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IRREDUCIBLE COMPONENTS OF EXOTIC SPRINGER FIBRES II: THE EXOTIC ROBINSON-SCHENSTED ALGORITHM

VINOTH NANDAKUMAR, DANIELE ROSSO, AND NEIL SAUNDERS

ABSTRACT. Kato's exotic nilpotent cone was introduced as a substitute for the ordinary nilpotent cone of type C with nicer properties. The geometric Robinson-Schensted correspondence is obtained by parametrizing the irreducible components of the Steinberg variety (the conormal variety for the action of a semisimple group on two copies of its flag variety) in two different ways. In type A the correspondence coincides with the classical Robinson-Schensted algorithm for the symmetric group. Here we give an explicit combinatorial description of the geometric bijection that we obtained in our previous paper by replacing the ordinary type C nilpotent cone with the exotic nilpotent cone in the setting of the geometric Robinson-Schensted correspondence. This "exotic Robinson-Schensted algorithm" is a new algorithm which is interesting from a combinatorial perspective, and not a naive extension of the type A Robinson-Schensted bijection.

CONTENTS

1. INTRODUCTION

The classical Robinson-Schensted correspondence is an algorithmic bijection

$$S_n \stackrel{\sim}{\longleftrightarrow} \bigsqcup_{\lambda \in \mathcal{P}_n} \mathrm{Std}(\lambda) \times \mathrm{Std}(\lambda),$$

where S_n is the symmetric group of degree n, $\mathcal{P}_n$ denotes the set of partitions of n and $\mathrm{Std}(\lambda)$ denotes *standard Young tableaux* of shape λ. This bijection has many rich combinatorial features and many applications in representation theory: for instance, the resulting partition of S_n into subsets S_λ indexed by $\mathcal{P}_n$ recover the two-sided cells as defined by Kazhdan and Lusztig in [KL79, Section 5] and also leads to a classification of unipotent character sheaves of GL_n [Lus85, Section 18].

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In [Ste76], Steinberg gives a geometric construction of the Robinson-Schensted correspondence using Springer theory. Using Spaltenstein's main result from [Spa76], stating that the irreducible components of type A Springer fibres are in bijection with standard Young tableaux, Steinberg parametrises the irreducible components of the following variety

$$\mathcal{Z} := \{(x, V_\bullet, U_\bullet) \,|\, x \in \mathcal{N},\, V_\bullet, U_\bullet \in \mathcal{F}_x\},$$

the so-called Steinberg variety, in two different ways.

Above $\mathcal{N}$ denotes the nilpotent cone of $\mathfrak{gl}_n$ and $\mathcal{F}_x$ denotes the Springer fibre above $x \in \mathcal{N}$. By matching up these two different descriptions, in [Ste88] Steinberg showed that the resulting bijection coincides with the Robinson-Schensted correspondence.

While the bijection in [Ste76] is in fact defined for an arbitrary semisimple group, in classical types other than type A the resulting algorithm is more complicated and it was described by van Leeuwen, [vL], building on earlier work of Spaltenstein (Section II.6 of [Spa82]) describing irreducible components of Springer fibres in those types. In this paper, we examine the 'exotic Robinson-Schensted algorithm' - analogously obtained using the geometry of Kato's exotic nilpotent cone as a substitute for the ordinary nilpotent cone of type C. The resulting combinatorial algorithm is more tractable and is not related to other type B/C generalisations of the Robinson-Schensted algorithm appearing in the literature, such as the 'naive' extension first defined by Stanley in [Sta82] or the one involving domino tableaux that goes back to the work of Barbasch and Vogan [BV82]. This builds on our previous paper [NRS16], parametrising irreducible components of exotic Springer fibres. Note that this is different from the exotic Robinson-Schensted correspondence constructed by Henderson and Trapa in [HT12]. Our hope is that this algorithm resulting from the geometry of the exotic Springer fibres will have some representation theoretic consequences for the Hecke algebra of type C with unequal parameters just like the ordinary Robinson-Schensted algorithm does for type A. Some results and conjectures on using variations of the RS correspondence for Hecke algebras with unequal parameters can be found in [BGIL10]

We briefly describe the exotic type C setting. Let $\mathcal{N}(\mathfrak{gl}_{2n})$ be the nilpotent cone for GL_{2n} and let $\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathfrak{gl}_{2n}) \cap \mathcal{S}$, where $\mathcal{S}$ is the Sp_{2n}-complement to $\mathfrak{sp}_{2n}$ in $\mathfrak{gl}_{2n}$ viewed as an Sp_{2n}-module. Kato's exotic nilpotent cone for Sp_{2n} is the variety $\mathfrak{N} = \mathbb{C}^{2n} \times \mathcal{N}(\mathcal{S})$ which is the Hilbert nullcone of the Sp_{2n}-module $\mathbb{C}^{2n} \oplus \mathcal{S}$. In [Kat09], Kato constructs an *exotic* Springer correspondence, and showed that the Sp_{2n}-orbits on $\mathfrak{N}$ are in bijection with the bipartitions of n, which also parametrise the irreducible representations of the Weyl group of type C. In subsequent work, many other Springer theoretic results have been extended to the exotic setting - intersection cohomology of orbit closures, (see Achar and Henderson, [AH08], and Shoji-Sorlin, [SS14]), theory of special pieces (see Achar-Henderson-Sommers, [AHS11]), and the Lusztig-Vogan bijection (see [Nan13]). In many respects, the exotic nilpotent cone behaves more nicely than the ordinary nilpotent cone of type C, and our present paper is another illustration of this.

Let $\pi : \widetilde{\mathfrak{N}} \longrightarrow \mathfrak{N}$ be the exotic Springer resolution as defined in [Kat11]. In [NRS16], we showed that the irreducible components of the fibres of π are in bijection with *standard Young bitableaux*, using Spaltenstein's techniques from [Spa76]. This allowed us to define an analogous *exotic Steinberg variety* whose irreducible components are parametrised in two separate ways: one way by elements of the Weyl group $W(C_n) = C_2 \wr S_n$ (signed permutations), and the other by pairs of standard Young bitableaux. Hence, by matching up these two descriptions, this gives us a bijection:

$$W(C_n) \xleftrightarrow{\sim} \bigsqcup_{(\mu,\nu)\in\mathcal{Q}_n} \mathcal{T}(\mu,\nu) \times \mathcal{T}(\mu,\nu), \tag{1.1}$$

where $\mathcal{T}(\mu,\nu)$ is the set of standard Young bitableaux of shape given by a bipartition (μ,ν) and $\mathcal{Q}_n$ is the set of all bipartitions of n. In this paper, we will give an explicit combinatorial description of this bijection.

The organisation of the paper is as follows:

- In Section 2 we introduce our notation for bipartitions and recall facts about the exotic nilpotent cone that we will need.
- In Section 3, which is mostly independent of the previous section, we define the exotic Robinson-Schensted bijection, a bijection between the Weyl group $W(C_n)$ and pairs of standard Young bitableaux. We provide the insertion and reverse bumping algorithms; these are interesting new algorithms and not a naive extension of the usual Robinson-Schensted correspondence.
- In Section 4 we examine the exotic Springer fibres, understanding the restriction of exotic Jordan types of pairs of generic points of the fibre.
- In Section 5 we use the results from Section 4 and [NRS16] to construct the reverse bumping algorithm from the geometry of exotic Springer fibres and thus prove the main theorem.

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2. Background

2.1. **Partitions.** We recall some standard combinatorial definitions which we will need. We closely follow the notation of [NRS16, Section 2] in this exposition.

Definition 2.1. Let n be a non-negative integer. A *partition* of n is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. We write $\lambda \vdash n$ or $|\lambda| = n$ to denote that λ is a partition of n and write $\ell(\lambda) = k$ to say that λ has k parts, or length k. The set of all partitions of n is denoted by $\mathcal{P}_n$.

A *bipartition* of n is a pair of partitions (μ,ν) such that $|\mu| + |\nu| = n$. We let $\mathcal{Q}_n$ denote the set of bipartitions of n. Given a bipartition (μ,ν) of n, we let $\lambda := \mu + \nu = (\mu_1 + \nu_1, \mu_2 + \nu_2, \ldots)$ denote the corresponding partition of n whose i-th part is the sum of the i-th parts of μ and ν respectively.

Associated to a partition λ, we have a Young diagram consisting of λ_i boxes on row i. We say that the Young diagram has shape λ. Similarly, we have a pair of Young diagrams associated to each bipartition.

Definition 2.2. Fix a bipartition $(\mu, \nu) \in \mathcal{Q}_n$ and let $\lambda = \mu + \nu$ be the corresponding partition of n. Fix a positive integer $m \leq \ell(\lambda)$. We define the following sets:

$$\begin{aligned} \Lambda_m &= \{1 \leq i \leq \ell(\lambda) \,|\, \lambda_i = \lambda_m\}, \\ \Gamma_m &= \{1 \leq i \leq \ell(\lambda) \,|\, \mu_i = \mu_m\}, \\ \Delta_m &= \{1 \leq i \leq \ell(\lambda) \,|\, \nu_i = \nu_m\}. \end{aligned}$$

Moreover, define

$$\Delta_{\leq m} = \{ i \in \Delta_m \,|\, i \leq m \} \quad \text{and} \quad \Delta_{<m} = \{ i \in \Delta_m \,|\, i < m \},$$

with similar definitions for Γ_m and Λ_m.

Definition 2.3. Given a Young diagram of shape $\lambda \in \mathcal{P}_n$, we obtain a standard Young tableau by filling in the boxes with the integers 1 up to n in such a way that the numbers are increasing along rows and down columns. Similarly a bitableau of shape $(\mu, \nu) \in \mathcal{Q}_n$ is standard if every integer between 1 to n occurs exactly once, and the increasing condition is satisfied in each of the two tableaux. To match the conventions of [NRS16] and [AH08], we reverse the direction of the rows of the first tableau, so numbers are increasing along rows from right-to-left. We let $\mathcal{T}(\mu, \nu)$ denote the set of standard Young bitableaux of shape (μ, ν).

Example 2.4. An example of a standard Young bitableau of shape $((3,1),(2,2,1))$ is the following:

$$T = \left(\begin{array}{|c|c|c|} \hline 6 & 3 & 1 \\ \hline \multicolumn{2}{c|}{} & 2 \\ \cline{3-3} \end{array} , \begin{array}{|c|c|} \hline 4 & 7 \\ \hline 5 & 8 \\ \hline 9 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right).$$

However, for notational convenience, especially for when describing the algorithm, we will often draw this diagram as follows:

$$T = \begin{array}{ccc|cc} 6 & 3 & 1 & 4 & 7 \\ & & 2 & 5 & 8 \\ & & & 9 & \end{array}$$

Definition 2.5. For T a standard Young bitableau and $1 \leq s \leq n$, define T_s to be the truncated bitableau consisting of just the numbers 1 up to s. By definition this remains a standard Young bitableau. If T originally had shape $(\mu, \nu) \in \mathcal{Q}_n$ then T_s has shape $(\mu^{(s)}, \nu^{(s)}) \in \mathcal{Q}_s$.

Example 2.6. For T as in Example 2.4, we have

$$T_5 = \left(\begin{array}{|c|c|} \hline 3 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \end{array} , \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \right),$$

which has shape $((2,1),(1,1)) \in \mathcal{Q}_5$.

Definition 2.7. We define $W(C_n)$ as the group of signed permutations as follows: $w \in W(C_n)$ is a permutation on the set of $2n$ elements $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ that satisfies the property that $w(\bar{i}) = \overline{w(i)}$ (here we are using the convention that $\bar{\bar{i}} = i$). Given the symmetry with respect to the involution $^-$, we will usually just write a signed permutation as a word $w = w(1) \cdots w(n)$.

2.2. **The Exotic Nilpotent cone and components of Exotic Springer Fibres.** In this section we recall some fundamental properties of the exotic nilpotent cone and the relevant results from [NRS16] that will be needed to establish the exotic Robinson-Schensted correspondence. Readers only interested in the actual algorithm can skip this section.

Let $V \cong \mathbb{C}^{2n}$ be a vector space endowed with a symplectic form $\langle \cdot, \cdot \rangle$. Denote by $Sp_{2n}(\mathbb{C})$ the corresponding symplectic group and $\mathfrak{sp}_{2n}(\mathbb{C})$ its Lie algebra.

Definition 2.8. Define $\mathcal{S}$ and $\mathcal{N}(\mathcal{S})$ as follows, noting that $\mathfrak{gl}_{2n} = \mathfrak{sp}_{2n} \oplus \mathcal{S}$ as $Sp_{2n}(\mathbb{C})$-modules:

$$\mathcal{S} = \{x \in \operatorname{End}(V) \mid \langle xv, w\rangle - \langle v, xw\rangle = 0,\ \forall v, w \in V\}; \text{and}$$
$$\mathcal{N}(\mathcal{S}) = \{x \in \mathcal{S} \mid x \text{ is nilpotent }\}.$$

The *exotic nilpotent cone* is the singular variety $\mathfrak{N} = V \times \mathcal{N}(\mathcal{S})$. It carries a natural $\mathrm{Sp}_{2n}(\mathbb{C})$-action:

$$g \cdot (v, x) = (gv, gxg^{-1}),$$

for $g \in Sp_{2n}(\mathbb{C})$ and $(v, x) \in \mathfrak{N}$.

Theorem 2.9 ([AH08, Thm 6.1])**.** *The orbits of $\mathrm{Sp}_{2n}(\mathbb{C})$ on $\mathfrak{N}$ are in bijection with $\mathcal{Q}_n$. More precisely, given a bipartition $(\mu, \nu) \in \mathcal{Q}_n$, the corresponding orbit $\mathbb{O}_{(\mu,\nu)}$ contains the point (v, x) if and only if there is a 'normal' basis of V given by*

$$\{v_{ij}, v_{ij}^* \,|\, 1 \le i \le \ell(\mu+\nu), 1 \le j \le \mu_i + \nu_i\},$$

with $\langle v_{ij}, v_{i'j'}^*\rangle = \delta_{i,i'}\delta_{j,j'}$, $v = \sum_{i=1}^{\ell(\mu)} v_{i,\mu_i}$ and such that the action of x on this basis is as follows:

$$xv_{ij} = \begin{cases} v_{i,j-1} & \text{if} \quad j \ge 2 \\ 0 & \text{if} \quad j = 1 \end{cases} \qquad xv_{ij}^* = \begin{cases} v_{i,j+1}^* & \text{if} \quad j \le \mu_i + \nu_i - 1 \\ 0 & \text{if} \quad j = \mu_i + \nu_i \end{cases}$$

in particular the Jordan type of x is $(\mu + \nu) \cup (\mu + \nu)$.

Definition 2.10. If $(v, x) \in \mathbb{O}_{(\mu,\nu)} \subset \mathfrak{N}$, we say that (μ, ν) is the exotic Jordan type of (v, x) and we denote that by $\operatorname{eType}(v, x) = (\mu, \nu)$.

The flag variety for $Sp_{2n}(\mathbb{C})$, which we denote by $\mathcal{F}(V)$, is the variety consisting of all symplectic flags, that is sequences of subspaces

$$F_\bullet = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq F_{2n-1} \subseteq F_{2n} = V)$$

where $\dim(F_i) = i$ and $F_i^\perp = F_{2n-i}$.

Definition 2.11. The exotic Springer resolution is a map $\pi : \widetilde{\mathfrak{N}} \twoheadrightarrow \mathfrak{N}$, where

$$\widetilde{\mathfrak{N}} = \{(F_\bullet, (v, x)) \in \mathcal{F}(V) \times \mathfrak{N} \mid v \in F_n,\ x(F_i) \subseteq F_{i-1}\ \forall i = 1, \dots, 2n\} \quad \text{and}$$
$$\pi(F_\bullet, (v, x)) = (v, x).$$

The variety $\widetilde{\mathfrak{N}}$ is smooth and π is proper, making this a resolution of singularities.

Definition 2.12. Given $(v, x) \in \mathbb{O}_{(\mu,\nu)}$, define the exotic Springer fibre $\mathcal{C}_{(v,x)} = \pi^{-1}(v, x)$. Explicitly:

$$\mathcal{C}_{(v,x)} = \{(0 \subset F_1 \subset \ldots \subset F_{2n-1} \subset \mathbb{C}^{2n}) \,|\, \dim F_i = i, F_i^\perp = F_{2n-i}, v \in F_n, x(F_i) \subseteq F_{i-1}\}.$$

The main result of [NRS16] was the following:

Theorem 2.13. [NRS16, Theorem 2.12] *Let $(v, x) \in \mathbb{O}_{(\mu,\nu)}$, then there is an open dense subset $\mathcal{C}^\circ_{(v,x)} \subset \mathcal{C}_{(v,x)}$, and a surjective map $\Phi : \mathcal{C}^\circ_{(v,x)} \longrightarrow \mathcal{T}(\mu, \nu)$ which induces a bijection between irreducible components of $\mathcal{C}_{(v;x)}$ and standard Young bitableaux of shape (μ, ν):*

$$\begin{array}{rcl} \operatorname{Irr} \mathcal{C}_{(v,x)} & \overset{\sim}{\longleftrightarrow} & \mathcal{T}(\mu,\nu); \\ \overline{\Phi^{-1}(T)} & \longleftrightarrow & T, \end{array}$$

These irreducible components all have the same dimension: $b(\mu, \nu) = |\nu| + 2\sum_{i\geq 1}(i - 1)(\mu_i + \nu_i)$.

Remark 2.14. A standard bitableau of shape (μ, ν) is the same thing as a *nested* sequence of bipartitions ending at (μ, ν), that is, a sequence of bipartitions

$$(\varnothing, \varnothing), (\mu^{(1)}, \nu^{(1)}), \ldots, (\mu^{(n)}, \nu^{(n)}) = (\mu, \nu)$$

such that $(\mu^{(i+1)}, \nu^{(i+1)})$ is obtained from $(\mu^{(i)}, \nu^{(i)})$ by adding one box. The identification is given by tracing the order in which the boxes are added according to the increasing sequence of numbers $1, 2, \ldots, |\mu| + |\nu|$.

Example 2.15. The standard bitableau $\left(\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 5 \\ \cline{2-2} \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \end{array}\right)$ corresponds to the nested sequence

$$(\varnothing, \varnothing), \left(\varnothing, \begin{array}{|c|}\hline \\ \hline \end{array}\right), \left(\begin{array}{|c|}\hline \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \end{array}\right), \left(\begin{array}{|c|c|}\hline & \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \end{array}\right), \left(\begin{array}{|c|c|}\hline & \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right), \left(\begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2} \end{array}, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right).$$

Remark 2.16. The map Φ of Theorem 2.13 is defined as follows: for $F_\bullet \in \mathcal{C}_{(v,x)}$, define

$$\Phi(F_\bullet) = (\operatorname{eType}(v + F_i, x|_{F_i^\perp/F_i}))_{i=0}^n.$$

For $F_\bullet \in \mathcal{C}^\circ_{(v,x)}$ we have then that $\Phi(F_\bullet)$ is a nested sequence of bipartitions, which defines a standard bitableau.

Remark 2.17. From Travkin [Tra09, Thm 1 and Cor 1], or Achar-Henderson [AH08, Thm 6.1], we know that $\operatorname{eType}(v, x) = (\mu, \nu)$ if and only if the following conditions hold:

$$\begin{array}{rcl} \operatorname{Type}(x, V) & = & (\mu_1 + \nu_1, \mu_1 + \nu_1, \mu_2 + \nu_2, \mu_2 + \nu_2, \ldots), \quad \text{and} \\ \operatorname{Type}(x, V/\mathbb{C}[x]v) & = & (\mu_1 + \nu_1, \mu_2 + \nu_1, \mu_2 + \nu_2, \mu_3 + \nu_2, \ldots). \end{array}$$

Here Type denotes the partition corresponding to the Jordan type of the corresponding nilpotent endomorphism.

One final important definition that we need to state our main theorem (Theorem 3.10 below) is the relative positions of two points in the flag variety.

Definition 2.18. Given two flags $F_\bullet, G_\bullet \in \mathcal{F}(V)$, we say that $F_\bullet$ and $G_\bullet$ are in *relative position* $w \in W(C_n)$ and write $w(F_\bullet, G_\bullet) = w$, if there is a basis $\{v_1, \ldots, v_n, v_{\bar{n}}, \ldots, v_{\bar{1}}\}$ such that $\langle v_i, v_j\rangle = \langle v_{\bar{i}}, v_{\bar{j}}\rangle = 0$ and $\langle v_i, v_{\bar{j}}\rangle = \delta_{ij}$ such that for $1 \leq i, j, \leq n$ we have

$$\begin{array}{lcl} F_i = \mathbb{C}\{v_n, \ldots, v_{n-i+1}\} & \text{and} & F_{2n-i} = F_i^\perp, \quad \text{and} \\ G_j = \mathbb{C}\{v_{w(n)}, \ldots, v_{w(n-j+1)}\} & \text{and} & G_{2n-j} = G_j^\perp. \end{array}$$

3. The Algorithm

In this section we present the exotic Robinson-Schensted algorithm in both directions, a 'reverse bumping' direction from bitableaux to Weyl group elements and then an 'insertion' algorithm in the other direction.

3.1. **Reverse bumping Algorithm.** The algorithm that we present now takes as an input a pair of standard Young bitableaux and produces an element in the Weyl group $W(C_n)$. Let T and R be two standard Young bitableaux. We will think of T as the "insertion" bitableau and R as the "recording" bitableau and produce a word $\mathrm{eRS}(T, R) \in W(C_n)$ as follows.

Definition 3.1. Let $1 \leq s \leq n$ and let T be a bitableau that is increasing going away from the centre and going down (standard condition) and does not contain the number s. An *available position* for s in T is a position such that the number in that position is smaller than s and if you replace that number with s, the increasing standard condition is still satisfied.

Example 3.2. Let $s = 13$, $T = \left(\begin{array}{|c|c|c|} \hline 10 & 3 & 1 \\ \hline 14 & 6 & 5 \\ \hline 15 & 11 & 9 \\ \hline \end{array} \, ; \, \begin{array}{|c|c|} \hline 4 & 16 \\ \hline 8 & 17 \\ \hline 18 \\ \cline{1-1} \end{array} \right)$, then the available positions for 13 are the ones containing the numbers 10, 11 and 8.

Definition 3.3. Given a bitableau T, of shape (μ, ν), let $\mathscr{R}$ be the set of rows of the two tableaux that comprise T, corresponding to the parts of the partitions μ and ν. We number the rows as follow,

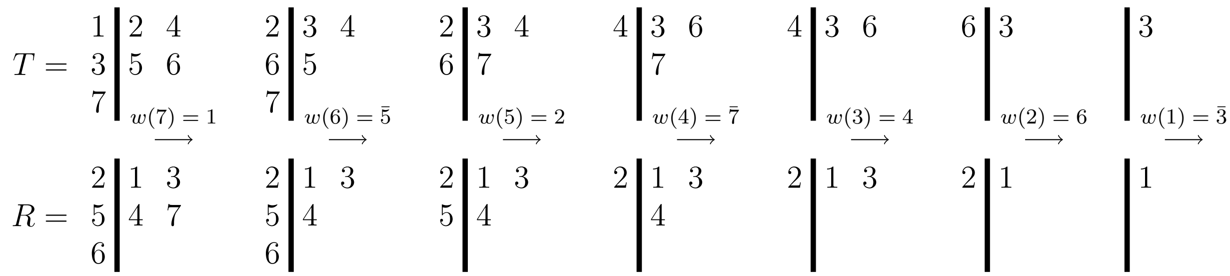
$$\mu_1 = 1, \quad \nu_1 = 2, \quad \mu_2 = 3, \quad \nu_2 = 4, \quad \mu_3 = 5, \ldots$$

So the first row of the left tableau is first, followed by the first row of the right tableau, followed by the second row of the left tableau and so on.

The Algorithm. Let T, R be two standard bitableaux with n boxes.

(1) Start with $k = n$.
(2) Find the position in R containing k. Let s be the number in the same position in the bitableau T and $m \in \mathscr{R}$ be the row in which s appears. Let R' be the bitableau obtained from R by removing k and T' the bitableau obtained from T by removing s.
(3) If $m = 1$, set $w(k) = s$. If $k = 1$ stop: the algorithm has ended, otherwise, return to (2) with k replaced by $k - 1$, and the bitableaux T and R replaced by T' and R'.
(4) If $m > 1$, consider all the available positions for s in T' that are in rows $\{r \in \mathscr{R} \mid r \geq m - 1\}$.
(5) If there are no available positions in those rows, let $w(k) = \bar{s}$ and return to (2) with k replaced by $k - 1$, and the bitableaux T and R replaced by T' and R'.
(6) Otherwise, if there are available positions, consider the one in the smallest numbered of those rows (notice that there can be at most one available position per row). Let T' be the bitableau obtained by replacing the number in that position with s, and let s' be the number that has been displaced. Let $m \in \mathscr{R}$ be the row where s' was. Return to (3) with s replaced by s'.

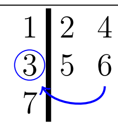
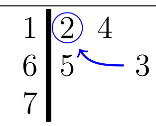
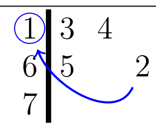
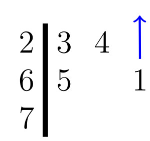
Example 3.4. Let $T = \left(\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 7 \\ \hline \end{array} \, ; \, \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \right)$, $R = \left(\begin{array}{|c|} \hline 2 \\ \hline 5 \\ \hline 6 \\ \hline \end{array} \, ; \, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & 7 \\ \hline \end{array} \right)$. We compute what element in $W(C_7)$ corresponds to this pair of bitableaux. For ease of notation we simply write the contents of the bitableau without the boxes in an array divided by the wall.



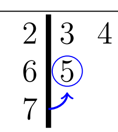
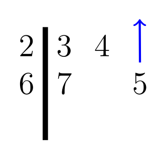
So in this case we have $\mathrm{eRS}(T, R) = w = \bar{3}64\bar{7}2\bar{5}1$.

We explain in more detail the first two steps of this algorithm which yields $w(7) = 1$ and $w(6) = \bar{5}$.

At the first stage, 7 is in the fourth row of R and 6 is in the corresponding position of T. The following table shows where 6, and then subsequent numbers, move to according to the rules.

(s, m)	Bitableau	Comment
$(6, 4)$		The smallest available position is in the box occupied by 3
$(3, 3)$		The smallest available position in the box occupied by 2
$(2, 2)$		The smallest available position is in the box occupied by 1
$(1, 1)$		Algorithm stops here, $w(7) = 1$

For the second step, the number 6 is in row 5 of R and 7 is in the corresponding position of T. As above, we track where the numbers move.

(s, m)	Bitableau	Comment
$(7, 5)$		The smallest available position is in the box occupied by 5
$(5, 4)$		There are no available positions in rows ≥ 3, so the algorithm stops here and $w(6) = \bar{5}$

Remark 3.5. A similar calculation shows that $\mathrm{eRS}(R, T) = w^{-1} = 75\bar{1}3\bar{6}2\bar{4}$. The fact that exchanging the (bi)tableaux gives the inverse element of the Weyl group is a feature of the ordinary Robinson-Schensted correspondence in Type A and is true also in our setting. One can deduce this from Theorem 3.10 because exchanging the two bitableaux corresponds to exchanging the two flags, and the relative position of the exchanged flags is the inverse signed permutation.

3.2. **Insertion Algorithm.** We present the algorithm that takes a signed permutation word in $W(C_n)$ and produces a pair of standard bitableaux.

Definition 3.6. Let $1 \leq s \leq n$ and let T be a bitableau that is increasing going away from the centre and going down (standard condition) and does not contain the number s. An *insertable position* for s is a position that is either outside of T but adjacent to a box of T, or in T such that the number in that position is bigger than s, and such that if you insert s there (possibly replacing a number), the resulting shape is still a bipartition and the increasing standard condition is still satisfied.

Example 3.7. Let $s = 13$, $T = \left(\begin{array}{|c|c|c|} \hline 10 & 3 & 1 \\ \hline 14 & 6 & 5 \\ \hline 15 & 11 & 9 \\ \hline \end{array} ; \begin{array}{|c|c|} \hline 4 & 16 \\ \hline 8 & 17 \\ \hline 18 \\ \cline{1-1} \end{array} \right)$, then the insertable positions for 13 are to the left of the number 10, under the number 9, in addition to the ones containing the numbers 14, 16 and 18.

Remember that the rows of a bitableau are ordered as in Definition 3.3.

The Algorithm. Let $w = w_1 w_2 \dots w_n$, with $w_i \in \{1, 2, \dots, n\} \cup \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ be a signed permutation with n letters.

(1) We set T, R to be two empty standard bitableaux, and we start with $k = 1$.
(2) If $w_k = s \in \{1, 2 \dots, n\}$, add a box containing s in the insertable position (which always exists) in row 1 of T.
(3) If $w_k = \bar{s} \in \{\bar{1}, \bar{2}, \dots, \bar{n}\}$, add a box containing s in the insertable position in either the first column to the left or the first column to the right of the wall of T, whichever one has the highest row number. (Notice there there is always an insertable position in those columns.)
(4) Set m to be the number of the row where s was inserted.
(5) If s was added to an empty position, add a box with the number k to the same position in R. If $k = n$ stop, otherwise, go back to (2) with k replaced by $k + 1$.
(6) Otherwise, if a number s' was replaced by s, consider the insertable positions for s' in T that are in rows $\{r \in \mathscr{R} \mid r \leq m + 1\}$ (there is always at least one). Add a box containing s' in the position among those with the highest row number. Replace s with s' and return to (4).

Theorem 3.8. *The reverse bumping algorithm and insertion algorithm are mutual inverses of each other.*

Proof. It is clear that step 2 of the insertion algorithm and step 3 of the reverse bumping algorithm are inverses of each other, inserting or removing a number in row 1 (corresponding to μ_1) which is not barred. Now suppose that we have a number s that was originally in row $m > 1$ of a bitableau T that satisfies the standard increasing conditions, and has been displaced during the reverse bumping process by a higher number $t > s$, let T' denote the bitableau with s replaced by t. There are two possibilities, depending on whether there are available positions for s in rows $r \geq m - 1$ of T'.

Case 1: Suppose there are no available positions, the reverse bumping algorithm then says that the number s is removed from the bitableau and $w(k) = \bar{s}$ for some k. But if there are no available positions for s, this means that all the numbers (if there are any) in rows $m-1, m+1, m+3, \ldots$ are bigger than s, and same for the rows $m+2, m+4, \ldots$. It also means that s was in the first column of row m of T, otherwise there would be an available position in that row. We want to show then that the insertion algorithm, starting with the bitableau T', and with $w(k) = \bar{s}$, would insert s in the place occupied by t according to step 3. Since $t > s$, and since any number in row $m-2$ above t has to be smaller than s (because T was an increasing bitableau), the position containing t is insertable for s. Also, the insertable position in the first column on the other side of the bitableau has to be in a row $r \leq m-1$ (because any elements in that column starting from row $m-1$ have to be bigger than s). It follows that the insertable position with highest row number in the first column of either the left or right tableau is indeed the one on row m, which is what we wanted. Reversing this argument, we can also see that if we start with $w(k) = \bar{s}$ and we insert s in the insertable position in the first column (either left or right) and row m of a bitableau T' to obtain a new bitableau T, then $m > 1$ (there is always an insertable position in row 2) and that there will be no available positions for s in rows $r \geq m-1$ of T'.

Case 2: Now suppose that there is an available position for s in T' and, by Step 6 of the reverse bumping algorithm, let $m' \geq m-1$ be the lowest numbered row with an available position and let T'' be the bitableau obtained by moving s to that position (displacing another number $u < s$). Notice that for any row $r \geq m-1$, there are no available positions for s in row r of T' if and only if there are no insertable positions for s in row $r+2$ (which is the row directly below r) of T'' (this happens exactly when there are the same number of columns in rows r and $r+2$ of T' containing numbers smaller than s). By minimality of m', there are no available positions in rows $m' > r \geq m-1$ of T', which is equivalent to the fact that there are no insertable positions in rows $m'+1 \geq r+2 > m$ of T''. This proves that, by doing Step 6 of the insertion algorithm on the bitableau T'', where s needs to move from row m', we would find that row m is the maximal with an insertable position among rows that are less or equal than $m'+1$, hence we would get back T'. Reversing the argument, since the statement about available positions in T' and insertable positions in T'' is an equivalence, we get that Step 6 of each algorithm is the inverse of the other one, which concludes the proof. $\square$

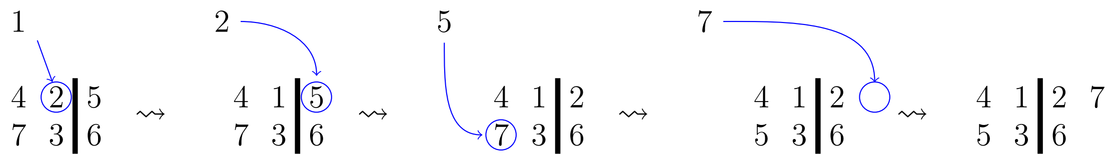
Example 3.9. Consider the element $w = 275\bar{6}4\bar{3}1$. We construct a pair of bitableaux using the insertion algorithm.

$$
\begin{array}{c}
\begin{array}{c|}2\\ \\ \end{array}\\[2ex] \begin{array}{c|}1\\ \\ \end{array}
\end{array}
\xrightarrow{7}
\begin{array}{c}
\begin{array}{cc|}7&2\\ && \end{array}\\[2ex] \begin{array}{cc|}2&1\\ && \end{array}
\end{array}
\xrightarrow{5}
\begin{array}{c}
\begin{array}{cc|c}5&2&7\\ &&\end{array}\\[2ex] \begin{array}{cc|c}2&1&3\\ &&\end{array}
\end{array}
\xrightarrow{\bar{6}}
\begin{array}{c}
\begin{array}{cc|c}5&2&7\\ &6&\end{array}\\[2ex] \begin{array}{cc|c}2&1&3\\ &4&\end{array}
\end{array}
\xrightarrow{4}
\begin{array}{c}
\begin{array}{cc|c}4&2&5\\ 7&6&\end{array}\\[2ex] \begin{array}{cc|c}2&1&3\\ 5&4&\end{array}
\end{array}
\xrightarrow{\bar{3}}
\begin{array}{c}
\begin{array}{cc|c}4&2&5\\ 7&3&6\end{array}\\[2ex] \begin{array}{cc|c}2&1&3\\ 5&4&6\end{array}
\end{array}
\xrightarrow{1}
\begin{array}{c}
\begin{array}{cc|cc}4&1&2&7\\ 5&3&6&\end{array}\\[2ex] \begin{array}{cc|cc}2&1&3&7\\ 5&4&6&\end{array}
\end{array}
$$

Therefore

$$
w = 275\bar{6}4\bar{3}1 \mapsto \left(\left(\begin{array}{|c|c|}\hline 4&1\\ \hline 5&3\\ \hline \end{array} ; \begin{array}{|c|c|}\hline 2&7\\ \hline 6& \\ \hline \end{array} \right), \left(\begin{array}{|c|c|}\hline 2&1\\ \hline 4&3\\ \hline \end{array} ; \begin{array}{|c|c|}\hline 3&7\\ \hline 6& \\ \hline \end{array} \right) \right).
$$

The following diagram explains the last step of the algorithm, where 1 is inserted into the bitableau $\left(\begin{array}{|c|c|}\hline 4&2\\ \hline 7&3\\ \hline \end{array} ; \begin{array}{|c|}\hline 5\\ \hline 6\\ \hline \end{array} \right)$.



Notice that here the numbers 2 and 5 both get bumped to the next row (from m to $m+1$ in the notation of the algorithm) but the number 7 does not have an insertable position in rows $m+1$ nor m, so it actually gets bumped 'up' to row $m-1$, according to step (6) of the algorithm.

From [NRS16, Section 6] it was shown that the *exotic Steinberg variety*

$$\mathfrak{Z} := \widetilde{\mathfrak{N}} \times_{\mathfrak{N}} \widetilde{\mathfrak{N}} := \{(F_\bullet, G_\bullet, (v,x)) \in \mathcal{F}(V) \times \mathcal{F}(V) \times \mathfrak{N} \mid F_\bullet, G_\bullet \in \mathcal{C}_{(v,x)}\},$$

has its irreducible components parametrised in two ways: one way by elements of the Weyl group $W(C_n)$ and the other by irreducible components of the exotic Springer fibres, or in other words, by pairs of standard Young bitableaux. This gives rise to a bijection

$$W(C_n) \overset{\sim}{\longleftrightarrow} \coprod_{(\mu,\nu)\in\mathcal{Q}_n} \mathcal{T}(\mu,\nu) \times \mathcal{T}(\mu,\nu)$$

defined geometrically, as in [NRS16, Corollary 7.1].

Theorem 3.10 (Main Theorem)**.** *Let $F_\bullet, G_\bullet \in \mathcal{C}^\circ_{(v,x)}$ be generic flags, with $\Phi(F_\bullet) = T$ and $\Phi(G_\bullet) = R$. Then $w(F_\bullet, G_\bullet) = \mathrm{eRS}(T,R)$.*

The proof of the theorem will be given in Section 5.

4. Intersecting Generic Hyperplanes

This section is devoted to answering the following question: given two one dimensional spaces X and W contained in $\ker(x) \cap (\mathbb{C}v)^\perp$, and such that

$$\mathrm{eType}(v+W, x_{|W^\perp/W}) = \mathrm{eType}(v+X, x_{|X^\perp/X}),$$

putting $Y = X + W$, what is $\mathrm{eType}(v+Y, x_{|Y^\perp/Y})$?

The answer to this question is summarised in Theorem 4.6 below, which is an analogue to Lemma 3.2 in [Ste88], and is the key step in allowing us to describe the steps in the reverse bumping algorithm. We fix some notation for throughout this section. Let $(v,x) \in \mathfrak{N}$ with $\mathrm{eType}(v,x) = (\mu,\nu)$ and let $\{v_{ij}, v^*_{ij}\}$ be a corresponding normal basis as described in Theorem 2.9. Let (μ',ν') be a bipartition obtained from (μ,ν) by decreasing either μ_m or ν_m by 1, where m is defined in the summation in (4.1). Define the variety

$$\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')} := \{W \subset \ker(x) \cap (\mathbb{C}[x]v)^\perp \mid \mathrm{eType}(v+W, x_{|W^\perp/W}) = (\mu',\nu')\}.$$

This variety was a key object in proving Theorem 2.13. Below we record the various properties that generic points in this variety satisfy.

Proposition 4.1. Let W be a generic point in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$. Then $W = \mathbb{C}w$ where

$$w = \sum_{i=1}^{m} \alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i}, \tag{4.1}$$

satisfies the following:

- if $\mu'_m = \mu_m - 1$, then $\sum_{i\in\Delta_{\leq m}} \beta_i = 0$;
- if $\nu'_m = \nu_m - 1$, then $\sum_{i\in\Delta_{\leq m}} \beta_i \neq 0$ if $\mu_m > \mu_{m+1}$, otherwise there is no condition on the β_i.

Proof. See Propositions 4.13 and 4.18 of [NRS16]. □

Remark 4.2. In the case where $\mu'_m = \mu_m - 1$, and $\nu_{m-1} > \nu_m$, this implies that $\beta_m = 0$ in the expression (4.1) for the spanning vector w of W. We also remark that the m we use in this section is different from the m in Section 3 because here the parts corresponding to the two bipartitions will be considered separately, as opposed to the numbering of all the rows together in Definition 3.3.

The next proposition and corollary precisely describe the bipartitions where pairs of generic points are perpendicular with respect to the symplectic form.

Proposition 4.3. Let (μ',ν') obtained from (μ,ν) by decreasing μ_m or ν_m by 1 and let $W, X \in \mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$ be two generic points. Then $X \subset W^\perp$ if and only if

(a) $\lambda_m \geq 2$; or
(b) $\lambda_m = \mu_m = 1$, $\nu_{m-1} > \nu_m = 0$ and $\mu'_m = \mu_m - 1$.

Proof. Let $W = \mathbb{C}w$ and $X = \mathbb{C}\mathtt{x}$ with

$$w = \sum_{i=1}^m \alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i} \quad \text{and} \quad \mathtt{x} = \sum_{i=1}^m \gamma_i v_{i,1} + \delta_i v^*_{i,\lambda_i}. \tag{4.2}$$

Then

$$\langle w, \mathtt{x}\rangle = \sum_{i\leq m\,:\,\lambda_i=1} (\alpha_i\delta_i - \beta_i\gamma_i). \tag{4.3}$$

If $\lambda_m \geq 2$, then the above sum is an empty sum, so $\langle w,\mathtt{x}\rangle = 0$ always. In the case where $\lambda_m = \mu_m = 1$ and $\nu_{m-1} > \nu_m = 0$ with $\mu'_m = \mu_m - 1$, by Remark 4.2 we have $\beta_m = \delta_m = 0$ for generic points, so

$$\langle w,\mathtt{x}\rangle = \alpha_m \cdot 0 - \gamma_m \cdot 0 = 0.$$

In all other cases the equation (4.3) will generically give a non-zero result. □

Corollary 4.4. With W and X as in Proposition 4.3, we have that $X \not\subseteq W^\perp$ if and only if:

(a) $\lambda_m = \mu_m = 1$ and $\nu_{m-1} = 0$; or
(b) $\lambda_m = \nu_m = 1$ (which implies $\mu_m = 0$).

Corollary 4.5. Let X and W as above, and suppose (μ',ν') is obtained from (μ,ν) by decreasing μ_1 by 1. Then $X = W$.

Proof. In this case, since $m = 1$, Proposition 4.1 implies that $\beta_1 = 0$ in (4.1), hence the variety $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$ consists of the single point $\{\mathbb{C}v_{1,1}\} = W = X$. □

From now on, we will assume that $X \subset W^\perp$ and so the bipartition (μ,ν) will satisfy the conditions of Proposition 4.3.

Theorem 4.6. *Suppose W and X are two generic points in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$ where (μ',ν') is obtained from (μ,ν) by decreasing one part of μ or ν by 1 and that (μ,ν) is a bipartition such that $X \subset W^\perp$. Put $Y = X + W$ and $\mathrm{eType}(v + Y, x_{|Y^\perp/Y}) = (\mu'',\nu'')$.*

(1) Suppose that $\nu' = \nu$ and $\mu'_m = \mu_m - 1$, with $m > 1$ (the case $m = 1$ is in Cor 4.5). Then

(a) $\nu''_{m-1} = \nu'_{m-1} - 1$ *if* $\nu_{m-1} > \nu_m$;
(b) $\mu''_m = \mu'_m - 1$ *if* $\nu_{m-1} = \nu_m$ *and* $\mu_m - 1 > \mu_{m+1}$;
(c) $\nu''_{\max\Delta_{m-1}} = \nu'_{\max\Delta_{m-1}} - 1$ *if* $\nu_{m-1} = \nu_m$, $\mu_m - 1 = \mu_{m+1}$ *and* $\max\Gamma_{m+1} > \max\Delta_m$;
(d) $\mu''_{\max\Gamma_m} = \mu'_{\max\Gamma_m} - 1$ *if* $\nu_{m-1} = \nu_m$, $\mu_m - 1 = \mu_{m+1}$ *and* $\max\Gamma_{m+1} \leq \max\Delta_m$.

(2) *Suppose that* $\mu' = \mu$ *and* $\nu'_m = \nu_m - 1$. *Then*
(a) $\mu''_m = \mu'_m - 1$ *if* $\mu_m > \mu_{m+1}$;
(b) $\nu''_m = \nu'_m - 1$ *if* $\mu_m = \mu_{m+1}$ *and* $\nu_m - 1 > \nu_{m+1}$;
(c) $\mu''_{\max\Gamma'_m} = \mu'_{\max\Gamma'_m} - 1$ *if* $\mu_m = \mu_{m+1}$, $\nu_m - 1 = \nu_{m+1}$ *and* $\max\Gamma_m \leq \max\Delta_{m+1}$;
(d) $\nu''_{\max\Delta'_m} = \nu'_{\max\Delta'_m} - 1$ *if* $\mu_m = \mu_{m+1}$, $\nu_m - 1 = \nu_{m+1}$ *and* $\max\Gamma_m > \max\Delta_{m+1}$.

Remark 4.7. Here we indicate how Theorem 4.6 corresponds to the reverse bumping algorithm of Section 3.1. We suppose that a number s is being displaced, so the bipartition (μ, ν) is the shape of the bitableau T_s and (μ', ν') the shape of the bitableau T_{s-1}, and the position where s was can be determined by the difference of those two shapes. If $\widetilde{T}$ is the bitableau obtained after s has displaced some other number $u < s$, then (μ', ν') is also the shape of $\widetilde{T}_s$ and (μ'', ν'') corresponds to the shape of $\widetilde{T}_{s-1}$, hence the difference between those shapes tells us where the number s has moved.

Suppose that the number s that is moving was in the row corresponding to the part μ'_m.

- If $m = 1$, s does not displace another number but is simply removed from the bitableau, so the shape does not change, this corresponds to the case of Corollary 4.5.
- If $m > 1$, then we are in the situation of Theorem 4.6, case (i).
- In (a), $\nu_{m-1} > \nu_m$, (i.e. $\nu'_{m-1} > \nu'_m$), and there is an "available position" in the first possible row: namely, the row corresponding to ν'_{m-1}. Replacing that label with s corresponds to decreasing ν'_{m-1} by 1.
- In (b), $\nu_{m-1} = \nu_m$, (i.e. $\nu'_{m-1} = \nu'_m$) so there are no "available positions" in the row corresponding to ν'_{m-1}. Since $\mu_m - 1 > \mu_{m+1}$ (i.e. $\mu'_m > \mu'_{m+1}$), the first "available position" is adjacent to s in the same row and replacing that label with s corresponds to decreasing μ'_m by 1.
- In (c), $\nu_{m-1} = \nu_m$ and $\mu_m - 1 = \mu_{m+1}$. If $\max\Gamma_{m+1} > \max\Delta_m$, then the first "available position" is at the end of row $\max\Delta_m$. Replacing that label with s corresponds to decreasing $\nu'_{\max\Delta_m}$ by 1.
- In (d), $\nu_{m-1} = \nu_m$ and $\mu_m - 1 = \mu_{m+1}$. If $\max\Gamma_{m+1} \leq \max\Delta_m$, then the first "available position" is at the beginning of row $\max\Gamma_{m+1}$. Replacing that label with s corresponds to decreasing $\mu'_{\max\Gamma_{m+1}}$ by 1.

One can similarly verify that Case (ii) (a), (b), (c), (d) match up with the reverse bumping algorithm when the number s starts from the part corresponding to ν'_m.

4.1. **Preliminaries on Restricting Jordan Types.** For the rest of this section, we can assume that $W \neq X$. By the Travkin and Achar-Henderson criterion, to calculate $\mathrm{eType}(v + Y, x_{|Y^\perp/Y})$, we need to know the following Jordan types: $\mathrm{Type}(x, Y^\perp/Y)$ and $\mathrm{Type}(x, Y^\perp/(\mathbb{C}[x]v + Y))$. Since $X \subset W^\perp$, we have $Y = X + W \subset X^\perp \cap W^\perp = Y^\perp$ and so we may regard Y/W as a 1-dimensional subspace of $\ker(x_{|W^\perp/W}) \cap (\mathbb{C}[x]v + W)^\perp/W \subset W^\perp/W$. The following two lemmas will be useful in this regard.

Lemma 4.8. *Let* σ *be the Jordan type of* x *restricted to the space* $W^\perp/(\mathbb{C}[x]v + W)$. *Then the Jordan type* σ' *of the induced nilpotent* x *on* $W^\perp/(\mathbb{C}[x]v + X + W)$ *is determined by the maximal*

k such that

$$X \subseteq x^{k-1}(W^\perp) + \mathbb{C}[x]v + W.$$

If $X \subset \mathbb{C}[x]v + W$, then $\sigma' = \sigma$ (and there is no maximal k), otherwise we have that σ' is obtained by removing the last box at the bottom of the k-th column of σ.

Proof. Suppose $X \not\subset \mathbb{C}[x]v + W$. Since $W^\perp/(\mathbb{C}[x]v + X + W)$ is the quotient of the space $W^\perp/(\mathbb{C}[x]v + W)$ by the one-dimensional subspace $(\mathbb{C}[x]v + X + W)/(\mathbb{C}[x]v + W)$, we know by [Spa76, page 1] that $\mathrm{Type}(x, W^\perp/(\mathbb{C}[x]v + X + W))$ is given by the maximal k such that

$$(\mathbb{C}[x]v + X + W)/(\mathbb{C}[x]v + W) \subseteq x^{k-1}(W^\perp/(\mathbb{C}[x]v + W)).$$

Translating this back to a condition on X, we find that

$$\begin{aligned} \mathbb{C}[x]v + X + W &\subseteq x^{k-1}(W^\perp) + \mathbb{C}[x]v + W \quad \text{and so} \\ X &\subseteq x^{k-1}(W^\perp) + \mathbb{C}[x]v + W \end{aligned}$$

as required. □

Lemma 4.9. *Let ρ be the Jordan type of x restricted to the space $W^\perp/(\mathbb{C}[x]v + X + W)$. Then the Jordan type ρ' of the induced nilpotent x on $Y^\perp/(\mathbb{C}[x]v + X + W) = Y^\perp/(\mathbb{C}[x]v + Y)$ is determined by the maximal l such that*

$$X^\perp \supseteq (x^{l-1})^{-1}(\mathbb{C}[x]v + W + X) \cap W^\perp.$$

Then ρ' is obtained by deleting the last box of the l-th column of ρ

Proof. Since $Y^\perp/(\mathbb{C}[x]v + Y)$ is an x-stable hyperplane in $W^\perp/(\mathbb{C}[x]v + Y)$, we know by [vL00, Lemma 1.4] that $\mathrm{Type}(x, Y^\perp/(\mathbb{C}[x]v + Y)$ is given by the maximal l such that

$$Y^\perp/(\mathbb{C}[x]v + Y) \supseteq \mathrm{im}(x_{|W^\perp/(\mathbb{C}[x]v+X+W)}) + \ker(x^{l-1}_{|W^\perp/(\mathbb{C}[x]v+X+W)}).$$

Now

$$\mathrm{im}(x_{|W^\perp/(\mathbb{C}[x]v+X+W)}) = \mathrm{im}(x_{|W^\perp}) + \mathbb{C}[x]v + X + W = x(W^\perp) + \mathbb{C}[x]v + X + W,$$

and

$$\ker(x^{l-1}_{|W^\perp/(\mathbb{C}[x]v+X+W)}) = (x^{l-1})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp.$$

Thus we require the maximal l such that

$$Y^\perp \supseteq x(W^\perp) + (x^{l-1})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp + \mathbb{C}[x]v + X + W. \tag{4.4}$$

Now since $X \subset W^\perp \cap \ker(x)$, we have $X \subset \ker(x_{|W^\perp})$ and so $X^\perp \supseteq x(W^\perp)$. Also, since $Y = X + W \subseteq (\mathbb{C}[x]v)^\perp$, we have that $Y^\perp \supseteq \mathbb{C}[x]v$, so (4.4) simplifies to the maximal l such that

$$Y^\perp \supseteq (x^{l-1})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp.$$

Translating this back to a condition on X and W, we require the maximal l such that

$$X^\perp \cap W^\perp \supseteq (x^{l-1})^{-1}(\mathbb{C}[x]v + W + X) \cap W^\perp.$$

Now the condition that $W^\perp \supseteq (x^{l-1})^{-1}(\mathbb{C}[x]v + W + X) \cap W^\perp$ is redundant and so we require

$$X^\perp \supseteq (x^{l-1})^{-1}(\mathbb{C}[x]v + W + X) \cap W^\perp$$

and the proof is complete. □

Now we know that $\mathrm{Type}(x, Y^\perp/(\mathbb{C}[x]v+Y))$ is obtained from $\mathrm{Type}(x, W^\perp/(\mathbb{C}[x]v+W))$ by decreasing two parts, we will see below that these are two consecutive parts, which completely determines $\mathrm{eType}(v+Y, x_{|Y^\perp/Y})$.

For what follows, we will always have $W = \mathbb{C}\{w\}$ and $X = \mathbb{C}\{\mathtt{x}\}$ where

$$w = \sum_{i=1}^{m} \alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i} \quad \text{and} \quad \mathtt{x} = \sum_{i=1}^{m} \gamma_i v_{i,1} + \delta_i v^*_{i,\lambda_i},$$

and m is the index where either μ or ν is decreased by 1 to obtain $\mathrm{eType}(v+W, x_{|W^\perp/W})$ (which obviously coincides with $\mathrm{eType}(v+X, x_{|X^\perp/X})$).

Proposition 4.10. The maximal k_2 such that $X \subseteq x^{k_2-1}(W^\perp)+W$, is

$$k_2 = \begin{cases} \lambda_{m-1} & \text{if} \quad \mu'_m = \mu_m - 1 \text{ and } \nu_{m-1} > \nu_m, \\ \lambda_m - 1 & \text{otherwise.} \end{cases}$$

Proof. Since X and W are generically chosen in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$, it follows that $X \subseteq x^{\lambda_m-1}(V)$ but not $x^{\lambda_m}(V)$. Therefore since $x(V) \subset W^\perp$, it immediately follows that $X \subseteq x^{\lambda_m-2}(W^\perp)$, so the maximal $k_2 \geq \lambda_m - 1$.

Consider the set $\mathcal{S} := \{v_{i,1}, v^*_{i,\lambda_i} \mid i \in \Lambda_m\}$. We claim that $S \subset x^{\lambda_m-1}(W^\perp)$ if and only if $\nu_{m-1} > \nu_m$. It is clear that the set of vectors $S' := \{v_{i,\lambda_i}, v^*_{i,1} \mid i \in \Lambda_m\}$ maps onto S via x^{λ_m-1}. Moreover it is easily seen that $S' \not\subseteq W^\perp$ whenever $|\Lambda_m| \geq 2$ since W is generically chosen implying that α_i's and the β_i's are non-zero as i runs through Λ_m. Indeed since W and X are chosen generically, we have $S' \subseteq W^\perp$ if and only if $|\Lambda_m| = 1$ and either $\alpha_m = 0$ or $\beta_m = 0$ in the expression for w given in (4.1). But for a generic W in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$, since $\mu'_m = \mu_m - 1$, we can only assume that $\beta_m = 0$ precisely when $\nu_{m-1} > \nu_m$ by Remark 4.2 (which also implies that $\delta_m = 0$ in the expression for $\mathtt{x}$). This establishes the claim.

Now we can quickly see that if $\nu_{m-1} = \nu_m$ then S and hence X is not contained in $x^{\lambda_m-1}(W^\perp)$. Moreover since $|S| > 1$ and X is generically chosen, it is impossible for X to be contained in $x^{\lambda_m-1}(W^\perp)+W$ as adding W only the dimension by 1. Therefore the maximal k_2 in this case is $\lambda_m - 1$.

Now suppose that $\nu_{m-1} > \nu_m$. Then the expressions for the spanning vectors w and $\mathtt{x}$ are as in (4.2) and we claim that the set of vectors $\{v_{i,1}, v^*_{i,\lambda_i} \mid 1 \leq i \leq m-1\}$ are contained in $x^{\lambda_{m-1}-1}(W^\perp)$. This is clear for all i such that $\lambda_i > \lambda_{m-1}$, so we only need to check this for $i \in \Lambda_{m-1}$, that is the i such that $\lambda_i = \lambda_{m-1}$. Notice that the vectors of the form $\alpha_i v_{i,\lambda_i} + \beta_i v^*_{i,1}$ are all contained in $W^\perp$ and so

$$x^{\lambda_{m-1}-1}(\alpha_i v_{i,\lambda_i} + \beta_i v^*_{i,1}) = \alpha_i v_{i,1} \in x^{\lambda_{m-1}-1}(W^\perp),$$

for $i \in \Lambda_{m-1}$. Therefore $v_{i,1}, v^*_{i,\lambda_i}$ are contained in $x^{\lambda_{m-1}-1}(W^\perp)$ for $1 \leq i \leq m-1$ and so

$$X \subseteq \mathbb{C}\{v_{i,1}, v^*_{i,\lambda_i} \mid 1 \leq i \leq m-1\} \oplus \mathbb{C}\{v_{m,1}\} \subseteq x^{\lambda_{m-1}-1}(W^\perp) + W.$$

Since the vectors $v_{i,1}$ and v^*_{i,λ_i} for $i \in \Lambda_{m-1}$ are not contained in $x^{\lambda_{m-1}}(V)$, they cannot be contained in $x^{\lambda_{m-1}}(W^\perp)$ and in this case, adding the space W cannot account for these missing vectors as well as $v_{m,1}$. So we have shown that the maximal k_2 in this case is λ_{m-1} which completes the proof. ☐

Proposition 4.11. Let X and W be two generic points of $\mathcal{B}_{(\mu,\nu)}$. Then the maximal l_2 such that

$$X^\perp \supseteq (x^{l_2-1})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp$$

is $\lambda_m - 1$ except when $\mu'_m = \mu_m - 1, \nu' = \nu$ and $\nu_{m-1} > \nu_m$.

Proof. Consider the vector $w' = \sum_{i=1}^{m} \alpha_i v_{i,\lambda_m} + \beta_i v^*_{i,\lambda_i-\lambda_m+1}$. We have that $x^{\lambda_m-1}(w') = w$, and $w' \in W^\perp$ because

$$\langle w, w' \rangle = \sum_{i\in\Lambda_m} (\alpha_i\beta_i - \beta_i\alpha_i) = 0.$$

But $\langle \mathtt{x}, w' \rangle = \sum_{i\in\Lambda_m} \alpha_i\delta_i - \beta_i\gamma_i$, which is generically non-zero except when $\mu' \preceq \mu, \nu' = \nu$ and $\nu_{m-1} > \nu_m$, which forces $|\Lambda_m| = 1$ and so $\beta_m = \delta_m = 0$ by Remark 4.2. Thus when this is not the case, we have $(x^{\lambda_m-1})^{-1}(W) \nsubseteq X^\perp$. Furthermore, since $X \subseteq x^{\lambda_m-1}(V)$, it follows that $X^\perp \supseteq \ker(x^{\lambda_m-1}) \supseteq \ker(x^{\lambda_m-2})$, and it is easily seen that $X^\perp \supseteq x^{\lambda_m-2}(W)$. Therefore $X^\perp \supseteq (x^{\lambda_m-2})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp$. □

Remark 4.12. We will deal with the case $\mu'_m = \mu_m - 1$, $\nu' = \nu$ and $\nu_{m-1} > \nu_m$ in Proposition 4.15.

4.2. **Box Removed from the Left.** We are now getting to the proof of Theorem 4.6. We first consider the case where $\mu'_m = \mu_m - 1$.

4.2.1. *$\mu'_m = \mu_m - 1, \nu' = \nu$ and $\nu_{m-1} = \nu_m$.* Recall that the condition $\nu_{m-1} = \nu_m$ is equivalent to $|\Delta_{\leq m}| \geq 2$. Also recall from Proposition 4.1 that a generic point in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$ has the following conditions on the coefficients of its spanning vector; namely for a spanning vector w as in (4.2) we have:

$$\sum_{i\in\Delta_{\leq m}} \beta_i = 0.$$

Proposition 4.13. Let X and W be generic points in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$. Then the maximal k_2 such that

$$X \subseteq x^{k_2-1}(W^\perp) + \mathbb{C}[x]v + W$$

is $\lambda_m - 1$.

Proof. We know by Proposition 4.10 that $k_2 \geq \lambda_m - 1$. To show that $k_2 = \lambda_m - 1$ we show that the vectors $v_{i,\mu_i-\mu_m+1}$ for $i \in \Delta_{\leq m}$ are not contained in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v$ which implies the vectors v^*_{i,λ_i} are not contained in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v + W$ for $i \in \Delta_{\leq m}$ and so $X \nsubseteq x^{\lambda_m}(W^\perp) + \mathbb{C}[x]v + W$.

Consider the vectors $\beta_m v_{i,\lambda_i} - \beta_i v_{m,\lambda_m}$ for $i \in \Delta_{\leq m}$. These vectors are clearly contained in $W^\perp$ and so $\beta_m v_{i,\mu_i-\mu_m+1} - \beta_i v_{m,1} = x^{\lambda_m-1}(\beta_m v_{i,\lambda_i} - \beta_i v_{m,\lambda_m})$ is contained in $x^{\lambda_m-1}(W^\perp)$. Now if each $v_{i,\mu_i-\mu_m+1}$ were in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v$, we would require that the vector

$$v' = \sum_{i\in\Delta_{\leq m}} v_{i,\mu_i-\mu_m+1} = x^{\mu_i-\mu_m}(v) - \sum_{i=1}^{m'} v_{i,\mu_i-\mu_m+1},$$

where $m' = \min \Delta_m - 1$ lies outside the span of $\beta_m v_{i,\mu_i-\mu_m+1} - \beta_i v_{m,1}$ as i runs through $\Delta_{\leq m}$. However we calculate:

$$\begin{aligned}\sum_{i\in\Delta_{<m}} \beta_m v_{i,\mu_i-\mu_m+1} - \beta_i v_{m,1} &= \beta_m \sum_{i\in\Delta_{<m}} v_{i,\mu_i-\mu_m+1} - \left(\sum_{i\in\Delta_{<m}} \beta_i\right) v_{m,1} \\ &= \beta_m \sum_{i\in\Delta_{<m}} v_{i,\mu_i-\mu_m+1} + \beta_m v_{m,1} \qquad \text{since } \sum_{i\in\Delta_{\leq m}} \beta_i = 0 \\ &= \beta_m \sum_{i\in\Delta_{\leq m}} v_{i,\mu_i-\mu_m+1} \\ &= \beta_m v'.\end{aligned}$$

Hence we cannot obtain the individual basis vectors $v_{i,\mu_i-\mu_m+1}$ in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v$, for $i \in \Delta_{\leq m}$. In particular, we cannot obtain the vectors $v_{i,1}$ for $i \in \Lambda_m$ by adding $\mathbb{C}[x]v$ which implies that we cannot obtain the vectors v^*_{i,λ_i} for $i \in \Lambda_m$ by adding $\mathbb{C}[x]v + W$. Therefore we cannot obtain X as a subspace of $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v + W$. □

of Theorem 4.6: Cases 1(b),(c),(d). We can now describe exactly how $\text{Type}(x, Y^\perp/(\mathbb{C}[x]v + Y) = (\mu'', \nu'')$ is obtained from $\text{Type}(x, W^\perp/(\mathbb{C}[x]v + W))$ when $\mu'_m = \mu_m - 1, \nu' = \nu$ and $\nu_{m-1} = \nu_m$. In this case we have by Proposition 4.13 and Proposition 4.11, $k_2 = l_2 = \lambda_m - 1$. Let

$$\sigma = \text{Type}(x, W^\perp/(\mathbb{C}[x]v + W) \quad \text{and} \quad \sigma' = \text{Type}(x, Y^\perp/(\mathbb{C}[x]v + Y))$$

We explicitly determine which parts of σ are decreased by 1 in order to obtain σ'. We have

$$\sigma = (\mu_1 + \nu_1, \mu_2 + \nu_1, \ldots, \mu_{m-1} + \nu_{m-1}, \mu_m - 1 + \nu_{m-1}, \mu_m - 1 + \nu_m, \mu_{m+1} + \nu_m, \ldots)$$

and we know that that σ' is obtained from σ by decreasing the last two parts of size $\lambda_m - 1 = \mu_m + \nu_m - 1$. Since $\nu_{m-1} = \nu_m$, we know that there certainly are at least two parts of σ that have this size, namely $\sigma_{2(m-1)} = \mu_m - 1 + \nu_{m-1}$ and $\sigma_{2m-1} = \mu_m - 1 + \nu_m$.

Case 1(b): If $\mu_m - 1 > \mu_{m+1}$, then $\mu_m - 1 + \nu_m > \mu_{m+1} + \nu_m$ and so $\sigma_{2(m-1)}$ and σ_{2m-1} are the last two consecutive parts of size $\lambda_m - 1$, and so in that case we have

$$\sigma' = (\mu_1 + \nu_1, \mu_2 + \nu_1, \ldots, \mu_{m-1} + \nu_{m-1}, \mu_m - 2 + \nu_{m-1}, \mu_m - 2 + \nu_m, \mu_{m+1} + \nu_m, \ldots).$$

Therefore $\text{eType}(v+Y, x_{|Y^\perp/Y})$ is obtained from $\text{eType}(v+W, x_{|W^\perp/W})$ by decreasing $\mu'_m = \mu_m - 1$ by 1. Thus we have

$$\text{eType}(v, x) \xrightarrow{\text{decrease } \mu_m \text{ by } 1} \text{eType}(v + W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \mu'_m \text{ by } 1} \text{eType}(v + Y, x_{|Y^\perp/Y}).$$

Case 1(c): Suppose that $\mu_m - 1 = \mu_{m+1}$, and $\max \Gamma_{m+1} > \max \Delta_m$, then the last two consecutive parts of σ of size $\lambda_m - 1$ are

$$\sigma_{2\max\Delta_m - 1} = \mu_{\max\Delta_m} + \nu_{\max\Delta_m} \quad \text{and} \quad \sigma_{2\max\Delta_m} = \mu_{\max\Delta_m + 1} + \nu_{\max\Delta_m}.$$

Therefore in this case $\text{eType}(v+Y, x_{|Y^\perp/Y})$ is obtained from $\text{eType}(v+W, x_{|W^\perp/W})$ by decreasing $\nu_{\max\Delta_m}$ by 1. Thus we have

$$\text{eType}(v, x) \xrightarrow{\text{decrease } \mu_m \text{ by } 1} \text{eType}(v + W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \nu_{\max\Delta_m} \text{ by } 1} \text{eType}(v + Y, x_{|Y^\perp/Y}).$$

Case 1(d): Suppose that $\mu_m - 1 = \mu_{m+1}$, and $\max \Gamma_{m+1} \leq \max \Delta_m$. Then the last two consecutive parts of σ of size $\lambda_m - 1$ are:

$$\sigma_{2(\max\Gamma_{m+1}-1)} = \mu_{\max\Gamma_{m+1}} + \nu_{\max\Gamma_{m+1}-1} \quad \text{and} \quad \sigma_{2\max\Gamma_{m+1}-1} = \mu_{\max\Gamma_{m+1}} + \nu_{\max\Gamma_{m+1}}.$$

Therefore $\mathrm{eType}(v+Y, x_{|Y^\perp/Y})$ is obtained from $\mathrm{eType}(v+W, x_{|W^\perp/W})$ by decreasing $\mu_{\max\Gamma_{m+1}}$ by 1. Thus we have

$$\mathrm{eType}(v,x) \xrightarrow{\text{decrease } \mu_m \text{ by } 1} \mathrm{eType}(v+W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \mu_{\max\Gamma_{m+1}} \text{ by } 1} \mathrm{eType}(v+Y, x_{|Y^\perp/Y}).$$

□

4.2.2. $\mu'_m = \mu_m - 1, \nu' = \nu$ *and* $\nu_{m-1} > \nu_m$. In this case Remark 4.2 forces us to have the coefficients of v^*_{m,λ_m}, namely β_m and δ_m to be 0, and so a generic w and $\mathtt{x}$ have the form

$$w = \sum_{i=1}^{m-1}(\alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i}) + \alpha_m v_{m,1} \quad \text{and} \quad \mathtt{x} = \sum_{i=1}^{m-1}(\gamma_i v_{i,1} + \delta_i v^*_{i,\lambda_i}) + \gamma_m v_{m,1}$$

with $\alpha_m \neq 0 \neq \gamma_m$. Note it is possible for $\nu_m = 0$ as we only require that $\nu_{m-1} > \nu_m$.

Proposition 4.14. Let W and X be generic points in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$. Then maximal k_2 such that $X \subset x^{k_2-1}(W^\perp) + \mathbb{C}[x]v + W$ is λ_{m-1}.

Proof. By Proposition 4.10, we know that λ_{m-1} is the largest integer such that $X \subset x^{\lambda_{m-1}-1}(W^\perp) + W$, so $k_2 \geq \lambda_{m-1}$. Also by the proof of Proposition 4.10, we know that the set of vectors $\{v_{i,1}, v^*_{i,\lambda_i} \,|\, 1 \leq i \leq m-1\} \subset x^{\lambda_{m-1}-1}(W^\perp)$ whereas the set of vectors $\{v_{i,1}, v^*_{i,\lambda_i} \,|\, i \in \Lambda_{m-1}\} \cup \{v_{1,m}\} \not\subseteq x^{\lambda_m}(W^\perp) + W$. It once again follows that the set of vectors $\{v_{i,1}, v^*_{i,\lambda_i} \,|\, i \in \Lambda_{m-1}\} \cup \{v_{1,m}\}$ are not contained in $x^{\lambda_m}(W^\perp) + \mathbb{C}[x]v + W$ since we cannot obtain the individual vectors $\{v_{i,1}\}$ for $i \in \Lambda_{m-1}$ and the vector $\{v_{1,m}\}$ by adding the subspace $\mathbb{C}[x]v$ and so we cannot then obtain the individual vectors v^*_{i,λ_i} for $i \in \Lambda_{m-1}$ either. Therefore $X \not\subseteq x^{\lambda_m}(W^\perp) + W$ and $k_2 = \lambda_{m-1}$ completing the proof. □

Proposition 4.15. Let W and X be generic points in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$. Then the maximal l_2 such that

$$X^\perp \supseteq (x^{l_2-1})^{-1}(\mathbb{C}[x]v + X + W) \cap W^\perp$$

is $\mu_m + \nu_{m-1} - 1$.

Proof. Consider the vector

$$u := x^{\mu_m-1}(\alpha_m v) - w = \sum_{i=1}^{m-1} \alpha_m v_{i,\mu_i-\mu_m+1} - (\alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i}) \in \mathbb{C}[x]v + X + W.$$

Observe that this vector is not supported on $v_{m,1}$. We define

$$u' := \sum_{i=1}^{m-1} \alpha_m v_{i,\mu_i+\nu_{m-1}} - (\alpha_i v_{i,\mu_m+\nu_{m-1}} + \beta_i v^*_{i,\lambda_i-\mu_m-\nu_{m-1}+1})$$

and observe that $x^{\mu_m+\nu_{m-1}-1}(u') = u$. Now, we let

$$u'' := u' - \frac{\langle w, u'\rangle}{\alpha_m} v^*_{m,1}$$

such that $x^{\mu_m+\nu_{m-1}-1}(u'')=u$ and

$$\langle w,u''\rangle=\langle w,u'\rangle-\frac{\langle w,u'\rangle}{\alpha_m}\langle w,v^*_{m,1}\rangle=\langle w,u'\rangle\left(1-\frac{\alpha_m}{\alpha_m}\right)=0$$

so $u''\in W^\perp$. Therefore $u''\in(x^{\mu_m+\nu_{m-1}-1})^{-1}(\mathbb{C}[x]v+X+W)$.

However, an easy calculation shows that

$$\langle \mathtt{x},u''\rangle=\begin{cases}\sum\limits_{i\in\Lambda_{m-1}}\alpha_m\delta_i-\gamma_m\beta_i & \text{if}\quad \mu_{m-1}>\mu_m,\\ \sum\limits_{i\in\Gamma_m\cap\Delta_{m-1}}(\alpha_i-\alpha_m)\delta_i+(\gamma_m-\gamma_i)\beta_i & \text{if}\quad \mu_{m-1}=\mu_m,\end{cases}$$

which is generically non-zero. Hence $u''\notin X^\perp$ and so $X^\perp\not\supseteq(x^{\mu_m+\nu_{m-1}-1})^{-1}(\mathbb{C}[x]v+X+W)\cap W^\perp$.

It remains to show that $X^\perp\supseteq(x^{\mu_m+\nu_{m-1}-2})^{-1}(\mathbb{C}[x]v+X+W)\cap W^\perp$. Since $\mu_m+\nu_{m-1}-2\leq\mu_{m-1}+\nu_{m-1}-2$, the only vector in $\ker(x^{\mu_{m-1}+\nu_{m-1}-2})$ with non-zero inner product with $\mathtt{x}$ is $v^*_{m,1}$. However this vector is not contained in $W^\perp$ and so $X^\perp\supseteq\ker(x^{\mu_{m-1}+\nu_{m-1}-2})\cap W^\perp$. Finally let $T:=\mathbb{C}(\{v_{i,\lambda_i},v^*_{i,1}\,|\,1\leq i\leq m-1\}\cup\{v^*_{m,1}\})$; this is the span of the basis elements that have non-zero inner product with w and $\mathtt{x}$. Then any vector in V that maps onto u under $x^{\mu_{m-1}+\nu_{m-1}-2}$ can easily be seen not to be supported on $T\cap W^\perp$ and so we have shown that $X^\perp\supseteq(x^{\mu_m+\nu_{m-1}-2})^{-1}(\mathbb{C}[x]v+X+W)\cap W^\perp$. Therefore $l_2=\mu_m+\nu_{m-1}-1$. □

of Theorem 4.6: Case 1(a). We now explicitly describe which parts of σ are removed to obtain σ'. Since $\nu_{m-1}>\nu_m$, the last two consecutive parts of σ of sizes $k_2=\lambda_{m-1}=\mu_{m-1}+\nu_{m-1}$ and $l_2=\mu_m+\nu_{m-1}-1$ are $\sigma_{2(m-1)-1}$ and $\sigma_{2(m-1)}$. Therefore σ' is obtained from σ by decreasing ν_{m-1} by 1; explicitly we have

$$\sigma'=(\mu_1+\nu_1,\mu_2+\nu_1,\ldots,\mu_{m-1}+\nu_{m-1}-1,\mu_m-2+\nu_{m-1},\mu_m-1+\nu_m,\mu_{m+1}+\nu_m,\ldots)$$

and so

$$\mathrm{eType}(v,x)\xrightarrow{\text{decrease }\mu_m\text{ by }1}\mathrm{eType}(v+W,x_{|W^\perp/W})\xrightarrow{\text{decrease }\nu_{m-1}\text{ by }1}\mathrm{eType}(v+Y,x_{|Y^\perp/Y}).$$

□

4.3. Box Removed from the Right.

4.3.1. $\mu'=\mu,\nu'_m=\nu_m-1$, $\mu_m>\mu_{m+1}$. Throughout this section, the maximal l_2 is λ_m-1 as given in Proposition 4.11.

Proposition 4.16. Suppose $\mathrm{eType}(v+W,x_{|W^\perp/W})$ is obtained from (μ,ν) by decreasing ν_m by 1, $\mu_m>\mu_{m+1}$, and $\nu_{m-1}=\nu_m$. Then the maximal k_2 such that $X\subseteq x^{k_2-1}(W^\perp)+\mathbb{C}[x]v+W$ is $\lambda_m=\mu_m+\nu_m=\mu_m+\nu_{m-1}$.

Proof. In this case we are required to have $\sum_{i\in\Delta_{\leq m}}\beta_i\neq 0$ for otherwise $\mathrm{eType}(v+W,x_{|W^\perp/W})$ would be obtained from (μ,ν) by decreasing μ_m by 1. It is clear that the vectors $v_{i,1},v_{i,\lambda_i^*}$ are contained in $x^{\lambda_m-1}(W^\perp)$ for all $i\notin\Delta_m$ and that the vectors $v_{i,1},v_{i,\lambda_i^*}$ are not contained in $x^{\lambda_m-1}(W^\perp)$ for all $i\in\Delta_m\setminus\Gamma_m$. We show that $v_{i,1}$ are contained in $x^{\lambda_m-1}(W^\perp)+\mathbb{C}[x]v$ for all $i\in\Delta_m$.

As in Proposition 4.13, we have $\beta_m v_{i,\lambda_i}-\beta_i v_{m,\lambda_m}\in W^\perp$ for $i\in\Delta_m$, and so

$$\beta_m v_{i,\mu_i-\mu_m+1}-\beta_i v_{m,1}=x^{\lambda_m-1}(\beta_m v_{i,\lambda_i}-\beta_i v_{m,\lambda_m})$$

is contained in $x^{\lambda_m-1}(W^\perp)$ for all $i \in \Delta_m$. In this case however, we must have that

$$v' = x^{\mu_m-1}(v) - \sum_{i=1}^{\min\Delta_m-1} v_{i,\mu_i-\mu_m+1} = \sum_{i\in\Delta_{\leq m}} v_{i,\mu_i-\mu_m+1}$$

lies outside the span of the vectors $\beta_m v_{i,\mu_i-\mu_m+1} - \beta_i v_{m,1}$ for $i \in \Delta_{\leq m}$ since $\sum_{i\in\Delta_{\leq m}} \beta_i \neq 0$. Therefore we have $\mathbb{C}\{v_{i,1} \,|\, i \in \Delta_m\} = \mathbb{C}\{\beta_m v_{i,\mu_i-\mu_m+1} - \beta_i v_{m,1} \,|\, i \in \Delta_m\} \oplus \mathbb{C}v'$ and so we have

$$U := \mathbb{C}\{v_{i,1}, v^*_{j,\lambda_j} \,|\, 1 \leq i \leq m,\, 1 \leq j \leq \min\Delta_m - 1\} \subseteq x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v.$$

We now show that the vectors v^*_{i,λ_i} for $i \in \Delta_m$ are contained in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v + W$. We have that $\alpha_m v^*_{i,1} - \alpha_i v^*_{m,1} \in W^\perp$ and so

$$\alpha_m v^*_{i,\lambda_i-\lambda_m+1} - \alpha_i v^*_{m,\lambda_m} = x^{\lambda_m-1}(\alpha_m v^*_{i,1} - \alpha_i v^*_{m,1}) \in x^{\lambda_m-1}(W^\perp).$$

Therefore we have

$$U' := \mathbb{C}\{v_{k,1}, v^*_{j,\lambda_j}, \alpha_m v^*_{i,\lambda_i-\lambda_m+1} - \alpha_i v^*_{m,\lambda_m} \,|\, 1 \leq k \leq m,\, 1 \leq j \leq \min\Delta_m - 1,\, i \in \Delta_m\}$$

is contained in $x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v$. Hence $U' + W \subseteq x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v + W$. Now since W is generically chosen in $\mathcal{B}^{(\mu,\nu)}_{(\mu',\nu')}$, we have that $w' := \sum_{i\in\Delta_m} \beta_i v^*_{i,\lambda_i}$, the part of w that is not supported on U, is not contained in $\mathbb{C}\{\alpha_m v^*_{i,\lambda_i-\lambda_m+1} - \alpha_i v^*_{m,\lambda_m} \,|\, i \in \Delta_m\}$. Therefore by counting dimensions we have

$$\mathbb{C}\{\alpha_m v^*_{i,\lambda_i-\lambda_m+1} - \alpha_i v^*_{m,\lambda_m} \,|\, i \in \Delta_m\} \oplus \mathbb{C}w' = \mathbb{C}\{v^*_{i,\lambda_i} \,|\, i \in \Delta_m\}$$

and so we have

$$X \subseteq \mathbb{C}\{v_{i,1}, v^*_{i,\lambda_i} \,|\, 1 \leq i \leq m\} = U' + W \subseteq x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v + W = x^{\lambda_m-1}(W^\perp) + \mathbb{C}[x]v$$

since $W \subset x^{\lambda_m-1}(W^\perp)$.

By repeating the argument at the end of the proof in Proposition 4.13, this shows that $k_2 = \lambda_m$ is maximal and so we are done. □

Proposition 4.17. Suppose eType$(v + W, x_{|W^\perp/W})$ is obtained from (μ,ν) by decreasing ν_m by 1, $\mu_m > \mu_{m+1}$, and $\nu_{m-1} > \nu_m$. If $m = 1$, then $X \subset \mathbb{C}[x]v + W$. Otherwise, if $m > 1$, the maximal k_2 such that $X \subseteq x^{k_2-1}(W^\perp) + \mathbb{C}[x]v + W$ is $k_2 = \mu_m + \nu_{m-1}$.

Proof. We first deal with the case $m = 1$. In this case, the condition $\nu_{m-1} > \nu_m \neq 0$ should be ignored. Then our vectors w and $\mathtt{x}$ have the form

$$w = \alpha v_{11} + \beta v^*_{1,\lambda_1} \quad \text{and} \quad \mathtt{x} = \gamma v_{11} + \delta v^*_{1,\lambda_1}.$$

Moreover, since $\max\Gamma_m = \max\Delta_m = 1$, we have $v_{11} = x^{\mu_1-1}(v)$ and so it follows that $X \subset \operatorname{span}\{x^{\mu_1-1}(v), w\} \subset \mathbb{C}[x]v + W$. Therefore there is no maximal k_2.

Now suppose that $m > 1$. Clearly the vectors $v_{i,1}$ and v^*_{i,λ_i} are all contained in $x^{\mu_m+\nu_{m-1}-1}(W^\perp)$ for $1 \leq i \leq \min\Lambda_{m-1} - 1$. We now show that the vectors $v_{i,\mu_i+\mu_m+1}, v^*_{i,\lambda_i}$ are contained in $x^{\mu_m+\nu_{m-1}-1}(W^\perp)$ for $i \in \Delta_{m-1}$. Using the vectors $v_{m,1}$ and v^*_{m,λ_m} as pivots we see that the vectors $\beta_m v_{i,\lambda_i} - \beta_i v_{m,\lambda_m}$ and $\alpha_m v^*_{i,1} - \alpha_i v^*_{m,1}$ are contained in $W^\perp$ for $i \in \Delta_{m-1}$. Hence

$$\beta_m v_{i,\mu_i-\mu_m+1} = x^{\mu_m+\nu_{m-1}-1}(\beta_m v_{i,\lambda_i} - \beta_i v_{m,\lambda_m}) \in x^{\mu_m+\nu_{m-1}-1}(W^\perp)$$

and

$$\alpha_m v^*_{i,\lambda_i} = x^{\mu_m+\nu_{m-1}-1}(\alpha_m v^*_{i,1} - \alpha_i v^*_{m,1}) \in x^{\mu_m+\nu_{m-1}-1}(W^\perp)$$

for $i \in \Delta_{m-1}$. Now $x^{\mu_m-1}(v) = \sum_{i=1}^m v_{i,\mu_i-\mu_m+1}$ and since $v_{i,\mu_i-\mu_m+1} \in x^{\mu_m+\nu_{m-1}-1}(W^\perp)$ for all $1 \le i \le m-1$, we conclude that $v_{m,1} \in x^{\mu_m+\nu_{m-1}-1}(W^\perp) + \mathbb{C}[x]v$ as well. Hence

$$\mathbb{C}\{v_{i,1}, v^*_{i,\lambda_i} \,|\, 1 \le i \le m-1\} \oplus \mathbb{C}\{v_{m,1}\} \subset x^{\mu_m+\nu_{m-1}-1}(W^\perp) + \mathbb{C}[x]v. \tag{4.5}$$

Now adding W to each side of (4.5), we see that

$$\begin{aligned}\mathbb{C}\{v_{i,1}, v^*_{i,\lambda_i} \,|\, 1 \le i \le m\} &= \mathbb{C}\{v_{i,1}, v^*_{i,\lambda_i} \,|\, 1 \le i \le m-1\} \oplus \mathbb{C}\{v_{m,1}\} + W \\ &\subseteq x^{\mu_m+\nu_{m-1}-1}(W^\perp) + \mathbb{C}[x]v + W,\end{aligned}$$

and so $X \subseteq x^{\mu_m+\nu_{m-1}-1}(W^\perp) + \mathbb{C}[x]v + W$. Therefore $k_2 \ge \mu_m + \nu_{m-1}$. But it is also clear that $\mu_m + \nu_{m-1} - 1$ is the highest power of x such that $v_{i,\mu_i-\mu_m+1}$ is contained in $x^{\mu_m+\nu_{m-1}-1}(W^\perp)$ for $i \in \Delta_{m-1}$, and without these vectors, we cannot obtain the vector $v_{m,1} \in x^{\mu_m+\nu_{m-1}-1}(W^\perp) + \mathbb{C}[x]v$. Hence $k_2 = \mu_m + \nu_{m-1}$ and we are done. □

of Theorem 4.6: Case 2(a). We now explicitly determine eType$(v+Y, x_{|Y^\perp/Y})$ by determining $\sigma' = \mathrm{Type}(x, Y^\perp/(\mathbb{C}[x]v + Y))$. We have

$$\begin{aligned}\sigma &= \mathrm{Type}(x, W^\perp/(\mathbb{C}[x]v + W)) \\ &= (\mu_1+\nu_1, \mu_2+\nu_1, \dots, \mu_m+\nu_{m-1}, \mu_m+\nu_m-1, \mu_{m+1}+\nu_m-1, \mu_{m+1}+\nu_{m+1}, \dots)\end{aligned}$$

Suppose that $m > 1$, then by Propositions 4.11 and either 4.16 or 4.17 (depending on whether $\nu_{m-1} = \nu_m$ or $\nu_{m-1} > \nu_m$) we have that $l_2 = \lambda_m - 1 = \mu_m + \nu_m - 1$ and $k_2 = \mu_m + \nu_{m-1}$. Observe that $\nu_{m-1} > \nu_m - 1$. Therefore the parts of σ of size k_2 and l_2 are:

$$\sigma_{2(m-1)} = \mu_m + \nu_{m-1} = \lambda_m \quad \text{and} \quad \sigma_{2m-1} = \mu_m + \nu_m - 1 = \lambda_m - 1.$$

It follows that σ' is obtained from σ by decreasing μ_m by 1. Thus we have

$$\mathrm{eType}(v,x) \xrightarrow{\text{decrease } \nu_m \text{ by } 1} \mathrm{eType}(v+W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \mu_m \text{ by } 1} \mathrm{eType}(v+Y, x_{|Y^\perp/Y}).$$

Now suppose that $m = 1$. Then Propositions 4.11 and 4.17 tell us that σ' is obtained by reducing the last part of σ of size $\lambda_1 - 1$ by 1. Since $\max \Gamma_1 = \max \Delta_1$, this part must be $\sigma_1 = \lambda_m - 1 = \mu_1 + \nu_1 - 1$. Therefore σ' is obtained from σ by decreasing μ_1 by 1 and so in this case we have:

$$\mathrm{eType}(v,x) \xrightarrow{\text{decrease } \nu_1 \text{ by } 1} \mathrm{eType}(v+W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \mu_1 \text{ by } 1} \mathrm{eType}(v+Y, x_{|Y^\perp/Y}).$$

□

4.3.2. $\mu' = \mu, \nu'_m = \nu_m - 1$ *and* $\mu_m = \mu_{m+1}$.

Proposition 4.18. Suppose eType$(v+W, x_{|W^\perp/W})$ is obtained from (μ, ν) be decreasing ν_m by 1 and $\mu_m = \mu_{m+1}$. Then the maximal k_2 such that $X \subseteq x^{k_2-1}(W^\perp) + \mathbb{C}[x]v + W$ is $\lambda_m - 1$.

Proof. In this case we can assume generically that the β_i and δ_i are non-zero, and Proposition 4.10 applies to tell us that $k_2 \ge \lambda_m - 1$. Let $S = \mathrm{span}\{v_{i,1},\ v^*_{i,\lambda_i} \mid i \in \Gamma_{\ge m}\}$, since $\mu_m = \mu_{m+1}$, we have $\dim(S) \ge 4$, unless $\mu_m = 0 = \nu_{m+1}$, in which case we can say that $\dim(S) \ge 2$. If $k \ge \lambda_m$, then $x^{k-1}(W^\perp) \cap S = 0$. Then, to have $X \subseteq x^{k-1}(W^\perp) + \mathbb{C}[x]v + W$ we would need $x' = \sum_{i\in\Gamma_{\ge m}} \gamma_i v_{i,1} + \delta_i v^*_{i,\lambda_i}$ to be contained in the span of $w' = \sum_{i\in\Gamma_{\ge m}} \alpha_i v_{i,1} + \beta_i v^*_{i,\lambda_i}$ (coming from W) and $v' = \sum_{i\in\Gamma_{\ge m}} v_{i,1}$ (coming from $\mathbb{C}[x]v$ and only in the case when $\mu_m > 0$). However, if $\mu_m > 0$, inside a space S that is at least 4 dimensional, generically x' will not be contained in a 2 dimensional subspace with no additional conditions. In the same way, if $\mu_m = 0$, inside the space S that has dimension at least 2, x' will not be contained in the span of w'. This proves that $k_2 < \lambda_m$, so indeed $k_2 = \lambda_m - 1$. □

of Theorem 4.6: Case 2(b), (c), (d). **Case 2(b):** First suppose that $\nu_m - 1 > \nu_{m+1}$. By Propositions 4.11 and 4.18 we have $k_2 = l_2 = \lambda_m - 1$ and the last two parts σ of this size are

$$\sigma_{2m-1} = \mu_m + \nu_m - 1 \quad \text{and} \quad \sigma_{2m} = \mu_{m+1} + \nu_m - 1.$$

Therefore it follows that σ' is obtained from σ by decreasing $\nu'_m = \nu_m - 1$ by 1. Thus we have

$$\mathrm{eType}(v, x) \xrightarrow{\text{decrease } \nu_m \text{ by } 1} \mathrm{eType}(v + W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \nu'_m \text{ by } 1} \mathrm{eType}(v + Y, x_{|Y^\perp/Y}).$$

Now suppose that $\nu_m - 1 = \nu_{m+1}$. Then $\lambda_m - 1 = \mu_m + \nu_m - 1 = \mu_{m+1} + \nu_{m+1} = \lambda_{m+1}$.

Case 2(c): Suppose $\nu_m - 1 = \nu_{m+1}$, $\max \Gamma_m \leq \max \Delta_{m+1}$. Then the last two parts of σ of size $\lambda_m - 1$ are:

$$\sigma_{2(\max \Gamma_m - 1)} = \mu_{\max \Gamma_m} + \nu_{\max \Gamma_m - 1} \quad \text{and} \quad \sigma_{2 \max \Gamma_m} = \mu_{\max \Gamma_m} + \nu_{\max \Gamma_m}.$$

Note that it is possible for $\nu_{\max \Gamma_m} = 0$ but $\nu_{\max \Gamma_m - 1} \neq 0$. In any case we obtain σ' from σ be decreasing $\mu_{\max \Gamma_m}$ by 1. We have

$$\mathrm{eType}(v, x) \xrightarrow{\text{decrease } \nu_m \text{ by } 1} \mathrm{eType}(v + W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \mu_{\max \Gamma_m} \text{ by } 1} \mathrm{eType}(v + Y, x_{|Y^\perp/Y}).$$

Case 2(d): Suppose $\nu_m - 1 = \nu_{m+1}$, $\max \Gamma_m > \max \Delta_{m+1}$. Then the last two parts of σ of size λ_{m+1} are

$$\sigma_{2 \max \Delta_{m+1} - 1} = \mu_{\max \Delta_{m+1}} + \nu_{\max \Delta_{m+1}} \quad \text{and} \quad \sigma_{2 \max \Delta_{m+1}} = \mu_{\max \Delta_{m+1} + 1} + \nu_{\max \Delta_{m+1}},$$

and so σ' is obtained from σ by decreasing $\nu_{\max \Delta_{m+1}}$ by 1. Note in the case where $\mu_m = 0$ we have $\mu_{m+1} = 0$ as well and so $\mu_{\max \Delta_{m+1}} = \mu_{\max \Delta_{m+1} + 1} = 0$. We have

$$\mathrm{eType}(v, x) \xrightarrow{\text{decrease } \nu_m \text{ by } 1} \mathrm{eType}(v + W, x_{|W^\perp/W}) \xrightarrow{\text{decrease } \nu_{\max \Delta_{m+1}} \text{ by } 1} \mathrm{eType}(v + Y, x_{|Y^\perp/Y}).$$

□

5. Proof of Main Theorem

5.1. **Setup.** In this subsection we go over the notation needed for the proof of Theorem 3.10.

Definition 5.1. Given $(v, x) \in \mathbb{O}_{(\mu,\nu)}$, we say that a flag $F_\bullet \in \mathcal{C}_{(v,x)}$ is 'good' if the sequence of bipartitions $\Phi(F_\bullet)$ below is a nested sequence (that is, one box is removed at each stage):

$$\Phi(F_\bullet) = (\mathrm{eType}(v, x), \mathrm{eType}(v + F_1, x|_{F_1^\perp/F_1}), \mathrm{eType}(v + F_2, x|_{F_2^\perp/F_2}), \ldots).$$

Notice that by Theorem 2.13 the set of good flags is open dense in $\mathcal{C}_{(v,x)}$.

Definition 5.2. Let $F_\bullet, G_\bullet \in \mathcal{C}_{(v,x)}$ be two good flags:

$$\begin{aligned} F_\bullet &= (0 \subset F_1 \subset F_2 \subset \ldots \subset F_2^\perp \subset F_1^\perp \subset \mathbb{C}^{2n}), \\ G_\bullet &= (0 \subset G_1 \subset G_2 \subset \ldots \subset G_2^\perp \subset G_1^\perp \subset \mathbb{C}^{2n}), \end{aligned}$$

We define two flags in the smaller exotic fibre $\mathcal{C}_{(\overline{v},\overline{x})}$ where $\overline{x} = x|_{G_1^\perp/G_1}$ and $\overline{v} = v + G_1 \in G_1^\perp/G_1$.

$$\widetilde{F_\bullet} = \left(\ldots \subset \frac{G_1 + F_i \cap G_1^\perp}{G_1} \subset \ldots \right)_{i=0}^{2n}, \tag{5.1}$$

$$\widetilde{G_\bullet} = (0 = G_1/G_1 \subset G_2/G_1 \subset \ldots \subset G_2^\perp/G_1 \subset G_1^\perp/G_1 \cong \mathbb{C}^{2(n-1)}). \tag{5.2}$$

Notice that the flag $\widetilde{F_\bullet}$ will have redundancies in two places, that is, there are two numbers k such that $\widetilde{F}_k = \widetilde{F}_{k+1}$. If $F_n \subset G_1^\perp$ (or, equivalently, if $F_n \supset G_1$) we call this a Type 1 redundancy.

Let r be minimal such that $F_{n+r} \not\subset G_1^\perp$. It follows then that $F_{n+r-1} = F_{n+r} \cap G_1^\perp$ and also that $G_1 + F_{n-r} = F_{n-r+1}$. In this case, the flag $\widetilde{F_\bullet}$ looks like:

$$\widetilde{F_\bullet} = \left(0 \subset \frac{G_1+F_1}{G_1} \subset \dots \subset \frac{G_1+F_{n-r-1}}{G_1} \subset \frac{F_{n-r+1}}{G_1} \subset \dots \subset \frac{F_{n+r-1}}{G_1} \subset \frac{F_{n+r+1}\cap G_1^\perp}{G_1} \subset \dots \subset \frac{G_1^\perp}{G_1}\right).$$

In this flag the indices $n+r$ and $n-r$ are missing.

If $F_n \not\subset G_1^\perp$ ($F_n \not\supset G_1$), we call this a Type 2 redundancy. Let r be minimal such that $F_{n+r} \supset G_1$, we have $F_{n+r-1} + G_1 = F_{n+r}$ and $F_{n-r-1} = F_{n-r} \cap G_1^\perp$, and:

$$\begin{aligned}\widetilde{F_\bullet} = (0 \subset \frac{G_1+F_1}{G_1} \subset \dots \subset \frac{G_1+F_{n-r-1}}{G_1} \subset \frac{G_1+F_{n-r+1}\cap G_1^\perp}{G_1} \subset \dots \\ \subset \frac{G_1+F_{n+r-1}\cap G_1^\perp}{G_1} \subset \frac{F_{n+r+1}\cap G_1^\perp}{G_1} \subset \dots \subset \frac{G_1^\perp}{G_1}).\end{aligned}$$

Again the indices $n+r$ and $n-r$ are missing from the labelling. Notice that $\widetilde{F_\bullet}$, as just defined, is not necessarily a good flag.

Remark 5.3. For a fixed flag $G_\bullet \in \mathcal{C}_{(v,x)}$, consider the map

$$\pi_G : \mathcal{C}_{(v,x)} \longrightarrow \mathcal{C}_{(\bar{v},\bar{x})}; \qquad \pi_G(F_\bullet) = \widetilde{F_\bullet},$$

then π_G is surjective, so $\pi_G^{-1}\left(\mathcal{C}^\circ_{(\bar{v},\bar{x})}\right)$ is dense in $\mathcal{C}_{(v,x)}$, by Theorem 2.13. This shows that, for a generic choice of $F_\bullet$ and $G_\bullet$, the flag $\widetilde{F_\bullet}$ will also be a good flag. If $\widetilde{F_\bullet}$ is a good flag, then by keeping track of the redundancies in the labelling, we can say that the nested sequence of bipartitions $\Phi(\widetilde{F_\bullet})$ is a bitableau that satisfies the increasing (standard) condition, containing the numbers $1,\dots,n$ excluding r.

Recall from Definition 2.18 that if $w = w(F_\bullet, G_\bullet)$, then we can choose an orthonormal basis $\{v_1,\dots,v_n,v_{\bar{n}},\dots,v_{\bar{1}}\}$ (that is, so that $\langle v_i, v_{\bar{j}}\rangle = \delta_{i,j} = -\langle v_{\bar{j}}, v_i\rangle$, $\langle v_i, v_j\rangle = \langle v_{\bar{i}}, v_{\bar{j}}\rangle = 0$), with

$$F_i = \begin{cases}\mathbb{C}\{v_n,\dots,v_{n-i+1}\}, & \text{if } 1 \le i \le n\\ \mathbb{C}\{v_n,\dots,v_1,v_{\bar{1}},\dots,v_{\overline{i-n}}\}, & \text{if } n+1 \le i \le 2n\end{cases}$$

and

$$G_j = \begin{cases}\mathbb{C}\{v_{w(n)},\dots,v_{w(n-j+1)}\}, & \text{if } 1 \le j \le n\\ \mathbb{C}\{v_{w(n)},\dots,v_{w(1)},v_{w(\bar{1})},\dots,v_{w(\overline{j-n})}\}, & \text{if } n+1 \le j \le 2n.\end{cases}$$

Using this basis we can explicitly describe the flags $\widetilde{G}_\bullet$ and $\widetilde{F}_\bullet$, while being careful regarding the type of redundancy that determines $\widetilde{F}_\bullet$. Firstly, we have

$$\widetilde{G}_j = \begin{cases}\mathbb{C}\{v_{w(n-1)},\dots,v_{w(n-j)}\}, & \text{if } 1 \le j \le n-1\\ \mathbb{C}\{v_{w(n-1)},\dots,v_{w(1)},v_{w(\bar{1})},\dots,v_{w(\overline{j-n+1})}\}, & \text{if } n \le j \le 2n-2.\end{cases} \tag{5.3}$$

where, by abuse of notation, we regard the basis elements as their images in the quotient space, and we will continue this convention in what follows.

Now for $\widetilde{F}_\bullet$ we had $\widetilde{F}_i = \frac{G_1+F_i\cap G_1^\perp}{G_1}$ for each i. In the Type 1 redundancy we had $F_n \subseteq G_1^\perp$ and chose r minimal such that $F_{n+r} \not\subseteq G_1^\perp$. Therefore

$$F_{n+r} = \mathbb{C}\{v_n,\dots,v_1,v_{\bar{1}},\dots,v_{\bar{r}}\} \not\subseteq G_1^\perp \quad \text{but} \quad F_{n+r-1} = \mathbb{C}\{v_n,\dots,v_1,v_{\bar{1}},\dots,v_{\overline{r-1}}\} \subseteq G_1^\perp,$$

so this implies that $w(\bar{n}) = \bar{r}$, or equivalently $w(n) = r$. For the Type 2 redundancy, we chose r minimal such that $F_{n+r} \supseteq G_1$, so in this case:

$$F_{n+r} = \mathbb{C}\{v_n, \ldots, v_1, v_{\bar{1}}, \ldots, v_{\bar{r}}\} \supseteq G_1 = \mathbb{C}\{v_{w(n)}\}, \quad \text{but}$$
$$F_{n+r-1} = \mathbb{C}\{v_n, \ldots, v_1, v_{\bar{1}}, \ldots, v_{\overline{r-1}}\} \not\supseteq G_1,$$

which implies $w(n) = \bar{r}$. Explicitly, in either case, we have $w(n) \in \{r, \bar{r}\}$ and:

$$\widetilde{F}_i = \begin{cases} \mathbb{C}\{v_n, \ldots, \hat{v}_r, \ldots, v_{n-i+1}\} & \text{if } 1 \le i \le n,\ i \ne n+1-r \\ \mathbb{C}\{v_n, \ldots, \hat{v}_r, \ldots, v_1, v_{\bar{1}}, \ldots, \hat{v}_{\bar{r}}, \ldots, v_{\overline{i-n}}\} & \text{if } n+1 \le i \le 2n,\ i \ne n+r \end{cases} \tag{5.4}$$

where $\hat{v}_p$ denotes omission of v_p.

Definition 5.4. Let $F_\bullet \in \mathcal{C}_{(v,x)}$ be a good flag, with $\Phi(F_\bullet) = T$. By Theorem 2.13, for any s with $1 \le s \le n-1$, we have a bijective map

$$\Phi_s : \operatorname{Irr} \mathcal{C}_{(v+F_{n-s}, x_{|F_{n+s}/F_{n-s}})} \longrightarrow \mathcal{T}(\epsilon^{(s)}, \eta^{(s)}) \tag{5.5}$$

where $(\epsilon^{(s)}, \eta^{(s)}) = \mathrm{eType}(v + F_{n-s}, x_{|F_{n+s}/F_{n-s}})$. We define the *s-truncated flag*

$$F^s_\bullet := \left(0 = \frac{F_{n-s}}{F_{n-s}} \subset \frac{F_{n-s+1}}{F_{n-s}} \subset \ldots \subset \frac{F_{n+s-1}}{F_{n-s}} \subset \frac{F_{n+s}}{F_{n-s}} \cong \mathbb{C}^{2s}\right)$$

In terms of our basis we can also write

$$F^s_i = \begin{cases} \mathbb{C}\{v_s, \ldots, v_{n-i+1}\} & \text{if } 1 \le i \le s, \\ \mathbb{C}\{v_s, \ldots, v_1, v_{\bar{1}}, \ldots, v_{\overline{i-s}}\} & \text{if } s+1 \le i \le 2s. \end{cases}$$

We have that $F^s_\bullet \in \mathcal{C}_{(v+F_{n-s}, x_{|F_{n+s}/F_{n-s}})}$ is a good flag, and that $\Phi_s(F^s_\bullet) = T_s$, the bitableau obtained from T by only considering the number 1 up to s (see Definition 2.5).

Remark 5.5. Using the above notation, the flag defined in (5.2) is a truncated flag $\widetilde{G}_\bullet = G^{n-1}_\bullet$, so if $\Phi(G_\bullet) = R$, then $\Phi_{n-1}(\widetilde{G}_\bullet) = \Phi_{n-1}(G^{n-1}_\bullet) = R_{n-1}$.

Definition 5.6. Let $F_\bullet$, $G_\bullet \in \mathcal{C}_{(v,x)}$ be two good flags such that $\widetilde{F}_\bullet \in \mathcal{C}_{(\bar{v},\bar{x})}$ is also a good flag. If $\Phi(F_\bullet) = T$, we define $\widetilde{T}$ to be the standard bitableau obtained by relabelling the entries of $\Phi(\widetilde{F_\bullet})$ with numbers from 1 up to n with r missing, as discussed above. We define the flag $\widetilde{F^s_\bullet} = (\widetilde{F_\bullet})^s$ to be the truncation (which is a good flag if the original one was), so we have $\Phi_s(\widetilde{F^s_\bullet}) = (\widetilde{T})_s$.

Remark 5.7. Using the basis we fixed before, we can describe the truncation of $\widetilde{F}_\bullet$ as follows:

$$\widetilde{F}^s_i = \begin{cases} \mathbb{C}\{v_s, \ldots, \hat{v}_r, \ldots, v_{n-i+1}\} & \text{if } 1 \le i \le s,\ i \ne n+1-r, \\ \mathbb{C}\{v_s, \ldots, \hat{v}_r, \ldots, v_1, v_{\bar{1}}, \ldots, \hat{v}_{\bar{r}}, \ldots, v_{\overline{i-s}}\} & \text{if } s+1 \le i \le 2s,\ i \ne n+r. \end{cases}$$

5.2. **Inductive step.** We now use the techniques developed by Steinberg in [Ste88] to determine the exotic Robinson-Schensted algorithm. This section is devoted to the proof of the following key Proposition (analogous to Lemma 1.2 from [Ste88]), which is the inductive step in the proof.

Proposition 5.8. Let $F_\bullet$ and $G_\bullet$ be two generic points in the exotic Springer fibre $\mathcal{C}_{(v,x)}$; in particular we can assume that $F_\bullet, G_\bullet, \widetilde{F}_\bullet$ are all good flags. Let T and R be the bitableaux corresponding to $F_\bullet$ and $G_\bullet$ and let $\widetilde{T}$ and R_{n-1} be the bitableaux corresponding to $\widetilde{F}_\bullet$ and $\widetilde{G}_\bullet$. Then the pair $(\widetilde{T}, R_{n-1})$ is obtained from (T, R) by the first iteration of the reverse bumping algorithm described in Section 3.1.

To prove the proposition, we need to describe the bitableau $\widetilde{T}$, which contains the numbers from 1 to n, with r missing. This follows the ideas of Steinberg's proof, but we break the steps into smaller lemmas.

Lemma 5.9. *The numbers* $1, \dots, r-1$ *occupy the same positions in* $\widetilde{T}$ *as they do in* T.

Proof. It suffices to show that for $k < r$, the flags $F_\bullet^k$ and $\widetilde{F_\bullet^k}$ are naturally isomorphic. The flag $F_\bullet^k$ consists of spaces of the form $F_{n\pm k'}/F_{n-k}$, where $k' \leq k$. Suppose first that we have a Type 1 redundancy. Then the spaces in the flag $\widetilde{F_\bullet^k}$ have the form

$$\frac{F_{n\pm k'}}{G_1}\bigg/\frac{F_{n-k}}{G_1} \cong F_{n\pm k'}/F_{n-k},$$

where $k' \leq k$ and so $F_\bullet^k$ and $\widetilde{F_\bullet^k}$ map to the same bitableau.

Now suppose that we are in a Type 2 redundancy. Then a subspace in $\widetilde{F_\bullet^k}$ has the form

$$\frac{G_1 + F_{n\pm k'} \cap G_1^\perp}{G_1}\bigg/\frac{G_1 + F_{n-k} \cap G_1^\perp}{G_1} \cong \frac{G_1 + F_{n\pm k'} \cap G_1^\perp}{G_1 + F_{n-k} \cap G_1^\perp},$$

where $k' \leq k$. Now since G_1 is not contained in F_{n-k} we have

$$\frac{G_1 + F_{n\pm k'} \cap G_1^\perp}{G_1 + F_{n-k} \cap G_1^\perp} \cong \frac{F_{n\pm k'} \cap G_1^\perp}{F_{n-k} \cap G_1^\perp}.$$

Also, there is a natural map $F_{n\pm k'} \cap G_1^\perp \longrightarrow F_{n\pm k'}/F_{n-k}$, which is the composition of the natural inclusion $F_{n\pm k'} \cap G_1^\perp \hookrightarrow F_{n\pm k'}$ and the natural projection $F_{n\pm k'} \to F_{n\pm k'}/F_{n-k}$, whose kernel is $F_{n-k} \cap G_1^\perp$. Therefore

$$\frac{F_{n\pm k'} \cap G_1^\perp}{F_{n-k} \cap G_1^\perp} \cong F_{n\pm k'}/F_{n-k},$$

and so again, $F_\bullet^k$ and $\widetilde{F_\bullet^k}$ map to the same bitableau. □

Now suppose that s is a number that is not in the same position in $\widetilde{T}$ as it was in T, then by Lemma 5.9 we have $s \geq r$.

Lemma 5.10. *Let* s *be as just described.*

(1) If s *was in row* 1 *of* T*, then* $s = r$ *and* $w(n) = r$.
(2) If s *was in row* $m > 1$ *of* T *with no available positions in rows* $m' \geq m-1$*, then* $s = r$ *and* $w(n) = \bar{r}$.
(3) If s *was in row* $m > 1$ *of* T *with available positions in rows* $m' \geq m-1$*, then* $s > r$ *and* s *will occupy in* $\widetilde{T}$ *the available position with smallest row number, displacing a smaller number.*

Proof. Consider the truncated flags $F_\bullet^s$ and $\widetilde{F_\bullet^s}$ defined above. These flags map to bitableaux T_s containing the numbers 1 up to s in the first case and $\widetilde{T_s}$ containing the numbers 1 up to s with r removed in the second case.

Since $s \geq r$, we have that $G_1 \not\subset F_{n-s}$ and that $F_{n+s} \cap G_1^\perp$ is contained in F_{n+s} as a hyperplane. Moreover, both of these latter spaces contain F_{n-s} as a subspace. Therefore

$$\frac{F_{n+s} \cap G_1^\perp}{F_{n-s}} \subset F_{n+s}/F_{n-s}$$

as a hyperplane and, inside the symplectic space F_{n+s}/F_{n-s}, we have that

$$\left(\frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}}\right)^{\perp} = \frac{F_{n-s}+G_1}{F_{n-s}},$$

which is a 1-dimension subspace of F_{n+s}/F_{n-s} contained in the $\ker(x|_{F_{n+s}/F_{n-s}})$.

Now

$$\frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}}\Big/\frac{F_{n-s}+G_1}{F_{n-s}} \cong \frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}+G_1},$$

which is the last term in the flag $\widetilde{F^s_\bullet}$. Hence the shape of $\widetilde{T}_s$ is given by

$$(\tilde{\mu}^s,\tilde{\nu}^s) = \mathrm{eType}\left(v+F_{n-s}+G_1, x|_{\frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}+G_1}}\right).$$

Since the flag $G_\bullet$, and in particular the space G_1, was chosen generically, the bipartition $(\tilde{\mu}^s,\tilde{\nu}^s)$ is obtained from

$$(\mu^s,\nu^s) = \mathrm{eType}\left(v+F_{n-s}, x|_{\frac{F_{n+s}}{F_{n-s}}}\right),$$

which is the shape of T_s, by removing a box.

Suppose that $s>r$, then the number s in $\widetilde{T}$ is contained in $\widetilde{T}_s$ which is contained in the shape of T_s. Since s does not mantain the same position and $s>r$, s moves to a position that was occupied by a smaller number in T, hence the position of s in $\widetilde{T}$ is contained in the shape of T_{s-1}. For the same reason, any other number s', with $r<s'\leq s$, that changes position will end up within the shape of T_{s-1}. By counting the number of boxes, this proves that the shape of $\widetilde{T}_s$ is the same as the shape of T_{s-1}, which is to say that

$$\mathrm{eType}\left(v+F_{n-s}+G_1, x|_{\frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}+G_1}}\right) = \mathrm{eType}\left(v+F_{n-s+1}, x|_{\frac{F_{n+s-1}}{F_{n-s+1}}}\right). \tag{5.6}$$

Hence $W=\frac{F_{n-s}+G_1}{F_{n-s}}$ and $X=F_{n-s+1}/F_{n-s}$ are two 1-dimensional subspaces contained in $\ker(x|_{F_{n+s}/F_{n-s}})$, with $X\subset W^{\perp}$ because $F_{n-s+1}\subset G_1^{\perp}$ for $s>r$. So the spaces W and X satisfy the conditions of Theorem 4.6 with (μ',ν') being the shape of T_{s-1}. Since $s>r$, we also have that $W\neq X$, therefore part 1 of Theorem 4.6 implies that s was in a row $m>1$ of T.

We now have that $Y:=X+W=\frac{F_{n-s+1}+G_1}{F_{n-s}}\subset\frac{F_{n+s-1}\cap G_1^{\perp}}{F_{n-s}}=X^{\perp}\cap W^{\perp}=Y^{\perp}$ is such that $Y^{\perp}/Y$ is the last space in the truncated flag $\widetilde{F}_\bullet^{(s-1)}$, so

$$(\mu'',\nu'') = \mathrm{eType}\left(v+(F_{n-s+1}+G_1), x|_{\frac{F_{n+s-1}\cap G_1^{\perp}}{F_{n-s+1}+G_1}}\right)$$

gives the shape of $\widetilde{T}_{s-1}$, hence by comparison with the shape of $\widetilde{T}_s$ we obtain the position of s inside $\widetilde{T}$. But now Theorem 4.6 tells us exactly that s has moved to the appropriate available position, as described in Remark 4.7, which proves (3). Notice that if $s=r$ we also have (5.6): this is because when $s=r$ and $w(n)=r$, we are in a Type 1 redundancy, so we have $F_{n+r}\cap G_1^{\perp}=F_{n+r-1}$ and $F_{n-r}+G_1=F_{n-r+1}$. Therefore

$$F_{n+r}\cap G_1^{\perp}/F_{n-r}+G_1 = F_{n+r-1}/F_{n-r+1}.$$

When $s = r$ and $w(n) = \bar{r}$ we have $G_1 = \mathbb{C}\{v_{\bar{r}}\}$ and $F_{n+r} = \mathbb{C}\{v_n, \dots, v_1, v_{\bar{1}}, \dots, v_{\bar{r}}\}$. We have then $F_{n+r} \cap G_1^{\perp} = \mathbb{C}\{v_n, \dots, \hat{v}_r, \dots, v_1, v_{\bar{1}}, \dots, v_{\bar{r}}\}$ and $F_{n-r} + G_1 = \mathbb{C}\{v_n, \dots, v_{r+1}, v_{\bar{r}}\}$. On the other hand, $F_{n+r-1} = \mathbb{C}\{v_n, \dots, v_1, v_{\bar{1}}, \dots, v_{\overline{r-1}}\}$ and $F_{n-r+1} = \mathbb{C}\{v_n, \dots, v_r\}$. We have then an obvious isomorphism

$$F_{n+r} \cap G_1^{\perp} / F_{n-r} + G_1 \cong F_{n+r-1}/F_{n-r+1}.$$

Now, for case (1), suppose that s was in row $m = 1$ of T, then by the previous argument we have $s = r$, because $s > r$ implies $m > 1$. Corollary 4.5, applied to $W = \frac{F_{n-r}+G_1}{F_{n-r}}$ and $X = F_{n-r+1}/F_{n-r}$ gives then that $W = X$, that is $\frac{F_{n+r}\cap G_1^{\perp}}{F_{n-r}} = F_{n+r-1}/F_{n-r}$, and so $F_{n+r} \cap G_1^{\perp} = F_{n+r-1}$. Therefore $F_n \subset F_{n+r-1} \subset G_1^{\perp}$ and so we are in a Type 1 redundancy, that is $w(n) = r$.

To conclude, suppose that s was in row $m > 1$ of T, with no available positions. As in the proof of Theorem 3.8, this implies that all the numbers in rows $m-1, m+1, m+2, m+3 \dots$ are bigger than s and that s is in the first column of row m. Then if (μ, ν) is the shape of T_s, it satisfies the conditions of Corollary 4.4 with the spaces $W = \frac{F_{n-s}+G_1}{F_{n-s}}$ and $X = F_{n-s+1}/F_{n-s}$. We then get $F_{n-s+1}/F_{n-s} \not\subseteq \frac{F_{n+s}\cap G_1^{\perp}}{F_{n-s}}$. Therefore $F_{n-s+1} \not\subseteq F_{n+s} \cap G_1^{\perp}$, which implies that $F_{n-s+1} \not\subseteq G_1^{\perp}$ and so we are in a Type 2 redundancy, which means that $w(n) = \bar{r}$. Rewriting this condition as $F_{n+s-1} \not\supseteq G_1$ and remembering the choice of r in this case, we conclude that $n + s - 1 \le n + r - 1$, that is $s \le r$. But our original assumption is that $s \ge r$ and so we have $s = r$ which proves (2). □

Proof of Proposition 5.8. First of all, notice that the shape of $\widetilde{T}$ is the same as the shape of R_{n-1} because they are both the bipartition given by eType$\left(v + G_1, x|_{G_1^{\perp}/G_1}\right)$. Consider the position of n in R, and let s be the number in the same position in T, then clearly s cannot be in the same position in $\widetilde{T}$ as it was in T. By Lemmas 5.9 and 5.10, we know that in the transition from T to $\widetilde{T}$, either s disappears from the bitableau or moves and displaces a smaller number, which in turn displaces a smaller number and so on, until eventually r disappears from the bitableau (from the first row in the Type 1 redundancy with $w(n) = r$, or from the first column to the left or right of the dividing wall in the Type 2 redundancy with $w(n) = \bar{r}$). This chain of displacements is governed by the available positions in T, by Lemma 5.10. It also follows by Lemma 5.10 that no other number in T causes a chain of displacements as that would imply that different numbers in T would move to the same position of $\widetilde{T}$, which is impossible. It follows that the first iteration of the exotic Robinson-Schensted bijection has been achieved. □

5.3. **Completing the proof.** While it may seem that the main theorem follows from Proposition 5.8 by induction, there is a subtlety: $\widetilde{F}_\bullet$ may not be generic in its irreducible component inside $\mathcal{C}_{(\bar{v},\bar{x})}$, even though $F_\bullet, G_\bullet$ are. In this section we remedy this by adapting the argument on page 528 of [Ste88].

Remark 5.11. Recall the definition of relative position of two symplectic flags $F_\bullet, G_\bullet \in \mathcal{F}(V)$ from Definition 2.18. The Bruhat decomposition states that the orbits of $Sp_{2n}(\mathbb{C})$ on $\mathcal{F}(V) \times \mathcal{F}(V)$ are determined by the relative positions of the two flags. We have the Bruhat ordering on $W(C_n)$, which satisfies the following: for each $w \in W(C_n)$, the set of all flags $(F_\bullet, G_\bullet)$ that satisfy $w(F_\bullet, G_\bullet) \le w$ is a closed set.

Definition 5.12. Given $w \in W(C_n)$, define $w^- \in W(C_{n-1})$ by eliminating the last letter of the signed permutation word and relabeling the rest with the numbers $1, \dots, n-1$. More precisely, if $w(n) \in \{r, \bar{r}\}$, then for $1 \leq i \leq n-1$, define

$$w^-(i) = \begin{cases} w(i) & \text{if } w(i) \in \{a, \bar{a}\},\ 1 \leq a < r, \\ a-1 & \text{if } w(i) = a,\ r < a \leq n, \\ \overline{a-1} & \text{if } w(i) = \bar{a},\ r < a \leq n. \end{cases}$$

Given $w \in W(C_{n-1})$ and $1 \leq r \leq n$, define w^{+r} (respectively $w^{+\bar{r}}$) by relabelling the numbers in w as $\{1, \dots, n\} \setminus \{r\}$ and adding r (respectively $\bar{r}$) at the end. More precisely

$$w^{+r}(n) = r, \qquad w^{+\bar{r}}(n) = \bar{r}$$

and for $1 \leq i \leq n-1$,

$$w^{+r}(i) = w^{+\bar{r}}(i) = \begin{cases} w(i) & \text{if } w(i) \in \{a, \bar{a}\},\ 1 \leq a < r, \\ a+1 & \text{if } w(i) = a,\ r \leq a \leq n-1, \\ \overline{a+1} & \text{if } w(i) = \bar{a},\ r \leq a \leq n-1. \end{cases}$$

Remark 5.13. The operations defined in Definition 5.12 preserve the Bruhat order. This means, if $u, w \in W(C_n)$, $u \leq w$, then $u^- \leq w^-$. Also, if $u, w \in W(C_{n-1})$, $u \leq w$, then $u^{+r} \leq w^{+r}$ and $u^{+\bar{r}} \leq w^{+\bar{r}}$.

Lemma 5.14. *Let $F_\bullet, G_\bullet \in \mathcal{F}(V)$, with $w = w(F_\bullet, G_\bullet)$, and $\tilde{w} = w(\widetilde{F_\bullet}, \widetilde{G_\bullet})$. Then $\tilde{w} = w^-$ and $w = \tilde{w}^{+w(n)}$.*

Proof. This follows immediately by comparing the basis elements spanning the flags $\widetilde{G_\bullet}$ and $\widetilde{F_\bullet}$ in equations (5.3) and (5.4). □

Proof of Theorem 3.10. To finish off the proof it only remains for us to examine the case where $\widetilde{F}_\bullet$, is not generic relative to the bitableau $\widetilde{T}$, even if $F_\bullet$ and $G_\bullet$ were chosen generically relative to T and R respectively.

Let $\mathrm{eRS}(T, R) = w$, with $w(n) \in \{r, \bar{r}\}$. By induction on n, the relative position of a generic element lying in $\Phi_{n-1}^{-1}(\widetilde{T})$, and a generic element lying in $\Phi_{n-1}^{-1}(R_{n-1})$ is equal to $\mathrm{eRS}(\widetilde{T}, R_{n-1}) := w^-$ (potentially up to relabelling).

By Remark 5.11, we may then assume that $w(\widetilde{F}_\bullet, \widetilde{G}_\bullet) = \widetilde{w} \leq w^-$, since if $\widetilde{F}_\bullet$ is not generic relative to $\widetilde{T}$, it lies in the closure of all flags $\widetilde{F'_\bullet}$ in the exotic Springer fibre that are generic. Now, by Remark 5.13 and Lemma 5.14, we have

$$w(F_\bullet, G_\bullet) = \widetilde{w}^{+w(n)} \leq (w^-)^{+w(n)} = w.$$

As (T, R) varies over all the bitableaux, the relative positions $w(F_\bullet, G_\bullet)$ pass over each permutation exactly once; a proof of this fact about the exotic Steinberg variety can be found in [NRS16, Lemma 6.3]. Furthermore, their images under the exotic Robinson-Schensted bijection $w = \mathrm{eRS}(T, R)$ also sweep out each permutation exactly once, since the correspondence is a bijection. It now follows that the inequality $w(F_\bullet, G_\bullet) \leq \mathrm{eRS}(T, R)$ is in fact an equality (by a backwards induction on the length of the word $\mathrm{eRS}(T, R)$, the base case being the long word w_0). Hence the main theorem is proved. □

Appendix A. The exotic Robinson-Schensted bijection for $n = 3$

Here we give the complete exotic Robinson-Schensted bijection for $n = 3$. The $n = 2$ case was given in [NRS16]. We will give this correspondence in such a way that the exotic cells in the Weyl group are clear.

Element of $W(C_3)$	**Bitableaux**	**Element of $W(C_3)$**	**Bitableaux**
123	$((\begin{smallmatrix}3&2&1\end{smallmatrix};-),(\begin{smallmatrix}3&2&1\end{smallmatrix};-))$	$\bar{3}\bar{2}\bar{1}$	$((-;\begin{smallmatrix}1&2&3\end{smallmatrix}),(-;\begin{smallmatrix}1&2&3\end{smallmatrix}))$
$1\bar{2}\bar{3}$	$((\begin{smallmatrix}1\\2\\3\end{smallmatrix};-),(\begin{smallmatrix}1\\2\\3\end{smallmatrix};-))$	$\bar{1}\bar{2}\bar{3}$	$((-;\begin{smallmatrix}1\\2\\3\end{smallmatrix}),(-;\begin{smallmatrix}1\\2\\3\end{smallmatrix}))$
$1\bar{2}3$	$((\begin{smallmatrix}3&1\\&2\end{smallmatrix};-),(\begin{smallmatrix}3&1\\&2\end{smallmatrix};-))$	$\bar{1}\bar{3}\bar{2}$	$((-;\begin{smallmatrix}1&3\\2&\end{smallmatrix}),(-;\begin{smallmatrix}1&3\\2&\end{smallmatrix}))$
$12\bar{3}$	$((\begin{smallmatrix}2&1\\&3\end{smallmatrix};-),(\begin{smallmatrix}2&1\\&3\end{smallmatrix};-))$	$\bar{2}\bar{1}\bar{3}$	$((-;\begin{smallmatrix}1&2\\3&\end{smallmatrix}),(-;\begin{smallmatrix}1&2\\3&\end{smallmatrix}))$
$1\bar{3}2$	$((\begin{smallmatrix}2&1\\&3\end{smallmatrix};-),(\begin{smallmatrix}3&1\\&2\end{smallmatrix};-))$	$\bar{2}\bar{3}\bar{1}$	$((-;\begin{smallmatrix}1&2\\3&\end{smallmatrix}),(-;\begin{smallmatrix}1&3\\2&\end{smallmatrix}))$
$13\bar{2}$	$((\begin{smallmatrix}3&1\\&2\end{smallmatrix};-),(\begin{smallmatrix}2&1\\&3\end{smallmatrix};-))$	$\bar{3}\bar{1}\bar{2}$	$((-;\begin{smallmatrix}1&3\\2&\end{smallmatrix}),(-;\begin{smallmatrix}1&2\\3&\end{smallmatrix}))$
132	$((\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$3\bar{2}1$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}))$
213	$((\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{1}32$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}))$
$\bar{1}23$	$((\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{2}\bar{1}3$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}))$
312	$((\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{2}31$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}))$
231	$((\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$3\bar{1}2$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}))$
$\bar{3}12$	$((\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{3}\bar{2}1$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}))$
$23\bar{1}$	$((\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}2&1\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$3\bar{2}\bar{1}$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2&3\end{smallmatrix}))$
$\bar{2}13$	$((\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{3}\bar{1}2$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}))$
$2\bar{1}3$	$((\begin{smallmatrix}3&2\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}3&1\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{2}3\bar{1}$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1&2\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1&3\end{smallmatrix}))$
$1\bar{3}\bar{2}$	$((\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$21\bar{3}$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}))$
321	$((\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{1}2\bar{3}$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}))$
$\bar{3}2\bar{1}$	$((\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{1}\bar{2}3$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}))$
$31\bar{2}$	$((\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{2}1\bar{3}$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}))$
$2\bar{3}1$	$((\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$2\bar{1}\bar{3}$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}))$
$\bar{3}1\bar{2}$	$((\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}),(\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{2}\bar{3}1$	$((\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}))$
$2\bar{3}\bar{1}$	$((\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}1\\2\end{smallmatrix};\begin{smallmatrix}3\end{smallmatrix}))$	$3\bar{1}\bar{2}$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}),(\begin{smallmatrix}1\end{smallmatrix};\begin{smallmatrix}2\\3\end{smallmatrix}))$
$\bar{3}21$	$((\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}),(\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}))$	$\bar{1}\bar{3}2$	$((\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}),(\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}))$
$32\bar{1}$	$((\begin{smallmatrix}2\\3\end{smallmatrix};\begin{smallmatrix}1\end{smallmatrix}),(\begin{smallmatrix}1\\3\end{smallmatrix};\begin{smallmatrix}2\end{smallmatrix}))$	$\bar{1}3\bar{2}$	$((\begin{smallmatrix}3\end{smallmatrix};\begin{smallmatrix}1\\2\end{smallmatrix}),(\begin{smallmatrix}2\end{smallmatrix};\begin{smallmatrix}1\\3\end{smallmatrix}))$

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