Challenging the classical notion of time in cognition: a quantum perspective

James M. Yearsley & Emmanuel M. Pothos
Department of Psychology
City University London

Please address correspondence regarding this article to James Yearsley or Emmanuel Pothos Department of Psychology, City University London EC1V 0HB. Electronic mail can be sent to james.yearsley.1@city.ac.uk or e.m.pothos@gmail.com.

Abstract

All mental representations change with time. A baseline intuition is that mental representations have specific values at different time points, which may be more or less accessible, depending on noise, forgetting processes etc. We present a radically alternative, motivated by recent research using the mathematics from quantum theory for cognitive modelling. Such cognitive models raise the possibility that certain possibilities or events may be incompatible, so that perfect knowledge of one necessitates uncertainty for the others. In the context of time dependence, in physics, this issue is explored with the so-called temporal Bell (TB) or Leggett-Garg inequalities. We consider in detail the theoretical and empirical challenges involved in exploring the TB inequalities in the context of cognitive systems. One interesting conclusion is that we believe the study of the TB inequalities to be empirically more constrained in psychology, than in physics. Specifically, we show how the TB inequalities, as applied to cognitive systems, can be derived from two simple assumptions, Cognitive Realism and Cognitive Completeness. We discuss possible implications of putative violations of the TB inequalities for cognitive models and our understanding of time in cognition in general. Overall, the paper provides a surprising, novel direction, in relation to how time should be conceptualized in cognition.

1. Introduction

Consider a cognitive variable, such as affect or interpretation, in relation to a stimulus, e.g., how much one likes eating chocolate. All cognitive variables can, in principle, change with time and how they do so is a key consideration in psychological theory, e.g., models of memory. The fundamental, though tacit, assumption regarding change in time is that of a classical trajectory. A cognitive variable has specific values at different time points, but of course these values are not always readily accessible or they may be accessed, but in a noisy way (e.g., Howe & Courage, 1997; Raaijmakers & Shiffrin, 1992; Shiffrin, 1970). Such intuitions seem straightforward and uncontroversial. With this paper, we challenge the notion that cognitive variables (always) have a specific, well-defined value at all times (cf. Raijmakers & Molenaar, 2004). The alternative possibility we present is that certainty about the value of a cognitive variable at a specific time will create uncertainty about the value at (most) other time points; the act of inquiring (e.g., through a psychological process of recall) about value, from time point to time point, may be constructive, so that it would be impossible to create a table of possible values of the cognitive variable at all time points; such a table would no longer exist. Such issues can be addressed in a technical way, using the mathematics developed for quantum theory in physics.

Recent work with quantum cognitive models has offered a comprehensive challenge to many established intuitions about basic properties of cognitive models, in a way analogous to the application of quantum theory in physics. By quantum theory (or quantum probability, QP, theory) we mean the rules for assigning probabilities to events from quantum mechanics, without any of the physics (e.g., Atmanspacher, Romer, & Wallach, 2006). QP theory is in principle applicable to any area where there is a need to formalize uncertainty. In psychology, classical probability (CP) theory is by far the most dominant approach for dealing with uncertainty (e.g., Oaksford & Chater, 2009; Tenenbaum et al., 2011), but empirical findings often challenge classical prescription. QP theory has enabled the development of compelling cognitive models for cases for which CP theory appears inadequate, for example, in conceptual combination (Aerts & Gabora, 2005), decision making (Busemeyer et al., 2011; Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011; Wang & Busemeyer, 2013), and memory (Bruza et al., 2009). We stress that these applications of QP theory to cognition are consistent with a fully classical brain and do not require a quantum brain (this latter hypothesis is very controversial). An important contribution of this research programme has been the introduction of explanatory concepts in psychology, with no prior analogue, such as incompatibility, superposition, and entanglement. Such concepts have enabled new insights about the principles underlying cognitive processes (for overviews see Busemeyer & Bruza, 2011; Khrennikov, 2010; Pothos & Busemeyer, 2013; for an early example see Aerts & Aerts, 1995).

Our present focus is on the implications from QP theory on how to understand time dependence in cognitive models. Atmanspacher and Filk (2010) have presented a pioneering analysis, wherein they argued that the process of perceiving a stimulus, which can have one of two stable perceptual interpretations (bistable perception), can be described with a quantum model, in a way which challenges classical notions of time dependence. Specifically, they presented conditions, which enabled an interpretation of what they called "temporal nonlocality", by which they meant that "… events cannot be uniquely fixed in time" (p.314). Their derivation is based on the temporal Bell (TB) inequalities (Leggett & Garg, 1985), also known as the Leggett-Garg inequalities. Briefly, in physics, the TB inequalities are based on a combination of two-time correlation functions, at different time points, for the value of a physical quantity which can be observed (such quantities are called, surprisingly enough, observables). Define realism to be the property that a system with two or more states will be at all times in one of these states. A TB inequality will be satisfied by all realist systems, provided they can be measured in a non-invasive way (we clarify measurement issues below).

The mathematical simplicity and elegance of the TB inequalities make them extremely appealing as tests of the necessity of a quantum description for a system. Indeed, historically, a violation of the (non-temporal) Bell inequalities is considered to be the ultimate proof of the failure of classical physics to describe the physical world (Bell, 2004). Observed violations of the Bell inequalities in physics rule out not just a single physical model, but rather an entire class of models, those satisfying 'local causality' (Bell, 2004). In physics, the corresponding empirical demonstration by Aspect and colleagues (e.g., Aspect, Graingier, & Roger, 1981) has been one of the most compelling events in the history of science. In a similar way, our purpose is to develop the necessary conceptual tools to consider violations of the TB inequalities in psychology; such violations, if observed, would rule out an entire class of cognitive models, those having the property of 'Cognitive Realism', which we define below.

However, in physics, tests of violations of the TB inequalities are fraught with empirical and conceptual difficulties and doubt has been cast on whether they are in principle possible (e.g., Ballentine, 1987; Yearsley, under review; Wilde & Mizel, 2012; see Peres, 1988, for early objections). Furthermore, the derivation and testing of the TB inequalities in psychology presents challenges different from those encountered in physics. The most important difference is that tests of the Bell and TB inequalities in physics are (with important caveats) designed to rule out the possibility of a realist account of fundamental physics. In contrast, the consensus in psychology is that cognitive function can be reduced (at least in principle) to the workings of a classical brain, and thus realism, in the sense that a physicist understands it, is a presumption of any cognitive model. It is thus important to establish exactly what would be proven by any purported violation of the TB inequalities in psychology. Put differently, if the TB inequalities are violated by a cognitive system, but we are assuming a classical brain, what exactly is it that is 'quantum'? Another issue concerns the generality of existing quantum models violating TB inequalities in psychology, such as the one of Atmanspacher and Filk (2010). Perhaps violations of TB inequalities do occur, but for such specific cognitive quantum models, that general implications for cognition are limited.

The purpose of this paper is to bridge the disciplinary gap between physics and psychology, in relation to the considerations for testing for and interpreting putative violations of the TB inequalities. There are several conceptual and interpretative issues to address in this effort. Nevertheless, we offer the promise of a radical reconceptualization of the construct of time in psychology and, indeed, potentially our understanding of memory. We hope that this 'bridging' paper will stimulate research in this novel, exciting research direction.

2. The assumptions underlying the TB inequalities

In this section we motivate a derivation of the TB inequalities in the context of cognitive models. To do this, we will need to specify in a fairly precise mathematical way two assumptions about the set of cognitive models under consideration. Then, the derivation of a TB inequality is fairly straightforward (see the Appendix).

Consider a cognitive model of a simple two-valued system such as, to follow from a famous example in decision making, participants' judgement about whether Linda is or is not a bank teller (Tversky & Kahneman, 1983). Cognitive models work by isolating a small set of judgments or thoughts (cf. Fodor, 1983) and assuming they can be modelled, without *detailed* knowledge of the underlying neuropsychological states of the participants' brains. Consistency between modelling at the cognitive level and the underlying neurophysiology usually concerns just assumptions about computability restrictions for the former from the latter, though there are exceptions (for a recent discussion see Jones & Love, 2011). The consideration of the putative psychological relevance of TB inequalities requires us to be more precise about such issues. We suggest that there are two implicit assumptions in all typical cognitive models, concerning the relation between cognitive states and neurophysiological ones. We argue that these assumptions are reasonable and, moreover, sufficient to derive the TB inequalities, as applied to cognitive systems. Therefore, an empirically observed violation of a TB inequality for a cognitive system would rule out the large class of cognitive models, consistent with these assumptions.

The first assumption implicit in all cognitive models may be called Cognitive Realism. This is the assumption that the reason for any judgment at the cognitive level is ultimately (in principle, if not in practice) reducible to processes at the neurophysiological level. We assume that the neurophysiology of the brain is classical (e.g., beim Graben & Atmanspacher, 2009), as arguments to the contrary remain controversial. Thus, we assume that, for example, if it were possible to read out the exact state of a person's brain at the neural level, this would be sufficient to uniquely determine the person's decisions. Of course, such a mapping between the neuropsychological and the cognitive level is likely to be enormously complicated and impossible to implement in practice. In some sense, this is the whole raison d'être of cognitive models. However, all we need assume presently is that such a mapping exists.

Mathematically, this means that the expected outcome of a particular judgement B in a cognitive model may be written as

$$\langle B \rangle = \sum_{\lambda} B(\lambda)\, \rho(\lambda)$$

Here the λ denote the possible neuropsychological states of the brain, $B(\lambda)$ tells us the judgement of a participant or group of participants, given that their neuropsychological state is λ, and $\rho(\lambda)$ denotes the probability distribution of the participants' neuropsychological states over the possible λ.

The neuropsychological states λ are like the 'hidden variables' in the physics context, to be distinguished from what we can call cognitive variables (which relate only to the cognitive state). The hidden (neurophysiological) variables represent the information that would be needed to fully determine both the cognitive state and its dynamics, i.e., to predict all future relevant decisions of a participant (at least up to classical noise arising from imperfect measurement). Thus each alternative configuration of the neuropsychological state λ determines the value of the judgment B, for participants with this particular neuropsychological state. This formalism is easily adapted to multiple judgments or to time-dependent cognitive variables. Cognitive variables are typically directly observable, whereas neurophysiological variables are not. Our uncertainty about the exact neuropsychological state of the participant is expressed by the fact that $\rho(\lambda)$ is a probability distribution, which may give non-zero probabilities for many possible states.

The assumption of Cognitive Realism may also be expressed in the following important way: For any set of judgements, and at all times, an observer has a definite opinion about all judgments. Cognitive Realism, together with the assumption of Cognitive Completeness (explained shortly), imply that participants' judgments reflect pre-existing preferences and so cannot be 'constructive'. Note, quantum cognition models do not satisfy the assumption of Cognitive Realism.

The second assumption, which we suggest is implicit in all standard cognitive models, can be called Cognitive Completeness. Consider a cognitive model to predict responses for an arbitrary set of judgments, for example, following again from Tversky and Kahneman's (1983) example, 'is Linda a feminist?', 'is Linda a bank teller?' etc. Cognitive Completeness is the assumption that the cognitive state of a person responding to such a set of judgments can be *entirely* determined by the probabilities for the judgment outcomes. That is, observing participant behaviour can fully determine the underlying cognitive state, without the need to invoke neurophysiological variables. It is possible that different neurophysiological states give rise to the same behaviour or not. Regardless, Cognitive Completeness means that knowledge of the relevant cognitive state (and its dynamics), in relation to a set of judgments, can fully occur, without the knowledge of neurophysiological variables.

Mathematically, this assumption means that every cognitive model defines a set of similarity classes on the set of all probability distributions over the neuropsychological variables, with two distributions $\rho(\lambda)$ and $\rho'(\lambda)$ being similar, $\rho(\lambda) \sim \rho'(\lambda)$, if they lead to the same predictions, for all judgments produced by the cognitive model.

This assumption has a crucial consequence. Consider any stimuli presented to, or measurement made on, a group of participants, which does not change the probabilities for the outcomes of any future judgment, in the relevant cognitive model. Let us call such measurements non-disturbing. Whether or not a measurement is non-disturbing can be established empirically. Call measurements which affect the neurophysiological variables invasive, by analogy with physics, whereby invasive measurements are ones which affect hidden variables (invasive measurements could, e.g., change the dynamics of a system, but in such a way that the probabilities for future measurements are the same). In physics, a fundamental challenge in any attempt to demonstrate violations of the TB inequality is that it is possible to empirically establish whether a measurement is disturbing or not, but this is not so for whether it is invasive or not (George et al, 2013, Palacios-Laloy et al 2010, Yearsley, under review). In psychology, with the assumption of Cognitive Completeness, we avoid this problem: Cognitive Completeness means that, as long as a measurement is non-disturbing, it can be assumed to be non-invasive as well, that is, that it has no effect on the neurophysiological state of a participant. This is because Cognitive Completeness tells us that the cognitive state of the participants may be fully determined by knowledge of the outcomes of all judgments in the relevant cognitive model. Thus, at most, a non-disturbing measurement may change the underlying neurophysiological state in a way that gives rise to the *same* cognitive state. But, any such change is undetectable by any measurement relevant to the cognitive model and thus we can simply assume that no change in the neurophysiological state occurred.

Let us recap the two assumptions that define the class of cognitive models we are considering. Cognitive Realism tells us that the outcomes of all judgements in a cognitive model are ultimately determined, doubtless in an extremely complicated way, by the participants' neuropsychological states. This expression of Cognitive Realism is uncontroversial, but in practice it rarely impacts on the specification of cognitive models. Of relevance to cognitive models is the implication from Cognitive

Realism that, for any set of judgments, and at all times, a definite outcome exists. Cognitive Completeness tells us that the cognitive state relevant to a particular set of judgments may be determined entirely from the probabilities for outcomes of those judgments and thus, that different neurophysiological states, which give rise to the same probabilities for these judgments, may be considered identical. In brief, Cognitive Completeness means that non-disturbing measurements can be assumed to be non-invasive. These assumptions are simple, plausible, and central, implicitly or explicitly, to most existing cognitive models.

A final caveat is that our motivation for Cognitive Completeness is partly based on considering the only plausible hidden variables to be neurophysiological ones. Why not consider the possibility of *cognitive* hidden variables, that is, the possibility of augmenting a cognitive model with more judgments, in the hope of identifying a larger set of judgments, such that the corresponding model satisfies both Cognitive Realism and Cognitive Completeness? If such additional judgments could be measured in a non-disturbing way, then we could get the marginal probability distribution for the original judgments by summing them out. But, in such a case, an observed violation of TB would tell us that this marginal does not exist, and therefore neither can the joint probability distribution for the original plus additional judgments. This implies that *any* cognitive hidden variable can never be measured in a non-disturbing way. However, the existence of a cognitive variable, which is impossible to measure without altering the probabilities for the outcomes of future judgments, indeed feels very much like an expression of 'quantumness' in a cognitive model.

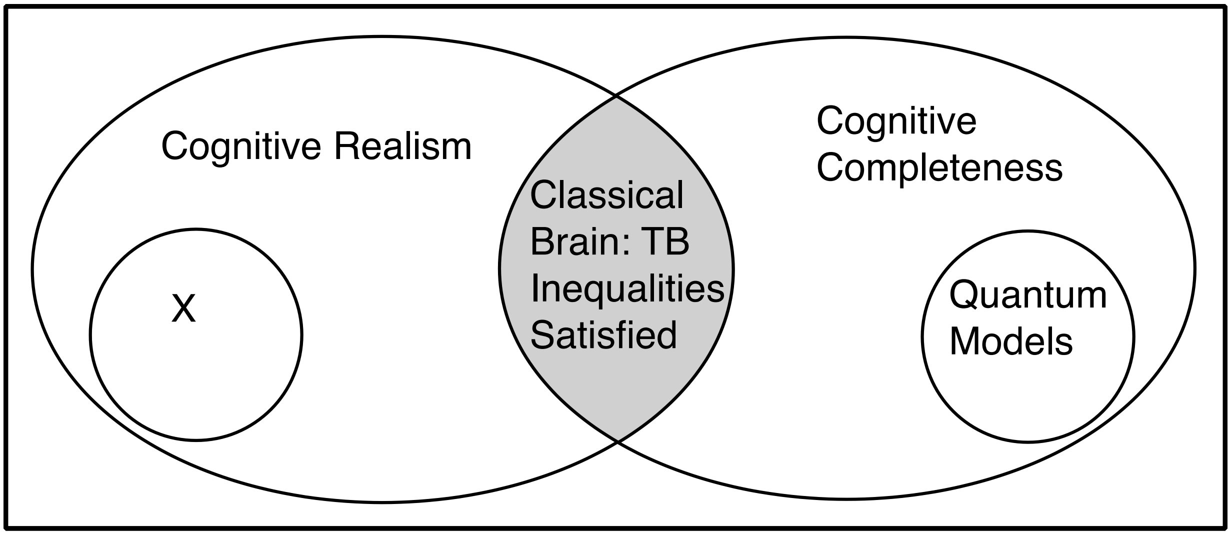

Figure 1: Venn diagram showing the relationship between the assumptions of Cognitive Realism and Cognitive Completeness and their overlap, which defines classical cognitive models. Quantum models satisfy Cognitive Completeness but not Cognitive Realism, and a model in the class 'X' would satisfy Cognitive Realism but not Cognitive Completeness.

Given the assumptions of Cognitive Realism and Cognitive Completeness, it is possible to derive a simple form of the TB inequality, as relevant to cognitive systems (see the Appendix). Consider a two level time dependent observable $Q(t)$, with two possible values ± 1. The definition of an observable in psychology is entirely analogous to that in physics; e.g., in psychology, an observable could correspond to a participant's impression of whether Linda is a bank teller or not. Let $\langle Q(t_1)Q(t_2)\rangle$ denote the two time correlation functions, by which we mean the expected value of the product of the observable at t_1 and the observable at t_2. Then, given our two assumptions, one can derive a TB inequality of the following form,

$$|\langle Q(t_2)Q(t_1)\rangle - \langle Q(t_4)Q(t_1)\rangle| \leq 2 \pm [\langle Q(t_3)Q(t_2)\rangle + \langle Q(t_4)Q(t_3)\rangle].$$

We note here a difference between the inequality above and the version in Atmanspacher and Filk (2010). The inequality we present involves correlations between the values of the observable Q at four different times, in contrast to Atmanspacher and Filk's (2010) one, which involves three. The derivation of the three time version involves the extra assumption that the possible values of $Q(\lambda)$, ie the measured value of Q, given that the neurophysiological state is λ, can only take the values ± 1 (see the Appendix). In psychological terms, this means demanding, first, that the judged value of Q follows deterministically, given a particular neurophysiological state (plausible, but an assumption which we would rather not require) and, second, that the experimental set up is such that the *measured* value of Q is perfectly correlated with the judged value of Q, i.e., there is no noise in the measurement. Both of these are strong assumptions and it seems better to use a framework which does not depend on them, as is the case for the four time version of the temporal Bell inequalities.

3. Planning for violations of the TB inequality

Classical cognitive models satisfy both Cognitive Realism and Cognitive Completeness, and so the TB inequalities. Quantum cognitive models may violate the TB inequalities, allowing us to consider whether Cognitive Realism or Cognitive Completeness might be rejected in cognitive explanation. However, this speculation is meaningless, unless it is possible to specify quantum cognitive models, which would guide prediction, regarding the time points when putative violations of TB inequalities are expected. In this section, we discuss how a dynamic quantum model can be developed, for a particular set of situations, which arise fairly often in cognitive modelling, that of bi-valued judgments, regarding a single question (e.g., an evaluation of positive vs. negative affect or risky vs. safe choice etc.)

We assume that we are dealing with a closed set of judgments, by which we mean that there is no obvious way to regard the judgments as some subset of a larger set of possibilities. (This assumption can be relaxed at the expense of requiring a more complicated model.) The main aspect of the specification of a quantum dynamical model then concerns the Hamiltonian, *H*, the operator which determines how a quantum system changes with time, via Schrödinger's equation, $i\frac{d\psi}{dt} = H \cdot \psi$. To simplify computations, we assume that H is independent of time and that we are working with dimensionless units. The solution to Schrödinger's equation is $\psi(t) = U(t - t_0) \cdot \psi(t_0) = e^{-iH(t-t_0)} \cdot \psi(t_0)$, where t_0 is the initial time. Note that we use the word 'time' here in a formal way. For certain types of stimuli, the 'time' in the solution of Schrödinger's equation may be the length of time for which the stimuli was presented, but for other, discrete stimuli, it may be proportional to the number of stimuli presented or even to the 'strength' of the stimuli in some sense (e.g. if the stimuli are quantities of money, t might be proportional to the amount of money).

Generally, it is difficult to a priori motivate a suitable Hamiltonian. However, for a two level system, any Hamiltonian must be a weighted sum of the three Pauli matrices ($\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) and the identity. The effect of the identity is just to introduce an overall phase factor onto the state, so it can be ignored (this phase factor cancels out when we compute probabilities). In the standard Bloch sphere representation of a two-level quantum system, there are three directions, x, y, z. Let us choose the direction z to correspond to the psychological variable of interest (recall, we are talking about a bi-valued observable, e.g., whether a hypothetical person is a bank teller or not), so that the projection operators to the two possibilities of interest can be set as $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (which correspond to the eigenstates of σ_z). As we are only concerned with projection along the z-axis, we can drop one of σ_x and σ_y, and we eliminate the latter. The Hamiltonian for such a system would then be determined by σ_x and σ_z. Our purpose here is not to specify the most general (reasonable) Hamiltonian, rather demonstrate how to derive optimal times for when to expect violations of the TB inequalities. So, for simplicity, we also eliminate σ_z (note that, in physics, σ_z controls the difference between the energies of the two psychologically relevant states, i.e., $\Delta E \sim \langle + | \sigma_z | + \rangle - \langle - | \sigma_z | - \rangle$, a function which is of arguable relevance in psychology).

Given these simplifying assumptions, $H = \omega\sigma_x$, where ω is a constant affecting the rate of change of the psychological state (ω could be determined through calibration experiments). While this model is not the most general one, even for a cognitive model for bi-valued judgments, the simplifying assumptions are reasonable and we think it would be useful in at least some cognitive modelling situations. Indeed, this model has the same form as that derived by Atmanspacher and Filk (2010).

We now put the model to good use, showing how it can guide empirical tests for putative violations of the TB inequalities. Specifically, we show that some control is needed over the times between measurements in order to generate a TB violation, and this model can guide us in our choice of measurement times. For the above quantum model it is easy to show that $\langle Q(t_1)Q(t_2)\rangle = Cos\big(2\omega(t_2 - t_1)\big)$. Taking the intervals between the measurements to be all equal to T means the TB inequality for this system reduces to

$$3\ Cos(2\omega T) - Cos(6\omega T) \leq 2$$

which is maximally violated for $T = \pi/8\omega$, when the left hand side is equal to $2\sqrt{2}$, but which is violated to a lesser extent for all times between measurements in the interval $(0, T_{max})$ where $T_{max} \sim 0.6/\omega$.

Thus, we see how it is possible to derive specific expectations regarding the measurement times, which can lead to violations of the TB inequalities. We note that the control over the measurement times need not be too precise, which makes an experimental test plausible. In the next section, we consider some operational details for such an experimental test.

4. Operational prescription

In physics, for a violation of a TB inequality to be interesting, one needs to demonstrate that a measurement is non-disturbing and non-invasive. However, in psychology, the assumption of Cognitive Completeness implies that all non-disturbing measurements may be considered non-invasive as well.

Thus, in psychology, we need only examine whether measurements are non-disturbing, and so the empirical challenge is simplified. A disturbing measurement changes the cognitive state and thus the expected probability distribution of future measurements.

We rephrase the necessary condition as one which will help with operational prescription: We seek to control against measurements, which have an influence on the results of future measurements. If such a possibility is not eliminated, it is possible to produce violations of TB, even for classical systems. It is easy to see why this is the case: Consider a version of Table 1, but such that the outcome at t_2 depends on whether a measurement was performed at t_1. Then, $Prob(+, t_2) \neq \sum_{+,-} Prob(+, t_2 \wedge measurement\ outcome\ t_1)$. It would be like having two separate columns for the outcome of the t_2 measurement in Table 1, depending on whether a measurement at t_1 had taken place or not. So, a dependence of measurements on the existence of previous measurements has to be precluded. (This is similar to the possibility of signalling between subsystems, which must be eliminated in tests of the standard Bell inequalities.)

Consider three measurement time points, t_1, t_2, and t_3, and three stimuli A, B, and C, one presented at each measurement point (it is simpler to discuss the operational prescription in terms of three time points and the extension to the required four is straightforward). The three stimuli can be thought of as determining the time evolution of the relevant observable. For example, the observable may be whether there is 'red' on a computer screen, as judged by a naïve observer, and the stimuli may be three colour patches, which are red to different degrees. Such a scheme translates easily to the matrix of possible observable values in Table 1. Then, we can easily specify a template for a cognitive experiment to examine putative violations of the TB inequality. Observe first that Table 1 implies (with simple set theory) that $N_-(t_1, t_3) \leq N_-(t_1, t_2) + N_-(t_2, t_3)$, where N_- indicates changes in the value of the system across corresponding time points (cf., Atmanspacher & Filk, 2010; in the Appendix we show how this inequality can be derived from the temporal Bell one in Section 2). Note that such an inequality makes sense only if we have a classical system, in which case all system values are assumed to be possessed. Then, we can arrange an experimental set-up, so that any change across successive time steps (t_1, t_2 and t_2, t_3) is small, so that a participant does not report a change. But, accumulatively, the change across t_1, t_3 is large enough for a change to be reported. Therefore, we would have that $N_-(t_1, t_3) \leq N_-(t_1, t_2) + N_-(t_2, t_3)$ translates into $High\ value \leq low\ value + low\ value$, and so a violation of the TB inequality.

There are some necessary controls. First, we must establish that the difference in the observable value, across stimuli A, B, C is, in principle, detectable. As noted, a clever design will ensure that participants are unlikely to report a difference between A,B and B,C, but this should not be due to a psychophysical inability to discriminate between the stimuli (T. Filk, personal communication, August 2013). This can be explored with a simple 2-alternative forced choice task, in which participants are shown the stimuli e.g. sequentially and have to decide which stimulus is more red. Second, we need consider whether measurements are non-disturbing (cf. the idea of adroit measurements in Wilde & Mizel, 2012) or not. One can compare the probability distribution of responses at t_2, following a measurement at t_1, and, likewise, at t_3, following measurements at t_2 and t_1. If the distributions are the same, this would be an indication that the measurements are not disturbing, in the above sense of earlier measurements not affecting later ones. Note that it is possible that a measurement on the actual value of the observable at the different time points is disturbing, but a measurement of whether there is a *change* across different time points (i.e., counting $N_-\left(t_x, t_y\right)$ statistics) is not (cf. Atmanspacher & Filk, 2010). Change measurements might be less disturbing, if it is possible to have a sense of a change in an observable, without knowledge of exact values.

The issue of controlling against disturbing measurements is certainly not trivial, as one needs a paradigm such that a question at t_1 would not affect the measurement outcomes at subsequent time points, e.g., t_2, t_3. Yet, in cognitive psychology, there have been other similar empirical challenges, whereby the influence of one judgment must not extend to other, related judgments (e.g., in studying violations of the law of total probability with within participants designs; cf. Shafir & Tversky, 1992). Such challenges have often been overcome through the judicious use of e.g. filler items and it is hoped that similar designs would enable the study of violations of the TB inequality for cognitive systems.

5. The implications of TB violation for a classical brain

We have discussed how the TB inequalities can apply to cognitive models. Consider a bi-valued system, at the cognitive level (e.g., whether a person is a bank teller or not). If we cannot conduct non-disturbing measurements, then the outlined approach fails (perhaps this indicates an inherent 'quantumness', though this cannot be established with the present analysis). Suppose then that we know we can conduct non-disturbing measurements. This is a fairly standard claim in psychology, and at any rate it is empirically verifiable, so it does not constitute a serious assumption of the same type as, say, Cognitive Realism. Suppose we conduct the non-disturbing measurements at different, appropriate time points, and we find a violation of the TB inequality. What are we to conclude?

We have proven that any cognitive model satisfying Cognitive Realism and Cognitive Completeness must respect the TB inequalities (assuming non-disturbing measurements), so we are forced to abandon one of these assumptions. The crucial question is, which one?

One might think that a conservative response is to abandon the assumption of Cognitive Completeness, that is, the idea that a cognitive state can be fully determined from the probability for all relevant judgments. This implies that the cognitive model in question, as specified, needs to be augmented with extra variables. Note, because of the assumption of a classical brain, we know that all cognitive models are incomplete, that is, it is always possible to provide a description of a cognitive process, in terms of purely classical (neurophysiological) variables, which does not violate any TB inequality. For example, a characterization of a person as a bank teller must be reducible to a very complicated function of the underlying brain state. However, there are at least two problems with such an approach. The first is that it is difficult to imagine how to extend a given cognitive model in an appropriate way. We noted in Section 2 that putative hidden variables for cognitive models cannot be cognitive, but, for the sake of argument, let us consider this possibility here. What could such hidden cognitive variables possibly be? For example, given the example of Linda discussed above, what other cognitive variables might be appropriate to include, in order to extend a cognitive model, based on beliefs about properties Linda may or may not have? There are no clear prescriptions. Alternatively, we could attempt to augment a cognitive approach with neurophysiological variables, but, manifestly, this is impractical and, indeed, currently impossible (many researchers have rightly pointed out the need for consistency between so-called computational and algorithmic levels of description, e.g., Jones & Love, 2011; but this is different from requiring a full specification of cognitive variables with neurophysiological ones). The second problem is that such a solution in a sense defeats the objective of cognitive models, which is to decide in advance on a small set of decisions, to be modelled in isolation (note, not all researchers accept this assumption; e.g., Fodor, 1983), and to avoid discussing other thoughts, stimuli, judgments etc. and, indeed, the supporting neurophysiology. In a very real sense, the assumption of Cognitive Completeness is fundamental for cognitive models, even more so than realism.

If we refuse to abandon the assumption of Cognitive Completeness, then a putative violation of a TB inequality, would force us to reassess the assumption of Cognitive Realism. So far, our discussion of the TB inequalities in cognitive models has been based on the assumption that these cognitive models are classical (realist). Without the assumption of Cognitive Realism, we have to adopt non-realist cognitive models, such as ones based on quantum theory. Adopting non-realist cognitive models means that we 'forget' about the underlying classical neurophysiology of the brain and so reject the key implication of Cognitive Realism, that for the relevant set of judgments an observer can have a definite opinion about all judgments at all time points.

Such quantum cognitive models have, in fact, provided simple and intuitive explanations for important cognitive phenomena, which have persistently resisted explanations using CP principles. For example, in the famous conjunction fallacy (Tversky & Kahneman, 1983), a hypothetical person, Linda, is judged more likely to be a bank teller and a feminist, than just a bank teller (i.e., Prob(bank teller & feminist) > Prob (bank teller)). Busemeyer et al. (2011) proposed that the possibilities of bank teller and feminist are incompatible with each other, in the quantum sense, so that certainty about one possibility creates uncertainty about/a unique perspective for the other. The explanation for the conjunction fallacy is then based on the idea that the probability of a bank teller, from the perspective of having accepted Linda as a feminist rises (feminists can have all sort of professions), compared to from the baseline perspective. In this and related research, considerable effort is devoted in motivating an assumption of incompatibility and considering relevant empirical tests (cf. Pothos & Busemeyer, 2013).

That quantum cognitive models do not satisfy the assumption of Cognitive Realism is one of their defining features. This arises because there are certain cognitive states, superpositions, in which a decision maker cannot be thought of as having a definite opinion about, e.g. whether Linda is a bank teller or not. Thus, such quantum cognitive models can violate the TB inequalities, without the need to assume additional, unknown variables (i.e., without having to abandon the assumption of Cognitive Completeness).

In summary then, a violation of the TB inequalities implies that one of the two assumptions of Cognitive Realism and Cognitive Completeness must be dropped. In other words, the observation of such a violation would indicate a failure of the top-down approach to cognition, in a classical, realist way. This presents theorists with two options. First, classical cognitive models can be augmented with additional variables. But, we have argued that this option is (currently at least) not feasible and, indeed, undesirable. Second, quantum theory can be employed to model the relevant cognitive system in a non-realist way, since violations of the TB inequalities are typical for any quantum system. This is an interesting conclusion, and mostly robust, but some qualifications are needed.

A violation of the TB inequality proves that a classical cognitive model is not possible for the corresponding cognitive system, without additional variables. This, however, does not quite prove that a quantum model will be adequate. Specifically, the violation of a TB inequality involving a particular observable, at different time points, implies that it is impossible to have a joint probability distribution for the (assumed possessed) value of the observable across all these time points. This important idea, that it is impossible to concurrently fix the observable values across all time points, *suggests* (but does not prove) a key property motivating the use of quantum models, that of incompatibility (as applied to considering the same observable at different time points). Incompatibility has been at the heart of what makes many current cognitive models work, e.g., through the finding that certainty about particular properties (e.g., that Linda is a feminist) facilitates the transition to other, incompatible, properties (e.g., that Linda is a bank teller), which are unlikely from a baseline perspective (Busemeyer et al., 2011; Trueblood & Busemeyer, 2011; Wang & Busemeyer, 2013).

Relatedly, the TB inequalities may also be employed as a test of whether a quantum model be adequate to describe a system. This is because QP theory allows a violation of the TB inequality only up to a certain constant ($2 \cdot \sqrt{2}$; this is the analogue of the Tsirelson bound, in the study of the Bell inequalities; Tsirelson, 1980). Thus, the TB inequality could, in principle, disprove the applicability of not only a CP theory model, but of a QP theory model too, thereby introducing a rigorous falsifiability test.

t1	t2	t3
+	+	+
+	+	-
+	-	+
+	-	-
-	+	+

-	+	-
-	-	+
-	-	-

Table 1. The values of a bi-valued (+,-) observable at different time points. A violation of the TB inequality means that it is impossible to specify such a table, for the corresponding cognitive system.

6. Discussion

We have argued that a violation of the TB inequalities in a cognitive system would demonstrate a limit to classical top-down modelling. Arguably, the whole point of cognitive psychology is to study cognition without getting embroiled in the detailed neurophysiology of the brain and so treat everything at the level of thoughts; this idea is more formally expressed with the assumption of Cognitive Completeness. Violations of the TB inequalities mean any classical (realist) model of cognition must distinguish between different states of the brain, corresponding to the same set of thoughts. Thus, any realist model of cognition would be basically forced to include detail about neurophysiology (assuming this is how classicality arises). Quantum cognitive models, on the other hand, can overcome this problem and still model cognition purely at the level of thoughts, although one pays a price of having to accept properties such as incompatibility, superposition, and entanglement, which introduce a certain level of vagueness about exactly what is going on at any given time. Our main conclusion is that putative violations of the TB inequalities could be accounted for, whilst retaining the assumption of Cognitive Completeness, by rejecting the assumption of Cognitive Realism.

The fundamental motivation for this discussion is understanding the role of time in cognition. Mental representations change in time, but how are we to understand this putative time-dependence? A classical trajectory is the most straightforward intuition, whereby a cognitive observable has specific values across different time points. The use of QP theory in cognitive modelling provides a radically different possibility, since quantum models (or indeed any model inconsistent with Cognitive Realism) can violate the TB inequalities. If a violation of the TB inequalities for the relevant cognitive system can be established, then a well-defined history for the cognitive observable does not exist. This is not about classical uncertainty, which may arise due to noise, forgetting, etc. but, rather about the fact that the copies of the observable at different time points are incompatible with each other and so a tabulation of values at different time points, as in Table 1, is impossible (cf. Fine, 1982). For example, a specific value of the observable at, e.g., t_2 requires uncertainty about the observable both at most future time points, t_3, and earlier ones, e.g., t_1. Recalling a judgment about an observable last week, potentially makes me uncertain about the same judgment the week before, and vice versa. Equally, unless I specifically probe (e.g., with a recall process) my memory of an observable on Monday, it is very possible that this memory does not exist at all; memory recall would have to be a constructive process (the idea that measurements are 'constructive' has a long history in quantum theory; see e.g. Jammer, 1974). A sequence of memory recalls would thus be subject to interference or order effects and reveal uncertainty relations. Is this part of the process which leads to false memories? This discussion does not take into account explicit bias, which may arise from a desire to be consistent in answering the same question across successive judgments. Nonetheless, if such biases can be eliminated, there is obvious potential for a complete reconceptualization of how mental representations depend on time.

A violation of the TB inequality is sometimes said to indicate entanglement in time. The term is borrowed from the discussion of the Bell inequality. In a typical experimental set-up to study the Bell inequalities, two sub-systems are separated in a way that ensures there is no interaction. However, despite the absence of interaction, quantum theory allows for the existence of states, whose representation for the overall system is not the (tensor) product of the representations for the subsystems. For such states, the behaviour of the full system is not factorizable into what happens in each separate subsystem. The two sub-systems are said to be entangled, which in turn means that the correlations between the measurement outcomes in each subsystem may exceed classical bounds. A violation of the TB inequality can be said to reflect entanglement in time, in an analogous sense, that is, the correlations between the outcomes of measurements at different time points may exceed classical bounds. The implications for cognitive theory (e.g., theories of memory) are potentially profound.

In sum, we have discussed in precise terms what a violation of the TB inequality would mean for cognitive systems and the conditions for a robust experimental demonstration. There are clearly many conceptual and empirical challenges, but, overall, we think that a successful resolution is possible (arguably, more so in psychology than in physics). We hope therefore that this paper will further motivate research in this novel, exciting research direction.

Acknowledgements

We would like to thank Harald Atmanspacher, and Thomas Filk for helpful discussions. EMP and JMY were supported by Leverhulme Trust grant RPG-2013-00. Further, EMP was supported by Air Force Office of Scientific Research (AFOSR), Air Force Material Command, USAF, grants FA 8655-13-1-3044. The U.S Government is authorized to reproduce and distribute reprints for Governmental purpose notwithstanding any copyright notation thereon.

References

Aerts, D. and Aerts, S. (1995). Applications of quantum statistics in psychological studies of decision processes. Foundations of Science, 1, pp. 85-97.

Aerts, D., & Gabora, L. (2005). A theory of concepts and their combinations II: A Hilbert space representation. Kybernetes, 34, 192-221.

Aspect, A., Graingier, P., & Roger, G. (1981). Experimental tests of realist local theories via Bell's theorem. Physical Review Letters, 47, 460-467.

Atmanspacher, H., & Filk, T. (2010). A proposed test of temporal nonlocality in bistable perception. Journal of Mathematical Psychology, 54, 314-321.

Atmanspacher, H., Romer, H., & Wallach, H. (2006). Weak quantum theory: formal framework and selected applications. Weak quantum theory: complementarity and entanglement in physics and beyond. Foundations of Physics, 32, 379-406.

beim Graben, P. & Atmanspacher, H. (2009). Extending the philosophical significance of the idea of complementarity. In H. Atmanspacher & H. Primas (Eds.). Recasting Reality. Wolfgang Pauli's Philosophical Ideas and Contemporary Science (pp. 99-113). Berlin: Springer.

Bell, J.S. (2004). Speakable and Unspeakable in Quantum Mechanics. Cambridge University Press: Cambridge, UK.

Bruza, P. D., Kitto, K., Nelson, D. & McEvoy, C. L. (2009). Is there something quantum-like about the human mental lexicon? Journal of Mathematical Psychology, vol. 53, pp. 362-377.

Busemeyer, J. R. & Bruza, P. (2011). Quantum models of cognition and decision making. Cambridge University Press: Cambridge, UK.

Busemeyer, J. R., Pothos, E. & Franco, R., Trueblood, J. S. (2011) A quantum theoretical explanation for probability judgment 'errors'. Psychological Review, 118 (2), 193-218.

Fine, A. (1982). Joint distributions, quantum correlations, and commuting observables. Journal of Mathematical Physics, 23, 1306-1310.

Fodor, J. A. (1983). The modularity of mind. Cambridge, MA: The MIT press.

George, R, Robledo, L, Maroney, O, Blok, M, Bernien, H, Markham, M, Twitchen, D, Morton, J, Briggs, A & Hanson, R (2013). Proceedings of the National Academy of Sciences, 110, 3777-3781.

Howe, M. L. & Courage, M. L. (1997). The emergence and early development of autobiographical memory. Psychological Review, 104, 499–523.

Jammer, M. (1966). The conceptual development of quantum mechanics. New York: McGraw Hill.

Jones, M. & Love, B. C. (2011). Bayesian fundamentalism or enlightenment? On the explanatory status and theoretical contributions of Bayesian models of cognition. Behavioral and Brain Sciences, 34, 169, 231.

Khrennikov, A. Y. (2010). Ubiquitous quantum structure: From psychology to finance. Springer.

Leggett, A. J. & Garg, A. (1985). Quantum mechanics versus macroscopic realism: is the flux there when nobody looks? Physical Review Letters, 54, 857-860.

Oaksford, M. & Chater, N. (2009). Précis of Bayesian rationality: the probabilistic approach to human reasoning. Behavioral and Brain Sciences, 32, 69-120.

Palacios-Laloy, A., Mallet, F., Nugyen, F., Bernet, P., Vion, D., Esteve, D. & Korotkov, AN., Experimental violation of a Bell’s inequality in time with weak measurement. Nature Phys. 6, 442 (2010).

Pothos, E. M. & Busemeyer, J. R. (2009). A quantum probability explanation for violations of 'rational' decision theory. Proceedings of the Royal Society B, 276, 2171-2178.

Pothos, E. M. & Busemeyer, J. R. (2013). Can quantum probability provide a new direction for cognitive modeling? Behavioral & Brain Sciences, 36, 255-327.

Raijmakers, M. E. J. & Molenaar, P. C. M. (2004). Modeling developmental transitions in adaptive resonance theory. Developmental Science, 7, 149-157.

Raaijmakers, J. G., & Shiffrin, R. M. (1992). Models for recall and recognition. Annual Review of Psychology 43: 205-234.

Shafir, E. & Tversky, A. (1992). Thinking through uncertainty: nonconsequential reasoning and choice. Cognitive Psychology, 24, 449-474.

Shiffrin, R. M. (1970). Forgetting, trace erosion or retrieval failure? Science, 168, 1601-1603.

Tenenbaum, J. B, Kemp, C., Griffiths, T. L., & Goodman, N. (2011). How to grow a mind: statistics, structure, and abstraction. Science, 331, 1279-1285.

Trueblood, J. S. & Busemeyer, J. R. (2011). A comparison of the belief-adjustment model and the quantum inference model as explanations of order effects in human inference. Cognitive Science, 35, 1518-1552.

Tsirelson, B.S. (1980). Quantum Generalizations of Bell’s Inequality. Letters in Mathematical Physics, 4, 93.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjuctive fallacy in probability judgment. Psychological Review, 90, 293-315.

Wang, Z., & Busemeyer, J. R. (2013). A quantum question order model supported by empirical tests of an a priori and precise prediction. Topics in Cognitive Science.

Wang, Z., Busemeyer, J. R., Atmanspacher, H., & Pothos, E. M. (2013). The potential of using quantum theory to build models of cognition. Topics in Cognitive Science, 1-17.

Wilde, M. M. & Mizel, A. (2012). Addressing the clumsiness loophole in a Leggett-Garg test of macrorealism. Foundations of Physics, 42, 256-265.

Yearsley, J. M. (under review). The Leggett-Garg inequalities and non-invasive measurability. Pre-print available at arXiv.org/abs/1310.2149

Appendix: Derivation of the TB Inequalities

We provide a derivation of the TB inequalities, starting from the two assumptions of Cognitive Realism and Cognitive Completeness. We review the definitions below.

Cognitive Realism

The outcomes of any measurement of a cognitive variable (Q), at a given time t can be expressed (in principle, not necessarily in practice) in terms of the values of the underlying neurophysiological variables (λ) in the following way,

$$< Q(t) >= \sum_{\lambda} Q(\lambda, t)\, \rho(\lambda) \quad (*)$$

(Throughout this appendix we will consider only measurements that are non-disturbing. As noted in Section 4 this is something that can be empirically verified, so it does not constitute an extra assumption.) Note, we assume that the relevant hidden variables are neurophysiological, since, as noted in the main text, cognitive hidden variables can be discounted. Here, the λ set of variables denote the possible neurophysiological states of the participants' brains, $Q(t, \lambda)$ tells us the judgment of a participant or group of participants, at time t, given that their neurophysiological state is λ, and $\rho(\lambda)$ denotes the probability distribution of the participants' neurophysiological states, over the possible λ.

Cognitive Completeness

The cognitive state of the participants may be fixed entirely in terms of the probabilities for the judgments contained within the cognitive model. Thus, for the purposes of computing probabilities for judgments, within a cognitive model different neurophysiological states $\rho(\lambda)$ and $\rho'(\lambda)$, which lead to the same predictions for the probabilities of judgments, may be considered equivalent and interchanged where desired.

The assumption of Cognitive Completeness has an important corollary. Consider a measurement, which we make of a cognitive variable $Q(t)$, which does not change the probabilities for future measurements of Q. As noted in the main text, such measurements are called non-disturbing. However, in general, non-disturbing measurements will still have an effect on the neurophysiological state. Suppose a measurement of $Q(t_1)$ changes the neurophysiological state from $\rho(\lambda)$ to $\rho(\lambda, t_1)$. (We use the time that a measurement was carried out, rather than a dash, to denote the new state, to allow for the possibility of multiple measurements and changes.) A non-disturbing measurement of $Q(t_1)$ followed by a measurement of $Q(t_2)$ will give,

$$< Q(t_2)Q(t_1) > = \sum_{\lambda} Q(\lambda, t_2)Q(\lambda, t_1)\rho(\lambda, t_1)$$

However since the probabilities for the outcomes of the measurement of $Q(t_2)$ are not changed by the measurement of $Q(t_1)$ (this being the definition of a non-disturbing measurement) we can also write,

$$< Q(t_2)Q(t_1) > = \sum_{\lambda} Q(\lambda, t_2)Q(\lambda, t_1)\rho(\lambda, t_1) = \sum_{\lambda} Q(\lambda, t_2)Q(\lambda, t_1)\rho(\lambda) \quad (**)$$

In other words $\rho(\lambda)$ and $\rho(\lambda, t_1)$ are equivalent for any non-disturbing measurement performed at t_1. Note that is a considerably weaker requirement than the usual assumption of 'non-invasive measurability', which is that $\rho(\lambda, t_1) = \rho(\lambda)$.

Given these two assumptions we may derive a TB inequality as follows.

Consider a set of four times $\{t_1, t_2, t_3, t_4\}$ and suppose we can perform a non-disturbing measurement of $Q(t)$ at at least the first three times. Then we have,

$$< Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) >= \sum_{\lambda} [Q(\lambda, t_2)Q(\lambda, t_1) - Q(\lambda, t_4)Q(\lambda, t_1)]\rho(\lambda, t_1)$$

Here, we employ the assumption of Cognitive Realism, to write the expected value for the outcome of a measurement of Q at a series of time points, as a summation across all possible specific values of the observable, in terms of the possible neurophysiological states, weighted by corresponding probabilities.

Next, we notice we can rewrite this as,

$$< Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) >$$

$$= \sum_\lambda Q(\lambda,t_2)Q(\lambda,t_1)[1 \pm Q(\lambda,t_4)Q(\lambda,t_3)]\rho(\lambda,t_1)$$

$$- \sum_\lambda Q(\lambda,t_4)Q(\lambda,t_1)[1 \pm Q(\lambda,t_3)Q(\lambda,t_2)]\rho(\lambda,t_1)$$

taking the absolute value of both sides and using the triangle inequality gives,

$$| < Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) > |$$

$$\leq \sum_\lambda [1 \pm Q(\lambda,t_4)Q(\lambda,t_3)]|Q(\lambda,t_2)Q(\lambda,t_3)|\rho(\lambda,t_1)$$

$$+ \sum_\lambda [1 \pm Q(\lambda,t_3)Q(\lambda,t_2)]|Q(\lambda,t_4)Q(\lambda,t_1)|\rho(\lambda,t_1)$$

$$| < Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) > |$$

$$\leq \sum_\lambda [1 \pm Q(\lambda,t_4)Q(\lambda,t_3)]\rho(\lambda,t_1) + \sum_\lambda [1 \pm Q(\lambda,t_3)Q(\lambda,t_2)]\rho(\lambda,t_1)$$

$$| < Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) > |$$

$$\leq 2 \pm [\sum_\lambda Q(\lambda,t_4)Q(\lambda,t_3)\rho(\lambda,t_1) + \sum_\lambda Q(\lambda,t_3)Q(\lambda,t_2)\rho(\lambda,t_1)$$

It is important to note that the previous two steps are purely algebraic, i.e. up to this point the only assumption we have made is that of Cognitive Realism. However, an examination of the terms on the right hand side of the above equation shows that they are not of the correct form, to be regarded as two-time correlation functions. This is because, in each of the products for the observable values, the first measurement is at t_2 or t_3, but the neurophysiological state has been altered not as a result of a measurement at these times, but rather by an apparently unperformed measurement at t_1. However, if we assume Cognitive Completeness then, as noted above, we can drop the dependence of the neurophysiological state on the measurements and conclude that

$$| < Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) > \leq 2 \pm [\textstyle\sum_\lambda Q(\lambda, t_4)Q(\lambda, t_3)\rho(\lambda) + \sum_\lambda Q(\lambda, t_3)Q(\lambda, t_2)\rho(\lambda)]$$

$$| < Q(t_2)Q(t_1) > - < Q(t_4)Q(t_1) > | \leq 2 \pm [< Q(t_4)Q(t_3) > + < Q(t_3)Q(t_2) >]$$

The final line is the desired four time temporal Bell inequality.

Let us finally briefly discuss the derivation of a three-time TB inequality, since this involves an extra assumption, touched on in the main text. To do this, we first set $t_4 = t_3$ to get,

$$| < Q(t_2)Q(t_1) > - < Q(t_3)Q(t_1) > | \leq 2 \pm [< Q(t_3)Q(t_3) > + < Q(t_3)Q(t_2) >]$$

Now consider the quantity

$$< Q(t_3)Q(t_3) > = \sum_\lambda Q(\lambda, t_3)^2 \rho(\lambda)$$

If we assume that $Q(\lambda, t)$ can only take the values ± 1 for any λ and t then it follows that

$$Q(\lambda, t)^2 = 1$$

Therefore we have

$$< Q(t_3)Q(t_3) > = \sum_\lambda \rho(\lambda) = 1$$

and thus the three time temporal Bell inequality becomes ,

$$| < Q(t_2)Q(t_1) > - < Q(t_3)Q(t_1) > | \leq 2 \pm [1 + < Q(t_3)Q(t_2) >]$$

$$< Q(t_2)Q(t_1) > + < Q(t_3)Q(t_2) > \leq 1 + < Q(t_3)Q(t_1) > \qquad (***)$$

Atmanspacher and Filk (2010) gave their TB inequality in terms of a different variable, $N^{-}(t_1, t_2)$, which is defined as +1 if the state changes between t_1 and t_2 and 0 otherwise. It is easy to see that $N^{-}(t_1, t_2)$ may be written as,

$$N^{-}(t_1, t_2) = \frac{1}{2}(1 - < Q(t_1)Q(t_2) >)$$

and so (***) may be written in these new variables as,

$$N^{-}(t_1, t_3) \leq N^{-}(t_1, t_2) + N^{-}(t_2, t_3)$$

This is the form given in Atmanspacher and Filk (2010).

Note though, as mentioned in the main text, the assumption that $Q(\lambda, t) = \pm 1$ is rather strict because it implies, firstly, that the map between neurophysiological states and cognitive judgments is deterministic and, secondly, that the measurements are noise free (both assumptions are required to assume that multiple copies of $Q(\lambda, t)$ produce the same value). In particular $< Q(t)Q(t) >= 1$ implies that, for every participant, a subsequent measurement preformed immediately after the first one will yield exactly the same result. In practice, this is unlikely to be the case and thus the derivation of the three-time TB inequality from the four-time one will not hold for a realistic experimental set up.