Verifying Robust Frequency Domain Properties of Non Linear Oscillators using SMT

Hafiz ul Asad
City University London
EC1 0HB London,UK
Email: hafiz.ul-asad.1@city.ac.uk

Kevin D. Jones
City University London
EC1 0HB London,UK
Email: kevin.jones.1@city.ac.uk

Frederic Surre
City University London
EC1 0HB London,UK
Email: frederic.surre.1@city.ac.uk

***Abstract*—We present a novel mixed time and frequency domain approach to the formal verification of oscillators properties which are specified in the frequency domain. We use robust periodogram specification to specify the oscillator behaviour in the close vicinity of the limit cycle. Using SAT modulo ODE (SMO) for Bounded Model Checking (BMC) of the non-linear hybrid automata, we show that the oscillator hybrid timed traces satisfy frequency domain specifications.**

I. Introduction

Significant time is spent, in the industry, verifying analog and mixed signal (AMS) circuits using SPICE simulations. Formal methods have been successfully used to verify digital circuits and could provide better solutions for more reliable, less time consuming AMS circuits design too.

This paper describes the formal verification of the frequency domain properties of a non-linear oscillator when it operates in the close proximity of its limit cycle. We propose a mixed time and frequency domain approach for this purpose, and show that the hybrid timed traces of an oscillator, robustly belongs to the frequency domain power spectral envelop specified as constraints on periodogram at harmonic frequencies. We model an oscillator circuit by the non-linear hybrid automaton and use the recent SMO technique for BMC of hybrid automata [1],[2], to compute the periodic invariant set (Limit Cycle). This limit cycle is verified against the robust frequency domain properties specification represented as constraints on its periodogram [3] at frequencies of interest.

A survey of the recent formal Analog and Mixed Signal (AMS) verification approaches can be found in [4]. Frequency domain approaches have been limited to the small signal AC analysis of a more approximate linearized model around an equilibrium point [5],[6].

II. Preliminaries

A. Non-linear Dynamical Systems as Hybrid Automata

Definition 1 (Non-linear Hybrid Automata)**.**
A Non-linear Hybrid Automata [7] is a tuple,
H=(Loc, Var, Flow, Inv, Trans) where,

- Loc is a finite set of locations.
- Var is a set of continuous variables,
  Var= $\{x_1, x_2.....x_n\} \subset \mathbb{R}^n$.
- Flow is the set of vector fields, i.e. Flow(ℓ) is an autonomous subsystem for each $\ell \in$ Loc and is of the form,

$$\dot{x} \quad = \quad f_\ell(x, u) \tag{1}$$

  $f_\ell \,:\, D_\ell^n \times \mathcal{U}^m \;\mapsto\; D_\ell^n$ is a non-linear but at least locally Lipschitz function of continuous vector $x \in D_\ell^n$, and a non deterministic vector $u \in \mathcal{U}^m$ of inputs and parameters.
- Inv is a constraint on the domain D_ℓ^n of each location $\ell \in$ Loc,

$$\text{Inv}(\ell) \quad = \quad \mathsf{I}_\ell(x(t), u) \geq 0 \tag{2}$$

- Trans is a set of discrete transitions; Each transition $\tau \in$ Trans, is a tuple $\tau = (\ell, guard_\tau, r_\tau, \ell')$; where $(\ell$,$\ell') \in$ Loc are the pre and post modes respectively, and $guard_\tau$ is a switching conditions given by system of equations,

$$guard_\tau \quad = \quad G_\tau(x(t), u) = 0 \tag{3}$$

  here $(guard_\tau \subset D_\ell^n) \in \mathcal{G}$, $\mathcal{G}$ being the set of guards. When a guard condition is met, a discrete transition takes place. $r_\tau \in \mathcal{R}$ is a reset, where for each $\tau \in$ Trans, it is a relation between elements of $guard_\tau$ and elements of $D_{\ell'}^n$, i.e., $r_\tau \subset guard_\tau \times D_{\ell'}^n$. Here $\mathcal{R}$ is the set of resets.

We use $D = \bigcup_\ell D_\ell^n$.

B. Non-linear Hybrid Automata Verification Using SAT modulo ODE

Andreas et al. in [1], presented SMO technique for the non-linear hybrid automata verification. Essentially, it is a technique based on the BMC of the non-linear hybrid automata, encoded as a large number of constraints; involving boolean, linear and non-linear algebraic, and non-linear ODE constraints. Establishing reachability of a target region (interval), predicative encoding of the hybrid transition system is used, i.e.,

$\Phi \;=\; DECL[0] \wedge .. \wedge DECL[N] \wedge Init[0] \wedge Trans[0,1] \wedge .. \wedge Trans[(N-1), N] \wedge Target[N]$.

This is a N-step unfolding of the transition system; where $DECL[N]$ are the simple bounds on variables in the N-th step, $Init[0]$ is the predicate for initial conditions at the 0-th step, $Trans(N, N-1)$ is the transition relation between variables during N-th and (N-1)th step, and $Target[N]$ is the instantiation of the target predicate at the N-th step.

C. Limit Cycles in Hybrid Systems

Here we introduce concepts of the limit sets, periodic orbits, and the limit cycles in hybrid automata. We define a map $\Phi^H : \mathbb{R} \times D \mapsto D$, which describes piecewise smooth flow over the hybrid domain D.

Definition 2 (Hybrid Limit Sets)**.**
A point $z \in D$ is called an ω-limit point of $y \in D$ if there is a sequence $t_n \to \infty$ for which, $lim_{n\to\infty}\Phi^H(t_n, y) = z$. The set of all such points of z, is the hybrid ω-limit set $\mathcal{L}^H_\omega(y)$.

Definition 3 (Hybrid periodic Orbits)**.**
An orbit η is a closed periodic orbit if, for some $x \in \eta$, it is not an equilibrium (i.e. $\Phi^H(t, x) \neq x$), and $\Phi^H(T, x) = x$, for some smallest $T \neq 0$. T is called the fundamental period of η. If η belongs to multiple domains D_ℓ, then it is called a hybrid periodic orbit.

Definition 4 (Hybrid Limit cycle)**.**
A closed hybrid orbit η, is called a hybrid limit cycle if, $\eta \subset \mathcal{L}^H_\omega(y)$ for some $y \notin \eta$.

III. Frequency Domain Properties Specification of Hybrid Limit Cycle

This section introduces robust frequency domain properties specification of the hybrid limit cycle using periodogram based power spectral envelop.

A. Robust Specification of a Periodic Function in Frequency Domain

A scalar function g is periodic with period T if $g(t) = g(t+nT), \forall t \in \mathbb{R}$ and $\forall n \in \mathbb{Z}$. We denote by $\mathcal{P}$, the set of all functions, which apart from being T periodic, also have the property of square sumability over a period T, i.e., $\mathcal{P} \subset L^2[0, T]$. All such periodic functions $g(t) \in \mathcal{P}$ can be represented by the sum of an infinite number of T-periodic sinusoids as,

$$g(t) \quad = \quad \sum_{k=0}^{\infty}(a_k \cos\omega_k t + b_k \sin\omega_k t) \tag{4}$$

where $\omega_k = 2\pi k/T$, $a_k, b_k \in \mathbb{R}$. Instead of an infinite series representation of the periodic functions, we use notion of the almost periodic functions [8], which are represented by at most a countable number of sinusoids. We denote such set of almost periodic functions by $A\mathcal{P}$, and therefore $g(t) \in \mathcal{P}$ is represented by its approximation $S_k(t) \in A\mathcal{P}$,

$$S_K(t) \quad = \quad \sum_{\omega_k \in \Omega_K}(a_k \cos\omega_k t + b_k \sin\omega_k t) \tag{5}$$

where Ω_K is the set of K frequencies. The finite series representation $S_K(t)$ is the best approximation of $g(t)$, and it has a least mean square error property. Let $\varepsilon_K = max\|g(t) - S_K(t)\|$ represent the maximum approximation error, then $g(t)$ can be conservatively represented by $S_K(t) - \varepsilon_K \leq g(t) \leq S_K(t) + \varepsilon_K$. The set $\mathcal{F}$ of all pairs $\{(a_0, b_0), ...(a_k, b_k)\}$, of the Fourier coefficients is called the frequency domain representation of a periodic function $g(t)$. Instead of specifying a periodic function $g(t)$ in the frequency domain in terms of the set $\mathcal{F}$, we use the periodogram specification which is defined below.

Definition 5 (Periodogram)**.**
The energy content of a signal at each frequency ω_k is called a periodogram, and is given by, $\mathsf{p}_k = (a_k^2 + b_k^2)$. We denote by $P = \{\mathsf{p}_0, ...\mathsf{p}_K\}$, the set of all periodograms at frequencies $\omega_k \in \Omega_K$.

To cater for parameter variations, temperature and uncertainty in initial conditions, we introduce the idea of robust periodogram specification.

Definition 6 (Robustness of Periodogram)**.**
We specify P such that pairs of the Fourier series coefficients (a_k, b_k) for all $\omega_k \in \Omega_K$, result in the function $S_K(t)$ (Eq. 5), which is the approximate representation of the periodic function $g(t)$ and satisfy the inequality constraint $S_K(t) - \varepsilon_K \leq g(t) \leq S_K(t) + \varepsilon_K$. We say that $\mathsf{p}'_k \in P$ has ϵ_k degree of robustness, if it can tolerate an ϵ_k amount of perturbation such that, $\exists \mathsf{p}_k \in P : \{\mathsf{p}_k - \epsilon_k \leq \mathsf{p}'_k \leq \mathsf{p}_k + \epsilon_k\}$.

B. Encoding Membership of the Limit Cycle in the Robust Power Spectral Envelop

Let η is a vector of scalar valued functions of time, $\eta(t) := \{\eta_1(t), ..., \eta_n(t)\} : \mathbb{R} \mapsto D$. In other words, $\eta(t) = \Phi^H_t(x); \forall x \in \eta$, i.e., $\eta(t)$ represent the information about the hybrid limit cycle η at each time t. We define a power spectral envelop $H(\omega_k) : \Omega_K \mapsto \mathbb{R}^+$, which maps each discrete frequency $\omega_k \in \Omega_K$ to a periodogram p_k. The set $A\mathcal{P}_{\epsilon_k}$ of all almost periodic functions belongs to the power spectral envelop $H(\omega_k)$ with ϵ_k degree of robustness, if the Fourier series coefficients satisfy the following constraints [9],

- $\forall k \in \mathbb{N},\ (\omega_k > \omega_K) \implies \mathsf{p}_k = 0$,
- $\forall k \in \mathbb{N},\ H(\omega_k) - \epsilon_k \leq \mathsf{p}_k \leq H(\omega_k) + \epsilon_k$, such that $0 \leq \omega_k \leq \omega_K$.

We require that for each $S_{n,K}(t) \in cl(A\mathcal{P}_{\epsilon_k})$, the scalar periodic orbit $\eta_n(t)$ satisfies the constraint $S_{n,K}(t) - \varepsilon_{n,K} \leq \eta_n(t) \leq S_{n,K}(t) + \varepsilon_{n,K}$. Here $cl(A\mathcal{P}_{\epsilon_k})$ denotes closure of $A\mathcal{P}_{\epsilon_k}$. We encode this by introducing the following set of constraints for the vector $\eta(t)$,

$$\psi_1 = \bigwedge_{n=1}^{N}\left(\bigwedge_{k=0}^{K}(H_n(\omega_k) - \epsilon_{n,k} \leq \mathsf{p}_{n,k} \leq H_n(\omega_k) + \epsilon_{n,k})\right),$$

$$\psi_2 = \bigwedge_{n=1}^{N}\left(\forall t \in [t_{min}, t_{max}]\right.$$
$$\left.\left(S_{n,K}(t) = \sum_{k=0}^{K}(a_{n,k}\cos\omega_k t + b_{n,k}\sin\omega_k t)\right)\right),$$

$$\psi_3 = \bigwedge_{n=1}^{N} \Bigg(\forall t \in [t_{min}, t_{max}]$$
$$\bigg(S_{n,K}(t) - \varepsilon_{n,K} \leq \eta_n(t) \leq S_{n,K}(t) + \varepsilon_{n,K} \bigg) \Bigg).$$

Here the first constraint ψ_1 puts upper and lower bounds on the periodograms at K frequencies in the presence of $\epsilon_{n,k}$ perturbation for N scalar periodic functions. The second constraint ψ_2 ensures that for all time t all the N periodic variables are approximated by K sinusoids. The last constraint ψ_3 conservatively over-approximate the periodic function η_n taking in to consideration the error generated by the almost approximate periodic function $S_{n,K}$. The universal quantifications in the last two constraints are implicit, i.e. the BMC algorithm using SAT modulo ODE verify, whether there is any time instant t, at which any of these constraints are violated.

C. Membership as BMC Target Predicate

We determine the membership of the hybrid timed traces in the robust power spectral envelop by incorporating the additional set of constraints $\psi_1, \psi_2, , \psi_3$, in the BMC algorithm discussed in section II.B. The initial conditions to the BMC is given in the form of a box $B_{initial}$ (Considering two dimensional system). Apart from the BMC ODE constraints, we add the set of constraints $\psi_1, \psi_2, , \psi_3$, for each scalar variable x_n to the BMC algorithm. In the ‘Target’ of the BMC algorithm, we introduce the following predicate, i.e.,

$\neg(time > 0 \wedge time <= t_{max} \wedge x_n \in B_{initial}) \vee$
$\neg(\|\eta_n(t) - x_n(t)\| \leq \sigma)$

This target predicate is actually a disjunction of two predicates. The predicate $\neg(time > 0 \wedge time <= t_{max} \wedge x_n \in B_{initial})$, ensures that starting in the box $B_{initial}$, the trajectories would return back to the same box before the maximum time limit is elapsed. A satisfiable valuation of this predicate is a counterexample of the periodicity property. The second predicate $\neg(\|\eta_n(t) - x_n(t)\| \leq \sigma)$, ensures that for all the time, the distance of the hybrid timed traces from the possible time domain periodic trajectories obtained from the frequency domain specification, must be less than a user defined error. A satisfiable valuation of this predicate indicates the violation of the frequency domain specification implicitly.

IV. Experimental Evaluation

A. Evaluation Methodology

We have used Tunnel diode Oscillator (TDO) and Voltage Controlled Oscillator (VCO) benchmarks for the evaluation of our proposed methodology Figs. 1, that have been taken from [10],[11]. Equations Eq. 6, and Eq. 7, represent the non-linear ODE model of the TDO, where $I_d(V_d)$ is the non-linear model of the tunnel diode. Mathematical model of VCO is given in Eqs. 8, 9, 10, where $I_{DS}(V_{GS}, V_{DS})$ is the Schichman-Hodges PMOS model [11]. For TDO we have used parameters, $C = 1nF \pm 2\%$, $L = 1mH \pm 2\%$, $R = 0.2$Ohm and V$_{in} \in$

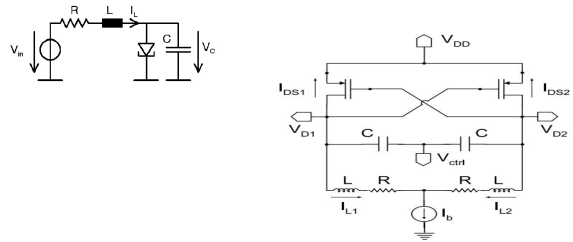


Figure 1: *Oscillators Circuit Diagrams,Left:TDO,Right:VCO*

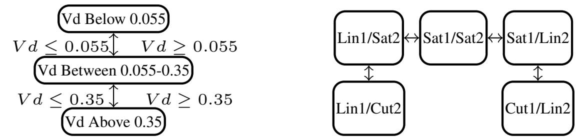


Figure 2: *Hybrid Automaton,Left:TDO,Right:VCO*

$[0.35, 0.36]$. Similarly, for VCO we have set, $C = 3.43nF \pm 2\%, L = 2.85mH \pm 2\%, V_{ctr} = 0$ and $V_{DD} \in [1.8, 1.85]$. We have used the SMO solver iSAT-ODE [1], to exercise BMC formulations of the non-linear hybrid automata, and Matlab [12] to compute periodogram specifications. We have used a 2.6 GHZ Intel(R) Core(TM) i5 machine with 4 GB of memory for all the experiments.

$$\dot{V_d} = \frac{1}{C}(-I_d(V_d) + I_L) \quad (6)$$
$$\dot{I_L} = \frac{1}{L}(-V_d + I_L.R + V_{in}) \quad (7)$$

$$\dot{V}_{D1} = \frac{-1}{C}(I_{DS1}(V_{D2} - V_{DD}, V_{D1} - V_{DD}) + I_{L1}) \quad (8)$$
$$\dot{V}_{D2} = \frac{-1}{C}(I_{DS2}(V_{D1} - V_{DD}, V_{D2} - V_{DD}) + I_b - I_{L1}) \quad (9)$$
$$\dot{I}_{L1} = \frac{1}{2L}(V_{D1} - V_{D2} - R(2I_{L1} - I_b)) \quad (10)$$

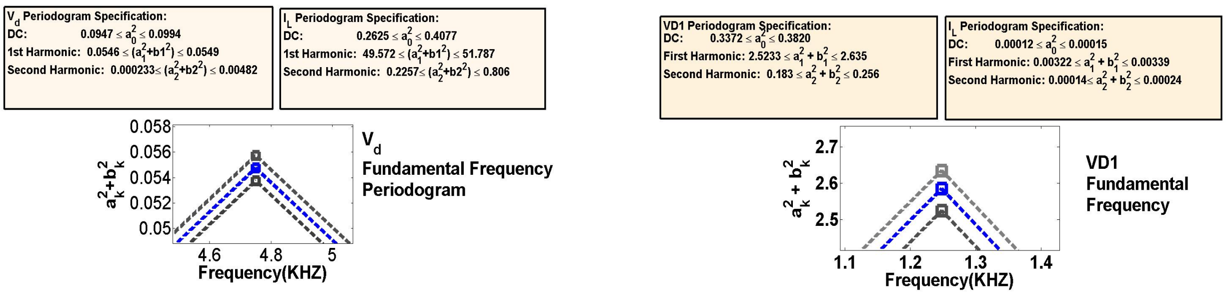
B. Results

Based on the non-linear diode and PMOS models in [10],[11], we got the non-linear hybrid automatons of TDO and VCO Fig. 2. Simulation traces are shown in Fig. 3a, Fig. 3b, whereas periodogram specifications for these traces are in Fig. 4a,. 4b, for TDO and VCO respectively. Here we have only shown specification for the fundamental frequency of the variables (V_d for TDO, and $VD1$ for VCO). The upper and lower bounds on these periodograms have been found based on the designer judgement, i.e., we chose random values in the parameter spaces and correspondingly varied the “ power spectral envelop" and arrived at these bounds. Taking $V_d \in [0.55, 0.58], I_L = 0.0$ as the initial conditions for the state variables, we model checked the TDO hybrid automaton for eight unwindings of the BMC formula Tab. IIa. Similarly for VCO, we considered initial conditions $VD1 \in [-1.5, -1.4]volts, VD2 \in [-0.9, -0.8]volts, I_L = 0.06mA$ and obtained the BMC results for eight unwindings of the formula Tab. IIb.

(a) TDO Limit Cycle Simulation

(b) VCO Limit Cycle Simulation

Figure 3: *Simulation Traces of Hybrid Automata*



(a) TDO Robust Periodogram Specification

(b) VCO Robust Periodogram Specification

Figure 4: *Frequency Domain Properties Specifications*

Depth	Decision	Time(Seconds)
0	Unsatisfiable	0
1	Unsatisfiable	81.07
2	Unsatisfiable	83.22
3	Unsatisfiable	304.37
4	Unsatisfiable	352.44
5	Unsatisfiable	1299.64
6	Unsatisfiable	1448.71
7	Unsatisfiable	26779.75
8	Unsatisfiable	27096.21

(a) TDO Verification Results

Depth	Decision	Time(Seconds)
0	Unsatisfiable	0
1	Unsatisfiable	6.13
2	Unsatisfiable	206.45
3	Unsatisfiable	538.39
4	Unsatisfiable	947.10
5	Unsatisfiable	2237.89
6	Unsatisfiable	3457.43
7	Unsatisfiable	11672.11
8	Unsatisfiable	15892.13

(b) VCO Verification Results

Table I: *Experimental Results.*

V. Conclusion

In this paper we have presented a novel mixed time and frequency domain approach to verify frequency domain properties of oscillators when they operate in the close vicinity of the limit cycle.

References

[1] Andreas Eggers, Martin Fränzle, and Christian Herde. Sat modulo ode: A direct sat approach to hybrid systems. In *Automated Technology for Verification and Analysis*, pages 171–185. Springer, 2008.

[2] Sicun Gao, Soonho Kong, and Edmund M Clarke. Satisfiability modulo odes. In *Formal Methods in Computer-Aided Design (FMCAD), 2013*, pages 105–112. IEEE, 2013.

[3] Michail Vlachos, Philip Yu, and Vittorio Castelli. On periodicity detection and structural periodic similarity. In *SIAM International Conference on Data Mining*, pages 449–460, 2005.

[4] Mohamed H Zaki, Sofiène Tahar, and Guy Bois. Formal verification of analog and mixed signal designs: A survey. *Microelectronics Journal*, 39(12):1395–1404, 2008.

[5] Lars Hedrich and Erich Barke. A formal approach to verification of linear analog circuits wth parameter tolerances. In *Proceedings of the conference on Design, automation and test in Europe*, pages 649–655. IEEE Computer Society, 1998.

[6] William Denman, Behzad Akbarpour, Sofiene Tahar, Mohamed H Zaki, and Lawrence C Paulson. Formal verification of analog designs using metitarski. In *Formal Methods in Computer-Aided Design, 2009. FMCAD 2009*, pages 93–100. IEEE, 2009.

[7] Rajeev Alur, Costas Courcoubetis, Nicolas Halbwachs, Thomas A Henzinger, P-H Ho, Xavier Nicollin, Alfredo Olivero, Joseph Sifakis, and Sergio Yovine. The algorithmic analysis of hybrid systems. *Theoretical computer science*, 138(1):3–34, 1995.

[8] Kenneth S Kundert and Alberto Sangiovanni-Vincentelli. Finding the steady-state response of analog and microwave circuits. In *Custom Integrated Circuits Conference, 1988., Proceedings of the IEEE 1988*, pages 6–1. IEEE, 1988.

[9] Aleksandar Chakarov, Sriram Sankaranarayanan, and Georgios Fainekos. Combining time and frequency domain specifications for periodic signals. In *Runtime Verification*, pages 294–309. Springer, 2012.

[10] Goran Frehse, Bruce H Krogh, Rob A Rutenbar, and Oded Maler. Time domain verification of oscillator circuit properties. *Electronic Notes in Theoretical Computer Science*, 153(3):9–22, 2006.

[11] Goran Frehse, Bruce H Krogh, and Rob A Rutenbar. Verifying analog oscillator circuits using forward/backward abstraction refinement. In *Proceedings of the conference on Design, automation and test in Europe: Proceedings*, pages 257–262. European Design and Automation Association, 2006.

[12] MATLAB. *Version 8.2 (R2013b)*. The MathWorks Inc., 2013.