ON THE POSSIBILITY OF A LAGRANGIAN DERIVATION OF NONSINGULAR EQUATIONS FOR THE GRAVITATIONAL FIELD

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DECLARATION

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ABSTRACT

Einstein's theory of general relativity is empirically verified to be the most successful model of gravitational phenomena. Its theoretical structure is both elegant and simple. On the other hand, in its strong-field limit, the theory has two major flaws. First, it is isolated from the mainstream of physical laws and in particular, it is not amenable to quantization. Second, by being essentially a singular theory it paradoxially predicts the inevitable gravitational collapse with its attendant formation of "black holes" the evidence for the existence of which is rather weak.

This setback of Einstein's model has motivated us to search for an alternative that will be both nonsingular and more amenable to quantization or at least possesses one of these features. In this thesis we explore the possibility of developing such an alternative using a variational approach based on a Lagrangian which is a nonlinear function of the scalar curvature.

We start with a general review and critique of the existing situation which puts our own contribution in its proper perspective. This contribution commences with a new derivation of the field equations which leads to a necessary condition on the structure of the Lagrangians.

By concentrating on static, isotropic free-field metrics, we obtain coupled ordinary differential equations for the metric coefficients and the scalar curvature in terms of the Lagrangian and its derivatives. This enables us to fix the asymptotic properties of the corresponding specetimes and hence to single out and classify viable Lagrangians.

Several examples are developed including the quadratic and more general Lagrangians. By appealing to the classical limit of an underlying quantum theory (i.e. the finiteness and smallness of Planck's constant) we find solution spacetimes which have good behaviour in both strong- and weak-field limits.

We then extend the quantum connection to consider solutions which become complex-valued in the strong-field domain and obtain an interesting result.

The thesis ends with a résumé and general outlook.

PART I

Synopsis

*In the introduction to this thesis the status quo of the gravitational theory is stated, and discussed from our point of view.

*In Chapter One the theory of general relativity is presented.

*In Chapter Two the viability of weak-field theories competing with Einstein's model is considered. Our critical viewpoint is given in the end of this chapter.

*The solution of general relativity as well as its applications to weak-field phenomena are given in Chapter Three. The singularities, and the gravitational collapse are discussed, whence in the end, we criticize the whole issue of applying general relativity to strong-field gravity.

*Chapter Four is concerned with the modifications of general relativity aimed to suit strong-field gravitation. A critical review is given within which we establish a strong criteria for a perfect theory of gravitation, the main goal which is successfully achieved in Part II of this thesis.

CHAPTER ZERO

INTRODUCTION

(0.0) On Einstein's theory of general relativity

The history of scientific development indicates that different theories of physics, at least those dealing with a certain phenomenon, are destined to be unified into one generalised theory, which, in its turn, may be invalidated if a more general situation is to be considered where only a more generalised theory can be properly applied.

The advent of the twentieth century witnessed a remarkable progress in physics in two directions; Newtonian mechanics has been generalised microscopically to quantum theory, and macroscopically to Einstein's general theory of relativity (GTR). Hence, it is quite logical to think of a unifying theory that may embody both quantum mechanics and general relativity as a further step of scientific development. But the isolated geometric representation of GTR with respect to the rest of physics, makes the attempts to such a unification, if not impossible, a very complicated task. This implies that either Einstein's theory or the rest of physics is incorrect. But the accumulated experimental evidence on the validity of quantum mechanics will leave no doubt about the incorrectness of general relativity or, say, its imperfection.

Further, the discovery [1] of quasi-steller objects, quasars, in the early sixties, has raised interest in the old, theoretically suggested in 1939, idea by J. Robert Oppenheimer [2] and others on the existence of exotic objects with powerful radio sources of energy, one of which is claimed to be "black hole", an assumedly massive body trapped in gravitational collapse.

If the nature of these quasars is to be explained in general relativistic terms where gravitational fields within these massive bodies would play a dominant role, then the gravitational attraction leading to collapse will possibly be the only powerful source of energy [3] through conversion of mass into the latter. Nevertheless, it is still possible that gravitational collapse may have no relation whatsoever with the existence of quasars.

However, the Schwarzschild solution of Einstein's equations in the strong-field region exhibits a singular behaviour which may be interpreted as the occurrence of black holes encompassed by a horizon of a closed, trapped surface. Evidently this kind of interpretation, giving singularity a physical meaning, is objected to by physicists who see the appearance of singular behaviour as a setback to any theory. But Penrose and Hawking [4] proved that gravitational collapse is an inevitable result of the existence of geometrical singularities in space, and hence, the black holes corresponding to these singularities should exist. In other words, if the geometrodynamical theory of Einstein is valid, then massive bodies are destined to collapse into black holes. But here we have to admit that the interpretation of X-ray radiation coming from Cygnus X-1 [5] as an indication of a black hole in this binary system, serves as weak evidence of its existence and the whole issue is still controversial.

On the other hand, Einstein's free-field equations, besides being repeatedly described as the most elegant and beautiful theory, was tested and found to be very successful in explaining a number of weak-field gravitational phenomena, like the red shift, the deflection of light by the sun, the precession of the perihelion of the orbit of Mercury, etc.

To resolve this paradoxial situation we may suggest that GTR is good and correct in the weak-field region and free space, but when one comes to areas with strong fields, i.e. high curvature, where possibly quantum effects may play an essential role, the theory is incorrect. Einstein himself thought that the concepts would be meaningless in regions of high field strength [6].

We therefore think that a modified model involving quantum corrections may be a priori nonsingular, and hence the idea of the yet unconfirmed existence of black holes being related to singularity is to be questioned and reconsidered if not entirely rejected.

To sum up the situation of GTR we have to quote here its positive and negative aspects.

(0.1) <u>Advantages and disadvantages of Einstein's model</u>

The theory is elegant and simple. It was tested to be in agreement with a number of weak gravitational field experiments and observations. It reduces to Newtonian Laws in the weak field limit and to the special relativity in the absence of gravity. It is complete in the sense that it is able to give answers to all questions related to gravitational phenomena and it is self-consistent since it will not lead to conflicting results if different methods based on it are used. But the model has its shortcomings. In the strong gravitational field it leads to a singular behaviour of the metric which is usually looked on as a limitation of any theory and, consequently, it predicts "black holes" whose hypothetical existence is surrounded by controversy and doubts, since the experimental evidence is still unreliable. The theory is isolated from the rest of physics in its geometrical picture and thus cannot be unified with other field theories and therefore, it is very probably non-quantizable.

Now, in order to accommodate the beauty and successfulness of Einstein's geometrodynamical theory with the avoidance of singularities "inevitably" leading to gravitational collapse, and to overcome other defects of this model, we suggest that there are two ways of thinking of gravitational phenomena.

(0.2) Modification of Einstein's model.

It seems fairly reasonable to think along the lines of modifying the geometrodynamical equations of general relativity in a way that it preserves its physical significance in both weak and strong field domains.

This has been done by different authors, in different manners, by attempting to quantize GTR equations [7], by deforming the solution of Einstein's model in order to eliminate singularities [8], by introducing spin and torsion to avert gravitational collapse [9], and generally by utilizing a nonlinear general lagrangian density in a generalized gravitational field equations which would have tended to Einstein's version in the case when this Lagrangian turns to be the scalar curvature R.

We will review these types of modifications in Chapter Four of this thesis, but here we briefly quote some of them:

a. Limiting curvature theory

By exploiting the generalized field equation first derived by Lanczos [10], Müller et al [11] used a nonlinear Lagrangian to establish their model. Being guided by the Born-and-Infeld [12] idea of limiting the electromagnetic field strength to eliminate the divergence of Maxwell's thoery, they have endeavoured to avoid gravitational collapse by postulating an upper bound to the curvature scalar, and in an attempt to ensure both the weak and strong field behaviours, the nonlinear Lagrangian has been chosen in a way that for the region of vanishing curvature it goes over to Einstein's field Lagrangian. In general this Lagrangian contains nonlinear terms and unlike Einstein's model, the scalar curvature is not a constant or zero, but dependent of the radius r from the gravitational centre. A critical review of this model is given in Chapter Four of this work.

b. Kilmister - Yang Model

This attempt, made by Kilmister [13], Yang [14] and Stephenson [15] and developed by them is based on considering the covariant derivative of Ricci tensor $R_{\mu\nu;\lambda}$, instead of the tensor $R_{\mu\nu}$ itself. Some studies [16] [17] of this kind of equation yield interesting results, especially in the strong field domain. The model has also Schwarzschild and Kerr spacetimes as special solution in the free-field case. It has the advantage of being amenable to quantization. The model will be introduced in Chapter Four of this thesis together with other strong field theories.

c. Lagrangian approach to nonsingular gravity.

This is a novel approach which represent our main contribution to modelling strong-field gravity. The model is based on a Lagrangian which we derive from general physical principles. The metric has been obtained by utilizing generalized field equations in the static isotropic space. The behaviour of the metric is controlled by constant coefficients and parameters which, we believe, will acquire their meaning in the quantized version of the model. It is hoped that the model, being nonsingular, is quantizable.

Schwarzschild space, with its inherent singularities, appears to be incorrect since it comes out as a result of further approximation of our model.

Three chapters of Part II of this thesis are mainly devoted to considering and discussing this new and promising approach.

d. A model with complex curvature

This is another contribution and an alternative approach towards a nonsingular quantizable gravitational theory. The scalar curvature is considered to be complex, which in its turn complexifies the metric coefficients. This complexity is motivated by the presumably close relation between the required nonsingular solution in strong energy regions and the expected quantum structure of the equation. We utilized a Lagrangian purely quadratic in the scalar curvature R which was long advocated for unified field theory [18]. In Chapter Nine this model is discussed.

(0.3) Duality between geometry and gravity

An alternative line of thinking towards interpreting the gravitational phenomena is based on the idea that Einstein's geometrodynamical theory is neither a unique nor a basic description of gravitation, and therefore, we should not always expect that by modifying it the inherent defects, like the appearance of horizons and singularities can always be remedied.

This would imply that any attempt to build a unified field theory stemming from purely geometrical concepts is likely to be accompanied by failure; and that might have explained the failure of attempts made either to geometrize, e.g. electromagnetism [19], or to quantize the geometrodynamical equations of GTR [20].

Therefore, and without discrediting GTR, one would suggest a complementary to it - "nongeometrical" model - as an alternative theory of gravity, and which will be preferred to explain strong gravitational field, while, in contrast to that, general relativity will be used, preferably in describing weak gravity. (By non-geometrical we mean, here, the theory is not expressed in the geometrodynamical language of general relativity.)

The above-mentioned situation goes in line with what Poincaré [21] thought of, that all geometries are equally true, i.e. both Einstein's and the proposed nongeometrical models are, but alternative ways of describing gravity, and it is a matter of convenience to choose this or that model to explain the gravitational phenomena with respect to this or that area of the gravitational energy spectrum. Hence, Einstein's general relativity theory should be regarded as the mathematically convenient way of describing gravitational phenomena occurring in certain regions, especially in free space and weak field, while a proposed nongeometrical theory would be more advantageous when one considers regions of interaction with matter.

This situation would have led to thinking of explaining gravitation in a dualistic language of geometry-gravity, in the same way the light phenomena were interpreted in a dualism of wave-corpuscular theory [22].

The Huyghens wave picture [23] successfully explained light phenomena, like reflection refraction etc., while it utterly failed to explain the interaction of matter with light waves, where this particular phenomenon was well-interpreted in terms of Newton's corpuscular theory.

As only quantum theory succeeded, later, in combining both wave and photon features of light by starting from an entirely new quantum concept, therefore the required candidate theory of gravitation might not be achieved by just adding a correction to Einstein's model, but by rather reconsidering the very basic concept of general relativity, namely the influence of geometry on matter and vice versa.

(0.4) Modification or Renovation

Whether the line of thinking discussed in (0.2) or alternatively in (0.3), is the most promising way of establishing a successful model, we have no option, in the absence of a viable non-geometrical theory competitive to Einstein's, but to start from the validity of GTR and modify it towards improving its outcome, provided that the physical significance will not be violated at any part of the energy spectrum.

As for the question concerning renovation by suggesting a dual model as described in (0.3) or by a whole departure from the framework of geometric metric theories to nongeometric non-metric ones, is left open for future development.

By metric theory we mean space-time model based on the principle of equivalence [24], the basic concept of GTR to which we will come back in Chapter One.

Thus we see that the sought theory should not necessarily emerge from the equivalence principle.

Throughout the last decades a number of theories [25], metric and nonmetric, had been established, one of which, i.e. Brans-Dicke theory [26], is in a good conformity with general relativity, while others mainly lead to unreasonable predictions [27].

The criteria for a theory to be good and viable whether metric or nonmetric has been put down by Thorne et al [28], where they suggested that three essential conditions should be satisfied by any candidate model: (i) completeness, (ii) self-consistency and (iii) agreement with past experiments.

In Chapter Two of this thesis, some different types of gravitational theories will be reviewed, among which GTR was tested and found to be the best in satisfying the above-mentioned requirements.

(0.5) <u>Nonsingularity and quantizability</u>

As we mentioned before in Section (0.0), in classical physics, mathematical singularities of any theory used to be reragded as a defect in any physical theory, and hence, attempts were usually made to eliminate them. We have some examples for similar situations in physics; the aforementioned in (0.2)a introduction of upper bounds to prevent divergence or collapse, the elimination of Rayleigh-Jeans catastrophe by Planck's quantization of blackbody radiation [29], and the prevention of collapse in Rutherford's orbital model of the atom by the semi-quantum, semi-classical theory of Bohr [30].

Although it is not one of our objectives in this thesis to quantize the gravitational field, we may be encouraged by these examples to think that by constructing a model that will be amenable to quantization we may approach a nonsingular solution or at least comprehend the nature of any possible singular behaviour.

On the other hand, it is strange to notice that, in contrast to other theories describing natural phenomena, all non-singular solutions of general relativity are unphysical, while physical solutions are singular.

These situations have motivated us to look for a nonsingular solution which will correspond to a physical situation and hopefully be amenable to quantization.

We notice that in the weak-field area, quantum effects are observed only in microscopic domains, while they cease to appear within macroscpoic scales. In strong-field regions, where general relativity is defective, we expect that quantum effects will be dominating and hence a quantizable gravitational theory should be the right candidate to explain strong-field phenomena. A proposed model should coincide with general relativity in a certain area, where the classical solution can be matched together with the quantum one, giving indications as to how a generalized theory can be constructed in order to successfully explain both weak as well as strong field behaviour.

Since all efforts to quantize general relativity did not achieve any significant success, we may be encouraged to believe that Einstein's model is not quite correct, even in its present classical form. This incorrectness, we believe, is the reason for the existence of singularities in the nonlinear part of the solution of the gravitational equations.

We therefore look for an alternative model which will be correct and free of any singularity and which, we expect, will contain certain parameters to be interpreted as classical limits of quantum gravitational quantities, analogously to those in Bohr's model, which had served an intermediate step between classical physics and quantum mechanics.

The proposed model will hopefully be the real viable classical candidate for a strong field theory and will successfully explain the weak field phenomena. Towards this aim we have devoted this present thesis.

(0.6) Presentation of the thesis

Together with this introductory chapter, which we denoted as Chapter Zero, the rest of our thesis, which is divided into two parts, will be comprising ten chapters altogether.

Part I consists of the first five chapters which are mainly a presentation of general relativity and its competitors, where the current situation of the theory of gravitation is reviewed and discussed.

Part II, containing solely our own contribution, comprises five chapters of the work.

The thesis ends with a résumé and a general outlook where a possible direction for further research is proposed.

A list of references is provided for each chapter separately at the end of the thesis.

In Chapter One, which is devoted to GTR, an historical background of the theory of gravitation is given in brief, where also the basic principle of general relativity, namely the equivalence principle is presented, together with Riemannian geometry, being the vehicle for Einstein's geometrodynamical model of gravitation. And owing to the geometrical structure of the theory, tensor calculus, the only relevant mathematical language, has been employed and some useful tensorial relations and definitions have been given.

Then physical and mathematical principles are used to obtain the general relativistic equations in two different ways.

In Chapter Two, we review some gravitational theories competitive to general relativity, mainly those dealing with the weak-field phenomena. The viability of metric and nonmetric theories is discussed and a classification of metric ones is given, on the top of which stands Brans-Dicke theory as the most viable alternative for general relativity.

Chapter Three, will deal with the solution of general relativistic gravitational equations, namely, the symmetrized Schwarzschild and Kerr metrics. The singularity problem is discussed and a critique on the speculations about black holes being related to the singularity is given.

In Chapter Four, we introduce different schemes made for modification of GTR with the intent to remedy its singular behaviour in the strong energy spectrum. Although this chapter is mainly concerned with strong-field models, we have also introduced some weak-field viable theories for the sake of generality. Cartan's general relativity plus torsion [9] [31], as well as scalar-tensor (Brans-Dicke) [26] and, vector-tensor [32] theories are briefly reviewed. We also quote other methods aiming to overcome the singularity in the strong-field region, either by attempting the difficult task of quantization [7], or by just deforming the solution of GTR in such a way that singularity will be eliminated [18].

Kilmister and Yang model [13] [14] is introduced as a good strong-field nonsingular model which is quantizable.

Also, we present theories with general and nonlinear Lagrangians where the possibility of utilizing Lagrangians with quadratic and linear in R terms, as well as applying to them different variational methods is discussed.

Finally, generalized 4th order in $g_{\mu\nu}$ differential equations due to Lanczos [33] [10] have been rederived by us to complete the picture of the Lagrangian formalism of gravitational theories. A nonlinear Lagrangian, due to Müller et al [11] and, used in these generalized equations with a postulated upper limit of the curvature R aiming to prevent a gravitational collapse, is introduced, and the model is criticized.

The following five chapters contained in Part II are purely our own contribution to the theory of gravitation which we present in the next section in more detail.

(0.7) <u>A novel approach</u>

(i) In Chapter Five we present a new derivation for the generalized field equations of the 4th differential order in $g_{\mu\nu}$. We obtain the same equations as those derived by Lanczos, though we think that our derivation has certain advantages. Firstly, it starts from very general assumptions, since our invariant function of the action integral is constructed from all possible derivatives of $g_{\mu\nu}$ while Lanczos considered as variational variables only the metric tensor $g_{\mu\nu}$ and Ricci tensor $R_{\mu\nu}$. Secondly, we are able to prove that the differential order of the equations cannot be higher than the 4th, whilst such proof is not apparent in Lanczos' derivation, and besides that, we think that our derivation is more transparent and straightforward.

(ii) We also managed to derive the most general Lagrangian in a polynomial form of R with certain coefficients and parameters which, we believe, will have their origin in quantum gravity. The derivation is based on the validity of the generalized field equations and Brans-Dicke scalar-tensor theory. By this our Lagrangian will combine the successes of both of these theories. In other words it includes both Einstein principle of equivalence which requires a Riemannian geometry, with Mach's principle of gravitation. Machian principle tells us about how the general distribution of matter affects the space geometry, whilst Einstein's equivalence principle speaks of how gravity is a mere manifestation of geometry. Hence, our Lagrangian does confirm the inseparability of gravity and geometry.

The polynomial structure of the Lagrangian is, in fact, corresponding to the highly nonlinear interconnectedness between geometry and gravity. The degree of nonlinearity depends on the value of the parameters which, in their turn, may be dependent on, say, quantum effects. Our Lagrangian contains all existing constructed Lagrangians as special cases and because of this highly general structure, we believe that a gravitational theory based on it will constitute the most viable and the most reliable model of gravity.

(iii) Another contribution to gravitational field theories, which we think is more interesting, is the important result we obtain in Chapter Six. Utilizing the generalized gravitational equations in the static isotropic coordinates, we are able to derive a number of useful relations of the field variables, which eventually lead to exact expressions for the metric. The metric coefficients g_{rr} and g_{tt} turned out to be Lagrangian-dependent. This implies that a certain choice of the Lagrangian will be sufficient to cure any possible singular behaviour at small distances from the source and any asymptotically non-flatness. It is very interesting to notice that we discovered that the Schwarzschild solution is but an approximated value resulting only when a constant linear Lagrangian is adopted, and therefore, we conclude that the appearance of horizons and singularities is a direct consequence of the imperfection of Einstein's GTR.

By applying our general polynomial Lagrangian of Chapter Five, the model will be the most nonsingular classical candidate for a viable and, hopefully, quantizable theory of gravity.

(iv) In Chapter Seven we confine ourselves to a special case, by choosing a Lagrangian containing only the square of the scalar curvature R. Such kind of Lagrangian, historically, had a certain appeal in the "unified-field theory" [18], and in spite of some objections against it [34], we feel that it still has to play an important role in the theory of gravitation.

We obtain useful relations for this kind of Lagrangian and through certain approximations, we manage to obtain fairly reasonable descriptions of gravitation for a point mass. We believe that even in their approximate form the R^2- equations are less singular than the Schwarzschild metric. This indicates that a further generalized Lagrangian will totally beat the singularities.

(v) Chapter Eight is a further step in our model described in Chapter Six, towards more concrete formulation of a nonsingular gravity.

We specialize here to the Lagrangian with both quadratic and linear in R terms plus a nongeometrical component. By imposing certain physical conditions on the constant coefficients we obtained very useful relationships which enabled us to formulate some theorems based on this type of Lagrangian and governing the behaviour of gravity by physical laws.

An expression for a nonconstant scalar curvature is first obtained and a possibility of the complex metric is stated.

(vi) In Chapter Nine our contribution is concerned with the possibility of establishing a theory with complex metric coefficients, which we have already introduced in (0.2)d of this introduction.

Again, in this particular attempt we exploit the R^2- equations of Chapter Seven, where by further derivation we come to highly nonlinear in R equations of the 3rd differential order with respect to r coordinates. Eventually we were able to reduce these equations to the first order, but still with the same non-linearity.

Investigation of this metric indicates that at least one of its real or imaginary parts will exhibit nonsingular behaviour at the points where Schwarzschild's singularities used to occur.

A further consideration of such models may elucidate much of the nature of the singularities inherent in GTR in strong-field domains.

In the end we reckon that our above-counted contributions to the theory of gravitation are novel and the obtained results will serve in furthering the research towards a comprehensive picture of gravity.

CHAPTER ONE

The Gravitational Field and the Theory of General Relativity.

(1.0) Historical Background

(1.1) Space and Physics

The history of mechanics witnesses that the question about the relation between physics and geometry of space was widely discussed. It led to philosophical speculations and arguments about whether space has a physical significance of its own and that the mechanical theory should not be depending on which frame of reference the motion is observed - a view strongly advocated by Leibnitz [1].

A counterargument was raised by Newton [2] based on his classical idea of absolute space and supported by his well-known bucket experiment [3] which led him to believe that space is not empty of physical meaning, but on the contrary, it exerts forces on material bodies.

It is understood from the Newton's law of inertia that a nonuniform motion of a body is always caused by a force and straight line ascribed to the uniform motion with constant speed of a material point is an indication of being free from any force.

This relation between physical force and space geometry is well illustrated by the creation of the so-called "apparent" Coriolis and centrifugal inertial forces [4], which may appear or disappear according to the choice of the appropriate frame of reference. Even the force of gravity was described by Einstein with regard to his famous elevator experiment [5] as an apparent force which can be made, vanishing in an especially chosen system of coordinates.

(1.2) The Principle of Equivalence

The fact that the gravitational force can be equally cancelled away by transformation to a proper coordinate system as does the inertial force, make gravity indistinguishably equivalent to inertia; this constitutes the meaning of the Principle of Equivalence [6].

In fact, the cancellation of gravitational force by an inertial one is possible, in particular, only in static homogeneous gravitational fields, but in time-dependent or inhomogeneous fields, this cancellation can be achieved only if a sufficiently small region was chosen in which time-dependence or inhomogeneity will be negligibly small. Thus the Principle of Equivalence can be formulated as follows:

At every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial system of reference, such that, within an infinitesmally small region of that point the laws of motion of freely-falling particles (or, more strongly, the laws of nature) take the same form as in unaccelerated Cartesian systems of coordinate (i.e. the special-relativistic form) in the absence of gravitation.

The principle of equivalence leads to the interesting fact by identifying gravitational and inertial masses, which was verified experimentally by Dicke and collaborators [7], [8] who improved on Eötvös method [9].

The equivalence of inertia, which is so connected to geometry, with gravity which is perceived as a physical phenomenon establishes a strong relation between geometry and physics, or, in other words, geometry becomes a physical reality.

Mach's principle (1883) [10] which was raised as a refutation of Newton's concept of absolute space, states that the geometry and inertial system of reference are determined by the mass distribution of celestial bodies in the entire space. GTR [11], while partially accepting this principle which proclaims the influence of matter on geometry, it disagrees with it for not recognising the effect of geometry on matter [12].

Since physical phenomena and the geometry of space are interconnected according to the Equivalence Principle, one sees that this mutual relation is the basis for GTR and, therefore, all the consequences of the application of this principle, whether leading to success or failure, would reflect how truly founded was the Principle of Equivalence. This is especially so if we know, as was discussed in the introduction to this thesis, that there is a breakdown in the conditions for directly applying this principle at space-time singularities, e.g. at the end-point of gravitational collapse [13].

(1.3) Riemannian Geometry

The Euclid's geometry, shown in his famous Elements [14] presented the space in which Newton built the concepts of the theory of gravitation.

Nearly two millenia had passed before the advent of a non-Euclidian geometry which started developing mainly due to the works of Gauss [15], Bólyai [16] and Labochevski [17]. Gauss, who first made a great emphasis on the inner properties of surfaces, had realised that the essential inner property of any surface is the metric function, that which gives the distance along the shortest path between two points on that surface.

The local inner property, i.e. metric function, for instance, for a cylinder is the same as that for a plane since the first can be unrolled to the latter without distortion, i.e. maintaining the same metric function unchanged but it is different from spherical surfaces for which the metric function should be different.

Gauss first conceived a metric space that includes a broad class of ordinary and non-ordinary curved spaces and which allows, in an infinitesmally small region, the possibility of finding a locally Euclidian coordinate system.

The axiom made by Gauss to be the basis of a non-Euclidian geometry resembles the Equivalence Principle which admits the possibility of finding a locally inertial system at any point in space.

The two dimensional space of Gauss used in determining metric functions was expanded to N-dimensions, and the complicated problem related to it was solved by Bernhard Riemann, who established a complete geometry of space, i.e. the Riemann Geometry [18], [19].

Therefore, since Einstein's principle of equivalence is in a deep analogy with Gauss and Riemann geometry, we would conceive of gravitation as a manifestation of geometry or, equally, of the curvature of space as an indication of matter distribution in this space [20].

In the light of this relationship and the equivalence principle, the accelerated systems in an infinitesmally small region can be regarded unaccelerated and, correspondingly, the geodesics would coincide with the straight line in a locally Lorentzian frame.

By Lorentz frame we mean that,in which matter satisfies the laws of Special Relativity [21],[22] and by geodesics we mean the analogue of a straight line of Lorentzian frame, in the general curved space-time geometry. Geodesics give the extremum distance between two end points in a Riemannian space.

The geometry of the abstract 4-dimensional space of Special Relativity is well interpreted in real physical terms. The point in such a space, which is termed world-point, represents a physical event, and a world-line will represent a continuous series of such events. The interval between two infinitesmally close events is represented by an arc length along a world-line, called line element, which is a geometrical invariant.

(1.4) METRIC AND CURVATURE TENSORS.

In the 4-dimensional space the line element (the interval) is defined by

$$ds^2 = -\chi_{\alpha\beta}\, d\eta^{\alpha} d\eta^{\beta} \qquad [1.4.1]$$

where

$\eta^{\alpha}, \eta^{\beta}$ $(\alpha\, \beta = 0,1,2,3)$ are coordinates

in a Lorentz space and,

$$\chi_{\alpha\beta} \equiv \begin{cases} 1 & \alpha = \beta = 1,2,3 \\ 0 & \alpha \neq \beta \\ -1 & \alpha = \beta = 0 \end{cases} \qquad [1.4.2]$$

is called Minkowskian tensor.

The non-negative line-element is called time-like interval. In the special case when we choose a proper Lorentz coordinate system where the 3-dimensional separation between events are zero, i.e. when we deal with a fixed spatial point, the line element ds^2 is purely temporal, and ds coincides with the proper time measured by a clock attached to that coordinate system [23].

As the Equivalence Principle teaches us, the laws of Physics should be equivalent, not only among inertial systems of coordinates, but also among accelerated ones, which means that the proper time interval is invariant under transformation not only between inertial systems, but also between any space-time accelerated coordinates (say, in Riemann space). Thus the invariant generalization of [1.4.1] yields the GTR, and the invariant general proper-time will be defined as

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \qquad [1.4.3]$$

where x^μ are any other coordinates not necessarily Lorentzian and $g_{\mu\nu}$ is the metric tensor defined by

$$g_{\mu\nu} = \frac{\partial\eta^\alpha}{\partial x^\mu}\frac{\partial\eta^\beta}{\partial x^\nu}\chi_{\alpha\beta} \; . \qquad [1.4.4]$$

It is obvious that when $\chi = \eta$ that means $g_{\mu\nu} = g_{\alpha\beta} = \chi_{\alpha\beta}$. Therefore, the principle of equivalence reads:

At any point ζ in an arbitrary strong gravitational field (a curved space), we can find a locally inertial coordinate system (a flat space), such that

$$g_{\alpha\beta}(\zeta) = \chi_{\alpha\beta} \qquad [1.4.5]$$

$$\left.\frac{\partial g_{\alpha\beta}^{(x)}}{\partial x^\gamma}\right|_{x=\zeta} = 0 \; . \qquad [1.4.6]$$

As we notice, Riemannian geometry and gravitation are so tightly related that, in order to build a theory of gravitation by utilizing geometrical objects, we should know what other tensors may be constructed from $g_{\mu\nu}$ and its derivatives to enter in the equation of the gravitational field.

The equation should be in tensorial form and hence will be invariant under general coordinate transformation [24][25].

We here write certain relations and definitions in terms of $g_{\mu\nu}$.

We see from [1.4.3.] that $g_{\mu\nu}$ is symmetric with respect to its indices, i.e.

$$g_{\mu\nu} = g_{\nu\mu} \qquad [1.4.7]$$

We define

$$\Gamma_{\gamma\mu\nu} \equiv \frac{1}{2} \left\{ \frac{\partial g_{\nu\gamma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\gamma}} \right\} \equiv [\mu\nu,\gamma] \qquad [1.4.8]$$

[1.4.8] is called Christoffel symbol of the 1st kind and the Christoffel symbol of the 2nd kind is defined by:

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\gamma} [\mu\nu,\gamma] \equiv \left\{ {\lambda \atop \mu\nu} \right\} = \Gamma^{\lambda}_{\nu\mu} \qquad [1.4.9]$$

As we know, scalars and vectors are considered tensors of zero and 1st rank respectively. The tensorial description in a general Riemann space can be reduced to scalars in specially chosen space. That means by virtue of the equivalence principle we will be able to come to measurable quantities by which the correctness of the gravitational theory can be checked.

Tensor quantities of nth rank has n different indices, and under a coordinate transformation $x \to x'$ a tensor e.g. of a third rank $T^{\mu\lambda}_{\nu}$ transforms as,

$$T'^{\mu\lambda}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} T^{\gamma\sigma}_{\rho} \qquad [1.4.10]$$

Tensors with indices upstairs or downstairs are called respectively contravariant and covariant, otherwise they are called mixed tensors like [1.4.10].

$g^{\mu\nu}$ is the inverse contravariant tensor of $g_{\mu\nu}$ which is a covariant one. Contracting $g_{\mu\nu}$ by $g^{\mu\nu}$ yield a mixed tensor (Kronecker symbol) i.e.

$$g^{\mu\lambda} g_{\nu\mu} = \delta^{\lambda}_{\nu} \qquad [1.4.11]$$

We see that [1.4.9] does not transform by the law [1.4.10], which means that $\Gamma^{\lambda}_{\mu\nu}$ is not a tensor. In fact, by the Equivalence Principle if $\Gamma^{\lambda}_{\mu\nu}$ were a tensor in, say, Minkowski-Lorentz space where by [1.4.5,6,8,10] it must vanish, therefore it would vanish everywhere else.

By further using of tensor analysis [26], we introduce the covariant differentiation of tensors. We write here a number of useful formulae and definitions. The covariant derivative for a mixed tensor $T^{\mu\nu}_{\lambda}$ is defined by

$$T^{\mu\nu}_{\lambda;\gamma} = \frac{\partial}{\partial x^{\gamma}} T^{\mu\nu}_{\lambda} + \Gamma^{\mu}_{\gamma\sigma} T^{\sigma\nu}_{\lambda} + \Gamma^{\nu}_{\gamma\sigma} T^{\mu\sigma}_{\lambda} - \Gamma^{\rho}_{\lambda\gamma} T^{\mu\nu}_{\rho} \qquad [1.4.12]$$

We write also the covariant derivative for the contravariant and the covariant tensors $T^{\mu\nu}$ and $T_{\mu\nu}$ respectively;

$$T^{\mu\nu}_{;\gamma} = \frac{\partial}{\partial x_{\gamma}} T^{\mu\nu} + \Gamma^{\mu}_{\gamma\sigma} T^{\sigma\nu} + \Gamma^{\nu}_{\gamma\sigma} T^{\mu\sigma} \qquad [1.4.13]$$

$$T_{\mu\nu;\gamma} = \frac{\partial}{\partial x^{\gamma}} T_{\mu\nu} - \Gamma^{\sigma}_{\mu\gamma} T_{\sigma\nu} - \Gamma^{\sigma}_{\nu\gamma} T_{\mu\sigma} \qquad [1.4.14]$$

These covariant derivatives are tensors and they reduce to ordinary derivatives in the absence of gravitation i.e. when $\Gamma = 0$.

By setting $\lambda = \nu$ in [1.4.12] it will be reduced to

$$T^{\mu\nu}_{\ \ \nu;\gamma} = \frac{\partial T^{\mu\sigma}_{\ \ \nu}}{\partial x\gamma} + \Gamma^{\mu}_{\gamma\sigma} T^{\sigma\nu}_{\ \ \nu} \qquad [1.4.15]$$

By the use of the equivalence principle and the following relation between $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$

$$\frac{\partial g_{\mu\nu}}{\partial x\gamma} = \Gamma^{\sigma}_{\gamma\mu} g_{\sigma\nu} + \Gamma^{\sigma}_{\gamma\nu} g_{\sigma\mu} ; \qquad [1.4.16]$$

and the relation between $g_{\mu\nu}$ and $g^{\mu\nu}$ and their derivatives, which can be obtained from [1.4.11];

$$g_{\mu\nu} \frac{\partial g^{\nu\mu}}{\partial x^{\gamma}} = -g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x\gamma} ; \qquad [1.4.17]$$

we obtain for the covariant derivative of the metric tensor defined by [1.4.13,14] and hence, for Kronecker tensor in [1.4.11] the following;

$$g^{\mu\nu}_{\ \ ;\gamma} = g_{\mu\nu;\gamma} = \delta^{\mu}_{\ \nu;\gamma} = 0 \qquad [1.4.18]$$

Further, the following relations are useful

$$\Gamma^{\mu}_{\mu\lambda} = \tfrac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} = \tfrac{1}{2} \frac{\partial}{\partial x^{\lambda}} \ln g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\lambda}} \sqrt{g} \qquad [1.4.19]$$

$$g = -\mathrm{Det}\left[g_{\rho\mu}\right] \qquad [1.4.20]$$

[1.4.19] can be used to obtain the covariant divergence of vector $\vec{V}$ and tensor $T^{\mu\nu}$; that yield

$$\mathrm{Div}\ \vec{V} = V^{\mu}_{\ ;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{g}\, V^{\mu}) \qquad [1.4.21]$$

and $\text{Div}\, T = T^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g}\, T^{\mu\nu}) + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$ [1.4.22]

or if $T^{\mu\lambda}$ is antisymmetric [1.4.22] will be reduced to:

$$\left.\begin{aligned} T^{\mu\nu}_{;\mu} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g}\, T^{\mu\nu}) \\ \text{with} \qquad T^{\mu\nu} &= -T^{\nu\mu} \end{aligned}\right\} \qquad [1.4.23]$$

Another relation comes from [1.4.14] for the covariant derivatives of the covariant tensor $T_{\mu\nu}$, if it is an anti-symmetric in its indices. The cyclical permutation of indices lead to:

$$T_{\mu\nu;\lambda} + T_{\lambda\mu;\nu} + T_{\nu\lambda;\mu} = T_{\mu\nu,\lambda} + T_{\lambda\mu,\nu} + T_{\nu\lambda,\mu} \qquad [1.4.24]$$

where the RHS of [1.4.24] is ordinary derivatives of the tensor $T_{\mu\nu}$, i.e. $T_{\mu\nu,\lambda} = \frac{\partial}{\partial x\lambda} T_{\mu\nu}$

The covariant curl coincides with the ordinary one and can be obtained from [1.4.14]

$$V_{\mu;\nu} - V_{\nu;\mu} = V_{\mu,\nu} - V_{\nu,\mu} \qquad [1.4.25]$$

Furthermore, we introduce the following important tensor, Riemann-Christoffel curvature tensor

$$R^{\lambda}_{\mu\nu\gamma} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\lambda}_{\mu\gamma}}{\partial x^{\nu}} + \Gamma^{\rho}_{\mu\nu} \Gamma^{\lambda}_{\gamma\rho} - \Gamma^{\rho}_{\mu\gamma} \Gamma^{\lambda}_{\nu\rho} \qquad [1.4.24]$$

which transforms according to the law in [1.4.10]. This tensor is the only tensor which can be constructed from the metric tensor and its 1st and 2nd derivatives. There are other forms contracted from $R^{\lambda}_{\mu\nu\gamma}$; i.e. Ricci tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\gamma}R_{\lambda\mu\gamma\nu} = R_{\nu\mu} \qquad [1.4.25]$$

and the scalar curvature,

$$R = g^{\mu\nu}R_{\mu\nu} \qquad [1.4.26]$$

The covariant form of $R^{\lambda}_{\mu\nu\gamma}$ reads;

$$R_{\delta\mu\nu\gamma} = g_{\delta\lambda}R^{\lambda}_{\mu\nu\gamma} = \tfrac{1}{2}\left[\frac{\gamma^2 g_{\delta\nu}}{\partial x^{\gamma}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\gamma}\partial x^{\delta}} - \frac{\partial^2 g_{\delta\gamma}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\gamma}}{\partial x^{\nu}\partial x^{\delta}}\right] +$$

$$+ g_{\rho\sigma}\left(\Gamma^{\rho}_{\nu\delta}\,\Gamma^{\sigma}_{\mu\gamma} - \Gamma^{\rho}_{\gamma\delta}\,\Gamma^{\rho}_{\mu\nu}\right) \qquad [1.4.27]$$

[1.4.27] shows that the curvature tensor has the following algebraic properties:

$$R_{\delta\mu\nu\gamma} = R_{\nu\gamma\delta\mu} \qquad [1.4.28]$$

$$R_{\delta\mu\nu\gamma} = - R_{\mu\delta\nu\gamma} = -R_{\delta\mu\gamma\nu} = R_{\mu\delta\gamma\nu}, \qquad [1.4.29]$$

$$R_{\delta\mu\nu\gamma} + R_{\delta\gamma\mu\nu} + R_{\delta\nu\gamma\mu} = 0 \qquad [1.4.30]$$

The covariant derivative of this tensor can be calculated in a locally-inertial system of coordinates where all Γ vanish, leading to the so-called Bianchi identities;

$$R_{\delta\mu\nu\gamma;\lambda} + R_{\delta\mu\lambda\nu;\gamma} + R_{\delta\mu\gamma\lambda;\nu} = 0 \qquad [1.4.31]$$

which will hold in any other coordinate by the principle of equivalence. Contracting [1.4.31] by the metric tensor and using [1.4.18] yield the following equations, which are equivalent to each other

$$(R^{\mu}_{\lambda} - \tfrac{1}{2}\,\delta^{\mu}_{\lambda}\,R)_{;\mu} = 0 \qquad [1.4.32]$$

$$(R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}R)_{;\mu} = 0 \qquad [1.4.33]$$

Further, we consider the 2nd covariant derivative of vectors V_{μ} and V^{λ} we obtain the following two commutation relations,

$$V_{\mu;\nu;\gamma} - V_{\mu;\gamma;\nu} = -V_{\rho}R^{\rho}_{\ \mu\nu\gamma} \qquad [1.4.34]$$

and

$$V^{\lambda}{}_{;\nu;\gamma} - V^{\lambda}{}_{;\gamma;\nu} = V^{\rho}R^{\lambda}_{\ \rho\nu\gamma} \qquad [1.4.35]$$

Similar formulae hold for any tensor, for instance,

$$T_{\mu\lambda;\nu;\gamma} = T_{\mu\lambda;\gamma;\nu} + T_{\mu\sigma}R^{\sigma}_{\ \lambda\nu\gamma} - T_{\sigma\lambda}R^{\sigma}_{\ \mu\nu\gamma} \qquad [1.4.36]$$

$$T^{\lambda}_{\mu;\nu;\gamma} = T^{\lambda}_{\mu;\gamma;\nu} + T^{\sigma}_{\mu}R^{\lambda}_{\ \sigma\nu\gamma} - T^{\lambda}_{\sigma}\,R^{\sigma}_{\ \mu\nu\gamma} \qquad [1.4.37]$$

The transformation to a locally inertial system of coordinate will make the covariant derivatives commute and therefore the curvature tensor vanished when the gravitational field vanishes. This expresses the close relation between space geometry (the curvature tensor) and the existence of gravitation field.

1.5 THE PRINCIPLE OF GENERAL COVARIANCE

This principle directly follows from the principle of equivalence. It says that the equation of physics is called generally covariant if it holds in the absence of gravitation as well as in its presence and it preserves its form under any general coordinate transformation.

That means if the equation is true in one coordinate system, so it will be true in all other coordinates. But by the principle of equivalence we can always find among these other coordinate systems a system which can be locally inertial; that means systems where the effect of gravitation vanishes, i.e. where [1.4.5] and [1.4.6] hold. The equation therefore holds in both locally inertial and any other coordinate system. It is therefore called generally covariant [27]. The principle of general covariance indicates the effects of gravitation on any system. In brief the general covariance states,

The Laws of Nature hold, in a general gravitational field, provided that they are generally covariant, and that they still hold in the absence of gravitational field. By exploiting this principle one can choose physical laws as simple and elegant as possible, and being guided by it we will be able to build the equations that describe the gravitational field and hence explain gravitational phenomena.

(1.6) EINSTEIN'S EQUATION FOR GRAVITATIONAL FIELD

Starting from the principle of general covariance which stems from the Einstein's Equivalence Principle, we require that the following criteria should necessarily hold for gravitational field equations to be constructed.

I - THE FIELD EQUATION SHOULD BE IN A COVARIANT TENSOR FORM

This is because, in accordance with the principle of general covariance the physical law should be indistinguishable in both accelerated and inertial coordinate systems. That means the coordinates do not enter in the equation; therefore, the the laws should be tensorial.

II - IT SHOULD BE A PARTIAL DIFFERENTIAL EQUATION OF $g_{\mu\nu}$

In Newtonian mechanics we have the differential equations [28];

$$\frac{d^2x_i}{dt^2} = -\nabla_i\phi \qquad [1.6.1]$$

which describe the motion of matter in a gravitational field with a potential ϕ, and the Poisson-Laplace partial differential equation

$$\vec{\nabla}^2\phi = \begin{cases} 4\pi G.\rho \\ 0 \quad (\text{if } \rho=0) \end{cases} \qquad [1.6.2]$$

which shows how the presence $(\rho\neq 0)$ or the absence $(\rho=0)$ of matter determines the gravitational field. In [1.6.2] ρ is the mass density of matter and the gravitational constant is given by

$$G \equiv 6.67\times10^{-8}\text{dyne} \cdot \text{cm}^2\text{g}^{-2}. \qquad [1.6.3]$$

We use the units where the speed of light is a unity. For weak static gravitational fields in the Newtonian limit it can be shown that $\frac{1}{2}g_{00} \sim -\phi$. Since the Laplace equation involves the second derivative of g_{00}, and because of the tensor form of our laws and according to criteria (I), all the components of $g_{\mu\nu}$ should enter our general relativistic equation.

Knowing that the energy density for non-relativistic matter T_{00} coincides with the mass density ρ, we can generalize the Poisson equation [1.6.2] to

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu} \qquad [1.6.4]$$

where $T_{\mu\nu}$ is the general energy momentum tensor, and $G_{\mu\nu}$, called the Einstein tensor, is a combination of possible derivatives of $g_{\mu\nu}$ and their products, which has yet to be constructed.

III - IT SHOULD CONTAIN DERIVATIVES NOT HIGHER THAN THE SECOND DERIVATIVE OF $g_{\mu\nu}$

Because Poisson's equation is of the 2nd order in g_{00} and due to criteria (I), $G_{\mu\nu}$ in [1.6.4] should contain the 2nd derivative of $g_{\mu\nu}$, and since the equation should be uniform in scale, therefore derivatives of order higher than the second in $g_{\mu\nu}$ will not be contained in the equation.

IV - IT SHOULD NOT INCLUDE $\Gamma^{\lambda}_{\mu\nu}$

Indeed, by (I), (II) and (III), $G_{\mu\nu}$ may contain $g_{\mu\nu}$, $g^{\mu\nu}$, $R_{\mu\nu}$ or their products $g_{\mu\nu}R$, R but not $\Gamma^{\lambda}_{\mu\nu}$, since the latter contradicts the covariant tensor form of the equation.

V - IN THE ABSENCE OF MATTER $g_{\mu\nu} \to \chi_{\mu\nu}$

This is by virtue of the general covariance principle for locally inertial coordinate systems the general space-time metric reduces to Lorentzian metric in this particular case. This means that the first derivative of the metric tensor, which is the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ is zero. (See [1.4.5,6,8].) The equation, therefore, must admit a Lorentz metric as a particular solution.

VI - THE SECOND DERIVATIVE OF $g_{\mu\nu}$ SHOULD ENTER LINEARLY

For our equation to have a unique solution, $\frac{\partial^2}{\partial x^\lambda \partial x^\gamma} g_{\mu\nu}$ should be uniquely determined by $\frac{\partial}{\partial x^\lambda} g_{\mu\nu}$ and $g_{\mu\nu}$, i.e. $\frac{\partial^2}{\partial x^\lambda \partial x^\gamma} g_{\mu\nu}$ should enter linearly.

It is clear that we cannot construct our equation in its covariant tensor form from only $g_{\mu\nu}$ and its first derivative, since the latter is excluded by (IV).

The only tensor that can be constructed from $g_{\mu\nu}$ and its first derivative, and linearly from the second derivative is the Riemann-Christoffel curvature tensor [1.4.27] and which can be contracted to R and $R_{\mu\nu}$ by [1.4.25,26]. The property [1.4.29] shows that R and $R_{\mu\nu}$ are the only tensors that can be contracted from R, $R_{\mu\nu}$ and $g_{\mu\nu}$.

VII - THE TENSOR SHOULD BE SYMMETRIC

In [1.6.4] since $T_{\mu\nu}$ is symmetric w.r. to indices it means that $G_{\mu\nu}$ is also symmetric. Indeed, since $G_{\mu\nu}$ is constructed of $g_{\mu\nu}$ and $R_{\mu\nu}$ which are themselves symmetric, it is therefore symmetric in its indices.

VIII - THE TENSOR IS CONSERVED

Since $T_{\mu\nu}$ is conserved, therefore by [1.6.4] $G_{\mu\nu}$ will be divergenceless in the sense of covariant differentiation. i.e.

$$\frac{\partial G^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} G^\mu_\nu + \Gamma^\lambda_{\mu\nu} G^\mu_\lambda = G^\mu_{\nu;\mu} = 0 \qquad [1.6.5]$$

IX - THE EQUATION SHOULD BE NONLINEAR

The gravitational field equation should be nonlinear partial differential equation (different from the Maxwellian equations and hence it does not satisfy the principle of superposition). The nonlinearity arises from the fact that the gravitational field (unlike the electromagnetic) has an effect on its source. However, the first approximation corresponding to weak fields is linear and therefore the superposition principle is valid there.

X - IT SHOULD REDUCE TO STATIONARY WEAK FIELD LIMIT FOR $T_{\mu\nu} \to T_{00}$.

That is, by [1.6.2], [1.6.4] becomes

$$G_{00} \sim \vec{\nabla}^2 g_{00} \qquad [1.6.6]$$

which is representing the Newtonian limit.

The above criteria and requirements should be satisfied in order to construct the gravitational field equation. The fulfillment of the first nine requirements leads to the well-known Einstein's field equation.

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \qquad [1.6.7]$$

from [1.6.4] Λ is called the cosmological constant. In order to satisfy condition (X), we should set $\Lambda = 0$.

The Einstein's field equation then reads:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}\ , \qquad [1.6.8]$$

and by contracting with $g^{\mu\nu}$ one gets

$$R = 8\pi G T^{\mu}_{\mu} \qquad [1.6.9]$$

or

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}). \qquad [1.6.10]$$

(1.7) The Variational Principle

The principle of Invariance of Relativity is automatically satisfied by the methods [29] of variational calculus because these methods deal with calculating the minimum of scalar quantities which do not depend on the coordinates in which they are measured.

The differential Riemannian geometry provides us with quantities invariant under any coordinate transformations. The "action" of the variational principle can be chosen to be invariant and General Relativity Principle is therefore satisfied and the required field equation can be easily set up. [30].

In the preceding section we have constructed the gravitational field equations based on certain physical requirements. By utilizing the variational principle the same equation can be derived.

The suitable invariant will be the Lagrangian density $\mathcal{L}(x)\sqrt{g}$.

We consider the variation of the action integral

$$I = \int \mathcal{L}(x)\sqrt{g}\, d\Omega, \qquad [1.7.1]$$

requiring that,

$$\delta I = 0 \qquad [1.7.2]$$

where $-g$ is the determinant of $g_{\mu\nu}$ defined by [1.4.20]; and $d\Omega$ is the 4-volume element. For Newtonian space these turn to be: $g = 1$ and $d\Omega = d^3x dt$.

We are using $g_{\mu\nu}$ with positive signature and with $\sqrt{g}$ real. By signature we mean the set of the diagonal elements

(+1, +1, +1, -1) denoting the signs of the eigenvalues of the symmetric matrix with coefficients $g_{\mu\nu}$. The sum of these elements gives the signature which in our case is +2. (Other authors [31] use other sign conventions and the quantity under the square-root can be $-g$ and the sign of the metric [1.4.1] can be positive. In either case the signature should be identified with that of the Lorentz metric of special relativity.)

Now the total action for a gravitational field is the sum of the action of the field I_g, and the action of the matter I_m i.e.

$$\delta I = \delta I_g + \delta I_m . \qquad [1.7.3]$$

By putting in mind the analogy to electromagnetic field [32] we will expect that the resulting gravitational equations of the Euler-Lagrange type [5.1;3], will contain, according to criteria III of the previous section, derivatives of $g_{\mu\nu}$ no higher than the second. Here $g_{\mu\nu}$ corresponds to the field potential.

We therefore assume that the integral should not contain derivatives of $g_{\mu\nu}$ higher than the first. But owing to criteria IV and VI of section six and the fact that the Lagrangian density is an invariant, $\mathcal{L}$ may be replaced by the scalar curvature R[32] which, although it contains up to the second derivative of $g_{\mu\nu}$, still leaves the integral [1.7.1] invariant.

In fact the variations of the integrands $R\sqrt{g}$ and $\mathcal{L}\sqrt{g}$ will differ by a quantity in a form of a divergence, which can be transformed away by integration, i.e.

$$\int \mathcal{L}\sqrt{g}d\Omega = \int R\sqrt{g}d\Omega + \int \frac{\partial}{\partial x^\lambda}(\sqrt{g}\,\mathcal{G}^\lambda)d\Omega \qquad [1.7.4]$$

where $\mathcal{G}^{\lambda}$ will be shown to be

$$\mathcal{G}^{\lambda} \equiv g^{\mu\lambda} \, \delta\Gamma^{\lambda}_{\mu\lambda} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} . \qquad [1.7.5]$$

In spite of $\mathcal{L}$ being composed of only the metric tensor and its first derivative, we see that, due to the vanishing of the integral with $\mathcal{G}^{\lambda}$, we are allowed to write

$$\delta\int \mathcal{L}\sqrt{g}\, d\Omega = \delta\int R\sqrt{g}\, d\Omega \qquad [1.7.6]$$

This is because, by Gauss theorem the volume integral of the divergence can be transformed to an integral over a hyper-surface surrounding the 4-volume of integration and the variation of the field on this boundary surface is zero.

Let us carry the variation of the action of the purely gravitational field. By using [1.4.19] and [1.4.26] we get,

$$\delta(R\sqrt{g}) = \sqrt{g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \sqrt{g}\, \frac{R}{2} g^{\mu\nu}\delta g_{\mu\nu} + \sqrt{g} g^{\mu\nu}\, \delta R_{\mu\nu} . [1.7.7]$$

The RHS second term in [1.7.7] will be rewritten so that $\delta g_{\mu\nu}$ be expressed in terms of $\delta g^{\mu\nu}$ according to [1.4.17].

The RHS last term will give the change due to the variation of Ricci tensor [1.4.24];

$$\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\lambda}}\, \delta\Gamma^{\lambda}_{\mu\nu} - \frac{\partial}{\partial x^{\nu}}\, \delta\Gamma^{\lambda}_{\mu\nu} +$$

$$+ \delta\Gamma^{\rho}_{\mu\nu}\, \Gamma^{\lambda}_{\nu\rho} + \Gamma^{\rho}_{\mu\nu}\, \delta\Gamma^{\lambda}_{\nu\rho} - \delta\Gamma^{\rho}_{\mu\lambda}\, \Gamma^{\lambda}_{\nu\rho} + \Gamma^{\rho}_{\mu\lambda}\, \delta\Gamma^{\lambda}_{\nu\rho} \qquad [1.7.8]$$

where the variation of $\Gamma^{\lambda}_{\mu\nu}$ defined by [1.4.8,9] is given by

$$\delta\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2}g^{\lambda\rho}\left[\frac{\partial\delta}{\partial x^{\nu}}g_{\rho\mu} + \frac{\partial\delta}{\partial x^{\mu}}g_{\rho\nu} - \frac{\partial\delta}{\partial x^{\rho}}g_{\mu\nu}\right] - \Gamma^{\sigma}_{\mu\nu}g^{\lambda\rho}\delta g_{\rho\sigma}.$$

[1.7.9]

Since [1.7.9] transforms like a tensor, it is a tensor in spite of $\Gamma^{\lambda}_{\mu\nu}$ being not a tensor by itself. The variation w.r. to $\Gamma^{\lambda}_{\mu\nu}$ carried independently of $g_{\mu\nu}$ is due to Palatini [33].

The use of the covariant derivative for $\delta g_{\mu\nu}$ in [1.7.9] according to [1.4.14] yields:

$$\delta\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2}g^{\lambda\rho}\left[(\delta g_{\rho\nu})_{;\mu} + (\delta g_{\rho\mu})_{;\nu} - (\delta g_{\mu\nu})_{;\rho}\right] \quad [1.7.10]$$

Also, the difference of the covariant derivatives of tensor $\delta\Gamma^{\lambda}_{\mu\nu}$ in [1.7.9] will lead to expression [1.7.8], which will have the form of the following, first obtained by Palatini [34], identity:

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}. \quad [1.7.11]$$

By taking into account [1.4.18], that the covariant derivative of the metric tensor is zero, we can now write the R.H. last term in [1.7.7] using [1.7.11]. We thus obtain:

$$\sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}\,[(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}] \quad [1.7.12]$$

or by [1.4.21] it turns to be:

$$\sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}) - \frac{\partial}{\partial x^{\lambda}}(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}) \quad [1.7.13]$$

which agrees with [1.7.4] and [1.7.5].

This term will vanish according to Gauss theorem when integration is carried over all the hyperspace. Finally we obtain after these considerations

$$\delta Ig = C\int\sqrt{g}(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}d\Omega \qquad [1.7.14]$$

Where C is a constant number which can be determined by reducing the resulting equation to the Newtonian limit.

Furthermore, we know [35] that the energy momentum tensor of any physical system can be calculated if the action integral is given. And since the energy-momentum tensor $T^{\mu\nu}$ for any material system for which the action I_m is given is the functional derivative of I_m with respect to the components of the metric tensor $g_{\mu\nu}$, we therefore define:

$$\delta I_m = \tfrac{1}{2}\int T^{\mu\nu}\sqrt{g}\delta g_{\mu\nu}d\Omega \qquad [1.7.15]$$

where we consider that the change in the action is induced by the change in $g_{\mu\nu}$. On the other hand, the field Lagrangian density will depend, as in the case of electromagnetic field, on the field variables, which, here, are $g^{\mu\nu}$ and its first derivative. This means

$$\mathcal{L} = \mathcal{L}\left(g^{\mu\nu}, \frac{\partial g^{\mu\nu}}{\partial x^\lambda}\right). \qquad [1.7.16]$$

By subjecting [1.7.16] to variation one obtains,

$$\delta I_g = \int\left\{\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{g}\,\mathcal{L}\right)\delta g^{\mu\nu} + \frac{\partial\left(\sqrt{g}\,\mathcal{L}\right)}{\partial\left(\frac{\partial g^{\mu\nu}}{\partial x^\lambda}\right)}\,\delta\left(\frac{\partial g^{\mu\nu}}{\partial x^\lambda}\right)\right\}d\Omega$$

By integrating by parts one comes to

$$\delta I_g = \int \left\{ \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{g}\mathcal{L}) - \frac{\partial}{\sigma x^{\lambda}} \left[\frac{\partial(\sqrt{g}\mathcal{L})}{\partial\left(\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}}\right)} \right] \right\} \delta g^{\mu\nu} d\Omega \quad [1.7.17]$$

Now by using in [1.7.15]

$$T^{\mu\nu}\delta g_{\mu\nu} = - T_{\mu\nu}\delta g^{\mu\nu}, \quad [1.7.18]$$

then the comparison of [1.7.15] with [1.7.17] will give:

$$T_{\mu\nu} \equiv \frac{-1}{\sqrt{g}} \left\{ \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{g}\mathcal{L}) - \frac{\partial}{\partial x^{\lambda}} \left[\frac{\partial(\sqrt{g}\mathcal{L})}{\partial\left(\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}}\right)} \right] \right\} \quad [1.7.19]$$

The symmetry of this expression and the fact that I_m is a scalar justify the definition [1.7.15].

The combination of [1.7.15] and [1.7.14] with the use of [1.7.19] will yield Einstein's equations [1.6.8] which will cause the total action to be stationary.

The constant C is determined by the gravitational constant G as:

$$C = - \frac{1}{16\pi G} \quad [1.7.20]$$

In fact, Einstein's equations can be obtained by using different methods of derivation [36] provided that the Equivalence Principle is fulfilled. The use of the Lagrangian formalism supplementary to the afore-given in the preceding section, derivation will certainly confirm the validity of GTR and will allow us to introduce other Lagrangians which can then be classified with respect to their structure in relation to the Lagrangian of general relativity. We will return to these kind of Lagrangians in Chapter Four.

(1.8) Conclusion

As for the possibility of having in the Eurler-Lagrange equation a gravitational field other than $g_{\mu\nu}$, one assumes the contrary, i.e. if otherwise another field $\tilde{g}_{\mu\nu}$ exists, there would have been a local Lorentz frame of $g_{\mu\nu}$ in which $\tilde{g}_{\mu\nu}$ will not completely vanish, violating the local validity of special relativity. Therefore, only $g_{\mu\nu}$ can enter the gravitational equations [37].

Further, the 16 differential equations of [1.6.8] are reduced by the symmetry relation [1.4.7] to only 10, and in the vacuum case, i.e. when $T_{\mu\nu} = 0$, the equations become

$$R_{\mu\nu} = 0 \qquad [1.8.1]$$

But the vanishing of $R_{\mu\nu}$ does not necessarily mean the vanishing of the field, due to [1.4.25] and hence a true gravitational field may exist in empty space. Furthermore, equation [1.8.1] is the simplest form of Einstein's graviational field equation, though it is too complicated to be solved. In Chapter Three, the solution of the equations is considered in a symmetrized form. This solution constitutes the basis for all tests of general relativity which proved to be successful. As we explained in the introductory chapter of this thesis, the theory has its setbacks which raised the need for alternative theories. In the next chapter we introduce alternative models which compete with Einstein's theory in the weak-field regions. In Chapter Four we introduce the strong-field competitors which are basically modifications of GTR.

CHAPTER TWO

The Gravitational Theories and the Weak-Field Gravity

(2.0) GTR and its Competitors

As it has been discussed in the introduction to this thesis, Einstein's theory of general relativity though regarded as the most successful theory of gravity, may not be the only one. A large number of competing theories have been made by different authors [1], [2] since the early twenties [3] but the more theories have been invented, the more they confirmed the supremacy of Einstein's general relativity, especially in the weak-field regions.

In this chapter we give a general account for different categories of these theories starting by considering their viability.

(2.1) The Viability of a Gravitational Theory

Among the numerous theories of gravity only few of them may compete with general relativity, namely those considered viable ones. For a theory to be essentially viable there are certain requirements to be satisfied [4]. These requirements are summarized in the following basic criteria.

(i) Agreement with observation and experiment

In order that any candidate theory be viable it should necessarily agree with observations and past experiments. This agreement should be more strongly confirmed with the improvement of the observational techniques. At least it is necessary that a proposed theory satisfies the tests of general relativity like gravitational red shift, perihelion shift, deflection of light, and radar echoes time delay [5].

(ii) Correct Newtonian limit

In the limit when gross motion of bodies, say planets, is considered the theory should agree with Newton's theory.

(iii) Being relativistic

This implies that when the effects of gravitational field be so negligible, the nongravitational laws derived from the theory should agree with those of special relativity.

(iv) Completeness

The theory should mesh and incorporate a complete set of physical laws like those of electromagnetism, quantum mechanics and particle physics or any other nongravitational laws and therefore be able to analyse by them any experimental result. It should also be capable of explaining all astronomical phenomena including those of strong gravity like quasars, pulsars, neutron star, etc. The theory that lacks this capability is considered incomplete.

(v) Self-consistency

That it should not lead to controversial results if different methods, based on the theory, are used.

Any theory that violates any of these criteria is basically nonviable. Newton's theory is an example of nonviable theories since it violates at least (i), (iii) and (iv). Birkhoff's theory [6] violates, for instance (ii) since it predicts that sound travels with the speed of light. Though it satisfies the basic tests of general relativity. The Whitehead theory [7] of gravity although it copes with the general relativity tests, it was contrarily realised [8] [9] that it contradicts the everyday observations that the ebb and flow of ocean tides are independent of time!

The Milne kinematical theory of relativity [10] does not satisfy (iv) because it is mathematically incapable of saying anything about the gravitational red shift of light. Kustaanheimo's theory [11] violates (v) since it gives two different predictions for the red shift, if light is regarded as an electromagnetic wave, there will be no red shift, but if the photon nature of light is considered it leads to nonzero shift.

There are many other theories that fail to fulfill the above requirements and hence are nonviable, but those which are viable divide into two categories, the so-called, "metric" and "non-metric" theories of gravity.

(2.2) Metric and non-metric theories

By metric theory of gravity, generally speaking, we mean that one in which gravitation is identified by curvature of space-time, or more precisely, it is that theory which incorporates the following two principles:

(i) Space-time has a metric, i.e. a fundamental geometric object describing gravity.

(ii) This metric satisfies the equivalence principle.

Being based on these principles, the following definition due to Thorne and Will [12] was given.

Definition:

A theory of gravity is a "metric theory" if and only if it can be given a mathematical representation in which two conditions hold:

Condition i. There exists a metric of signature 2, which governs proper length and proper time measurements in the usual manner of special and general relativity as in [1.4.3]:

$$ds^2 = -g_{\mu\nu} dx^\mu \, dx^\nu \qquad [2.2.1]$$

Condition ii Stressed matter and fields being acted upon by gravity respond in accordance with the equation

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad [2.2.3]$$

where $T^{\mu\nu}$ is the total stress-energy tensor for all matter and nongravitational fields, and where the covariant derivative with respect to $g_{\mu\nu}$ is denoted by the semi-colon (;). As it was explained in (1.7) the signature may be as well -2 depending on the adopted convention.

Einstein's general relativity among other gravitational theories [13] which fulfilled the above stated conditions, serves the best example of a metric theory.

Metric theories differ from each other in the way matter affects space-time metric. While nonmetric theories do not relate gravity and matter with the space-time curvature and therefore do not satisfy the conditions [2.2.1] and [2.2.3].

The basically viable theories are the metric ones since they satisfy the criteria stated in the preceding section. All nonmetric theories are basically nonviable except, say, Cartan's [14], [15] which agrees with experiment to the same accuracy as Einstein's general relativity does. Though, it is claimed that Cartan's theory violates the viability criterion of being relativistic, i.e. it does not reduce to the laws of special relativity in the absence of gravitation. In fact, a nonmetric theory usually violates at least one of the viability criteria.

The strongest criterion for testing the viability of any theory is its agreement with experiment therefore, in order to test the viability of nonmetric theory the following two so-called foundational experiments are utilized.

The Foundational Experiments

(a) EÖTVÖS EXPERIMENT [16]. This experiment which had been first conducted by Eötvös and later on improved to a higher accuracy by Dicke [17] and by Braginsky [18], demonstrates the indistinguishability of mass in any gravitational field with that in an accelerated system, i.e. by comparing the ratio of the gravitational mass to the inertial mass M_G/M_I for various materials, or their free-fall rates.

This experiment gives an experimental verification of the so-called weak principle of equivalence which states that the trajectories of electrically neutral small bodies are independent of their composition.

WEAK PRINCIPLE OF EQUIVALENCE.

According to Dicke [19] it reads [20]:

> If an uncharged test body is placed at an initial event in space-time, and is given an initial velocity there, then its subsequent world line will be independent of its internal structure and composition.

This is an alternative weak version of the statement given in Section 2 of the first chapter, where here, by uncharged test body we mean that it is shielded from external fields or particles and that the self-gravitational field of the body is negligibly small.

SCHIFF CONJECTURE

L. Schiff made a conjecture stating that every nonmetric theory violates the weak equivalence principle and therefore it is nonviable, since, conversely, the basically viable gravitational theory agreeing with the weak equivalence principle is necessarily metric. It reads [21]:

> Any complete and self-consistent gravitational theory that obeys the weak equivalence principle must also, unavoidably, obey Einstein's equivalence principle.

It was shown by Lightman and Lee [22] that Schiff's conjecture together with Eötvös-Dicke-Braginsky experiment can be used to rule out many nonmetric theories from being viable.

(b) RED SHIFT EXPERIMENT [23]. This experiment measures the shift of the light spectrum towards the red end due to the presence of gravitational field.

Experiments [24] [25] [26] had been conducted with a high precision to measure the red shift of the earth's gravitational field. It was found that every metric theory with the right

Newtonian limit predicts the same red shift which has been confirmed by these experiments. However, different nonmetric theories generally predict different red shifts, even though they have the correct Newtonian limit.

The red shift experiment, therefore, like Eötvös -type experiments, besides confirming GTR, they, infact, test the validity of nonmetric theories and hence their viability.

THE UNIVERSALITY OF GRAVITATIONAL RED-SHIFT

Schild [27] regarded the gravitational red shift as an evidence of a curving of space and hence he concluded that the correct theory of gravity must be metric.

To be more rigorous, we refer to the previous definition for metric theories in (2.2) which implies that

(i) trajectories of freely falling test bodies are geodesics of $g_{\mu\nu}$.

(ii) in local freely falling systems the physical laws coincide with those of special relativity

i.e. locally, the frequencies of atomic clocks measured in these freely falling systems will not be affected by external gravitational field and will depend only on the universal constants like the velocity of light c, Planck's constant $\hbar$, and the electronic charge e. The gravitational red shift will, therefore, be universal because the comparison of the frequencies is determined solely by the comparison of the trajectories which are universal. We then write the following conjecture as worded by Clifford Will [28].

> Any complete, self-consistent and relatavistic theory of gravity that embodies the universality of gravitational red shift is necessarily a metric theory.

Nonmetric theories violate the above mentioned conditions and hence they violate the universality of gravitational red-shift, and therefore, they may predict a gravitational red shift which depends on the nature of the clock to be used in the red shift experiment.

The two above-mentioned experiments in (a) and (b) are considered complementing each other in testing the viability of nonmetric theory. In fact, both of them are interlinked to the principle of equivalence, Eötvös experiment through Schiff's conjecture and the red shift experiment through the conjecture stated by Clifford Will on the universality of gravitational red shift.

Up to now there is, with one exception, no nonmetric theory which does not fail the test by at least one of the two foundational experiments and therefore be filtered out from the class of viable theories. The exception is Cartan's theory of gravity [14] which although it is nonmetric it, as argued, obeys the Einstein's equivalence principle. See (2.6) and (4.6).

As an example of nonmetric theory which is well developed and agreeing with some experiments and yet ruled out from being viable, we have:

- <u>Belinfante-Swihart theory of gravity.</u>

This theory was constructed by Belinfante and Swihart [29] [30] [31] in 1957 to be Lorentz invariant and quantizable.

Like general relativity the theory is Lagrangian-based and agree with the three classical tests of gravitation; red shift of light, bending of light and perihelion shift of Mercury. But it violates Eötvös-type experiment and partially agrees with the weak equivalence principle, i.e. does not cope with Einstein's principle of equivalence as it is expected by Schiff's conjecture.

The theory has been analysed by Lee and Lightman [32] where attempts were made to put it in the metric form, though this did not save the theory from being nonviable. Because of this nonviability we are not going to consider the theory further. Due to the defectiveness of nearly all nonmetric theories the whole attention now is confined to the metric theories of gravity which are regarded as the most viable models.

A brief account of Cartan's theory will be given in Chapter Four together with the strong-field models. This is sometimes called Cartan-Einstein theory [15] different from Cartan's theory with Newtonian spacetime (1923) [33] which is also nonmetric but nonviable.

In this last theory Cartan introduced his idea of stratifying Newton's Euclidean space into slices with flat geometry and constant universal time which he regarded as a scalar field function.

We will come back to the idea of stratification in (2.4) when we briefly consider the stratified metric theories of gravity.

(2.3) Parametrized Post-Newtonian Formalism

THE SUPER METRIC THEORY

Each viable metric theory of gravity differs from others by the way the metric is generated, i.e. by types of the gravitational fields and the mode of their interaction. On the other hand, but parallel to that, theories differ among themselves in how far everyone agrees with experiment.

It was also noticed [34] that just beyond the Newton's limit, i.e. post-Newton approximation, all metric theories have the same form and can, therefore, be unified in one SUPERMETRIC theory [34] whose most general mathematical formula will contain certain coefficients-parameters.

Adding corrections $K_{\mu\nu}$ to the components of the metric tensor, i.e.

$$g_{\mu\nu} = \delta_{\mu\nu} + K_{\mu\nu} \qquad [2.3.1]$$

and imposing certain constraints [35] on the form of these corrections, will lead [36] to the following expressions for the components of the metric coefficients $g_{0\nu}$ and g_{00} [37];

$$g_{0\nu} = K_{0\nu} = -\frac{7}{2}\Delta_1 V_\nu - \tfrac{1}{2}\Delta_2 W_\nu + O(\varepsilon^5), \qquad [2.3.2]$$

and

$$g_{00} = -1 + 2\gamma\phi + K_{00} = -1 + 2\gamma\phi - 2\beta\phi^2$$
$$+ \beta_1\psi_1 + \beta_2\psi_2 + \beta_3\psi_3 + \beta_4\psi_4 - \zeta P - \eta D \qquad [2.3.3]$$

where ϕ, ψ, Δ_1, Δ_2, P,D are integrals for the fields [38] which die out not slower than $\frac{1}{r}$ far from the solar system. ϕ stands for the Newtonian potential defined by Poisson equation [1.6.2] and having the solution:

$$\phi(\bar{x},t) = -G \int \frac{\rho(\bar{x},t)}{|\bar{x}-x'|} d^3x' \qquad [2.3.4]$$

THE PARAMETRISED POST-NEWTONIAN PARAMETERS

Other integrals in [2.3.2, 3] represent the gravitational potential and the ten coefficients γ, β, β_1, β_2, β_3, β_4, ζ, η D_1, D_2 are unknown constants called Paramatrized Post Newtonian (PPN) parameters first introduced by Nordtvedt [39].

Every individual metric theory can be obtained by a certain choice of numerical values of these parameters or, conversely, these parameters can be determined by comparing the coefficients of the equations of the given theory with those in [2.3.2, 3].

A physical significance is attributed to each of the ten PPN parameters.

γ- indicates how strongly a unit of mass curves the space-time.

β - denotes the nonlinearity in gravitational laws due to the combination of gravitational potentials of different bodies.

η - characterizes the effects due to anisortropy in pressure.

The revised parameters [40] $\alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ which are defined by the following combination of parameters:

$$\left.\begin{aligned} \alpha_1 &\equiv 7\Delta_1 + \Delta_2 - 4\gamma - 4 \\ \alpha_2 &\equiv \Delta_2 + \zeta - 1 \\ \alpha_3 &\equiv 4\beta_1 - 2\gamma - 2 - \zeta \\ \zeta_1 &\equiv \zeta \\ \zeta_2 &\equiv 2\beta + 2\beta_2 - 3\gamma - 1 \\ \zeta_3 &\equiv \beta_3 - 1 \\ \zeta_4 &\equiv \beta_4 - \gamma \end{aligned}\right\} \qquad [2.3.5]$$

are also indicating certain effects: α_1, α_2 and α_3 describe the phenomena connected with the so-called "preferred" frame of reference where the result of observation depends on the velocity of the observer. If at least any one of these α is not zero the theory is a preferred-frame theory [41] [42]. Parameters α_1, α_2 and α_3 measure the size and nature of the effects connected with such frames.

In general relativity all $\alpha = 0$, i.e. the theory does not recognise such "preferred" frames. In fact the preferred frame effect is in conflict with Dicke's [43][44] so-called strong equivalence principle.

THE STRONG EQUIVALENCE PRINCIPLE

This principle states that

(a) The weak principle is valid.

(b) The result of ANY local test experiment whether gravitational or nongravitational (stronger than Einstein's principle which is confined to nongravitational ones) is independent of when and where in the universe it is performed and independent of the velocity of the freely falling apparatus.

As an example of this the dimensionless ratios of non-gravitational constants are independent of location, time and velocity.

The Cavendish experiment [45] measuring the variability of the gravitational constant G, satisfies the Einstein's principle but, the result would violate the strong Principle of Equivalence if it indicated that G was not constant.

The parameters ζ_1, ζ_2, ζ_3, ζ_4 and α_3 indicate the extent to which the laws of conservation of energy, momentum and angular momentum are violated. The non-violation of conservation laws i.e. $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0$ characterizes the so-called conservative theories such as GTR and Brans-Dicke theory. These theories also are free from preferred-frame effects since all $\alpha = 0$.

We notice that Einstein's gravitational theory has a significant position among various theories of gravity and the simple form of PPN parameters ($\gamma=\beta=1$ & $\alpha=\zeta=0$) reflects its consistency with the laws of nature and therefore its viability.

The coefficient η is usually neglected by considering the stress at the post-Newtonian limit as isotropic pressure.

An extra parameter appeared in the calculation of some theories [46] is associated with gravitational effect due to the presence of nearby, say galactic, matter when induced in our solar system. Such effect is not predicted by GTR and many other theories.

(2.4) Classification of Metric Theories of Gravity

By introducing the PPN formalism, Nordtvedt made it possible to analyse experiments and to study and classify different metric theories of gravity according to the numerical values given to the PPN parameters [47], into nearly eight categories.

The following are the known classes:

1. General Relativity
2. Scalar-Tensor theories (Brans and Dicke) [48] [49]
3. Vector-Tensor theories (Nordvedt, Helling and Will) [50].
4. Tensor-Tensor theory (Lee and Lightman) [51].
5. Conformally flat theories (Nordström) [52].
6. Stratified theories (Rosen-type) (Rosen) [53].
7. Stratified theories (Ni-type) (Ni) [2].

8. Quasi-linear theories (Whitehead) [54].

Of these classes only the first three are viable. The others violate specific solar system experiments. It can be shown [2] that this classification is also based on the types of gravitational fields included in the theory and on the way they interact with each other.

Apart from general relativity where only one gravitational field is contained in the theory, other theories deal with a scalar field Φ being generated by matter and nongravitational field, via a wave equation. Examples for these are the Brans-Dicke scalar-tensor theory and conformally flat theories and stratified field theories.

These last two classes are nonviable because of their violent disagreement with certain solar system experiments and observations [52]. We give, here, a brief idea about these two kinds of theories and more details about the ones listed in the above-given compendium can be found in reference [2].

Conformally flat theories [52]

This theory possesses, besides the scalar field ϕ, a flat background metric $d\bar{n}^2$ and a conformally flat physical metric ds^2.

The generated field ϕ and $d\bar{n}^2$ generate the physical metric by the conformal relation

$$ds^2 = e^{-2f(\Phi)}d\bar{n}^2 \qquad [2.4.1]$$

By conformally flat metric we mean that one which is flat for photons and curved for particles. These theories predict no deflection of light therefore disagree with experimental results. In order to avoid this disagreement the following kind of theories is proposed:

<u>The Stratified theories with conformally flat space slices</u> [55].

In this modified version of preceding conformally flat theories a "preferred" universal frame of reference and conformally flat space slices (strata) of this frame are postulated [55]. The whole space-time is not flat.

In these kinds of theories the field Φ generated by the matter and nongravitational field via a wave equation combines with the universal time-coordinate t and $d\bar{n}^2$ to generate the physical metric ds^2 through the following relation:

$$ds^2 = e^{-2f_1(\Phi)}d\bar{n}^2+[e^{2f(\Phi)}- e^{-2f_1(\Phi)}]dt^2 \qquad [2.4.2]$$

In the rest frame of the universe this metric reduces to

$$ds^2 = e^{2f(\Phi)}dt^2 - e^{-2f_1(\Phi)}(dx^2+dy^2+dz^2) \qquad [2.4.3]$$

Different stratified theories will correspond to different choices of the functions f and f_1 and by the field equation for Φ. In these theories the background metric and the universal time coordinate do not change by changing the distribution of the gravitating masses. This is in contrast with the principle of covariance and that is why GTR and Brans-Dicke theory are free of these geometrical aspects which are named "prior geometry" [56].

The "strata" concept and the universal frame had been introduced in the stratified theories in order to remedy the zero deflection of light, but unfortunately the theory still is in conflict with experiment and therefore nonviable.

(2.5) Brans-Dicke-Jordan Theory

The theory was first formulated by Jordan (1948, 1955) [57], Thirry (1948) [58], independently, and also by Brans and Dicke (1961) [59].

This scalar-tensor theory comes immediately after Einstein's as the most viable competitor and hence is considered as the strongest alternative to GTR.

In contrast with general relativity the theory is based on the Machian Principle [60] which states that inertia arises from accelerations with respect to the general mass distribution of the universe.

By this, the inertial masses of particles should represent the interactions of these particles with a long-range field ϕ coupled to the mass density of the universe ρ_M by

$$\nabla^2 \phi \sim \rho_M \sim T_{M\circ}^{\circ} \qquad [2.5.1]$$

Or in the generally covariant form

$$\phi_{;}{}^{\sigma}{}_{;\sigma} \equiv \Box^2 \phi = 4\pi \varkappa \, T_{M\mu}^{\mu} \qquad [2.5.2]$$

where $\Box^2 \phi$ is the invariant d'Alembertian.

T_M is the stress-energy tensor of matter and nongravitational fields.

$\varkappa$ is a coupling constant.

As it was mentioned in (2.4) ϕ is generated by matter and nongravitational fields and T_M acts with ϕ to generate spacetime curvature, i.e. the metric.

The total Lagrangian for the whole system can be constructed and the variational principle will be

$$\delta \int \sqrt{g} \, (\phi R + 16\pi \mathcal{L}_M - \omega \frac{\phi_{;\mu} \phi^{;\mu}}{\phi}) \, d\Omega = 0 \qquad [2.5.3]$$

where R is the scalar curvature and $\mathcal{L}_M$ is the matter Lagrangian including nongravitational fields.

It is noticed that if ϕ^{-1} in [2.5.3] be replaced by the gravitational constant G, it will yield the variational principle of GTR given in (1.7).

Thus the resulting field equations read,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{\phi}T_{M\mu\nu}$$

$$-\frac{\omega}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{;\sigma}\phi^{;\sigma}\right)$$

$$-\frac{1}{\phi}\left(\phi_{;\mu;\nu} - g_{\mu\nu}\ \Box^2\phi\right) \qquad [2.5.4]$$

If ω, being a dimensionless constant, is given by

$$\omega = \frac{1}{\varkappa} - \frac{3}{2}\ ; \qquad [2.5.5]$$

called Dicke's coupling constant, then the wave equation [2.5.2] becomes:

$$\phi^{;\sigma}{}_{;\sigma} = \Box^2\phi = \frac{8\pi}{3 + 2\omega}T_{M\mu}{}^{\mu} \qquad [2.5.6]$$

For ω large, then $\Box^2\phi \sim 0\left(\frac{1}{\omega}\right)$. It can be also shown [61] that then, we can write:

$$\phi = \frac{1}{G} + 0\left(\frac{1}{\omega}\right) \qquad [2.5.7]$$

This will make [2.5.4].

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} + 0\left(\frac{1}{\omega}\right) \qquad [2.5.8]$$

and that means for $\omega \to \infty$ [2.5.4] will reduce to Einstein's gravitational field equation [1.6.8]. It is obvious from [2.5.6] that the theory breaks down at $\omega = -3/2$.

It is seen from [2.5.4] that the metric represented by $G_{\mu\nu}$ is generated by $T_{\mu\nu}$ and the scalar field ϕ but the latter has no effect on the matter.

The post-Newton approximation was obtained [2] and the PPN parameters are given by:

$$\left.\begin{array}{lll} \gamma = \dfrac{1+\omega}{2+\omega}, & \beta = 1, & \zeta = 0 \\ \beta_1 = \dfrac{3+2\omega}{4+2\omega}, & \beta_2 = \dfrac{1+2\omega}{4+2\omega}, & \beta_3 = 1 \\ \beta_4 = \dfrac{1+\omega}{2+\omega}, & \Delta_1 = \dfrac{10+7\omega}{14+\omega}, & \Delta_2 = 1 \end{array}\right\} \quad [2.5.9]$$

Where ω may run the values between $-\frac{3}{2}$ and ∞.

Thus ω can be adjusted to fit the best experimental data. For $\omega \to \infty$ parameters in [2.5.9] and consequently in [2.3.5] will coincide with those of general relativity.

Brans-Dicke-Jordan theory, therefore, satisfies the equivalence principle and reasonably agrees with nearly all experiments that test Einstein's theory of gravitation.

We will come back again to this theory in Chapter Four within the frame of scalar-tensor theories (4.4), where its validity in strong gravity is considered.

For brevity this theory is sometimes referred to as simply Brans-Dicke theory.

(2.6) Summary and Critique

In the preceding sections of this chapter we considered different aspects of the gravitational theories those which serve alternative models to general relativity in the weak-field gravity. The viability criteria (2.1) ruled out many theories from consideration, especially those whose mathematical formulation cannot be put in a metric form, i.e. nonmetric theories. Schiff conjecture gives credibility to only the metric ones which are defined, more or less, in accordance with general relativity. By this, Einstein's model has been anticipated to be the standard theory of gravitation with respect to which the validity of other models are to be assessed. But the unreliability of GTR in the areas of strong-field gravity, discussed in Chapter Zero, makes this distinctive position of Einstein's theory questionable.

Moreover, the fact that GTR as well as its sister theories are mostly singular in the vicinity of massive bodies, whereas in the weak gravitational field they predict Newtonian limit, would suggest that Newtonian inverse-square law might break down as from at a certain field strength along the energy spectrum. Fujii-O'Hanlon theory of gravity [62], [63], which satisfies all but one viability requirement due to its predicting a non-Newtonian potential even in the weak field limit, may confirm this situation. Consequently, extra restrictions in the conditions should be imposed so that a gravitational theory can still be viable in the strong-field domains.

Hence the viability conditions should be amended to accommodate strong-field gravity. For instance, the completeness conditions should be understood in the classical sense that the theory be capable of giving definite answers, whether correct or incorrect, in predicting all gravitational as well as nongravitational phenomena, without having being meshed with quantum laws. Otherwise a certain indefiniteness in the predictions will be allowed when quantum mechanical laws are incorporated in the theory.

Another example, Cartan's theory [14], which is considered nonmetric, though accepted as viable because its experimental predictions are almost identical to those of GTR. But this viability is inconsistent with its being nonrelativistic, since the torsion generated by spin does not represent a gravitational field which, by the general covariance principle, disappears in local systems. Trautman [64] refuted this by arguing that since in the real world spin effects decrease with the decrease of mass in the laboratory, therefore the spin-induced torsion automatically disappears with the absence of gravitation. That means the theory can be locally reduced to special relativistic laws. We think that the cogency of either arguments can be reconciled if we look to the torsion as a microscopic effect which arises in strong-field gravity where the applicability of the equivalence principle breaks down as far as that relates to GTR. In weak gravity the spin cancels out each other within the bulk of matter whilst the equivalence principle is meaningful.

In the end we notice in the catalogue of metric theories which differ among themselves with accordance to the values of PPN parameters, GTR possesses an outstanding position by having the simple values 0 and 1 for these parameters. This implies that other viable metric theories are but a modification of Einstein's model whose success will be always judged by how far from general relativity they stand in terms of their predictions or in other words, with respect to their PPN parameters. But we also notice that the PPN formalism has nothing to do with either nonmetric theories or with strong-field gravity where the superiority of GTR over its weak-field competitors is meaningless.

(2.7) Concluding Viewpoint

The above-noted modification of general relativity towards improving its outcome will gain no further success if this modification is made by just adding an auxilliary scalar field additional to Einstein's Lagrangian R. Firstly, because this additional field is already a weak field it will not be having any significant effect in the strong-field gravity. Secondly, it is imposed into the gravitational equation with a strange assumption of being included in the source and at the same time not affecting the equality of inertial and gravitational mass [65], i.e. again sticking to general relativity in one way or another.

Now, since the whole success lies with the geometrical picture of Einstein's theory we think that the modification of GTR will lead to fruitful results if we maintain the geometrical representation by replacing $\mathcal{L} = R$ by a nonlinear geometric Lagrangian. This can be done, in a way, by visualizing the additional field, like that of Brans-Dicke theory, in terms of Riemannian geometry. In other words, by generalizing GTR so that Brans-Dickes additional field may be not just an auxilliary one but a priori embodied in the metric space. We will come back to this issue in Chapter Four when we will deal with strong-field models and in Chapter Five in relation to the generalized gravitational field equations and the generalized Lagrangian.

In the next chapter we consider the solution of the gravitational equations, particularly the metric of Einstein space which represents the data base for all applications and tests of general relativity and comparatively of other weak-field models.

CHAPTER THREE

Gravitational Equations

(3.0) Solutions of the Gravitational Field Equations

Being nonlinear, equations of gravitational field like Einstein's [1.6.8] do not have the exact general solution.

But in certain cases an approximation can be allowed and a sufficiently adequate solution can be obtained while in other cases the equation itself can be simplified and the exact solution can be achieved.

In the case of anisotropic nonstatic systems, two methods of approximation are utilised; the weak-field approximation [1] dealing with matter of relativistic velocities, i.e. gravitational radiation, and the post-Newtonian approximation [2] referred to in (2.3), which is adapted for nonrelativistic bodies e.g. of the solar system.

As for the case of symmetric and static systems, the gravitational field equation can be simplified by assuming either isotropy [3] or axilally symmetry [4] of space, leading to an exact solution due to Schwarzschild [5] and Kerr [6] respectively.

In domains where neither this kind of simplification nor approximation schemes are applicable, namely, in the intensive gravitational radiation, e.g. during the formation of Neutron stars, the exact general solution remains to be the only alternative.

Besides the analytic solutions, computational analysis may play a greater role in solving nonlinear equations of gravity as the numerical methods develop [7].

Finally, although some progress has been achieved [8][9] in attempting to obtain exact solutions for Einstein's gravitational equations, the symmetric-static metrics still have the significance of being the most useful solutions.

In this chapter we confine ourselves to these symmetrized metrics where we consider the empty-space Scwarzschild solution as well as its generalized and transformed versions. We shall not go into the detailed derivations of the metrics and the resulting application formulae, which we regard as a textual information obtainable from any standard literature on general relativity (see for example [10] or [11]).

As we are concerned with a nonsingular gravity in our present work, we deem necessary to be well acquainted with the singular behaviour of Schwarzschild metric, the metric relative to which we may assess the validity of any other alternative model. The concept of "black holes" [12] as being related to these singularities is discussed and criticized.

(3.1) Solution of Einstein Free-Field Equation

In the free space outside the matter generating the gravitational field, Einstein's equation will be having the simple form [1.7.20]; though it is too complicated to be solved in general terms as being of second order partial differential equations for $g_{\mu\nu}$. This is clear from the definition of the Ricci tensor [1.4.25] which can be contracted from [1.4.24] to have the form:

$$R_{\mu\nu} = \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} \qquad [3.1.1]$$

But since we know that the equation is built from the metric tensor, therefore we will rather look for the structure of $g_{\mu\nu}$, i.e. the solution of the metric tensor in terms of space coordinates. Thus, in order to study the field represented by the metric we shall study the metric and for this purpose we choose the simplest form with space symmetry and independence of time, i.e. isotropic, or axially symmetric, static metrics.

a. Static Isotropic Metric

In this metric the gravitational field will depend on the spatial coordinates in the form $\bar{X}^2$, $d\bar{X}^2$, $\bar{X}\cdot d\bar{X}$ being rotational invariants.

The invariant proper time interval [1.4.3] should be the same for all points in symmetrical positions.

By using the spherical coordinates one can come to the following static isotropic metric [13]:

$$ds^2 = -g_{\mu\nu} dX^\mu dX^\nu$$
$$= -[A(r)dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2 - B(r)dt^2] \qquad [3.1.2]$$

where,

$$\left.\begin{aligned} g_{rr} &= A(r) \\ g_{\theta\theta} &= r^2 \\ g_{\phi\phi} &= g_{\theta\theta}\sin^2\theta \\ g_{tt} &= -B(r) \\ g_{\mu\nu} &= 0 \qquad \text{for} \quad \mu \neq \nu \end{aligned}\right\} \qquad [3.1.3]$$

and owing to [1.4.11]

$$g_{\mu\mu} = (g^{\mu\mu})^{-1} \qquad [3.1.4]$$

The function A and B can be determined from the solution of the field equation.

Further, by applying [3.1.3] to [1.4.9] and [1.4.8] or [1.4.19] one obtains

$$\left.\begin{aligned} &\Gamma^r_{rr} = \frac{\dot{A}}{2A}, \ \Gamma^r_{\theta\theta} = \frac{-r}{A}; \ \Gamma^r_{\phi\phi} = \frac{-r\sin^2\theta}{A}, \ \Gamma^r_{tt} = \frac{\dot{B}}{2A} \\ &\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \\ &\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \\ &\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{\dot{B}}{2B} \\ &\Gamma^\lambda_{\mu\nu} = 0 \quad \text{for all other indices} \end{aligned}\right\} \qquad [3.1.5]$$

and by using [3.1.5] in [3.1.1] the components of the Ricci tensor will have the following form,

$$
\left.\begin{aligned}
R_{rr} &= \frac{\ddot{B}}{2B} - \frac{1}{4}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{B} - \frac{1}{r}\frac{\dot{A}}{A} \\
R_{\theta\theta} &= \frac{R_{\phi\phi}}{\sin^2\theta} = -1 + \frac{r}{2A}\left(-\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{1}{A} \\
R_{tt} &= -\frac{\ddot{B}}{2A} + \frac{1}{4}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{A} - \frac{\dot{B}}{rA} \\
R_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu
\end{aligned}\right\} \qquad [3.1.6]
$$

where

$$\dot{A} \equiv \frac{dA}{dr}, \quad \text{and} \quad \dot{B} \equiv \frac{dB}{dr}$$

From [3.1.6] one obtains the following relation:

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \qquad [3.1.7]$$

The scalar curvature can be calculated from [1.4.26] and [3.1.4,5] to give

$$R = \frac{R_{rr}}{A} - \frac{R_{tt}}{B} + \frac{2R_{\theta\theta}}{r^2} \qquad [3.1.8]$$

and by using [3.1.6] we come to the following expression for R

$$R = \frac{\ddot{B}}{AB} - \frac{1}{2A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{B} + \frac{2}{rA}\left(-\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{2}{r^2A}(1-A) \qquad [3.1.9]$$

i.e. the scalar curvature in the static isotropic metric does not depend on either the time or θ and ϕ, it is a function of only r,

$$R = R(r). \qquad [3.1.10]$$

The Schwarzschild Metric

In the vacuum Einstein's equation [1.7.20] gives,

$$R_{rr} = R_{\theta\theta} = R_{tt} = 0 \qquad [3.1.11]$$

which by [3.1.7] yields

$$\frac{d}{dr}(AB) = 0 \qquad [3.1.12]$$

$$AB = \text{constant}$$

$$A(\infty) = B(\infty) = 1 \qquad [3.1.13]$$

being the flatness condition at large r, and therefore

$$A(r) = B^{-1}(r) \qquad [3.1.14]$$

Using [3.1.14] in [3.1.6] will yield

$$\frac{d}{dr}\left(\frac{r}{A}\right) = 1$$

and,

$$\dot{B} = C_1 r^{-2},$$

which by integration yield the following solutions for A and B;

$$\left.\begin{aligned} A(r) &= \left(1 + \frac{C_1}{r}\right)^{-1} \\ B(r) &= 1 + \frac{C_1}{r} \end{aligned}\right\} \qquad [3.1.15]$$

The gravitational Newtonian potential of source mass M,

$$\phi = \frac{-MG}{r} \qquad [3.1.16]$$

which is related to g_{tt} by

$$-g_{tt} = B \xrightarrow{r \to \text{large}} 1 + 2\phi , \qquad [3.1.17]$$

if entered in [3.1.15] the following metric space due to Schwarzschild (1916) [14], results from [3.1.2]:

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad [3.1.18]$$

By transforming the time coordinate in the form,

$$\tau = t + 2MG\ln|1-r/2M| \qquad [3.1.19]$$

and accordingly the metric tensor components Eddington (1924) [15] obtained the following expression for [3.1.18],

$$ds^2 = d\tau^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{2MG}{r}(d\tau + dr)^2 \qquad [3.1.20]$$

where the last term represents the nonflatness of space.

The Schwarzschild metric [3.1.19] although having singularities at $r = 0$ and $r = 2MG$, was very useful in studying celestial and astrophysical problems, namely those investigating static gravitational field exterior to spherically symmetric bodies like, say, stars.

b - Axially symmetric metric

Besides the Schwarzschild solution which is confined to static gravitational fields, Lense and Thirring [16] (1918) tackled the problem of gravitational field generated by a rotating spherical body by using a perturbational method. But their solution, being approximate, is not useful for describing the strong field of rotating stars.

The problem of axially symmetric stationary gravitational field was successfully treated by Kerr (1963) [17]. By stationary we mean time-independent but not static.

The Kerr solution will be a generalization for Schwarzschild space when the stationary field of a rotating body is considered instead of static gravitational fields. The theory accounting for electromagnetic effect, by incorporating the electric charge, and including the intrinsic angular momentum of the rotating body will be having a metric space called Kerr-Newman [18] geometry, which will reduce to the so-called Reissner-Nortström [19][20] form for nonrotating objects, or to Kerr metric for the vanishing electric charge.

It is obvious that in the case of zero electric charge and zero angular momentum, i.e. static body, the Kerr-Newman solution coincides with the Schwarzschild's.

The Kerr Metric

Commencing with Eddington (Eddington-Finkelstein [21]) form [3.1.20] of the Schwarzschild metric, in terms of the Boyer-Linquist coordinates [22] [23], one can obtain [24] for the axially symmetric rotating body, the following solution of Einstein's free field equations;

$$ds^2 = - A(\rho,\theta)d\rho^2 - B(\rho,\theta)dt^2 - C(\rho,\theta)d\theta^2 - D(\rho,\theta)d\phi^2 - E(\rho,\theta)dtd\phi, \qquad [3.1.21]$$

with

$$\left.\begin{aligned}
A &\equiv \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2-2MG\rho}\ , \\
B &\equiv \frac{2MG\rho}{\rho^2+a^2\cos^2\theta} - 1, \\
C &\equiv \rho^2 + a^2\cos^2\theta, \\
D &\equiv (\rho^2+a^2)\sin^2\theta + \frac{2MG\rho a^2\sin^4\theta}{\rho^2+a^2\cos^2\theta}\ , \\
E &\equiv \frac{4MG\rho a\sin^2\theta}{\rho^2+a^2\cos^2\theta}\ ,
\end{aligned}\right\} \quad [3.1.22]$$

and where the radial marker ρ is defined in terms of the radius r and the constant parameter a, as:

$$\rho^2 = r^2 - a^2\sin^2\theta \qquad [3.1.23]$$

It is clear that when $a = 0$, $\rho = r$ and in [3.1.22] $E = 0$, i.e. the last cross term in [3.1.21] disappears and the metric will then reduce to the Schwarzschild's space time.

[3.1.21] was first obtained by Kerr (1963) [25] for axially symmetric time-independent gravitational field exterior to a rotating object.

This axially symmetry and time-independence is manifestly shown by the absence of both ϕ and t in the coefficients definitions of [3.1.22].

There are some other features of solution [3.1.21] which we can summarize as follows.

(i) If $t \to -t$ and $\phi \to -\phi$, ds^2 will not be changed. This physically means the equivalence of running time backwards with a negative spin direction to running time forward with positive spin.

(ii) If $a = 0$, ds^2 reduces by [3.1.22] to the static isotropic metric, this gives indication that the parameter a is a measure of the angular momentum per unit mass of the source.

(iii) Also the change of $\phi \to -\phi$ and $a \to -a$ will leave ds^2 unchanged which implies that a specifies spin direction.

(3.2) Post-Schwarzschild metric

The fact that Einstein's is not the only successful theory owing to its having strong competitors, the Brans-Dicke-Jordan theory, make us think that Schwarzschild metric which is derived from the equations of general relativity may not be unique, at least, in describing the gravitational field generated by static spherically symmetric bodies. The metric may, therefore, be modified in such a way that the metric coefficients $A(r)$ and $B(r)$ in [3.1.18] will contain small corrections. To do so we expand these metric coefficients as power series in the small value MG/r being the gravitational Newtonian potential [3.1.16]. That yields:

$$B(r) = 1 - \frac{2\alpha MG}{r} + 2(\beta-\gamma\alpha)\frac{M^2G^2}{r^2} + \qquad [3.2.1]$$

$$A(r) = 1 + 2\gamma\frac{MG}{r} + \ldots. \qquad [3.2.2]$$

where α,β,γ are dimensionless constants. This expansion, which was first made by Eddington [26] and Robertson [27] reduces to Schwarzschild's solution of Einstein's equation when

$$\alpha = \beta = \gamma = 1 \qquad [3.2.3]$$

Indeed, since the centrepetal acceleration g of a planet moving with speed v around the sun (with $M_\odot G/r<<1$ and $v^2<<1$, i.e. far and slowly) is determined by:

$$g = \frac{M_\odot G}{r^2} \qquad [3.2.4]$$

whereas from [1.4.8] and [1.4.9] we have

$$g = \Gamma^r_{tt} = -\tfrac{1}{2}\frac{\partial g_{tt}}{\partial r} = \alpha\frac{M_\odot G}{r^2} \qquad [3.2.5]$$

By comparing this with [3.2.4] one gets $\alpha = 1$. As for the values of β and γ being equal unity, one can verify that by virtue of general relativity tests [28].

Brans-Dicke-Jordan theory [2.5], in its post-Newtonian approximation [2.5.9], has the same form as Robertson isotropic metric:

$$ds^2 = \left[1-2\alpha \frac{MG}{r} + (2\beta-2\alpha\gamma) \frac{M^2G^2}{r^2} + \cdots\right]dt^2$$

$$-(1+2\gamma \frac{MG}{r} + \cdots)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\ \phi^2 \qquad [3.2.6]$$

with $\alpha = \beta = 1$ agreeing with general relativity, but

$$\gamma = \frac{\omega+1}{2+\omega}. \qquad [3.2.7]$$

Here, Brans-Dicke-Jordan theory has the advantage over Einstein's, since the parameter ω can be always adjusted to fit the best experimental data.

Finally, the metric [3.2.6] as well as Schwarzschild's metric indicates that the solutions for the gravitational field equation at any point outside the static spherically symmetrical body depend only on its mass and not on any other of its characteristics.

As for the stationary gravitational field at any point exterior to an axially symmetric rotating body the solution will not depend on any other property of this body other than its mass and angular momentum.

In the following sections we shall consider the applications of expressions [3.2.1] and [3.2.2] to the weak-field of the solar system gravitation. We shall also consider the consequence of applying such a metric to the strong gravitational fields.

(3.3) Application of Schwarzschild Solution to Solar System's Gravitational Phenomena.

Being the data-base for all weak-field predictions of general relativity, Schwarzschild solution is quite applicable to the gravitational field of the sun, where it describes well the behaviour of the metric space at distances from the sun $r > r_\odot > 2M_\odot G$, with $r_\odot$ denoting the sun's radius and $2M_\odot G$ the Schwarzschild's gravitational radius for the sun at which space geometry becomes infinite. For simplicity the sun can be considered a correctly shaped sphere and static, that means any possible anomalous effect produced by the solar oblateness can be ignored.

We introduce in brief the classic tests of general relativity that are based on Schwarzschild's description. As we have stated before in (3.0) we shall not elaborate on the derivations of the formulae predicting different gravitational phenomena, which are obtainable from any standard text book on general relativity, but rather confine ourselves to how far these predictions agree with experiment.

Let us first obtain the equation which describes the motion of a particle in static isotropic gravitational field represented by the metric [3.1.18].

By referring to [1.4.1] defining the proper time ds, the equation of motion of a particle, fixed to a freely falling coordinate η^μ in a purely gravitational field, w.r. to this (locally Lorentzian) coordinate reads:

$$\frac{d^2}{ds^2}\eta^\mu = 0. \qquad [3.3.1]$$

Now by considering the coordinates η^μ as a function of other coordinates, say x^ν, equation [3.3.1] will lead to

$$\frac{d}{ds}\left(\frac{\partial\eta^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}\right) = \frac{\partial\eta^\mu d^2x^\nu}{\partial x\, ds^2} + \frac{\partial^2\eta^\mu}{\partial x^\mu \partial x^\nu} \cdot \frac{dx^\nu}{ds}\frac{dx^\mu}{ds} = 0$$

or

$$\frac{d^2x^\lambda}{ds^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \qquad [3.3.2]$$

where the coefficient $\Gamma^\lambda_{\mu\nu}$ of the transformation of a coordinate from one point in one manifold to another, is defined by

$$\frac{\partial x^\lambda}{\partial\eta^\alpha}\frac{\partial^2\eta^\alpha}{\partial x^\mu \partial x^\nu} \equiv \Gamma^\lambda_{\mu\nu}. \qquad [3.3.3]$$

It is called the affine connection which is a nontensor, and can be identified with the afore-given in [1.4.9] Christoffel symbol [28]

Now by using the nonvanishing components of $\Gamma^\lambda_{\mu\nu}$ from [3.1.5] and hence the Robertson expansions [3.2.1, 2] of the static isotropic space in [3.3.2], one can derive a relationship expressing the shape of a particle's orbit represented by r, as a function of the coordinate ϕ or conversely,

$$\phi = \phi(r) \qquad [3.3.4]$$

The explicit form of $\phi(r)$ (see [10]) will be employed in predicting the following two phenomena of the gravitational field of the sun.

1. PRECESSION OF THE PERIHELIA OF THE PLANET

Consider a particle or a planet moving in an orbit around the sun. It is predicted, by virtue of [3.3.4] that the planetary orbit precesses in the same direction of the planet's motion, forming an angle

$$\Delta\phi = \frac{6\pi GM_\odot}{L}\left(\frac{2-\beta+2\gamma}{3}\right) \text{ radians per revolution} \qquad [3.3.5]$$

where L, called semilatus rectum, determines the dimension of the eliptical orbit and can be found from astronomical tables for the eccentricity **e**, and the semi-major axis **a** of any known planet by the formula

$$L = (1-e^2)a \qquad [3.3.6]$$

For general relativity the parameters $\gamma = \beta = 1$, therefore [3.3.3] reads

$$\Delta\phi_{GTR} = \frac{6\pi GM_\odot}{L} \text{ radians per revolution} \qquad [3.3.7]$$

As we have $GM_\odot = 1.475$ KM and for Mercury $L = 5.54\times10^7$ KM and since it makes 415 revolutions around the sun per century, then we get for this planet

$$\Delta\phi_{GTR} = 43.03'' \text{ per century.} \qquad [3.3.8]$$

The observed value for the centennial precession of Mercury was found by Clemence [29] to be

$$\Delta\phi_{obs} = 43.11 \pm 0.45'' \text{ per century} \qquad [3.3.9]$$

which is an excellent agreement with the predicted value. This high accuracy results from the analysis of the accurate observations accumulated since the last two centuries. Usually astronomers use light rays which, as we shall see below, are affected by the sun's gravitational field and hence systematic errors will occur, but because the precession is cumulative, this kind of error will be more insignificant, the more revolutions are counted. For this situation this test is considered the most important in verifying general relativity. The Brans-Dicke version gives by [3.2.7] and with $\beta = 1$ the following formula.

$$\Delta\phi_{BD} = \frac{3\omega+4}{3\omega+6}\,\Delta\phi_{GTR}\ . \qquad [3.3.10]$$

The parameter ω in this relation and β and γ in [3.3.5] can be well estimated by virtue of [3.3.9].

As for other planets like Venus, the earth and Icarus, the accuracy of the observations will become less as the planet is orbiting far from the sun and the uncertainty in estimating the errors will be greater.

2 - DEFLECTION OF LIGHT BY THE SUN.

In this phenomena we consider as a particle the photon of light which isnot bound in an orbit around the sun. The light coming from a great distance where the metric is Minkowskian, and approaching the sun, could continue travelling in straight lines, but because of the solar gravitational field, it will be deflected towards the sun forming an angle,

$$\Delta\phi = 2\ |\ \phi(r_0) - \phi_\infty\ | - \pi \qquad [3.3.11]$$

where $\phi(r)$ is the explicit expression for [3.3.4] and $\phi = \phi_\infty$ as $A(\infty) = B(\infty) = 1$, and r_0 the distance of the closest approach to the sun which will coincide with the solar radius $r_\odot$ as the light rays graze the sun's surface. This yields

$$\Delta\phi = \frac{4GM_\odot}{r_\odot}\left(\frac{1+\gamma}{2}\right) = 0.8755(1+\gamma) \qquad [3.3.12]$$

GTR with $\gamma = 1$ predicts

$$\Delta\phi_{GTR} = \frac{4GM_\odot}{r_\odot} = 1.75'', \qquad [3.3.13]$$

and Brans-Dicke theory with γ given by [3.2.7] yields,

$$\Delta\phi_{BD} = \frac{2\omega+3}{2\omega+4}\,\Delta\phi_{GTR} \qquad [3.3.14]$$

The deflection of light by any other body less massive than the sun will be negligibly small.

The deflection angle is measured by comparing the apparent positions of stars during the eclipse of the sun when they appear to lie next to the solar disc with their positions half a year later when their observed images will be mostly unaffected by the sun's gravity. Such a measurement was made during the 1952 eclipse in the Sudan [30] and was in good agreement with GTR giving the following deflection angle.

$$\left.\begin{array}{l} \Delta\phi_{obs} = 1.70 \pm 0.10'' \\ \text{while } \gamma \text{ is found to have the value} \quad \gamma = 0.94 \pm 0.12 \end{array}\right\} \qquad [3.3.15]$$

Beside the optical observations radio astronomical means have been recently utilized with a potentially greater accuracy in measuring the deflection of radio waves by the solar gravitational field. This yielded [31]:

$$\left.\begin{aligned} \Delta\phi_{obs} &= 1.77 \pm 0.20'' \\ \text{with} \qquad \gamma &= 1.02 \pm 0.24 \end{aligned}\right\} \qquad [3.3.16]$$

Thus one may see that despite the technical and observational difficulties [10] surrounding these kinds of experiments, the results obtained fairly confirm Einstein's general relativity. Further, besides the afore-discussed gravitational phenomena which are predicted by general relativity we introduce the following third test of Einstein's theory, i.e.

3 - RADAR ECHO DELAY

This time general relativity predicts a delay in time a radar signal takes to travel to an inner planet (Mercury) and back again. By exploiting the geodesic equation [3.3.2] we obtain an equation governing the time history of the photon trajectory whose solution will determine the motion as a function of time [10]. Then one can calculate the time required for a light to travel from position r_1 to position r_2 via r_0, i.e

$$t_{12} = t(r_1,r_0) + t(r_2,r_0) \qquad [3.3.17]$$

The delay in time will be maximum when a light signal or the radio echo, grazes the solar surface in its round-trip. This yields the following excess time delay;

$$(\Delta t)_{max} = 4GM_{\odot}\left\{1 + \left(\frac{1+\gamma}{2}\right)\ln\left(\frac{4r_{\oplus}r_{☿}}{r_{\odot}}\right)\right\}$$

$$= 5.9KM \left|1 + 11.2\left(\frac{1+\gamma}{2}\right)\right| \qquad [3.3.18]$$

where $r_{\oplus}$ and $r_{☿}$ are the distances to the sun from the earth and Mercury respectively. Then general relativity (γ=1) would predict.

$$(\Delta t)_{max} \simeq 72 \text{ KM} = 240\ \mu\text{Sec}. \qquad [3.3.19]$$

Series of experiments have been carried out by Shapiro and collaborators [32] who used Mercury as a reflector when it was at superior conjunction. In other experiments [33] the artificial satellites Mariner 6 and 7 were used as reflectors for the electromagnetic signals. Although these experiments encountered considerable difficulties [34], the results obtained are comparable with those obtained from the deflection of light of distant stars by the solar gravity. These experiments gave the following values for the parameter γ.

$\gamma = 1.03 \pm 0.04$ by using Venus and Mercury as reflectors (1967-70),

and $\gamma = 1.00 \pm 0.028$ by using the spacecrafts Mariner 6 and 7 as reflectors (1969-71).

Furthermore, we add here two other experiments 4 and 5 that test the validity of general relativity, but differently from the preceding tests, are not based on Schwarzschild's solution.

4 - PRECESSION OF ORBITING GYROSCOPES

In this test a gyroscope was proposed [35] to be placed in an orbit around the earth. The precession of the spin vector S_μ can be used to measure the fine details of the earth's gravitational field. Owing to the rotation of the earth the system is not static and not isotropic and therefore the Schwarzschild metric is not applicable. By treating the gyroscope as a point particle the frame can be regarded as locally inertial in which the spin does not precess. In a general coordinate system spin precesses according to the equation

$$\frac{dS_\mu}{ds} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{ds} \qquad [3.3.20]$$

The angle θ between the direction of the spin of the gyroscope $\vec{S}_g$ and the velocity vector of light rays $\vec{U}_g$ coming from a distant star is given by

$$\theta = \cos^{-1}\left(\frac{\vec{S}_g \cdot \vec{U}_g}{|\vec{S}_g|\,|\vec{U}_g|}\right) \qquad [3.3.21]$$

It can be measured by focussing the star's image on photoelectric cells fixed to the gyroscope in such a way that any change of θ will cause the image to move over the cells. Consequently, the photoelectric current will be produced describing the precession $\vec{\Omega}$ of the spin with which a value for the constant γ can be obtained according to the relation [36]

$$\vec{\Omega} = -\left(\tfrac{1}{2} + \gamma\right)(\vec{\nabla} \times \vec{\nabla}\Phi) \qquad [3.3.22]$$

where Φ the gravitational potential of the earth.

We note that, although the metric under consideration is not static-isotropic, this experiment has been introduced here to test general relativity for only the sake of generality.

5 - THE GRAVITATIONAL RED SHIFT.

This test, like the last one, is not an application of Schwarzschild's solution. It is rather a test for the validity of the equivalence principle and hence confirms the GTR. We have already discussed in (2.2) the universality of the gravitational red shift and how that is related to the universality of the trajectories of the photons. We consider that **n** light waves with frequency $\nu_{\odot}$ are emitted, within $ds_{\odot}$ seconds, on the sun where the potential is $\Phi_{\odot}$. The same number of waves will be received on the earth where the potential is $\Phi_{\oplus}$. Now since the interval between two events is given by:

$$ds^2 = -g_{\mu\lambda}(x)dx^{\mu}dx^{\lambda} = -g_{tt}dt^2 - g_{ik}dx^i dx^k \qquad [3.3.23]$$

then the actual time interval between two events for a given point will yield

$$ds^2 = -g_{tt}dt^2 \quad \text{with} \quad dx^i = dx^k = 0. \qquad [3.3.24]$$

Then by taking into account [3.1.17], the **n** waves received during $ds_{\oplus}$ seconds on the earth will be having frequency $\nu_{\oplus}$ which is different from $\nu_{\odot}$ and hence

$$\mathbf{n} = \nu_{\odot} ds_{\odot} = \nu_{\oplus} ds_{\oplus} \qquad [3.3.25]$$

i.e.

$$\nu_{\odot}\sqrt{1+2\Phi_{\odot}} = \nu_{\oplus}\sqrt{1+2\Phi_{\oplus}} \qquad [3.3.26]$$

Now since the field is weak then we can expand to the first order in Φ and this yields

$$\nu_{\oplus} = \frac{(1+\Phi_{\odot})}{(1+\Phi_{\oplus})}\nu_{\odot} \qquad [3.3.27]$$

or finally,

$$\frac{\nu_{\oplus} - \nu_{\odot}}{\nu_{\odot}} \equiv \frac{\Delta\nu}{\nu_{\odot}} = \Delta\Phi \equiv \Phi_{\odot} - \Phi_{\oplus} \,. \qquad [3.3.28]$$

This formula perdicts the gravitational shift of the spectral lines to the red end. Evidently, since the sun is a massive body, its large potential which is negative w.r. to the earth will make $\Delta\nu$ negative. It means that the frequency would decrease by leaving the sun, and by being received on the earth would shift towards the red end of the light spectrum. This can be explained by the relatively slower vibrations of the atoms on the sun by comparison with those on the earth where the gravitational field is less intense than that on the sun.

Now by calculating the sun's gravitational potential on its surface,

$$\Phi_{\odot} = -\frac{GM_{\odot}}{r_{\odot}} = -2.12\times10^{-6} \quad . \qquad [3.3.29]$$

The frequency of light from the sun [3.3.27] will be shifted towards the red by about 2.12 parts per million.

There are practical difficulties in measuring the red shift due to the Doppler effect caused by the motion of the source, and due to the convection of gasses in the solar atmosphere [37]. However, an empirical evidence that supports the afore-predicted value for the red shift was obtained by Pound and Rebka [38] who performed a terrestrial experiment of highest precision by using Mössbauer effect [39]. In this case [3.3.28] yields,

$$\frac{\Delta\nu}{\nu} = -2.46\times10^{-15} \qquad [3.3.30]$$

which is in excellent agreement with the experimental value [40],

$$\frac{\Delta\nu}{\nu} = (2.57 \pm 0.26)\times10^{-15}. \qquad [3.3.31]$$

For further information about the experimental data concerning solar system experiments the reader may refer to [41] [42][11].

Now at the end of this section we conclude that the last two tests are not relevant to the theme of the present chapter which is mostly confined to the solutions of the gravitational equations and in particular, to the Schwarzschild metric. However, as we noted before, these have been supplemented here for the sake of generality in presentation. Finally, we admit that in the previous sections we mainly investigated Schwarzschild's metric in regions where gravity is considerably weak. It will be rather interesting to consider, in the next section, the behaviour of Schwarzschild's space in its nonlinear part where singular points are exhibited.

(3.4) The Schwarzschild Singularity.

The Schwarzschild metric [3.1.18] corresponding to the solution of Einstein's free-field equation is obviously singular at $r = 0$, and at $r = 2GM$, which is called the Schwarzschild radius of the mass M.

The singularity at the origin is real, since physically, there is no point mass whose gravitational field is *infinite*, and mathematically it is coordinate-independent, i.e. it is unremovable by the use of any coordinate transformation. As for the singularity that appears to occur at $r = 2GM$, one can prove that it does not affect the well-behaved nonzero curvature invariants at this point. By curvature invariants we understand quantities composed of Riemann tensor, its contractions and the metric tensor, those constituting certain constructions of the so-called Weyl's conformal tensor [43][44]. There are only four such nonvanishing invariants. Had the singularity been present w.r. to any of these invariants at the Schwarzschild radius, it would not have disappeared in all coordinates. Thus because the curvature invariants are singular at $r = 0$ and non-singular at $r = 2GM$, this situation will remain so in all coordinate systems. Therefore, in contrast to the real singularity at $r = 0$, the singularity at $r = 2GM$ is "apparent" because, physically, it does not correspond to any breakdown in the laws of nature, and, mathematically, it can be abolished by employing a new set of coordinates like the following one due to Kruskal [45] and Szekeres [46] which describe an unusual topology [47], i.e.

$$\left.\begin{aligned} t' &= C\left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} \cosh\left(\frac{t}{4GM}\right) \exp\left(\frac{r}{4GM}\right) \\ r' &= C\left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} \sinh\left(\frac{t}{4GM}\right) \exp\left(\frac{r}{4GM}\right) \end{aligned}\right\} r > 2GM \qquad [3.4.1]$$

and

$$\left.\begin{aligned} t' &= C\left(1 - \frac{r}{2GM}\right)^{\frac{1}{2}} \sinh\left(\frac{t}{4GM}\right)\exp\left(\frac{r}{4GM}\right) \\ r' &= C\left(1 - \frac{r}{2GM}\right)^{\frac{1}{2}} \cosh\left(\frac{t}{4GM}\right)\exp\left(\frac{r}{4GM}\right) \end{aligned}\right\} r < 2GM \qquad [3.4.2]$$

or

$$r'^2 + t'^2 \equiv \frac{2t'r'}{\tanh\left(\frac{t}{2GM}\right)} \qquad [3.4.3]$$

and

$$r'^2 - t'^2 \equiv C^2\left(1 - \frac{r}{2GM}\right)\exp\left(\frac{r}{2GM}\right) \qquad [3.4.4]$$

where C is an arbitrary constant and where the prime signifies the new coordinates. The resulting from [3.1.18] metric, therefore, becomes,

$$ds^2 = \frac{32G^3M^3C^{-2}}{r} \exp\left(\frac{-r}{2GM}\right)\left(dt'^2 - dr'^2\right) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi \qquad [3.4.5]$$

which now, is singularity-free at the Schwarzschild's radius and still singular at $r = 0$. In fact, at the origin the scalar curvature [3.1.9] will be infinite irrespective of which coordinate system is used as the metric.

Apart from Kruskal-Szekeres, Novikov [48] had suggested in (1963) a well behaved metric. He used the so-called co-moving coordinates [49] in which the radially moving particle remains always at rest and hence the proper time interval between two points coincides with the time measured by the clock attached to the (freely falling) particle. Novikov coordinates though easily visualized, are not conveniently usable. They transform from Schwarzschild metric by very complicated formulae.

More detailed discussions and illustrations about Novikov and Kruskal-Szekeres modifications of Schwarzschild geometry can be found in reference [46].

Now although the singularity at r = 2GM is not real, Schwarzschild solution is still singular, like other symmetrized solutions of general relativity [50]. Then what causes this singularity to happen, say, in Schwarzschild space? The answer can be either because Schwarzschild solution is not the correct general solution of gravitational equations, or because GTR itself is not the perfect model of gravitation. The first possibility could be thought of as due to the symmetry conditions imposed on the space-time, as was claimed by Lifshitz and Khalatnikov [50]. However, Penrose and Hawking [51], employing topological methods, proved in a number of theorems, that space-time singularities can occur under general assumptions of asymmetry.

Hence, it became clear that time-independence and isotropy of Schwarzschild space are not responsible of its singular behaviour. Thus, as the validity of general relativity is a basic assumption of Penrose-Hawking theory, it is obvious that we are left with the second possibility that Einstein's model is not perfect.

In this thesis we show that because of the imperfection of general relativity, the Schwarzschild solution has its limitation in strong field region one of which is the occurrence of singularity and the prediction of gravitational collapse and black holes.

(3.5) Gravitational Collapse and Black Holes

In the introduction to this thesis, Section (0.0), we have mentioned the probably close relation between the quasars and the gravitational collapse being the mechanism which may explain the powerful source of energy indicated by their great optical luminosity.

Now in order to know how the gravitational collapse may affect the geometry of space, let us imagine Schwarzschild metric generated by a star after the latter has shrunk to $r = 0$?. In Kruskal-Szekeres coordinates this situation can be visualized as "wormhole"-like shape connecting two asymptotically flat universes through two singularities at $r = 0$ one in each universe. This implies that in despite of the pathological behaviour of the metric at zero, the external geometry will be unaffected by the disappearance of the star generating the gravitational field.

- BIRKHOFF THEOREM

Birkhoff formulated in (1923) [52] a theorem which states that a spherically symmetric gravitational field in vacuum must be static with a metric given by the Schwarzschild solution. This means that the region of any spherically symmetric empty space-time does belong to Schwarzschild geometry. Thus Birkhoff theorem shows how Schwarzschild geometry can be applied to a collapsing star. It says that although the collapse is a dynamical process the outer region will still be static implying that no radiation can escape to the exterior. Similarly, an electrically neutral axially symmetric body would not affect its external Kerr metric [53] by undergoing collapse, the moment it has settled down to a stationary state.

- COLLAPSE OF A MASSIVE STAR

In astrophysics one recognizes that a massive star whose fuel has been exhausted, would cool and contract inward. The most massive white dwarf can exist if its mass is less than a critical value of 1.2 solar mass called Chandrasekhar limit [54] before it becomes stable. However, if it reaches the end of its thermonuclear evolution with a mass above that limit, its internal pressure will fail to support it and therefore it collapses radially until, possibly, it heats up and explode to a supernova. Then matter might be blown off so that the remnant mass drops below the Chandrasekhar limit and collapses into a neutron star. If however the collapsing mass does not reach an equilibrium and does not drop below the so-called Oppenheimer-Volkoff limit of 0.7 solar mass [55], it is thought that further collapse will take place. Now if the following possibilities like; ejection of matter from the collapsing star, its explosion into fragments with small enough mass so that stable neutron stars or white dwarfs can be formed, or the existence of a limiting curvature, are ruled out, then by general relativity further collapse is predicted to be unavoidable.

HORIZONS AND BLACK HOLES

Once the collapsing star is doomed to further collapse under the gigantic gravitational force then by passing its Schwarzschild radius the gravitational field will be so large that any incoming or outgoing particles including photons will be captured inward. An event horizon of a (trapped surface) is thought to be formed and no light or radio waves can escape from the region $r < 2GM$, which for this reason is called black hole.

The formation of black holes might take place, as believed by astrophysicists, in two other different situations; one of which can be caused by the possible coalition of massive stars, say, at the centre of star cluster in the galactic nuclei. The other, is due to certain perturbations in the early stage of the universe where the so-called primordial black holes were produced. The latter were suggested by Hawking [56] in (1971) to be of a wide mass range starting from about 10^{-5} gram for mini-holes and up to perhaps 10^{15} solar masses [57]. The huge primordial black holes are thought to be grown from mini-holes by continual accretion of matter. They might have constituted the core around which elliptical galaxies condensed [58]. For more information about the progress of the black hole hypothesis the reader may refer to the review by M. J. Rees [59].

On the other hand, inspite of different speculations advocating the realness of black holes, search for them since (1964) has not yet reached the stage to tell decidedly whether or not they do realistically exist [60]. It was hoped that the binary system Cygnus X-1 would be the site where a black hole can most probably be detected. By studying the spectral red shift of the visible component of this system, whose mass and orbital inclination are known, the mass of the invisible component orbiting relatively to the former, can be estimated. Up to now (the writing up of this thesis), it is not decided whether the estimated mass of the collapsed component of this system will represent a neutron star, a white-dwarf or a black hole. This uncertainty will not allow us to consider the outcome of such observations an experimental evidence for the existence of black holes. However, the lack of experimental verification should not prevent us from considering further the theoretical model that lead to their prediction.

Beside the black holes, the concept of "white holes" was introduced [61] as representing an undressed singularities where matter is expelled out into the universe. White holes are thought to have been originated in the chaotic situation after what cosmologists believe to be the initial state of the big bang. Now since it is obvious that this kind of singularity is unobservable the idea about the white holes remains a mere speculation.

Further, it can be understood that the spherical collapse or the collapse with slightly perturbed sphericity is characterized by the following stages, instability, implosion, horizon and according to Penrose [62] a singularity should take place. Now if we assume the formation of an event horizon hiding a black hole, then the field exterior to it, as we noted in (3.2), is uniquely determined by the mass of the collapsing star, in the case of Schwarzschild's geometry, or by the mass, charge and angular momentum in the case of Kerr-Newman geometry [18].

This implies that the black hole has no other characteristics, and that is why it was termed as "black hole without hair" [63].

In the realistic terms, the collapse of a star is nonspherical with net charge either positive or negative, and therefore the resulting black hole is somewhat asymmetric, differently from Schwarzschild's. It was also explained, in heurestic terms [64] that the external geometry, i.e. the gravitational field of a black hole would still be uniquely determined by the same characteristics (mass, charge and angular momentum).

As for the interior geometry beyond Schwarzschild's or any other asymmetric horizon where singularity is believed to occur, we should first define the meaning of the singularity and thence explain what will happen inside the event horizon and how the realistic collapse is thought to come to its final state.

(3.6) <u>On the Penrose-Hawking Singularity</u>

In space-time manifolds we recognize three kinds of geodesics:

1 - TIME-LIKE when the metric is negative, this represents the path of a freely falling observer.

2 - SPACE-LIKE when the metric is positive - it represents the path of the tachyons (hypothetical particles moving with speed greater than the speed of light).

3 - NULL GEODESICS the path of photons.

Besides that, the time-like curves with bounded acceleration (path of a moving observer).

We recognize also that if a regular point has been cut out of the space-time the geodesic will be incomplete. For the time-like geodesics incompleteness would imply that there could be a freely falling observer or a particle whose history terminates after the lapse of a finite proper-time.

If we assume that the manifold is inextensible beyond this termination point then we call this point together with its neighbouring points a "singularity." This definition was given by Schmidt [65].

Similarly, the space-time which is null-geodesically incomplete is also regarded as singular because it represents the history of zero-mass particles. As for space-like geodesics, since we cannot immediately attach any physical meaning to tachyons moving on space-like curves, we may confine ourselves to time-like and null-geodesics. Thus we require that for a space-time to be singularity-free, it should be, at least, non-space-like geodesically complete, or conversely for a space-time to be singular it should be at least, non-space-like geodesically incomplete.

We notice that according to Schmidt's definition of the singularity the Schwarzschild radius does not represent a singular point since the manifold is extensible beyond $r = 2GM$.

Moreover, by examining the two-dimensional spherical surfaces $(r, t) = \text{const.}$, inside Schwarzschild horizon, Penrose devised the concept of trapped surface, a closed two-dimensional space-like surface for which both tne outgoing and the ingoing families of future-directed null geodesics orthogonal to the surface are converging.

In Schwarzschild's geometry this surface must be spherical, however, Hawking and Penrose [66] suggested that the same holds for asymmetric space-times. The convergence of light from both inward and outward directions at the trapped surface can be attributed to the intense gravitational pull where photons of light are sucked into the singularity.

Now having elucidated the concepts of the singularity and the trapped surface we introduce in the following an epitome for the singularity theorem due to Penrose and Hawking. Within the frame of the present thesis, we shall be mainly concerned with the content and the physical significance of the theorem rather than with its formulation and proof for which the reader is referred to the authors' original works, Hawking and Ellis (1973) [67], Hawking and Penrose (1969)[68] and Geroch (1971) [69].

- THE SINGULARITY THEOREM: It states that;

The space-time manifold $\mathcal{M}$ is singular (in the Schmidt sense) by necessarily containing incomplete inextendable non-space-like geodesics if:

(i) the equations of general relativity hold,

(ii) for every non-space-like vector **V** the following inequality holds:

$$R_{\mu\nu} V^{\mu}V^{\nu} \geq 0$$

(iii) $\mathcal{M}$ is general i.e. sufficiently non-symmetric,

(iv) $\mathcal{M}$ contains no closed non-space-like curves

(v) $\mathcal{M}$ contains a closed trapped surface.

The implications of these conditions may be explained in the following:

Condition (i) which obviously means the validity of GTR on the space-time manifold would imply that certain modifications of general relativity will essentially violate the singularity theorem.

Condition (ii) indicates that matter has a focussing effect on geodesics, it also means the positivity of energy and by (i) it yields the equivalent condition on the stress-energy tensor at each event on $\mathcal{M}$, i.e:

$$(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) V^{\mu} V^{\nu} \geq 0$$

Condition (iii), called the generic condition; requires that $\mathcal{M}$ should be not too highly symmetric, in the sense that on every non-spacelike geodesic there is a point at which the tangent vector V satisfies.

$$V_{[\alpha} R_{\beta]\mu\nu[\gamma} V_{\lambda]} V^{\mu} V^{\nu} \neq 0$$

Generality means that the manifold should be curved space-time differently from Minkowsky space which is flat and complete and therefore nonsingular. It also means, as we noted before that the symmetry of Schwarzschild metric has nothing to do with the appearance of the singularity.

Condition (iv) constitutes a reasonable causality condition which states that, the causal future $J^+(P)$ and the causal past $J^-(P)$ of the event P comprise the set

$$J^+(P) \cap J^-(P) \quad P \in \mathcal{M}$$

which should be empty i.e. P is not on a closed non-space-like curve. Where by $J^-(P)$ and $J^+(P)$ we mean the set of events that precede and follow P respectively, and by stating that the event Q causally follows P we mean that there is at least one future-directed causal non-space-like curve that extends from P to Q.

Condition (iv) is also a chronology condition, since chronology and causality coincide when by non-space-like curve we mean only time-like curve. That means the causality condition holds on $\mathcal{M}$ if there are no closed non-space-like curves whereas chronology condition holds if there are no closed time-like curves.

As for Condition (v) there is no evidence that real massive stars in realistic space-time may develop a trapped surface. Though it seems reasonable to accept the idea of trapped surfaces once the general relativity is accepted to be applicable in strong field gravity. This means that Condition (v) is a consequence of Condition (i). The other three conditions (ii), (iii) and (iv) seem to be quite reasonable for any physically realistic space-time.

- THE SINGULARITY AND THE REALISTIC COLLAPSE

If we admit that the above-given conditions are satisfied for a real situation then the theorem will tell us that, singularity does take place beyond which space-time is inextendable and, that massive stars are doomed to collapse into black holes.

Then one would ask how a realistic gravitational collapse might come to its final state, and, what is the nature of the singularity at the end point? Before answering these questions we should realize that all full information about the interior of the collapsing system is hidden beyond the event horizon which constitutes a one-way membrane such that no naked singularity can appear. Therefore due to this sort of, what was called by Penrose, "Cosmic Censorship", [70] one can do nothing but speculate. That is, if we accept the validity of general relativity at all as a strong field model of gravitation, then perhaps one of the following possibilities may be admitted to explain the nature of the singularity at the end point of the realistic collapse:

(1) Infinite curvature at the singularity, i.e. infinite tidal force crushes the collapsing body to infinite density. This would imply that the laws of physics breakdown near the singular point including the general relativistic law. Consequently this would suggest that GTR is not the right model for strong gravity, and therefore the predicted singularities are but a result of a wrong theory, which would contradict our basic assumption on the validity of general relativity.

(2) At the singularity the curvature grows to a certain limit after which matter re-explodes, not outwardly through the horizon which is a one-way membrane, but into another region of space-time forming e.g. a "wormhole" in the manifold.

(3) A reverse classical effect may take place in the neighbourhood of the singularity, e.g. a repulsive gravity may develop that would counteract any further collapse.

(4) Near the singularity, quantum fluctuations in the space-time curvature become dominantly large and the classical laws of general relativity cease to hold.

Beside these possibilities, the space-time singularity may not necessarily be associated with the infinitely large gravitational force and gravitational collapse as one may learn from Taub-Nut space-time [71]. However, from the afore-counted possible situations we learn that it is not matter that is collapsing by approaching singularity, it is rather general relativity which is destined to collapse.

(3.7) Singular and Non-Singular Gravity

Discussion and Critique

In the preceding section we came to the controversial conclusion that by predicting the gravitational collpase general relativity is, in fact, predicting its own collapse. To resolve this paradox we proceed as in the following. Firstly, there is a possibility that the singularity theorem is not quite well established and its prediction of the gravitational collapse is questionable for the following reasons:

(1) As we have explained before, in the proximity of the singular point the classical laws of general relativity are no longer valid.

(2) It was shown by Geroch [72] that there is no quite suitable definition for the singularity.

(3) Even in the Schmidt sense the incompleteness of the non-space-like geodesics does not necessarily imply the singularity, and as we mentioned before, Taub-Nut space-time has a locally non-singular region in which the geodesics are incomplete [73].

(4) If a closed trapped surface is assumed to exist with slightly deformed symmetry it becomes unreasonable to assume the same with large departures from sphericity.

(5) Neither these trapped surfaces nor the singularity are verifiable by experiment and, in addition, there is no reliable evidence of the existence of black holes.

For these reasons and because of the incompatibility of general relativity as a strong field model and its prediction of the gravitational collapse, we decidedly believe that the solution of the gravitational equation must be non-singular. In the introduction to this thesis (0.5) we gave a justification to this belief.

Secondly, a non-singular solution can be achieved in two prospects:

(a) by modifying Schwarzschild metric which proved to be a successful weak-field solution, so that its singular features at strong-field energies be eliminated. Recently, by deforming Schwarzschild space-time, Osborne [74] managed to avert the essential singularities in the strong-field domain without harming the successful predictions of general relativity in the weak-field area.

(b) by modifying general relativity itself so that the theory becomes well-behaved at the strong-field end of the energy spectrum. Part II of this thesis is devoted to this purpose where we obtain our generalized non-singular metric. Schwarzschild's space-times with its singular features come out as a special case within certain restrictions imposed on the curvature scalar R.

It will be shown in [6.3.2] that the solution,

$$A_0 = g_{rr} = \frac{1}{1 + \frac{K}{r} + \gamma_0 r^2} \qquad [3.7.1]$$

$$R = R_0 = \text{constant}$$

constitutes an intermediate situation between space-time described by our generalized metric and the Schwarzschild singular geometry. It yields Schwarzschild space-time by setting $\gamma_0 = 0$, and $K = -1$, and by setting $K = 0$ and $\gamma_0 = -\frac{1}{3}\Lambda$, we have the following cosmological solution,

$$g_{rr} = \frac{1}{1 - \frac{1}{3}\Lambda r^2} \qquad [3.7.2]$$

the so-called De Sitter space-time for an Einstein static universe with a perfect fluid distribution [75], where Λ represents the cosmic constant of Einstein's space

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \qquad [3.7.3]$$

We will show in Chapters Five and Six that by generalizing Einstein's vacuum equations so that more geometrical terms would appear in the R.H. side of [3.7.3], the corresponding solution will be more general and non-singular.

In chapter Four we briefly review other ways and schemes of modification of general relativity that aimed to lead to a non-singular gravitation.

CHAPTER FOUR

Generalized Theories of Gravitation and Strong-Field Models

(4.0) Modification of general relativity

In spite of the great successes of general relativity theory in describing weak gravitational field phenomena, there are problems of space-time singularity and event horizons which are predicted to occur in strong gravitational field, as described in (3.5).

The postulated existence of black holes and speculations related to them, encouraged many authors to think of alternative theories. These theories are required to be capable of describing gravitational phenomena and at the same time not be defeated by the appearance of singularities in the strong-field region.

Besides this, the incompatibility of Einstein's picture with the already well established nongravitatioal theories like electromagnetic theory, made some theorists think of even abandoning the geometrical representation of general relativity altogether.

Many theories had been invented, some of which are based on the same geometrical grounds as Einstein's, called metric, and as described in (2.2); others are non-metric. The latter had mostly failed the viability test by not satisfying the experimental requirements and therefore were deleted from being alternative theories of gravitation (2.1) even in the weak-field regions. The viable metric theories are but a modification of general relativity model.

Because Einstein's theory was founded on solid grounds by its fulfilment the physical requirements (1.6), modifiers of general relativity always try to adhere totally or partially to these requirements. To do that a number of approaches and schemes had been adopted to modify GTR with a hope that strong-field gravity will be well described, both physically and mathematically.

In the following sections of this chapter, we review different schemes of modification starting in the next section by quoting a number of these approaches.

(4.1) Generalized gravitational models

We provide here a compendium of different categories of approaches towards a generalized theory of gravitation that is intended to be successful in strong-field as well as weak-field regions of energy spectrum, i.e.

(a) Theories with general and nonlinear Lagrangian,

(b) Theory with a limiting curvature and nonlinear Lagrangian,

(c) Theories with additional fields,

(d) Kilmister-Yang model,

(e) General Relativity plus torsion,

(f) Quantization of general relativity

(g) Model with a deformed Schwarzschild space.

Besides these we introduce our own approach which will be presented in detail in the next five chapters of this thesis as a novel model towards a nonsingular gravity. The above-counted theories, though aimed to be strong-field models of gravity, are not all entirely successful in that area; nevertheless, they do at least pave the route towards a comprehensive picture of gravitation that will explain well all features of energy domains.

We briefly review these theories in the following sections without going into their derivations to which we refer the reader to their sources, but putting emphasis on the motivations and ideas behind them, though a derivation of a general Lagrangian is provided to serve as a comparison with our own derivation in the next chapter.

(4.2) Theories with General and Nonlinear Lagrangian

a – ELECTROMAGNETIC ANALOGY OF GRAVITY.

There are certain aspects analogous between electromagnetism and gravitation which are linked by the deep coherence of the mass with the charge. In the real world every electromagnetic effect implies the presence of gravitation whilst gravitational effects are not always accompanied by electromagnetic field. This irreversible situation may elucidate more the contrasts between gravitational and electromagnetic phenomena. However, there are a number of common grounds between them. First, as we mentioned in (1.7), there is a certain correspondence between the metric tensor components and the potential of the electromagnetic field, whereas the Riemannian tensor [1.4.27] which has the form:

$$R_{\lambda\mu\nu\gamma} = g_{\lambda\sigma}R^{\sigma}_{\mu\nu\gamma} \qquad [4.2.1]$$

corresponds to the electromagnetic field strength:

$$F_{\mu\nu} = g_{\mu\lambda}g_{\nu\gamma}F^{\lambda\gamma} \qquad [4.2.2]$$

Second, one observes another feature of analogy between the Einsteinian gravito-geometrodynamical free field equations:

$$R_{\mu\nu} = 0 \qquad [4.2.3]$$

and the homogeneous Maxellian electrodynamical equations [1]

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0 \qquad [4.2.4]$$

where both of them represent a system of coupled partial differential equations of the second order.

Third, the resemblance between the inverse-square law of Newtonian gravitational and Coulombian electrostatic fields is a common feature between gravitational and electromagnetic interactions, at least in the weak field areas. But as we noted in (1.6)-IX, there is a major difference. Mathematically, the gravitational equations which describe a tensor field are nonlinear, and physically the gravity influences its source, whereas Maxwell's equations that describe a vector field are linear in the field variables, and the electric charge is not affected by its field. Contrary to that, Einstein's equations [4.2.3] are derived from a variational principle by using a Lagrangian linear in R (1.7), whilst the electromagnetic equations [4.2.4] are derivable by utilizing a Lagrangian quadratic in the field strength [2] i.e.

$$\mathcal{L} \sim F^{\mu\nu}F_{\mu\nu} \qquad [4.2.5]$$

Therefore, it seems very logical due to the correspondence between $F_{\mu\nu}$ and $R_{\mu\nu\lambda\gamma}$ to employ for the gravitational field, analogously, a Lagrangian

$$\mathcal{L} \sim R^{\mu\nu\gamma\rho}R_{\mu\nu\gamma\rho} \qquad [4.2.6]$$

Moreover, according to Weyl's guage invariance [3], the action integral should be not only invariant w.r. to arbitrary coordinate transformation but also w.r. to the arbitrary units in which the length is measured [4]. Hence, it is demanded that for the action integral to be rational it should be gauge invariant, that means it should be a pure number. This implies that the Lagrangian should be quadratic in the scalar curvature R or any other contracted form of Riemann tensor. This is different from Einstein's Lagrangian where the action integral has the dimension of the square of length [3].

In (2.7) we pointed out the rationality of modifying Einstein's equations by using a Lagrangian containing nonlinear R, but still free from additional nongeometrical terms.

(i) General quadratic Lagrangian

Due to the afore-discussed analogy between gravity and electromagnetism and the rationality behind the utilization of non-linear Lagrangians in R, quadratic Lagrangians were recommended to be a basis for gravitational equations. It was also learnt that general relativity theory which is far from being amenable to quantization (0.5) may be improved towards being quantizable (similar to the electromagnetic theory) if a quadratic term is included in the Lagrangian rather than having alone the scalar curvature R.

By being thus motivated, authors employed variational principles with a Lagrangian containing quadratic invariants of Riemannian-Christoffel tensor and its contractions. In fact, quadratic Lagrangians were suggested to be considered in gravitational theory even in the early decades of general relativity [5]. Later, special interest was made by Lanczos who devoted a number of papers [6] [7] advocating the use of quadratic Lagrangians to construct a unified field theory.

Let us denote the following Lagrangian densities as

$$\left.\begin{aligned} L_1 &\equiv \sqrt{g}\, R^2 \\ L_2 &\equiv \sqrt{g}\, R_{\mu\nu} R^{\mu\nu} \\ L_3 &\equiv \sqrt{g}\, R_{\mu\nu\gamma\rho} R^{\mu\nu\gamma\rho} \\ L_4 &\equiv \sqrt{g}\, \varepsilon^{\sigma\rho\mu\nu} R_{\sigma\rho\lambda\gamma} R^{\lambda\gamma}{}_{\mu\nu} \end{aligned}\right\} \qquad [4.2.7]$$

where

$$\varepsilon^{\sigma\rho\mu\nu} = \begin{cases} +1 & \text{if } \sigma\rho\mu\nu \text{ even permutation of } 0123 \\ -1 & \text{if } \sigma\rho\mu\nu \text{ odd permutation of } 0123 \\ 0 & \text{if } \sigma\rho\mu\nu \text{ are not all different} \end{cases} \qquad [4.2.8]$$

is the Levi-Civita tensor density components [8] and 0123 is a sequence of t,x,y,z or t,r,θ,ϕ.

These four invariants in [4.2.7] are algebraically independent and as it was shown by Lanczos [9] only two of them are variationally independent [10], since

$$\delta \int L_4 \, d\Omega \equiv 0 \qquad [4.2.9]$$

$$\delta \int (L_1 - 4L_2 + L_3) d\Omega \equiv 0 \qquad [4.2.10]$$

The field equations derived from [4.2.10] can be therefore equally derived from a linear combination of only L_1 and L_2, viz

$$\delta \int \left(\alpha_1 L_1 + \alpha_2 L_2\right) d\Omega = 0 \qquad [4.2.11]$$

with α_1 and α_2 constant coefficients. Moreover, the most general combination will include a matter Lagrangian density L_M and the gaussian scalar curvature R to ensure the Newtonian correspondence. Thus the field equations will result from the variation;

$$\delta \int (\alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \beta R + \eta L_M)\sqrt{g}d\Omega = 0 \qquad [4.2.12]$$

where β and η are also constants.

(ii) Lagrangian quadratic in scalar curvature alone

Gravitational theories based on Lagrangians containing the square of the scalar curvature, R^2, were well studied by many authors and much effort was made by Lanczos [11], Buchdahl [12], Stephenson [13], Bicknell [14] and others.

Special attention was paid to Lagrangians consisting of the square of the scalar curvature alone which because of its similarity to other field theories' Lagrangians it has some appeal. Some authors [15] [16] exploited combination $\alpha R^2 + \beta R$ rather than R^2 alone as a Lagrangian to secure the correct Newtonian limit. Bicknel has explained that field equations derived from the Lagrangian R^2 alone, may have physically reasonable solutions but the predictions of the theory were found to disagree with observations when the matter term was introduced into the equations [14]. Though R^2-field equations are satisfied by spaces $R = 0$ and $R = \text{constant}$. In the case $R \neq 0$ it was proved [12] that asymptotically flat solutions do not exist for equations based on R^2. Notwithstanding, these solutions are cosmologically reasonable; for instance, those belonging to Einstein's space

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \qquad [4.2.13]$$

like the De Sitter solution quoted in (3.7) where the metric coefficients g_{tt} and g_{rr} have the form:

$$g_{tt} = 1 - \frac{1}{3}\Lambda r^2 = g^{rr} \qquad [4.2.14]$$

Λ is the Einstein's cosmological constant.

Further, as far as the variational methods are concerned, Stephenson has stressed that when dealing with Lagrangians other than the linear scalar curvature, one should distinguish between two variational devices: the g-variation (or Hilbert method), and P-variation (or Palatini method). It can be shown that in the case of $\mathcal{L} = R$ the two methods are indistinguishable [17]. Below we give a brief account of these two variations.

b – HILBERT AND PALATINI VARIATIONS

In 1915 Hilbert, first and independently obtained [18] the Einstein's geometrodynamical law by utilizing the Hamiltonian action principle (1.7) where he used as a Lagrangian the 4-dimensional scalar gaussian curvature R.

The action integral [1.7.1] is subjected to small variations $\delta g^{\mu\nu}$ in the components of the contravariant metric tensor $g^{\mu\nu}$, whilst the Cristoffel connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ are not considered independent variables. $\Gamma^{\lambda}_{\mu\nu}$ are generally called affine connections, but can be identified with Cristoffel connections (or symbols) [1.4.9].in some special cases [19].

Palatini (1919) [20] subjected to variation not only $g^{\mu\nu}$, but also the arbitrary symmetric affine connection $\Gamma^{\lambda}_{\mu\nu}$, both of which were treated as independent variables in the stationary action integral.

$$\delta \int \sqrt{g}\,\mathcal{L}\; d\Omega = 0 \qquad [4.2.15]$$

and after variation $\Gamma^{\lambda}_{\mu\nu}$ was specialized to the Cristoffel connection.

If the Lagrangian function $\mathcal{L}$ is given the form:

$$\mathcal{L} = \beta R + \gamma \qquad [4.2.16]$$

where γ corresponds to a nongravitational field and R- is the scalar curvature, the empty space gravitational equations derived from [4.2.15] will be equivalently the same in both P- and g-variational methods.

As for Lagrangians other than [4.2.16], the Palatini method has been severely criticized by Buchdahl [21] [22] who has shown that the P-variation will lead to strange results. Thus we are left with the g-variation in considering Lagrangians that are nonlinear in R.

C - GENERALIZED EQUATIONS OF THE GRAVITATIONAL FIELD

In subsection (a) we considered the introduction of Lagrangian quadratic in R into the theory of gravitation as one way of modifying general relativity. The resulting equations are no longer those of Einstein's theory. Now instead of dealing separately with differently constructed Lagrangians we would rather assume a very general one that will allow classification for all its possible forms. This will enable us to choose from among various possible constructions the most perfect Lagrangian form which will be required to lead to a complete and self-consistent theory of gravitation that hopefully:

(1) Shares with GTR all its successes, i.e. it reduces to general relativity in the weak field areas and hence agrees with experiment and admits correct Newtonian and Minkowskian limits,

(2) does not exhibit any pathological behaviour anywhere, especially in strong-field domains,

(3) becomes amenable to quantization.

In the next chapter we will introduce our derivation based on a certain variational principle and which will yield a fourth-order in $g_{\mu\nu}$ partial differential equations, different from general relativistic equations which are of the second order in the metric tensor derivatives.

For the sake of comparison with our variation we will herein give a derivation due to C. Lanczos who first obtained this kind of generalized equation.

(i) Lanczos Variation

In his (1932) paper [23] C. Lanczos employed the Hamiltonian principle of least action [1.7.2] for the action integral [1.7.1], this would give:

$$\delta I \equiv \delta\int H d\Omega = 0 \qquad [4.2.17]$$

where function H being invariant will contain the curvature tensor $R_{\mu\nu}$ as well as the metric tensor $g_{\mu\nu}$ or their contravariant forms, i.e.

$$H = H(R_{\mu\nu}, g_{\mu\nu}) \qquad [4.2.18]$$

and, where, owing to [1.4.26] and [1.4.11] $R^{\mu\nu}$ and $g^{\mu\nu}$ are related to $R_{\mu\nu}$ and $g_{\mu\nu}$ respectively by:

$$R^{\mu\nu} = g^{\rho\mu}g^{\nu\sigma}R_{\rho\sigma} \qquad [4.2.19]$$

and

$$g^{\mu\nu} = g^{\mu\lambda}g^{\nu\gamma}g_{\lambda\gamma} \qquad [4.2.20]$$

The condition of the minimum action for the integral [4.2.17] over the volume Ω will be carried out, with assumption that $R_{\mu\nu}$ and $g_{\mu\nu}$ are, at first, independent variables.

We also consider the change $\delta R_{\mu\nu}$ in $R_{\mu\nu}$, caused by infinitesmally small change $\delta g_{\mu\nu}$ in $g_{\mu\nu}$, infinitesmally small and we denote them by:

$$\delta R_{\mu\nu} \equiv \rho_{\mu\nu} \qquad [4.2.21]$$

and

$$\delta g_{\mu\nu} \equiv \gamma_{\mu\nu} \qquad [4.2.22]$$

Then by considering the function H as the Lagrangian density:

$$H = \sqrt{g}\mathcal{L}(R) \qquad [4.2.23]$$

as it was given in [4.2.15], the variation of the action integral with respect to $g_{\mu\nu}$ will yield:

$$\int \delta(\sqrt{g}\mathcal{L})\, d\Omega = \int (\sqrt{g}\ \frac{\partial \mathcal{L}}{\partial R}\delta R + \mathcal{L}\delta\sqrt{g})\, d\Omega = 0 \qquad [4.2.24]$$

Further, by knowing from [1.4.26] that, $R = g^{\mu\nu}R_{\mu\nu}$, and by using [1.4.19] which yields

$$\delta\sqrt{g} = \frac{\sqrt{g}}{2}\, g^{\mu\rho}\delta g_{\rho\mu} \qquad [4.2.25]$$

and by denoting the derivative of $\mathcal{L}$ w.r. to R as

$$\mathcal{L}' \equiv \frac{\partial \mathcal{L}}{\partial R} \qquad [4.2.26]$$

one will obtain [4.2.24] in the form,

$$\int \sqrt{g}(\mathcal{L}' g^{\mu\nu}\delta R_{\mu\nu} + \mathcal{L}' R_{\mu\nu}\delta g^{\mu\nu} + \tfrac{1}{2}\mathcal{L} g^{\mu\nu}\delta g_{\mu\nu})\, d\Omega = 0 \qquad [4.2.27]$$

Now, by the aid of the following relation:

$$\delta g^{\mu\nu} = -g^{\mu\lambda}g^{\gamma\nu}\delta g_{\lambda\gamma} \qquad [4.2.28]$$

obtained from [1.4.17] and by using [4.2.19,21,22] integral [4.2.27] will have the general form:

$$\int \sqrt{g}(U^{\mu\nu}\rho_{\mu\nu} - \upsilon^{\mu\nu}\gamma_{\mu\nu})\, d\Omega = 0 \qquad [4.2.29]$$

where

$$U^{\mu\nu} = \mathcal{L}'(R)g^{\mu\nu}, \qquad [4.2.30]$$

and

$$V^{\mu\nu} = \mathcal{L}'(R)R^{\mu\nu} - \tfrac{1}{2}\mathcal{L}(R)g^{\mu\nu} \qquad [4.2.31]$$

Expression for $\rho_{\mu\nu}$, defined in [4.2.21] and being infinitely small change in $R_{\mu\nu}$, caused by infinitely weak deformation $\gamma_{\mu\nu}$ of $g_{\mu\nu}$, was calculated by Lanczos [24] (1923), [25] (1925).

Following Lanczos' derivation we rewrite the Ricci tensor [3.1.1].

$$R_{\mu\nu} = \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \theta_{\mu\nu} \qquad [4.2.32]$$

where

$$\theta_{\mu\nu} \equiv \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} \qquad [4.2.33]$$

will vanish if a locally inertial system of coordinate was adopted, since in this case all Γ will be zero. It will be shown also that the variation $\delta\theta_{\mu\nu}$ will give no contribution to $\rho_{\mu\nu}$.

Let us calculate the variations of $\theta_{\mu\nu}$ and Γ, i.e.

$$\delta\theta_{\mu\nu} = \delta\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} + \Gamma^{\eta}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\eta} - \delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\eta} \qquad [4.2.34]$$

where Γ is defined in [1.4.8,9] and as given in [1.7.9]

$$\delta\Gamma^{\eta}_{\mu\lambda} = \tfrac{1}{2}\, g^{\eta\alpha}\left(\frac{\partial\delta g_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial\delta g_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial\delta g_{\mu\lambda}}{\partial x^{\alpha}}\right) + \frac{\delta g^{\eta\alpha}}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}}\right) \qquad [4.2.35]$$

Now by using [4.2.22,28] and [1.4.8,9] one gets

$$\delta\Gamma^{\eta}_{\mu\lambda} = \tfrac{1}{2}\, g^{\eta\alpha}\left(\frac{\partial\gamma_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial\gamma_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial\gamma_{\mu\lambda}}{\partial x^{\alpha}}\right) - \tfrac{1}{2}\gamma_{\rho\alpha}g^{\alpha\eta}\Gamma^{\rho}_{\mu\lambda} \qquad [4.2.36]$$

$\gamma_{\alpha\mu}$ being a difference of two tensors will constitute a tensor and $\delta\Gamma^{\eta}_{\mu\lambda}$ is also a tensor since it transforms by the tensor law [1.4.10]. Further, by using the covariant differentiation according to [1.4.13,14] in [4.2.36] and due to the symmetry property of both $\Gamma_{\mu\nu}$ and $\gamma_{\mu\nu}$, one obtains the same expression [1.2.10] while all the terms with Γ cancel amongst themselves. We, thus, have:

$$\left.\begin{aligned}
\delta\Gamma^{\eta}_{\mu\lambda} &= \tfrac{1}{2}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\lambda} + \gamma_{\alpha\lambda;\mu} - \gamma_{\mu\lambda;\alpha}) \\
\delta\Gamma^{\lambda}_{\nu\eta} &= \tfrac{1}{2}\, g^{\lambda\alpha}(\gamma_{\alpha\nu;\eta} + \gamma_{\alpha\eta;\nu} - \gamma_{\nu\eta;\alpha}) \\
\delta\Gamma^{\eta}_{\mu\nu} &= \tfrac{1}{2}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\nu} + \gamma_{\alpha\nu;\mu} - \gamma_{\mu\nu;\alpha}) \\
\delta\Gamma^{\lambda}_{\lambda\eta} &= \tfrac{1}{2}\, g^{\lambda\alpha}(\gamma_{\alpha\lambda;\eta} + \gamma_{\alpha\eta;\lambda} - \gamma_{\lambda\eta;\alpha})
\end{aligned}\right\} \qquad [4.2.37]$$

By substituting [4.2.37] into [4.2.34] we get:

$$\left.\begin{aligned}
2\delta\theta_{\mu\nu} = {} & \Gamma^{\lambda}_{\nu\eta}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\lambda} + \gamma_{\alpha\lambda;\mu} - \gamma_{\mu\lambda;\alpha}) \\
& + \Gamma^{\eta}_{\mu\lambda}\, g^{\lambda\alpha}(\gamma_{\alpha\nu;\eta} + \gamma_{\alpha\eta;\nu} - \gamma_{\nu\eta;\alpha}) \\
& - \Gamma^{\lambda}_{\lambda\eta}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\nu} + \gamma_{\alpha\nu;\mu} - \gamma_{\mu\nu;\alpha}) \\
& - \Gamma^{\eta}_{\mu\nu}\, g^{\lambda\alpha}(\gamma_{\alpha\lambda;\eta} + \gamma_{\alpha\eta;\lambda} - \gamma_{\lambda\eta;\alpha})
\end{aligned}\right\} \qquad [4.2.38]$$

By making the change $\lambda \to \nu$, the first term bracket cancels with the third bracket and the second cancels with the fourth, giving for [4.2.38] the value:

$$\delta\theta_{\mu\nu} = 0 \qquad [4.2.39]$$

It will be convenient to adopt the locally inertial coordinate system all through the derivation and in addition $g_{\mu\nu}$ should be brought to its standard orthogonal form.

Now, in virtue of [4.2.22,25] and the notation

$$\gamma \equiv g^{\lambda\alpha}\gamma_{\lambda\alpha} \qquad [4.2.40]$$

we will have

$$\frac{\delta g}{g} = \gamma = \gamma^{\lambda}_{\lambda} \qquad [4.2.41]$$

and accordingly

$$\delta g^{\mu\nu} = -g^{\alpha\mu}g^{\lambda\nu}\gamma_{\lambda\alpha} = -\gamma^{\mu\nu} \qquad [4.2.42]$$

and

$$\gamma_{\lambda\nu} = g_{\lambda\mu}\gamma^{\mu}_{\nu} = g_{\lambda\lambda}\gamma^{\lambda}_{\nu} \qquad [4.2.43]$$

Then [4.2.41] and [1.4.19] yield

$$\delta\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}\,\frac{\partial\gamma}{\partial x^{\mu}} \qquad [4.2.44]$$

which could be obtained also from any of equations [4.2.37] with the use of [4.2.43].

Therefore, by taking into account [4.2.39,44], the variation $\delta R_{\mu\nu}$ of [4.2.32] will become

$$\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\,\delta\Gamma^{\lambda}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\,\delta\Gamma^{\lambda}_{\mu\nu} + \delta\theta_{\mu\nu}$$

$$= \frac{1}{2}\,\frac{\partial^{2}\gamma}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial\delta\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} \qquad [4.2.45]$$

The use of [4.2.37] and covariant differentiation will result into:

$$\delta R_{\mu\nu} = \tfrac{1}{2}\gamma_{;\mu;\nu} + \tfrac{1}{2} g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} - \tfrac{1}{2} g^{\lambda\lambda}(\gamma_{\nu\lambda;\mu;\lambda} + \gamma_{\mu\lambda;\nu;\lambda})$$

$$- \tfrac{1}{2} g^{\lambda\lambda}\left(2\gamma_{\lambda\rho} \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}} + \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda} \frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}} - \gamma_{\nu\rho} \frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}}\right) - \gamma^{\lambda\alpha} \frac{\partial}{\partial x^{\lambda}}[\mu\nu,\alpha], \qquad [4.2.46]$$

where $[\mu\nu,\alpha]$ is defined in [1.4.8], and the covariant derivatives are used instead of the ordinary ones to secure the invariant form of $\delta R_{\mu\nu}$.

Further, with the help of [1.4.37] and [4.2.43] together with the obvious relationship:

$$\gamma_{;\mu;\nu} = \gamma_{;\nu;\mu} \qquad [4.2.47]$$

expression [4.2.46] becomes:

$$\delta R_{\mu\nu} = g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} - \tfrac{1}{2}(\gamma^{\lambda}_{\nu;\lambda;\mu} - \tfrac{1}{2}\gamma_{;\nu;\mu} + \gamma^{\lambda}_{\mu;\lambda;\nu}$$

$$- \tfrac{1}{2}\gamma_{;\mu;\nu}) - \tfrac{1}{2}\gamma^{\sigma}_{\nu}R_{\sigma\mu} - \tfrac{1}{2}\gamma^{\sigma}_{\mu}R_{\sigma\nu}$$

$$+ \tfrac{1}{2}\gamma^{\lambda}_{\sigma}(R^{\sigma}_{\nu\mu\lambda} + R^{\sigma}_{\mu\nu\lambda}) - \gamma^{\lambda\alpha} \frac{\partial}{\partial x^{\lambda}}[\mu\nu,\alpha]$$

$$- \frac{g^{\lambda\alpha}}{2}\left(2\gamma_{\lambda\rho} \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}} + \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda} \frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}} - \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\nu\rho} \frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}}\right). \qquad [4.2.48]$$

(ii) Lanczos Generalised Field Equations

By introducing the following notations:

$$\gamma^{\lambda}_{\nu;\lambda} - \tfrac{1}{2}\gamma_{;\nu} \equiv X_{\nu}, \qquad [4.2.49]$$

$$g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} \equiv \Box^{2}\gamma_{\mu\nu}, \qquad [4.2.50]$$

and the relation,

$$\gamma^{\sigma}_{\nu}R_{\sigma\mu} = \gamma_{\nu\lambda}g^{\lambda\sigma}R_{\sigma\mu} = \gamma_{\nu\lambda}R^{\lambda}_{\mu} \qquad [4.2.51]$$

resulting from [4.2.43], where

$$R^{\mu}_{\nu} = g^{\mu\sigma}R_{\sigma\nu}; \qquad [4.2.52]$$

into [4.2.48], then due to [4.2.21] one obtains,

$$2\rho_{\mu\nu} = \Box^{2}\gamma_{\mu\nu} - (X_{\mu;\nu} + X_{\nu;\mu}) + 2\gamma^{\lambda\sigma}R_{\mu\nu\lambda\sigma}$$

$$+ R^{\sigma}_{\mu}\gamma_{\sigma\nu} + R^{\sigma}_{\nu}\gamma_{\sigma\mu} \equiv D(\gamma_{\mu\nu}) \qquad [4.2.53]$$

where,

$$2R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma} \equiv -g^{\lambda\alpha}\left(\gamma_{\mu\rho}\frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda}\frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}} - \gamma_{\mu\rho}\frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\nu\rho}\frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}} + 2\gamma_{\lambda\rho}\frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}}\right) - 2\gamma^{\lambda\alpha}\frac{\partial}{\partial x^{\lambda}}[\mu\nu,\alpha]$$

$$- R^{\lambda}_{\mu}\gamma_{\nu\lambda} - R^{\lambda}_{\nu}\gamma_{\mu\lambda}. \qquad [4.2.54]$$

The R.H. side of [4.2.54] can be rearranged to give the L.H. side by utilizing the definition of $R_{\mu\nu\lambda\sigma}$ which is given in [1.4.27].

Furthermore, we write the following property of the "adjoint" differential expressions

$$\int d\Omega[u^{\mu\nu}D(\gamma_{\mu\nu}) - \gamma^{\mu\nu}D^{+}(U_{\mu\nu})] = \oint \qquad [4.2.55]$$

where $\oint$ is the integral over the surface containing the volume Ω, which according to Gauss vanishes by variation [26] and, D^+ is the adjoint of D defined as:

$$D^{+}(a_{\mu\nu}) \equiv D(a_{\mu\nu}) + g^{\mu\nu}a^{\alpha\beta}_{;\alpha;\beta} - (g^{\alpha\beta}a_{\alpha\beta})_{;\mu;\nu} . \qquad [4.2.56]$$

Now, according to [4.2.55] we can transform the term with $\rho_{\mu\nu}$ in [4.2.29] and we eventually get,

$$\delta I = \int \sqrt{g}\, [\tfrac{1}{2}D^{+}(U^{\mu\nu}) - V^{\mu\nu}]\gamma_{\mu\nu} d\Omega = 0 \qquad [4.2.57]$$

Therefore, since $\gamma_{\mu\nu}$ is arbitrary, we obtain the following form for the field equation:

$$D^{+}(U^{\mu\nu}) = 2V^{\mu\nu}. \qquad [4.2.58]$$

Then, substituting for $U_{\mu\nu}$ and $V_{\mu\nu}$ from [4.2.30,31] into [4.2.56,58] yields the following 4th order in $g_{\mu\nu}$ differential equation.

$$\begin{aligned} H_{\mu\nu} \equiv & \mathcal{L}'''(R)(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\sigma}R_{;\lambda}g^{\sigma\lambda}) \\ & + \mathcal{L}''(R)(R_{;\mu;\nu} - g_{\mu\nu}\Box^{2}R) \\ & + \mathcal{L}'(R)R_{\mu\nu} - \tfrac{1}{2}\mathcal{L}(R)g_{\mu\nu} = 0, \end{aligned} \qquad [4.2.59]$$

where the covariant derivatives for the scalar curvature R are given by:

$$R_{;\mu} = \frac{\partial R}{\partial x^{\mu}} , \qquad [4.2.60]$$

$$R_{;\mu;\nu} = \frac{\partial}{\partial x^\nu}\left(\frac{\partial R}{\partial x^\mu}\right) - \Gamma^\lambda_{\mu\nu}\frac{\partial R}{\partial x^\lambda} \qquad [4.2.61]$$

It is clear from [4.2.59] that the field equations are divergence-free since:

$$H^\mu_{\nu;\mu} = 0 \, . \qquad [4.2.62]$$

(4.3) Theory with a nonlinear Lagrangian and a limiting curvature.

The following nonlinear Lagrangian has been chosen by A. Müller et al [27];

$$\mathcal{L}(R) \equiv -\frac{R_0}{m}\left[\left(1-\frac{R}{R_0}\right)^m - 1\right], m < 1 \, , \qquad [4.3.1]$$

in order to fulfill certain necessary requirements like coinciding with Einstein's general relativity in the weak field area i.e. when $R \to 0$, and satisfying the asymptotic flatness.

This Lagrangian was chosen to be successful also in the strong field domain, where, in order to avoid gravitational collapse an upper limiting bound of the curvature has been postulated i.e.

$$R \leqslant R_0 \qquad [4.3.2]$$

As we mentioned in (0.2) the idea of the limiting bound is motivated by other field theories [28] [29].

By employing equation [4.2.59] with the Lagrangian [4.3.1], the following expressions for the scalar curvature and the digonal components; $g_{rr} = A$ and $-g_{tt} = B$ in the static isotropic metric result;

$$\ddot{R} = \frac{(2-m)}{R - R_0} \dot{R}^2 + \left(\frac{\dot{A}}{2A} - \frac{2}{r}\right)\dot{R} + \frac{R-R_0}{m-1} \left(\frac{AR}{2} + \frac{\dot{A}}{rA} + \frac{1}{r^2} + \frac{A}{r^2}\right)$$

$$- \frac{A(R-R_0)^2}{2m(m-1)} \left(1 - \frac{R}{R_0}\right)^{-m} [(1 - \frac{R}{R_0})^m - 1] \qquad [4.3.3]$$

$$\frac{\dot{A}}{A} = \frac{m-1}{3(R-R_0)} \left[-\frac{\dot{B}}{B} r + 2\right] \dot{R} + \frac{1}{3} \left[\frac{\dot{B}}{B} + \frac{4}{r} - A\left(Rr + \frac{4}{r}\right)\right] \qquad [4.3.4]$$

$$\frac{\dot{B}}{B} = \frac{1}{1 + \frac{(m-1)\dot{R}r}{2(R-R_0)}} \left\{ \frac{2(m-1)}{R_0 - R} \dot{R} + \frac{ARr}{2} + \frac{1}{r} - \frac{A}{r} + \frac{rAR_0}{2m} \frac{[(1 - \frac{R}{R_0})^{m-1}]}{\left(1 - \frac{R}{R_0}\right)^{m-1}} \right\} \qquad [4.3.5]$$

These expressions can be obtained by direct substitution of [4.3.1] into the generalized equations [6.2.35, 32, 34] respectively where,

$$\dot{R} \equiv \frac{d}{dr} R \qquad [4.3.6]$$

Equations [4.3.3,4,5], as we will show in (6.4), admit the Schwarzschild solution with its Minkowskian flat space limit as special cases and, by choosing special values of the parameter m cosmological solutions with asymptotically non-flat metric can also be obtained, like

$$R = \frac{8}{9} R_0 \quad \text{at} \quad m = \tfrac{1}{2} \qquad [4.3.7]$$

which would correspond to Einstein's equation

$$G_{\mu\nu} = \frac{2}{g} R_0 g_{\mu\nu} \qquad [4.3.8]$$

Apart from that, the model has a free space solution with $R = R(r)$ which exhibits no pathological behaviour of the metric at any value of r, and which is limited by

$$\lim_{r \to o} R(r) = R_0 \qquad [4.3.9]$$

In their paper [27] Müller et al investigated and discussed the metric by employing global numerical solutions. They imposed certain constraints by requiring that the metric should be finite everywhere with correct Newtonian limit, and that the speed of light stays limited as r goes to zero.

In spite of the criticism [30] against this work,in which it was suggested that it be revised, the idea of utilizing the nonlinear Lagrangian to develop a nonsingular model of gravitation looks promising and very attractive. In (5.4) we will show that [4.3.1] is a special case of a very general and rational Lagrangian form.

Comparing the solutions of this limiting curvature theory with those of GTR, both in the free-field region as well as in the interior of a star, suggests that the model is a good candidate for a nonsingular description of strong gravity.

The difficulty, in this theory, arose in transition from a solution with a point source to an extended hydrodynamical model for the star, whence equations [4.3.3,4,5] will contain components of the energy-momentum tensor,

$$T^{\nu}_{\mu} = Tr\left(\rho, \frac{-P}{C^2}, \frac{-P}{C^2}, \frac{-P}{C^2}\right) \qquad [4.3.10]$$

where, ρ the density and P the hydrostatic pressure of the incompressible fluid modelling a sphere of the steller matter. Here the velocity of light C will be set to unity(see Adler et al. [31]). This is because equations [4.2.59] are now inhomogeneous by having $T_{\mu\nu}$ in the R.H.S. Now, such a difficulty of transition is, perhaps, due to the big contrast between the physically imperceptible point mass, that are supposed to generate an isolated gravitational field, and a physical star, and also due to the sophisticated mechanism that allows this transition.

We would always expect that quantum effects have a certain role to play in domains of strong gravity, namely inside the star or within its immediate vicinity. Therefore, the author's conclusion about the unavoidability of the collapse of a heavy star to a point singularity in spite of the limiting curvature being introduced, should be re-thought.

(4.4) Theories with additional fields

The idea of having beside the gravitational tensor field an additional field, as we have noted in (2.5), is aiming towards bringing together the principle of equivalence with Mach's principle [32]. Inspite of our criticism in (2.7) to the way an additional field had to be incorporated into the field equation, theories with additional fields beside being, mostly, viable, lead to interesting results. Among the theories listed in (2.4) the only viable ones are those which contain auxiliary fields in addition to $g_{\mu\nu}$ - field. We have three categories of these

Scalar-tensor theories e.g. Brans-Dicke-Jordan theory [33],
Vector-tensor theory due to Nordvedt, Helling and Will [34].
Tensor-tensor theory due to Lee and Lightman [35].

All these theories are metric and Lagrangian-based. By Lagrangian-based we roughly mean that the theory possesses a generally co-variant representation in a similar way to general relativity [36]. To the first category belong, certain generalizations of the Brans-Dicke-Jordan theory made by Bergmann [37], Wagoner [38], and Nordvedt [39]. We have also the scalar-tensor theories due to Tupper [40] and Yilmaz [41] and the Fujii-O'Hanlon theory [42][43]. The last one exhibits a non-Newtonian gravity at small radius and thence violates the viability condition. As we are concerned, in this chapter, with the modifications of GTR which are aimed to improve its strong-field behaviour, we shall pay less attention to the weak-field aspect of the theories under consideration. Our aim is to point out which theory might be considered as a good candidate for strong-field gravity. We give, here, a concise survey of scalar-tensor, and vector-tensor theories as examples for modified versions of general relativity.

A - SCALAR-TENSOR THEORIES

We have been aquainted with auxilliary scalar field Φ being introduced into the Einstein's Lagrangian density [2.5.3] of the Brans-Dicke-Jordan scalar-tensor theory in the form:

$$\Phi R - \frac{\Phi_{;}^{\ \nu}\ \Phi_{;\nu}}{\Phi} \qquad [4.4.1]$$

Terms, such as:

$$\Phi, \Phi^2, \Phi^2 R, \Phi^{\mu}_{;}\Phi_{;\nu}, \Phi^{\mu}_{;}\Phi_{;\nu}R, \Phi^{-1}\Phi^{\mu}_{;}\Phi_{;\nu}, \Phi^{-1}\Phi^{\mu}_{;}\Phi_{;\nu}R \qquad [4.4.2]$$

can also, at least for generality, be incorporated in the Lagrangian function.

Similarly, as in [1, 7, 17], the Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial \Phi(X)} - \left[\frac{\partial \mathcal{L}}{\partial \Phi(X)^{\nu}_{;}}\right]_{;\mu} = 0 \qquad [4.4.3]$$

will reduce the power of Φ giving rise to a term not depending on Φ and can be representing the Φ - field source.

Lagrangians containing terms, such as those in [4.4.1] and [4.4.2] in addition to the scalar curvature R were constructed and developed by Thirry (1948) [44], Jordan (1948, 1955) [45][46], Bergmann (1968); Wagoner (1970), Nordtved (1979), and by Brans and Dicke (1962) [47] [48].

In the following, we briefly introduce the Brans-Dicke-Jordan model as a main example for the modification of general relativity together with some other scalar-tensor theories which are related to it.

(i) Brans-Dicke-Jordan Theory

As we stated in Section (2.5), this theory constitutes the strongest competitor to GTR. The field equations are given by [2.5.4] and [2.5.6]. The theory is primarily, viable and in particular, it satisfies well all the classic tests of general relativity. The predicted values according to this theory for the solar system's experiments; i.e. the perihelion precession, the bending of light, and the radar echo delay can be respectively obtained from [3.3.10, 14, 18] with [3.2.7]. As the theory reduces to general relativity when the parameter $\omega \rightarrow \infty$, the Schwarzschild solution of the vacuum equations is admitable. Cosmological solutions, like those of GTR, are also allowed in this model. In the strong field region Brans-Dicke-Jordan theory, like GTR, predicts singularities and black holes. Though, recently some solution of this theory have been found in discrepancy with the singularity theorem [49]. Particularly, the energy condition stated in Section (3.6) is not always satisfied and this would imply that non-singular solutions can be obtained. Therefore, irrespective of the lack of adequate physical significance of this kind of solution we think that their very existence indicates that by modifying general relativity its singular features might disappear. This would mean that by further modification i.e. further departures from GTR we will, probably, be able to achieve, within the classical frame, a non-singular gravity.

(ii) <u>Tupper and Yilmaz Theories</u> [40] [41]

These two scalar-tensor theories, though, steming from different motivations are interlinked so that their static solutions may coincide with each other. On the other hand they are related to Brans-Dicke-Jordan theory and share with it its being viable and its satisfying the equivalence principle.

Tupper's theory is based on the idea that the Einstein's free-field equations

$$R_{\mu\nu} = 0 \qquad [4.4.4]$$

are not necessary to guarantee the agreement with the weak-field tests of general relativity, whilst Yilmaz's is motivated by the viewpoint that there is a lack of correspondence between the gravitational potential and the components of the metric tensor tensor $g_{\mu\nu}$. Thus the field equation [4.4.4] was generalized by Tupper to have the form:

$$R_{\mu\nu} = \lambda \Psi_{;\mu}\Psi_{;\nu} \qquad [4.4.5]$$

where λ is a constant and Ψ is the scalar field satisfying the Laplace equation, i.e.

$$\Box^2 \Psi = 0 \qquad [4.4.6]$$

Due to the simplicity of these equations it is sufficient to use Tupper's version to generate static solutions which will be representing both theories. The relationship between Tupper and Yilmaz theories on one hand and Tupper and Bran-Dicke-Jordan theories on the other hand will certainly make it easy to transform the solution of one theory to the other [40]. Thus we see that Tupper and Yilmaz theories can be regarded as alternative versions of Brans-Dicke-Jordan model, the common feature among which is, that they all assume additional fields coupled to massless scalarons. Apart from that, an additional scalar field with such hypothetical particles, but which are massive, was postulated in the Fujii-O'Hanion theory which follows in the next sub-section.

(iii) Fujii-O'Hanlon Theory [42] [43]

This theory satisfies the principle of equivalence, but unlike other scalar tensor theories it possesses, as we have noted before, an additional field associated with non-zero-mass particles. This latter causes the theory to exhibit a non-Newtonian gravity and subsequently violate one of the viability criteria.

The field equations read

$$G_{\mu\nu} = \Phi^{-1}\left[8\pi T_{\mu\nu} + \Phi_{;\mu;\nu} - g_{\mu\nu}\left(\Box^2\Phi - \frac{m^2}{2}\cdot f(\Phi)\right)\right] \qquad [4.4.7]$$

where $f(\Phi)$ is an undetermined function of the scalar field which satisfies

$$\Box^2\Phi = \frac{8\pi}{3}T + \frac{m^2}{3}\left(\Phi\frac{\partial f}{\partial\Phi} - 2f\right) \qquad [4.4.8]$$

and m denotes the non-zero mass of a dilaton, a boson, that couples with the matter to produce the gravitational potential Φ_g. Within the weak-field approximation the gravitational potential of the point mass M was found to be:

$$\Phi_g(r) = -\frac{G_oM}{r} - \frac{G_oM}{3}\frac{e^{-mr}}{r} \qquad [4.4.9]$$

in which the first term represents the Newtonian potential and the second term which is non-Newtonian would vanish fastly as

$r \to \infty$. As $r \to 0$ the gravitational potential would tend to be

$$\Phi_g(r) \to -\frac{GM}{r} \qquad [4.4.10]$$

with $$G \equiv \frac{4}{3} G_0 \qquad [4.4.11]$$

The coupling "constant", and G_0 the background value of the gravitational constant. It is noticed here that G reflects a certain variability over a short range of r. Such a variability can be verified by Cavendish type experiment [50].

Now, so far, without having the exact solution for [4.4.7.8] the potential form [4.4.9] will only point towards the interesting properties of, possibly, the non-Newtonian gravity in the strong-field domains. One interesting remark which we make here is that in [4.4.9] the second term of the gravitational potential resembles the Yukawa nuclear potential [51] of the mesonic field,

$$\Phi_Y = \Phi_Y^0 \frac{e^{-\alpha r}}{r} \qquad [4.4.12]$$

which as $\alpha \to 0$ reduces to the Coulomb potential of the electrostatic field. Analogously, we can think of the Fujii-O'Hanlon gravitational potential as reducing to Newtonian potential as $m \to 0$, and since α gives the range of the Yukawan potential, one may think of m as indicating a range for a non-classical parameter [52].

Further, in the next sub-section we introduce another example of the modification of GTR in which the additional field is still associated to massless particles like that of Barns-Dicke-Jordan theory, but alternatively, it is now a vector field.

B - VECTOR-TENSOR THEORY [34] [53]

In this model the Einstein's Lagrangian density was amended to include terms that describe the interaction between the gravitational tensor field and the auxilliary vector field. The following reconstruction for the system's Lagrangian density was given [34] that yields,

$$\sqrt{g}(16\pi G_0)^{-1} R + \sqrt{g}\,\mathcal{L}_{field} +$$

$$\sqrt{g}(16\pi G_0)^{-1}(a V_\mu V^\mu R + bV^\mu V^\nu R_{\mu\nu} + ct_{\mu\nu} t^{\mu\nu}) \qquad [4.4.13]$$

where V_μ is the vector field component and a, b, and c are constants. The terms in brackets represent the vector-tensor interaction, and the curl of the vector $\vec{V}$, as given by [1.4.25], as defined by

$$\text{curl } \vec{V} \equiv V_{\mu;\nu} - V_{\nu;\mu} = V_{\mu,\nu} - V_{\nu,\mu} = t_{\mu\nu} \qquad [4.4.14]$$

The resulting Euler-Lagrange equations will lead to field equations which by contraction yield

$$R = 8\pi G_0 T - (3P + \frac{1}{2} q) \Box^2 \Phi - q(V^\mu V^\nu)_{;\mu;\nu} \qquad [4.4.15]$$

where, in this theory, Φ is defined by

$$\Phi \equiv V_\mu V^\mu \qquad [4.4.16]$$

and where G_0 the background value of the gravitational constant and, p and q are dimensionless parameters.

The weak-field approximation in the case of the static field will predict values of the PPN parameters that can be made identical to those of GTR by a certain choice of the dimensionless parameters p and q. For this reason the theory will successfully satisfy all the experimental tests of general relativity. It also satisfies other viability requirements. Moreover, like scalar-tensor theories, this model demonstrates the variability of the gravitational coupling constant. This variability is caused by the presence of the additional field and supported by the afore-mentioned Cavendish experiment. The dependence of the gravitational constant, for instance, on the position in the space, would lead to departure from the geodesic motion of freely-falling particles due to the anomalous acceleration associated with the spatial variation. Such a departure from the geodesic motion, called Nordtvedt effect [54], is unpredictable in general relativity [55]. Though, we believe that the possibility of observing such a phenomenon would not be so great that general relativity or any other viable theory might be disproved.

Finally, we conclude that whether a real field of massless vectorons exists or not, we reckon that further investigation of this model will lead to new results. If an exact solution can be reached then we would expect that interesting features of the strong-field gravity will be manifested. But before this is carried out no statement about the advantages of scalar-tensor or vector-tensor theories can be made. We, therefore, think that it will be feasible to modify general relativity, not by introducing additional fields whose nature is nongeometrical, but by rather sticking to the geometrical representation of Einstein's model. This is the viewpoint we advocated in Section (2.7) and within Chapter Five of the present thesis.

In the next section we introduce an example of a modification of general relativity which is free from any additional nongeometrical elements.

(4.5) Kilminster-Yang Model

By adhering to the idea, that, the most complete and self-consistent theory of gravity must be a quantum one, thought has been directed towards building a quantum theory of gravitation whose classical limit would undoubtedly be Einstein's general relativity. In other words, the general relativistic equations should be quantized so that microscopic and strong-field gravitational effects find their correct interpretation. But, as it will be stated in (4.7), general relativity is, disappointingly, not amenable to quantization. Therefore a need had arisen for a theory that beside preserving the advantages of general relativity in the weak-field region will be hopefully quantizable. That means one should first achieve a classical generalization for GTR and then look for the quantized version of that.

One alternative generalized model of general relativity which is considered a good candidate for a quantizable theory is due to Kilmister and Yang. Like Einstein's theory, this model is purely geometric but alternatively the free-field Lagrangian density is quadratic in the contracted Riemannian tensor components i.e.

$$\mathcal{L} = \frac{1}{4}\sqrt{g}\, g^{\alpha\mu} g^{\beta\nu} R^{\gamma}_{\mu\nu\lambda} R^{\lambda}_{\alpha\beta\gamma} \qquad [4.5.1]$$

As we explained in (4.2), such a Lagrangian has certain appeal and historically it was suggested and advocated by both Weyl [56] and Lanczos [57] to be a basis for a unified-field theory of gravitation and electromagnetism and more later it was utilized by Yang in his gauge theory of gravitation [58].

Now, in order to derive the field equations the Lagrangian density [4.5.1] should be subjected to variation. But because of the unconformity between the p- and g-variations stated in (4.2) – b, and in view of the point raised by Stephenson [21]; that the p-variation method is highly dubious when it is applied to quadratic Lagrangian, one would adopt only the g-variation to derive the field equation. Otherwise, a straightforward derivation due to Kilmister will lead to the gravitational equation. By assuming that the space time is pseudo-Riemannian, and that the Riemann tensor is divergenceless, then contracting the Bianchi identities [1.4.31] and using the symmetry properties of the Riemann tensor [1.4.29] yield,

$$R_{\delta\nu;\lambda} - R_{\delta\lambda;\nu} = 0 \qquad [4.5.2]$$

These equations were first obtained by Stephenson [59] and derived and studied by Kilmister [60] [61] and by Yang [58]. They represent a set of coupled partial differential equations of the third order for $g_{\mu\nu}$. Constituting a generalization of Einstein's vacuum equation $R_{\mu\nu} = 0$, which are of the second differential order, the equation [4.5.2] will be, therefore, more informative than GTR in the domains of higher curvature.

In the static isotropic coordinates Kilmister - Yang equations, unlike Einstein's, yield a variety of solutions [62] [63], including Schwarzschild space-times, as special cases. Hence this theory satisfies all the viability requirements. The conservation of the curvature tensor makes necessary, the constancy of the scalar curvature R of all solutions. Moreover, solution of cosmological nature, as well as other interesting spacetimes were also, obtained [64]. In particular, the theory admitted a well-behaved metric in the strong-field region, and a metric with periodic character, where, gravity may be repulsive, was also allowed.

Now, considering the above-counted merits of Kilmister - Yang theory may allow us to believe that this model is a big step towards a deep comprehension of gravitation, especially, in its strong-field limit. As we stated in (3.7), the defectiveness of GTR is primarily caused by its imperfection. The best remedy for that, we believe, is not by destorting its geometrical nature which was tested to be correct in the weak-field gravity, but by rather "promoting" it, i.e. by raising its differential order so that it becomes capable of accommodating higher nonlinear effects. Thus, Kilmister - Yang modification of GTR goes in the right path towards perfection. It is our goal in Part II of this thesis to develop the most accepted model of gravity by taking the furthest step in this line of generalization i.e. by utilizing gravitational equations of the highest possible differential order.

(4.6) General Relativity plus Torsion

This is the earliest attempt to modify the general relativistic laws, in which an additional geometrical feature was introduced into Einstein's theory of general relativity. In 1922 É. Cartan [65] put forward his idea in which he proposed a space-time manifold with a metric tensor and a linear connection. The torsion of the connection was required to be related to the density of the intrinsic angular momentum. By varing the metric, the frames, and the linear connection independently of each other it can be shown [66] that Einstein's equations may be written in two equivalent forms called Einstein-Cartan equations. Similar ideas towards generalizing GTR were proposed by other authors [67]. In support of Cartan's theory, A. Trautman [66] suggested that space geometry can be affected by the spin in the same way it is affected by the mass. Thus, by introducing torsion related to the intrinsic angular momentum, spin effects on space geometry can be admitted.

In Section (2.2) we stated that, although Cartan's theory is not metric it is, exceptionally, considered viable. In fact, the theory is experimentally indistinguishable from general relativity and this confirms its viability [68]. However, some authors [69] argue that this theory violates one of the viability criteria by claiming that it is irreducible to special relativistic laws in the absence of gravity. Nonetheless, as we showed in Section (2.6), this argument was cogently refuted by Trautman [69] who believed that in the real world spin effects disappear automatically with the disappearance of gravity and thus the Cartan's theory is still linked to special relativity. In fact it can be realized that, in contrast to the masses which are essentially additive quantities, the spins of individual particles usually cancel each other when the system is considered as a whole. Therefore, in such a circumstance no torsion effect on space will be observed. But in the world of particles the influence of spin on space geometry is thought to be greater than that made by their masses. Thus, it is expected that in the case of, say, neutron stars where the spin density may play an effective role the torsion should be taken in a great account and, as suggested by Trautman, gravitational collapse may be avoided. By this, and by knowing that some interesting results based on this theory have been obtained [70] we conclude that Cartan's model serves a good approach to strong gravity. On the other hand since spin effects are basically of a quantum nature one would believe that a proper modification of general relativity should be achieved by quantization - what we shall introduce in the following section.

(4.7) Quantization of General Relativity

As it was learnt before, the strong evidence of the viability of Einstein's theory in the weak gravity, is counterbalanced by the weak evidence of its validity in strong gravity. This situation necessitates that general relativity should be reconstructed in a way that strong field effects may be primarily accounted for. Presumably this may be achieved, as many used to believe [71], by leaving the framework of classical physics, i.e. by introducing a quantum-mechanical modification of Einstein's gravitational model.

We have already stated in the introduction to this thesis in Section (0.5) that, certain motivations had encouraged this sort of thinking, since by the quantization of classical theories of the electromagnetic field unreasonable behaviours in physics were averted or eliminated. For instance, the collapse of the classical Rutherford's atom, the Rayleigh-Jeans catastrophe were respectively remedied by Bohr's semi-quantum model [72] and the Planck quantization of black body radiation [73]. In spite of this similarity the nonlinear character of the gravitational field and the geometrical picture of general relativity had made Einstein's theory stand apart from the mainstream of physics. Hence, all attempts at quantizing GTR run into great difficulties.

Since, nearly, the last three decades a diversity of frameworks, and a great deal of studies had been made by Wheeler [74], DeWitt [75], Feynman, Schwinger and others [76]. Unfortunately, all these approaches are either ambiguous or, arbitrary or both and, have unmanageable technical problems [77]. In particular, the nonrenormalizability of the quantized general relativity nearly closes the gates against any further progress in this direction [78].

As we noted in the beginning of Section (4.5) the unquantizability of GTR as well as its singular character are but a consequence of its imperfection. Whence, we believe, that the road to the quantized gravity should firstly pass through a certain classical modification of Einstein's theory.

In fact, ultimately, we are not looking for a quantized theory of gravity within the framework of this thesis, but for rather a classical alternative that will be hopefully amenable to quantization.

(4.8) A Model with a Deformed Schwarzschild Solution

In this model, instead of modifying general relativity, its Schwarzschild solution is amended so that the resulting metric can be made free from singular features, say, in the strong gravity, whereas it retains the usual weak-field behaviour. The idea of this approach is to associate dynamical systems with the flow of the geodesic equations. As it was learnt from Section (3.6) space-time singularities are defined by the incompleteness of the geodesic. Then since the dynamical system is defined for all the values of the proper-time, representing the flow parameters, the geodesic will be complete. Therefore, the singularity on the original manifold will be disconnected from the space-times of all observers [79].

In Section (3.7) we suggested that the singularity can be eliminated by either deforming Schwarzschild space-time or by deforming Einstein's gravitational equations themselves. It seems to us that both ways must be equivalently meaningful, since any deformation of the solution would generally correspond to a deformation of the equation and vice versa. We, therefore, introduce this model to complete the picture of different modifications of general relativity that required to lead to a non-singular strong-field behaviour.

(4.9) Summary and Critique

In the preceding sections we reviewed different approaches aiming to establishing a theory which should fulfil the following main requirements:

1 - Satisfying the viability conditions;

2 - Not predicting the gravitational collapse,

3 - Not leading to ambiguities or arbitrariness, when subjected to quantization.

The first requirement is almost satisfied by all kinds of the afore-given modifications of GTR. The fulfilment of the second requirement is our main target in this chapter. In fact all strong-field models are designed to, primarily, satisfy this requirement. There are strong indications from our previous discussions, especially in Section (4.5) that collapse can be prevented if the following two criteria are observed:

i - The geometrical representation of GTR is preserved.

ii - The gravitational equation of the modified theory are of a differential order higher than that of general relativity.

This implies that a Lagrangian with nonlinear term in R should be employed. As we noted in Section (4.2) -a the quadratic Lagrangian seems to be more important in considering strong-field gravity or when we need to bridge the gap to quantum physics. In the next chapter we will show that the most general and the most rational form of the Lagrangian should be nonlinear in R with coefficients whose significance will be revealed in Chapter Eight. As for the objection raised against the quadratic Lagran-Lagrangian, and mentioned in (4.2) - a, we believe that it is weak. Apart from our critique which will be given in Section (7.5) this objection is not well substantiated since it relies solely on the weak-field correspondence. It is illogical to throw away the valuable (in strong-field gravity) quadratic terms of the curvature because it does not satisfy the (weak-field) flat space limit. Lanczos, arguing on that, had suggested, that there exists a highly agitated metrical plateau of such high frequency that, for all macroscopic purposes, only the average value of the $g_{\mu\nu}$ are at our disposal which are thus constants, although they hide the existence of very high curvatures [80].

As for the third requirement concerning the quantizability of general relativity, some modifications like Kilmister-Yang model are promising. Again any achievement in this direction can be secured by sticking to the geometrical picture of general relativity and by utilizing nonlinear Lagrangian which leads to gravitational equations of a differential order higher than that of GTR. Only then the gravitational theory may be brought under the umbrella of quantum mechanics.

Further, since we know that the unquantizable Einsteinian model is essentially singular, we can think of a modification which is non-singular to be quantizable. And since the gravitational collapse is due to the existence of singularities it seems to be sufficient to require that a model, viable in the weak-field gravity, should be non-singular in the strong-field gravity. In the example explained by Trautman for a non-singular strong-field gravity, based on Cartan's theory (4.6) the gravitational collapse is prevented by the spin-induced torsion. The prevention of collapse by assuming certain strong-field effects seems to be very interesting. For instance, the repulsive potential in Fujii-O'Hanlon theory (4.4) - (iii), or the theory with a limiting curvature (4.3) may serve examples for these kinds of effects. In [5.4.34] we will show how the limiting curvature may be related to, possibly, quantum parameters - which will perhaps have something to do with the spin effects. Such an example gives strong indications of the deeply close relation between the nonlinearity of the Lagrangian, the nonsingularity of the metric and, probably, the quantizability of such a modified theory.

Furthermore, it was explained in (2.6) and (4.6) that Cartan's theory is still considered viable, though it seems to be nonrelativistic (in one sense). This discrepancy arises because in the areas where spin effects vanish the principle of equivalence is applicable, and in the areas where this principle breaks down the spin is dominant. But because of the fact that the vanishing of spin does not mean the absence of gravity, whereas the absence of gravity automatically means the absence of spin, and in view of Trautman's argument given before, Cartan's theory is, in fact, relativistic and hence viable. This is also consistent with the above-quoted argument raised by Lanczos.

Moreover, the above-mentioned strong-field effects can be also accounted for as it is proposed in (4.8) by deforming Schwarzschild solution. This idea seems to be useful in distinguishing which modification of GTR is more promising. If a deformed solution fully agrees with the experiment, thence it will give an indication of how the Lagrangian, and consequently, the gravitational equations, can be modified. Thus a correct generalization of general relativity may be obtained.

As for the introduction of the additional fields into general relativity as an alternative way of modification, we see that it is less encouraging. Though, this approach would indicate how the theory should be modified in order to reasonably explain strong-field behaviour. We have noticed that theories with additional fields predict certain unusual effects like the variability of gravitational coupling "constant" G, and the non-Newtonian potential as it was shown in Section (4.4). The variability of G leads to Nordtvedt effect which contradicts the equivalence principle because of the anomalous acceleration that forbids the geodesic motion. In fact such an effect is due to Mach's principle. We notice that general relativity does not fully agree with the Machian hypothesis, since the Minkowskian space which constitutes one of the solutions of GTR is non-Machian.

On the other hand, the non-Newtonian potential [4.4.9] whose form coincides with the Yukawan potential [4.4.12] may lead to a very interesting idea. Namely, one may think that the exact solution of Fujii-O'Hanlon potential is nothing but the potential of the nuclear scalar mesonic field (see for example reference [7]). This would imply that nuclear forces are somehow linked to the strong gravitational field phenomena.

To sum up, we end this chapter by concluding that all the afore-presented modifications of GTR give strong indications that:

1 - General relativity is not the perfect theory of gravitation.

2 - Modification of GTR is the route to strong-field nonsingular quantizable gravity.

3 - A successful modification must be within the frame of the geometrical picture of general relativity.

4 - A perfect version of a successful modification should exhaust the maximum geometrical content of spacetime.

Being guided by these considerations we devote the following five chapters of this thesis to the establishing of a perfect model of gravitation.

PART II

Synopsis

This part is purely our contribution to the gravitational theory.

*In Chapter Five, we give a new derivation of the 4th order gravitational equations from which we establish our theory on the generalized Lagrangian. We derive the most reasonable form of the Lagrangian function that will be the basis of our generalized theory of gravitation. We also obtain a relationship that should be obeyed by any Lagrangian describing gravitation.

*In Chapter Six, we obtain useful relations governing the metric coefficients and the scalar curvature in static isotropic space times. A classification rule for different Lagrangians is given. Our generalized metric is introduced which is nonsingular and reducible to cosmological and thence to Schwarzschild solutions by imposing a weak-field constraint.

*Chapter Seven deals with a special case of the generalized metric where we first obtain an exact expression for a variable scalar curvature.

*The advanced model of gravity is given in Chapter Eight where we establish three useful theoremsgoverning this kind of metric. We set an upper limit for the scalar curvature beyond which general relativity should be not applicable. We also first obtain an exact solution for our metric which allows complex values in the strong-field limit that may characterize quantum effects.

*More consideration of the complexity of the metric is provided in Chapter Nine.

- A résumé and a general outlook come at the end of the thesis.

CHAPTER FIVE

The Most Generalized Gravitational Field Equations and the Most General Lagrangian

(5.0) Prelude

In this chapter we present an alternative derivation of the Lanczos 4th order in $g_{\mu\nu}$ generalized equation [4.2.59] Our method, we believe, is more advantageous in comparison with Lanczos' derivation.

We started from very general assumptions in constructing the Lagrangian function of our action integral by allowing all possible derivatives of $g_{\mu\nu}$, in contrast to Lanczos', whose invariant function consists of only $g_{\mu\nu}$ and $R_{\mu\nu}$. Therefore we are able to answer the question, why the equations cannot have a differential order higher than the fourth? Such an answer is, at least, not directly clear in the previous derivation. Lanczos excluded from his invariant function the 1st derivative of the metric tensor by employing the principle of general covariance. We conversely show that the inclusion of this derivative will give no contribution to the resulting equations and this confirms the equivalence principle. Our derivation, besides that, is more transparent and straightforward.

Moreover, we exploited the resulting gravitational equations together with the Brans-Dicke-scalar-tensor theory to derive the most general physical Lagrangian in a polynomial form of R. The arbitrary parameters of this polynomial will be utilized, we suggest, to establish the long-awaited generalized theory of gravity which will hopefully be able to successfully explain the weak-field as well as the strong-field gravitational phenomena.

We commence by employing Euler-Lagrange equations resulting from the Hamiltonian principle of the least action, to derive the generalized gravitational field equations.

(5.1) Euler-Lagrange Partial Differential Equations:

Let us consider the variation δI of the following integral

$$I \equiv \int_{\alpha}^{b} dXf(x,y,y', y'') \qquad [5.1.1]$$

w.r. to y, where $y = y(x)$ and $y' = \frac{dy}{dx}$, and where a and b are some constant numbers. As has been discussed before, for f being an action function the variation δI will be zero, i.e.

$$\delta I = \int_a^b dx\delta f(x,y,y',y'') = \int_a^b dx\left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \frac{\partial f}{\partial y''}\delta y''\right) = 0$$

[5.1.2]

where the variations,

δy, $\delta y'$ and $\delta y''$ are infinitesmally small. Integrating twice by parts will immediately lead to the following well-known Euler-Lagrange differential equations [1]

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \qquad [5.1.3]$$

with boundary term

$$\delta I = \left[\left(\frac{\partial f}{\partial y'} - \frac{d}{dX}\left(\frac{\partial f}{\partial y''}\right)\right)\delta y + \frac{\partial f}{\partial y''}\,\delta y''\right]_a^b \qquad [5.1.4]$$

[5.1.3] is the necessary and sufficient condition for [5.1.1] to be stationary with given boundary conditions. If the integral involves more than one independent variable, say, $X^\lambda \equiv \{x^1, x^2, x^3, x^4\}$ i.e. $\lambda = 1,2,3,4$ then the following integral

$$I = \iiiint dx^1 dx^2 dx^3 dx^4 f(x^1, x^2, x^3, x^4, y, y_1', y_1', y_1'' \ldots y_2' y_2'' \ldots y_{12}'' \ldots)$$

$$\equiv \int d^4x_\lambda f(x^\lambda, y, y_\lambda', y_{\lambda\gamma}'') \qquad [5.1.5]$$

will lead to the following equations for y.

$$\frac{\partial f}{\partial y} = \sum_{\lambda,\gamma=1}^{4}\left[\frac{d}{dx^\gamma}\left(\frac{\partial f}{\partial y_\lambda'}\right) - \frac{d^2}{dx^\gamma dx^\lambda}\left(\frac{\partial f}{\partial y_{\lambda\gamma}''}\right)\right] \qquad [5.1.6]$$

with $y_\lambda' \equiv \dfrac{\partial y}{\partial x^\lambda}$ and $y_{\lambda\gamma}'' \equiv \dfrac{\partial^2 y}{\partial x^\gamma \partial x^\lambda}$.

In the next chapter we will apply equations [5.1.3, 4, 6] to obtain the differential equations that govern the behaviour of the gravitational field.

(5.2) The Generalized Equations of Gravitation. An Alternative Derivation.

In the derivation of equations [4.2.59] the Ricci tensor $R_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ were regarded, at first, as independent of each other and the Lagrangian density [4.2.23] was subjected to variation. The change $\delta R_{\mu\nu}$ corresponding to the infinitesmal deformation $\delta g_{\mu\nu}$ was calculated to give the expression [4.2.48] representing derivatives of $g_{\mu\nu}$, higher than those of general relativity equations.

The expressions have been simplified during derivation by bringing $g_{\mu\nu}$ to its orthogonal form, and by choosing locally inertial coordinate systems in which the first derivatives of $g_{\mu\nu}$ vanishes.

It will be shown here that we are able to derive the same generalised equations [4.2.59] alternatively by simply applying the afore-given Euler-Lagrange differential equations.

Let us take as variational variables [5.1.1] the metric tensor and its 1st and 2nd mixed derivatives. We thus consider the integral over 4-volume $\Omega(x^\lambda)$,

$$I \equiv \int d\Omega\, f\left(x^\lambda, g_{\mu\nu}(x^\lambda), \frac{\partial g_{\mu\nu}}{\partial x^\lambda}, \frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right) \qquad [5.2.1]$$

The variation will yield the following system of equations,

$$\frac{\partial f}{\partial g_{\mu\nu}} = \sum_{\gamma,\lambda=1}^{4}\left\{\frac{d}{dX^{\lambda}}\left[\frac{\partial f}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right)}\right] - \frac{d^2}{dx^{\gamma}dx^{\lambda}}\left[\frac{\partial f}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^{\gamma}\partial x^{\lambda}}\right)}\right]\right\}$$

[5.2.2]

Function f which we consider invariant is containing ordinary derivatives of $g_{\mu\nu}$ of the first and second order with respect to coordinate X. The first derivatives of $g_{\mu\nu}$ which do not constitute a tensor will acquire the tensorial form by being subjected to variation. Now by using the following notation,

$$f(x) = \sqrt{g(x)}\,\mathcal{L}(x) \equiv f(R(x)) \qquad [5.2.3]$$

the L.H. side of [5.2.2] yields:

$$\frac{\partial f}{\partial g_{\mu\nu}} = \sqrt{g}\,\frac{\partial\mathcal{L}}{\partial R}\frac{\partial R}{\partial g_{\mu\nu}} + \mathcal{L}\frac{\partial\sqrt{g}}{\partial g_{\mu\nu}} \qquad [5.2.4]$$

Also, by the aid of the definition [1.4.26] for R where $R_{\mu\nu}$ is assumed to be independent of $g_{\mu\nu}$ and by using [4.2.22,42,52], one gets

$$R_{\mu\nu}\delta g^{\mu\nu} = -R^{\mu\nu}\delta g_{\mu\nu} \qquad [5.2.5]$$

Then by employing the relation $R = g^{\mu\nu}R_{\mu\nu}$ and by using [5.2.5] together with [4.2.25] into [5,2,4] , one obtains for the L.H. side of [5.2.2] the following expression

$$\frac{\partial f}{\partial g_{\mu\nu}} = -\sqrt{g}\,(\mathcal{L}'R^{\mu\nu} - \tfrac{1}{2}\mathcal{L}g^{\mu\nu}). \qquad [5.2.6]$$

We then calculate the R.H. side of [5.2.2] starting with the first term, i.e.

$$\frac{d}{dx^\lambda}\left[\frac{\partial f}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\right] = \frac{d}{dx^\lambda}\left[\frac{\partial f}{\partial R}\;\frac{\partial R}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\right]$$

$$= \frac{d}{dR}\left(\frac{\partial f}{\partial R}\right) R_{;\lambda}\;\frac{\partial R}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)} + \frac{\partial f}{\partial R}\,\frac{d}{dx^\lambda}\left[\frac{\partial R}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\right] \qquad [5.2.7]$$

where, as in [4.2.60] the covariant derivative of the scalar curvature R, coincides with its ordinary derivative.

Now by using [5.2.3] and the following notation:

$$C^{\mu\nu}_{\lambda} \equiv \frac{\partial R}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)} \qquad [5.2.8]$$

we will get

$$\frac{d}{dx^\lambda}\left[\frac{\partial f}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\right] = \sqrt{g}\left[C^{\mu\nu}_{\lambda}\mathcal{L}'' R_{;\lambda} - \frac{d}{dx^\lambda}C^{\mu\nu}_{\lambda}\mathcal{L}'\right] \qquad [5.2.9]$$

Further, the last term in [5.2.2] yields:

$$\frac{d^2}{dx^\gamma dx^\lambda}\left[\frac{\partial f}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right)}\right] = \frac{d}{dx^\gamma}\left\{\frac{d}{dR}\left[f'\,\frac{\partial R}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right)}\right]R_{;\lambda}\right\}$$

$$= \frac{d}{dx^\gamma}\left\{\frac{d}{dR}\left[f'\;\frac{\partial R}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right)}\right]\right\} R_{;\lambda} + \frac{d}{dR}\left[f'\;\frac{\partial R}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right)}\right] R_{;\lambda;\gamma}$$

[5.2.10]

where, in [5.2.10] the ordinary differentiation was replaced by the covariant form, and where the expression in the brackets { } has no tensorial features, in particular, w.r. to the indices λ and γ that means,

$$\frac{d}{dX^\gamma}\left\{\ \right\} = \left\{\ \right\}_{;\gamma}$$

In fact, introducing covariant differentiation will secure the covariant form of the resulting field equations by causing to disappear any first derivatives of $g_{\mu\nu}$.

Now by denoting

$$C^{\mu\nu}_{\gamma\lambda} \equiv \frac{\partial R}{\partial\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\lambda}\right)} \qquad [5.2.11]$$

which implies

$$\frac{d}{dR} C^{\mu\nu}_{\gamma\lambda} = 0, \qquad [5.2.12]$$

the R.H. side of [5.2.10] becomes:

$$\frac{d}{dX^\gamma}\left\{\frac{d}{dR}\left[f' C^{\mu\nu}_{\lambda\gamma}\right]\right\} R_{;\lambda} + \frac{d}{dR}\left[f' C^{\mu\nu}_{\lambda\gamma}\right] R_{;\lambda;\gamma}$$

$$= \sqrt{g}\, C^{\mu\nu}_{\lambda\gamma} \left(\mathcal{L}''' R_{;\lambda} R_{;\gamma} + \mathcal{L}'' R_{;\lambda;\gamma}\right) \qquad [5.2.13]$$

where [5.2.4] has been used, i.e.

$$\frac{\partial f}{\partial R} = \sqrt{g}\mathcal{L}' \qquad [5.2.14]$$

and according to [5.2.12] $C^{\mu\nu}_{\gamma\lambda}$ is independent of R.

Now we calculate the coefficient $C^{\mu\nu}_{\lambda}$ in [5.2.8].

First, by recalling definitions [1.4.8,9,26,25,24] we rewrite the following relations

$$R = R(x) = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\rho\sigma}R_{\rho\mu\sigma\nu}$$

$$= \frac{g^{\mu\nu}g^{\rho\sigma}}{2}\left[\frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\rho\nu}}{\partial x^\sigma \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho}\right]$$

$$+ g^{\mu\nu}g^{\rho\sigma}g_{\eta\delta}\left[\Gamma^{\eta}_{\sigma\rho}\Gamma^{\delta}_{\mu\nu} - \Gamma^{\eta}_{\nu\rho}\Gamma^{\delta}_{\mu\sigma}\right] \qquad [5.2.15]$$

with

$$\Gamma^{\lambda}_{\mu\nu} \equiv \tfrac{1}{2}g^{\sigma\lambda}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right) \qquad [5.2.16]$$

Then [5.2.8] yields:

$$C^{\mu\nu}_{\lambda} \equiv \frac{\partial R}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)} =$$

$$= g^{\mu\nu\ \rho\sigma}g_{\eta\delta}\left[\frac{\partial\Gamma^{\eta}_{\sigma\rho}}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\Gamma^{\delta}_{\mu\nu} + \Gamma^{\eta}_{\sigma\rho}\frac{\partial\Gamma^{\delta}_{\mu\nu}}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)} - \frac{\partial\Gamma^{\eta}_{\nu\rho}}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)}\Gamma^{\delta}_{\mu\sigma}\right.$$

$$\left. - \Gamma^{\eta}_{\nu\rho}\frac{\partial\Gamma^{\delta}_{\mu\sigma}}{\partial\left(\frac{g_{\mu\nu}}{x^\lambda}\right)}\right] = g^{\mu\nu}g^{\rho\sigma}g_{\eta\delta}\left[\frac{\Gamma^{\delta}_{\mu\nu}}{2}g^{\eta\alpha}\left(\delta^{\alpha}_{\mu}\delta^{\sigma}_{\nu}\delta^{\rho}_{\lambda}\right.\right.$$

$$\left. + \delta^{\alpha}_{\mu}\delta^{\rho}_{\nu}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}\delta^{\alpha}_{\lambda}\right) + \frac{\Gamma^{\eta}_{\sigma\rho}g^{\delta\alpha}}{2}\left(\delta^{\alpha}_{\mu}\delta^{\mu}_{\nu}\delta^{\nu}_{\lambda}\right.$$

$$\left. + \delta^{\alpha}_{\mu}\delta^{\nu}_{\nu}\delta^{\mu}_{\lambda} - \delta^{\mu}_{\mu}\delta^{\nu}_{\nu}\delta^{\alpha}_{\lambda}\right) - \frac{\Gamma^{\delta}_{\mu\sigma}g^{\eta\alpha}}{2}\left(\delta^{\alpha}_{\mu}\delta^{\nu}_{\nu}\delta^{\rho}_{\lambda}\right.$$

$$\left. + \delta^{\alpha}_{\mu}\delta^{\rho}_{\nu}\delta^{\nu}_{\lambda} - \delta^{\nu}_{\mu}\delta^{\rho}_{\nu}\delta^{\alpha}_{\lambda}\right) - \frac{\Gamma^{\eta}_{\nu\rho}g^{\delta\alpha}}{2}\left(\delta^{\alpha}_{\mu}\delta^{\mu}_{\nu}\delta^{\sigma}_{\lambda}\right.$$

$$\left.\left. + \delta^{\alpha}_{\mu}\delta^{\sigma}_{\nu}\delta^{\mu}_{\lambda} - \delta^{\alpha}_{\mu}\delta^{\mu}_{\nu}\delta^{\sigma}_{\lambda}\right)\right] = \tfrac{1}{2}g^{\mu\nu}\left[2\Gamma^{\mu}_{\mu\nu}g^{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}g^{\nu\mu} + \Gamma^{\lambda}_{\sigma\rho}g^{\sigma\rho}\right.$$

$$+ \Gamma^{\lambda}_{\sigma\rho}g^{\rho\sigma} - \Gamma^{\lambda}_{\sigma\rho}g^{\sigma\rho} - 2\Gamma^{\mu}_{\mu\sigma}g^{\lambda\sigma} + \Gamma^{\lambda}_{\mu\rho}g^{\mu\rho} - \Gamma^{\nu}_{\nu\rho}g^{\rho\lambda} - \Gamma^{\lambda}_{\nu\rho}g^{\rho\nu}$$

$$\left. + \Gamma^{\nu}_{\nu\rho}g^{\rho\lambda}\right] = 0. \qquad [5.2.17]$$

This means that

$$C^{\mu\nu}_{\lambda} = \frac{d}{dX} C^{\mu\nu}_{\lambda} = 0 \qquad [5.2.18]$$

Now we have in [5.2.9]

$$\frac{d}{dX^{\lambda}}\left[\frac{\partial f}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right)}\right] = 0 \qquad [5.2.19]$$

which yield the following interesting result: that our function f, and hence by [5.2.3] the Lagrangian function is either linearly dependent of $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}$ or not dependent at all. The latter, of course, is completely consistent with the principle of general covariance which excludes the first derivatives of $g_{\mu\nu}$ from the Lagrangian. However, the result,

$$\frac{\partial f}{\partial\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right)} = \text{const.} = C \qquad [5.2.20]$$

will always allow the inclusion of $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}$, at first, in the invariant Lagrangian, since the constant C can be set zero. Also, the result [5.2.19] indicates that although R is constructed from $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}$ according to [5.2.15,16] the terms containing Γ would give no contribution if the change in R w.r. to $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}$ has been considered. This situation is equivalent to employing the principle of equivalence by working in a locally inertial system of coordinates where all Γ vanish.

This implies that either [5.2.18] is an expression for the validity of the general covariance principle or else is in accordance with it.

Now we return to equation [5.2.7,9] where the following boundary term corresponding to [5.1.4] is given for [5.2.9].

$$\delta I = \sqrt{g} \sum_{\lambda,\gamma=1}^{4} \left\{ \left[C^{\mu\nu}_{\lambda} \mathcal{L}' - \frac{d}{dX^{\lambda}} (C^{\mu\nu}_{\lambda\gamma} \mathcal{L}') \right] \delta g_{\mu\nu} + \mathcal{L}' C^{\mu\nu}_{\lambda\gamma} \delta \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right) \right\}_{b}^{a} \tag{5.2.21}$$

Then because of

$$\frac{d}{dR} C^{\mu\nu}_{\lambda\gamma} = 0, \tag{5.2.22}$$

we get

$$\frac{d}{dX^{\lambda}} (C^{\mu\nu}_{\lambda\gamma} \mathcal{L}') = R_{;\lambda} \frac{d}{dR} (C^{\mu\nu}_{\lambda\gamma} \mathcal{L}') = C^{\mu\nu}_{\lambda\gamma} R_{;\lambda} \mathcal{L}'' \tag{5.2.23}$$

Therefore, in virtue of [5.2.18,22,23] [5.2.21] becomes:

$$\delta I = -\sqrt{g} \sum_{\gamma}^{4} \left[\mathcal{L}'' C^{\mu\nu}_{\lambda\gamma} R_{;\lambda} \delta g_{\mu\nu} - \mathcal{L}' C^{\mu\nu}_{\lambda\gamma} \delta \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right) \right]_{a}^{b} \tag{5.2.24}$$

and this will give zero, since $\delta g_{\mu\nu}$ are assumed to be vanishing at the boundaries. Furthermore, by collecting the terms [5.2.6,7] and also [5.2.19] together with [5.2.13], then by using them in [5.2.2] the following field equations result:

$$\mathcal{L}' R^{\mu\nu} - \frac{\mathcal{L}}{2} g_{\mu\nu} + \sum_{\gamma,\lambda=1}^{4} C^{\mu\nu}_{\gamma\lambda}(\mathcal{L}''' R_{;\lambda} R_{;\gamma} + \mathcal{L}'' R_{;\lambda;\gamma})$$

$$\equiv H^{\mu\nu} = 0 \qquad [5.2.25]$$

We then utilize [5.2.15] to calculate the terms with the coefficients $C^{\mu\nu}_{\lambda\gamma}$ under the summation sign. It is convenient to adopt the usual tensorial summation notation by dropping the $\sum$ in [5.2.25]. Thus, we will have:

$$C^{\mu\nu}_{\gamma\lambda} R_{;\lambda} R_{;\gamma} = \frac{g^{\mu\nu}}{2} (g^{\rho\sigma} \delta^{\rho}_{\mu} \delta^{\lambda}_{\nu} \delta^{\gamma}_{\mu} \delta^{\sigma}_{\nu}$$

$$- g^{\rho\sigma} \delta^{\mu}_{\mu} \delta^{\sigma}_{\nu} \delta^{\lambda}_{\nu} \delta^{\gamma}_{\rho} - g^{\rho\sigma} \delta^{\rho}_{\mu} \delta^{\nu}_{\nu} \delta^{\lambda}_{\sigma} \delta^{\gamma}_{\mu}$$

$$+ g^{\rho\sigma} \delta^{\mu}_{\mu} \delta^{\nu}_{\nu} \delta^{\gamma}_{\rho} \delta^{\lambda}_{\sigma}) R_{;\lambda} R_{;\gamma}$$

$$= \frac{g^{\mu\nu}}{2} (R_{;\mu} R_{;\nu} g^{\mu\nu} - R_{;\nu} R_{;\rho} g^{\rho\nu}$$

$$- R_{;\mu} R_{;\sigma} g^{\mu\sigma} + R_{;\sigma} R_{;\rho} g^{\rho\sigma}) \qquad [5.2.26]$$

By renaming the repeated indices in the last three terms in [5.2.26] as follows: $\nu \to \sigma$ in the second term, $\mu \to \rho$ in the third term and $\sigma \to \nu$, $\rho \to \mu$ in the fourth term. This yields:

$$C^{\mu\nu}_{\gamma\lambda} R_{;\lambda} R_{;\gamma} = g^{\mu\nu}(R_{;\mu} R_{;\nu} g^{\mu\nu} - R_{;\sigma} R_{;\rho} g^{\rho\sigma}) \qquad [5.2.27]$$

which by lowering the indices μ and ν of $C^{\mu\nu}_{\gamma\lambda}$, becomes:

$$C^{\gamma\lambda}_{\mu\nu} R_{;\lambda} R_{;\gamma} = R_{;\mu} R_{;\nu} - g_{\mu\nu} R_{;\sigma} R_{;\rho} g^{\rho\sigma} \qquad [5.2.28]$$

Similarly we obtain

$$C^{\gamma\lambda}_{\mu\nu}R_{;\lambda;\gamma} = R_{;\mu;\nu} - g_{\mu\nu}R_{;\sigma;\rho}g^{\rho\sigma} \qquad [5.2.29]$$

$$= R_{;\mu;\nu} - g_{\mu\nu}\,\Box^2 R \qquad [5.2.30]$$

where the following invariant D'Alembert-Laplace operator was used:

$$\Box^2 \equiv {}_{;\sigma;\rho}\, g^{\rho\sigma} \qquad [5.2.31]$$

Finally, by entering [5.2.28] and [5.2.30] into [5.2.25] we obtain the following set of 16 gravitational field equations in its covariant form:

$$H_{\mu\nu} = \mathcal{L}'''(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\sigma}R_{;\rho}g^{\rho\sigma}) + \mathcal{L}''(R_{;\mu;\nu} - g_{\mu\nu}\,\Box^2 R) + \mathcal{L}'R_{\mu\nu} - \tfrac{1}{2}\mathcal{L}g_{\mu\nu} = 0 \qquad [5.2.32]$$

which is similar to [4.2.59] derived by Lanczos. The equation is divergenceless and automatically reduces to Einstein's equation of general relativity for linear $\mathcal{L}$.

The contribution of the first two brackets in [5.2.32] makes the equation of the fourth order in $g_{\mu\nu}$ and it vanishes for the Lagrangian $\mathcal{L} = R$.

We will employ equations [5.2.32] in our further consideration of the gravitational field that will lead to a generalized theory of gravity.

(5.3) Commentary and Remarks

It seems again interesting to notice that in deriving [5.2.32] the part of R containing Γ did not give any contribution to the structure of the equation, since had it given this, it would have contradicted the principle of general covariance.

As for the possibility of generalizing [5.2.32] to contain derivatives of $g_{\mu\nu}$ up to the nth order, we will view that as follows.

If we include in the function f of [5.2.1] derivatives of $g_{\mu\nu}$ of order n, higher than the second, we would have expected in [5.2.2] terms like:

$$\frac{d^n}{dX^n}\left(\frac{\partial f}{\partial\left(\frac{\partial^n g_{\mu\nu}}{\partial x..\partial x^n}\right)}\right),$$

and correspondingly, a derivative of the type,

$$\frac{\partial R}{\partial\left(\frac{\partial^n g_{\mu\nu}}{\partial x...\partial x^n}\right)}.$$

But in view of definition [5.2.15] of R and its contractions being constructed of $g_{\mu\nu}$ derivatives not higher than the second, no such extra terms will occur.

Therefore, we come to the conclusion that, within the frame of Riemannian geometry, being characterized by the curvature tensor or its contractions, we cannot construct gravitational field equations of order higher than the fourth derivative in $g_{\mu\nu}$.

This result makes our derivation more advantageous than that made by Lanczos, where such an explanation is not immediately apparent.

On the other hand, since the invariant function [4.2.18], proposed by Lanczos is built solely from $R_{\mu\nu}$ and $g_{\mu\nu}$, where $R_{\mu\nu}$ implicitly contains the first and the second derivatives of $g_{\mu\nu}$, it must be equivalent to our Lagrangian function of [5.2.3] which explicitly contains these derivatives, and hence it looks natural that the two derivations will lead to the same generalized field equation.

In the end we would like to admit that the inclusion of $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}$ in our Lagrangian did not contribute to the resulting equation which justifies the choice of the Lagrangian made by Lanczos in his derivation.

We also add here a concluding point about Equations [5.2.32]. By constituting a further generalization of GTR within the frame of Riemannian geometry, and by having the maximum possible differential order, these equations are compatible with the requirements set in the end of the preceding chapter. Thus, they will serve as a good basis for developing our generalized model of gravitation in the following chapters.

(5.4) Most General Lagrangian Derived

Since equation [5.2.32] has been derived as a result of more general assumptions about the structure of the Lagrangian density, therefore, it is regarded as a modification of general relativity in the case when Lagrangians other than R are to be employed, and hence it should contain terms representing Einstein's model and should have Einstein's spaces as special solutions.

Let us rewrite equations [5.2.32] with the use of Einstein's tensor $G_{\mu\nu}$ [1.6.8]; we then have the following vacuum equation:

$$G_{\mu\nu} = \frac{\mathcal{L}'''}{\mathcal{L}}(g_{\mu\nu}R_{;\rho}R_{;\sigma}g^{\rho\sigma} - R_{;\mu}R_{;\nu})R + \frac{\mathcal{L}''}{\mathcal{L}}(g_{\mu\nu}\Box^2 R - R_{;\mu;\nu})R - (\frac{\mathcal{L}'R}{\mathcal{L}} - 1)R_{\mu\nu} \qquad [5.4.1]$$

Contracting [5.4.1] by $g^{\mu\nu}$ yields:

$$g^{\mu\nu}G_{\mu\nu} = -R = -(\frac{\mathcal{L}'R}{\mathcal{L}} - 1)R + \frac{3R\mathcal{L}''}{\mathcal{L}}\Box^2 R + \frac{\mathcal{L}'''}{\mathcal{L}}(4R_{;\rho}R_{;\sigma}g^{\rho\sigma} - g^{\mu\nu}R_{;\mu}R_{;\nu})R \qquad [5.4.2]$$

i.e.

$$\Box^2 R = \frac{-2\mathcal{L}}{3\mathcal{L}''} + \frac{\mathcal{L}'R}{3\mathcal{L}''} - \frac{\mathcal{L}'''}{3\mathcal{L}''}(4R_{;\rho}R_{;\sigma}g^{\rho\sigma} - g^{\mu\nu}R_{;\mu}R_{;\nu}) \qquad [5.4.3]$$

It is obvious that for $\mathcal{L} = R$ [5.4.1] reduces to

$$G_{\mu\nu} = 0 \qquad [5.4.4]$$

which is general relativity free field-equation, and by [5.4.2] it gives

$$R = 0 \qquad [5.4.5]$$

It is interesting to notice that [5.4.1],[5.4.3] can be interpreted as Brans-Dicke-Jordan type equations: [2.5.4] and [2.5.6] , where the R.H. side of [5.4.1] will represent the contribution brought in by the zero mass additional scalar field R coupled to them and satisfying the wave equation [5.4.3].

This situation, while explaining why the Brans-Dicke theory got the best reputation after Einstein's model, will suggest that modifying GTR by deriving generalized field equations is equivalent, in a certain sense, to introducing an auxilliary scalar field. This modification may be looked to as a geometrization of the scalar field of Brans and Dicke theory, as [5.2.32] is constructed solely from geometrical objects.

Further, for the Lagrangian quadratic in R,[5.4.1], and [5.4.3] yield respectively:

$$G_{\mu\nu} = - R_{\mu\nu} + \frac{2}{R}(g_{\mu\nu}\Box^2 R - R_{;\mu;\nu})$$

$$= -\tfrac{1}{4}g_{\mu\nu}R + \frac{1}{R}(g_{\mu\nu}\Box^2 R - R_{;\mu;\nu}) \qquad [5.4.6]$$

and

$$\Box^2 R = 0 \qquad [5.4.7]$$

If we define R in terms of Brans and Dicke scalar field density as

$$R = -4C\phi \qquad [5.4.8]$$

where C is a constant. Then we get from [5.4.6,7] the following

$$G_{\mu\nu} = C\phi g_{\mu\nu} - \frac{1}{\phi}\,\phi_{;\mu;\nu} \qquad [5.4.9]$$

Comparison with [2.5.4] suggests that this equation is nothing but Brans-Dicke vacuum equation in the case $\omega = 0$, and the term with $g_{\mu\nu}$ is a cosmological term. But we can, after using [2.5.6], rewrite [2.5.4] in the following manner:

$$G_{\mu\nu} = -\frac{(3+2\omega)g_{\mu\nu}\Box^2\phi}{4\,\phi} - \frac{\omega}{2\phi^2}\, g_{\mu\nu}\,\phi_{;\rho}\phi_{;\sigma}g^{\rho\sigma}$$

$$+ \frac{\omega}{\phi^2}(g_{\mu\nu}\phi_{;\rho}\phi_{;\sigma}g^{\sigma\rho} - \phi_{;\mu}\phi_{;\nu})$$

$$+ \frac{1}{\phi}\,(g_{\mu\nu}\Box^2\phi - \phi_{;\mu;\nu}) \qquad [5.4.10]$$

where the first two terms may be regarded as cosmological terms, and where, we used the following conformal relation

$$\phi^{;\rho} = \phi_{;\sigma}g^{\rho\sigma} \qquad [5.4.11]$$

In view of the common features of [5.4.1, 10], we can add to ϕ a term $k = k(\omega)$ which will ensure that when $k(\omega \to 0) \to 0$, $\mathcal{L} \to R^2$ as the case in [5.4.6 $\to$ 9]. Thus we assume that:

$$R = -\,4C\phi - k(\omega) \qquad [5.4.12]$$

Now by using this relation, the term-by-term comparison between [5.4.1] and [5.4.10] yields:

$$\frac{\mathcal{L}''' R}{\mathcal{L}} = \frac{\omega}{(R+k)^2} \qquad [5.4.13]$$

$$\frac{\mathcal{L}'' R}{\mathcal{L}} = \frac{1-2\omega}{4(R+k)} \qquad [5.4.14]$$

and

$$\frac{(\mathcal{L}' R-\mathcal{L})}{\mathcal{L}} R_{\mu\nu} = \frac{R}{4}\frac{(\mathcal{L}' R-\mathcal{L})}{\mathcal{L}} g_{\mu\nu} = \frac{\omega}{2(R+k)^2} R_{;\rho} R_{;\sigma} g^{\rho\sigma} g_{\mu\nu}$$

[5.4.15]

i.e.

$$\frac{R}{2}\frac{\mathcal{L}' R-\mathcal{L}}{\mathcal{L}} = \frac{\omega}{(R+k)^2} R_{;\rho} R_{;\sigma} g^{\rho\sigma} \qquad [5.4.16]$$

Now by using [5.4.13] we get:

$$\tfrac{1}{2}(\mathcal{L}' R-\mathcal{L}) = \mathcal{L}''' R_{;\rho} R_{;\sigma} g^{\rho\sigma} = g^{\rho\sigma} \frac{\partial R}{\partial x^\rho} \frac{\partial R}{\partial x^\sigma} \frac{\partial^2 \mathcal{L}'}{\partial R^2}$$

$$\Box^2 \mathcal{L}' = \tfrac{1}{2}(\mathcal{L}' R-\mathcal{L}) \qquad [5.4.17]$$

This equation can be looked to as a wave equation with a source term. Also, [5.4.13] and [5.4.14] give

$$\frac{\mathcal{L}'''}{\mathcal{L}''} = \frac{4(n-2)}{R+k} \qquad [5.4.18]$$

with

$$n = n(\omega) \equiv \frac{2-3\omega}{1-2\omega} \qquad [5.4.19]$$

Then by integrating [5.4.18] over R, one gets the following general form for the Lagrangian:

$$\mathcal{L}(R) = \frac{C_1}{2(2n-3)} (R+k)^{2(2n-3)} + \beta R + \gamma \qquad [5.4.20]$$

where C_1, β, γ are arbitrary constants.

On the other hand, differentiating [5.4.14] w.r. to R and then using [5.4.13] will lead to the equation:

$$\frac{\mathcal{L}'}{\mathcal{L}} = \frac{1}{R} + \frac{4n-7}{R+k} \qquad [5.4.21]$$

which is satisfied by:

$$\mathcal{L}(R) = aR(R+k)^{4(n-2)+1} \qquad [5.4.22]$$

where a is a constant.

Now by setting in [5.4.20] the following value for constant C_1

$$C_1 = 2\alpha(2n-3) \qquad [5.4.23]$$

where α is a new constant, we will obtain

$$\mathcal{L}(R) = \alpha(R+k)^{2(2n-3)} + \beta R + \gamma \qquad [5.4.24]$$

The difference appearing in the two expressions [5.4.22,24] describing the same Lagrangian might have been caused by the incomplete correspondence in comparison between ϕ- and R-equations. However, the two formulae can be identified by imposing certain values on the arbitrary constants.

Now, by knowing that,

$$\left.\begin{array}{ll} \text{for } \mathcal{L} = R^2 & n(0) = 2 \\ \text{for } \mathcal{L} = R & n(\infty) = \frac{3}{2} \end{array}\right\} \qquad [5.4.25]$$

we will find which of [5.4.22] and [5.4.24] give exactly the correct expression for quadratic and linear in R Lagrangians. Thus, if in general we assume that

$$k(0) \sim 0 \qquad [5.4.26]$$

and γ is sufficiently small, as it will be shown by [5.5.3], we will get for instance from [5.4.22],

$$\text{for} \quad \omega \to 0 \quad \mathcal{L} = a\,R^2 + a k(0) R \approx aR^2, \qquad [5.4.27]$$

$$\text{for} \quad \omega \to \infty \quad \mathcal{L} = a\,\frac{R}{R+k} \approx a, \text{with } \frac{k(\infty)}{R} \to 0, \qquad [5.4.28]$$

and, from [5.4.24],

$$\text{for} \quad \omega \to 0 \quad \mathcal{L} = \alpha R^2 + \beta R + \gamma \qquad [5.4.29]$$

$$\text{for} \quad \omega \to \infty \quad \mathcal{L} = \alpha + \beta R + \gamma \qquad [5.4.30]$$

These expressions for $\mathcal{L}$ show that for $\omega \to 0$ the correct Lagrangian is obtained in both [5.4.22,24] while in the case $\omega \to \infty$ [5.4.22] is not satisfactory. We therefore accept [5.4.24] as the most reliable Lagrangian. Furthermore, we notice that our Lagrangian can be presented in any form of the type [4.2.12] with certain values of the constants. The free constant appearing in [5.4.24,29,30] can be regarded as in [4.2.16] standing for non-gravitational field. Also, by using for the constants in [5.4.24] the following substitutions

$$\left.\begin{aligned} \alpha &= (-1)^{m+1}\frac{R_0^{1-m}}{m} \\ \beta &= m(-1)^m R_0^{m-1}\alpha + 1 \\ \gamma &= (-1)^{m+1} R_0 \alpha \end{aligned}\right\} \qquad [5.4.31]$$

where $-R_0 = k$ is a new constant and,

$$m = 2(2n-3) \qquad [5.4.32]$$

then, the Lagrangian will be converted to the form:

$$\mathcal{L} = -\frac{R_0}{m}\left[\left(1 - \frac{R}{R_0}\right)^m - 1\right] \qquad [5.4.33]$$

$m \leq 2$

which is but the Lagrangian proposed in [4.3.1] with two differences: that m is not a mere number but dependent of the parameter ω via [5.4.32] and [5.4.19], and that R_0 which was defined by [4.3.2] as an upper bound for R, now, accordingly is depending on ω through $m(\omega)$ owing to the following relation:

$$R_0 = R_0(\omega) = \left[\frac{\beta\,(-1)^m + (-1)^{m-1}}{m\cdot\alpha}\right]^{\frac{1}{m-1}} \qquad [5.4.34]$$

and conversely for $k = k(\omega)$ which by [5.4.31,34] becomes:

$$k(\omega) = (-1)^{\frac{m}{m-1}}\left[-R_0^{\,m-1} + \frac{(-1)^{m+1}}{m}\,\frac{\beta}{\alpha}\right]^{\frac{1}{m-1}} \qquad [5.4.35]$$

In these relations, since β is the coefficient securing correct Newtonian limits, it should be:

$$\beta \equiv \frac{1}{16\pi G} \qquad [5.4.36]$$

and since R_0 depends on ω, it will be possible to estimate this limiting curvature value by choosing certain values of $m(\omega)$ and $k(\omega)$, provided that β is given by [5.4.36]. For finding out how R might depend on ω, we substitute $\mathcal{L}$ in the equation [5.4.17] from [5.4.24] to get the following wave equation for R.

$$\Box^2 R = \frac{(2-m)g^{\mu\nu}}{R+k} \cdot \frac{\partial R}{\partial x^\mu}\frac{\partial R}{\partial x^\nu} - \frac{(R+k)^m}{2m(m-1)} + \frac{(R+k)^{m-1}R}{2(m-1)} - \frac{\gamma(R+k)^{2-m}}{2C_1(m-1)} \quad [5.4.37]$$

$m \neq 0, 1$.

where by [5.4.23] and [5.4.32] the constant C_1 is defined by:

$$C_1 \equiv \alpha(\omega)m(\omega) \quad [5.4.38]$$

with

$$\alpha \equiv \alpha(\omega) = C_1(\tfrac{1}{2}-\omega) \quad [5.4.39]$$

and,

$$m \equiv m(\omega) = -(\omega-\tfrac{1}{2})^{-1} \quad [5.4.40]$$

(5.5) Remarks

The R.H. side of [5.4.37] which represents the source term is quite informative as we notice the following:

(i) The first term with the derivatives of R describes how the source couples with the gradient of R, where R indicates the strength of the gravitational field.

(ii) The expression for R will depend on the value of the parameter ω via $m(\omega)$ and on the constants γ and C_1.

(iii) The coefficient β cancelled out indicating that there is no given matter in the source term.

(iv) The last term reflects the contribution of the non-gravitational part of the Lagrangian brought into the field by γ.

(v) If R is constant somewhere in the spacetime we will be left with the last three terms which yield:

$$(R+k)^{2m} - mR(R+k)^{2m-1} + \frac{m\gamma}{C_1}(R+k)^2 = 0 \qquad [5.5.1]$$

This would allow imaginary part for R at certain values of m.

(vi) In [5.5.1] if somewhere in the space R is set to be zero one would get, by the aid of [5.4.39,40], the following expression for the parameter k.

$$k \equiv K(\omega,\gamma) = \left[\frac{-\gamma}{\alpha(\omega)}\right]^{\frac{1}{2\left[\frac{C_1}{\alpha(\omega)}-1\right]}} = \left[\frac{-\gamma m(\omega)}{C_1}\right]^{\frac{1}{2[m(\omega)-1]}} \qquad [5.5.2]$$

This means that k describes the interaction between the nongravitational part γ, and the nonlinear part represented by $\alpha(\omega)$ and $m(\omega)$ of the general Lagrangian [5.4.24]. As ω vanishes which corresponds to quadratic R, we will have,

$$K(o,\gamma) = \sqrt{\frac{-2\gamma}{C_1}} \qquad [5.5.3]$$

what is, now, purely nongravitational, and for $\gamma \to 0$ it agrees with our previous assumption in [5.4.26] and for positive γ it becomes a complex quantity.

(vii) When $\omega = 0$ i.e. $m = 2$ this corresponds to the Lagrangian with quadratic R, the first term disappears signifying that the source does not couple with the gradient of the field and also reflects the special position of this kind of Lagrangian. Now [5.4.37] becomes

$$\Box^2 R = \frac{R^2-k^2}{4} - \frac{\gamma}{2C_1} \qquad [5.5.4]$$

which can be reduced by [5.5.3] to

$$\Box^2 R = \frac{R^2}{4} \qquad [5.5.5]$$

for $R = R(r)$, and for constant R it yields Einstein equation i.e. $R=0$.

(viii) The parameter $m(\omega)$ reflects the nonlinearity of the Lagrangian and ω is likely indicating the quantum effects that are related to this nonlinearity and which are presumably considerable in the strong gravitational field.

(5.6) The Generalized Metric

By generalized metric we mean that which is derived from generalized equations of gravitation and based on the most general Lagrangian of the type [5.4.24]. In the following chapters we shall exploit this kind of Lagrangian to demonstrate the possibility of establishing an updated metric by virtue of a certain choice of the Lagrangian's constant coefficients and parameters. As we notice from [5.4.38] and [5.5.2] the parameters k and m and the coefficients α and γ are deeply interlinked, which will impose strong restrictions on the choice of the Lagrangian. In Chapter Eight we will show that due to some physical requirements the coefficient β of the general Lagrangian must be determined by the other two coefficients α and γ.

Indeed, these restrictions emerge from our original idea of bringing together the scalar-tensor theory with the generalized gravitational field into the geometrical picture of Riemann. Thus we get rid of the duality between R and ϕ fields in the Lagrangian of Brans-Dicke theory and at the same time maintain the validity of the equivalence principle as we have noted in sections (1.8) and (2.7) of this thesis. We see that the generalization of Einstein's vacuum equations to the 4th differential order in $g_{\mu\nu}$ is equivalent to incorporating a massless source term as in [5.4.1]. On the other hand, introducing an additional field that shares the stage with $g_{\mu\nu}$ in generating the metric but does not enter the equation of motion, suggests that these kinds of fields must have a zero mass. (This latter concept, due to Dicke [2], [3], is motivated by Mach's principle (1.2) [4] which regards the inertial masses of particles in the universe not as fundamental constants, but rather as an effect caused by their interaction with some cosmic field [5] .)

Therefore, we identify the two above-noted concepts by regarding the generalized equations as an embodiment of Brans-Dicke auxilliary field in Riemannian geometry.

In fact, the generalized equations had been first derived by Lanczos as a result of applying a variational method that accounts for weak deformations of the metric tensor $g_{\mu\nu}$, whereas Mach's idea can have its precise significance only in the case of weak perturbations of a given metric field. The two concepts, therefore, seem equivalent.

Now the metric resulting from this unification of $g_{\mu\nu}$ and Φ-fields is generated by everything in the universe including any test mass say, a star under consideration.

GTR ignores all fields other than $g_{\mu\nu}$ of the given gravitating mass whilst Brans-Dicke theory gives these fields a secondary role.

As there are no isolated masses in the universe, we think that the source of our gravitational field is extended all over the space-time continuum, and that there is no privileged location in space where equation [5.4.37] does not hold. The cancellation of the mass in the source term does not mean its vanishing but, rather, it means that we are not considering a given mass. The presence of a derivative of R with respect to the coordinates even in the R.H. side of [5.4.37] would imply the existence of a source or (absorber) which causes the change of the field.

Finally we would say that [5.4.37] describes the entire field in the cosmos. It says nothing about the metric of space-time manifold due to a particular mass and the whole theory is concerned with constructing the most physical and the most rational form of the Lagrangian. It is only by employing this Lagrangian in a metric derived from the generalized equations, that we can consider the field generated by a given material object. We will devote the following chapters to investigate and study metrics based on general quadratic Lagrangians

(5.7) Discussion

Now the expression [5.4.24] , we believe, constitutes the most general possible classical Lagrangian which is not just constructed to fulfill certain physical requirements, but is mathematically derived from correct relations, based on the validity of Riemannian geometry and Mach principle, namely the generalized gravitational field equation and Brans-Dicke theory. This Lagrangian has the advantage of having a self-emerging linear term in R, that ensures the correct Newtonian limit. Usually such a term is added forcibly to avoid bad behaviour at that limit.

The constant coefficients can be adjusted to make predominant this or that term and the parameter ω which, clearly, has a certain effect on R, can be chosen to give the most adequate and successful result.

The fact that ω has a certain role to play in the structure of the Lagrangian suggests that ω may have a quantum origin. There are some indications of that . We notice that when the Lagrangian [5.4.24] is reduced to the Einstein Lagrangian $\mathcal{L} = \beta R$ parameters, $m(\omega)$ and $k(\omega)$ disappear from the equation while the effect of ω

and $k(\omega)$ on the equation for R is obvious for any other form of $\mathcal{L}$.

Furthermore, we notice that the R.H. side of [5.4.37] which would represent the source of the gravitational field, the more strongly it is affected by ω through m and k, the more significant is the field. All these indications make our Lagrangian a possible candidate for a strong field gravitational theory. Moreover, we would like to admit that since the Lagrangian [5.4.24] contains all possible powers of R it will provide us with more freedom, and by virtue of the arbitrary constants and parameter ω, to satisfy all possible requirements for a successful theory of gravitation. Our incentive was to combine the successes of all existing models, especially the viable Einstein's and Brans-Dicke theories which are already here correlated via the generalized field equations of gravitation. In the end, if we accept the idea of scalar curvature having an upper bound R_0 in the Lagrangian [5.4.24] and hence the relation [5.4.34] explaining R_0 as depending on the parameter ω, we therefore are admitting that $R \leq R_0$ and owing to [5.4.35] k, can have imaginary parts at certain values of m, i.e. at certain ω.

Because of this and what we noted with respect to [5.5.1] we will believe that the scalar curvature can be regarded a complex function. It looks as though the complexity of R has something to do with the quantum significance of the parameter ω which we think is bearing information which, we hope, can be explained later if a successful quantum theory be formulated. In Chapter Nine we will elaborate more on the possibility of the scalar curvature being complexified.

(5.8) The Cosmic Evolution

The inseparability of matter and space-time on one hand, and the motion being the product of their interaction on the other hand, raises the question, whether or not the entire Universe can be considered a dynamical system. Einstein admitted as one solution of his general relativistic equations, the one describing a static homogeneous universe [6], where he introduced an additional cosmological term [see (1.6)] into the gravitational equations. But the observations made by Hubble [7] lead cosmologists to believe that the universe is undergoing an evolutionary expansion.

In this last section we consider this situation in the light of our theory of the generalized Lagrangian. We recall Equation [5.4.17] which constitutes a condition on any Lagrangian of any dynamical system in relation to the scalar curvature R. It relates the space geometry characterized by R to the gravitational energy represented by the Lagrangian $\mathcal{L}$ and transmitted through the space according to the wave equation with a non-vanishing source, i.e.

$$\Box^2\tilde{\Phi}(R) = - 4\pi\rho G(R) \qquad [5.8.1]$$

where we denoted in [5.4.17] the following

$$\mathcal{L}'(R) \equiv \tilde{\Phi}(R) \quad [5.8.2]$$

$$R\mathcal{L}' - \mathcal{L} \equiv 8\pi\rho G(R) \quad [5.8.3]$$

with $\tilde{\Phi}(R)$ a certain wave function and G(R) represents the source, and ρ the background density in the universe.

Further, as we have noted in (5.6) that because of the non-localizability of a certain gravitating centre for the whole universe, the entire distribution of matter within the space-time manifold is serving as the source of gravitation. Generally the varying nature of the wave equation which describes a dynamic process necessitates the existence of a definitely allocated source in the space. But because, in our case, the entire space-time is coherently connected with matter distribution, therefore, we interpret [5.8.1] as describing the evolution of the universe, which, if traced back along the time dimension, it would have converged to the maximum initial concentration of matter, when and where, possibly, the Big Bang [8] could have taken place.

Moreover, by using [5.4.24, 32] in [5.8.3] the function G becomes,

$$G(R) = G_m(R) = -\gamma + \alpha(R+K)^{m-1}[(m-1)R-K] \quad [5.8.4]$$

For Lagrangians linear in R i.e. m = 0, 1 G reduces to a constant which may be identified with the gravitational constant,

$$G = G_0 = -\gamma - \alpha_0 = G_1 + \alpha_1 K_1 - \alpha_0 \quad [5.8.5]$$

For the nonlinear Lagrangians G develops a certain dependence on R. We envisaged such kind of variability in Section (4.4) where we considered theories with additional fields. Thus it becomes evident that in the region dominated by nonlinear terms of the Lagrangian, say, in the direct vicinity of a given neutronic star, the G-dependence on R may be observed.

In the end, and as we noted before our previous consideration of the generalized Lagrangian says nothing about the behaviour of a specific gravitational field. In the next chapter we shall utilize this generalized Lagrangian to construct the metric that describes the field generated by a given gravitating mass.

CHAPTER SIX

Generalized Metric in the Static Isotropic Space-Time

(6.0) Overview

In Chapter Three Einstein's gravitational field equation was considered under the conditions of spatial and temporal symmetries which resulted in Schwarzschild and Kerr solutions. In this chapter we consider, instead, the generalized field equations in a static isotropic metric for a very general Lagrangian where we will be able to obtain useful relations governing this Lagrangian with respect to the field variables R and $g_{\mu\nu}$. By utilizing these relations we get expressions for the metric tensor components g_{rr} and g_{tt}, which by specializing to our Lagrangian [5.4.24], will yield the most perfect form for the generalized metric that we discussed before in (5.6). These relations also allow us to classify different Lagrangians from which we can single out the unreasonable ones.

The generalized metric will certainly contain Schwarzschild's space-time as a special solution, whereas in general the solution should be far more advanced than those of Einstein's spaces. We aim to achieve a metric that explains well, within the classical limitations, all possible gravitational phenomena in both weak and strong parts of the energy spectra. Thus, our metric is expected to be free from any singular features at any distance from the source of gravity.

(6.1) Generalized Equations of Gravitation in the Static Isotropic Metric

Since by [3.1.10] the scalar curvature is a function of only r, therefore the covariant derivatives $R_{;\mu} = \frac{\partial R}{\partial x^{\mu}}$ will vanish for all values of μ other than r, i.e.

$$\left.\begin{aligned} & R_{;\theta} = R_{;\phi} = R_{;t} = 0 \\ & R_{;r} = \dot{R} \neq 0 \end{aligned}\right\} \qquad [6.1.1]$$

Hence, by the aid of [4.2.61] and [3.1.5] one obtains

$$\left.\begin{aligned} & R_{;r;r} = \frac{\partial R_{;r}}{\partial r} - \Gamma^{r}_{rr} R_{;r} = \ddot{R} - \tfrac{1}{2}\frac{\dot{A}}{A}\dot{R} \\ & R_{;\theta;\theta} = \frac{r}{A}\dot{R} = \frac{R_{;\phi;\phi}}{\sin^2\theta} \\ & R_{;t;t} = -\frac{\dot{B}}{2A}\dot{R} \\ & R_{;\mu;\nu} = R_{;\nu;\mu} = -\Gamma^{r}_{\mu\nu}\dot{R} = 0 \quad {\scriptstyle \mu \neq \nu,\ \mu,\nu=\theta,\phi,t} \end{aligned}\right\} \qquad [6.1.2]$$

where $\ddot{R} \equiv \frac{d}{dr}\dot{R} = \frac{d^2}{dr^2}R$.

Now we return to [5.2.32] which constitute a set of 16 differential field equations, out of which only 3 are independent while all other components of $H_{\mu\nu}$ identically go to zero.

Thus, by [6.1.1,2] we will have

$$H_{rr} = \mathcal{L}''(R_{;r;r} - g_{rr}g^{\sigma\lambda}R_{;\sigma;\lambda}) + \mathcal{L}'R_{rr}$$

$$-\frac{\mathcal{L}}{2}g_{rr} = -\mathcal{L}''g_{rr}(g^{\theta\theta}R_{;\theta;\theta} +$$

$$+ g^{\phi\phi}R_{;\phi;\phi} + g^{tt}R_{;t;t}) + \mathcal{L}'R_{rr} - \frac{\mathcal{L}}{2}g_{rr} = 0$$

$$H_{\theta\theta} = \frac{H_{\phi\phi}}{\sin^2\theta} = -\mathcal{L}''' g_{\theta\theta}g^{rr}R^2_{;r} + \mathcal{L}''(R_{;\theta;\theta}$$

$$- g_{\theta\theta}g^{\sigma\lambda}R_{;\sigma;\lambda}) + \mathcal{L}'R_{\theta\theta} - \frac{\mathcal{L}}{2}g_{\theta\theta}$$

$$= -\mathcal{L}''' g_{\theta\theta}g^{rr}\dot{R}^2 - \mathcal{L}''g_{\theta\theta}(g^{rr}R_{;r;r}$$

$$+ g^{\phi\phi}R_{;\phi;\phi} + g^{tt}R_{;t;t}) + \mathcal{L}'R_{\theta\theta} - \frac{\mathcal{L}}{2}g_{\theta\theta} = 0$$

$$H_{tt} = -\mathcal{L}''' g_{tt}g^{rr}R^2_{;r} + \mathcal{L}''(R_{;t;t} - g_{tt}g^{\sigma\lambda}R_{;\sigma;\lambda})$$

$$+\mathcal{L}'R_{tt} - \frac{\mathcal{L}}{2}g_{tt} = -\mathcal{L}''' g_{tt}g^{rr}\dot{R}^2$$

$$-\mathcal{L}''g_{tt}(g^{rr}R_{;r;r} + g^{\theta\theta}R_{;\theta;\theta} + g^{\phi\phi}R_{;\phi;\phi})$$

$$+\mathcal{L}'R_{tt} - \frac{\mathcal{L}}{2}g_{tt} = 0 \qquad [6.1.3]$$

contracting these equations by $g^{\mu\nu}$ and taking into account that $R = g^{\mu\nu}R_{\mu\nu}$ will lead to the following equation:

$$H = g^{\mu\nu}H_{\mu\nu}$$

$$= -3\mathcal{L}''' g^{rr}\dot{R}^2 - 3\mathcal{L}''(g^{rr}R_{;r;r} + 2g^{\theta\theta}R_{;\theta;\theta}$$

$$+ g^{tt}R_{;t;t}) + \mathcal{L}' R - 2\mathcal{L} = 0. \qquad [6.1.4]$$

Again, by the aid of [6.1.2] together with [3.1.3] equations [6.1.3,4] become:

$$H_{rr} = -\mathcal{L}''\dot{R}\left(\frac{\dot{B}}{2B} + \frac{2}{r}\right) + \mathcal{L}' R_{rr} - \frac{\mathcal{L}}{2} A = 0$$

$$H_{\theta\theta} = -\mathcal{L}'''\frac{\dot{R}^2 r^2}{A} - \frac{\mathcal{L}'' r^2}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A}\right.\right.$$

$$\left.\left. - \frac{\dot{B}}{2B} - \frac{1}{r}\right)\right] + \mathcal{L}' R_{\theta\theta} - \frac{\mathcal{L} r^2}{2} = 0$$

$$H_{\phi\phi} = \sin^2\theta H_{\theta\theta} = 0$$

$$H_{tt} = \frac{\mathcal{L}''' B\dot{R}^2}{A} + \frac{\mathcal{L}'' B}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{2}{r}\right)\right]$$

$$+ \mathcal{L}' R_{tt} + \frac{\mathcal{L}}{2} B = 0 \qquad [6.1.5]$$

and

$$H = -3\frac{\mathcal{L}''' \dot{R}^2}{A} - 3\frac{\mathcal{L}''}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{2}{r}\right)\right]$$

$$+ \mathcal{L}' R - 2\mathcal{L} = 0 \qquad [6.1.6]$$

we have also from [6.1.5] the following relations

$$H_{rr} + \frac{A}{B}H_{tt} = \mathcal{L}''' \dot{R}^2 + \mathcal{L}''\ddot{R} - \frac{\mathcal{L}''\dot{R}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \mathcal{L}'\left(R_{rr} + \frac{A}{B}R_{tt}\right) = 0$$

[6.1.7]

$$H_{rr} - \frac{A}{r^2}H_{\theta\theta} = \mathcal{L}'''\dot{R}^2 + \mathcal{L}''\ddot{R} - \frac{\mathcal{L}''\dot{R}}{2}\left(\frac{\dot{A}}{A} + \frac{2}{r}\right) + \mathcal{L}'\left(R_{rr} - \frac{A}{r^2}R_{\theta\theta}\right) = 0 \qquad [6.1.8]$$

$$H_{tt} + \frac{B}{r^2}H_{\theta\theta} = \frac{\mathcal{L}''\dot{R}B}{2A}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) + \mathcal{L}'\left(R_{tt} + \frac{B}{r^2}R_{\theta\theta}\right) = 0 \qquad [6.1.9]$$

(6.2) Differential Equations for g_{rr}, g_{tt} and R

By virtue of equations [6.1.5,–,9] we can obtain non-linear differential equations for the metric coefficients $A = g_{rr}$, $B = -g_{tt}$ and the scalar curvature R together with the Lagrangian $\mathcal{L}$ and its derivatives w.r. to R. Thus, by using [3.1.7] in [6.1.7] we will obtain.

$$\mathcal{L}'''\dot{R}^2 + \mathcal{L}''\ddot{R} - \left(\frac{\mathcal{L}''\dot{R}}{2} + \frac{\mathcal{L}'}{r}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 \qquad [6.2.1]$$

Also from [6.1.6] one gets

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{2\mathcal{L}'''\dot{R}}{\mathcal{L}''} + \frac{2\ddot{R}}{\dot{R}} + \frac{4}{r} + \frac{2}{3}\frac{A}{\dot{R}\mathcal{L}''}(2\mathcal{L} - R\mathcal{L}') \qquad [6.2.2]$$

from which one obtains

$$\dot{R} = \frac{c}{r^2}\sqrt{\frac{A}{B}}\,\psi_{\mathcal{L}}(r) \qquad [6.2.3]$$

where c is an arbitrary constant and $\psi_{\mathcal{L}}$ is defined by:

$$\frac{d}{dr}\ln\psi_{\mathcal{L}} \equiv \frac{A}{3\mathcal{L}''\dot{R}}(\mathcal{L}'R - 2\mathcal{L}) - \frac{\mathcal{L}'''\dot{R}}{\mathcal{L}''} \qquad [6.2.4]$$

or what is the same:

$$\frac{d}{dr}\ln[\psi_{\mathcal{L}}(r)\mathcal{L}''] = \frac{A}{3\mathcal{L}''\dot{R}}(\mathcal{L}'R - 2\mathcal{L}) \qquad [6.2.5]$$

By substituting $\ddot{R}$ from [6.2.1] in [6.2.2] one gets:

$$\mathcal{L}''\dot{R}\left(\frac{\dot{B}}{B}+\frac{2}{r}\right)+\frac{\mathcal{L}'}{r}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)+\frac{A}{3}(2\mathcal{L}-\mathcal{L}'R)=0 \qquad [6.2.6]$$

i.e.

$$\frac{\dot{A}}{A}=-\frac{\dot{B}}{B}-\frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}\left(\frac{\dot{B}}{B}r+2\right)+\frac{Ar}{3\mathcal{L}'}\left(R\mathcal{L}'-2\mathcal{L}\right) \qquad [6.2.7]$$

inserting the expression for $\frac{\dot{B}}{B}$ from [6.2.2] into [6.2.7]

yields the following:

$$\frac{\dot{A}}{A}=-\frac{\mathcal{L}''\dot{R}r}{2\mathcal{L}'}\frac{\dot{A}}{A}+\frac{\mathcal{L}''\ddot{R}r}{\mathcal{L}'}+\frac{\ddot{R}}{\dot{R}}+\frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}+\frac{2}{r}$$

$$-\frac{A}{3}(R\mathcal{L}'-2\mathcal{L})\left[\frac{r}{2\mathcal{L}'}-\frac{1}{\mathcal{L}''\dot{R}}\right]$$

$$+\mathcal{L}'''\left(\frac{\dot{R}}{\mathcal{L}''}+\frac{\dot{R}^2r}{\mathcal{L}'}\right) \qquad [6.2.8]$$

and substituting [6.2.7] into [6.2.2] yields:

$$\frac{\dot{B}}{B}=-\frac{\mathcal{L}''r\dot{R}\dot{B}}{2\mathcal{L}'B}-\left(\frac{\mathcal{L}''}{\mathcal{L}'}+\frac{\mathcal{L}'''}{\mathcal{L}''}\right)\dot{R}-\frac{\ddot{R}}{\dot{R}}-\frac{2}{r} \qquad [6.2.9]$$

$$+\frac{A}{3}\left[\mathcal{L}'R-2\mathcal{L}\right]\left[\frac{1}{\mathcal{L}''\dot{R}}+\frac{r}{2\mathcal{L}'}\right]$$

or by eliminating $\frac{\dot{B}}{B}$ from the R.H. side of [6.2.9] by [6.2.2]

one gets

$$\frac{\dot{B}}{B} = -\frac{\mathcal{L}''\dot{R}r}{2\mathcal{L}'}\frac{\dot{A}}{A} + \frac{\mathcal{L}''}{\mathcal{L}'}(\ddot{R}r+\dot{R}) - \frac{\ddot{R}}{\dot{R}} - \frac{2}{r} + \mathcal{L}'''\dot{R}\left(\frac{\dot{R}r}{\mathcal{L}'} - \frac{1}{\mathcal{L}''}\right)$$

$$+ \frac{A}{3}(\mathcal{L}'R - 2\mathcal{L})\left(\frac{1}{\mathcal{L}''\dot{R}} - \frac{r}{2\mathcal{L}'}\right) \qquad [6.2.10]$$

Moreover, by using the relation $R = g^{\mu\nu}R_{\mu\nu}$; and with the help of [3.1.3] , [6.1.9] will lead to:

$$AR = \frac{\mathcal{L}''\dot{R}}{2\mathcal{L}'}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) + \frac{3}{r^2}AR_{\theta\theta} + R_{rr} \qquad [6.2.11]$$

which by [3.1.6,9] becomes

$$AR = \frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) - \frac{4}{r^2}(A-1) + \frac{1}{r}\frac{\dot{B}}{B} - \frac{3}{r}\frac{\dot{A}}{A}$$

or

$$A\left(\frac{Rr}{4} + \frac{1}{r}\right) = \frac{\mathcal{L}''\dot{R}r}{4\mathcal{L}'}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) + \frac{1}{r} - \frac{3}{4}\frac{\dot{A}}{A} + \frac{1}{4}\frac{\dot{B}}{B} \qquad [6.2.12]$$

Now if we get rid of the term $\frac{\mathcal{L}''\dot{R}}{4\mathcal{L}'}\frac{\dot{B}}{B}r$ in this equation by using [6.2.7] we will have:

$$A\left(\frac{Rr}{4} + \frac{1}{r}\right) = \frac{\dot{B}}{2B} - \frac{\dot{A}}{2A} + \frac{1}{r} + \frac{\mathcal{L}''\dot{R}}{\mathcal{L}'} - \frac{Ar}{12\mathcal{L}'}(R\mathcal{L}' - 2\mathcal{L}) \qquad [6.2.13]$$

Then using [6.2.2] in [6.2.13] to eliminate $\frac{\dot{B}}{B}$ and $\frac{\dot{A}}{A}$ will result in expressing A in terms of only r, $\dot{R}$, $\mathcal{L}$ and their derivatives, i.e.

$$A\left(\frac{Rr}{4} + \frac{1}{r}\right) = -\frac{\ddot{R}}{\dot{R}} - \frac{1}{r} + \frac{\mathcal{L}''\dot{R}}{\mathcal{L}'} - \frac{\mathcal{L}'''\dot{R}}{\mathcal{L}''} + A(\mathcal{L}'R-2\mathcal{L})\left(\frac{1}{3\dot{R}\mathcal{L}''} - \frac{r}{12\mathcal{L}'}\right) \qquad [6.2.14]$$

or,

$$A(r) = \frac{-\frac{d}{dr}\ln\left(\frac{\mathcal{L}''}{\mathcal{L}'}\dot{R}r\right)}{\frac{Rr}{4} + \frac{1}{r} - (R\mathcal{L}'-2\mathcal{L})\left[\frac{1}{3\dot{R}\mathcal{L}''} - \frac{r}{12\mathcal{L}'}\right]} \qquad [6.2.15]$$

Further, by multiplying $H_{\theta\theta}$ by $-\frac{3}{r^2}$ in [6.1.5], then by [6.1.6] we get:

$$\frac{2r\dot{R}}{A} + \frac{2\mathcal{L}'R_{\theta\theta}}{\mathcal{L}''} - \frac{r^2\mathcal{L}}{\mathcal{L}''} + \frac{2r^2}{3\mathcal{L}''}(2\mathcal{L} - \mathcal{L}'R) = 0 \qquad [6.2.16]$$

and by multiplying H_{tt} by $\frac{3}{B}$ in [6.1.5] and using [6.1.6] and will have

$$-\frac{2r\dot{R}}{A} + \frac{4r\mathcal{L}'}{\mathcal{L}''}\frac{R_{tt}}{\dot{B}} + \frac{2r\mathcal{L}}{\mathcal{L}''}\frac{B}{\dot{B}} - \frac{4rB}{3\dot{B}\mathcal{L}''}(2\mathcal{L} - \mathcal{L}'R) = 0 \qquad [6.2.17]$$

Now by adding [6.2.16,17] and using [3.1.6] we obtain

$$\frac{1}{A}\left(-\ddot{B} + \frac{\dot{B}^2}{B} - \frac{\dot{B}}{r}\right) - \frac{\dot{B}}{r} + (2B - r\dot{B})\left(\frac{R}{3} - \frac{\mathcal{L}}{6\mathcal{L}'}\right) = 0 \qquad [6.2.18]$$

That is:

$$\frac{d}{dr}\left(\frac{\dot{B}r}{B}\right) + \frac{A}{r}\left[\frac{\dot{B}r}{B} - r^2\left(2 - \frac{\dot{B}r}{B}\right)\zeta_{\mathcal{L}}(R)\right] = 0 \qquad [6.2.19]$$

or

$$\dot{X} + A\left(\frac{1}{r} + r\zeta_{\mathcal{L}}\right)X - 2A\zeta_{\mathcal{L}}r = 0 \qquad [6.2.20]$$

where

$$X \equiv \frac{\dot{B}r}{B} \qquad [6.2.21]$$

and

$$\zeta_{\mathcal{L}}(R) \equiv \frac{1}{3}\left(R - \frac{\mathcal{L}}{2\mathcal{L}'}\right) \qquad [6.2.22]$$

Now by [6.2.19] we can write the following expression for the metric coefficient A in terms of $r, R, \frac{\dot{B}}{B}, \mathcal{L}$ and $\mathcal{L}'$, i.e.

$$A(r) = \frac{-r\frac{d}{dr}\left(\frac{\dot{B}r}{B}\right)}{\frac{\dot{B}r}{B} - r^2\left(2 - \frac{\dot{B}r}{B}\right)\zeta_{\mathcal{L}}(R)} = \frac{-\frac{d}{dr}\ln\left(\frac{\dot{B}r}{B}\right)}{\frac{1}{r} + r\left(1 - \frac{2}{\frac{\dot{B}}{B}r}\right)\zeta_{\mathcal{L}}(R)} \qquad [6.2.23]$$

Here we notice that in this expression A does not contain explicitly derivatives like $\mathcal{L}''$ and $\mathcal{L}'''$ and R while in [6.2.15] A includes $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ and $R, \dot{R}, \ddot{R}$ and does not depend on B or $\dot{B}$.

We also notice that $\zeta_{\mathcal{L}}(R)$ which does not depend on either $\mathcal{L}''$ nor $\mathcal{L}'''$ will have different values for different Lagrangians and it will indicate the contribution of the given Lagrangian to the value of A.

Now using [6.2.22] into [6.2.23] will yield:

$$\frac{A}{r}\left(1 + \frac{Rr^2}{4} - \frac{r^2\mathcal{L}}{8\mathcal{L}'} + \frac{\zeta r^2}{4} - \frac{2r}{\frac{\dot{B}}{B}}\zeta\right) = -\frac{d}{dr}\ln\left(\frac{\dot{B}r}{B}\right) \qquad [6.2.24]$$

and by subtracting this equation from [6.2.14], the following relationship yields

$$A\left[(R\mathcal{L}' - 2\mathcal{L})\left(\frac{r}{6\mathcal{L}'} - \frac{1}{3\dot{R}\mathcal{L}''}\right) - 2\zeta_{\mathcal{L}}\frac{B}{\dot{B}}\right] = \frac{d}{dr}\ln\left(\frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}\frac{B}{\dot{B}}\right) \qquad [6.2.25]$$

or

$$A\left[\frac{2\mathcal{L}'}{\dot{R}\mathcal{L}''}\zeta_{\mathcal{L}} - \frac{\mathcal{L}'}{\mathcal{L}''\dot{R}}\frac{\dot{B}}{B}(R\mathcal{L}' - 2\mathcal{L})\left(\frac{r}{6\mathcal{L}'} - \frac{1}{3\dot{R}\mathcal{L}''}\right)\right] =$$

$$= \frac{d}{dr}\left(\frac{\mathcal{L}'\dot{B}}{\mathcal{L}''\dot{R}B}\right) \qquad [6.2.26]$$

Further, it is obvious that an expression for $B(r)$ can be obtained from [6.2.15] or [6.2.23] with the aid of [6.2.3] where the following relation holds:

$$B(r) = \frac{C^2\psi_{\mathcal{L}}^2}{r^4\dot{R}^2}A(r) \qquad [6.2.27]$$

and $\psi_{\mathcal{L}}(r)$ is such, that in order to satisfy the flat space conditions, should behave at $r \to \infty$ (i.e. $A = B = 1$) in the following manner:

$$\lim_{r\to\infty} \psi_{\mathcal{L}}(r) \sim r^2\dot{R} \qquad [6.2.28]$$

Furthermore, by using [6:1.5, 6] we calculate the vanishing combination:

$$H_{\theta\theta} - \frac{r^2H}{3} = 0$$

and with the help of [3.1.6] we get

$$\frac{\dot{A}}{A} = \frac{\dot{B}}{B} + \frac{2}{r} - \frac{2A}{r} - \frac{2ARr}{3} + \frac{Ar\mathcal{L}}{3\mathcal{L}'} + \frac{2\mathcal{L}''\dot{R}}{\mathcal{L}'} \qquad [6.2.29]$$

Also, equations [6.2.12] together with [6.2.7] lead to

$$\frac{\dot{A}}{A} = \frac{1}{r} - \frac{A}{r} - \frac{ARr}{4} - \frac{\mathcal{L}''\dot{R}r}{2\mathcal{L}'}\frac{\dot{B}}{B} + \frac{Ar}{12}(R\mathcal{L}' - 2\mathcal{L}) \qquad [6.2.30]$$

which can be written as:

$$\frac{d}{dr}\left(\frac{r}{A}\right) = 1 + \frac{r^2}{12}\,(3R + 2\mathcal{L} - R\mathcal{L}') + \frac{\dot{R}\mathcal{L}''}{2A\mathcal{L}'}\,\frac{\dot{B}r^2}{B} \qquad [6.2.31]$$

To sum up, we derive the following useful expressions starting with [6.2.12] which will read:

$$\frac{\dot{A}}{A} = -\frac{\mathcal{L}''\dot{R}}{3\mathcal{L}'}\left(\frac{\dot{B}r}{B} - 2\right) + \frac{1}{3}\left[\frac{\dot{B}}{B} + \frac{4}{r} - A\left(Rr + \frac{4}{r}\right)\right] \qquad [6.2.32]$$

Also, adding and subtracting [6.2.7] and [6.2.29] with each other will respectively yield:

$$\frac{\dot{A}}{A} = \frac{1}{r} - \frac{A}{r} - \frac{Ar}{6}\left(R + \frac{\mathcal{L}}{\mathcal{L}'}\right) - \frac{\mathcal{L}''\dot{R}}{2\mathcal{L}'}\,\frac{\dot{B}r}{B} \qquad [6.2.33]$$

and

$$\left(\frac{\mathcal{L}''\dot{R}}{2} + \frac{\mathcal{L}'}{r}\right)\frac{\dot{B}}{B} = \frac{2\mathcal{L}''\dot{R}}{r} + \frac{AR\mathcal{L}}{2} - \frac{A\mathcal{L}}{2} + \frac{A\mathcal{L}'}{r^2} - \frac{\mathcal{L}'}{r^2} \qquad [6.2.34]$$

which, if employed in [6.2.1] will give

$$\ddot{R} = \frac{-\mathcal{L}'''\dot{R}^2}{\mathcal{L}''} + \left(\frac{\dot{A}}{2A} - \frac{2}{r}\right)\dot{R} + \frac{\mathcal{L}'}{\mathcal{L}''}\left(\frac{AR}{2} + \frac{\dot{A}}{rA} - \frac{A\mathcal{L}}{2\mathcal{L}'} + \frac{A}{r^2} - \frac{1}{r^2}\right) \qquad [6.2.35]$$

Only two out of equations [6.2.32, 34, 35] are independent. These equations will be useful in considering different Lagrangians. We also add here that alternatively to [6.2.35] a differential equation of $\dot{R}$ can be written from [6.2.14], i.e.

$$\ddot{R} + \left[\frac{d}{dr}\ln\left(\frac{\mathcal{L}''}{\mathcal{L}'}\right) + A\left(\frac{Rr}{3} + \frac{1}{r} - \frac{r\mathcal{L}}{6\mathcal{L}'}\right) + \frac{1}{r}\right]\dot{R} - \frac{A}{3\mathcal{L}''}(\mathcal{L}'R - 2\mathcal{L}) = 0 \qquad [6.2.36]$$

where

$$\left(\frac{\mathcal{L}'''}{\mathcal{L}''} - \frac{\mathcal{L}''}{\mathcal{L}'}\right)\dot{R}^2 = \frac{d}{dr} \ln\left(\frac{\mathcal{L}''}{\mathcal{L}'}\right)\dot{R} \qquad [6.2.37]$$

Now we conclude that the above derived equations are valid for any Lagrangian under the conditions of time-independence and isotropy of space. The solution of such equations can be sought only if one specializes to a particular Lagrangian. Before doing so we consider the simplified case when the scalar curvature R does not depend on r, everywhere.

(6.3) The solution of the gravitational equation with constant scalar curvature.

Let us employ equation [6.2.29] to obtain the following expression for B with globally constant curvature $R \equiv R_0$:

$$B(R_0) \equiv B_0 = \frac{C_1 A_0}{r^2} \exp\left\{\int A_0\left[\frac{2}{r} + \frac{2R_0 r}{3} - \frac{\mathcal{L}_0 r}{3\mathcal{L}_0'}\right]dr\right\} \qquad [6.3.1]$$

Equation [6.2.31] will also be reduced to give at $R = R_0$, the following Köttler-type solution for A, [1] i.e.

$$A_0(r) = \frac{1}{1 + \frac{K_1}{r} + \gamma(R_0)r^2} \qquad [6.3.2]$$

which makes [6.3.1] having the form:

$$B_0(r) = \frac{C_1}{r^2\left[1 + \frac{K_1}{r} + \gamma(R_0)r^2\right]} \exp\left\{\int \frac{[2 + G(R_0)r^2]}{K_1 + r + \gamma_0 r^3} dr\right\} \qquad [6.3.3]$$

with C_1 & K_1 constant numbers and, where we denoted:

$$\gamma_0 = \gamma(R_0) \equiv \frac{1}{36}\left[3R_0 + 2\mathcal{L}_0 - R_0 \mathcal{L}'_0\right] \qquad [6.3.4]$$

and

$$G(R_0) \equiv \frac{2R_0}{3} - \frac{\mathcal{L}_0}{3\mathcal{L}'_0} = 2\zeta_{\mathcal{L}_0}(R_0) \qquad [6.3.5]$$

We notice here if $\mathcal{L}_0 = R_0 = 0$ equations [6.3.2,3] will reduce to Schwarzschild solution [3.1.15]. However, the flat space limit can be satisfied specially for A_0 not only by $R_0 = 0$ but also by setting $\mathcal{L}_0 = -3R_0 < 0$, this will make $\gamma(R_0)$ vanish and $G(R_0) = \frac{R_0}{3}$. Therefore, we get the Schwarzschild form for A_0 i.e.

$$A_0(r) = \frac{r}{r+K_1} \qquad [6.3.6]$$

with K_1 a negative constant.

As for $B_0(r)$ we obtain the following expression:

$$B_0(r) = \left(1+\frac{K_1}{r}\right) \exp\left\{\frac{R_0}{3}\left[\frac{r^2}{2} - K_1 r + K_1^2 \ln(r+K_1) + \text{const.}\right]\right\} \qquad [6.3.7]$$

It is obvious that for $R_0 = 0$ we come again to Schwarzschild metric with its observed singularities. We think that the exponential factor in [6.3.3] might play a significant role in deciding the asymptotic behaviour of B and also whether the singularities at $r = -K_1$ and $r = 0$ will be stronger or weaker or may somehow be cancelled away. For this a non-zero non-constant scalar curvature should be assumed. However, for static and stationary universe [6.3.2] and [6.3.7] will serve good cosmological solutions.

(6.4) General Lagrangians, Classified.

The comparison of equations [6.2.30] with [6.2.33] will lead to interesting relations between the Lagrangian $\mathcal{L}(R)$ and the scalar curvature $R(r)$ which help in classifying different Lagrangians, i.e.

$$(\mathcal{L}'-1)(R\mathcal{L}'-2\mathcal{L}) = 0 \qquad [6.4.1]$$

There are three possibilities of satisfying this condition:

(1) if
$$\left.\begin{aligned} \mathcal{L}' &= 1 \\ R\mathcal{L}' &= 2\mathcal{L} \end{aligned}\right\} \qquad [6.4.2]$$

This will yield a linear Lagrangian with constant R

$$\mathcal{L} = R_0 + \gamma = \frac{R_0}{2} \qquad [6.4.3]$$

which corresponds to Einstein's spaces

$$R_0 = -2\gamma \qquad [6.4.4]$$

and if $\gamma = 0$ $\qquad R_0 = 0$. $\qquad$ [6.4.5]

(2) If
$$\left.\begin{aligned} \mathcal{L}' &= 1 \\ \text{but} \quad R\mathcal{L}' &\neq 2\mathcal{L} \end{aligned}\right\} \qquad [6.4.6]$$

This gives $\qquad \mathcal{L} = R(r) + \gamma \qquad$ [6,4,7]

(3) If
$$\left.\begin{aligned} \mathcal{L}' &\neq 1 \\ \text{but} \quad R\mathcal{L}' &= 2\mathcal{L} \end{aligned}\right\} \qquad [6.4.8]$$

and this yields

$$\mathcal{L} = \alpha R^2 \qquad [6.4.9]$$

being the unique solution for [6.4.8] where γ and α are arbitrary constants.

The relation [6.4.8] had been obtained also by Stephenson [2] who subjected to variation the Lagrangian $\mathcal{L}$ which he considered as a function of only R. Now [6.4.1] can be used to classify different forms of Lagrangian and exclude those with unreasonable meaning.

We see that the combination of the Lagrangians [6.4.3,7,9] or more generally, our Lagrangians [5.4.24] will satisfy condition [6.4.1] if only certain restrictions were imposed on the coefficients α, β, γ and the parameters m and K. For instance, [6.4.8] will be satisfied by [5.4.24] i.e.

$$(R+K)^{m-1}[\alpha(m-2)R-2\alpha K] = \beta R+2\gamma \qquad [6.4.10]$$

provided that,

$$\beta = -2\alpha K, \qquad [6.4.11]$$

$$\gamma = -\alpha K^2, \qquad [6.4.12]$$

and

$$m \equiv 2(2n-3) = 2. \qquad [6.4.13]$$

We notice that [6.4.12,13] are fully consistent with [5.5.2] and [5.4.38] and that [6.4.11,12] relate the coefficients α,β,γ by,

$$\beta^2 = -4\alpha\gamma \qquad [6.4.14]$$

We also notice that [6.4.13] will represent the general Lagrangian with quadratic and linear in R terms.

We will also see that any Lagrangian of any power P, i.e.

$\mathcal{L} = R^P$ will reduce to quadratic R, by [6.4.8] without imposing any constraints on R itself, while for any Lagrangian other than $\mathcal{L} = R^2$, R will be affected by [6.4.8]. For instance, if one chooses as a Lagrangian,

$$\mathcal{L}(R) = \sin a(R) \qquad [6.4.15]$$

it will result in the following

$$R = 2\left(\frac{\partial a}{\partial R}\right)^{-1} \tan a(R) \qquad [6.4.16]$$

If $a(R) = R^P$, this yields

$$R^P = \frac{2}{P} \tan R^P \qquad [6.4.17]$$

which for $P = 2$ leads to $R \to o$ since

$$R^2 = \tan R^2 \qquad [6.4.18]$$

The Lagrangian $\mathcal{L} = R$ will imply, in view of [6.4.8], that R is not any constant but identically zero, i.e.

$$R \equiv 0 \qquad [6.4.19]$$

This situation gives the Lagrangian $\mathcal{L} = R^2$ a particular position among other Lagrangians. Moreover, as another example we use the Lagrangian [4.3.1] in [6.4.1] that yields,

$$R_0\left[\left(\left(1-\frac{R}{R_0}\right)^{m-1} - 1\right)\left(\left(\frac{2}{m}-1\right)\left(1-\frac{R}{R_0}\right)^m + \left(1-\frac{R}{R_0}\right)^{m-1} - \frac{2}{m}\right)\right] = 0, \qquad [6.4.20]$$

which, as we have given in [4.3.7,8] and w.r. to cases [6.4.2,6,8] will be satisfied by Einstein-Schwarzschild space $R = 0$ as well as by Einstein's cosmological spaces [3],

$$R = \frac{8}{9} R_0 \quad \text{for} \quad m = \tfrac{1}{2} \qquad [6.4.21]$$

$$R = \tfrac{1}{2} R_0 \quad \text{for} \quad m = -1 \qquad [6.4.22]$$

(6.5) On the possibility of physical solutions of the generalized equations.

By physical solution we mean that which satisfies flat space limit and preferably free from any apparent singularity. As it was mentioned in (4.2) a (ii), the Lagrangian $\mathcal{L} = R^2$ leads to solutions which are not asymptotically flat. Therefore, since the flatness condition is a physical requirement, the Lagrangian ought to be chosen in such a way that this requirement be fulfilled. In other words, to ensure the asymptotically good behaviour of A(r), B(r) and R(r) we suggest that constraints should be imposed on the general Lagrangian as it has been done in [6.2.28].

We, therefore, in view of [6.2.15] and [6.2.23], require that the following relationships should be fulfilled in order that the flatness condition be satisfied:

$$\frac{1}{r^2}\frac{d}{dr}\ln\left(\frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}\right) \overset{r\to\infty}{=} \frac{R\mathcal{L}'-2\mathcal{L}}{12r\mathcal{L}'} - \frac{R\mathcal{L}'-2\mathcal{L}}{3C\mathcal{L}''} - \frac{R}{4r} \qquad [6.5.1.]$$

$$\frac{1}{r}\frac{d}{dr}\ln\left(\frac{\dot{B}r}{B}\right) \overset{r\to\infty}{=} -\zeta_{\mathcal{L}}(R) \qquad [6.5.2]$$

where $\zeta_{\mathcal{L}}(R)$ is defined in [6.2.22]. But since we do not know the structure of $\dot{R}$ or $\frac{\dot{B}}{B}$ in terms of $\mathcal{L}$ and R we will not be able to solve the equation to find out the desired Lagrangian, or to try different Lagrangians and see which one will satisfy the equation.

We should, first, discover the structure of $\dot{R}$ as depending on R, $\mathcal{L}$ and the derivatives of $\mathcal{L}$ w.r. to R.

To do that we proceed in the following way:

From [6.3.1] we get

$$B_0 = B(R_0) = \frac{C_1 A_0}{r^2} e^{2\int \frac{A_0 dr}{r}} \cdot \exp\left[\frac{2}{3}\int A_0 \left(R_0 - \frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right) r dr\right] \qquad [6.5.3]$$

Also from [6.2.3,5] we have

$$B(R) = \frac{C^2 A}{r^4 \dot{R}^2 \mathcal{L}''^2} \exp\left[\frac{2}{3}\int \frac{A}{\dot{R}\mathcal{L}''}(\mathcal{L}' R - 2\mathcal{L}) dr\right] \qquad [6.5.4]$$

Therefore, if we consider the correspondence between [6.5.3] and [6.5.4] we have:

$$\frac{C_2 C^2 A}{\dot{R}^2 r^4 \mathcal{L}''^2} \exp\left[\frac{2}{3}\int \frac{A}{\mathcal{L}''\dot{R}}(\mathcal{L}' R - 2\mathcal{L}) dr\right]$$

$$\overset{R\to R_0}{\Longrightarrow} \frac{C_1 A_0}{r^2} e^{2\int \frac{A_0 dr}{r}} \exp\left[\frac{2}{3}\int A_0 \left(R_0 - \frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right) r dr\right] \qquad [6.5.5]$$

where C, C_1, C_2 are arbitrary constants.

Now by comparison in [6.5.5] we get

$$\frac{C_2 C^2 A}{r^4 \dot{R}^2 \mathcal{L}''^2} \overset{R\to R_0}{\Longrightarrow} \frac{C_1 A_0}{r^2} e^{2\int \frac{A_0 dr}{r}} \qquad [6.5.6]$$

and

$$\frac{A(\mathcal{L}' R - 2\mathcal{L})}{\mathcal{L}''\dot{R}} \overset{R\to R_0}{\Longrightarrow} A_0 \left(R_0 - \frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right) r \qquad [6.5.7]$$

Then we can set,

$$\frac{C_2 C^2}{r^4 \dot{R} \mathcal{L}''(\mathcal{L}' R - 2\mathcal{L})} = \frac{C_1 e^{2\int \frac{A_0 dr}{r}}}{\left(R_0 - \frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right) r^3} \qquad [6.5.8]$$

This will yield for any Lagrangian the following relation:

$$r\dot{R}\mathcal{L}''(\mathcal{L}'R-2\mathcal{L}) = C^2C_3\left(R_0-\frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right)\exp\left(-2\int\frac{A_0 dr}{r}\right) \quad [6.5.9]$$

with $C_3 = \frac{C_2}{C_1}$, a new arbitrary constant.

We notice that when $C^2 = 0$ [6.5.9] will give an Einstein space $R = R_0 = 0$.

From [6.5.9] we get the following expression for $\dot{R}$ in terms of r, R and $\mathcal{L}$ with its derivatives, i.e.

$$\dot{R} = \frac{3C_3C^2\zeta_{\mathcal{L}_0}(R_0)}{r(\mathcal{L}'R-2\mathcal{L})\mathcal{L}''}\, e^{-2\int\frac{A_0 dr}{r}} \quad [6.5.10]$$

where, A_0 is given by [6.3.2] and $\zeta_{\mathcal{L}_0}$ is given by [6.2.22]

$$\zeta_{\mathcal{L}_0}(R_0) = \frac{1}{3}\left(R_0-\frac{\mathcal{L}_0}{2\mathcal{L}_0'}\right) \quad [6.5.11]$$

We then have:

$$\frac{d}{dr}\ln\left(\frac{\mathcal{L}''\dot{R}r}{\mathcal{L}'}\right) = -\frac{2A_0}{r} - \frac{d}{dr}\ln[\mathcal{L}'(\mathcal{L}'R-2\mathcal{L})]$$

$$= \frac{-2A_0}{r} - \left[\frac{\mathcal{L}''}{\mathcal{L}'} + \frac{\mathcal{L}''R-\mathcal{L}'}{\mathcal{L}'R-2\mathcal{L}}\right]\dot{R} \quad [6.5.12]$$

Now by inserting [6.5.10] and [6.5.12] into [6.2.15] we obtain the following expression for $g_{rr} = A(r)$, i.e.

$$A(r) = \frac{2A_0(r) + C^2 Q_{\mathcal{L}}(R) e^{-2\delta_0}}{1 + \left[\frac{\mathcal{G}_{\mathcal{L}}(R)}{C^2} e^{2\delta_0} + \mathcal{D}_{\mathcal{L}}(R)\right] r^2} \qquad [6.5.13]$$

where,

$$\delta_0(r) \equiv \int \frac{A_0(r)\,dr}{r} \qquad [6.5.14]$$

$$Q_{\mathcal{L}}(R) \equiv \frac{3C_3 \zeta_{\mathcal{L}_0}(R_0)}{\mathcal{L}''(\mathcal{L}' R - 2\mathcal{L})} \left[\frac{\mathcal{L}''}{\mathcal{L}'} + \frac{\mathcal{L}'' R - \mathcal{L}'}{\mathcal{L}' R - 2\mathcal{L}}\right] \qquad [6.5.15]$$

$$\mathcal{G}_{\mathcal{L}}(R) \equiv \frac{(R\mathcal{L}' - 2\mathcal{L})^2}{9C_3 \zeta_{\mathcal{L}_0}(R_0)} \qquad [6.5.16]$$

$$\mathcal{D}_{\mathcal{L}}(R) \equiv \frac{1}{6}\left(R + \frac{\mathcal{L}}{\mathcal{L}'}\right) \qquad [6.5.17]$$

If we set $R = R_0$ in [6.5.13] we will have, in view of [6.3.2], the following:

$$\frac{1}{1 + \frac{K_1}{r} + \gamma_0 r^2} = A_0(r) = \frac{-C^2 Q_{\mathcal{L}_0}(R_0) e^{-2\delta_0(r)}}{1 - \left[\frac{\mathcal{G}_{\mathcal{L}0}(R_0)}{2} e^{2\delta_0(r)} + \mathcal{D}_{\mathcal{L}0}(R_0)\right] r^2} \qquad [6.5.18]$$

We notice in this equation, that since $\delta_0(0) = \text{Const.}$, owing to definition [6.5.14], therefore we have:

$$A_0(0) = A_0(\infty) = 0 \qquad [6.5.19]$$

and

$$Q_{\mathcal{L}}(R) = 0 \text{ at } r = 0 \qquad [6.5.20]$$

This result will lead to a good estimate for the value of the scalar curvature at the centre of the gravitating mass.

Certainly, since by [6.5.9] and [6.5.11], $r = 0$, would mean that $\zeta_{\mathcal{L}_0}(R_r) = 0$, therefore [6.5.20] would yield,

$$2\mathcal{L}' R_0 = \mathcal{L}_0 \quad [6.5.21]$$

which will be satisfied by,

$$R(0) = R_0 = \text{const.}\, \mathcal{L}_0^2(R\ ) \quad [6.5.22]$$

This implies that R depends on the choice of the Lagrangian. For instance, if we use the quadratic form of [5.4.24] i.e.

$$\mathcal{L}(R_0) = \alpha R_0^2 + \beta R_0 + \gamma \quad [6.5.23]$$

Then [6.5.21] will become

$$3\alpha R_0^2 + \beta R_0 - \gamma = 0 \quad [6.5.24]$$

which will have the following solution:

$$R_0 = \frac{\beta \pm \sqrt{\beta^2 + 12\alpha\gamma}}{6\alpha} \quad [6.5.25]$$

This solution can be adjusted with a certain choice of the constants α, β and γ to have a physically accepted value.

We also expect imaginary values for $R_0(0)$ which is due to the introducing of the quadratic form of the Lagrangian and which we will discuss later in Chapter Nine. Thus, since the quadratic Lagrangian is associated with the quantization of the gravitational field as we have mentioned in (4.2) and (5.7) one would admit that the imaginary part of R_0 and hence any R may be interpreted in quantum mechanical terms as quantum gravity theories develop. Moreover, $R_0(0)$ can be given a physical meaning if we set in [6.5.24] $\alpha = 0$, $\gamma = \frac{1}{2}T_0$ where T_0 represents the matter density, and if β is set to fulfill the Newtonian limit, i.e. $\beta = \dfrac{1}{16\pi G}$ as we did before in [5.4.36]. We then have,

$$R_0(0) = 8\pi G T_0 \qquad [6.5.26]$$

relating the scalar curvature at the centre of mass with matter density. But if α is set not to be zero, then by [6.5.22] or [6.5.25] $R_0(0)$ would have an imaginary part indicating, possibly, the quantum behaviour within and in the vicinity of the material centre.

Further, in order to complete the picture of our static isotropic generalized metric which we aim to be the successful alternative to Schwarzschild's, we have to obtain the generalized expression for the metric tensor component $g_{tt} = -B$. Therefore, by the use of [6.5.4] and in view of [6.5.10,16] we obtain the following expression:

$$B(r) = \frac{A\,\mathcal{G}_{\mathcal{L}}(R)\,e^{4\delta_0}}{G C^2 \zeta_{\mathcal{L}_0}(R_0)\, r^2} \exp\left[\frac{2}{C^2}\int A\,\mathcal{G}_{\mathcal{L}}\, e^{2\delta_0}\, r\,dr\right] \qquad [6.5.27]$$

This expression, together with [6.5.13] will define our generalized metric space.

(6.6) Discussion

Now we return to the expression [6.5.13] and [6.5.27] where we can deduce certain interesting results.

We immediately notice that since r enters quadratically in the expression, therefore no singularity at Schwarzschild's or any finite radius will be observed in $A(r)$ or $B(r)$. In fact, Schwarzschild's singularity arises in $A_0(r)$ not in $A(r)$, because in Einstein's model, vacuum equation makes R identically zero.

As for Lagrangians with variable $R(r)$, the term $A_0(r)$ defined in [6.3.2] will appear as an approximation to $A(r)$ in the case when $R \not\simeq R_0$. Thus, at $r = 0$, although $A_0(r)$ will vanish, we see that $A(0)$ may be made non zero, i.e.

$$A(0) = C^2 Q_{\mathcal{L}}(R) \neq 0 \qquad [6.6.1]$$

if only a certain Lagrangian, say $\mathcal{L} = R^2$, was chosen, that may compete with $\zeta_{\mathcal{L}_0}(R_0)$ in [6.5.15]. The possible link between quadratic Lagrangian and quantum mechanics will suggest that the non-vanishing of A at zero may be explained as a quantum effect due to the strength of the gravitational field in the centre of the matter. This situation may at least weaken the Schwarzschild's singularity at $r = 0$.

However, $A(0)$ may vanish according to the choice of the Lagrangian or by setting the arbitrary constants C or C_3 in $Q_{\mathcal{L}_0}$ and $\mathcal{G}_{\mathcal{L}_0}$, to be zero.

But generally the value of $A(r)$ results from the competition among the different terms which are subjected to the choice of the Lagrangian. Therefore, the Lagrangian can be chosen in a way that in the weak energy regions the term A_0 will be dominant, while at strong field areas, i.e. when r is in the vicinity of a massive body, the nonlinearity in the Lagrangian, combined with the exponential, will give the essential contribution.

Thus, if we utilize the Lagrangian [5.4.24], then by a special choice of the parameters m, k and the constant coefficients, α, β and γ in the Lagrangian as well as the constants C_3 and C in [6.5.10, 13], it will be possible to make the terms with r^2 behave in such a way that at $r \to \infty$, $A(r)$ will go over to the flat space limit.

$$A(\infty) = 1 \qquad [6.6.2]$$

Further, the factor 2 in $2A_0$ appears as a result of applying the generalized gravitational field equation. It disappears in Einstein's theory because there, one deals only with A_0. This factor may be interpreted as double counting of the gravitational effect due to the mutual interaction between the gravitational field and its source. That is why, where there is no such mutual interaction at $r = 0$ and $r = \infty$, $A_0(r)$ automatically vanishes as in [6.5.19].

In [6.5.13] Einstein's GTR is represented by only A_0 which, as described in (6.3), will reduce to Schwarzschild's solution [6.3.6] by a certain choice of the Lagrangian in

[6.3.2] [6.3.4].

This implies:

$$A \xrightarrow{\mathcal{L}(R)\to\mathcal{L}(R_0)} A_0 \xrightarrow{\mathcal{L}=R_0=0} A_0 \text{ Schwarzschild} \qquad [6.6.3]$$

Similarly, [6.5.27] reduces to [6.3.7] and hence to Schwarzschild space according to

$$B(r) \xrightarrow{\mathcal{L}(R)\to\mathcal{L}(R_0)} B_0(r) \xrightarrow{R_0=0} B_{\text{Schwarzschild}} \qquad [6.6.4]$$

As for the behaviour of $B(r)$ as $r \to 0$ or as $r \to \infty$, we note that it will be subject to the competition between different factors of expression [6.5.27].

Furthermore, it is interesting to notice that in [6.5.13] the term with $Q_{\mathcal{L}}$ decreases with large r due to the decreasing of R from which the Lagrangian is constructed and also due to the exponential. This will signify that far away from the source of gravity no quantum effects may be observed, but a weak gravitational field in a form of radiation whose wave nature may be described by the exponential factor and which carries the gravitational energy within $Q_{\mathcal{L}}$ and $\mathcal{G}_{\mathcal{L}}$ that may play the role of the amplitudes of the gravitational waves.

By this we sum up our discussion by concluding that it is always possible to obtain a nonsingular solution for $A(r)$ and $B(r)$ with a good behaviour at Minkowskian limit, provided that an appropriately chosen Lagrangian is used. The

Lagrangian selected this way to fulfill the requirements of asymptotically flat geometry and nonsingularity of space will constitute the most perfect classical choice and the chosen constants and parameters may acquire their full meaning in a quantum mechanical interpretation.

(6.7) Conclusion

The fact that expression [6.5.13] and [6.5.27] for A and B are dependent on the Lagrangian, provides us with a freedom to choose the suitable Lagrangian form so that the physical requirements can be satisfied by our metric space. The asymptotically flat space and nonsingular behaviour can be achieved, and Einstein's and Schwarzschild's spaces come as a special case. Also cosmological and conformal solutions can be obtained at certain choices of $\mathcal{L}$ and can be derived from exact formulae. Being approximated values of A and B, the functions A_0 and B_0 describe the interaction between the gravity and its source, and therefore bear more physical significance than does Schwarzschild metric which is regarded as a further approximation and hence, is defected by singularity.

Since the decrease of the exponential power in [6.5.13] is accompanied by increase in A and vice versa, we hopefully believe that our model will be good enough to explain both weak and strong field features of gravitational phenomena.

We conclude that since our model is exactly derived from generalized equations of gravitation, and by this no singular solution is observed, we think that it will constitute a good basis for a quantizable gravitational theory, especially if we take into the account the indications, those mentioned in the preceding section. In the next chapters we will elaborate in considering this generalized metric with certain choices of the Lagrangian function.

CHAPTER SEVEN

Lagrangian Quadratic in R

(7.0) Foreword

Although we believe that the most general choice of a Lagrangian should have the form [5.4.24], we would still reckon that it is interesting to consider a Lagrangian quadratic in only the scalar curvature. In other words, we look into the general Lagrangian [5.4.24] with conditions that $K = \beta = \gamma = 0$, $\alpha = 1$ and $m = 2$. In spite of the objections against this kind of Lagrangian [1][2] stated in (4.2), due to its lack of asymptotically flat metric, and the disagreement with observation of its predictions especially when a matter term is incorporated in it, we do admit that the Lagrangian $\mathcal{L} = R^2$ has an important role to play in the generalized theory of gravitation.

As it was shown towards the end of (6.4), we notice that this Lagrangian has a special position amongst other Lagrangians, and beside that it gives a dominating contribution in the generalized Lagrangian $\mathcal{L} = \alpha R^2 + \beta R + \gamma$.

In this chapter we will show that, in spite of all objections raised against the Lagrangian $\mathcal{L} = R^2$, we are able to obtain a metric that is more advantageous than Schwarzschild's though it is derived from some approximated relations in which Schwarzschild's metric is but a further approximation.

(7.1) R^2 - Equations

By directly substituting for $\mathcal{L} = R^2$ in the generalized equations and relations of section (6.2), we will correspondingly obtain the following relations and expressions for A(r), B(r) and R(r). Starting from [6.2.1] to [6.2.30] we will have respectively the following relevant relationships: i.e.,

$$\ddot{R} = \left(\frac{\dot{R}}{2} + \frac{R}{r}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \qquad [7.1.1]$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{2\ddot{R}}{\dot{R}} + \frac{4}{r} \qquad [7.1.2]$$

$$\dot{R} = \frac{C}{r^2}\sqrt{\frac{A}{B}} \qquad [7.1.3]$$

$$\frac{\dot{A}}{A} + \frac{\dot{B}}{B} = -\frac{\dot{R}}{R}\left(\frac{\dot{B}}{B}r + 2\right) \qquad [7.1.4]$$

$$\frac{\dot{A}}{A} = -\frac{r}{2}\frac{\dot{R}}{R}\frac{\dot{A}}{A} + \frac{\ddot{R}}{R}r + \frac{\ddot{R}}{\dot{R}} + \frac{\dot{R}}{R} + \frac{2}{r} \qquad [7.1.5]$$

$$\frac{\dot{B}}{B} = -\frac{r}{2}\frac{\dot{R}}{R}\frac{\dot{B}}{B} - \frac{\dot{R}}{R} - \frac{\ddot{R}}{\dot{R}} - \frac{2}{r} \qquad [7.1.6]$$

$$\frac{\dot{B}}{B} = -\frac{r}{2}\frac{\dot{R}}{R}\frac{\dot{A}}{A} + \frac{\ddot{R}}{R}r + \frac{\dot{R}}{R} - \frac{\ddot{R}}{\dot{R}} - \frac{2}{r} \qquad [7.1.7]$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{2\dot{R}}{R} + \frac{2}{r} - \frac{2A}{r}\left(1 + \frac{Rr^2}{4}\right) \qquad [7.1.8]$$

$$A(r) = -\left[\frac{1}{r} + \frac{rR}{4}\right]^{-1}\frac{d}{dr}\ln\left(\frac{\dot{R}r}{R}\right) \qquad [7.1.9]$$

$$-\ddot{B} + \frac{\dot{B}^2}{B} - \frac{\dot{B}}{r} - \frac{\dot{B}A}{r} + \frac{1}{2}\left(B - \frac{\dot{B}r}{2}\right)RA = 0 \qquad [7.1.10]$$

$$\zeta(R) = \frac{R}{4} \qquad [7.1.11]$$

$$\mathcal{L}=R^2$$

$$A(r) = \frac{-\frac{d}{dr}\ln\left[\frac{\dot{B}r}{B}\right]}{\frac{1}{r}+\frac{rR}{4}\left(1-\frac{2}{\frac{\dot{B}r}{B}}\right)} = \frac{-r\frac{d}{dr}\left[\frac{\dot{B}r}{B}\right]}{\frac{\dot{B}r}{B}-\frac{r^2R}{4}\left(2-\frac{\dot{B}r}{B}\right)} \qquad [7.1.12]$$

$$\frac{d}{dr}\ln\left(\frac{\dot{R}}{R}\frac{B}{\dot{B}}\right) = -\frac{AR}{2}\frac{B}{\dot{B}} \qquad [7.1.13]$$

$$\frac{d}{dr}\left(\frac{R}{\dot{R}}\frac{\dot{B}}{B}\right) = \frac{AR^2}{2\dot{R}} = \frac{r^4R^2\dot{R}}{2C}B \qquad [7.1.14]$$

where we used,

$$B = \frac{C}{r^4\dot{R}^2}A \qquad [7.1.15]$$

and hence,

$$\frac{d}{dr}\ln\left(\frac{R}{\dot{R}}\frac{\dot{B}}{B}\right) = \frac{AR}{2}\frac{B}{\dot{B}} \qquad [7.1.16]$$

$$\frac{d}{dr}\ln\left(\frac{\dot{R}}{R}\frac{r^2}{A}\right) = \frac{\dot{R}r}{2R}\frac{\dot{B}}{B}\text{, i.e.} \qquad [7.1.17]$$

$$\frac{d}{dr}\left(\frac{RA}{\dot{R}r^2}\right) = \frac{A}{2r}\frac{\dot{B}}{B} = \frac{-r^3\dot{R}^2}{2C}\dot{B} \qquad [7.1.18]$$

$$\dot{A} = \frac{A}{r} - \frac{A^2}{r} - \frac{A^2R}{4}r - \frac{A}{2}\frac{\dot{R}}{R} - \frac{\dot{B}}{B}r \qquad [7.1.19]$$

We have also from [6.2.32,34,35] the following relationships:

$$3\frac{\dot{A}}{A} = \frac{-\dot{R}}{R}\left(\frac{\dot{B}r}{B} - 2\right) + \frac{\dot{B}}{B} + \frac{4}{r} - A\left(Rr + \frac{4}{r}\right) \qquad [7.1.20]$$

$$\left(\frac{\dot{R}}{R} + \frac{2}{r}\right)\frac{\dot{B}r}{B} = \frac{4\dot{R}}{R} - \frac{2}{r} + 2A\left(\frac{Rr}{4} + \frac{1}{r}\right) \qquad [7.1.21]$$

$$\ddot{R} = \dot{R}\left(\frac{\dot{A}}{2A} - \frac{2}{r}\right) + R\left[\frac{\dot{A}}{rA} - \frac{1}{r^2} + \frac{A}{r}\left(\frac{Rr}{4} + \frac{1}{r}\right)\right] \qquad [7.1.22]$$

Further, from [6.3.2,4] one gets:

$$A_0(r) = \left[1 + \frac{K_1}{r} + \frac{R_0 r^2}{12}\right]^{-1} \qquad [7.1.23]$$

and from [7.1.1] and [7.1.2] one obtains for $\frac{\dot{A}}{A}$ and $\frac{\dot{B}}{B}$ the following:

$$\frac{\dot{A}}{A} = \frac{2\ddot{R}}{\dot{R}}\left(\frac{\dot{R}r + R}{\dot{R}r + 2R}\right) + \frac{2}{r} \qquad [7.1.24]$$

$$\frac{\dot{B}}{B} = -\frac{\ddot{R}}{\dot{R}\left(\frac{\dot{R}r}{2R} + 1\right)} - \frac{2}{r} \qquad [7.1.25]$$

From which one has:

$$AB = \text{const.}\ \exp\left[2\int \frac{\ddot{R}r\,dr}{\dot{R}r + 2R}\right] \qquad [7.1.26]$$

and by [7.1.3] this yields:

$$A = C\dot{R}r^2\exp\left[\int \frac{\ddot{R}rdr}{\dot{R}r + 2R}\right] \qquad [7.1.27]$$

with C a constant.

The flatness condition hence reads:

$$\lim_{r'\to\infty} \int_0^{r'} \frac{\ddot{R}rdr}{\dot{R}r + 2R} = \text{constant} \qquad [7.1.28]$$

Finally, we notice that by [7.1.3]

$$\dot{R} \underset{r\to\infty}{\sim} \frac{1}{r^2} \quad \text{i.e.} \quad R \underset{r\to\infty}{\sim} \frac{1}{r} \qquad [7.1.29]$$

which is quite reasonable.

Now, all the relations between the metric tensor components and the scalar curvature given in this section are obtained from the generalized equations of section (6.2), almost, by direct substitution of the Lagrangian $\mathcal{L} = R^2$. They are presented here to demonstrate their nonlinearity in comparison with the relations obtained from general relativity.

The solution of these equations will be involving a Schwarzschild version as a special case. The various expressions relating A, B and R will be useful in facilitating the search for an analytic, or otherwise a computational solution. We will utilize these expressions partially in the present chapter and in Chapter Nine.

(7.2) An approximate solution for R^2-equations.

Because of the nonlinearities of the equations given above which are based on $\mathcal{L} = R^2$, we use a certain approximation to a certain extent that the resulting solution will differ from that based on the linear Lagrangian, $\mathcal{L} = R$ and then we look as to how the introduction of nonlinear Lagrangians may improve the solution of the gravitational equation.

If the approximated solution leads to any improvement then we expect that the exact one will surely lead to a better result. And this gives indications that the use of a more general Lagrangian with adjustable parameters may allow us to obtain the most adequate solutions. Thus, we employ equations [7.1.3] and [7.1.8] and by denoting:

$$\frac{A(r)}{B(r)} \equiv h^2(r) \qquad [7.2.1]$$

we get:

$$\dot{R}(r) = \frac{C}{r^2} h(r) \qquad [7.2.2]$$

i.e.

$$R(r) = C\int \frac{h(r)dr}{r^2} + R_0 \qquad [7.2.3]$$

with $R_0 = R(0) = \text{constant}$,

and

$$\frac{\dot{h}}{h} = \frac{\dot{R}}{R} + \frac{1}{r} - \frac{A}{r}\left(1 + \frac{Rr^2}{4}\right) \qquad [7.2.4]$$

Since C in [7.2.3] is an arbitrary constant, therefore by setting $C = 0$, that is when $R = R_0$, and with the help of [7.1.1] or [7.1.4] we shall have:

$$A_0(r)B_0(r) = K^2 = \text{const.} \qquad [7.2.5]$$

which by [7.2.1] yields

$$A_0(r) = \pm K h_0(r) \qquad [7.2.6]$$

Therefore, [7.2.4] will have the form:

$$\frac{r}{h_0}\frac{d}{dr}\left(\frac{h_0}{r}\right) = -K\frac{h_0}{r}\left(1 + \frac{R_0 r^2}{4}\right) \qquad [7.2.7]$$

or what is the same,

$$\frac{d}{dr}\left(\frac{r}{h_0}\right) = K\left(\frac{R_0 r^2}{4} + 1\right). \qquad [7.2.8]$$

The following solution now results;

$$h_0(r) = \frac{1}{K + \frac{K_1}{r} + \frac{K R_0 r^2}{12}} = \frac{A_0(r)}{K} \qquad [7.2.9]$$

In general $h(r)$ is given by [7.2.1] for nonconstant R; it is only within our approximation that $h \to h_0$ which is presented in [7.2.9]. Further approximation would yield Schwarzschild's form where $R_0 = 0$ and $K = 1 = -K_1$, i.e.

$$h_0(r) \approx \frac{1}{1 - \frac{1}{r}} \approx A_0(r) \qquad [7.2.10]$$

Thus [7.2.9] will yield for the scalar curvature [7.2.3], the following approximated form:

$$R(r) \approx -\frac{C}{K_1} \ln\left|K + \frac{K_1}{r} + \frac{K R_0}{12} r^2\right| + R_0 \qquad [7.2.11]$$

Now by using [7.2.2,9,11] in [7.1.9] we obtain for A(r) and hence by [7.2.1] for B(r) the following approximate expressions

$$A(r) \simeq \frac{\dfrac{K}{K+\dfrac{K_1}{r}+\dfrac{KR_0r^2}{12}} - \dfrac{K_1}{\left(Kr+K_1+\dfrac{KR_0r^3}{12}\right)\left(\dfrac{-K_1R_0}{C}+\ln\left|K+\dfrac{K_1}{r}+\dfrac{R_0r^2}{12}\right|\right)} + \dfrac{\dfrac{KR_0}{4}r^2}{K+\dfrac{K_1}{r}+\dfrac{KR_0r^2}{12}}}{1-\dfrac{Cr^2}{4K_1}\left(\dfrac{-K_1}{C}R_0+\ln\left|K+\dfrac{K_1}{r}+\dfrac{KR_0r^2}{12}\right|\right)} \qquad [7.2.12]$$

and

$$B(r) = h^{-2}(r)A(r) \simeq \left(K+\frac{K_1}{r}+\frac{KR_0r^2}{12}\right)^2 A(r) \qquad [7.2.13]$$

We notice that the above expressions for R(r), A(r) and B(r) exhibit no singularity at Schwarzschild's or any finite radius $r = \frac{-K_1}{K}$, and that at zero the singularity is much weaker than that in the solution of general relativity since

$$\left.\begin{aligned} &\lim_{r\to 0} A(r) \sim \frac{1}{\ln r} \to 0 \\ \text{and} \quad &\lim_{r\to 0} B(r) \quad \frac{1}{r^2 \ln r} \to \infty \end{aligned}\right\} \qquad [7.2.14]$$

We notice also that at asymptotically large r the metric is, in contrast with Scharzschild's geometry, not flat. This situation makes us think that such a defect can be remedied if we apply the exact solution for R^2-equations or at least by adopting a stronger method of approximation and consequently the singularity at zero can be made much weaker or may be totally abolished. That is, because we believe that the constant quantities like R_0, C, K and K_1 appearing in the expressions [7.2.11,12,13] as a result of utilizing the Lagrangian R^2, must have a certain role to play in deciding the behaviour of the metric. For instance, R_0 which represents the scalar curvature at the strongest area of gravitational energy (as $R_0 = R(0)$), should depend on quantum effects that may be described by certain parameters. We have already noted such a connection between R_0 and the parameter ω in [5.4.34], and it will be shown in [7.3.25] of the next section and in [8.4.7] when a more general Lagrangian will be considered, that R_0 is linked with parameters that govern the metric space. Thus we reckon that these constants are but the classical limits of quantum parameters. Therefore we assume that the constant quantity, say, R_0 would become significantly large whenever quantum feature of gravity be dominant, and conversely, it would diminish to zero as long as no quantum effects are observed. (In GTR $R_0 = 0$ and the constant $K = 1$.) This assumption will secure the fulfilment of the flatness condition at asymptotic distances, i.e.

$$\left.\begin{aligned} R(\infty) &= 0 \\ A(\infty) = B(\infty) &= 1 \end{aligned}\right\} \qquad [7.2.15]$$

We may assume that the constants, K, K_1 and C can be also influenced by quantum effects near the gravitational source, so that a well behaved metric can be established through assigning certain proper values for these constant parameters.

Now if we consider the exact solution of R^2-equation instead of the afore-given approximated one, we will notice that more terms will be present that may compete with each other in deciding the form of the metric and that other parameters will emerge, as we shall see in the next section, whose significance may elucidate the picture of the gravitational field in all energetic domains. In fact, the most general and the most informative metric should be based on the most general Lagrangian. However, we have justified, in the beginning of this chapter, our choice for the Lagrangian R^2.

Further, the approximated expressions for A and B are not aimed to be necessarily the reliable source of information about the space geometry, they are rather indicating how the metric may behave under certain constraints. Let us adopt a weaker approximation by constraining R_0 to be zero, i.e. by ignoring any quantum effect that could occur. This will, of course, allow a straight-forward comparison with Schwarzschild's metric. Thus [7.2.11,12,13] reduces, for $R_0 = 0$, to the following:

$$R(r) \approx -\frac{C}{K_1} \ln\left|K + \frac{K_1}{r}\right| \qquad [7.2.16]$$

$$A(r) \approx \frac{1 - \dfrac{K_1}{Kr+K_1}\left[1 + \left[\ln\left|K + \dfrac{K_1}{r}\right|\right]^{-1}\right]}{1 - \dfrac{Cr^2}{4K_1}\ln\left|K + \dfrac{K_1}{r}\right|} \qquad [7.2.17]$$

$$B(r) \simeq \left(K + \frac{K_1}{r}\right)^2 A(r) \qquad [7.2.18]$$

These expressions are good in the absence of quantum effects which we characterize by, say, setting $R_0 = C = 0$, and this will explain why such a constraint on the parameters leads to satisfying Minkowskian flatness [7.2.15] related to weak-field areas where definitely only classical features of gravity are prevailing.

Therefore, [7.2.16,17,18] as well as Schwarzschild metric are not applicable in the area where $R_0 \neq 0$, i.e. in strong gravity, it is only the exact solution which will account for strong-field region by virtue of the quantum parameters which will govern the field behaviour. Furthermore, the constant K_1 can be determined by exploiting the Newtonian limit to be $K_1 = -2MG$. but as we showed in (5.8), that G is not necessarily constant specially in strong gravitational fields, we will admit that consequently K_1 is not necessarily constant and besides that, for infinitismally small mass, point source, $K_1 \to 0$ and hence the singularity of $B(r)$ at $r = 0$ may not be extremely big as in [7.2.14]. This yields:

$$\infty > B(0) > A(0) = 0 \qquad [7.2.19]$$

Moreover, by furthering the approximation we can come to Schwarzschild's metric. We set $C = 0$ $K = 1$ in [7.2.16,17,18], to get:

$$\left.\begin{aligned} R(r) &\approx 0 \\ A(r) &\approx \frac{1}{1+\frac{K_1}{r}} - \frac{K_1}{r\left(1+\frac{K_1}{r}\right)\ln\left|1+\frac{K_1}{r}\right|} \\ B(r) &\approx 1 + \frac{K_1}{r} - \frac{K_1\left(1+\frac{K_1}{r}\right)}{r\ln\left|1+\frac{K_1}{r}\right|} \end{aligned}\right\} \qquad [7.2.20]$$

where the first terms in both $A(r)$ and $B(r)$ represent Schwarzschild's solution, whereas the other terms would have disappeared had we employed Einsteinian Lagrangian $(R = 0)$, instead of $R^2(r) \neq 0$. We notice that even in this last approximation the behaviour of $A(r)$ and $B(r)$ is better than in Schwarzschild's geometry and yet sharing with GTR its experimental successes by including the Schwarzschild term.

We also notice that $B(r)$ is large and positive at $r = 0$. This would suggest that the gravitational potential, there, will be maximum since g_{tt} - the metric tensor component - is related to the potential by

$$-g_{tt}(r) = B(r) \sim \Phi(r) \sim \frac{-2MG}{r} \qquad [7.2.21]$$

which is a reasonable result.

It is obvious from relation [7.2.18] that A and B coincide only if

$$1 + \frac{K_1}{r} = \pm 1$$

that implies either $K_1 \to 0$ i.e. extremely small mass, or at large distance from the source, i.e. $r \to \infty$, or otherwise when

$$r = -\frac{K_1}{2} = MG.$$

Unlike Schwarzschild's space the scalar curvature in this metric is finite at $r = 0$, and non-zero for small r. The points of intersection of the curves representing A and B with the curve representing R can serve as additional conditions ensuring the consistency amongst our approximated values. This would enable us to reconstruct an expression for R which will be more adequate than the one in [7.2.11]. That can be achieved by a special choice of the constant parameters K, K_1, R_o and C.

In Figure 7-I we sketch a graph for the expressions of R, A and B given in [7.2, 11, 12, 13] where we set $K = -K_1 = 1$, $R_o = 1.5$, and $C = -0.1$, being average values of these parameters.

It is clear from the plotted curves that our modified R^2-metric is better behaved than the Schwarzschild's version in the strong-field area. The peak appearing at the gravitational radius would represent the remnant of Schwarzschild's singularity and no singular feature of B(r) at the zero. As R(o) is finite, this would suggest that there is a limiting curvature beyond which space geometry cannot be influenced by matter. This last statement is equivalent to the existence of a short-range gravitational field that would overcome any force that might cause the matter to collapse. The constant parameters symbolize the quantum aspect of this field. Later we shall suggest that they develop an r-dependence in the proximity of the source due to the quantum effects of gravity. Thus, by knowing the exact values of these parameters we may completely abolish the irregular behaviour of the metric, e.g. the peak at $r = 1$, in Figure 7-I would shrink to the minimum.

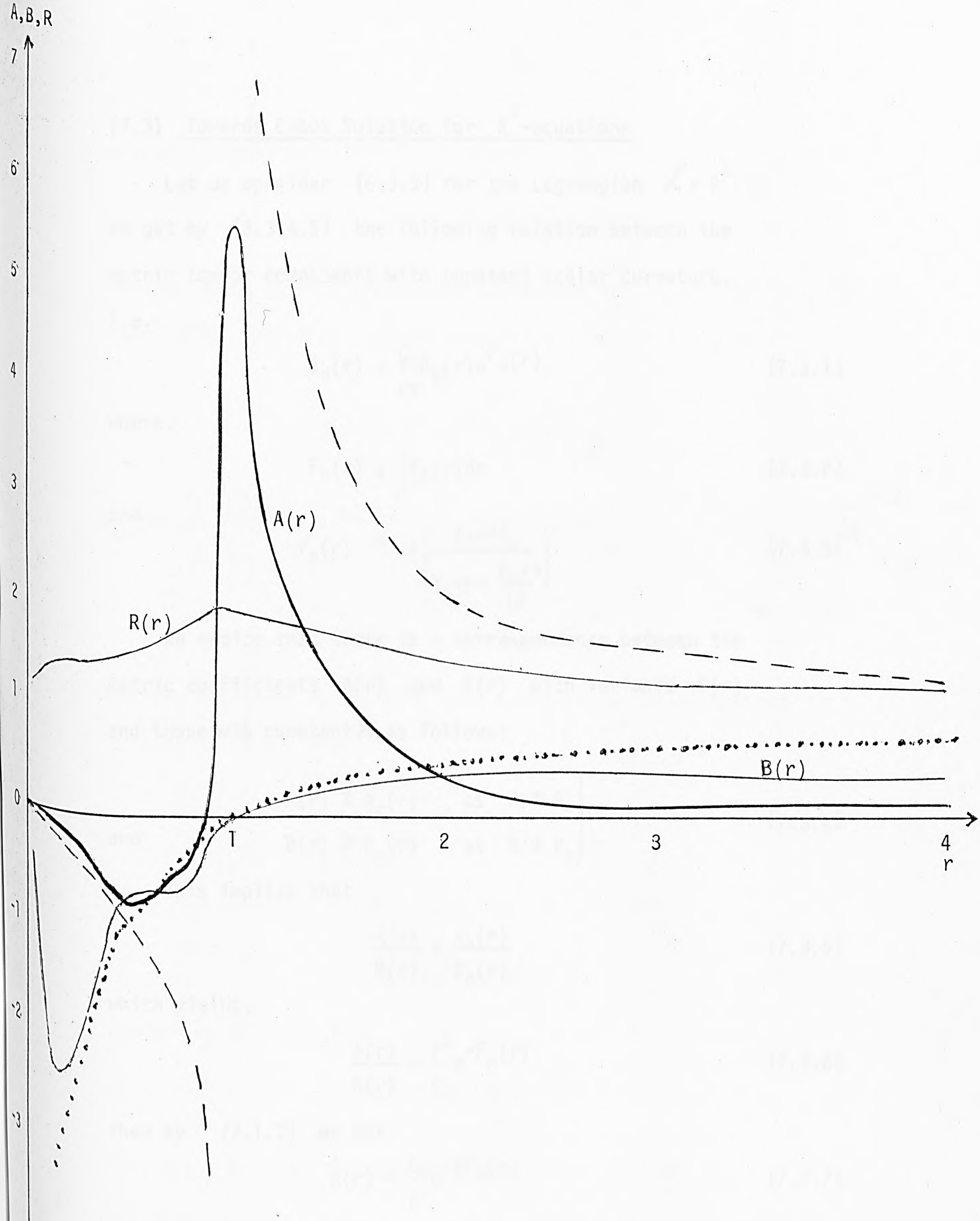

Figure 7_I An R^2-metric versus the Schwarzschild's metric.
The broken and dotted curves represent, respectively, the metric coefficients A and B in Schwarzschild's geometry.

(7.3) Towards Exact Solution for R^2-equations

Let us consider [6.3.3] for the Lagrangian $\mathcal{L} = R^2$. We get by [6.3.4,5] the following relation between the metric tensor components with constant scalar curvature, i.e.

$$B_0(r) = \frac{C_1}{r^2} A_0(r) e^{F_0(r)} \qquad [7.3.1]$$

where,

$$F_0(r) \equiv \int f_0(r)\,dr \qquad [7.3.2]$$

and

$$f_0(r) \equiv \frac{1}{2}\left(\frac{4+r^2R_0}{K_1+r+\frac{R_0r^3}{12}}\right) \qquad [7.3.3]$$

We notice that there is a correspondence between the metric coefficients $A(r)$ and $B(r)$ with variable $R(r)$ and those with constant R, as follows:

$$\left.\begin{array}{lll} A(r) \mp A_0(r) & \text{as} & R \mp R_0 \\ \text{and} \quad B(r) \mp B_0(r) & \text{as} & R \mp R_0 \end{array}\right\} \qquad [7.3.4]$$

This implies that

$$\frac{A(r)}{B(r)} = \frac{A_0(r)}{B_0(r)} \qquad [7.3.5]$$

which yields,

$$\frac{A(r)}{B(r)} = \frac{r^2}{C_1} e^{-F_0(r)} \qquad [7.3.6]$$

Then by [7.1.3] we get

$$\dot{R}(r) = \frac{C_2}{r} e^{-\frac{1}{2}F_0(r)} \qquad [7.3.7]$$

with

$$C_2 \equiv \frac{\pm C}{\sqrt{C_1}} \qquad [7.3.8]$$

Now by integrating [7.3.7] it gives:

$$R(r) = C_2\int \exp[-\tfrac{1}{2}F_0(r)]\frac{dr}{r} + R(0) \qquad [7.3.9]$$

Then we ought to obtain an exact expression for $F_0(r)$ so that $R(r)$ and $\dot{R}(r)$ can be calculated. Thus [7.3.2,3] yield,

$$F_0(r) = 2\delta_0(r) + \frac{R_0}{2}\int\frac{r^2\,dr}{K_1+r+\frac{R_0 r^3}{12}} \qquad [7.3.10]$$

where, as defined in [6.5.14], we will have

$$\delta_0(r) \equiv \int\frac{A_0(r)}{r}dr = \int\frac{dr}{K_1+r+\frac{R_0 r^3}{12}} \qquad [7.3.11]$$

and the integrand in the R.H. side integral can be factorized to make [7.3.10] read

$$\int\frac{r^2\,dr}{K_1+r+\frac{R_0}{12}r^3} = \frac{4}{R_0}\ln\left|\left(K_1+r+\frac{R_0 r^3}{12}\right)\right| - \frac{4}{R_0}\delta_0(r) \qquad [7.3.12]$$

hence, by this, [7.3.10] reduces to

$$F_0(r) = 2\ln\left|K_1+r+\frac{R_0 r^3}{12}\right| \qquad [7.3.13]$$

Thence [7.3.7] becomes

$$\dot{R}(r) = \frac{C_2}{r}\left(K_1+r+\frac{R_0}{12}r^3\right)^{-1} \qquad [7.3.14]$$

This means that our approximated expression [7.2.9] is correct.

Further, equation [7.3.9] will have the form

$$R(r) = C_2 \int \frac{dr}{r\left(K_1 + r + \frac{R_0 r^3}{12}\right)} + R(0) \qquad [7.3.15]$$

Now the integral in [7.3.15] can be calculated as follows; we use the identity:

$$\frac{1}{r\left(K + r + \frac{R_0 r^3}{12}\right)} \equiv \frac{1}{K_1}\left(\frac{1}{r} - \frac{1 + \frac{R_0 r^2}{12}}{K_1 + r + \frac{R_0 r^3}{12}}\right) \qquad [7.3.16]$$

Then, by [7.3.11,12] we shall have

$$\int \frac{dr}{r\left(K_1 + r + \frac{R_0}{12} r^3\right)} = \frac{-1}{3K_1} \ln\left|1 + \frac{K_1}{r} \frac{R_0 r^2}{12}\right| - \Delta(r) \qquad [7.3.17]$$

where $\Delta(r)$ is defined by

$$\Delta(r) \equiv \frac{2}{3K_1}\left[\delta_0(r) - \ln C_3 r\right] \qquad [7.3.18]$$

with C_3 an arbitrary constant.

Therefore, [7.3.15] yields

$$R(r) = -\frac{C_2}{3K_1} \ln\left|1 + \frac{K_1}{r} + \frac{R_0 r^2}{12}\right| - C_2\Delta(r) + R_0 \qquad [7.3.19]$$

The R.H. side first term represents our approximated value [7.2.11] of the preceding section where we have neglected $\Delta(r)$ term. We shall see whether or not such neglect can be justified in certain regions of gravitational field. Before that we have to calculate the exact expression for the function $\delta_0(r)$ the form of which will define $\Delta(r)$. In order to do that we proceed by employing the following identity for the algebraic rational functions [3],

$$\frac{1}{(K+d\cdot r)(a+br+cr^2)} \equiv \frac{1}{ad^2+cK^2-bKd}\left[\frac{d^2}{K+d\cdot r} + \frac{cK-bd-dcr}{a+br+cr^2}\right] \qquad [7.3.20]$$

where K,a,b,c,d are constants which are, generally, complex.

Let us set

$$\left.\begin{aligned} bd &\equiv -Kc \\ Kb &\equiv 1-ad \\ K_1 &\equiv aK \\ R_0 &\equiv 12cd \end{aligned}\right\} \qquad [7.3.21]$$

where, $-K_1$ is the Schwarzschild's gravitational radius and the constant scalar curvature

$$R_0 = R(0) \qquad [7.3.22]$$

we then get for [7.3.20] the form,

$$\frac{1}{K_1+r+\frac{R_0r^3}{12}} \equiv \frac{1}{3ad^2-2d}\left[\frac{d}{\frac{K_1}{ad}+r} + \frac{\frac{K_1R_0}{6a} - \frac{R_0r}{12}}{a - \frac{K_1R_0r}{12ad^2} + \frac{R_0r^2}{12d}}\right] \qquad [7.3.23]$$

which by being integrated results into the following, somewhat, lengthy expression for [7.3.11], i.e.

$$\delta_0(r) = \frac{1}{3ad-2}\ln\left(\frac{r+\frac{K_1}{ad}}{\sqrt{a - \frac{K_1R_0}{12ad^2}r + \frac{R_0}{12d}r^2}}\right) -$$

$$- \frac{K_1}{4ad(3ad-2)} \ln \left| a - \frac{K_1R_0}{12ad^2} r + \frac{R_0}{12d} r^2 \right| +$$

$$+ \frac{K_1R_0(8d^2-K_1)}{48ad^3(3ad-2)\sqrt{\left(\frac{K_1R_0}{12ad^2}\right)^2 - \frac{aR_0}{3d}}} \ln \left| \frac{\frac{R_0}{6d}r - \frac{K_1R_0}{12ad^2} - \sqrt{\left(\frac{K_1R_0}{12ad^2}\right)^2 - \frac{aR_0}{3d}}}{\frac{R_0}{6d}r - \frac{K_1R_0}{12ad^2} + \sqrt{\left(\frac{K_1R_0}{12ad^2}\right)^2 - \frac{aR_0}{3d}}} \right|$$

[7.3.24]

The constants appearing in this relationship a,b and d may be determined through K_1 and R_0 from [7.3.21] by,

$$K_1R_0 = -12abd^2 \qquad [7.3.25]$$

This would give a, b and d a physical meaning and at the same time, would impose restrictions on the parameters. Now, if as we did in (3.1), go to the Newtonian limit, we can set,

$$K_1 = -2MG \qquad [7.3.26]$$

and by making use of [6.5.26] which relates R_0 at $r = 0$ with the mass density T_0, i.e.

$$R(0) = R_0 = 8\pi G\, T_0 \qquad [7.3.27]$$

Then by using [7.3.26,27] in [7.3.25] we get

$$abd^2 = \frac{4\pi}{3} MG^2T_0 \qquad [7.3.28]$$

Indeed, this will constitute a physical condition on the parameters a, b and d and at the same time, may uncover their nature. For instance, we notice that for a large density T_0

the product abd^2 will be large, and vice versa, that would, certainly, indicate together with [5.4.34] and [7.3.25] that a, b and d are characterizing the strong gravitational field with its presumably quantum properties.

Further, by analysing the expression [7.3.19] we find out that the terms dominated by the constant parameters bring a significant contribution to $R(r)$ mainly through [7.3.24] in which we believe quantum mechanical features might be hidden. Indeed, since by [7.3.27] R_0 responds to the mass density, so whenever T_0 vanishes, i.e. in the vacuum, which is equivalent to setting $R_0 = 0$, the last term of [7.3.24] will disappear. The same terms will equally vanish at large r, whilst it will reduce, together with other terms of $\delta_0(r)$ as $r = 0$ to a logarithmic form containing R_0, K_1, a and d and indicating that there is no strong-field (quantum) effect far from the source of gravitation and that for small r the parameters of strong-field gravity will be effectively contributing to space curvature. In the preceding section we have ignored terms dominated by these parameters so that a direct comparison with Schwarzschild's singular metric can be permitted in weak-field areas.

Now owing to [7.1.3] and [7.1.9] the behaviour of the metric components $A(r)$ and $B(r)$ will be fully determined by [7.3.19]. We therefore note here that due to the arbitrariness of the parameters and their being generally complex, we can adjust them to satisfy conditions that will lead to physically and mathematically accepted nonsingular metric. We would also admit, that by virtue of the constant parameters which we regard as indicating the classical limit of quantum gravity, a quantizable model based on R^2-gravitational equations can be developed. Moreover, we notice that because of the complexity of the parameters the expressions for $R(r)$ and consequently for $A(r)$ and $B(r)$ will contain imaginary parts, and since $R(r)$ is a purely logarithmic function we can calculate its real and imaginary parts by utilizing the following useful formula [3].

$$\ln(x \pm iy) = \tfrac{1}{2} \ln(x^2+y^2) \pm i \tan^{-1} \frac{y}{x} \qquad [7.3.29]$$

where x and y represent the combination of both the real and the imaginary parts of the argument of the logarithmic function expressing $R(r)$.

Thus the scalar curvature can be presented as

$$R(r) = \mathrm{Re}R(r) + i\, \mathrm{Im}\, R(r) \qquad [7.3.30]$$

Therefore, by calculating the metric coefficients based on the [7.3.30] we can achieve an exact solution for gravitational equations based on the Lagrangian $\mathcal{L} = R^2$.

Finally, as we noted before in sections (5.7) and (6.5), the presence of an imaginary in the formulation of the metric influenced by the complexity of $R(r)$ would intuitively be connected with quantum effects of gravity. We shall discuss this situation again in (8.5, 6) with respect to a more general Lagrangian. In Chapter Nine we shall elaborate more on the complex aspect of the metric.

(7.4) The Effective Potential and the R^2-Theory

The magnitude of the 4-vector of energy-momentum is related to the rest mass M_0 of a particle moving in a gravitational field by the following equation [4]:

$$M_0^2 = -g_{\mu\nu} P^\mu P^\nu \qquad [7.4.1]$$

where,

$$P^\mu = g^{\mu\nu} P_\nu \qquad [7.4.2]$$

and,

$P_0 \equiv -E$ (Energy) $P_\phi \equiv \pm L$ (Angular Momentum)

$P \equiv \frac{dr}{ds}$ (Radial Momentum) ds - the proper time interval

$P^\theta \equiv \frac{d\theta}{ds}$ - Zero Momentum, i.e. in θ-direction.

Now to calculate P^μ per unit rest mass we will simply set $M_0^2 = 1$, then [7.4.1] yields:

$$1 = -g^{00} E^2 - g_{rr}\left(\frac{dr}{ds}\right)^2 - g^{\phi\phi} L^2 \qquad [7.4.3]$$

By the help of [3.1.3,4] we get,

$$A\left(\frac{dr}{ds}\right)^2 + 1 + \frac{L^2}{r^2} = \frac{E^2}{B}$$

i.e.

$$AB\left(\frac{dr}{ds}\right)^2 + B\left(1 + \frac{L^2}{r^2}\right) = E^2 \qquad [7.4.4]$$

The term $B\left(1 + \frac{L^2}{r^2}\right)$ is called the effective potential Φ^2_{eff} and it is defined as the full energy per unit mass for a test particle moving in spherically symmetric fields of a massive source when the radial kinetic energy becomes zero. The effective potential therefore grows quickly in the proximity of the source, and it tends to unity at asymptotic distances. At weak field area, owing to [7.2.21] and when the velocity vanishes, we would have:

$$\phi_{eff} = E = B^{\frac{1}{2}}\left(1 + \frac{L^2}{r}\right)^{\frac{1}{2}} \approx \left(1 - \frac{MG}{r}\right)\left(1 + \frac{L^2}{2r^2}\right) \qquad [7.4.5]$$

i.e.

$$\phi_{eff} \sim -1 - \frac{MG}{r} - \frac{MGL^2}{2r^3} + \ldots\ldots \qquad [7.4.6]$$

where, the first terms correspond to Newton potential. Moreover, the factor AB appearing in [7.4.4], by virtue of [7.2.18] reads:

$$AB = \left(1 + \frac{K_1}{r}\right)^2 A^2 \qquad [7.4.7]$$

and in the Newtonian approximation will tend to MG, therefore we suggest to call AB the "effective gravitational mass". In Schwarzschild's geometry AB = 1, and the effective potential reads

$$\phi_{eff} = \left(1 - \frac{2MG}{r}\right)^{\frac{1}{2}}\left(1 + \frac{L^2}{r^2}\right)^{\frac{1}{2}} \qquad [7.4.8]$$

The effective potential depends on the spacetime metric. This is why it will be useful in comparing different models. Because of the competition between $\frac{M}{r^2}$ and A near $r \rightarrow 0$ the effective potential and the effective mass in our R^2-model behave better than those in Schwarzschild space and are more promising even in the strong field region.

(7.5) Résumé and Critique

In the preceding sections we obtained useful relationships between the metric coefficients $A(r)$ and $B(r)$ and the scalar curvature $R(r)$ derived from the generalized field equations in the static isotropic space and based on the Lagrangian R^2. Equation [7.1.9] expresses $A(r)$ in terms of $R(r)$ and [7.1.3] relates A with B. Although these equations are too complicated to solve analytically, where usually computational methods are adopted, we are able to achieve an exact solution for R^2-equations. The problem was to solve the equation for $R = R(r)$ and hence by [7.1.9] and [7.1.3] get expressions for $A(r)$ and $B(r)$ respectively. In fact, the equations for R, as we shall see in Chapter Nine, are highly non-linear and very complicated, but we avoided their directly difficult solution by introducing a method of correspondence.

The results obtained in Section (7.2) are based on approximated expressions for $R(r)$ which would represent the dominant term of the exact solution in the weak-field areas and thus it is regarded as more advantageous by comparison with Schwarzschild's version.

In spite of these rather interesting results, there is what seemed to be a valid objection to our Lagrangian $\mathcal{L} = R^2$. The objection that we mentioned in the beginning of this chapter made Bicknell [1] believe that theories of gravity based on such a Lagrangian should be excluded from the class of viable gravitational theories.

Also, by investigating the gravitational field equations arising from this Lagrangian, Buchdahl [2] came to the following conclusion, that there exists no static solution of the field equations generated by R^2 which have asymptotically flat behaviour and the curvature does not vanish everywhere.

On the other hand, this Lagrangian has a certain appeal and a certain position. Its similarity to Lagrangians of quantizable nongravitational fields raised the hope that it would pave the route to quantum gravitational theory. In addition to that, the results obtained by us give strong indications that the above raised points against the R^2-theory may be wrong. To explain this, we note that the success of GTR is manifest in the free-field when the matter term is zero, as for inside, or, in the vicinity of a massive body, i.e. in strong-field region, we do not believe that the way matter is incorporated in Einstein's Lagrangian is necessarily indisputable. Therefore, instead of objecting to the quadratic Lagrangian when matter is incorporated in it, one would rather object to the way matter is usually assumed to generate the R^2-curvature in the strong-field area, in which case, we think it is not similar to that in general relativity. Thus, we do not accept the very idea of incorporating the matter to the R^2-Lagrangian, since in our model described in sec. (5.6), the matter term is a priori included in our derived Lagrangian [5.4.24] through the nongeometrical term γ. i.e.

$$\mathcal{L}(R) = \alpha R^2 + \beta R + \gamma$$

where α, β and γ are restricted by physical constraints as we will show in the next chapter. Now, setting $\frac{\gamma}{\alpha} = \frac{\beta}{\alpha} = 0$, would imply that we are concerned about a particular area where the contribution of R^2-term is dominantly large. Therefore, we see that the objection made by Bicknell against R^2-Lagrangian with matter term is, at least, irrelevant to our choice of $\mathcal{L} = R^2$.

As for the point raised by Buchdahl, we notice that if we use equations [6.1.2] or [6.1.6] we will get for the static isotropic space the following result,

$$|\square|^2 R = g^{\mu\nu} R_{;\mu;\nu} = \frac{1}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{2}{r}\right)\right] = 0 \, . \qquad [7.5.1]$$

This condition is satisfied by setting $\dot{R} = 0$, i.e. $R =$ constant. It was found [2],[5] if R was assumed to be depending on r then asymptotically the spacetime will not be flat as $r \to \infty$. However, the choice of coordinate made by Buchdahl, by which he came to this conclusion, is not necessarily reasonable. We regard it as an imposition of constraints on the metric. Our result [7.1.29] derived from [7.1.3] and based on the validity of Minkowskian limit, would impose on $R(r)$ the condition [7.1.28] which we can satisfy by a proper choice of our parameters in [7.3.9].

Thus the Buchdahl choice which resulted in $\ddot{R} = 0$ and hence $R \sim kr$ and leads to nonflatness, must be rejected. Further, differently from other authors, we do not consider Schwarzschild metric as an alternative solution, but, rather as an approximation to a correct metric and hence there is no fear from the singular features which are removable in our metric, based on R^2-equation and derived from very general formulae.

Furthermore, since in the weak-field approximation [7.2.20] Schwarzschild terms are present, we expect our metric to reproduce all the well-known results of general relativity including the correct Newtonian limit.

Moreover, we first obtain the exact solution with adjustable parameter for the scalar curvature which is, in general, a complex function. We introduced the concept of complexity of $R(r)$ to be an access to quantum gravity, the formulation of which, generally, includes imaginary parts. By doing so, we were guided by the well-established fact that any correct classical theory must be quantizable. The solution which we have obtained in Section (7.3), we believe, should revive with great optimism the interest of the gravitational theory based on the Lagrangian quadratic in the scalar curvature R.

CHAPTER EIGHT

An Advanced Model of Gravity

(8.0) Introductory Remark

In Chapter Seven we considered a special case of the generalized Lagrangian where we were concerned solely about the contribution brought into the metric by R^2 and resulted in very useful relationships between the field variables. In fact, the application of R^2-Lagrangian indicates directly to the secrets of strong-field gravity through a number of parameters.

In this chapter we do not, a priori, impose any conditions on the coefficients of the generalized Lagrangian, which we discover, are restricted by certain physical constraints.

The most general form of Lagrangian

$$\mathcal{L}(R) = \alpha(R+K)^m + \beta(R+q) + \gamma \qquad [8.0.1]$$

which is similar to [5.4.24], would probably reveal more about the metric at any distance from the gravitating source. However, a form of the Lagrangian with R and R^2 plus a nongeometrical term will fairly serve the purpose. This kind of Lagrangian is quite interesting since it differs from Einstein's one by being raised only one power higher and from the Lagrangian $\mathcal{L} = R^2$ by the presence of the nongeometrical term γ_1, i.e.

$$\mathcal{L}(R) = (\alpha_1 R + \gamma_1)^2 = \alpha_1^2 R^2 + 2\alpha_1\gamma_1 R + \gamma_1^2 \qquad [8.0.2]$$

and since it will lead to expressions that allow direct comparison with general relativity.

By using a Lagrangian of this type in [6.5.13,27] we would be able to analyse the behaviour of the metric at any radius from the source. We shall commence in the next section by considering the metric at the boundary values $r = 0$ and $r = \infty$ which represent, respectively, the strong and weak-field regions.

(8.1) The Metric at Weak and Strong-field Limits

Let us employ the Lagrangian [8.0.1] in the following special case when $m = 2$ and $K = q = 0$, i.e.

$$\mathcal{L}(R) = \alpha R^2 + \beta R + \gamma \qquad [8.1.1]$$

We recall the metric coefficients $A(r)$ and $B(r)$ in [6.5.13] and [6.5.27] together with the functions [6.3.4] and [6.5.11], then by substituting [8.1.1] in $A_0(r)$, $Q_{\mathcal{L}}(R)$, $\mathcal{G}_{\mathcal{L}}(R)$ and $\mathcal{D}_{\mathcal{L}}(R)$ defined in [6.3.2] and [6.5.15,17] respectively we obtain

$$A(r) = \frac{\dfrac{2}{1+K_1/r+\gamma(R_0)r^2} - \dfrac{3C^2C_3\zeta(R_0)(4\alpha\beta R+2\alpha\gamma+\beta^2)}{2\alpha(\beta R+2\gamma)^2(2\alpha R+\beta)}e^{-2\delta_0(r)}}{1+\left[\dfrac{(\beta R+2\gamma)^2}{gC_3C^2\zeta(R_0)}e^{2\delta_0(r)}+\dfrac{(3\alpha R^2+2\beta R+\gamma)}{6(2\alpha R+\beta)}\right]r^2} \qquad [8.1.2]$$

and

$$B(r) = \frac{A(r)(\beta R+2\gamma)^2\, e^{2\delta_0(r)}}{3C^2C_3\zeta(R_0)r^2}\exp\left[\frac{2}{gC^2C_3\zeta(R_0)}\int A(r)(\beta R+2\gamma)^2e^{2\delta_0(r)}r\,dr\right]$$

[8.1.3]

where,

$$\gamma(R_0) = \frac{1}{36}\left[(\beta+3)R_0+2\gamma\right] \qquad [8.1.4]$$

and

$$\zeta(R_o) = \frac{3\alpha R_0^2+\beta R_0-\gamma}{3(4\alpha R_0+2\beta)} \qquad [8.1.5]$$

Now since $B(r)$ depends on $A(r)$, therefore we will confine ourselves at first in investigating the expression [8.1.2] by imposing physical requirements on the Lagrangian coefficients α,β, and γ. Then having found the physical structure of $A(r)$ we look to how it would affect the expression for $B(r)$. To avoid lack of consistency in fulfilling the physical conditions between $A(r)$ and $B(r)$ we will be able to impose further constraints on the constants, otherwise we should employ the Lagrangian in its most general form where more parameters are involved. That will provide us with more freedom to adjust our constants to the most correct values.

We consider the behaviour of $A(r)$ at $r = 0$ and at $r = r_\infty \to \infty$. Then by [6.5.14] and [6.5.19] we get

$$A(0) = -\frac{3C^2C_3\zeta(R_0)\left[4\alpha\beta R(0)+2\alpha\gamma+\beta^2\right]}{2\alpha\left[\beta R(0)+2\gamma\right]^2\left[2\alpha R(0)+\beta\right]} \qquad [8.1.6]$$

for any $\gamma(R_0)$,

and,
$$A(\infty) = \frac{-\dfrac{3C^2C_3\zeta(R_0)(2\alpha\gamma+\beta^2)}{8\alpha\beta\gamma^2}\, e^{-2\delta(\infty)}}{1+\left[\dfrac{4\gamma^2 e^{2\delta_0(\infty)}}{9C_3C^2\zeta(R_0)}+\dfrac{\gamma}{6\beta}\right] r_\infty^2} \qquad [8.1.7]$$

for any $\gamma(R_0) = 0$, where we have assumed that $R(\infty) \equiv 0$.

Now to guarantee the asymptotically flat space limit we set

$$e^{2\delta(\infty)} = \frac{-3C_3C^2\zeta(R_0)(2\alpha\gamma+\beta^2)}{8\alpha\beta\gamma^2} = \frac{-3C_3C^2\zeta(R_0)}{8\beta\gamma} \qquad [8.1.8]$$

and this means that,

$$\beta = \pm i\sqrt{\alpha\gamma} \, . \qquad [8.1.9]$$

This relation is equivalent to the flatness condition,

$$A(\infty) = B(\infty) = 1 \, . \qquad [8.1.10]$$

It imposes constraints on the constant coefficients of the Lagrangian where any two of which will determine the third constant. Evidently it is these constants which will decide the structure of our model, its validity and viability, by being adjusted to satisfy the physical requirements.

(8.2) Construction of Different Lagrangians

The requirement of the asymptotic flatness of space geometry [8.1.10] will be formulated in terms of α,β,γ in the following theorem.

THEOREM 1.

For the space geometry to be flat at asymptotically large distance from the source of static isotropic gravitational field, it is necessary that the constant coefficients in the general Lagrangian,

$$\mathcal{L}(R) \equiv \alpha R(r)^2 + \beta R(r) + \gamma$$

be chosen in such a way that the following relation holds, i.e.

$$\beta^2 \equiv -\alpha\gamma \,. \qquad [8.2.1]$$

Provided that the scalar curvature is such that

$$R(\infty) = 0. \qquad [8.2.2]$$

This restriction will allow us to classify different Lagrangians with respect to α,β,γ each value of which may be positive, negative or zero either real or even imaginary.

Thus we can build the Lagrangian in different ways within the condition [8.1.9], some of which will lead to either pathological or trivial solutions, whilst on the other hand the Lagrangian may be complex or even purely imaginary. But if we required the Lagrangian to be real, we would impose further conditions of the metric. Here we count ten possible constructions of the Lagrangian, i.e.

$$\left.\begin{array}{lllll}
1. & \gamma = -\eta, & \alpha\eta > 0\,; & \beta = \pm\sqrt{\alpha\eta}\,; & \mathcal{L} = \alpha R^2 \pm\sqrt{\alpha\eta}\,R - \eta. \\
2. & \alpha = -\lambda, & \lambda,\gamma > 0\,; & \beta = \pm\sqrt{\lambda\gamma}\,; & \mathcal{L} = -\lambda R^2 \pm\sqrt{\lambda\gamma}\,R + \gamma \\
3. & \alpha = \pm\lambda,\ \gamma = \pm\eta & \text{i.e. } \alpha\gamma > 0\,; & \beta = \pm i\sqrt{\lambda\eta}\,; & \mathcal{L} = \pm\lambda R^2 \pm\eta \pm i\sqrt{\lambda\eta}\,R \\
4. & \alpha = \pm i\lambda,\ \gamma = \pm i\eta & \alpha\gamma < 0; & \beta = \mp\sqrt{\lambda\eta}\,; & \mathcal{L} = \mp\sqrt{\lambda\eta}\,R \pm i(\lambda R^2 + \eta) \\
5. & \alpha = \pm i\lambda,\ \gamma = \mp i\eta & \alpha\gamma > 0; & \beta = \pm\sqrt{\lambda\eta}\,; & \mathcal{L} = \pm i(\lambda R^2 + \sqrt{\lambda\eta}\,R \pm \eta)
\end{array}\right\} \qquad [8.2.3]$$

where the signs are taken in a respective way.

If we require the Lagrangian to be real, then the cases 3, 4 and 5, listed above, should be excluded. Otherwise, the scalar curvature R will have a restricted complex form.

Evidently, a complex Lagrangian will cause the metric components A(r) and B(r) to be complex.

In cases 3 and 4, since α and γ have the same sign, the Lagrangian will be complex. Consequently expressions [8.1.2, 3] for both A(r) and B(r) will contain imaginary parts.

In case 5, the purely imaginary Lagrangian will lead to complex $\gamma_0(R_0)$ and hence (see [8.1.2, 4], [6.5.14] and [6.3.2]) to complex values of $A_0(r)$ and $\delta_0(r)$.

(8.3) Lagrangian Coefficients and Physical Significance.

We notice that since the correct Newtonian limit is guaranteed solely by the linear in R terms of the Lagrangian, then by adjusting β in [8.2.1] to fit that limit as in [5.4.36], i.e. by setting,

$$\alpha \equiv - \frac{1}{(16\pi G)^2 \gamma} \quad \gamma > 0, \tag{8.3.1}$$

we will give the Lagrangian [8.1.1] the following form:

$$\mathcal{L}(R) = - \frac{\gamma^{-1}}{(16\pi G)^2} R^2 + \frac{1}{16\pi G} R + \gamma. \tag{8.3.2}$$

It is obvious that the term with quadratic R in $\mathcal{L}(R)$ will give the bigger contribution, the smaller we set the nongeometric term γ and vice versa, and due to the positiveness of γ it will always be negative.

Further, since we assumed as in [8.2.2] that $R(\infty) = 0$, then the Lagrangian will reduce to just a constant which, by condition [8.1.9], becomes

$$\mathcal{L}(R(\infty)) = \gamma = -\frac{\beta^2}{\alpha}. \qquad [8.3.3]$$

Then due to the positiveness of γ, this implies that the suitable Lagrangian can be constructed in only two different ways for any value of $R(r)$, i.e.

$$\mathcal{L}(R) = -\lambda R^2 \pm bR + \gamma \qquad [8.3.4]$$

or

$$\mathcal{L}(R) = \alpha R^2 \pm ibR + \gamma \qquad [8.3.5]$$

where λ, α, b and γ are real positive numbers. Now, [8.3.4] and [8.3.5] belong to 2. and 3. of [8.2.3] respectively.

Therefore, by imposing the following conditions,

1. the metric should be flat at $r \to \infty$;
2. the correct Newtonian limit should be satisfied;
3. the nongeometric term γ should be positive ;

we will exclude from [8.2.3] all other ways of constructing the Lagrangian leaving only those of the type [8.3.4] and [8.3.5].

If we require that $\mathcal{L}(R)$ must be real we would have to reject [8.3.5], otherwise R would be imaginary, and in this latter case, we would have to reject [8.3.4]. In either case the signs of R and R^2 are just interchanged, which suggests that both cases are equivalent. We notice here that the construction [8.3.2] is the most reliable form of Lagrangian since it fulfills all the above requirements with a certain choice of the constant γ. We also notice that the complexity arising in the Lagrangian is due to the requirement of asymptotic flatness expressed by condition [8.1.9].

In order to summarize the above discussed ideas we reformulate Theorem 1. in the following statements.

THEOREM - 2

Every solution with asymptotically flat behaviour, of gravitational equations describing static isotropic gravity and based on the Lagrangian $\mathcal{L} = \alpha R^2 + \beta R + \gamma$, with $\gamma > 0$, will not have correct Newtonian limit , if this Lagrangian is irreducible to a positive constant at asymptotic distance.
We also introduce the following theorem,

THEOREM - 3

Every nonzero constant Lagrangian of the static isotropic gravitational field will lead to gravitational equations whose solution will neither be asymptotically flat, nor satisfying Newtonian limit.

As an example to verify this last theorem, we take the solutions [6.3.2] and [6.3.7] which are physically unaccepted because of their vanishing at infinity and their violation of Newton's limit. Moreover, we add that equations [8.3.1,2] describe the relation between the general distribution of matter in the universe represented by γ which will be negligibly small in the vicinity of a given star and which, therefore, causes α to be exceedingly large and the term with R^2 be dominant. Conversely, this implies that far from a given star this quadratic term will be of a diminishing effect, and this explains why the Lagrangian quadratic in R is so linked with quantum and strong-field gravity. Furthermore, if instead of [8.1.1] we employ the Lagrangian [8.0.1] in the case m = 2:

$$\mathcal{L}(R) = \alpha(R+K)^2 + \beta(R+q) + \gamma \qquad [8.3.6]$$

which will reduce to the form [8.1.1] as,

$$\mathcal{L}(R) = \alpha R^2 + \tilde{\beta} R + \tilde{\gamma} \qquad [8.3.7]$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ are new constants

$$\left.\begin{aligned} \tilde{\beta} &\equiv \beta + 2\alpha K \\ \tilde{\gamma} &\equiv \gamma + \beta q + \alpha K^2 \end{aligned}\right\} \qquad [8.3.8]$$

then the flatness condition will become

$$(\beta+2\alpha K)^2 = -\alpha(\gamma+\beta q+\alpha K^2) \qquad [8.3.9]$$

i.e.

$$\beta^2 = -\alpha\gamma - \alpha\beta q - 4\alpha\beta K - 4\alpha^2 K^2 \qquad [8.3.10]$$

This will yield [8.2.1] as $K = q = 0$.

If we set $\alpha = 0$, then by [8.3.9] or [8.3.10] β becomes zero, but differently from [8.2.1] if γ was set to be zero β will not vanish. This situation will give more freedom in choosing the coefficients which will satisfy the physical requirements.

Now for $\tilde{\gamma} = 0$, $\tilde{\beta}$ will be zero, $\beta = -2\alpha k$ and $\mathcal{L}(R)$ will become

$$\mathcal{L}(R) = \frac{\gamma}{k(2q-k)} (R+k)^2 - 2\alpha k(R+q) + \gamma \qquad [8.3.11]$$

for $\alpha = 0$, $\tilde{\beta} = \beta = 0$ and $\mathcal{L}(R) = \gamma$. [8.3.12]

Let us now consider which gravitational field has to be regarded as weak and which is strong. For this we use the Lagrangian [8.3.7] in the form [8.3.4]

$$\mathcal{L}(R) = -\lambda R^2 \pm \tilde{b}R + \tilde{\gamma} \qquad [8.3.13]$$

with

$$\tilde{b} = \sqrt{\lambda\tilde{\gamma}} = \pm \tilde{\beta} \qquad [8.3.14]$$

where $\tilde{\gamma}$ and $\tilde{\beta}$ are defined in [8.3.8] and $\lambda = -\alpha$.

The scalar curvature is positive, and the constants λ and γ and hence b are positive and real.

We note that the term with quadratic R, assumed to represent strong gravity, should cease to be dominant as from a certain distance from the gravitational centre. The value of $R = R(r)$ at that distance will serve an upper limit beyond which the weak-field behaviour of gravity diminishes. Correspondingly, the positive and negative terms of [8.3.13] would cancel out giving for the scalar curvature the following value,

$$R_\ell = \frac{\pm \tilde{b} + \sqrt{\tilde{b}^2 + 4\lambda\tilde{\gamma}}}{2\lambda} > 0 \qquad [8.3.15]$$

Then the scalar curvature describing a weak-field phenomena should not exceed R_ℓ. Thus by [8.3.14] it yields;

$$R(r) \leq R_\ell = \frac{1}{2}(\sqrt{5} \pm 1)\sqrt{\frac{\tilde{\gamma}}{\lambda}} \qquad [8.3.16]$$

which by the help of [8.2.1] and [8.3.8] becomes;

$$R(r) \leq \frac{1}{2}(\sqrt{5} \pm 1)\sqrt{\frac{\gamma}{\lambda} \pm k\sqrt{\frac{\gamma}{\lambda}} - k^2} \qquad [8.3.17]$$

where we set $q = k$

This limiting value of the scalar curvature leads to a very interesting result, since it determines the domain for the applicability of general relativity. It indicates that any value of R not restricted by [8.3.17] would predict unphysical situation. Thus due to this limitedness of the scalar curvature a suitable choice of the parameter k will curb R from growing large. This implies that the gravitational collapse can be prevented.

We also notice that [8.3.17] allows imaginary part for R which we have admitted in our previous discussions and which will be considered again in Chapter Nine.

The square root in [8.3.17] will play an important role for values of R beyond the limiting value R_ℓ. It indicates to how far the quantum effects might dominate strong-field gravity. In fact, as we explained in (5.5), the parameter k and consequently λ, which describe the interaction between matter and its field, are connected with quantum features of the gravity. From [5.5.3] one gets

$$k(0,\gamma) = \sqrt{\frac{\gamma}{\lambda(0)}} \qquad [8.3.18]$$

and hence [8.3.17] may become

$$R(r) \leq \frac{1}{2}(\sqrt{5} \pm 1)k(0,\gamma) \qquad [8.3.19]$$

Thus the parameter k which can be regarded as a classical limit of a quantum-mechanical function turns to be the upper limit of the scalar curvature. Further, by using [8.3.1, 18] in [8.3.19] and by denoting the matter density as;

$$T \equiv (\sqrt{5} \pm 1)\gamma \qquad [8.3.20]$$

we obtain

$$R(r) \leq 8\pi GT \qquad [8.3.21]$$

or

$$R_{\ell} = 8\pi GT_{\ell} \qquad [8.3.22]$$

where T_{ℓ} is the limiting matter density that corresponds to the limiting curvature, and Equation [8.3.22] is the Einstein's general relativistic equation.

In the end we conclude that the new result obtained in this section confirms our belief which we advocated throughout this thesis that strong-field gravity is nonsingular, quantizable and undescribable by Lagrangian linear in R.

(8.4) <u>The Scalar Curvature</u> (Approximated)

In order to know the behaviour of A and B in [8.1.2,3] at any distance from a given centre we ought to, first, get an explicit expression for the scalar curvature, then by an appropriate choice of the constant coefficients the metric will be fully defined.

We notice in [6.5.10] that the expression for R must depend not only on r, but on the choice of the Lagrangian. The mutual dependence between R and $\mathcal{L}$ is evident from the very definition of the Lagrangian [8.1.1] and it arises from the fact that gravitational energy affects the space geometry and in return the curvature of space affects the path of energy.

Now by [8.1.1] expression [6.5.10] reads

$$(\beta R + 2\gamma)\dot{R} = \tilde{C}(\mathcal{L}_0)\,\frac{e^{-2\delta_0(r)}}{r} \qquad [8.4.1]$$

where $\delta_0(r)$ is defined in [6.5.14] and given by [7.3.24] and

$$\tilde{C}(\mathcal{L}_0) \equiv -\frac{C_3 C^2(3\alpha R_0^2+\beta R_0-\gamma)}{2\alpha(4\alpha R_0+2\beta)} \qquad [8.4.2]$$

Equations [8.4.1] can now be written as

$$\frac{d}{dr}(\beta R+2\gamma)^2 = 2\beta\tilde{C}(\mathcal{L}_0)\,\frac{e^{-2\delta_0(r)}}{r} \qquad [8.4.3]$$

that yields

$$\beta R + 2\gamma = \pm\beta P(r) \qquad [8.4.4]$$

or

$$R(r) = -\frac{2\gamma}{\beta} \pm P(r) \qquad [8.4.5]$$

where the function $P(r)$ is defined as follows,

$$P(r) \equiv \left[\frac{2\tilde{C}(\mathcal{L}_0)}{\beta} \int \frac{e^{-2\delta_0(r)}}{r} dr + \frac{C_0}{\beta^2}\right]^{\frac{1}{2}} \tag{8.4.6}$$

and C_0 is an arbitrary constant of integration.

Further, by recalling [8.1.4] and using [8.2.1] we will see that if α and β were chosen to be satisfying:

$$R_0 = \frac{2\beta^2}{\alpha(\beta+3)} \tag{8.4.7}$$

then setting $\gamma(R_0) \to 0$ is equavalent to $R_0 \to \dfrac{2\beta^2}{\alpha(\beta+3)}$,

and accordingly [8.4.2] gives,

$$\tilde{C}(R_0) \underset{\gamma_0(R_0)\to 0}{\longrightarrow} \frac{-3C_3C^2\beta(5\beta^2+4\beta+3)}{4\alpha^2(5\beta^2+18\beta+9)} \tag{8.4.8}$$

and $\delta_0(r)$ becomes

$$\delta_0(r) \equiv \int \frac{A_0(r)}{r} dr = \int \frac{dr}{r+k_1+\gamma_0(R_0)r^3} \xrightarrow{\gamma_0(R_0)\to 0} \ln\left[C_{01}^{-1}(R+k_1)\right] \tag{8.4.9}$$

Therefore in this case we get:

$$\int \frac{e^{-2\delta(r)}}{r} dr \simeq C_{01}^2 \int \frac{dr}{r(r+k_1)^2} = \frac{C_{01}^2}{k_1(r+k_1)} - \frac{C_{01}^2}{k_1^2} \ln\left(1 + \frac{k_1}{r}\right) \tag{8.4.10}$$

where C_{01} is a constant of integration.

Now the function [8.4.6] will become,

$$P(r) \simeq \left\{\frac{-3C_3C^2C_{01}}{2\alpha^2}\frac{(5\beta^2+4\beta+3)}{(5\beta^2+18\beta+9)}\left[\frac{1}{K_1(r+K_1)} - \frac{1}{K_1^2}\ln\left(1+\frac{K_1}{r}\right)\right] + \frac{C_0}{\beta^2}\right\}^{\frac{1}{2}} \qquad [8.4.11]$$

Then because of the vanishing of $R(r)$ at infinity and by [8.4.5] and [8.4.11] the constant C_0 is determined as,

$$C_0 = 4\gamma^2 \qquad [8.4.12]$$

We now conclude that if the constant curvature R_0 was chosen to be in the proximity of the value [8.4.7] then the following expression results for $R(r)$:

$$R(r) \simeq -\frac{2\gamma}{\beta} \pm \left\{\frac{3C^2C_{01}^2C_3}{2\alpha^2}\frac{(5\beta^2+4\beta+3)}{(5\beta^2+18\beta+9)}\left[\frac{1}{K_1^2}\ln\left(1+\frac{K_1}{r}\right) - \frac{1}{K_1(r+K_1)}\right] + \frac{4\gamma^2}{\beta^2}\right\}^{\frac{1}{2}} \qquad [8.4.13]$$

It is clear that in view of [8.1.9] $R(r)$ will be, in general, a complex function. However, a real R can be obtained by a certain choice of the constants.

Let us consider the case when $\mathcal{L} = R^2$, i.e. when we set $\gamma = 0$, i.e. $\beta = 0$ and $\alpha = 1$. Thus [8.4.13] will become:

$$R(r) \simeq \sqrt{\frac{C^2C_{01}^2C_3}{2}}\left[\frac{1}{K_1^2}\ln\left(1+\frac{K_1}{r}\right) - \frac{1}{K_1(r+K_1)}\right]^{\frac{1}{2}} \qquad [8.4.14]$$

or

$$R^2(r) = C^0\ln\left(1+\frac{K_1}{r}\right) - C^0\frac{K_1}{r+K_1} \qquad [8.4.15]$$

with

$$C^0 \equiv \frac{C^2C_{01}^2C_3}{2K_1^2}\,. \qquad [8.4.16]$$

Comparing [8.4.14, 15] with our approximated function [7.2.16] and the exact expression [7.3.19] obtained from R^2-equations, we find that the square root expression results from utilizing the full Lagrangian instead of $\mathcal{L} = R^2$. The latter is a restricted form of [8.1.1] where $\alpha = 1$, $\gamma = 0$ and by [8.2.1] $\beta = 0$. Such constraints on the coefficients should be accepted with certain care, since α and γ are interconnected by [8.3.1].

(8.5) An Exact Form of the Scalar Curvature

The afore-given results have been obtained on the basis of certain restrictions on the Lagrangian coefficients α and β and on the constant R_0. This was done in order to simplify the integral [8.4.6] expressing the scalar curvature R(r), and hence allow to understand the mode of its behaviour within these restrictions. However a full general expression of the scalar curvature results when one utilizes in [8.4.6] the expression [7.3.24] for $\gamma_0(r)$ with one difference; the constant $\frac{R_0}{12}$ is now replaced by $\gamma(R_0)$ defined in [8.1.4], i.e.

$$\delta_0(r) = \frac{2}{3K_1 e}\left[\ln(r+p) - \frac{1}{2}(1+\frac{1}{2}p)\ln(a - \sigma r + \rho r^2) + \frac{(8d^2 - K_1)\sigma}{2(p_+ - p_-)d}\ln\left(\frac{2\rho r - p_+}{2\rho r - p_-}\right) - e\ln C_3\right] \qquad [8.5.1]$$

where we denoted

$$\left.\begin{aligned}
e &\equiv 3ad - 2 \\
\sigma &\equiv \frac{K_1\gamma_o}{a\,d^2} \\
\rho &\equiv \frac{\gamma_o}{d} \\
\gamma_o = \gamma_o(R_o) &\equiv \frac{R_o}{12} + \frac{\beta R_o + 2\gamma}{36} \\
p &\equiv \frac{K_1}{ad} \\
p_+ &\equiv \sigma + \sqrt{\sigma^2 - 4a\rho} \\
p_- &\equiv \sigma - \sqrt{\sigma^2 - 4a\rho}
\end{aligned}\right\} \qquad [8.5.2]$$

This yields for the following exponential function the form;

$$e^{-2\delta_o(r)} = \tilde{C}_3 \frac{(a - \sigma r + \rho r^2)^{p_2}}{(r + p)^{p_1}} \left[\frac{2\rho r - p_-}{2\rho r - p_+}\right]^{p_3} \qquad [8.5.3]$$

with

$$\left.\begin{aligned}
p_1 &\equiv \frac{4}{3K_1 e} \\
p_2 &\equiv \frac{1}{2}\, p_1(1 + \frac{1}{2}\, p) \\
p_3 &\equiv p_1 \frac{\sigma(8d^2 - K_1)}{2(p_+ - p_-)d} \\
\tilde{C}_3 &= \text{constant}
\end{aligned}\right\} \qquad [8.5.4]$$

It is clear that if γ_0 is set to be zero, $p_1 = 2$, and $p = K_1$, [8.5.3] will reduce to

$$e^{-2\delta_0(r)} = \tilde{C}_3 \frac{a^{\tilde{2}}}{(r + K_1)^2} \qquad [8.5.5]$$

which will lead to [8.4.10] and hence the expression [8.4.13] for the scalar curvature can be obtained.

The general form of R(r) can be obtained by solving the integral

$$\int \frac{e^{-2\delta_0(r)}}{r} dr = \tilde{C}_3 \int \frac{(a - \sigma r + \rho r^2)^{p_2}}{r(r + p)^{p_1}} \left[\frac{2\rho r - p_-}{2\rho r - p_+}\right]^{p_3} dr \qquad [8.5.6]$$

for certain choices of the constants $\tilde{C}_3, \sigma$, ρ, p, p_1, p_2 and p_3. Now, by [8.4.5, 6] the full expression for the scalar curvature reads

$$R(r) = -\frac{2\gamma}{\beta} \pm \left\{\frac{4\gamma^2}{\beta^2} + \frac{2\tilde{C}_3 \tilde{C}}{\beta} \cdot \int \frac{(a - \sigma r + \rho r^2)^{p_2}}{r(r + p)^{p_1}} \left(\frac{2\rho r - p_-}{2\rho r - p_+}\right)^{p_3} dr\right\}^{\frac{1}{2}} \qquad [8.5.7]$$

By using this in [8.1.2, 3] the exact full solution for the metric coefficients A(r) and B(r) can be obtained. Then by taking into account the relationships [8.2.1] and [8.3.1] based on certain physical requirements, and by suitably chosing values for the parameters and the constants in [8.5.2, 4] a non-singular metric can be obtained.

If we assume that the quadratic terms of the Lagrangian is dominantly large within a certain domain due to a large α, then by view of [8.3.1] we can approximately set $\gamma \simeq 0$, and [8.5.7] will reduce to the square root of the integral form which can be reduced further to [8.4.14].

Now let us set $\alpha = -\lambda$ and the arbitrary constant $\tilde{C}_3 = \beta$, then by using [8.2.1] and [8.4.2] expression [8.5.7] reads

$$R(r) = \pm 2k \mp \left\{ 4k^2 + C_4 \frac{(3R_o^2 - kR_o - k)}{2(2R_o - k)} \cdot \int \frac{(a - \sigma r + \rho r^2)^{p_2}}{r(r+p)^{p_1}} \cdot \left(\frac{2\rho r - p_-}{2\rho r + p_+} \right)^{p_3} dr \right\}^{\frac{1}{2}} \quad [8.5.8]$$

where similar to [8.3.18] we denote

$$k \equiv \sqrt{\frac{\gamma}{\lambda}} \quad [8.5.9]$$

and

$$C_4 \equiv \frac{C_3\, c^2}{\lambda} \quad [8.5.10]$$

(8.6) Strong Gravity Domain

The expression [8.5.7] or [8.5.8] constitutes the very general exact solution for R(r), based on the Lagrangian [8.1.1]. It is required that this form of the scalar curvature will reasonably describe the weak-field as well as the strong-field behaviours of the gravitational field.

The integral under the square root bracket is difficult to solve for general values of the parameters p_1, p_2 and p_3, defined in [8.5.4, 2] by undetermined constants. However we know that [8.5.7] should reduce to [7.3.19] as the full Lagrangian $\mathcal{L}_f = \alpha R^2 + \beta R + \gamma$ reduces to $\mathcal{L} = \alpha R^2$. But [7.3.19] is primarily aimed to describe strong-field gravity that is, when the ratio $\frac{\gamma}{\alpha}$ is negligibly small. Therefore by taking $\alpha \doteq 1$ and setting to zero the coefficients γ and, consequently by [8.2.1], β, one obtains from [8.5.7];

$$R(r) = \frac{1}{2}\sqrt{3C_4R_0}\left[\int \frac{(a - \tilde{\sigma}r + \tilde{\rho}r^2)^{p_2}}{r(r+p)^{p_1}} \cdot \left(\frac{2\tilde{\rho}r - \tilde{p}_-}{2\tilde{\rho}r - \tilde{p}_+}\right)^{\tilde{p}_3} dr\right]^{\frac{1}{2}} \qquad [8.6.1]$$

$$\mathcal{L}_f \rightarrow R^2$$

where now according to [8.5.2, 4] we have,

$$\left.\begin{aligned}
\tilde{p}_3 &= \frac{p_1\tilde{\sigma}}{2d} \cdot \frac{(8d^2 - K_1)}{(\tilde{p}_+ - \tilde{p}_-)} \\
\tilde{p}_\pm &= \tilde{\sigma} \pm \sqrt{\tilde{\sigma}^2 - 4a\tilde{\rho}} \\
\tilde{\sigma} &\doteq \frac{K_1\,\tilde{\gamma}_0}{ad^2} \\
\tilde{\rho} &= \frac{\tilde{\gamma}_0}{d} \\
\text{and } \tilde{\gamma} &= \frac{R_0}{12}
\end{aligned}\right\} \qquad [8.6.2]$$

Now [8.6.1] can be equated with [7.3.19] from which it seems possible to assess the values of p_1, p_2 and p_3. It is clear that [8.6.1] represents the contribution to the expression for R [8.5.8], of the quadratic term. This contribution corresponds to strong gravity which is therefore, expected to disappear at large distances from the gravitational centre. Such a situation suggests that the constant coefficients and parameters are somehow attached to strong-field effect that would presumably diminish with distance from the centre. We also learn from [8.6.1] that the square root would indicate that the strong-field gravity is generally described by complex functions.

This description seems to be essentially nonsingular or at least can be made nonsingular by virtue of certain choices of the constant coefficients and parameters. It again confirms our belief concerning the close link between the nonsingularity of the metric, the complexity of the field variables and the quadraticity of the Lagrangian in strong-field gravity. Then the afore-mentioned effect in strong gravity is thought to be of a quantum nature and hence described by complex functions. Thus, it is a quantum effect, and as we noted in Sections (4.8), (5.7) and (7.3) it should be represented by a nonlinear (quadratic) Lagrangian. Such an effect will certainly abolish any singularity that could have occurred had a linear Lagrangian been employed.

Now in order to explain how strong gravity can be manifested through the coefficients and parameters we visualize R_0 and, k and C_4 given by [8.5.9, 10], as classical limits of quantum-mechanical quantities. Being guided by the relationships [5.4.34, 39] and [5.5.2] where R_0 and $\lambda = -\alpha$ are presumably reflecting certain quantum mechanical features of gravity, we admit the following.

At very small distances from the gravitational centre where strong gravity is prevailing R_0, C_4 and k develop an effective dependence on r such that, for instance:

$$R_0(\omega, r) = R_0 \cdot \left(1 - e^{-\frac{\hbar f_1(\omega)}{r^{\ell}}}\right) \qquad [8.6.3]$$

$$C_4(\omega, r) = C_4 \cdot \left(1 - e^{-\frac{\hbar f_2(\omega)}{r^{m}}}\right) \qquad [8.6.4]$$

$$k(\omega, r) = k \cdot e^{-\frac{\hbar f_3(\omega)}{r^{n}}} \qquad [8.6.5]$$

where $\hbar$ is the Planck constant characterizing the quantum corrections and $f(\omega)$ characterizes the field strength, and ℓ, m, and n are certain positive constants.

Such choice will guarantee that the scalar curvature [8.5.8] vanishes, at asymptotically large distances whereas it becomes of a significant value in strong field regions. The same will hold for the expression [7.3.19] if only the coefficient C_2 is to be identified with C_4 given by [8.6.4].

(8.7) Conclusion

The choice [8.6.4] implies that the quantity C_3C^2 given by [8.5.10] is some function of the (quantum) parameter ω i.e,

$$C_3C^2 \equiv C_5(\omega) \qquad [8.7.1]$$

which would compete with λ to secure the correct behaviour of either the strong or the weak gravitational fields.

Now the use of [8.6.3, 4, 5] in [8.5.8] yields

$$R(o) = \frac{1}{2}\sqrt{3C_4R_0}\, C_3^{-1} \lim_{r\to o} \left[\int_0^r e^{-2\delta_0(r)}dr\right]^{\frac{1}{2}} \qquad [8.7.2]$$

where $\delta_0(r)$ is defined by [8.5.3] , and also,

$$R(\infty) = \lim_{r\to\infty} R(r) = 0 \qquad [8.7.3]$$

Thus R(r) satisfies the condition of the asymptotic flatness at large distances, and in the proximity of the gravitational centre it becomes finite, though, in general, non-real. The complexity of R, as we noted before, would correspond to a quantum mechanical effect.

Then if [8.7.2, 3] are substituted into [8.1.6, 7] respectively, a well-behaved A(r) will be obtained in the strong-field region, which will satisfy the Minkowski limit in the weak-field zones. The same can be proved to hold for the metric coefficient B(r) in [8.1.3] by virtue of the appropriate choice of α and γ and also p_1, p_2, p_3

appearing in [8.5.3]. Though, a strong singularity can occur at $r = 0$ if such a choice is not made. This situation imposes additional constraints on these quantities which will lead to a precise definition of their nature.

Further study of the expression for $R(r)$ [8.5.8] and consequently expressions [8.1.2, 3] for $A(r)$ and $B(r)$ will provide us with the full picture of the gravitational interaction at any distance from the source. However the detailed expression of $R(r)$ can be obtained only by evaluating the integral under the square root in [8.5.8]. But as we have pointed out in the preceding section the expression [7.3.19] can be identified with the strong-field limit of R^2, that is, with the integral in [8.6.1]. Thus an explicit form of R in terms of r can be readily given without going to the tedious task of detailed calculations. Here it is suggested that the coefficient $\frac{C_2}{K_1}$ in [7.3.19] should behave in a way similar to $C_4(r, \omega)$ in [8.6.4]. This result will evidently provide us with a very useful information about the nonlinear aspect of the gravitational field in the proximity of, say, a star with a gigantic density. However, our approach is still classical and restricted to the static isotropic spacetimes. We therefore believe that eventually the departure from the classical and symmetrical metric to a more realistic situation will certainly elucidate more the enigmatic features of gravity in the limit of strong energy spectrum. It is obvious that such a departure should not disprove our static model. It will rather add to our metric certain quantum dynamical corrections.

In the end, we conclude this section by admitting that we are able to derive and obtain, in this chapter, an essentially nonsingular **advanced** metric based on the Lagrangian containing terms both linear and quadratic in R as well as a nongeometrical term. The coefficients in the Lagrangian form proved to bear certain physical significance whose full interpretation may be well understood in quantum gravity. However the interesting results we obtain in Sections (8.2), (8.3) and (8.5) reveal the importance of the role these coefficients can play in strong gravitational field domains. As far as quantum mechanical effects are concerned, we believe that the emerging of an imaginary part in the expression for R(r) in areas with small r, reflects the nonclassical character of strong-field gravity. In the next chapter we shall elaborate more on the complexity of the scalar curvature and consequently of the metric space.

CHAPTER NINE

The Complex Metric

(9.0) A New Possibility

In this chapter we draw the attention to a new possibility of the solution for the gravitational equations. Namely, by splitting the scalar curvature R to a real and imaginary parts, we can clear out the controversy arising when one tries to combine strong and weak gravity behaviours by one law. Such a controversy might arise due to ignoring either the imaginary or the real parts of R in either strong or weak-field areas of the gravitational field.

As it was noticed from the preceding chapter the imaginary part of R emerges as a result of employing the Lagrangian with a quadratic term. Then by complexifying the scalar curvature the resulting equations will lead to expressions for the metric hopefully explaining gravity at both small and asymptotically large distances from the source. For instance, the imaginary parts of R, and consequently of the metric coefficient g_{rr}, might explain well the strong-field gravity whilst the weak-field gravitation can be appropriately described by their real parts.

Now by admitting this new possibility of solution we are, in fact, reviving the interest in the quadratic Lagrangian which as we mentioned before, has been abandoned now for some time.

(9.1) The Physical Meaning of complexity

We have mentioned repeatedly in Sections (5.7), (6.5), (7.3, 5) and (8.6) that a close relation exists between the scalar curvature R having an imaginary part and the effects of a strong gravitational field in the vicinity of its source. It was shown before that the complexity of R can be imposed by constant coefficients of physically chosen Lagrangians. We also pointed out that the metric coefficients will consequently be complex and their behaviour at $r \to 0$ will be essentially nonsingular. In (8.2) we gave examples of Lagrangians that cause the metric to have an imaginary part. Towards the end of the last chapter we admitted the existence of a complex value of the metric coefficients A and B near $r = 0$, which in our opinion is due to quantum nature of gravitation in the strong gravity limit.

General relativity which is based on the linear Lagrangian $\mathcal{L} = R$ will not have imaginary parts in its formulation and hence is not applicable to strong-field quantum effects. The singular behaviour of GTR in the region $r \to 0$, is, in fact, due to this lack of applicability.

It was noted in Section (4.2)i that the quantization of the gravitational field was thought to be linked with the quadratic-R-equations. This link becomes, now stronger due to the interrelation between the quadraticity and the complexity on one hand, and the quantization on the other hand.

Further, in the Lagrangian

$$\mathcal{L}(R) = - \lambda R^2 \pm \sqrt{\lambda\gamma}\, R + \gamma; \quad \lambda, \gamma > 0, \qquad [9.1.1]$$

since R becomes infinitesimally small as $r \to \infty$, i.e. as the field tends to be flat, and by virtue of the property of the coefficient λ described by (8.5.9) and (8.6.5) the term λR^2 will be negligible. The Lagrangian will, therein, be dominated by the linear R-term. Contrarily, in the areas with $r \to 0$ the term with quadratic R will be dominant.

Also, it can be noticed that the disappearance of the R^2-term by setting $\lambda = 0$ will cause the term with linear R to disappear whereas the contrary is not necessarily valid. It is, therefore, possible to choose a Lagrangian purely quadratic in R that will lead to the equations obtained in (7.1). In the next sections we consider these equations with more detail with an intent to obtain a general solution for R which will be of a complex form.

At the beginning we shall not assume the complexity of the scalar curvature which we expect to be manifested by approaching the gravitational source which in our case is a point mass. We will rather look for the expression for R at the zero point to see what it looks like. Such value of R will constitute an initial point for calculating the curvature at any distance from the point source, Moreover, we regard the complexity of R as corresponding to a short-range effect of strong gravity and hence we will be less interested in the asympotic behaviour of this complex metric.

(9.2) Further derivation of R^2-equations

In Chapter Seven we obtained useful relations between the metric coefficients $A(r)$ and $B(r)$ and the scalar curvature $R(r)$. In order to investigate $R(r)$ we utilize these relations to eliminate A and B, but at their expense we will have to deal with highly nonlinear third order in r differential equations.

We commence by employing [7.1.9] and [7.1.12] to get rid of A: i.e.

$$\frac{\frac{d}{dr}\left(\frac{\dot{B}}{B}r\right)}{\frac{\dot{B}}{B}-\left(2-\frac{\dot{B}}{B}r\right)\frac{Rr}{4}} = \frac{\dot{R}r\,\frac{d}{dr}\left(\frac{R}{\dot{R}r}\right)}{R\left(\frac{Rr}{4}+\frac{1}{r}\right)} \qquad [9.2.1]$$

By using [7.1.25] on the L.H. side, we eliminate $\frac{\dot{B}}{B}$ and the following equation results by differentiation:

$$\begin{aligned} &r^2R^2(Rr^2+4)(2R+r\dot{R})\dot{R}\dddot{R}-r^2R^2(Rr^2+4)(4R+3r\dot{R})\ddot{R}^2 \\ &+rR[(4+r^2R)(2R^2+r^2\dot{R}^2)+(2+r^2R)(R+r\dot{R})(2R+r\dot{R}) \\ &-2(2+r^2R)(2R+r\dot{R})^2]\dot{R}\ddot{R}+2r^3(2+r^2R)\dot{R}^5+6r^2R(2+r^2R)\dot{R}^4 \\ &-8(2+r^2R)R^3\dot{R}^2 = 0. \end{aligned}$$

By differentiating $\ln A$ of [7.1.9] we get *after equating that to [7.1.24]*,

$$\frac{d}{dr}\ln\left(\frac{\dot{R}}{R}-\frac{\ddot{R}}{\dot{R}}-\frac{1}{r}\right)-\frac{d}{dr}\ln\left(\frac{rR}{4}+\frac{1}{r}\right) = \frac{2\ddot{R}(\dot{R}r+R)}{\dot{R}(\dot{R}r+2R)}+\frac{2}{r} \qquad [9.2.3]$$

which results into the equation,

$$\begin{aligned} &r^2R^2(4+r^2R)(2R+r\dot{R})\dot{R}\dddot{R}-r^2R^2(4+r^2R)(4R+3r\dot{R})\ddot{R}^2 \\ &+[(4+r^2R)(-2rR^3-8rR^3\dot{R}-4r^2R^2\dot{R}^2+r^3R\dot{R}^3)+rR^2(4\dot{R}-r^2R\dot{R} \\ &-r^3\dot{R}^2)(2R+r\dot{R})]\ddot{R}+2r^3(2+r^2R)\dot{R}^5+6r^2R(2+r^2R)\dot{R}^4 \\ &-8(2+r^2R)R^3\dot{R}^2 = 0. \end{aligned} \qquad [9.2.4]$$

Also, the use of [7.1.14] and [7.1.9] yields,

$$\frac{d}{dr}\left(\frac{R}{\dot{R}}\frac{\dot{B}}{B}\right) = \frac{2r^2R}{(4+r^2R)}\frac{d}{dr}\left(\frac{R}{\dot{R}r}\right) \qquad [9.2.5]$$

and by [7.1.25] this leads to,

$$r^2R^2(4+r^2R)(2R+r\dot{R})\dot{R}\ddot{R}-r^2(4+r^2R)(4R+3r\dot{R})R^2\ddot{R}^2$$
$$-rR^3(4+r^2R)(16R^2+8r^2R^2+12rR\dot{R}+7r^3R\dot{R}-4r^2\dot{R}^2)\dot{R}^2\ddot{R}$$
$$+2r^3(2+r^2R)\dot{R}^5+6r^2R(2+r^2R)\dot{R}^4-8R^3(2+r^2R)\dot{R}^2= 0. \qquad [9.2.6]$$

Now equations [9.2.2] and [9.2.4] reduce to the cubric 1st order nonlinear equation:

$$r^2\dot{R}^3-rR(1+r^2R)\dot{R}^2-R^2(10+3r^2R)\dot{R}-R^2(4+r^2R) = 0 \qquad [9.2.7]$$

and, equations [9.2.2] and [9.2.6] reduce to:

$$4r^2R^2(4+r^2R)\dot{R}^3-[2r^2+(4+r^2R)(7r^3+12r)R^3]\dot{R}^2$$
$$+[5r(2+r^2R)R-8(2+r^2)(4+r^2R)R^4]\dot{R}+4(1+r^2R)R^2= 0 \qquad [9.2.8]$$

Eliminating $\dot{R}^3$ by using [9.2.7] in [9.2.8] yields the following quadratic in $\dot{R}$ equation:

$$(4r^5R^5-7r^5R^4+8r^3R^4-28r^3R^3-32rR^3-2r^2)\dot{R}^2+$$
$$+(12r^4R^6-8r^4R^5+72r^2R^5-32r^2R^4+96R^4+ 5r^3R^2+$$
$$+10rR)\dot{R}+4R^2(r^4R^4+8r^2R^3+16R^2+r^2R+1) = 0 . \qquad [9.2.9]$$

Equation [9.2.7] is satisfied by the trivial solution $R=0$ which represents Einstein's space. It satisfies also the solution

$$R(r) = \frac{-4}{r^2} \qquad [9.2.10]$$

This solution would have meant, due to [7.1.3] , that the metric coefficients $A(r)$ and $B(r)$ are related by:

$$\frac{A(r)}{B(r)} = \frac{64}{C^2r^2} \qquad [9.2.11]$$

which gives bad asymptotic behaviour at large r. Otherwise the flat space limit can be satisfied only if C is infinitesmally small so that $r \to r_\infty \equiv \frac{8}{C} \to \infty$. However, [9.2.10] is not the unique solution and hence $R(r)$ and $\dot{R}(r)$ are not necessarily singular at $r = 0$. In fact, being cubic, equation [9.2.7] allows other solutions which may be complex.

(9.3) Scalar Curvature Complexified

Equations [9.2.7] and [9.2.9] are still too complicated to be analytically solved. But, here, in this chapter, we are less interested in the solution itself which we have already obtained in the preceding chapter. For more general cases, we are rather investigating the complex structure of this solution in areas where the behaviour of the metric is either singular or incompatible with physical requirements in some other theories.

Now, in order to avoid any violation of flat space limit, we assume that $R(0)$ and $\dot{R}(0)$ are finite. Therefore, at $r = 0$ [9.2.7] and [9.2.9] yield:

$$\left.\begin{aligned} \dot{R}(0) &= -0.4 \\ R(0) &= \pm i \frac{1}{\sqrt{6.4}} \end{aligned}\right\} \qquad [9.3.1]$$

This indicates the complex nature of $R(r)$ and suggests that the solution for $\tilde{R}$-equation should be sought in the form of a complex function, i.e.

$$R(r) = Ke^{-i\alpha(r)} = K[\cos\alpha(r) - i\sin\alpha(r)] \quad [9.3.2]$$

and

$$\dot{R}(r) = -i\dot{\alpha}(r)R(r) = -K\dot{\alpha}(r)[\sin\alpha(r) + i\cos\alpha(r)] \quad [9.3.3]$$

where K is a constant.

The use of [9.3.1] in [9.3.2] and [9.3.3] yields:

$$\left.\begin{aligned} \cos\alpha(0) &= 0 \\ \sin\alpha(0) &= \pm\frac{1}{K\sqrt{6.4}} \end{aligned}\right\} \quad [9.3.4]$$

$$K = \pm\frac{1}{\sqrt{6.4}} \quad [9.3.5]$$

$$\dot{\alpha}(0) = \pm 0.4\sqrt{6.4} \quad [9.3.6]$$

Thus we write $R(r)$ in the form:

$$R(r) = \pm\frac{1}{\sqrt{6.4}}e^{-i\alpha(r)} = \pm\frac{1}{\sqrt{6.4}}[\cos\alpha(r) - i\sin\alpha(r)] \quad [9.3.7]$$

i.e. the real and imaginary parts of $R(r)$ will read,

$$\left.\begin{aligned} ReR(r) &= \pm\frac{1}{\sqrt{6.4}}\cos\alpha(r) \\ ImR(r) &= \mp\frac{1}{\sqrt{6.4}}\sin\alpha(r). \end{aligned}\right\} \quad [9.3.8]$$

Now by considering A and B, the metric coefficients, being complex functions because of the complexity of R, we get, by using [7.1.9] the following expressions for real and imaginary parts of $A(r)$, i.e.

$$\mathrm{Re}A(r) = \frac{-z(r)[1 + \frac{Kr^2}{4}\cos\alpha(r)]}{1 + \frac{Kr^2}{2}\cos\alpha(r) + \frac{K^2r^4}{16}} \qquad [9.3.9]$$

$$\mathrm{Im}\,A(r) = \frac{Kr^2\sin\alpha(r)}{4 + Kr^2\cos\alpha(r)}\,\mathrm{Re}A(r) \qquad [9.3.10]$$

where

$$Z(r) \equiv 1 + \frac{r\ddot{\alpha}(r)}{\dot{\alpha}(r)} \qquad [9.3.11]$$

Further, we denote

$$\mathcal{K}(r) \equiv \frac{r_\infty^4\dot{\alpha}^2(\infty)}{r^4\dot{\alpha}^2(r)} = \begin{cases} 1 & r=r_\infty \\ \text{finite} & r<r_\infty \end{cases} \qquad [9.3.12]$$

where $r_\infty \equiv \lim_{r\to\infty} r$, i.e. r_∞ is sufficiently large value for r.

Then, by using the flat space condition that $A(\infty)B^{-1}(\infty) = 1$, and the relation [7.1.3], we obtain for the metric coefficient $B(r)$ the following expressions:

$$\mathrm{Re}B(r) = \mathcal{K}(r)\left[\cos(\alpha(\infty)-\alpha(r)) + \sin(\alpha(\infty)-\alpha(r))\frac{\sin\alpha(r)}{\left[\frac{4}{Kr^2} + \cos\alpha(r)\right]}\right]\mathrm{Re}A(r) \qquad [9.3.13]$$

and,

$$\mathrm{Im}B(r) \equiv \mathcal{K}(r)\left[\cos(\alpha(\infty)-\alpha(r))\frac{\sin\alpha(r)}{\left[\frac{4}{Kr^2} + \cos\alpha(r)\right]} - \sin(\alpha(\infty)-\alpha(r))\right]\mathrm{Re}A(r) \qquad [9.3.14]$$

Now since by [9.3.9] and [9.3.10] we have:

$$A(r) = \left[1 + i \frac{\sin\alpha(r)}{\frac{4}{Kr^2} + \cos\alpha(r)} \right] \mathrm{Re}A(r) \qquad [9.3.15]$$

therefore,

$$A(0) = \mathrm{Re}A(0) = Z(0) \qquad [9.3.16]$$

and,

$$A(\infty) = \left[1 + i \frac{\sin\alpha(\infty)}{\cos\alpha(\infty)} \right] \mathrm{Re}A(\infty). \qquad [9.3.17]$$

But to satisfy the flatness condition we ought to set:

$$\left. \begin{aligned} \sin\alpha(\infty) &= 0 \\ \cos\alpha(\infty) &= \pm 1 \end{aligned} \right\} \qquad [9.3.18]$$

and

$$A(\infty) = \mathrm{Re}A(\infty) = 1 \qquad [9.3.19]$$

By [9.3.9] and [9.3.11] this would make $Z(\infty)$ behave as:

$$\pm Z(\infty) = \pm 2 + \frac{K}{4} r_\infty^2 \qquad [9.3.20]$$

$$\pm \frac{r_\infty \ddot{\alpha}(\infty)}{\dot{\alpha}(\infty)} = \frac{K}{4} r_\infty^2 \pm 3 \qquad [9.3.21]$$

We have also, owing to [9.3.10] and [9.3.18], the following

$$\mathrm{Im}A(0) = 0 \qquad [9.3.22]$$

$$\mathrm{Im}A(\infty) = 0 \qquad [9.3.23]$$

Further, by using [9.3.4] and [9.3.18] together with [9.3.12], [9.3.19] in [9.3.13] and [9.3.14] one gets:

$$\left. \begin{aligned} \mathrm{Re}B(0) &= \mathcal{K}(0)Z(0)\cos(\alpha(\infty)-\alpha(0)) \\ \mathrm{Im}B(0) &= \mathcal{K}(0)Z(0)\sin(\alpha(\infty)-\alpha(0)) \end{aligned} \right\} \qquad [9.3.24]$$

and,

$$\left.\begin{aligned} \mathrm{Re}B(\infty) &= \mathrm{Re}A(\infty) = 1 \\ \mathrm{Im}B(\infty) &= 0 \end{aligned}\right\} \qquad [9.3.25]$$

where the trigonometric functions in [9.3.24] will have, due to [9.3.4,18] the following values.

$$\left.\begin{aligned} \cos(\alpha(\infty) - \alpha(0)) &= \cos\alpha(\infty)\cos\alpha(0) + \sin\alpha(\infty)\sin\alpha(0) = 0 \\ \sin(\alpha(\infty) - \alpha(0)) &= \sin\alpha(\infty)\cos\alpha(0) - \cos\alpha(\infty)\sin\alpha(0) = \pm 1 \end{aligned}\right\} \qquad [9.3.26]$$

We notice from [9.3.13,14,15] that if $\mathrm{Re}A(r) = 0$ then both $A(r)$ and $B(r)$ will automatically vanish, which is meaningless. This implies that $\mathrm{Re}A(r)$ should be always nonzero.

Now to calculate A and B we need to obtain expressions for $Z(r)$ and $\mathfrak{K}(r)$. For this we use [9.3.2] into [9.2.7] to yield the following two coupling equations:

$$\left.\begin{aligned} &Kr^2\dot{\alpha}^3(r) - K^2r^3\sin\alpha(r)\dot{\alpha}^2(r) + K[3Kr^2\cos\alpha(r) + 10]\dot{\alpha}(r) = 4\sin\alpha(r) \\ &Kr(1+Kr^2\cos\alpha(r)]\dot{\alpha}^2(r) + 3K^2r^2\sin\alpha(r)\dot{\alpha}(r) = 4\cos\alpha(r) + Kr^2 \end{aligned}\right\}$$

[9.3.27]

By multiplying the 2nd equation of [9.3.27] by $\dot{\alpha}(r)$ and then by solving the two equations together one can eliminate $\dot{\alpha}^3(r)$, then by again eliminating $\dot{\alpha}^2(r)$ by the 2nd equation of [9.3.27] it yields:

$$\dot{\alpha}(r) = 4\,\frac{u(r)}{v(r)}\,\sin\alpha(r) \qquad [9.3.28]$$

where we denote

$$\left.\begin{aligned} u &= u(r) \equiv 5kr^2 \cos\alpha(r) + k^2r^4 + 1 \\ v &= v(r) \equiv y(r) + y_1(r)\cos\alpha(r) + y_2(r)\cos^2\alpha(r) \\ y &= y(r) \equiv 12k^3r^4 + kr^3 + 10K \\ y_1 &= y_1(r) \equiv k^2r^5 + 13k^2r^2 + 4r \\ y_2 &= y_2(r) \equiv 4kr^3 - 9k^3r^4 \end{aligned}\right\} \qquad [9.3.29]$$

By differentiating [9.3.28] one gets

$$\frac{\ddot{\alpha}(r)}{\dot{\alpha}(r)} = \left[\frac{\cos\alpha}{\sin\alpha} - \frac{5Kr^2\sin\alpha}{u} + \frac{2y_2\cos\alpha\sin\alpha + y_1\sin\alpha}{v}\right]\dot{\alpha}(r) + \frac{10kr\cos\alpha + 4k^2r^3}{u} - \frac{X + X_1\cos\alpha + X_2\cos^2\alpha}{v} \qquad [9.3.30]$$

With

$$\left.\begin{aligned} X &= X(r) \equiv 48k^3r^3 + 3kr^2 \\ X_1 &= X_1(r) \equiv 5k^2r^4 + 26k^2r + 4 \\ X_2 &= X_2(r) \equiv 12kr^2 - 36k^3r^3 \end{aligned}\right\} \qquad [9.3.31]$$

The relation [9.3.28] yields

$$\dot{\alpha}(\infty) = 0 \qquad [9.3.32]$$

Relations [9.3.28, 30] can be used to compute $\mathcal{K}(r)$ and $Z(r)$ defined by [9.3.12, 11] respectively, and thence the real and imaginary parts of both A and B can be computed.

A preliminary study of the scalar curvature R and the metric coefficients A and B is made. The curves for the real and the imaginary parts of R(r), A(r) and B(r) are illustrated in Figures 9.I, II and III respectively. For a qualitative description we exploited the Euler's computational method run on a Fortran programme. Although this method is not quite accurate for these kind of equations we reckon it will be fairly sufficient for assessment within a short range of r.

(9.4) Discussion

The resulting expression for R, A and B in the preceding section are derived from equations based on a very restricted form of the Lagrangian [9.1.1], viz, $\mathcal{L} = R^2(r)$. This form is essentially irrelevant for asymptotically large r from the source, since as we mentioned earlier, it may represent the strong-field contribution. Thus, for our present purpose distances lying just beyond the range of the strong gravitational field of the point mass may be regarded as asymptotically far from the gravitational centre. This is what we meant by subjecting A and B to the flatness condition [9.3.19, 25]. It is interesting to notice from these relations and from Figures 9-II, III that it is the real parts of A and B which can be subjected to the flatness limit. As for the imaginary parts of R, A and B, Figures 9-I, IIb and IIIb respectively reveal that they describe a short range field in the proximity of the source. The inconsistency between Figure 9-IIIa and the flatness of Re B(r) at large r is due to the limitation of the computational method we used, caused by the accumulation of the error. In contrast with Schwarzschild's metric the curvature is finite at r = 0 and, in view of [9.3.1] and Figure 9-I, is purely imaginary.

Thus, we conclude this chapter by the following:

1. The gravitational equations based on the Lagrangian $\mathcal{L} = R^2$ in a static isotropic space lead to the complexity of the metric.

2. The imaginary part of this metric would correspond to a short-range gravitational field.

3. The presence of such a short-range field cause the metric to be essentially nonsingular.

Finally, we admit that due to the inaccuracy of the computational method used by us, the curves illustrated in Figures 9-II, III will not be quite reliable for assessment of the theory with a complex metric. However, analytically the boundary values of R, A and B consistently ensure the correctness of this theory.

We believe that a further investigation of the complex metric with the aid of a more accurate computational method would lead to more interesting results.

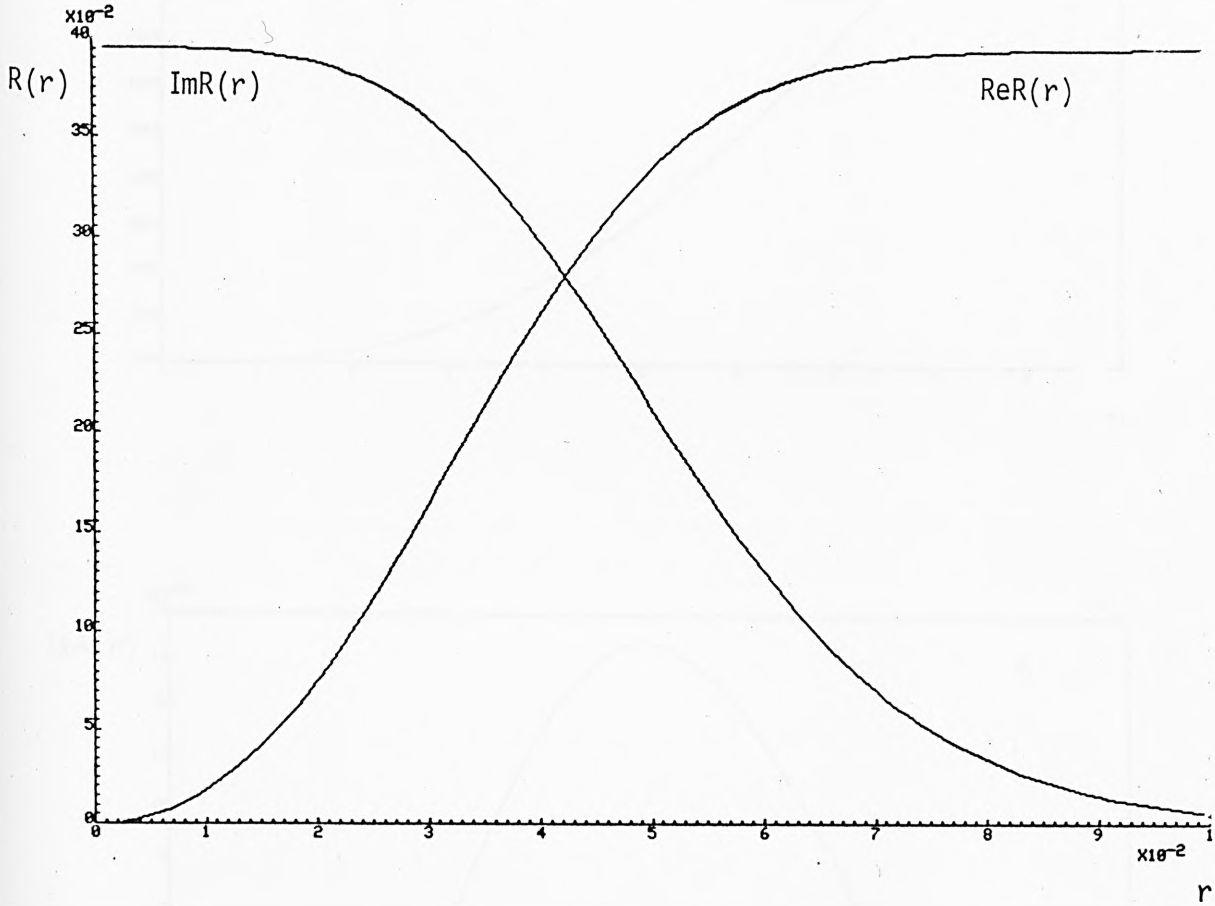

Figure 9–I The scalar curvature R in the vicinity of the gravitational source.

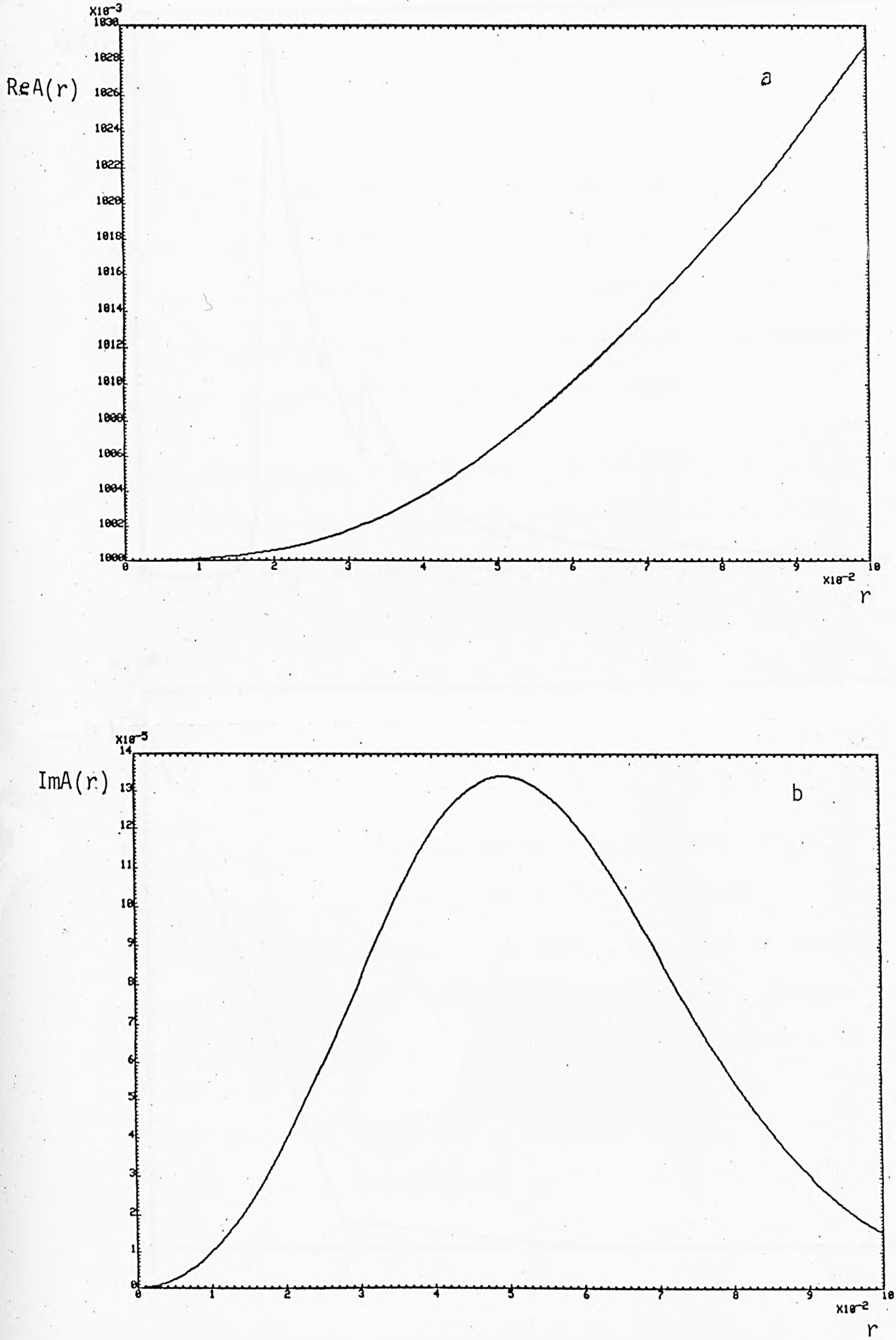

Figure 9-IIa,b The metric component A in the vicinity of the gravitational source.

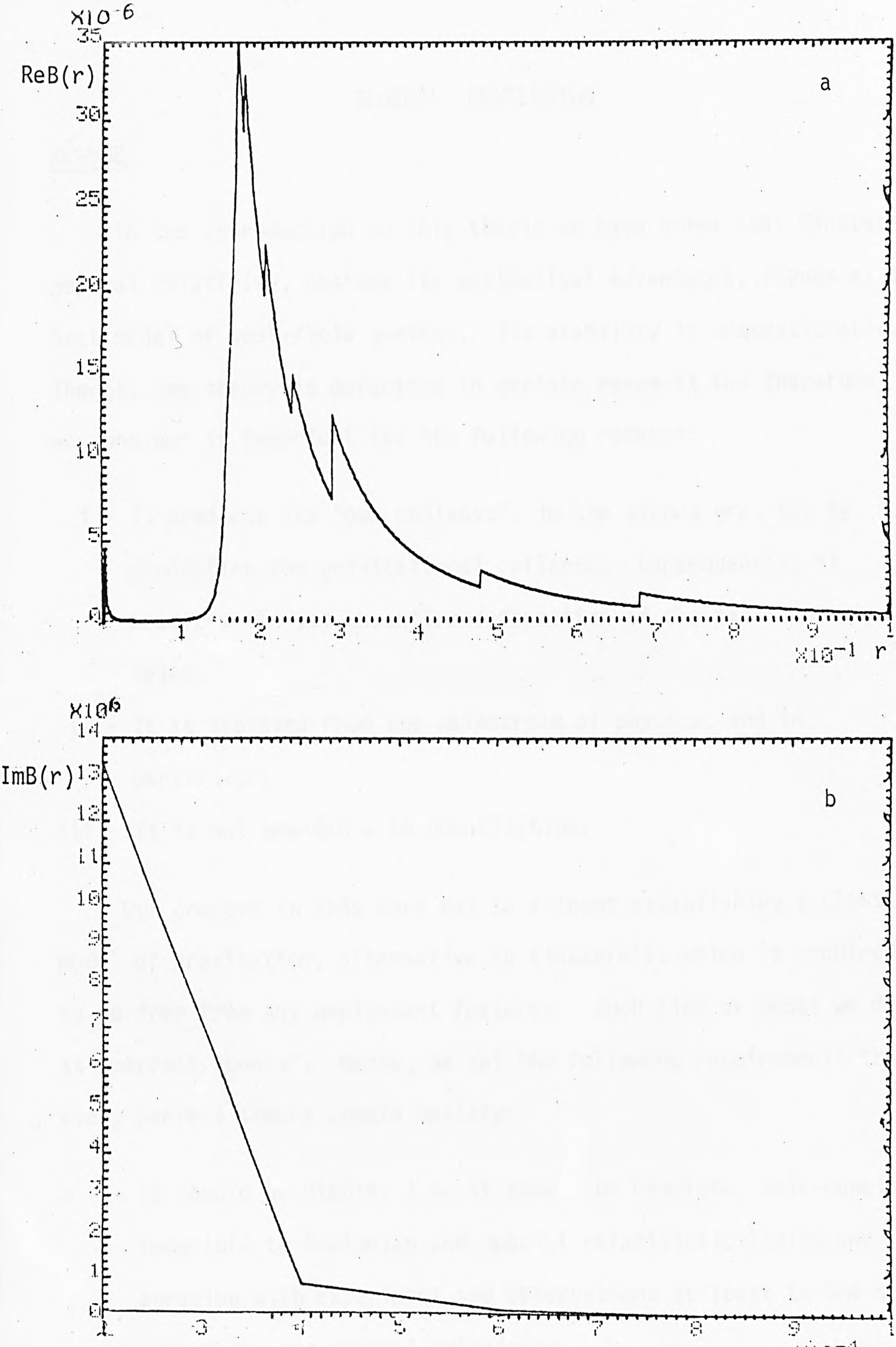

Figure 9-IIIa,b The metric component B in the vicinity of the gravitational source.

GENERAL CONCLUSION

RÉSUMÉ

In the introduction to this thesis we have noted that Einstein's general relativity, besides its aesthetical advantages, stands as the best model of weak-field gravity. Its viability is unquestionable. Though, the theory is defective in certain respects and therefore we consider it imperfect for the following reasons:

i - It predicts its "own collapse", in the strong gravity, by predicting the gravitational collapse. Consequently, it predicts the yet unconfirmed formation of the exotic black holes.

ii - It is isolated from the mainstream of physics, and in particular;

iii - It is not amendable to quantization.

Our concern in this work was to attempt establishing a classical model of gravitation, alternative to Einstein's, which is required to be free from any unpleasant features. Such kind of model we define as "perfect theory". Hence, we set the following requirements that every perfect theory should satisfy:

a - It should be viable, i.e. it should be complete, self-consistent, reducible to Newtonian and special relativistic limits and agreeing with experiment and observations at least to the same extent as does general relativity.

b - It should not contradict itself by predicting the breakdown of its laws, i.e. should not be singular.

c - It should be quantizable i.e. it should not lead to arbitrary or ambiguous description when subjected to quantization.

d - It should share with the rest of physics common grounds i.e. should be not isolated from the general picture of physical laws.

A theory coping with these requirements will be certainly challenging all the existing gravitational models.

Most of the various attempts introduced in Chapter Four, at constructing a candidate theory congruent with the afore-set conditions, achieved no significant success. However, some like Kilmister-Yang model, and the limiting curvature theory, whose viability is not disputable, are quite promising.

Then in the light of our previous critical views and assessments made mainly in the last sections of each chapter of this thesis we came to the following important conclusions. These conclusions summarize the previously discussed ideas and lay criteria that should be taken into account in establishing a perfect theory of gravitation. Thus, we conclude that:

1 - Although general relativity is imperfect it is still regarded as the best basis for any further modification.

2 - Any generalization of GTR should be within the picture of Riemannian geometry.

3 - Gravitational equations of the highest possible (i.e. the 4th) differential order should be exploited.

4 - The successfully modified version of general relativity should utilize a Lagrangian containing both linear and nonlinear terms of the scalar curvature R.

5 - Quantum gravity may be reached not by directly subjecting GTR to quantization, but by quantizing its generalized form.

6 - The resulting metric space should allow a complex structure in which the imaginary part of R would correspond to strong-field quantum effects.

7 - Such a complexity is caused by the contribution of the quadratic term in the Lagrangian, $\mathcal{L} = \alpha R^2 + \beta R + \gamma$.

8 - This Lagrangian has the most physical and the most rational form. As it has been elucidated in Sections (8.2, 3) the Lagrangian coefficients are physically inter-connected. Therefore, by a certain choice of them, the metric may reduce to Einstein and Schwarzschild spaces in the weak gravity or, for another choice it would fairly describe strong gravity without exhibiting any pathological behaviour. This property of the Lagrangian coefficients suggests that they would represent classical limits of quantum mechanical quantities as we have advocated that in (8.6).

9 - In view of No. 8 a Lagrangian dominated by the quadratic R-term could bridge the gap with other fields' theories and with quantum mechanics. As we pointed out in (7.5) such a Lagrangian will not be relevant to describe the spacetime at asymptotic distances from the gravitational centre.

Now we notice that the gravitational model developed by us throughout the last five chapters of this thesis is consistently adapted to the above-counted criteria and ideas. Therefore it is worthy of being considered, to a certain extent, as a perfect classical theory of gravitation. Theories which partially agree with these criteria will have limited merits.

AN OUTLOOK

Physics does not come to an end. We learn from the history of physics that there is no "absolutely perfect" theory. A physical theory is called perfect with respect to the utmost knowledge human intellect can produce and comprehend. Every law of physics considered as general at one time will need to be further generalized after it reaches the boundary of its domain of applicability and so forth. It is this chain of generalizations which makes the history of science continue.

As we have noted in the introduction to this thesis, the early decades of this century had witnessed a great revolution in physics. Newtonian mechanics was generalized in two directions, quantum theory, and special relativity. The latter was immediately generalized into GTR. It is therefore very rational to think of a further modification that may accommodate a generalized version of "general" relativity with say, quantum theory. Hence we believe that the generalized metric which we have founded in the present work is a step forward that will underlie certain prospects towards success. However, though our model has been generalized, further generalization would open new horizons for exploring the highly complicated nature of the gravitational field. We propose here four avenues for a further study of our generalized metric, i.e.

I - By employing the most general nonlinear Lagrangian (5.4.24).

II - By studying the complex metric of Chapter Nine in more details.

III - By departing from the static isotropic coordinates.

IV - By subjecting the model to quantization.

Then we expect that more information about the subtle features of the gravitational field may be revealed which may generate new ideas for a new theory. This will be a new threshold in the realm of physical laws.

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