AUGMENTED INTEGRATED SYSTEM OPTIMISATION AND PARAMETER ESTIMATION TECHNIQUES FOR ON-LINE HIERARCHICAL CONTROL OF LARGE SCALE INDUSTRIAL PROCESSES

BY

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DECLARATION

The author grants powers of discretion to the University Librarian to allow this thesis to be copied in whole or part without further reference to her. This permission covers only single copies made for study purposes, subject to normal conditions of acknowledgement.

PUBLICATIONS

The following papers, based on the work described in this thesis, have been published :

(1) Extended hierarchical augmented lagrangian adaptive technique for optimising control of large scale systems [1].

(2) An augmented model based double loop iterative technique for integrated system optimisation and parameter estimation of large scale industrial processes [13].

(3) Comparison of some algorithms for hierarchical steady-state optimising control of interconnected industrial processes [49].

ABSTRACT

This research investigates on-line steady-state optimisation techniques based on the augmented Lagrangian function. The augmented Lagrange function is applied to a class of hierarchical Integrated System Optimisation and Parameter Estimation (ISOPE) algorithms. The ISOPE algorithm employs model equations and information feedback from the process. In order to overcome the model uncertainty, the algorithm needs to perform both optimisation and identification tasks. The identification task is to iteratively update the model parameter value which will be used to improve the modified optimisation problem.

Two different methods of augmentation are developed for augmenting the ISOPE algorithm : the variable and constraint augmentation. The applicability of these new versions of augmented algorithms are extended to non-convex problems. The constraint augmented algorithms with double iterative coordination strategies are found to be the most efficient algorithms in reducing the number of required set-point adjustments compared to the other algorithms which have been proposed.

The optimality and convergence properties of these augmented algorithms are derived. Based on the convergence studies and simulation results, a comparative study on the applicability of the algorithms are performed.

Among the proposed augmented ISOPE algorithms, the model based double loop algorithm is the most suitable for on-line application and futher work can be performed to investigate the on-line behaviour of the algorithm. The author also hopes that this work will provide valuable information for futher development of the ISOPE algorithms.

LIST OF SYMBOLS

$A(.,.)$	algorithm mapping for system based double
	loop technique
$\tilde{A}$	lipschitz constant
A_*	reality lipschitz constant
a_*	reality manotonic constant
$\underset{\sim}{a}$	monotonic constant
Arg	argument
$b(.)$	positive real-valued upper semicontinuous
$B(.)$	positive real-valued function
c, c_i	control (set-point)
$\hat{c}(.)$	control solution mapping of optimisation
C, C_i	feasible set of control
CUY, CUY_i	feasible set of control
$D(.)$	Dual function
$d(.)$	manipulated input mapping
$f(.)$	output mapping
$F(.)$, $F_{*i}(.)$	model input-output mapping

$\tilde{g}(.)$, $\tilde{g}_i(.)$	constraint mapping
$\tilde{g}_*(.)$, $\tilde{g}_{*i}(.)$	reality constraint mapping
$G(.)$, $G_{ij}(.)$	constraint mapping
$G_*(.)$	reality constraint mapping
$g(.)$, $g_i(.)$, $g_{ij}(.)$	model interaction imbalance
$g_*(.)$, $g_{*i}(.)$, $g_{*ij}(.)$	reality interaction imbalance
$h(.)$	set-point mapping
H, H_i, H_{ij}	interconnection matrices
inf	infimum
I	identity matrix
$k_{v,w}$	gain matrix for set-point and input
$K_*(.)$, $K_{*i}(.)$	reality control-output mapping
lim	limit
$L(.)$	lagrange function
$L_*(.)$	reality lagrange function
$\tilde{m}$, $\tilde{m}_i$	manipulated input
m	number of output
n	number of control input

N	number of subsystem
p, p_i, p_{ij}	price vector
$q(.)$, $q_i(.)$	performance index
$q_*(.)$, $q_{*i}(.)$	reality performance index
$Q(.)$, $Q_i(.)$	performance index
r	lipschitz constant
R^a	a-dimensional real space eg., a=m,n, or t
R_x	gain matrix for set-point and input
R_p	gain matrix for price
sup	supremum
t, t_i, t_{ij}	lagrange multiplier
u, u_i, u_{ij}	interaction input
$\hat{u}(.)$	interaction input solution mapping of optimisation
$\tilde{v}$	observation
v, v_i	control vector (applied to process)
w, w_i	input vector (applied to process)
$\tilde{x}(.)$, $\tilde{x}_i(.)$	state variable vector
y, y_i, y_{ij}	output vector (interaction)

y_*, y_{*i}, y_{*ij}	reality output vector
$\tilde{z}$	disturbance vector
α, α_i	model parameter vector
$\hat{\alpha}(.)$	solution mapping of parameter estimation
$\gamma(.)$	component mapping of system based double loop technique
δ	lipschitz constant
∂	partial derivative
ε	gain coefficient
ε_v	gain coefficient for set-point
ε_ξ	gain coefficient for lagrange multiplier
ε_p	gain coefficient for price
ε_a	gain coefficient for approximation loop
$\zeta_{[to,t]}(.)$	output mapping
η	lagrange multiplier
λ, λ_i, λ_{ij}	modifier vector (lagrange multiplier)
λ_{min}, λ_{max}	minimal eigenvalue of
μ	lagrange multiplier
ξ	lagrange multiplier

$\xi(.)$	lagrange multiplier associated with optimisation solution
π	projection operator from $R^n \times R^m$ to R^n
ϱ	penalty multiplier
τ	positive scalar
$\psi_{v,i}(.)$	iterative strategy for set-point
$\psi_{w,i}(.)$	iterative strategy for input
$\psi_{\xi,i}(.)$	iterative strategy for lagrange multiplier
$\omega(.)$	component mapping of system based double loop algorithm
$\varphi_{[to,t]}(.)$	state variable mapping
Ω	solution set

CHAPTER 1 - INTRODUCTION

1.1 Control Systems

In recent years significant progress has been made in control systems. These control systems have played an important role in meeting increasing demands for greater output, lower cost and improved quality of product. The advances made in digital computers and more recently in microcomputers have also contributed to the popularity and importance of control systems in all industries.

Digital computers are employed entensively for simulation and computation of control systems. Digital computer simulations are used to conduct the analysis and design of complex systems which cannot be solved by any of the established analytical methods. Another important application of digital computers is their use as controllers. The use of digital computers in control systems has become increasingly important because of their computing speed, storage, capacity and flexibility. Digital computers are also used in direct on-line control systems.

Most large scale industrial processes consist of interconnected subsystems and the number of variables involved is large. Designing a centralised control system for such a complex process may be very difficult or almost impossible. For instance, there may be some difficulties for a single computer to handle such a large amount of computations due to the limited storage, and the solution process may be time consuming. However, this problem can be overcome by decomposition techniques and hierarchical structures. Employing static optimisation algorithms to determine the optimal process variable can also lead to substantial economic saving.

A common practice in controlling an industrial process is to split the control action into parts : the follow-up or direct control and the supervisory control. The task of the direct control is to keep the chosen process variables at their desired values or set-points. The supervisory control, which performs steady-state optimising control, is responsible for computing optimal values of the set-points.

1.2 Optimising Control

The objective of the optimising control is to optimise a specific performance criterion for the controlled system subject to changing external or process conditions, assuming that the optimal operating condition changes slowly as external factors change.

Generally, the optimising control can be implemented through a direct on-line optimisation approach using measurements, or the model approach. In practice, algorithms for the direct on-line optimisation approach are not easy to implement because of the nature of process dynamics which are often slow, and in the presence of noise causing the performace of such algorithms to deteriorate.

Model methods provide an alternative approach to optimising control. The model refers to the set of relationships and equality constraints which describe the process behaviour. The model structure and model parameter values can be obtained using various identification techniques. The resulting model based optimisation problem can be readily solved on digital computers using mathematical programming together with hierarchical decomposition techniques.

In practice, the mathematical model used in the model based optimisation problem will generally deviate from the actual system, resulting a in sub-optimal solution. The approximations in the mathematical model can be compensated through model adaptation. This technique, sometimes refered to as parameter estimation, involves a periodic adjustment of the parameters of the model.

Attempts have been made to couple the optimisation problem with parameter estimation problem to form an integrated iterative technique (Durbeck, 1965 ; Foord, 1974 ; Youle and Duncanson, 1970 ; Roberts, 1979). Durbeck (1965) and Foord (1974) formulated a sufficient condition that at the final converged optimal model based solution the derivatives of model outputs with respect to the controller set-points should match exactly the corresponding derivatives in the real process. An algorithm, known as the modified two-step technique, proposed by Roberts (1979), which is also applicable to problems with model-reality differences, gives the real optimal solution. The modified two-step technique is an improved version of the two-step approach.

1.3 Scope and Aim of the Thesis

The two-step approach is an iterative technique which involves successive solutions of the system optimisation and model parameter estimation problems. The algorithm computes optimum values of feedback controller set-points using a steady-state mathematical model of the process. In practice, accurate mathematical models of the processes are rarely available, and therefore a model parameter estimation problem is used to update the models.

The two-step approach is a sub-optimal technique because the derivatives of the model outputs with respect to the controller set-points do not match exactly the corresponding derivatives in the real process at the final converged solution of the model optimisation (Durbeck 1965 ; Foord 1974). Roberts (1979) proposed the modified two-step approach which is an improved version of the two-step approach technique, in that it incorporates a modifier to compensate for mismatch in derivatives. This method has been successfully applied to many example problems (Roberts and Williams, 1981 ; Ellis and Roberts, 1981 ; Stevenson, Brdyś and Roberts, 1985 ; Brdyś , Ellis and Roberts 1987). Since the technique couples the two problems together, the optimisation and the parameter estimation problems, the above algorithms is sometimes called the integrated system optimisation and parameter estimation (ISOPE) algorithm. The modified two-step approach has been extended to a class of problems where process inequality constraints depend on process outputs (Brdyś, Chen and Roberts 1986) and the global convergence condition for the algorithm, without output dependent inequality constraints has also been derived (Brdyś and Roberts, 1987).

Michalska, Ellis and Roberts (1985) extended the modified two-step approach to large-scale interconnected processes, by combining the modified two-step algorithm with the price correction mechanism. Brdyś and Roberts (1986), using a similar but more systematic approach, developed a group of hierarchical adaptive optimal algorithms. Three techniques were also proposed to solve the above hierarchical adaptive optimal algorithm : the single loop, system based double loop and the model based double loop techniques (Chen, Brdyś and Roberts (1986) ; Brdyś, Abdullah and Roberts (1986)). The model based double loop technique was based on the previous work of Shao and Roberts (1983), and has the advantage of reducing the number of set-point changes and consequently the time for determining the optimal operating condition when applied to industrial processes with the usual slow dynamics. All the ISOPE algorithms, discussed to date, are based on the normal lagrangian. The applicability of the ISOPE algorithm is thought to be wider if augmentation is applied to the algorithm.

In this thesis, research will be concentrated on the augmented ISOPE algorithm based on the previous works (Brdyś, Abdullah and Roberts (1986) ; Chen, Brdyś and Roberts (1986)). The formulation of the augmented ISOPE algorithm together with the optimality and convergence conditions will be derived. A comprehensive simulation study of the augmented algorithm will also be investigated.

1.4 The Outline of the Thesis

Chapter 2 introduces the basic concepts of the multilevel and multilayer structures used in control systems. Applying the multilayer concept, the control action is split into layers according to the frequency of occurance of disturbances, where each layer has different time horizons. The optimisation and control problems are divided into coordinated subsystems to form a multilevel structure. These concepts provide a background for developing hierarchical control used in designing complex control systems for large scale industrial processes.

The two-step approach of the integrated system optimisation parameter estimation (ISOPE) algorithm is introduced in chapter 3. A class of optimal ISOPE algorithms based on the improved version of the two-step approach is defined. The algorithms are also extended to large scale interconnected processes.

All the ISOPE algorithms are formulated based on a normal lagrangian. The applicability of the ISOPE algorithms can be extended by introducing augmentation to the normal lagrangian function. Chapter 4 presents a systematic formulation of the augmented ISOPE algorithms for a large scale interconnected process using output feedback. Two version of augmentation are introduced : the variable and the constraint augmentation.

Chapter 5 provides optimality and global convergence conditions of the augmented ISOPE algorithms decribed in Chapter 4. The convergence theories in this chapter are based on the following papers (Abdullah, Brdyś and Roberts (1986) ; Tatjewski, Abdullah and Roberts (1986)). It is shown that, when augmentation is applied, the method is not restricted to the situation where the number of model parameters is equal to the number of measured outputs as in the case of the ISOPE algorithms based on the normal lagrangian. At the same time the model performance function does not have to be convex.

In chapter 6 the augmented ISOPE algorithms based on input and output feedback are derived. The optimality and convergence conditions are derived, and this chapter is based on the works of Brdyś, Abdullah and Roberts (1986) , and Tatjewski, Abdullah and Roberts (1986)). The algorithms are also applicable to the problem where the number of model parameters is not equal to the number of measured outputs and to the case of a non-convex model performance index.

The augmented ISOPE algorithms presented in Chapters 4 and 6 were simulated using two sample examples of different nature. The results of the simulation are discussed in chapter 7. The applicability, sensitivity and convergence rate of the various algorithms are compared. Finally, conclusions of this research and suggestions for futher work are presented in chapter 8.

CHAPTER 2 - DECOMPOSITION AND COORDINATION OF HIERARCHICAL CONTROL

2.1 Introduction

A large scale industrial process is an example of a complex system which consists of interconnected subsystems as shown in Fig. 2.1. The subsystems represent unit processes in which outputs of one subsystem are connected with inputs of another subsystem. The design of a control system for such a process is very complex due to the following reasons :

i) It is often impossible for a single computer to solve the control and optimisation problems for the entire process due to its limited information handling capacities.

ii) The transmission of information between subsystems is expensive and subject to distortion.

However, these difficulties can be overcome by decomposing the control and optimising problems into a number of smaller problems in such a way leading to hierarchical dependence. By using computers, each one of the subsystems solves the control and the optimising problems independently. It is important to solve these individual problems in such a manner that the overall goal will be achieved. If they are in conflict, a coordinating agent is necessary.

The popularity of hierarchical control in industrial control applications is due to the computer control technology. The advent of microprocessors has made control computers so cheap and handy that they are now being widely used in all levels of industrial control. In this chapter we will be looking at ways of dealing with large scale problems through hierarchical control and coordination.

2.2 Control of Process

Let us begin by looking at some basic notions and formulations relating to control systems as proposed by Findeisen et. al (1980). A typical scheme of a control system is shown in Fig. 2.2, where the task of the control unit is to determine the value of manipulated input, $\tilde{m}$, that achieves a certain goal such that it meets a certain specification on the behaviour of the controlled system. The control unit makes use of the system model and the observation $\tilde{v}$ to shape its control decisions.

In many controlled systems, the output y(t) at a particular time t, depends not only on the input u(t), the disturbance input $\tilde{z}(t)$ at the same instant but also on the state of a system $\tilde{x}(t)$ which describes all their past values for all t ranging from infinity to the present. The vector of state variable $\tilde{x}(t_o)$ at value t_o and the inputs m and z over the interval $[t_o, t]$ determine the state $\tilde{x}(t)$ uniquely.

The state can now be written as

$$\tilde{x}(t) = \varphi_{[to,t]}(\tilde{x}(t_o), \tilde{m}_{[to,t]}, \tilde{z}_{[to,t]}) \qquad (2.1)$$

and the output can be expressed as

$$y(t) = g(\tilde{x}(t),\tilde{m}(t),\tilde{z}(t)) \qquad (2.2)$$

Combining (2.1) and (2.2) yeilds :

$$y(t) = \zeta_{[to,t]}(\tilde{x}(t_o),\tilde{m}_{[to,t]},\tilde{z}_{[to,t]}) \qquad (2.3)$$

When a steady-state condition is enforced by using manipulated inputs m, the output y which describes a static time-varying system can be written as folllows :

$$y(t) = g(\tilde{x}_s,\tilde{m}(t),\tilde{z}(t)) \qquad (2.4)$$

where $\tilde{x}_s$ is the value of state parameter in this dependence and it changes over the time because of the external input $\tilde{z}(t)$.

The observation $\tilde{v}(t)$ and manipulated input $\tilde{m}(t)$ are assumed to be in the forms :

$$\tilde{v}(t) = h(\tilde{x}(t),\tilde{m}(t),\tilde{z}(t))$$

and

$$\tilde{m}(t) = d(\tilde{v}_{[to,t]})$$

respectively.

The goal of the control unit can be expressed in terms of control decisions that maximize or minimize a scalar-valued performance index Q, which can be written in two equivalent formulas :

$$Q = \int_{t_o}^{t_f} q(t)dt \quad , \quad Q = \frac{1}{t_f - t_o} \int_{t_o}^{t_f} q(t)dt$$

where $q(t)$ is the value of the performance rate at t, t_o is the initial time and t_f is the final time.

Let us now describe the complex system of Fig.2.1. For the subsystem i, $\tilde{x}_i$ denotes the state vector, $\tilde{m}_i$ the manipulated input, $\tilde{z}_i$ the disturbance, u_i the input from other subsystems and y_i the output connected to other subsystems. The subsystem state equation is (compare Eq.(2.1))

$$\tilde{x}_i(t) = \varphi_{i,[to,t]}(\tilde{x}_i(t_o),\tilde{m}_{i,[to,t]},u_{i,[to,t]},\tilde{z}_{i,[to,t]}) \quad (2.5)$$

Assume that Eq.(2.5) is in the form of an ordinary differential equation :

$$\dot{\tilde{x}}_i(t) = f_i(\tilde{x}_i(t),\tilde{m}_i(t),u_i(t),\tilde{z}_i(t)) \quad (2.6)$$

The output y_i will be related to (x_i,m_i,u_i,z_i) by the output equation :

$$y_i(t) = g_i(\tilde{x}_i(t),\tilde{m}_i(t),u_i(t),\tilde{z}_i(t)) \quad (2.7)$$

Introducing the first-layer or direct controls to the subsystem, as in Fig. 2.3, the following is enforced :

$$c_i(t) = h_i(\tilde{x}_i(t),\tilde{m}_i(t),u_i(t)) = c_{di}(t) \quad (2.8)$$

where functions h(.) relates c(t) to the values of plant variable $\tilde{x}(t)$ and $\tilde{m}(t)$ at the same instant in time. The values of c are directly measured (observed). At steady-state, $\dot{\tilde{x}}_i(t) = 0, \ \forall t$, $\tilde{x}_i(t) = \tilde{x}_{si}$ = a constant ; the functions $h_i(.)$ have been chosen properly so as to ensure uniqueness of the state $\tilde{x}_{si}$ and manipulated output $\tilde{m}_i(t)$ in response to the imposed $c_i(t)$ and $u_i(t)$, with $\tilde{z}_i(t)$ as a parameter. Then Eq.(2.6) becomes

$$f_i(\tilde{x}_{si}, \tilde{m}_i(t), u_i(t), \tilde{z}_i(t)) = 0 \qquad (2.9)$$

and Eq.(2.9) along with (2.8) provide for $\tilde{x}_{si}$, $\tilde{m}_i(t)$ to be functions of $c_i(t)$. Therefore Eq.(2.7) becomes the following input-output dependence :

$$y_i(t) = F_i(c_i(t), u_i(t), \tilde{z}_i(t)) \qquad (2.10)$$

Equation (2.10) is a relation between instantaneous values, where the system is assumed to be in steady state $\tilde{x}(t) = \tilde{x}s$ = a constant. Additionally, if we use notation y_i, c_i, u_i, z_i to express time functions, and the dynamics of the subsystem are suppressed, then by dropping the disturbance input Eq.(2.10) becomes

$$y_i = F_i(c_i, u_i) , \qquad i \in \overline{1,N} \qquad (2.11)$$

Assuming that the system is in steady state, the interconnections in the system are described by

$$u_i = H_i y = \sum_{i=1}^{N} H_{ij} y_i , \quad \text{so that} \quad u = Hy \qquad (2.12)$$

where H_i is part of matrix H:

$$H = \begin{vmatrix} H_1 \\ \text{----} \\ H_N \end{vmatrix}$$

It may assummed that a resourse constraint is imposed on the system as a whole

$$\sum_{i=1}^{N} r_i(c_i,u_i) \leqslant r_o \qquad (2.13)$$

and also that some local constraints (c_i,u_i) may exist

$$(c_i,u_i) \in CU_i \ , \quad i \in \overline{1,N} \qquad (2.14)$$

We further assume that a local performance index is associated with the system

$$Q_i(c_i,u_i) \ , \quad i \in \overline{1,N} \qquad (2.15)$$

and a global system performance is also defined as

$$Q = \psi(Q_1,Q_2,...,Q_N) \qquad (2.16)$$

where the function ψ is assumed to be strictly order-preserving.

2.3 Decomposition of the Control Problem

In a large-scale system, it is often imposible to design a single control system for the entire process. Consequently, it is necessary to decompose the problem into a number of smaller problems where the individual subsystem control problems can be solved more easily. It is important to solve the individual problems in such a way that the overall goal will be achieved.

The above problem can be decomposed by use of some special hierarchical structures. There are two classical concepts in hierarchical control : The multilayer concept (Lefkowitz, 1966), where the control of an object is split into algorithms, or layers, each of which act at different time intervals ; and the multilevel concept (Mesarovic et. al, 1970), where the control of an interconnected, complex system is divided into local goals, and local control units are introduced where their action is coordinated by an additional supremel unit.

In this section, we will be looking at three different approaches to decomposition of on-line systems as proposed by Mesarovic (1970) which is based on the above two classical concepts of decomposition. They are

i) The multilevel approach.

ii) The stratum approach

iii) The multilayer approach

It is also important to note that all three approaches are often present in one system.

2.3.1 Multilevel Approach

Decomposition on the basis of structure results in a multilevel control system, where a system is decomposed into a number of smaller subsystems, with each subsystem having its own control system, which are all coordinated by an upper level in order to optimise the overall system objective. Fig.2.4 illustrates a multilevel decomposition where the local decision units and the coordinator make their decisions based on the mathematical models of the system, and may also use information feedback from the system.

Fig.2.5 represents a multilevel, multi-goal system. The controllers are distributed in levels and they are arranged in a hierarchy having a pyramid structure. The controllers on the first level control each of the interconnected subsystems, whereas the controllers on the second level are then assigned the task of coordinating groups of first-level controllers. Similarly, third-level controllers may in turn control the second-level units which results in a hierarchy. The higher level in the hierarchy must act in such a way that the global solution is obtained.

2.3.2 Stratum Approach

Decomposition by the stratum approach refers to a decomposition of the control system based on the level of influence ; the problem is separated into a number of smaller better defined subproblems and each of the subproblems are solved separately. There are three characteristics of the decomposition on the basis of strata :

i) Individual strata have different tasks.

ii) A priority is associated with each stratum with the higher strata having priority over the lower.

iii) Each of the strata considers a different time horizon, the higher strata having the longer horizon.

Fig. 2.6 illustrates the decomposition based on stratum, where all the strata are acting in parallel and, in general, the higher the level the less often the control action takes place.

2.3.3 Multilayer Approach

Multilayer decomposition is based on the decomposition of levels of control. This implies that the control objective is divided into separate parts according to relative frequency of occurance of the disturbance. Thus faster disturbances are assigned to one layer of the control and slower disturbances assigned to another layer of the control and even slower disturbances to yet another. Individual layers are solved by a different methodology.

An example of a process using the multilayer concept to control a process is shown in Fig. 2.7. Firstly, the disturbances are classified into four frequency ranges, then control systems corresponding to the four frequency ranges are designed in the form of a multilayer system, with each layer of control devoted to each frequency range. The layers are :

i) Regulation or direct control

ii) Optimisation

iii) Adaption

iv) Self-organisation

The above functional four-layer hierarchy will be described in the following sections.

i) The regulation or direct control layer

The task of this layer is to maintain the chosen process variables at their desired values (set-points) despite fast disturbances acting upon the process.

ii) The supervisory or optimising layer

The task of this layer is to determine the optimal values of the set-points by optimising some performance function subject to operating constraints and assuming some fixed parameters in the model of the plant or the environment or both.

iii) The adaptive layer

This layer is concerned with adapting or updating the parameter values used in the mathematical models employed in the optimisation layer.

iv) Self organisation layer

This layer is responsible for selecting the structure, functions and strategies for the lower layers such that an overall goal is achieved.

2.4 Coordinator

Optimising a single subsystem in a large system without taking account of the effects of interaction can result in a significant loss of efficiency in overall performance. In this section, we will look at two different approaches which allow decomposition of a static optimisation problem into independent subproblems which will give the overall system optimum when solved independently.

i) The model-coordination method (the feasible method)

ii) The goal-coordination method (the dual feasible method)

First, let us consider the following model based optimising problem :

$$\min \sum_{i=1}^{N} q_i(c_i, u_i, y_i)$$

subject to :

$$y_i = f_i(c_i, u_i, \tilde{z}_i) \qquad i \in \overline{1,N}$$

$$\tilde{g}_i(c_i, u_i, \tilde{z}_i) \leqslant 0 \qquad i \in \overline{1,N} \qquad (2.17)$$

$$u_i = \sum_{j=1}^{N} H_{ij} y_j \,, \qquad i \in \overline{1,N}$$

where

c_i is a vector of set points to subsystem i.

u_i is a vector of inputs to subsystem i which are composed of outputs from other subsystems.

y_i is a vector of outputs from subsystem i.

$\tilde{z}_i$ is a vector of disturbance inputs to subsystem i.

2.4.1 The Model-Coordination Method

In this approach the coordinator prescribes the values of the subprocess interconnected outputs and inputs. The optimisation problem (2.17) can now be decomposed into a first-level problem and a second-level (coordinator) problems as follows :

First-level problem

$$\min_{c_i} \; q_i(c_i, y)$$

$$\text{subject to : } \quad y_i - f_i(c_i, y, \tilde{z}_i) = 0$$

$$\tilde{g}_i(c_i, y, \tilde{z}_i) \leqslant 0 \qquad (2.18)$$

where y are specified by the second level problem.

Second level problem

$$\min_{y} \quad Q(c, y) = \sum_{i=1}^{N} q_i(c_i, y) \qquad (2.19)$$

The coordination variables y are obtained by performing one iteration of the second level problem, where c_i, $i \in \overline{1,N}$ are the specified solutions of the first level problems at the previous iteration. For the particular case of two subprocesses the method can be illustrated as shown in Fig. 2.8.

2.4.2 The Goal Coordination Method

In this method the interaction between the subsystems is removed by cutting all links between the subsystem. This may be done as indicated in Fig.2.9.

An additional penalty term is introduced to penalize the performance of the system if the interconnections do not balance.

The modified performance index is

$$L(c,u,y,\lambda) = \{ \sum_{i=1}^{N} q_i(c_i,u_i,y_i) + p_i(u_i - \Sigma H_{ij}y_j) \} \qquad (2.20)$$

where p is a vector of lagrange multipliers, known as the price vector.

The modified performance index may be decomposed to form individual modified optimisation indices, one for each subsystem. Hence, the i-th infimal unit problem becomes :

$$\min_{c_i,u_i} \{ q_i(c_i,u_i,y_i) + p_i u_i - \Sigma p_k H_{ki} y_i \}$$

subject to : $y_i = f_i(c_i,u_i,\tilde{z}_i)$

$\tilde{g}_i(c_i,u_i,\tilde{z}_i) \leqslant 0$

where values of p are specified by the coordinator. The value of c_i and u_i are directly obtained by the optimisation. The coordinator task is to choose the coordination variables p to force interaction balance which is achieved when the interaction constraints :

$$u_i - \sum_{j=1}^{N} H_{ij} y_i = 0 \qquad (2.21)$$

Using Lagrange Duality Theory, an algorithm for the coordinator can be determined and the following overall dual function can be obtained :

$$D(\lambda) = \sum_{i=1}^{N} D_i(\lambda) = \min L(c,u,y,\lambda) \qquad (2.22)$$

Fig.2.10 illustrates the goal-coordination method for a two subsystems example.

2.5 Summary

A technique of controlling a large scale process through hierarchical control has been examined. The control problem is decomposed into a number of smaller problems such that the individual design or implementation is straightforward. Three methods of decomposition have been discussed in decomposing the control problem. The model-coordination and the goal-coordination methods for multi-level decomposition are introduced in order to counter the effects of interaction between the subsystems so that the overall system goal is achieved.

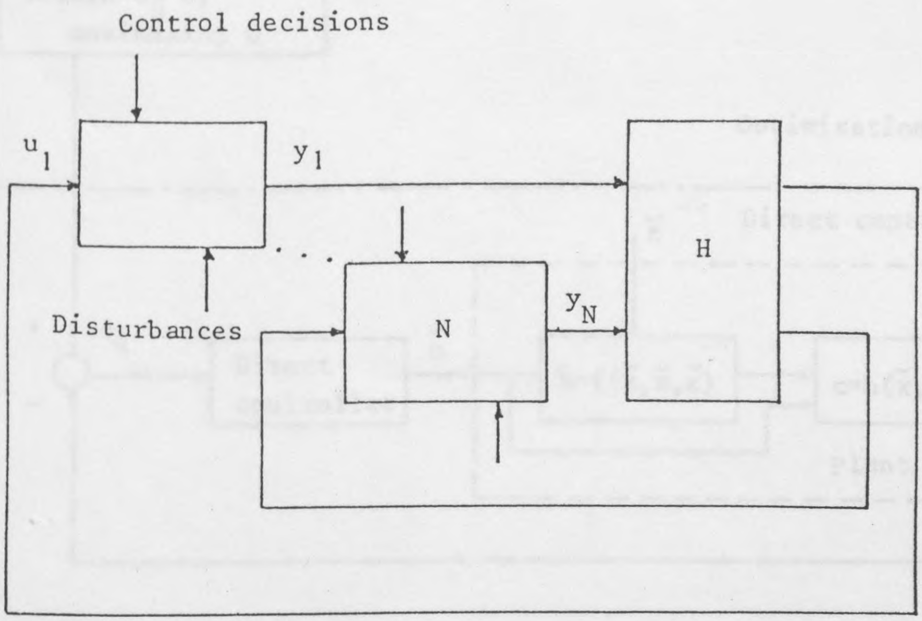

Figure 2.1 The large scale system with an ordering matrix H.

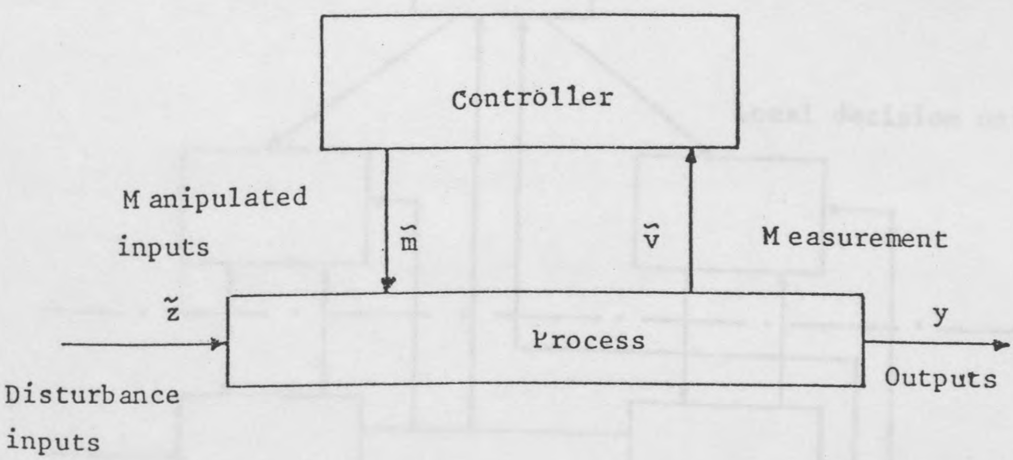

Figure 2.2 The control system.

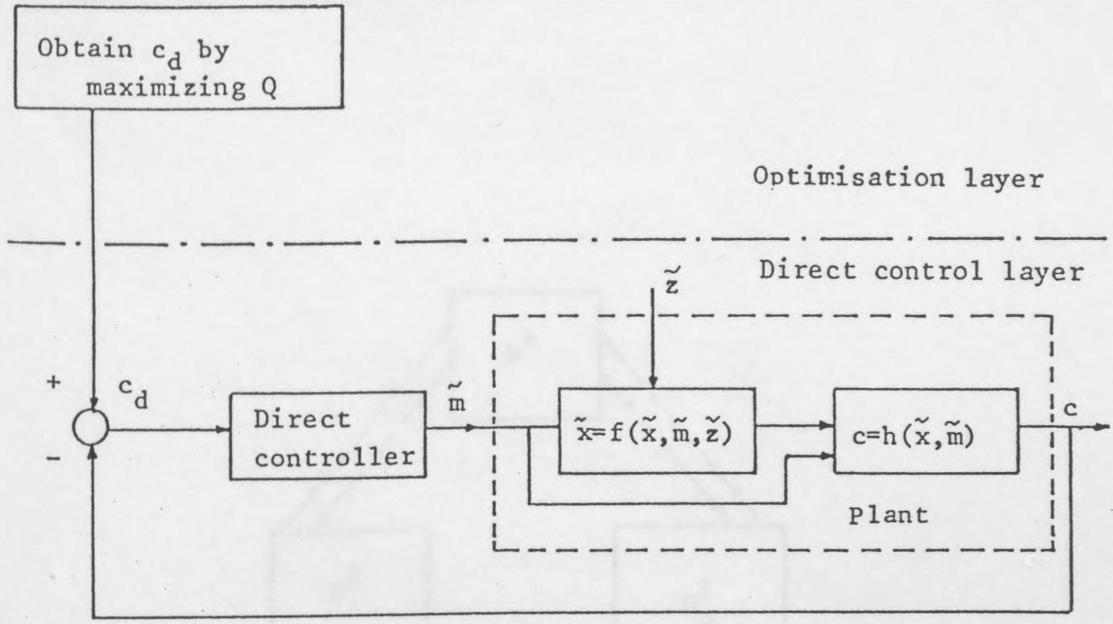

Figure 2.3 A two-layer system.

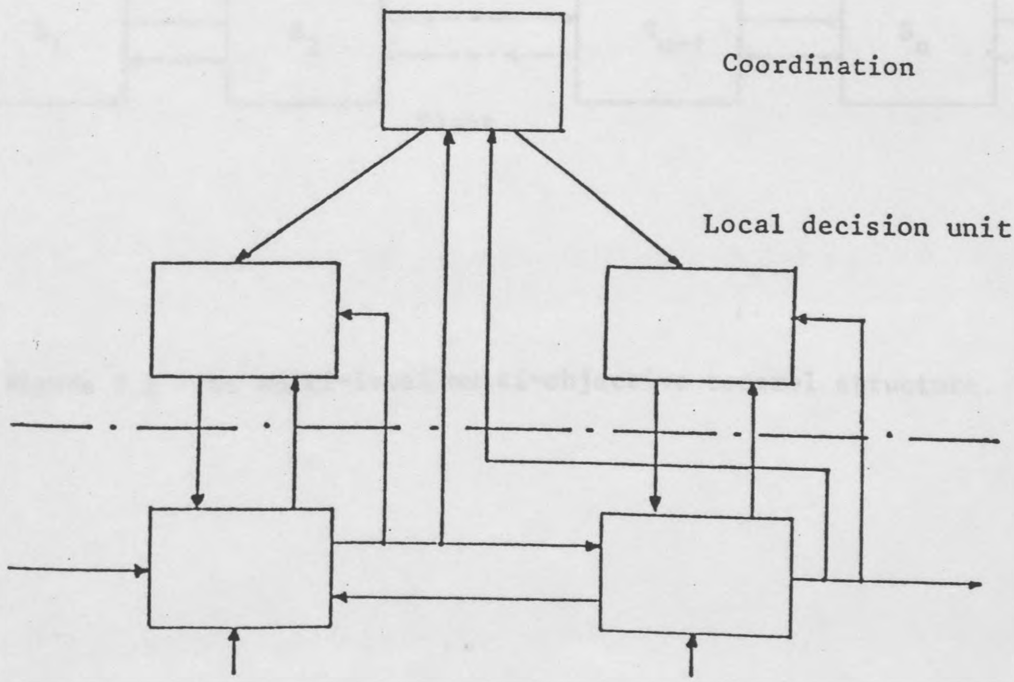

Figure 2.4 Multilevel control.

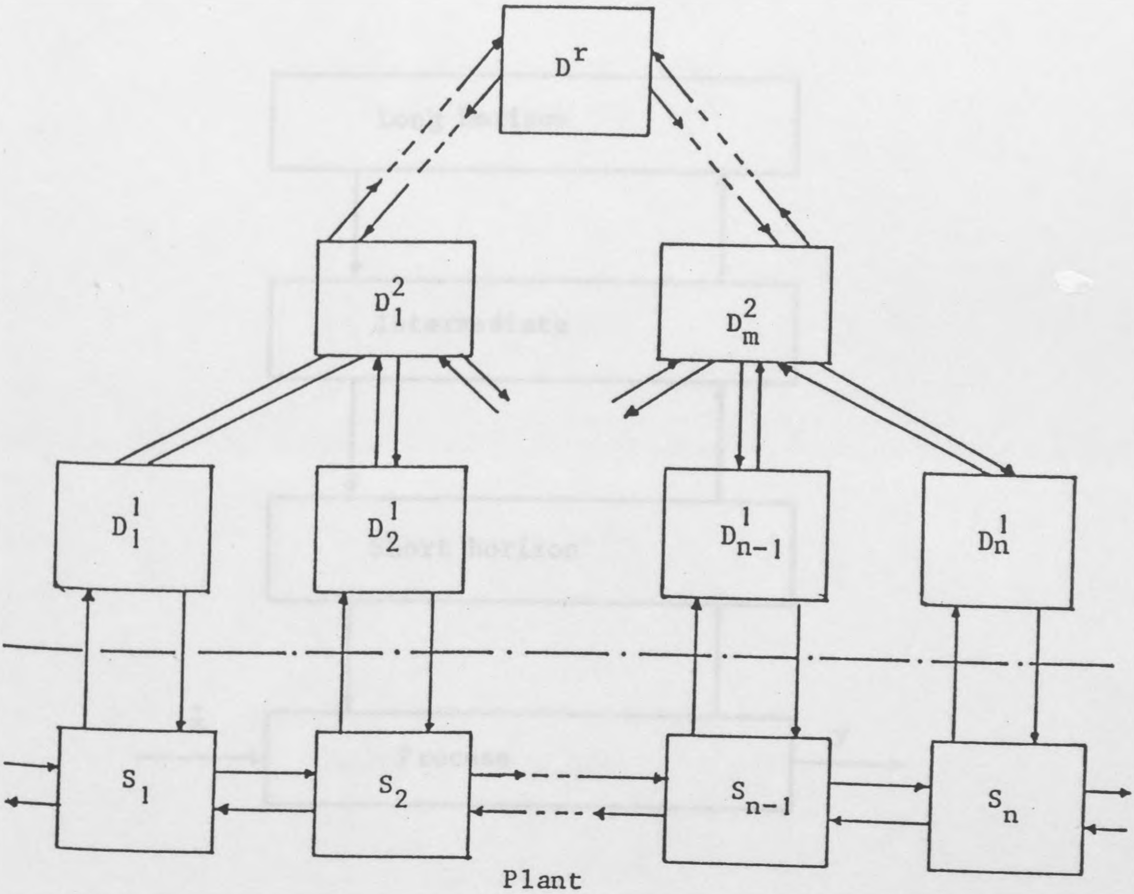

Figure 2.5 The multi-level multi-objective control structure.

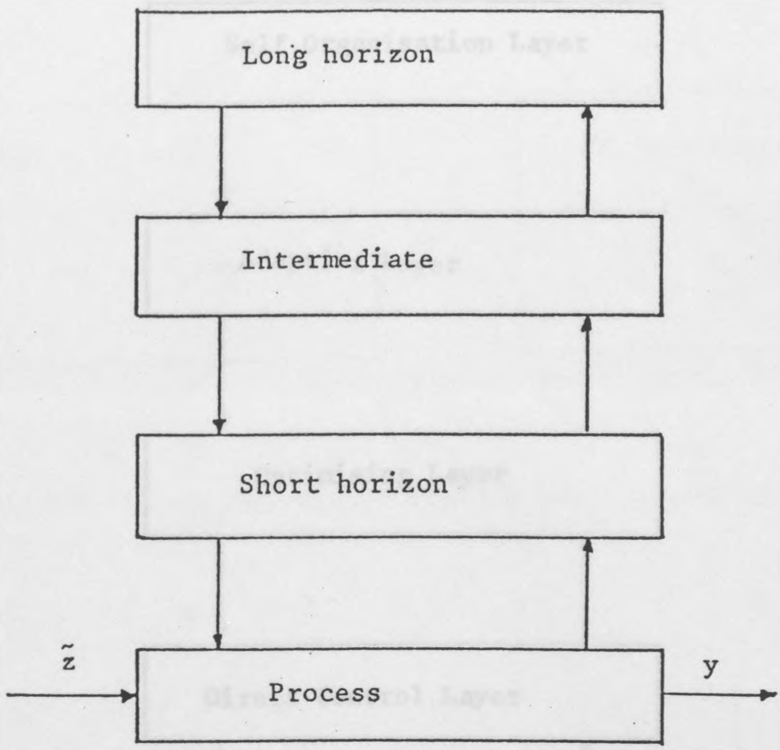

Figure 2.6 The decomposition of control system on the basis of strata.

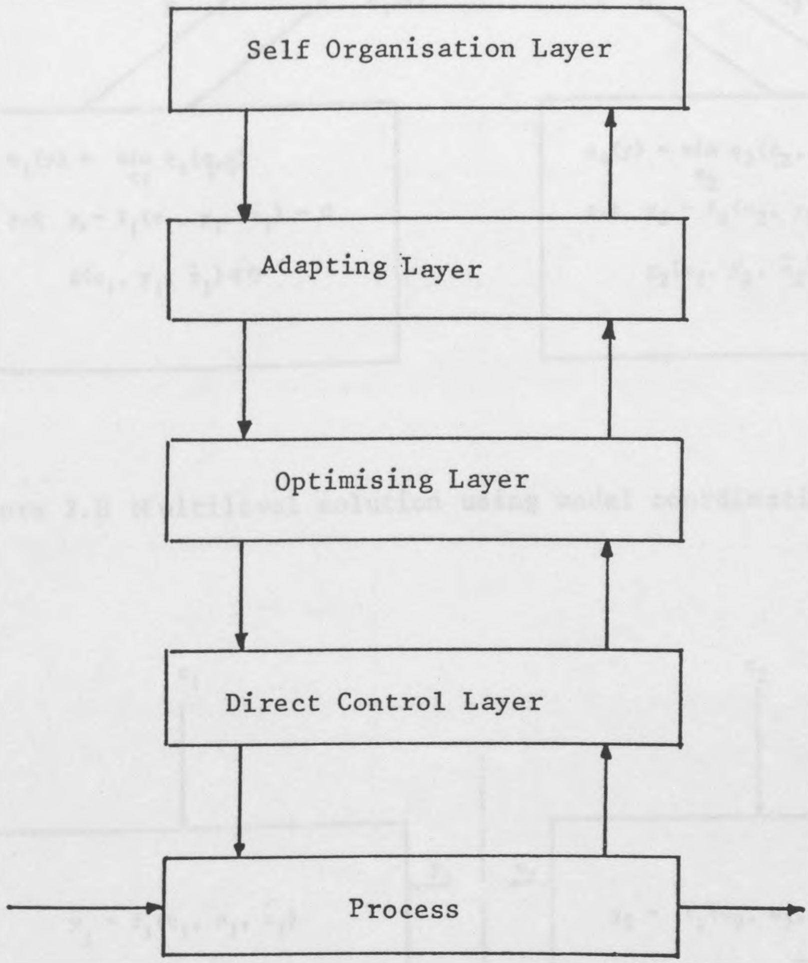

Figure 2.7 Multilayer control.

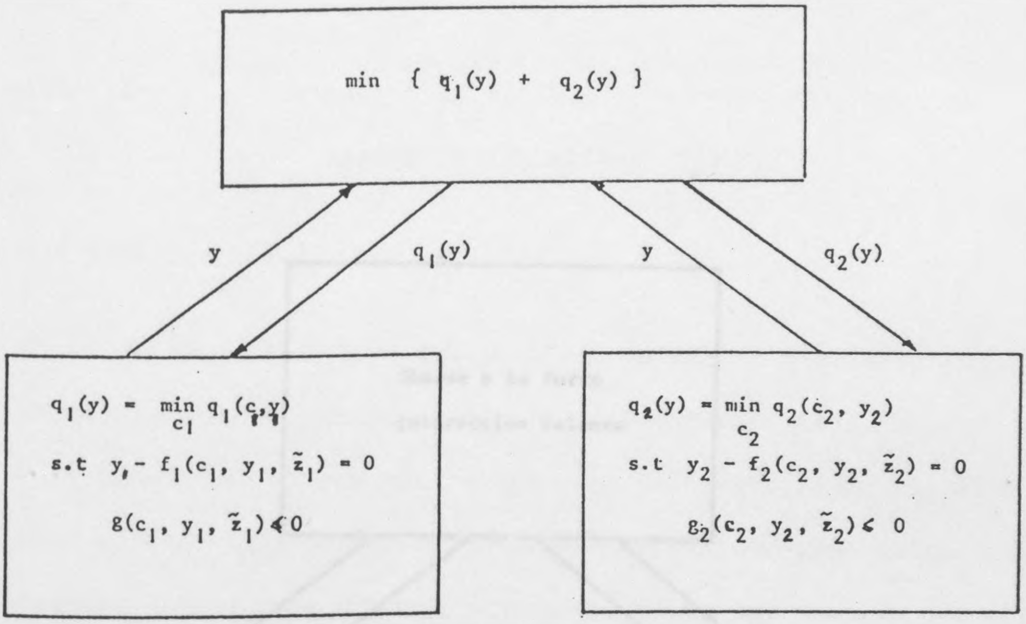

Figure 2.8 Multilevel solution using model coordination.

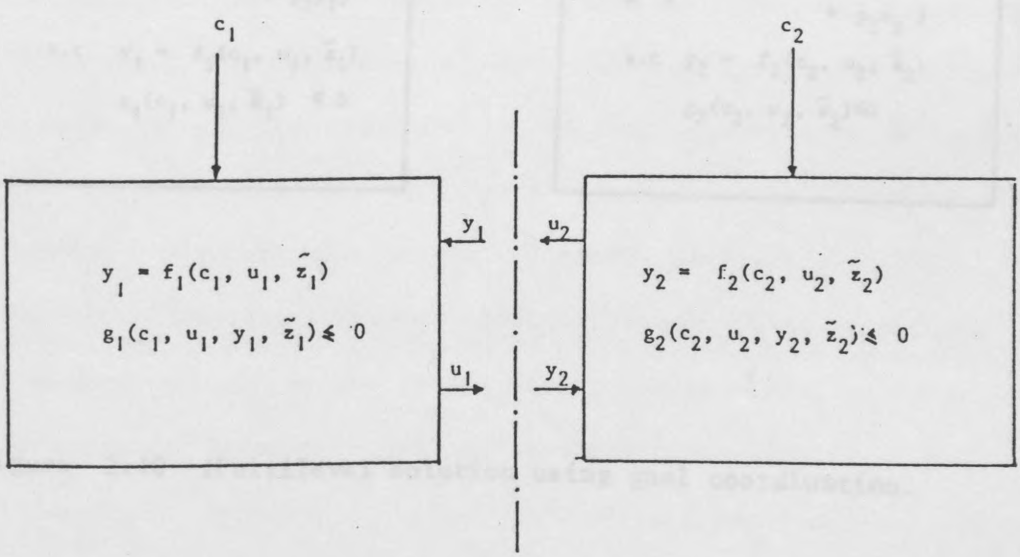

Figure 2.9 Decoupled system.

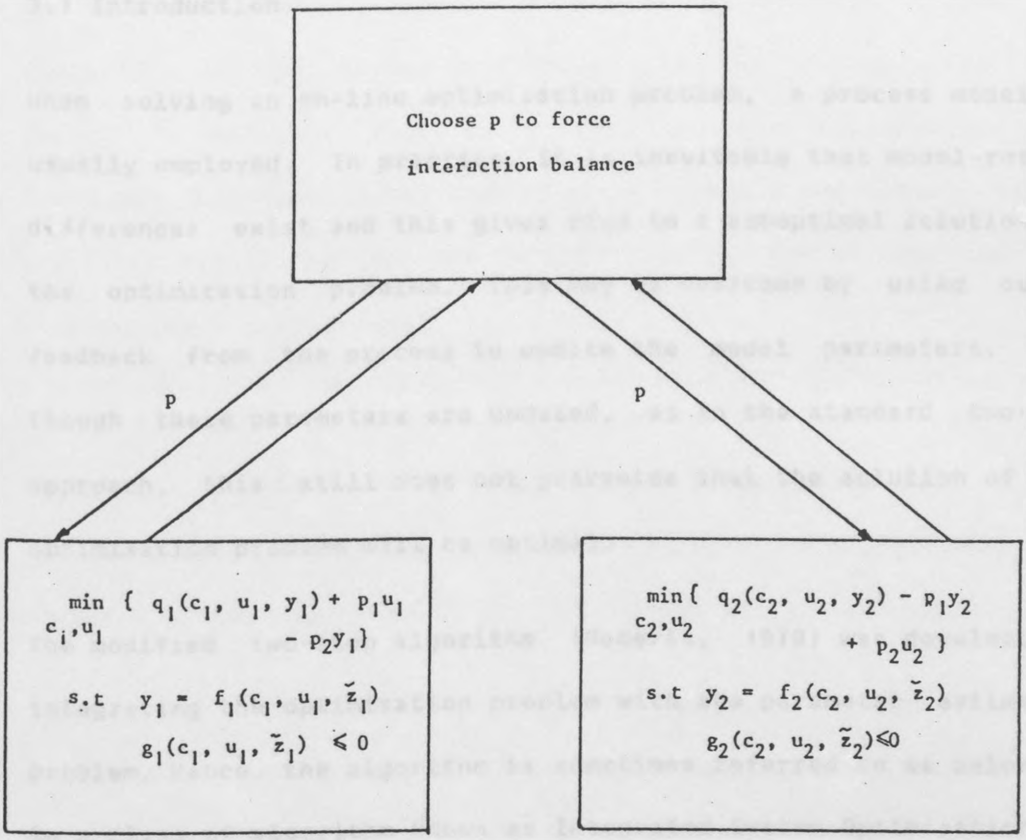

Figure 2.10 Multilevel solution using goal coordination.

CHAPTER 3 - ON-LINE INTEGRATED SYSTEM OPTIMISATION AND PARAMETER ESTIMATION (ISOPE)

3.1 Introduction

When solving an on-line optimisation problem, a process model is usually employed. In practice, it is inevitable that model-reality differences exist and this gives rise to a suboptimal solution in the optimisation problem. This may be overcome by using output feedback from the process to update the model parameters. Even though these parameters are updated, as in the standard two-step approach, this still does not guarantee that the solution of the optimisation problem will be optimal.

The modified two-step algorithm (Roberts, 1979) was developed by integrating the optimisation problem with the parameter estimation problem. Hence, the algorithm is sometimes referred to as belonging to a class of algorithm known as Integrated System Optimisation and Parameter Estimation (ISOPE). In spite of model deficiency, the algorithm converges to the correct optimum. This is due to the combined nature of the method which compensates for derivative differences between model and reality. The algorithm has been succesfully applied to solve many example problems (Roberts and William (1981) ; Ellis and Roberts (1981) ; Stevenson, Brdyś and Roberts (1985)). Brdyś and Roberts (1987) extended the applicability of the modified two-step method to a problem where the system inequality constraints are output dependent. Brdyś, Chen and Roberts (1986) has also proved that the algorithm converged to the real optimum.

The drawback of the modifed two-step algorithm is the requirement to evaluate real process derivatives, which will affect the robustness of the method when the process is contaminated with noise. In order to avoid using real process derivatives, the two-level type method has been derived by Brdyś and Roberts (1986) and investigated by Chen and Roberts (1984), but at the cost of introducing a second level.

Michalska, Ellis and Roberts (1985) extended the application of the two-step approach to interconnected large-scale systems by combining the two-step approach with the price technique, and this combined method is known as the joint coordination method.

Applying a similar approach as in the joint coordination methods, Brdyś and Roberts (1986) formulated several optimal adaptive structures. In these structures, the available process measurement feedbacks are employed efficiently. The algorithm is also applicable to problems with output dependent inequality constraints.

3.2 The ISOPE Technique for a Single Process

In this section we will be looking at methods of determining the optimal set-points of the process feedback controllers using integrated system optimisation and parameter estimation (ISOPE) techniques. The integrated techniques to be examined are :

i) The two-step technique

ii) The modified two-step technique

Consider the following optimising control problem

$$\begin{aligned} \min_{c,y} \quad & Q(c,y) \\ \text{s.t} \quad & y = F_*(c) \qquad\qquad (3.1) \\ & c \in C = \{ c : G(c) \leqslant 0 \} \end{aligned}$$

Where $c \in R^n$ and $y \in R^m$ are vectors of controls (controller set-points) and real process outputs respectively. The performance of the system is described by the mapping $Q : R^n \times R^m \dashrightarrow R$, and $F_* : R^n \dashrightarrow R^n$ is the process input-output mapping . $G : R^n \dashrightarrow R^s$ is an inequality constant mapping imposed on the system. Assuming the process output mapping F_* is not available, its approximate model $F: R^n \times R^m \dashrightarrow R^m$ will be used instead

$$y = F(c,\alpha) \qquad\qquad (3.2)$$

where $\alpha \in R^m$ is a vector of parameters of the process model.

It is assumed that the model equation (3.2) is point-parametric on C, that is for every $c \in C$ there is an $\alpha(c)$ such that $F_*(c) = F(c,\alpha(c))$ (Brdys, 1983), and that mappings F, F_*, Q, G are twice continously differentiable.

3.2.1 The Two-step Technique

By substituting model equation (3.2) into equation (3.1), the optimising control problem (3.1) can be written in an equivalent form :

$$\min_{c,\alpha} \; q(c,\alpha)$$

$$\text{s.t} \;\; F(c,\alpha) = F_*(c) \qquad (3.3)$$

$$\tilde{g}(c,\alpha) \leqslant 0$$

where $q(c,\alpha) = Q(c,F(c,\alpha))$

$\tilde{g}(c,\alpha) = G(c,F(c,\alpha))$

The process model optimisation and parameter estimation problems are treated separately and solved repetitively, one after another, until an optimal solution is found.

The technique proceeds as follows :

Choose some initial set point $c^o \in C$ and $\beta < 0$.

Step 1 : Apply c^k to the real process and measure its output $F_*(c^k)$. Determine the new parameter value $\hat{\alpha}(c^k)$ by solving the estimation problem

$$F(\hat{c}^k,\hat{\alpha}^k) = F_*(c^k)$$

Step 2 : Solve the optimisation problem (3.3) to obtain $\hat{c}^k$ for a given value of α^k. If the condition $\|\hat{c}^k - c^k\| < \beta$ is satisfied then stop, otherwise update the control c^k as follow

$$c^{k+1} = \hat{\psi}_c(c^k, c^k)$$

and goto step 1.

A simple relaxation algorithm can be used :

$$c^{k+1} = c^k + \varepsilon(\hat{c}^k - c^k)$$

Simulation results by Brdyś, Chen and Roberts (1986) have shown that this technique gives a sub-optimal real performance. This is due to the difficulties in getting an exact match between process and corresponding model output derivatives with respect to the set-points at the optimal point, when an inaccurate model is used (Durbeck, 1965).

3.2.2 The Modified Two-step Technique

Since the conditions suggested by Durbeck are difficult to satisfy, Roberts (1979) proposed a modified two-step technique in which the optimisation and the parameter estimation tasks are combined together. Later, a new version of the modified two-step method was formulated (Brdyś and Roberts (1987)) for problems in which inequality constraints are output dependent.

In order to decouple the optimisation and parameter estimation tasks, additional variables $v \in R^n$ are futher introduced converting (3.3) into :

$$\min_{c,v,\alpha} \; q(c,\alpha)$$

$$\text{s.t} \quad G(c,\alpha) \leqslant 0 \qquad (3.4)$$

$$F(v,\alpha) = F_*(v)$$

$$v = c$$

The lagrangian function for (3.4) is

$$L(c,v,\alpha,\lambda,\mu) = q(c,\alpha) + \lambda^T(v-c) + \mu^T(F(v,\alpha)-F_*(v)) + \xi^T g(c,\alpha) \qquad (3.5)$$

where $\lambda \in R^n$, $\mu \in R^r$ are lagrangian or Kuhn-Tucker multipliers. The problem (3.5) is reformulated in the following form ;

$$\min_{c} \; \{ q(c,\hat{\alpha}(v)) - \lambda^T(v)c \}$$

$$\text{s.t} \quad \tilde{g}(c,\alpha(v)) \leqslant 0 \qquad (3.6)$$

where (Brdyś, Chen and Roberts (1986))

$$\lambda(v) = [F'_c(v,\hat{\alpha}(v))-F'_*(v)]^T[Q'^T_y(v,F(v,\alpha(v))+G'^T_y(v,\hat{\alpha}(V))\xi] \qquad (3.7)$$

and where $\hat{\alpha}(v)$ is a parameter value such that

$$F(v,\hat{\alpha}(v)) = F_*(v) \qquad (3.8)$$

The technique can be implemented as follow :

Choose some initial set-points $v^o \in C$, $\xi \in R^r$ and β

Step 1 : Apply v^k to the real process and measure its output $F_*(v^k)$. Determine a new parameter value $\hat{\alpha}^k(v)$ by solving the estimation problem

$$F(v^k, \hat{\alpha}^k(v)) = F_*(v^k)$$

Perform perturbations around v^k and, based on the corresponding process outputs, evaluate finite difference approximations $\dot{F}_*(v^k)$. Then calculate $\lambda^k(v)$ using Eq.(3.7).

Step 2 : Solve the model-based modified optimisation problem (3.6) to obtain $\hat{c}^k$ for given α^k and λ^k. If $\|\hat{c}^k - v^k\| < \beta$ and $\|\hat{\xi}^k - \xi\| < \beta$ are satisfied then stop, otherwise update the control v^k and ξ^k as follow

$$v^{k+1} = \psi_v(\hat{c}^k, v^k) \quad , \quad \xi^{k+1} = \psi_\xi(\hat{\xi}^k, \xi^k) \qquad (3.9)$$

and goto to step 1.

Simple relaxation formulae were used for (3.9)

$$v^{k+1} = v^k + \varepsilon_v(\hat{c}^k - v^k)$$

$$\xi^{k+1} = \xi^k + \varepsilon_\xi(\hat{\xi}^k - \xi^k)$$

This technique has been successful in solving many example problems and gives optimal solutions as proved by Brdys, Chen and Roberts (1984).

3.2.3 Two-level Type Method

A two-level type method investigated by Chen and Roberts (1984) requires an additional introduction of a second level to prescribe modifiers by minimizing the performance index. The advantage of the proposed algorithm is that process derivatives are not required. Therefore, the algorithm is applicable to problems, where the constraint $G(c,y)$ is non-differentiable with respect to y and under noisy conditions.

The model optimisation problem at the infimal level has the following form :

$$\min_{c} \; \{ Q(c,F(c,\alpha)) - \lambda^T c \} \tag{3.10}$$

with λ given by the supremal level, and the solution of the infimal level is denoted by $\hat{c}(\lambda)$.

The task of the supremal level is to determine a λ_{opt} such that

$$\lambda_{opt} = \arg \min_{\lambda \varepsilon \hat{A}} q(\lambda) \tag{3.11}$$

where $\hat{A}$ consists of all λ for which $\hat{c}(\lambda)$ exists, and

$$q(\lambda) = Q(\hat{c}(\lambda),F(\hat{c}(\lambda),\alpha(\hat{c}(\lambda)))) = Q(\hat{c}(\lambda),F_*(\hat{c}(\lambda)))$$

Simulation results by Chen and Roberts (1984) have shown that the technique is as accurate as the modified two-step technique but at the cost of increasing on-line computing time.

3.3 The ISOPE for Large Scale Industrial Process

Brdyś and Roberts (1986) has extended the centralised ISOPE to interconnected industrial processes. This is done by incorporating the modified two-step technique with the price method in a similar manner as in the Mutually Interacting Approach (Brdyś (1983)) and Joint Coordination method (Michalska, Ellis and Roberts (1985)).

Brdyś and Roberts (1984) have proposed several iterative optimal adaptive schemes, which utilise the available real process measurements efficiently and some of the schemes do not require real process derivative information. In this section, two of the proposed schemes will be examined : namely, the structure with output feedback and the structure with input and output feedbacks.

3.3.1 Preliminaries

By employing a similar technique as Findeisen and co-workers (1980), it is assumed that the controlled system together with its follow-up controllers are described in a decomposed manner by the set of subsystem input-output equations :

$$y_i = F_{*i}(c_i, u_i) \quad i \in \overline{1,N}$$

where $F_{*i} : C_i \times U_i \longrightarrow Y_i$, $i \in \overline{1,N}$ is the i-th subsystem input-output mapping and N denotes the number of subsystems. C_i, U_i and Y_i are finite dimensional spaces. The variables c_i, u_i and y_i are the i-th subsystem control, interaction input and interaction output, respectively, and also $c_i \in C_i$, $u_i \in U_i$, and $y_i \in Y_i$.

The subsystems are interconnected with assumed structure equations :

$$u_i = H_i y = H_{ij} y_j \quad , \quad i \in \overline{1,N}$$

where H_i and H_{ij} are interconnected matrices.

Let us denote

$$c \stackrel{\Delta}{=} (c_i, \ldots\ldots, c_N) \quad \varepsilon \quad C_1 \times \ldots\ldots \times C_N \stackrel{\Delta}{=} C$$

$$u \stackrel{\Delta}{=} (u_i, \ldots\ldots, u_N) \quad \varepsilon \quad U_1 \times \ldots\ldots \times U_N \stackrel{\Delta}{=} U$$

$$y \stackrel{\Delta}{=} (y_i, \ldots\ldots, y_N) \quad \varepsilon \quad Y_i \times \ldots\ldots \times Y_N \stackrel{\Delta}{=} Y$$

then the subsystem equations and interactions can be written in the global form :

$$y = F_*(c,u) \quad , \quad u = Hy \qquad (3.12)$$

where

$$F_*: C \times U \dashrightarrow Y, F_*(c,u) = (F_{*1}(c_1, u_1), \ldots, F_{*N}(c_N, u_N)), H = \{H_{ij}\}_{i,j \in \overline{1,N}}$$

Assuming that for each $c \in C$, there exists only one solution of the equation :

$$y = F_*(c, Hy)$$

the global system mapping is given by :

$$K_* : C \dashrightarrow Y$$

i.e $\quad y = K_*(c) = (K_{*1}(c), \ldots\ldots, K_{*N}(c))$

In practice, approximate models are used due to uncertainty of the real system relations. The models are :

$$F_i = C_i \times U_i \times A_i \text{ ----> } Y_i, \quad i \in \overline{1,N}$$

$$y_i = F_i(c_i, u_i, \alpha_i), \quad i \in \overline{1,N}$$

where α_i is a finite dimensional space and $\alpha_i \in A_i$ is the i-th subsystem model parameter variable. However, the interconnection relationships are assumed to be known exactly.

Each subsystem is assumed to be subjected to local constraints

$$(c_i, u_i, y_i) \in CUY_i \overset{\Delta}{=} \{(c_i, u_i, y_i) \in C_i \times U_i \times Y_i \; : \; G_{ij}(c_i, u_i, y_i) \leqslant 0, \; j \in J_i\}$$

These constraints can be written jointly as :

$$(c,u,y) \in CU \overset{\Delta}{=} \{(c,u,y) \in C \times U \times Y \; : \; G(c,u,y) \leqslant 0\} \qquad (3.13)$$

where

$$G(c,u,y) \overset{\Delta}{=} (G_1(c_1,u_1,y_1), \ldots\ldots, G_N(c_N,u_N,y_N))$$

$$G_i(c_i,u_i,y_i) \overset{\Delta}{=} (G_{i1}(c_i,u_i,y_i), \ldots\ldots, G_{ij}(c_i,u_i,y_i))$$

With each subsystem, a known local performance function is associated :

$$Q_i : C_i \times U_i \times Y_i \text{ ----> } R^1, \quad i \in \overline{1,N}$$

The overall performance index of the system is assumed to have the additive form :

$$Q(c,u,y) = \sum_{i=1}^{N} Q_i(c_i,u_i,y_i) \qquad (3.14)$$

The algorithm is applicable when the following assumptions are satisfied :

A1. The system model is point parametric on a set $\pi_{CxU}(CUY)$ (Brdyś 1983). That is for every point $(\bar{c},\bar{u}) \in \pi_{CxU}(CUY)$ there exists $\alpha \in A$ such that $F_*(\bar{c},\bar{u}) = F(\bar{c},\bar{u},\bar{\alpha})$.

A2. : Mappings $F(.)$, $F_*(.)$, $K_*(.)$, $K(.)$, $Q(.)$ and $G(.)$ are continuously Frechet differentiable on their domains.

The system optimising control problem (OCP) can be defined as follows :

$$
\text{(OCP)} \quad \begin{aligned} & \min_{c,u,y} Q(c,u,y) \\ \text{s.t} \quad & y = F_*(c,u) \\ & u = Hy \\ & (c,u,y) \in CUY = \{ G(c,u,y) \leqslant 0 \} \end{aligned} \qquad (3.15)
$$

where $G(c,u,y) = [G_1(c_1,u_1,y_1,\ldots,G_N(c_N,u_N,y_N)]$.

The above optimising control problem (3.15) consists of minimising Eq.(3.14) with respect to the set-point vector c (controls) subject to Eqs. (3.12) and (3.13).

Using an approximate model, OCP is replaced by the equivalent problem (Brdyś, 1983)

$$
\text{(OCP)}_1 \quad \begin{aligned} & \min_{c,u,\alpha} q(c,u,\alpha) \\ \text{s.t} \quad & F(c,u,\alpha) = K_*(c) \\ & u = HF(c,u,\alpha) \\ & \tilde{g}(c,u,\alpha) \leqslant 0 \end{aligned} \qquad (3.16)
$$

where $q(c,u,\alpha) \triangleq Q(c,u,F(c,u,\alpha))$

$\tilde{g}(c,u,\alpha) \triangleq G(c,u,F(c,u,\alpha)$

3.3.2 Structure with Output Feedback

Assuming the real system output measurements are available, the $(OCP)_1$ is expanded by introducing new variables v and w as follows (Brdys and Roberts (1986)) :

$$\min_{c,u,v,w,\alpha} q(c,u,\alpha)$$

$$u = HF(c,u,\alpha) \qquad (3.17)$$

$$\tilde{g}(c,u,\alpha) \leqslant 0$$

$$F(v,w,\alpha) = K_*(v)$$

$$v = c$$

$$w = u$$

The Lagrangian function for (3.17) is

$$L(c,u,v,w,\alpha,p,\lambda,t,\xi,\mu) = q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)] + \lambda^T(v-c)$$

$$+ t^T(w-u) + \xi^T\tilde{g}(c,u,\alpha) + \mu^T[F(v,w,\alpha)-K_*(v)] \qquad (3.18)$$

where $\lambda \in R^n$, $t \in R^r$ and $\xi \in R^m$ are lagrangian or Kuhn-Tucker multipliers.

For given values of v, w, α and price vector $p \in U$, Lagrangian analysis provides the following optimising problem (Brdyś and Roberts , 1985).

$$\min [q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)] - \lambda^T c - t^T u] \quad (3.19)$$

where

$$\lambda(v,w,p,\xi) = [\partial^T \frac{K_*(v)}{\partial v} - \partial^T \frac{F(v,w,\alpha)}{\partial v}][\partial^T \frac{F(v,w,\alpha)}{\partial \alpha}]^{-1}$$

$$[- \partial^T \frac{q(c,u,\alpha)}{\partial \alpha} + \partial^T \frac{F(c,u,\alpha)}{\partial \alpha} - \partial^T \frac{\tilde{g}(c,u,\alpha)\xi}{\partial \alpha}] \quad (3.20)$$

$$t^T(v,w,p,\xi) = - \partial^T \frac{F(v,w,\alpha)}{\partial w}[\partial^T \frac{F(v,w,\alpha)}{\partial \alpha}]^{-1} [- \partial^T \frac{q(c,u,\alpha)}{\partial \alpha}$$

$$+ \partial^T \frac{F(c,u,\alpha)}{\partial \alpha} - \partial^T \frac{\tilde{g}(c,u,\alpha)\xi}{\partial \alpha}] \quad (3.21)$$

The parameter $\hat{\alpha}(v,w)$ is determined by the parameter estimation problem by satisfying

$$F(v,w,\hat{\alpha}(v,w)) = K_*(v) \quad (3.22)$$

It is assumed that any required inverse of a matrix exists.

The overall problem (3.17) can be solved by solving the optimisation problem (3.19) provided by the Lagrange analysis associated with problem (3.17), and variables c and u are obtained as the solution. The parameter α is calculated by solving (3.22), whereas the variables v, w, ξ and p will be adjusted, in an appropriate way, to solve the following equations :

$$\hat{c}(v,w,\xi,p) = p \quad (3.23)$$

$$\hat{u}(v,w,\xi,p) = w \quad (3.24)$$

$$\hat{\xi}(v,w,\xi,p) = \xi \quad (3.25)$$

$$\hat{u}(v,w,\xi,p) = HF(\hat{c}(v,w,\xi,p),\hat{u}(v,w,\xi,p),\hat{\alpha}(v,w)) \quad (3.26)$$

The optimising problem (3.19) can be decomposed into the following form :

$$\min_{c,u} q_i(c_i,u_i,\hat{\alpha}_i(v,w_i)) + p_i^T u_i - \sum_{j=1}^{N} p_j^T H_{ij} F_i(c_i,u_i,\hat{\alpha}_i(v,w_i))$$

$$- \lambda_i^T c_i - t_i^T u_i$$

$$\text{s.t} \quad \tilde{g}_i(c_i,u_i,\hat{\alpha}_i(v,w_i)) \leqslant 0 \quad i \in \overline{1,N} \qquad (3.27)$$

The subsystem model parameter value $\hat{\alpha}_i(v,w_i)$ is determined by the i-th subsystem model parameter estimation, which is made up of N independent local estimation units, by solving :

$$F_i(v_i,w_i,\alpha_i) = K_{*i}(v) \quad , \quad i \in \overline{1,N} \qquad (3.28)$$

The optimum set-points c_{opt} can be determined by solving equations (3.23),..,(3.26) by three different strategies, as proposed by Brdyś, Abdullah and Roberts (1985) :

i) Single loop iterative technique is obtained by solving equations (3.23),...,(3.26) with the same frequency.

ii) System based double loop iterative technique is obtained by solving (3.23), (3.24) and (3.25) in the inner loop for a given iteration of the outer loop, while Eq.(3.26) is solved in the outer loop to determine the value of price p.

iii) Model based double loop iterative technique consisting of separating Equations (3.23), (3.24) and (3.25) from Equation (3.26). The price p is determined by solving Eq.(3.26) in the inner loop for given values of v, w and ξ which are prescribed by the outer loop. The values of v, w and ξ are obtained by solving equations (3.23), (3.24) and (3.25) respectively in the outer loop.

Numerical simulation results by Brdyś, Abdullah and Roberts (1986) have shown that the model based double loop technique significantly reduces the real set-point changes in comparison with the other two techniques.

3.3.3 Structures with Input-Output Feedback (ISOPE)

It is assumed that the real system output and the real system input measurements are both available. The optimisation problem described by Eq.(3.16) can be expanded by introducing a new variable v as follows :

$$\begin{aligned} &\min_{c,u,v,\alpha} \; q(c,u,\alpha) \\ &u = HF(c,u,\alpha) \\ &\tilde{g}(c,u,\alpha) \leqslant 0 \\ &F(v,HK_*(v),\alpha) = K_*(v) \\ &v = c \end{aligned} \qquad (3.29)$$

The Lagrangian associated with problem (3.29) is

$$L(c,u,v,\alpha,p,\lambda,\xi,\mu) = q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)] + \lambda^T(v-c) + \xi^T g(c,u,\alpha) + \mu^T[F(v,HK_*(v),\alpha)-K_*(v)] \qquad (3.30)$$

For a given value of v, α and price vector $p \in U$, Lagrangian analysis provides the following optimisation problem (Brdyś and Roberts, 1987) :

$$\min_{c,u} [q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)] - \lambda^T c] \qquad (3.31)$$

where

$$\lambda(v,w,p,\xi) = [\partial^T \frac{K_*(v)}{\partial v} - \partial^T \frac{F(v,HK_*(v),\alpha)}{\partial v}][\partial^T \frac{F(v,HK_*(v),\alpha)}{\partial \alpha}]^{-1}$$

$$[- \partial^T \frac{q(c,u,\alpha)}{\partial \alpha} + \partial^T \frac{F(c,v,\alpha)}{\partial \alpha} H^T p - \partial^T \frac{\hat{g}(c,u,\alpha)}{\partial \alpha}\xi] \qquad (3.32)$$

The parameter value $\hat{\alpha}(v)$ is determined by solving

$$F(v,HK_*(v),\hat{\alpha}(v)) = K_*(v) \qquad (3.33)$$

Let $\hat{c}(v,\xi,p)$, $\hat{u}(v,\xi,p)$ be the solution of (3.31) and $\hat{\xi}(v,\xi,p)$ be a corresponding Lagrangian multiplier vector. Similarly, as in section 3.3.2 , the values of v, ξ and p are adjusted so as to satisfy the following set of equations :

$$\hat{c}(v,\xi,p) = v \qquad (3.34)$$

$$\hat{\xi}(v,\xi,p) = \xi \qquad (3.35)$$

$$\hat{u}(v,\xi,p) = HF(\hat{c}(v,\xi,p),\hat{u}(v,\xi,p),\hat{\alpha}(v)) \qquad (3.36)$$

The optimisation problem (3.31) can be decomposed into the following form :

$$\min_{c,u} q_i(c_i,u_i,\hat{\alpha}_i(v_i)) + p_i^T u_i - \sum_{j=1}^{N} p_j^T H_{ij} F_i(c_i,u_i,\hat{\alpha}_i(v_i)) - \lambda_i^T c$$

$$\text{s.t} \quad \tilde{g}_i(c_i,u_i,\alpha_i(v_i)) \leqslant 0 \qquad i \in \overline{1,N} \qquad (3.37)$$

where the parameter value $\alpha_i(v_i)$ is determined by solving

$$F_i(v_i,HK_*(v_i),\alpha_i) = K_{*i}(v_i) \quad , \; i \in \overline{1,N} \qquad (3.38)$$

Similarly Equations (3.34),...,(3.36) can be solved by the three different schemes described in section 3.3.2.

Brdyś, Chen and Roberts (1986); and Brdyś et. al (1987) have demonstrated that these techniques have solved many example problems successfully with optimal results, and have also shown that these structures are superior than the structures with output feedback only. Among the three techniques, in both structures, the model based double loop method proved to be the most efficient in reducing the number of set-point changes, but at the cost of increasing the amount of information exchange between the decision making units.

3.4 Summary

Brdyś and Roberts (1984) extented the applicabilty of the modified two-step approach (Roberts 1983) to problems with output dependent inequality constraints. Furthermore, Michalska; Ellis and Roberts

(1985) ; Brdyś and Roberts (1984) combined the modified two-step approach with the price method for solving large scale interconnected industrial processes.

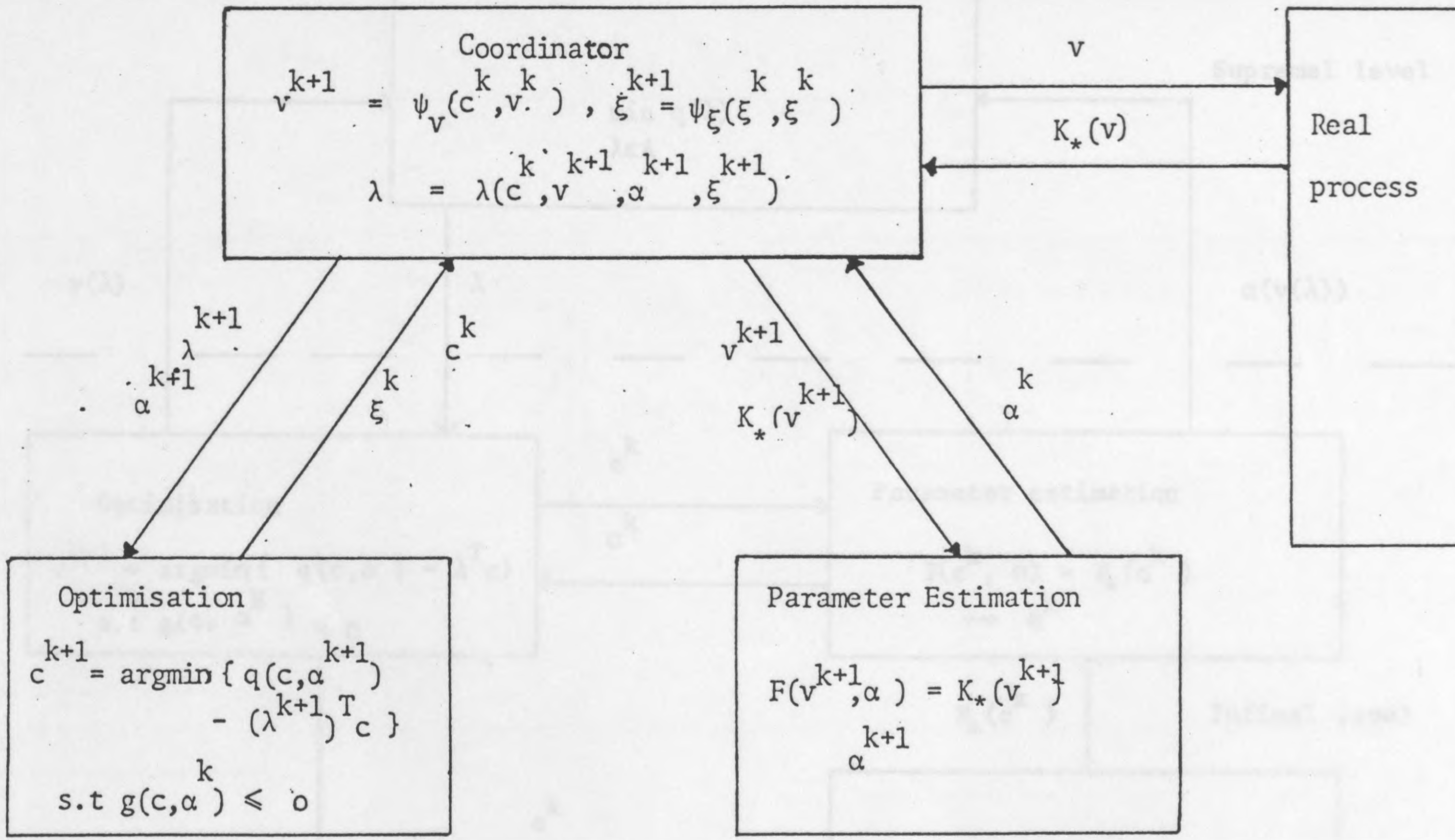

Figure 3.1 Modified two - step algorithm.

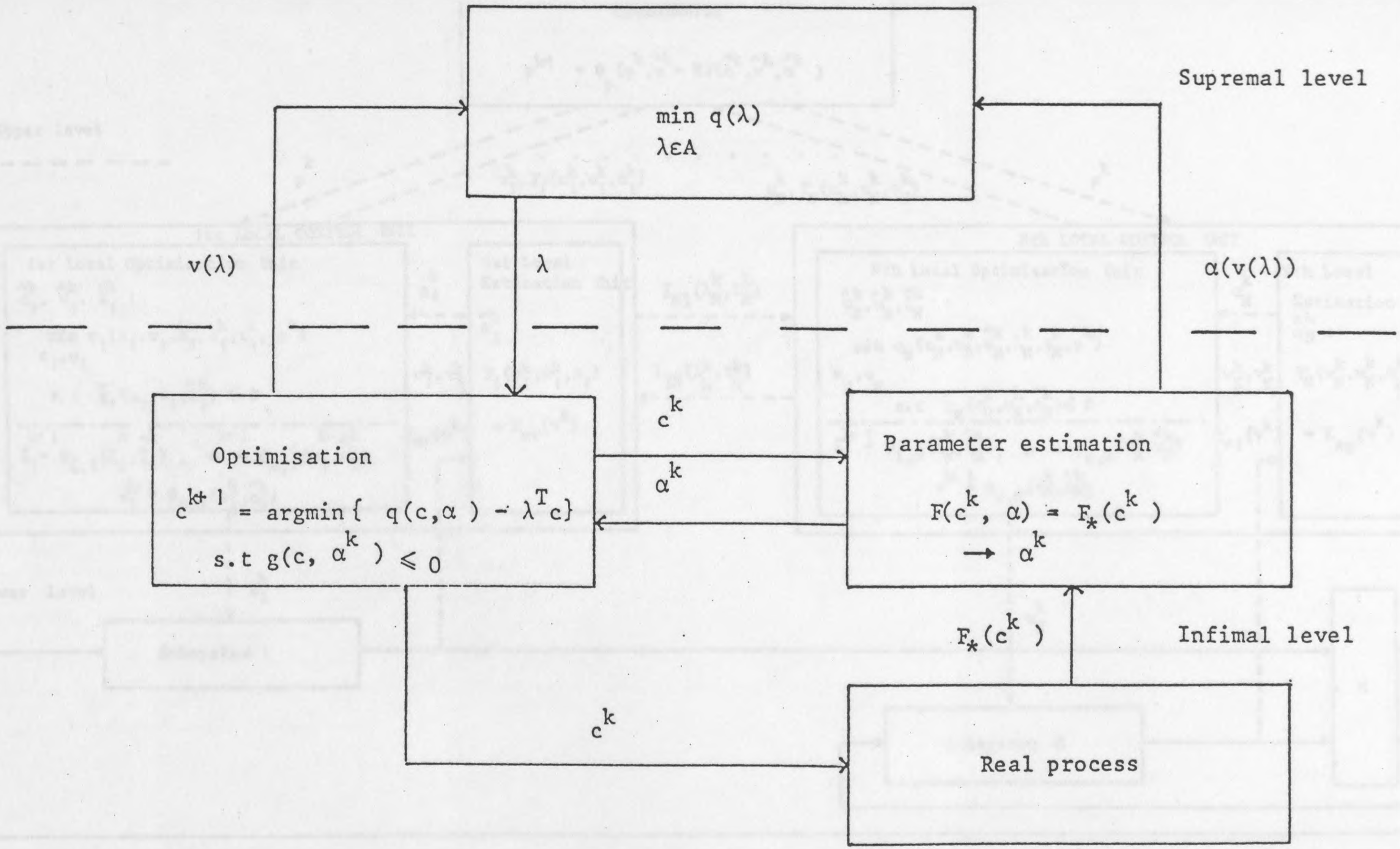

Figure 3.2 Two - level type algorithm.

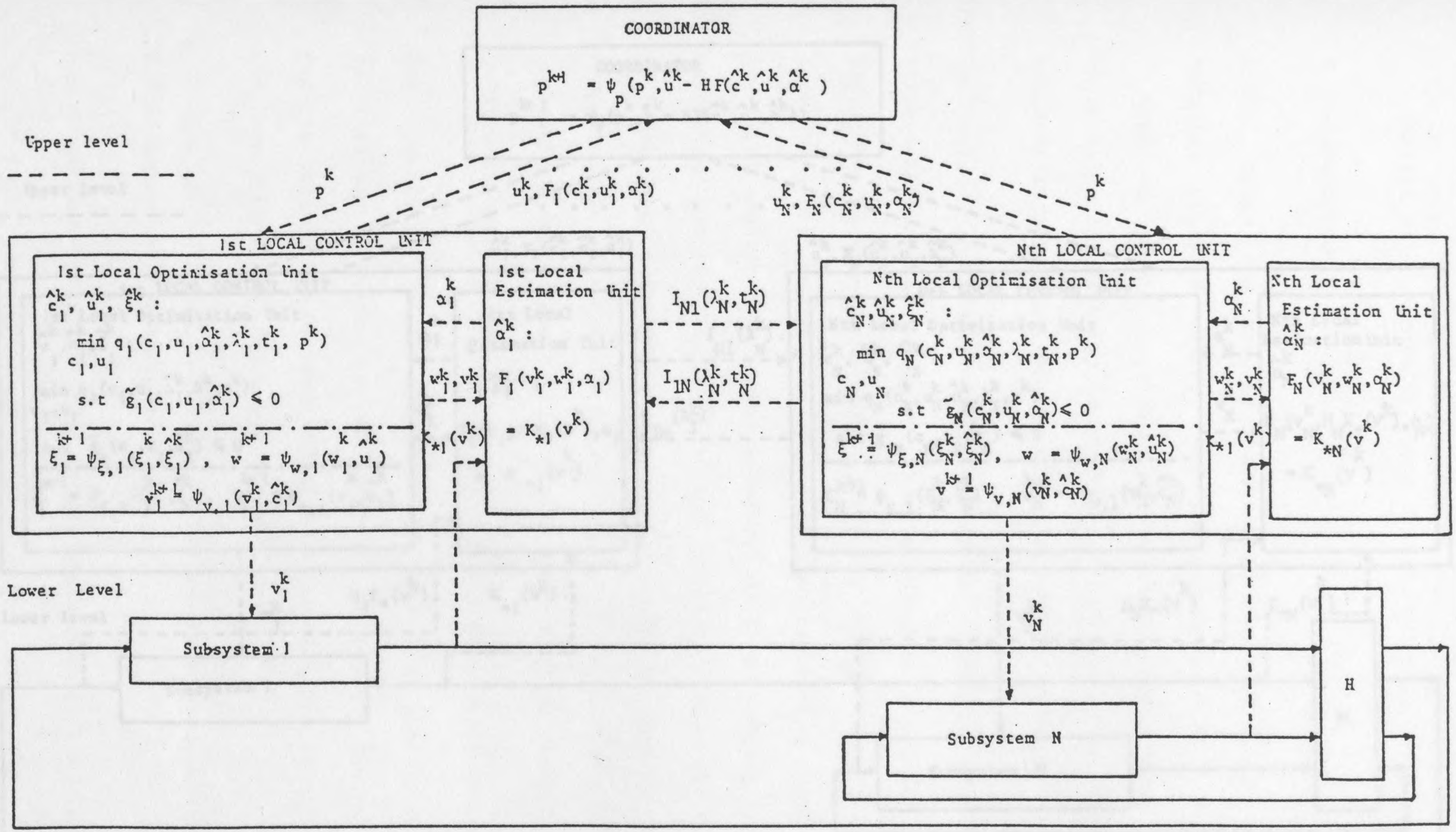

Figure 3.3 A two-level structure with output information feedback.

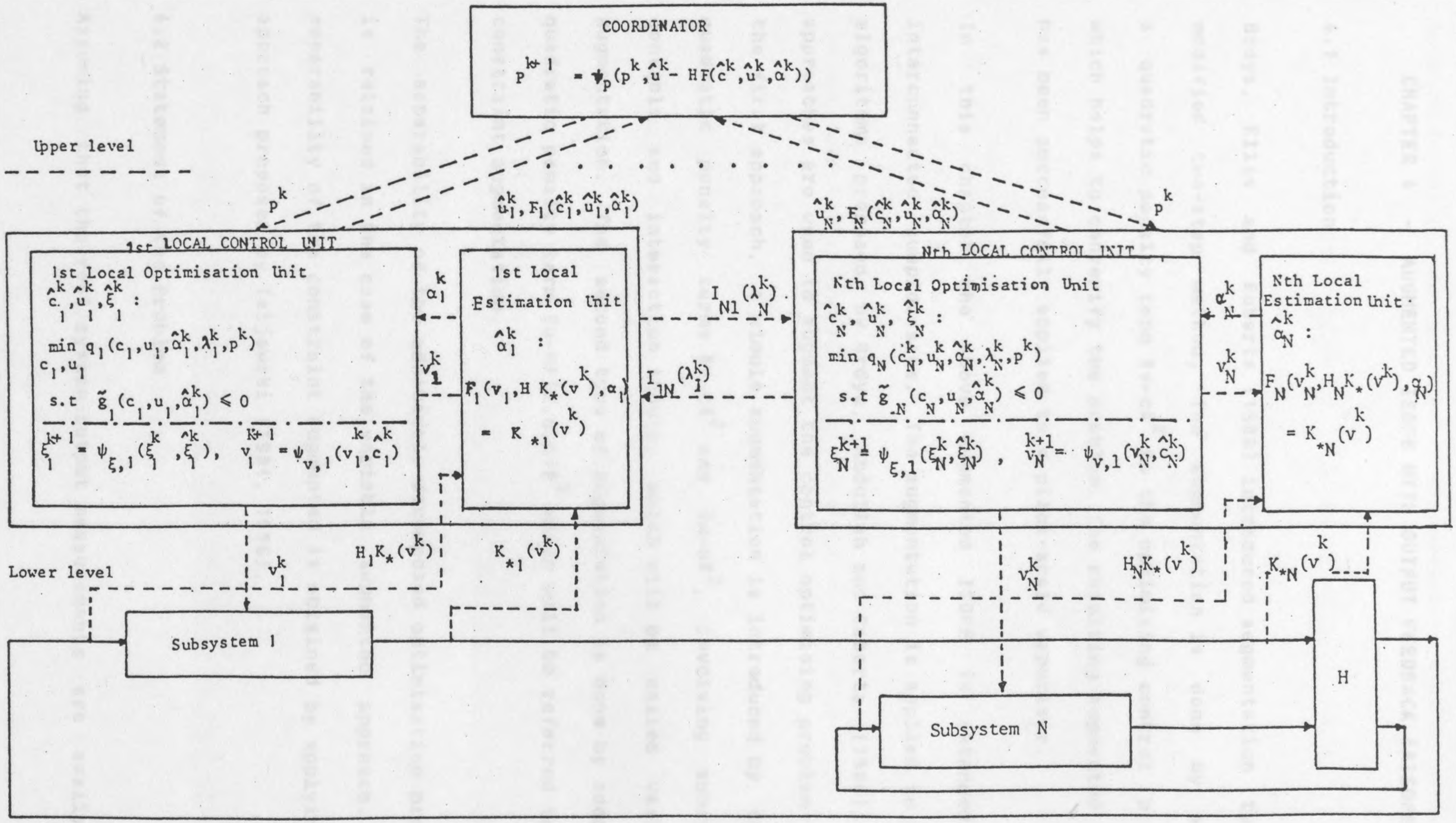

Figure 3.4 A two-level structure with input-output information feedback.

CHAPTER 4 - AUGMENTED ISOPE WITH OUTPUT FEEDBACK (AISOPE)

4.1 Introduction

Brdys, Ellis and Roberts (1986) introduced augmentation to the modified two-step method. The augmentation is done by adding a quadratic penalty term $\|v-c\|^2$ to the optimising control problem which helps to convexify the problem. The resulting augmented ISOPE has been successfully applied to a pilot-scale vaporisor.

In this chapter the above augmented ISOPE is extended to interconnected subprocesses. The augmentation is applied to three algorithms proposed by Brdys, Abdullah and Roberts (1986). Two approaches are used to augment the control optimising problem ; In the first approach, a simple augmentation is introduced by adding quadratic penalty terms $\|v-c\|^2$ and $\|w-u\|^2$, involving subsystem controls and interaction inputs, which will be called variable augmentation. The second type of augmentation is done by adding a quadratic penalty term $\|u-HF(c,u,\alpha)\|^2$ which will be referred to as constraint augmentation.

The separability of the additively decomposed optimisation problem is retained in the case of the variable augmented approach. The separability of the constraint augmented is obtained by applying an approach proposed by Tatjewski (1984, 1986).

4.2 Statement of the Problem

Assuming that the real system output measurements are available, problem (3.18) can be augmented in two different ways as follows :

$$\min_{c,u,v,w,\alpha} \{ q(c,u,\alpha) + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|w-u\|^2 \} \qquad (4.1)$$

and

$$\min_{c,u,v,w,\alpha} \{ q(c,u,\alpha) + \frac{1}{2} r \|u-HF(c,u,\alpha)\|^2 \} \qquad (4.1a)$$

$$\text{s.t} \quad u = HF(c,u,\alpha)$$

$$\tilde{g}(c,u,\alpha) \leqslant 0$$

$$F(v,w,\alpha) = K_*(v)$$

$$v = c, \quad w = u$$

The combined augmented Lagrangian function for problem (3.18), see chapter 3, is :

$$L(c,v,u,w,\alpha,p,\lambda,t,\mu,\xi,\varrho,r) = q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)]$$
$$+ \lambda^T(v-c) + t^T(w-u) + \mu^T[K_*(v) - F(v,w,\alpha)] + \frac{1}{2}\varrho_1\|v-c\|^2$$
$$+ \frac{1}{2}\varrho_2\|w-u\|^2 + \frac{1}{2}\varrho_3\|K_*(v)-F(v,w,\alpha)\|^2 + \frac{1}{2}r\|u-HF(c,u,\alpha)\|^2$$
$$+ \frac{1}{2}\sum_{j=1}^{M} \varrho_{3+j}[(\max(0,\tilde{g}(c,u,\alpha) + \frac{\xi_j}{\varrho_{3+j}}))^2 - (\frac{\xi_j}{\varrho_{3+j}})^2] \qquad (4.2)$$

where

$\varrho = [\varrho_1,\varrho_2,\varrho_3,\ldots\ldots,\varrho_{M+3}]$, $j > 0$ for every $j = 1,2,3,..,M+3.$

and r are penalty coefficients ; λ, t, μ and ξ are vectors of Lagrangian multipliers.

For given values of ϱ and r the stationary points of

$L(\ldots,p,\lambda,t,\mu,\xi,\varrho,r)$ are defined as follows :

$$\Delta_c L = \frac{\partial q^T(c,u,\alpha)}{\partial c} + \frac{\partial^T[u-HF(c,u,\alpha)]p}{\partial c} - \lambda - \varrho_1(v-c)$$

$$- r\frac{\partial^T[u-HF(c,u,\alpha)]}{\partial c}[u-HF(c,u,\alpha)]^T + \frac{\partial P^T(c,u,\alpha,\xi)}{\partial c} = 0 \tag{4.3}$$

$$\Delta_u L = \frac{\partial q^T(c,u,\alpha)}{\partial u} + \frac{\partial^T[u-HF(c,u,\alpha)]p}{\partial u} - t - \varrho_2(w-u)$$

$$- r\frac{\partial F^T[u-HF(c,u,\alpha)]}{\partial u}[u-HF(c,u,\alpha)]^T + \frac{P^T(c,u,\alpha,\xi)}{\partial u} = 0 \tag{4.4}$$

$$\Delta_v L = \lambda + [\frac{\partial^T K_*(v)}{\partial v} - \frac{\partial^T F(v,w,\alpha)]\mu}{\partial v} + \varrho_1(v-c)$$

$$+ \varrho_3 [\frac{\partial K_*(v)}{\partial v} - \frac{\partial F(v,w,\alpha)}{\partial v}]^T[K_*(v) - F(v,w,\alpha)]^T = 0 \tag{4.5}$$

$$\Delta_w L = t - \frac{\partial^T F(v,w,\alpha)}{\partial w} + \varrho_2(w-u) -$$

$$\varrho_3 \frac{\partial F^T(v,w,\alpha)}{\partial w}(K_*(v)-F(v,w,\alpha))^T = 0 \tag{4.6}$$

$$\Delta_\alpha L = \frac{\partial q^T(c,u,\alpha)}{\partial \alpha} - \frac{\partial^T F(c,u,\alpha)}{\partial \alpha}H^T p - \frac{\partial F^T(c,u,\alpha)}{\partial \alpha}$$

$$- r\frac{\partial F^T(c,u,\alpha)}{\partial \alpha}(u-HF(c,u,\alpha))^T + \frac{\partial P^T(c,u,\alpha,\xi)}{\partial \alpha} = 0 \tag{4.7}$$

$$\Delta_p L = u - HF(c,u,\alpha) = 0 \tag{4.8}$$

$$\Delta_\lambda L = v - c = 0 \tag{4.9}$$

$$\Delta_\mu L = K_*(v) - F(v,w,\alpha) = 0 \tag{4.10}$$

$$\Delta_\xi L(c,v,u,w,p,\lambda,t,\xi,\mu,\varrho,r) = 0 \tag{4.11}$$

where

$$P(c,u,\alpha,\xi) = \frac{1}{2}\sum_{j=1}^{M} \varrho_{3+j}\left[\left(\max(0,\tilde{g}_j(c,u,\alpha) + \frac{\xi_j}{\xi_{3+j}})\right)^2 - \left(\frac{\xi_j}{\xi_{3+j}}\right)^2\right] \tag{4.12}$$

4.3 Variable Augmentation

Equations (4.3),....,(4.11) can be used to reformulate optimality conditions for (4.1) in the form of an ISOPE algorithm, and the corresponding optimising control problem is :

$$\min \left[q(c,u,\hat{\alpha}(v,w)) - \lambda^T c - t^T u + p^T[u-HF(c,u,\hat{\alpha}(v,w))) + \frac{1}{2}\varrho_1\|v-c\|^2 + \frac{1}{2}\varrho_2\|w-u\|^2\right] \tag{4.13}$$

$$\text{s.t} \quad \tilde{g}(c,u,\alpha(v,w)) \leqslant 0$$

where

$$\lambda^T(v,w,\xi,p,\varrho) = [K_*'^T(v) - F_v'^T(v,w,\alpha)][F_\alpha'^T(v,w,\alpha)]^{-1}[-q_\alpha'(c,u,\alpha) + F_\alpha'^T(c,u,\alpha)H^T p - P_\alpha'^T(c,u,\alpha,\xi)] \tag{4.14}$$

and

$$t^T(v,w,\xi,p,\varrho) = -F_w'^T(v,w,\alpha)[F_\alpha'^T(v,w,\alpha)]^{-1}[-q_\alpha'(c,u,\alpha) + F_\alpha'^T(c,u,\alpha)H^T p - P_\alpha'^T(c,u,\alpha,\xi)] \tag{4.15}$$

and assuming that the inverse $[F_{\alpha}^{'T}(v,w,\alpha)]^{-1}$ exists.

Since

$$[F_{\alpha}^{'T}(v,w,\alpha)]^{-1} q_{\alpha}^{'T}(v,w,\alpha) = Q_{y}^{'T}(v,w,F(v,w,\alpha))$$

and

$$[F_{\alpha}^{'T}(v,w,\alpha)]^{-1} p_{\alpha}^{'T}(v,w,\alpha,\xi) = \sum_{j=1}^{M} \varrho_{3+j}[\max(0,\tilde{g}_j(c,u,\alpha)) + \frac{\xi_j}{\varrho_{3+j}}].$$

$$.(G_j)_y^{'T}(v,w,\alpha)$$

Therefore, λ and t can be transformed to

$$\lambda^T(v,w,\xi,p,\varrho) = [p^T H - \sum_{j=1}^{M} \varrho_{3+j}\max(0,\tilde{g}_j(c,u,\hat{\alpha}(v,w)) + \frac{\xi_j}{\varrho_{3+j}}).$$

$$.(G_j)_y^{'T}(v,w,\hat{\alpha}(v,w)) - Q_y^{'}(v,w,K_*(v))][K_*^{'}(v) - F_v^{'}(v,w,\hat{\alpha}(v,w))] \quad (4.16)$$

$$t^T(v,w,\xi,p,\varrho) = -[p^T H - \sum_{j=1}^{M} \varrho_{3+j}\max(0,\tilde{g}_j(c,u,\hat{\alpha}(v,w)) + \frac{\xi_j}{\varrho_{3+j}}).$$

$$.(G_j)_y^{'T}(v,w,\hat{\alpha}(v,w)) - Q_y^{'T}(v,w,K_*(v))]F_u^{'}(v,w,\hat{\alpha}(v,w)) \quad (4.17)$$

The parameter $\hat{\alpha}(v,w)$ can be estimated by solving

$$F(v,w,\hat{\alpha}(v,w)) = K_*(v) \quad (4.18)$$

The solution of (4.13) is described by $\hat{c}(v,w,\xi,p)$ and $\hat{u}(v,w,\xi,p)$, and the corresponding Lagrangian multiplier by $\hat{\xi}(v,w,\xi,p)$.

The following is a set of coordinating conditions for (4.13) :

$$\hat{c}(v,w,\xi,p) = v \qquad (4.19)$$

$$\hat{u}(v,w,\xi,p) = w \qquad (4.20)$$

$$\hat{\xi}(v,w,\xi,p) = \xi \qquad (4.21)$$

$$\hat{u}(v,w,\xi,p) = HF(\hat{c}(v,w,\xi,p),\hat{u}(v,w,\xi,p),\hat{\alpha}(v,w)) \qquad (4.22)$$

If the regularity conditions are satisfied at every point of a set CU, then a part (v_*,w_*) of any solution (v_*,w_*,ξ_*,p_*) of the set of equations (4.19), (4.20), (4.21) and (4.22) satisfies the first order necessary Kuhn-Tucker (K-T) conditions corresponding to the expanded problem (3.18) (see Brdyś and Roberts, 1987).

If the process inequality constraints are output independent i.e $G(c,u,y) = G(c,u)$, equation (4.19),..,(4.22) can be simplified to

$$\hat{c}(v,w,p) = v \qquad (4.23)$$

$$\hat{u}(v,w,p) = w \qquad (4.24)$$

$$\hat{u}(v,w,p) = HF(\hat{c}(v,w,p),\hat{u}(v,w,p),\hat{\alpha}(v,w)) \qquad (4.25)$$

and the corresponding λ and t are

$$\lambda^T(v,w,p,\varrho) = [K_*'^T(v) - F_v'^T(v,w,\alpha)][F_\alpha'^T(v,w,\alpha)]^{-1}.$$
$$.[- q_\alpha'^T(c,u,\alpha) + F_\alpha'^T(c,u,\alpha)H^T p] \qquad (4.26)$$

$$t^T(v,w,p,\varrho) = - F_w'^T(v,w,\alpha)[F_\alpha'^T(c,u,\alpha)]^{-1}.$$
$$.[- q_\alpha'^T(c,u,\alpha) + F_\alpha'^T(c,u,\alpha)H^T p] \qquad (4.27)$$

The above set of equations (4.19),...,(4.22) have a two-level hierarchical structure with information exchange between local control units and with information feedback from the system to each control unit as shown in fig.4.1 (Brdyś and Roberts, 1987).

Brdyś and Roberts (1987) proposed an iterative scheme for finding the optimum solution through solving equations (4.19),...,(4.22). This hierarchical structure consists of N local units, where each i-th control unit is composed of two parts : namely, a local parameter identification unit and a local optimisation unit. The task of the local parameter identification unit is to estimate the i-th subsystem and model parameter α_i^k by solving

$$F_i(v_i^k, w_i^k, \alpha_i^k) = K_{*i}(v^k) \qquad (4.28)$$

for given values v_i^k, w_i^k and p_i^k.

The local optimisation unit task, for given values of α_i^k and p_i^k from the parameter identification and coordinator respectively, is to perform the following modified optimisation problem

$$\min_{c,u} \; \Sigma\, \tilde{q}_i(c_i, u_i, \hat{\alpha}(v,w), \lambda_i, t_i, p_i, \varrho)$$

$$\text{s.t} \;\; \tilde{g}_i(c_i, u_i, \hat{\alpha}_i(v_i, w_i)) \leqslant 0 \quad , i \in \overline{1,N}$$

where

$$\tilde{q}_i(c_i, u_i, \hat{\alpha}_i(v_i, w_i), \lambda_i, t_i, p, \varrho) = q_i(c_i, u_i, \hat{\alpha}_i(v_i, w_i)) + p_i^T u_i$$

$$- \sum_{j=1}^{N} p_j^T H_{ji} F_i(c_i, u_i, \hat{\alpha}(v_i, w_i)) - \lambda_i^T c_i - t_i^T u_i + \frac{1}{2}\varrho_1 \|v_i - c_i\|^2$$

$$+ \frac{1}{2}\varrho_2 \|w_i - u_i\|^2 \qquad (4.29)$$

producing corresponding control $\hat{c}_i^k$ and interaction input $\hat{u}_i^k$. Where the vectors λ_i^k, t_i^k and ξ_i^k are i-th components of the vectors

$\lambda^T = \lambda(v^k, w^k, \alpha^k, p^k, \xi^k, \varrho)$, $t^k(v^k, w^k, \alpha^k, p^k, \xi^k, \varrho)$

and $\xi^k = \xi(v^k, w^k, \alpha^k, p^k, \xi^k, \varrho)$ (4.30)

respectively.

Information exchange between the units is required to compute λ_i^k, t_i^k and ξ_i^k. The derivatives needed for evaluating λ_i^k, t_i^k are determined experimentally by performing additional perturbations around v_i^k, $i \in \overline{1,N}$.

The local units are coordinated by a coordinator whose task is to adjust the new price vector according to the rule :

$$p^{k+1} = \psi_p(p^k, \hat{u}^k - HF(\hat{c}^k, \hat{u}^k, \hat{\alpha}^k)) \qquad (4.31)$$

while the new values of v_i^{k+1}, w_i^{k+1}, ξ_i^{k+1}, $i \in \overline{1,N}$ are iterated for each local unit according to

$$v_i^{k+1} = \psi_{v,i}(v_i^{k+1}, \hat{c}_i^k) \quad , \quad w_i^{k+1} = \psi_{w,i}(w_i^k, \hat{u}_i^k)$$

and $\quad p_{\xi_i}^{k+1} = \psi_{\xi,i}(\xi_i^k, \hat{\xi}_i^k) \qquad (4.32)$

where the mappings $\psi_{p,i}$, $\psi_{v,i}$, $\psi_{w,i}$ and $\psi_{\xi,i}$, $i \in \overline{1,N}$ have to be properly chosen in order to ensure convergence.

The key element of the algorithm structure is the coordinator algorithm for adjusting v, w and p. Brdyś, Abdullah and Roberts (1986) proposed 3 different schemes for adjusting v, w and p for solving problem (4.13) which resulted in the following algorithms :

i) Single loop technique

ii) System based double loop technique

iii) Model based double loop technique

In the next section, we will be looking at problems with output independent inequality constraints.

4.3.1 Single loop technique

An augmented single loop technique is obtained by iterating all the coordinating variables in equations (4.23),..,(4.25) simultaneously, with mappings $\psi_{p,i}(.)$, $\psi_{v,i}(.)$ and $\psi_{w,i}(.)$ chosen in such a way that

$$p^{k+1} = p^k + \varepsilon_p^k[\hat{u}^k - HF(\hat{c}^k, \hat{u}^k, \hat{\alpha}^k)] \qquad (4.33)$$

$$v^{k+1} = v^k + \varepsilon_v^k[\hat{c}^k - \hat{v}^k] \qquad (4.34)$$

$$w^{k+1} = w^k + \varepsilon_w^k[\hat{u}^k - w^k] \qquad (4.35)$$

where ε_p, ε_v and ε_w are positive constants, known as gain coefficients. In order to speed up convergence, equation (4.25) is replaced by (see Brdys, Abdullah and Roberts, 1986).

$$\hat{u}(v,w,p) = HF(\hat{c}(v,w,p),\hat{u}(v,w,p),\hat{\alpha}(c(v,w,p),\hat{u}(v,w,p))) \qquad (4.36)$$

The sets of equations (4.23), (4.24), (4.25) and (4.23), (4.24) and (4.36) are equivalent.

Applying the same iterative strategy used previously to (4.23), (4.24) and (4.36), we obtain exactly the same formulae as (4.33), (4.34) and (4.35) for updating p, v and w, except for the parameter value in (4.33) which now becomes :

$$\hat{\alpha}^k = \hat{\alpha}^k(\hat{c}^k,\hat{u}^k)$$

The convergence properties of the technique will be analysed in chapter 5 with gain coefficients possibly varying during the iteration but under the assumption that $\varepsilon_v^k = \varepsilon_w^k = \varepsilon^k$. The structure of the single loop algorithm can be summarised as follows :

1^o. The initial values of the coordinating variables v^o, w^o and p^o are prescribed. The accuracy values β_v, β_w and β_p are appropriately chosen and k is set as k = 0 .

2^o. The set-point v_i^k are applied to the system and outputs $K_{*i}(v)$ are measured. The derivatives $K_{*i}(v)$ are estimated by applying a perturbation around v_i^k.

3^o. The parameter values $\alpha_i^k(v)$ are estimated by solving equation (4.28). These tasks are performed by each of the i-th local parameter estimation units.

4^o The modifier λ_i^k and t_i^k are evaluated according to (4.26) and (4.27) respectively and performed by the i-th local control unit. The required information to evaluate λ^k and t^k is obtained from step 2^o and 3^o and the rest of the information comes from information exchange between the local control units.

5^o For given values of λ_i^k, t_i^k and price p_i^k, each of the i-th local optimisation units solves its local augmented modified problem (4.29), yielding set-point control $\hat{c}_i(v,w,p)$ and input interaction $\hat{u}_i(v,w,p)$.

6^o Each of the i-th local control units adjusts v_i, w_i and p_i according to (4.33), (4.34) and (4.35) respectively. The updating of price p is performed by the coordinator.

7^o Convergence of the coordinating variables is checked, i.e if conditions $|v^{k+1} - v^k| < \beta_v$, $|w^{k+1}-w^k| < \beta_w$ and $|p^{k+1}-p^k| < \beta_p$ are satisfied the algorithm stops, otherwise the whole procedure is repeated.

A second version of the augmented single loop technique is formulated with slightly less restrictive sufficient conditions.

The problem (4.13) can be written as follows :

$$\min \{ q(c,u,\hat{\alpha}(v,w)) + p^T g(c,u,\hat{\alpha}(v,w)) - [\lambda^T_{(\varepsilon,\varepsilon p)}(v,w,p), t^T_{(\varepsilon,\varepsilon p)}(v,w,p)] \begin{vmatrix} c \\ u \end{vmatrix} + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|w-u\|^2 \} \quad (4.37)$$

where

$$[\lambda^T_{(\varepsilon,\varepsilon p)}(v,w,p,\varrho), t^T_{(\varepsilon,\varepsilon p)}(v,w,p,\varrho)] = q^{\cdot}_{c,u}(v,w,\hat{\alpha}(v,w)) - \varepsilon q_*(v,w) + p^T[g^{\cdot}_{c,u}(v,w,\hat{\alpha}(v,w)) - \varepsilon g_*(v,w)] \quad (4.38)$$

and where ε is a positive number.

Using similar equations (4.23), (4.24) and (4.36), and applying the previously presented algorithm with $\varepsilon_p^k = \varepsilon_p$ and $\varepsilon_v^k = \varepsilon_w^k = 1$, the following algorithm is obtained :

$$p^{k+1} = p^k + \varepsilon_p[\hat{u}^k - HF(\hat{c}^k, \hat{u}^k, \hat{\alpha}^k)] \quad (4.39)$$

$$v^{k+1} = \hat{c}^k \quad (4.40)$$

$$w^{k+1} = \hat{u}^k \quad (4.41)$$

The mappings defined by (4.37) and (4.13) are different, therefore the sequence of points generated by this algorithm is different from the previous algorithm.

The algorithm is iterated in the same manner as in the first version of the augmented single loop method, except that (4.37) and (4.38) are used instead of (4.13), (4.26) and (4.27).

4.3.2 The System Based Double Loop Technique

The proposed technique involves an iterative procedure of solving equations (4.23) and (4.24) in the inner loop for a given value of price p. The outer loop task is to evaluate the price p such that equation (4.25) is satisfied and this is equivalent to solving the following equation :

$$w(p) = HF(v(p),w(p),\hat{\alpha}(v(p),w(p))) \qquad (4.42)$$

where v(p) and w(p) are the solution of the inner loop problem under given p. The outer loop task is performed by the coordinator. The following strategy is proposed for updating the price.

$$p^{s+1} = p^{s} + \varepsilon_{p}(w(p^{s}) - H(v(p^{s}),w(p^{s}),\hat{\alpha}(v(p^{s}),w(p^{s}))] \qquad (4.43)$$

where ε_p is a positive number suitably chosen to preserve convergence and s denotes an iteration number.

The properties of the inner loop solution are stated in the proof of Lemma 5.2 (see Chapter 5). Let us consider the inner loop problem with the iterative strategy :

$$\begin{vmatrix} v^{k+1} \\ w^{k+1} \end{vmatrix} = \begin{vmatrix} v^{k} \\ w^{k} \end{vmatrix} + \varepsilon^{k}\left(\begin{vmatrix} \hat{c}^{k} \\ \hat{u}^{k} \end{vmatrix} - \begin{vmatrix} v^{k} \\ w^{k} \end{vmatrix} \right) \qquad (4.44)$$

where ε^{k} is a positive number suitably chosen at the k-th iteration to preserve convergence.

Sufficient conditions for convergence of the iterative scheme (4.43), expressed in terms of the properties of $L_{*}(.)$, can be derived from the general results, as reported in Findeisen et. al (1980). The optimality and convergence analysis of the algorithm is presented in Chapter 5.

The procedure of implementing the algorithm are summarised as follows :

Steps 1^o to 4^o are similar to the single loop technique.

5^o For given values of v_i^k, w_i^k and p_i^k, the i-th local optimisation unit performs the minimisation of the augmented modified optimisation problem (4.13), giving rise to $\hat{c}_i(v,w,p)$ and $\hat{u}_i(v,w,p)$.

6^o Each i-th local unit adjusts v and w according to (4.44).

7^o Convergence of the set-point v and iteraction input w are checked, i.e if condition $|v^{k+1}-v^k| < \beta_v$ and $|w^{K+1}-w^k| < \beta_w$ are satisfied then the algorithm stops, otherwise return to step 5^o.

8^o Coordinator updates price p according to (4.43) which uses the appropriate information from all the local control units.

9^o Price convergence is checked, if equation (4.42) is satisfied to the prescribed accuracy β_p the algorithm is stopped, otherwise the whole procedure is repeated from step 2^o.

4.3.3 Model Based Double Loop Technique

The model based double loop technique is derived by solving equation (4.25) saparately from equations (4.23) and (4.24). The outer loop task is to produce the new values of v and w by satisfying equations (4.23) and (4.24). The values of v and w, obtained from the outer loop, are sent to the inner loop to solve equation (4.25) with respect to price p. The new value of the price p is sent back to the next outer loop iteration and the whole procedure is repeated until optimal values of the set-point are achieved.

Employing a similar technique by Brdyś, Abdullah and Roberts (1986) the coordinating equations (4.23), (4.24) and (4.25) are modified in order to make the method more suitable for application. Let us replace problem (4.13) by the following :

$$\min \{ q(c,u,\hat{\alpha}(v,w)) + p_1^T g(c,u,\hat{\alpha}(v,w)) - [\lambda^T(v,w,p_2), t^T(v,w,p_2)] \begin{vmatrix} c \\ u \end{vmatrix} + \frac{1}{2}\rho_1 \|v-c\|^2 + \frac{1}{2}\rho_2 \|w-u\|^2 \} \quad (4.45)$$

The price affecting term $p^T g(c,u,\hat{\alpha}(v,w))$ is distinguished from the price affecting term $[\lambda^T(v,w,p), t^T(v,w,p)] \begin{vmatrix} c \\ u \end{vmatrix}$.

The solution of (4.45) with respect to c and u under given p_1 and p_2, v and w is denoted by $\hat{c}(v,w,p_1,p_2)$ and $\hat{u}(v,w,p_1,p_2)$.

Therefore, the set of coordinating equations (4.23), (4.24) and (4.25) is equivalent to

$$\hat{c}(v,w,p_1,p_2) = v \quad (4.46)$$

$$\hat{u}(v,w,p_1,p_2) = w \quad (4.47)$$

$$\hat{u}(v,w,p_1,p_2) = HF(\hat{c}(v,w,p_1,p_2),\hat{u}(v,w,p_1,p_2),\hat{\alpha}(v,w)) \quad (4.48)$$

The resulting inner loop problem is to solve the following equation

$$\hat{u}(v,w,p_1,p_2) = HF(\hat{c}(v,w,p_1,p_2),\hat{u}(v,w,p_1,p_2),\hat{\alpha}(v,w)) \tag{4.49}$$

Equation (4.49) can be solved using the Interaction Balance Method (see Findeisen et. al., 1980). Therefore the inner loop iteration is performed in the following way :

$$p_1^{s+1} = p_1^{s} + \gamma[\hat{u}^s - HF(\hat{c}^s,\hat{u}^s,\hat{\alpha}^k)] \tag{4.50}$$

where $\gamma > 0$, $\hat{u}^s = \hat{u}(v^k,w^k,p_1^s,p_2^k)$, $\hat{c}^s = \hat{c}(v^k,w^k,p_1^s,p_2^k)$, $\hat{\alpha}^k = \hat{\alpha}(v^k,w^k)$ and where s denotes the number of the inner loop iterations corresponding to the k-th iteration of the outer loop. The following iterative scheme is proposed for the outer loop problem.

$$\begin{vmatrix} v^{k+1} \\ w^{k+1} \end{vmatrix} = \begin{vmatrix} v^{k} \\ w^{k} \end{vmatrix} + R_x \begin{vmatrix} \hat{c}^k - v^k \\ \hat{u}^k - w^k \end{vmatrix} \tag{4.51}$$

$$p_2^{k+1} = p_2^{k} + R_p[\hat{p}_1^{k} - \hat{p}_2^{k}] \tag{4.52}$$

where matrices R_x and R_p have inverses R_x^{-1} and R_p^{-1} respectively, and are suitably chosen to preserve convergence, and where

$\hat{c}^k = \hat{c}(v^k,w^k,p_2^k,\hat{p}_1^k(v^k,w^k,p_2^k))$, $\hat{u}^k = \hat{u}(v^k,p_2^k,\hat{p}_1^k(v^k,w^k,p_2^k))$

and $\hat{p}_1^{k} = \hat{p}_1(v^k,w^k,p_2^k)$.

The following procedure summarises the model based double loop algorithm :

1^o The initial values of v^o, w^o, $p_1{}^o$ and $p_2{}^o$ are prescribed. The accuracy of β_v, β_{p1} and β_{p2} are appropriately chosen, and $k = 0$.

Steps 2^o and 3^o are similar to the single loop technique.

4^o For given $(p_2)_i^k$, the modifiers λ_i^k and t_i^k are evaluated according to (4.26) and (4.27) respectively, where the price $(p_1)_i^k$ is substituted for p.

5^o For given values of $(p_2)_i^s$, λ_i^k and t_i^k, the i-th local optimisation unit determines the local optimum augmented modified performance of (4.45), producing $\hat{c}_i^k(v,w,p_1,p_2)$ and $\hat{u}_i^k(v,w,p_1,p_2)$.

6^o The price $(p_1)_i^s$ is updated according to (4.50).

7^o Convergence of the price $(p_1)_i^s$ is checked ; i.e if condition $|p_1{}^{s+1}-p_1{}^s| < \beta_{p1}$ is satisfied then proceed to step 8^o, otherwise return to step 5^o.

8^o The i-th local control unit updates v_i^k and w_i^k according to (5.51), and $(p_2)_i^k$ is adjusted according to (5.52).

9^o Convergence of v_i^k, w_i^k and $(p_2)_i^k$ are checked ; if conditions $|v_i^{k+1}-v_i| < \beta_v$, $|w_i^{k+1}-w_i| < \beta_w$ and $|p_2{}^{k+1}-p_2{}^k| < \beta_{p2}$ are satisfied then stop, otherwise set $k = k+1$ and repeat from step 2^o.

4.4 Constraint Augmentation

Using a similar approach to the formulation of problem (4.13), the constraint augmentation problem (4.2) can be reformulated as follows :

$$\min [\, q(c,u,\hat{\alpha}(v,w)) - \lambda^T c - t^T c - p^T[u-HF(c,u,\hat{\alpha}(v,w)) + \frac{1}{2} r\|u-HF(c,u,\alpha\|^2)\} \qquad (4.53)$$

$$\text{s.t } \tilde{g}(c,u) \leqslant 0$$

where

$$\lambda^T(v,w,p,r) = [K_*^{'T}(v) - F_c^{'T}(v,w,\alpha(v,w))[F_\alpha^{'T}(v,w,\alpha)]^{-1}.$$

$$.[- q_\alpha^{'}(v,w,\alpha) + F_\alpha^{'T}(c,u,\alpha)H^T p + F_\alpha^{'T}(c,u,\alpha)H^T r[u-HK_*(v))] \qquad (4.54)$$

$$t^T(v,w,p,r) = - F_u^{'T}(v,w,\hat{\alpha}(v,w))[F_\alpha^{\dagger}(v,w,\alpha)]^{-1}.$$

$$.[- q_\alpha^{'}(c,u,\alpha) + F_\alpha^{'T}(c,u,\alpha)H^T p + F_\alpha^{'T}(c,u,\alpha)H^T r(w-HK_*(v))] \qquad (4.55)$$

The constraint augmenting term $\frac{1}{2}r\|u-HF(c,u,\alpha)\|^2$ is not separable,

$$\|u-HF(c,u,\alpha)\|^2 \simeq \|u\|^2 + \|HF(c,u,\alpha)\|^2 - 2u^T HF(c,u,\alpha) \qquad (4.56)$$

because each variable in the term $2u^T HF(c,u,\alpha)$ depends on each corresponding input or output variable, hence the term is not a sum of N terms. Separability can be obtained by linearisation of this term around some point (c^s,u^s) as proposed by Stephanopoulos and Westerberg (1975), which results in the following approximation

$$u^T HF(c,u,\alpha) = (u^s)^T HF(c,u,\alpha) + u^T HF(c^s,u^s,\alpha) - (u^s)^T HF(c^s,u^s,\alpha) \quad (4.59)$$

In order to solve $\hat{c}^s$ and $\hat{u}^s$, we have to introduce an additional loop known as an approximation loop which can be summarised as follows :

i) Set the initial values of (c^o, u^o) and the accuracy of $\delta_a > 0$ is appropriately chosen, and set s =1.

ii) For given $\lambda(v,w,p,r)$, $t(v,w,o,r)$, p and r the following saparable modified optimisation problem

$$\min \{q(c,u,\hat{\alpha}(v,w)) + p^T[u-HF(c,u,\hat{\alpha}(v,w))] - \lambda^T(v,w,p,r)c - t^T(v,w,p,r)u + [\tfrac{1}{2}r\|u\|^2 + \|F(c,u,\alpha(v,w))\|^2 - 2(u^s)^T pHF(c,u,\hat{\alpha}(v,w)) - 2u^T HF(c^s,u^s,\hat{\alpha}(v,w))]\} \quad (4.58)$$

is solved to obtain $\hat{c}^s$ and $\hat{u}^s$.

iii) The convergence is checked, i.e if condition $\|(\hat{c}^s,\hat{u}^s) - (c^s,u^s)\| \leqslant \delta_a$ is satisfied then set $\hat{c}(v,w,p) = \hat{c}^s$, $\hat{u}(v,w,p) = \hat{u}^s$, where $(\hat{c}(v,w,p),\hat{u}(v,w,p))$ denotes a solution of (4.58), otherwise, $\hat{c}^s$ and $\hat{u}^s$ are updated as follows

$$(c^{s+1},u^{s+1}) = (c^s,u^s) + \varepsilon_a[(\hat{c}^s,\hat{u}^s) - (c^s,u^s)] \quad (4.59)$$

and set s = s+1 and go to (ii).

When the constraint augmentation is applied to the three proposed algorithms, the iterative procedures described in sections 4.3.1, 4.3.2 and 4.3.3 remain the same, except that the optimising control step is replaced by the approximation loop. As an example, when the augmented single loop technique is applied, the iterative procedure in section 4.3.1 is modified as follows :

Step 1^o to 4^o remain the same.

5^o The optimising control step is replaced by the approximation loop which consists of steps (i) to (iii). If the convergence condition $(\hat{c}^s, \hat{u}^s) - (c^s, u^s) \leqslant \beta_a$ is satisfied then set $\hat{c}(v,w,p) = \hat{c}^s$ and $\hat{u}(v,w,p) = \hat{u}^s$ and proceed to step 6^o.

4.5 Summary

An extended hierarchical Augmented ISOPE method with output measurement feedback has been augmented using variable and constraint augmentation. Hence, six versions of the hierarchical ISOPE algorithm are obtained :

i) Variable augmented single loop technique

ii) Variable augmented system based double loop technique

iii) Variable augmented model based double loop technique

iv) Constraint augmented single loop technique

v) Constraint augmented system based double loop technique

vi) Constraint augmented model based double loop technique

A significant reduction in the number of the system iterations is achieved with these two types of augmentation and this has been confirmed by simulation results in Chapter 7. A comparative study of these algorithms will also be presented in Chapter 7. The optimality and convergence conditions of these algorithms will be analysed in Chapter 5.

CHAPTER 5 - OPTIMALITY AND CONVERGENCE ANALYSIS FOR AUGMENTED ISOPE WITH OUTPUT FEEDBACK

5.1 Introduction

In this chapter the variable and constraint augmented ISOPE algorithms with output feedback will be presented. The algorithms are studied with the inequality constraints being output dependent. The optimality and convergence of the variable augmented ISOPE algorithm which will be presented has been developed by Abdullah, Brdyś and Roberts (1986). The convergence studies of the constraint augmented ISOPE performed by Tatjewski, Abdullah and Roberts (1986) will also be included in the chapter. These augmented ISOPE algorithms are extended versions of the normal lagrangian ISOPE algorithms formulated by Brdyś, Abdullah and Roberts (1986).

Zangwill's Theorem (1986) will be employed in the convergence proof of the proposed augmented integrated system optimisation and parameter estimation AISOPE algorithms. The techniques used by Cohen (1980) will also be applied in the convergence proof of the augmented single loop technique and the augmented model based double loop technique.

5.2 Optimality Properties of The Variable Augmented Algorithms

Let Ω be a set of all points (v,w,p,ξ) solving the set of equations (4.19), (4.20), (4.21) and (4.22). The optimality properties of the points which belong to Ω will be examined. The first key step is to apply equations (4.14) and (4.15) to problem (4.13) which converts problem (4.13) into another equivalent form:

$$
\begin{aligned}
\min \{ \; & Q(c,u,F(c,u,\hat{\alpha}(v,w)) + Q_y'(v,w,K_*(v)). \\
& .[K_*'(v)-F_c'(v,w,\hat{\alpha}(v,w))]c - Q_y'(v,w,K_*(v))F_u'(v,w,\hat{\alpha}(v,w))u \\
& + p^T[u-HF(c,u,\hat{\alpha}(v,w))] + p^T H[F_c'(v,w,\hat{\alpha}(v,w))-K_*'(v)]c \\
& + p^T HF_u'(v,w,\hat{\alpha}(v,w))u + \sum_{j=1}^{M} \varrho_{3+j} \max(0,\tilde{g}_j(c,u,\hat{\alpha}(v,w)+\frac{\xi_j}{\varrho_{4+j}}). \\
& .(G_j)_y'(v,w,\hat{\alpha}(v,w)).F_u'(v,w,\hat{\alpha}(v,w))u \\
& + \sum_{j=1}^{M} \varrho_{3+j} \max(0,\tilde{g}_j(c,u,\hat{\alpha}(v,w))+\frac{\xi_j}{\varrho_{4+j}})(G_j)_y'(v,w,\alpha(v,w)). \\
& .[F_c'(v,w,\hat{\alpha}(v,w))-K_*'(v)]c + \frac{1}{2}\varrho_1\|v-c\|^2 + \frac{1}{2}\varrho_2\|w-u\|^2 \} \qquad (5.1)
\end{aligned}
$$

Let us define

$$q_*(c,u) \stackrel{\Delta}{=} Q(c,u,K_*(c)) \qquad (5.2)$$

$$g_*(c,u) \stackrel{\Delta}{=} u - HK_*(c) \qquad (5.3)$$

$$g(c,u,\alpha) \stackrel{\Delta}{=} u - HF(c,u,\alpha) \qquad (5.4)$$

$$P_*(c,u,\xi) \stackrel{\Delta}{=} \frac{1}{2} \sum_{j=1}^{M} \varrho_{3+j}((\max(0,\tilde{g}_{*j}(v,w)+\frac{\xi_j}{\varrho_{4+j}})^2-(\frac{\xi_j}{\varrho_{4+j}})^2) \qquad (5.5)$$

Equations (4.14), (4.15) and (4.18) can now be expressed in the following form :

$$[\lambda^T(v,w,p,\varrho),t^T(v,w,p,\varrho)] = q'_{c,u}(v,w,\hat{\alpha}(v,w)) - q'_*(v,w)$$
$$+ \ P'_{c,u}(v,w,\hat{\alpha}(v,w),\xi) - P'_*(v,w,\xi)$$
$$+ \ p^T[g'_{c,u}(v,w,\hat{\alpha}(v,w)) - g'_*(v,w)\ ] \qquad (5.6)$$

Therefore problem (4.13) can be expressed as follows :

$$\min_{(c,u)\varepsilon CU} \{\ q(c,u,\hat{\alpha}(v,w)) + p^T g(c,u,\hat{\alpha}(v,w))$$
$$- [\lambda^T(v,w,p,\varrho),t^T(v,w,p,\varrho)] \begin{vmatrix} c \\ u \end{vmatrix}$$
$$+ \frac{1}{2}\varrho_1\|v-c\|^2 + \frac{1}{2}\varrho_2\|w-u\|^2 \ \} \qquad (5.7)$$

$$\text{s.t} \quad \tilde{g}(c,u,\alpha) \leqslant 0$$

The optimal control problem, OCP, can be written as

$$\min_{(c,u)\varepsilon CU} q_*(c,u)$$
$$\text{s.t} \quad g_*(c,u) = 0 \qquad (5.8)$$
$$\tilde{g}_*(c,u) \leqslant 0$$

$$\text{where} \quad \tilde{g}_*(c,u) \overset{\Delta}{=} G(c,u,F_*(c,u))$$

Let $(\bar{v},\bar{w},\bar{p},\bar{\xi})$ ε Ω and define a Lagrangian function corresponding to Eq.(5.8) as

$$L_*(c,u,p,\alpha) = q_*(c,u) + p^T g_*(c,u) \qquad (5.9)$$

Then $\tilde{g}(\bar{v},\bar{w},\hat{\alpha}(\bar{v},\bar{w})) \leq 0$ and let us assume that the regularity conditions are satisfied at each point of $CU=\{(c,u):\tilde{g}_*(c,u)\leq 0\}$.

Lemma 5.1

If the set CU is convex then $(\bar{v},\bar{w},\bar{p},\bar{\xi})$ is a stationary point of $L_*(.)$ on CU×U. If, additionally, the function $L_*(.,\bar{p},\bar{\xi})$ is convex on CU then $\bar{v}$ is a solution of the OCP.

<u>Proof</u>

It follows from the definition of $\hat{c}(.)$, $\hat{u}(.)$, $\hat{\xi}(.)$, $\hat{\alpha}(.)$ and Eq.(5.7) that (Luenberger, 1984):

$$\begin{aligned}&[q'_{c,u}(\hat{c}(\bar{v},\bar{w},\bar{p},\bar{\xi}),\hat{u}(\bar{v},\bar{w},\bar{p},\bar{\xi}),\hat{\alpha}(\bar{v},\bar{w}))\\ &+ p^T g'_{c,u}(\hat{c}(\bar{v},\bar{w},\bar{p},\bar{\xi}),\hat{u}(\bar{v},\bar{w},\bar{p},\bar{\xi}),\hat{\alpha}(\bar{v},\bar{w}))\\ &- \varrho[\bar{v} - \hat{c}(\bar{v},\bar{w},\bar{p},\bar{\xi}),\ \bar{w} - \hat{u}(\bar{v},\bar{w},\bar{p},\bar{\xi})]\\ &- [\lambda^T(\bar{v},\bar{w},\bar{p},\bar{\xi},\varrho),t^T(\bar{v},\bar{w},\bar{p},\bar{\xi},\varrho)]\left(\begin{vmatrix} c \\ u \end{vmatrix} - \begin{vmatrix} \bar{v} \\ \bar{w} \end{vmatrix}\right) \geq 0\end{aligned}$$

for all $(c,u)\varepsilon CU$

Let us notice that

$$[\lambda^T(\bar{v},\bar{w},\bar{p},\bar{\xi},\varrho),t^T(\bar{v},\bar{w},\bar{p},\bar{\xi},\varrho)] = q'_{c,u}(\bar{v},\bar{w},\hat{\alpha}(\bar{v},\bar{w}))$$

$$- q'_*(\bar{v},\bar{w}) + P'_{c,u}(\bar{v},\bar{w},\hat{\alpha}(\bar{v},\bar{w}),\bar{\xi}) - P'_*(\bar{v},\bar{w},\bar{\xi})$$

$$+ p^T(g'_{c,u}(\bar{v},\bar{w},\hat{\alpha}(\bar{v},\bar{w})) - g'_*(\bar{v},\bar{w})]$$

Therefore, due to the fact that $(\bar{v},\bar{w},\bar{p},\bar{\xi}) \in \Omega$ the following holds :

$$\left.\begin{array}{l} L'_{*c,u}(\bar{v},\bar{w},\bar{p},\bar{\xi}) \begin{vmatrix} c - \bar{v} \\ u - \bar{w} \end{vmatrix} \geqslant 0 , \\ \qquad \text{for all } (c,u) \in CU \\ \text{Moreover , Eq.(4.18) and (4.22) imply that} \\ \qquad g_*(\bar{v},\bar{w}) = 0 \\ \text{and} \\ \xi^T{}_j \hat{g}_{*j}(\bar{v},\bar{w}) = 0 ,\ \xi_j \geqslant 0,\ \hat{g}_{*j}(\bar{v},\bar{w}) \leqslant 0 , \\ \qquad \text{for } j=1,\ldots,M \end{array}\right\} \quad (5.10)$$

and a proof of the first part of the Lemma is completed.

If $L_*(.,\bar{p},\bar{\xi})$ is convex on CU then $(\bar{v},\bar{w})$ minimizes $L_*(.,\bar{p},\bar{\xi})$ on this set which, together with (5.10), means that $(\bar{v},\bar{w},\bar{p},\bar{\xi})$ is a saddle point of $L_*(.)$ on CUxU (Lasdon, 1970). Hence $\bar{v}$ is a solution of OCP.

Q.E.D

The following theorem provides sufficient conditions under which, for every solution $\bar{c}$ of OCP, there exists $\bar{u}$, $\bar{p}$ and $\bar{\xi}$ such that $(\bar{c},\bar{u},\bar{p},\bar{\xi})$ belong to Ω.

Let us define the model based Lagrangian as

$$L(c,u,\alpha,p) \overset{\Delta}{=} q(c,u,\alpha) + p^T g(c,u,\alpha)$$

Let $\bar{c}$ be a solution of OCP and let $\bar{\alpha} \varepsilon \hat{\alpha}(\bar{c},\bar{u})$

Theorem 5.1

Assume

i) The function G_{ij} is convex on $C_i \times U_i$ for every $j \varepsilon J_i$ and $i \varepsilon \underline{1,N}$ and, for every $\alpha \varepsilon A$, $p \varepsilon U$ and $\xi \varepsilon R^{\sum_{i=1}^{N} J_i}$ the function $\{L(c,u,\alpha,p) + \frac{1}{2}\varrho_1 \|c-\bar{c}\|^2 + \frac{1}{2}\varrho_2 \|u-\bar{u}\|^2\}$ is convex on CU with respect to (c,u).

ii) At a point $(\bar{c},HK_*(\bar{c}))$ the regularity conditions for OCP are satisfied.

iii) The set $\{(c,u) \varepsilon C \times U : \tilde{g}(c,u,\hat{\alpha}(\bar{c},HK_*(\bar{c})) \leqslant 0\}$ is convex.

Then, for every solution of the OCP, there exists values of interaction input $\bar{u} \varepsilon U$, price $\bar{p} \varepsilon U$ and $\bar{\xi} \varepsilon R^{\sum_{i=1}^{N} J_i}$ such that $(\bar{c},\bar{u},\bar{p},\bar{\xi}) \varepsilon \Omega$.

Proof

Due to assumption (ii) there exists a vector $\bar{\xi} \varepsilon R^{\sum_{i=1}^{N} J_i}$ such that

$$\left.\begin{aligned} & \dot{L}_{*c}(\bar{c},\bar{u}) + (\dot{P}_*)_c(\bar{c},\bar{u},\bar{\xi}) = 0 \\ & \xi_j^T \tilde{g}_{*j}(\bar{v},\bar{w}) = 0 \ , \ \bar{\xi}_j \geqslant 0 \ , \ \tilde{g}_{*j}(v,w) \leqslant 0, \\ & \qquad\qquad \text{for } j=1,\ldots,M) \end{aligned}\right\} \quad (5.11)$$

Let us consider problem (5.7) for given values of $v = \bar{c}$, $w = \bar{u}$, $\xi = \bar{\xi}$ and $\alpha = \bar{\alpha}$; Its derivative with respect to c and u at a point $(\bar{c},\bar{u},\bar{\alpha},\bar{\xi})$ is equal to

$$[\dot{q}_{c,u}(\bar{c},\bar{u},\bar{\alpha}) + \bar{p}^T \dot{g}_{c,u}(\bar{c},\bar{u},\bar{\alpha}) - \dot{q}_{c,u}(\bar{c},\bar{u},\bar{\alpha})$$

$$+ \dot{q}_*(\bar{c},\bar{u}) - \bar{p}^T(\dot{g}_{c,u}(\bar{c},\bar{u},\bar{\alpha}) - \dot{g}_*(\bar{c},\bar{u}))$$

$$\dot{P}_{c,u}(\bar{c},\bar{u},\bar{\alpha},\bar{\xi}) + \dot{P}_*(\bar{c},\bar{u},\bar{\xi}) + [\rho_1(c-\bar{c}) , \rho_2(u-\bar{u})]$$

$$= \dot{q}_*(\bar{c},\bar{u}) + \bar{p}^T \dot{g}_*(\bar{c},\bar{u}) + \dot{P}_*(\bar{c},\bar{u},\bar{\xi})$$

Therefore, due to the above and the relations (5.11), the point $(\bar{c},\bar{u},\bar{p},\bar{\xi})$ constitutes the Kuhn-Tucker point for problem (5.7) where $v = \bar{c}$, $w = \bar{u}$, $\xi = \bar{\xi}$ and $\alpha = \bar{\alpha}$.

Q.E.D

The assumptions of Theorem 5.1 are much weaker compared with the assumption of Lemma 4 of Brdyś and Roberts (1984) due to the fact that the existence of $[\dot{F}_{\alpha}^T(\bar{c},\bar{u},\bar{\alpha})]^{-1}$ is not required.

This means that the method is not restricted to situations where the number of model parameters is equal to the number of measured outputs. At the same time the function $q(.,.,\alpha)$ does not have to be convex.

Assuming that $q(.,.,\alpha)$ and $F(.,\alpha)$ are twice continuously Frechet differentiable for $\alpha = \hat{\alpha}(\bar{c},\bar{u})$ and that the set CU is convex and compact, it can be shown that assumption (i) is satisfied with the function $\{L(c,u,p,\alpha) + \frac{1}{2}\varrho_1\|c-\bar{c}\|^2 + \frac{1}{2}\varrho_2\|u-\bar{u}\|^2\}$ is not only convex but is also a uniformly convex function if ϱ_1 and ϱ_2 are sufficiently large.

5.3 Convergence of Variable Augmented Algorithm

In this section the theoretical studies of the algoritm are made under the simplifying assumption that the process inequality constraint are output independent.

5.3.1 System Based Double Loop Technique

The properties of the inner loop solution are stated in the following Lemma.

Lemma 5.2

Assume that CU is convex and compact and that for given $p \in U$ a function $L_*(.,p)$ is convex on CU. Then

1^o. The inner loop problem solution exists and the point $(v(p),w(p))$ minimizes $L_*(.,p)$ on CU.

2^o. A dual function $\varphi_*(.)$ corresponding to $L_*(.)$ is sub-differentiable at p and its subgradient $\partial\varphi_*(p)$ at p can be determined by

$$\partial\varphi_*(p) = w(p) - HF(v(p),w(p),\hat{\alpha}(v(p),w(p))) \qquad (5.12)$$

Moreover, if the inner loop problem has a unique solution then $\varphi_*(.)$ is Frechet differentiable at p and (5.12) represents the formula for $\varphi_*(p)$.

Proof

Proof of part 1^o follows immediately from Lemma 5.1. Due to the assumptions of Lemma 5.2 and according to 1^o the following holds (see Findeisen et. al, 1980):

$$\partial\varphi_*(p) = w(p) - HK_*(v(p))$$

This , together with the definition of $\hat{\alpha}(.)$, implies that (5.12) holds. The last statement of the Lemma results from general properties of the dual function (Findeisen et. al, 1980).

The scheme (4.43) proposed for updating the price in the coordinator is a gradient type iterative strategy. Sufficient conditions for convergence of this iterative scheme (4.43) expressed in terms of the properties of $L_*(.)$ can be derived from the general results presented by Findeisen et. al (1980).

Sufficient conditions for iterative strategy (4.44) of the inner loop are formulated in the following theorem.

The algorithm gain (see (5.15)) is allowed to change during the iterations such that

$$\tau \leqslant K^{i}_{v,w} \leqslant B(v^{i},w^{i})$$

where $\tau > 0$ and $B : CU \longrightarrow R^{1}$ is an appropriately defined function of v and w.

Let us define the following point-to-set mapping :

$$\omega : CU \longrightarrow 2^{CU\times CU}$$

$$\omega(v,w) \overset{\Delta}{=} ((v,w),(\hat{c}(v,w,p),\hat{u}(v,w,p)))$$

and

$$\gamma : CU\times CU \longrightarrow 2^{CU}$$

$$\gamma((v,w),(c,u)) \overset{\Delta}{=} \{(v,w) + k_{v,w}(c-v,u-w) : \tau\leqslant k_{v,w}\leqslant B(v,w) \}$$

Therefore, without performing a stop criterion operation, the i-th iteration of the algorithm can be described as finding v^{k+1}, w^{k+1} such that

$$(v^{k+1},w^{k+1}) \in A(v^{k},w^{k})$$

where A(.,.) is named an algorithmic mapping and it is defined as a composition of ω and γ, i.e.,

$$(v,w) \longrightarrow A(v,w) \in 2^{CU}$$

$$A(v,w) \overset{\Delta}{=} \{(v,w) + k_{v,w}(c-v,u-w) : \tau \leqslant k_{v,w} \leqslant B(v,w),$$

$$c \in \hat{c}(v,w,p),\ u \in \hat{u}(v,w,p),\ \hat{\alpha}(v,w) \} \qquad (5.13)$$

Let us define

$$\hat{x} \overset{\Delta}{=} (\hat{c},\hat{u}),\ x \overset{\Delta}{=} (c,u),\ z \overset{\Delta}{=} (v,w),\ \varrho \overset{\Delta}{=} (\varrho_1,\varrho_2)$$

and

$$L_\varrho(.,p,\alpha) \overset{\Delta}{=} [q(.,p,\alpha) + p^T g(x,\alpha) + \frac{1}{2}\varrho\|z-x\|^2]$$

Theorem 5.2

For given $p \varepsilon U$ assume,

i) The set CU is compact and convex.

ii) The set A is compact.

iii) The mappings $F_*(.)$ and $F(.,\alpha)$ are continuous on CU and CU×A respectively.

iv) The function $L_*(.,.,p)$ is Frechet differentiable on CU.

v) For every $\alpha \in A$ the function $L_\varrho(.,p,\alpha)$ is twice Frechet differentiable on CU and $L_\varrho(.,p,\alpha)$, $\dot{L}_{\varrho x}(.,p,\alpha)$ and $(\ddot{L}_\varrho)_{xx}(.,p,\alpha)$ are continuous on CU×A.

vi) A positive constant ϱ is chosen (at least one choice is possible) such that

$$\varrho > - \min_{\alpha \varepsilon A} b(\alpha)$$

where $b(\alpha) = \min\limits_{x \in CU} \lambda_{min}((L_\varrho)_{xx}(x,p,\alpha))$ (5.14)

vii) The point-to-set mapping $\hat{\alpha}(.)$ is open on A

Then,

I. If $$B(v,w) \overset{\Delta}{=} \min \left\{ 1,\ \frac{2 \inf\limits_{\alpha \in \hat{\alpha}(v,w)} b(\alpha) + 2\varrho}{\delta + \varepsilon} \right\} \qquad (5.15)$$

where $$\delta \overset{\Delta}{=} \max_{x \in CU} \| \dot{L}_*(x,p) \| \qquad (5.16)$$

and if τ and $\varepsilon > 0$ are chosen (at least one choice is possible) such that

$$0 \leqslant \tau \leqslant \min \left\{ 1,\ \frac{2 \inf\limits_{\alpha \in A} b(\alpha) + 2\varrho}{\delta + \varepsilon} \right\} \qquad (5.17)$$

then the algorithm mapping A(.,.) is well defined on CU for every v,w ε CU.

II. There is at least one cluster point of the sequence $\{(v^k, w^k)\}$ generated by the inner loop iterative scheme. Each cluster point belongs to the algorithm solution set Ω .

Proof

According to assumptions (i), (ii), (iv), (v) and A1, the mappings $\hat{c}(.,p)$, $\hat{u}(.,p)$ and $\hat{\alpha}(.)$ are well defined on CU, respectively. Therefore, a mapping ω(.) is also well defined.

Let us now consider a mapping $\gamma(.)$. Due to assumptions (i) and (v), the function $b(.)$ is continuous on A which is a compact set (see assumption (ii)). Hence, there exists $\bar{\varrho} > 0$ such that for all $\varrho > \bar{\varrho}$ the assumption (vi) is satisfied.

Assumption (iv) implies that a number δ (see(5.16)) is finite and not negative. Therefore there exists numbers τ and ε such that inequalities (5.17) are satisfied. Since

$$\inf_{\alpha \varepsilon A} b(\alpha) + 2\varrho \leqslant \inf_{\alpha \varepsilon \hat{\alpha}(v,w)} b(\alpha) + 2\varrho$$

for every $v,w \varepsilon CU$ then (5.17) implies that the mapping $\gamma(.)$ with $B(.)$ defined by (5.15) is well defined on CU.

Therefore, the algorithm mapping $A(.)$ is well defined on CU (see (5.13)). Due to (5.16) and according to assumption (i), $A(v,w) \varepsilon 2^{CU \times CU}$. A proof of the assertion I has now been completed.

We will now show that $A(.)$ is closed at every point $((v,w),(c,u)) \varepsilon CU \times CU$ such that $v \neq c$ and $w \neq u$, by proving closeness of these mapping components and then by applying suitable composition theorems. Let us start with mapping $\hat{\alpha}(.)$, for given $v,w \varepsilon CU$

$$\hat{\alpha}(v,w) = \operatorname{Arg} \min_{\alpha \varepsilon A} \| F_*(.) - F(v,w,\alpha) \|$$

Since the set A is compact (see assumption (ii)) and function $\|F_*(.) - F(.,.,\alpha)\|$ is continuous on $CU \times A$ (see assumption (iii)) then the mapping $\hat{\alpha}(.)$ is closed on CU (Hogan, 1973).

Let us consider a function

$$CU\times(CU\times A) \ni (c,u,v,w,p,\alpha) \;--->\; L_{\varrho}(c,u,v,w,p,\alpha) - $$

$$- [\lambda^{T}(v,w,p), t^{T}(v,w,p)] \begin{vmatrix} c \\ u \end{vmatrix} \;\varepsilon\; R^{1}$$

Let v, w, p and α be fixed in CU and A, respectively. The Hessian of this function is equal to

$$(\ddot{L}_{\varrho})_{xx}(c,u,p,\alpha) + \varrho I$$

and

$$h^{T}(\ddot{L}_{\varrho xx}(c,u,p,\alpha) + \varrho)h \;\geqslant\; (\varrho + \min_{\alpha\varepsilon A} b(\alpha)) \| h \|^{2}$$

where $h \;\varepsilon\; R^{n}$.

Hence, due to assumption (vi) problem (5.7) consists of minimizing a uniformly convex function on a convex and compact set at given values of v, w, p and α. Consequently, $\hat{c}(v,w,p,\alpha)$ and $\hat{u}(v,w,p,\alpha)$ consists of a single point. The continuity of the considered function (assumptions (iv) and (v)) on CU×(CU×A) and compactness of CU imply that $\hat{c}(.,p,\alpha)$ and $\hat{u}(.,p,\alpha)$ is a continuous function on CU×A.

Let us consider the following mapping

$$CU \ni (v,w) \;---->\; \{(v,w,p,\alpha) : \hat{\alpha}(v,w)\} \;\varepsilon\; 2^{CU\times A}$$

Since α(.) is closed on CU, this mapping is also closed on CU. Hence, because CU×A is compact then we can employ a closeness of mapping composition theorem (Zangwill, 1969) and obtain that the mapping

$$CU \ni (v,w) \longrightarrow \{ \hat{c}(v,w,\rho,\alpha), \hat{u}(v,w,\rho,\alpha) ; \hat{\alpha}(v,w) \} \in 2^{CU}$$

is closed on CU.

Finally, applying a closeness of mapping arithmetic composition theorem (Zangwill, 1969) to the above mapping and to an identity mapping on CU, we conclude that the mapping $\omega(.)$ is closed on CU.

The continuity of $(\ddot{L}_{\varrho})_{xx}(.,.,\alpha)$ on CU×A and compactness of CU (assumptions (v) and (i) respectively) imply that b(.) is upper-semicontinuous on A (see (5.14)). Therefore, owing to assumption (vii) the following function

$$CU \ni (v,w) \longrightarrow \inf_{\alpha \in \alpha(v,w)} b(.) \quad \in \quad R^1$$

is upper semicontinuous on CU (Hogan, 1973). Thus B(v,w) is upper semicontinuous on CU (see (5.15)).

We will now show that $\gamma(.)$ is closed at every point ((v,w),(c,u)) $\in$ CU×CU such that $v \neq c$ and $w \neq u$.

Let

$$CU \times CU \ni ((v^k,w^k,(c^k,u^k)) \underset{k \to \infty}{\longrightarrow} ((v,w),(u,w)),\ v \neq c,\ w \neq u$$

and

$$\gamma(v^k,w^k),(c^k,u^k)) \ni x^k = (v^k,w^k) + \varepsilon^k(c^k-v^k,u^k-w^k) \underset{k \to \infty}{\longrightarrow} x_i$$

Thus, for sufficiently large k, the following holds :

$$\varepsilon^k = \frac{\| x^k - (v^k,w^k) \|}{\| (c^k-v^k, u^k-w^k) \|} \xrightarrow[k \to \infty]{} \frac{\| x - (v,w) \|}{\| (c-v,u-w) \|} \overset{\Delta}{=} \varepsilon_x$$

Clearly,

$$x = (v,w) + \varepsilon_x (c-v, u-w)$$

Since

$$\tau \leqslant k^k_{v,w} \leqslant B(v^k,w^k)$$

and due to assumption (v), the function B(.) is upper semicontinuous then

$$\tau \leqslant \varepsilon_x \leqslant \limsup_{k \to \infty} B(v^k,w^k) \leqslant B(v,w)$$

Therefore, $x \in \gamma((v,w),(c,u))$ and consequently, $\gamma(.)$ is closed at ((v,w),(c,u)). Hence, the algorithmic mapping A(.,.) is closed at every point which is not a solution of an inner loop problem.

To show that $A(v,w) \in 2^{CU \times CU}$, let us notice that due to (5.15), $B(v,w) \leqslant 1$ and $\tau > 0$. Hence, according to the convexity of CU (see assumption (i)), $A(v,w) \in 2^{CU \times CU}$. Therefore, a proof of part II of the theorem has now been completed.

Moreover, defining the algorithm solution set as a set of all solutions of the inner loop problem, we can now state that the following assumptions of Zangwill's convergence theorem are satisfied and that all the points generated by the algorithm are in a compact set CU and the algorithm mapping is closed outside an algorithm solution set. To verify the remaining assumptions of Zangwill's theorem it is necessary to show that there is continuous function $Z : CU \longrightarrow R^1$ such that

-- if (v,w) is not a solution then for any $(c,u) \in A(v,w)$

$$Z(c,u) < Z(v,w)$$

and

-- if (v,w) is a solution then for any $(c,u) \in A(v,w)$

$$Z(c,u) \leqslant Z(v,w)$$

Let $(c,u) \in A(v,w)$ where v,w are arbitrarily chosen from CU. We shall prove that the above conditions hold if $Z(.) \overset{\Delta}{=} L_*(.,p)$. Due to both assumptions (i) and (iv), $\dot{L}_*(.,p)$ is Lipshitz continuous on CU with constant δ (see (5.16)) and, due to the convexity of CU, the following inequality holds (see Kantorovich and Akilov, 1964).

$$L_*(v,w,p) - L_*(c,u,p) \geqslant \dot{L}_*(v,w,p)\begin{vmatrix} v-c \\ w-u \end{vmatrix} - \frac{1}{2}\delta\|(c-v,w-u)\|^2$$

for all $c,u \in CU$

But

$$(c,u) = (v,w) + k_{v,w}[(\hat{c}(v,w,p)-v),(\hat{u}(v,w,p)-w))$$

for some

$$\underline{k} \leqslant k_{v,w} \leqslant B(v,w)$$

Therefore,

$$L_*(v,w,p) - L_*(c,u,p) \geqslant k_{(v,w)}\dot{L}_*(v,w,p)\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix}$$

$$- \frac{1}{2}k^2_{(v,w)}\delta\|(v-\hat{c}(v,w,p),(w-\hat{u}(v,w,p)\|^2 \qquad (5.18)$$

In order to estimate the first term on the right hand side of (5.18), we will utilize the definition of c(.), u(.), problem (5.7) and the fact that CU is convex. Therefore the following holds (see Luenberger, 1973):

$$\left(\dot{L}_{c,u}(\hat{c}(v,w,p),\hat{u}(v,w,p),p,\hat{\alpha}(v,w)) - [\lambda^T(v,w,p),t^T(v,w,p)] - \varrho(v-c,w-u)\right)\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix} \geqslant 0$$

together with (5.6) implies that

$$\dot{L}_*(v,w,p)\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix} \geqslant [\dot{L}_{c,u}(v,w,p,\alpha) - \dot{L}_{c,u}(\hat{c},\hat{u},p,\alpha)].$$

$$.\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix} + \varrho\| v - \hat{c}(v,w,p), w - \hat{u}(v,w,p) \|^2 \qquad (5.19)$$

Assumptions (i) and (v) imply that there is $0 \leqslant \theta \leqslant 1$ such that (see (5.14))

$$[\dot{L}_{c,u}(v,w,p,\alpha) - \dot{L}_{c,u}(\hat{c},\hat{u},p,\alpha)]\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix} =$$

$$\begin{vmatrix} v - \hat{c}(v,w,p) \\ w - \hat{u}(v,w,p) \end{vmatrix}^T \ddot{L}_{xx}(\hat{c}(v,w,p,\alpha) + \theta(v - \hat{c}(v,w,p)),\hat{\alpha}(v,w)).$$

$$\cdot\left|\begin{matrix} v-\hat{c}(v,w,p) \\ w-\hat{u}(v,w,p)\end{matrix}\right| \geqslant \lambda_{min}(L_{xx}(\hat{c}(v,w,p)+\theta(v-\hat{c}(v,w,p)),\hat{\alpha}(v,w)).$$

$$\left\|\begin{matrix} v-\hat{c}(v,w,p) \\ w-\hat{u}(v,w,p)\end{matrix}\right\|^2 \geqslant b(\alpha)\|v-\hat{c}(v,w,p),w-\hat{u}(v,w,p)\|^2$$

Therefore

$$L_*'(v,w,p)\left|\begin{matrix} v-\hat{c}(v,w,p) \\ w-\hat{u}(v,w,p)\end{matrix}\right| \geqslant (b(\alpha)+\varrho)\|v-\hat{c}(v,w,p),w-\hat{u}(v,w,p)\|^2 \tag{5.20}$$

Finally, combining (5.19) and (5.20) we obtain

$$L_*(v,w,p) - L_*(c,u,p) \geqslant \tau(b(\alpha)+\varrho-\tfrac{1}{2}k_{v,w}\delta).$$

$$.\| v - \hat{c}(v,w,p), w - \hat{u}(v,w,p)\|^2$$

Since

$$k_{(v,w)} \leqslant B(v,w) \leqslant \frac{2b(\alpha) + 2\varrho}{\delta+\varepsilon}$$

then $b(\alpha) + \varrho - \frac{1}{2}k_{(v,w)}\delta \geqslant 0$

Hence,

$$L_*(c,u.,p) \leqslant L_*(v,w,p)$$

and

$$L_*(c,u,p) < L_*(v,w,p) \quad \text{if } (v,w) \text{ is not a solution}$$

Q.E.D

5.3.2 Single-loop Technique

Zangwill's theorem will be used to prove convergence of the algorithm. The techniques developed by Cohen (1980), in the convergence proof of his algorithm 6.1, will also be utilized to some extent to find the Z function.

Let Ω be a set of all solutions of (4.23), (4.24) and (4.25). It is assummed that the algorithm is stopped iff $(v^k, w^k, p^k) \in \Omega$, which is equivalent to $v^{k+1} = v^k$, $w^{k+1} = w^k$ and $p^{k+1} = p^k$, for some k, since $\varepsilon^k \neq 0$ and $\varepsilon \neq 0$.

Let us define the following notation for simplicity

$$\hat{x} \overset{\Delta}{=} (\hat{c}, \hat{u}), \quad z \overset{\Delta}{=} (v, w)$$

and

$$\tilde{q}(x, \alpha, \varrho) \overset{\Delta}{=} q(x, \alpha) + \frac{1}{2}\varrho\|x\|^2$$

Theorem 5.3

Let the assumptions of Theorem 5.1 be satisfied, and assume

i) The set CU and A are compact and the function $q(.,\alpha)$ is twice continuously Frechet differentiable on CU for every $\alpha \varepsilon A$.

ii) The mapping $\dot{q}_*(.)$ is uniformly monotone on CU with constant $a_* > 0$.

iii) The mappings $F_*(.)$ and $F(.,\alpha)$ are linear for every $\alpha \varepsilon A$.

iv) There exists numbers $\tilde{A}$ and $\underset{\sim}{a}$ such that, for every $\alpha \varepsilon A$,

$$\tilde{A} \geqslant A(\alpha) \quad \text{and} \quad a(\alpha) \geqslant \underset{\sim}{a} > 0 \tag{5.21}$$

v) A value of ϱ is chosen (at least one such choice exists) such that there exists numbers $\tilde{A} > 0$ and $\underset{\sim}{a} > 0$ satisfying the following inequalities :

$$\tilde{A} \geqslant \max_{\substack{x \varepsilon CU \\ \alpha \varepsilon A}} \| \dot{q}_*(x,\alpha) \| + \varrho \tag{5.22}$$

and

$$\min_{\substack{\alpha \varepsilon A \\ x \varepsilon CU}} \lambda_{min} (\ddot{q}_{xx}(x,\alpha)) + \varrho \geqslant \underset{\sim}{a} \tag{5.23}$$

vi) The system mathematical model and value of ϱ are chosen such that

$$\tilde{A} \leqslant 2a_* - \frac{A_*}{2} \quad \text{and} \quad \frac{3A_*}{4} < \underset{\sim}{a} \leqslant A_* \tag{5.24}$$

where A_* is a Lipschitz constant of $\dot{q}_*(.)$ on CU.

vii) The class of systems considered is restricted in such a manner that

$$a_* \geqslant \frac{5A_*}{8} \tag{5.25}$$

Then

1^o There exists a unique solution $(\bar{v},\bar{w})$ of the OCP. Every point belonging to Ω is of a form $(\bar{v},\bar{w},\tilde{p})$.

2^o There exist such numbers $\underline{\varepsilon}_p$, $\underline{\varepsilon}$ and $\bar{\varepsilon}$, such that $\underline{\varepsilon} \neq \bar{\varepsilon}$ and $[\underline{\varepsilon}, \bar{\varepsilon}] \subset (0,1)$ and the algorithm either stops at Ω or generates a sequence $\{v^k, w^k\}$ convergent to $(\bar{v}, \bar{w})$ provided that

$$0 < \underline{\varepsilon}_p \leqslant \varepsilon_p^k < \frac{a}{\tilde{s}} \quad \text{and} \quad \varepsilon_p^k \in [\underline{\varepsilon}, \bar{\varepsilon}]$$

and $\tilde{s} = \max\{1, r^2\|H\|^2\}$ where r is a Lipschitz constant of $\tilde{K}_*(.)$ on CU.

Proof

Assumption (ii) implies that the functional $q_*(.)$ is uniformly monotone on a compact and convex set CU. Hence, there is a unique point $\bar{z}$ solving the OCP and due to Theorem 5.1, there is $\bar{p}$ such that $(\bar{z}, \bar{p})$ belongs to Ω . Let $(\tilde{v}, \tilde{w}, \tilde{p}) \in \Omega$.

Since $q_*(.)$ is convex (assumption (ii)) and because $F_*(.)$ is linear, then Lemma 5.1 implies that $\tilde{v} = \bar{v}$ and $\tilde{w} = \bar{w}$. Therefore, the proof of part 1^o is completed.

Let us consider the k-th iteration of the algorithm. Due to the fact that the set CU is convex and to condition (5.22), the function $\{q_x'(., \hat{\alpha}^k) + \varrho(.)\}$ is uniformly monotone on the convex set CU and a unique $\hat{x}^k$ exists. Therefore, the following condition holds (see Luenberger, 1973)

$$[q_x'(\hat{x}^k, \hat{\alpha}^k) + (p^k)^T g_x'(\hat{x}^k, \hat{\alpha}^k) - q_x'(z^k, \hat{\alpha}^k) + q_*'(z^k)$$
$$- (p^k)^T (g_x'(z^k, \hat{\alpha}^k) - g_*'(z^k)) - \varrho(z^k - x^k)](\bar{z} - \hat{x}^k) \geqslant 0 \tag{5.25}$$

Furthermore, because $\hat{x}^k \in CU$ and defining $(\bar{z},\bar{p})$ as the solution, we obtain

$$[q_*'(\bar{z}) + \bar{p}^T g_*'(\bar{z})](\hat{x}^k - \bar{z}) \geq 0 \qquad (5.26)$$

Adding (5.26) and (5.25) leads to the following inequality

$$q_x'(\hat{x}^k,\hat{\alpha}^k)(\bar{z}-\hat{x}^k) + q_x'(z^k,\hat{\alpha}^k)(\hat{x}^k-\bar{z}) + q_*'(z^k)(\bar{z}-\hat{x}^k)$$

$$+ q_*'(\bar{z})(\hat{x}^k-\bar{z}) + (p^k)^T g_x'(\hat{x}^k,\hat{\alpha}^k)(\bar{z}-\hat{x}^k) + (p^k)^T g_x'(z^k,\hat{\alpha}^k)(\hat{x}^k-\bar{z})$$

$$+ (p^k)^T g_*'(z^k)(\bar{z}-\hat{x}^k) + \bar{p}^T g_*'(\bar{z})(\hat{x}^k-\bar{z}) + \varrho z^{kT}(\hat{x}^k-\bar{z})$$

$$+ \varrho \hat{x}^{kT}(\bar{z} - x^k) \geq 0 \qquad (5.27)$$

Since $\varepsilon^k \in (0,1)$ then $z^{k+1} \in CU$.

Inquality (5.27) can be written as

$$\tilde{q}_x'(\hat{x}^k,\hat{\alpha}^k,\varrho)(\bar{z}-\hat{x}^k) + \tilde{q}_x'(z^k,\hat{\alpha}^k,\varrho)(\hat{x}^k-\bar{z})$$

$$+ q_*'(z^k)(\bar{z}-\hat{x}^k) + q_*'(\bar{z})(\hat{x}^k-\bar{z}) + (p^k)^T g_x'(\hat{x}^k,\hat{\alpha}^k)(\bar{z}-\hat{x}^k)$$

$$+ (p^k)^T g_x'(z^k,\hat{\alpha}^k)(\hat{x}^k-\bar{z}) + (p^k)^T g_*'(z^k)(\bar{z}-\hat{x}^k)$$

$$+ (\bar{p})^T g_*'(\bar{z})(\hat{x}^k-\bar{z}) \geq 0 \qquad (5.28)$$

Let us notice that, due to assumption (i), for a given α the corresponding numbers

$$\max_{x \in CU} \| \dot{q}_x(\cdot,\alpha) \| + \varrho ,$$

and

$$\min \lambda_{min} (\ddot{q}_{xx}(\cdot,\alpha)) + \varrho$$

are lipschitz and monotonicity constants respectively of the function $\tilde{q}(\cdot,\alpha)$.

The first and the second term of (5.28) can be expressed as follows :

$$\tilde{q}_x{}'(\hat{x}^k,\hat{\alpha}^k,\varrho)(\bar{z}-\hat{x}^k) \leqslant \tilde{q}(\bar{z},\hat{\alpha}^k,\varrho) - \tilde{q}(\hat{x}^k,\hat{\alpha}^k,\varrho)$$

$$-\frac{1}{2}a(\hat{\alpha}^k)\|\hat{x}^k-\bar{z}\|^2 \leqslant \tilde{q}(\bar{z},\hat{\alpha}^k,\varrho) - \tilde{q}(\hat{x}^k,\hat{\alpha}^k,\varrho) + \frac{1}{\tilde{2}}a\|\hat{x}^k-\bar{z}\|^2 ,$$

$$\tilde{q}_x{}'(z^k,\hat{\alpha}^k,\varrho)(\hat{x}^k-\bar{z}) = \tilde{q}_x{}'(z^k,\hat{\alpha}^k,\varrho)(\hat{x}^k-z^k) + \tilde{q}_x{}'(z^k,\hat{\alpha}^k)(z^k-\bar{z})$$

$$\leqslant \tilde{q}(\hat{x}^k,\hat{\alpha}^k,\varrho) - \tilde{q}(z^k,\hat{\alpha}^k,\varrho) - \frac{1}{\tilde{2}}a\|\hat{x}^k-z^k\|^2 + \tilde{q}(z^k,\hat{\alpha}^k,\varrho)$$

$$- \tilde{q}(\bar{z},\hat{\alpha}^k,\varrho) + \frac{1}{2}\tilde{A}\|z^k-\bar{z}\|^2 ,$$

Applying assumption (ii) to the third and fourth terms of (5.28) gives :

$$q_*'(z^k)(\bar{z}-x^k) = q_*'(z^k)(\bar{z}-x^k) + q_*'(z^k)(z^k-z^{k+1})$$

$$+ q_*'(z^k)(z^{k+1}-\hat{x}^k)$$

$$\leqslant q_*(\bar{z}) - q_*(z^k) - \frac{1}{2}a_*\|\bar{z}-z^k\|^2 + q_*'(z^k)(z^k-z^{k+1})$$

$$+ q_*'(z^k)(z^{k+1}-\hat{x}^k) \quad ,$$

and

$$q_*'(\bar{z})(\hat{x}^k-\bar{z}) = q_*'(\bar{z})(\hat{x}^k-z^{k+1}) + q_*'(\bar{z})(z^{k+1}-\bar{z})$$

$$\leqslant q_*'(\bar{z})(\hat{x}^k-z^{k+1}) + q_*(z^{k+1}) - q_*(\bar{z}) - \frac{1}{2}a_*\|z^{k+1}-\bar{z}\|^2$$

Application of assumption (iii) to a sum of the last four terms in (5.28) yields :

$$(p^k)^T[g(\bar{z},\hat{\alpha}^k) - g(\hat{x}^k,\hat{\alpha}^k)] + (p^k)^T[g(\hat{x}^k,\hat{\alpha}^k)-g(\bar{z},\hat{\alpha}^k)] +$$

$$(p^k)^T[g_*(\bar{z})-g_*(\hat{x}^k)] + \bar{p}^T[g_*(\hat{x}^k)-g_*(\bar{z})]$$

$$= (p^k - \bar{p})^T[g_*(\bar{z})-g_*(\hat{x}^k)]$$

Applying the above relations to inequality (5.28) we obtain

$$q_*(z^{k+1}) - q_*'(z^k) + q_*'(z^k)(z^k - z^{k+1})$$

$$+ [q_*'(z^k) - q_*'(\bar{z})](z^{k+1} - \hat{x}^k) + \frac{1}{2}a_*\|z^k - \bar{z}\|^2$$

$$- \frac{1}{2}a_*\|z^{k+1} - \bar{z}\|^2 - \frac{1}{2\tilde{}}a\|\hat{x}^k - \bar{z}\|^2 - \frac{1}{2\tilde{}}a\|\hat{x} - z^k\|^2$$

$$+ \frac{1}{2}\tilde{A}\|z^k - \bar{z}\|^2 + (p^k - \bar{p}^k)^T[g_*(\bar{z}) - g_*(\hat{x}^k)] \geq 0 \qquad (5.29)$$

Notice that

$$q_*(z^{k+1}) - q_*(z^k) + q_*'(z^k)(z^k - z^{k+1}) \leq \frac{1}{2}(\varepsilon^k)^2 A_*\|z^k - \hat{x}^k\|^2$$

$$(5.30)$$

and that

$$[q_*'(z^k) - q_*'(\bar{z}^k)](z^{k+1} - \hat{x}^k) \leq (1-\varepsilon^k)\|q_*'(\bar{z}) - q_*'(z^k)\|\|z^k - \hat{x}^k\|$$

$$\leq (1-\varepsilon)A_*\|\bar{z} - z^k\|\|z^k - \hat{x}^k\| \leq \frac{1}{2}(1-\varepsilon^k)A_*(\|z^k - \bar{z}\|^2 + \|z^k - \hat{x}^k\|^2)$$

$$(5.31)$$

Let us consider the last term in (5.30), since $g_*(\bar{z}) = 0$ and

$$p^{k+1} = p^k + \varepsilon_p^k g(\hat{x}^k, \hat{\alpha}^k) = p^k + \varepsilon_p^k g_*(\hat{x}^k)$$

the following holds :

$$\|p^{k+1} - \bar{p}\|^2 = \|(p^k - \bar{p})^T + \varepsilon_p^k[g_*(\hat{x}^k) - g_*(\bar{z})]\|^2$$

and

$$(p^k-\bar{p})^T[g_*(\bar{z})-g_*(\hat{x}^k)] = \frac{1}{2\varepsilon_p^k}\|\bar{p}^k-p\|^2 - \frac{1}{2\varepsilon_p^k}\|p^{k+1}-\bar{p}\|^2$$

$$+ \frac{1}{2}\varepsilon_p^k\|g_*(\hat{x}^k)-g_*(\bar{z})\|^2$$

Using definitions (5.3) and (5.4) and part 2^o

$$\|g_*(\hat{x}^k)-g_*(\bar{z})\|^2 = \|\hat{u}^k - HK_*(\hat{c}^k) - \bar{u} + HK_*(\bar{c})\|^2$$

$$\leqslant \|\hat{u}^k - \bar{u}\|^2 + r^2\|H\|^2\|\hat{c}^k-\bar{c}\|^2 \leqslant \underset{\sim}{s}\|\hat{x}^k-\bar{z}\|^2$$

Therefore

$$(p^k-\bar{p})^T[g_*(\bar{z})-g_*(\hat{x}^k)] \leqslant \frac{1}{2\varepsilon_p^k}\|p^k-\bar{p}\|^2 - \frac{1}{2\varepsilon_p^k}\|p^{k+1}-\bar{p}\|^2$$

$$+ \frac{1}{2}\underset{\sim}{s}\varepsilon_p^k\|\hat{x}^k-\bar{z}\|^2 \qquad (5.32)$$

Finally, applying inequalities (5.30), (5.31) and (5.32) to inequality (5.26), we obtain :

$$a_*\|z^k-\bar{z}\|^2 + \frac{1}{2\varepsilon_p^k}\|p^k-\bar{p}\|^2 - a_*\|z^{k+1}-\bar{z}\| - \frac{1}{\varepsilon_p^k}\|p^{k+1}-\bar{p}\|^2$$

$$\geqslant [2a_*-\tilde{A}+(\varepsilon^k-1)A_*]\|z^k-\bar{z}\|^2 + (\underset{\sim}{a}-\underset{\sim}{s}\varepsilon_p^k)\|x^{\hat{k}}-\bar{z}\|^2$$

$$+ [\underset{\sim}{a} - A_*((\varepsilon^k)^2 - \varepsilon^k + 1)]\|z-\hat{x}^k\|^2 \qquad (5.33)$$

Notice that if

$$2a_* - \tilde{A} + (\varepsilon^k - 1)A_* > 0 \qquad (5.34)$$

$$\underset{\sim}{a} - \underset{\sim}{s}\varepsilon_p^k > 0 \qquad (5.35)$$

$$\underset{\sim}{a} - A_*((\varepsilon^k)^2 - \varepsilon^k - 1) > 0 \qquad (5.36)$$

then the right hand side of (5.33) is not negative.

Conditions (5.21) and (5.24) are sufficient for an existence of such a number $\underline{\varepsilon}$, $\bar{\varepsilon} \in [0,1]$, $\underline{\varepsilon} \neq \bar{\varepsilon}$ such that inequalities (5.34), (5.35) and (5.36) hold provided that $\varepsilon^k \in [\underline{\varepsilon}, \bar{\varepsilon}]$ and $0 < \varepsilon_p^k < \frac{\underset{\sim}{a}}{\underset{\sim}{s}}$.

Since $\tilde{A} \geqslant A(\alpha) \geqslant a(\alpha) \geqslant \underset{\sim}{a}$, then condition (5.25) is sufficient and necessary for the existence of the pair $(\tilde{A}, \underset{\sim}{a})$ satisfying inequalities (5.24).

A Z function could be easily derived from inequality (5.33), then Zangwill's theorem can be used to prove that $\bar{z}$ is only a cluster point of this sequence. However, since inequality (5.24) is already expressed in analytical form, therefore instead of applying Zangwill's theorem, it is easier to use inequality (5.33) directly to prove that a sequence $\{z^k\}$ converges to $\bar{z}$.

Let us consider the infinite sequence $\{z^k, p^k\}$ generated by the algorithm. Since $(z^k, p^k) \in \Omega$ for every k = 1, 2, 3, there are two possible cases to be considered : $z^k \neq z^{k+1}$ or $p^k = p^{k+1}$. Since $\varepsilon^k \neq 0$ then $z^k \neq \hat{x}^k$, in the former case. In the second case, since $\varepsilon \neq 0$ then $g(\hat{x}^k, \hat{\alpha}^k) \neq 0$ and consequently, $g_*(\hat{x}^k) \neq 0$. Hence, $\hat{x}^k = \bar{z}$.

Let us define a function

$$T(z,p,\varepsilon_p) \overset{\Delta}{=} a_* \|z-\bar{z}\|^2 + \frac{1}{\varepsilon_p} \|p-\bar{p}\|^2 \qquad (5.37)$$

It follows from the above that the sequence $\{T(z^k,p^k,\varepsilon_p^k)\}$ and the inequality (5.33) is strictly decreasing and it is bounded below by zero. Thus, the sequence $\{T(z^k,p^k,\varepsilon_p^k)\}$ is convergent and, consequently, a left hand side of the inequality (5.33) tends to zero. Therefore, the right side of inequality (5.33), which is not negative also tends to zero. Finally, we obtain

$$\|z^k-\bar{z}\|^2 \xrightarrow[k\to\infty]{} 0$$

and part 2° of the theorem has now been completed.

Let us consider $\{p^k\}$. A sequence $\{ \frac{1}{\varepsilon_p^k} (\bar{p}^k - p) \}$ is convergent since $\{T(z^k,p^k,\varepsilon_p^k)\}$ and $\{z^k - \bar{z}\}$ are convergent. From part 2° of the theorem $0 < \underline{\varepsilon}_p \leq \varepsilon_p^k < \frac{a}{\tilde{s}}$ therefore $\{p^k\}$ is bounded. By using standard arguments (Bensoussan, Lions and Teman (1972)) it can be shown that any cluster point $\hat{p}$ (at least one exists) of the sequence $\{p^k\}$ constitutes together with $\bar{z}$ a saddle point of $L_*(.,.)$ on CUxU.

Q.E.D

The sufficient condition for convergence of the single loop technique are much stronger than those required by the system based double loop technique. However, the uniform monotonicity assumption is now on derivatives of the augmented mappings and thus can be readily satisfied. Due to assumption (iii), then the algorithm is restricted to a linear system. Condition (5.25) limits the technique applicabilty while condition (5.24) makes the choice of the system model difficult.

The convergence conditions of the second version of the augmented single loop technique will be presented in the following theorem.

The algorithm solution is equal to the optimal solution set Ω. The algorithm is stopped iff $(v^k, w^k, p^k) \varepsilon\ \Omega$.

Theorem 5.4

Let the assumptions of Theorem 5.1 and assumptions (i), (ii), (iii), (iv) and (v) of Theorem 5.3 be satisfied.

Assume in addition :

vi) The numbers ε and ε_p and the value of ϱ are chosen such that

$$0 < \underline{\varepsilon}_p \leqslant \varepsilon_p \leqslant \frac{2a_*}{\underset{\sim}{s}} \qquad (5.38)$$

$$\frac{\tilde{A} - \underset{\sim}{a}}{2a_* - \varepsilon_p s} < \varepsilon < \frac{\underset{\sim}{a}}{A_*} \qquad (5.39)$$

$$0 < \underline{\varepsilon} \leqslant \varepsilon \qquad (5.40)$$

where A_* is a Lipschitz constant of $\dot{q}_*(.)$ on CU, and

$$\tilde{s} = \max \{ 1, r^2 \|H\|^2 \}$$

where r is a Lipschitz constant of $K_*(.)$ on CU.

The system mathematical model is chosen such that the above choice of ε and ε_p is possible.

Then

1° There exists a unique solution $(\bar{v},\bar{w})$ of the OCP. Every point belongs to Ω is of the form $(\bar{v},\bar{w})$.

2° The algorithm either stops at Ω or generates a sequence $\{v^k,w^k,p^k\}$ such that $\{v^k,w^k\}$ is convergent to $(\bar{v},\bar{w})$.

Proof

Proof of part 1° is exactly the same as in Theorem 5.3. The symbols $\hat{x}$, z and $\tilde{q}(.)$ used previously in Theorem 5.3 will be used again. Using exactly the same arguments as before, we conclude that

$$\tilde{q}_x'(\hat{x}^k,\hat{\alpha}^k,\varrho)(\bar{z}-\hat{x}^k) + \tilde{q}_x'(z^k,\hat{\alpha}^k,\varrho)(\hat{x}^k-\bar{z}^k) +$$

$$\tilde{q}_x'(z^k)(\bar{z}-\hat{x}^k) + q_*'(\bar{z})(\hat{x}^k-\bar{z}) +$$

$$(p^k)^T g_x'(\hat{x}^k,\hat{\alpha}^k)(\bar{z}-\hat{x}^k) + (p^k)^T g_x'(z^k,\hat{\alpha}^k)(\hat{x}^k-\bar{z}) +$$

$$(p^k)^T g_x'(z^k)(\bar{z}-\hat{x}^k) + p^T g_*'(\bar{z})(\hat{x}^k-\bar{z}) \geqslant 0 \qquad (5.41)$$

The first and the second terms of (5.41) can be expressed as follows :

$$\tilde{q}_x'(\hat{x}^k,\hat{\alpha}^k,\rho)(\bar{z}-\hat{x}^k) \leqslant \tilde{q}(\bar{z},\hat{\alpha}^k,\rho) - \tilde{q}(\hat{x}^k,\hat{\alpha}^k,\rho) - \frac{1}{2}\tilde{a}\|\hat{x}^k-\bar{z}\|^2 ,$$

$$\tilde{q}_x'(z^k,\hat{\alpha}^k,\rho)(\hat{x}^k-\bar{z}) \leqslant \tilde{q}(\hat{x}^k,\hat{\alpha}^k,\rho) - \tilde{q}(z^k,\hat{\alpha}^k,\rho) -$$

$$\frac{1}{2}\tilde{a}\|\hat{x}^k-z^k\|^2 + \tilde{q}(z^k,\hat{\alpha}^k,\rho) - \tilde{q}(\bar{z},\hat{\alpha}^k,\rho) + \frac{1}{2}\tilde{A}\|z^k-\bar{z}\|^2$$

Applying assumption (ii) to each of the third and fourth terms of (5.41) in an appropriate manner, yields

$$\varepsilon q_*'(z^k)(\bar{z}-\hat{x}^k) = \varepsilon q_*'(z^k)(\bar{z}-z^k) + \varepsilon q_*'(z^k)(z^k-\hat{x}^k)$$

$$\leqslant \varepsilon q_*(\bar{z}) - \varepsilon q_*(z^k) - \frac{\varepsilon}{2}a_*\|z^k-\bar{z}\|^2 + \varepsilon q_*(z^k) -$$

$$- \varepsilon q_*(\hat{x}^k) + \frac{\varepsilon}{2}A_*\|z^k-\hat{x}^k\|^2 ,$$

$$\varepsilon q_*'(\bar{z})(\hat{x}^k-\bar{z}) = \varepsilon q_*(\hat{x}^k) - \varepsilon q_*(\bar{z}) - \frac{\varepsilon}{2}a_*\|\hat{x}^k-\bar{z}\|^2$$

Applying assumption (iii) to a sum of the last four terms of (5.41), shows that the sum is equal to

$$\varepsilon(p^k-\bar{p})^T[g_*(\bar{z}) - g_*(\hat{x}^k)]$$

Introducing the above relations to inequality (5.41) yields

$$-\frac{1}{2}\underset{\sim}{a}\|\hat{x}^k-\bar{z}\|^2-\frac{1}{2}\underset{\sim}{a}\|\hat{x}^k-z^k\|^2+\frac{1}{2}\tilde{A}\|z^k-\bar{z}\|^2$$

$$+\frac{\varepsilon}{2}A_*\|z^k-\hat{x}^k\|^2-\frac{\varepsilon}{2}a_*\|z^k-\bar{z}\|^2-\frac{\varepsilon}{2}a_*\|\hat{x}^k-\bar{z}\|^2$$

$$+\varepsilon(p^k-\bar{p})^T[g_*(\bar{z})-g_*(\hat{x}^k)] \geqslant 0 \qquad (5.42)$$

Since (see derivation of (5.32))

$$\varepsilon(p^k-\bar{p})^T[g_*(\bar{z})-g_*(\hat{x}^k)] \leqslant \frac{\varepsilon}{2\varepsilon_p}\|p^k-\bar{p}\|^2-\frac{\varepsilon}{2\varepsilon_p}\|p^{k+1}-\bar{p}\|^2$$

$$+\frac{\varepsilon}{2}\underset{\sim}{s}\varepsilon_p\|\hat{x}^k-\bar{z}\|^2$$

then the following inequality holds :

$$(\tilde{A}-\varepsilon a_*)\|z^k-\bar{z}\|^2+\frac{\varepsilon}{\varepsilon_p}\|p^k-\bar{p}\|^2-(\bar{A}-\varepsilon a_*)\|\hat{x}^k-\bar{z}\|^2-\frac{\varepsilon}{\varepsilon_p}\|p^{k+1}-\bar{p}\|^2$$

$$\geqslant(\underset{\sim}{a}+2\varepsilon a_*-\tilde{A}-s\varepsilon\varepsilon_p)\|\hat{x}^k-\bar{z}\|^2+(\underset{\sim}{a}-\varepsilon A_*)\|z^k-\hat{x}^k\|^2$$

Conditions (5.21), (5,38) and (5.39) imply that

$$\tilde{A}-\varepsilon a_*>0$$

$$\underset{\sim}{a}+2\varepsilon a_*-\tilde{A}-s\varepsilon\varepsilon_p>0$$

$$\underset{\sim}{a}-\varepsilon A_*>0$$

Taking into account that $\hat{x} = z^{k+1}$ (see (4.40) and (4.41)) and using the arguments used in the proof of Theorem 5.1 , we conclude that in the case of an infinite sequence $\{z^k, p^k\}$ generated by the algorithm, the sequence $\{ z^k \}$ is convergent to $(\bar{v}, \bar{w})$.

The properties of $\{p^k\}$ have been discussed previously . Hence, the proof is completed.

Q.E.D

5.3.3 Model Based Double Loop Technique

The convergence properties of the augmented model based double loop technique will be studied by considering a linear quadratic problem

$$Q(x,y) = (x-d)^T M(x-d) + (y-e)^T E(y-e) \tag{5.43}$$

where $x = (c,u)$

$$F(x,\alpha) = D_1 c + D_2 u + P(\alpha) \tag{5.44}$$

and

$$F_*(x) = D_{*1} c + D_{*2} u + d_* \tag{5.45}$$

Matrices D_1, D_2, D_{*1}, D_{*2} and P(.) are such that assumptions A1 and A2 are satisfied. Furthermore, equation ($u - HD_1 c - HD_2 u = 0$) and $g_*(x) = 0$ are both linearly independent.

Let us define the following matrices :

$$B = [- HD_1 , I - HD_2] \tag{5.46}$$

where I is an identity matrix,

$$B_* = [\ (HD_{*2} - I)^{-1}HD_{*1},\ I] \qquad (5.47)$$

$$\tilde{M} = M + D^TED \qquad (5.48)$$

$$M_* = M + D_*ED_* \qquad (5.49)$$

where $D = [D_1, D_2]$, $D_* = [(I-D_{*2}H)^{-1}D_{*1}, 0]$.

The iterations are terminated when $(v^k, w^k, p^k) \in \Omega$.

Theorem 5.5

i) $$(Rp^{-1})^TB = BR^{-1} \qquad (5.50)$$

ii) $$(\tilde{M} + \varrho)R^{-1} = (R^{-1})^T(\tilde{M}+\varrho) \qquad (5.51)$$

iii) $$(\tilde{M}+\varrho)R^{-1} - \frac{1}{2}M_* > 0 \qquad (5.52)$$

iv) $$B_*M_*^{-1}R_p^{-1} + (Rp^{-1})^TBM_*^{-1}B^T -$$

$$B_*M_*^{-1}((\tilde{M}+\varrho)R_x^{-1} + \frac{1}{2}M_*)M_*^{-1}{}_*B^T{}_* > 0 \qquad (5.53)$$

Then

1^o There exists a unique solution $(\bar{v},\bar{w})$ of the OCP. The algorithm solution set Ω consists of a single point $\{\bar{v},\bar{w},\bar{p}\}$.

2^o The algorithm either stops at Ω or generates a sequence $\{(v^k, w^k, p_2{}^k)\}$ which is convergent to $(\bar{v},\bar{w},\bar{p})$.

Proof

By using (5.43) , (5.44) and (5.45) we obtain

$$q(x,\alpha) = \frac{1}{2}(x-d)^T M(x-d) + \frac{1}{2}(Dx-e)^T E(Dx-e) + P^T(\alpha)E(Dx-e)$$

$$+ P^T(\alpha)EP(\alpha) \qquad (5.54)$$

$$g_*(x) = B_* x - b_*$$

where

$$b_* = (I - HD_{*2})^{-1} Hd_* , \qquad (5.55)$$

$$g(x,\alpha) = Bx - HP(\alpha) \qquad (5.56)$$

$$q_*(x) = \frac{1}{2}(x-d)^T M(x-d) + \frac{1}{2}(D_* x-f)^T E(D_* x-f) \qquad (5.57)$$

where $f = e - d$

Let us denote

$$\hat{x} \triangleq (\hat{c},\hat{u}) \quad \text{and} \quad z \triangleq (v,w)$$

Let (z^k, p_2^k) be given.

Since $\{q(.,\alpha^k) + \frac{1}{2}\varrho\|\hat{x}-z\|^2\}$ is a uniformly convex function if ϱ is sufficiently large and $g(.,\alpha^k)$ is linear then necessary and sufficient conditions for $(\hat{c}^k, \hat{u}^k, \hat{p}^k)$ to be a solution of the corresponding inner loop problem can be written as follows (see (5.46), (5.47) and (5.49)) :

$$(\tilde{M}+\varrho)(\hat{x}^k - z) + M_* z^k + B^T(\hat{p}_1^k - p_2^k) + B_*^T p_2^k - Md - D_*^T Ef = 0 \tag{5.58}$$

and

$$B\hat{x}^k - HP(\hat{\alpha}^k) = 0 \tag{5.59}$$

Since $g(z^k, \hat{\alpha}^k) = g_*(z^k)$ (see (4.18), (5.3) and (5.4)), then $HP(\hat{\alpha}^k) = Bz^k - B_* z^k + b_*$ and condition (5.59) can be written as

$$B(\hat{x}^k - z^k) + B_* z^k - b_* = 0 \tag{5.60}$$

Due to the assumptions $\tilde{M} > 0$ and matrix B has full rank ,

$$\det \begin{pmatrix} (\tilde{M}+\varrho) & B^T \\ B & 0 \end{pmatrix} = \det(\tilde{M}+\varrho) \det(-B(\tilde{M}+\varrho)B^T)$$

Therefore the solution of (5.58) and (5.59) is unique with respect to $(\hat{x}^k, \hat{p}_1^k)$. Hence, the iterative scheme (4.51) and (4.52) is well defined. A point z is a solution of OCP iff there is $p \varepsilon U$ such that the following holds :

$$M_* \bar{z} + B_*^T \bar{p}_2 - Md - D_*^T Ef = 0 \tag{5.61}$$

and

$$B_* \bar{z} - b_* = 0 \tag{5.62}$$

Since $M_* > 0$ and B_* has full rank, employing a similar approach as previously, there is exactly one pair $(\bar{z}, \bar{p}_2)$ satisfying (5.61) and (5.62). Next, using similar arguments as in the proof of part 1^o of Theorem 5.3, the proof of part 1^o of the theorem is completed.

Conditions (5.58) and (5.60) constitute a basis for further consideration. By using (4.51) and (4.52), these conditions can be expressed in terms of points generated by the outer loop as follows:

$$(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) + M_* z^k + B^T R_p^{-1}(p_2^{k+1}-p_2^k) + B_*^T p_2^k - Md - D_*^T Ef = 0 \qquad (5.63)$$

$$BR_x^{-1}(z^{k+1}-z^k) + B_* z^k - b_* = 0 \qquad (5.64)$$

Next, let us consider an approach used by Cohen (1980) in his Theorem 5.1 . Then , we will utilize Cohen's approach as a basis for further consideration.

Multiplying equality (5.61) by $(z^k - z^{k+1})^T$ and equality (5.64) by $(p_2^{k+1} - p_2^k)^T$, then adding the resulting equalities and next employing assumption (i), we obtain

$$- (z^{k+1}-z^k)(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) - (z^k)^T M_*(z^{k+1}-z^k)$$

$$- (z^{k+1}-z^k)^T B_*^T p^k_2 + (z^{k+1}-z^k)(Md-D_*^T Ef)$$

$$+ (z^k)^T B_*^T (p_2^{k+1}-p_2^k) - b_*^T (p_2^{k+1}-p_2^k) = 0 \qquad (5.65)$$

Due to (5.61) and (5.62) the following holds :

$$b_*^T(p_2^{k+1}-p_2^{k}) = \bar{z}^T B^T(p_2^{k+1}-p_2^{k})$$

and

$$(z^{k+1}-z^k)^T(Md-D_*^T Ef) = (z^{k+1}-z^k)^T(M_*\bar{z}+B_*^T\bar{p}_2)$$

Applying these equalities to equality (5.64) and performing suitable ordering, we obtain :

$$(\bar{z}-z^k)^T M_*(z^{k+1}-z^k) + (\bar{p}_2-p_2^k)^T B_*(z^{k+1}-z^k) +$$
$$(p_2^{k+1}-p_2^k)^T B_*(z^k-\bar{z}) - (z^{k+1}-z^k)^T(\hat{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) = 0 \tag{5.66}$$

The first term in (5.66) can be expressed as

$$\frac{1}{2}(z^k-\bar{z})^T M_*(z^k-\bar{z}) - \frac{1}{2}(z^{k+1}-\bar{z})^T M_*(z^{k+1}-\bar{z})$$
$$+ \frac{1}{2}(z^{k+1}-z^k)^T M_*(z^{k+1}-z^k) \tag{5.67}$$

The second term in (5.66) is equal to

$$(\bar{p}_2-p_2^k)^T B_*(\bar{z}-z^k) - (\bar{p}_2-p_2^{k+1})^T B_*(\bar{z}-z^{k+1}) +$$
$$(p_2^{k+1}-p^k{}_2)B_*(x^{k+1}-x^k) + (p_2^k-p_2^{k+1})^T B_*(\bar{z}-z^k) \tag{5.68}$$

Let us now compute the term $B_*(\bar{z}-z^k)$

Inequalities (5.61) and (5.62) imply that

$$z^k + M_*^{-1}(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) + M_*^{-1}B^T R_p^{-1}(p_2^{k+1}-p^k{}_2) +$$

$$M_*^{-1}B^T{}_* p^k - M_*^{-1}(Md+D_*{}^T Ef) = 0$$

and

$$\bar{z} + M_*^{-1}B^T{}_*\bar{p}_2 - M_*^{-1}(Md+D_*{}^T Ef) = 0$$

Hence,

$$B_*(\bar{z}-z^k) = B_* M_*^{-1}(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) +$$

$$B_* M_*^{-1}B^T R_p^{-1}(p_2^{k+1}-p_2^k) + B_* M_*^{-1}B^T{}_*(p_2^k-\bar{p}_2)$$

$$2(p_2^k-p_2^{k+1})^T B_*(\bar{z}-z^k) =$$

$$+ \; 2(p_2^k-p_2^{k+1})^T B_* M_*^{-1}(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k) +$$

$$+ \; 2(p_2^k-p_2^{k+1})^T B_* M_*^{-1}B^T R_p^{-1}(p_2^{k+1}-p_2^k) +$$

$$+ \; 2(p_2^k-p_2^{k+1})B_* M_*^{-1}B_*^T(p_2^k-\bar{p}) =$$

$$- \; 2(p_2^{k+1}-p^k{}_2)^T B_* M_*^{-1}(\tilde{M}+\varrho)R_x^{-1}(z^{k+1}-z^k)$$

$$- \; 2(p_2^{k+1}-p^k{}_2)^T B_* M_*^{-1}B^T R_p^{-1}(p_2^{k+1}-p^k{}_2)$$

$$+ \; (p^k{}_2-\bar{p}_2)^T B_* M_*^{-1}B_*^T(\bar{p}_2{}^k-\bar{p}_2)$$

$$- \; (p_2^{k+1}-\bar{p}_2)^T B_* M_*^{-1}B_*^T(p_2^{k+1}-\bar{p})$$

$$+ \; (p_2^{k+1}-\bar{p}_2{}^k)^T B_* M_*^{-1}B_*^T(p_2^{k+1}-p_2^k) \qquad (5.69)$$

Finally, let us define the following functional:

$$T(z,\bar{p}_2) \overset{\Delta}{=} \frac{1}{2}(p_2-\bar{p}_2)^T B_* M_*^{-1} B_*^T (p_2-\bar{p}_2)$$

$$+ \frac{1}{2}[M_*(z-\bar{z}) + B_*^T(p_2-\bar{p}_2)]^T M_*^{-1}[M_*(z-\bar{z}) + B_*^T(p_2-\bar{p}_2)]$$

(5.70)

Applying (5.69), (5.68) and (5.67) to (5.66), we conclude that

$$T(z^k,p_2^k) - T(z^{k+1},p_2^{k+1}) = (z^k-z^{k+1})^T[(\tilde{M}+\varrho)R_x^{-1} - \frac{1}{2}M_*](z^k-z^{k+1})$$

$$+2(p_2^k-p_2^{k+1})^T[B_*M_*^{-1}(\tilde{M}+\varrho)R_x^{-1} - \frac{B_*}{2}](z^k-z^{k+1})$$

$$+ (p_2^k-p_2^{k+1})^T[2B_*M_*^{-1}B^T R_p^{-1} - B_*M_*^{-1}B_*^T](p_2^{k+1}-p_2^k)$$

(5.71)

Employing assumptions (ii) and (iii), and utilizing the following equality

$$a^T Xa + 2b^T Ya = [Xa + Y^T b]^T X^{-1}[Xa + Y^T b] - b^T Y X^{-1} Y^T b$$

where a and b are the vectors while X and Y are the matrices such that X^{-1} exists and $X = X^T$, from (5.71), we obtain the following :

$$T(z^k,p_2^k) - T(z^{k+1},p_2^{k+1}) = [(z^k-z^{k+1}) + M_*^{-1}B_*^T(p_2^k-p_2^{k+1})]^T.$$

$$.((\breve{M}+\varrho)R_x^{-1} - \tfrac{1}{2}M_*)[(z^k - z^{k+1}) + M_*^{-1}B_*^T(p_2^k - p_2^{k+1})]$$

$$+(p_2^k - p_2^{k+1})^T(2B_*M_*^{-1}B^TR_p^{-1} - B_*M_*^{-1}((\breve{M}+\varrho)R_x^{-1} + \tfrac{M_*}{2})M_*^{-1}B_*^T)].$$

$$.(p_2^k - p_2^{k+1}) \qquad (5.72)$$

Let us consider an finite sequence $\{z^k, p^k\}$ generated by the algorithm. Due to assumption (iii) and (iv), therefore $(z^k, p^k) \in \Omega$ for every k = 1, 2,.., and the sequence $\{T(z^k, p_2^k)\}$ is strictly decreasing (see (5.72)). Since $B_*M_*^{-1}B_*^T > 0$ and $M_*^{-1} > 0$ then (5.70) implies that the sequence is bounded. Hence, $T(z^k, p_2^k)$ converges and consequently $T(z^k, p_2^k) - T(z^{k+1}, p_2^{k+1}) \dashrightarrow 0$. Therefore, (5.72) implies that the sequence

$$\{ z^k - z^{k+1}, p^k - p_2^{k+1} \}$$

is convergent to zero.

By applying (5.61), (5.62), (5.63) and (5.64), we conclude eventually that

$$\{(z^k, p^k)\} \xrightarrow[k \to \infty]{} \{\bar{z}, \bar{p}\}$$

Q.E.D

5.4 Optimality and Convergence Analysis of Constraint Augmentation

The optimality properties of the augmented variable augmentation technique can easily be extended to the constraint augmentation method. The algorithms are optimal in the sense that a point found by their application satisfies necessary optimality of the original optimising control problem (OCP). If the problem is convex, then the obtained point is a desired optimal point. For non-convex problems some kind of augmentation should be used, and the obtained point is expected to be at least locally optimal.

Let us begin with the convergence analysis of the augmented system based method (see (4.3a) and (4.40)), and let us define

$$g(c,u,\alpha) = u - HF(c,u,\alpha) \tag{5.72}$$

$$g_*(c,u) = u - HK_*(c) \tag{5.73}$$

$$L_{*r}(c,u,p) = q_*(c,u) + p^T g_*(c,u) + \frac{1}{2} r \| g_*(c,u) \|^2 \tag{5.74}$$

$$L_r(c,u,\alpha,p) = q(c,u,\alpha) + p^T g(c,u,\alpha) + \frac{1}{2} r \| g(c,u,\alpha) \|^2 \tag{5.75}$$

The properties of the inner loop solution are stated in the following lemma

Lemma 5.3

Assume that CU is convex and compact and that, for a given $p \varepsilon U$ and every $\alpha \varepsilon A$, $L_r(.,.,\alpha,p)$ is convex on CU. Then the inner loop solution exists. If $L_*(.,.,p)$ is convex on CU, then every inner loop solution $\hat{z}(p) = (\hat{v}(p), \hat{w}(p))$ minimizes $L_{*r}(.,.,p)$ on CU. Moreover, if $\hat{z}(p)$ is unique, then the augmented dual function,

$$d_{*r}(p) = \min_{(c,u)\varepsilon CU} L_{*r}(c,u,p) \qquad (5.76)$$

is differentiable and

$$d_{*r}(p) = \hat{w}(p) - HF(\hat{z}(p), \hat{\alpha}(\hat{z}(p))) \qquad (5.77)$$

The proof of this Lemma is given by Tatjewski, Abdullah and Roberts (1986) and it is attached as Appendix B.

In the case of non-convex problems $L_r(.,\alpha,p)$ can always be made strictly convex at least locally (in a neighbourhood of the optimum), provided the following model optimisation problem

$$\min \{ q(c,u,\hat{\alpha}) - \hat{\lambda}^T c - \hat{t}^T u \}$$

$$\text{s.t} \quad g(c,u,\alpha) = 0, \quad G(c,u) \leqslant 0 \qquad (5.78)$$

satisfies the second order sufficient optimality conditions (see Findeisen et. al. 1980). These conditions are known to be weak and almost necessary. Similarly, $L_{*r}(.,p)$ can always be made strictly convex, at least locally, provided that the system optimisation problem , OCP,

$$\min q_*(c,u) \tag{5.79}$$

$$\text{s.t } g_*(c,u) = o \ , \quad G(c,u) \leqslant 0$$

satisfies the second order sufficient optimality conditions. The statement that $\hat{z}(p)$ minimizes $L_{*r}(.,p)$ is very favourable, since then taking $\varepsilon_p = r$ in the outer-loop adjustment formula (4.39) makes it equivalent to the Hestenes-Powell multiplier rule, whose convergence properties are well known (see Findeisen et. al 1980).

Sufficient conditions for convergence of the inner loop iteration (5.40) are given in the following theorem.

Theorem 5.6

For given p and r assume :

i) CU is compact and convex.

ii) $\hat{\alpha}(.)$ is point-to-point continuous mapping on CU.

iii) For given $\alpha \in A$, $L_r(.,\alpha,p)$ is twice continuously differentiable on CU, with $\ddot{L}_{r(x,x)}(x,\alpha,p)$ satisfying, for some constant b>0 and every $x \varepsilon CU$, $h \varepsilon CU$.

$$(x+h)^T [\ddot{L}_{r(x,x)}(x,\alpha,p) + \varrho I](x+h) \geqslant b\|h\|^2 \tag{5.80}$$

iv) $\dot{L}_{*r(x)}(.,p)$ is Lipschitz continuous on CU with Lipschitz constant $\delta > 0$.

Then,

1^o The inner loop iteration scheme (5.40) with $\varepsilon = \varepsilon^k$ is well defined on CU provided

$$\underline{\varepsilon} \leqslant \varepsilon^k \leqslant \{ 1, \frac{2b}{\delta} - \tau \} \qquad (5.81)$$

where $\tau > 0$ and $1 \geqslant \underline{\varepsilon} > 0$ are any sufficiently small constants chosen such that

$$\underline{\varepsilon} \leqslant \frac{2b}{\delta} - \tau \qquad (5.82)$$

2^o There is at least one cluster point of the sequence $\{z^k\}$ generated by (5.40), and each such point is an inner loop problem solution $\hat{z}(.)$.

The proof of this theorem is given by Tatjewski, Abdullah and Roberts (1986) and is given in Appendix C.

Assumption (iii) of Theorem 5.6 seems to be restrictive. Condition (5.78) is satisfied at least locally in the neighbourhood of the optimum provided that the modified optimisation problem (4.49) satisfies second order sufficient optimality conditions.

Let us consider the augmented single loop technique, see (4.31), (4.32) and (4.33), and define

$$q_r(x,\alpha) = q(x,\alpha) + \frac{1}{2}r\|g(x,\alpha)\|^2 \qquad (5.83)$$

$$q_{*r}(x) = q_*(x) + \frac{1}{2}r\|g_*(x)\|^2 \qquad (5.84)$$

A set of sufficient convergence conditions for the constraint augmented single loop technique is presented in the following theorem :

Theorem 5.7

i) The set CU is compact and convex.

ii) The derivative $\dot{q}_{*r}(x)$ is uniformly monotone on CU with constant $a_{*r} > 0$.

iii) For every $\alpha \in A$ the mapping $\dot{q}_{rx}(x,\alpha)$ is uniformly monotone on CU with constant $a_r(\alpha) > 0$.

iv) The mappings $F_*(.)$ and $F(.,\alpha)$ are linear , for $\alpha \in A$.

v) There exists values $\underset{\sim}{a}_r$ and $\tilde{A}_r$ such that for every $\alpha \in A$.

$$\tilde{A}_r \geqslant A_r(\alpha) , \qquad a_r(\alpha) \geqslant \underset{\sim}{a}_r$$

where $A_r(\alpha)$ is a Lipschitz constant of $\dot{q}_{rx}(.,\alpha)$ on CU.

vi) The system , its mathematical model and values of r, ε_p and ε, $0 < \varepsilon \leqslant 1$, are such that

$$\tilde{A}_r < 2a_{*r} + (\varepsilon-1)A_{*r} \tag{5.85}$$

$$\underset{\sim}{a}_r > (\varepsilon^2 - \varepsilon + 1)A_{*r} \tag{5.86}$$

$$0 < \varepsilon_p < \frac{a_{*r}}{\underset{\sim}{s}} \tag{5.87}$$

where $\underset{\sim}{s} = \max\{2,\ 2\|HK_*(.)\|^2\}$ and A_{*r} is a Lipschitz constant of $\dot{q}_{*r}(.)$ on CU.

Then the algorithm (4.31), (4.32) and (4.33) either stops at a point $(\bar{z},\bar{p})$ satisfying the coordinating conditions, i.e., (compare (4.23) - (4.25)).

$$\hat{x}(\bar{z},\bar{p}) = \bar{z} \tag{5.88}$$

$$\hat{u}(\bar{z},\bar{p}) - HF(\hat{x}(\bar{z},\bar{p}),\hat{\alpha}(\bar{z})) = 0 \tag{5.89}$$

or generates a sequence converging to $(\bar{z},\bar{p})$.

The proof of the Theorem is presented by Tatjewski, Abdullah and Roberts (1986) and given in Appendix D.

The assumptions of Theorem 5.7 are restrictive. However, the uniform monotonicity assumptions are now on derivatives of the augmented mappings and thus they can be readily satisfied, at least locally.

The second version of the augmented single loop technique is fomulated sligtly different, and a less restrictive sufficient condition is obtained as follows :

Let us take $\varepsilon = 1$ in the updating formulae (4.31) and (4.32). Using (5.2), (5.3) and (5.4), the formulae (4.50), (4.51) can be expressed in the following form :

$$(\lambda_\varepsilon,t_\varepsilon)(z,p,r)^T = \dot{q}_x(z,\hat{\alpha}(z)) - \varepsilon\dot{q}_*(z) + [p+rg_*(z)]^T[\dot{g}_x(z,\hat{\alpha}(z)) - \varepsilon\dot{g}_*(z)] \tag{5.90}$$

Thus, for the second version, $(\lambda_\varepsilon, t_\varepsilon)$ is used instead of (λ, t), together with the updating formulae (compare (4.31) - (4.33))

$$z^{k+1} = \hat{x}(z^k, p^k) \quad (5.91)$$

$$p^{k+1} = p^k + \varepsilon_p g(\hat{x}(p^k, z^k), \hat{\alpha}(z^k)) \quad (5.92)$$

We have the following convergence theorem.

Theorem 5.8

Assume that assumptions (i) - (v) of Theorem 5.7 are satisfied, and that

vi) The values of ε, ε_p and r are such that

$$0 < \frac{A_r - a_r}{2a_{*r} - \varepsilon_p s} < \varepsilon < \frac{a_r}{A_{*r}} \quad (5.93)$$

$$0 < \varepsilon_p < \frac{2a_{*r}}{s} \quad (5.94)$$

Then the algorithm (5.89) - (5.90) either stops at a point $(\bar{z}, \bar{p})$ satisfying the required coordinating conditions, or generates a sequence converging to $(\bar{z}, \bar{p})$.

The theorem is a rather straightforward generalization of Theorem 5.4, and the proof is analogous.

Assumption (vi) is different and less restrictive than in Theorem 5.7, since (5.91) and (5.92) can always be satisfied if ε_p is sufficiently small and ϱ sufficiently large.

Convergence analysis of the constraint augmented model based double loop technique is omitted here due to its restrictive applicability. Convergence conditions for the approximation loop will be presented in Chapter 6.

Summary

Optimality and convergence proof of the variable and the constraint augmented techniques are presented. The proposed algorithms are optimal in terms that a point generated by the algorithms satisfies the necessary optimality conditions of the original optimising problem (OCP).

CHAPTER 6 - THE AUGMENTED ISOPE WITH INPUT-OUTPUT FEEDBACK

6.1 Introduction

In the present chapter the variable and constraint augmented ISOPE algorithms using input-output feedbacks are presented. The optimality properties of the algorithms are studied with the inequality constraints being output dependent.

Since the augmented model based double loop technique is the most efficient algorithm in reducing set-point changes compared to the other two algorithms, a detailed analysis of this technique will be studied in this chapter. The convergence analysis for the variable augmented model based double loop technique presented by Abdullah, Brdys and Roberts (1986) will be examined.

However, convergence analysis for the approximation loop portion of the constraint augmented ISOPE algorithms has been achieved as presented by Tatjewski, Abdullah and Roberts (1986) and will also be included in this chapter.

6.2 Problem Statement and formulation

Let us assume that both system output and input measurements are available. Similarly, problem (3.80) (see chapter 3) can be augmented in two different ways as follows :

$$\min \{ q(c,u,\alpha) + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|u-HK_*(v)\|^2 \} \qquad (6.1)$$

and

$$\min \{ q(c,u,\alpha) + \frac{1}{2} r \|u-HF(c,u,\alpha)\|^2 \} \qquad (6.2)$$

$$\begin{aligned} \text{s.t} \quad & u = HF(c,u,\alpha) \\ & \tilde{g}(c,u,\alpha) \leq 0 \\ & F(v,HK_*(v,\alpha)) = HK_*(v) \\ & v = c \end{aligned}$$

The Lagrangian function for problem (3.30) is

$$L(c,v,u,\alpha,p,\lambda,\mu,\xi,\varrho,r) = q(c,u,\alpha) + p^T[u-HF(c,u,\alpha)] + \lambda^T(v-c)$$

$$+ \mu^T[K_*(v) - F(v,HK_*(v),\alpha)] + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|u-HK_*(v)\|^2$$

$$+ \frac{1}{2}\varrho_3 \|K_*(v)-F(v,HK_*(v),\alpha)\|^2 + \frac{1}{2} r\|u- HF(c,u,\alpha)\|^2$$

$$+ \frac{1}{2}\sum_{j=1}^{M} \varrho_{3+j} \left[\left(\max(0,\tilde{g}_j(c,u,\alpha) + \frac{\xi_j}{\varrho_{3+j}})\right)^2 - \left(\frac{\xi_j}{\varrho_{3+j}}\right)^2 \right] \qquad (6.3)$$

where

$\varrho = [\varrho_1,\varrho_2,\varrho_3,\ldots,\varrho_{M+3}]$, $j > 0$ for every $j=1,2,3,\ldots,M+3$

and r are vectors of penalty coefficients ; λ, μ and ξ are vectors of Lagrangian multipliers.

For given values of ϱ and r, the stationary points of $L(.,.,.,p,\lambda,\mu,\xi,\varrho,r)$ are defined as follows :

$$\Delta_c L = \frac{\partial q^T(c,u,\alpha)}{\partial c} + \frac{\partial^T[u-HF(c,u,\alpha)]}{\partial c}p - \lambda + \varrho_1(v-c) + r\frac{\partial^T[u-HF(c,u,\alpha)]}{\partial c}[u-HK(c,u,\alpha)]^T + \frac{\partial P^T(c,u,\alpha,\xi)}{\partial c} = 0 \qquad (6.4)$$

$$\Delta_u L = \frac{\partial^T q(c,u,\alpha)}{\partial u} + \frac{\partial^T[u-HF(c,u,\alpha)]}{\partial u}p + \varrho_2\frac{\partial^T[u-HK_*(v)]}{\partial u}[u-HK_*(v)]^T + r\frac{\partial^T[u-HF(c,u,\alpha)]}{\partial u}[u-HF(c,u,\alpha)]^T + \frac{\partial P^T(c,u,\alpha,\xi)}{\partial u} = 0 \qquad (6.5)$$

$$\Delta_v L = \lambda + \frac{\partial^T[K_*(v)-F(v,HK_*(v),\alpha)]}{\partial v}\mu + \varrho_1(v-c) + \varrho_3\frac{\partial^T[K_*(v)-F(v,HK_*(v),\alpha)]}{\partial v}[K_*(v)-F(v,HK_*(v),\alpha)]^T + \varrho_2\frac{\partial^T[u-HK_*(v)]}{\partial v}[u-HK_*(v)]^T = 0 \qquad (6.6)$$

$$\Delta_\alpha L = \frac{\partial^T q(c,u,\alpha)}{\partial \alpha} + \frac{\partial^T[u-HF(c,u,\alpha)]}{\partial \alpha}p + \frac{\partial^T[K_*(v)-F(v,HK_*(v),\alpha)]}{\partial \alpha}\mu + \varrho_3\frac{\partial^T[K_*(v)-F(v,HK_*(v),\alpha)]}{\partial \alpha}[K_*(v)-F(v,HK_*(v),\alpha)]^T + r\frac{\partial^T[u-HF(c,u,\alpha)]}{\partial \alpha}[u-HF(c,u,\alpha)]^T + \frac{\partial P^T(c,u,\alpha,\xi)}{\partial \alpha} = 0 \qquad (6.7)$$

$$\Delta_p L = u - HF(c,u,\alpha) = 0 \qquad (6.8)$$

$$\Delta_\lambda L = v - c = 0 \qquad (6.9)$$

$$\Delta_{\mu} L = K_*(v) - HF(v, HK_*(v), \alpha) = 0 \qquad (6.10)$$

$$\Delta_{\xi} L(c, v, u, \alpha, p, \lambda, \mu, \xi, \varrho, r) = 0 \qquad (6.11)$$

where $p(c,u,\alpha,\xi)$ is the same equation as defined in (4.12) (see chapter 4).

6.2.1 Variable Augmentation

Employing a similar approach as the one presented previously, equation (6.3),...,(6.11) can be used to formulate optimality conditions for the variable augmented problem (6.1), giving

$$\min_{(c,u)\varepsilon CU} [q(c,u,\alpha) - \lambda^T c + p^T[u - HF(c,u,\alpha)] + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|u - HK_*(v)\|^2]$$

$$\text{s.t} \qquad \hat{g}(c,u,\alpha) \leqslant 0 \qquad (6.12)$$

for given values of v, $\alpha = \hat{\alpha}(v)$, price p and ϱ.

Where

$$\lambda(v,\xi,p,\varrho) = [K_*^{'T}(v) - F_c^{'T}(v, HK_*(v), \alpha)][F_\alpha^{'T}(v, HK_*(v), \alpha)]^{-1}$$

$$.[- q_\alpha^{'T}(c,u,\alpha) + F_\alpha^{'T}(c,u,\alpha)H^T p - P_\alpha^{'T}(v,u,\alpha,\xi)] \qquad (6.13)$$

and assuming that the inverse $[F_\alpha^{'T}(v, HK_*(v), \alpha)]^{-1}$ exists.

Since

$$[F_\alpha^{'T}(v,HK_*(v),\alpha)]^{-1} q_\alpha^{'T}(v,w,\alpha) = Q_y^{'T}(v,HK_*(v),\alpha)$$

and

$$[F_\alpha^{'T}(v,HK_*(v),\alpha]^{-1} P_\alpha^{'T}(v,u,\alpha,\xi) =$$

$$\sum_{j=1}^{M} \varrho_{3+j}[\max(0,\hat{g}_j(c,u,\alpha))+ \frac{\xi_j}{\varrho_{3+j}}].(G_j)_y^{'T}(v,HK_*(v),\alpha)$$

λ can be transformed to

$$\lambda(v,\xi,p,\varrho) = [p^T H - \frac{1}{2} \sum_{j=1}^{M} \varrho_{3+j}\max(0,\hat{g}_j(c,u,\alpha) + \frac{\xi_j}{\varrho_{3+j}}).$$

$$.(G_j)_y^{'T}(v,HK_*(v),\alpha)][K_*^{'}(v) - F_\alpha^{'T}(v,HK_*(v),\alpha)] \qquad (6.14)$$

The formula (6.14) does not require an existence of $[F_\alpha^{'T}(v,HK_*(v,\alpha)]^{-1}$, and therefore the number of process model parameters does not have to be the same as the number of the outputs. The parameter α can be estimated by solving

$$F(v,HK_*(v),\alpha) = K_*(v) \qquad (6.15)$$

The solution of (6.1) is denoted by $\hat{c}(v,p,\xi)$, $\hat{u}(v,p,\xi)$ and the corresponding Lagrangian multiplier vector by $\hat{\xi}(v,p,\xi)$. The following is a set of coordinating conditions associated with problem (6.1) :

$$\hat{c}(v,p,\xi) = v \qquad (6.16)$$

$$\hat{\xi}(v,p,\xi) = \xi \qquad (6.17)$$

$$\hat{u}(v,p,\xi) = HF(\hat{c}(v,p,\xi),\hat{u}(v,p,\xi),\hat{\alpha}(v)) \qquad (6.18)$$

The control set-point vector v_* of any solution (v_*, p_*, ξ_*) of the set of the equations (6.16), (6.17) and (6.18), which corresponds to the expanded problem (3.30), satisfies the Kuhn-Tucker first order necessary conditions if the regularity conditions are satisfied at every point of a set CU (see Brdyś and Roberts 1984). If the process inequality constraints are output independent, equations (6.16), (6.17) and (6.18) and λ can be simplified to

$$\hat{c}(v,p) = v \qquad (6.19)$$

$$\hat{u}(v,p) = HF(\hat{c}(v,p), \hat{u}(v,p), \hat{\alpha}(v)) \qquad (6.20)$$

and

$$\lambda(v,p) = [K_*^{'T}(v) - F_c^{'T}(v, HK_*(v), \alpha)][F_\alpha^{'T}(v, HK_*(v), \alpha)]^{-1} .$$

$$.[- q_\alpha^{'T}(c,u,\alpha) + F_\alpha^{'T} p(c,u,\alpha) H^T p] \qquad (6.21)$$

respectively.

The general structure of the algorithm is identical to the previous algorithm considered in Chapter 4. The modified optimisation (6.12) and parameter estimation (6.15) are both separable for fixed values of v and p, where each problem separates into N independent smaller problems.

By structuring the coordinator algorithm differently, problem (6.12) can be solved in a similar scheme as proposed in Chapter 4 for the augmented ISOPE algorithm using output feedback :

i) An augmented single loop coordination technique is obtained when v and p are adjusted simultaneously,

$$v^{k+1} = v^k + \varepsilon(\hat{c}(v^k,p^k) - v^k) \qquad (6.22)$$

$$p^{k+1} = p^k + \varepsilon_p[\hat{u}(v^k,p^k) - HF(\hat{c}(v^k,p^k,\xi^k),\hat{u}(v^k,p^k,\xi^k),\hat{\alpha}(v^k))] \qquad (6.23)$$

where ε and ε_p are positive numbers suitably chosen to ensure convergence and k denotes an iteration number.

ii) An augmented system based double loop technique is obtained when p is adjusted in an outer loop,

$$p^{n+1} = p^n + \varepsilon_p[\hat{u}(p^n) - HF(\hat{c}(p^n),\hat{u}(p^n),\hat{\alpha}(v(p^n)))] \qquad (6.24)$$

while in the inner loop the set-point vector v is adjusted, for fixed p^n, according to

$$v^{k+1} = v^k + \varepsilon(\hat{c}(v^k,p^n) - v^k) \qquad (6.25)$$

iii) An augmented model based double loop technique is obtained when v is adjusted in an outer loop,

$$v^{k+1} = v^k + \varepsilon(\hat{c}(v^n)-v^n), \qquad (6.26)$$

while in the inner loop the price p^k is adjusted, for fixed v^n, according to

$$p^{k+1} = p^k + \chi[\hat{u}(v^n,p^k) - HF(\hat{c}(v^n,p^k),\hat{u}(v^n,p^k),\hat{\alpha}(v^n))] \tag{6.27}$$

where $\chi > 0$.

Since the price p enters the performance index (6.12) through the modifier $\lambda(v,\hat{\alpha}(v,p),\rho)$ (see equation (6.21)) strategy (6.27) is not of a gradient type (relaxed method) and the existance of the inner loop solutions may be restricted. In order to overcome these difficulties, equation (6.19) and (6.20) are modified in the following manner. The optimisation (6.12) is replaced by

$$\min \{ q(c,u,\hat{\alpha}(v)) + \tfrac{1}{2}\rho_1\|v-c\|^2 + \tfrac{1}{2}\rho_2\|u-HK_*(v)\|^2 + p_1^T(c,u,\hat{\alpha}(v)) - \lambda^T(v,p_2)c \} \tag{6.28}$$

where, in comparison with Eq. (6.12), the price affecting term $p^T g(c,u,\hat{\alpha}(v))$ is distinguished from the price affecting term $\lambda^T(v,\hat{\alpha}(v),p)c$. The solution of Eq. (6.28) with respect to c and u under given p_1, p_2 and v is denoted by $\hat{c}(v,p_1,p_2)$ and $\hat{u}(v,p_1,p_2)$. Clearly, the set of Eqs. (6.19) and (6.20) is equivalent to :

$$\hat{c}(v,p_1,p_2) = v \tag{6.29}$$

$$p_1 = p_2 \tag{6.30}$$

$$\hat{u}(v,p_1,p_2) = HF(\hat{c}(v,p_1,p_2),\hat{u}(v,p_1,p_2),\hat{\alpha}(v)) \tag{6.31}$$

Equation (6.31) is solved in the inner loop with respect to p_1 for given values of v and p_2, and the solution is denoted as $p_1(v,p_2)$. The outer loop solves Eqs. (6.29) and (6.30) with respect to v and p_2. The price p_1 in the inner loop, now adjusted according to (6.27), is of the gradient type of the interaction balance iterative scheme (Findeisen and co-workers, 1980). An additional iteration of the price vector in the outer loop is required :

$$p_2^{k+1} = p_2^k + \varepsilon_p [\hat{p}_1^k - \hat{p}_2^k] \qquad (6.32)$$

where ε_p is a positive number suitably chosen in order to guarantee convergence ($\varepsilon_p \neq \varepsilon$ in general).

The structure of the algorithms is identical to the previous augmented ISOPE method described in Chapter 4. For example in the case of the augmented single loop technique, it is iterated in a similar manner as decribed in section 4.3.1 with the modified optimisation problem (6.12) and parameter estimation problem (6.15) used instead of (4.29) and (4.28) respectively. The set-point v and the price p are adjusted according to equations (6.22) and (6.23) respectively. The modifier λ is evaluated by solving equation (6.21) instead of equation (4.26). Similarly, by applying appropriate changes the augmented system based double loop and the augmented model based double loop techniques can be iterated as described in Chapter 4.

6.2.2 Constraint Augmentation

Using a similar approach to the formulation of problem (6.12), the constraint augmentation problem (6.2) can be reformulated as follows :

$$\min \{ q(c,u,\hat{\alpha}(v)) + \frac{1}{2} r \| u-HF(c,u,\hat{\alpha}(v)) \|^2$$

$$+ p^T (u-HF(c,u,\hat{\alpha}(v))) - \lambda(v,p)^T c \}$$

$$\text{s.t} \quad \tilde{g}(c,u) \leqslant 0 \qquad (6.33)$$

for given value of v, $\alpha = \hat{\alpha}(v)$, price p, ϱ and where $\lambda(v,p,r)$ is evaluated by equation (6.21).

The constraint augmenting term $\frac{1}{2} r \| u-HF(c,u,\alpha) \|$ in problem (6.33) is linearized around some point (c^s,u^s) in the same way as proposed in section 3.4 (see Chapter 3), in order to obtain the separability of the term.

The approximation loop can now be iterated as follows :

i) Set the initial values of (c^o,u^o), an appropriate choice of $\beta > 0$, and set s = 1.

ii) For given $\lambda(v,p,r)$, p and r the following separable modified optimisation problem

$$\min_{(c,u)\varepsilon CU} \{ q(c,u,\hat{\alpha}(v)) + p^T (u-HF(c,u,\hat{\alpha}(v))) - \lambda(v,p,r)^T$$

$$+ \frac{1}{2} r [\| u \|^2 + \| F(c,u,\hat{\alpha}(v)) \|^2 - 2(u^s)^T HF(c,u,\hat{\alpha}(v))$$

$$+ 2u^T HF(c^s,u^s,\hat{\alpha}(v))]\} \qquad (6.34)$$

is solved to obtain $\hat{c}^s$ and $\hat{u}^s$.

ii) The convergence is checked, i.e if condition $(\hat{c}^s,\hat{u}^s) - (c^s,u^s) \leqslant \beta_a$ is satisfied then set $\hat{c}(v,p) = c^s$, $\hat{u}(v,p) = u^s$, where $(\hat{c}(v,p), \hat{u}(v,p))$ denotes a solution of (6.34), otherwise, $\hat{c}^s$ and $\hat{u}^s$ are updated as follows

$$(c^{s+1}, u^{s+1}) = (c^s,u^s) + \varepsilon_a[(\hat{c}^s,\hat{u}^s) - (c^s,u^s)] \quad (6.35)$$

and set s = s + 1 and go to (ii).

When the single loop, the system based double loop and the model based double loop techniques are applied to the constraint augmented algorithm, the general iterative procedure remains the same as prevously described in section 6.3, and the approximation loop is iterated in a similar manner as in the AISOPE structure with output feedback described in section 3.4 (see Chapter 3).

6.3 Optimality and Convergence Analysis

In this section we will be examining the optimality properties of AISOPE algorithms with input and output feedback. The convergence analysis is restricted to the model based double loop technique for the variable augmentation and to the approximation loop for the constraint augmentation.

6.3.1 Optimality

Let Ω be a set of all points (v,p) solving Equations (6.16), (6.17) and (6.18). In this section we investigate the optimality properties of points which belong to Ω. The first step is to apply equation (6.13) to problem (6.12) and then transform it to another equivalent problem as follows :

$$\min \{ Q(c,u,F(c,u,\hat{\alpha}(v)) + \dot{Q}_y(v,HK_*(v),K_*(v)).$$

$$.[\dot{F}_c(v,HK_*(v),\hat{\alpha}(v))-\dot{K}_*(v)]c - p^T H[\dot{K}_*(v)-\dot{F}_c(v,HK_*(v),\hat{\alpha}(v))]c$$

$$+ \; p^T[u - HF(c,u,\hat{\alpha}))]$$

$$+ \sum_{j=1}^{M} \varrho_{3+j} \max(0,\tilde{g}_j(c,u,\hat{\alpha}(v)) + \frac{\xi_j}{\varrho_{4+j}})\dot{G}_{jy}(v,HK_*(v),\hat{\alpha}(v)).$$

$$.[\dot{F}_c(v,HK_*(v),\hat{\alpha}(v)) - \dot{K}_*(v)]c + \frac{1}{2}\varrho\|v-c\|^2$$

$$+ \; \frac{1}{2}\varrho\|u-HK_*(v)\|^2 \} \tag{6.35a}$$

Let us define

$$q_*(c,u) \overset{\Delta}{=} Q(c,u,K_*(c)) \tag{6.36}$$

$$g_*(c,u) \overset{\Delta}{=} u - HK_*(c) \tag{6.37}$$

$$g(c,u) \overset{\Delta}{=} u - HF(c,u,\alpha) \tag{6.38}$$

$$P_*(c,u,\xi) \overset{\Delta}{=} \frac{1}{2}\sum_{j=1}^{M} \varrho_{3+j}((\max(0,\tilde{g}_{*j}(v,HK_*(v)) - \frac{\xi_j}{\varrho_{3+j}})^2 - (\frac{\xi_j}{\varrho_{3+j}})^2) \tag{6.39}$$

Equation (6.13) for the modifier $\lambda(v,p,\xi,\varrho)$ together with Eq.

(6.15) can be expressed as :

$$\lambda^T(v,p,\xi,\varrho) = q_c'(v,HK_*(v),\hat{\alpha}(v)) - q_{*c}'(v,.HK_*(c))$$
$$+ [q_u'(v,HK_*(v),\hat{\alpha}(v)) - q_u'(v,HK_*(v))]HK_*(v)$$
$$+ p^T([g_c'(v,HK_*(v),\hat{\alpha}(v)) - g_{*c}'(v,HK_*(v))]$$
$$+ [g_u'(v,HK_*(v),\hat{\alpha}(v)) - g_{*u}'(v,HK_*(v))]HK_*'(v)$$
$$+ P_c'(v,HK_*(v),\hat{\alpha}(v),\xi) - P_{*c}'(v,HK_*(v),\xi) \tag{6.40}$$
$$+ [P_u'(v,HK_*(v),\hat{\alpha}(v),\xi) - P_{*u}'(v,HK_*(v),\xi)]HK_*'(v)$$

Therefore, problem (6.13) can be expressesed as :

$$\min_{(c,u)\varepsilon CU} \{ q(c,u,\hat{\alpha}(v)) + p^T g(c,u,\hat{\alpha}(v)) - \lambda^T(v,p,\xi,\varrho)c + \frac{1}{2}\varrho_1 \|v-c\|^2 + \frac{1}{2}\varrho_2 \|u-HK_*(v)\|^2\} \tag{6.41}$$

The optimal control problem, OCP, can be written as

$$\min_{(c,u)\varepsilon CU} q_*(c,u) \tag{6.42}$$
$$\text{s.t } g_*(c,u) = 0$$

Let $(\bar{v},\bar{p}) \; \varepsilon \; \Omega$ and define a Lagrangian corresponding to OCP, Eq.(6.39), as

$$L_*(c,u,p) \overset{\Delta}{=} q_*(c,u) + p^T g_*(c,u) \tag{6.43}$$

we also define

$$\bar{u} \overset{\Delta}{=} u(\bar{v},\bar{p}) \tag{6.44}$$

$$\tilde{p} \overset{\Delta}{=} [g_{*u}'^T(\bar{v},HK_*(\bar{v}))\bar{p} + F_u'^T(\bar{v},HK_*(\bar{v}),\bar{\alpha})Q_y'^T(\bar{v},HK_*(\bar{v}),K_*(\bar{v})) + P_{*u}'^T(\bar{v},HK_*(\bar{v}),\xi)] \tag{6.45}$$

Lemma 6.1

Assume that a point $(v,HK_*(v)) \varepsilon CU$ the Kuhn-Tucker regularity conditions are satisfied. Then the point satisfies the Kuhn-Tucker necessary optimality conditions for OCP while the price p is the corresponding Lagrange multiplier.

Proof

Due to the definition of $\hat{\alpha}(.)$ and according to Eq.(6.18) the following holdss :

$$\hat{u}(\bar{v},\bar{p},\bar{\xi}) = HK_*(\bar{v}) \qquad (6.46)$$

Utilising the definition of $\hat{c}(.)$ and $\hat{u}(.)$ and employing (6.46) and Eq.(6.16) we obtain that there exists such a vector $\xi \geqslant 0$ that the following holds :

$$q'_{c,u}(\bar{v},HK_*(\bar{v}),\bar{\alpha}) + \bar{p}^T g'_{c,u}(\bar{v},HK_*(\bar{v}),\bar{\alpha}) - [\lambda^T(\bar{v},\bar{\alpha},\bar{p}),0] + P'^T_{c,u}(\bar{v},HK_*(\bar{v}),\bar{\alpha},\bar{\xi}) = 0 \qquad (6.47)$$

and $\xi_j \tilde{g}_{ij}(c_i,u_i)=0$, for $i \varepsilon \overline{1,N}$, $j \varepsilon J$ (6.48)

Combining (6.16), (6.47) and (6.48) and performing straight forward manipulations we conclude that

$$q'_{*c,u}(\bar{v},HK_*(\bar{v})) + \tilde{p}^T g'_{*c,u}(\bar{v},HK_*(\bar{v})) + P'_{*c,u}(\bar{v},HK_*(\bar{v}),\bar{\xi}))=0$$

and

$$\bar{\xi}_j \tilde{g}_{ij}(\bar{v}_i,H_iK_*(\bar{v}))=0 \quad , \text{ for } i \varepsilon \overline{1,N} , j\varepsilon J$$

which completes the proof of the Lemma.

Q.E.D

Sufficient conditions for optimality of the solution set Ω are given in the next Lemma.

Let us define the system model based Lagrangian as

$$L(c,u,\alpha,p) = q(c,u,\alpha) + p^T g(c,u,\alpha) \qquad (6.49)$$

and let us denote

$\alpha_{opt} = \alpha(\hat{c}_{opt})$, $u_{opt} = HK_*(c_{opt})$, $y_{opt} = K_*(c_{opt})$,

and $\xi_{opt} = \xi(\hat{c}_{opt})$.

We assume that the point (c_{opt}, u_{opt}) satisfies the Kuhn-Tucker regularity conditions for the OCP. Let us denote by p_{opt} and ξ_{opt} the corresponding Lagrange multipliers associated with interaction input and inequality constraints, respectively. It is also assumed that the inverse $[g_u'^T(c_{opt}, u_{opt}, \alpha_{opt}]^{-1}$ exists.

Lemma 6.2

Assume that the set CU is convex and the function $L(.,.,\alpha_{opt}, \bar{p})$ is convex on CU, where

$$\bar{p} = [g_u'^T(c_{opt}, u_{opt}, \alpha_{opt})]^{-1}(p_{opt} - F_u'^T(c_{opt}, u_{opt}, \alpha_{opt}).$$
$$.Q_y'^T(c_{opt}, u_{opt}, y_{opt}) - P_u'^T(c_{opt}, u_{opt}, \alpha_{opt}, \xi_{opt})$$

Then $(c_{opt}, p) \in$.

Let us consider the modified model based algorithm problem under $v = c_{opt}$, $\alpha = \alpha_{opt}$, $\xi = \xi_{opt}$ and $p = \bar{p}$. Performing straightforward computations we obtain that a derivative of the minimised function taken at a point (c_{opt}, u_{opt}) is equal to

$$q_*(c_{opt}, u_{opt}) + \bar{p}^T g_*'(c_{opt}, u_{opt}) + P_*'^T(c_{opt}, u_{opt}, \xi_{opt})$$

Therefore, a triplet $(c_{opt}, u_{opt}, \xi_{opt})$ is the Kuhn-Tucker point for the considered modified model based optimisation problem. Hence, due to the Lemma assumptions on convexity, the point (c_{opt}, u_{opt}) is a solution to this problem. Clearly, this solution satiesfies Eqs.(6.16), (6.17) and (6.18). Therefore, $(c_{opt}, \bar{p}, \bar{\xi}) \in \Omega$. A proof of the Lemma has now been completed.

Q.E.D

The optimality studies for the constraint augmented algorithm can be performed in a similar manner, exept that the modified optimisation problem (6.33) is used instead of (6.12).

6.3.2 Convergence of the Model Based Double Loop Technique

In this thesis, no attempt is made to analyse the convergence of the single loop technique and the system based double loop technique. Brdys, Abdullah and Roberts (1986) presented a convergence analysis for the model based double loop technique but only for a case with a quadratic performance Q, linear system and model mappings F_* and with no inequality constraints. Even for such a simplified case, the proof is lengthy and the convergence conditions are rather complex and difficult to check. When the algorithm is applied to nonlinear problems, the convergence results are valid in a local sense only. The performance index is considered to have the form :

$$Q(x) = \frac{1}{2}(x-d)^T M(x-d) \qquad (6.51)$$

where $x \triangleq (c,u)$

$$F(c,u,\alpha) = D_1 c + D_2 u + P(\alpha) \qquad (6.52)$$

$$F_*(c,u) = D_{*1} c + D_{*2} u - d_* \qquad (6.58)$$

where M is a symmetric matrix, and matrices D_1, D_2 and $P(\alpha)$ are chosen so that the model is point parametric. It is assumed that the inverses

$$[HD_{*2} - I]^{-1} \quad \text{and} \quad [HD_2 - I]^{-1}$$

exist and consequently the following matrices are well defined :

$$B \overset{\Delta}{=} [\, -HD_1 \,,\, I - HD_2 \,] \tag{6.54}$$

$$B_* \overset{\Delta}{=} [\, [HD_{*2} - I]^{-1} HD_{*1},\, I] \tag{6.55}$$

$$B \overset{\Delta}{=} [I - HD_2]B_* \tag{6.56}$$

Assuming that the matrix B_* has full rank, and that the second order sufficient conditions for optimality hold for the optimising control problem (3.16), i.e

$x^T Mx > 0$ for every $x \varepsilon C \times U$, $x \neq 0$ such that $B_* x = 0$ (6.57), then there is a unique solution $\bar{c}$ to the control problem and a unique corresponding price $\bar{p}$ associated with the contraints $g_*(c,u) = 0$. Consequently, there is a unique solution $(\bar{c}, \tilde{p})$ of the set of Eqs. (6.19) and (6.20),

where

$$\tilde{p} = [\, (I - HD_2)^{-1}]^T p \tag{6.58}$$

and the non-convex case is included from the considerations. It is assumed that the value of ϱ in (6.12) is chosen such that

$$M_\varrho \overset{\Delta}{=} M + \varrho I > 0 \tag{6.59}$$

The following matrices are defined

$$L \stackrel{\Delta}{=} \begin{vmatrix} I_{cxc} \\ (I-HD_2)^{-1}HD_1 \end{vmatrix} \qquad (6.60)$$

and

$$L_* \stackrel{\Delta}{=} \begin{vmatrix} I_{cxc} \\ (I-HD_{*2})^{-1}HD_{*1} \end{vmatrix} \qquad (6.61)$$

Theorem 6.1

Assume

i) $BM_\varrho^{-1}B^T > 0$ (6.62)

ii) $L_*^T M_\varrho L > 0$ (6.63)

Then :

1. The inner loop problem described by Eq. (6.20) is well defined. There exists a number $\bar{\chi} > 0$ such that for any $\chi \in (0,\bar{\delta})$, the iterative scheme (6.27) is convergent for every v.

2. The outer loop iterative scheme described by Eq. (6.26) is convergent to the point $\bar{c}$ while the corresponding sequence of price vectors is convergent to $\breve{p}$ for every value of ε such that,

$$L_*^T M_\varrho L - \frac{\varepsilon}{2} L_*^T M L_* > 0 \qquad (6.64)$$

A proof of this theorem is given in Appendix E.

The condition (6.62) is only required to preserve convergence of the inner loop iterations while condition (6.63) is needed to preserve convergence of the outer loop iterations. By choosing a sufficiently small value for ε, condition (6.64) can always be satisfied. Since $\breve{B} = B$ and $L_* = L$ if the mathematical model is perfect, conditions (6.62) and (6.63) are satisfied in that particular situation.

The second double iterative loop technique with the relaxed inner loop problem will now be considered.

Let us define a matrix M_γ in the following way :

$$M_\gamma \overset{\Delta}{=} M + \gamma \tilde{B}^T \tilde{B} \qquad (6.65)$$

where $\gamma > 0$ is chosen such that $M_\gamma > 0$.

Notice that, according to (6.52) and due to assumption (6.53), such a choice always exists (e.g Luenberger 1973). It is assumed, additionally, that the matrix B has full rank.

Theorem 6.2

assume

$$\begin{vmatrix} \frac{1}{\varepsilon} L_*^T M_\varrho L - \frac{1}{2} L_*^T M_\gamma L_* & \frac{1}{2\varepsilon} \tilde{B} M_\gamma^{-1} M_\varrho L + \frac{1}{2\varepsilon} B L_* \\ \frac{1}{2\varepsilon} \tilde{B} M_\gamma^{-1} M_\varrho L + \frac{1}{2\varepsilon} B L_* & B M_\gamma^{-1} \tilde{B}^T - \frac{1}{2\varepsilon} \tilde{B} M_\gamma^{-1} \tilde{B}^T \end{vmatrix} > 0 \qquad (6.66)$$

Then the algorithm described by Eqs. (6.26) and (6.32) is well defined and generates a sequence $\{v^k, p^k\}$ which is convergent to $\{\bar{c}, \tilde{p}\}$.

The proof of Theorem 6.2 is given in Appendix F.

Condition (6.62) implies that condition (6.63) is satisfied and condition (6.62), in which a matrix M_ϱ^{-1}, is replaced by matrix M_γ^{-1}, is also satisfied.

Condition (6.66) is only required to preserve convergence of the outer loop. The convergence properties of the relaxed inner loop are better, but the overall convergence conditions of the outer loop seem to be more restrictive in the linear quadratic case. The second technique is expected to be more efficient in some general non-linear problems due to better efficiency of the inner loop.

let us examine condition (6.66) and the conditions (6.62) and (6.63) in a convex case, i.e, when $M > 0$ and assume that $\varepsilon=\varepsilon_p$. The value $\varrho=0$ and $\gamma=0$, the conditions (6.62) and (6.63) and (6.66) take the form

$$BM^{-1}\tilde{B}^T > 0, \quad L_*^T ML > 0$$

and

$$\begin{vmatrix} \frac{1}{\varepsilon}L_*^T ML - \frac{1}{2}L_*^T ML_* & \frac{1}{2\varepsilon}(\tilde{B}L + BL_*) \\ \frac{1}{2\varepsilon}(\tilde{B}L + BL_*) & BM^{-1}\tilde{B}^T - \frac{1}{2}\tilde{B}M^{-1}\tilde{B}^T \end{vmatrix} > 0$$

respectively.

It can be easily verified that $\tilde{B}L + BL_* = 0$. Therefore, condition (6.66) is equivalent to conditions (6.62) and (6.63) in the convex case.

6.3.3 Convergence of the Approximation Loop of Constraint Augmented algorithms

We will be looking only at the convergence conditions for the approximation loop used within AISOPE with input and output feedbacks which has been presented by Tatjewski, Abdullah and Roberts (1986) and later extended to AISOPE algorithms with output feedback.

Let us denote by $\hat{x} = (\hat{c},\hat{u})$ the optimal point of the original optimising control problem (3.16) (see Chapter 3), and by $\hat{p}$ the optimal Lagrange multiplier for the constraints $u-HF(c,u,\alpha)=0$ which is equivalent to formulation (3.17) (see Chapter 3), and also note that $\hat{v}=\hat{c}$, $\hat{\alpha}=\hat{\alpha}(v)$, $\hat{\lambda}=\hat{\lambda}(\hat{v},\tilde{\hat{p}})$. Let us formulate the following non-augmented model optimisation problem

$$\begin{aligned} &\min \{ q(c,u,\hat{\alpha}) - \hat{\lambda}^T c \} \\ &\text{s.t} \;\; g(c,u,\hat{\alpha}) = 0,\; G(c,u) \leqslant 0 \end{aligned} \tag{6.67}$$

where $\hat{\lambda}$ and $\hat{\alpha}$ denote values corresponding to the optimal point. Finally, let us denote by (v,p) a current point of an AISOPE algorithm with input and output feedbacks, i.e., $(v,p) = (v^k,p^k)$ for the single loop technique, $(v,p)=(v^n,p^k)$ for the system based double loop technique. The local convergence conditions are formulated in the following theorem:

Theorem 6.3

Assume that

i) Mappings Q, F and G are twice continuously differentiable, locally in some neighbourhood of the optimal point, and the problem (6.67) satisfies at $(\hat{c},\hat{u})$ second order sufficient optimality conditions with strict complementarity (see Findeisen et. al. 1980).

ii) The point $(\hat{c},\hat{u})$ is a regular point of (6.67) i.e., at $(\hat{c},\hat{u})$ gradients of all active constraints are linearly independent.

iii) K_* is continuously differentiable in some neighbourhood of $\hat{v}$, and $F'_\alpha(\hat{c},\hat{u},\hat{\alpha})$ is nonsingular.

Then, provided (v,p) is sufficiently close to $(\hat{v},\hat{p})$ and r is sufficiently large, there is an $\bar{\varepsilon}_a>0$ such that for every $\varepsilon_a \in (0,\bar{\varepsilon}_a)$ the approximation algorithm (6.35) is locally linearly convergent to the solution $(\hat{c}(v,p),\hat{u}(v,p))$ of the constraint augmented optimisation problem (6.34).The proof of the theorem is given in Appendix G.

The assumptions of Theorem 6.2 are rather weak and the second order sufficient optimality conditions are known to be "almost necessary" (see Bertsekas 1982). It can be shown that $\bar{\varepsilon}_a>1$, but usually values $\tilde{\varepsilon}_a=1$ or a little smaller result in good convergence (Tatjewski 1985). The convergence conditions for AISOPE algorithms using output feedback are identical, except that the optimisation problem (4.58) must be considered instead of (6.34).

6.4 Summary

Optimality and convergence analysis of the constraint and variable augmented ISOPE algorithms using input-output feedback has been investigated. The convergence analysis has only been performed on the variable augmented model based double loop technique and the approximation loop portion of the constraint augmented ISOPE algorithms. The convergence analysis of the model double loop technique is restricted, however, to a quadratic performance Q, linear system and model equations, and with only equality constraints. Even with a simplified example, the convergence analysis is very complex, and it was found that, at present, the complete convergence analysis for the more general situation including constraint augmentation, is intractable. Simulation results and a comparative study of these algorithms will be discussed in Chapter 7.

CHAPTER 7 - DISCUSSION OF THE SIMULATION RESULTS

7.1 Introduction

In this chapter we will be discussing the simulation results of the Augmented Integrated System Optimisation and Parameter Estimation (AISOPE) algorithms. All together we will be looking at twelve versions of the augmented ISOPE algorithm as summarised in Table 7.1. In order to avoid confusion and simplify matters, we will be applying abbreviation, for example the augmented ISOPE with input and output feedbacks will be referred to as AISOPE1 whereas the other structure with only output feedback will be known as AISOPE2.

The number of system iterations (IS), i.e, iterations on the system (set-point changes) required to achieve the desired final accuracy from a given initial starting point, will be used as a comparison criterion to study the effectiveness of the algorithms. Sensitivity of the convergence speed of each algorithm to its parameters, such as step length coefficients ε and ε_p and penalty coefficients r and ρ, will be used as a means of providing an insight into practical applicability of the algorithms. Both criteria as mentioned above describe the practical effectiveness of each algorithm. Lastly, the total number of solutions of the modified optimisation problem (IT) will roughly give us the time of pure computation taken by various algorithms to perform model based optimisation.

7.2 Discussion

In order to examine the behaviour of the proposed algorithms, the following examples have been simulated :

7.2.1 Examples

Example 1

Subsystem 1

$$F_{*1}(c_1,u_2) = 2.1c_1 + u_1 + 0.5c_1u_1$$
$$F_1(c_1,u_1,\alpha_1) = 2c_1 + u_1 + \alpha_1$$
$$Q_1(c_1,u_1,y_1) = 32c_1^2 - 16c_1 + (y_1 - 1)^2$$
$$CU_1 = \{ c_1,u_1 : 2c_1 + u_1 \leqslant 2.25 \}$$

Subsystem 2

$$F_{*2}(c_2,u_2) = 0.6c_2 + 0.55u_2$$
$$F_2(c_2,u_2,\alpha_2) = 0.5c_2 + 0.5u_2 + \alpha_2$$
$$Q_2(c_2,u_2,y_2) = 10c_2^2 + 4c_2u_2 - 8(y_2)^2$$
$$CU_2 = R^2$$

Structure equation

$$\begin{vmatrix} u_1 \\ \\ u_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ & \\ 1 & 0 \end{vmatrix}\begin{vmatrix} y_1 \\ \\ y_2 \end{vmatrix}$$

Example 2

Subsystem 1

$$F_{*1}(c_1,u_1) = 1.3c_{11} - c_{12} + 2.0u_{11} + 0.15u_{11}c_{11}$$

$$F_2(c_1,u_1,\alpha_1) = c_{11} - c_{12} + 2.0u_{11} + \alpha_{11}$$

$$Q_1(c_1,u_1) = (u_{11} - 1.0)^4 + 5.0(c_{11} + c_{12} - 1.0)^2$$

$$CU_1 = \{(c_1,u_1) \in R^3 : c_{11}^2 + c_{12}^2 \leqslant 1.0, \quad 0.0 \leqslant u_{11} \leqslant 0.5 \}$$

Subsystem 2

$$F_{*2}(c_2,u_2) = c_{21} - c_{22} + 1.2u_{21} - 3.0u_{22} + 0.1c_{22}^2$$

$$F_2(c_2,u_2,\alpha_2) = c_{21} - c_{22} + u_{21} - 3.0u_{22} + \alpha_{21}$$

$$F_{*22}(c_2,u_2) = 2.0c_{22} - 1.25c_{23} - u_{21} + u_{22} + 0.25c_{22}c_{23} + 0.1$$

$$F_{22}(c_2,u_2,\alpha_2) = 2.0c_{22} - c_{23} - u_{21} + u_{22} + \alpha_{22}$$

$$Q_2(c_2,u_2) = 4.0u_{21}^2 + u_{22}^2 + 2.0(c_{21} - 1.0)^2 + c_{22}^2 + 3.0c_{23}^2$$

$$CU_2 = \{ (c_2,u_2) \in R^5 : 0.5c_{21} + c_{22} + 2.0c_{23} \leqslant 1.0,$$

$$4.0c_{21}^2 + 2.0c_{21}u_{21} + 0.4u_{21} + c_{21}c_{23} + 0.5c_{23}^2 + u_{21}^2 \leqslant 4.0 \}$$

Subsystem 3

$$F_{*3}(c_3,u_3) = 0.8c_{31} + 2.5c_{32} - 4.2u_{31}$$

$$F_3(c_3,u_3) = c_{31} + 2.5c_{32} - 4.0u_{31} + \alpha_{31}$$

$$Q_3(c_3,u_3) = (u_{31} - 1.0)^2 + (c_{31} + 1.0)^2 + 2.5c_{32}^2$$

$$CU_3 = \{(c_3,u_3) \in R^3 : c_{31} + u_{31} + 0.5 \geqslant 0.0, \; 0.0 \leqslant c_{32} \leqslant 1.0 \}$$

Structure equations

$$\begin{vmatrix} u_{11} \\ u_{21} \\ u_{22} \\ u_{31} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} y_{11} \\ y_{21} \\ y_{22} \\ y_{31} \end{vmatrix}$$

Starting points were chosen as :

Example 1

$v^o = [0.5, 0.25]^T$

$w^o = [1.25, 2.25]^T$

$p^o = [8.0, 13.0]^T$

Example 2

$v^o = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]^T$

$w^o = [0.0, 0.0, 0.0, 0.0]^T$

$p^o = [0.0, 0.0, 0.0, 0.0]^T$

In order to test the effectiveness of the techniques the real optimal solutions have been determined as follows :

Example 1

$\bar{v} = [0.347, 0.251]$

$\bar{w} = [1.555, 2.555]$

$\bar{p} = [-5.461, 8.711]$

$\bar{Q} = -15.447$

Example 2

$\bar{v} = [0.481, 0.876, 0.986, -0.179, 0.0360, -0.512, 0.342]$

$\bar{w} = [0.137, 0.0316, 0.356, 0.0200]$

$\bar{p} = [-0.552, -1.563, 0.663, -0.282]$

$\bar{Q} = 6.326$

7.2.2 AISOPE Algorithms with Input-Output Feedback

Extensive simulation studies have been performed on two examples. Example 1 is a small dimensional example, where modified local optimisation can be easily analytically solved. It is taken from Tatjewski (1987), but with a non-linear term $0.5c_1u_1$ additionally introduced to F_{*1}. The problem is non-convex, with a duality gap, where the minimal necessary convexification requires r and ρ be greater than 4.0.

Example 2 has been taken from Findeisen and co-workers (1980), and has non-linear and non quadratic performance Q, non-linear output mapping F_{*i} with linear models F_i, i=1,..N, and some nonlinear inequality constraints. The NAG Library subroutine E04VBF was used to obtain the subsequent solution of the modified local optimisation problems, for each of the tested AISOPE algorithms.

First, we will be looking at simulation results obtained from Example 1. The stopping criterion, for every algorithm, is satisfied if the Euclidean norm of the coordinating equations becomes not greater than $\delta = 10^{-4}$. Selected results for Example 1, presenting the number of system iterations IS for all combinations of ε and ε_p, are given in Tables 7.2 to 7.9. Number 111 in Table 7.2 indicates fast divergence and number 100 implies slow divergence. It can readily be seen from Tables 7.2 and 7.3 that the constraint augmented single loop (CA - SL1) algorithm is less sensitive and always performs well in a region near $\varepsilon \cong 1$, $\varepsilon_p \cong 0.5r$. In the case of the variable augmented single loop

(VA-SL1) algorithm, the optimal value of ε_p can also be chosen in the region near $\varepsilon \cong 0.5\varrho$, but a suitable value of ε has to be properly chosen in order to ensure convergence, which depends very much on the value of ϱ being used.

The optimal values of ε and ε_p are chosen so as to produce the smallest number of system iterations IS required to achieve a prescribed final accuracy δ. Tables 7.4 and 7.5 indicate that at an optimal value of ε and ε_p for each algorithm, the constraint augmented single loop (CA-SL1) technique requires about 9 system iterations IS, which is about half those needed by the variable augmented single loop (VA-SL1) technique. Comparing the constraint augmented single loop (CA-SL1) and the variable augmented single loop (VA-SL1) techniques, it appears that the constraint augmented single loop (CA-SL1) technique is superior, both in the value of IS and its sensitivity to the value of the penalty coefficient. But the results in Table 7.4 and 7.5 show also the disadvantage of the constraint augmented algorithm due to the increased total number of optimisations IT. Table 7.6 demonstrates that combining the variable augmentation with the constraint augmentation to the single loop technique did not improve the convergence.

Representative results obtained for the model based double loop technique for Example 1 are presented in Tables 7.7 to 7.9 and Figs. 7.1 and 7.2, where IIL denotes the number of times the multiplier p is updated in the first inner loop. Fig. 7.2 shows that the penalty coefficient ϱ improves the inner loop convergence of the model based double loop technique for the variable augmented

(VA-MBDL1) algorithm, if the value of ϱ is chosen large enough and also allows a higher value of ε_p to be employed. However, the value of ε in the outer loop is sensitive to the value of ϱ, as shown in Fig.7.1. Therefore a suitable value of ϱ has to be chosen such that it improves the convergence of the inner loop, but does not affect the outer loop convergence. Sensitivity to the choice of ε seems to be better with the constraint augmented version, and practically does not depend on r, see Fig. 7.1 . The variable augmented version indicates significantly greater numbers of optimisation IT for smaller values of the penalty coefficient (ϱ = 5.0, 6.0), and smaller numbers of IT for medium and large values of ϱ. Medium range values of the penalty coefficient r yield the smallest IT numbers for the constraint augmented version.

Comparing the number of system iterations IS for both types of augmentation, see Tables 7.7 and 7.8, it appears that the constraint augmented model based double loop (CA-MBDL1) technique is slightly better than the variable augmented version. Table 7.9 indicates that the combination of the two types of augmentation did not improve the convergence.

Simulations of Example 1 were not performed with the system based double loop technique. This is due to the reason that the convergence of the price p in the outer loop is slower compared to the convergence of the set-point in the inner loop. Therefore, the system based double loop (SBDL1) technique is not competitive with respect to the set-point change or system iterations IS when compared to the other two algorithms.

Results obtained from the simulation study for Example 2 are presented in Tables 7.10 to 7.15. Tables 7.10 and 7.11 present the results for the single loop algorithms. The variable augmented single loop technique (VA-SL1) will converge for $\varrho < 3.0$, if the step coefficient ε_p chosen is very small, and the algorithm gives the best result for values of step coeficients around $\varepsilon \simeq 1$, $\varepsilon_p(\varrho) \simeq 0.6$. The constraint augmented single loop technique (CA-SL1) is convergent for a much wider range of the penalty coefficient r and step coefficient ε_p. The number of the system iterations IS decreases as r gets larger. It requires approximately half the number of system iterations IS and about twice the number of optimisations compared to those required by the variable augmented single loop technique (VA-SL1). The algorithm gives the best result for value of step coefficient about $\varepsilon \simeq 1$, $\varepsilon_p(r) \simeq 1$.

In example 2, when simulated with the model based double loop technique, dynamic accuracy is employed in the approximation loop. In the model based double loop technique, both inner and approximation loop are performed on the model only, and the use of dynamic accuracy for the approximation loop is possible and advisable, which should significantly reduce the number of optimisations IT for the constraint augmented algorithm. Hence, for the constraint augmented model based double loop technique (CA-MBDL1) a dynamic accuracy δ_a^k of the approximation loop was also applied

i.e $\quad \delta_a^{k+1} = \gamma \| g(\hat{x}(v^n, p^k), \hat{\alpha}(v^n) \|, \; k = 0,1,..,$

with $\gamma = 0.5$

Results obtained for the model based double loop techniques are presented in Tables 7.12 and 7.13. Simulation results show that the model based double loop techniques are better than the single loop techniques. The constraint augmented model based technique (CA-MBDL1) is significantly better than the variable augmented model based technique (VA-MBDL1), both with respect to the number of system iterations IS and to the number of optimisations IT, especially when using the dynamic accuracy technique in the approximation loop, see Tables 7.12 and 7.13. The best coefficient values are similar to those previously found for the single loop technique case.

Example 2 was also simulated using the system based double loop technique with a dynamic inner loop accuracy β_i^n for the coordinating equation, according to

$$\beta_i^{n+1} = \sigma \| g(\hat{x}(p^n), \hat{\alpha}(v(n^n))) \| \quad (7.1)$$

where $\sigma > 0$ is some scaler. In the case of the variable augmented system based double loop technique (VA-SBDL1), for sufficiently large σ the algorithm behaves as a single loop technique and decreasing σ only causes deterioration in the number of system iterations IS, see Tables 7.14 and 7.10. However, for the constraint augmented system based double loop technique (CA-SBDL1) similar or even slightly improved results, when compared with the constraint augmented single loop technique (CA-SL1), were obtained for larger values of r, see Tables 7.15 and 7.11. The value $\sigma = 0.5$ was used in (7.1) but the results were similar.

The augmented single loop technique of version 2 were also simulated using Examples 1 and 2 and, since the performance of the algorithms were almost similar to the augmented single loop technique of version 1, the result is not included in the thesis.

7.2.3 AISOPE2 Algorithms with Output Feedback

The AISOPE2 algorithms were also applied to Examples 1 and 2 and the results are shown in Tables 7.16 to 7.20. The variable augmented single loop technique (VA-SL2) works with Example 1, but the convergence is very poor. Table 7.16 presents the optimised variable augmented algorithm (VA-SL2) with the numbers of system iterations IS required. They are much worse compared with the other variable augmented single loop technique (VA-SL1) of the AISOPE1 algorithm (see Table 7.4). The constraint augmented single loop technique (CA-SL2), when applied to Example 1, did not satisfy the prescribed tolerance even after 150 iterations of IT, for all values of r. The algorithm converged very slowly only for a range of values around r = 20.

The results obtained for the model based double loop technique for Example 1 are shown in Tables 7.17 and 7.18, for optimised variable and constraint augmented algorithms respectively. Both versions of the augmented model based double loop technique work better than the single loop ones, but the results are still significantly worse than when they were applied to the AISOPE1 algorithms (compare Tables 7.7 and 7.8). The variable augmented algorithms seem to give better results than those with the constraint augmented algorithms, in contrast to the case when they are applied to AISOPE1 algorithms.

Tables 7.19 and 7.20 represent the simulation results obtained with Example 2, for the variable augmented single loop (VA-SL2) and system based double loop (VA-SBDL2) techniques respectively. The variable augmented single loop technique (VA-SL2) converges much slower than in the case of the AISOPE1 algorithms (compare Table7.10). In the case of the variable augmented system based double loop technique (VA-SBDL2), when simulated with inner loop dynamic accuracy, the algorithm behaves as a single loop technique (see Table 7.20).

Applying the variable augmented model based double loop technique (VA-MBDL2) to Example 2, the algorithm did not converge. This is due to the fact that Example 2 does not satisfy part (iv) of Theorem 5.5 (see Chapter 5).

The constraint augmented version of AISOPE2 was also applied to Example 2. The results are not presented, since the algorithm converged significantly slower, or even failed to converge at all for many choices of the penalty and step length coefficients.

7.2.4 Comparison

We will be comparing the efficiency of both types of augmentation based on the simulation results of Example 1 and 2. The best results selected for each of the single loop or model based double loop techniques of AISOPE1 and AISOPE2 algorithms are tabulated in

Table 7.21 and 7.22, respectively. Since the system based double loop technique behaves as the single loop technique, they are not included in the comparison.

In the case of AISOPE1 algorithms, the constraint augmented model based double loop technique (CA-MBDL1) gave the best performance and also the best sensitivity characteristic. The constraint augmented single loop technique (CA-SL1) and the variable augmented model based double loop technique (VA-MBDL1) gave a similar performance with Example 1 but the constraint single loop technique (CA-SL1) is superior in example 2. It is observed that the constraint augmented algorithms are less sensitive to the choice of values of penalty coefficients and also the choice of step length coefficients. Based on the obtained results, a suitable value of ε_p can be chosen for the constraint augmented algorithms from the proposed formula $\varepsilon_p = \gamma.r$, where $\gamma \in (0.5, 1.25)$.

When applying AISOPE2 algorithms, the variable augmented model based double loop technique (VA-MBDL2) gave the best performance with Example 1 but the algorithm did not converge with Example 2, (see Tables 7.21 and 7.22). In the case of AISOPE2 algorithms, the variable augmented algorithms are superior than the constraint augmented algorithms. In both examples, the constraint augmented algorithms failed to converge.

7.2.5 Effect of Noise on the AISOPE Algorithms

The calculation of modifiers λ and t involves the evaluation of the derivatives of process outputs with respect to the set-points.

These derivatives are evaluated by perturbing each set-point around v and measuring all the interconnected outputs, $K_*(v)$, and then performing finite difference computations.

If measurement is taken under noisy conditions, the derivatives will be distorted, and will deteriorate the performance of the algorithms. Howerer, introducing simple filter techniques can significantly reduce the influence of noise. The modifier vectors are filtered by using a first order low-pass digital filter (Ellis, 1981) :

$$\bar{\lambda}^k = \varepsilon\bar{\lambda}^{k-1} + (1 - \varepsilon_\lambda)\lambda^k \tag{7.2}$$

and

$$\bar{t}^k = \varepsilon\bar{t}^{k-1} + (1 - \varepsilon_t)t^k \tag{7.3}$$

where $(\bar{\lambda}^k, \bar{t}^k)$ and (λ^k, t^k) are filtered and unfiltered values of the modifier vectors, respectively, and $0 \leqslant \varepsilon < 1$. Furthermore, the effect of zero mean value noise can be reduced by employing a simple averaging technique, for instance take T measurements and apply the averaging formula

$$\bar{y}^* = \frac{1}{T}\sum_{i=1}^{T} y^*(i) \tag{7.4}$$

Both versions of the augmented based double loop algorithms were applied to Example 2 in the presence of noise. The NAG library subroutine G05DDF is used to generate the noise vector. Figs. 7.5 and 7.6 illustrate the behaviour of the constraint and variable augmented algorithms, respectively, when the measurements are contaminated with noise with a standard deviation of 0.02 and representing a noise to signal power ratio of 10%. The performance of both versions of augmented algorithms has deteriorated compared to those in the absence of noise (see Figs. 7.4 and 7.4). Figs. 7.5 and 7.6 show that a significant improvement is achieved by using simple digital filter techniques (7.2), (7.3) and (7.4), where T is taken as 50 measurements.

Algorithm	Technique	Basic version		Variable constraint augmented
		Variable augmented	Constraint augmented	
ISOPE1 (using input feedback)	Single loop	VA-SL1	CA-SL1	VCA-SL1
	System based double loop	VA-SBDL1	CA-SBDL1	VCA-SBDL1
	Model based double loop	VA-MBDL1	CA-MBDL1	VCA-MBDL1
ISOPE2 (using input & output feedback)	Single loop	VA-SL2	CA-SL2	VCA-SL2
	System based double loop	VA-SBDL2	CA-SBDL2	VCA-SBDL2
	Model based double loop	VA-MBDL2	CA-MBDL2	VCA-MBDL2

Table 7.1 : Different versions of the augmented ISOPE algorithms.

ϱ = 15	$\varepsilon_{v,w}$									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
15.0	111	111	111	111	111	111	111	111	46	95
13.5	111	111	111	111	111	111	111	111	44	96
12.0	100	100	111	111	111	111	111	48	40	98
10.5	100	100	100	100	100	100	100	46	32	99
9.0	100	100	100	100	100	100	100	48	38	100
7.5	100	100	100	100	79	50	47	50	51	100
6.0	81	73	66	58	53	49	49	51	61	100
4.5	73	71	66	60	55	58	56	59	70	100
3.0	74	74	73	72	70	67	66	66	84	91
1.5	100	100	100	100	100	100	100	100	100	100

ϱ = 30	$\varepsilon_{v,w}$									
ε_p =	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	1.0
30.0	111	111	111	111	111	111	111	111	67	100
27.0	111	111	111	111	111	111	111	45	68	100
24.0	111	100	111	111	111	111	32	47	70	100
21.0	100	100	100	100	100	19	34	48	71	100
18.0	100	100	38	20	20	26	36	50	74	100
15.0	100	40	21	23	24	29	38	52	76	100
12.0	100	23	27	28	29	32	40	54	79	100
9.0	30	31	31	31	31	31	43	57	83	100
6.0	49	49	49	49	49	49	49	61	89	100
3.0	100	100	100	100	100	100	100	100	100	100

ϱ = 60	$\varepsilon_{v,w}$									
ε_p	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
60.0	111	111	111	111	111	111	111	111	96	100
54.0	111	111	111	111	111	111	111	70	98	100
48.0	111	111	111	111	111	111	111	72	100	100
42.0	111	111	111	111	44	49	59	74	100	100
36.0	100	100	100	75	44	51	61	77	100	100
30.0	100	100	100	41	45	53	63	80	100	100
24.0	100	100	94	41	47	55	66	81	100	100
18.0	100	100	41	43	49	58	70	89	100	100
12.0	100	100	48	48	51	61	75	96	100	100
6.0	100	98	98	98	98	98	98	100	100	100

Table 7.2 : Sensitivity of the variable augmented single loop technique (VA-SL1) of AISOPE1 (Example 1).

r = 15	ε_v									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
ε_p =										
15.0	100	60	45	37	32	27	24	21	25	50
13.5	65	50	41	35	31	28	25	23	25	49
12.0	47	42	36	33	31	29	26	25	26	48
10.5	37	36	34	32	31	29	28	27	27	48
9.0	31	32	31	30	30	31	30	30	31	47
7.5	27	27	28	29	31	31	33	33	35	47
6.0	24	25	26	28	30	31	34	38	41	50
4.5	22	21	24	26	29	33	38	43	49	61
3.0	28	26	27	31	33	35	41	50	61	83
1.5	64	64	64	64	63	62	58	67	81	100
r = 30	ε_v									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
ε_p =										
30.0	100	36	32	23	18	17	16	18	21	48
27.0	43	30	27	21	19	18	17	18	21	48
24.0	33	27	22	19	20	19	19	19	21	47
21.0	27	23	20	20	19	18	19	20	23	47
18.0	21	18	19	19	20	19	21	21	27	46
15.0	17	17	16	18	19	21	22	22	30	45
12.0	13	14	17	16	17	19	21	25	32	44
9.0	18	18	18	19	20	22	25	29	36	48
6.0	26	27	27	28	29	29	30	31	43	60
3.0	48	48	47	47	48	51	54	58	62	86
r = 60	ε_v									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
ε_p										
60.0	48	32	28	24	30	100	100	14	100	50
54.0	38	26	20	18	22	20	100	100	100	88
48.0	29	22	17	17	17	19	16	12	100	45
42.0	19	17	14	15	15	14	18	18	20	45
36.0	16	11	15	14	15	15	19	22	20	44
30.0	10	11	12	13	14	17	20	16	19	44
24.0	9	11	14	15	17	19	22	17	23	44
18.0	16	15	15	16	15	15	24	20	26	44
12.0	25	25	25	24	24	23	25	27	32	48
6.0	54	54	54	53	53	52	51	48	48	67

Table 7.3 : Sensitivity of the constraint augmented single loop (VA-SL1) of AISOPE1 (Example 1).

ϱ	5.	6.	8.	10.	15.	20.	30.	40.	60.	80.	100.
ε_p		3.	4.	8.	10.5	8.	21.	16.	18.	16.	30.
ε		0.1	0.1	0.2	0.2	0.9	0.5	0.8	0.8	0.8	0.7
IS		97	61	48	32	27	19	27	41	48	59

Table 7.4 : Variable augmented single loop technique (VA-SL1) results of AISOPE1 (Example 1).

r	5.	6.	8.	10.	15.	20.	30.	40.	60.	80.	100.
ε_p	2.5	4.2	8.0	10.	4.5	6.	12.	16.	24.	40.	40.
ε	0.2	0.1	0.2	0.2	0.9	1.0	1.0	1.0	1.0	0.9	0.7
IS	88	56	34	26	21	16	13	12	9	9	10
IT	312	192	126	103	87	73	67	65	53	63	69

Table 7.5 : Constraint augmented single loop technique (CA-SL1) results of AISOPE1 (Example 1).

r	20.						60.					
ϱ	1.	2.	3.	5.	10.	20.	1.	2.	3.	5.	10.	20.
ε_p	10.	12.	14.	16.	16.	20.	36.	42.	54.	54.	42.	16.
ε	1.	1.	1.	1.	0.9	0.7	0.8	1.	1.	0.9	0.8	0.9
IS	17	16	18	19	21	24	10	11	11	12	14	21
IT	77	73	82	87	80	86	71	63	66	84	69	80

Table 7.6 : Variable and constraint augmented single loop technique (VCA-SL1) results of ISOPE1 (Example 1).

ϱ	5.	6.	8.	10.	15.	20.	30.	40.	60.	80.	100.
ε_p	0.5	2.4	4.8	7.	12.	16.	24.	28.	42.	64.	70.
ε	0.9	0.8	0.8	0.8	0.8	0.7	0.3	0.4	0.4	0.5	0.5
IS	8	8	9	10	12	14	13	9	11	14	9
IT	352	231	127	96	88	48	60	49	51	45	40

Table 7.7 : Variable augmented model based double loop technique (VA-MBDL1) results of AISOPE1 algorithm (Example 1).

r	5.	6.	8.	10.	15.	20.	30.	40.	60.	80.	100.
ε_p	3	4.2	7.2	10.	15.	20.	24.	24.	24.	32.	30.
ε	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
IS	6	7	7	7	7	7	7	7	7	7	6
IT	259	165	134	123	102	109	109	110	128	118	138

Table 7.8 : Constraint augmented model based double loop technique (CA-MBDL1) results of AISOPE1 for Example 1 (r=20).

ϱ	1.	2.	3.	5.	10.
ε_p	20.	20.	20.	20.	20.
ε	0.7	0.7	0.7	0.8	0.8
IS	7	7	8	8	10
IT	100	104	105	133	146

Table 7.9 : Variable and constraint augmented model based double loop technique (VCA-MBDL1) results of AISOPE1 for Example 1 (r=20).

ϱ	ε	ε_p	IS	IT	final Q_*
< 3	for all algorithm to converge the step coefficient ε_p has to be very small				
4.0	1.0	0.7	not convergent		
		0.6	20	20	6.3261
		0.4	31	31	6.3260
	0.8	0.6	22	22	6.3264
6.0	1.0	0.7	not convergent		
		0.6	24	24	6.3260
		0.4	37	37	6.3258
	0.8	0.6	24	24	6.3258
8.0	1.0	0.7	not convergent		
		0.6	28	28	6.3259
		0.4	44	44	6.3259
	0.8	0.6	29	29	6.3259
10.0	1.0	0.7	not convergent		
		0.6	33	33	6.3260
	0.8	0.6	33	33	6.3259

Table 7.10 : Variable augmented single loop (VA-SL1) technique results for Example 2.

r	ε	ε_p	IS	IT	final Q_*
0.25	1.0	0.6	21	83	6.3266
		0.4	23	78	6.3263
0.5	1.0	1.0	21	92	6.3269
		0.75	13	48	6.3265
	0.8	0.75	13	48	6.3266
1.0	1.0	1.5	9	44	6.3280
		1.2	10	45	6.3274
		1.0	12	50	6.3273
	0.8	1.5	9	44	6.3280
2.0	1.0	3.0	12	62	6.3297
		2.5	7	36	6.3309
		2.0	8	37	6.3302
	0.8	2.5	7	37	6.3303
3.0	1.0	4.0	9	48	6.3351
		3.5	6	34	6.3343
		3.0	6	32	6.3341
	0.8	3.0	7	41	6.3341
4.0	1.0	5.0	8	43	6.3392
		4.0	6	31	6.3397
6.0	1.0	7.0	7	43	6.3538
		6.0	6	35	6.3538
	0.8	6.0	6	35	6.3538

Table 7.11 : Constraint augmented single loop technique (CA-SL1) results for Example 2.

ρ	ε	ε_p	IS	IT	final Q_*
≤ 4.0		for algorithm to converge the step coefficient ε_p has to be very small			
6.0	1.0	0.7	not convergent		
		0.6	12	123	6.3267
		0.4	12	161	6.3263
		0.2	12	345	6.3264
	0.8	0.6	16	141	6.3263
8.0	1.0	0.7	not convergent		
		0.6	15	164	6.3266
		0.4	14	246	6.3259
		0.2	14	488	6.3258
10.0	1.0	0.7	not convergent		
		0.6	17	185	6.3269
		0.4	16	278	6.3259
		0.2	16	552	6.3259
	0.8	0.6	20	204	6.3264
3.0	1.0	4.0	9	48	6.3351

Table 7.12 : Variable augmented model based double loop technique (VA-MBDL1) results for Example 2.

r	ε	ε_p	dynamic $\delta_a = \delta_a^k$			$\delta_a = 10^{-4}$	
			IS	IT	final Q_*	IS	IT
0.25	1.0	0.5	5	56	6.3284	5	141
		0.25	5	102	6.3282	5	263
	0.8	0.5	6	55	6.3269	6	141
0.5	1.0	0.5	5	59	6.3282	5	172
	0.8	0.5	6	59	6.3271	6	169
1.0	1.0	1.0	5	38	6.3294	5	120
	0.8	1.0	6	38	6.3276		
2.0	1.0	2.0	5	33	6.3323	5	95
	0.8	2.0	6	34	6.3306	6	93
4.0	1.0	4.0	6	33	6.3397	5	80
	0.8	4.0	6	34	6.3306	6	78
6.0	1.0	6.0	6	37	6.3530	5	72
	0.8	6.0	6	34	6.3540	6	71

Table 7.13 : Constraint augmented model based double loop technique (CA-MBDL1) results for Example 2.

			dynamic $\delta_i = \delta_i^n$		
ϱ	ε	ε_ρ	IS	IT	final Q_*
4.0	1.0	0.6	20	23	6.3261
		0.4	31	34	6.3260
	0.8	0.6	19	23	6.3257
8.0	1.0	0.6	32	32	6.3260
		0.5	38	38	6.3259
	0.8	0.6	33	33	6.3258
10.0	1.0	0.6	36	36	6.3258
		0.5	43	43	6.3258
	0.8	0.6	37	37	6.3258

Table 7.14 : Variable augmented system based double loop technique (VA-SBDL1) results for Example 2.

r	ε	ε_p	IS	IT	final Q_*
0.25	1.0	0.25	39	144	6.3263
0.5	1.0	1.0	20	75	6.3267
		0.5	22	75	6.3267
	0.8	1.0	20	87	6.3267
2.0	1.0	3.0	12	66	6.3296
		2.0	8	38	6.3301
		1.5	12	51	6.3301
	0.8	2.0	8	36	6.3301
4.0	1.0	5.0	7	43	6.3409
		4.0	6	33	6.3398
		3.0	9	43	6.3396
	0.8	4.0	6	32	6.3397
6.0	1.0	8.0	11	66	6.3544
		6.0	5	29	6.3537
		4.0	9	47	6.3531
	0.8	6.0	5	31	6.3537

Table 7.15 : Constraint augmented system based double loop technique (CA-SBDL1) results for Example 2.

ρ	6.	6.5	7.	8.	10.	15.	20.	30.
ε_p		4.55	6.3	7.2	9.	10.5	12.	15
ε		0.1	0.1	0.1	0.1	0.1	0.1	0.1
IS		133	114	115	120	140	>150	>150

Table 7.16 : Variable augmented single loop technique (VA-SL2) results of AISOPE2 for Example 1.

ρ	5.	6.	8.	10.	15.	20.	30.	40.
ε_p	1.5	2.4	4.	4.	4.5	6.	6	8
ε	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2
IS	26	26	27	33	62	72	920	>100
IT	565	343	176	284	353	418	703	>745

Table 7.17 : Variable augmented model based double loop technique (VA-MBDL2) results of AISOPE2 for Example 1.

r	5.	6.	8.	10.	15.	20.	30.	40.
ε_p			4.8	5.	7.5	8.	9.	7.
ε			0.1	0.1	0.1	0.3	0.2	0.1
IS			90	97	95	75	80	>100
IT			623	524	576	2757	2357	>1323

Table 7.18 : Constraint augmented model based double loop technique (CA-MBDL2) results of AISOPE2 for Example 2.

ϱ	ε	ε_p	IS	IT	final Q_*
<0.5	convergent if ε_p is taken very small				
1.0	1.0	1.0	91	91	6.3266
		0.8	88	88	6.3264
	0.8	1.0	< 130		
2.0	1.0	1.0	45	45	6.3263
3.0	1.0	1.0	57	57	6.3265
5.0	1.0	1.0	slow convergent		

Table 7.19 : Variable augmented single loop (VA-SL2) technique results of AISOPE2 for Example 2.

ϱ	ε	ε_p	dynamic $\delta_i = \delta_i^n$		final Q_*
			IS	IT	
<0.5	convergent if ε_p is taken very small				
1.0	1.0	0.9	55	55	6.3264
2.0	1.0	0.9	50	50	6.3264
3.0	1.0	0.9	65	65	6.3265

Table 7.20 : Variable augmented system based double loop (VA-SBDL2) technique results of AISOPE2 for Example 2.

Algorithm	Example 1		Example 2	
	IS	IT	IS	IT
VA-SL1	19	19	20	20
CA-SL1	9	53	6	32
VA-DLM1	8	231	12	123
	9	49		
CA-DLM1	6	259	5	120*
	9	102*	5	38**

Table 7.21 : Comparison of best results of AISOPE1 algorithms.

* - fixed accuracy for the approximation loop

** - dynamic accuracy for the approximation loop

Algorithm	Example 1		Example 2	
	IS	IT	IS	IT
VA-SL2	114	114	45	45
CA-SL2	-	-	-	-
VA-DLM2	26	26	-	-
CA-DLM2	-	-	-	-

Table 7.22 : Comparison of best results of AISOPE2 algorithms.

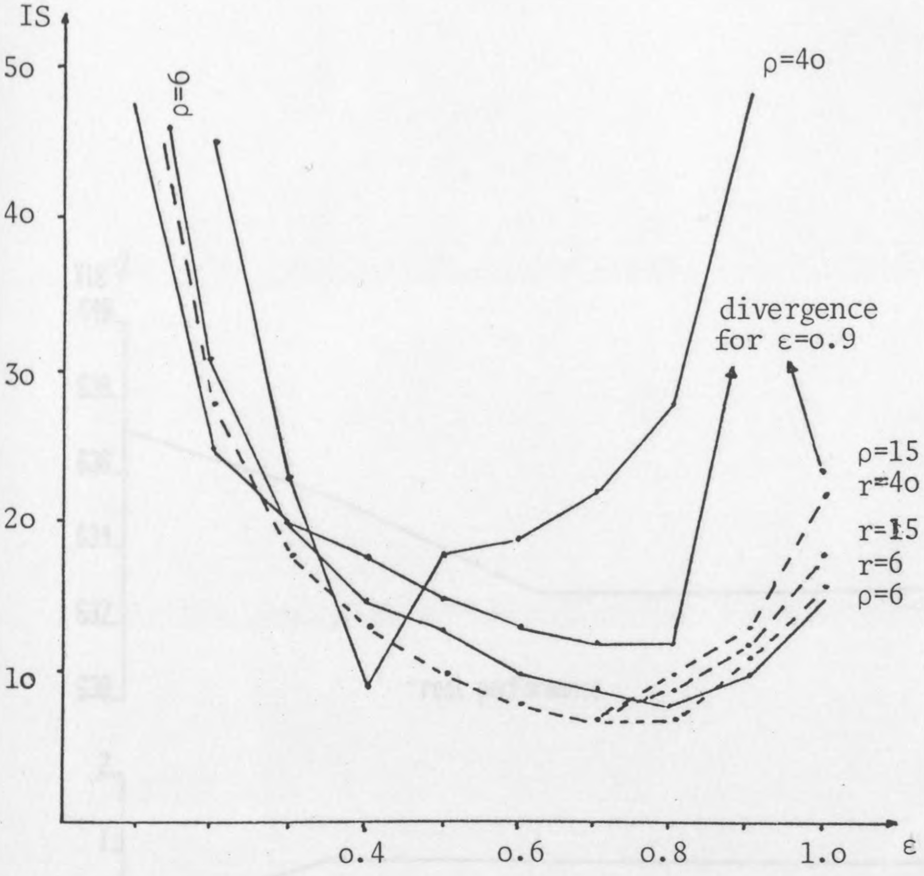

Figure 7.1 Sensitivity to ε of model based double loop (MBDL1) techniques (Example 1).

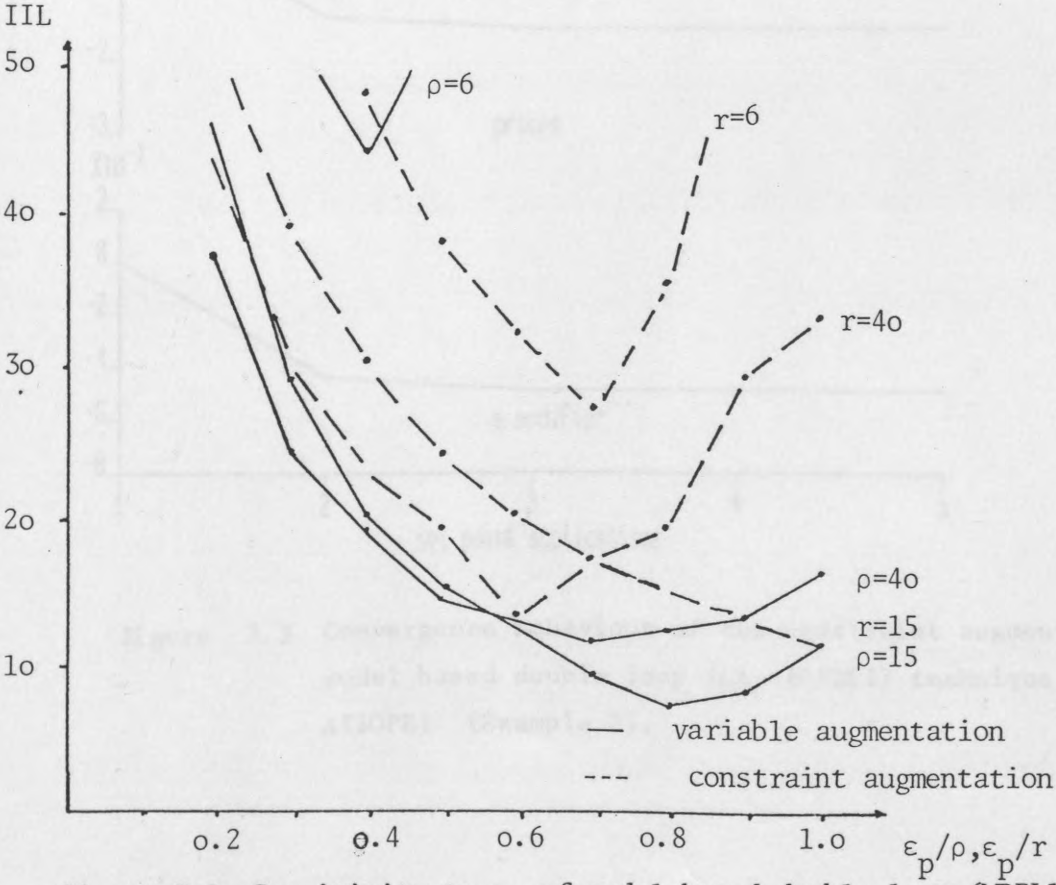

Figure 7.2 Sensitivity to ε_p of model based double loop (MBDL1) technique (Example 1).

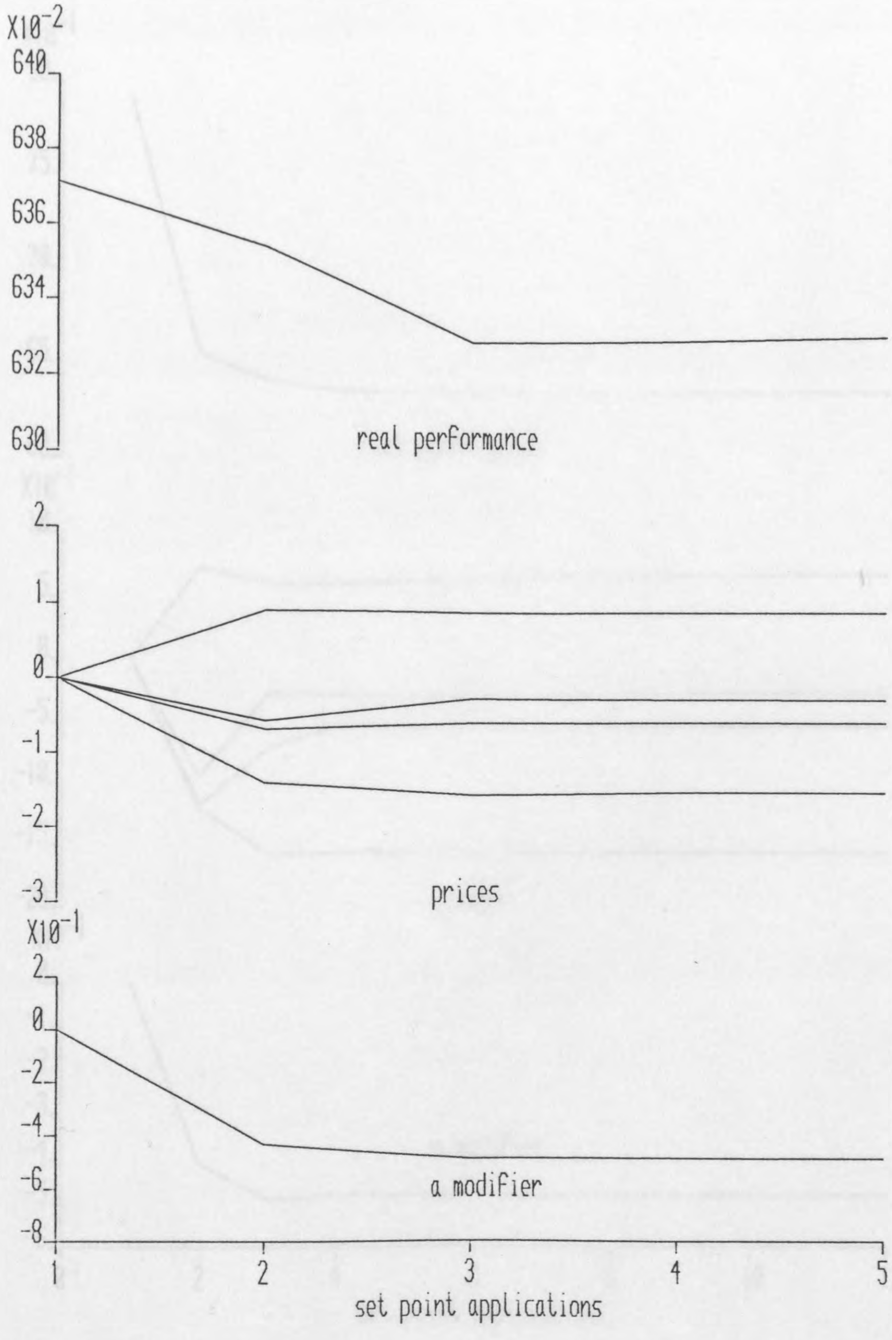

Figure 7.3 Convergence behaviour of the constraint augmented model based double loop (CA - M BDL1) technique of AISOPE1 (Example 2).

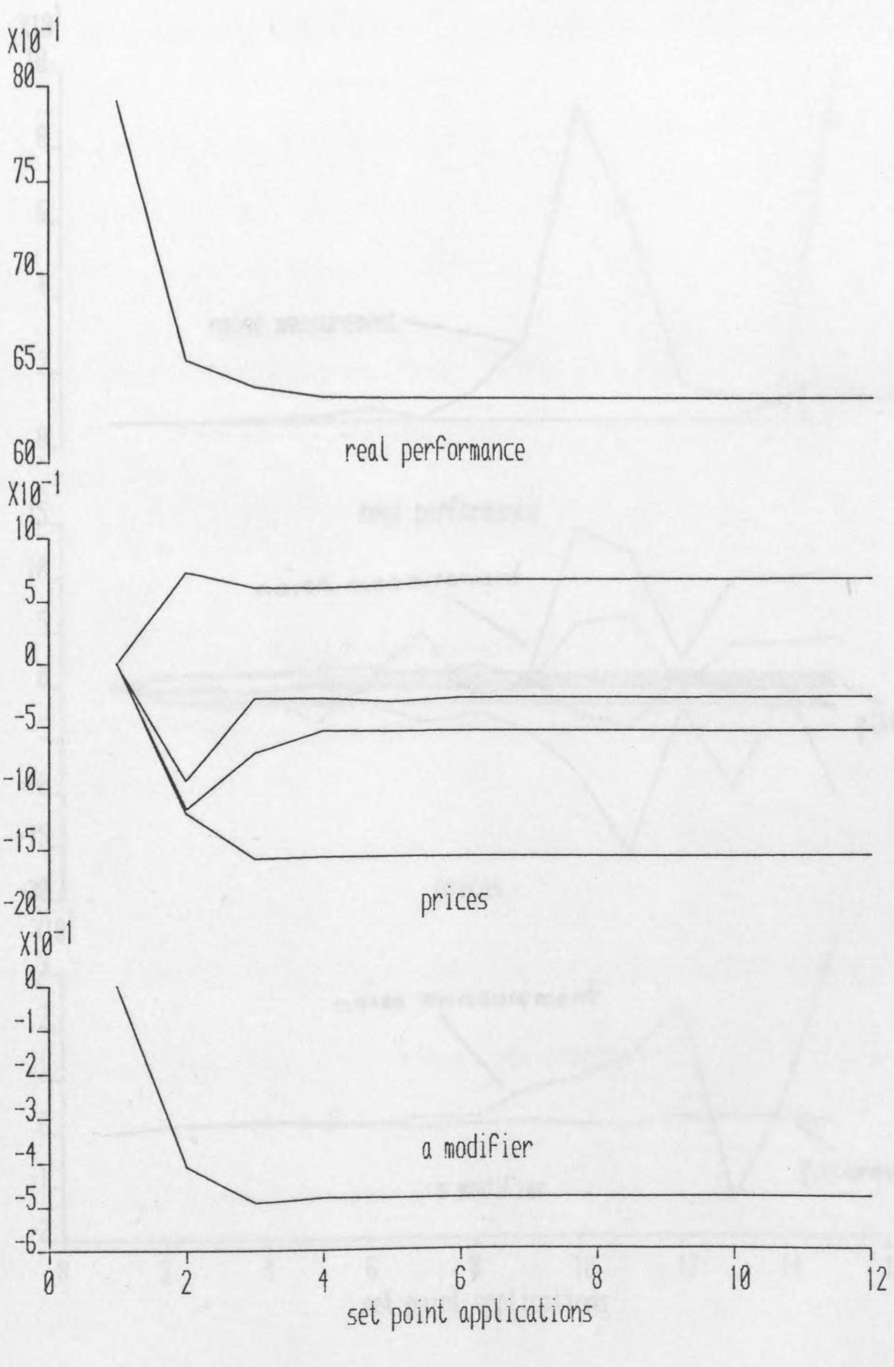

Figure 7.4 Convergence behaviour of the variable augmented model based double loop (VA - M BDL2) technique of AISOPE2 (Example 2).

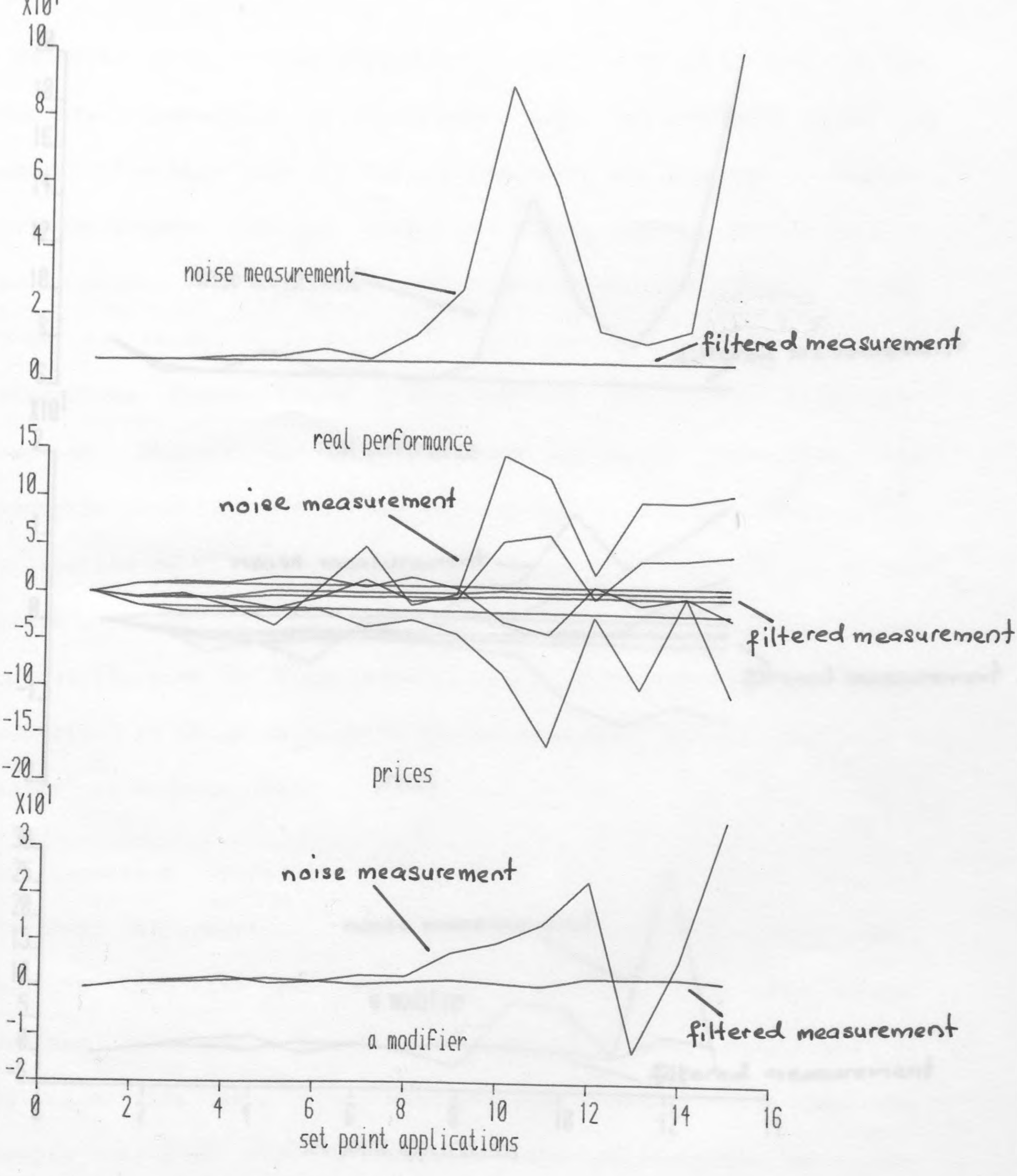

Figure 7.5 Behaviour of the constraint augmented model based double loop (CA-MBDL1)technique of AISOPE1 in the presence of noise (Example 2).

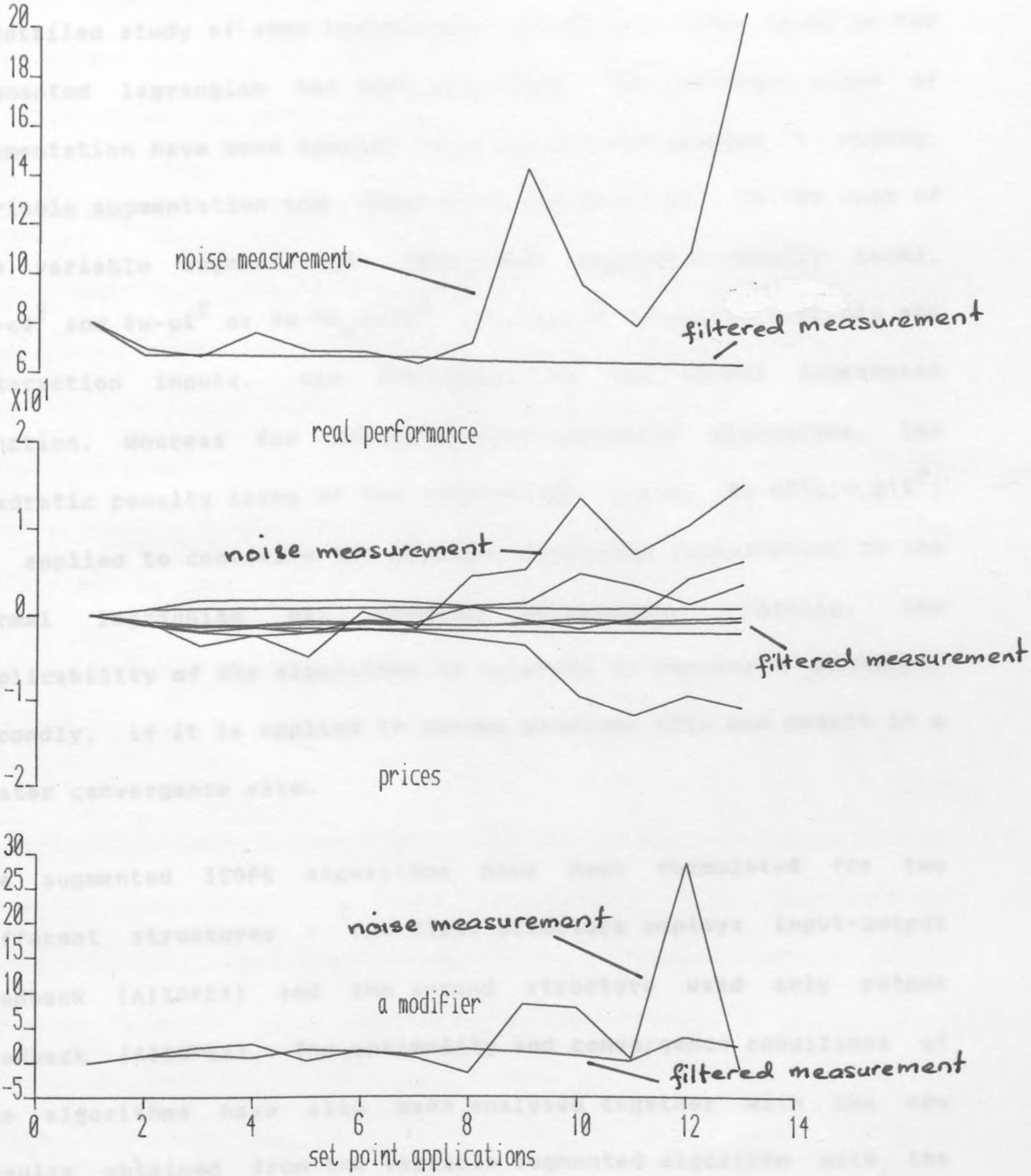

Figure 7.6 Behaviour of the variable augmented model based double loop (VA-MBDL2) technique of AISOPE2 in the presence of noise (Example 2).

CHAPTER 8 - CONCLUSIONS

A detailed study of some hierarchical ISOPE algorithms based on the augmented lagrangian has been presented. Two different kinds of augmentation have been applied to convexify the problem : namely, variable augmentation and constraint augmentation. In the case of the variable augmentation, additional quadratic penalty terms, $\|v-c\|^2$ and $\|w-u\|^2$ or $\|u-HK_*(v)\|^2$, involving subsystem controls and interaction inputs, are introduced to the normal lagrangian function. Whereas for the constraint augmented algorithms, the quadratic penalty terms of the interaction input, $\|u-HF(c,u,\alpha)\|^2$, is applied to convexify the problem. Employing augmentation to the normal lagrangian has several advantages. Firstly, the applicability of the algorithms is extended to nonconvex problems. Secondly, if it is applied to convex problems this can result in a faster convergence rate.

The augmented ISOPE algorithms have been formulated for two different structures : the first structure employs input-output feedback (AISOPE1) and the second structure used only output feedback (AISOPE2). The optimality and convergence conditions of the algorithms have also been analysed together with the new results obtained from the variable augmented algorithm with the output feedback.

Based on the simulation examples, a comparative study of the AISOPE algorithm has been presented. These results have shown that the model based double loop technique was the most efficient algorithm in reducing the number set-point changes compared to the single-loop and the system based double loop techniques. The system based double loop technique is as good as the single-loop technique, if the dynamic accuracy is employed in its inner loop. The simulation results have also shown that the AISOPE1 were more efficient than the AISOPE2. Among the AISOPE1 algorithms the constraint augmented model based double loop technique was the best with respect to all the considered comparison criteria and also giving a better convergence rate. The constraint augmented single loop of AISOPE1 and the variable augmented model based double loop of AISOPE2 showed similar performance. The constraint augmented algorithms are less sensitive to the values of their coefficients which have to be prescribed by the user.

The requirement for the real output derivatives with respect to the set-points is an important practical drawback of using AISOPE algorithms. However, within a tolerable noise level the algorithms will work provided the modifer vector is filtered. This can be done by using simple filter technique and averaging output measurements which will also significantly improve the performance of the AISOPE algorithms under noisy measurement conditions.

Recently, Lin, Hendawy and Roberts (1987a, and 1987b) have developed another version of ISOPE which is based on the normal lagrangian with output dependent constraints. The applicability of these new versions of ISOPE algorithms can be extended by introducing augmentation. These can be done by employing a similar technique of augmentation as used in this thesis.

All the algorithms derived in the thesis use simple relaxation formulae. However, it is considered that these formulae may not be utilised in the most efficient way compared to the information contained in the subsequently updated model results of their optimisation. An improved version of the modified two-step algorithm for a single process has been successfully implemented where the set-points and lagrangian multipliers are updated using a Newton-like formula (Tatjewski and Roberts, 1987). Currently, research has been carried out in the control Engineering centre of The City University to apply simple Newton-like formulae to the AISOPE Algorithms developed in this thesis.

Further work is required to extend the application of the AISOPE algorithms to on-line steady-state and dynamic processes. Other versions of ISOPE algorithms which are based on a normal lagrangian, have been succesfully applied to the on-line control of a vaporiser (Bakali, 1986), hence applying AISOPE based on the augmented lagrangian might improve the efficiency of its control system. The modified two-step algorithm for a single process has also been successfully applied to the on-line control of a pilot scale travelling load furnace, which is a batch dynamic plant (Stevensen, 1985). Similarly, after decomposing the travelling load furnace into several subsystems, the AISOPE could be used to control the furnace.

REFERENCES

[1] Abdullah N., M. Brdyś and P.D Roberts (1986). Extended hierarchical augmented lagrangian adaptive technique for optimising control of large scale systems. Research Memorandum, CEC/NA-MB-PDR/48, Control Engineering Centre, City University, London.

[2] Bakalis, P.S., (1986). On-line Hierarchical Control. Ph.D. Thesis, Control Engineering Centre, City University, London.

[3] Bensoussan, A., Lions, J.L. and R. Teman (1972). Sur les Methodes de decomposition, de Decentralisation, et de Coordination et Aplications. Methodes Numeriques d'Analyse de systems, Vol. 2, Cahier IRIA, No. 11.

[4] Bertsekas P.D (1982). Constrained Optimisation and Lagrange Multiplier Methods. Academic Press, New York.

[5] Brdyś, M. (1983). Hierarchical optimising control of steady-state scale systems under model-reality difference of mixed type - a mutually interacting approach. Proc.3rd. IFAC Symp. on Large Scale Theory and Applications. Warsaw, pp 49-57.

[6] Brdyś, M., S. Chen, and P.D Roberts (1986). An extension to the modified two-step algorithm for steady state system optimisation and parameter estimation. Int. J Systems Science. Vol. 17, No. 8, pp 1299 - 1243.

[7] Brdyś, M., W. Findeisen, and P. Tatjewski (1980). Hierarchical

control for systems operating in steady-state. Large Scale Systems. Vol. 1, pp 193 - 213.

[8] Brdyś M., and P.D Roberts (1986). Optimal structures for steady-state adaptive optimising control of large scale industrial processes. Int. J. System Science, Vol. 17, No.10, pp 1449 - 1479.

[9] Brdyś M., and P.D.Roberts (1987). Convergence and optimality of modified two-step algorithm for integrated system optimisation and parameter estimation. Int. J. Systems Science, Vol. 18, No. 7, pp 1305 - 1322.

[10] Brdyś M., and P.D Roberts (1985). Hierarchical single and double iterative techniques for optimising control of large scale industrial processes ; derivation, applicability and convergence. IEE Conference, Control 85. Publication No. 252, pp 89 - 94.

[11] Brdyś M., P.D Roberts, M.M Badi, I.C Kokkinos and N. Abdullah (1987). Model based double iterative strategy for integrated system optimisation and parameter estimation of large scale industrial processes. Research Memorandum, CEC/MB-PDR-MMB-ICK-NA/62, Control Engineering Centre, City University, London.

[12] Brdyś M., N. Abdullah and P.D Roberts (1986). Hierarchical adaptive technique for optimising control of large scale steady state systems : Optimality, iterative strategies and their convergence. Research Memorandum, CEC/MB-NA-PDR/13. Control Engineering Centre, City University, London.

[13] Brdyś M., N. Abdullah and P.D Roberts (1987). An augmented

model based double loop iterative technique for integrated system optimisation and parameter estimation of large scale industrial processes. Proc.at 10th. IFAC World Congress on Automation Control. Munich, F R Germany.

[14] Brdyś M., J.E Ellis and P.D Roberts (1987). Augmented integrated system optimisation and parameter estimation technique : derivation, optimality and convergence. IEE Proceeding, Vol. 134, pt. D, No. 3, pp 201 - 209.

[15] Chen S., Brdys M., and P.D. Roberts (1986). An integrated system optimisation and parameter estimation technique for hierarchical control of steady-state systems. Int. J. Systems Sci., Vol. 17, No. 8, pp 1209 - 1228.

[16] Chen S., P.D. Roberts, and M. Brdys (1984). Comparison of several on-line integrated system optimisation and parameter estimation methods for steady-state systems. Research Memorandum. CEC/SC-PDR-MB/15, Control Engineering Centre, City University, London.

[17] Cohen G. (1980). Auxiliary problem principle and decomposition of optimisation problems. JOTA Vol. 32, No. 3, pp 277 - 305.

[18] Durbeck R.C (1965). Principles for Simplication of optimising Control Models. Ph.D. Thesis, Case Western Reserve University, Cleveland, Ohio, U.S.A.

[19] Ellis J.E., and P.D Roberts (1981). Simple models for

integrated optimision and parameter estimation. Int. J. Systems Science. Vol. 12, No. 4, pp 455 - 472.

[20] Ellis J.E., and P.D Roberts (1982). Measurement and modelling trade-off for integrated system optimization and parameter estimation. Large Scale Systems, Vol. 3, pp 191 - 204.

[21] Ellis J.E., and P.D Roberts (1985). On the practical viability of integrated system optimisation and parameter estimation. IEE International Conference 'Control 85', University Cambridge, England, Publication No. 252, pp 281 - 285.

[22] Findeisen W., F.N Bailey, M. Brdys, K. Malinowski, P. Tatjewski, and A. Wozniak (1980). Control and Coordination in Hierarchical Systems. Wiley, London.

[23] Foord, A.G. (1974). On-line Optimisation of a Petrochemical Complex. Ph.D. Thesis, University of Cambridge, England.

[24] Haimes, Y.Y., and D.D. Wismer (1972). A computational approach to the combined problem of optimisation and parameter identification. Automatica. Vol. 8, pp 337 - 347.

[25] Hogan, W.W (1973). Point-to-point maps in mathematical programming. SIAM Review. Vol. 15, No. 3, pp 591 - 603.

[26] Kantorovich L.V., and G.P. Akilov (1982). Functional Analysis. (second edition) Pergamon Press, Oxford.

[27] Lefkowitz, I (1966). Multilevel approach applied to control

system design. Trans. ASME, Series D. J. of Basic Eng., Vol. 88, No. 2, pp 392 - 398.

[28] Lin J., S Chen and P.D Roberts (1987). A modified algorithm for steady state integrated system optimisation and parameter estimation. Research Memorandum, CEC/JL-SC-PDR/47. Control Engineering Centre, City University, London.

[29] Lin J., Z.M Hendawy and P.D Roberts (1987). An extension of integrated system optimisation and parameter estimation to hierarchical control of steady state system with output dependent constraints, accepted for publication by Int. J. Control.

[30] Lin J., Z.M Hendawy and P.D Roberts (1987). A new model based double loop iterative strategy for integrated system optimisation and parameter estimation of large scale industrial processes. Research Memorandum, CEC/JL-ZHM-PDR/63. Control Engineering Centre. City University, London.

[31] Luenberger D.G (1973). Introduction to Linear and Non-linear Programming. Addison - Wesley, Reading, Massachusetts.

[32] Luenberger D.G (1984). Linear and Non-linear Programming (second edition). Addison - Wesley, Reading, Massachussetts.

[33] Michalska H., J.E Ellis, and P.D Roberts (1985). Joint coordination method for the steady - state control of large scale systems. Int. J. Systems Sci., Vol. 16, No. 5, pp 605 - 618.

[34] Mesarovic, M.D., D. Macko, and Y. Takahara (1970). Theory of Hierarchical, Multilevel Systems. Academic Press, New York.

[35] Ortega, J.M. and W.C. Rheinboldt(1970). Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York.

[36] Roberts P.D (1977a). Multilevel approaches to the combined problem of system optimisation and parameter identification. Int. J. System Sci., Vol. 8, No. 3, pp 273 - 299.

[37] Roberts P.D (1977b). Integrated system optimisation and parameter estimation of a chemical reactor using a multilevel approach. Preprints of the 5th IFAC/IFIP Conference on Digital computer Applications to Process Control, Hague, Netherlands, pp 551 - 557.

[38] Roberts P.D (1979). An algorithm for steady - state system optimisation and parameter estimation. Int. J. Systems Sci., Vol. 10, No. 7, pp 719 - 734.

[39] Roberts P.D., and J.E Ellis (1981). Refinements to an algorithm for combined system optimisation and parameter estimation. Proc. of the 1981 UKSC Conference on Computer Simulation. Harrogate, England.

[40] Roberts P.D., and T.W.C William (1981). On an algorithm for combined system optimisation and parameter estimation. Automatica, Vol. 17, pp 199 - 209.

[41] Shao F.Q., and P.D Roberts (1983). A price correction mechanism with global feedback for hierarchical control of steady - state systems. Large Scale Systems, Vol. 4, pp 67 - 80.

[42] Stevenson I.A., M Brdyś and P.D Roberts (1985). Integrated system optimisation and parameter estimation for travelling load furnace control. In H.A Barker and P.c Young (Eds) Identification and System Parameter Estimation 1985. Proc. 7th IFAC Symposium, Pergamon Press, Oxford - New York.

[43] Tatjewski P. and P.D Roberts (1987). Integrated optimisation and parameter estimation algorithms for interconnected systems with global resource - type constraints. Research Memorandum. CEC/PT-PDR/60. Control Engineering Center. City University, London.

[44] Tatjewski P. (1984). A hierarchical algorithm for large scale system optimisation problems with duality gaps. In Thoft - Christansen (Ed). Modelling and Optimisation, Proc. 11th Conf. Springer, Berlin, pp 662 - 671.

[45] Tatjewski P. (1985). On-line hierarchical control of steady-state systems using the augmented interaction balance method with feedback. Large Scale System. Vol. 8 pp 1 - 18.

[46] Tatjewski P. (1986). New dual decomposition algorithms for non-convex separable optimisation problems. Prepr. 4th IFAC Symp. Large Scale Systems Theory and Applications. Zurich, Switzerland, pp 296 - 303.

[47] Tatjewski P., P.D Roberts (1986). Newton-type integrated system optimisation and parameter estimation algorithms for hierarchical steady-state control of large - scale processes : Part 1 - the single loop algorithm and part 2 - The double loop algorithm and simulation results. Research memorandum

CEC/PT-PDR/59. Control Engineering Centre. City University, London.

[48] Tatjewski, P., Abdullah N., and P.D. Roberts (1986). Comparative study and development of integrated optimisation and parameter estimation algorithm for hierarchical steady-state control. Research Memorandum, CEC/PT-NA-PDR/49. Control Engineering Centre, City University, London.

[49] Tatjewski, P., Abdullah N., and P.D Roberts (1988). Comparison of some algorithms for hierarchical steady-state optimising control of interconnected industrial processes. This paper is accepted for presentation in the International Conference 'Control 88'. University of Oxford (13 - 15 April, 1988).

[50] Wu, N.Q., F.Q. Shao and R.H. Li (1986). A jointed method of estimation and optimisation for large scale systems. The 4th IFAC/IFORS Symposium of Large Scale Systems : Theory and Application, Zurich, July.

[51] Youle, P.V., and L.A. Duncanson (1970). On-line control of olefine plant. Chemical and Process Engineering, Vol. 51, pp 49-52.

[52] Zangwill W.I (1969). Nonlinear Programming a Unified Approach. Prentice - Hall Inc. Englewood Cliffs. N.J.

[53] NAG FORTRAN Library Manual Mark 10, 1983.

APPENDIX A : MATHEMATICAL PRELIMINARIES FOR CHAPTER 5

Definition A.1 : Point-to-set map

A point-to-set map F from a set X into a set Y is a map which associates a subset Y with each point of X.

Definition A.2 : Algorithm

An algorithm is an iterative process consisting of sequence of point-to-set maps $\varphi_k : X \longrightarrow 2^X$ and the iteration of the algorithm generates a sequence $\{ x^k \}$ such that $x^{k+1} \in \varphi_k(x^k)$. The mappings φ_k are called the algorithmic mappings.

Definition A.3 : Open mapping

F is open at a point $\bar{x}$ in X if $\{ x^k \} \subset X$, $x^k \longrightarrow \bar{x}$, and $\bar{y} \in F(\bar{x})$ imply the existence of an integer m and a sequence $\{ y^k \} \subset Y$ such that $y^k \in F(x^k)$ for $k \geq m$ and $y^k \longrightarrow \bar{y}$.

Definition A.4 : Closed mapping

F is closed at a point $\bar{x}$ in X if $\{ x^k \} \subset X$, $x^k \longrightarrow \bar{x}$, $y^k \in F(x^k)$, and $y^k \longrightarrow \bar{y}$ imply that $\bar{y} \in F(\bar{x})$.

Definition A.5 : Continuous mapping

F is continuous at point $\bar{x}$ in X if it is both open and closed at $\bar{x}$.

Defination A.6 : Composite Mapping

Let F : X ---> Y and G : Y ---> Z be point-to-set mappings. The composite mapping H = GF is defined as the point-to-set mapping H : X ---> Z with

$$H(x) \overset{\Delta}{=} \bigcup_{y \in F(x)} G(y)$$

Definition A.7 : Compactness

A set X is compact if any sequence (or subsequence) contains a convergence subsequent. Given a subsequence $\{ z^k \}_K$ in X , where X is compact, there exists a $K' \subset K$ such that

$$z^k \longrightarrow z^\infty \qquad k \in K'$$

and z^∞ is in X.

<u>Remark</u> : In Euclidean space compact sets correspond to closed and bounded sets.

Definition A.8 : Monotone

A mapping $F : D \subseteq X \longrightarrow Y$ is monotone on $D_o \subset D$ if

$$[F(x) - F(y)]^T (x-y) \geq 0, \quad \forall \; x,y \in D_o \qquad (A.1)$$

F is strictly monotone on D_o if strict inequality holds in (A.1) whenever $x \neq y$ and uniformly monotone if there is a $r > 0$ such that

$$[F(x) - F(y)]^T (x-y) \geq r(x-y)^T (x-y), \quad \forall \; x,y \in D_o$$

Definition A.9 Upper semicontinuous

A real-valued function $f : X \longrightarrow R$ is upper semicontinous at $x \in X$ if $x^k \longrightarrow x$, $k \longrightarrow \infty$ implies $\overline{\lim_{k\to\infty}} f(x^k) \leq f(x)$, where

$$\overline{\lim_{k\to\infty}} f(x^k) \overset{\Delta}{=} \lim_{k\to\infty} \sup f(x^k)$$

Definition A.10 : Point-to-point mapping

The mapping F is a point-to-point mapping if for each $x \in X$ the set $F(x)$ consists of a single point in Y.

<u>Remark</u> : If $x_k \longrightarrow x$ then $F(x_k) \longrightarrow F(x)$, and it follows that F is closed at x. Thus for point-to-point mappings continuity implies closedness.

The following two theorems are due to Hogan (1973):

Let us define a real valued function $f : W \times Y \longrightarrow R$ and a point-to-point mapping $H : W \longrightarrow Y$. Let

$$d(w) \overset{\Delta}{=} \inf \{f(w x y) : y \in H(w)\} \text{ and } \Omega(w) \overset{\Delta}{=} \{ y \in H(w) : C(w) = f(w,y) \}$$

The function C is termed the infimal value function and the point-to-set map Ω describes its solution set.

Theorem A.1

If H is open at w and f is upper semicontinuous on $w x H(w)$, then d is upper semicontinuous.

Theorem A.2

Suppose H is continuous on w, w is connected, Y is compact, f is continuous on W×Y, and Ω is single-valued at w. Then Ω is continuous at $\bar{x}$.

Theorem A.3 (Zangwill 1969, Luenberger 1984)

Let C : W ---> X and B : X ---> Y be point-to-point mappings. Suppose C is closed at w and B is closed at C(w). Then the composition mapping A = BC is closed at w if any of the following three conditions holds :

a) If w^k ---> w and y^k ε C(w^k), then there is a y such that, for some subsequence { y^{ki} }, y^{ki} ---> y.

b) Y is compact.

c) C is a point-to-point mapping continuous at x.

Theorem A.4 (Zangwill, 1969)

Let C : W ---> X and B : X ---> Y be point-to-set mappings closed at w ε W. Then the sum map A = C + B : W ---> Y is closed at w if any one of the following three conditions holds :

a) Either B or C is a point-to-point mapping continuous at x.

b) If w^k ---> w and y^k ε C(w^k), then there is a y such that, for some subsequence { y^{ki} }, y^{ki} ---> y.

c) Y is compact.

Theorem A.5 (Luenberger 1984)

Let the real-valued function f be a convex function defined on the convex set C and assume that f is continuously differentiable. If x^* is the global minimum point of f over C, then for all $y \epsilon C$

$$\Delta f(x^*)(y - x^*) \geqslant 0$$

where $\Delta f(x)$ denotes $\frac{\delta f(x)}{\delta x}$.

Zangwill's convergence theorem (Zangwill 1969, Luenberger 1984)

Let the point-to-set mapping A : X ---> X, and suppose that, given x_0 the sequence $\{ x_k \}_{k=0}^{\infty}$ is generated satisfying

$$x_{k+1} \;\epsilon\; A(x_k)$$

Let a solution set $\Omega \subset X$ be given, and suppose

i) All point x_k are contained in a compact set $s \;\epsilon\; X$.

ii) There is a continuous function Z on X such that

a) if $x \;\epsilon\; \Omega$, then $Z(y) < Z(x)$ for all $y \;\epsilon\; A(x)$

b) if $x \;\epsilon\; \Omega$, then $Z(y) \leqslant Z(x)$ for all $y \;\epsilon\; A(x)$

iii) The mapping A is closed at points outside Ω . Then either the algorithm stops at a solution or the limit of any convergent subsequence of $\{ x_k \}$ is a solution.

Appendix B - Covergence Proof of Lemma 5.3 (Chapter 5)

It follows from the Weierstrass theorem that there exists a point $\tilde{z} = (\tilde{v},\tilde{w})$ minimising $L_{*r}(.,p)$ on CU. Then (recall $x = (c,u)$).

$$L^{'}_{*r(x)}(\tilde{z},p)(x-\tilde{z}) \geqslant 0 \text{ for all } x \in CU \qquad (B.1)$$

using (5.72)- (5.75) the formulae (4.54), (4.55) for modifers can be expressed in the form

$$(\lambda,t)(z,p,r) = q_*^{'}(z,\alpha(z)) - q_*^{'}(z) + [p + rg_*(z)]^T. \\ .[g_*^{'}(z,\alpha(z)) - g_*^{'}(z)] \qquad (B.2)$$

Using (B.1) we obtain

$$L_{*r(x)}(z,p) = q_*^{'}(\tilde{z}) + [p + rg_*(\tilde{z})]^T pg_*^{'}(\tilde{z}) =$$

$$q_x^{'}(\tilde{z},\alpha(z)) + [p + rg(\tilde{z},\alpha(z))]^T pg_x^{'}(\tilde{z},\alpha(z)) -$$

$$- (\lambda,t)(z,p,r)^T = L^{'}_{r(x)}(\tilde{z},\alpha(z),p) \qquad (B.3)$$

Due to the convexity of $L_r(.,\alpha,p)$, (B.1) and (B.3) imply that $\hat{x}(\tilde{z},p) = \tilde{z}$ is a solution to the modified optimisation problem

$$\min_{x \in CU} \{ q(x,\hat{\alpha}(z)) + p^T g(x,\hat{\alpha}(z)) + \frac{1}{2} r \| g(x,\alpha(z)) \|^2 \\ - (\lambda,t)(z,p,r)^T x \} \qquad (B.4)$$

Thus, $\hat{z}$ is an inner loop solution $\hat{z}(p)$. An argument analogous to the above, but with L_r and L_{*r} interchanged, proves that every inner loop solution minimises $L_{*r}(.,p)$ on CU if $L_{*r}(.,p)$ is convex.

If $\hat{z}(p)$ is unique, then, see (Findeisen and others 1980).

$$\Delta d_{*r}(p) = g_*(\hat{z}(p)),$$

which completes the proof since $g_*(\hat{z}(p)) = g(\hat{z}(p), \hat{\alpha}(\hat{z}(p)))$.

Q.E.D.

Appendix C : Convergence Proof of Theorem 5.6 (Chapter 5)

Despite a somewhat different formulation Theorem 5.6 is a generalisation of Theorem 2 from (Brdys, Abdullah and Roberts 1986) and the proof is analogous. The only significant difference is that now $L_{*r}(.,p)$ should be used as a Zangwill function, and estimation of the term $L_{*r}(z,p)(z-\hat{x}(z,p))$ is somewhat different. It will be shown now. The optimality condition for the modified optimisation problem (4.53), with the variabe augmentation added, can be stated as follows :

$$[L'_{r(x)}(\hat{x},\hat{\alpha},p) + \varrho(\hat{x}-z)^T p - (\lambda,t)^T(z,p,r)(z-\hat{x}) \geqslant 0, \tag{C.1}$$

where $\hat{x} \triangleq \hat{x}(z,p)$, $\hat{\alpha} \triangleq \hat{\alpha}(z)$, for notational simplicity.

Using (B.2), and then assumption (iii), we obtain from (C.1)

$$L'_{*r(x)}(z,p)(z-\hat{x}) \geqslant [L'_{r(x)}(z,\hat{\alpha},p) - L'_{r(x)}(\hat{x},\hat{\alpha},p)$$

$$+ \varrho(z - \hat{x})^T](z - \hat{x})$$

$$= (z - \hat{x})^T[L''_{r(x,x)}(\hat{x} + \theta(z - \hat{x}) + \varrho I](z - \hat{x}) \geqslant b\|z - \hat{x}\|^2,$$

where $o \leqslant \theta \leqslant 1$. The rest of the proof is analogous.

Q.E.D.

Appendix D : Convergence Proof of Theorem 5.7 (Chapter 5)

The Therem 5.7 is a generalization of Theorem 3 from (Brdys, Abdullah and Roberts 1986), although the formulation is slightly different. Thus, the proof can be made quite analogously, by taking q_{*r} and q_r instead of q_* and q.

APPENDIX E - CONVERGENCE PROOF OF THEOREM 6.1 (CHAPTER 6)

Using (6.51) and (6.52) we obtain

$$g_*(c,u) = B_* \begin{vmatrix} c \\ u \end{vmatrix} - b_*$$

$$g(c,u,\alpha) = B \begin{vmatrix} c \\ u \end{vmatrix} - HP(\alpha)$$

where $b_* = -[I - HD_{*2}]^{-1} Hd_*$

Let v^k be given, and denote $u_*^k = HK_*(v^k)$. Since

$$g(v^k, u_*^k, \hat{\alpha}^k) = g_*(v^k, u_*^k) \text{ then}$$

$$HP(\hat{\alpha}^k) = B \begin{vmatrix} v^k \\ u_*^k \end{vmatrix}$$

and the condition

$$g(\hat{c}^k, \hat{u}^k, \hat{\alpha}^k) = 0$$

can be written as

$$B \left| \begin{vmatrix} c^k \\ u^k \end{vmatrix} - \begin{vmatrix} v^k \\ u_*^k \end{vmatrix} \right| = 0 \qquad \text{(E.1)}$$

Let us denote

$$\hat{x}^k \overset{\Delta}{=} \begin{vmatrix} c^k \\ u^k \end{vmatrix} \quad \text{and} \quad z^k \overset{\Delta}{=} \begin{vmatrix} v^k \\ u_*^k \end{vmatrix}$$

Since $M_\rho > 0$ (see Eq. (6.57)) then the sufficient and necessary condition for $\hat{x}^n$ to be a solution of the inner loop infimal problem described by Eqs. (6.12) and (6.13) can be written as follows :

$$M_{\varrho}(\hat{x}^{n} - z^{k}) + Mz^{k} + \tilde{B}^{T}p^{n} - Md = 0 \qquad (E.2)$$

where n denotes the number of the inner loop iteration corresponding to the k^{th} iteration of the outer loop.

Eq. (E.2) yields that

$$\hat{x}^{n} = - M_{\varrho}^{-1}\tilde{B}^{T}p^{n} + (I - M_{\varrho}^{-1}M)z^{k} + M_{\varrho}^{-1}Md \qquad (E.3)$$

Therefore, and due to (E.1), the iteration scheme (6.27) can be written as :

$$p^{n+1} = p^{n} + \chi B(\hat{x}^{n} - z^{k}) = [I - \chi BM_{\varrho}^{-1}B^{T}]p^{n}$$
$$+ (I - M_{\varrho}^{-1}M - \chi B)z^{k} + M_{\varrho}^{-1}Md \qquad (E.4)$$

Since the last two terms on the right side od Eq. (E.4) are constant the sequence $\{p^{n+1}\}$ is convergent to p^{k} (if it is convergent then, due to the structure of (6.27) it must converge to p^{k}) from an arbitrary chosen starting point if and only if the following discrete dynamic system :

$$p^{n+1} = [I - \chi BM_{\varrho}^{-1}\tilde{B}^{T}]p^{n} \qquad (E.5)$$

is asymptotically stable in the large at a point $p = 0$.

Since the matrix $BM_{\varrho}^{-1}\tilde{B}^{T}$ is not symmetric then a positiveness of this matrix does not immediately guarantee an existence of a suitable value of χ sufficient for stability. Let us define

$$z(p) \overset{\Delta}{=} \| p \|^2 \qquad \text{(E.6)}$$

as a candidate Liapunov function.
Let us denote

$$A(\chi) \overset{\Delta}{=} [I - \chi B M_{\varrho}^{-1} \tilde{B}^T]$$

and

$$W \overset{\Delta}{=} B M_{\varrho}^{-1} \tilde{B}^T$$

Then

$$Z(A(\chi)p) = p^T p - \chi p^T W p - \chi p^T W^T p + \chi^2 p^T W p =$$
$$= p^T p - \chi p^T (W^T + W) p + \chi^2 p^T W^T W p \leqslant \| p \|^2 - \delta\lambda_{min}(W + W^T)\|p\|^2$$
$$+ \chi^2 \lambda_{max}(W^T W)\|p\|^2 = \zeta(\chi)\|p\|^2$$

where

$$\zeta(\chi) \overset{\Delta}{=} 1 - \delta\lambda_{min}(W + W^T) + \chi^2 \lambda_{max}(W^T W)$$

Since $\zeta(0) = 1$ and because $\lambda_{min}(W + W^T) > 0$ and $\lambda_{max}(w^T w) > 0$ then one can easily verify that exists such a value χ that for every $\chi \in (0, \bar{\chi}]$ the following holds :

$$0 < \zeta(\chi) < 1$$

and consequently

$$Z(A(\chi)p) < \|p\|^2 = Z(p)$$

for such value of χ.

It has now been proved that for $\chi \in (0, \chi]$ the function Z(.)

described by Eq.(E.6) is the Liapunov function corresponding to the system described by Eq.(E.5).

Therefore, assertion 1 has now been proved.

Under a value z^k of z prescribed by the outer loop, the inner loop generates the solution (x^k, p^k) satisfying the following equations :

$$M_\varrho(\hat{x}^k - z^k) + Mz^k + \tilde{B}^T p^k - Md = 0 \qquad (E.7)$$

and

$$B(\hat{x}^k - z^k) = 0 \qquad (E.8)$$

The optimising control problem solution $\bar{z} = (\bar{c}, HK_*(\bar{c}))$ satisfies the following equation :

$$M\bar{z} + B_*^T\bar{p} - Md = 0 \qquad (E.9)$$

and

$$B_*\bar{z} - b_* = 0 \qquad (E.10)$$

Eq. (E.9) can also be written as (see EQs. (6.58) and (6.55))

$$M\bar{z} + \tilde{B}^T\tilde{p} - Md = 0 \qquad (E.11)$$

Conditions (E.7), (E.8), (E.10) and (E.11) constitute a basis for further considerations. Using (6.26) and (E.8) we can express (E.7) in terms of points generated by the outer loop as follows :

$$\frac{1}{\varepsilon}M_\varrho L(v^{k+1} - v^k) + Mz^k + \tilde{B}^T p^k - Md = 0 \qquad (E.12)$$

Multiplying the equality (E.12) by $(z^k - z^{k+1})^T$ and utilizing the fact that $\tilde{B}(z^k - z^{k+1}) = 0$ we obtain

$$-\frac{1}{\varepsilon}(v^{k+1} - v^k)^T L_*^T M_\varrho L(v^{k+1} - v^k) + (v^k - v^{k+1})^T L_*^T M z^k$$

$$- (z^k - z^{k+1})^T M d = 0 \qquad (E.13)$$

Note that owing to (E.11) the following holds :

$$(z^{k+1} - z^k)^T M d = (z^{k+1} - z^k)^T (M\bar{z} + \tilde{B}^T\tilde{p}) = (z^{k+1} - z^k)^T M\bar{z}$$

which together with (E.13) and (6.61) yield

$$-\frac{1}{\varepsilon}(v^{k+1}-v^k)^T L_*^T M_\varrho L(v^{k+1}-v^k) + (\bar{c}-v^k)^T L_*^T M L_*(v^{k+1}-v^k)$$

$$= -\frac{1}{\varepsilon}(v^{k+1}-v^k)^T L_*^T M_\varrho L(v^{k+1}-v^k) + \frac{1}{2}(v^k-\bar{c})^T L_*^T M L_*(v^k-\bar{c})$$

$$-\frac{1}{2}(v^{k+1}-\bar{c})^T L_*^T M L_*(v^{k+1}-\bar{c}) + \frac{1}{2}(v^{k+1}-v^k)^T L_*^T M L_*(v^{k+1}-v^k)$$

$$(E.14)$$

Since $B_*L_*(v^k - \bar{c}) = 0$ and $B_*L_*(v^{k+1} - \bar{c}) = 0$ then (E.14) can be written in the following form :

$$\frac{1}{2\varepsilon}(v^{k+1}-v^k)^T[\varepsilon L_*^T M L_* - 2L_*^T M_\varrho L](v^{k+1}-v^k)$$

$$+ \frac{1}{2}(v^k-\bar{c})^T L_*^T M_\gamma L_*(v^k-\bar{c}) - \frac{1}{2}(v^{k+1}-\bar{c})^T L_*^T M_\gamma L_*(v^{k+1}-\bar{c}) = 0$$

$$(E.15)$$

where

$$M_\gamma \overset{\Delta}{=} M + \gamma B_*^T B_*, \qquad \gamma > 0 \qquad (E.16)$$

Let us notice that due to assumption (6.56) there exists such a value of γ that (Luenberger (1973))

$$M_\gamma > 0 \tag{E.17}$$

Finally, let us define the following function :

$$T(v) \overset{\Delta}{=} \frac{1}{2}(v-\bar{c})^T L_*^T M_\gamma L_*(v-\bar{c}) \tag{E.18}$$

where γ is chosen to satisfy Eq. (E.17).

The matrix L_*^T has full rank (see (6.61)). Hence, the matrix $L_*^T M_\gamma L_*$ is positively defined. Therefore, according to (E.15) and due to assumption (6.64), the sequence $\{T(z^k)\}$ is decreasing and bounded above by $T(z^o)$. This and (E.18) imply that the sequence $\{v^k\}$ is bounded and consequently the sequences $\{z^k\}$ and $\{x^k\}$ are also bounded (see (E.8)). The matrix B_* has full rank. Hence, the matrix $\tilde{B}$ also has full rank. Therefore, the inverse $[\tilde{B}M_\varrho^{-1}\tilde{B}^T]^{-1}$ exists and (E.12) yeilds

$$p^k = [\tilde{B}M_\varrho\tilde{B}^T]^{-1}[Md - \frac{1}{\varepsilon}M_\varrho(v^{k+1} - v^k) - Mz^k]$$

which implies that the sequence $\{p^k\}$ is also bounded. Therefore, the sequence $\{z^k,p^k\}$ has at least one convergent subsequence. Eqs. (E.7) and (E.8) show that a limit of any subsequence of the sequence $\{z^k,p^k\}$ satisfies Eqs. (E.10) and (E.11). It has been proved, however, that $(\bar{z},\tilde{p})$ is the only single point to satisfy these inequalities. Therefore, a proof of assertion 2 has now been completed.

APPENDIX F - CONVERGENCE PROOF OF THEOREM 6.2 (CHAPTER 6)

The necessary and sufficient conditions for x^k and p^k to be a solution of the inner loop problem, corresponding to a prescribed outer loop value (z^k, p_2^k) of (z, p_2) can be written as follows :

$$M_\varrho(\hat{x}^k - z^k) + Mz^k + B^T(\hat{p}_1^k - p_2^k) + \tilde{B}^T p_2^k - Md = 0 \tag{F.1}$$

and

$$B(\hat{x}^k - z^k) = 0 \tag{F.2}$$

According to the assumptions that $M_\varrho > 0$ and that the matrix B has full rank, therefore, there is a unique solution of (F.1) and (F.2) with respect to $(\hat{x}^k, \hat{p}_1^k)$. Hence, the iterative scheme (6.26), (6.32) is well defined.

Conditions (F.1) and (F.2), together with definitions (6.60) and (6.62), constitute a basis for further considerations. Using (6.26), (6.32) and (F.2), (see (6.60)), we express (F.1) in terms of points generated by the outer loop as follows :

$$\frac{1}{\varepsilon}M_\varrho L(v^{k+1} - v^k) + Mz^k + \frac{1}{\varepsilon_p}B^T(p_2^{k+1} - p_2^{k+1}) + \tilde{B}^T p_2^k - Md = 0 \tag{F.3}$$

Utilizing (E.11) and the equality $\tilde{B}(z^k - \bar{z}) = 0$ we transform

(F.2) to the following form :

$$\frac{1}{\varepsilon} M_{\varrho} L(v^{k+1} - v^{k}) + M_{\gamma}(z^{k} - z) + \frac{1}{\varepsilon_p} B^{T}(p_2^{\,k+1} - p_2^{\,k}) + \tilde{B}^{T}(p_2^{\,k} - p) = 0 \qquad (F.4)$$

Multiplying Eq.(F.4) by $(z^{k} - z^{k+1})^{T}$ and utilizing (6.61) we obtain :

$$- \frac{1}{\varepsilon}(v^{k+1}-v^{k})^{T} L_{*}^{T} M_{\varrho} L(v^{k+1}-v^{k}) + (\bar{c}-v^{k})^{T} L_{*}^{T} M_{\gamma} L_{*}(v^{k+1}-v^{k}) + (\tilde{p}-p_2^{\,k})^{T} \tilde{B} L_{*}(v^{k+1}-v^{k}) + \frac{1}{\varepsilon_p}(p_2^{\,k+1}-p_2^{\,k})^{T} B L_{*}(v^{k}-v^{k+1}) = 0 \qquad (F.5)$$

The second term in (F.5) can be expressed as :

$$\frac{1}{2}(v^{k}-\bar{c})^{T} L_{*}^{T} M_{\gamma} L_{*}(v^{k}-\bar{c}) - \frac{1}{2}(v^{k+1}-\bar{c})^{T} L_{*}^{T} M_{\gamma} L_{*}(v^{k+1}-\bar{c}) + \frac{1}{2}(v^{k+1}-v^{k})^{T} L_{*}^{T} M_{\varrho} L_{*}(v^{k+1}-v^{k}) \qquad (F.6)$$

The third term in (F.5) is equal to

$$(\tilde{p}-p_2^{\,k})^{T} \tilde{B} L_{*}(\bar{c}-v^{k}) - (\tilde{p}-p_2^{\,k+1})^{T} \tilde{B} L_{*}(\bar{c}-v^{k+1}) + (p_2^{\,k+1}-p_2^{\,k})^{T} \tilde{B} L_{*}(v^{k+1}-v^{k}) + (p_2^{\,k}-p_2^{\,k+1})^{T} \tilde{B} L_{*}(\bar{c}-v^{k}) \qquad (F.7)$$

Let us now compute the term $\tilde{B} L_{*}(\bar{c} - v^{k})$. The equalities (F.4) and (F.11) imply that

$$\bar{z} - z^{k} = \frac{1}{\varepsilon} M_{\gamma}^{-1} M_{\varrho} L(v^{k+1} - v^{k}) + \frac{1}{\varepsilon_p} M_{\gamma}^{-1} B^{T}(p_2^{\,k+1} - p_2^{\,k}) + M_{\gamma}^{-1} \tilde{B}^{T}(p_2^{\,T} - \tilde{p})$$

and

$$\breve{B}L_*(\bar{c} - v^k) = \frac{1}{\varepsilon}\breve{B}M_\gamma^{-1}M_\varrho L(v^{k+1} - v^k)$$

$$+ \frac{1}{\varepsilon_p}\breve{B}M_\gamma^{-1}B^T(p_2^{k+1} - p_2^k) + \breve{B}M_\gamma^{-1}\breve{B}^T(p_2^k - \breve{p})$$

Hence

$$(p_2^k - p_2^{k+1})^T\breve{B}L_*(\bar{c} - v^k) = \frac{1}{\varepsilon}(p_2^k - p_2^{k+1})^T\breve{B}M_\gamma^{-1}M_\varrho L(v^{k+1} - v^k)$$

$$+ \frac{1}{\varepsilon_p}(p_2^k - p_2^{k+1})^T\breve{B}M_\gamma^{-1}B^T(p_2^{k+1} - p_2^k) +$$

$$+ (p_2^k - p_2^{k+1})^T\breve{B}M_\gamma^{-1}\breve{B}^T(p_2^{k+1} - \breve{p}) =$$

$$= -\frac{1}{\varepsilon}(p_2^{k+1} - p_2^k)^T\breve{B}M_\gamma^{-1}M_\varrho L(v^{k+1} - v^k) -$$

$$-\frac{1}{\varepsilon_p}(p^{k+1} - p^k)^T\breve{B}M_\gamma^{-1}B^T(p_2^{k+1} - \breve{p}_2^k) +$$

$$+ \frac{1}{2}(p_2^k - \breve{p})^T\breve{B}M_\gamma^{-1}\breve{B}^T(p_2^k - \breve{p})$$

$$- \frac{1}{2}(p_2^{k+1} - \breve{p})^T\breve{B}M_\gamma^{-1}\breve{B}^T(p_2^{k+1} - \breve{p}) +$$

$$+ \frac{1}{2}(p_2^{k+1} - p_2^k)^T\breve{B}M_\gamma^{-1}\breve{B}^T(p_2^{k+1} - p_2^k) \qquad (F.8)$$

Finally, let us define the following function :

$$T(v, p_2) \overset{\Delta}{=} \frac{1}{2}[M_\gamma L_*(v - \bar{c}) + \breve{B}^T(p_2 - \bar{p})]^T M_\gamma^{-1}[M_\gamma L_*(v - \bar{v})$$

$$+ B^T(p_2 - \breve{p})] \qquad (F.9)$$

We shall now utilize a technical idea from Cohen's proof of his Theorem 5.1 (cohen 1980). Namely, applying (F.8) and (F.7) and (E.6) to (F.5) we conclude that

$$T(v^k, p_2^k) - T(v^{k+1}, p_2^{k+1}) =$$

$$+ (v^k - v^{k+1})^T [\frac{1}{\varepsilon} L_* M_\varrho L - \frac{1}{2} L_*^T M_\gamma L_*](v^k - v^{k+1})$$

$$+ (p_2^k - p_2^{k+1})^T [\frac{1}{\varepsilon} \tilde{B} M_\gamma^{-1} M_\varrho L + \frac{1}{\varepsilon_p} B L_*](v^k - v^{k+1})$$

$$+ (p_2^k - p_2^{k+1})^T [\frac{1}{\varepsilon_p} \tilde{B} M_\gamma^{-1} B^T - \frac{1}{2} \tilde{B} M_\gamma^{-1} \tilde{B}^T](p^{k+1} - p^k) \qquad (F.10)$$

According to (F.10) and due to assumption (6.66) the sequence $\{T(v^k, p_2^k)\}$ is not increasing. Because $M_\gamma^{-1} > 0$, (F.9) then implies that the sequence is bounded below. Hence, $\{T(v^k, p_2^k)\}$ converges and consequently

$$T(v^k, p_2^k) - T(v^{k+1}, p_2^{k+1}) \xrightarrow[k \to \infty]{} 0$$

Therefore, (F.10) implies that

$$\{(v^k - v^{k+1}, p_2^k - p_2^{k+1})\} \xrightarrow[k \to \infty]{} 0$$

and consequently (see (F.2))

$$\{(x^k - z^{k+1}, p_2^k - p_2^{k+1})\} \xrightarrow[k \to \infty]{} 0$$

Since $B_* z^k - b_* = 0$ then

$$\tilde{B} z^k - (I - HD_2) b_* = 0 \qquad (F.11)$$

Eqs. (F.1) and (F.11) imply that the sequence $\{z^k, p_2^{\,k}\}$ converges to a point $(\bar{\bar{z}}, \bar{\bar{p}}_2)$ satisfying the following equations :

$$M\bar{\bar{z}} + B^T\bar{\bar{p}}_2 - Md = 0 \qquad (F.12)$$

and

$$\hat{B}\bar{\bar{z}} - (I - HD_2)b_* = 0, \text{ or } B_*\bar{\bar{z}} - b_* = 0 \qquad (F.13)$$

Because there is only one solution of (F.12) and (F.13) then

$$\bar{\bar{z}} = \bar{z}$$

and

$$\bar{\bar{p}}_2 = \hat{p}$$

and a proof of Theorem F has now been completed.

Q.E.D.

APPENDIX G - CONVERGENCE PROOF OF THEOREM 6.3 (CHAPTER 6)

The separable modified optimisation problem (6.34) can be easily transformed to the following equivalent form

$$\min_{(c,u)\in C\bar{U}} \{ L_r(c,u,\hat{\alpha}(v),p) - \lambda(v,p)^T c + r(u - u^s)H[F(c,u,\hat{\alpha}(v)) - F(c^s,u^s,\hat{\alpha}(v))]\} \quad (G.1)$$

Let us restrict the analysis to a neighbourhood of the optimal point $(\hat{c},\hat{u})$ such that precisely those inequality constraints which are active in this neighbourhood are also active at $(\hat{c},\hat{u})$. Such a neighbourhood exists due to the strict complementarity assumption. Let us denote by $G_A(c,u) \leqslant 0$ the vector of active constraints. Then, necessary optimality conditions for (5.1) are

$$\begin{cases} \dot{L}_{r(c,u)}(c,u,\hat{\alpha}(v),p)^T - \begin{vmatrix} \lambda(v,p) \\ 0 \end{vmatrix} \\ \quad + r[F'_{(c,u)}(c,u,\hat{\alpha}(v))H^T p(u - u^s) \\ \quad + I_o H(F(c,u,\hat{\alpha}(v)) - F(c^s,u^s,\hat{\alpha}(v))] + \dot{G}_A(c,u)^T \tau_A = 0 \\ G_A(c,u) = 0 \end{cases} \quad (G.3)$$

where $I_o = \Delta_{(c,u)}[u]$, and μ_A is a Kuhn-Tucker multiplier. Due to assumption (iii) $\lambda(v,p)$ and $\hat{\alpha}(v)$ are continuously differentiable mappings, (see (6.15)), in a neighbourhood of $\hat{\lambda} = \lambda(\hat{v},\hat{p})$, $\hat{\alpha} = \hat{\alpha}(\hat{v})$. Eqs. (G.2) and (G.3) are satisfied for $(c,u) = (\hat{c},\hat{u})$ if $(c^s,u^s) = (\hat{c},\hat{u})$, $v = \hat{v}$ and $p = \hat{p}$. Due to assumption (i) and (ii) it can be shown using the implicit function theorem in a similar way as in Theorem 2 in (Tatjewski 1985), that the solutions $(c,u,\mu_A)(v,p,c^s,u^s)$ of Eqs. (G.2) - (G.2) are locally unique minimizing points for (6.34), continously dependent on (v,p,c^s,u^s).

Keeping v and p constant let us denote $s_* = (c,u,\mu_A)$, thus $s_*^s = (c^s,u^s,\mu_A^s)$. Then Eqs. (F.2) - (F.3) can be treated as defining an operator $\Psi(\hat{s}_*(s^s),s^s) = 0$, when μ_A^s is an auxillary variable. Hence, formula (6.35) can be written in an extended form.

$$s_*^{s+1} = s_*^s + \varepsilon_a(\hat{s}_*(s_*^s) - s_*^s) = \Psi(s_*^s) \qquad (G.4)$$

The aim of iteration (G.4) is to find a fixed point $\hat{s}_{*vp}$ of $\Psi(.)$, $\hat{s}_{*vp} = (\hat{c}(v,p),\hat{u}(v,p),\hat{\mu}_A(v,p))$. According to Ostrowski's Theorem, see e.g. (Ortega and Rheinbodt 1970), local convergence of (G.4) will be assured if

$$\| \Psi'(\hat{s}_{*vp})\|_s < 1,$$

where $\|.\|_s$ denotes the spectral norm. Using (G.4) we have

$$\Psi(\frac{1}{\varepsilon_a}\Psi(s_*^s) + (1 - \frac{1}{\varepsilon_a})s_*^s, s_*^s) = 0$$

and, due to continuous differentiability of Ψ,

$$\Psi_s'(\hat{s}_{*vp},\hat{s}_{*vp})[\frac{1}{\varepsilon_a}\Psi'(\hat{s}_{*vp}) + (1 - \frac{1}{\varepsilon_a})I] + \Psi_2'(\hat{s}_{*vp},\hat{s}_{*vp}) = 0, \qquad (G.5)$$

where Ψ_i' denotes the partial derivative of Ψ with respect to the i-th argument. From (G.5) we obtain

$$\Psi'(\hat{s}_{*vp}) = \varepsilon_a P(\hat{s}_{*vp},\hat{s}_{*vp}) + (1 - \varepsilon_a)I \qquad (G.6)$$

where

$$p(\hat{s}_*,\hat{s}_*) \overset{\Delta}{=} - \Psi_1'(\hat{s}_*,\hat{s}_*)^{-1}\Psi_2'(\hat{s}_*,\hat{s}_*)$$

Due to (local) continuity of all mappings it is sufficient to show that $\Psi_1'(\hat{s}_*,\hat{s}_*)$ is nonsingular and $\|\Psi'(\hat{s}_*)\|_s < 1$ to complete the proof, provided that v, p are sufficiently close to $\hat{v}$, $\hat{p}$, where $\hat{s}_* = \hat{s}_{*vp} = (\hat{c},\hat{u},\hat{\mu}_A)$. Matrices $\Psi_1'(\hat{s}_*,\hat{s}_*)$ and $P(\hat{s}_*,\hat{s}_*)$ are precisely equal to matrices M(ϱ) and P(ϱ) from Theorem 2 in (Tatjewski 1985), and it was shown there that M(ϱ) is nonsingular and P(ϱ) has all eigenvalues real and less than 1 provided r is sufficiently large. If v_i is an eigenvalue of $P(\hat{s}_*,\hat{s}_*)$, then due to (G.5) there is an eigenvalue of δ_i of $\Psi(\hat{s}_*)$ such that

$$\varepsilon_a v_i = \delta_i - (1 - \varepsilon_a)$$

Thus, since $v_i < 1$ for all i, $\delta_i \in (-1, +1)$ for all i provided $\varepsilon_a \in (0,\bar{\varepsilon}_a)$, where $\bar{\varepsilon}_a = \dfrac{2}{(1 - v_m)}$, v_m being the smallest eigenvalue of $P(\hat{s}_*,\hat{s}_*)$.