepsilon(0,m\Delta t)\right]$$

$$\int_0^{\tau_m} s(\tau)d\tau = \Delta t\left[\frac{1}{2}s(0)+s(\Delta t)+s(2\Delta t)+...+s((m-1)\Delta t)+\frac{1}{2}s(m\Delta t)\right]$$

$$\int_0^{\tau_m} s(\tau)^{2\gamma-1}d\tau = \Delta t\left[\frac{1}{2}s(0)^{2\gamma-1}+s(\Delta t)^{2\gamma-1}+s(2\Delta t)^{2\gamma-1}+...+s((m-1)\Delta t)^{2\gamma-1}+\frac{1}{2}s(m\Delta t)^{2\gamma-1}\right]$$



At time 0, we separate the integral $\int_0^{s(0)} \varepsilon(r,0)dr$ into two components as follows with

$n_0\Delta r < s(0) < (n_0 + 1)\Delta r$

$$\int_0^{s(0)} \varepsilon(r,0)dr = \int_0^{n_0\Delta r} \varepsilon(r,0)dr + \int_{n_0\Delta r}^{s(0)} \varepsilon(r,0)dr$$

We discretize each of the two integrals using the implicit trapezium rule as follows:

$$\int_0^{n_0\Delta r} \varepsilon(r,0)dr \approx \Delta r\left[\frac{1}{2}\varepsilon(0,0) + \varepsilon(\Delta r,0) + \varepsilon(2\Delta r,0) + \ldots.\varepsilon((n_0 - 1)\Delta r,0) + \frac{1}{2}\varepsilon(n_0\Delta r,0)\right]$$

$$\int_{n_0\Delta r}^{s(0)} \varepsilon(r,0)dr \approx \frac{(s(0) - n_0\Delta r)}{2}\left[\varepsilon(n_0\Delta r,0) + E\right]$$

Combining the above two discretizations gives us:

$$\int_0^{s(0)} \varepsilon(r,0)dr = \Delta r\left[\frac{1}{2}\varepsilon(0,0) + \varepsilon(\Delta r,0) + \varepsilon(2\Delta r,0) + \ldots + \varepsilon((n_0 - 1)\Delta r,0) + \frac{1}{2}\varepsilon(n_0\Delta r,0)\right]$$
$$+ \frac{(s(0) - n_0\Delta r)}{2}\left[\varepsilon(n_0\Delta r,0) + E\right]$$

At time τ_m, as at time 0, we separate the integral $\int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr$ into two components

as follows with $n_m\Delta r < s(\tau_m) < (n_m + 1)\Delta r$

$$\int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr = \int_0^{n_m\Delta r} \varepsilon(r,\tau_m)dr + \int_{n_m\Delta r}^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

We discretize each of the two integrals using the implicit trapezium rule as follows:

$$\int_0^{n_m\Delta r} \varepsilon(r,\tau_m)dr \approx \Delta r\left[\frac{1}{2}\varepsilon(0,\tau_m) + \varepsilon(\Delta r,\tau_m) + \varepsilon(2\Delta r,\tau_m) + \ldots .\varepsilon((n_m - 1)\Delta r,\tau_m) + \frac{1}{2}\varepsilon(n_m\Delta r,\tau_m)\right]$$

$$\int_{n_m\Delta r}^{s(\tau_m)} \varepsilon(r,0)dr \approx \frac{(s(\tau_m) - n_m\Delta r)}{2}\left[\varepsilon(n_m\Delta r,0) + E\right]$$

Combining the above two discretizations gives us:

$$\int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr \approx \Delta r\left[\frac{1}{2}\varepsilon(0,\tau_m) + \varepsilon(\Delta r,\tau_m) + \varepsilon(2\Delta r,\tau_m) + \ldots + \varepsilon((n_m - 1)\Delta r,\tau_m) + \frac{1}{2}\varepsilon(n_m\Delta r,\tau_m)\right]$$
$$+ \frac{(s(\tau_m) - n_m\Delta r)}{2}\left[\varepsilon(n_m\Delta r,0) + E\right]$$

For the integral $-\frac{\sigma^2}{2}\int_0^{\tau_m} \frac{\partial\varepsilon(0,\tau)}{\partial r}d\tau$ we first discretize $\frac{\partial\varepsilon(0,\tau)}{\partial r}$ using the forward difference approximation such that:

$$\frac{\partial\varepsilon(0,\tau)}{\partial r} \approx \frac{\varepsilon(\Delta r,\tau) - \varepsilon(0,\tau)}{\Delta r}$$

Substituting the above expression into the original integral gives us:

$$-\frac{\sigma^2}{2}\int_0^{\tau_m}\frac{\partial\varepsilon(0,\tau)}{\partial\tau}dt = -\frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(\Delta r,\tau)d\tau + \frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau$$

Discretizing each of the components of the above equation gives gives us:

$$\frac{-\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau \approx \frac{-\sigma^2\Delta t}{2\Delta r}\left[\frac{1}{2}\varepsilon(0,0)+\varepsilon(0,\Delta t)+\varepsilon(0,2\Delta t)+....+\varepsilon(0,(m-1)\Delta t)+\frac{1}{2}\varepsilon(0,m\Delta t)\right]$$

$$\frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau \approx \frac{\sigma^2\Delta t}{2\Delta r}\left[\frac{1}{2}\varepsilon(\Delta r,0)+\varepsilon(0,\Delta t)+\varepsilon(\Delta r,2\Delta t)+....+\varepsilon(\Delta r,(m-1)\Delta t)+\frac{1}{2}\varepsilon(\Delta r,m\Delta t)\right]$$

Combining the above two discretizations gives us:

$$-\frac{\sigma^2}{2}\int_0^{\tau_m}\frac{\partial\varepsilon(0,\tau)}{\partial\tau}dt = -\frac{\sigma^2\Delta t}{2\Delta r}\left[\begin{array}{l}\frac{1}{2}\left(\varepsilon(0,0)-\varepsilon(\Delta r,0)\right)+\left(\varepsilon(0,\Delta t)-\varepsilon(\Delta r,\Delta t)\right)+... \\ +\left(\varepsilon(0,(m-1)\Delta t)-\varepsilon(\Delta r,(m-1)\Delta t)\right)+\frac{1}{2}\left(\varepsilon(0,m\Delta t)-\varepsilon(\Delta r,m\Delta t)\right)\end{array}\right]$$

To discretize the double integrals we first change the order of integration as follows:

$$\int_0^{\tau_m}\int_0^{s(\tau)}(k+\lambda-r)\varepsilon(r,\tau)drd\tau = \int_0^{s(\tau)}\left[\int_0^{\tau_m}(k+\lambda-r)\varepsilon(r,\tau)d\tau\right]dr$$

$$\int_0^{\tau_m}\int_0^{s(\tau)}r^{2\gamma-2}\varepsilon(r,\tau)drd\tau = \int_0^{s(\tau)}\left[\int_0^{\tau_m}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr$$

We now discretize the above double integrals at successive time steps

First at time period Δt :

$$\int_0^{s(\tau)}\left[\int_0^{\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr = \frac{\Delta t}{2}\int_0^{s(0)}(k-r)\varepsilon(r,0)dr + \frac{\Delta t}{2}\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{\Delta t} r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr = \frac{\Delta t}{2}\int_0^{s(0)} r^{2\gamma-2}\varepsilon(r,0)dr + \frac{\Delta t}{2}\int_0^{s(\Delta t)} r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

At time period $2\Delta t$:

$$\int_0^{s(\tau)}\left[\int_0^{2\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr = \frac{\Delta t}{2}\int_0^{s(0)}(k+-r)\varepsilon(r,0)dr + \Delta t\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$+\frac{\Delta t}{2}\int_0^{s(2\Delta t)}(k-r)\varepsilon(r,2\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{2\Delta t} r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr = \frac{\Delta t}{2}\int_0^{s(0)} r^{2\gamma-2}\varepsilon(r,0)dr + \Delta t\int_0^{s(\Delta t)} r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

$$+\frac{\Delta t}{2}\int_0^{s(\Delta t)} r^{2\gamma-2}\varepsilon(r,2\Delta t)dr$$

At time period $m\Delta t$:

$$\int_0^{s(\tau)}\left[\int_0^{m\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}(k-r)\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$+\Delta t\int_0^{s(2\Delta t)}(k-r)\varepsilon(r,2\Delta t)dr+\ldots+\Delta t\int_0^{s((m-1)\Delta t)}(k-r)\varepsilon(r,(m-1)\Delta t)dr+\frac{\Delta t}{2}\int_0^{s(m\Delta t)}(k-r)\varepsilon(r,m\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{m\Delta t}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}r^{2\gamma-2}\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

$$+\Delta t\int_0^{s(2\Delta t)}r^{2\gamma-2}\varepsilon(r,2\Delta t)dr+\ldots\Delta t\int_0^{s((m-1)\Delta t)}r^{2\gamma-2}\varepsilon(r,(m-1)\Delta t)dr+\frac{\Delta t}{2}\int_0^{s(m\Delta t)}r^{2\gamma-2}\varepsilon(r,m\Delta t)dr$$

We note that the above integrals are similar to $\int_0^{s(\tau_m)}\varepsilon(r,\tau_m)dr$ and hence discretized as follows:

$$Y_m=\int_0^{s(m\Delta t)}(k-r)\varepsilon(r,m\Delta t)dr=\begin{bmatrix}\frac{1}{2}k\varepsilon(0,\tau_m)+(k-\Delta r)\varepsilon(\Delta r,\tau_m)\Delta r\\+(k-2\Delta r)\varepsilon(2\Delta r,\tau_m)\Delta r\\+\ldots+(k-(n_m-1)\Delta r)\varepsilon((n_m-1)\Delta r,\tau_m)\Delta r\\+\frac{1}{2}(k-m\Delta r)\varepsilon(n_m\Delta r,\tau_m)\Delta r+\\\left(\frac{s(m\Delta t)-m\Delta r}{2}\right)\times\\((k-m\Delta r)\varepsilon(n_m\Delta r,\tau_m)+(k-s(m\Delta t))E)\end{bmatrix}$$

$$Z_m = \int_0^{s(m\Delta t)} r^{2\gamma-2}\varepsilon(r, m\Delta t)dr = \left[\begin{array}{l} (\Delta r)^{2\gamma-2}\varepsilon(\Delta r, \tau_m)\Delta r + (2\Delta r)^{2\gamma-2}\varepsilon(2\Delta r, \tau_m)\Delta r \\ +\ldots+((n_m - 1)\Delta r)^{2\gamma-2}\varepsilon((n_m - 1)\Delta r, \tau_m)\Delta r \\ +\frac{1}{2}(m\Delta r)^{2\gamma-2}\varepsilon(n_m\Delta r, \tau_m)\Delta r + \\ \left(\frac{s(m\Delta t) - m\Delta r}{2}\right)\times \\ \left((m\Delta r)^{2\gamma-2}\varepsilon(n_m\Delta r, \tau_m) + (s(m\Delta t))^{2\gamma-2}E\right) \end{array}\right]$$

Thus summarizing both the above double integrals,we have:

$$\int_0^{\tau_m}\int_0^{s(\tau)}(k + \lambda - r)\varepsilon(r, \tau)drd\tau = \frac{\Delta t}{2}Y_1 + \Delta t Y_2 +\ldots+\Delta t Y_{m-1} + \frac{\Delta t}{2}Y_m$$

$$\int_0^{\tau_m}\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r, \tau)drd\tau = \frac{\Delta t}{2}Z_1 + \Delta t Z_2 +\ldots+\Delta t Z_{m-1} + \frac{\Delta t}{2}Z_m$$

4.4. Locating the Free Boundary

At the maturity date of the contingent claim we define the following function discretized at interest rate point r_k:

$$\phi_k = E - B(r_k)$$

If we let r_{k-2}, r_{k-1}, r_k and r_{k+1} be interest rate points and ϕ_{k-2}, ϕ_{k-1}, ϕ_k and ϕ_{k+1} be values of the above function at these interest rates. Then, we can derive the following polynomial:

$$A(r) = \sum_{l=k-2}^{k+1} \phi_l L_l(r)$$

where

$$L_l(r) = \prod_{l=k-2, l\neq k}^{k+1} \frac{r - r_l}{r_k - r_l}$$

with the following property:

$$\phi_l = A(r_l) \qquad l = k-2,\ k-1,\ k,\ k+1$$

We now use Newton-Raphson iteration, to derive the critical interest rate $s(0)$ at expiry date of the put option.

$$s(0) = r - \frac{A(r)}{\frac{d}{ds}[A(r)]}$$

At general time step $m\Delta t$, the free boundary is located by solving for the zero of the function:

$$\phi = \phi_{LHS} - \phi_{RHS}$$

where:

$$\phi_{LHS} = LHS_0 + LHS_1 + LHS_2 + LHS_3 + LHS_4 + LHS_5 + LHS_6$$

$$\phi_{RHS} = RHS_0 + RHS_1$$

Numerical experimentation indicates that Newton-Raphson is not suitable except at the maturity date of the option. Thus at general time step $m\Delta t$, we start with a value of $s(m\Delta t)$ which by examination of the grid at this time step is known to be lower than the actual value of $s(m\Delta t)$. To estimate a more accurate $s(m\Delta t)$ we iterate upwards at interest rate steps of $\Delta r/20$ until the following criterion is met:

$$|\phi_{LHS} - \phi_{RHS}| = |10^{-5}|$$

Once this criterion is met we move to the next time step to calculate $s((m+1)\Delta t)$ and so on until we reach to end of the grid at time step $M\Delta t$.

We now investigate the nature of the free boundary of American put options based on widely used single factor term structure models. In particular we consider the Vasicek model $(\gamma = 0)$, CIR model $\left(\gamma = \frac{1}{2}\right)$ and Brennan-Schwartz model $(\gamma = 1)$. All three models are of course enclosed by the more general CKLS model. We investigate the free boundary both for short expiry and long expiry put options. The short expiry options are based on bonds with 5-year maturity bond and expiry of 1 year. The longer expiry put options are based on 10-year bonds and expiry of 5 years. The bonds are zero coupon and have a face value of 100.00. The parameters take the following values: $\sigma = 0.5$, $k = 0.1$, $\theta = 0.08$. On the grid the interest rate spacing is $\Delta r = 0.05$ and the time intervals of $\Delta t = 0.002$.

4.5. Analysis

We plot the free boundaries for $\gamma = 0$ (Vasicek), $\gamma = 0.5$ (CIR) and $\gamma = 1$ (Brennan-Schwartz). For each γ value two sets of free boundaries are plotted at different exercise prices. The terms to expiry of the put options are either 1 year or 5 years.

The 1 year put options are priced on a 5-year bond during the last year before it matures. The 5 year put options are priced on a 10-year bond during the last 5-year before it matures. All the free boundaries are plotted backwards in time that is, we start plotting from the expiry date of the options to current date at which the put option is written.

For $\gamma = 0$, 1 year put option (Figure 1) the critical interest rate increases rapidly. However, as the current date of the option approaches, the critical interest rate increases asymptotically; such that by the current date the free boundary is almost flat. For 5 year options (Figure 2), the critical interest rate increases rapidly close to the expiry date of the option as with $\gamma = 0$. Although the free boundary is almost flat by the current date careful examination of the graph indicates that critical interest rate actually start to decrease as the current date of the put option approaches. This is in contrast to the free boundary of the one-year option.

For $\gamma = 0.5$, 1 year put option (Figure 3), the free boundary evolves in the same way as for $\gamma = 0$. For 5 year put option (Figure 4), the free boundary increases close to the maturity date of the option. However as the current date of the option approaches, the critical interests show a noticeable decline. The end result is that for a 5 year put option, the free boundary initially increases and then declines asymptotically.

For $\gamma = 1$, 1 year put option (Figure 5), the free boundary initially increases close to the maturity date and the declines as the current date approaches. This is in contrast to the behavior of free boundaries for $\gamma = 0$ and $\gamma = 0.5$. For 5 year put option (Figure

6), the critical interest initially increases, but then quickly declines. Although the free boundary in this case shows the same overall behaviour as the free boundaries for $\gamma = 0$ and $\gamma = 0.5$, there is in this case two distinct observable differences. First the critical interest rate starts to decline much closer to the maturity date than for $\gamma = 0$ and $\gamma = 0.5$. Secondly the rate of decline i.e. the downward steepness of the free boundaries is greater than for $\gamma = 0$ and $\gamma = 0.5$.

Figure1 - Figure 6 all exhibit discontinuities at the expiry date and close to the expiry date of the options. This is due to an inconsistency in our model at maturity because at maturity we assume $\frac{\partial \varepsilon(s(0),0)}{\partial r} = 0$, when $\frac{\partial \varepsilon(P(s(0)),0)}{\partial r} \neq 0$. Further, although none of the free boundaries show any discontinuities except at and near the expiry date, the free boundaries nonetheless do exhibit small oscillations. This oscillation is due to the approximations we have made in setting up the grid and secondly the small errors in the critical interest rate from previous time periods feeding through to the critical interest rate at the current time period.

4.6. Conclusion

Since Courtadon (1982) used a linear interpolation approach to track the free boundary of interest rate contingent claims, no further research has been done to extend this work. In this chapter we have provided a new method to check and track the free boundary. We have applied this new approach to check the free boundary of short dated and long dated American put options based on widely used one factor interest rate models. Our finding suggests that the shape of the free boundary varies from model to model and with the term to expiry of the options. Generally, we observe that the risk boundary increases asymptotically towards the current date, such that by the current date the free boundary is almost flat or slightly declining.

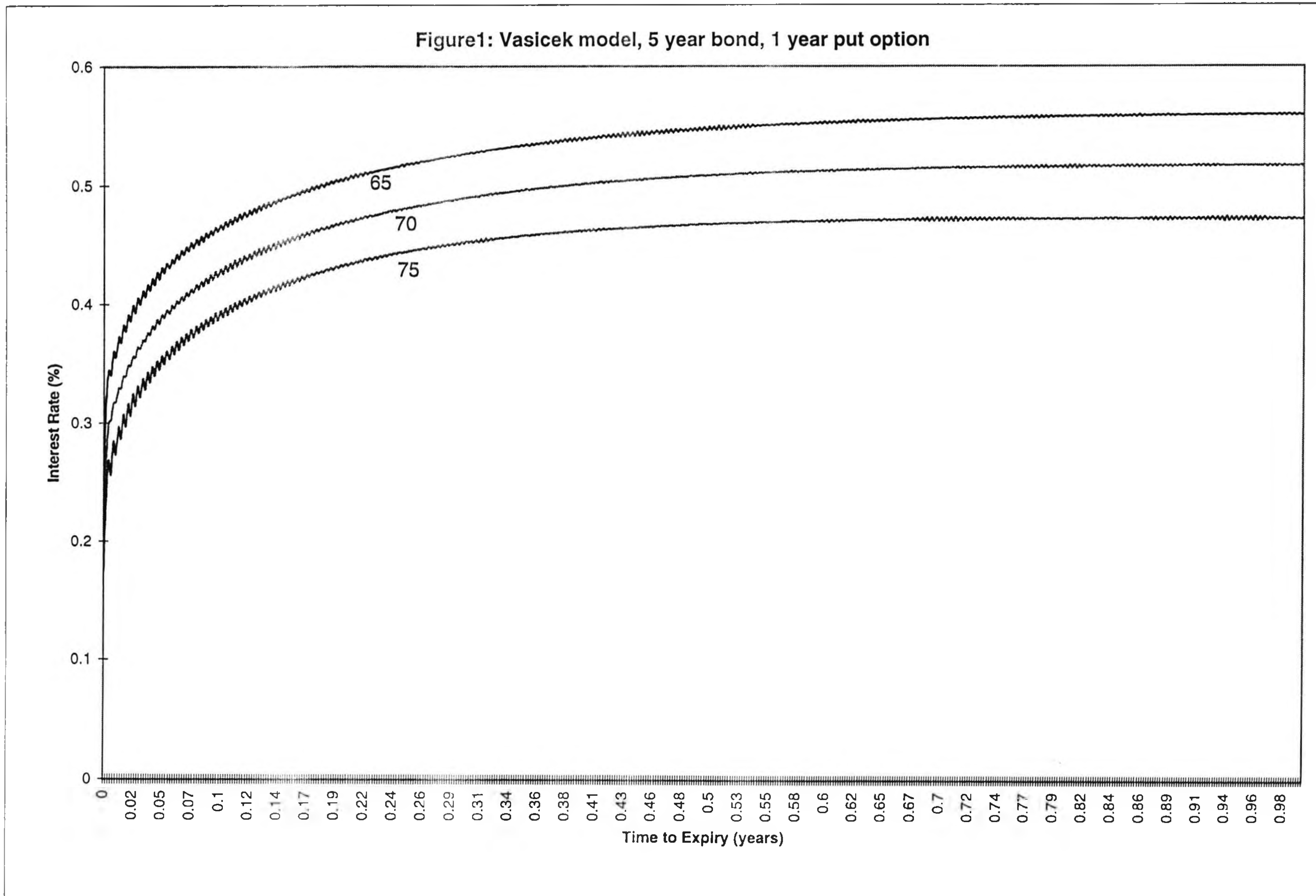


Figure1: Vasicek model, 5 year bond, 1 year put option
Interest Rate (%)
0.6
0.5
0.4
0.3
0.2
0.1
0
65
70
75
0
0.02
0.05
0.07
0.1
0.12
0.14
0.17
0.19
0.22
0.24
0.26
0.29
0.31
0.34
0.36
0.38
0.41
0.43
0.46
0.48
0.5
0.53
0.55
0.58
0.6
0.62
0.65
0.67
0.7
0.72
0.74
0.77
0.79
0.82
0.84
0.86
0.89
0.91
0.94
0.96
0.98
Time to Expiry (years)

Figure 2: Vasicek model, 10 year bond, 5 year put option

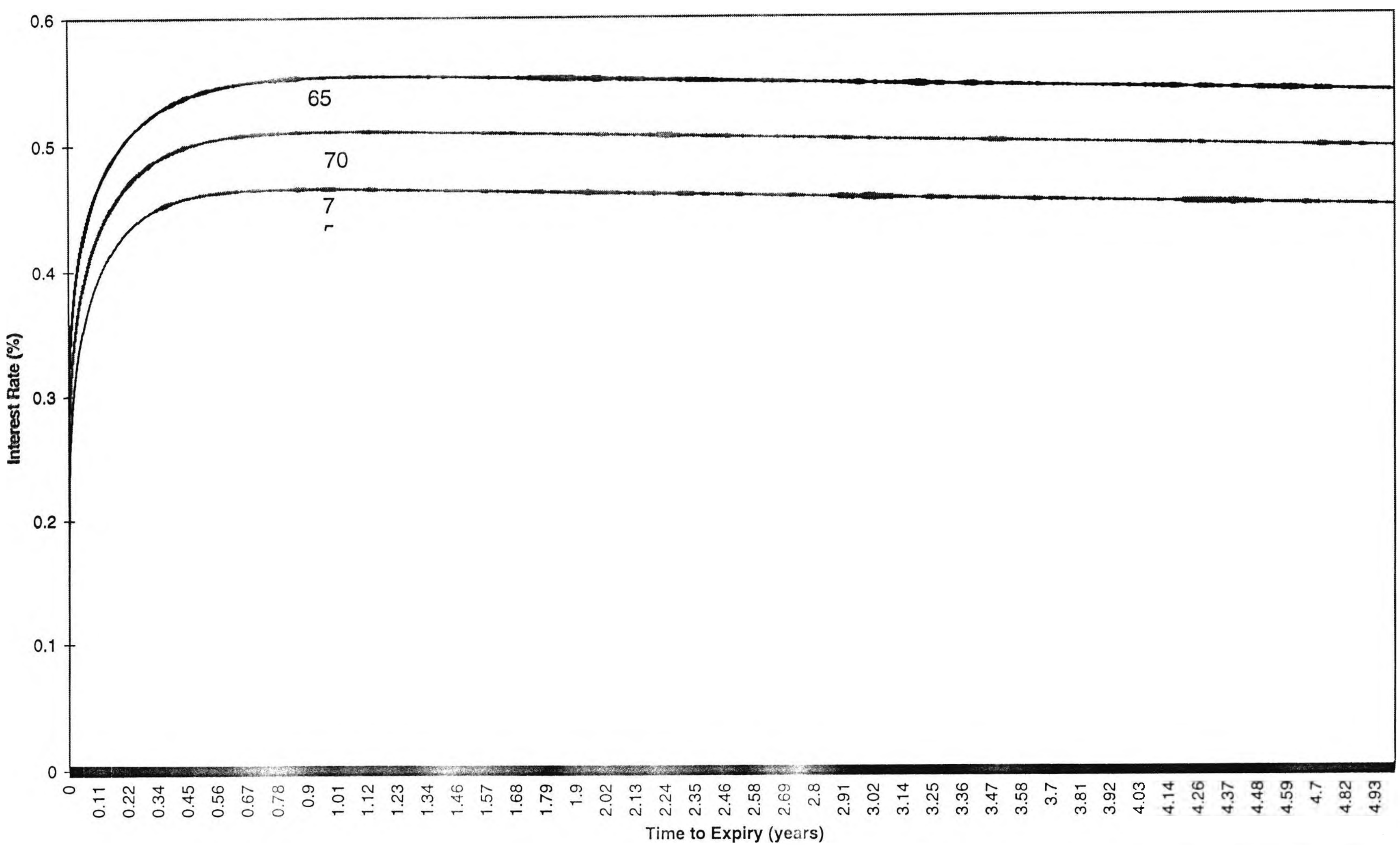
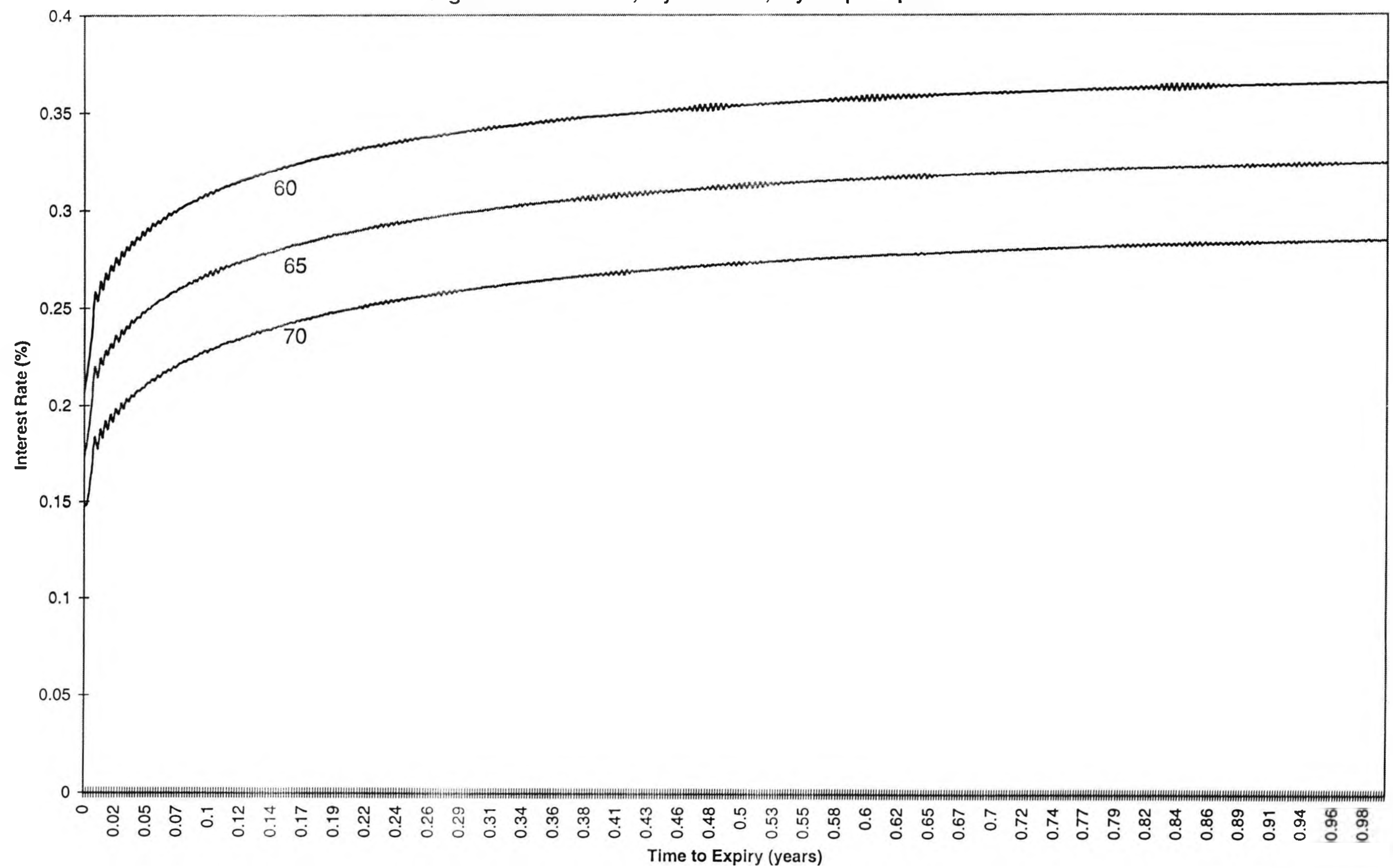


Figure 3: CIR model, 5 year bond, 1 year put option



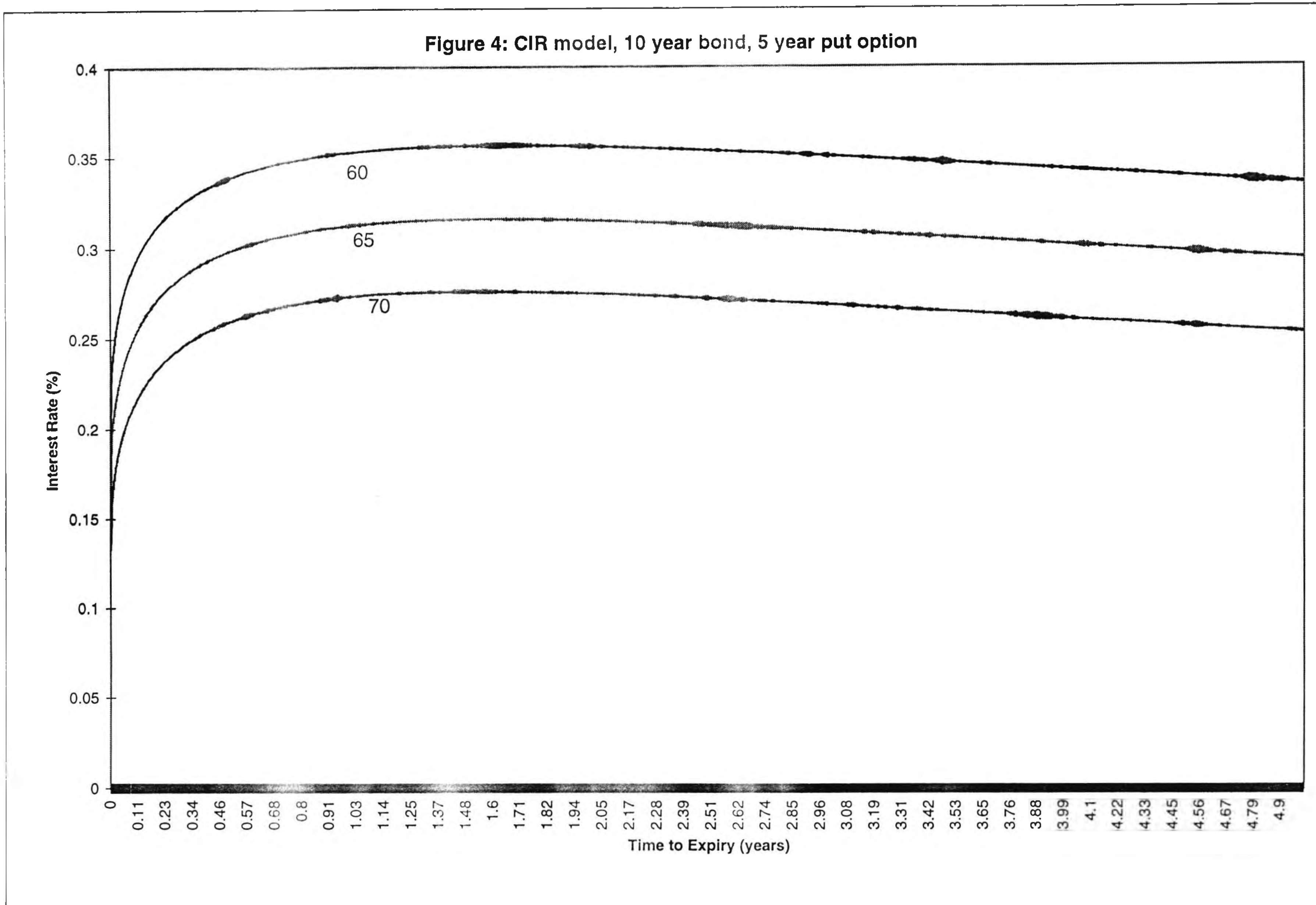


Figure 4: CIR model, 10 year bond, 5 year put option
Interest Rate (%)
Time to Expiry (years)
60
65
70
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0

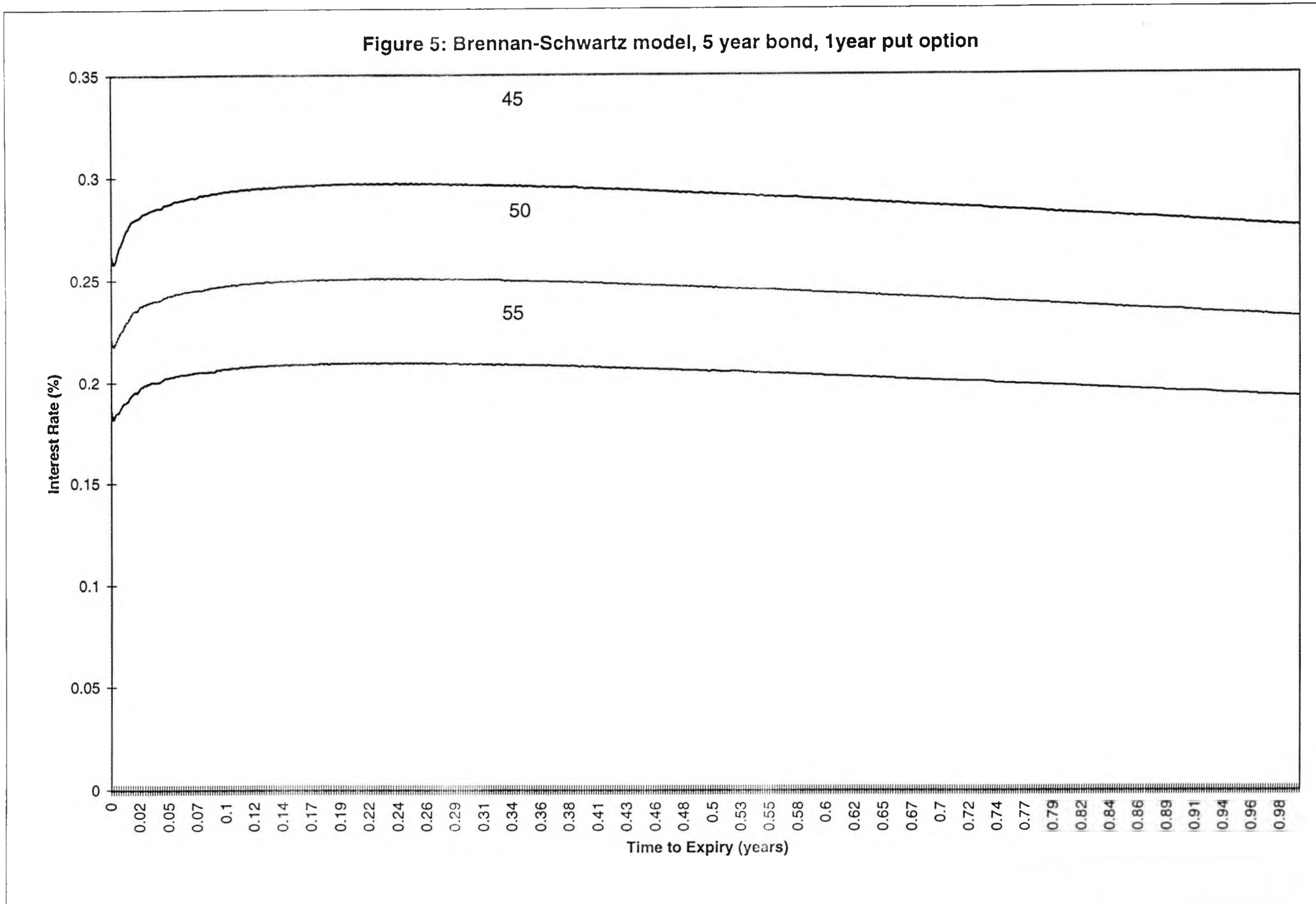


Figure 5: Brennan-Schwartz model, 5 year bond, 1year put option
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Interest Rate (%)
45
50
55
0
0.02
0.05
0.07
0.1
0.12
0.14
0.17
0.19
0.22
0.24
0.26
0.29
0.31
0.34
0.36
0.38
0.41
0.43
0.46
0.48
0.5
0.53
0.55
0.58
0.6
0.62
0.65
0.67
0.7
0.72
0.74
0.77
0.79
0.82
0.84
0.86
0.89
0.91
0.94
0.96
0.98
Time to Expiry (years)

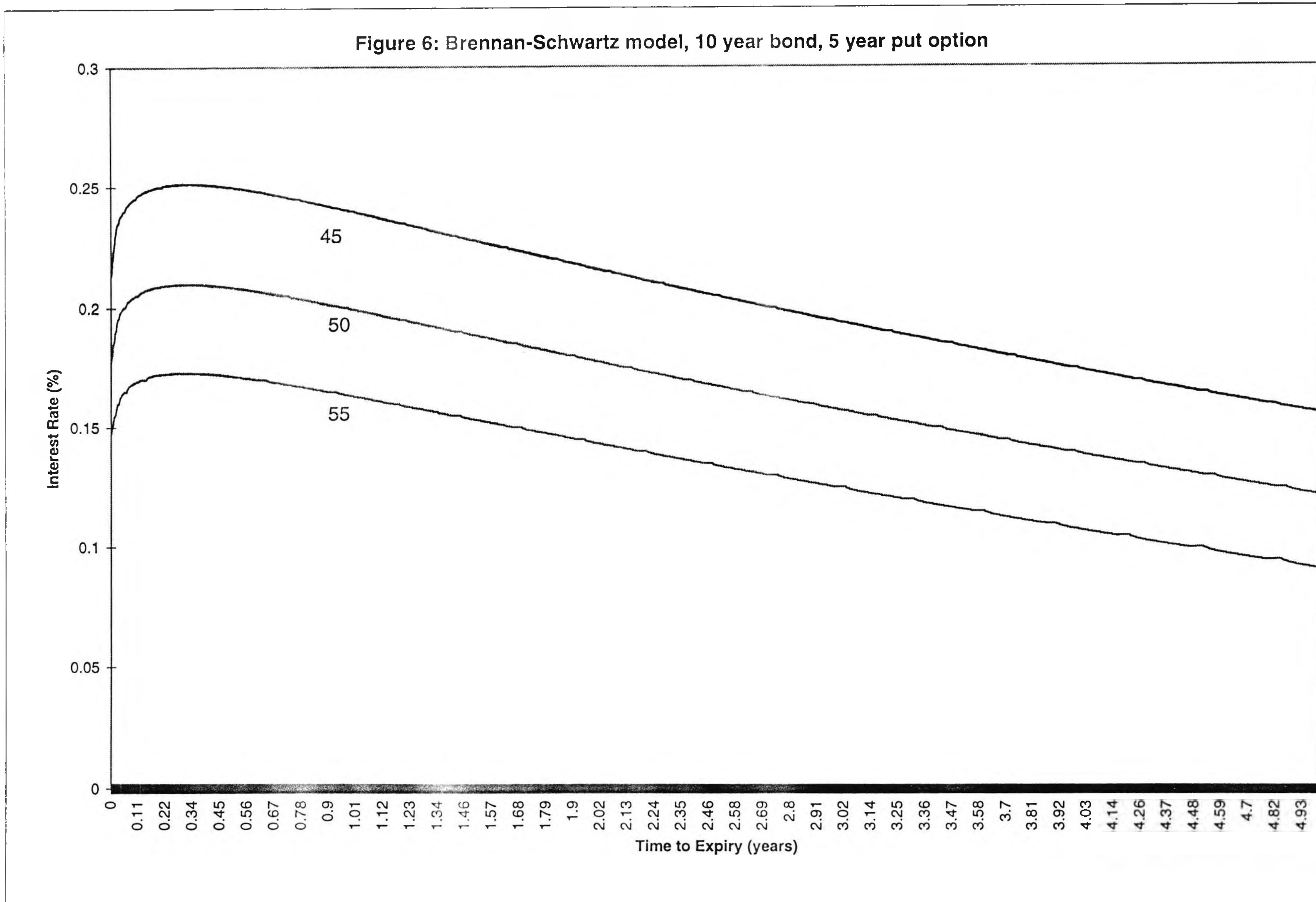


Figure 6: Brennan-Schwartz model, 10 year bond, 5 year put option
Interest Rate (%)
0.3
0.25
0.2
0.15
0.1
0.05
0
45
50
55
0
0.11
0.22
0.34
0.45
0.56
0.67
0.78
0.9
1.01
1.12
1.23
1.34
1.46
1.57
1.68
1.79
1.9
2.02
2.13
2.24
2.35
2.46
2.58
2.69
2.8
2.91
3.02
3.14
3.25
3.36
3.47
3.58
3.7
3.81
3.92
4.03
4.14
4.26
4.37
4.48
4.59
4.7
4.82
4.93
Time to Expiry (years)

CHAPTER 5.

AN EVALUATION OF CONTINGENT CLAIMS USING THE CKLS INTEREST RATE MODEL: AN ANALYSIS OF AUSTRALIA, CANADA, HONG KONG, JAPAN, U.K. , AND U.S.A

5.1. Introduction

In Chapter 3, we compared three numerical methods using assumed parameter values. Our main finding was that only the Box method converged to produce accurate bond and contingent claim prices for all combination of parameters. In this chapter using historical estimates of the CKLS model obtained for Australia, Canada, Hong Kong, Japan, U.K. and U.S.A., we calculate implied bond and contingent claim prices. The outline of this Chapter is as follows: Section 2 describes the data used in the study and Section 3 presents the implied bond and contingent claim prices. Section 4 contains a summary and conclusion.

5.2. Data

Over the years interest rate researchers have used different estimation methods. The most recent of these estimation methods is the non-parametric estimation method introduced by Ait-Sahalia (1996). This method is used primarily to test any non-linearity in the drift. As the CKLS (1992) model assumes that the drift is linear non-parametric method is not considered. The most widely used estimation method by researchers is GMM as used by CKLS (1992), Gibbons and Ramaswamy (1986) amongst other researchers. CKLS (1992) used an approximation in their estimation which introduced a bias term. The Gaussian method of Nowmna (1997a) reduces this effect of the bias term by using an analytical expression. Thus to estimate the CKLS model historically we use the approach of Nowman (1997a) who estimated the CKLS model on US and UK data. The discrete model used for estimation by Nowman (1997a) was derived by Bergstrom (1984, Theorem 2) and modified for heteroskedasticity in Nowman (19997a) given by equation (5.2.1) below.

$$r(t) = e^{\beta} r(t-1) + \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + \eta_t \qquad (t = 1,2,\ldots\ldots,T) \qquad (5.2.1)^1$$

where η_t (t = 1,2,.....,T) satisfies the conditions given Nowman (1997a). Following Bergstrom (1983) we let $L(\theta)$ be minus twice the logarithm of the Gaussian likelihood function where the complete vector of parameters is $\theta = [\alpha, \beta, \gamma, \sigma^2]$. The Gaussian estimates are obtained from equation (5.2.2) where m_{tt}^2 was given in Nowman (1997a).

$$L(\theta) = \sum_{t=1}^{T} \left[2 \log m_{tt} + \frac{\left\{ r(t) - e^{\beta} r(t-1) - \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) \right\}^2}{m_{tt}^2} \right] \qquad (5.2.2)$$

CKLS use the one-month Treasury bill yield as the proxy. However, Duffie (1996) finds Eurodollar rates are more suitable. The short-term interest rates used in this study are monthly one and three month Euro-currency rates for

Australia, Canada, Hong Kong, Japan, UK and US currencies (middle rate) obtained from *Datastream*. Table 5.1 reports the summary statistics. The mean and standard deviations of the different series are as follows: Australian one and three month means are (0.09822) and (0.09881) respectively with standard deviations of (0.04152) and (0.04191); Canadian one and three month means are (0.08992) and (0.09108) respectively with standard deviation of (0.03924) and (0.03859); Hong Kong one and three month means are (0.05928) and (0.06105) respectively with standard deviations of (0.02123) and (0.02029); Japanese one and three month means are (0.04693) and (0.04714) respectively with standard deviations of (0.02421) and (0.02440); UK one and three month means are (0.10009) and (0.10050) respectively with standard deviations of (0.03112) and (0.03063) and finally US one and three month means are (0.07645) and (0.07770) respectively with standard deviations of (0.03371) and (0.03419). The highest mean is for the UK and the lowest for Japan. The standard deviations of Hong Kong and Japan are the lowest.

5.3. Analysis of Results

In this section we discuss the results. The tables are organised such that in the first section of the table we analyse the bond prices. Bond prices are calculated for maturities ranging from 5 to 15 years and across short-term interest rates from 5% to 11%. Bond prices are calculated using the Box method for the Vasicek model $(\gamma = 0)$, Cox, Ingersoll and Ross (CIR) model $(\gamma = 0.5)$, Brennan

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[1] CKLS (1992) take an approximation of this expression

and Schwartz model $(\gamma = 1)$ and the actual market γ. Further we also calculate analytical bond prices for the CIR model using the formula in the original CIR paper. In the second part of each table we calculate both American type call and put options based on the zero coupon bonds. Note that as the underlying instrument is a zero coupon bond the value of the American call option is the same as European call option. We exploit this feature to check the accuracy of our numerical CIR[2] call price. We calculate analytical call prices using the formula provided by CIR in their original paper. We calculate both short dated and long dated call options. The short dated call options are based on a 5-year bond with an expiry date of 1 year and is during the last year before the bond matures. Similarly long dated options are based on 10-year bond with an expiry date of 5 years during the last 5-year's of the bond. Finally call and put option prices are calculated across a wide range of exercise prices. The exercise prices are chosen so as to highlight the variation of contingent claim prices across the standard models. We take the market price of risk to be zero. The analysis is based on annualised estimates in the tables to make it consistent with the grid. Table 5.2 contains the estimates of the historical parameters of the different countries considered.

5.3.1. Australia

The results for Australia dollar imply an unrestricted estimate of $\gamma = 1.4052$ for the one month and $\gamma = 1.0515$ for the three months rate. These results compare

[2] We also attempted to calculate the analytical prices for the Vasicek model. However, we found that the analytical formula did not lead to meaningful prices except in the case of Hong

to Tse's (1995) estimate for three-month money market date of 0.6763 and implies that the volatility of rates has become more dependent on the rate level in recent years. The three-month rate is very close to the assumed value of the Brennan and Schwartz model.

With regard to Table 5.3 the market γ bond prices differ enormously when compared with the standard models. The discrepancy increases as the term to maturity of the bond increases. For example, if we consider a 15-year bond at 11% interest rate, we see that market γ price is 33.1099, $\gamma = 1$ price is 61.2186, $\gamma = 0.5$ price is 81.4529 and $\gamma = 0$ price is 85.0643. For $\gamma = 0.5$ and $\gamma = 0$ bond prices are very similar across both interest rate and maturity dates. Both call and put option prices vary widely depending on which model is used. Market γ call prices are close to zero indicating that for the exercise prices chosen, the options are out of the money. For $\gamma = 0.5$ call prices vary widely indicating that the exercise prices chosen ensure that the call options are both in the money and out of the money. For market γ put prices we find the exercise prices chosen lead to the puts being deeply in the money and as a result the intrinsic value dominates.

Turning to Table 5.4, we find that market γ and $\gamma = 1$ bond prices are similar irrespective of the term to maturity of the bonds. For $\gamma = 0.5$ and $\gamma = 0$ bond prices are very similar whereas between $\gamma = 1$ and $\gamma = 0.5$ they are not. As a result we find that market γ and $\gamma = 1$, puts and calls are similar.

Kong. For example for 1 month Australia, 5% interest rate, 5 year maturity, bond price using the analytical formual is 9.8 x 10^{10}.

5.3.2. Canada

The results for the Canadian dollar imply an unrestricted estimate of $\gamma = 0.3912$ for the one month rate and $\gamma = 0.3700$ for the three-month rate. These results compare to Tse's (1995) estimate for three-month money market data of –0.3600, which was not statistically different from zero.

Turning to Table 5.5 the market γ bond prices are similar to $\gamma = 0.5$ bond prices. For $\gamma = 1$ bond prices collapse as the term to maturity increases. For example, for $\gamma = 1$, a 15 year bond at 11% is only valued at 9.8069. As a result we find that market γ and $\gamma = 0.5$ option prices are very similar and $\gamma = 1$ and $\gamma = 0$ option prices are substantially different.

Turning to Table 5.6 market γ bond prices are similar to $\gamma = 0.5$ bond prices. As before $\gamma = 1$ bond prices collapse as the term to maturity increases, for example for $\gamma = 1$, a 15-year bond at 11% is only valued at 10.2977. As a result we find that the market γ and $\gamma = 0.5$ option prices are similar whilst $\gamma = 1$ and $\gamma = 0$ option prices differ substantially from market γ prices.

5.3.3. Hong Kong

The results for the Hong Kong dollar imply an unrestricted estimate of $\gamma = 0.0076$ for the one month and $\gamma = 0.3221$ for the three months rate. These results compare to Tse's (1995) estimate for three-month market date of 1.5997 and implies that the volatility of rates has become less dependent on the rate level in recent years.

Turning to Table 5.7 as the term to maturity increases bond prices collapse for all models. For example, for a 15-year bond at11%, the market γ bond price is 2.9151, $\gamma = 0.5$ bond price is 0.6895 and $\gamma = 0$ bond price is 2.9967. There was no convergence for $\gamma = 1$, this is not surprising we take into that actual market $\gamma = 0.0076$. Further this is the only model where the analytical formula for default free bonds derived by Vasicek (1977) produces acceptable bond prices. These are given below:

	5%	8%	11%
5	51.4756	45.2315	39.7448
10	22.1594	17.7003	14.1385
15	13.3821	9.9633	7.4179

This indicates than only when market γ is close to zero will numerical prices be of the same order as analytical Vasicek prices. Market γ bond prices are similar to $\gamma = 0$ bond prices. This results with $\gamma = 0$ option prices being similar to market γ prices. This is in sharp contrast to $\gamma = 0.5$ option prices.

In Table 5.8 as the term to maturity increases bond prices collapse except for $\gamma = 0$. For $\gamma = 0.5$ bond prices are the closest to market γ prices. For

$\gamma = 1$ bond prices are considerably lower than market γ bond prices, whereas $\gamma = 0$ bond prices are higher than market γ bond prices. This results with the $\gamma = 0$ option prices being substantially different from the prices of other models.

5.3.4. Japan

The results for Japanese yen imply an unrestricted estimate of $\gamma = 0.3985$ for the one-month rate and $\gamma = 0.3870$ for the three-month rate. These results compare to Tse's (1995) estimate for three-month money market data of 0.6187, Shoji and Ozaki's (1996) estimate of 1.5443 for the one-month CD rate; Hiraki and Takezawa's (1996) estimates using offshore rates of 0.392 for the one-month rate and 0.367 for the three-month rate. Nowman (1997b) reports using also the Euro-currency one-month rate as used here an estimate of 0.9838 indicating the volatility has fallen over the last two years. Finally Chan et al (1992b) using the Gensaki rate reported $\gamma = 2.4353$.

Turning to Table 5.9 there is wide difference in bond price amongst the models. With $\gamma = 1$ bond prices are always lower than the market γ price and $\gamma = 0$ bond price always higher than market γ bond prices. This difference leads to the $\gamma = 0$ option prices being higher than the option prices of other models. In Table 5.10 we have the same trends as for Table 5.9.

5.3.5. United Kingdom

The results for British sterling pound imply an unrestricted estimate of $\gamma = 1.0461$ for the one-month rate and $\gamma = 1.3564$ for the three-month rate. These results compare to Tse's (1995) estimate for three-month money market data of 0.1132, Dahlquist's (1996) estimate of 0.1562 using monthly one-month Euro-currency rates, and Nowman's (1997a) estimate using monthly one-month interbank rates of 0.2898. This implies the volatility of rates has become more dependent on the level of rates in recent years.

In Table 5.11 market γ and $\gamma = 1$ bond prices are very similar across all range of maturities considered. For $\gamma = 0.5$ and $\gamma = 0$ bond prices are higher than actual market γ prices across all maturity ranges. These differences translates onto option prices, with market γ and $\gamma = 1$ option prices being substantially different than $\gamma = 0.5$ and $\gamma = 0$ option prices.

Turning to Table 5.12 we see that all models yield bond prices, which are substantially higher than market γ bond prices. This leads to option prices for market γ which are substantially different.

5.3.6. United States

The results for U.S. dollar imply an unrestricted estimate of $\gamma = 1.122$ for the one-month rate and $\gamma = 1.2660$ for the three months rate. These results compare to Tse's (1995) estimate for three month money market data of 1.7283, Shoji and Ozaki's (1996) estimate of 1.1473 for the one month US T. bill rate and

CKLS's estimate of 1.4999 using one US T. bill data. Nowman (1997b) who also used the one-month Euro-currency rate used here reported an estimate of 1.0519 indicating only a marginal increase in volatility over the last two years.

In Table 5.13 all models yield bond prices which are higher than the market γ bond prices. However, $\gamma = 1$ is reasonably close to market γ bond prices. This leads to market γ and $\gamma = 1$ option prices being different order from $\gamma = 0.5$ and $\gamma = 0$ option prices.

In Table 5.14 all models yield bond prices which are higher than market γ bond prices. This leads to market γ option prices, which are of different order from the options of other models.

5.4. Conclusion

In this Chapter we have applied the Box method to value default free bonds and contingent claims starting from the CKLS model. Using the Box method and historical estimates of the CKLS model obtained for Australia, Canada, Hong Kong, Japan, UK and US we calculated implied bond and contingent claims prices for these currencies. Our results indicate that the Box method can be used to value default free bonds and contingent claims in a wide range of economies. Secondly that default free bond prices and contingent claim prices are sensitive to the underlying interest rate model used.

Table 5.1.

Summary Statistics

r(t)	T	Mean	Standard Deviation
Australia	May'86-Dec'97		
1-Month	140	0.09822	0.04152
3-Month	140	0.09881	0.04191
Canada	Feb'81-Dec'97		
1-Month	203	0.08992	0.03924
3-Month	203	0.09108	0.03859
Hong Kong	Feb'86-Dec'97		
1-Month	143	0.05928	0.02123
3-Month	143	0.06105	0.02029
Japan	Feb'81-Dec'97		
1-Month	203	0.04693	0.02421
3-Month	203	0.04714	0.02440
UK	Feb'81-Dec'97		
1-Month	203	0.10009	0.03112
3-Month	203	0.10050	0.03063
US	Feb'81-Dec'97		
1-Month	203	0.07645	0.03371
3-Month	203	0.07770	0.03419

Table 5.2.

Gaussian Estimates of CKLS short-term Interest Rate Model

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma} dZ$$

	α	β	σ	γ
AUSTRALIA				
1-Month	0.0008	-0.0164	0.1415	1.4052
	(0.0009)	(0.0132)	(0.0510)	(0.1477)
3-Month	0.0008	-0.0157	0.0636	1.0515
	(0.0009)	(0.0127)	(0.0212)	(0.1367)
CANADA				
1-Month	0.0015	-0.0240	0.0180	0.3912
	(0.0011)	(0.0129)	(0.0046)	(0.1001)
3-Month	0.0014	-0.0227	0.0166	0.3700
	(0.0011)	(0.0127)	(0.0041)	(0.0962)
HONG KONG				
1-Month	0.0046	-0.0755	0.0086	0.0076
	(0.0016)	(0.0295)	(0.0040)	(0.0020)
3-Month	0.0030	-0.0455	0.0161	0.3221
	(0.0017)	(0.0283)	(0.0088)	(0.1891)
JAPAN				
1-Month	-0.0001	-0.0061	0.0125	0.3985
	(0.0003)	(0.0078)	(0.0021)	(0.0489)
3-Month	-0.0002	-0.0034	0.0090	0.3870
	(0.0002)	(0.0059)	(0.0016)	(0.0519)
UK				
1-Month	0.0015	-0.0183	0.0719	1.0461
	(0.0012)	(0.0138)	(0.0347)	(0.2046)
3-Month	0.0013	-0.0161	0.1403	1.3564
	(0.0011)	(0.0136)	(0.0636)	(0.1925)
US				
1-Month	0.0014	-0.0258	0.0927	1.1122
	(0.0007)	(0.0124)	(0.0216)	(0.0858)
3-Month	0.0011	-0.0203	0.1224	1.2660
	(0.0006)	(0.0110)	(0.0305)	(0.0927)

Table 5.3.

1 Month Australia, $\alpha = 0.0096, \beta = -0.1968, \sigma = 1.6980$, Market $\gamma = 1.4052$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	94.1696	92.7282	92.6344	83.9364	74.1262
		8			92.3077	90.4855	90.3680	80.0498	67.8383
		11			90.4503	88.2970	88.1467	76.6817	62.9405
10		5		Bond	91.3229	89.1659	89.0415	74.9913	54.2983
		8			89.5173	87.0093	86.8546	71.4881	49.0954
		11			87.7160	84.9048	84.7197	68.4581	45.2101
15		5		Bond	88.5622	85.7404	85.5797	67.0614	39.8596
		8			86.8112	83.6667	83.4778	63.9284	35.9873
		11			85.0643	81.6431	81.4529	61.2186	33.1099
5	1	8	80	Call	16.7400	15.8928	15.7271	7.8492	0.3622
			85		12.0468	11.3012	11.1458	4.1801	0.0079
			90		7.3563	6.7189	6.5839	1.2390	0.0000
			95		2.6684	2.1460	2.0516	0.0000	0.0000
5	1	8	80	Put	3.0857		3.0497	3.7514	12.1617
			85		3.5472		3.4680	5.8361	17.1617
			90		4.0927		4.4230	9.9502	22.1617
			95		4.8022		5.6209	14.9502	27.1617
10	5	8	80	Call	15.8279	14.8870	14.7196	7.8593	0.1340
			85		11.2369	10.3977	10.2418	4.1469	0.0001
			90		6.6482	5.9107	5.7764	0.8707	0.0001
			95		2.0617	1.4260	1.3357	0.0000	0.0001
10			80	Put	4.4157		4.4594	8.6752	30.9046
			85		5.0738		5.3348	13.5119	35.9046
			90		5.8459		6.4897	18.5119	40.9046
			95		6.8889		8.3942	23.5119	45.9046

Table 5.4.

3 Month Australia, $\alpha = 0.0096, \beta = -0.1884, \sigma = 0.7632$, Market $\gamma = 1.0515$

$\Delta t = 0.05, \Delta r = 0.5\%$:

All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	90.7201	85.9614	86.0467	75.0143	74.2908
		8			87.6580	81.5589	81.6436	68.1457	67.1817
		11			84.6165	77.3818	77.4660	62.3195	61.1899
10		5		Bond	86.1611	78.9200	79.0170	56.8685	54.7985
		8			83.2495	74.8442	74.9355	50.7740	48.5007
		11			80.3574	70.9789	71.0649	45.7964	43.4342
15		5		Bond	81.8357	72.5001	72.6114	43.5093	40.6854
		8			79.0702	68.7557	68.8606	38.7542	35.8845
		11			76.3233	65.2047	65.3036	34.8906	32.0511
5	1	8	70	Call	23.7108	19.4171	19.1793	6.2054	5.2449
			75		19.2642	15.4070	15.1467	3.2831	2.5084
			80		14.8279	11.4843	11.2158	1.2407	0.7711
			85		10.4018	7.6545	7.4007	0.2205	0.0871
5	1	8	70	Put	3.8296		3.9824	3.9211	4.0445
			75		4.4566		4.9987	6.9928	7.8183
			80		5.1606		6.2253	11.8543	12.8183
			85		5.9674		7.7421	16.8543	17.8183
10	5	8	65	Call	26.5689	22.6061	22.3030	8.0958	6.6762
			70		22.2407	18.6597	18.3316	5.4035	4.1845
			75		17.9180	14.7242	14.3840	3.0356	2.1054
			80		13.6010	10.7997	10.4677	1.1890	0.6420
10			65	Put	4.7372		5.4942	14.2260	16.4993
			70		5.5304		5.5304	19.2260	21.4993
			75		6.4148		6.4148	24.2260	26.4993
			80		7.4116		7.4116	29.2260	31.4993

Table 5.5.

1 Month Canada, $\alpha = 0.0180, \beta = -0.2880, \sigma = 0.2160$, Market $\gamma = 0.3912$

$\Delta t = 0.05, \Delta r = 0.5\%$:

All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	77.4645	68.8495	69.0133	66.0166	70.6581
		8			71.7250	61.6145	61.7695	58.3866	63.6266
		11			66.1612	55.1397	55.3021	51.6178	57.2416
10		5		Bond	61.9798	43.2411	43.5902	34.2938	47.8235
		8			57.0224	37.2618	37.5371	28.1646	41.8660
		11			52.2247	32.1094	32.3367	23.1715	36.5803
15		5		Bond	49.7840	26.8032	27.1909	15.6126	32.3979
		8			45.7877	22.8898	23.1939	12.3479	28.2204
		11			41.9203	19.5478	19.7924	9.8069	24.5293
5	1	8	55	Call	23.8433	12.1899	12.3481	7.9888	14.5963
			60		20.1493	8.5584	8.7027	3.7660	11.0045
			65		16.5821	5.4748	5.6014	0.8500	7.8398
			70		13.1413	3.0633	3.1730	0.0342	5.1693
5	1	8	55	Put	3.8737		1.5050	0.1345	2.0154
			60		5.1250		2.8802	1.6134	3.4001
			65		6.6569		5.1228	6.6134	5.4014
			70		8.2051		8.5185	11.6134	8.1450
10	5	8	30	Call	35.9476	19.0532	19.4050	10.7078	23.2370
			35		32.5606	16.2058	16.6030	7.9148	20.3405
			40		29.2155	13.4767	13.9254	5.3098	17.5416
			45		25.9134	10.8972	11.3868	3.0653	14.8553
10			30	Put	2.7330		2.2915	1.9042	2.4904
			35		3.8220		3.8464	6.8354	3.9014
			40		5.0956		6.0102	11.8354	5.7270
			45		6.5625		8.9170	16.8354	8.0192

Table 5.6.

3 Month Canada, $\alpha = 0.0168, \beta = -0.2724, \sigma = 0.1992$, Market $\gamma = 0.3700$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	77.5266	69.2875	69.4359	66.7966	71.2748
		8			71.5406	61.8539	61.9960	59.0473	64.0704
		11			65.7542	55.2179	55.3704	52.1523	57.5299
10		5		Bond	62.1990	43.7179	44.0510	35.5976	48.9440
		8			56.9476	37.3973	37.6562	29.1381	42.6671
		11			51.8819	31.9965	32.2035	23.8665	37.1065
15		5		Bond	50.1424	27.1536	27.5319	16.6542	33.6852
		8			45.8877	22.9614	23.2519	13.0768	29.1930
		11			41.7841	19.4164	19.6464	10.2977	25.2320
5	1	8	55	Call	23.5800	12.2405	12.3663	8.5947	14.9027
			60		19.9144	8.5452	8.6473	4.2432	11.2751
			65		16.3891	5.4146	5.4952	1.0370	8.0771
			70		13.0148	2.9879	3.0484	0.0453	5.3750
5	1	8	55	Put	3.7454		1.2252	0.0774	1.8705
			60		5.0281		2.5621	1.1209	3.2002
			65		6.5149		4.7591	5.9527	5.1585
			70		8.2149		8.2057	10.9527	7.8508
10	5	8	30	Call	35.9930	19.1415	19.4095	11.4567	23.8893
			35		32.5621	16.2951	16.5813	8.5882	20.9576
			40		29.2347	13.5702	13.8767	5.8685	18.1343
			45		25.9528	10.9977	11.3234	3.4692	15.4217
10			30	Put	2.6941		2.0724	1.2275	2.3599
			35		3.7854		3.5777	5.8619	3.3703
			40		5.0684		5.7208	10.8619	5.5026
			45		6.5442		8.6497	15.8619	7.7349

Table 5.7.

1 Month Hong Kong, $\alpha = 0.0552, \beta = -0.9060, \sigma = 0.1032$, Market $\gamma = 0.0076$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	49.3405	44.0102	44.0411	NC	49.2033
		8			44.0230	38.8527	39.0229	NC	43.8838
		11			39.1195	34.2995	34.5347	NC	38.9862
10		5		Bond	15.5610	8.4583	8.6736	NC	15.3684
		8			13.2429	6.9280	7.1481	NC	13.0627
		11			11.1568	5.6746	5.8780	NC	10.9931
15		5		Bond	4.3126	1.0339	1.1000	NC	4.2131
		8			3.6193	0.8142	0.8721	NC	3.5266
		11			2.9967	0.6412	0.6895	NC	2.9151
5	1	8	35	Call	13.4635	6.0934	7.6302	NC	13.3044
			40		10.0609	1.7358	3.7029	NC	9.8959
			45		7.2149	0.1845	1.0842	NC	7.0538
			50		4.9226	0.0047	0.1287	NC	4.7776
5	1	8	35	Put	1.4596		0.2309	NC	1.4309
			40		3.0037		1.5966	NC	2.9715
			45		5.2790		5.9711	NC	5.2579
			50		8.2663		10.9771	NC	8.2737
10	5	8	5	Call	11.0529	4.5204	5.1978	NC	10.8739
			10		8.9893	1.3814	3.2971	NC	8.8215
			15		7.1710	0.2023	1.6942	NC	7.0104
			20		5.6240	0.0164	0.6634	NC	5.4738
10			5	Put	0.2410		0.0604	NC	0.3032
			10		1.6606		2.8517	NC	1.6638
			15		4.2928		7.8517	NC	4.3238
			20		7.8396		12.8517	NC	7.9133

Table 5.8.

3 Month Hong Kong, $\alpha = 0.0360, \beta = -0.5460, \sigma = 0.1932$, Market $\gamma = 0.3221$

$\Delta t = 0.05, \Delta r = 0.5\%$:

All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	66.0368	56.6229	56.7703	53.1297	59.4860
		8			60.7445	50.5590	50.7199	47.8498	53.5423
		11			55.6462	45.1445	45.3261	42.4730	48.1267
10		5		Bond	40.7025	22.8525	23.1141	16.8569	28.5970
		8			37.0723	19.5387	19.7607	14.3074	24.9714
		11			33.5850	16.7055	16.9000	11.8751	21.7433
15		5		Bond	25.0103	8.3681	8.5367	4.0908	13.2388
		8			22.7633	7.0683	7.2057	3.3770	11.4917
		11			20.6052	5.9703	6.0847	2.7182	9.9431
5	1	8	45	Call	21.8209	10.0866	10.4764	7.0021	13.7844
			50		18.2355	6.3735	6.8299	2.8566	10.2257
			55		14.8568	3.4336	3.8681	0.3794	7.1437
			60		11.6778	1.4579	1.7862	0.0013	4.6044
5	1	8	45	Put	2.9890		1.0245	0.1218	1.6763
			50		4.3158		2.4151	2.1502	3.0647
			55		5.9212		5.0085	7.1502	5.1335
			60		7.8045		9.2801	12.1502	8.0001
10	5	8	15	Call	28.1466	11.7942	12.2602	7.1439	17.0902
			20		25.3090	9.2431	9.9233	4.8335	14.6294
			25		22.5543	6.8774	7.7563	2.7561	12.3036
			30		19.8858	4.8874	5.8102	1.1845	10.1350
10			15	Put	1.2475		0.9354	0.8160	1.0924
			20		2.3727		2.5892	5.6926	2.4534
			25		3.8505		5.6047	10.6926	4.4975
			30		5.6702		10.2393	15.6926	7.2716

Table 5.9.

1 Month Japan, $\alpha = 0.0012, \beta = -0.0732, \sigma = 0.1500$, Market $\gamma = 0.3985$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	86.5602	80.0104	80.0837	78.4301	81.1516
		8			78.1691	69.3743	69.4783	67.2302	70.7928
		11			70.1713	60.1521	60.2847	57.7059	61.6848
10		5		Bond	86.2354	71.1612	71.1976	62.7055	75.0389
		8			76.5730	56.1848	56.2645	45.9946	60.8920
		11			67.4128	44.3603	44.4623	34.0260	49.2056
15		5		Bond	87.8041	69.7597	69.9713	51.6878	75.3962
		8			77.8565	52.7459	52.7237	32.6555	59.7490
		11			68.4305	39.8817	39.8941	21.1593	47.0821
5	1	8	60	Call	25.7471	14.5862	14.6609	11.8349	16.3565
			65		22.1565	10.6595	10.7115	7.2535	12.5458
			70		18.7228	7.2251	7.2504	3.0271	9.1591
			75		15.4340	4.4330	4.4464	0.1530	6.2842
5	1	8	60	Put	4.0374		0.8353	0.0054	1.3276
			65		5.3729		1.8000	0.1402	2.4274
			70		6.9166		3.5237	2.7700	4.1283
			75		8.6601		6.3345	7.7700	6.5779
10	5	8	50	Call	38.3420	22.6866	22.7716	12.4916	26.9504
			55		34.5845	19.7091	19.8071	9.3148	23.9027
			50		30.8320	16.8580	16.9711	6.3699	20.9476
			65		27.0830	14.1485	14.2778	3.8324	18.0880
10			50	Put	5.2752		4.9353	4.0050	5.4247
			55		6.3225		6.8136	9.0050	7.1204
			60		7.4643		9.1267	14.0050	9.1096
			65		8.7095		11.9137	19.0050	11.4086

Table 5.10

3 Month Japan, $\alpha = 0.0024, \beta = -0.0408, \sigma = 0.1080$, Market $\gamma = 0.3870$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	84.5601	78.4267	78.4973	77.4189	79.3790
		8			74.3960	66.4860	66.5787	65.1272	67.6277
		11			64.9313	56.3633	56.4857	54.8531	57.5575
10		5		Bond	84.5174	66.2278	66.2307	59.3104	70.6543
		8			71.5535	47.7951	47.8932	39.9810	52.7677
		11			59.6631	34.4927	34.6251	27.2007	39.1409
15		5		Bond	88.5659	63.7351	63.2743	45.4144	71.7604
		8			74.5918	40.9450	40.8290	23.3814	50.0626
		11			61.8033	26.3042	26.3112	12.4685	34.4933
5	1	8	80	Call	22.1959	11.6245	11.7131	9.7972	13.1574
			85		18.8417	7.7638	7.8319	5.2307	9.4569
			90		15.7259	4.5931	4.6405	1.3488	6.3256
			95		12.8401	2.3100	2.3423	0.0569	3.8775
5	1	8	80	Put	4.0638		0.7165	0.0069	1.2271
			85		5.6427		1.8781	0.4063	2.5256
			90		7.5216		4.2494	4.8729	4.6835
			95		9.6915		8.2414	9.8729	7.9135
10	5	8	80	Call	43.0354	21.8158	21.9882	13.9483	26.7320
			85		39.6159	18.8823	19.0906	10.7476	23.8325
			90		36.2180	16.0979	16.3542	7.6686	21.0591
			95		32.8387	13.4872	13.7714	4.8705	18.4202
10			80	Put	4.8246		3.1224	1.0533	3.7789
			85		6.0024		4.8174	5.0189	5.3850
			90		7.2861		7.0367	10.0189	7.3461
			95		8.6748		9.8347	15.0189	9.6760

Table 5.11

1 Month United Kingdom, $\alpha = 0.0180, \beta = -0.2196, \sigma = 0.8628$,
Market $\gamma = 1.0461$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	88.3718	83.1961	83.3082	71.1805	70.4013
		8			85.6650	79.4725	79.5805	65.5213	64.5544
		11			82.9737	75.9154	76.0196	60.6559	59.5615
10		5		Bond	80.9287	72.4601	72.5842	49.2257	47.3657
		8			78.4483	69.2038	69.3213	44.9324	42.9781
		11			75.9821	66.0939	66.2051	41.3137	39.3229
15		5		Bond	74.1143	63.1236	63.2563	34.0566	31.8186
		8			71.8428	60.2868	60.4127	31.0681	28.8460
		11			69.5842	57.5776	57.6970	28.5525	26.3746
5	1	8	70	Call	21.6485	17.3674	17.0338	4.3360	3.5122
			75		17.1993	13.3294	12.9697	1.8205	1.2585
			80		12.7634	9.3696	9.0080	0.3768	0.1801
			85		8.3404	5.4924	5.1745	0.0055	0.0000
5	1	8	70	Put	3.8383		4.0644	5.1649	5.3652
			75		4.4839		5.0644	9.4787	10.4456
			80		5.2289		6.3499	14.4787	15.4456
			85		6.0924		8.0166	19.4787	20.4456
10	5	8	60	Call	27.4331	22.4265	22.0480	6.9716	5.7875
			65		23.2282	18.6232	18.2035	4.3937	3.3994
			70		19.0319	14.8357	14.3902	2.1953	1.4811
			75		14.8445	11.0641	10.3174	0.6282	0.2932
10			60	Put	4.9006		5.3994	15.0676	17.0219
			65		5.7938		6.6727	20.0676	22.0219
			70		6.7898		8.2021	25.0676	27.0219
			75		7.9089		10.0722	30.0676	32.0219

Table 5.12

3 Month United Kingdom, $\alpha = 0.0156, \beta = -0.1932, \sigma = 1.6836$,

Market $\gamma = 1.3564$

$\Delta t = 0.05, \Delta r = 0.5\%$:

All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	92.5920	90.7854	90.6874	81.1154	71.8653
		8			90.7693	88.6007	88.4719	77.4573	66.5086
		11			88.9510	86.4685	86.3089	74.2719	62.2237
10		5		Bond	88.1049	85.2138	85.0566	68.9817	50.1569
		8			86.3705	83.1631	82.9786	65.8482	46.1385
		11			84.6403	81.1617	80.9499	63.1239	42.9961
15		5		Bond	83.8352	79.9841	79.7754	58.6978	34.9876
		8			82.1849	78.0592	77.8265	56.0314	32.1724
		11			80.5385	76.1807	75.9237	53.7131	29.9738
5	1	8	80	Call	15.4304	14.3181	14.0803	5.7209	0.1194
			85		10.7616	9.7609	9.5379	2.3874	0.0000
			90		6.0970	5.2152	5.0260	0.2302	0.0000
			95		1.4365	0.6811	0.5722	0.0000	0.0000
5	1	8	80	Put	3.2926		3.2749	4.4959	13.4914
			85		3.8020		3.9547	7.5658	18.4914
			90		4.4141		4.8765	12.5427	23.4914
			95		5.2596		6.5945	17.5427	28.4914
10	5	8	75	Call	18.5126	17.0879	16.8360	8.2122	0.2589
			80		14.0109	12.7115	12.4677	4.6470	0.0002
			85		9.5127	8.3388	8.1149	1.4632	0.0002
			90		5.0182	3.9699	3.7870	0.0000	0.0002
10			75	Put	4.7439		4.7001	9.1680	28.8615
			80		5.4735		5.6375	14.1518	33.8615
			85		6.3115		6.8236	19.1518	38.8615
			90		7.3363		8.4858	24.1518	43.8615

Table 5.13

1 Month United States, $\alpha = 0.0168, \beta = -0.3096, \sigma = 1.1124$, Market $\gamma = 1.1122$
$\Delta t = 0.05, \Delta r = 0.5\%$:
All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	90.1635	86.3600	86.4224	73.6895	71.0441
		8			87.8043	83.2381	83.2943	68.6494	65.4985
		11			85.4549	80.2291	80.2792	64.3319	60.8538
10		5		Bond	84.0257	77.8916	77.9583	54.8833	49.1124
		8			81.8268	75.0738	75.1344	50.9480	44.9661
		11			79.6371	72.3579	72.4125	47.6111	41.5605
15		5		Bond	78.3061	70.2562	70.3260	40.9693	33.9649
		8			76.2569	67.7146	67.7785	38.0266	31.0842
		11			74.2162	65.2649	65.3231	35.5325	28.7208
5	1	8	75	Call	18.4560	15.7300	15.4355	3.6074	1.6030
			80		13.9148	11.4766	11.1844	1.2746	0.2698
			85		9.3818	7.2629	7.0003	0.1276	0.0013
			90		4.8568	3.0901	2.9095	0.0000	0.0000
5	1	8	75	Put	3.8303		4.0949	6.7019	9.5015
			80		4.4385		5.0214	11.3506	14.5015
			85		5.1428		6.1948	16.3506	19.5015
			90		6.0033		7.8773	21.3506	24.5015
10	5	8	65	Call	25.0704	21.6319	21.3475	7.4253	4.1488
			70		20.7389	17.5804	17.2774	4.5778	1.9505
			75		16.4134	13.5738	13.2293	2.1096	0.4706
			80		12.0941	9.5041	9.2108	0.3986	0.0052
10			65	Put	4.8400		5.1287	14.0520	20.0339
			70		5.6493		6.2302	19.0520	25.0339
			75		6.5542		7.5556	24.0520	30.0339
			80		7.5797		9.9186	29.0520	35.0339

Table 5.14

3 Month United States, $\alpha = 0.0132, \beta = -0.2436, \sigma = 1.4688$, Market $\gamma = 1.2660$

$\Delta t = 0.05, \Delta r = 0.5\%$:

All options are written on zero coupon bonds with a face of $100.00

Maturity of Bond	Expiry of Option	r(%)	Exercise Price	Asset/ Option	$\gamma = 0$	Analytic $\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 1$	Market γ
5		5		Bond	92.6265	90.4255	90.3982	79.9692	72.9355
		8			90.6158	87.9164	87.8714	75.6841	67.3088
		11			88.6110	85.4770	85.4143	71.9969	62.7650
10		5		Bond	88.4428	84.9967	84.9470	67.4096	52.4752
		8			86.5230	82.6382	82.5275	63.7304	48.0856
		11			84.6087	80.3450	80.2634	60.5775	44.6253
15		5		Bond	84.4481	79.8941	79.8246	56.9219	37.8037
		8			82.6150	77.6771	77.5933	53.8142	34.6238
		11			80.7871	75.5216	75.4235	51.1512	32.1213
5	1	8	80	Call	15.6071	14.2479	14.0360	4.8363	0.3809
			85		10.9657	9.7628	9.5658	1.9100	0.0047
			90		6.3286	5.2941	5.1324	0.1980	0.0000
			95		1.6958	0.8421	0.7631	0.0000	0.0000
5	1	8	80	Put	3.5286		3.6328	5.5414	12.6912
			85		4.0702		4.3956	9.3159	17.6912
			90		4.7183		5.4151	14.3159	22.6912
			95		5.5982		7.2349	19.3159	27.6912
10	5	8	75	Call	18.7922	17.1202	16.9107	7.6450	0.8868
			80		14.2993	12.7838	12.5790	4.2811	0.0491
			85		9.8100	8.4515	8.2641	1.3697	0.0001
			90		5.3243	4.1232	3.9758	0.0004	0.0001
10			75	Put	4.8146		4.9415	11.2696	26.9144
			80		5.5514		5.9323	16.2696	31.9144
			85		6.3900		7.1798	21.2696	36.9144
			90		7.4208		8.9109	26.2696	41.9144

CHAPTER 6.

CONCLUSIONS AND FUTURE RESEARCH

6.1. Summary

In this research we have examined numerical issues in the valuation of default free bonds and American interest rate contingent claims. The main focus has been on the problems that arise in the pricing of default-free bonds and American interest rate contingent claims based on the single factor CKLS short term interest rate model.

One of the major contributions of this work has been the introduction of a new numerical method. By making suitable transformations, we were able to develop the Box Method. This allowed us to value default-free bonds and American interest rate contingent claims based on the single factor CKLS model.

This thesis by focusing on the CKLS short term interest rate model, makes the following contributions to the numerical methods for the valuation of default-free bonds and American interest rate contingent claims.

First, we found that the use of Tian's Simplified Binomial lattice did not always lead to meaningful values of default-free bonds and interest rate contingent claims. We found that the value of γ is critical for the stability of the lattice. Theoretically we could achieve convergence when $\gamma > \frac{1}{2}$, however, in such circumstances we need a large number of time steps. From a practical viewpoint, we found that for a certain combination of parameters, convergence is achieved around $\gamma = 0.7$.

Second we introduced a new numerical method. We extended the Method of Lines to value default free bonds. Further it was not possible to calculate any contingent claim values using the Method of Lines, due to the difficulty of locating the free boundary. The Crank-Nicholson bond prices and Box Method prices were close to each other. However, we found that where analytical prices were available Box bond prices where closer to the analytical bond prices than the Crank-Nicholson bond price. This lead to the Crank-Nicholson bond price being radically different from analytical call option prices for certain combination of parameters. As for general matrix valuation, we found that overall Successive Over Relaxation (SOR) approach was superior to Gaussian elimination. The SOR approach for all combination of parameters lead to sensible bond and hence contingent claim prices.

Third using the Box Method as the basis, we developed a new procedure both to track and check the free boundary associated with American interest rate put options. By setting up the American pricing problem as an obstacle problem, we derived an integral equation. We used this scheme to track the free boundaries of both short and long dated put options based on commonly used single factor interest rate models. We found that the nature of the free boundary is dictated by the term to expiry of the put option as well as the underlying interest rate model used.

Fourth, this thesis explores prices of default-free bonds and interest rate contingent claims based on the estimates of the CKLS model obtained for Australia, Canada, Hong Kong, Japan, U.K. and U.S.A. using the Box Method. We compare the default free bond prices and contingent claim prices implied by the market γ with those implied by the widely used single factor models; namely Vasicek, CIR and Brannan and Schwartz. We also calculate the analytical default-free bond prices and call prices for the CIR model. This allowed us to check analytical default-free bond prices and calls with numerical default-free bond prices and calls. Clearly any significant discrepancy between the two would indicate that our numerical scheme has broken down. Our analysis firstly, suggests that both default-free bond prices and interest rate contingent claim prices are sensitive to the underlying short-term interest rate model used. We find for example, that the actual γ prices vary significantly from those of the standard models. Secondly we find that the Box Method is robust enough to be applied to a wide range of γ values.

6.2. Issues for further research

In this study we have introduced a new numerical method for the valuation of default-free bond prices and interest rate contingent claim prices. We have developed the algorithm such that it can be applied to a wide range of single factor interest rate models. We have further demonstrated that the Box Method outperforms all the existing numerical schemes.

Thus a clear extension of our work would be to extend the Box Method to two factor models. For example, we can use the Box Method to value default-free bonds and interest rate contingent claims based on an extended form of the Brennan and Schwartz (1979) model. In this instance term the CKLS process models interest rate and the long-term interest rate is taken to be the yield on the consol bond.

The checking procedure of Chapter 4 can be further expanded to track the free boundary surface associated with two factor American interest rate contingent claim. Indeed as numerical schemes for two factors are more complicated than for single factors, a checking procedure may be vital to ensure that the numerical scheme has not broken down.

Recently a number of papers have been published which have attempted to expand the use of Monte-Carlo simulation to value American contingent claims. However, none of these papers have suggested a scheme on how to value American interest rate contingent claims. Of all the Monte-Carlo schemes suggested for the valuation of American interest rate contingent claims, perhaps the approach of Grant, Vora and Weeks holds the most promise. Grant, Vora and Weeks value a single factor American Asian option by linking forward moving simulation and backward moving recursion through an iterative search process. An obvious extension to their scheme would be to use it value default free bond prices and American interest rate contingent claims based on multi-factor models.

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