HAMILTON'S PRINCIPLE APPLIED TO DYNAMIC ANALYSIS OF BRIDGES

Thesis submitted for the degree of Doctor of Philosophy

by Edward Goodwin

Department of Civil Engineering
The City University
London

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SYNOPSIS

This investigation is concerned with the elastic response of bridge girders to high speed travelling loads. The loads are treated as masses in contrast to the common practice of treating them as forces.

A novel method of calculation is developed, based on Hamilton's Principle of Least Action. The initiating idea of this thesis was to examine the possibilities of employing this principle in engineering dynamics.

The types of bridge considered are as follows:

a) Simply supported beam

b) End cantilevers with central simply supported main span.

The loads considered comprise both single vehicles and trains of vehicles.

Possible benefits of "humping" the carriageway or railway track, to reduce the dynamic bending moments, are investigated.

Effects of damping, and elastic suspension of vehicles, are also evaluated.

CONTENTS

LIST OF SYMBOLS

L	Span length
t	Time
u	Speed of travelling load
x	Position co-ordinate - distance along a beam
y	Downward elastic deflection of a beam
z,w	Downward elastic deflections of cantilevers
Y,Z,W	Mid span value of y, free end values of z and w
$\mathcal{Z}$	Elastic deflection of vehicle springs (Chapter 7 only)
Y_o, Z_o, W_o	Values of Y,Z,W, at the beginning of a stage in a multi-stage calculation
m	Mass of a beam or girder, per unit length
M	Mass of a travelling load
EI	Flexural rigidity of a beam or girder
V	Elastic strain energy
P	Gravitational potential energy, P_1 for m, P_2 for M.
T	Kinetic energy, T_1 for m, T_2 for M.
n	Stage number in a multi stage calculation
Δ	Maximum possible static deflection
D	Dynamic Factor, i.e. ratio of dynamic deflection to Δ
β	A parameter to indicate the speed of a travelling load (see para. 1.1)
γ	Ratio of load mass to total beam mass (M/ml)
A,B,C	Coefficients in a displacement/time function
a,b,c	Amplitudes of small variations in displacement/time functions
q	Indicates the time occupied by the above variations (π/q)

δV, δP, δT	Changes in V,P, and T, due to the introduction of a small variation in a displacement/time function
y_M, w_M, z_M	Downward elastic deflection of a beam at the instantaneous position of a moving load M
ω	"Circular" frequency of a vibration = frequency X 2Π
t_p	Periodic time of a vibration cycle = $2\Pi/\omega$
k	Stiffness of a spring or an elastic structure
E_o	Total energy of an elastic vibration
N	Total number of stages in a computation
ξ	Ratio of stiffness of vehicle suspension to midspan stiffness of beam
ν	expresses the duration of a variation as a proportion of a half period of natural vibration

INTRODUCTION

This work was embarked upon in order to study the feasibility of employing Hamilton's Principle as a basis for computer solution of problems in structural dynamics.

It was desired to extend the familiarity of Hamilton's Principle, there being no obvious reason why a Least Action principle should be so rarely employed compared to other concepts such as the Energy Conservation Principle and its derivatives.

The following guide lines were laid down at the outset:

(a) the feasibility study would be based on evaluating the response of bridge girders to travelling loads, and

(b) the travelling loads would be treated strictly as masses. It was felt that this would extend the interest of the results and at the same time, provide a more appropriate challenge for the new method.

As regards item (b) above, the following remarks are relevant. If travelling loads are regarded as masses then the force that each one exerts on the bridge is affected by the vertical accelerations arising from the dynamic deflections. This means that the travelling load exerts a continually varying force as it traverses the span and at certain points this force may differ considerably from the constant gravity force on which such analysis is often based. Furthermore, the magnitude of this discrepancy will increase considerably with the speed of transit.

In point of fact it became apparent, at a fairly early stage in the investigations, that the improvement in logic due to treating loads as masses would only be significant in the case of very high speeds indeed (perhaps in excess of current practice or probability). Nevertheless, this feature was retained because an idea of "track humping" to reduce dynamic deflections had been evolved and this could not be evaluated unless the loads were treated as masses. Also it was desired to investigate the case where the load is a vehicle with elastic suspension.

In addition to the computational application of Hamilton's Principle some further investigation was conducted with regard to its manifestation in natural vibration theory.

CHAPTER 1

PRELIMINARY CONSIDERATIONS

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1.1 ORIGINS OF VERTICAL ACCELERATIONS

If a bridge is vibrating, whilst the load is thereupon, it follows that the load will experience accelerated motion in a direction perpendicular to its direction of travel. Hence there will be a continuous variation of the downward force which the load exerts on the structure.

However, there are two other sources of vertical accelerations which may be equally significant in their contribution to this variation of imposed force:

(a) The path of the moving load will be curved in a vertical plane due to the elastic curvature produced by the load. This introduces a centripetal component of acceleration.

(b) Due to the change in gradient associated with the elastic dynamic deflections the load will experience a Coriolis component of acceleration.

If the dead load deflection has not been cambered out, the accelerations referred to in (a) above, may be increased considerably. On the other hand the structure may be over cambered or the carriageway may be "humped" relative to the deck. In such cases the centripetal acceleration may be reduced (even to a negative value) with a corresponding reduction in dynamic stresses due to fast moving loads. One of the principle benefits of developing a method which treats loads as masses, as distinct from forces assumed to be unaffected by vertical acceleration components, is the capability to evaluate such considerations.

1.2 CRITICAL SPEED OF TRAVELLING LOADS

The magnitude of dynamic stresses will depend on the speed of the vehicle which causes them. It is to be expected that higher vehicle speeds will give rise to higher dynamic stresses.

However, at very high vehicle speeds there may be insufficient time for the elastic response to be initiated, which implies that at some point the dynamic stresses begin to decrease with further increase of vehicle speed.

The need arises for some means of describing the speed of a vehicle in terms of the elastic characteristics of a bridge. This can be done most conveniently by measuring time in terms of the natural period of vibration of the structure.

Let τ be the time taken for the load to cross the bridge

Let t_p be the periodic time of free natural vibration of the structure (in fundamental mode)

Then $\beta = \tau/\frac{1}{2}t_p$

β will be adopted throughout this treatise to indicate the speed of travel of a moving load. High values of β will represent relatively slow speeds whilst, at the other extreme, β=0 would imply an infinitely high speed. In the first case stresses will be little greater than those due to a stationary load whilst in the second case the stresses must be zero because the load acts on the bridge for zero time.

In between these two extremes there must be a value of β which will cause the greatest stresses which can possibly be produced. The speed of travel of the load corresponding to this value of β will be referred to as a "critical speed".

A convenient way of defining β for any particular case will now be stated:

If the speed of the load is u and the span of the bridge is L:

$\tau = L/u$ (transit time).

Also $\frac{1}{2}t_p = \Pi/\omega$ where ω has its usual significance in vibration theory.

So, finally:

$$\beta = \omega L/\Pi u \tag{1.1}$$

Possible values of β for real bridges

Critical values of β are found to be 2.5 or less which, in conjunction with conventional bridges, implies vehicle speeds considerably beyond current expectations.

It is believed, on a basis of rudimentary design calculations, that $\beta = 3.0$ is the lowest value that could be anticipated in practice. Even then it is necessary to invoke some extreme assumption such as a speed of 200 km/hr or construction material of exceptionally high strength/modulus ratio.

1.3 POSSIBLE ACCELERATION DUE TO ELASTIC CURVATURE IN A REAL GIRDER

The radius of curvature of an elastic beam at a point where the bending moment is B_M is given by standard theory as:

$$R = \frac{B_M}{EI} = \frac{E.c}{\sigma}$$

where c is the half depth of the beam and σ is the maximum bending stress at the point in question.

The centripetal acceleration experienced by a load passing over this point at speed u will be:

$$\frac{u^2}{R} = \frac{u^2\sigma}{E.c} \qquad (1.2)$$

this being an upward acceleration causing an increase in the force exerted on the beam by the load.

If it is assumed that the stress σ is due to live load only, and that the beam is steel, then a probable value for σ/E is

$$80 \text{ N/mm}^2 \div 2 \times 10^5 \text{ N/mm}^2 = 4 \times 10^{-4}.$$

Also, if the highest plausible vehicle speed is taken to be 200 km/hr then u = 56m/sec and equation (1.2) becomes:

$$\frac{u^2}{R} = \frac{56^2 \times 4 \times 10^{-4}}{C} = \frac{1.24}{C} \qquad (1.3)$$

To examine the worst case it will be assumed that a beam one metre deep is the smallest that need be envisaged. Then c = 0.5 and equation (1.3) gives:

$$\frac{u^2}{R} = \frac{1.24}{0.5} = 2.48 \text{ m/sec}^2 = 0.25g.$$

Accelerations much higher than 0.25g are found in some critical cases (see graph 4.2 for example) but it must be remembered that centripetal acceleration is only one of the three components discussed in para. 1.1.

Another relevant point is that centripetal and Coriolis acceleration components are both proportional to the square of the vehicle speed (i.e. proportional to $1/\beta^2$) and that graph 4.2 refers to $\beta = 1.60$ compared to a probable maximum practical value $\beta = 3.0$, as already suggested in para 1.2.

CHAPTER 2

HAMILTON'S PRINCIPLE

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2.1 There have been, traditionally, two alternative approaches to the solution of problems in engineering dynamics, which begin with one or other of the following procedures:

(a) writing equations of motion directly based on Newton's Second Law of Motion, or

(b) writing energy equations based on the conservation principle.

It will now be proposed that Hamilton's Principle offers a third basic method of approach to such problems, although it appears to have been employed to a relatively negligible extent. It is one of the main aims of this treatise to investigate the viability of this proposition.

2.2 The types of energy involved are elastic strain energy (V), kinetic energy (T), and gravitational potential energy (P). Hamilton's Principle is expressed in terms of these quantities, but it is quite different to other energy concepts in that an integration, with respect to time, is involved. The basic statement of the Principle is that the following integral has a stationary value for any time limits t_1 and t_2:

$$\int_{t_1}^{t_2} (T - V - P)dt$$

To apply this concept to problem solution it is necessary to introduce a small variation into the displacement/time function and then to employ the fact that, since the Hamiltonian integral has a stationary value, the variation will have a negligible effect thereupon.

A suitable form of variation is $\delta x = a.\sin qt$ where δx is a small modification to the displacement x, 'a' is a small amplitude, and q defines the time occupied by the variation (Π/q).

As a result of this variation the energy quantities will alter by amounts δT, δV, and δP the first two involving terms in a^2 as well as 'a'. Since the variation is small these a^2 terms may be ignored and the aggregate of the other terms will have no effect on the value of the Hamiltonian integral. In terms of this procedure Hamilton's Principle can be stated as follows:

$$\int_0^{\Pi/q} (\delta T - \delta V - \delta P)dt = 0 \tag{2.1}$$

The integral of energy with respect to time may be regarded as a physical quantity in its own right which is known as <u>Action</u>.

The stationary value of the Hamiltonian integral is commonly a minimum and the principle could therefore be more fully entitled Hamilton's Principle of Least Action.

2.3 Although not relevant to the problems in view, it is interesting to note an example of the wider significance of the Least Action concept. For instance, consider the case of a body of mass M in motion with constant velocity v, so that the only type of energy involved is kinetic energy T. The displacement time function with a small variation will be:

$$x = vt + a \sin qt$$

velocity $\dot{x} = v + q\, a \cos qt$

$$T = \tfrac{1}{2}M(\dot{x})^2 = \tfrac{1}{2}M(v^2 + 2q\, a\, v \cos q\, t + q^2 a^2 \cos^2 q\, t)$$

So the Action in the period occupied by the variation is:

$$A = \int_0^{\Pi/q} T dt = \frac{M\Pi v^2}{2q} + 0 + \frac{M\Pi q a^2}{4}$$

The term involving 'a' has vanished (because $\int_0^{\Pi/q} \cos qt\, dt = 0$) leaving the other term as the only effect of introducing the variation. This remaining term will obviously always be positive and therefore it represents an increase in Action. Hence, the natural (Newtonian) motion at constant velocity can be regarded as a state of Least Action.

2.4 With regard to the characteristics of the variation a.sin qt it is relevant to point out that the situation of the mass M, after the variation has occurred, will be exactly the same in both space and time as if it had never been introduced.

Since the proposed variation is cyclic it could be allowed to extend over any number of half cycles without violating the aforementioned condition. However, it has been found, from the problems attempted in this thesis, that a single half cycle of a.sin qt is a suitable variation to employ in connection with Hamilton's Principle.

2.5 PROOF OF HAMILTON'S PRINCIPLE FOR A SINGLE MASS PARTICLE

Consider a particle of mass M acted on by a force F and restrained by a spring of stiffness k. If x is the displacement of the particle at any time t, after the spring begins to extend, then the equation of motion is:

$$F - k.x = M \frac{d^2x}{dt^2}$$

Multiplying this equation through by a.sin qt and integrating with respect to time, during the period of the variation, gives:

$$\underbrace{M.a \int_o^{\Pi/q} \frac{d^2x}{dt^2} \sin qt\, dt}_{I_1} + \underbrace{k.a \int_o^{\Pi/q} x \sin qt\, dt}_{I_2} - \underbrace{F.a \int_o^{\Pi/q} \sin qt\, dt}_{I_3} = 0 \qquad (2.2)$$

Consider the first integral in equation (2.2)

Integrating by parts:

$$I_1 = M.a \left[\frac{dx}{dt} \sin qt - q \int \frac{dx}{dt} \cos qt\, dt \right]_o^{\Pi/q}$$

$$\text{i.e.} \quad I_1 = - M\, aq \int_o^{\Pi/q} \frac{dx}{dt} \cos qt\, dt$$

Now consider the kinetic energy change due to varying x to x + a.sin qt. The total kinetic energy would become:

$$T + \delta T = \tfrac{1}{2}M\left(\frac{dx}{dt} + qa \cos qt\right)^2$$

$$\therefore \delta T = M\,aq\,\frac{dx}{dt}\cos qt \qquad \text{(neglecting } a^2\text{)}$$

Comparing this with I_1 in equation (2.2) it can be seen that:

$$I_1 = -\int_0^{\Pi/q} \delta T.dt$$

Now consider the change in strain energy V due to introducing the small variation:

$$V + \delta V = \tfrac{1}{2}k(x + a \sin qt)^2$$

$$\therefore \delta V = xka \sin qt \qquad \text{(neglecting } a^2\text{)}$$

Comparing this with I_2 in equation (2.2) it can be seen that:

$$I_2 = \int_0^{\Pi/q} \delta V\, dt$$

Now consider the change in work done by the force F due to the variation:

$$P + \delta P = -F(x + a \sin qt)$$

$$\therefore \delta P = -F\, a \sin qt$$

Comparing this with I_3 in equation (2.2) it can be seen that:

$$I_3 = \int_0^{\Pi/q} \delta P\, dt$$

So, finally, equation (2.2) may be written:

$$\int_{o}^{\Pi/q} (\delta T - \delta V - \delta P)dt = 0$$

which is a statement of Hamilton's Principle as in equation (2.1).

Now consider the case of a particle having its motion specified with reference to Cartesian co-ordinates, as follows:

$$F_x - k.x = M \frac{d^2x}{dt^2}$$

$$F_y - k.y = M \frac{d^2y}{dt^2}$$

$$F_z - k.z = M \frac{d^2z}{dt^2}$$

It is easily shown that equation (2.1) is applicable for any <u>one</u> of the following variations wherein a,b, and c are small:

$$x_a = x + a \sin qt$$

$$y_b = y + b \sin qt$$

$$z_c = z + c \sin qt$$

If all three variations are employed, then three independent equations will be obtained.

2.6 PROOF OF HAMILTON'S PRINCIPLE FOR A SYSTEM

A system has been chosen, for this purpose, which has some relevance to the problems which will be tackled in later chapters.

It is depicted in Figure 2.1 and consists of a mass M rolling, with constant speed u, along a rigid beam OA of mass m per unit length. This beam is supported, at one end only, by a spring controlled hinge, the restraining moment due to the spring being k per unit angle α. It will be assumed that α remains a small angle.

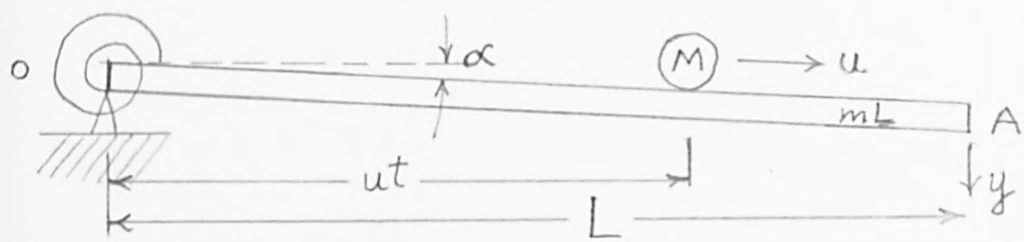


FIGURE 2.1

Let y_0 be the initial deflection at A due to the self weight mgL of the beam, and let y be the dynamic deflection.
Let y_M be the dynamic deflection of the beam at the instantaneous position of M.

$$\text{Then } y_M = y\frac{ut}{L}$$

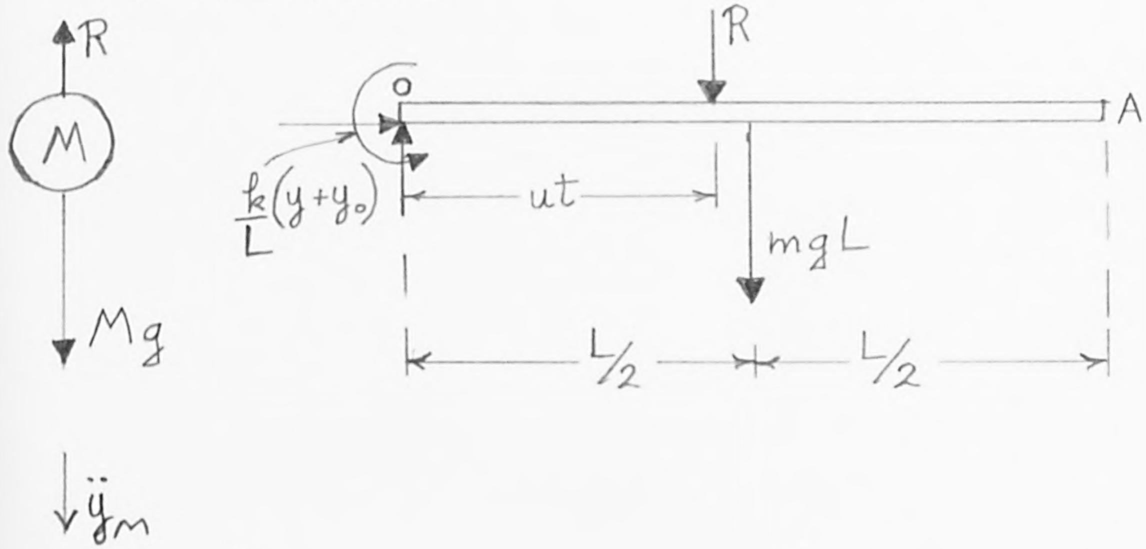
Double differentiation of this will give the vertical acceleration experienced by M, as follows:

$$\dot{y}_M = \dot{y}\frac{ut}{L} + y\frac{u}{L}$$

$$\ddot{y}_M = \frac{2\dot{y}u}{L} + \ddot{y}\frac{ut}{L}$$

The first term is a Coriolis acceleration as referred to in para. 1.2, whilst the second term is the acceleration of the beam itself at the point where M is instantaneously positioned.

"Free body" diagrams of the load and the beam are shown in Figures 2.2 (a) and (b).



(a) (b)

FIGURE 2.2.

From "free body" (a):

$$M\ddot{y}_M = M\left(\frac{2\dot{y}u}{L} + \frac{\ddot{y}ut}{L}\right) = Mg - R \qquad (2.3)$$

Considering moments about O in "free body" (b):

$$Rut + \frac{mgL^2}{2} - \frac{k}{L}(y + y_o) = \frac{mL^3}{3} \cdot \frac{\ddot{y}}{L} \qquad (2.4)$$

Now combine equations (2.3) and (2.4) to eliminate R and hence obtain the equation of motion for the system:

$$\ddot{y}\left(\frac{Mu^2t^2}{L^2} + \frac{mL}{3}\right) + \dot{y}\left(\frac{2Mu^2t}{L^2}\right) + \frac{k}{L^2}(y+y_o) = \frac{Mgut}{L} + \frac{mgL}{2} \qquad (2.5)$$

Now suppose the dynamic displacement y is modified by a small variation to $y + \delta y = y + a.\sin qt$.

Following the procedure used in para. 2.5 each term of the equation of motion (equation 2.5) will be multiplied by a.sin qt and then integrated from zero to Π/q giving:

$$\underset{①}{\frac{Mu^2a}{L^2}\int_o^{\Pi/q} \ddot{y}t^2 \sin qt\, dt} + \underset{②}{\frac{2Mu^2a}{L^2}\int_o^{\Pi/q} \dot{y}t \sin qt\, dt} + \underset{③}{\frac{mLa}{3}\int_o^{\Pi/q} \ddot{y} \sin qt\, dt}$$

$$+ \underset{④}{\frac{ka}{L^2}\int_o^{\Pi/q} (y+y_o)\sin qt\, dt} - \underset{⑤}{\frac{Mgua}{L}\int_o^{\Pi/q} t \sin qt\, dt} - \underset{⑥}{\frac{mgLa}{2}\int_o^{\Pi/q} \sin qt\, dt}$$

$$= 0 \qquad (2.6)$$

It is now necessary to show that equation (2.6) is equivalent to:

$$-\int_o^{\Pi/q} (\delta T - \delta V - \delta P)dt = 0$$

where δT, δV, and δP are the changes in kinetic energy, strain energy and potential energy due to introducting the variation a.sin qt into the dynamic displacement y of the system in Figure 2.1.

Consider the strain energy V of the system with the variation included:

$$V + \delta V = \frac{k}{2L^2}(y+y_o + a \sin qt)^2$$

$$\therefore \delta V = \frac{ka}{L^2}(y+y_o)\sin qt$$

Comparing δV with term (4) in equation (2.6) it can be seen that term (4) is equivalent to

$$\int_{o}^{\Pi/q} \delta V \, dt$$

Now consider the gravitational potential energy P of the system. With the variation included:

$$P + \delta P = -Mg(y + a \sin qt) \frac{ut}{L} - \frac{mgL}{2} (y + a \sin qt)$$

$$\delta P = \frac{-Mgua}{L} t \sin qt - \frac{mgLa}{2} \sin qt$$

Comparing δP with terms (5) and (6) in equation (2.6) it can be seen that these two terms together are equivalent to

$$+ \int_{o}^{\Pi/q} \delta P \, dt$$

Since $\frac{ky_o}{L} = \frac{mgL^2}{2}$ from the initial static equilibrium of the system it follows that, in equation (2.6), the y_o term within term (4) will cancel with term (6) . Hence, in practical calculations, for systems of this type, it will be a justifiable simplification to ignore both strain energy and potential energy associated with the <u>self weight</u> of the beam. For the same reason the equation of motion (2.5) can also be condensed as below:

$$\ddot{y}\left(\frac{Mu^2t^2}{L^2} + \frac{mL}{3}\right) + \dot{y}\left(\frac{2Mu^2t}{L^2}\right) + y\left(\frac{k}{L^2}\right) = \frac{Mgut}{L} \qquad (2.7)$$

Now consider term ③ in equation (2.6), which is concerned with the inertia of the beam itself. Let the kinetic energy of the beam itself be denoted by T_1.

$$\text{Then } T_1 = \frac{1}{2} \frac{mL^3}{3} \frac{\dot{y}^2}{L^2} = \frac{mL\dot{y}^2}{6}$$

So including the effect of the variation:

$$T_1 + \delta T_1 = \frac{mL}{6} (\dot{y} + qa \cos qt)^2$$

$$\delta T_1 = \frac{mLaq}{3} \dot{y} \cos qt$$

Now take term ③ from equation (2.6) and integrate by parts:

$$\frac{mLa}{3} \int_o^{\Pi/q} \ddot{y} \sin qt \, dt = \frac{mLa}{3} \Big[\dot{y} \sin qt - q \int \dot{y} \cos qt \, dt \Big]_o^{\Pi/q}$$

which is equivalent to $- \int_o^{\Pi/q} \delta T_1 \, dt$

since $\Big[\dot{y} \sin qt \Big]_o^{\Pi/q}$ is zero.

Now consider terms (1) and (2), in equation (2.6), which are concerned with the kinetic energy T_2 of the moving load M.

$$T_2 = \tfrac{1}{2}M\dot{y}_M^{\,2}$$

$$\text{Now } y_M = (y + a \sin qt)\,\frac{ut}{L}$$

$$\dot{y}_M = \frac{yu}{L} + \frac{\dot{y}ut}{L} + a \sin qt\,\frac{u}{L} + qa \cos qt\,\frac{ut}{L}$$

$$T_2 + \delta T_2 = \tfrac{1}{2}M\left(\frac{yu}{L} + \frac{\dot{y}ut}{L} + \frac{u}{L}\,a \sin t + \frac{ut}{L}\,qa \cos t\right)^2$$

$$\delta T_2 = \frac{Mu^2a}{L^2}\,(y \sin qt + \dot{y}t \sin qt + qyt \cos qt + q\dot{y}t^2 \cos qt)$$

$$\text{i.e. } \delta T_2 = \frac{Mu^2a}{L^2}\left(y\left(\sin qt + qt \cos qt\right) + \dot{y}t \sin qt + q\dot{y}t^2\cos qt\right)$$

$$\text{Now } \int(\sin qt + qt \cos qt)\,dt = t \sin qt$$

$$\int_0^{\Pi/q} y(\sin qt + qt \cos qt)dt = \Big[\, yt \sin qt - \int \dot{y}t \sin qt\,dt \,\Big]_0^{\Pi/q}$$

$$= -\int_0^{\Pi/q} \dot{y}t \sin qt\,dt$$

Hence, when δT_2 is integrated from zero to Π/q, the first three terms are collectively zero, leaving only:

$$\int_0^{\Pi/q} \delta T_2 dt = q\int_0^{\Pi/q} \dot{y}t^2\cos qt\,dt \qquad (2.8)$$

Term ① of equation (2.6) gives:

$$\int_0^{\Pi/q} \ddot{y}t^2 \sin qt\, dt = \Big[\underbrace{\sin qt \int \ddot{y}t^2 dt}_{\text{zero}} - q\int \cos qt \int \ddot{y}t^2 dt\, dt\Big]_0^{\Pi/q}$$

Term ② of equation (2.6) gives:

$$2\int_0^{\Pi/q} \dot{y}t \sin qt\, dt = \Big[\underbrace{\sin qt \int \dot{y}t\, dt}_{\text{zero}} - 2q\int \cos qt \int \dot{y}t\, dt\, dt\Big]_0^{\Pi/q}$$

Now $\int \ddot{y}t^2 dt = t^2\dot{y} - 2\int t\dot{y}\, dt$

So the sum of terms ① and ② is:

$$-q\int_0^{\Pi/q} \cos qt(t^2\dot{y} - 2\int t\dot{y}\, dt)dt - 2q\int_0^{\Pi/q} \cos qt \int t\dot{y}\, dt\, dt$$

$$= -q\int_0^{\Pi/q} \dot{y}t^2 \cos qt\, dt$$

$$= -\int_0^{\Pi/q} \delta T_2 dt, \text{ as per equation(2.8)}$$

All the six terms in equation (2.6) have now been examined to establish the complete equivalence of equations (2.6) and (2.1)

CHAPTER 3

ILLUSTRATIVE WORKED EXAMPLE

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3.1 The system discussed in Chapter 2 and shown in Figure 2.1 will now be employed as the basis for a fully worked example. The purpose is to illustrate the details of the proposed method of applying Hamilton's Principle to problems of moving mass loads on elastic beams.

A simplified version of the same problem, wherein the moving load is considered to exert an unvarying force Mg, will also be solved by the Hamiltonian method, and then again by direct solution of the equation of motion (2.7). This equation reduces to a tractable form, when the above mentioned simplification is adopted, and will thus provide an "exact" solution to check against.

3.2 The proposed method is to divide the time spent by the load in traversing the beam into a number of equal stages. During each stage it will be assumed that the angular acceleration of the beam OA is constant and hence that $\ddot{y}$ is constant. The displacement/time function applicable during any one of these short stages will then be:

$$y = At^2 + \dot{y}_o t + y_o + a \sin qt \qquad (3.1)$$

where A is a constant

y_o, $\dot{y}_o$ = deflection, and deflectional velocity, at point A, in Figure 2.1, at the end of the previous stage.

Using equation (3.1) expressions can be written for δT_1, δT_2, δV, δP, and then Hamilton's Principle can be employed in the form of equation (2.1) to give the value of the constant A for the first stage of the calculation, in which $y_o = \dot{y}_o = 0$. Then the values of y and $\dot{y}$ at the end of the first stage can be evaluated, from equation (3.1), and these will be employed as $y_o = \dot{y}_o$ for the second stage, and so on.

Clearly, the accuracy obtained will increase with the number of stages adopted and it will be of prime importance to discover how many stages are needed to obtain acceptable accuracy. For this reason the theory will now be developed in relation to a proposed computation with any desired number of stages N. By observing the alteration in results, due to an increase in N, it will be possible to estimate the number of stages necessary to give reliable results.

3.3 Expressions for δV, δP, δT_1, δT_2 will now be developed.

The time occupied by the whole problem, for a load having a speed u and a beam having a length L, will be L/u. Hence each stage will occupy a time L/Nu. The small variation a.sin qt will be made to occupy the same period by putting $q = N\Pi u/L$.

Strain Energy (V)

$$V = \tfrac{1}{2}k\left(\frac{y}{L}\right)^2 = \frac{k}{2L^2}\left(At^2 + \dot{y}_o t + y_o + a\sin\frac{N\Pi ut}{L}\right)^2$$

$$\therefore \quad \delta V = \frac{ka}{L^2}\left(At^2 \sin\frac{N\pi ut}{L} + \dot{y}_o t \sin\frac{N\pi ut}{L} + y_o \sin\frac{N\pi ut}{L}\right)$$

$$\int_o^{L/Nu} \delta V dt = \frac{ka}{L^2}\left(\frac{0.189304\ AL^3}{N^3u^3} + \frac{\dot{y}_o L^2}{\pi N^2 u^2} + \frac{2y_o L}{\pi Nu}\right) \qquad (3.2)$$

Gravitational Potential Energy (P)

In equation (3.1) t = 0 at the beginning of every stage, so the distance the load has moved on to the beam, from end 0, will have to be expressed as follows:

$$x = (n - 1)\frac{L}{N} + ut \qquad (3.3)$$

for the nth stage of a calculation wherein the total number of stages is N.

Now $P = -Mg\ y_M$ where y_M is the downward deflection of the beam at the instantaneous position of the load, so that $y_M = y\frac{x}{L}$ (see Figure 2.1)

$$P = -Mg\left(At^2 + \dot{y}_o t + y_o + a \sin\frac{N\pi ut}{L}\right)\left(\frac{n-1}{N} + \frac{ut}{L}\right),$$

using equations (3.1) and (3.3).

$$\therefore \delta P = -Mg\ a \sin \frac{N\Pi ut}{L}\left(\frac{n-1}{N} + \frac{ut}{L}\right)$$

$$\therefore \int_0^{L/Nu} \delta P dt = -Mg\ a\left[\frac{n-1}{N} \cdot \int_0^{L/Nu} \sin \frac{N\Pi ut}{L}\, dt + \frac{u}{L}\int_0^{L/Nu} t \sin \frac{N\Pi ut}{L}\, dt\right]$$

i.e. $$\int_0^{L/Nu} \delta P dt = -Mg\ a\ \frac{L}{u}\left(\frac{2(n-1)}{\Pi N^2} + \frac{1}{\Pi N^2}\right) \qquad (3.4)$$

Kinetic Energy of the beam itself (T_1)

$$T_1 = \frac{mL^3}{6}\left(\frac{\dot{y}}{L}\right)^2$$

$$\dot{y} = 2At + \dot{y}_o + qa \cos qt \qquad \text{from equation (3.1)}$$

Also $q = N\Pi u/L$ for an N stage calculation

$$\therefore T_1 = \frac{mL}{6}\left(2At + \dot{y}_o + \frac{N\Pi u}{L} \cos \frac{N\Pi ut}{L}\right)^2$$

$$\therefore \delta T_1 = \frac{N\Pi mu}{3}\left(2At \cos \frac{N\Pi ut}{L} + \dot{y}_o \cos \frac{N\Pi\ ut}{L}\right)$$

$$\therefore \int_0^{L/Nu} \delta T_1 dt = \frac{-4mAaL^2}{3\Pi Nu} \qquad (3.5)$$

Kinetic Energy of the moving Load (T_2)

$$T_2 = \tfrac{1}{2}M\dot{y}_M^{\,2}$$

Now $$y_M = y \cdot \frac{x}{L} = \left(\frac{n-1}{N}\right)y + \frac{yut}{L} \qquad \text{using equation (3.3)}$$

$$\therefore \quad \dot{y}_M = \left(\frac{n-1}{N}\right)\dot{y} + y\frac{u}{L} + \dot{y}\frac{ut}{L}$$

i.e.
$$\dot{y}_M = \left(\frac{n-1}{N}\right)(2At+\dot{y}_o+qa\cos qt) + \frac{u}{L}(At^2+\dot{y}_o t+y_o+a\sin qt)$$
$$+ \frac{u}{L}(2At^2+\dot{y}_o t + qt\,a\cos qt) \quad \text{using equation (3.1)}$$

$$\therefore \quad T_2 = \tfrac{1}{2}M\left[\left(\frac{n-1}{N}\right)(2At+\dot{y}_o+qa\cos qt)+\frac{u}{L}(At^2+\dot{y}_o t+y_o+a\sin qt) + \frac{u}{L}(2At^2+\dot{y}_o t+qta\cos qt)\right]^2$$

$$\therefore \quad \delta T_2 \div Ma =$$

$$2\left(\frac{n-1}{N}\right)^2 Aqt\cos qt + \left(\frac{n-1}{N}\right)^2 \dot{y}_o q\cos qt + \frac{3u^2}{L^2}At^2\sin qt + \frac{2u^2}{L^2}\dot{y}_o t\sin qt + \frac{u^2}{L^2}y_o\sin qt$$

$$+ \frac{3u^2}{L^2}Aqt^3\cos qt + \frac{2u^2}{L^2}\dot{y}_o qt^2\cos qt + \frac{2u}{L}\left(\frac{n-1}{N}\right)At\sin qt + \frac{u}{L}\left(\frac{n-1}{N}\right)\dot{y}_o\sin qt$$

$$+ 5A\frac{u}{L}\left(\frac{n-1}{N}\right)qt^2\cos qt + \frac{3u}{L}\left(\frac{n-1}{N}\right)\dot{y}_o qt\cos qt + \frac{u}{L}\left(\frac{n-1}{N}\right)y_o q\cos qt + \frac{u^2}{L^2}y_o qt\cos qt$$

Integrating and putting $q = \frac{N\pi u}{L}$ gives:

$$\int_0^{L/Nu} \delta T_2 dt = \frac{MAaL}{u}\left[-\frac{4}{\pi N}\left(\frac{n-1}{N}\right)^2 - \frac{8}{\pi N^2}\left(\frac{n-1}{N}\right) - \frac{35.218}{N^3\pi^3}\right.$$
$$\left. + M\dot{y}_o a\left[-\frac{4}{N\pi}\left(\frac{n-1}{N}\right) - \frac{2}{\pi N^2}\right]\right.$$

$$\text{i.e.} \quad \int_{o}^{L/Nu} \delta T_2 dt \;=\; \frac{-MAaL}{N^3 u}\left(\frac{4}{\Pi}(n-1)^2 + \frac{8}{\Pi}(n-1) + 1.13583\right)$$

$$\frac{-M\dot{y}_o a}{N^2}\left(\frac{4}{\Pi}(n-1) \;+\; 0.63662\right) \qquad (3.6)$$

The terms involving y_o have cancelled out because, in this simple problem, the beam has no elastic curvature.

3.4 Having now developed expressions for δV, δP, δT_1 and δT_2 it is necessary to substitute them in equation (2.1) as follows:

$$\int_{o}^{L/Nu} \delta T_1 dt \;+\; \int_{o}^{L/Nu} \delta T_2 dt \;-\; \int_{o}^{L/Nu} \delta V dt \;=\; \int_{o}^{L/Nu} \delta P dt$$

i.e.

$$ma\,\frac{AL^2}{u}\left(\frac{-4}{3\Pi N}\right) + Ma\,\frac{AL}{u}\left(-\frac{4}{\Pi N}\left(\frac{n-1}{N}\right)^2 - \frac{8}{\Pi N^2}\left(\frac{n-1}{N}\right) - \frac{35.22}{\Pi^3 N^3}\right)$$

$$+ \; Ma\dot{y}_o\left(\frac{-4}{\Pi N}\left(\frac{n-1}{N}\right) - \frac{2}{\Pi N^2}\right) - ka\,\frac{AL}{u^3}\left(\frac{.1893}{N^3}\right) - \frac{ka\dot{y}_o}{u^2 \Pi N^2}$$

$$- \; \frac{2kay_o}{uL\Pi N} \;=\; \frac{-MgaL}{u}\left(\frac{2(n-1)}{\Pi N^2} + \frac{1}{\Pi N^2}\right) \qquad (3.7)$$

The speed parameter β (see para. 1.2) can be introduced at this stage. For the system at present under consideration (Figure 2.1) the periodic time of natural vibration (for the unloaded beam) is:

$$t_p = 2\Pi \sqrt{\frac{mL^3}{3k}}$$

$$\text{Now } \beta = \frac{\text{time of transit}}{\text{half periodic time}} = \frac{2L}{ut_p} = \frac{wL}{\Pi u}$$

$$\therefore \beta^2 = \frac{4L^2 3k}{u^2(2\Pi)^2 mL^3}$$

i.e.

$$mL^2u^2 = \frac{0.30396kL}{\beta^2} \tag{3.8}$$

If equation (3.7) is multiplied through by $-u^3.N^3$ then equation (3.8) is a suitable form for introducing β. Also the live/dead load ratio $\gamma = M/mL$ can be introduced and finally equation (3.7) can be stated as follows:

$$A = \frac{\left[\gamma g(.63662n - .31831)N - 1.0472N\beta^2\left(\frac{\dot{y}_o u}{L}\right) - 2.0944N^2\beta^2\left(y_o\frac{u^2}{L^2}\right) - 2N\gamma(.63662n - .31831)\ \dot{y}_o\frac{u}{L}\right]}{.424413N^2 + \gamma(1.27324n^2 - .13734) + .62279\beta^2} \tag{3.9}$$

Evaluation of dynamic deflections

The computation will consist of repeated applications of equation (3.9). In the first stage (n=1), y_o and $\dot{y}_o$ are both zero, which means that the last three terms in the numerator of equation (3.9) have zero initial values. Hence, the first value of A will be computed as a coefficient of γg. Then equation (3.1) can be used to compute values of $y_o\frac{u^2}{L^2}$ and $\dot{y}_o\frac{u}{L}$ at the end of stage 1 when t = L/Nu. These values are then fed into the numerator of equation (3.9) for the second application, and so on.

The procedure will be sufficiently illustrated by the first stages of a five stage calculation (n=5) which are set out in detail as follows:

If β has the value 1.432 (shown later to have a special significance) then equation (3.9) becomes:

$$A = \frac{\gamma g(3.18n - 1.59) - 10.7\left(\dot{y}_o \frac{u}{L}\right) - 107\left(y_o \frac{u^2}{L^2}\right) - 2(3.18n - 1.59)\dot{y}_o\frac{u}{L}\gamma}{10.6 + \gamma(1.27n^2 - 0.14) + 1.28} \qquad (3.9a)$$

Stage 1 n=1 (also putting γ = 1)

$$A = \gamma g \frac{1.59}{13.01} = 0.122\ \gamma g$$

Now from (3.1) with $t = \frac{L}{Nu} = \frac{L}{5u}$ (to obtain y_o and $\dot{y}_o$ for stage 2):

$$y_o \frac{u^2}{L^2} = \frac{0.122\gamma g}{25} = 0.0049\gamma g \quad \text{(from equation (3.1))}$$

$$\dot{y}_o \frac{u}{L} = \frac{2 \times 0.122\gamma g}{5} = 0.049\gamma g \quad \text{(from equation (3.1) differentiated)}$$

Stage 2 n=2

$$A = \gamma g \frac{4.77 - 10.7(.049) - 107(.0049) - 2(4.77)(.049)}{16.8}$$

$$= 0.194\ \gamma g$$

Then again from (3.1), and its derivative, with $t = \frac{L}{5u}$:

$$y_o \frac{u^2}{L^2} = \frac{0.194\gamma g}{25} + \frac{0.049\gamma g}{5} + 0.0049\gamma g = 0.0225\gamma g$$

$$\dot{y}_o \frac{u}{L} = \frac{2 \times 0.194\gamma g}{5} + 0.049\gamma g = 0.1266\gamma g \quad \text{et cetera}$$

The intended value of γ must be inserted in the fourth numerator term, and the second denominator term, of equation (3.9) or (3.9a). $\gamma = 1$ was inserted at these points in the above example so that the obtained results are only valid for $\gamma = 1$.

Evaluation of the dynamic factor D

It can be seen that, at the end of every stage, the dynamic deflection y (at point A in Figure 2.1) will be evaluated as a coefficient of $\gamma g L^2/u^2$.

Now, from equation (3.8)

$$\gamma g L^2/u^2 = \frac{mgL^3\beta^2\gamma}{.30396k} = \frac{MgL^2\beta^2}{.30396k}$$

But $MgL^2/k = \Delta$ (maximum possible static deflection)

$$\therefore \gamma g L^2/u^2 = \frac{\beta^2\Delta}{.30396} = 3.29\beta^2\Delta$$

So the dynamic factor D will be the computed coefficient multiplied by $3.29\beta^2$.

A program to evaluate dynamic factors for $\gamma = 1$ but for any desired values of N and β is shown below.

```
          ZN=40.0
          B=0.432
          DO11 I=1,10
          XN=1.0
          YY=0.0
          Y=0.0
          DO10 J=1,40
          XA=0.63662*XN-0.31831                                    *
          AN=ZN*XA-1.0472*ZN*B*B*YY-2.0944*ZN*ZN*B*B*Y-2.0*ZN*XA*YY
          AD=0.424413*ZN*ZN+1.27232*XN*XN-0.13741ZO.622792*B*B
          A=AN/AD                       *
          Y=A/ZN*ZN)+YY/ZN+Y
          YY=2.0*A/ZN+YY
          DY=3.29*Y*B*B
          WRITE(6,1) DY
    1     FORMAT (F15.7)
    10    XN=XN+1.0
    11    B=B+1.0
          STOP
          END
```

The terms marked with an asterisk are concerned with the kinetic energy of the moving load M (i.e. T_2). If these terms are omitted the program gives a solution for the case where the load is regarded as a moving force M.g of constant magnitude. This modification was adopted to obtain special results for comparison with a direct solution of the equation of motion, which now follows.

3.5 SOLUTION OF THE EQUATION OF MOTION ($\gamma = 0$)

In the case where the load is regarded as a moving force of constant magnitude M.g, the equation of motion (2.7) can be reduced to the form shown below, (3.10), which is simple enough for direct solution. Such a solution will now be presented in order to obtain some "exact" results which can be compared with the Hamiltonian stage by stage computations.

$$\ddot{y} + \frac{3k}{mL^3} y = \frac{3Mgut}{mL^3} \qquad (3.10)$$

$$\therefore \; y = A \sin \omega t + B \cos \omega t + \frac{MgutL}{k}$$

$$\text{where} \quad \omega = \sqrt{\frac{3k}{mL^3}}$$

Since $y = 0$ when $t = 0$ $\quad \therefore B = 0$

Also $\dot{y} = 0$ when $t = 0$:

$$\therefore \; 0 = \omega A + \frac{MguL}{k} \; ; \quad \therefore \; A = \frac{-MguL}{k\omega}$$

So the final relevant solution of equation (3.10) is:

$$y = \frac{MguL}{k\omega} (\omega t - \sin \omega t)$$

Now introduce $\beta = \frac{L}{u} \div \frac{1}{2} tp = \frac{\omega L}{\Pi u}$

$$\text{i.e.} \quad \frac{uL}{\omega} = \frac{L^2}{\Pi\beta}$$

$$y = \frac{MgL^2}{\Pi\beta k} (\omega t - \sin \omega t)$$

But MgL^2/k is the maximum static deflection Δ

$$\text{dynamic factor } D = \frac{y}{\Delta} = \frac{t - \sin \omega t}{\Pi\beta} .$$

But $\omega t = \frac{\Pi\beta ut}{L}$

$$D = \frac{ut}{L} - \frac{\sin\left(\frac{\Pi\beta ut}{L}\right)}{\Pi\beta} \qquad (3.11)$$

To discover the maximum value D can ever attain put t = L/u in equation (3.11) and then differentiate:

$$D = 1 - \frac{\sin \Pi\beta}{\Pi\beta}$$

$$\frac{\partial D}{\partial \beta} = \frac{\Pi^2\beta \cos \Pi\beta - \Pi \sin \Pi\beta}{- \Pi^2\beta^2}$$

This will be zero when $\tan \Pi\beta = \Pi\beta$

i.e. when $\Pi\beta = 4.5$

i.e. $\beta = 1.432$

Substituting this value of β in equation (3.11) gives:

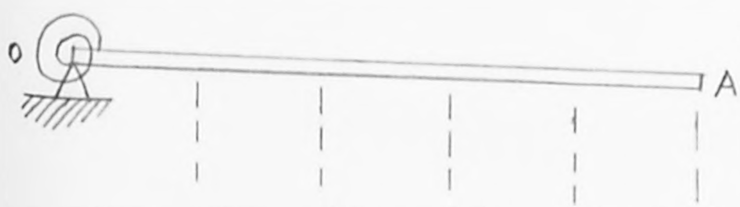
$$D = \frac{ut}{L} - \frac{\sin 4.5\ ut/L}{4.5}$$

Finally, substituting values of u.t/L = 0.2, 0.4, 0.6, 0.8, 1.0 will give values of D for five instantaneous positions of the load when it is travelling at its critical speed. The actual values are reported in the next paragraph, the fifth value being the greatest dynamic factor attainable in this problem whilst the load is on the beam. In fact the deflection will increase a little further after the load has run off the end of the beam.

3.6 ACCURACY OBTAINABLE WITH THE PROPOSED METHOD

Cases where the load is treated as a moving constant force M.g will be discussed first, because, in such cases, the "exact" solution of para. 3.5 can be used as a standard of accuracy.

Values of the dynamic factor D are shown below for five instantaneous positions of the load when it is travelling at the critical speed represented by $\beta = 1.432$.



D =	.026	.184	.505	.898	1.217	"Exact results from para 3.5
D =	.036	.191	.496	.877	1.202	From a five stage computation using the program of para 3.4
D =	.026	.184	.505	.898	1.217	From a forty stage computation

The forty stage computation shows that the Hamiltonian method is capable of producing the "exact" results. However, it is interesting to note that a surprising degree of accuracy is obtained with only five stages of computation. It has been found, from general experience with the Hamiltonian method, that it is most effective at low values of β (i.e. when the load is moving fast). So the success of the above five stage calculation is largely attributable to the fact that a low value of β is involved.

Nevertheless it is a convenient feature of the method that it operates best in the region of greatest interest.

With high values of β the calculation may fail to converge. However, this is not of much concern because a high value of β implies a "crawling" load and a dynamic factor differing little from unity.

Now the more interesting problem, where the load is treated as a mass, will be discussed. All the results shown below have been obtained using the program in para 3.4 and apply to the case where the mass of the load is equal to the mass of the beam itself ($\gamma = 1$). Only maximum values of the dynamic factor D are given, i.e. values applicable to the instant when the load arrives at the end A of the beam.

β	5 stage computation D	10 stage computation D	20 stage computation D	40 stage computation D
.432	.120	.116	.116	.116
1.432	.786	.791	.792	.793
2.432	.970	.955	.951	.951
3.432	.890	.888	.890	.891
4.432	.980	1.005	1.012	1.014
5.432	.990	.990	.971	.967
6.432		.962	.963	.970
7.432		1.007	1.017	1.018
8.432			.975	.967
9.432			.985	1.000

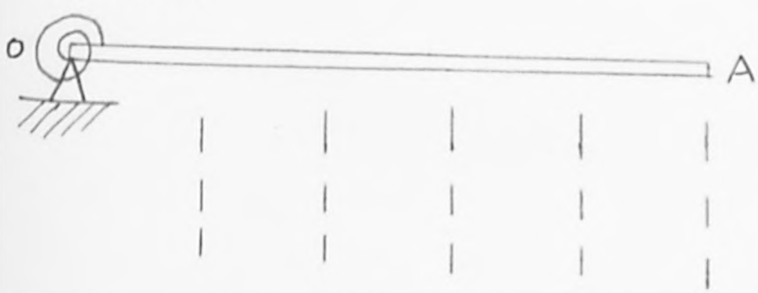
Table 3.1

The first two columns in Table 3.1 are incomplete because of the failure of the program to converge when N is small and β is large, a tendency already referred to above.

The columns for 20 and 40 stage computations show only very small discrepancies and thereby suggest that 20 stages yield adequate reliability. Obviously the maximum dynamic factor should tend to unit value as β increases.

Finally, since the main aim of these calculations is to compare dynamic deflections based (a) on constant force moving loads and (b) mass loads exerting varying forces, it is relevant to report the comparable results for the problem in hand.

The results shown below are dynamic factors corresponding to five instantaneous positions of the load. They were obtained from two forty stage applications of the program shown in para. 3.4



β = 1.432

D =	.026	.184	.505	.898	1.217	FORCE LOADING (γ = 0)
D =	.024	.148	.351	.576	.793	MASS LOADING (γ = 1)

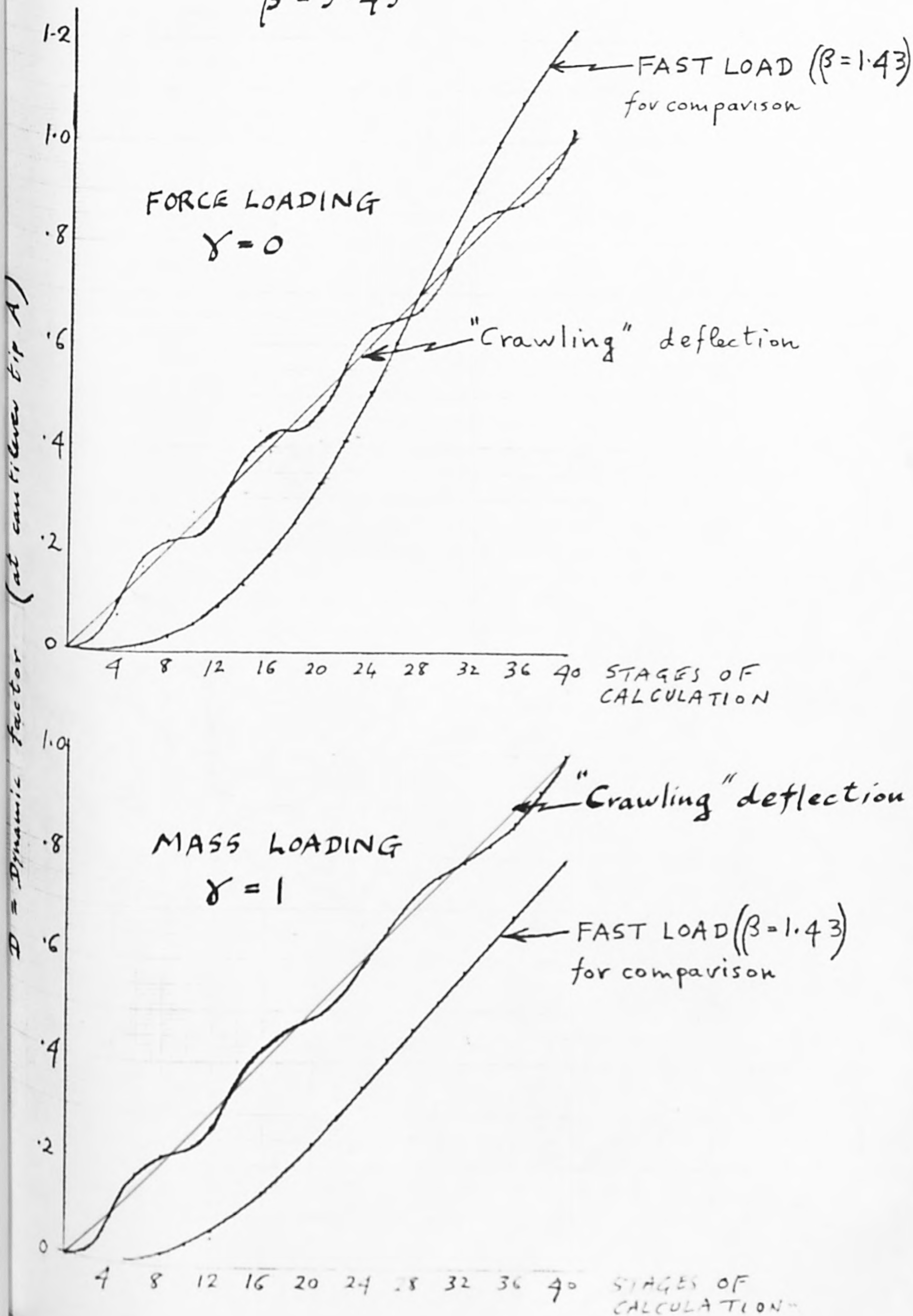
<u>Cause of decreasing accuracy with higher β values</u>

A large value of β implies a low speed of the moving load. The beam performs a significant number of cycles of vibration, during a slow transit, leading to many reversals of curvature in the dynamic deflection graph. This can be observed in graph 3.1 which is based on a 40 stage computation. It is not to be expected that a step by step calculation could trace this type of curve if only a small number of steps were employed.

RESULTS OF 40 STAGE COMPUTATION APPLIED TO A SLOW MOVING LOAD

$\beta = 9{\cdot}43$

GRAPH 3.1.



CHAPTER 4

BEHAVIOUR OF UNIFORM SIMPLY SUPPORTED BEAMS

CHAPTER 4

BEHAVIOUR OF UNIFORM SIMPLY SUPPORTED BEAMS

4.1 This type of problem will be tackled by a procedure almost identical to that illustrated in Chapter 3. The calculations will be programmed for computer solution so that a wide range of results can be obtained as follows:

(a) Maximum dynamic factors due to a single load traversing the span at various speeds (β) and accounting for different values of the ratio load mass/beam mass (γ).

(b) Detailed comparison of responses for various values of γ in selected cases.

(c) Effect of initial deflections whether sagging, due to dead load, or hogging due to track humping.

(d) Reconsideration of (a), (b) and (c) above, in terms of trains of successive loads, which may, or may not, be longer than the span.

The computation will be based on ten stages. (This small number of stages was originally adopted for a preliminary trial of the method. However, the results appeared to be adequate and have subseqently stood the test of comparison with limiting cases of the sixty stage computation reported in Chapter 7).

It will be assumed, as in Chapter 3, that during each of these stages the deflectional acceleration is constant. Then, in each stage, the following displacement/time function will be applicable:

$$y = (At^2 + \dot{Y}_o t + Y_o) \sin \frac{\Pi x}{L} \tag{4.1}$$

where Y_o and $\dot{Y}_o$ are the midspan deflection and velocity at the end of the previous stage.

Note that equation (4.1) gives the deflection at any point of the span at any instant, with the implication that t is taken to be zero at the beginning of every stage. The constant A will have a different value in each stage.

In order to employ Hamilton's Principle (equation 2.1) it will be necessary to introduce a small variation into equation (4.1) as follows:

$$y = (At^2 + \dot{Y}_o t + Y_o + a \sin \frac{10\Pi ut}{L}) \sin \frac{\Pi x}{L} \tag{4.2}$$

$$\dot{y} = (2At + \dot{Y}_o + \frac{10\Pi ua}{L} \cos \frac{10\Pi ut}{L}) \sin \frac{\Pi x}{L} \tag{4.3}$$

It is intended that the variation should only exist for a half cycle so that the time it occupies is L/10u which is the time taken for the load to travel the stage length L/10. Note that the variation does not affect the path of any particle but merely distorts its "timetable" by successive speeding up and slowing down. After the variation has occurred all particles will be at the same point in both time and space as if the variation had never occured.

If the variation is of small amplitude (so that a^2 is negligible) then the Hamilton integral (equation 2.1) will also be unaffected by its occurrence. This is the condition that enables the value of the constant A, in equation (4.1) to be determined.

4.2 Expressions for δV, δP, δT_1, δT_2, for substitution in equation (2.1) will now be developed.

Strain Energy (V)

In general strain energy due to bending is given by:

$$V = \tfrac{1}{2}EI\int (d^2y/dx^2)^2 dx$$

and from equation (4.2):

$$\frac{d^2y}{dx^2} = -\frac{\Pi^2}{L^2}\left(At^2 + \dot{Y}_o t + Y_o + a\sin\frac{10\Pi ut}{L}\right)\sin\frac{\Pi x}{L}$$

$$\therefore\ V = \tfrac{1}{2}EI\frac{\Pi^4}{L^4}\left(At^2 + \dot{Y}_o t + Y_o + a\sin\frac{10\Pi ut}{L}\right)^2\int_o^L \sin^2\frac{\Pi x}{L}dx$$

$$\text{i.e. } V = \frac{\Pi^4 EI}{4L^3}\left(At^2 + \dot{Y}_o t + Y_o + a\sin\frac{10\Pi ut}{L}\right)^2 \qquad (4.4)$$

To obtain δV from equation (4.4) it is necessary to extract only those terms which involve 'a', terms involving a^2 being neglected.

$$\therefore\ \delta V = \frac{\Pi^4 EIa}{2L^3}\left(At^2\sin\frac{10\Pi ut}{L} + \dot{Y}_o t\sin\frac{10\Pi ut}{L} + Y_o\sin\frac{10\Pi ut}{L}\right)$$

$$\int_o^{L/10u}\delta V dt = .0092207\frac{EIAa}{u^3} + .15503\frac{EI\dot{Y}_o a}{Lu^2} + 3.1006\frac{EIY_o a}{L^2 u} \qquad (4.5)$$

In equation (4.4), y should originate from a position of zero elastic distortion, which may involve a deflection due to the self weight of the beam. However, it has been shown in Chapter 2 that δV, and δP associated with structure weight, may be balanced one against the other and hence omitted from the calcualtions. This means that Y_o in equation (4.5) will always be zero in the first stage of the calculations, so far as strain energy is concerned.

Potential Energy (P)

$$P = -Mgy_M$$

where y_M is the local value of dynamic deflection beneath the moving load.

To convert y to y_M it is necessary to substitute for x, in equation (4.2), an expression which defines the position of the load at time t. The required expression is:

$$x = (n-1)\frac{L}{10} + ut \qquad (4.6)$$

this being applicable in the nth of the ten stages into which the transit of the span will be subdivided for calculation purposes. As before the implication is that t will be taken as zero at the beginning of every stage.

$$\therefore \quad y_M = \left(At^2 + \dot{Y}_o t + Y_o + a \sin\frac{10\Pi ut}{L}\right)\sin\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)$$

$$\therefore \quad \delta P = -Mg\ a \sin\frac{10\ \Pi ut}{L}\sin\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)$$

$$\int_0^{L/10u} \delta P dt = -Mga \int_0^{L/10u} \sin\frac{10\Pi ut}{L}\left(\sin\frac{(n-1)\Pi}{10}\cos\frac{\Pi ut}{L} + \cos\frac{(n-1)\Pi}{10}\sin\frac{\Pi ut}{L}\right)dt$$

i.e.
$$\int_0^{L/10u} \delta P dt = -\frac{MgLa}{u}\left(\frac{N}{\Pi(N^2-1)}\right)\left(\sin\left(\frac{n-1}{N}\right)\Pi + \sin\frac{n\Pi}{N}\right) \qquad (4.7)$$

Values of $\int \delta P dt$ for each of the ten stages are given below: $(N = 10)$

STAGE NUMBER n	$\int \delta P dt$ (coefficient of MgaL/u = p_n)
1	-.00993567
2	-.0288344
3	-.0449107
4	-.0565908
5	-.0627314
6	-.0627314
7	-.0565908
8	-.0449107
9	-.0288344
10	-.00993567

Table 4.1

Kinetic Energy of the beam itself (T_1)

In general, for a beam with uniformly distributed self mass:

$$T_1 = \tfrac{1}{2}m \int_0^L (\dot{y})^2 dx$$

i.e. $$T_1 = \tfrac{1}{2}m\left(2At + \dot{Y}_o + \frac{10\Pi u}{L} a \cos \frac{10\Pi ut}{L}\right)^2 \int_0^L \sin^2 \frac{\Pi x}{L} dx$$ (using equation 4.3)

i.e. $$T_1 = \frac{mL}{4}\left(2At + \dot{Y}_o + \frac{10\Pi u}{L} a \cos \frac{10\Pi ut}{L}\right)^2 \qquad (4.8)$$

To obtain δT_1 from equation (4.8) it is necessary to extract only those terms which involve a, terms involving a^2 being negligible.

$$\delta T_1 = \frac{mLa}{2}\left(\frac{20A\Pi u}{L} t \cos \frac{10\Pi ut}{L} + \frac{10\Pi u}{L} \dot{Y}_o \cos \frac{10\Pi ut}{L}\right)$$

$$\int_0^{L/10u} \delta T_1 dt = -0.063662 \frac{mL^2Aa}{u} \qquad (4.9)$$

Kinetic Energy of the moving load (T_2)

$$T_2 = \tfrac{1}{2}M(\dot{y}_M)^2$$

where y_M is the local value of dynamic deflection beneath the moving load.

To convert y to y_M it is necessary to substitute for x, in equation (4.2), an expression which defines the position of the load at time t (which was also done under the heading "Potential Energy").

So, combining equations (4.2) and (4.6):

$$y_M = \left(At^2 + \dot{Y}_o t + Y_o + a \sin \frac{10\Pi ut}{L}\right) \sin\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)$$

$$\therefore \quad \dot{y}_M = \left(2At + \dot{Y}_o + \frac{10\Pi ua}{L} \cos \frac{10\Pi ut}{L}\right) \sin\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)$$

$$+ \frac{\Pi u}{L}\left(At^2 + \dot{Y}_o t + Y_o + a \sin \frac{10\Pi ut}{L}\right) \cos\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)$$

$$\therefore \quad T_2 = \tfrac{1}{2}M(\dot{y}_M)^2$$

$$= \tfrac{1}{2}M\left[\left(2At + \dot{Y}_o + \frac{10\Pi ua}{L} \cos \frac{10\Pi ut}{L}\right) \sin\left((n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}\right)\right.$$

$$\left. + \frac{\Pi u}{L}\left(At^2 + \dot{Y}_o t + Y_o + a \sin \frac{\Pi ut}{L}\right) \cos\left((n-1)\frac{\Pi}{10} + \frac{ut\Pi}{L}\right)\right]^2$$

Now extract only those terms involving 'a' to obtain an expression for δT_2. For brevity let $\phi_n = (n-1)\frac{\Pi}{10} + \frac{\Pi ut}{L}$.

$$\delta T_2 = \tfrac{1}{2}M_a\left(\frac{40\Pi u}{L} At \cos \frac{10\Pi ut}{L} \sin^2\phi_n + \frac{20\Pi u}{L} \dot{Y}_o \cos \frac{10\Pi ut}{L} \sin^2\phi_n\right.$$

$$+ \frac{2\Pi^2u^2}{L^2} At^2 \sin \frac{10\Pi ut}{L} \cos^2\phi_n + \frac{2\Pi^2u^2}{L^2} Y_o \sin \frac{10\Pi ut}{L} \cos^2\phi_n$$

$$+ \frac{2\Pi^2u^2}{L^2} \dot{Y}_o t \sin \frac{10\Pi ut}{L} \cos^2\phi_n$$

$$+ \frac{4\Pi u}{L} At \sin \frac{10\Pi ut}{L} \sin\phi_n \cos\phi_n + \frac{2\Pi u}{L} \dot{Y}_o \sin \frac{10\Pi ut}{L} \sin\phi_n \cos\phi_n$$

$$+ \frac{20\Pi^2u^2}{L^2} At^2 \cos \frac{10\Pi ut}{L} \sin\phi_n \cos\phi_n + \frac{20\Pi^2u^2}{L^2} Y_o \cos \frac{10\Pi ut}{L} \sin\phi_n \cos\phi_n$$

$$\left. + \frac{20\Pi^2u^2}{L^2} \dot{Y}_o t \cos \frac{10\Pi ut}{L} \sin\phi_n \cos\phi_n\right)$$

$$\int_{0}^{L/10u} \delta T_2 dt = \tfrac{1}{2}Ma\left(\frac{40\Pi u}{L} AI_1 + \frac{20\Pi u}{L}\dot{Y}_o I_2 + \frac{2\Pi^2 u^2}{L^2} AI_3\right.$$

$$+ \frac{2\Pi^2 u^2}{L^2} Y_o I_4 + \frac{2\Pi^2 u^2}{L^2}\dot{Y}_o I_5$$

$$+ \frac{4\Pi u}{L} AI_6 + \frac{2\Pi u}{L}\dot{Y}_o I_7 + \frac{20\Pi^2 u^2}{L^2} AI_8$$

$$\left. + \frac{20\Pi^2 u^2}{L^2} Y_o I_9 + \frac{20\Pi^2 u^2}{L^2}\dot{Y}_o I_{10}\right) \qquad (4.10)$$

The ten integrals I_1 to I_{10} required for equation (4.10) all have different values in each of the ten stages of the calculation. The necessary 100 values were computed separately, as shown in Appendix 4. In terms of these print out values equation (4.10) will be modified as follows, after introducing correction factors $\frac{1}{10\Pi}$, $\frac{1}{100\Pi^2}$, or $\frac{1}{1000\Pi^3}$, as appropriate:

$$\int_{0}^{L/10u} \delta T_2 dt = \tfrac{1}{2}Ma\left[A\frac{L}{u}\left(.127324\ I_1' + .0006366\left(I_3' + 10\ I_8'\right) + .0127324\ I_6'\right)\right.$$

$$+\dot{Y}_o(2I_2' + 0.02\ I_5' + 0.20\ I_7' + 0.20\ I_{10}')$$

$$\left. +Y_o\frac{u}{L}\left(0.628318\left(I_4' + 10\ I_9'\right)\right)\right]$$

Or more briefly:

$$\int_{0}^{L/10u} \delta T_2 dt = Ma\left(A\frac{L}{u}(t_A) + \dot{Y}_o\ (t_{YY}) + Y_o\frac{u}{L}(t_Y)\right) \qquad (4.11)$$

Values for t_A, t_{YY}, and t_Y, for each of the ten stages are shown in Table 4.2.

STAGE	t_A	t_{YY}	t_Y
1	-.010854	-.060091	.017769
2	-.042872	-.153099	.130394
3	-.082483	-.181632	.314045
4	-.114556	-.134791	.497160
5	-.126841	-.030469	.610336
6	-.114646	+.091488	.610343
7	-.082627	+.184497	.497179
8	-.043017	+.213030	.314068
9	-.010944	+.166190	.130952
10	+.001342	+.061868	.017776

Table 4.2

4.3 The expressions derived in para. 4.2 for $\int\delta T_1$, $\int\delta T_2$, $\int\delta V$, and $\int\delta P$ will now be substituted in equation (2.1):

$$\int\delta T_1 dt + \int\delta T_2 dt - \int\delta V dt - \int\delta P dt = 0 \qquad (2.1)$$

i.e. $$-.063662\,\frac{mL^2Aa}{u} + t_A\,\frac{ML}{u}\,Aa + t_{YY}M\dot{Y}_o a + t_Y\,\frac{Mu}{L}\,Y_o a$$

$$-\ .0092207\,\frac{EI}{u^3}\,Aa - .15503\,\frac{EI\dot{Y}_o a}{Lu^2} - 3.1006\,\frac{EIY_o a}{L^2u} - p_n\,\frac{MgL}{u}\,a = 0$$

$$\therefore\quad A = \frac{-p_n MgLu^2 - .15503\,EI\dot{Y}_o\,\frac{u}{L} - 3.1006\,EIY_o\,\frac{u^2}{L^2} + t_{YY}Mu^3\dot{Y}_o + t_Y Mu^3\dot{Y}_o\,\frac{u}{L}}{.063662\,mL^2u^2 - t_A MLu^2 + .0092207\,EI}$$

At this stage the principal parameters γ and β will be introduced:

$$\underline{\beta^2 = \frac{\omega^2L^2}{\Pi^2u^2}}^{\,*} = \frac{\Pi^4}{L^4}\cdot\frac{EI}{m}\cdot\frac{L^2}{\Pi^2u^2} = \frac{\Pi^2}{L^2}\,\frac{EI}{mu^2}$$

i.e. $$mL^2u^2 = \Pi^2\,EI/\beta^2 \qquad (4.12)$$

Also $M = \gamma mL$

$$\therefore\quad A = \frac{MgLu^2}{EI}\left[\frac{-p_n - .15503\,\frac{EI}{MgLu^2}\,\dot{Y}_o\,\frac{u}{L} - 3.1006\,\frac{EI}{MgLu^2}\,Y_o\,\frac{u^2}{L^2} + \frac{t_{YY}}{g}\,Y_o\frac{u}{L} + \frac{t_Y}{g}\,Y_o\frac{u^2}{L^2}}{.063662\,\Pi^2/\beta^2 - t_A\,\gamma\Pi^2/\beta^2 + .0092207}\right]$$

(4.13)

* See para. 1.2 equation (1.1)

In the first stage (when $Y_o = \dot{Y}_o = 0$) all the terms inside the bracket, in equation (4.13) can be assigned numerical values. Hence the constant A can be computed as a coefficient of $MgLu^2/EI$.

i.e. $A' = A \div MgLu^2/EI$

where A' indicates the value actually computed.

(In the following development all symbols with primes are to be interpreted in this way)

Now values of Y and $\dot{Y}_o$, for the second stage, can be calculated as follows ($t = \frac{L}{10u}$ at the end of stage 1):

$$Y_o = At^2 = \frac{AL^2}{100u^2} = \frac{A'}{100} \cdot \frac{MgL^3}{EI} = Y_o' \frac{MgL^3}{EI} \qquad (4.14)$$

$$\dot{Y}_o = 2At = \frac{2AL}{10u} = \frac{A'}{5} \frac{MgL^2u}{EI} = \dot{Y}_o' \frac{MgL^2u}{EI} \qquad (4.15)$$

$$\ddot{Y}_o = 2A = 2A' \frac{MgLu^2}{EI} = 2A'\gamma g \frac{mL^2u^2}{EI} = 2A' \frac{\Pi^2}{\beta^2}\gamma g \qquad (4.16)\ )$$

Now substitute from equations (4.14) and (4.15) back into equation (4.13) to obtain the following expression for A which will be applicable to the second stage and all subsequent stages:

$$A = \frac{MgLu^2}{EI} \left[\frac{-p_n - .15503\ \dot{Y}_o' - 3.1006\ Y_o' + \frac{MgLu^2}{EI}\left(\frac{t_{YY}}{g}\dot{Y}_o' + \frac{t_Y}{g} Y_o'\right)}{.063662\ \Pi^2/\beta^2 - t_A\gamma\Pi^2/\beta^2 + .0092207} \right]$$

Now substitute from equations (4.12) to modify the terms involving t_{YY} and t_Y :

$$A' = \frac{-p_n - .15503\,\dot{Y}_o' - 3.1006\,Y_o' + t_{YY}\gamma\,\frac{\pi^2}{\beta^2}\,\dot{Y}_o' + t_Y\gamma\,\frac{\pi^2}{\beta^2}\,Y_o'}{(.063662 - t_A\gamma)\,\pi^2/\beta^2 + .0092207} \qquad (4.17)$$

Equation (4.17) is in a suitable form for automatic computing and is the basis of the programs shown in Appendix 4. Each computer run will be based on a specified pair of values for the principal parameters β and γ. Values of p_n are taken from Table 4.1 and values of t_A,t_Y and t_{YY} from Table 4.2.

Variables named in the computer list are as follows:

γ=G; β=B; π^2/β^2=BB; Y_o' =Y; $\dot{Y}_o'$ =YY; $\ddot{Y}_o'/\gamma g$ = YYY.

All variables carry additional symbols A, B, C etc. to indicate the stage of calculation,

e.g. YYE = value of $\dot{Y}_o'$ in Stage 5.

Consideration of initial dead load sag or track humping

In these cases the term $t_Y \gamma\ \Pi^2/\beta^2\ Y_o'$ in equation (4.17) will not be zero at the beginning of the first stage of calculation. Y_o' will be given an initial value to represent the "sag" or "hump" of the track.

The problems of this type which have been computed are reported on Figure 4.5. Reference to this graph and also to the programme list should suffice to clarify the details involved. This initial value of Y_o' is named UM in the list.

e.g. to introduce a sinusoidal hump with a maximum rise equal to the static live load deflection ad midspan ($\frac{MgL^3}{48EI}$) it would be necessary to employ UM = 1/48 = .02083. A zero value of UM implies that dead load deflection has been cambered out and that the beam is perfectly straight and level prior to the transit of the live load.

4.4 TRAVELLING DISTRIBUTED LOADS (See Graph 4.8)

This problem has been considered in terms of a train of loads each of mass M and separated from each other by a distance L/10. To deal with such a load system it is only necessary to "accumulate" the values of p_n, t_A, t_Y and t_{YY} given in Tables 4.1 and 4.2. For example, in the tenth stage of calculation these quantities would be assigned values equal to the sum of values in the tabulated columns. To deal with the possibility of trains longer than the span L, these values would then be held at their tenth stage values for as many more stages as the train length implies.

This type of problem is of greater practical significance, than that of the single load, because it is much more likely that the total live load will be equal to, or greater than, the mass of the beam itself. Results have been reported for $\gamma = 0.1$ and $\gamma = 0.2$. The former value implies that the greatest live load on the span, at any given time, will be ten loads each having a mass one tenth of that of the beam itself.

At very high transit speeds 'u' (i.e. low values of β) a steady state solution is possible with trains of infinite length. In such a case the beam attains a quasi static deflection, the elastic resistance of the beam being steadily balanced by the product of load masses with gravitational plus centripetal acceleration. At even higher speeds a state of instability or divergence can occur. No practical importance is attached to these matters because the transit speed 'u' would be far beyond any probable value.

However, this "steady state" situation yields quite readily to mathematical analysis and thus has been considered of sufficient interest to present in Appendix 3.

4.5 DETERMINATION OF MAXIMUM DYNAMIC DEFLECTION WHEN IT OCCURS AFTER THE LOAD HAS LEFT THE SPAN

In some cases the deflection of the beam will still be increasing when the load reaches the far end of the beam, so that the maximum deflection must occur after the load has run off the beam. These cases are easily spotted because the value of $\dot{Y}$, at the end of the tenth stage of computation, will be positive (YYJ in the programme list).

Once the beam is free of load its residual motion must be a free vibration at natural frequency which can be described, in terms of the symbols in the programme list as:

$$Y = \frac{YYJ}{\omega} \sin \omega t + YJ \cos \omega t$$

However, YYJ is a coefficient of $\frac{MgL^2u}{EI}$ (equation 4.15)

whilst Y and YJ are coefficients of $\frac{MgL^3}{EI}$ (equation 4.14)

So, for dimensional consistency the above equation must be modified to:

$$Y = YYJ \frac{u}{\omega L} \sin \omega t + YJ \cos \omega t = \frac{YYJ}{\beta \Pi} \sin \omega t + YJ \cos \omega t$$

The maximum amplitude of this vibration is:

$$Y_{MAX} = \sqrt{\left(\frac{YYJ}{\Pi \beta}\right)^2 + (YJ)^2} \qquad (4.18)$$

4.6 ACCELERATIONS EXPERIENCED BY THE LOAD DURING TRANSIT

(See also, Appendix 2, para. A2.1)

At any instant the deflection of the beam just beneath the moving load is given by $y_M = Y \sin \frac{\Pi ut}{L}$ and the acceleration, of the load, can be found from this by differentiating twice, as follows:

$$\dot{y}_M = \dot{Y} \sin \frac{\Pi ut}{L} + Y \frac{\Pi u}{L} \cos \frac{\Pi ut}{L}$$

$$\ddot{y}_M = \ddot{Y} \sin \frac{\Pi ut}{L} + 2\dot{Y} \frac{\Pi u}{L} \cos \frac{\Pi ut}{L} - \frac{\Pi^2 u^2}{L^2} Y \sin \frac{\Pi ut}{L} \qquad (4.19)$$

In terms of the computer programme symbols, euqation (4.19) may be rewritten as follows:

$$\ddot{y}_M = \gamma g(YYY) \sin \frac{\Pi ut}{L} + 2\gamma g(YY) \frac{\Pi m L^2 u^2}{EI} \cos \frac{\Pi ut}{L} - \gamma g(Y) \frac{\Pi^2 m L^2 u^2}{EI} \sin \frac{\Pi ut}{L}$$

Now introduce $\frac{mL^2u^2}{EI} = \frac{\Pi^2}{\beta^2}$ from equation (4.12), giving:

$$\ddot{y}_M = \gamma g(YYY) \sin \frac{\Pi ut}{L} + 2\gamma g \frac{\Pi^3}{\beta^2} (YY) \cos \frac{\pi ut}{L} - \gamma g \frac{\Pi^4}{\beta^2} Y \sin \frac{\Pi ut}{L} \qquad (4.20)$$

ACCN. OF BEAM   CORIOLIS ACCN.   CENTRIPETAL ACCN.

The following set of values was calculated from equation (4.20) using values of Y, YY and YYY from a print out obtained for $\beta = 1.60$ and $\gamma = 1$. Values for $\gamma = 1$, from Table A2.1 are also shown to illustrate the discrepancy between "mass" and "force" loading.

ut/L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Eqn. 4.20 $\ddot{y}_M$	0	+.23g	+.56g	+.56g	+.38g	+.06g	−.45g	−1.30g	−2.55g	−3.34g	−.42g
Table A2.1 $\ddot{y}_M$	0	+.4g	+1.1g	+1.4g	+0.7g	−0.9g	−2.5g	−3.0g	−1.8g	+0.8g	+3.3g

Table 4.3

SIMPLE BEAM BRIDGE

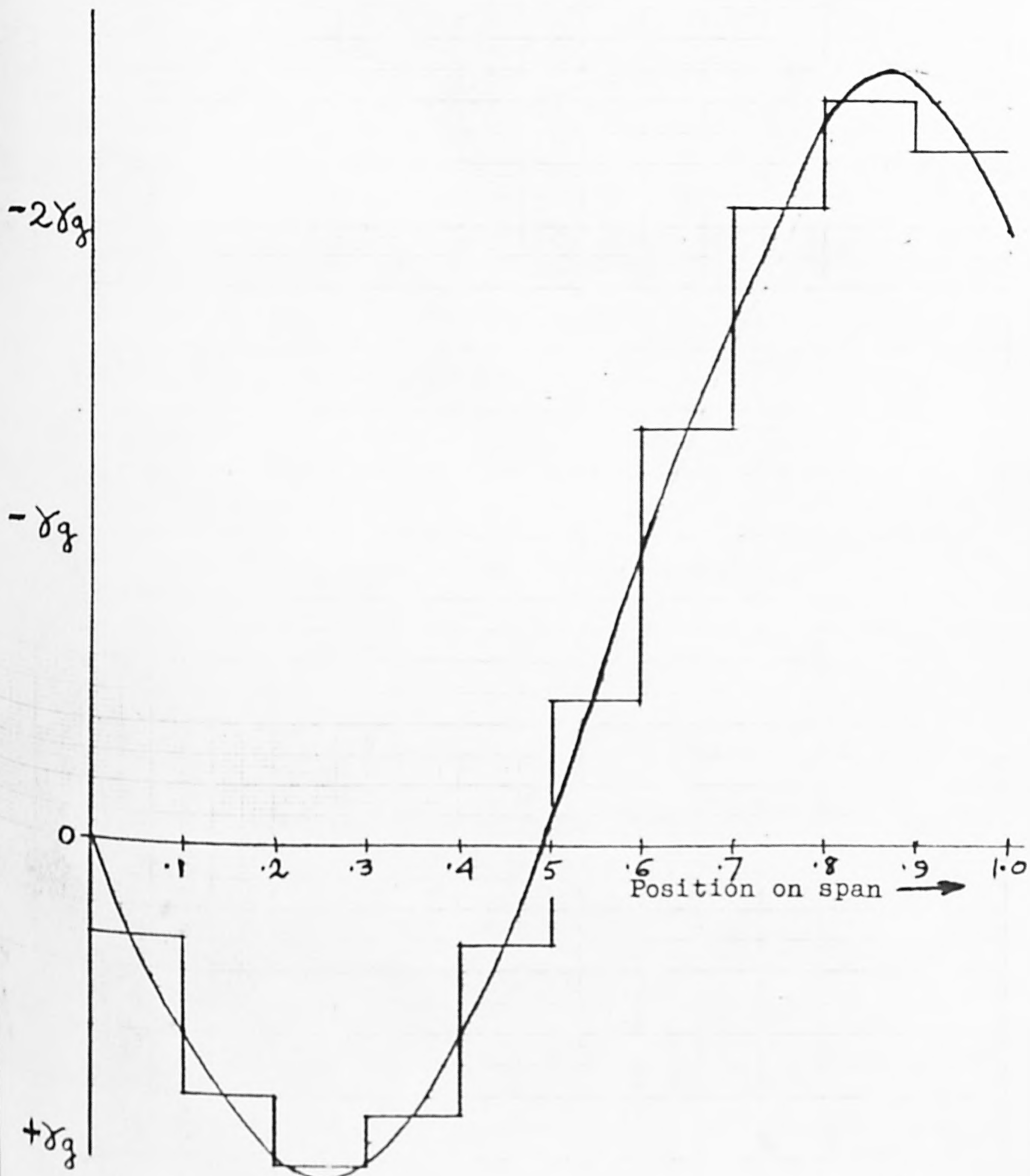
GRAPH 4.1

MIDSPAN ACCELERATION OF THE BEAM DUE TO A SINGLE LOAD CROSSING AT CRITICAL SPEED (β = 1.60)

Comparison of the proposed method with the classical method.

The curve is obtained from double differentiation of equation A2.5 (with x = L/2)

Straight line segments are the midspan accelerations in the ten stages of the proposed method (YYY in equation 4.20)

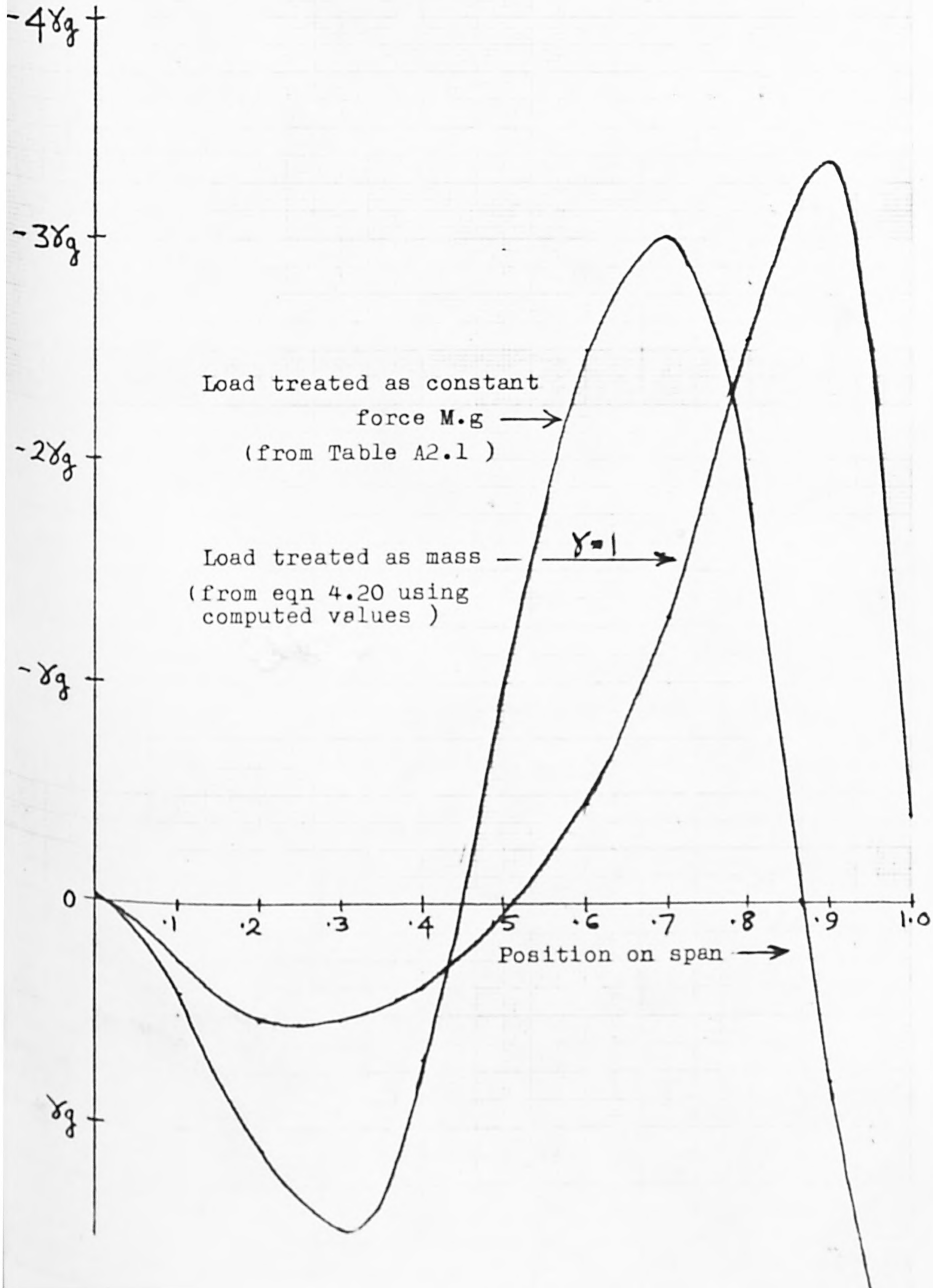


GRAPH 4.2.

SIMPLE BEAM BRIDGE

SINGLE LOAD CROSSING THE SPAN AT CRITICAL SPEED (β = 1.60)
COMPARISON OF ACCELERATIONS ($\ddot{y}_M$) EXPERIENCED BY THE LOAD
TREATED ALTERNATIVELY AS FORCE OR MASS

(See also table 4.3)

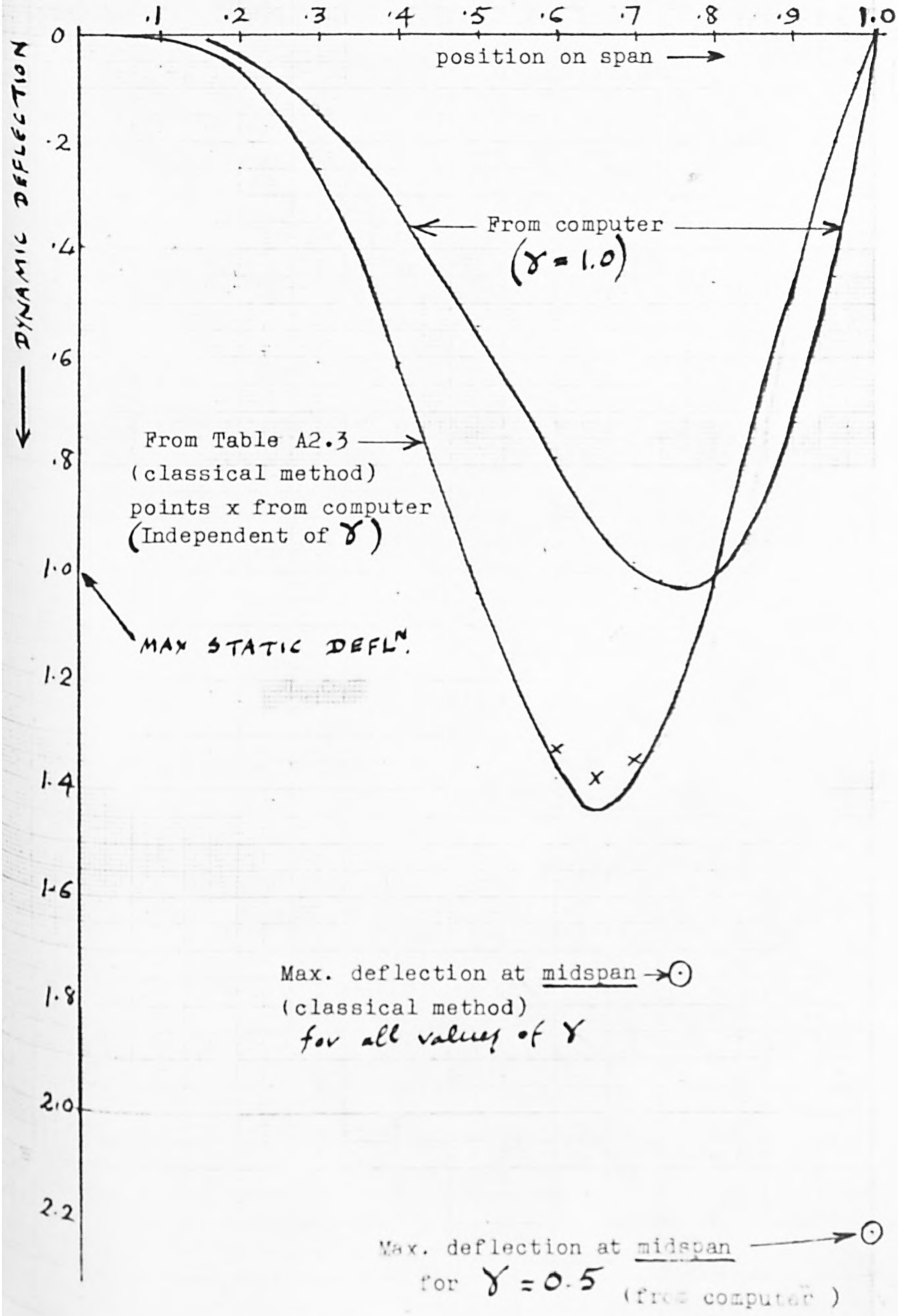


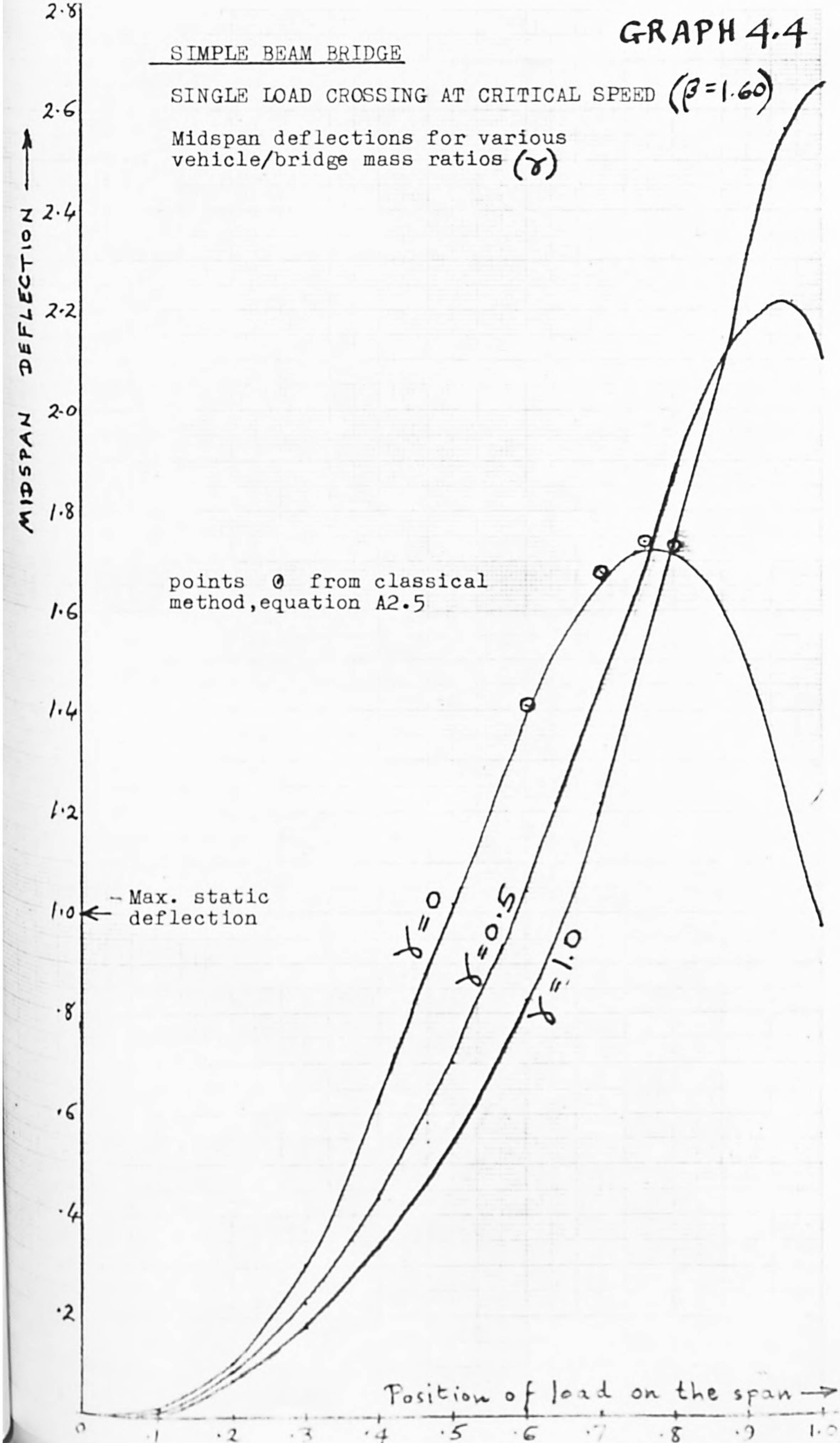
GRAPH 4.3

SIMPLE BEAM BRIDGE

SINGLE LOAD CROSSING THE SPAN AT CRITICAL SPEED (β = 1.60)

DEFLECTION OF THE BEAM BENEATH THE LOAD (Actual path of the load)




GRAPH 4.4
SIMPLE BEAM BRIDGE
SINGLE LOAD CROSSING AT CRITICAL SPEED (β=1.60)
Midspan deflections for various
vehicle/bridge mass ratios (γ)
points ⊙ from classical
method, equation A2.5
Max. static
deflection
γ=0
γ=0.5
γ=1.0
MIDSPAN DEFLECTION
Position of load on the span
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

GRAPH 4.5.

SIMPLE BEAM BRIDGE

SINGLE LOAD CROSSING AT CRITICAL SPEED ($\beta = 1.60$)

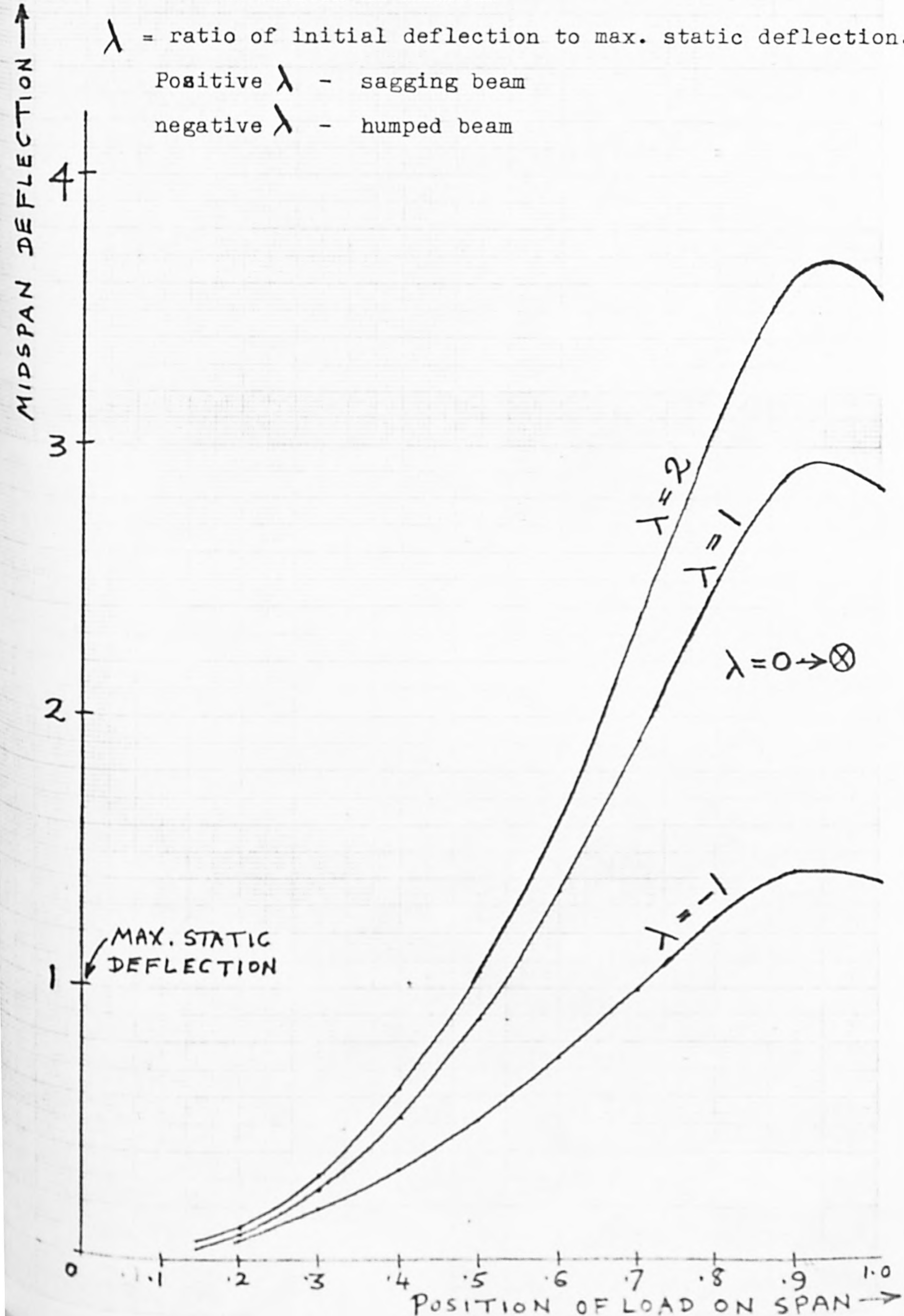
$\gamma = 0{\cdot}5$

EFFECT OF INITIAL DEFLECTION ON DYNAMIC DEFLECTION AT MIDSPAN

λ = ratio of initial deflection to max. static deflection.

Positive λ - sagging beam

negative λ - humped beam

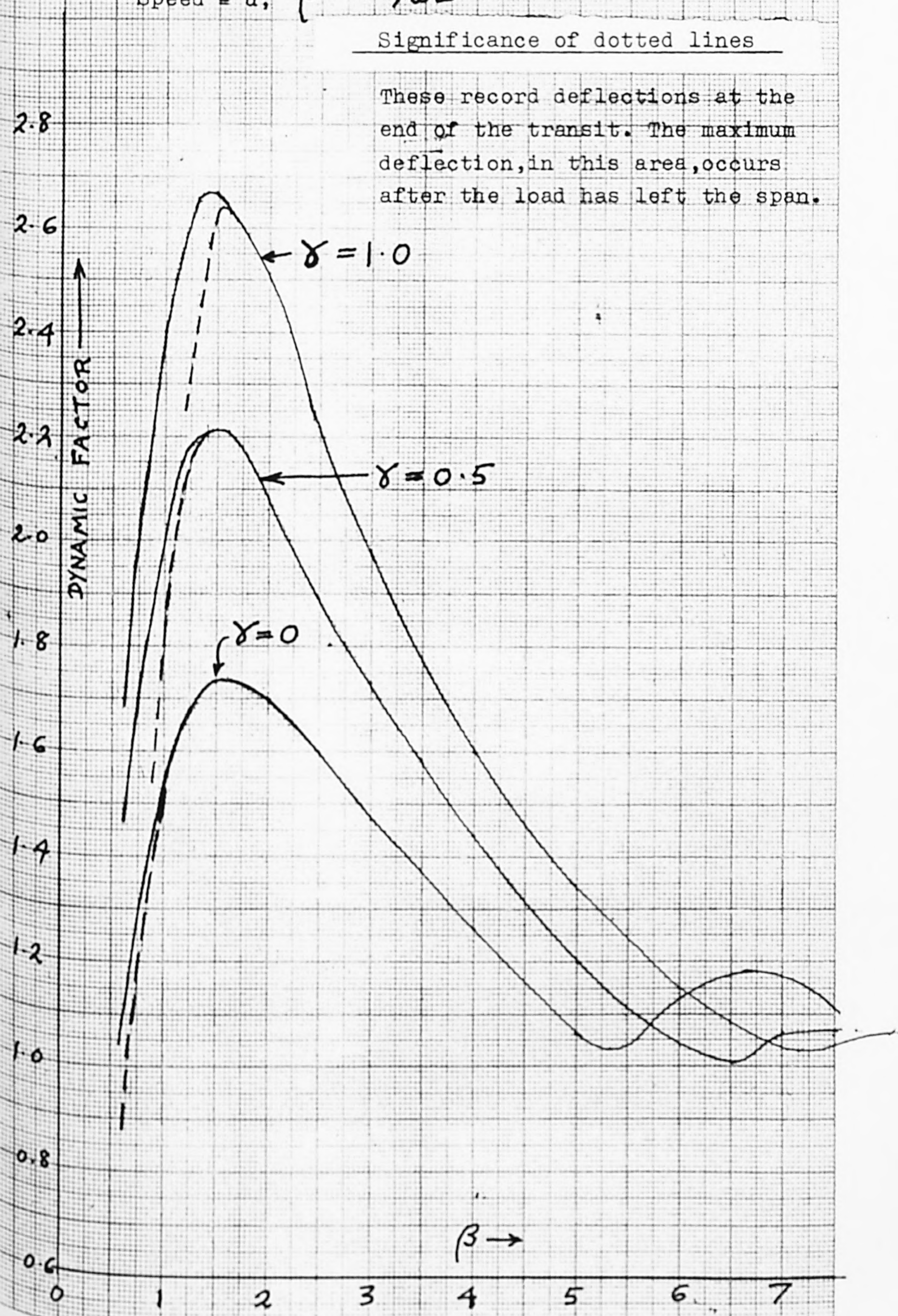


GRAPH 4.6

SIMPLE BEAM BRIDGE

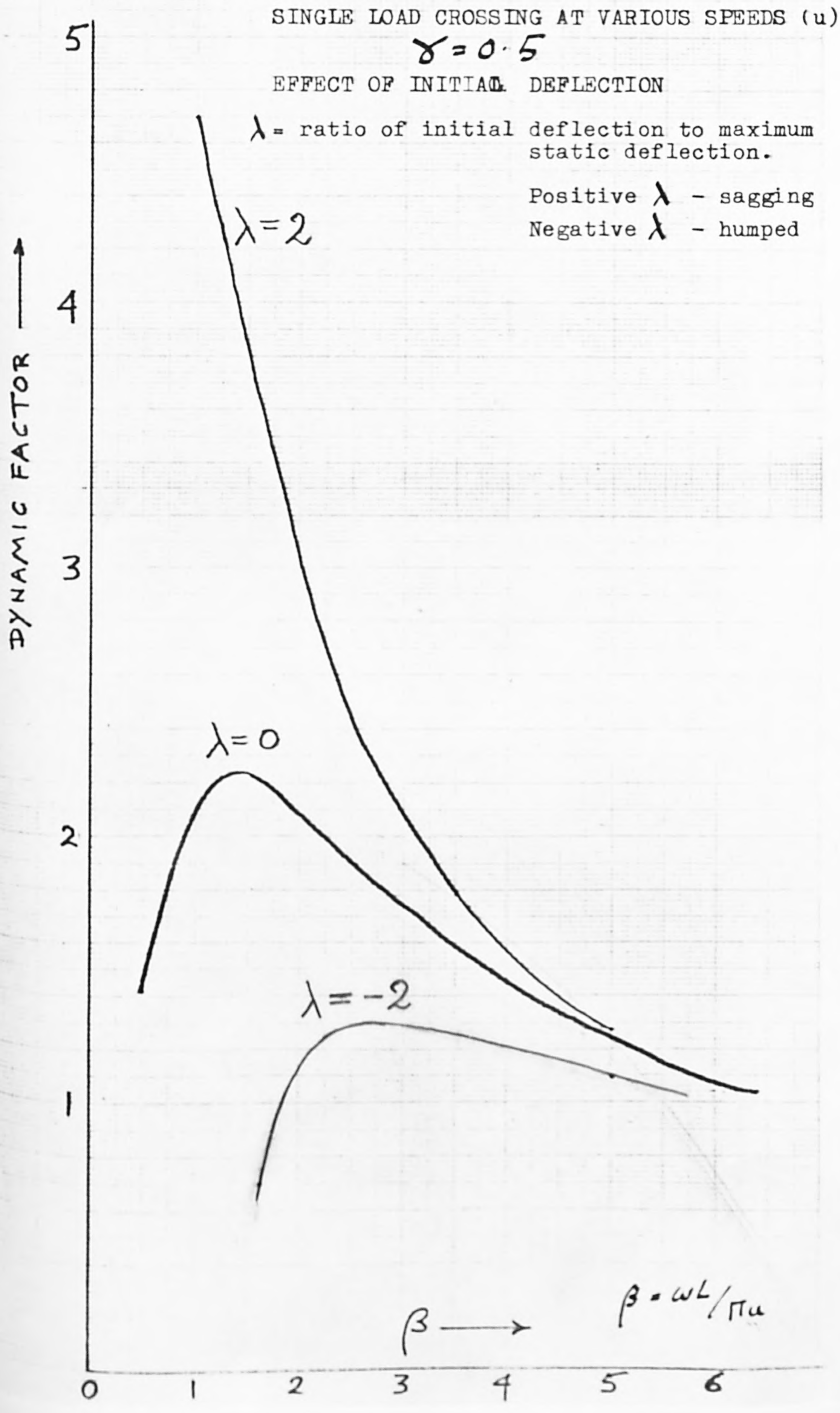
MAXIMUM EFFECTS OF A SINGLE LOAD CROSSING AT VARIOUS SPEEDS

Speed = u, $\beta = \pi u/\omega L$



SIMPLE BEAM BRIDGE

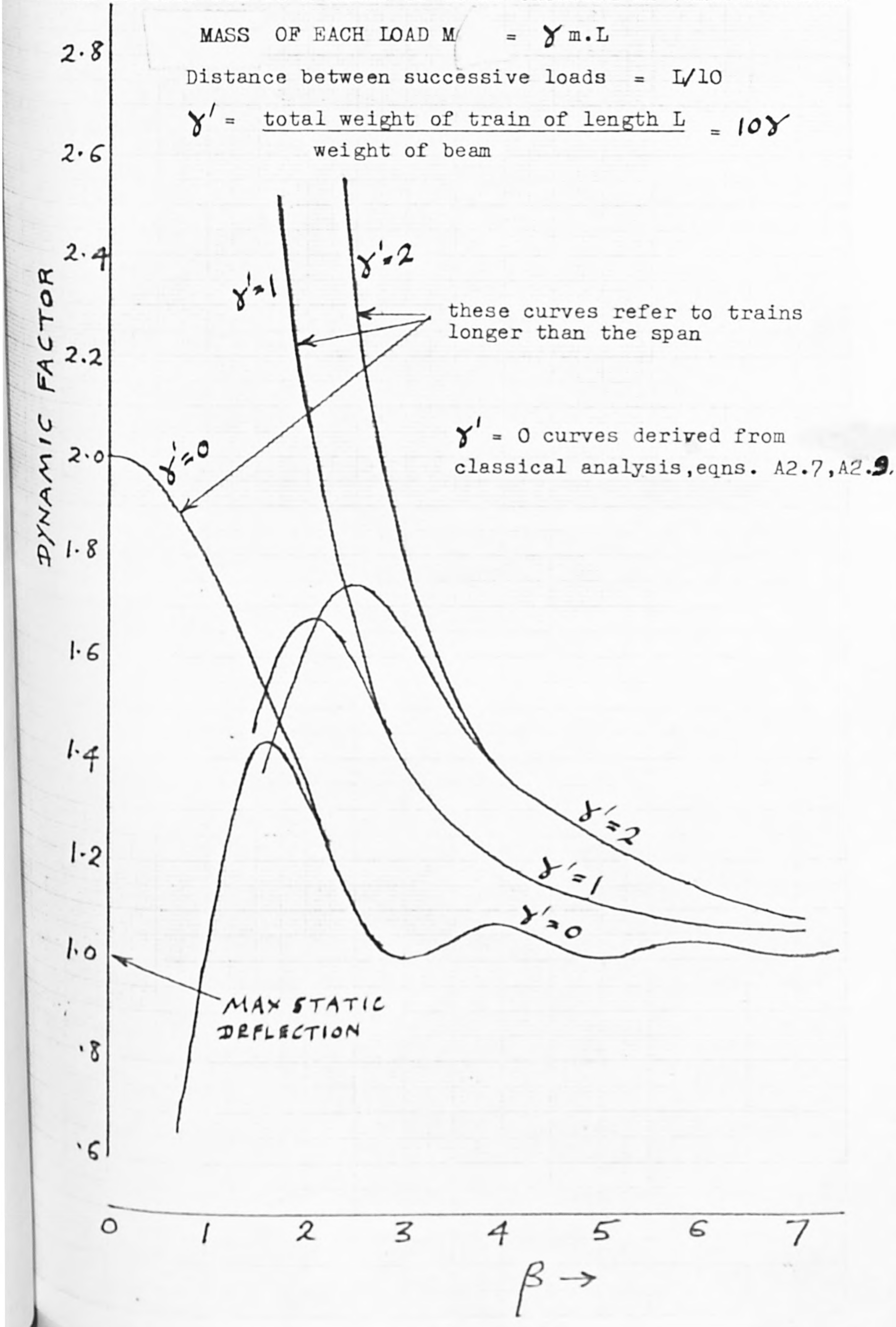
GRAPH 4.7.



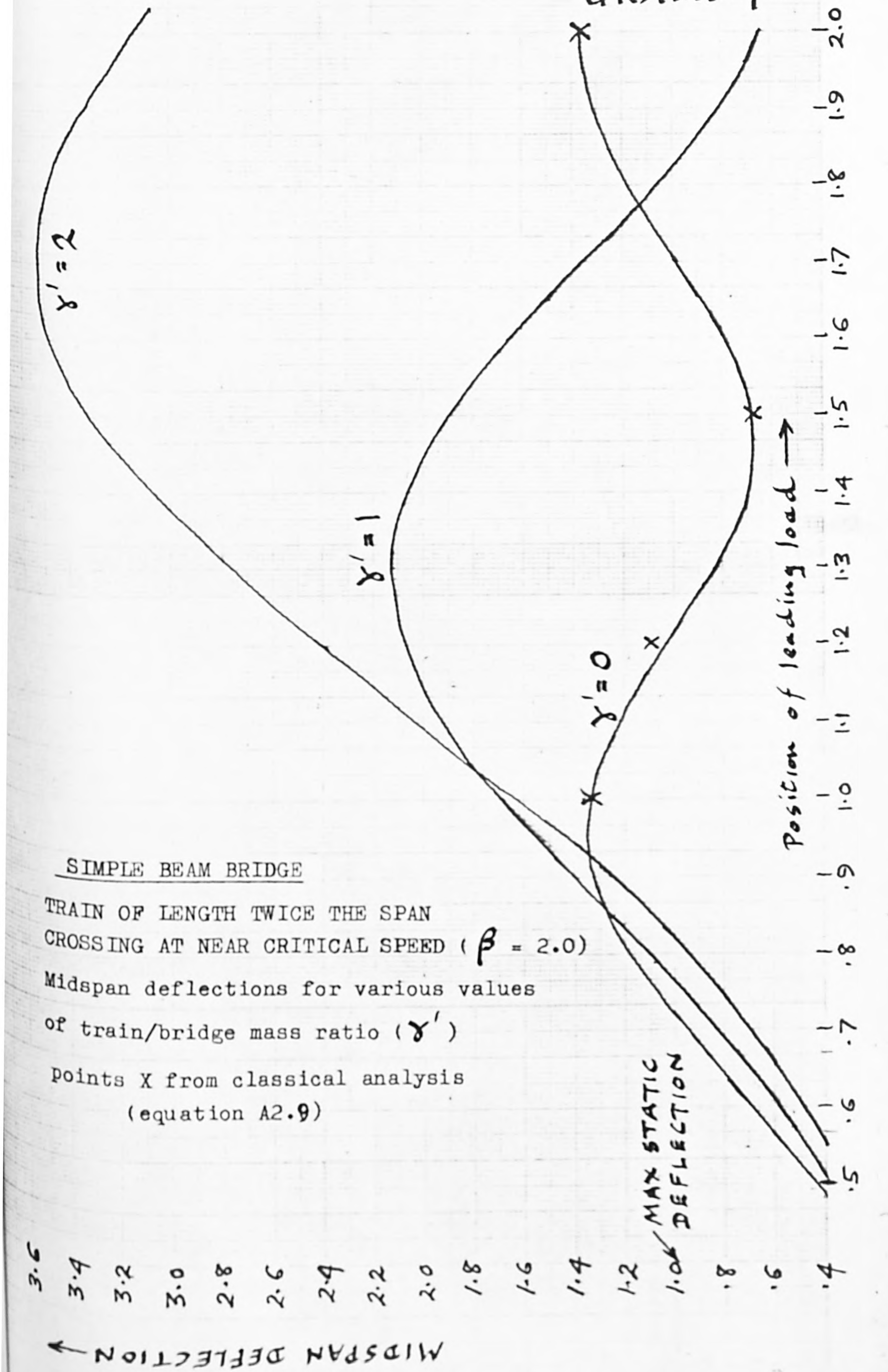
SIMPLE BEAM BRIDGE

GRAPH 4.8.

MAXIMUM EFFECTS DUE TO A TRAIN OF LOADS CROSSING AT VARIOUS SPEEDS



GRAPH 4.9



SIMPLE BEAM BRIDGE

TRAIN OF LENGTH TWICE THE SPAN CROSSING AT NEAR CRITICAL SPEED ($\beta = 2.0$)

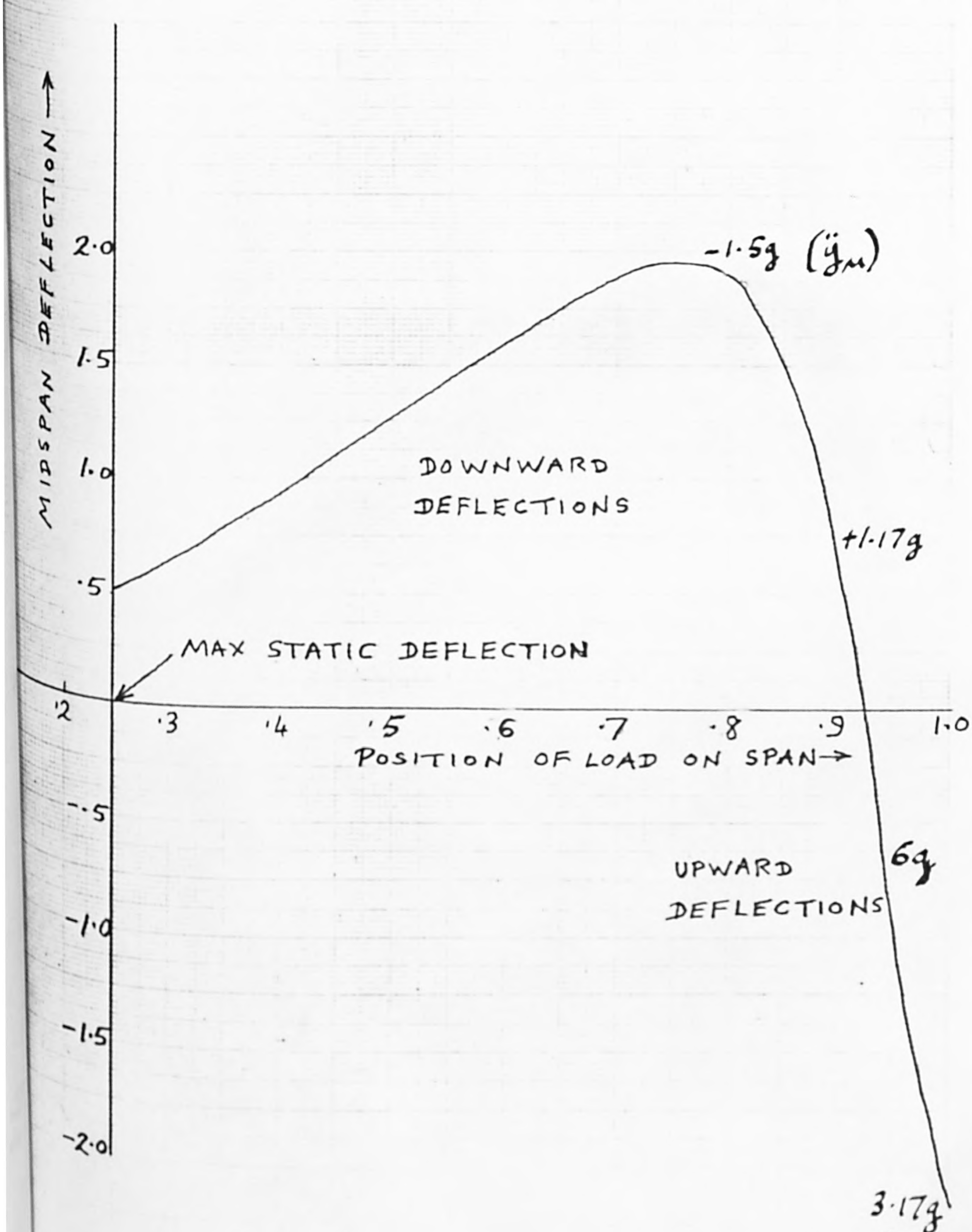
Midspan deflections for various values of train/bridge mass ratio (γ')

points X from classical analysis (equation A2.9)

GRAPH 4.10

SIMPLE BEAM BRIDGE

SINGLE VERY HEAVY LOAD ($\gamma = 5.0$) CROSSING AT A LOW SPEED ($\beta = 6.0$)



GRAPH 4.11.

SIMPLE BEAM BRIDGE

MAXIMUM EFFECTS OF A SINGLE LOAD CROSSING AT VARIOUS SPEEDS

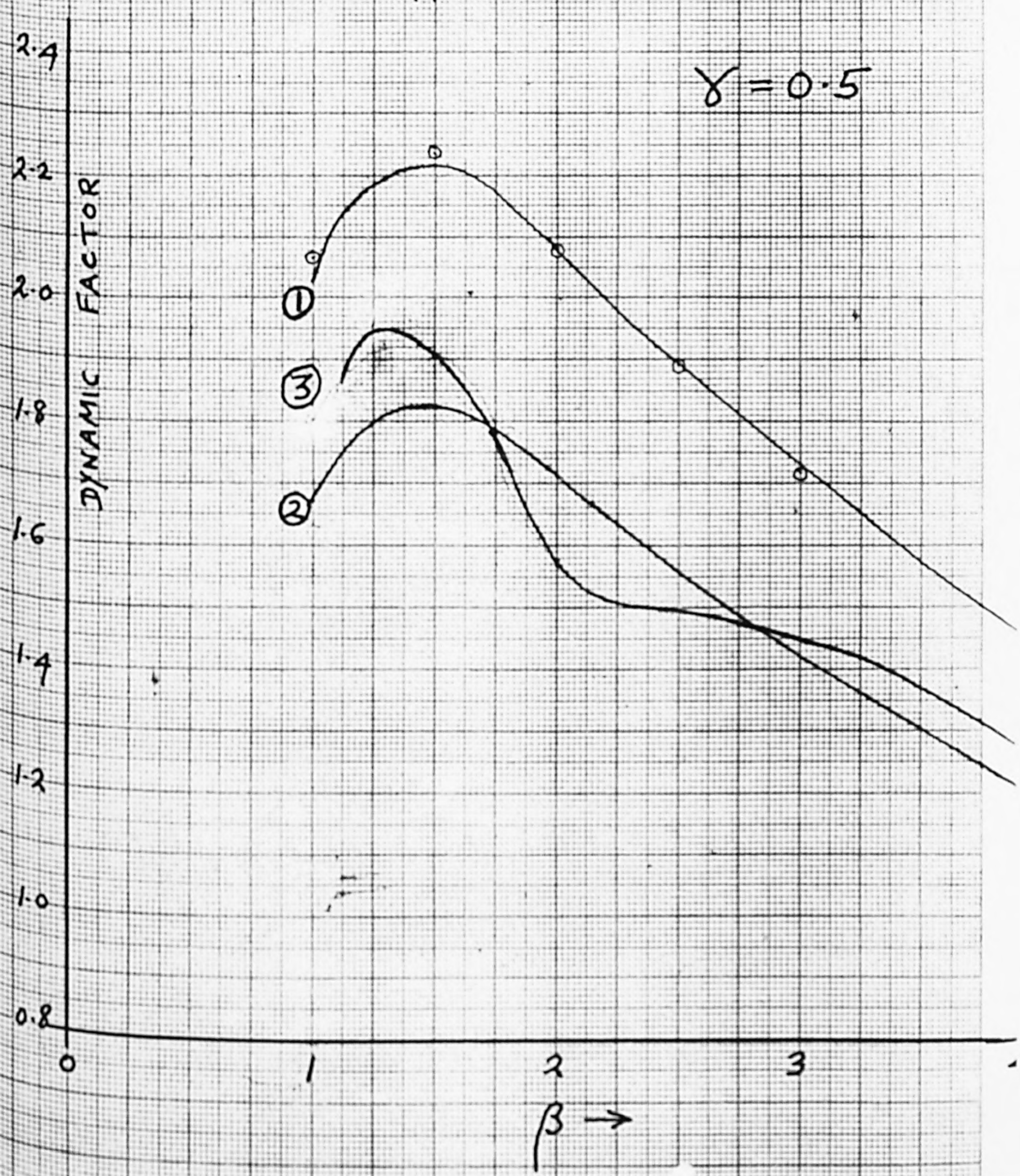
(1) Midspan deflection, as on Graph 4.6 (ten stage computation

(points ⊙ are from a sixty stage computation with two shape functions).

(2) Midspan bending moment (0.822 x ordinates (1)).

(3) Midspan bending moment from a sixty stage calculation with two shape functions.

See pp 56, 57, 58



GRAPH 4.12.

SIMPLE BEAM BRIDGE

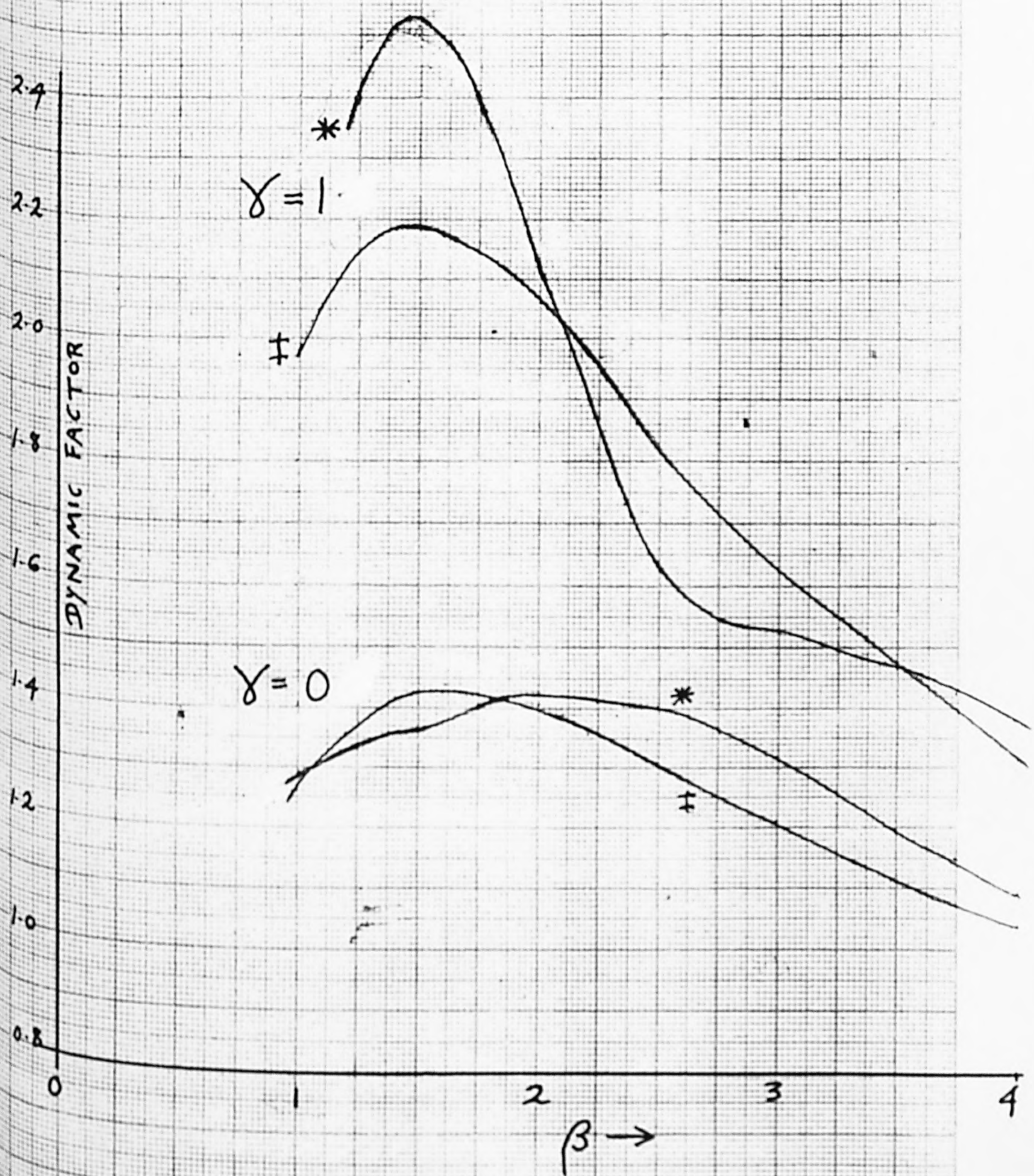
MAXIMUM EFFECTS OF A SINGLE LOAD CROSSING AT VARIOUS SPEEDS

MIDSPAN BENDING MOMENTS

‡ from ten stage calculation

* from sixty stage calculation with two shape functions

See pp 56, 57, 58



4.7 COMMENTS ON GRAPHS

Graph 4.1

This graph gives information about the accuracy obtainable with the ten stage computation. An assessment may be made by comparing the computed midspan accelerations of the beam with those from the classical method based on the response of the first mode only. Since the latter method treats the load as an unvarying force, the computed results have been deliberately limited in the same way (by assigning zero values to t_Y, t_{YY} and t_A in equation 4.17).

Graph 4.2

The accuracy revealed by Graph 4.1 was thought sufficiently encouraging to proceed to examination of the difference in dynamic response due to treating the load as a mass, this being a main object of interest.

Graph 4.2 reveals this difference, in the case of a transit at critical speed, in terms of the accelerations experienced by the load itself as it crosses the span. It may be seen that the classical method predicts unrealistically high downward accelerations near the beginning of the transit.

The classical curve would be the same for any value of the live/dead load ratio γ, but the computed curve would be different for any other value of γ, because of the more rigorous allowance for kinetic effects.

Graphs 4.3 and 4.4

Here the contrast between "force" and "mass" loading, for a critical speed of transit, is shown in terms of the instantaneous deflection of the beam at the position of the moving load (Graph 4.3) and at midspan (Graph 4.4).

In both graphs the curves for "force" loading are obtainable either from the classical method or from the computer programme. Discrepancies between these two alternatives are due to the fact that the programme is based on only ten stages, but it can now be seen that these discrepancies are much less than those between "force" and "mass" loading.

Since the dynamic deflections are expressed as a factor on static deflection, the classical results are independent of γ. Comparable computed points are labelled $\gamma = 0$ (in Graph 4.4) because the programme is so arranged that this statement eliminates kinetic effects of the load and, in effect, reduces it to an unvarying force.

Graphs 4.5 and 4.7

These graphs show the effect of initial deflection of the beam due to either under cambering or over cambering. It must be emphasised that the statement $\lambda = 0$ means that the cambering is perfect, i.e. that the apparent dead load deflection is zero and that the beam is therefore perfectly free from vertical curvature before the load runs on to it. The results shown on these graphs can only be obtained if the load is treated strictly as a mass.

Graphs 4.6 and 4.8

These graphs constitute a concise presentation of the results aimed for in this chapter. They may be allowed to speak for themselves with the following reservations:

(a) Their accuracy is dubious when $\beta > 5$ because they are based on a computation in only ten stages (see comments at the end of Chapter 3).

(b) In their evidently successful and convincing region ($1 < \beta < 5$) there will be some inherent error due to the simple sinusoidal shape function assumed in equation (4.1). This error will be similar to that arising from a classical analysis based on the response of the first mode only. Certainly the error will be small compared to the discrepancies arising from treating the load as a constant force (i.e. regarding the $\gamma = 0$ curves as the only possibilities).

(c) Some of the higher dynamic factors will be unattainable, in practice because the implied accelerations, which the load would experience, at some time would exceed a downward acceleration of g. This is expecially true in graph 4.8 where it concerns trains of loads longer than the beam itself. For instance when $\beta = 2.0$, $\gamma = 0.2$, a maximum dynamic factor 3.52 was computed for a train twice as long as the span. However, using equation (4.20) it can be shown that the eleventh load would need to have a downward acceleration of 1.5 g as it ran on to the beam which is, of course, not possible. Also, even higher upward accelerations are implied in some cases, this situation being well illustrated by graph 4.2.

Upward accelerations greater than "g" are not impossible, but nevertheless it can be seen that Grpahs 4.6 and 4.8 contain areas beyond the limits of practical probability. However, this is partly due to the high values of γ examined, and also it may be recalled that high dynamic factors can be alleviated by "humping" as shown, for example, in Graph 4.5. A modest amount of "humping" would seem to be a reasonable method of reducing effects of high speed traffic especially as it only needs to be applied to the track or carriageway and not to the basic structure.

Graph 4.10

This is presented as a "curiousity" being outside practical limits in that the load weighs five times as much as the beam itself. The downward (positive) acceleration of the load is very high as it nears the end of its transit and implies that the load would jump off the beam in this region. This effect must be due to Coriolis and centripetal components of acceleration arising from the sudden upward deflection of the beam in the final stages.

4.8 DYNAMIC BENDING MOMENTS

So far the dynamic response of beams has been evaluated only in terms of deflections. The response in terms of bending moment is of greater practical importance and will now be considered.

With the simple shape function, used up to now, the maximum deflection and bending moment are bound to occur simultaneously at midspan. Consider the basic displacement/time function (equation 4.1) which may be restated as follows:

$$y(x,t) = Y(t) \sin \frac{\Pi x}{L}, \text{ where Y is midspan deflection.}$$

Now
$$BM = - EI \frac{\partial^2 y}{\partial x^2} = \frac{\Pi^2 EI}{L^2} Y(t) \sin \frac{\Pi x}{L}$$

$$BM_{MAX} = \frac{\Pi^2 EI}{L^2} Y(t)_{MAX}$$

The static midspan bending moment can be expressed as $12\ EI\Delta/L^2$ where Δ is the midspan static deflection $MgL^3/48EI$.

So the dynamic factor for bending moment is

$$\frac{\Pi^2}{12} \frac{y_{MAX}}{\Delta} = \frac{\Pi^2}{12} D = 0.822\ D$$

where D is the dynamic factor for deflection as reported on Graphs 4.6 and 4.8.

However, it is wellknown that, in the classical method, dynamic bending moments can be markedly affected by the contribution of higher modes. To take this into account, in the Hamiltonian step by step method, it will be necessary to extend the displacement/time function beyond the simple form of equation 4.1.

It is now intended to introduce some further computations based on the following displacement/time function in lieu of equation 4.1.

$$y = W \sin \frac{\Pi x}{L} + Z \sin \frac{3\Pi x}{L} \qquad (4.21)$$

where $W = Bt^2 + \dot{W}_o t + W_o$

and $Z = Ct^2 + \dot{Z}_o t + Z_o$

The additional shape function $(\sin \frac{3\Pi x}{L})$ has been chosen so that the significant additional bending moment will also occur at midspan. We now have:

$$- EI \frac{\partial^2 y}{\partial x^2} = \frac{\Pi^2 EI}{L^2} (W \sin \frac{\Pi x}{L} + 9Z \sin \frac{3\Pi x}{L})$$

$\therefore$ Max. bending moment (at midspan) $= \frac{\Pi^2 EI}{L^2} (W(t) - 9Z(t))_{MAX}$

Now if W' and Z' are the representative numbers in the computer printout, then:

$$\text{Max. bending moment} = \frac{\Pi^2 EI}{L^2} (W' - 9Z')_{MAX} \left(\frac{MgL^3}{EI} \right)$$

$$= \Pi^2 MgL (W' - 9Z')_{MAX}$$

Max. static bending moment $= MgL/4$

Dynamic factor for bending moment

$$= 4\Pi^2 (W' - 9Z')_{MAX}$$

Computation Scheme

With two terms in the displacement/time function, as now proposed, it becomes necessary to introduce two small variations (one in each of the time functions W and Z). It follows that two simultaneous equations must be solved at the end of every computation stage in order to obtain current values of W' and Z'.

The preliminary procedure for handling the kinetic effect of the load becomes too complex to be dealt with by the direct method used heretofore and the approach used in Chapters 5 and 7 will be used instead. This matter is fully explained in those chapters. At the same time the number of calculation stages has been increased from ten to sixty.

A selection of the results obtained is given on Graphs 4.11 & 4.12.

Comments on Graphs 4.11 and 4.12

These graphs present information comparable to Graph 4.6 and therefore reveal the improvement obtained by employing the more rigorous displacement/time function (equation 4.21).

Graph 4.11 compares deflection and bending moment factors obtained by both the programs and it can be seen that, so far as deflections are concerned, the difference is not significant. The difference as regards bending moments, however, is considerable and it must be concluded that the more rigorous program is essential when the load is treated as a mass and very high speeds of transit are to be considered.

Graph 4.12 shows bending moment factors only, and caters for two other values of the live/dead load ratio γ. It can be seen that the need to employ the more rigorous program is more urgent in the case of higher γ values. In other words the dynamic bending moment contributed by higher modes is more important when the load is treated as a mass.

CHAPTER 5

BEHAVIOUR OF A CANTILEVER BRIDGE

CHAPTER 5

BEHAVIOUR OF A CANTILEVER BRIDGE UNDER THE ACTION OF MOVING MASS LOADS

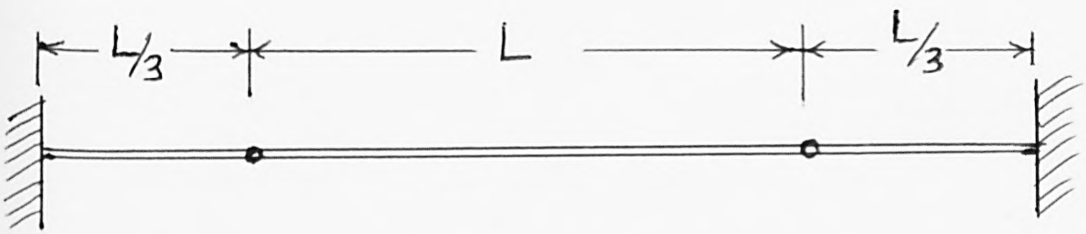


Figure 5.1

5.1 The bridge to be studied has the general layout shown in Figure 5.1. It will be assumed to have the same cross sectional properties throughout its length and to have a uniform mass m per unit length. Alternative data can be introduced without difficulty when specified.

The method of computation will be generally similar to that employed in Chapter 4, but there will be considerable differences of detail.

The most significant difference is that it will be necessary to adopt independent displacement/time functions for the three parts of the bridge (two cantilevers and one suspended span). This means that in applying Hamilton's Principle it will be necessary to introduce three variations instead of one, and to determine the separate effect of each variation on the energy quantities T,V and P. Then three equations (of the form of equation 2.1) will be produced which must be solved simultaneously at the end of each stage of the calculation.

In view of the greater complexity of the problem it has been decided to conduct it in one hundred stages (20 stages while the load, or the leading load, traverses each cantilever, and 60 stages while it traverses the main span).

The proposed displacement/time functions, including the small variations, are shown in detail in Figure 5.2.

Expressions for δV, δT_1, δT_2, δP, for substitution in equation (2.1) will now be developed. Suffixes, a, b, or c, will be appended to indicate which of the three variations is involved.

5.2 STRAIN ENERGY V

$$V = \tfrac{1}{2}EI(Ct^2 + \dot{Z}_o t + Z_o + c\sin qt)^2 \frac{\Pi^4}{16\ell^4}\int_o^\ell \cos^2\frac{\Pi x}{2\ell}\,dx$$

$$+ \tfrac{1}{2}EI(At^2 + \dot{Y}_o t + Y_o + a\sin qt)^2 \frac{\Pi^4}{L^4}\int_o^L \sin^2\frac{\Pi x}{L}\,dx \quad *$$

$$+ \tfrac{1}{2}EI(Bt^2 + \dot{W}_o t + W_o + b\sin qt)^2 \frac{\Pi^4}{16\ell^4}\int_o^\ell \cos^2\frac{\Pi x}{2\ell}\,dx$$

i.e.
$$V = \tfrac{1}{2}EI\left[\frac{\Pi^4}{32\ell^3}(Ct^2 + \dot{Z}_o t + Z_o + c\sin qt)^2 + \frac{\Pi^4}{32\ell^3}(Bt^2 + \dot{W}_o t + W_o + b\sin qt)^2 + \frac{\Pi^4}{2L^3}(At^2 + \dot{Y}_o t + Y_o + a\sin qt)^2\right]$$

* Main span strain energy is related to the elastic deflection. So the brackets containing B and C, in the general expression for y (Figure 5.2) are omitted, because they represent a "rigid body" movement carried over from the motion of the cantilevers.

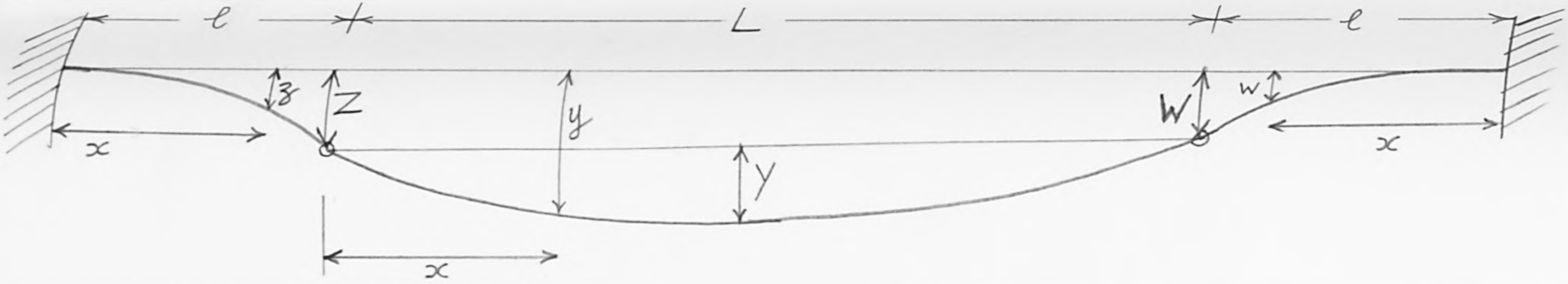


Figure 5.2

DISPLACEMENT/TIME FUNCTIONS FOR THE CANTILEVER BRIDGE

$$z = (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)(1 - \cos \frac{\Pi x}{2\ell})$$

$$y = (At^2 + \dot{Y}_o t + Y_o + a \sin qt)\sin \frac{\Pi x}{L}$$

$$+ (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)(1 - \frac{x}{L})$$

$$+ (Bt^2 + \dot{W}_o t + W_o + b \sin qt)\frac{x}{L}$$

$$w = (Bt^2 + \dot{W}_o t + W_o + b \sin qt)(1 - \cos \frac{\Pi x}{2\ell})$$

Notes : Suffix 0 indicates a value of a principal deflection at the beginning of a stage of calculation. (t = 0 at the beginning of every stage).

The three variations are, in effect, introduced successively, not simultaneously.

'x' has a different meaning in each of three expressions but its origin is obvious from the form of the shape functions.

$$\delta V_a = \frac{EI\Pi^4 a}{2L^3}(At^2 \sin qt + \dot{Y}_o t \sin qt + Y_o \sin qt)$$

$$\delta V_c = \frac{EI\Pi^4 c}{32\ell^3}(Ct^2 \sin qt + \dot{Z}_o t \sin qt + Z_o \sin qt)$$

$$\delta V_b = \frac{EI\Pi^4 b}{32\ell^3}(Bt^2 \sin qt + \dot{W}_o t \sin qt + W_o \sin qt)$$

$$\int_o^{\Pi/q} \delta V_a dt = \frac{EI\Pi^4 a}{2L^3}\left(\frac{5.870A}{q^3} + \frac{\Pi\dot{Y}_o}{q^2} + \frac{2Y_o}{q}\right)$$

$$\int_o^{\Pi/q} \delta V_c dt = \frac{EI\Pi^4 c}{32\ell^3}\left(\frac{5.870C}{q^3} + \frac{\Pi\dot{Z}_o}{q^2} + \frac{2Z_o}{q}\right)$$

$$\int_o^{\Pi/q} \delta V_b dt = \frac{EI\Pi^4 b}{32\ell^3}\left(\frac{5.870B}{q^3} + \frac{\Pi\dot{W}_o}{q^2} + \frac{2W_o}{q}\right)$$

NOW PUT $\ell = 1/3L$ and $q = 60\Pi u/L$ for a 100 stage calculation

$$\left(\frac{\text{total bridge length}}{100} = \frac{1\ 2/3\ L}{100} = \frac{L}{60}\right)$$

$$\int \delta V_a dt = \frac{EIa}{u^3}\left(.000042688A + .0043064\ \dot{Y}_o \frac{u}{L} + .516771\ Y_o \frac{u^2}{L^2}\right)$$

$$\int \delta V_c dt = \frac{EIc}{u^3}\left(.000072036C + .0072671\ \dot{Z}_o \frac{u}{L} + .872051\ Z_o \frac{u^2}{L^2}\right)$$

$$\int \delta V_b dt = \frac{EIb}{u^3}\left(.000072036B + .0072671\ \dot{W}_o \frac{u}{L} + .872051\ W_o \frac{u^2}{L^2}\right)$$

5.3 KINETIC ENERGY OF THE BEAM ITSELF (T_1)

FOR THE MAIN SPAN:

$$\dot{y} = \underbrace{(2At + \dot{Y}_o + qa \cos qt)}_{P} \sin \frac{\Pi x}{L}$$

$$+ \underbrace{(2Ct + \dot{Z}_o + qc \cos qt)}_{Q}(1 - \frac{x}{L}) + \underbrace{(2Bt + \dot{W}o + qb \cos qt)}_{R}\frac{x}{L}$$

$$\therefore (\dot{y})^2 = P^2 \sin^2 \frac{\Pi x}{L} + Q^2 (1 - \frac{x}{L})^2 + R^2 (\frac{x}{L})^2$$

$$+ 2PQ (1 - \frac{x}{L}) \sin \frac{\Pi x}{L} + 2PR \frac{x}{L} \sin \frac{\Pi x}{L} + 2QR \frac{x}{L} (1 - \frac{x}{L})$$

$$\therefore \int_o^L (\dot{y})^2 dx = \frac{L}{2}(2At + \dot{Y}_o + qa \cos qt)^2 + \frac{L}{3}(2Ct + \dot{Z}_o + qc \cos qt)^2$$

$$+ \frac{L}{3}(2Bt+\dot{W}_o+qb \cos qt)^2 + \frac{2L}{\Pi}(2At+\dot{Y}_o+qa \cos qt)(2Ct+\dot{Z}_o+qc \cos qt)$$

$$+\frac{2L}{\Pi}(2At+\dot{Y}_o+qa \cos qt)(2Bt+\dot{W}o+qb \cos qt)+\frac{L}{3}(2Ct+\dot{Z}_o+ qc \cos qt)(2Bt+\dot{W}_o + qb \cos qt)$$

$$\text{NOW } T_1 = \tfrac{1}{2}m \int_o^L (\dot{y})^2 dx$$

$$\therefore \delta T_{1a} = ma \,(ALqt \cos qt + \dot{Y}_o \frac{L}{2} q \cos qt + \frac{2L}{\Pi} Cqt \cos qt$$

$$+ \dot{Z}_o \frac{L}{\Pi} q \cos qt + \frac{2L}{\Pi} Bqt \cos qt + \dot{W}_o \frac{L}{\Pi} q \cos qt)$$

$$\delta T_{1c} = mc \,(\frac{2L}{3} Cqt \cos qt + \dot{Z}o \frac{L}{3} q \cos qt + \frac{2L}{\Pi} A\, qt \cos qt$$

$$+ \dot{Y}_o \frac{L}{\Pi} q \cos qt + \frac{L}{3} B\, qt \cos qt + \frac{L}{6} \dot{W}_o\, q \cos qt)$$

$$\delta T_{1b} = mb \,(\frac{2L}{3} B\, qt \cos qt + \dot{W}_o \frac{L}{3} q \cos qt + \frac{2L}{\Pi} A\, qt \cos qt$$

$$+ \dot{Y}_o \frac{L}{\Pi} q \cos qt + \frac{L}{3} C\, qt \cos qt + \frac{L}{6} \dot{Z}_o\, q \cos qt)$$

FOR THE LEFT CANTILEVER:

$$\dot{z} = (2Ct + \dot{Z}_o + qc \cos qt)(1 - \cos \frac{\Pi x}{2\ell})$$

$$(\dot{z})^2 = (2Ct + \dot{Z}_o + qc \cos qt)^2 (1 - \cos \frac{\Pi x}{2\ell})^2$$

$$\therefore T_1 = \tfrac{1}{2}m \int_0^{\ell} (\dot{z})^2 dx = \tfrac{1}{2}m(2Ct + \dot{Z}_o + qc \cos qt)^2 \; 0.2268\ell$$

$$\therefore \delta T_{1c} = mc(2C\, qt \cos qt) + \dot{Z}_o\, q \cos qt)\; 0.2268\ell$$

FOR THE RIGHT CANTILEVER:

$$\delta T_{1b} = mb\,(2Bqt \cos qt + \dot{W}_o\, q \cos qt)\; 0.2268\ell$$

EXPRESSIONS FOR δT_1 FOR THE WHOLE SYSTEM:

$$\delta T_{1c} = mc(C(\frac{2L}{3} + 0.4536\ell)qt \cos qt + \dot{Z}_o(\frac{L}{3} + 0.2268\ell)q \cos qt$$

$$+ \frac{2L}{\Pi} A\, qt \cos qt + \dot{Y}_o \frac{L}{\Pi} q \cos qt$$

$$+ \frac{L}{3} B\, qt \cos qt + \frac{L}{6} \dot{W}_o\, q \cos qt)$$

$$\delta T_{1b} = mb(B(\frac{2L}{3} + 0.4536\ell)qt \cos qt + \dot{W}_o(\frac{L}{3} + 0.2268\ell)q \cos qt$$

$$+ \frac{2L}{\Pi} A\, qt \cos qt + \dot{Y}_o \frac{L}{\Pi} q \cos qt$$

$$+ \frac{L}{3} C\, qt \cos qt + \frac{L}{6} \dot{Z}_o\, q \cos qt)$$

$$\delta T_{1a} = ma\,(ALqt \cos qt + \dot{Y}_o \frac{L}{2} q \cos qt + \frac{2L}{\Pi} C\, qt \cos qt$$

$$+ \dot{Z}_o \frac{L}{\Pi} q \cos qt + \frac{2L}{\Pi} B\, qt \cos qt + \dot{W}_o \frac{L}{\Pi} q \cos qt)$$

Now integrate δT_1 expressions from Π/q to 0 noting that

$$\int_0^{\Pi/q} qt \cos qt \, dt = -2/q \quad \text{and} \quad \int_0^{\Pi/q} q \cos qt \, dt = 0.$$

$$\int_0^{\Pi/q} \delta T_{1a} \, dt = \frac{-ma}{q} \left(2AL + \frac{4CL}{\Pi} + \frac{4BL}{\Pi} \right)$$

$$\int_0^{\Pi/q} \delta T_{1c} \, dt = \frac{-mc}{q} \left(\frac{4AL}{\Pi} + 2C\left(\frac{2L}{3} + 0.4536\ell \right) + \frac{2BL}{3} \right)$$

$$\int_0^{\Pi/q} \delta T_{1b} \, dt = \frac{-mb}{q} \left(\frac{4AL}{\Pi} + \frac{2CL}{3} + 2B\left(\frac{2L}{3} + 0.4536\ell \right) \right)$$

NOW PUT $\ell = \frac{L}{3}$ and $q = \frac{60\Pi u}{L}$ for a 100 stage calculation

$$\int \delta T_{1a} \, dt = \frac{-maL^2}{u} (.010610A + .0067547C + .0067547B)$$

$$\int \delta T_{1c} \, dt = \frac{-mcL^2}{u} (.0067547A + .0086778C + .0035368B)$$

$$\int \delta T_{1b} \, dt = \frac{-mbL^2}{u} (.0067547A + .0035368C + .0086778B)$$

5.4a KINETIC ENERGY (T_2) OF A MOVING LOAD ON THE LEFT HAND CANTILEVER

It is intended to regard t as zero at the beginning of each of the 100 stages of computation. Hence, during the 20 stages whilst the load is on the left hand cantilever, its position will be given by:

$$x_M = \left(\frac{n-1}{60}\right) L + ut$$

Note that, in the above expression, the length ratio of cantilevers and main span has been specified by putting L = 3ℓ.

The instantaneous deflection of the cantilever, at the position of the moving load, can now be expressed as follows:

$$z_M = (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)(1 - \cos \frac{3\Pi x_M}{2L})$$

$$\text{i.e. } z_M = (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)(1 - \cos(\Pi(\frac{n-1}{40}) + \frac{3\Pi ut}{2L}))$$

$$\therefore \dot{z}_M = \frac{3\Pi u}{2L}(Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)\sin(\Pi(\frac{n-1}{40}) + \frac{3\Pi ut}{2L})$$

$$+ (2Ct + \dot{Z}_o + qc \cos qt)(1 - \cos(\Pi(\frac{n-1}{40}) + \frac{3\Pi ut}{2L}))$$

$$\text{Let, } \sin(\Pi(\frac{n-1}{40}) + \frac{3\Pi ut}{2L}) = \sin\frac{3\Pi x_M}{2L} = \Theta\Theta\Theta$$

$$\text{and, } 1 - \cos(\Pi(\frac{n-1}{40}) + \frac{3\Pi ut}{2L}) = 1 - \cos\frac{3\Pi x_M}{2L} = QQQ$$

Since the computation is to be performed in 100 stages (and a larger number could be employed, if necessary) it is proposed to proceed as follows. The integration required to obtain $\int_o^{L/60u} T_2 dt$

will be approximated by splitting each stage into two subdivisions and then introducing average values ΘΘΘ and QQQ in each half stage. These average values can be expressed as follows:

$$\Theta\Theta\Theta 1 = \sin\Pi\left(\frac{4n-3}{160}\right) \;;\quad \Theta\Theta\Theta 2 = \sin\Pi\left(\frac{4n-1}{160}\right)$$

$$QQQ1 = 1 - \cos\Pi\left(\frac{4n-3}{160}\right); \quad QQQ2 = 1 - \cos\Pi\left(\frac{4n-1}{160}\right)$$

Now $T_2 = \frac{1}{2}M\,\dot{z}_M{}^2$

$$\frac{\delta T_{2c}}{M.c} = \left(\underbrace{\Theta\Theta\Theta 1^2}_{\Pi/2q \text{ to } 0} \;;\; \underbrace{\Theta\Theta\Theta 2^2}_{\Pi/q \text{ to } \Pi/2q}\right) \frac{9\Pi^2u^2}{4L^2}\left(Ct^2 + \dot{Z}_o t + \dot{Z}_o\right)\sin qt$$

$$+ \left(QQQ1^2 \;;\; QQQ2^2\right)(2Ct + \dot{Z}_o)q\cos qt)$$

$$+ \frac{3\Pi u}{2L}\left(\Theta\Theta\Theta 1.QQQ1;\; \Theta\Theta\Theta 2.QQQ2\right)(Ct^2 + \dot{Z}_o t + Z_o)q\cos qt$$

$$+ \frac{3\Pi u}{2L}\left(\underbrace{\Theta\Theta\Theta 1.QQQ1}_{\Pi/2q \text{ to } 0};\; \underbrace{\Theta\Theta\Theta 2.QQQ2}_{\Pi/q \text{ to } \Pi/2q}\right)(2Ct + \dot{Z}_o)\sin qt$$

VALUES OF INTEGRALS FOR A 100 STAGE COMPUTATION

	$\Pi/2q$ to 0	Π/q to $\Pi/2q$
$\int \sin qt\,dt$	.0053052 L/u	.0053052 L/u
$\int t \sin qt\,dt$	.00002814 L^2/u^2	.00006027 L^2/u^2
$\int t^2 \sin qt\,dt$	.000001704 L^3/u^3	.000007060 L^3/u^3
$\int q \cos qt\,dt$	1	- 1
$\int qt \cos qt\,dt$	.003028 L/u	- .013638 L/u
$\int qt^2 \cos qt\,dt$	.00001315 L^2/u^2	- .0001900 L^2/u^2

$$q = \frac{60\Pi u}{L}$$

$$\frac{1}{M.c} \int_{o}^{L/60u} \delta T_{2c} dt =$$

$$\frac{CL}{u} \Big(+.00003777(\theta\theta\theta 1)^2 + .00001565(\theta\theta\theta 2)^2$$
$$+.006359(QQQ1)^2 - .02864(QQQ2)^2$$
$$+.0003272(\theta\theta\theta 1)(QQQ1) - .0003272(\theta\theta\theta 2)(QQQ2) \Big)$$

$$+ \dot{Z}_o \Big(+.0006256(\theta\theta\theta 1)^2 + .001340(\theta\theta\theta 2)^2$$
$$+\overset{*}{1}.27(QQQ1)^2 - \overset{*}{1}.27(QQQ2)^2$$
$$+.03998(\theta\theta\theta 1)(QQQ1) - .04248(\theta\theta\theta 2)(QQQ2) \Big)$$

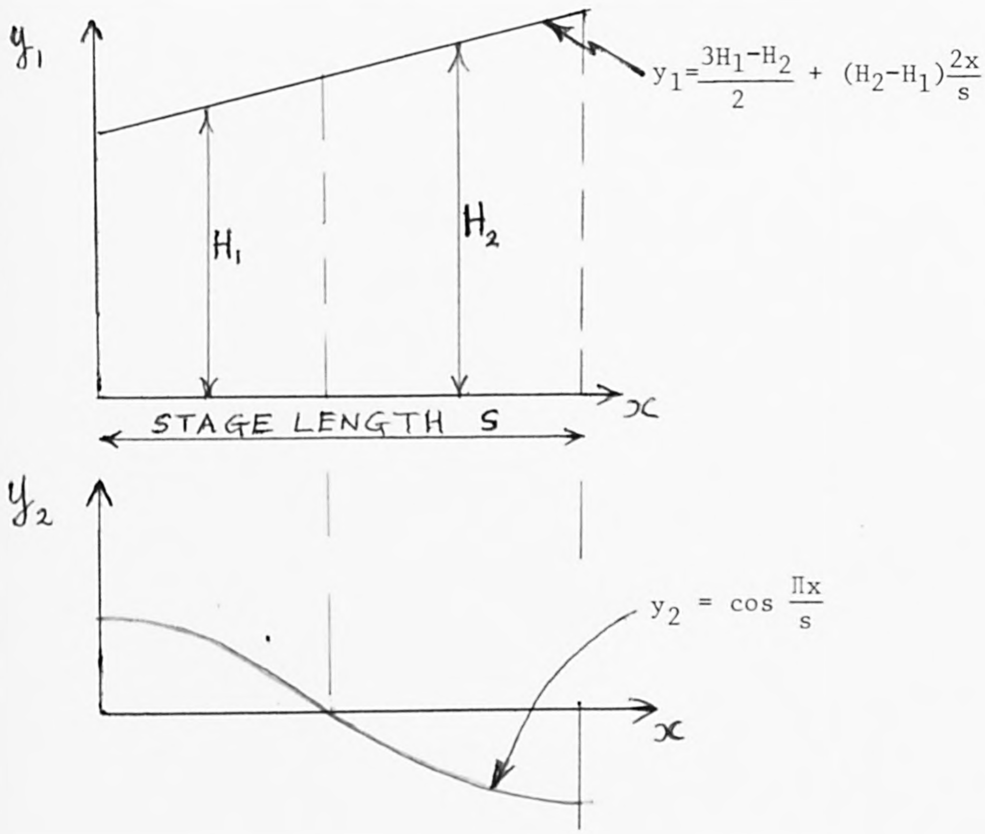
$$+ Z_o \frac{u}{L} \Big(.11781(\theta\theta\theta 1)^2 + .11781(\theta\theta\theta 2)^2$$
$$+ \overset{*}{6}(\theta\theta\theta 1)(QQQ1) - \overset{*}{6}(\theta\theta\theta 2)(QQQ2) \Big)$$

An "improvement factor" $4/\pi$ has been applied to terms* which are of the form:

$$H_1 \int_{o}^{\pi/2q} q \cos qt \, dt + H_2 \int_{\pi/2q}^{\pi/q} q \cos qt \, dt$$

where H_1 and H_2 are values at the quarter and three quarter points of a stage of the computation.

The situation may be represented graphically as shown below, assuming that H_1 and H_2 are ordinates of a straight line function (a good approximation if the stage length is very short).



The exact integral desired is:

$$\int_0^s y_1 y_2 dx = \left(\frac{3H_1 - H_2}{2}\right)\int_0^s \cos \frac{\Pi x}{s} dx + \frac{2}{s}\left(H_2 - H_1\right) \int_0^s x \cos \frac{\Pi x}{2} dx$$

$$= \frac{-4s}{\Pi^2} (H_2 - H_1)$$

The approximate method actually employed is:

$$\int_0^s y_1 y_2 dx = H_1 \int_0^{s/2} \cos \frac{\Pi x}{s} dx + H_2 \int_{s/2}^{s} \cos \frac{\Pi x}{s} dx = -\frac{s}{\Pi}(H_2 - H_1)$$

Multiplying the approximate result by $4/\Pi$ gives the exact integral.

Paragraphs 5.4b and 5.4c, which deal with kinetic energy of moving loads on the main span and the exit cantilever, have been transferred to Appendix 5. (page A.42 et seq)

5.5 POTENTIAL ENERGY OF A MOVING LOAD

Load on the Left Hand Cantilever

$$\delta P_{2a} = \delta P_{2b} = 0$$

$$\delta P_{2c} = - Mg\,c \sin qt\left(1 - \cos \frac{\Pi x_M}{2\ell}\right) = - Mg\,c\,\sin qt\,(QQQ)$$

(See paras. 5.4a) b) and c) for significance of QQQ1 and similar symbols used below)

$$\int_{o}^{\Pi/q} \delta P_{2c}\,dt = - Mg\,c\left(QQQ1 \int_{o}^{\Pi/2q} \sin qt\,dt + QQQ2 \int_{o}^{\Pi/q} \sin qt\,dt \right)$$

See para. 5.4a) for numerical values of integrals appropriate to a 100 stage calculation.

Load on the Main Span

$$\delta P_{2a} = - Mga \sin qt \sin \frac{\Pi x_M}{L}$$

$$\int_{o}^{\Pi/q} \delta P_{2a} dt = - Mga\left(QQ1 \int_{o}^{\Pi/2q} \sin qt\,dt + QQ2 \int_{\Pi/2q}^{\Pi/q} \sin qt\,dt \right)$$

$$\delta P_{2c} = - Mgc \sin qt \left(1 - \frac{x_M}{L}\right)$$

$$\int_{o}^{\Pi/q} \delta P_{2c} dt = -Mgc\left(RR1 \int_{o}^{\Pi/2q} \sin qt\,dt + RR2 \int_{o}^{\Pi/q} \sin qt\,dt \right)$$

$$\delta P_{2b} = -Mgb \sin qt \left(\frac{x_M}{L}\right)$$

$$\int_0^{\Pi/q} \delta P_{2b} dt = -Mgb\left(SS1 \int_0^{\Pi/2q} \sin qt\, dt + SS2 \int_{\Pi/2q}^{\Pi/q} \sin qt\, dt \right)$$

<u>Load on the Right Hand Cantilever</u>

$$\delta P_{2b} = -Mgb \sin qt \left(1 - \cos \frac{\Pi x_M}{2\ell} \right)$$

Note that x_M here varies from ℓ to 0, instead of from 0 to ℓ.

$$\int_0^{\Pi/q} \delta P_{2b} dt = -Mgb\left(QQ\theta 1 \int_0^{\Pi/2q} \sin qt\, dt + QQ\theta 2 \int_{\Pi/2q}^{\Pi/q} \sin qt\, dt \right)$$

$$\delta P_{2a} = \delta P_{2c} = 0$$

5.6 Having now obtained expressions for $\int \delta T dt$, $\int \delta V dt$, $\int \delta P dt$, in paras. 5.2 to 5.5, it now remains to substitute these expressions in the basic Hamiltonian equation(2.1)

viz: $$\int_{o}^{\Pi/q} (\delta T - SV - \delta P) dt = 0$$

Since the cantilever bridge problem employs three variations, equation (2.1) will yield three equations at the end of every stage of the computation. These three equations must then be solved simultaneously to obtain the necessary information to begin the next stage. They are shown below in the form in which they appear in the computer program. Symbols such as AA, YYA, YA, etc. represent the appropriate expressions in brackets from page 68 or Appendix 5. PA, PC, and PB signify expressions contained in similar brackets from para 5.5.

For the significance of Y, Z and W, see Figure 5.2. Suffix o indicates that the value is that appropriate to the beginning of a stage of computation.

For variation "a" :

$$A(-.010610 - .000042688\ \beta^2/\Pi^2 + AA.\gamma)$$
$$+C(-.0067547 + CA.\gamma) \quad + \quad B(-.0067547 + BA.\gamma)$$
$$=$$
$$+ \gamma.g.PA + .0043064\ \frac{u}{L}\ \dot{Y}_o\ \frac{\beta^2}{\Pi^2} \quad + .516771\ \frac{u^2}{L^2}\ Y_o\ \frac{\beta^2}{\Pi^2}$$
$$+ \gamma\ \frac{u}{L}\ \dot{Y}_o(-YYA) + \gamma\ \frac{u}{L}\ \dot{Z}_o\ (-ZZA) + \gamma\ \frac{u}{L}\ \dot{W}_o\ (-WWA)$$
$$+ \gamma\ \frac{u^2}{L^2}\ Y_o(-YA) + \gamma\ \frac{u^2}{L^2}\ Z_o\ (-ZA) + \gamma\ \frac{u^2}{L^2}\ W_o\ (- WA)$$

For variation "c" :

$$A(-\ .0067547 + AC.\gamma)$$

$$+\ C(-.0086778 - .000072036\ \beta^2/\Pi^2 + CC.\gamma)$$

$$+\ B(-\ .0035368 + BC.\gamma)$$

$$=$$

$$+\ \gamma.g.PC + .0072671\ \frac{u}{L}\ \dot{Z}_o\ \frac{\beta^2}{\Pi^2} + .872051\ \frac{u^2}{L^2}\ Z_o\ \frac{\beta^2}{\Pi^2}$$

$$+\ \gamma\ \frac{u}{L}\ \dot{Y}_o\ (-YYC) + \gamma\ \frac{u}{L}\ \dot{Z}_o\ (-ZZC) + \gamma\ \frac{u}{L}\ \dot{W}_o\ (-WWC)$$

$$+\ \gamma\ \frac{u^2}{L^2}\ Y_o\ (-YC) + \gamma\ \frac{u^2}{L^2}\ Z_o(-ZC) + \gamma\ \frac{u^2}{L^2}\ W_o(-WC)$$

For variation "b" :

$$A(-\ .0067547 + AB.\gamma)$$

$$+\ C(-\ .0035368 + CB.\gamma)$$

$$+\ B(-\ .0086778 - .000072036\ \frac{\beta^2}{\Pi^2} + BB.\gamma$$

$$=$$

$$\gamma.g.PB + .0072671\ \frac{u}{L}\ \dot{W}_o\ \frac{\beta^2}{\Pi^2} + .872051\ \frac{u^2}{L^2}\ W_o\ \frac{\beta^2}{\Pi^2}$$

$$+\ \gamma\ \frac{u}{L}\ \dot{Y}_o(-YYB) + \gamma\ \frac{u}{L}\ \dot{Z}_o(-ZZB) + \gamma\ \frac{u}{L}\ \dot{W}_o\ (-WWB)$$

$$+\ \gamma\ \frac{u^2}{L^2}\ Y_o\ (-YB) + \gamma\ \frac{u^2}{L^2}\ Z_o\ (-ZB) + \gamma\ \frac{u^2}{L^2}\ W_o\ (-WB)$$

5.7 INTERPRETATION OF PARAMETER β IN RELATION TO THE CANTILEVER BRIDGE

In setting up the three equations derived from equation (2.1) and programming the calculations, the parameter β has been taken to be identical with its definition in previous chapters.

i.e. $\beta = \frac{\omega L}{\Pi u}$ where $\omega/2\Pi$ is the fundamental frequency of a simply supported beam of span L.

$$\therefore \quad \omega^2 = \frac{\Pi^4}{L^4} \frac{EI}{m}$$

Now the fundamental frequency of the cantilever bridge is $0.644 \frac{\Pi^4}{L^4} \frac{EI}{m} = \omega_s^2$ and its total span is 5L/3.

So the equivalent value of β for the cantilever bridge will be

$$\beta_s = \frac{5}{3}\sqrt{0.644}.\beta = 1.337\beta.$$

If this equivalent value is employed then β_s will be the ratio:

Time of transit/half fundamental period

which was the original concept of the parameter (see para. 1.2)

5.8 DAMPING

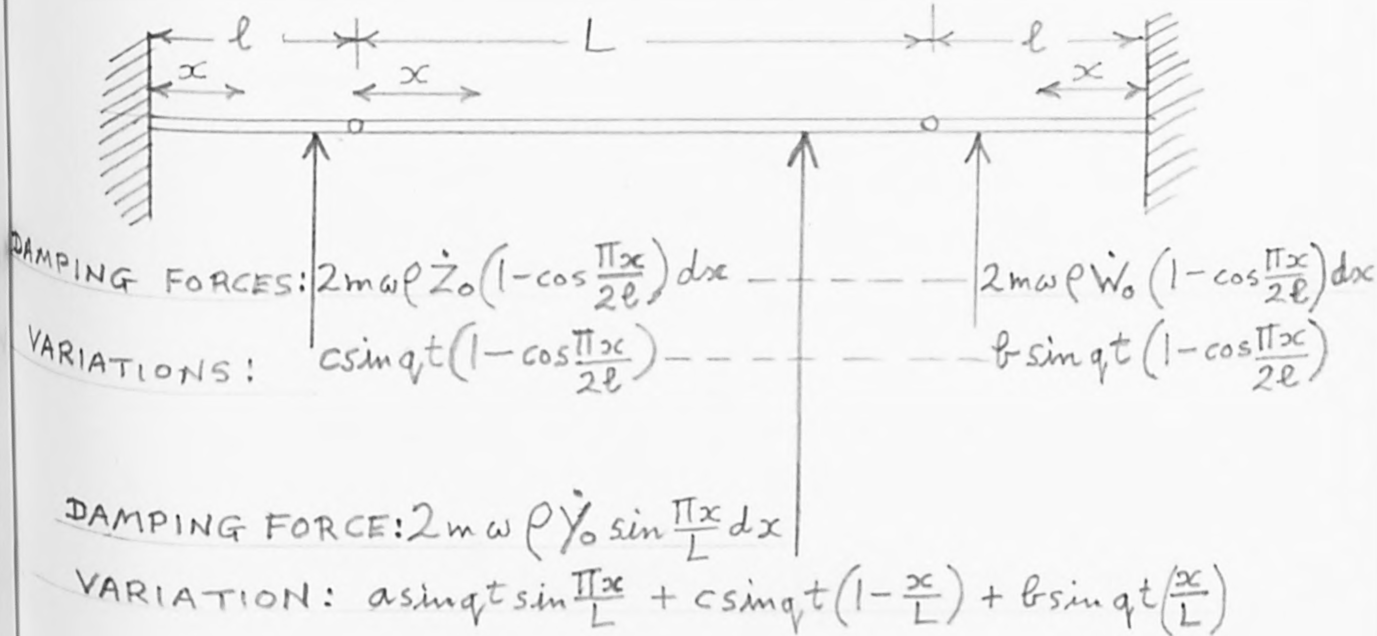


Figure 5.3

Resisting forces per unit length of span, to represent damping, will be introduced as indicated in Figure 5.3, which shows a typical damping force in each section of the cantilever bridge. The variations which will affect each force are also shown. Note that each of the damping forces is equal to the local velocity, at the beginning of a computation stage, multiplied by a constant $2m\omega\rho$, where ρ expresses the damping as a proportion of the critical value. No alteration in damping force occurs during a given stage, so that the damping forces may be treated as external applied forces.

Note that ω signifies the circular frequency of a simply supported beam of span L.

Inclusion of the damping forces in the Hamiltonian Equation (2.1)

In each case we require $\displaystyle\int_o^{\Pi/q} (\text{damping force})\text{x}(\text{variation})dt$

$$= \int_o^{\Pi/q} \delta P_d dt$$

"a" variation

$$\delta P_{d_a} = 2m\omega\,\rho\,\dot{Y}_o\ a \sin qt \int_o^L \sin^2 \frac{\Pi x}{L}\,dx = mL\omega\,\rho\,\dot{Y}_o\ a \sin qt$$

$$\therefore \int_o^{\Pi/q} \delta P_{d_a} dt = \frac{2mL\omega\rho\dot{Y}_o a}{q} = \frac{.010610 a\rho m L^2 \omega \dot{Y}_o}{u} \quad --- \ddagger$$

Putting $q = \frac{60\Pi u}{L}$, corresponding to division of the bridge, in Figure 5.3, into 100 stages when $\ell = \frac{L}{3}$.

"c" variation

$$\delta P_{d_c} = 2m\omega\rho\dot{Z}_o c \sin qt \int_o^{\ell} (1 - \cos \frac{\Pi x}{2\ell})^2\,dx$$

$$+ 2\,m\omega\rho\dot{Y}_o c \sin qt \int_o^L (1 - \frac{x}{L})\sin \frac{\Pi x}{L}\,dx$$

i.e. $\delta P_{d_c} = mL\omega\rho c \sin qt(.151174\ \dot{Z}_o + .63662\ \dot{Y}_o)$, if $\ell = \frac{L}{3}$

$$\therefore \int_o^{\Pi/q} \delta P_{d_c} dt = \frac{c\omega m L^2 \rho}{u}\left(.001604\ \dot{Z}_o + .002150\ \dot{Y}_o\right) --- \#$$

with $q = \frac{60\Pi u}{L}$

"b" variation

$$\int_{0}^{\Pi/q} \delta P_{d_b}\, dt = \frac{b\rho m L^2 \omega}{u} (.001604 \dot{W}_o + .002150 \dot{Y}_o) \text{ ——— } \flat$$

Note that $\int_{0}^{L} \frac{x}{L} \sin \frac{\Pi x}{L}\, dx = \int_{0}^{L} (1 - \frac{x}{L}) \sin \frac{\Pi x}{L}\, dx$

Program form of the damping terms

The damping terms will appear in the right sides of the three equations (para. 5.6) which are to be solved, at the end of each stage, to evaluate A, B, and C. Now the terms in these equations are all equivalent to $\frac{u}{mL^2}\int \delta E dt$. Hence the above damping expressions $\ddagger$, $\#$, $\flat$, would enter therein as:

$$(a,b,c)\rho\omega(n_y \dot{Y}_o,\ n_z \dot{Z}_o,\ n_w \dot{W}_o)$$

To produce a form compatible with para. 5.6, this can be re-written as:

$$(a,b,c)\ \frac{\rho\omega L}{u}\ (n_y \dot{Y}_o \frac{u}{L},\ n_z \dot{Z}_o \frac{u}{L},\ n_w \dot{W}_o \frac{u}{L})$$

$$= \quad (a,b,c)\rho\Pi\beta \quad (\qquad \text{ditto} \qquad)$$

where n_y, n_z and n_w, represent the numerical coefficients in $\ddagger$, $\#$, and $\flat$.

Furthermore the equations in para. 5.6 have all been multiplied through by Π^2/β^2, before programming, to make PA a coefficient of $\gamma g\ \Pi^2/\beta^2 = \frac{MgLu^2}{EI}$, thus putting the computing scheme on the same basis as that explained in chapter 4, para. 4.3.

So the items which need to appear in the actual computation are $\frac{\rho \Pi^3}{\beta}$ (n_y, n_z, n_w), which may be expressed in full as follows:

<u>"a" variation</u>

$$0.329\ \rho/\beta\ \dot{Y}_o{}' \qquad \text{from } ‡$$

<u>"b" variation</u>

$$\rho/\beta(0.04973\ \dot{Z}_o + 0.06667\ \dot{Y}_o) \qquad \text{from } \#$$

<u>"c" variation</u>

$$\rho/\beta(0.04973\ \dot{W}_o + 0.06667\ \dot{Y}_o) \qquad \text{from } \flat$$

5.9 EFFECT OF CAMBERING OR HUMPING

Cambering

In all cases, unless otherwise stated, results have been computed with the assumption that there is no initial deflection of the bridge. That is to say the deflection due to self weight has been cambered out, and the track, or carriageway, is perfectly straight and level when the load runs on to it.

Humping

Either excess camber, or track humping, will reduce the force exerted on the bridge by the mass load as it travels across the span. A modification of $\int \delta T_2 dt$ is needed, to account for this, which amounts to the introduction of initial values for Y_o, Z_o, W_o, in the last three terms on the right sides of the equations in para. 5.6 (whereas, in the case of perfect camber referred to above, these would be zero at the beginning of the first stage of computation).

It is proposed to introduce an initial value of Y_o alone which implies the existence of a sinusoidal hump extending only across the central suspended span. To fulfil the implied assumptions, however, it would be essential to provide entrance and exit profiles as shown in Figure 5.4. It appears in this diagram that Z_o, W_o, also have initial values, but these are ineffective in the computation, because they merely define the height of linear ramps. They are not principal ordinates of a shape function.

For example, an initial value $Y'_o = -.027$ would mean that the height Y_o of the hump is equal to the maximum midspan static deflection due to a single load M.g.(i.e. $0.027 \frac{MgL^3}{EI}$, when $\ell = \frac{L}{3}$).

Recall that a primed symbol, such as Y'_o above, indicates the numerical quantity in the computer program which represents its counterpart in the general theory.

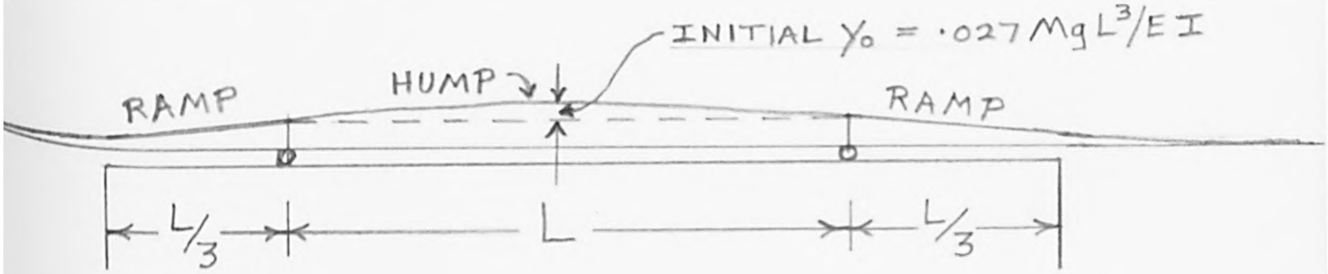


Figure 5.4

Natural Frequencies of the Cantilever Bridge

The first three natural frequencies were calculated to be as follows:

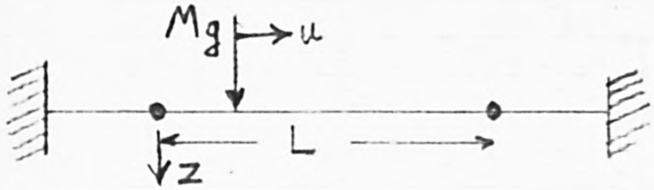
$$\omega_1^2 = 0.644 \frac{\Pi^4}{L^4} \frac{EI}{m}$$

$$\omega_2^2 = 3.48 \frac{\Pi^4}{L^4} \frac{EI}{m}$$

$$\omega_3^2 = 7.66 \frac{\Pi^4}{L^4} \frac{EI}{m}$$

GRAPH 5.1.

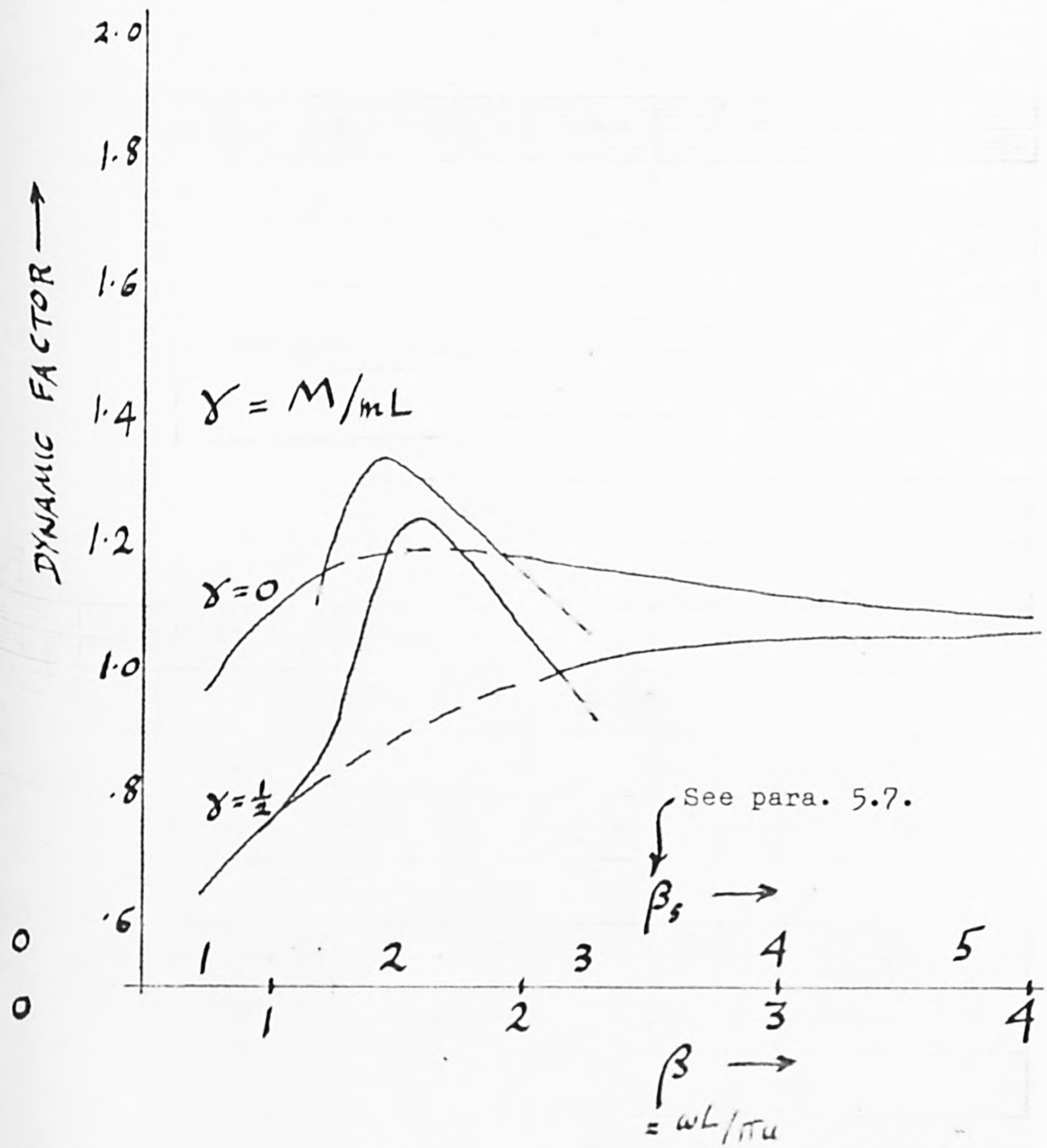
CANTILEVER BRIDGE



MAXIMUM DEFLECTIONS Z DURING TRANSIT OF A SINGLE

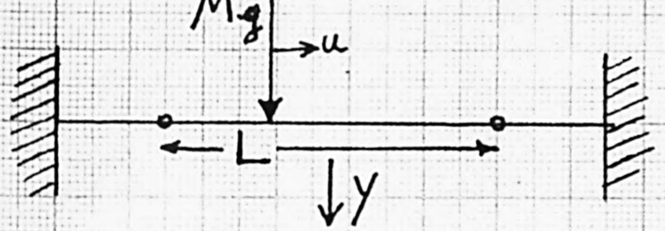
LOAD AT VARIOUS SPEEDS (u)

(No damping or humping)

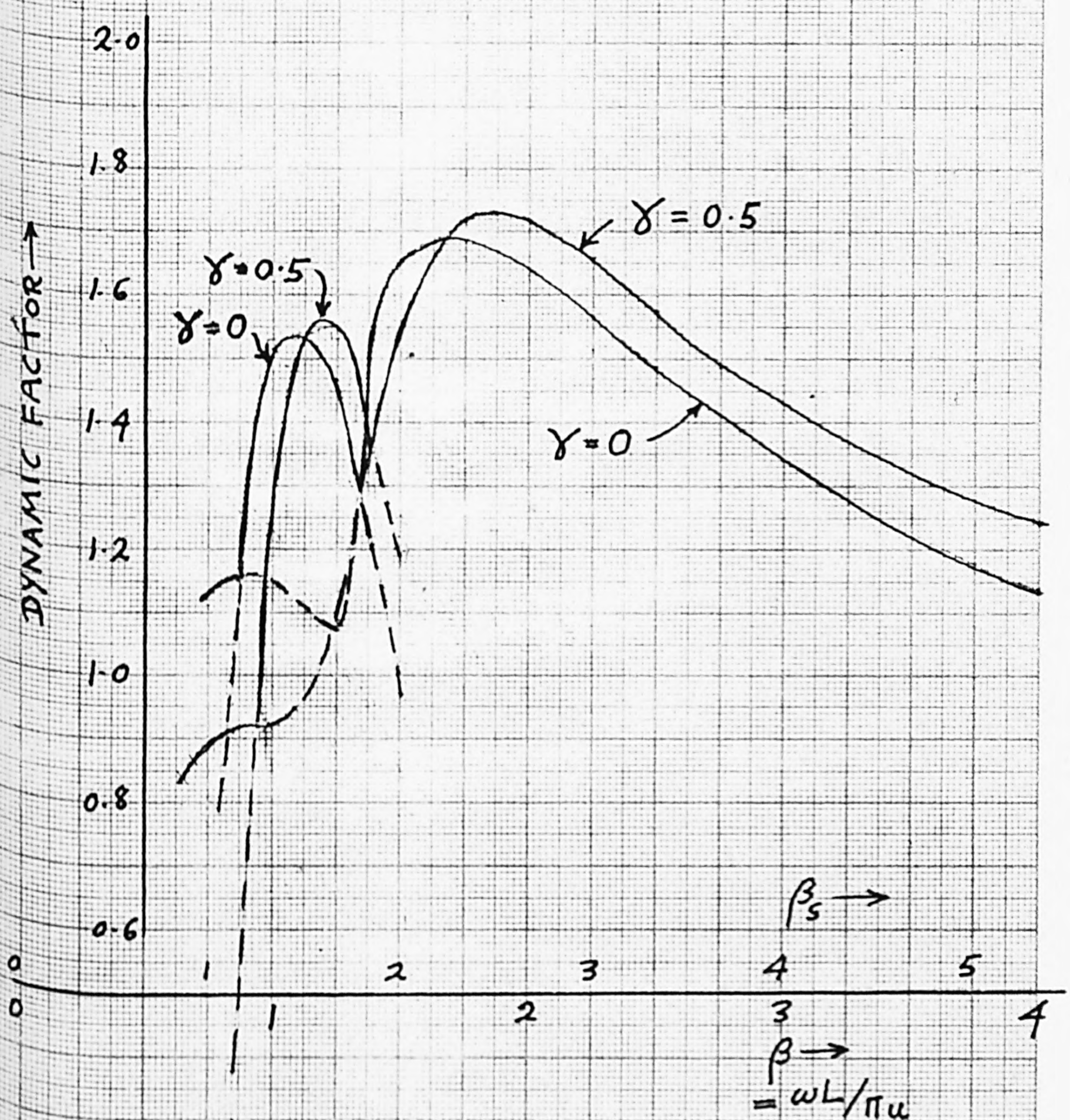


GRAPH 5.2.

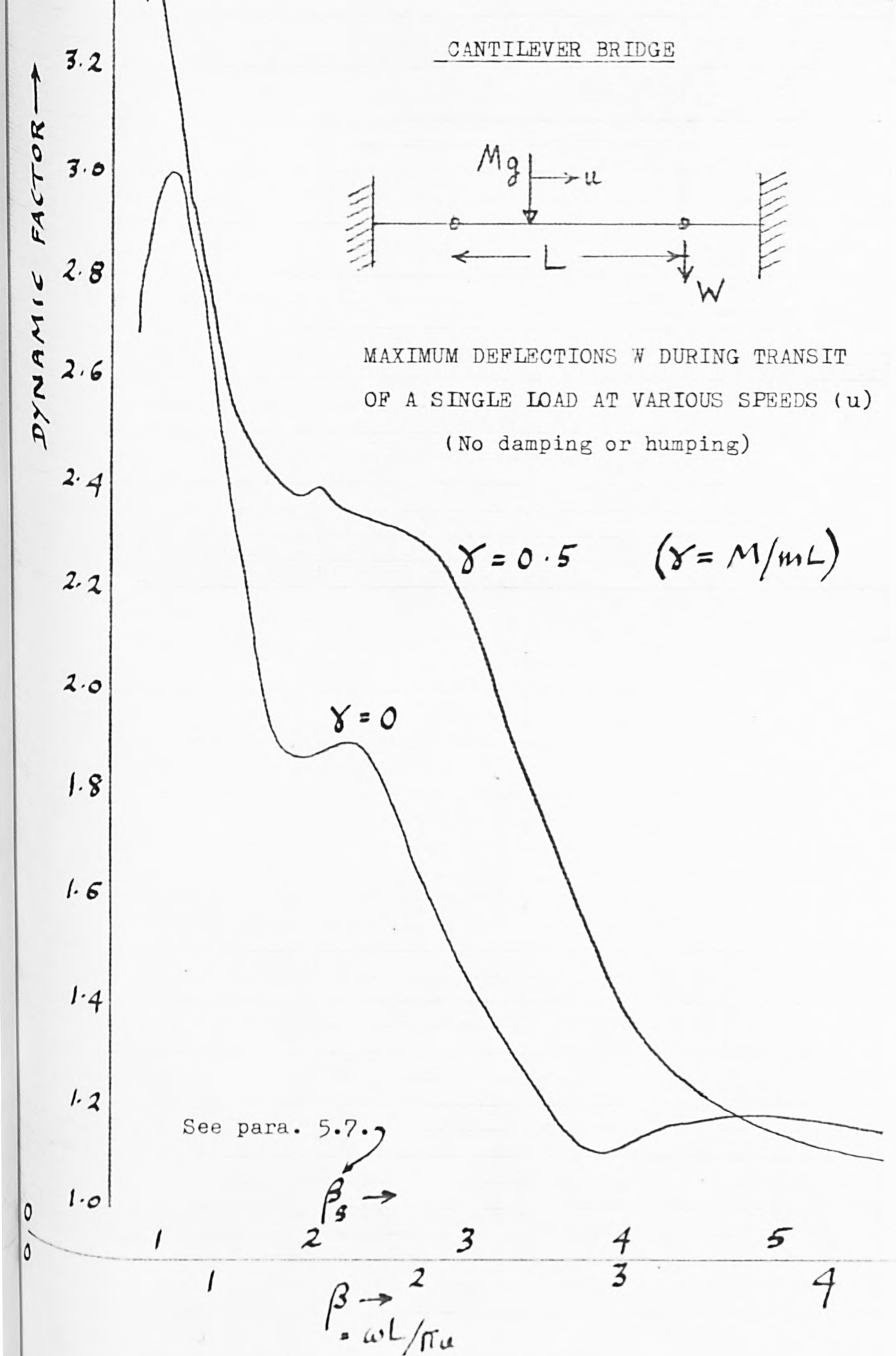
CANTILEVER BRIDGE



MAXIMUM DEFLECTIONS "Y" DUE TO TRANSIT
OF A SINGLE LOAD AT VARIOUS SPEEDS (u)
(No damping or humping)

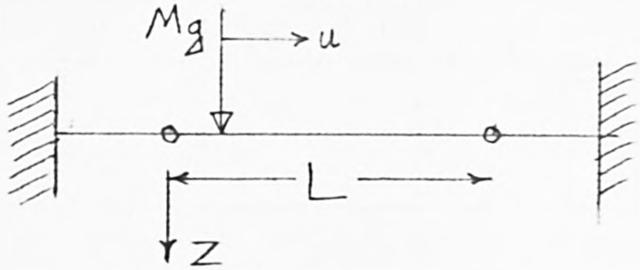


GRAPH 5.3.


CANTILEVER BRIDGE
Mg
u
L
W
MAXIMUM DEFLECTIONS W DURING TRANSIT
OF A SINGLE LOAD AT VARIOUS SPEEDS (u)
(No damping or humping)
γ = 0·5
(γ = M/mL)
γ = 0
DYNAMIC FACTOR →
3·4
3·2
3·0
2·8
2·6
2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
See para. 5.7.
βs →
0
1
2
3
4
5
0
1
2
3
4
β →
= ωL/πu

GRAPH 5.4.

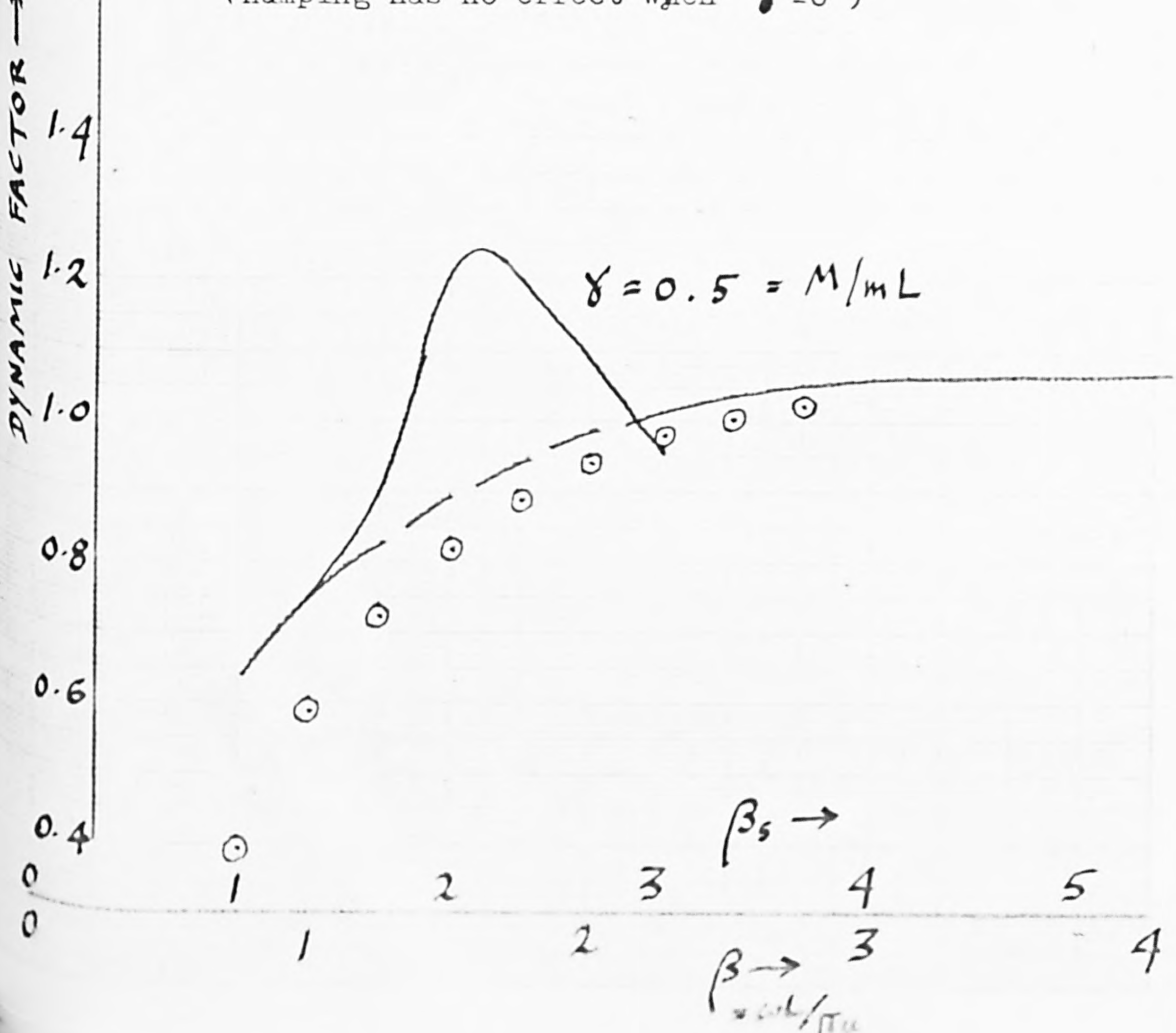
CANTILEVER BRIDGE



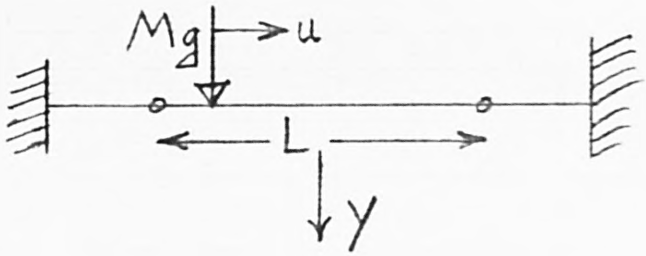
MAXIMUM DEFLECTIONS Z DURING TRANSIT OF
A SINGLE LOAD AT VARIOUS SPEEDS (u)

Points ⊙ refer to a humped deck such that Y_o in figure 5.4 is equal to the midspan static deflection under the same load. The full line graph is a repeat of the $\gamma = 0.5$ curve in Graph 5.1

(humping has no effect when $\gamma = 0$)



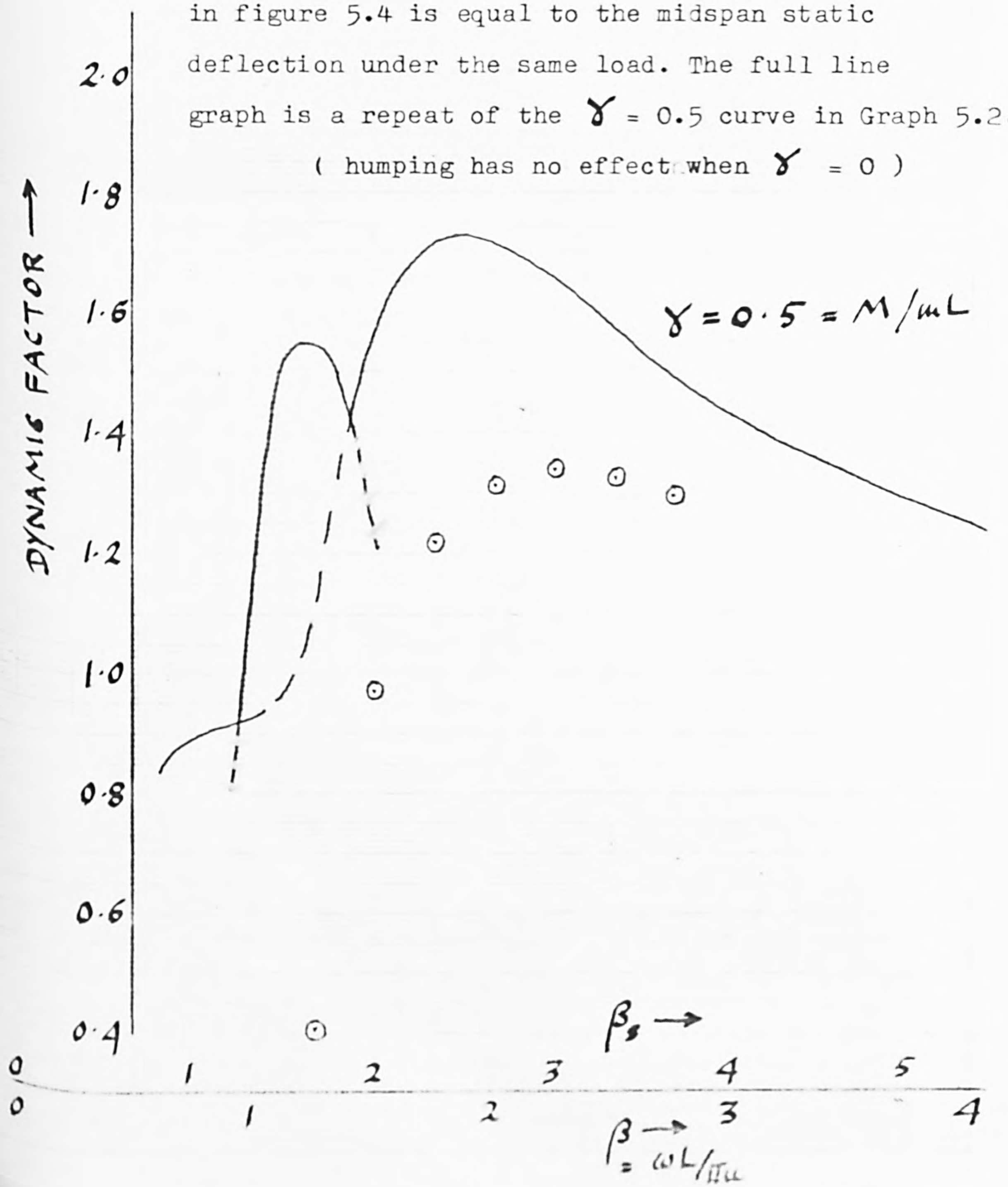
CANTILEVER BRIDGE



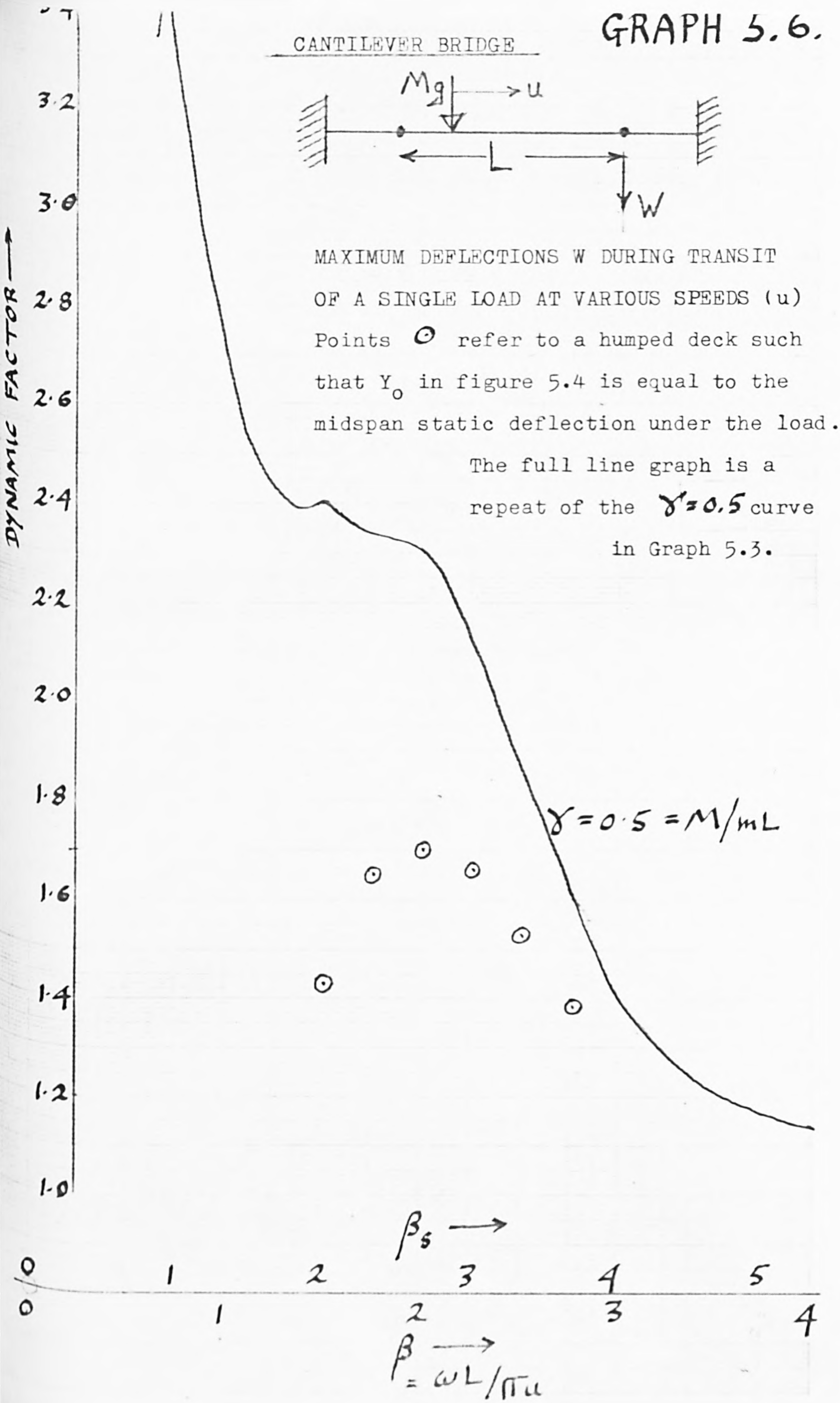
MAXIMUM DEFLECTIONS Y DURING TRANSIT OF A SINGLE LOAD AT VARIOUS SPEEDS (u)

Points ⊙ refer to a humped deck such that Y_0 in figure 5.4 is equal to the midspan static deflection under the same load. The full line graph is a repeat of the γ = 0.5 curve in Graph 5.2.

(humping has no effect when γ = 0)

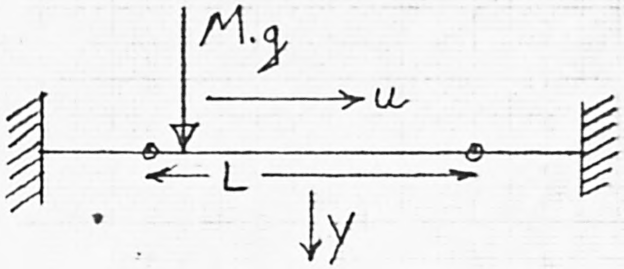


GRAPH 5.6.


CANTILEVER BRIDGE
Mg
u
L
W
MAXIMUM DEFLECTIONS W DURING TRANSIT OF A SINGLE LOAD AT VARIOUS SPEEDS (u)
Points ⊙ refer to a humped deck such that Y_0 in figure 5.4 is equal to the midspan static deflection under the load.
The full line graph is a repeat of the $\gamma = 0.5$ curve in Graph 5.3.
$\gamma = 0.5 = M/mL$
DYNAMIC FACTOR
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
β_s
0
1
2
3
4
5
0
1
2
3
4
$\beta = \omega L/\pi u$

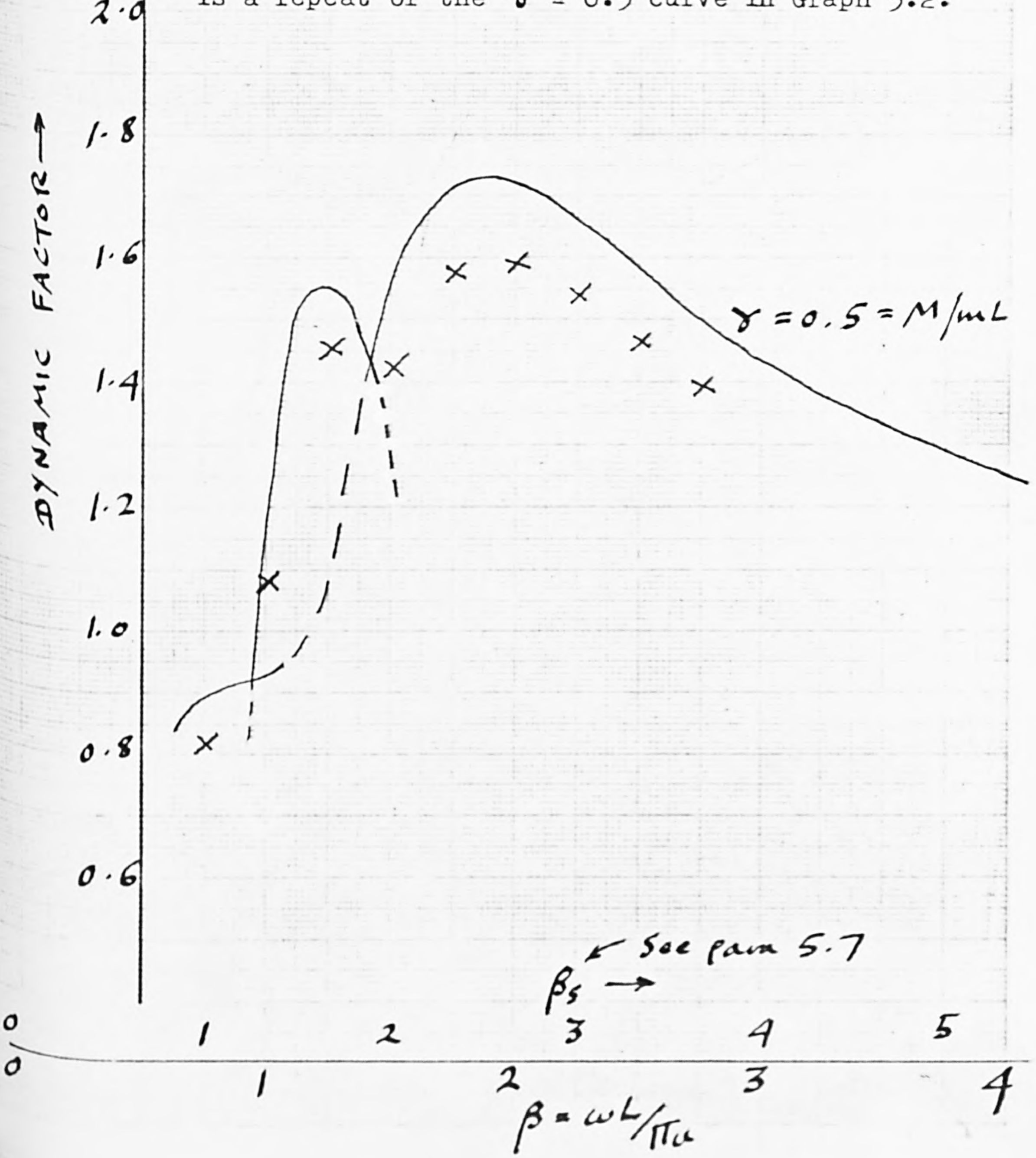
CANTILEVER BRIDGE

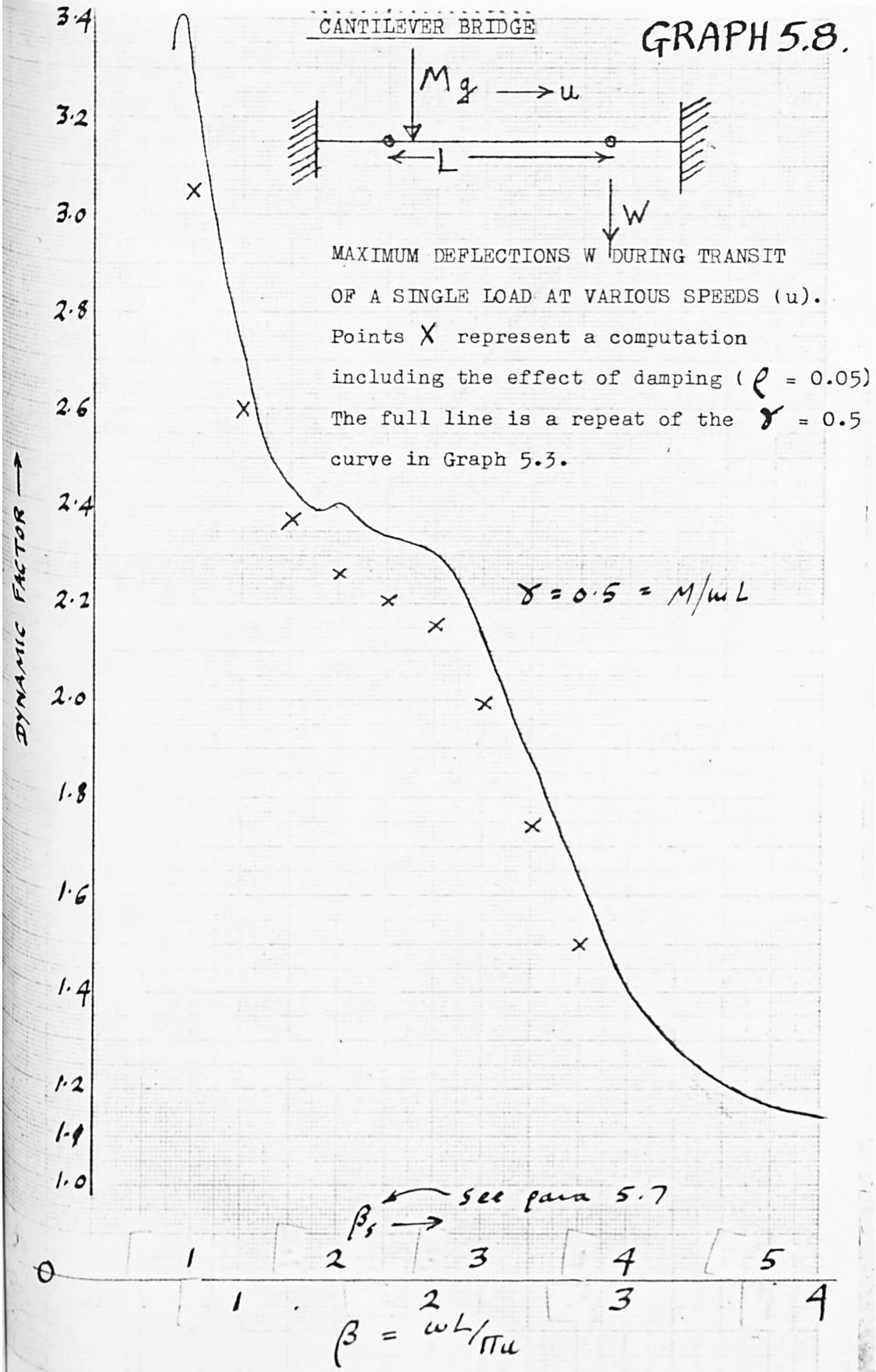
GRAPH 5.7.



MAXIMUM DEFLECTIONS Y DURING TRANSIT OF A SINGLE LOAD AT VARIOUS SPEEDS (u)

Points X represent a computation including the effect of damping (ρ = 0.05 - see para. 5.8 - approx 6% critical damping). The full line graph is a repeat of the γ = 0.5 curve in Graph 5.2.



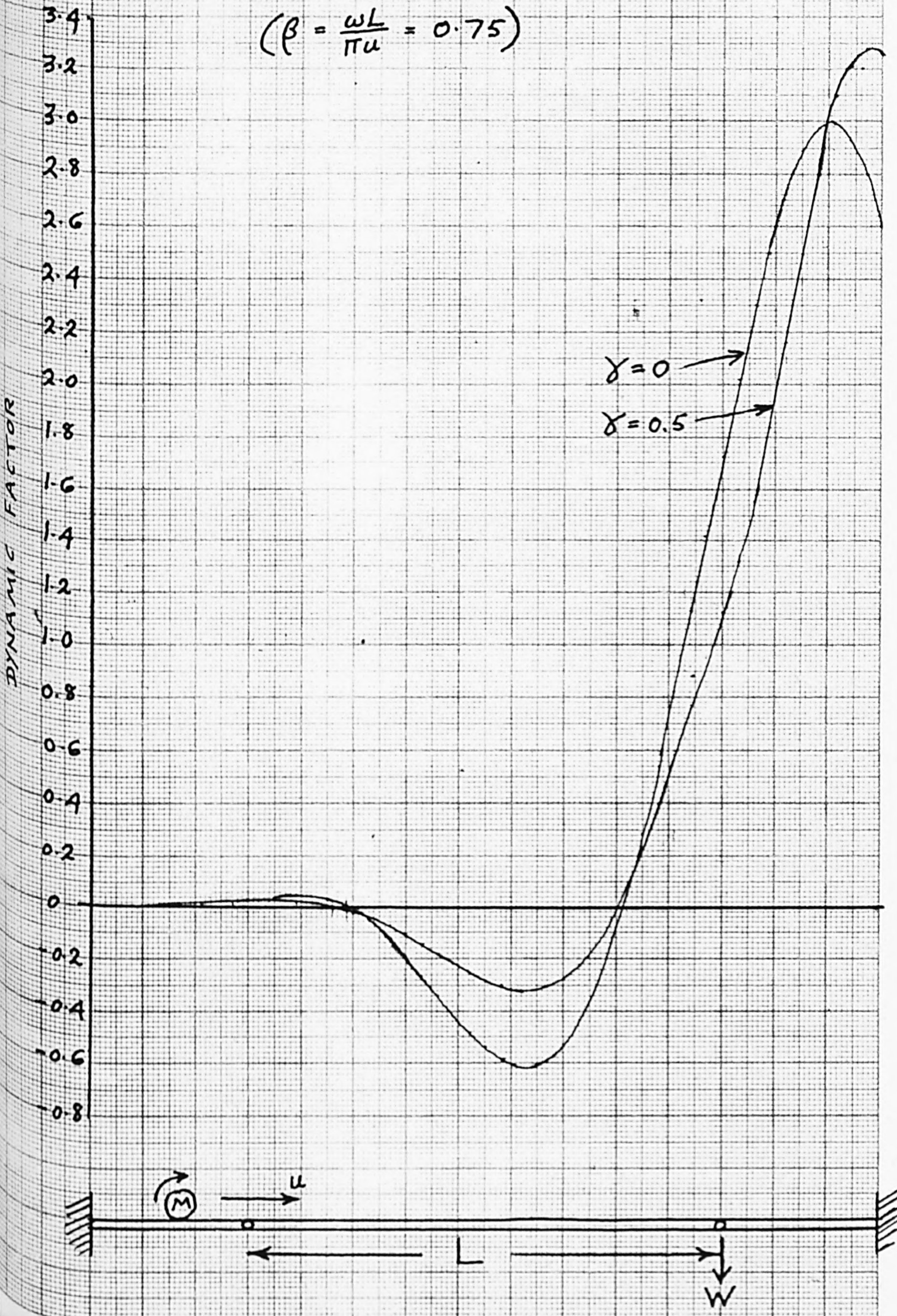

GRAPH 5.8.
CANTILEVER BRIDGE
Mg
u
L
W
MAXIMUM DEFLECTIONS W DURING TRANSIT
OF A SINGLE LOAD AT VARIOUS SPEEDS (u).
Points X represent a computation
including the effect of damping (ρ = 0.05)
The full line is a repeat of the γ = 0.5
curve in Graph 5.3.
γ = 0·5 = M/wL
DYNAMIC FACTOR →
see para 5.7
β₁ →
β = ωL/πu

CANTILEVER BRIDGE

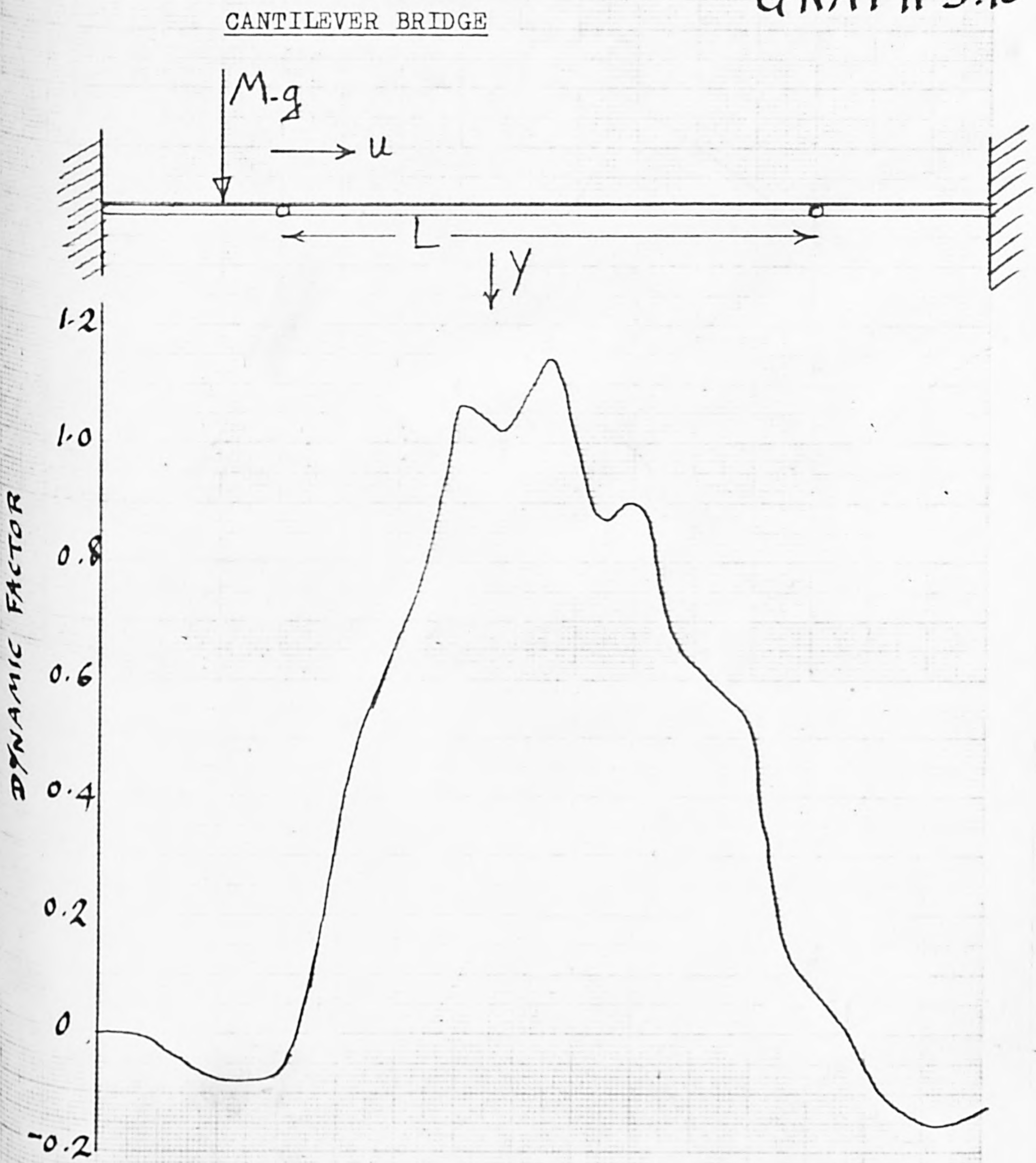
GRAPH 5.9.

DEFLECTION "W" DUE TO TRANSIT OF A SINGLE LOAD AT CRITICAL SPEED

$\left(\beta = \frac{\omega L}{\pi u} = 0.75\right)$



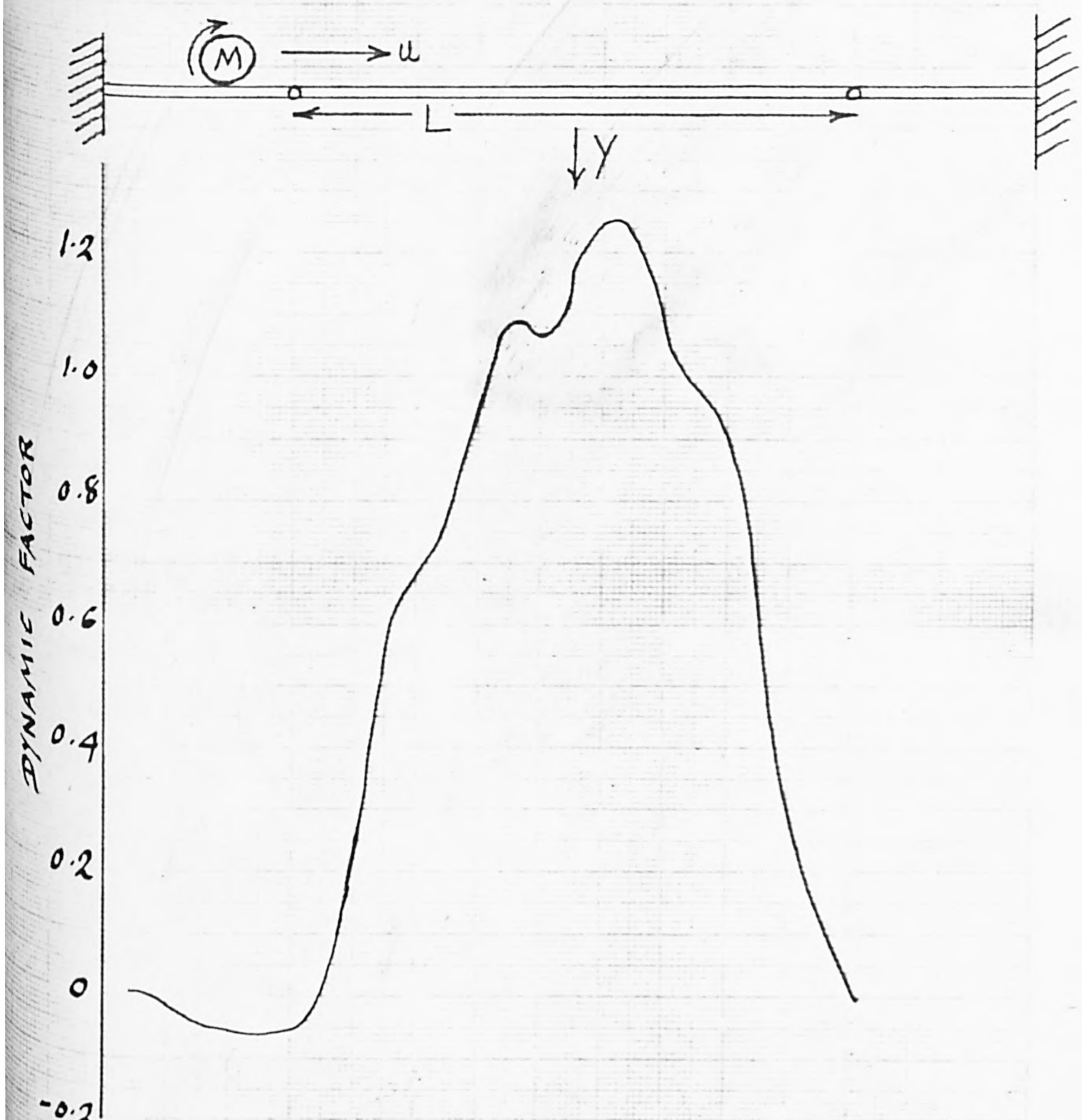
GRAPH 5.10



MIDSPAN DEFLECTION Y DURING TRANSIT OF A SINGLE LOAD
AT A SPEED u CORRESPONDING TO $\beta = 4$
(the load has been treated as a constant force M.g)

CANTILEVER BRIDGE

GRAPH 5.11.



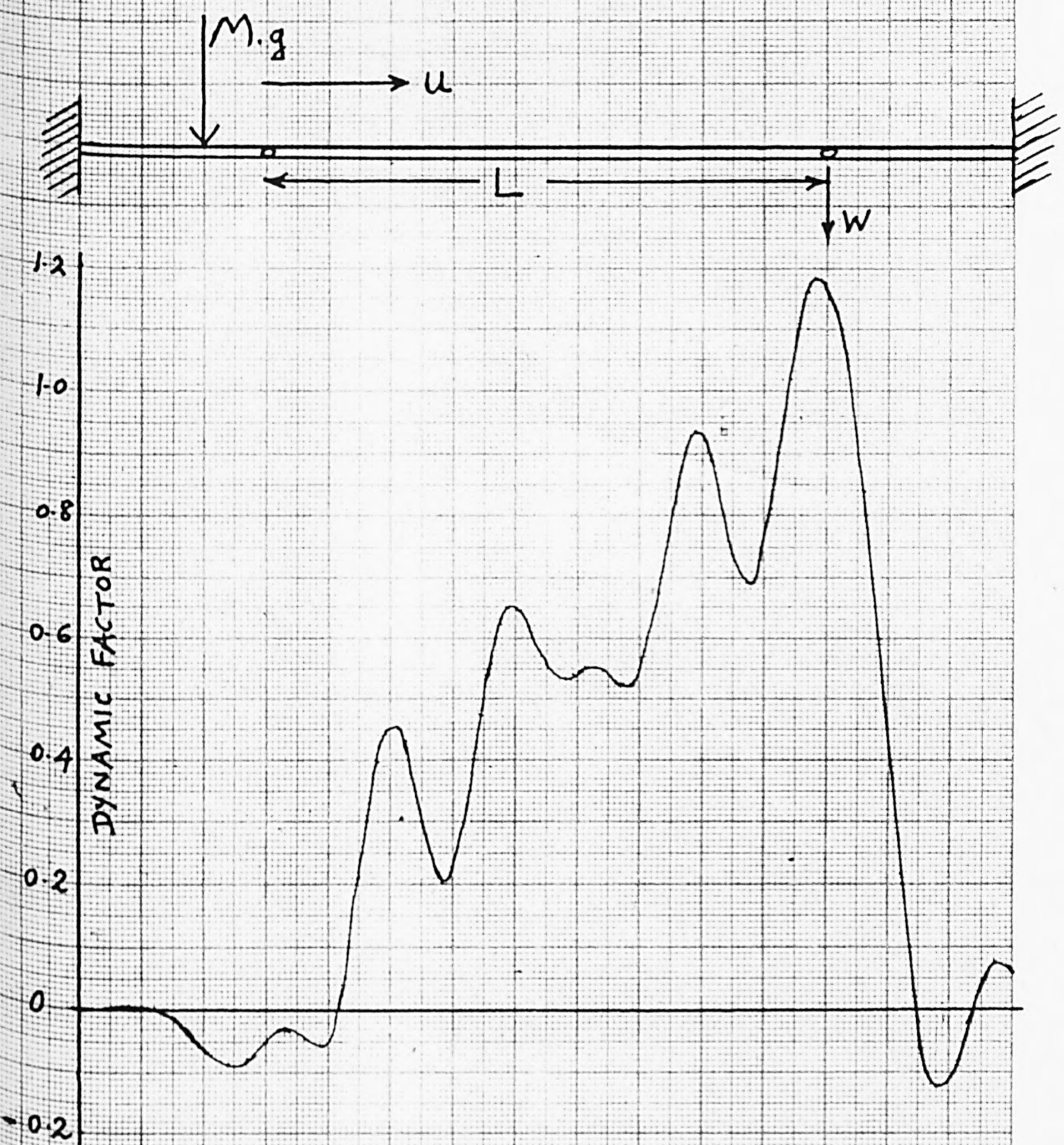
MIDSPAN DEFLECTION Y DURING TRANSIT OF A SINGLE LOAD AT A SPEED CORRESPONDING TO $\beta = 4$

(The load has been treated as a mass M equal to half the self mass of the main span)

i.e. $M/mL = \gamma = 0.5$

GRAPH 5.12.

CANTILEVER BRIDGE

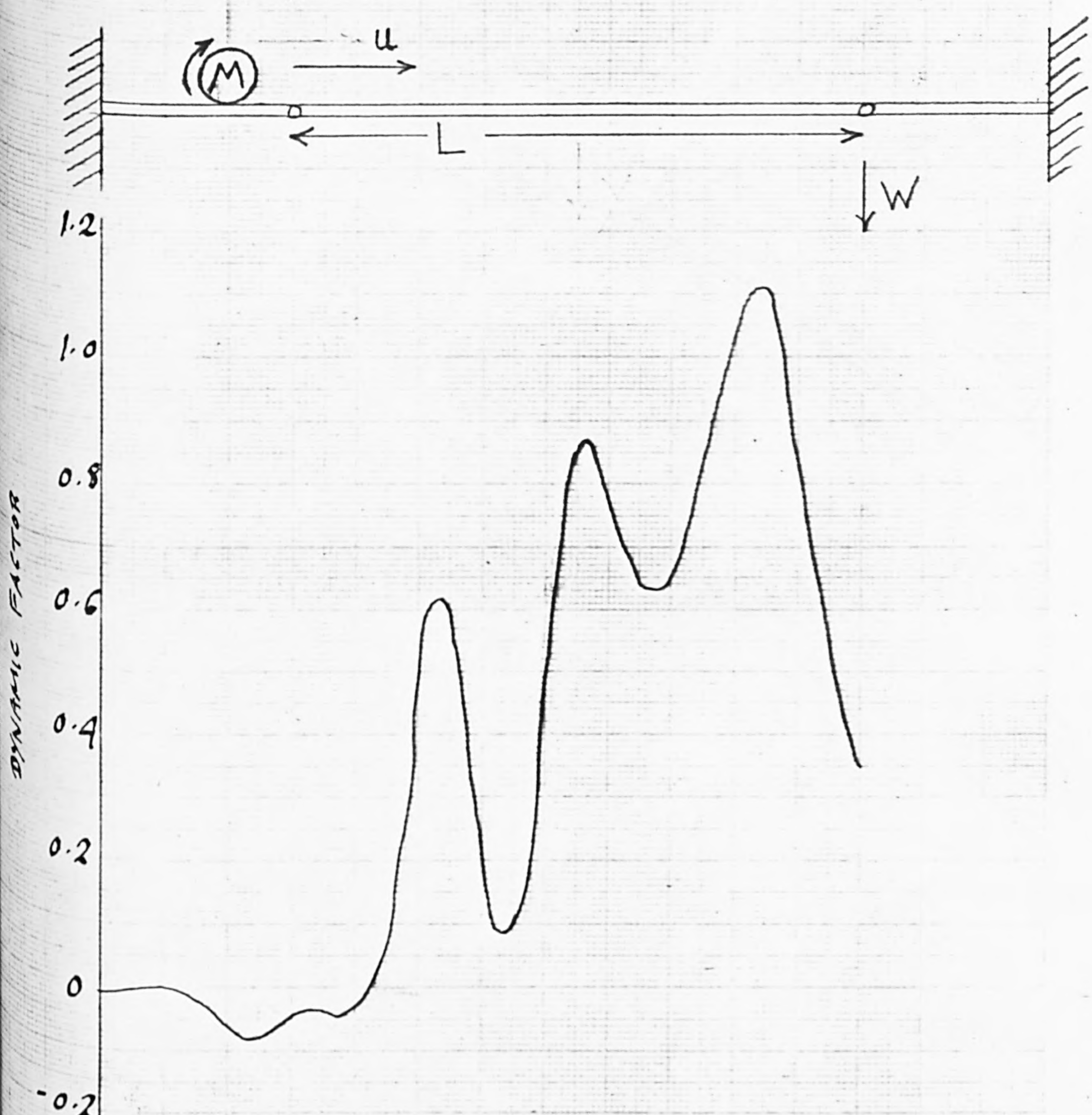


CANTILEVER DEFLECTION "W" DURING TRANSIT OF A SINGLE LOAD AT A SPEED CORRESPONDING TO $\beta = 4$

(The load has been treated as a constant force M.g)

CANTILEVER BRIDGE

GRAPH 5.13.



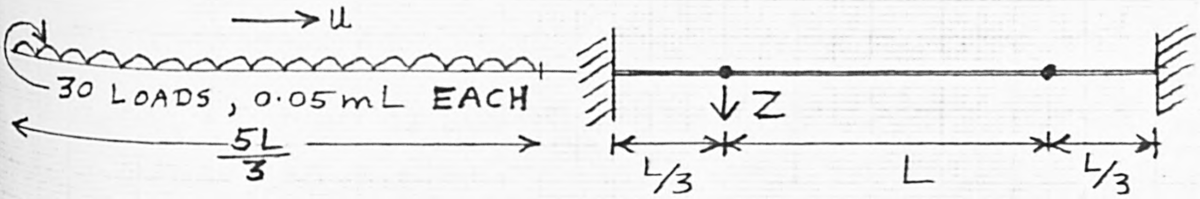
CANTILEVER DEFLECTION W DURING TRANSIT OF A SINGLE LOAD AT A SPEED u CORRESPONDING TO β = 4

(The load has been treated as a mass M equal to half the self mass of the main span)

i.e. $M/mL = \gamma = 0.5$

CANTILEVER BRIDGE

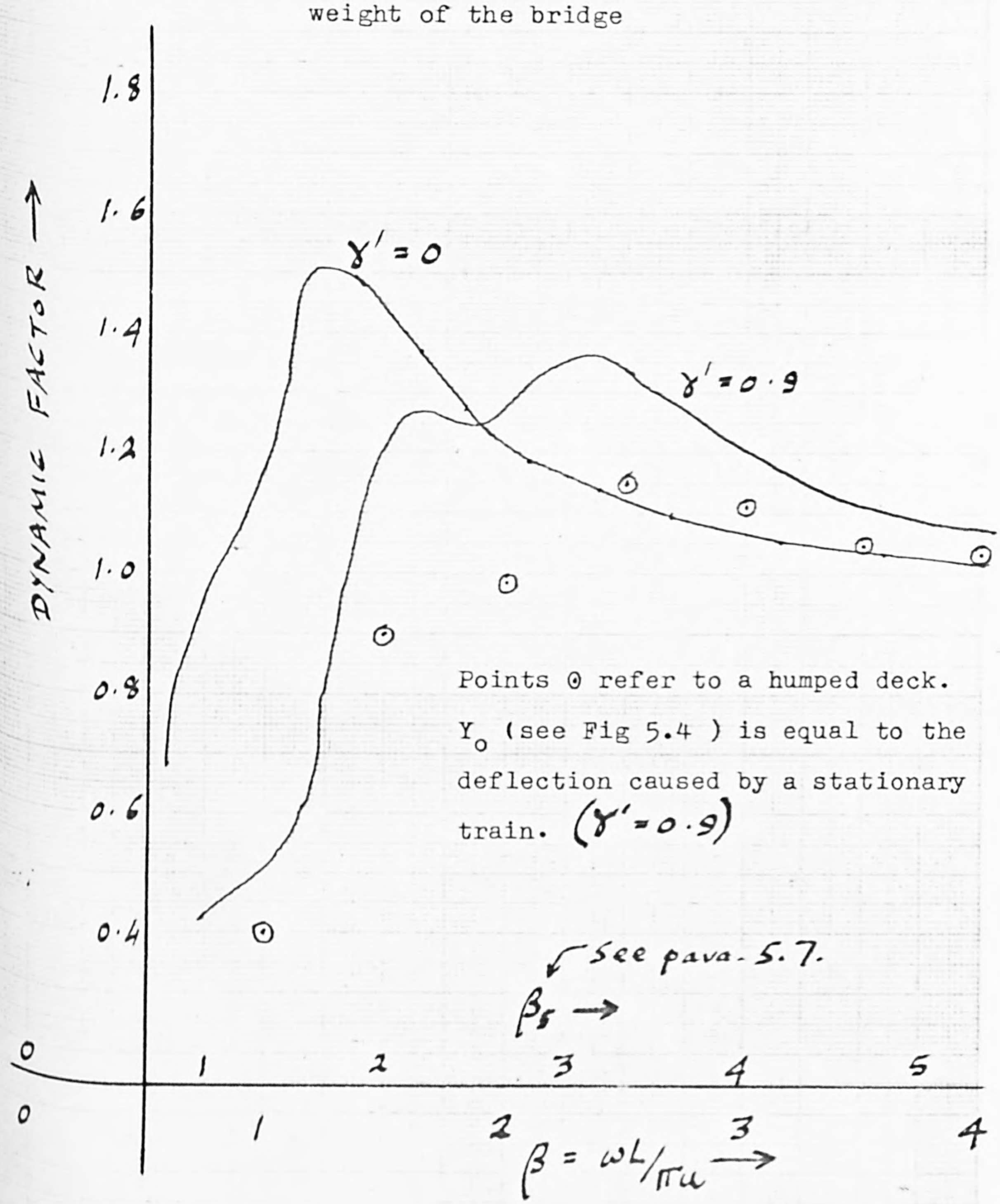
GRAPH 5.14



MAXIMUM DEFLECTIONS Z DURING TRANSIT OF A TRAIN OF LOADS

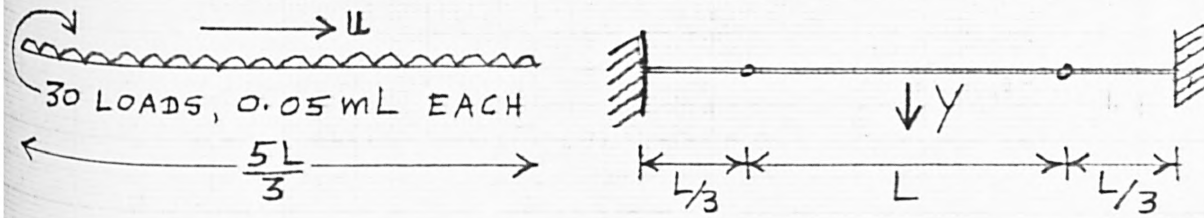
AT VARIOUS SPEEDS (u)

$$\gamma' = 0.9 = \frac{\text{total weight of a train equal in length to the bridge}}{\text{weight of the bridge}}$$



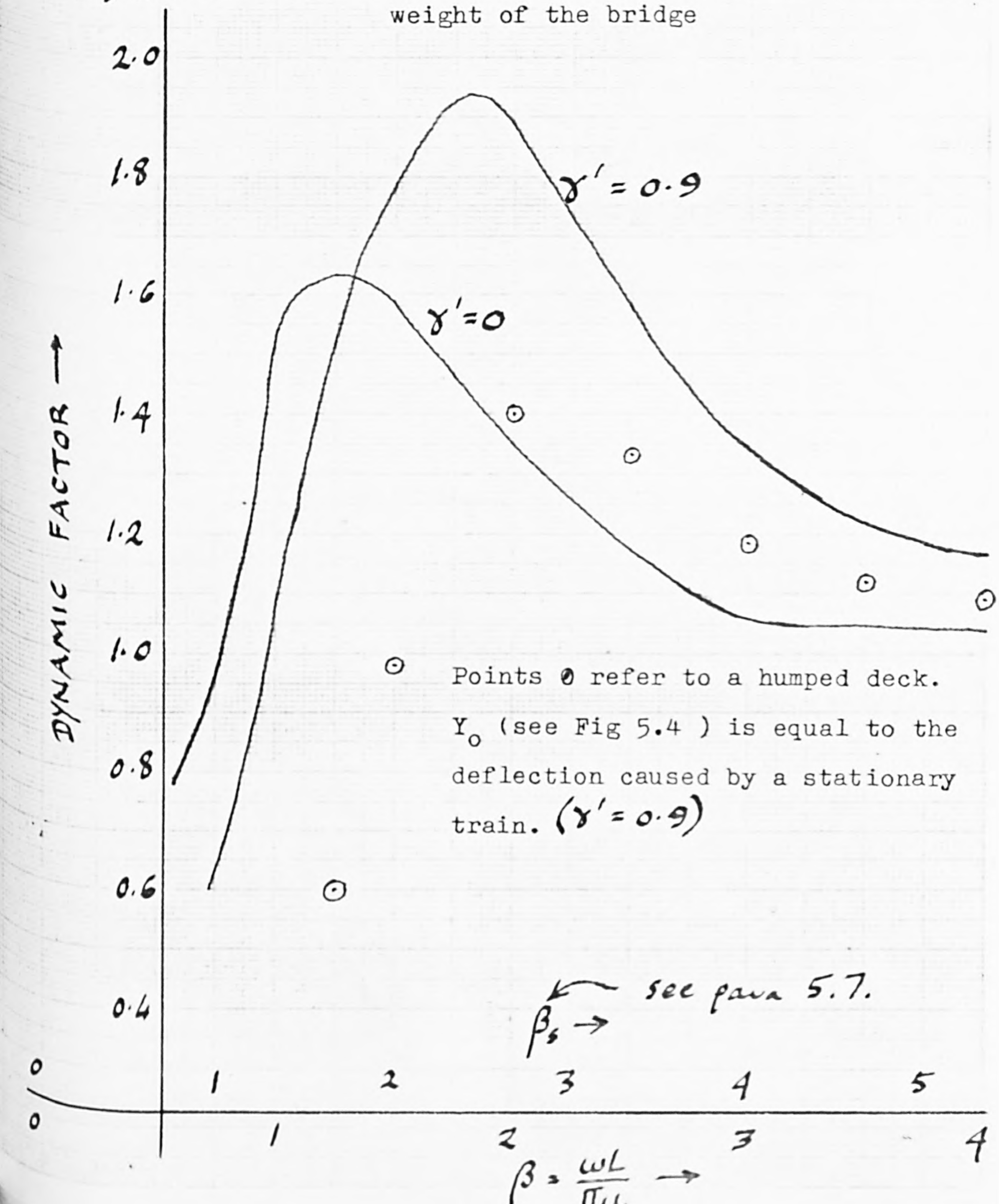
CANTILEVER BRIDGE

GRAPH 5.15



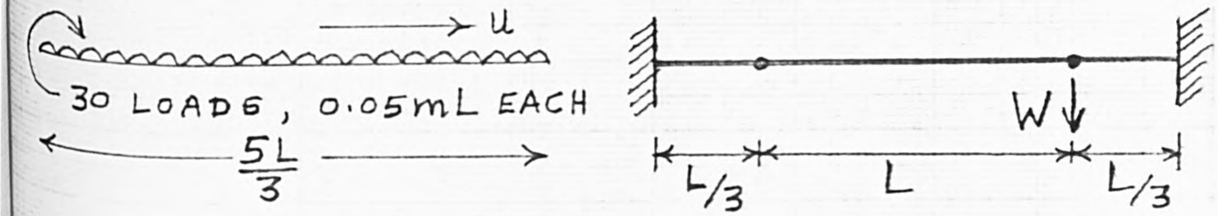
MAXIMUM DEFLECTIONS Y DURING TRANSIT OF A TRAIN OF LOADS AT VARIOUS SPEEDS (u)

$$\gamma' = 0.9 = \frac{\text{total weight of a train equal in length to the bridge}}{\text{weight of the bridge}}$$



CANTILEVER BRIDGE

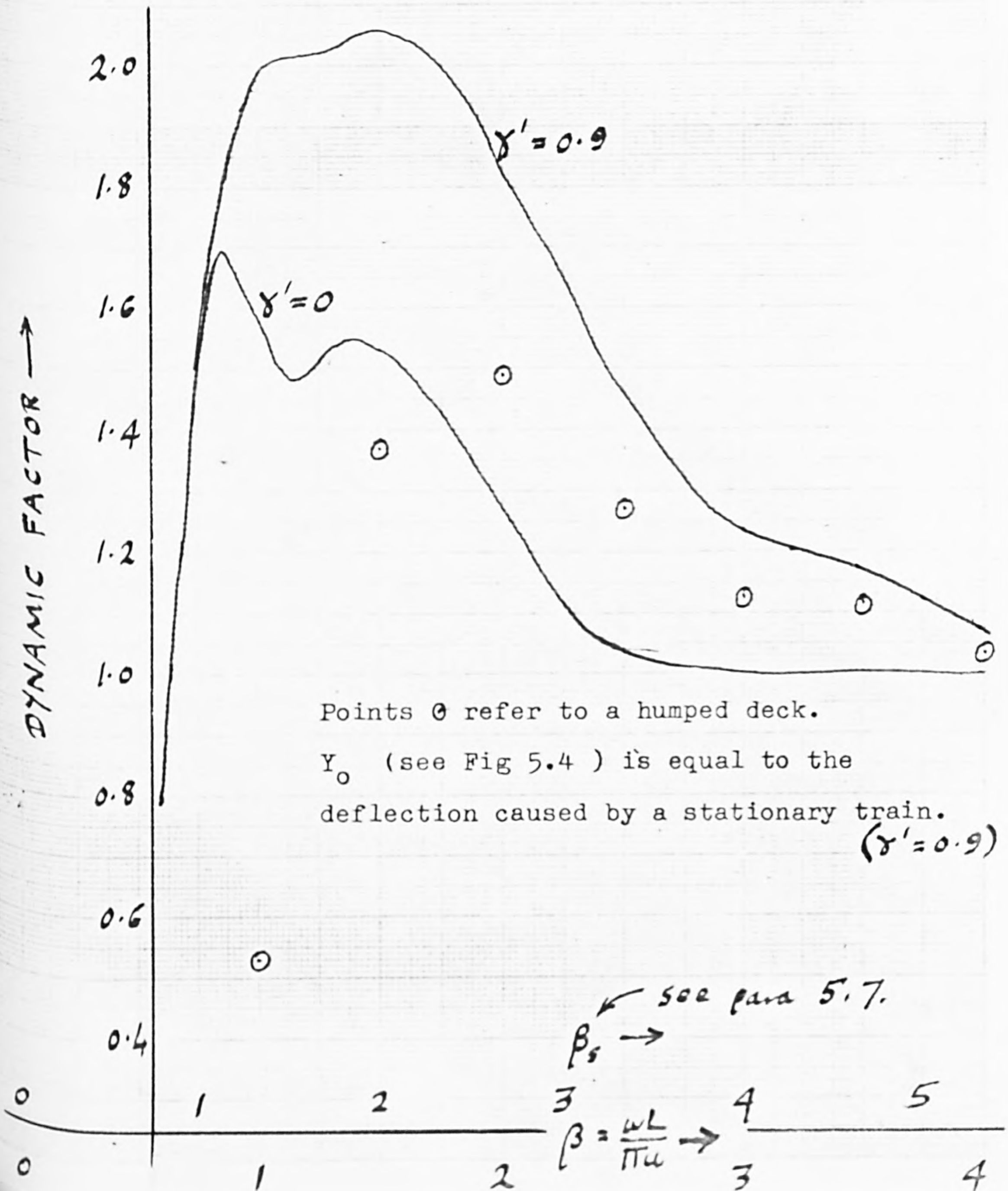
GRAPH 5.16.



MAXIMUM DEFLECTIONS W DURING TRANSIT OF A TRAIN OF LOADS

AT VARIOUS SPEEDS (u)

$$\gamma' = 0.9 = \frac{\text{total weight of a train equal in length to the bridge}}{\text{WEIGHT OF THE BRIDGE}}$$



ORDINATES FOR GRAPHS 5.1, 5.2, AND 5.3.

β	Y	*	Z	*	W	*
·75	1.131	67	·968	40	2.990	93
·76	1·136	67	·969	39	2.999	93
·77	1.140	67	·976	39	3·005	92
·78	1·145	66	·983	39	3·006	92
·79	1·148	66	·989	39	3·007	91
·80	·808 1·152	100 65	·994	38	3·004	90
1.00	1·154 1·513	58 100	1·090 ·644	35 84	2·578	80
1·25	1·072 1·467	51 79	1·156 1·192	31 91	1·969	67
1·50	·968 1·653	47 70	1.183 1·322	29 80	1·893	86
1·75	1·685	53	1·185 1·245	27 61	1·872	70
2.00	1·640	58	1·176 1·151	26 64	1·648	73
2.25	1·565	54	1·160 1·063	25 59	1·452	67
2·50	1·486	52	1·143	25	1·287	63
2·75	1·413	49	1·129	24	1·143	59
3.00	1·340	47	1·116	24	1·141	81
4.00	1.138	50	1·083	23	1·181	79
5.00	1·044	53	1·089	23	1·084	82
6·00	1·051	48	1·104	22	1·110	81

γ = 0

β	Y	*	Z	*	W	*
·65	·839	78	·583	46	3·35	100
·70	·868	75	·611	45	3·39	100
·75	·888	72	·637 ·643	43 98	3·28	99
·80	·901	70	·661 ·700	42 96	3·14	96
·85	·374 ·909	100 69	·684 ·736	41 95	2·99	94
1.00	·913 1·175	64 100	·743 ·768	38 90	2·68	89
1·25	1·086 1·545	80 100	·822 ·913	35 87	2·44	86
1·35	1·309 1·418	77 100	·848 1·068	34 87	2·39	87
1·40	1·402 1·345	76 100	·860 1·135	34 86	2·39	86
1·50	1·546 1·195	73 98	·884 1·214	33 84	2·40	87
1·75	1·713	68	·932 1·191	31 77	2·34	83
2·00	1·715	63	·972 1·058	30 70	2·30	77
2·25	1·655	60	1·000 ·927	29 65	2·11	72
2·50	1·575	56	1·021 ·850	28 60	1·85	67
2·75	1·487	53	1·008 ·880	26 76	1·60	63
3.00	1·43	64	1·04	27	1·39	60
4·00	1·24	55	1·06	24	1·12	70
5·00	1·09	51	1·11	25	·98	60
6·00	1·11	51	1·14	24	1·04	78

γ = 0.5

* Percentage of transit

ORDINATES FOR GRAPHS 5.14, 5.15, AND 5.16

β	Y	*	Z	*	W	*
.60	.78	100	.67	100	.78	100
.75	.90	"	.94	"	1.46	"
.85	1.09	"	1.02	"	1.69	"
1.00	1.45	"	1.15	"	1.59	"
1.10	1.60	"	1.30	"	1.49	"
1.25	1.62	"	1.50	"	1.50	"
1.35	1.60	"	1.49	"	1.54	"
1.50	1.59	"	1.44	90	1.53	97
1.60	1.54	"	1.39	87	1.50	97
1.75	1.45	93	1.31	83	1.43	93
2.00	1.35	83	1.20	77	1.28	90
2.25	1.26	80	1.15	87	1.12	87
2.50	1.18	77	1.12	80	1.03	90
3.00	1.06	77	1.05	73	1.00	87
4.00	1.04	80	1.01	80	.99	87

$\gamma = 0$

β	Y	*	Z	*	W	*
.75	.60	97	.42	60	1.51	100
.875	.75	100	.46	57	1.83	100
1.0	.99	100	.49	53	1.98	100
1.125	1.26	"	.55	100	2.01	"
1.25	1.49	"	.77	"	2.02	"
1.375	1.65	"	1.02	"	2.03	"
1.5	1.76	"	1.20	"	2.05	"
1.625	1.86	"	1.26	"	2.04	"
1.75	1.92	"	1.26	"	1.99	"
1.875	1.93	"	1.23	"	1.91	"
2.0	1.89	"	1.26	"	1.83	97
2.125	1.82	"	1.91	"	1.73	97
2.25	1.74	"	1.35	"	1.64	93
2.375	1.67	"	1.36	"	1.54	93
2.5	1.59	"	1.33	"	1.46	90
2.625	1.51	"	1.29	93	1.38	90
2.75	1.45	93	1.27	91	1.31	90
2.875	1.39	93	1.24	90	1.26	90
3.125	1.30	90	1.18	83	1.21	90
3.375	1.24	83	1.13	83	1.19	90
3.875	1.17	80	1.08	87	1.09	87

$\gamma = 0.05$

* Percentage of transit

Comments on Chapter Five

The computations reported in this chapter were all originally performed by a thirty stage program. However, no direct checks on the validity of the results (such as those derived from alternative analyses in Chapter 4) were available. Therefore the computations for a single travelling load were all repeated using a hundred stage program. Examination of the two sets of results revealed little difference and it was then decided that the computations for trains of loads need not be repeated. So in the outcome the graphs in this chapter represent 100 stage calculations for single loads and 30 stage calculations for load trains.

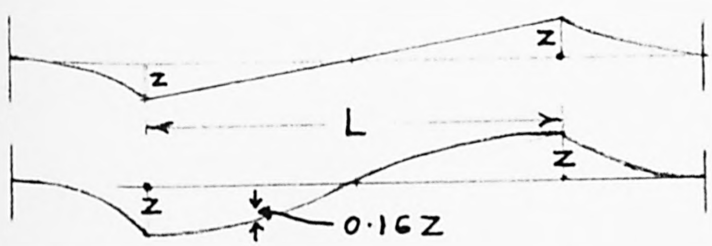
The principle approximation inherent in the calculations stems from the shape functions adopted (see Figure 5.2) to describe the dynamic/ elastic deformations. These are not true natural mode shapes neither can they be combined to produce exact mode shapes. Nevertheless, they are reasonable approximations which satisfy appropriate boundary conditions. Also they introduce three independent coordinates (Y,Z and W) and may therefore be expected to yield an accuracy approaching that of a modal analysis based on the response of the first three modes. The cyclic part of calculated responses will, however, be distorted, in relation to time, because the beam is being forced to adopt shapes which are not strictly compatible with simple harmonic motion. This is an error which does not arise in modal analysis, but it must be remembered that when the moving load is treated as a mass the natural mode shapes will be different for every stage of the computation and hence the cyclic response will not then be simple harmonic. So, in such cases, there would be no special logic in employing mode shapes which are only true for the unloaded beam. It is believed that the calculations in this chapter are adequate to reveral the extent of the difference in response due to treating the moving load as a mass rather than as a force.

Graphs 5.10, 5.11, 5.12, 5.13

These graphs are commented upon first because they enable the performance of the method to be observed in connection with the basic problem of tracing deflections during the transit of a load.

Graphs 5.10 and 5.12 are for $\gamma = 0$ and therefore represent the simple case in which the load is regarded as an unvarying force M.g. In this case the natural frequencies will also remain unaltered during the transit and it is to be expected that the response of three natural modes will be involved. Note that these four graphs have been drawn for a fairly slow transit (β=4) otherwise there is insufficient time for an interesting number of vibratory cycles to occur.

Graph 5.12 contains a notable proportion of mode 2 and the distortion, anticipated above, is observable. It was anticipated that this mode would suffer the most because it can only be formed, in the program, as shown in the first diagram below, whereas it should be more like the second diagram. The frequencies quoted are obtained from Rayleigh's method.



$$\omega^2 = 3.48 \frac{\Pi^4}{L^4} \frac{EI}{m}$$

$$\omega^2 = 3.16 \frac{\Pi^4}{L^4} \frac{EI}{m}$$

The mode 2 wavelength should be approximately 2.5 cm on Graph 5.12. calculated from the frequency and the value of β.

Graph 5.10 shows the midspan deflection relative to a straight line joining the cantilever ends. (Y on Figure 5.2). This exaggerates the response of the third mode and hence a mode 3 cycle is clearly visible on the graph having a wavelength agreeing within 9% with a calculation based on the third frequency (page 81) and the value of β.

The reason for computing the midspan deflection in this relative manner was to give an indication of the variation in midspan bending moment which is of more practical importance than deflection. If Graph 5.10 is replotted in the form of absolute deflection ($Y+\frac{1}{2}(Z+W)$) then these mode 3 cycles become scarcely visible and the negative response near the origin (which is a correct tendency in the case of bending moment) also disappears almost completly.

Graphs 5.11 and 5.13 refer to the more challenging case where the load is treated as a mass. These curves represent effects too complex for any purely observational analysis to be attempted.

Graph 5.9

The curve for $\gamma=0$, on this graph, evidently involves mode 2 to a considerable degree, its correct wave length being 13 cm on the horizontal scale of the graph. This numerical result was derived independently from the appropriate natural frequency and the value of β applicable to Graph 5.9.

Graphs 5.1, 5.2, 5.3, 5.14, 5.15, 5.16

These graphs summarise the principal results obtained in Chapter 5. They show, in the form of dynamic factors, the maximum deflections of cantilevers and main span for transits of loads at a range of speeds including the critical speeds. Graphs 5.1 to 5.3 refer to single loads whilst Graphs 5.14 to 5.16 refer to trains of loads.

Graphs 5.1 and 5.14 are concerned with the "entry" cantilever and show consistently smaller dynamic factors then the "exit" cantilever. For this reason they are of very limited interest because loads moving in the opposite direction must also be taken into account, which means that both cantilevers must be designed to cater for the "exit" case.

The discontinuities in Graph 5.2, for the midspan deflection, when β has vaules between 1 and 2, are due to the fact that the response curves for this region have two stationary points, sometimes the first and sometimes the second being dominant. However, these discontinuities occur at transit speeds far too high to be of practical importance. It is interesting that the increase in dynamic factors due to treating the load as a mass is much less here than in the case of s simply supported beam (Graph 4.6).

In Graph 5.3, for the exit cantilever, the result of treating the load as a mass is again much more marked. The other features of note on this graph are the very high values of dynamic factor at critical speed. The sppeds implied are too high to be of practical significance but it has been possible to obtain corroboration of these high dynamic factors by experimental measurement (see Chapter 6).

On Graph 5.6 it can be seen that the high dynamic factor can be obviated by humping the track. In fact the contrast between the two curves (with and without "humping") on this graph (and also on Graphs 5.5, 5.15 and 5.16) tends to confirm the preliminary conclusion, in para. 1.2, that $\beta=3$ is the lowest value that could have practical significance. It can be seen on all these graphs, that when $\beta<3$, the divergence between the two curves becomes drastic. However, it must be recalled that the graphs refer to a very high value of vehicle/bridge mass ratio (γ) and that accelerations experienced by the load are reduced in proportion to this parameter. It should also be noted that, in all cases where humping has been introduced, the maximum hump Y_o (see Figure 5.4) has been made equal to the midspan deflection due to the live load applied statically.

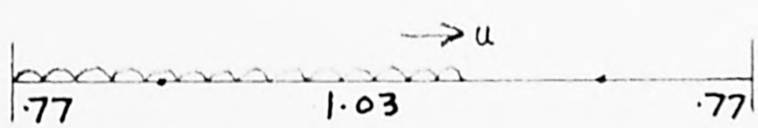
In the case of trains of loads it is more likely that high dynamic factors will be attained in practice because the vehicle/bridge mass ratio is much more likely to be equal to the value considered. The dynamic factors on Graphs 5.15 and 5.16 would be applicable to very heavy loads and these two graphs may be the most significant results of Chapter 5 from the design point of view.

In many of the computations, for load trains, the deflections are still increasing when the leading load reaches the far end of the bridge. The ultimate dynamic factor will thus be somewhat greater than the largest value in the computer printout. The extent to which it will exceed the value printed in the last stage of the computation depends on the length of the train in relation to the bridge. There are thus many possibilities and the values reported on Graphs 5.15 and 5.16 are simply the final printout values. This is considered to be acceptable since the speeds of transit are too high to be practical, anyway. At more reasonable speeds ($\beta > 2.5$) this difficulty does not arise.

Finally it must be pointed out that no final conclusions about cantilever bridges can be drawn until other values of the ratio main span/ cantilever span have been investigated, together with appropriate variations in EI and distribution of dead loading. The programs used in Chapter 5 could readily be modified to cater for these variables.

Dynamic Bending Moments

Although three shape functions have been employed, in the cantilever bridge problem, there is only one function for each of the three basic structural elements (two cantilevers plus one central span). As a result the dynamic factors for deflection and bending moment are related by simple ratios summarised below.



CHAPTER 6

EXPERIMENTS WITH A MODEL CANTILEVER BRIDGE

THE MODEL BRIDGE

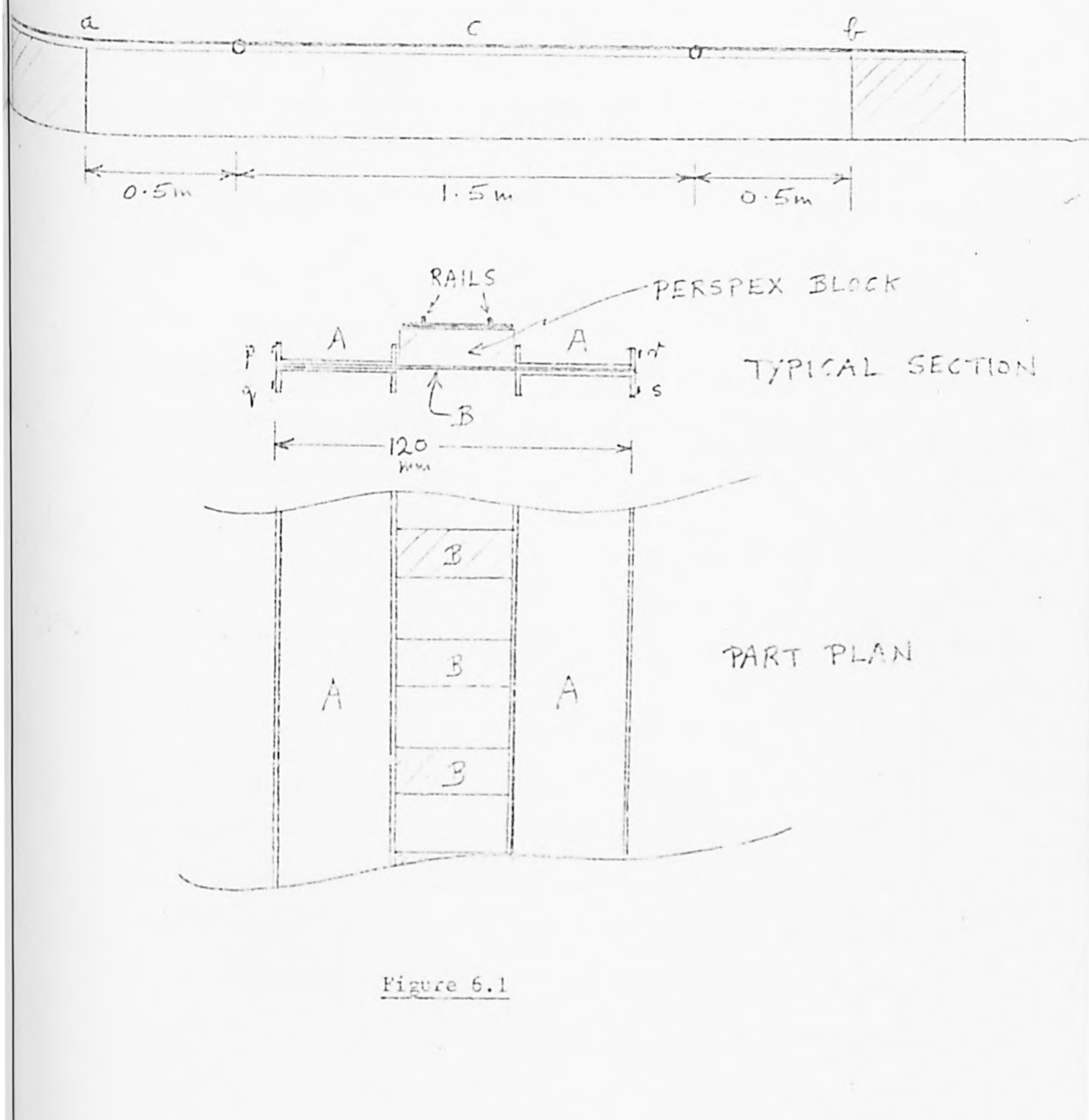


Figure 6.1

6.1 Experimental confirmation of computed results has been sought with the aid of a model cantilever bridge of which the principal features are shown in Figure 6.1.

The model is entirely fabricated from aluminium sheet and is mounted on a datum slightly off horizontal, so that a vehicle on the bridge will maintain a steady speed, once started. The frictional resistance is overcome by the downward gradient.

The test vehicle consists of a steel block provided with flanged wheels so that it can run on standard model railway track. This track is mounted on perspex blocks which rest, at intervals, on the transverse connecting strips B, as shown in Figure 6.1. No connection exists between the track system and the bridge itself so that the track will not add to the stiffness of the structure.

Strain gauges are affixed to the bridge at the cantilever roots a and b and also at midspan c. Their positions are indicated more precisely by p, q, r, s, in the sectional part of Figure 6.1.

The vehicle is projected across the span by means of an approach ramp (see photograph) the speed of transit depending on the distance up the ramp at which the vehicle is released. As the vehicle passes points a and b, in Figure 6.1, it triggers a magnetic pick up (without physical contact) which results in "blips" on the output trace of a U.V. recorder. By this means the speed of the vehicle (u) can be readily deduced.

The U.V. recorder traces the output from the strain gauges simultaneously.

6.2 Frequency control system

An obvious difficulty to be anticipated is that of achieving sufficiently high vehicle speeds to cover the intended range of investigation.

Apart from actually running the vehicle faster the only other way of attaining relatively high speed transits is to reduce the natural frequency of the model bridge, by adding dead load masses to it. Unfortunately this expedient can only be employed to a very limited extent because of the unrealistically large dead load sag which it causes.

So an arrangement was devised whereby the frequency of the model could be reduced to a considerable extent without actually attaching fixed loads to it. The underside of the deck was coupled by a series of push rods to a corresponding series of rocker beams. Any amount of mass could then be applied to these "see saws" whilst keeping them in balance so that the bridge would experience no static load on their account but only an inertial effect. In fact it was found to be possible, by slightly over balancing the rockers, to cancel out the deflection of the bridge due to its own weight. The details of this apparatus are shown in Figure 6.2, there being nine such units employed, uniformly spaced along the span.

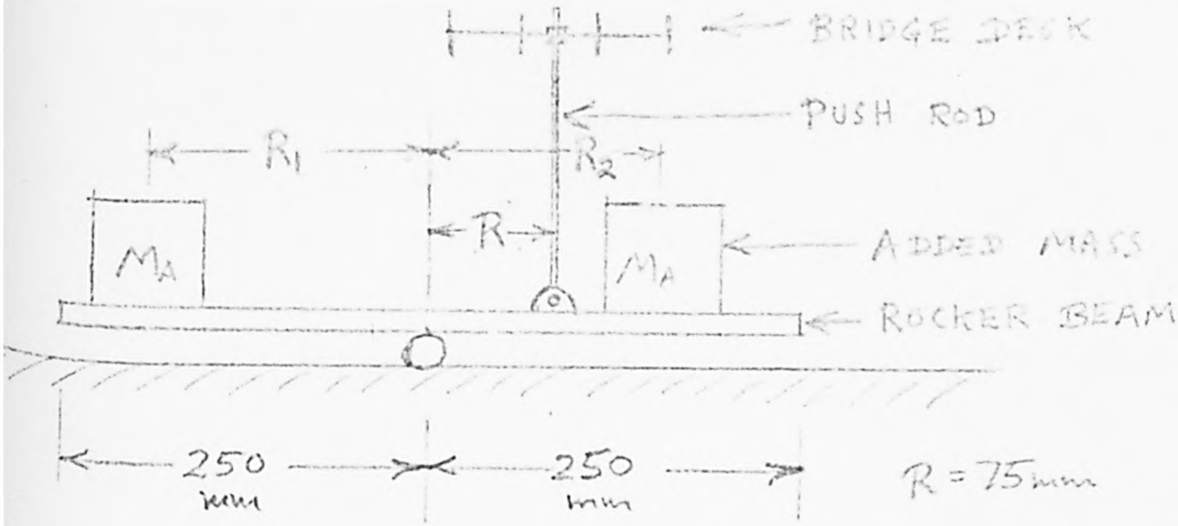


Figure 6.2

It is easily shown that the arrangement shown in Figure 6.2 is equivalent to placing a dead load mass M_D on the bridge itself, where:

$$M_D = \frac{I_b + M_A(R_1^2 + R_2^2)}{R^2} \tag{6.1}$$

With the values of R_1, R_2, and M_A actually employed, equation (6.1) gives $M_D = 15.7$ kg. So the total effective dead load mass of the model bridge is then 9 x 15.7 = 141 kg. The model vehicle has a mass of only 1.3 kg so that the effective live to dead load ratio γ is less than 1/100. This means that the experimental results will be comparable to theoretical results for $\gamma = 0$, in Chapter 5, and there is no possibility of introducing a model vehicle heavy enough to give comparability with the theoretical results for $\gamma = 0.5$. On the other hand, the great advantage of the rocker beam system is that the natural frequency of the bridge is thereby reduced to only 0.73 hertz. This means that results can be obtained for values of β down to 1.0 by employing a vehicle speed of only 3.7 m/sec. Otherwise the theoretical maximum dynamic factors could not be attained with the model, on account of the excessive vehicle speeds implied. In particular it was considered essential to obtain experimental corroboration for the very high dynamic factors predicted by graph 5.3 in Chapter 5.

6.3 Operation of the apparatus

Dynamic factors were evaluated by comparing the maximum deviation of the dynamic trace, on the recording paper, with the deviation produced by placing the vehicle statically at an appropriate position on the bridge. That is, at midspan for assessing main span dynamic factors, or at one end of the main span for assessing dynamic factors for the cantilevers.

Initially results were taken for a range of vehicle speeds equivalent to a variation in β from 1.0 to 3.0.

Subsequently the observations were concentrated in the region of maximum dynamic factors, in order to assess the accuracy of the most significant readings in terms of repeatability. The scatter of the readings, in this region, was found to be $\pm 3\%$ at worst. These carefully checked results are the only ones actually reported below. Each dynamic factor is the average of four actual observations.

Additionally a transducer was used, in a separate set of experiments, to produce a U.V. trace related to deflection beneath the exit end of the main span. By this means an independent check was obtained of the predicted large dynamic factors for the exit cantilever. This device, however, was somewhat less reliable and produced readings with a scatter of $\pm 6\%$.

6.4 EXPERIMENTAL RESULTS

Maximum dynamic factors for the entry cantilever

β	2.65	2.34	2.11	1.62
D_{max}	1.08	1.19	1.27	1.16

Maximum dynamic factors for the main span

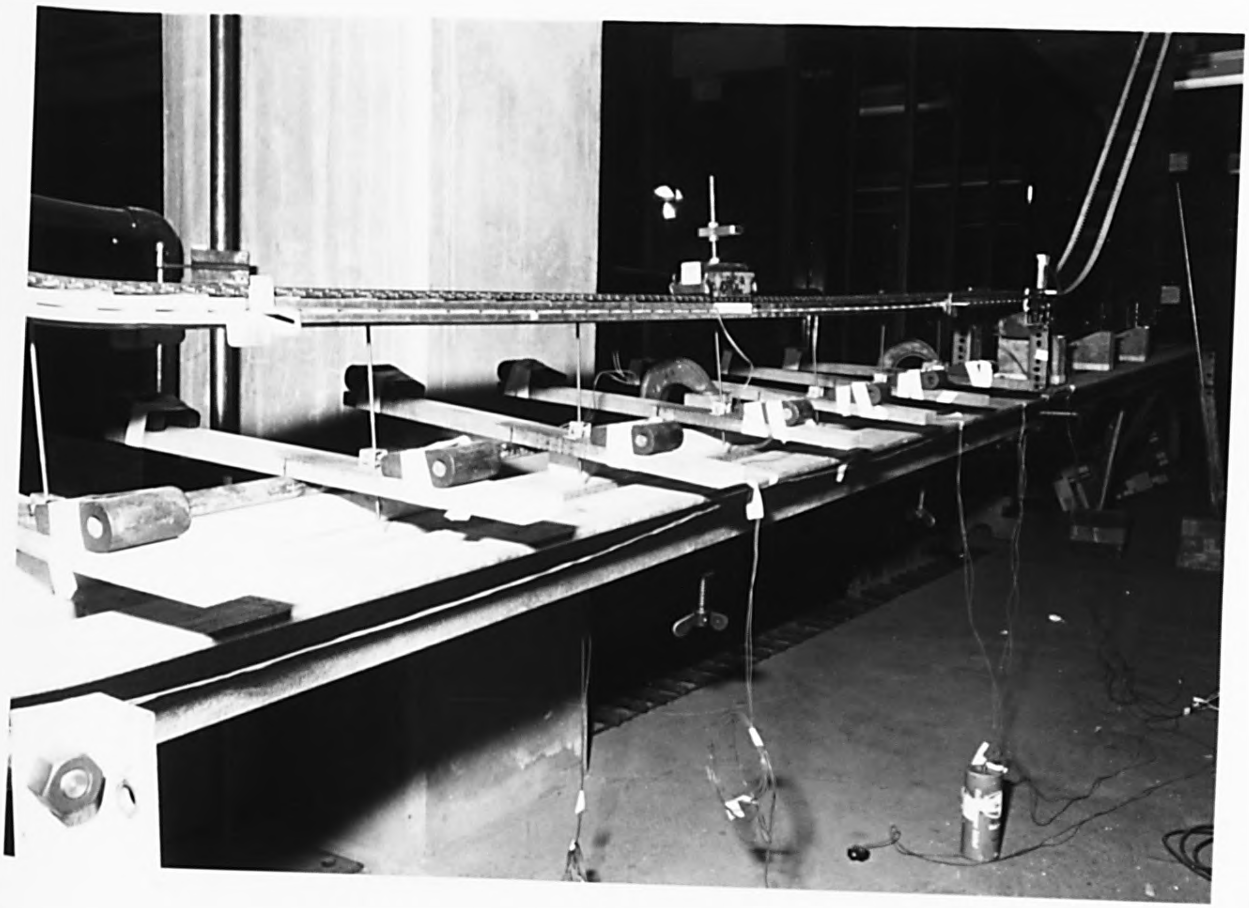
β		2.11	1.62	1.20
D_{max}		1.43	1.26	1.27

Maximum dynamic factors for the exit cantilever

β	1.04	0.99	0.94	0.91	0.86	0.77
D_{max}	3.37	3.44	3.46	3.46	3.43	3.44
* D_{max}	2.76	2.82	2.86	2.84	2.88	2.83

* Independent results based on the deflection transducer.

The above results are also shown on graphs 6.1, 6.2 and 6.3.



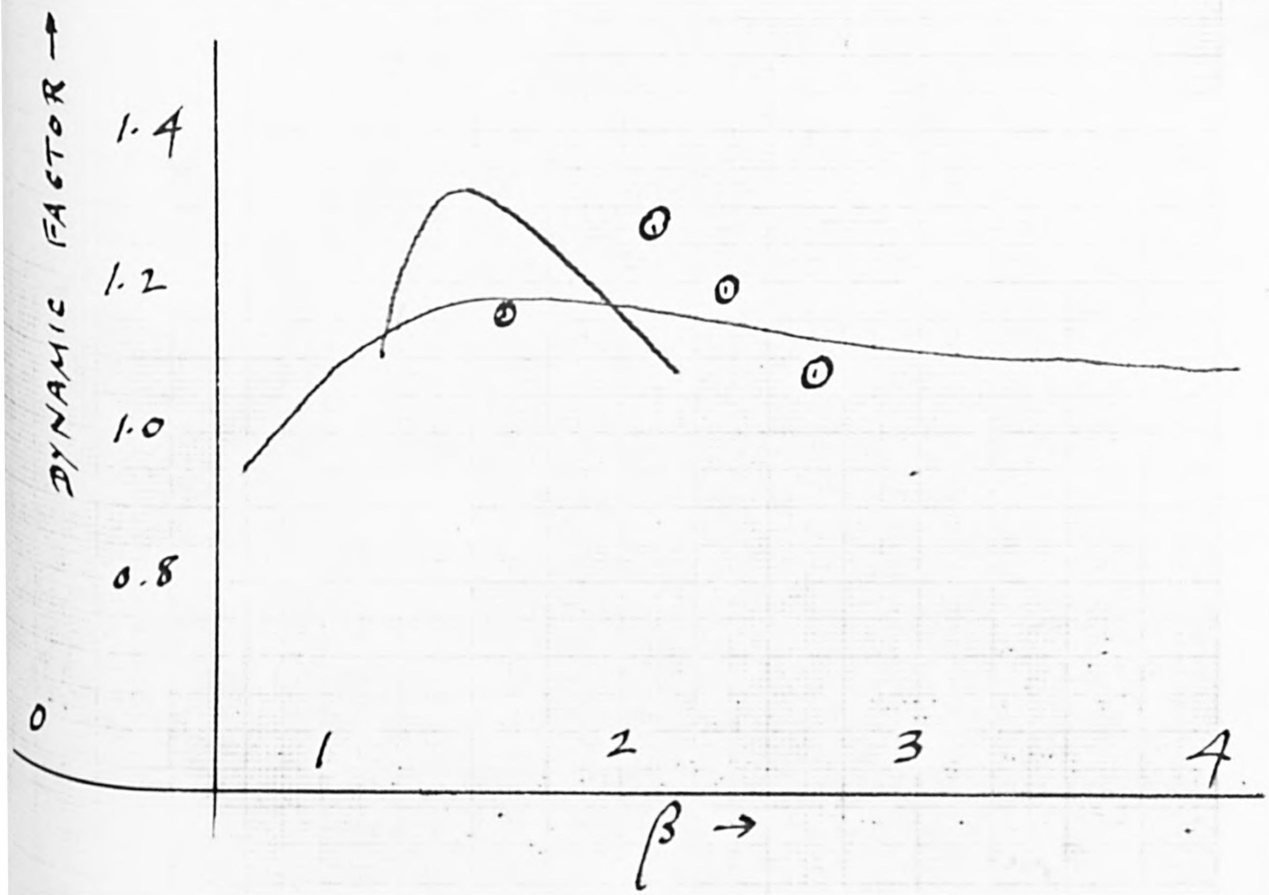
GRAPH 6.1.

EXPERIMENTAL RESULTS

MAXIMUM DYNAMIC FACTORS FOR THE ENTRY CANTILEVER

Points ʘ are from page 89 (obtained experimentally)

The full line graph is a repeat of the γ = 0 from graph 5.1.



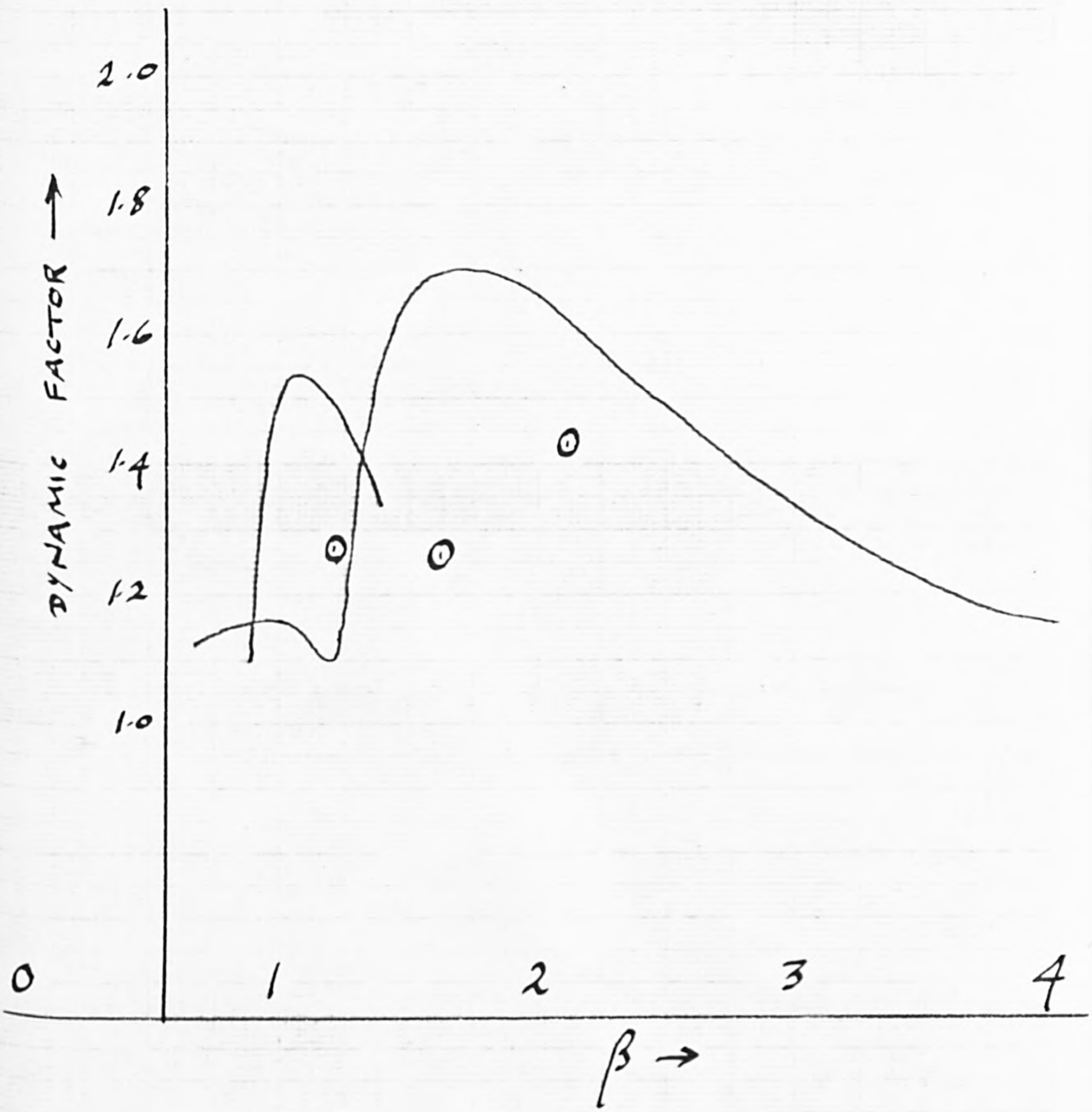
GRAPH 6.2.

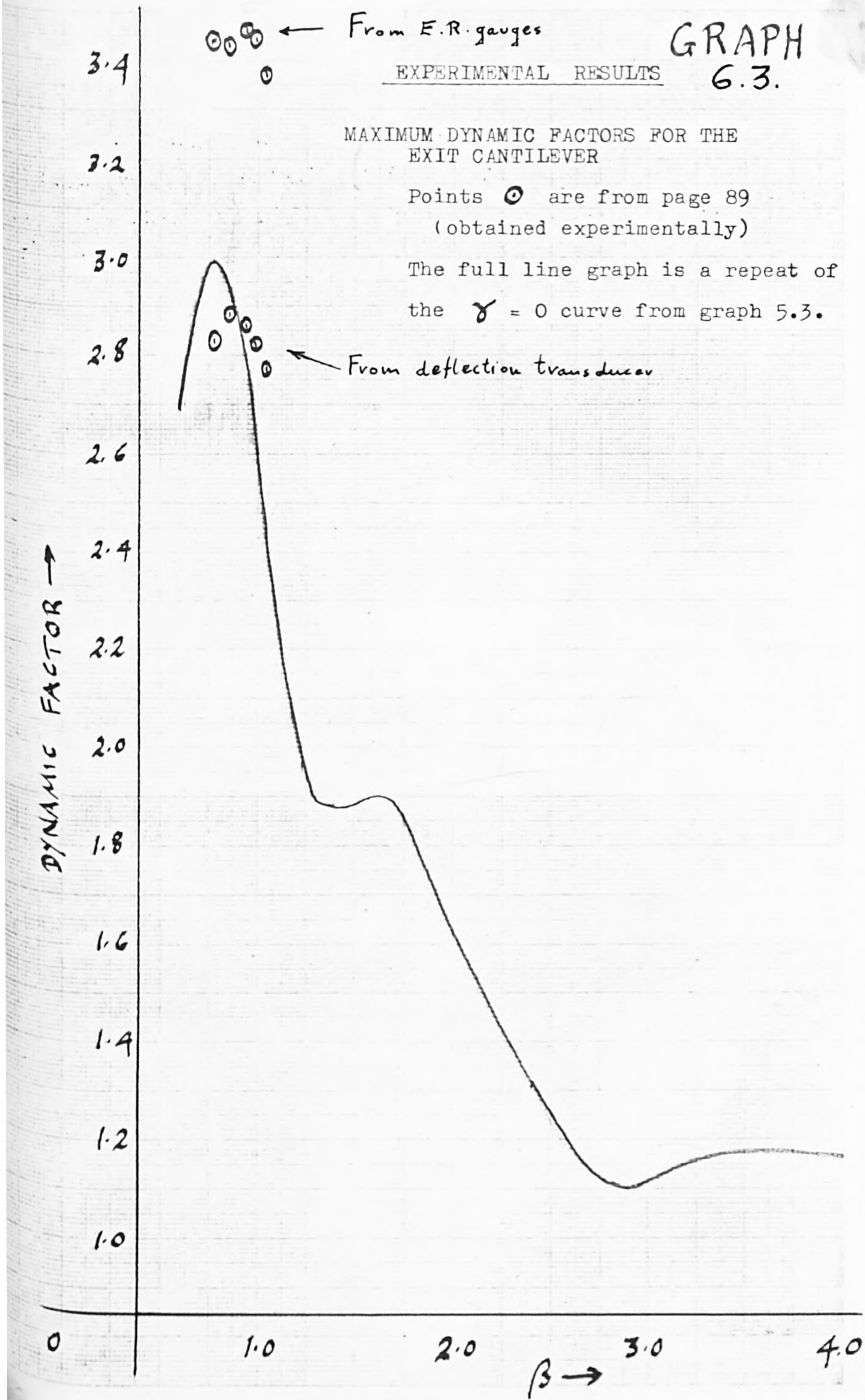
EXPERIMENTAL RESULTS

MAXIMUM DYNAMIC FACTORS FOR THE MAIN SPAN

Points ⊙ are from page 89 (obtained experimentally)

The full line graph is a repeat of the $\gamma = 0$ from graph 5.2.




GRAPH 6.3.
EXPERIMENTAL RESULTS
MAXIMUM DYNAMIC FACTORS FOR THE EXIT CANTILEVER
Points ⊙ are from page 89 (obtained experimentally)
The full line graph is a repeat of the γ = 0 curve from graph 5.3.
From E.R. gauges
From deflection transducer
DYNAMIC FACTOR →
β →
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
1.0
2.0
3.0
4.0

CHAPTER 7

EFFECT OF A VEHICLE WITH ELASTIC SUSPENSION

CHAPTER 7

EFFECT OF A VEHICLE WITH ELASTIC SUSPENSION TRAVERSING A SIMPLY SUPPORTED BEAM

7.1 If the moving load is mounted on springs its centre of mass will experience additional vertical accelerations, relative to the beam, due to the flexing of these springs.

To discover the extent to which maximum dynamic factors may be modified it is proposed to apply the Hamiltonian method to the idealised system shown in Figure 7.1. The results obtained will be compared with those obtained in Chapter 4 (see graph 4.6).

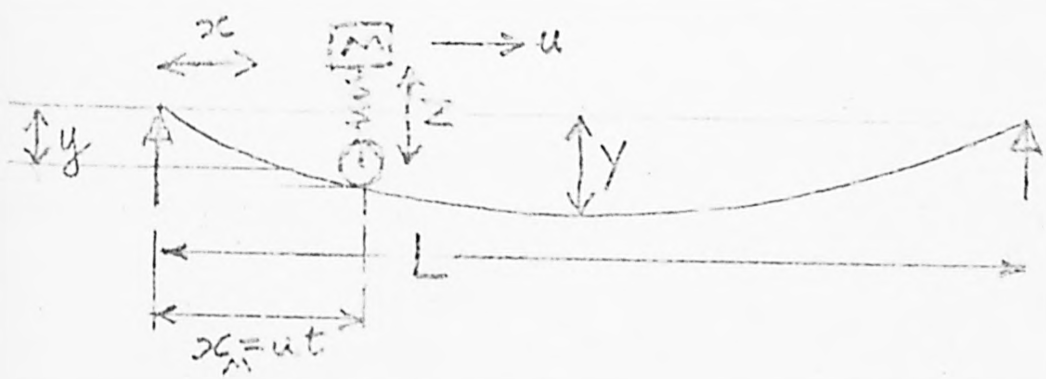


FIGURE 7.1

Hamilton's Principle will be applied with respect to the following displacement/time functions which include appropriate variations.

Deflection of the vehicle springs:

$$Z = Ct^2 + \dot{Z}_o t + Z_o + c \sin qt$$

Deflection of the midspan point of the beam:

$$Y = At^2 + \dot{Y}_o t + Y_o + a \sin qt$$

Deflection of the beam at any point:

$$y = Y \sin \frac{\Pi x}{L}, \quad y_M = Y \sin \frac{\Pi ut}{L}$$

Total deflection of the vehicle (as a function of time):

$$Z_M = Z + y_M = Z + Y \sin \frac{\Pi ut}{L}$$

The calculation will be performed in 60 stages.

7.2 STRAIN ENERGY (V)

Strain energy of the vehicle springs (V_2)

Let Z_1 be the initial deflection in these springs due to the static weight of the vehicle.

Then $V_2 = \frac{1}{2}k(Z + Z_1)^2 = \frac{1}{2}kZ^2 + \frac{1}{2}kZ_1{}^2 + kZ_1Z$

Now substitute for Z from previous para., and also put $kZ_1 = Mg$:

$$V_2 = \tfrac{1}{2}k(Ct^2+\dot{Z}_o t+Z_o+ c \sin qt)^2 + \tfrac{1}{2}kZ_1{}^2 + Mg(Ct^2+\dot{Z}_o t+Z_o + c \sin qt)$$

$$\therefore \delta V_{2c} = ck(Ct^2 + \dot{Z}_o t + Z_o)\sin qt + Mg\, c \sin qt$$

$$\therefore \int_0^{\Pi/q} \delta V_{2c} dt = ck\left(C \int_0^{\Pi/q} t^2 \sin qt dt + \dot{Z}_o \int_0^{\Pi/q} t \sin qt dt + Z_o \int_0^{\Pi/q} \sin qt dt \right)$$

(The term involving Mg has been dropped because it will cancel with a part of $\int \delta P dt$).

Strain energy of the beam itself (V_1)

The elastic deflection of the beam is:

$$y = (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \sin \frac{\Pi x}{L}$$

$$\therefore \quad V_1 = \tfrac{1}{2}EI \frac{\Pi^4}{L^4} (At^2 + \dot{Y}_o t + Y_o + a \sin qt)^2 \int_o^L \sin^2 \frac{\Pi x}{L} \, dx$$

$$\text{i.e.} \quad V_1 = \frac{\Pi^4 EI}{4L^3} (At^2 + \dot{Y}_o t + Y_o + a \sin qt)^2$$

$$\therefore \quad \delta V_{1a} = \frac{\Pi^4 EIa}{2L^3} (At^2 + \dot{Y}_o t + Y_o) \sin qt$$

$$\therefore \quad \int_o^{\Pi/q} \delta V_{1a} dt = \frac{\Pi^4 EIa}{2L^3} \left(A \int_o^{\Pi/q} t^2 \sin qt dt + \dot{Y}_o \int_o^{\Pi/q} t \sin qt dt + Y_o \int_o^{\Pi/q} \sin qt dt \right)$$

Numerical values for a 60 stage calculation ($q = \frac{60\Pi u}{L}$)

$$\text{Let} \quad \xi = \frac{\text{stiffness of vehicle springs}}{\text{stiffness of beam at midspan}}$$

$$\text{Then} \quad \int_o^{L/60u} \delta V_{2c} dt = \xi \, C \, \frac{EI}{u^3} (.000042685)$$

$$+ \xi \, \dot{Z}_o c \, \frac{EI}{u^2 L} (.004306)$$

$$+ \xi \, Z_o c \, \frac{EI}{uL^2} (.516775)$$

$$\int_0^{L/60u} \delta V_{1a} dt = Aa \frac{EI}{u^3} \ (.000042685)$$

$$+ \dot{Y}_o a \frac{EI}{u^2 L} \ (.004306)$$

$$+ Y_o a \frac{EI}{uL^2} \ (.516775)$$

7.3 KINETIC ENERGY OF THE BEAM ITSELF (T_1)

$$y = (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \sin \frac{\Pi x}{L}$$

$$\therefore \dot{y} = (2At + \dot{Y}_o + qa\cos qt) \sin \frac{\Pi x}{L}$$

$$T_1 = \tfrac{1}{2}m \int_0^L \dot{y}^2 dx = \tfrac{1}{2}m(2At + qa\cos qt)^2 \int_0^L \sin^2 \frac{\Pi x}{L} dx$$

i.e. $$T_1 = \frac{mL}{4} (2At + \dot{Y}_o + qa\cos qt)^2$$

$$\therefore \delta T_{1a} = \frac{amL}{2} (2Aqt\cos qt + \dot{Y}_o q\cos qt)$$

$$\therefore \int_0^{\Pi/q} \delta T_{1a} dt = aAmL \int_0^{\Pi/q} qt\cos qt dt, \left(\text{since} \int_0^{\Pi/q} \cos qt dt = 0\right)$$

For a 60 stage calculation, $q = \dfrac{60\Pi u}{L}$

$$\int_0^{L/60u} \delta T_{1a} dt = -0.010610 \ Ama \ L^2/u$$

7.4 POTENTIAL ENERGY OF THE VEHICLE (P)

$$P = -Mg\ Z_M$$

$$= -Mg \left\{ (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \sin \frac{\Pi ut}{L} + Ct^2 + \dot{Z}_o t + Z_o + c \sin qt \right\} \quad \dagger$$

$$\therefore \quad \delta P_a = -Mg\ a \sin qt \sin \frac{\Pi ut}{L}$$

Let QQ1 and QQ2 be the values of $\sin \frac{\Pi ut}{L}$ at the quarter and three quarter points of a stage calculation, respectively. Then for an approximate integration process (as used in Chapter 5):

$$\int_o^{\Pi/q} \delta P_a dt = -Mga \left\{ QQ1 \int_o^{\Pi/2q} \sin qt\ dt + QQ2 \int_{\Pi/2q}^{\Pi/q} \sin qt\ dt \right\}$$

$$\delta P_c = -Mgc \sin qt \quad \text{(from } \dagger \text{ above)}$$

So $\int \delta P_c dt$ will cancel with the Mg term in $\int \delta V_{2c} dt$, as noted previously.

For a 60 stage calculation, $q = 60\ \Pi\ u/L$:

$$\int_o^{L/60u} \delta P_a dt = -Mg \frac{L}{u} a(.0053052(QQ1) + .0053052(QQ2))$$

7.5 KINETIC ENERGY OF THE VEHICLE (T_2)

$$T_2 = \tfrac{1}{2} M \dot{Z}_M^2 \text{, and from para. 7.1:}$$

$$Z_M = (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt) + (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \sin \frac{\Pi ut}{L}$$

$$\dot{Z}_M = 2Ct + \dot{Z}o + qc \cos qt$$
$$+ (2At + \dot{Y}_o + qa \cos qt) \sin \frac{\Pi ut}{L}$$
$$+ \frac{\Pi u}{L}(At^2 + \dot{Y}_o t + Y_o + a \sin qt) \cos \frac{\Pi ut}{L}$$

The above expression will now be modified so that t=0 refers consistently to the beginning of the nth stage of calculation, viz:

$$t \rightarrow \left(\frac{n-1}{60} \cdot \frac{L}{u} + t\right) \text{ for a 60 stage calculation.}$$

Then $\sin \frac{\Pi ut}{L} \rightarrow \sin\left(\Pi \left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right)$

and $\cos \frac{\Pi ut}{L} \rightarrow \cos\left(\Pi \left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right)$

($\frac{\Pi ut}{L}$ varies from 0 to $\Pi/60$ during each stage.)

As in Chapter 5, it is intended to approximate by employing only the values of these quantities appropriate to the 1/4 and 3/4 points of each stage, when $\frac{\Pi ut}{L}$ has the values $\frac{\Pi}{240}$ and $\frac{3\Pi}{240}$, respectively. So the expressions actually employed will be as follows:

$$\sin \frac{\Pi ut}{L} = \sin\left(\Pi \left(\frac{n-1}{60}\right) + \frac{\Pi}{240}\right) = \sin \frac{(4n-3)\Pi}{240} = QQ1$$

$$\sin \frac{\Pi ut}{L} = \sin\left(\Pi \left(\frac{n-1}{60}\right) + \frac{3\Pi}{240}\right) = \sin \frac{(4n-1)\Pi}{240} = QQ2$$

$$\cos \frac{\Pi ut}{L} = \cos\left(\Pi \left(\frac{n-1}{60}\right) + \frac{\Pi}{240}\right) = \cos \frac{(4n-3)\Pi}{240} = \theta\theta 1$$

$$\cos \frac{\Pi ut}{L} = \cos\left(\Pi \left(\frac{n-1}{60}\right) + \frac{3\Pi}{240}\right) = \cos \frac{(4n-1)\Pi}{240} = \theta\theta 2$$

Now consider $T_2 = \frac{1}{2}M\dot{Z}_M^{\,2}$. The terms involving 'a' will be as follows, noting that $q = \frac{60\Pi u}{L}$:

Π/2q to 0 Π/q to Π/2q

$$\frac{\delta T_{2a}}{M.a} = (2At + \dot{Y}_o)q \cos qt \;(QQ1^2 ; \quad QQ2^2 \quad)$$

$$+ \frac{\Pi^2 u^2}{L^2} (At^2 + \dot{Y}_o t + Y_o) \sin qt \;(\theta\theta 1^2 ; \quad \theta\theta 2^2)$$

$$+ \frac{\Pi u}{L} (2At + \dot{Y}_o) \sin qt \;(QQ1.\theta\theta 1 ; \quad QQ2.\theta\theta 2)$$

$$+ \frac{\Pi u}{L} (At^2 + \dot{Y}_o t + Y_o)\, q \cos qt \;(QQ1.\theta\theta 1 ; \quad QQ2.\theta\theta 2$$

Π/2q to 0; Π/q to Π/2q

$$+ (2Ct + \dot{Z}_o)\dot{q} \cos qt \;(QQ1 \quad ; \quad QQ2)$$

$$+ \frac{\Pi u}{L}(2Ct + \dot{Z}_o) \sin qt \;(\theta\theta 1 \quad ; \quad \theta\theta 2)$$

Again considering $T_2 = \frac{1}{2}M\dot{Z}_M^{\,2}$, the terms involving 'c' are as follows:

Π/2q to 0 Π/q to Π/2q

$$\frac{\delta T_{2c}}{M.c} = (2At + \dot{Y}_o)q \cos qt \;(QQ1 \quad ; \quad QQ2 \quad)$$

$$+ \frac{\Pi u}{L} (At^2 + \dot{Y}_o t + \dot{Y}_o)\, q \cos qt \;(\theta\theta 1 ; \quad \theta\theta 2)$$

$$+ (2Ct + \dot{Z}_o)q \cos qt.$$

Now to obtain the required integral the procedure of Chapter 5 is again adopted as follows:

$$\int_{o}^{\Pi/q} \delta T_2 dt = \int_{o}^{\Pi/2q} \delta T_2 dt + \int_{\Pi/2q}^{\Pi/q} \delta T_2 dt$$

where $q = 60\Pi u/L$ for a 60 stage calculation

$\theta\theta 1$ and QQ1 will be introduced into the first integration and $\theta\theta 2$ and QQ2 into the second. These quantities have been defined to be values of $\sin\frac{\Pi ut}{L}$ and $\cos\frac{\Pi ut}{L}$ at the $\frac{1}{4}$ and $\frac{3}{4}$ points of each of the 60 stages of the calculation.

The net resulting values of $\int_{o}^{\Pi/q} \delta T_2 dt$ will be increased in accuracy by applying correction factors $4/\Pi$ and 1.05 to integrals involving cos qt and t cos qt, respectively (see para. 5.4a)

$$\frac{1}{M.a} \int_{o}^{L/60u} \delta T_{2a} dt = \qquad \text{(note } L/60u = \Pi/q\text{)}$$

$$A \frac{L}{u}\Big(.006359(QQ1)^2 - .02864(QQ2)^2$$
$$+ .000001682(\theta\theta 1)^2 + .000006968(\theta\theta 2)^2$$
$$+ .000218(\theta\theta 1).(QQ1) - .0002181(\theta\theta 2).(QQ2)\Big)$$

$$+ \dot{Y}_o\Big(1.273(QQ1)^2 - 1.273(QQ2)^2$$
$$+ .00027773(\theta\theta 1)^2 + .0005948(\theta\theta 2)^2$$
$$+ .026655(\theta\theta 1).(QQ1) - .028320(\theta\theta 2).(QQ2)\Big)$$

$$+ Y_o \frac{u}{L}\Big(.052360(\theta\theta 1)^2 + .052360\ (\theta\theta 2)^2$$
$$+ 4(\theta\theta 1)(QQ1) - 4(\theta\theta 2)(QQ2)$$

$$+ C\frac{L}{u}\Big(.006359(QQ1) - .028640\ (QQ2)$$
$$+ .0001768(\theta\theta 1) + .0003787\ (\theta\theta 2)\Big)$$

$$+ \dot{Z}_o\ \Big(1.273(QQ1) - 1.273(QQ2)$$
$$+ .016667(\theta\theta 1) + .016667(\theta\theta 2)$$

$$\frac{1}{M.c}\int_o^{L/60u} \delta T_{2c}dt\ = \qquad \left(\text{note } \frac{L}{60u} \cong \Pi/q\right)$$

$$A\frac{L}{u}\ \Big(.0063588(QQ1) - .028640\ (QQ2)$$
$$+ .0000413\ (\theta\theta 1) - .0005969(\theta\theta 2)\Big)$$

$$+ \dot{Y}_o\ \Big(1.273(QQ1) - 1.273(QQ2)$$
$$+ .009988(\theta\theta 1) - .044987(\theta\theta 2)\Big)$$

$$+ Y_o\frac{u}{L}\Big(4(\theta\theta 1) - 4(\theta\theta 2)\Big)$$

$$+ C\frac{L}{u}\ \Big(-.022281\Big)$$

$$+ \dot{Z}_o\ \Big(\ \text{ZERO}\ \Big)$$

7.6 Having now obtained expressions for $\int \delta T dt_1$, $\int \delta V dt$ and $\int \delta P dt$ in paras. 7.2 to 7.5, it now remains to substitute these expressions in the basic Hamiltonian equation 2.1.

viz $$\int_o^{\Pi/q} (\delta T - \delta V - \delta P) dt = 0 \qquad (2.1)$$

where Π/q is the time duration of the small variations a sin qt and c sin qt and also $\frac{\Pi}{q} = \frac{L}{60u}$ which is the length of each stage of the 60 stage computation.

Since there are two variations introduced into the present problem, equation (2.1) will yield two equations at the end of each of the 60 stages. The form of these two equations, which must be solved simultaneously to give the stage values of A and C, is shown below. Symbols such as AA, CA, YA, YYA, etc. represent the appropriate expressions in brackets from para. 7.5. For the significance of Y and Z, see Figure 7.1. A suffix 'o' indicates that the value is that at the beginning of a stage of calculation. All the terms have been multiplied by $\frac{u^3}{EI}$, and the transit speed parameter β has been introduced according to its definition in equation (4.12), together with the load mass parameter $\gamma = M/mL$.

FOR THE 'a' VARIATION:

$$A\left(-.01061\,\frac{\Pi^2}{\beta^2} + \gamma(AA)\,\frac{\Pi^2}{\beta^2} - .00004269\right) + C(CA)\,\gamma\,\frac{\Pi^2}{\beta^2}$$

$$= \frac{MgLu^2}{EI}\,(AP) - \frac{\Pi^2}{\beta^2}\,\gamma\left(\dot{Y}_o\,\frac{u}{L}\,(YYA) + Y_o\,\frac{u^2}{L^2}\,(YA) + \dot{Z}_o\,\frac{u}{L}\,(ZZA)\right)$$

$$+ \dot{Y}_o\,\frac{u}{L}\,(.004306) + Y_o\,\frac{u^2}{L^2}\,(.51677)$$

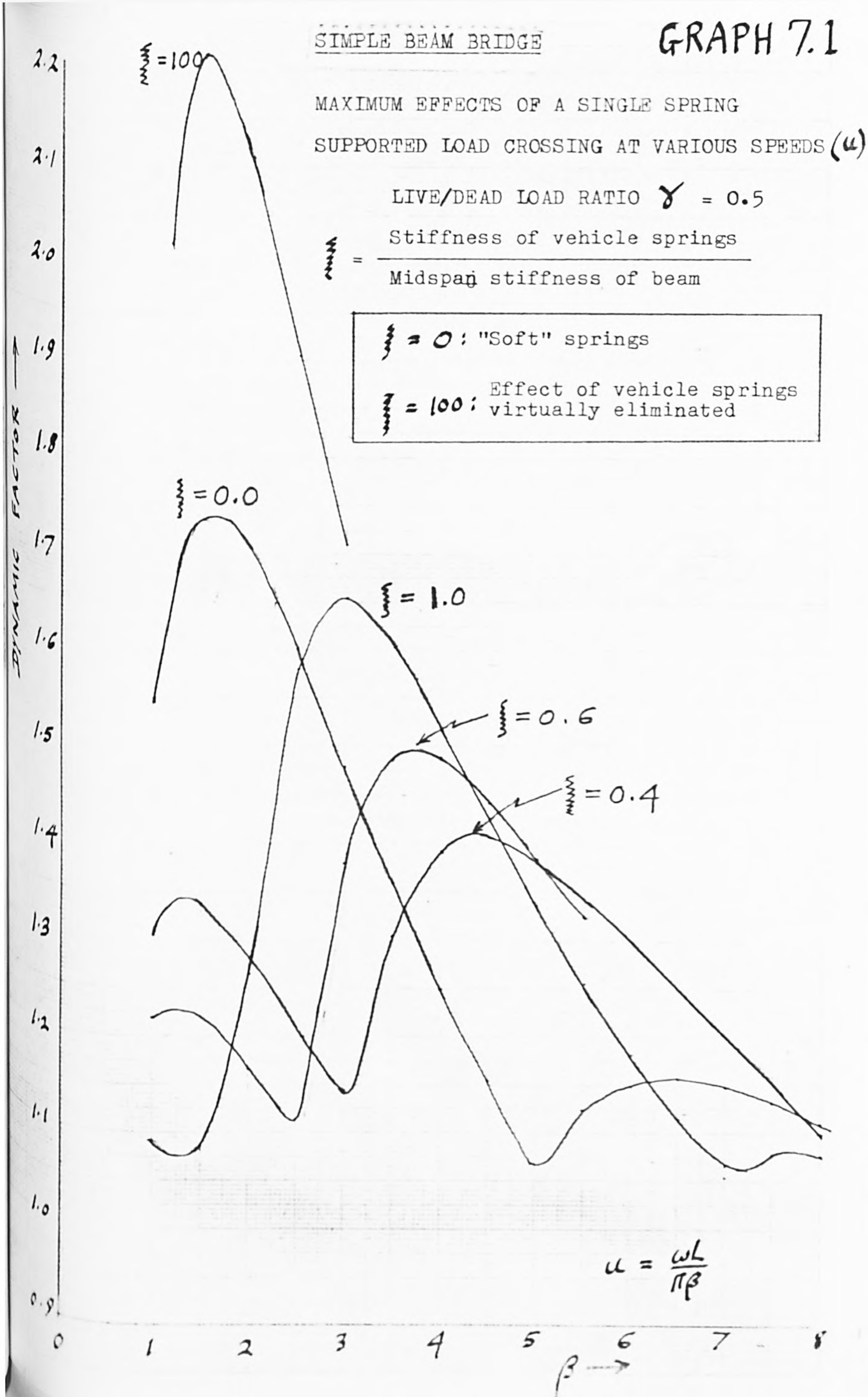
FOR THE 'c' VARIATION:

$$A(AC)\,\gamma\,\frac{\Pi^2}{\beta^2} + C\left(\gamma(CC)\,\frac{\Pi^2}{\beta^2} - .00004269\,\xi\right)$$

$$= -\frac{\Pi^2}{\beta^2}\,\gamma\left(\dot{Y}_o\,\frac{u}{L}\,(YYC) + Y_o\,\frac{u^2}{L^2}\,(YC)\right)$$

$$+ \dot{Z}_o\,\frac{u}{L}\,\xi(.004306) + Z_o\,\frac{u^2}{L^2}\,\xi\,(.51677)$$

$$\xi = \frac{\text{Stiffness of vehicle springs}}{\text{Stiffness of beam at midspan}}$$



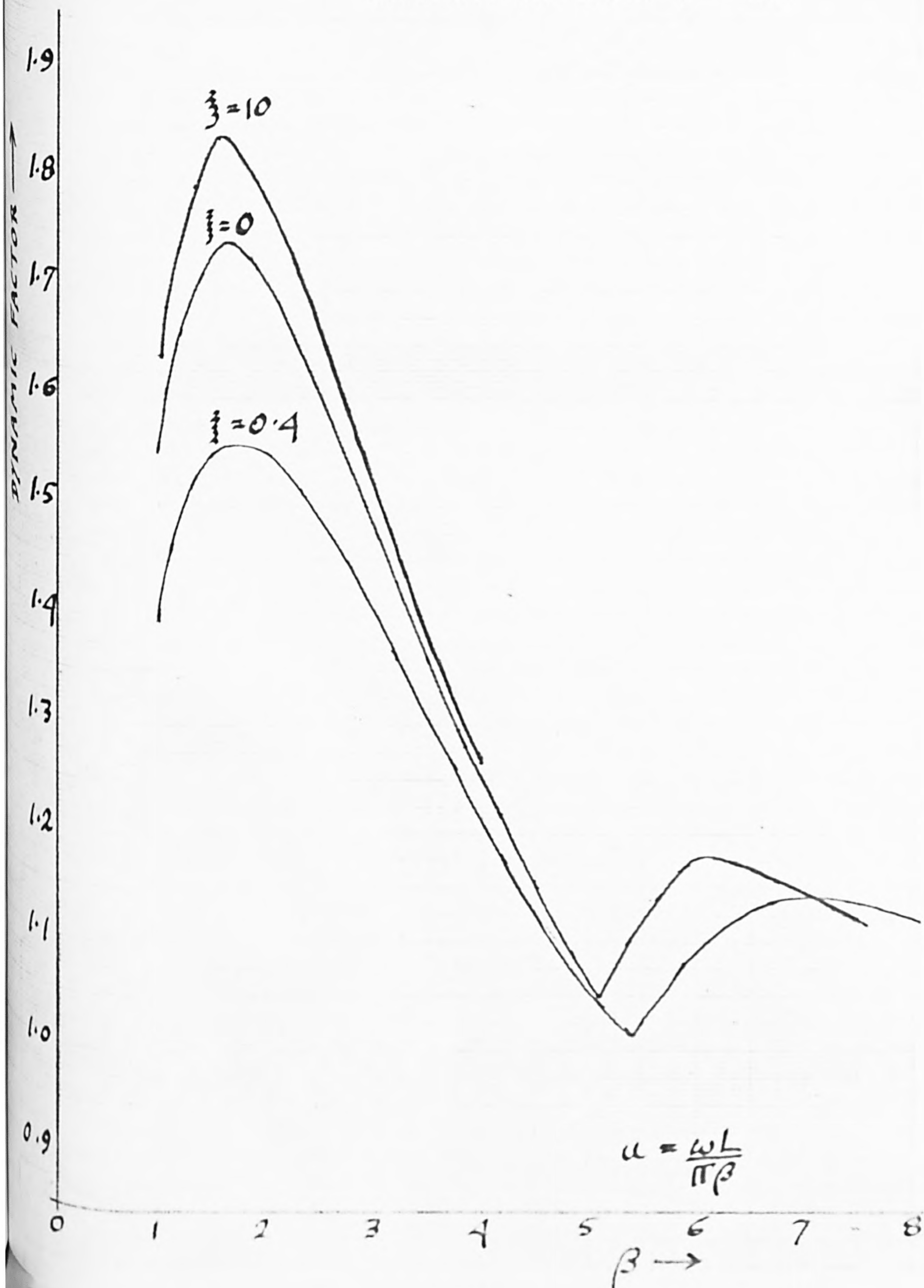
GRAPH 7.1
SIMPLE BEAM BRIDGE
MAXIMUM EFFECTS OF A SINGLE SPRING SUPPORTED LOAD CROSSING AT VARIOUS SPEEDS (u)
LIVE/DEAD LOAD RATIO γ = 0.5
$\xi = \frac{\text{Stiffness of vehicle springs}}{\text{Midspan stiffness of beam}}$
ξ = 0 : "Soft" springs
ξ = 100 : Effect of vehicle springs virtually eliminated
ξ=100
ξ=0.0
ξ= 1.0
ξ=0.6
ξ=0.4
DYNAMIC FACTOR →
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0
1
2
3
4
5
6
7
8
β →
$u = \frac{\omega L}{\pi \beta}$

GRAPH 7. 2.

SIMPLE BEAM BRIDGE

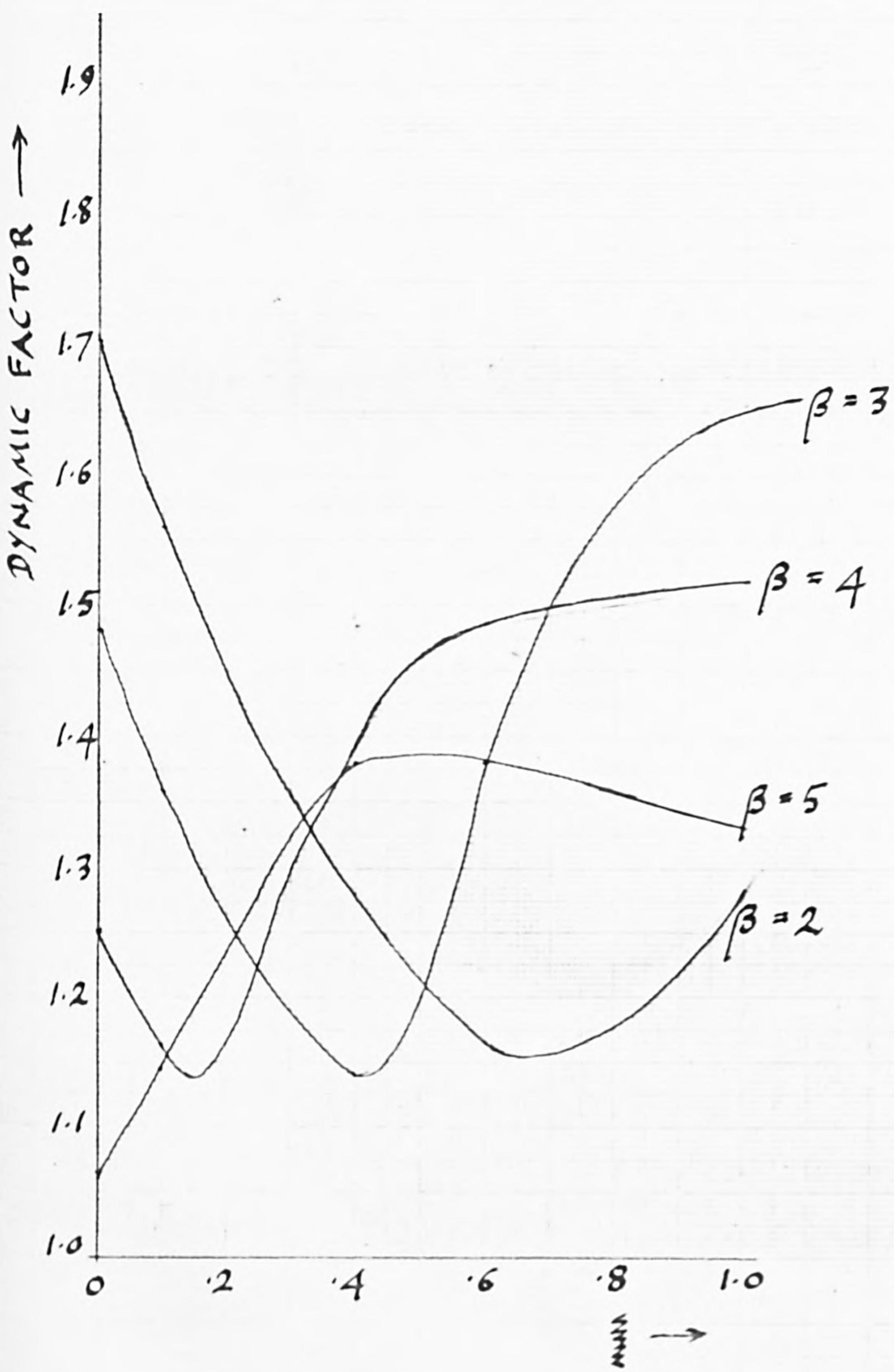
MAXIMUM EFFECTS OF A SINGLE SPRING SUPPORTED LOAD CROSSING AT VARIOUS SPEEDS (u)

LIVE/DEAD LOAD RATIO γ = 0.1



GRAPH 7.3.

CROSS PLOT OF GRAPH 7.1.



Comments on Chapter 7

In previous chapters treating the load as a mass has been regarded as a refinement but it becomes essential in the case of a load carried on springs.

The programme produced in this chapter can yield results for any vehicle/beam mass ratio (γ) and some typical results are presented on graph 7.1. All the curves on this sheet are for the same value of γ, (0.5), and the main object is to reveal the effect of different degrees of stiffness in the vehicle springs. This variation is expressed by the parameter ξ, where $\xi = 0$ represents infinitely soft springs, whilst $\xi = 1$ represents a spring system such that the deflection in the vehicle springs would be equal to the deflection of the bridge itself if the vehicle were standing at midspan.

Limiting cases may be successfully checked against results for an unsprung mass load in Chapter 4. For instance, when $\xi = 100$ (very hard springs), the results agree with those on graph 4.6 with $\gamma = 0.5$. When $\xi = 0$, (infinitely "soft" springs), the results are closely comparable with the $\gamma = 0$ curve on graph 4.6. This seems a logical outcome because relative movements between vehicle and axle, with "zero" springs, would be small compared to the initial static deflection in these springs. Hence the force on the axles would scarcely alter at all. Another way of expressing this situation is to say that the classical analysis (as in Appendix 2) which treats the load as an unvarying force, is correct for a vehicle with very soft springing.

The other curves on graph 7.1 suggest that there is an optimum value of ξ (round about 0.4) which leads to the greatest possible reduction in maximum dynamic factor, as compared to an unsprung load. It may also be noticed that the maximum dynamic factor then occurs at a higher value of β in such cases. That is to say the critical speed of the vehicle is lower and therefore more likely to be attained in practice.

However, any conclusions drawn from graph 7.1 are limited in value by the fact that the bridge designer has no control over the design of vehicle suspension systems and must design the bridge for the worst possiblity.

It appears, from the operations in this chapter, that the Hamiltonian method could well be adapted to record the behaviour of vehicle suspension systems whilst the vehicle is traversing "rough ground" as distinct from an elastically deformed beam. The vehicle would be treated as a rigid body supported by an appropriate number of spring connected axles. Damping forces would need to be introduced, of magnitude considerably greater than those usually envisaged in structural analysis.

CHAPTER 8

DISCUSSION AND CONCLUSIONS

DISCUSSION AND CONCLUSIONS

At this stage it is natural to attempt an assessment of the Hamiltonian method developed in this thesis and the main point to consider will be the accuracy of the calculations. The degree of accuracy obtainable depends on the following factors:

(a) the number of stages employed in each computation

(b) the number of shape functions introduced into the displacement/ time functions, and

(c) whether or not these shape functions are true mode shapes.

The most accurate calculations will be those illustrated in Graphs 4.11 and 4.12 which are based on sixty stages and two shape functions(which are both true mode shapes for the problem in question). This degree of accuracy was, at that point, considered necessary in order to evaluate dynamic bending moments as distinct from deflections which were obtained to a reasonable degree of accuracy from a much less rigorous program in the earlier part of Chapter 4.

In Chapter 5, for the cantilever bridge, a considerable number of stages were employed and three shape functions which, however, were not true mode shapes. The actual inaccuracy introduced on this account remains uncertain and could only be reduced by introducing more shape functions. It is thought that a fourth function (allowing the main span to bend in double curvature) would be a sufficient advance, so far as dynamic deflections are concerned, but that a fifth function (allowing triple curvature) would be desirable if midspan bending moments are sought. Inevitably these additions would increase the complexity of the program but the extent to which they would do so depends very much on whether the loads are to be regarded as forces or masses.

The treatment of loads as masses has been a principal feature of this thesis and it has been made evident that the complexity of the Hamiltonian method is greatly increased thereby (see Appendix 5). The results obtained have shown without doubt that increased dynamic factors are obtainable with mass loading but that this increase falls off considerably as the speed of transit is reduced. Also it has been shown (Chapter 7) that the increase will be alleviated when the mass load is carried on springs.

All in all it is believed to have been adequate to study the "mass" effect in calculations of limited accuracy and complexity in the belief that the proportional effect would be very similar in a more rigorous calculation. It has been common practice to ignore the "mass" effect altogether and it can be seen from results, obtained herein, that this may be justified on the understanding that critical speeds are never approached and that live/dead load ratios are not exceptional. If the mass effect is dismissed then the Hamiltonian method can be employed much more easily even when more shape functions are introduced. In fact, when these shape functions are true mode shapes (and $\gamma=0$) it will not be necessary to solve simultaneous equations at the end of every stage of the calculation. On the other hand it must be remembered that the lower the speed of transit the more important it becomes to use a generous number of stages in each computation. This, however, in no way increases the complexity of application of the Hamiltonian method.

It will now be appropriate to make a more specifical appraisal of the Hamiltonian method of calculation which has been developed and applied in this thesis. This will be done by comparing specific numerical results with those obtained by other workers using other methods. The principal source of such comparative results arises in the case of simply supported uniform beams which have received much attention.

Before proceeding to such comparisons it will be recalled that the Hamiltonian method has already been successfully checked against classical calculations for the case of "force" loading. To cater for "mass" loading requires a considerable increase in complexity, whatever method is used, and it is particularly in this area that comparisons will now be made.

References 8 and 9 present a rigorous algorithm for evaluating dynamic deflections of a simply supported beam subject to a moving mass load. Some of the numerical results quoted are for a live/dead load mass ratio equivalent to the author's $\gamma = 0.5$, so that a direct comparison with results in Chapter 4 of this thesis, is possible. This comparison is shown by Graph 8.1. Unfortunately reference 8 does not quote results in the region $\beta = 1.4$ to 1.6 which the author found to be the location of maximum dynamic factors. A maximum discrepancy of 3% is observable when $\beta = 2.0$ and it is estimated (by rough plotting of ref. 8 points) that the discrepancy in maximum response would be little more than this.

At values of $\beta \lesssim 1.0$ the points plotted in Graph 8.1 represent the deflection just as the load completes the transit. Actually, at these low values of β the maximum response occurs after the load has run off the span. This has been allowed for on the author's Graph 4.6 but is not discussed in Reference 8.

Another graph, in Reference 8, reports the deflection of the beam beneath the moving load for $\beta = 2.0$. This graph has been compared with results from the author's program developed in Chapter 4. The agreement is good, especially in the region of maximum ordinates where the discrepancy is only 2 or 3 percent. A graph of the same type, but for a critical value of $\beta = 1.6$ is Graph 4.3 in this thesis. Another point of general agreement, between Reference 8 and this thesis, is in the position of the load when maximum response occurs.

Reference 10 presents values of dynamic factor for $\gamma = 1.0$ as well as $\gamma = 0.5$, and includes values for $\beta = 1.33$ which is very close to the critical speed of transit. Points obtained from Reference 10 are also plotted on Graph 8.1 and can be seen to agree closely with the author's points. It seems worth remarking that the points in Reference 10 came from a calculation in 400 stages whereas the author's points are dependent on a ten stage calculation.

Another point of agreement is that both Reference 10 and Graph 4.6 of this thesis, show that the increase of dynamic factor with γ is approximately linear.

Reference 11 also presents results comparable to those discussed above but show some disagreement (about 8%). However, it is noteworthy that the authors of this paper emphasise the importance of allowing for all the components of acceleration experienced by the load, as outlined on Page 1 of this thesis.

The dynamic response of cantilever bridges is discussed in Reference 12, the results being calculated by standard modal analysis based on mode shapes obtained in Reference 13. Two cases are considered:

(a) a single vehicle treated as a constant force moving at a constant speed (three bridges considered)

(b) a simple single vehilce treated as a spring supported mass (one bridge considered).

The results quoted refer to particular existing bridges and the speeds of transit are much lower than critical. Also the configuration of these bridges is basically different to the one envisaged in Chapter 5 of this thesis, and travelling distributed loads are not considered, so that no really significant comparisons are possible. However, in the final conclusions of Reference 12 there are two statements that point in the same direction as the results of Chapter 5 and may be summarised as follows:

(a) the response of such bridges increases with the speed of transit and the "exit" cantilever invariably shows more response than the "entry" cantilever.

(b) the presence of "humps" has a marked effect on the responses.

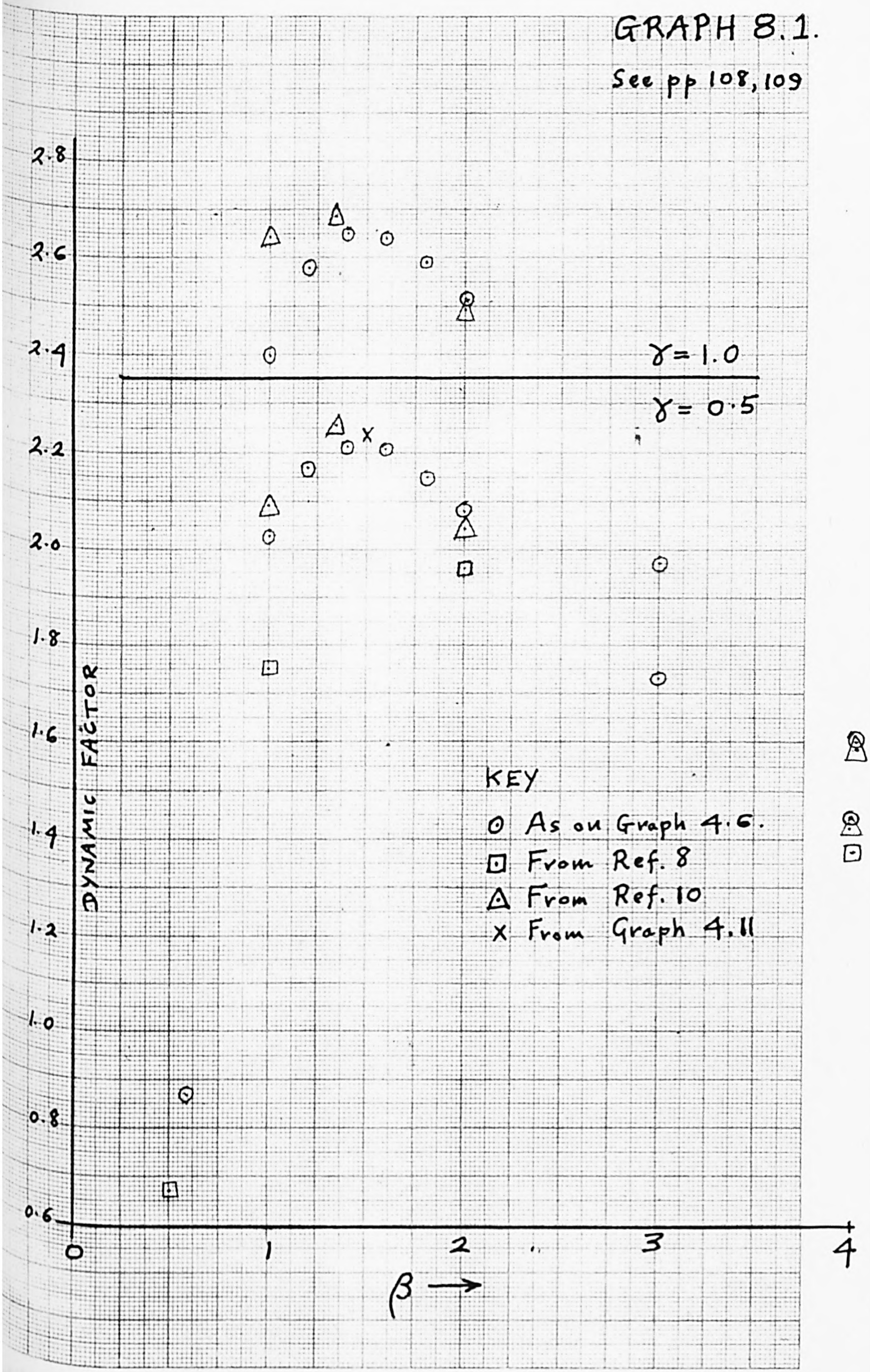
The cantilever bridge in Chapter 5 was devised by the author merely to serve as a basis for developing his Hamiltonian method in a situation more challenging than a simply supported beam. Having done this it was felt that the method could be applied equally well to any other type of cantilever bridge but that much more computing would be needed before any general conclusions could be drawn about the dynamic behaviour of this type of structure. The essential need is to discover the ratio of cantilever spans to main span that leads to the most favourable dynamic factors, also taking into account appropriate variations in flexural rigidity EI. The number of variables is rather daunting especially as the "answer" is likely to be different for every value of the live/dead load ratio.

Before applying the Hamiltonian method to such an extensive field of investigation it is thought that it could be beneficial to investigate the Hamiltonian method itself a little further, as suggested below.

It is thought that one of the most encouraging features of the Hamiltonian method has been the ability to produce some interesting and fairly accurate results from a program involving only ten stages of calculation. It would seem worth while to investigate further to discover how far this feature can be relied on when problems are being tackled which involve more structural elaboration than a simply supported uniform beam. For instance, although 100 stages of calcuation were used in Chapter 5 for the single load on the cantilever bridge, it was found that the results were very little different to those obtained with 30 stages, especially as regards maximum responses.

At the same time, however, it will be desirable to investigate, quite extensively, the loss of accuracy arising from the use of shape functions which are not true mode shapes. It is suggested that this should be done by computing detailed dynamic responses in free vibration and comparing the results with corresponding ones obtained from modal analysis. (The free vibration could be initiated by a computer input of initial deflections or velocities or both). The object of special interest, here, would be to observe the extent of the wave distortion which will occur on deflection/time graphs for particular points on the structure (referred to already in the "comments" section of Chapter 5). Also to note to what extent this type of error builds up with time.

Finally to turn attention again to the other object of this thesis which is to elucidate Hamilton's Principle itself. It is possible to conclude, from the work done, that Hamilton's Principle is merely another way of organising the arithmetic in order to produce results identical to those obtainable by a more orthodox approach. However, it must be pointed out that the same remark could be made about the use of energy equations and techniques in many areas of applied mechanics where they form an alternative approach to the use of force equations. Nevertheless, "Energy" is regarded as being a significant concept in its own right. The author felt, and still feels, that "Action" has also its own individual significance. The special significance of "Energy" arises from the fact that it is associated with a conservation principle, and if "Action" has a special significance it must presumably be because it is associated with a minimum principle. The reader is referred to Appendix I where the author has looked more closely at the manner in which "Action" manifests itself in connection with vibratory behaviour.


GRAPH 8.1.
See pp 108, 109
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
DYNAMIC FACTOR
γ = 1.0
γ = 0.5
0
1
2
3
4
β →
KEY
As on Graph 4.6.
From Ref. 8
From Ref. 10
X From Graph 4.11

APPENDIX

APPENDIX I

HAMILTON'S PRINCIPLE DISCUSSED IN TERMS OF THE NATURAL VIBRATION OF A SIMPLE SPRING/MASS SYSTEM

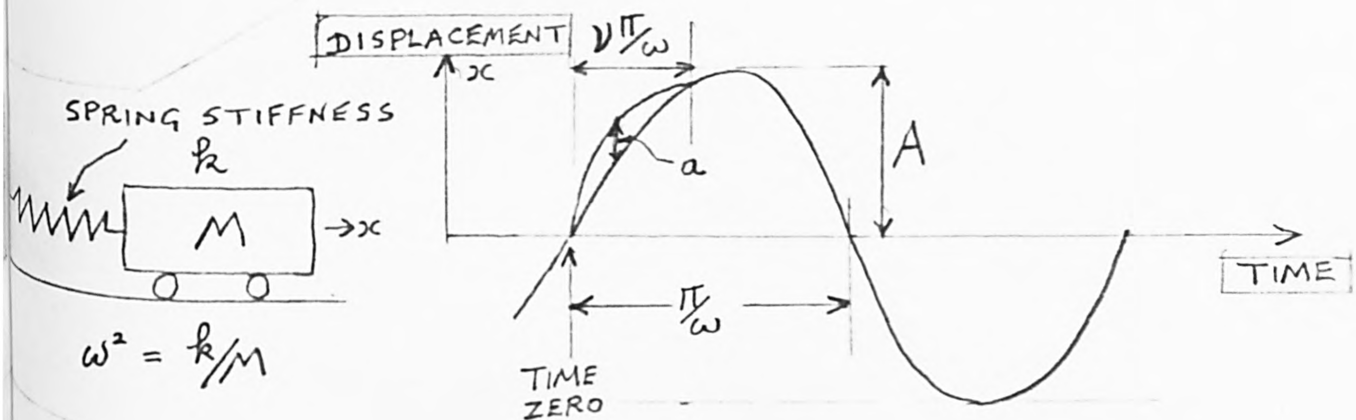


FIGURE A.1

A1.1 The displacement/time function for such a system including a small variation is:

$$x = A\sin\omega t + a\sin\frac{\omega t}{\nu}$$

$$\therefore \quad \dot{x} = \omega A\cos\omega t + \frac{\omega a}{\nu}\cos\frac{\omega t}{\nu} \quad \text{(velocity)}$$

$$\text{Strain energy } V = \tfrac{1}{2}kx^2 = \tfrac{1}{2}k\left(A^2\sin^2\omega t + 2Aa\sin\omega t\sin\frac{\omega t}{\nu} + a^2\sin^2\frac{\omega t}{\nu}\right)$$

$$\text{Kinetic energy } T = \tfrac{1}{2}M\dot{x}^2 = \tfrac{1}{2}M\omega^2\left(A^2\cos^2\omega t + \frac{2Aa}{\nu}\cos\omega t\cos\frac{\omega t}{\nu} + \frac{a^2}{\nu^2}\cos^2\frac{\omega t}{\nu}\right)$$

$$\int_0^{\frac{\nu\pi}{\omega}} V\,dt = \tfrac{1}{2}kA^2\left(\frac{\nu\pi - \frac{1}{2}\sin 2\nu\pi}{2\omega} + \frac{2a}{A}\cdot\frac{\nu\sin\nu\pi}{\omega(1-\nu^2)} + \frac{a^2\nu\pi}{2\omega A^2}\right)$$

$$\int_0^{\frac{\nu\pi}{\omega}} T\,dt = \tfrac{1}{2}M\omega^2A^2\left(\frac{\nu\pi + \frac{1}{2}\sin 2\nu\pi}{2\omega} + \frac{2a}{\nu A}\cdot\frac{\nu^2\sin\nu\pi}{\omega(1-\nu^2)} + \frac{a^2}{2\omega A^2}\cdot\frac{\pi}{\nu}\right)$$

Now $\frac{1}{2}kA^2 = \frac{1}{2}M\omega^2A^2 = E_o$ = energy of the vibration

$$\therefore \int_0^{\frac{\nu\Pi}{\omega}} (T-V)dt = \frac{E_o \sin 2\nu\Pi}{2\omega} + 0 + E_o \frac{\Pi a^2}{2\omega A^2}\left(\frac{1}{\nu} - \nu\right) \qquad \text{A1.1}$$

Considering the natural behaviour of the spring/mass system, the variation terms in equation A1.1 therefore being ignored, it can be seen that the value of the Hamiltonian integral, for any fraction ν of a half cycle is:

$$H = \int_0^{\frac{\nu\Pi}{\omega}} (T-V)dt = \frac{E_o \sin 2\nu\Pi}{2\omega} \qquad \text{A1.2}$$

where E_o is the energy of the vibration.

The maximum value of H is thus $\frac{E_o}{2\omega} = \frac{E_o tp}{4\Pi}$ where tp is the periodic time. It requires one eigth of a cycle to develop this value, the successive values of H during a half cycle being 0, $\frac{E_o}{2\omega}$, 0, $-\frac{E_o}{2\omega}$, 0.

However, the employment of Hamilton's Principle for problem solving depends only on the fact that all such values of H are <u>stationary</u> values. The truth of this is exhibited by the fact that the terms introduced into equation A1.1, on account of the variation $\delta x = a\sin\frac{\omega t}{\nu}$ have cancelled out, except for the terms involving "a^2". This means that if "a" is small then H is unaffected by the resulting departure from the natural "timetable" of the motion. Hence H may be said to be stationary during the natural motion, no matter what part of the cycle is examined.

Inspection of the a^2 terms reveals the effect of larger variations, and it can be seen that the effect of these terms depends on the value of $(\frac{1}{\nu} - \nu)$, which, in turn, depends on the duration of the variation $(\nu\Pi/\omega)$.

If $\nu = 1$ the variation becomes merely an increase in amplitude of the basic motion, so it is then not a genuine variation. Provided $\nu<1$, however, the a^2 terms will constitute an increase in the Hamiltonian integral which is thereby shown to be a minimum in the natural motion. Concerning the possibility of $\nu>1$ no definite conclusions can be drawn except, perhaps, that variations, to be meaningful, must be small in time as well as space.

Finally, it must be re-emphasised that the way in which Hamilton's Principle has been employed in this thesis depends only on the cancellation of the "a" terms. In other words it has only been employed in the form of equation (2.1).

HAMILTON'S PRINCIPLE DISCUSSED WITH REFERENCE TO NATURAL VIBRATIONS OF AN ELEMENTARY BRIDGE STRUCTURE

A1.2 It is now proposed to re-examine the operation of Hamilton's Principle, this time in connection with a system having more than one degree of freedom.

Figure A.2 shows an idealised cantilever bridge structure having its self mass "lumped" into three discrete positions, i.e. at midspan and both hinge positions.

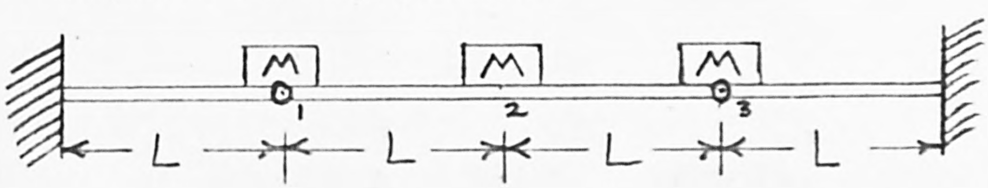


FIGURE A.2.

Stiffness coefficients for this system are:

$$k_{11} = k_{33} = \frac{9EI}{2L^3}, \quad k_{22} = \frac{6EI}{L^3}, \quad k_{13} = k_{31} = \frac{3EI}{2L^3},$$

$$k_{12} = k_{21} = k_{23} = k_{32} = \frac{-3EI}{L^3}$$

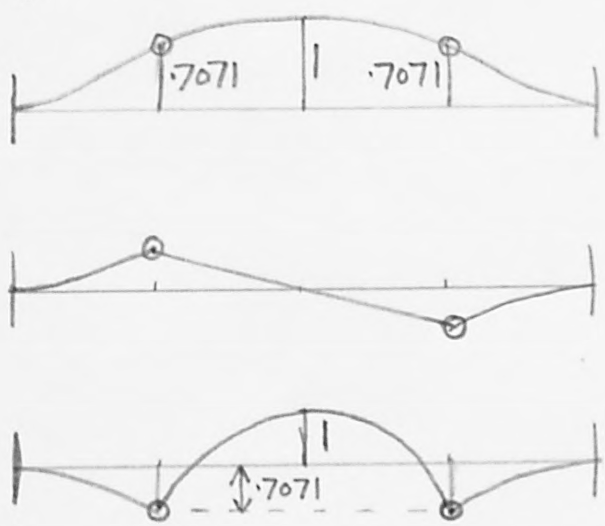
Hence the equations of motion in natural vibration are:

$$\begin{vmatrix} (\mu - \frac{3}{4}), & \frac{1}{2}, & -\frac{1}{4} \\ \frac{1}{2}, & (\mu - 1), & \frac{1}{2} \\ -\frac{1}{4}, & \frac{1}{2}, & (\mu - \frac{3}{4}) \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \qquad \text{where } \mu = \frac{M\omega^2 L^3}{6EI}$$

The eigenvalues and eigenvectors derivable from these equations of motion are as follows :

Natural frequencies
$$\begin{cases} \mu_a = .2929 \\ \mu_b = .5000 \\ \mu_c = 1.7071 \end{cases}$$

Mode Shapes



MODE a

MODE b

MODE c

It is intended to discuss only modes a and c. In order to evaluate relevant energy quantities the following shape functions will be employed:

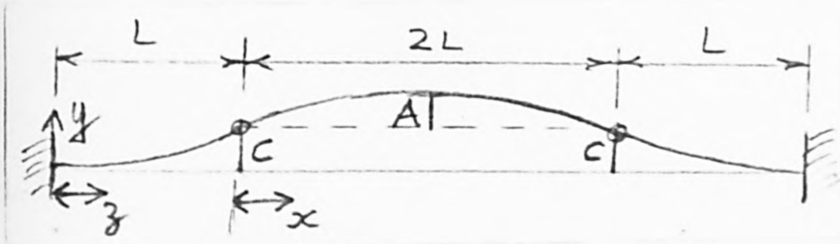


FIGURE A.3

$$y = C\left(1 - \cos\frac{\Pi z}{2L}\right) \text{ for the end cantilevers}$$

$$y = C + A\sin\frac{\Pi x}{2L} \quad \text{for the central span}$$

Energy of the vibration (E_o)

In terms of kinetic energy at zero elastic deflection:

$$E_o = 2 \times \tfrac{1}{2}M\omega^2 C^2 + \tfrac{1}{2}M\omega^2 (A + C)^2$$

i.e. $$E_o = \tfrac{1}{2}M\omega^2 (3C^2 + 2AC + A^2) \qquad \text{A1.3}$$

In terms of strain energy at maximum deflection:

$$E_o = \tfrac{1}{2}EI\left\{2\int_o^L \left(\frac{d^2y}{dz^2}\right)^2 dz + \int_o^{2L}\left(\frac{d^2y}{dx^2}\right)^2 dx\right\}$$

$$= 2 \times \tfrac{1}{2}EIC^2 \frac{\Pi^4}{16L^4}\int_o^L \cos^2\frac{\Pi z}{2L}\,dz + \tfrac{1}{2}EIA^2 \frac{\Pi^4}{16L^4}\int_o^{2L}\sin^2\frac{\Pi x}{2L}\,dx$$

i.e. $$E_o = \frac{\Pi^4 EI}{32L^3}(C^2 + A^2) \qquad \text{A1.4}$$

Equations A1.3 and A1.4 are equally applicable to mode shapes a and c. These mode shapes can be alternatively expressed in terms of A and C (Figure A3) as follows:

Mode a : $\frac{C}{A} = 2.4142$

Mode c : $\frac{C}{A} = -0.4142$

EXPRESSIONS FOR INSTANTANEOUS VALUES OF ENERGY

Since the purpose of these expressions is to facilitate study of the Hamiltonian integral, small variations will be incorporated.

Strain energy V : cantilever parts

$$y = C(1 - \cos\frac{\Pi z}{2L})\sin\omega t + c(1 - \cos\frac{\Pi z}{2L})\sin\frac{\omega t}{\nu}$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{\Pi^2}{4L^2}\cos\frac{\Pi z}{2L}\left(C\sin\omega t + c\sin\frac{\omega t}{\nu}\right)$$

$$\therefore V = 2 \times \tfrac{1}{2}EI\int_o^L\left(\frac{\partial^2 y}{\partial z^2}\right)^2 dz = \frac{\Pi^4 EI}{16L^4}\left(C\sin\omega t + c\sin\frac{\omega t}{\nu}\right)^2\int_o^L\cos^2\frac{\Pi z}{L}\,dz$$

$$\text{i.e. } V = \frac{EI\Pi^4}{32L^3}\left(C^2\sin^2\omega t + 2Cc\sin\omega t\sin\frac{\omega t}{\nu} + c^2\sin^2\frac{\omega t}{\nu}\right)$$

$$\therefore \int_0^{\frac{\nu\Pi}{\omega}} V dt = \frac{\Pi^4 EI}{32L^3}\left(\frac{C^2}{2\omega}(\nu\Pi - \tfrac{1}{2}\sin 2\nu\Pi) + \frac{2Cc\nu\sin\nu\Pi}{\omega(1-\nu^2)} + \frac{c^2\nu\Pi}{2\omega}\right) \qquad \text{A1.5}$$

for both cantilevers.

Strain energy V, main span

$$y = (C + A \sin \frac{\Pi x}{2L}) \sin \omega t + (c + a \sin \frac{\Pi x}{2L}) \sin \frac{\omega t}{\nu}$$

$$\frac{\partial^2 y}{\partial x^2} = - \frac{\Pi^2}{4L^2} \sin \frac{\Pi x}{2L} (A \sin \omega t + a \sin \frac{\omega t}{\nu})$$

$$\therefore V = \frac{\frac{1}{2}\Pi^4 EI}{16L^4} (A \sin \omega t + a \sin \frac{\omega t}{\nu})^2 \int_0^{2L} \sin^2 \frac{\Pi x}{2L} \, dx$$

$$\text{i.e. } V = \frac{\Pi^4 EI}{32L^3} \left(A^2 \sin^2 \omega t + 2Aa \sin \omega t \sin \frac{\omega t}{\nu} + a^2 \sin^2 \frac{\omega t}{\nu} \right)$$

$$\therefore \int_0^{\frac{\nu\Pi}{\omega}} V dt = \frac{\Pi^4 EI}{32L^3} \left(\frac{A^2}{2\omega} (\nu\Pi - \tfrac{1}{2}\sin 2\nu\Pi) + \frac{2Aa\nu \sin\nu\Pi}{\omega(1-\nu^2)} + \frac{a^2 \nu\Pi}{2\omega} \right) \qquad \text{A1.6}$$

Kinetic energy T : cantilevers

$$\dot{y} = C(1 - \cos \frac{\Pi z}{2L}) \, \omega \cos \omega t + c(1 - \cos \frac{\Pi z}{2L}) \frac{\omega}{\nu} \cos \frac{\omega t}{\nu}$$

$$T = 2 \times \tfrac{1}{2} M \dot{y}^2 (z=L) = 2 \times \tfrac{1}{2} M (\omega C \cos\omega t + \frac{\omega c}{\nu} \cos \frac{\omega t}{\nu})^2$$

$$\text{i.e. } T = M\omega^2 C^2 \cos^2 \omega t + 2MCc \frac{\omega^2}{\nu} \cos\omega t \cos \frac{\omega t}{\nu} + Mc^2 \frac{\omega^2}{\nu^2} \cos^2 \frac{\omega t}{\nu}$$

$$\therefore \int_0^{\frac{\nu\Pi}{\omega}} T dt = M\omega^2 \left(\frac{C^2}{2\omega} (\nu\Pi + \tfrac{1}{2}\sin 2\nu\Pi) + \frac{2Cc}{\nu\omega} \frac{\nu^2 \sin\nu\Pi}{(1-\nu^2)} + \frac{c^2 \Pi}{2\nu\omega} \right) \qquad \text{A1.7}$$

for both cantilevers

Kinetic energy T: main span

$$\dot{y} = \omega\cos\,\omega t(C + A\sin\frac{\Pi x}{2L}) + \frac{\omega}{\nu}\cos\frac{\omega t}{\nu}(c + a\sin\frac{\Pi x}{2L})$$

$$\dot{y}(x = L) = \omega\cos\,\omega t(C+A) + \frac{\omega}{\nu}\cos\frac{\omega t}{\nu}(c+a)$$

$$\therefore\quad T = \tfrac{1}{2}M\omega^2(C+A)^2\cos^2\omega t + M(C+A)(c+a)\frac{\omega^2}{\nu}\cos\omega t\cos\frac{\omega t}{\nu} + \tfrac{1}{2}M\frac{\omega^2}{\nu^2}(c+a)^2\cos^2\frac{\omega t}{\nu}$$

$$\therefore\quad \int_0^{\frac{\nu\Pi}{\omega}} T dt = \tfrac{1}{2}M\omega^2\left(\frac{(C+A)^2}{2\omega}(\nu\Pi+\tfrac{1}{2}\sin 2\nu\Pi) + \frac{2(C+A)(c+a)\nu^2\sin\nu\Pi}{\nu\omega(1-\nu^2)} + \frac{(c+a)^2\Pi}{2\nu\omega}\right)$$

A1.8

HAMILTONIAN INTEGRAL WITH VARIATION 'c' ONLY

From equations A1.5, A1.6, A1.7 and A1.8, but omitting terms with 'a':

$$\int_0^{\frac{\nu\Pi}{\omega}}(T-V)dt =$$

$$\tfrac{1}{2}M\omega^2\left((3C^2+2AC+A^2)\left(\frac{\nu\Pi+\tfrac{1}{2}\sin 2\nu\Pi}{2\omega}\right) + (6Cc+2Ac)\frac{\nu^2\sin\nu\Pi}{\nu\Pi(1-\nu^2)} + \frac{3c^2\Pi}{2\nu\omega}\right)$$

$$-\frac{EI\Pi^4}{32L^3}\left((C^2+A^2)(\frac{\nu\Pi - \tfrac{1}{2}\sin 2\nu\Pi}{2\omega}) + \frac{2Cc\nu\sin\nu\Pi}{\omega(1-\nu^2)} + \frac{c^2\nu\Pi}{2\omega}\right)$$

Now introduce E_o, the total energy of the vibration, by substituting from equations A1.3 and A1.4:

$$\int_0^{\frac{\nu\Pi}{\omega}} (T-V)dt = \frac{E_o \sin 2\nu\Pi}{2\omega}$$

$$+ \tfrac{1}{2}M\omega^2 \left(\frac{c\nu\sin\nu\Pi}{\omega(1-\nu^2)} \right) \overset{\#}{\left(6C + 2A - \frac{2C(3C^2+2AC+A^2)}{C^2+A^2} \right)}$$

$$+ \tfrac{1}{2}M\omega^2 \left(\frac{3}{\nu} - \frac{\nu(3C^2+2AC+A^2)}{C^2 + A^2} \right) \frac{c^2\Pi}{2\omega} \qquad \text{A1.9}$$

COMMENTS ON EQUATION A1.9

Reference to equation A1.2 shows that the basic Hamiltonian (first term on right side of A1.9) is the same as for a "single degree of freedom" system.

The bracket #, in equation A1.9, becomes zero when C/A = 2.4142 or -0.4142, which values have already been calculated as being characteristic of the two natural modes of vibration. Hence, in the natural motion the Hamiltonian integral is unaffected by the small variations of amplitude 'c'.

With the natural values of C/A the last term in equation A1.9 becomes, for modes 'a' and 'c' respectively:

$$\frac{\Pi}{4} M\omega c^2 \left(\frac{3}{\nu} - 3.414\nu \right) \quad \text{or} \quad \frac{\Pi}{4} M\omega c^2 \left(\frac{3}{\nu} - 0.586\nu \right)$$

HAMILTONIAN INTEGRAL WITH VARIATION 'a' ONLY

From equations A1.5, A1.6, A1.7 and A1.8, but omitting terms with 'c':

$$\int_{0}^{\frac{\nu\Pi}{\omega}} (T-V)\,dt =$$

$$\tfrac{1}{2}M\omega^2 \left((3C^2+2AC+A^2)\left(\frac{\nu\Pi+\frac{1}{2}\sin 2\nu\Pi}{2\omega}\right) + \frac{2(C+A)a\nu^2\sin\nu\Pi}{\nu\omega(1-\nu^2)} + \frac{a^2\Pi}{2\nu\omega} \right)$$

$$- \frac{EI\Pi^4}{32L^3} \left((C^2+A^2)\left(\frac{\nu\Pi-\frac{1}{2}\sin 2\nu\Pi}{2\omega}\right) + \frac{2Aa\nu\sin\nu\Pi}{\omega(1-\nu^2)} + \frac{a^2\nu\Pi}{2\omega} \right)$$

Now introduce E_o, the total energy of the vibration, by substituting from equations A1.3 and A1.4:

$$\int_{0}^{\frac{\nu\Pi}{\omega}} (T-V)\,dt = \frac{E_o \sin 2\nu\Pi}{2\omega}$$

$$+\tfrac{1}{2}M\omega^2 \left(\frac{a\nu\sin\nu\Pi}{\omega(1-\nu^2)}\right)\overset{b}{\left(2C + 2A - \frac{2A(3C^2 + 2AC + A^2)}{C^2 + A^2}\right)}$$

$$+\tfrac{1}{2}M\omega^2 \left(\frac{1}{\nu} - \frac{\nu(3C^2 + 2AC + A^2)}{C^2 + A^2}\right)\frac{a^2\Pi}{2\omega} \qquad \text{A1.10}$$

With the natural values of C/A ~~the~~ *the bracket b becomes zero whilst* the last term in equation A1.10 becomes, for modes 'a' and 'c' respectively:

$$\frac{\Pi}{4} M\omega a^2 \left(\frac{1}{\nu} - 3.414\nu\right) \quad \text{or} \quad \frac{\Pi}{4} M\omega a^2 \left(\frac{1}{\nu} - 0.586\nu\right)$$

NATURE OF THE STATIONARY HAMILTONIAN INTEGRAL - THE CONCEPT OF LEAST ACTION

The final expressions quoted on pages A8 and A9 represent the effect of the a^2 and c^2 terms arising from the proposed variations. These expressions will indicate that the Hamiltonian integral is a minimum if they represent an increase therein. It is clear that this will not be so unless the duration of the variation is arbitrarily restricted. The most severe restriction is implied by the first of the two expressions at the bottom of page A9 which represents an increase only if $\gamma < 0.54$, i.e. the action will only be a minimum for variations restricted in duration to approximately one quarter of a cycle.

However, it must be noted that the actual restriction needed to reveal a state of least action depends also on the nature of the variation itself and on the mode of vibration which is under consideration. For instance it would be possible to employ a variation wherein both 'a' and 'c' are simultaneously involved and, since the ratio a/c is also arbitrary, there is an infinite number of possibilities.

This matter has been studied in reference 3 wherein the concept of spatial and temporal variations has been introduced. A spatial variation is one that causes an alteration to the mode shape as well as the "timetable", whilst a temporal variation causes only a variation in "timetable" and preserves the mode shape.

The 'a' and 'c' variations discussed above thus have a spatial element unless they are employed simultaneously with the ratio a/c the same as the ratio a/c in the natural vibratory modes. In the latter event the variation becomes purely temporal and it is easily shown that its duration may then extend to half a cycle of the natural vibration without violating the least action concept.

This is the same conclusion that was reached with respect to a simple "one degree of freedom" vibration in para. A1.1 and it can be more generally illustrated by considering natural vibrations of a uniform simply supported beam with distributed mass m per unit length. The vibration of such a beam, in its nth mode, is given by the first term in the expression below, the second term being a purely temporal variation on the basic motion.

$$y = A \sin \frac{n\Pi x}{L} \sin n^2\omega t + a \sin \frac{n\Pi x}{L} \sin \frac{n^2\omega t}{\nu}$$

where $\omega^2 = \frac{\Pi^4}{L^4} \frac{EI}{m}$

The change in $\dot{y}$ due to the variation will be $\delta\dot{y} = \frac{n^2\omega}{\nu} a \sin \frac{n\Pi x}{L} \cos \frac{n^2\omega t}{\nu}$ and the consequential change in kinetic energy (due to the a^2 terms alone) will therefore be:

$$\delta T_{a^2} = \tfrac{1}{2}m \int_0^L (\delta y)^2 dx = \frac{a^2}{4} \frac{mLn^4\omega^2}{\nu^2} \cos^2\left(\frac{n^2\omega t}{\nu}\right)$$

The change in $\frac{\partial^2 y}{\partial x^2}$ due to the variation will be

$$\delta\left(\frac{\partial^2 y}{\partial x^2}\right) = -\frac{n^2\Pi^2}{L^2} a \sin \frac{n\Pi x}{L} \sin \frac{n^2\omega t}{\nu}$$

and the consequential change in strain energy (due to the a^2 terms alone) will therefore be:

$$\delta V_{a^2} = \tfrac{1}{2}EI \int_0^L \left(\delta\left(\frac{\partial^2 y}{\partial x^2}\right)\right)^2 dx = \frac{a^2}{4} \frac{n^4\Pi^4}{L^3} EI \sin^2\left(\frac{n^2\omega t}{\nu}\right)$$

So the net alteration in the action integral due to the a^2 terms arising from the variation will be:

$$\int_0^{\frac{\Pi\nu}{n^2\omega}} (\delta T_{a^2} - \delta V_{a^2})dt = \frac{\Pi\nu a^2}{8n^2\omega} \left[\frac{mLn^4\omega^2}{\nu^2} - \frac{\Pi^4 n^4 EI}{L^3} \right]$$

This represents an increase in action if:

$$\frac{1}{\nu^2} > \frac{\Pi^4 EI}{m\omega^2 L^4}$$

i.e. if $\nu < 1$

On the other hand it can be shown that a variation with a <u>spatial</u> component may not reveal a state of least action when $\nu < 1$.

A general way of introducing a spatial component is to replace the purely temporal variation just employed with $a.\sin\frac{(n+N)\Pi x}{L}\sin\frac{n^2\omega t}{\nu}$. It is easily shown that if N is a positive integer a state of least action will not be revealed without a more severe restriction on the duration of the variation ($\nu < (\frac{n}{n+N})^2$).

The above considerations are relevant with regard to the significance of Hamilton's Principle, but are not of any consequence in connection with the method of calculation employed in this thesis. These calculations are based on equation 2.1 which is true whether the action is minimum or not.

A1.3 RELATION BETWEEN HAMILTON'S PRINCIPLE, RAYLEIGH-RITZ METHOD METHOD, AND THE SIMPLE RAYLEIGH METHOD

If the δV and δT terms (that is to say the terms involving 'a' and 'c', but not a^2 or c^2) are extracted from equations A1.5, A1.6, A1.7 and A1.8, then the following two equations may be written, with equation 2.1 also in mind:

$$\int\delta V dt = \int\delta T dt$$

$$\frac{\Pi^4 EI}{32L^3}(2A) = \tfrac{1}{2}M\omega^2(2C + 2A) \quad \text{for the 'a' variation}$$

$$\frac{\Pi^4 EI}{32L^3}(2C) = \tfrac{1}{2}M\omega^2(6C + 2A) \quad \text{for the 'c' variation}$$

These are precisely the equations which would be obtained from a standard Rayleigh-Ritz procedure, in which the next step would be to arrange them as two homogeneous equations for A and C, as follows:

$$\begin{vmatrix} (2\mu-2) & ; & 2\mu \\ 2\mu & ; & (6\mu-2) \end{vmatrix} \begin{vmatrix} A \\ C \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad \text{A1.11}$$

where $\mu = \dfrac{M\omega^2 L^3}{6EI}$ (Note: $\dfrac{16}{\Pi^4} \doteqdot \dfrac{1}{6}$)

The condition for the determinant to be null is:

$$8\mu^2 - 16\mu + 4 = 0$$

From which $\mu = 0.2929$ or 1.7071, these being the values already calculated, by the stiffness method, in para. A1.2.

Now consider the simple Rayleigh method for evaluating frequencies which merely consists of equating the two expressions for E_0, from equations A1.3 and A1.4, as follows:

$$\tfrac{1}{2}M\omega^2(3C^2 + 2AC + A^2) = \frac{\Pi^4 EI}{32L^3}(C^2 + A^2)$$

$$\omega^2 = \frac{6EI}{ML^3}\left(\frac{C^2 + A^2}{3C^3 + 2AC + A^2}\right)$$

$$\text{i.e. } \mu = \frac{C^2 + A^2}{3C^2 + 2AC + A^2} \qquad \text{A1.12}$$

$$\text{i.e. } \mu = \frac{(C/A)^2 + 1}{3(C/A)^2 + 2\,C/A + 1}$$

This will have a stationary value if:

$$2\left(\frac{C}{A}\right)\left(3\left(\frac{C}{A}\right)^2 + 2\left(\frac{C}{A}\right) + 1\right) = \left(\left(\frac{C}{A}\right)^2+1\right)\left(6\left(\frac{C}{A}\right) + 2\right)$$

i.e. if $2\left(\frac{C}{A}\right)^2 - 4\left(\frac{C}{A}\right) - 2 = 0$

from which C/A = 2.414 or -0.414.

Substituting these values back into equation A1.12 then gives μ = .2229 or 1.7071.

Hence, in this particular problem, the Rayleigh quotient gives both the lowest and the highest natural frequencies provided the values accepted are the minimum and maximum values of the quotient.

To investigate this conclusion, concerning the Rayleigh quotient, in relation to the Rayleigh-Ritz method it is instructive to rearrange equations A1.11 as a pair of simultaneous equations in the two unknowns C/A and μ as follows:

$$\left.\begin{aligned} 2\mu A - 2A + 2\mu C &= 0 \\ 2\mu A + 6\mu C - 2C &= 0 \end{aligned}\right\} \qquad \text{A1.11 bis}$$

i.e. $2\mu - 2 + 2\mu\, C/A = 0$ (1)

$2\mu + 6\mu\, C/A - 2\, C/A = 0$ (2)

from (1) : $\mu = \dfrac{1}{1 + C/A}$

Substituting this in (2) gives:

$$\frac{2 + 6\, C/A}{1 + C/A} - 2\, C/A = 0 \text{ , i.e. } 2\left(\frac{C}{A}\right)^2 - 4\left(\frac{C}{A}\right) - 2 = 0$$

which was previously found to be the condition for the Rayleigh quotient to have maximum and minimum values.

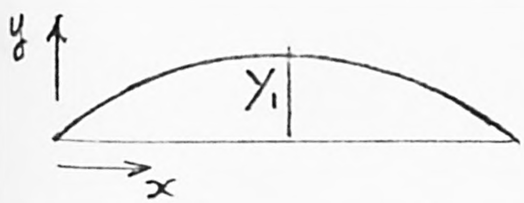
It has been shown, above, that the Rayleigh Ritz method, for calculating natural frequencies and mode shapes, is consistent with Hamilton's Principle.

It appears that recourse to Hamilton's Principle is in fact necessary to justify the acceptance of stationary values from Rayleigh quotients.

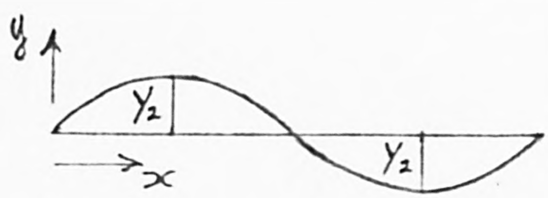
APPENDIX 2

CLASSICAL ANALYSIS OF SIMPLY SUPPORTED UNIFORM BEAMS SUBJECT TO MOVING LOADS OF CONSTANT MAGNITUDE

A2.1 Let the natural frequencies of the beam be $\frac{\omega_1}{2\pi}$, $\frac{\omega_2}{2\pi}, \ldots \frac{\omega_n}{2\pi}$ and its natural mode shapes by $Y_1 f_1(x)$, $Y_2 f_2(x), \ldots\ldots\ldots Y_n f_n(x)$ where Y_1, $Y_2 \ldots\ldots Y_n$, are the maximum deflections of each mode shape. For a simply supported beam the above statements may be shown diagrammatically as below:

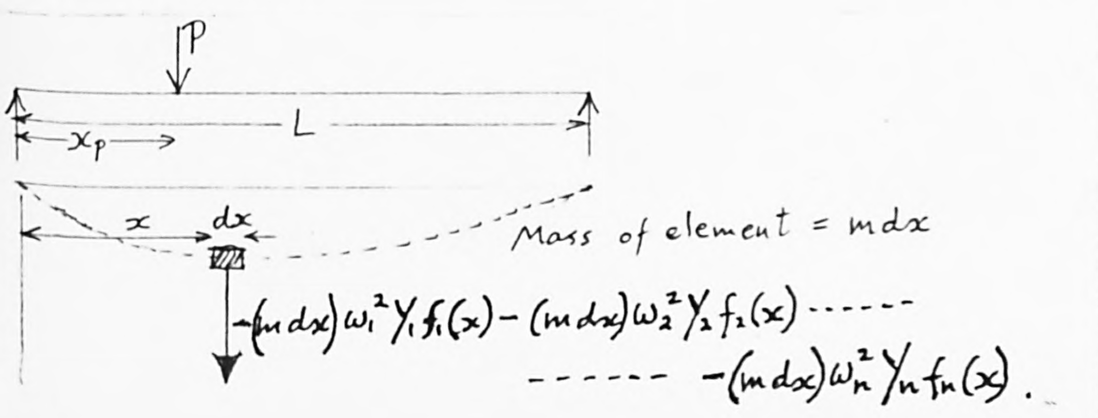


MODE 1



MODE 2

Now consider an elemental length of any uniform beam with mass 'm' per unit length. The instantaneous configuration of the beam, due to the action of an external dynamic load P, will be regarded as a superposition of all the natural mode shapes. The forces experienced by the element, on account of elasticity, can then be expressed in terms of free vibration parameters, as shown in the diagram below.



The equation of motion of the element is:

$$-(mdx)\omega_1^2 Y_1 f_1(x) - (mdx)\omega_2^2 Y_2 f_2(x) \ldots\ldots\ldots - (mdx)\omega_n^2 Y_n f_n(x)$$

$$= (mdx)\ddot{Y}_1 f_1(x) + (mdx)\ddot{Y}_2 f_2(x) \qquad \ldots\ldots\ldots + (mdx)\ddot{Y}_n f_n(x)$$

Multiplying this equation through by $f_1(x)$ and integrating to cater for the whole beam and also introducing the external load P:

$$Pf_1(x_p) - m\omega_1^2 Y_1 \int_o^L (f_1(x))^2 dx - m\omega_2^2 Y_2 \int_o^L f_1(x) f_2(x) dx \ldots\ldots\ldots$$

$$- m\omega_n^2 Y_n \int_o^L (f_1(x) f_n(x) dx$$

$$= m\ddot{Y}_1 \int_o^L (f_1(x))^2 dx + m\ddot{Y}_2 \int_o^L f_1(x) f_2(x) dx + \ldots\ldots\ldots\ldots$$

$$+ m\ddot{Y}_n \int_o^L f_1(x) f_n(x) dx$$

On account of the orthogonality relationship between natural mode shapes, all the above integrals are zero except $\int_o^L (f_1(x))^2 dx$, and the equation therefore reduces to:

$$\ddot{Y}_1 + \omega_1^2 Y_1 = \frac{Pf_1(x_p)}{m \int_o^L (f_1(x))^2 dx} \qquad \text{A2.1}$$

If the elemental equation of motion is multiplied through by $f_n(x)$, instead of $f_1(x)$ the result would be:

$$\ddot{Y}_n + \omega_n^2 Y_n = \frac{Pf_n(x_p)}{m \int_o^L (f_n(x))^2 dx} \qquad \text{A2.2}$$

Summation of the solutions of n equations A2.2 will give the dynamic deflection of the beam due to the load P:

$$y = Y_1 f_1(x) + Y_2 f_2(x) + \dots\dots\dots + Y_n f_n(x) \qquad \text{A2.3}$$

Now if the applied load P is moving across the beam at a constant speed u, a close approximation to the dynamic deflection can be obtained from the first term of equation A2.3, which, in turn, can be derived from a solution of equation A2.1 alone.

For a simply supported beam $f_1(x) = \sin \frac{\Pi x}{L}$ and $\int_o^L (f_1(x))^2 dx = \frac{1}{2}L$, so equation A2.1 becomes:

$$\ddot{Y}_1 + \omega_1^2 Y_1 = \frac{2P}{mL} \sin \frac{\Pi x_p}{L}$$

The position of the load P will be defined by $x_p = ut$, so the final form of the equation to obtain an approximate solution to the envisaged problem is:

$$\ddot{Y} + \omega^2 Y = \frac{2P}{mL} \sin \frac{\Pi ut}{L} \qquad \text{A2.4}$$

where $\omega^2 = \frac{\Pi^4}{L^4} \frac{EI}{m}$

Solution of equation A2.4:

$$Y = A\sin\omega t + B\cos\omega t + \frac{\frac{2P}{mL}\sin\frac{\Pi ut}{L}}{\omega^2 - \frac{\Pi^2 u^2}{L^2}}$$

when $t = 0$, $Y = 0$, $B = 0$

when $t = 0$, $\dot{Y} = 0$:

$$0 = \omega A + \frac{2P\Pi u}{mL^2} / \omega^2 - \frac{\Pi^2 u^2}{L^2}$$

$$A = -\frac{2P\Pi u}{m\omega L^2} / \omega^2 - \frac{\Pi^2 u^2}{L^2}$$

$$Y = \frac{\frac{2P}{mL} \left(\sin \frac{\Pi ut}{L} - \frac{\Pi u}{\omega L} \sin \omega t \right)}{\omega^2 - \Pi^2 u^2 / L^2}$$

$$\text{Now} \frac{2P}{mL\omega^2} = \frac{2P}{mL\left(\frac{\Pi^4}{L^4} \frac{EI}{m} \right)} = \frac{2PL^3}{\Pi^4 EI} \simeq \frac{PL^3}{48EI} = \Delta$$

where Δ is the greatest possible static deflection due to P.

$$Y = \frac{\Delta \left(\sin \frac{\Pi ut}{L} - \frac{\Pi u}{\omega L} \sin \omega t\right)}{1 - \Pi^2 u^2 / \omega^2 L^2} \qquad \text{(midspan deflection)}$$

$$\therefore y = \frac{\Delta \left(\sin \frac{\Pi ut}{L} - \frac{\Pi u}{\omega L}\right) \sin \frac{\Pi x}{L}}{1 - \Pi^2 u^2 / \omega^2 L^2}$$

Finally, introducing $\beta = \frac{\omega L}{\Pi u}$ from equation 1.1 the above expression for y becomes:

$$y = \frac{\Delta}{1 - 1/\beta^2} \left(\sin \frac{\Pi ut}{L} - \frac{1}{\beta} \sin \frac{\beta \Pi ut}{L} \right) \sin \frac{\Pi x}{L} \qquad \text{A2.5}$$

Vertical accelerations can now be found by differentiating equation A2.5 twice, with respect to time, t, but this will give merely the acceleration of the beam itself, at the point where the load acts upon it. In order to obtain the vertical acceleration experienced by the load it is necessary to replace x by u.t in equation A2.5 and then differentiate twice as follows:

$$y_M = \frac{\Delta}{1 - 1/\beta^2}\left(\sin^2 \frac{\Pi ut}{L} - \frac{1}{\beta}\sin\frac{\beta\Pi ut}{L}\sin\frac{\Pi ut}{L}\right) \qquad \text{A2.5a}$$

$$\dot{y}_M = \frac{\Delta}{1 - 1/\beta^2}\left(2\sin\frac{\Pi ut}{L}\cos\frac{\Pi ut}{L} - \frac{1}{\beta}\sin\frac{\beta\Pi ut}{L}\cos\frac{\Pi ut}{L} - \cos\frac{\beta\Pi ut}{L}\sin\frac{\Pi ut}{L}\right)\frac{\Pi u}{L}$$

A2.5b

$$\ddot{y}_M = \frac{\Pi^2 u^2 \Delta}{L^2(1-1/\beta^2)}\left(2\cos\frac{2\Pi ut}{L} - 2\cos\frac{\Pi ut}{L}\cos\frac{\beta\Pi ut}{L} + \left(\frac{1}{\beta} + \beta\right)\sin\frac{\Pi ut}{L}\sin\frac{\beta\Pi ut}{L}\right)$$

Now $\frac{\Pi^2 u^2}{L^2} = \frac{\omega^2}{\beta^2}$ from eqn. 1.1. Also $\omega^2 = \frac{\Pi^4}{L^4}\frac{EI}{m}$ and $\Delta = PL^3/48EI$

$$\therefore \frac{\Pi^2 u^2}{L^2}\Delta = \frac{1}{\beta^2}\cdot\frac{\Pi^4}{L^4}\cdot\frac{EI}{m}\cdot\frac{PL^3}{48EI}\cdot\frac{g}{g}$$

$$= 2.029\ \gamma g/\beta^2 \text{ where } \gamma = \frac{P}{mgL} = \frac{P}{mgL} = \frac{Mg}{mgL} = \frac{M}{mL}$$

where γ is the ratio of load mass to structure mass.

With the above substitutions the vertical acceleration experienced by the load can finally be written:

$$\ddot{y}_M = \frac{2.029\gamma g}{(\beta^2-1)}\left(2\cos\frac{2\Pi ut}{L} - 2\cos\frac{\Pi ut}{L}\cos\frac{\beta\Pi ut}{L} + \left(\frac{1}{\beta} + \beta\right)\sin\frac{\Pi ut}{L}\sin\frac{\beta\Pi ut}{L}\right)$$

A2.5c

Now it is well known that equation A2.5 gives a maximum result when $\beta = 1.62$. This gives a maximum deflection of 1.743Δ at mid-span which occurs when the load is 76% of the way across the span.

To obtain the vertical accelerations experienced by the load, in this critical case, it is only necessary to substitute $\beta = 1.62$

in equation A2.5c which then becomes:

$$y''_M = 1.246\ \gamma g(2\cos\frac{2\Pi ut}{L} - 2\cos\frac{\Pi ut}{L}\cos\frac{1.62\Pi ut}{L} + 2.238\sin\frac{\Pi ut}{L}\sin\frac{1.62\Pi ut}{L})$$

giving the following values.

TABLE A2.1

ut/L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y''_M	0	+ 0.4γg	+ 1.1γg	+ 1.4γg	+ 0.7γg	− 0.9γg	− 2.5γg	− 3.0γg	− 1.8γg	+ 0.8γg	+ 3.3γg

The further values, below, represent the acceleration of the beam itself at the point where the load acts. These values show the contrast between the accelerations experienced by the load, and the beam, at the same point and the same time.
(The values were calculated from double differentiation of equation A2.5).

TABLE A2.2

ut/L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y''	0	+ .18γ6	+ .57γg	+ .80γg	+ .58γg	− .11γg	− .95γg	− 1.47γg	− 1.37γg	− .73γg	0

If β = 1.60 is substituted into equation A2.5a the following values are obtained which represent the deflection of the beam at the moving load point. this information is also shown on graph 4.3, which thus depicts the actual path of the load in the critical case.

TABLE A2.3

ut/L	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.98
y_M	.004Δ	.06Δ	.25Δ	.61Δ	1.04Δ	1.36Δ	1.38Δ	1.03Δ	.46Δ	.20Δ	.07Δ

It may be noted that the vertical accelerations experienced by the moving load are proportional to the live/dead load ratio γ, and Table A2.1 shows that very high values can be obtained in extreme circumstances. Downward accelerations are positive and there is no force available to produce them, apart from the weight of the vehicle itself. So it can be deduced from Table 1 (if $\gamma = 1$, for instance) that the vehicle would momentarily rise off the bridge when about 25% of the way across.

In appraising this conclusion it is necessary to remember that $\beta = 1.62$ represents a <u>very</u> high vehicle speed, and that all acceleration components increase in proportion to the square of the vehicle speed.

The negative acceleration, e.g. when the load is 70% of the way across, implies that the force exerted on the beam would then be much greater than the weight of the vehicle.

A2.2 RESPONSE TO UNIFORMLY DISTRIBUTED LOAD

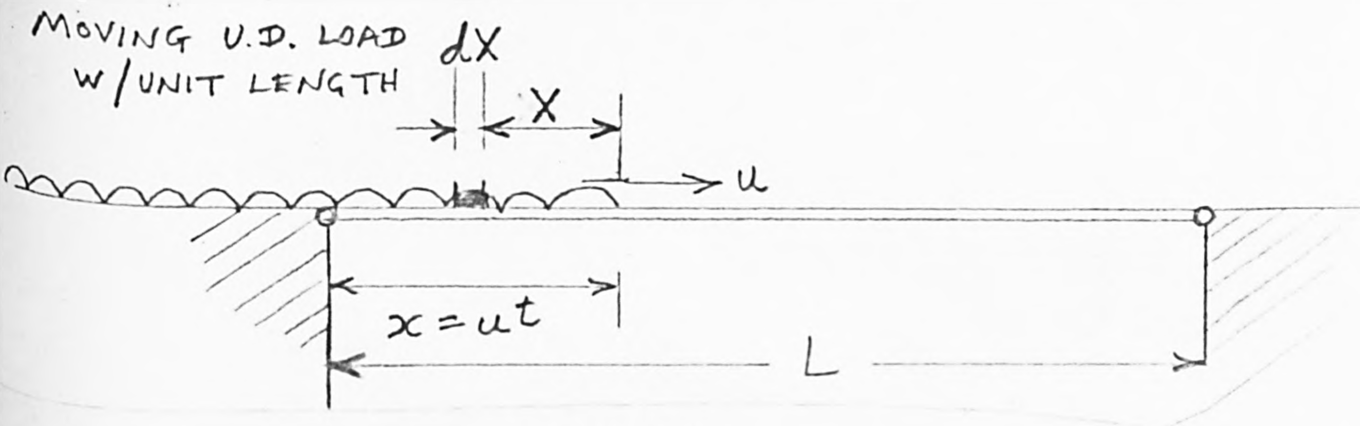


FIGURE A.4

Employing equation A2.3 but replacing P by w.dx:

$$\ddot{Y} + \omega^2 y = \frac{2w\,dx}{mL} \sin \frac{\Pi(x-X)}{L}$$

The solution of this equation would give the dynamic deflection at mid-span due to the element dx of the moving load acting at (x-X) from the beginning of the span.

Integrating the right hand side, to cater for all the load actually on the bridge:

$$\ddot{Y} + \omega^2 Y = \frac{2w}{mL} \int_{o}^{x} \sin \frac{\Pi(x-X)}{L}\, dx$$

$$\text{i.e.} \quad \ddot{Y} + \omega^2 Y = \frac{2w}{\Pi m} \left(1 - \cos \frac{\Pi x}{L} \right)$$

$$\text{i.e.} \quad \ddot{Y} + \omega^2 Y = \frac{2w}{\Pi m} \left(1 - \cos \frac{\Pi u t}{L} \right) \qquad \text{A2.6}$$

Solution of equation A2.6:

$$Y = A \sin \omega t + B \cos \omega t + \frac{2w}{\Pi m}\left(\frac{1}{\omega^2} - \frac{\cos \frac{ut}{L}}{\omega^2 - \Pi^2 u^2/L^2}\right)$$

Boundary conditions:

When t=0, $\dot{Y}=0$, A=0

When t=0, Y=0

i.e. $$0 = B + \frac{2w}{\Pi m}\left(\frac{1}{\omega^2} - \frac{1}{\omega^2 - \Pi^2 u^2/L^2}\right)$$

$$B = \frac{2w}{\Pi m}\left(\frac{\Pi^2 u^2/\omega^2 L^2}{\omega^2 - \Pi^2 u^2/L^2}\right)$$

So the final solution of A2.6 is:

$$Y = \frac{2w}{\Pi m}\left(\frac{1}{\omega^2} + \frac{\Pi^2 u^2/\omega^2 L^2 \cos\omega t}{\omega^2 - \Pi^2 u^2/L^2} - \frac{\cos \frac{\Pi ut}{L}}{\omega^2 - \Pi^2 u^2/L^2}\right)$$

Now introduce $\beta = \frac{\omega L}{\Pi u}$ and replace ut by x:

$$Y = \frac{2w}{\Pi m \omega^2}\left(1 + \frac{\frac{1}{\beta^2} \cos\beta \frac{x\Pi}{L} - \cos \frac{\Pi x}{L}}{1 - 1/\beta^2}\right)$$

Now introduce $\omega^2 = \frac{\Pi^4}{L^4} \frac{EI}{m}$ and note that then

$\frac{2w}{\Pi m \omega^2} = \frac{2wL^4}{\Pi^5 EI} \simeq 0.5\Delta$ where Δ is the maximum possible static deflection due to a uniformly distributed load w/unit length.

$$\therefore \quad Y = 0.5\Delta \left(1 + \frac{\frac{1}{\beta^2} \cos \frac{\beta \Pi x}{L} - \cos \frac{\Pi x}{L}}{1 - 1/\beta^2} \right)$$

$$\text{i.e.} \quad Y = \frac{0.5\,\Delta}{1 - 1/\beta^2} \left((1 - \cos \frac{\Pi x}{L}) - \frac{1}{\beta^2} (1 - \cos \frac{\beta \Pi x}{L}) \right) \qquad \text{A2.7}$$

(x defined by Figure A.4)

Equation A2.7 is valid for $0 \leq x \leq L$.

EFFECT OF A LOAD LONGER THAN THE SPAN

Now consider what happens after the nose of the train has reached the far end of the span, supposing that the train is long enough for the span to remain, for an appreciable time, loaded from end to end.

Under this fully loaded condition the equation of motion A2.6 becomes ($x = ut = L$, constant):

$$\ddot{Y} + \omega^2 Y = \frac{4w}{\Pi m}$$

$$Y = A \sin \omega t + B \cos \omega t + \Delta \qquad \text{A2.8}$$

since $\frac{4w}{\Pi m \omega^2} = \Delta$, as already noted.

The initial boundary conditions for solving equation A2.8 must be the values of Y and $\dot{Y}$ implied by equation A2.7 with $x = L$, which is the limit of validity of equation A2.7 and represents the instant when the nose of the train reaches the far end of the beam. So putting $t = 0$ in A2.8 and $x = ut = L$ in A2.7:

$$B + \Delta = \frac{0.5\Delta}{1-1/\beta^2} \left(2 - \frac{1}{\beta^2}(1 - \cos \beta\Pi)\right)$$

$$B = \frac{\Delta}{1-1/\beta^2} \left(\frac{1 + \cos \beta\Pi}{2\beta^2} \right) = \frac{0.5\Delta(1+\cos\beta\Pi)}{\beta^2-1}$$

Now differentiate A2.7, remembering that x = ut:

$$\dot{Y} = \frac{\frac{0.5\Pi u}{L}\Delta}{1-1/\beta^2} \left(\sin \frac{\Pi x}{L} - \frac{1}{\beta} \sin \frac{\beta\Pi x}{L} \right)$$

Now put x=L and equate to $\dot{Y}(t=0)$ from equation A2.8:

$$\frac{\frac{-0.5\Pi u}{L}\Delta}{1-1/\beta^2} \left(\frac{\sin \beta\Pi}{\beta} \right) = \omega A$$

$$A = \frac{-0.5 \ \Delta \ \sin \beta\Pi}{\beta^2 - 1}, \text{ since } \frac{\Pi u}{\omega L} = \frac{1}{\beta}$$

So the final solution of equation A2.8 is:

$$Y = \frac{0.5\Delta}{\beta^2-1} \left((1 + \cos\beta\Pi)\cos\omega t - \sin\beta\Pi\sin\omega t \right) + \Delta$$

Now put $\omega t = \frac{\beta\Pi ut}{L} = \frac{\beta\Pi x}{L}$; (x now signifies the distance the nose of the train has moved beyond the far end of the span)

$$Y = \frac{\Delta}{2(\beta^2-1)} \left(\cos \frac{\beta\Pi x}{L} + \cos \beta\Pi(1 + \frac{x}{L}) \right) + \Delta \qquad \text{A2.9}$$

This equation will not necessarily give greater values of Y than the maximum value obtainable from equation A2.7. However, the higher the spped u, of the moving UD load, the more likely it is that a greater value of Y will emerge from equation A2.9

instead of A2.7. For a very high speed (u) β will be very small, so that equation A2.9 will reduce to the following approximation:

$$Y \quad -\tfrac{1}{2}\Delta \left(2 \cos \frac{\beta \Pi x}{L}\right) + \Delta = \Delta\left(1 - \cos \frac{\beta \Pi x}{L}\right)$$

In arriving at this approximation it has been assumed that β is negligible whilst βx is not. This implies large values of x, i.e. very long trains, in which case Y can rise to a maximum value of 2Δ when $x = L/\beta$.

For a numerical example consider $\beta = 0.5$.

Then equation A2.9 becomes:

$$Y = \frac{-2}{3}\Delta \left(\cos \frac{\Pi}{2}\frac{x}{L} + \cos\left(\frac{\Pi}{2} + \frac{\Pi}{2}\frac{x}{L}\right)\right) + \Delta$$

This will be maximum when $\frac{\Pi}{2}\frac{x}{L} = \frac{3\Pi}{4}$

i.e. $\underline{Y_{MAX}} = \frac{2}{3}\Delta(0.7071 + 0.7071) + \Delta = \underline{1.943\ \Delta}$.

Note that this occurs when $\frac{x}{L} = 1\frac{1}{2}$ which means that the train must be $2\frac{1}{2}$ x the span of the beam for this maximum to be attained.

APPENDIX 3

STEADY STATE TRANSIT OF CONTINUOUS TRAVELLING UNIFORMLY DISTRIBUTED LOAD

When a travelling distributed load moves on to a beam it causes vibrations in the course of which a maximum dynamic deflection will occur, as has already been reported in Chapter 4 (see graph 4.8).

If, however, the load is "infinitely long" and therefore continues running across the beam after the occurence of the maximum deflection, then the vibrations will die out. A "steady state" will then exist in which the deflection, although greater than static, will remain constant.

The excess of this deflection, above the static value, will be due to the centripetal acceleration experienced by the load, this acceleration itself being due to the curvature associated with the deflection. This is a "feedback" situation which suggests a possibility of instability, and strongly diverging deflections have been observed in computer print outs for low β values.

It will now be shown, by independent analysis, that for any given value of live/dead load ratio (γ) there will be a critical transit speed (u) at which this type of instability must be expected to occur.

It will be assumed that the loading actually on the span consists of ten equally spaced point loads each having a mass γmL, where "m" is the mass per unit length of the beam itself. Then the equivalent UD loading is $10\gamma mg\ L/L = 10\ \gamma mg$.

(This gives consistency with the programme).

The radius of curvature of the beam due to elastic bending will be $R = \left(\frac{d^2y}{dx^2}\right)^{-1}$ and the differential equation for the steady state deflection can then be written as follows:

(centripetal acceleration = $\frac{u^2}{R} = u^2 \frac{dy^2}{dx^2}$)

(y positive upwards)

(y is positive downwards in all other sections of this report).

$$-EI \frac{d^4y}{dx^4} = 10\ \gamma mg + 10\ \gamma mu^2 \frac{d^2y}{dx^2} \qquad \text{A3.1}$$

$$\text{i.e. } \frac{d^4y}{dx^4} + \frac{10\ \gamma mu^2}{EI} \frac{d^2y}{dx^2} = -\frac{10\ \gamma mg}{EI}$$

Let $\alpha = \frac{10\ \gamma mu^2}{EI}$, then after integrating twice:

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{-5\ \gamma mgx^2}{EI} + A'x + B'$$

$$y = A\sin\alpha x + B\cos\alpha x + \frac{10\ \gamma mg}{EI\alpha^4} - \frac{5\ \gamma mgx^2}{EI\alpha^2} + \frac{A'x}{\alpha^2} + \frac{B'}{\alpha^2}$$

Boundary conditions

$$\left.\begin{matrix} x = 0 \\ \frac{d^2y}{dx^2} = 0 \end{matrix}\right\} \quad -\alpha^2 B - \frac{10\ \gamma mg}{EI\alpha^2} = 0 \qquad \therefore B = \frac{-10\ \gamma mg}{EI\alpha^4}$$

$$\left.\begin{matrix} x = L \\ \frac{d^2y}{dx^2} = 0 \end{matrix}\right\} \quad -\alpha^2 A\sin\alpha L + \frac{10\ \gamma mg}{\alpha^2 EI}\cos\alpha L - \frac{10\gamma mg}{\alpha^2 EI} = 0$$

$$\therefore A = \frac{-10\gamma mg}{EI\alpha^4}\left(\operatorname{cosec}\alpha L - \cot\alpha L\right)$$

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\} \quad B + \frac{10\,\gamma mg}{EI\alpha^4} + \frac{B'}{\alpha^2}, \therefore B' = 0$$

$$\left.\begin{array}{l} x = L \\ y = 0 \end{array}\right\} \quad A\sin\alpha L - \frac{10\gamma mg}{EI\alpha^4}\cos\alpha L + \frac{10\gamma mg}{EI\alpha^4} - \frac{5\gamma mgL^2}{EI\alpha^2} + \frac{A'L}{\alpha^2} = 0$$

$$\therefore A' = \frac{10\gamma mg\sin\alpha L}{\alpha^2 EIL}\left(\mathrm{cosec}\,\alpha L - \cot\alpha L\right) + \frac{10\gamma mg}{\alpha^2 EIL}\cos\alpha L$$

$$- \frac{10\gamma mg}{\alpha^2 EIL} + \frac{5\gamma mgL}{EI}$$

$$\therefore A' = \frac{5\gamma mgL}{EI}$$

So the complete solution of equation A3.1 is:

$$y = \frac{10\gamma mg}{EI\alpha^4}\left((\cot\alpha L - \mathrm{cosec}\,\alpha L)\sin\alpha x - \cos\alpha x + 1 + \frac{\alpha^2 Lx}{2} - \frac{\alpha^2 x^2}{2}\right)$$

The solution may be checked by noting that $\frac{dy}{dx} = 0$ when $x = L/2$.

Now put $x = L/2$ to obtain the maximum midspan value of y, at the same time putting all functions in terms of the half angle:

$$y_{MAX} = \frac{EIg}{10\gamma mu^4}\left(\frac{\sin\frac{1}{2}\alpha L(\cos^2\frac{1}{2}\alpha L - \sin^2\frac{1}{2}\alpha L) - \sin\frac{1}{2}\alpha L - 2\sin\frac{1}{2}\alpha L\cos^2\frac{1}{2}\alpha L}{2\sin\frac{1}{2}\alpha L\cos\frac{1}{2}\alpha L}\right.$$

$$\left. + 1 + \frac{10\gamma mu^2 L^2}{8EI}\right)$$

i.e. $$y_{MAX} = \frac{EIg}{10\gamma mu^4}\left(1 - \sec\tfrac{1}{2}\alpha L + \frac{10\gamma mu^2 L^2}{8EI}\right)$$

$$\text{i.e. } y_{MAX} = \frac{(EI)^2 gmL^4\gamma}{10\gamma^2 m^2 L^4 u^4 EI}\left(1 - \sec\tfrac{1}{2}\alpha L + \frac{10\gamma mu^2 L^2}{8EI}\right)$$

Now substitute $\frac{mL^2u^2}{EI} = \frac{\Pi^2}{\beta^2}$ and $\frac{(EI)^2}{m^2L^4u^4} = \frac{\beta^4}{\Pi^2}$ giving:

$$y_{MAX} = \frac{\beta^4}{10\Pi^4\gamma^2} \cdot \frac{\gamma mgL^4}{EI}\left(1 - \sec\tfrac{1}{2}\alpha L + \frac{10\gamma\Pi^2}{8\beta^2}\right)$$

Now the static midspan deflection due to ten equally spaced loads γmL is $0.1307\ \gamma mgL^4/EI = \Delta$

$$y_{MAX} = 0.956\Delta\left(\frac{0.8}{\gamma^2}\ \frac{\beta^4}{\Pi^4}\ (1 - \sec\tfrac{1}{2}\alpha L) + \frac{\beta^2}{\gamma\Pi^2}\right) \qquad \text{A3.2}$$

where $\alpha L = \sqrt{\frac{10\gamma\Pi^2}{\beta^2}}$

It can be seen from equation A3.2 that y_{MAX} will become infinite when $\sec\frac{1}{2}\alpha L = \sec\frac{\Pi}{2}$

i.e. when $\alpha L = \sqrt{\frac{10\gamma\Pi^2}{\beta^2}} = \Pi$

$$\text{i.e. when } \beta = 10\gamma \qquad \text{A3.3}$$

Equation A3.3 gives the transit speed at which instability occurs, in terms of the live/dead load ratio γ ($\beta = \frac{\omega L}{\Pi u}$).

Magnitude of the centripetal acceleration

To obtain $\frac{d^2y}{dx^2}$ differentiate (twice) the solution of equation A3.1 giving:

$$\frac{d^2y}{dx^2} = \frac{-10\gamma mg}{EI\alpha^2}\left(1 - \cos\alpha x + (\cot\alpha L - \operatorname{cosec}\alpha L)\sin\alpha x\right)$$

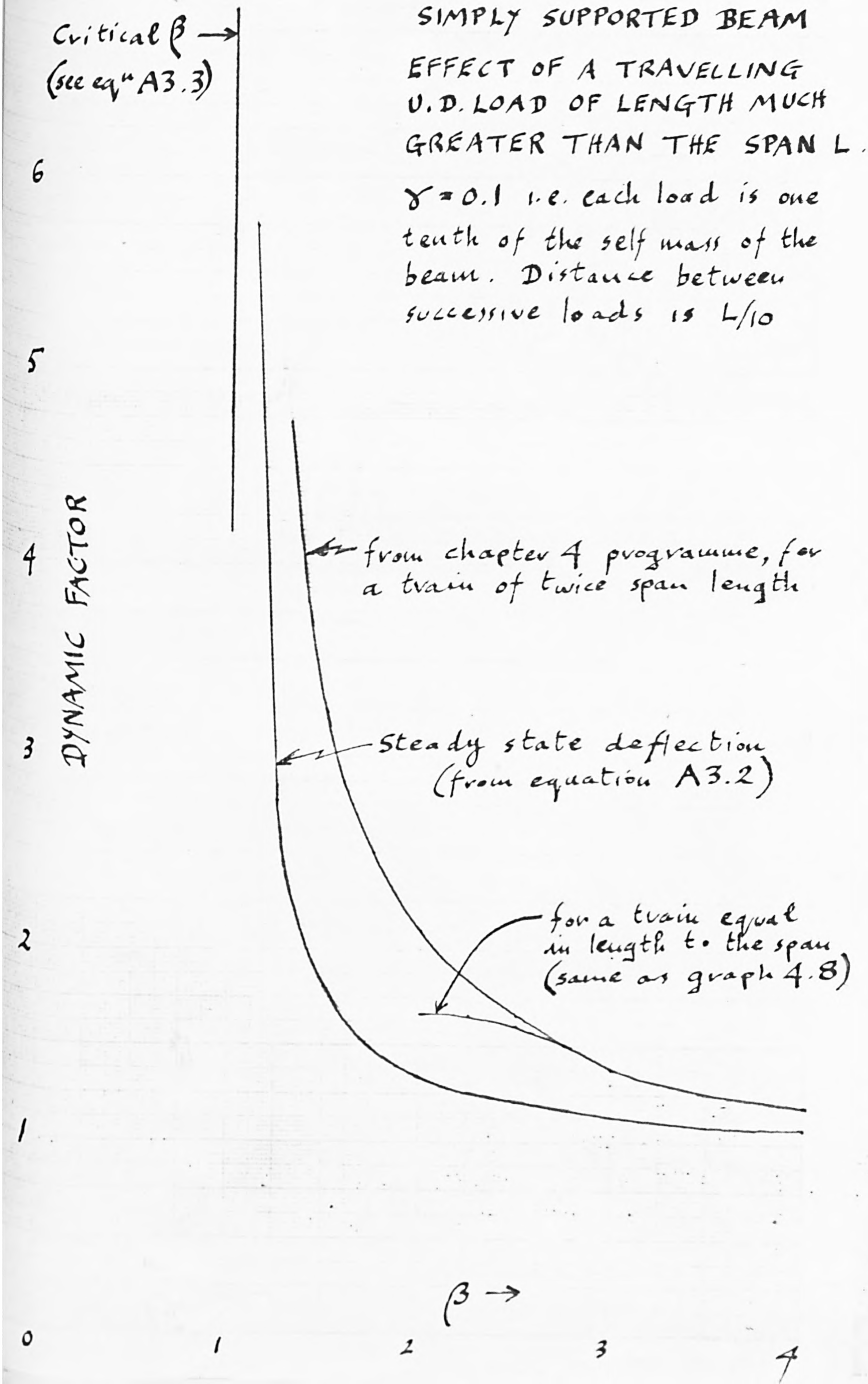
This will be maximum when $x = L/2$ and so the maximum centripetal acceleration is:

$$u^2 \frac{d^2y}{dx^2} MAX = \frac{-10\gamma mgu^2L^2}{EI(\alpha L)^2}(1 - \sec\tfrac{1}{2}\alpha L)$$

Now substitute $\frac{mL^2u^2}{EI} = \frac{\Pi^2}{\beta^2}$ and $\alpha L = \sqrt{\frac{10\gamma\Pi^2}{\beta^2}}$

$$\text{Then } u^2 \frac{d^2y}{dx^2} MAX = g(\sec\tfrac{1}{2}\alpha L - 1) \qquad \text{A3.4}$$

For example, if $\gamma = 0.1$, $\beta = 1.5$ then equation A3.4 gives a maximum centripetal acceleration of g. However, the acceleration will increase very rapidly as β approaches its critical value (which is $\beta = 1$ for $\gamma = 0.1$).



Critical β →
(see eqn A3.3)
SIMPLY SUPPORTED BEAM
EFFECT OF A TRAVELLING U.D. LOAD OF LENGTH MUCH GREATER THAN THE SPAN L.
γ = 0.1 i.e. each load is one tenth of the self mass of the beam. Distance between successive loads is L/10
from chapter 4 programme, for a train of twice span length
Steady state deflection (from equation A3.2)
for a train equal in length to the span (same as graph 4.8)
DYNAMIC FACTOR
β →
0
1
2
3
4
5
6

PROGRAMME USED IN CHAPTER 4

```
G=0.5
R=0.0
UM=-0.0416
B=2.0
DO10 I=1,5
BBB=3.14159/B
BB=BBB*BBB
AAN=0.00993567
1+0.017769*UM  *BB*G
AAD=(0.063662+0.010854*G)*BB+0.0092207
AA=AAN/AAD
YA=AA/100.0
YYA=AA/5.0
ABN=0.0288344-3.1006*YA-0.15503*YYA+0.130934*YA*BB*G
1-0.153099*YYA*BB*G
1+0.130934*UM*BB*G
1-1.974*R*YYA/B
ABD=(0.063662+0.042872*G)*BB+0.0092207
AB=ABN/ABD
YB=AB/100.0+YYA/10.0+YA
YYB=AB/5.0+YYA
ACN=0.0449107-3.1006*YB-0.15503*YYB+0.314045*YB*BB*G
1-0.181632*YYB*BB*G
1+0.314045*UM*BB*G
1-1.974*R*YYB/B
ACD=(0.063662+0.082483*G)*BB+0.0092207
AC=ACN/ACD
YC=AC/100.0+YYB/10.0+YB
YYC=AC/5.0+YYB
ADN=0.0565908-3.1006*YC-0.15503*YYC+0.49716*YC*BB*G
1-0.134791*YYC*BB*G
1+0.49716*UM*BB*G
1-1.974*R*YYC/B
ADD=(0.063662+0.114556*G)*BB +0.0092207
AD=ADN/ADD
YD=AD/100.0+YYC/10.0+YC
YYD=AD/5.0+YYC
AEN=0.0627314-3.1006*YD-0.15503*YYD+0.610336*YD*BB*G
1-0.030469*YYD*BB*G
1+0.610336*UM*BB*G
1-1.974*R*YYD/B
AED=(0.063662+0.126814*G)*BB+0.0092207
AE=AEN/AED
YE=AE/100.0+YYD/10.0+YD
YYE=AE/5.0+YYD
AFN=0.0627314-3.1006*YE-0.15503*YYE+0.610343*YE*BB*G
1+0.091488*YYE*BB*G
1+0.610343*UM*BB*G
1-1.974*R*YYE/B
AFD=(0.063662+0.11484*G)*BB+0.0092207
AF=AFN/AFD
YF=AF/100.0+YYE/10.0+YE
YYF=AF/5.0+YYE
AGN=0.056591-3.1006*YF-0.15503*YYF+0.497179*YF*BB*G
1+0.184497*YYF*BB*G
1+0.497179*UM*BB*G
1-1.974*R*YYF/B
AGD=(0.063662+0.082627*G)*BB+0.0092207
AG=AGN/AGD
YG=AG/100.0+YYF/10.0+YF
YYG=AG/5.0+YYF
AHN=0.0449107-3.1006*YG-0.15503*YYG+0.314068*YG*BB*G
1+0.21301*YYG*BB*G
```

Cont.

```
1+0.314068*UM*BB*G
1-1.974*R*YYG/B
 AHD=(0.063662+0.043017*G)*BB+0.0092207
 AH=AHN/AHD
 YH=AH/100.0+YYG/10.0+YG
 YYH=AH/5.0+YYG
 AIN=0.0288344-3.1006*YH-0.15503*YYH+0.130952*YH*BB*G+0.16619*YYH*
1BB*G
1+0.130952*UM*BB*G
1-1.974*R*YYH/B
 AID=(0.063662+0.010944*G)*BB+0.0092207
 AI=AIN/AID
 YI=AI/100.0+YYH/10.0+YH
 YYI=AI/5.0+YYH
 AJN=0.00993567-3.1006*YI-0.15503*YYI+0.017776*YI*BB*G
1+0.061868*YYI*BB*G
1+0.017776*UM*BB*G
1-1.974*R*YYI/B
 AJD=(0.063662-0.0013414*G)*BB+0.0092207
 AJ=AJN/AJD
 YJ=AJ/100.0+YYI/10.0+YI
 YYJ=AJ/5.0+YYI
 WRITE (6,2) YA,YB,YC,YD,YE,YF,YG,YH,YI,YJ
 FORMAT (F15.7)
 B=B+0.5
 STOP
 END
```

INTEGRALS REQUIRED FOR CHAPTER 4 PROGRAMME (SEE PAGE 42)

```
      AN=0.314159
      AX=-AN
      DO10 J=1,10
      XX=0.062832
      X=0.0
      Y=0.0
      AF=0.5
      AX=AX+AN
      DO11 I=1,51
      P=X*X
      Q=COS(X)
      SA=SIN(X*0.1+AX)
      SB=COS(X*0.1+AX)
      R=SA*SB
      Y=Y+P*Q*R*AF
      X=X+XX
11    AF=1.0
      Y=Y-0.5*P*Q*R
      Z=0.062832*Y
10    WRITE (2,1) Z
1     FORMAT (F15.7)
      STOP
      END
```

I_1'	I_2'
-0.17[illegible]7356	-0.06[illegible]2467
-0.6896682	-0.16[illegible]369[illegible]
-1.51[illegible]674	-0.1962042
-1.825781	-0.16[illegible]355
-2.02[illegible]3027	-0.06[illegible]2547
-1.828948	0.06[illegible]2593
-1.520355	0.16[illegible]3417
-0.6890093	0.1961963
-0.1769017	0.16[illegible]346[illegible]
0.0206817	0.06[illegible]2470

I_3'	I_4'	I_5'	I_6'
5.6019588	1.94[illegible]5097	3.02[illegible]1090	0.5668759
4.4026220	1.58[illegible]7960	2.409698[illegible]	1.5130417
2.6420088	0.999672[illegible]	1.477597[illegible]	1.5556712
0.9926124	0.4175463	0.58[illegible]898[illegible]	1.204088[illegible]
0.0844457	0.0577729	0.0621085	0.592565
0.2643955	0.0577714	0.119387[illegible]	-0.568372
1.4637287	0.417563[illegible]	0.73[illegible]857[illegible]	-1.513059[illegible]
3.2243400	0.999697[illegible]	1.062957[illegible]	-1.555671[illegible]
4.8737593	1.581793	2.559657[illegible]	-1.204090[illegible]
5.7819106	1.94[illegible]5087	3.078449[illegible]	-0.592589

I_7'	I_8'	I_9'	I_{10}'
0.3000417	-1.6732695	-0.186501	-0.596587
0.80[illegible]226[illegible]	-2.9221735	-0.1105022	-0.969691[illegible]
0.99[illegible]371	-3.0549083	-0.000003[illegible]	-0.972407[illegible]
0.80[illegible]2270	-2.0207740	0.110496[illegible]	-0.0036975
0.3000435	-0.2147730	0.186298[illegible]	-0.004395[illegible]
-0.300039	1.6732024	0.186501	0.596584
-0.80[illegible]225	2.9221703	0.1105027	0.969690[illegible]
-0.99[illegible]371	3.0549104	0.0000042	0.972408[illegible]
-0.80[illegible]229	2.0207804	-0.110495	0.0036997
-0.3000[illegible]6	0.2147822	-0.1862985	0.004398[illegible]

SECOND PROGRAMME USED IN CHAPTER 4.

See pp 56,57,58

```
      G=1.0
      BX=0.75
      DO10 I=1,10
      BBP=3.14159/BX
      BB=BBP*BBP
      W=0.0
      WW=0.0
      Z=0.0
      ZZ=0.0
      P=1.0
      DO11 J=1,60
      PP1=COS(3.14159*(4.0*P-3.0)/240.0)
      PP2=COS(3.14159*(4.0*P-1.0)/240.0)
      QQ1=SIN(3.14159*(4.0*P-3.0)/240.0)
      QQ2=SIN(3.14159*(4.0*P-1.0)/240.0)
      RR1=COS(3.14159*(12.0*P-9.0)/240.0)
      RR2=COS(3.14159*(12.0*P-3.0)/240. 0)
      SS1=SIN(3.14159*(12.0*P-9.0)/240.0)
      SS2=SIN(3.14159*(12.0*P-3.0)/240.0)
      BP=-.0053052*QQ1-.0053052*QQ2
      CP=- .0053052*SS1-.0053052*SS2
      BOB=.000001682*PP1*PP1+.000006968*PP2*PP2+.006359*QQ1*QQ1
     1-.02864*QQ2*QQ2+.0002181*PP1*QQ1-.0002181*PP2*QQ2
      WWB=.0002777*PP1*PP1+.0005948*PP2*PP2+1.2732*QQ1*QQ1
     1-1.2732*QQ2*QQ2+.02665*PP1*QQ1-.02832*PP2*QQ2
      WB=.05236*PP1*PP1+.05236*PP2*PP2+4.0*PP1*QQ1-4.0*PP2*QQ2
      COB=.00000501*PP1*RR1+.0000209*PP2*RR2+.0001239*QQ1*RR1
     1-.001791*QQ2*RR2+.0001763*PP1*SS1+.0003787*PP2*SS2
     1+.006359*QQ1*SS1-.02864*QQ2*SS2
      ZZB=.0008332*PP1*RR1+.001784*PP2*RR2+.02996*QQ1*RR1
     1-.13496*QQ2*RR2+.01667*PP1*SS1+.01667*PP2*SS2
     1+1.2732*QQ1*SS1-1.2732*QQ2*SS2
      ZB=.1571*PP1*RR1+.1571*PP2*RR2+12.0*QQ1*RR1-12.0*QQ2*RR2
      BOC=.00000504*PP1*RR1+.0000209*PP2*RR2+.00004131*PP1*SS1
     1-.000597*PP2*SS2+.0005304*QQ1*RR1+.001136*QQ2*RR2
     1+.00636*QQ1*SS1-.02864*QQ2*SS2
      WWC=.0008332*PP1*RR1+.001784*PP2*RR2+.00999*PP1*SS1
     1-.04499*PP2*SS2+.05*QQ1*RR1+.05*QQ2*RR2
     1+1.2732*QQ1*SS1-1.2732*QQ2*SS2
      WC=.1571*PP1*RR1+.1571*PP2*RR2+4.0*PP1*SS1-4.0*PP2*SS2
      COC=.00001514*RR1*RR1+.00006271*RR2*RR2+.006359*SS1*SS1
     1-.02864*SS2*SS2+.000654*RR1*SS1-.000654*RR2*SS2
      ZZC=.0025*RR1*RR1+.005353*RR2*RR2+1.2732*SS1*SS1
     1-1.2732*SS2*SS2+.08*RR1*SS1-.085*RR2*SS2
      ZC=.4712*RR1*RR1+.4712*RR2*RR2+12.0*RR1*SS1-12.0*RR2*SS2
      BBB=(-.01061+BOB*G)*BB-.00004268
      CCB=COB*G*BB
      CCC=(-.01061+COC*G)*BB-.003457
      BBC=BOC*G*BB
      RHSB=BP+.004306*WW+.51677*W-G*BB*(WWB*WW+WB*W+ZZB*ZZ+ZB*Z)
      RHSC=CP+.34879*ZZ+41.86*Z-G*BB*(WWC*WW+WC*W+ZZC*ZZ+ZC*Z)
      BCD=CCC*BBB-CCB*BBC
      BN=RHSB*CCC-RHSC*CCB
      CN=RHSC*BBB-RHSB*BBC
      B=BN/BCD
      C=CN/BCD
      W=B/3600.0+WW/60.0+W
      WW=B/30.0+WW
      Z=C/3600.0+ZZ/60.0+Z
      ZZ=C/30.0+ZZ
      D=48.0*(W-Z)
      DBM=39.5*(W-9.0*Z)
      WRITE (6,2) DBM
2         FORMAT(F15.7)
11        P=P+1.0
10        BX=BX+0.25
          STOP
          END
```

PROGRAMME USED IN CHAPTER 5

```
R=0.05
UM=0.0
G=0.5
BX=0.75
DO11 I=1,9
BBP=3.14159/BX
BB=BBP*BBP
Y=0.0
YY=0.0
Z=0.0
ZZ=0.0
W=0.0
WW=0.0
P=1.0
DO12 K=1,20
OOO1=SIN(3.14159*(4.0*P-3.0)/160.0)
QQQ1=1.0-COS(3.14159*(4.0*P-3.0)/160.0)
OOO2=SIN(3.14159*(4.0*P-1.0)/160.0)
QQQ2=1.0-COS(3.14159*(4.0*P-1.0)/160.0)
CP=-0.0053052*QQQ1-0.0053052*QQQ2
CC=0.000003777*OOO1*OOO1+0.00001565*OOO2*OOO2+0.006359*QQQ1*QQQ1
1-0.02864*QQQ2*QQQ2+0.0003272*OOO1*QQQ1-0.0003272*OOO2*QQQ2
ZZC=0.0006256*OOO1*OOO1+0.00134*OOO2*OOO2+1.27*QQQ1*QQQ1
1-1.27*QQQ2*QQQ2+0.03998*OOO1*QQQ1-0.04248*OOO2*QQQ2
ZC=0.11781*OOO1*OOO1+0.11781*OOO2*OOO2+6.0*OOO1*QQQ1-6.0*OOO2*QQQ2
AAA=-0.01061*BB-0.00004269
CCA=-0.0067547*BB
BBA=-0.0067547*BB
RHSA=0.516771*Y+0.0043064*YY
1+(0.329*R*YY)/BX
AAC=-0.0067547*BB
CCC=(-0.0086778+CC*G)*BB-0.000072036
BBC=-0.003537*BB
RHSC=CP-G*BB*(ZZC*ZZ+ZC*Z)+0.872051*Z+0.0072671*ZZ
1+((0.04973*ZZ+0.06667*YY)*R)/BX
AAB=-0.0067547*BB
CCB=-0.0035368*BB
BBB=-0.0086778*BB-0.000072036
RHSB=0.872051*W+0.0072671*WW
1+((0.04973*WW+0.06667*YY)*R)/BX
V1=CCC*BBB-CCB*BBC
V2=CCA*BBB-CCB*BBA
V3=CCA*BBC-CCC*BBA
AN=RHSA*V1-RHSC*V2+RHSB*V3
AD=AAA*V1-AAC*V2+AAB*V3
A=AN/AD
V4=AAC*BBB-AAB*BBC
V5=AAA*BBB-AAB*BBA
V6=AAA*BBC-AAC*BBA
CN=RHSA*V4-RHSC*V5+RHSB*V6
CD=CCA*V4-CCC*V5+CCB*V6
C=CN/CD
V7=AAC*CCB-CCC*AAB
V8=AAA*CCB-CCA*AAB
V9=AAA*CCC-CCA*AAC
BN=RHSA*V7-RHSC*V8+RHSB*V9
BD=BBA*V7-BBC*V8+BBB*V9
B=BN/BD
Y=A/3600.0+YY/60.0+Y
YY=A/30.0+YY
Z=C/3600.0+ZZ/60.0+Z
ZZ=C/30.0+ZZ
W=B/3600.0+WW/60.0+W
```

Cont.

```
      WW=B/30.0+WW
      DY=48.0*Y
      DZ=81.0*Z
      DW=81.0*W
      WRITE (6,2)  DY,DZ,DW
2     FORMAT(3(3X,F15.7))
12    P=P+1.0
      Q=1.0
      DO13 L=1,60
      001=COS(3.14159*(4.0*Q-3.0)/240.0)
      002=COS(3.14159*(4.0*Q-1.0)/240.0)
      QQ1=SIN(3.14159*(4.0*Q-3.0)/240.0)
      QQ2=SIN(3.14159*(4.0*Q-1.0)/240.0)
      RR1=1.0-(4.0*Q-3.0)/240.0
      RR2=1.0-(4.0*Q-1.0)/240.0
      SS1=(4.0*Q-3.0)/240.0
      SS2=(4.0*Q-1.0)/240.0
      CP=-0.0053052*RR1-0.0053052*RR2
      AC=-0.0000005353*001-0.000002218*002+0.00004131*001*RR1
     1-0.0005969*002*RR2-0.00005628*QQ1-0.0001205*QQ2+0.006359*QQ1*RR1
     1-0.02864*QQ2*RR2
      CC=0.0000008764-0.00006944*RR1+0.00006944*RR2
     1+0.006359*RR1*RR1-0.02864*RR2*RR2
      BC=-0.0000008764+0.00001315*RR1-0.00019*RR2-0.00005628*SS1
     1-0.0001205*SS2+0.006359*RR1*SS1-0.02864*RR2*SS2
      YYC=-0.0000884*001-0.0001893*002+0.009988*001*RR1-0.04499*002*RR2
     1-0.0053052*QQ1-0.0053052*QQ2+1.27*QQ1*RR1-1.27*QQ2*RR2
      ZZC=0.00008841-0.008485*RR1+0.009015*RR2+1.27*RR1*RR1-1.27*RR2*RR2
      WWC=-0.00008841+0.003179*RR1-0.01432*RR2-0.005305*SS1-0.005305*SS2
     1+1.27*RR1*SS1-1.27*RR2*SS2
      YC=-0.01667*001-0.01667*002+4.0*001*RR1-4.0*002*RR2
      ZC=0.01061-1.27*RR1+1.27*RR2
      WC=-0.01061+1.27*RR1-1.27*RR2
      AP=-0.0053052*QQ1-0.0053052*QQ2
      AA=0.000001682*001*001+0.000006968*002*002+0.006359*QQ1*QQ1
     1-0.02864*QQ2*QQ2+0.0002181*001*QQ1-0.0002181*002*QQ2
      CA=-0.0000005353*001-0.000002218*002-0.00001315*QQ1+0.00019*QQ2
     1+0.0001768*001*RR1+0.0003787*002*RR2+0.006359*QQ1*RR1
     1-0.02864*QQ2*RR2
      BA=0.0000005353*001+0.000002218*002+0.00001315*QQ1-0.00019*QQ2
     1+0.0001768*001*SS1+0.0003787*002*SS2+0.006359*QQ1*SS1
     1-0.02864*QQ2*SS2
      YYA=0.0002777*001*001+0.0005948*002*002+1.27*QQ1*QQ1-1.27*QQ2*QQ2
     1+0.02665*001*QQ1-0.02832*002*QQ2
      ZZA=-0.0000884*001-0.0001893*002-0.003179*QQ1+0.01432*QQ2
     1+0.01667*002*RR2+0.01667*001*RR1+1.27*QQ1*RR1-1.27*QQ2*RR2
      WWA=0.0000884*001+0.0001893*002+0.003179*QQ1-0.01432*QQ2
     1+0.01667*001*SS1+0.01667*002*SS2+1.27*QQ1*SS1-1.27*QQ2*SS2
      YA=0.05236*001*001+0.05236*002*002+4.0*001*QQ1-4.0*002*QQ2
      ZA=-0.01667*001-0.01667*002-1.27*QQ1+1.27*QQ2
      WA=0.016669*001+0.016666* 002+1.2699*QQ1-1.2699*QQ2
      BP=-0.0053052*SS1-0.0053052*SS2
      AB=0.0000005353*001+0.000002218*002+0.00004131*001*SS1
     1-0.0005969*002*SS2
     1+0.0005628*QQ1+0.0001205*QQ2+0.006359*QQ1*SS1-0.02864*QQ2*SS2
      CB=-0.0000008764-0.00001315*SS1+0.00019*SS2+0.00005628*RR1
     1+0.0001205*RR2+0.006359 *RR1*SS1-0.02864*RR2*SS2
      BB1=0.0000008764+0.00006946*SS1-0.00006946*SS2+0.006359*SS1*SS1
     1-0.02864*SS2*SS2
      YYB=0.00008841*001+0.0001893*002+0.009988*001*SS1-0.04499*002*SS2
     1+0.005305*QQ1+0.005305*QQ2+1.27*QQ1*SS1-1.27*QQ2*SS2
      ZZB=-0.00008841-0.003179*SS1+0.01432*SS2+0.005305*RR1
     1+0.005305*RR2+1.27*RR1*SS1-1.27*RR2*SS2
```

Cont.

```
 WWB=0.00008841+0.008485*SS1-0.009015*SS2+1.27*SS1*SS1-1.27*SS2*SS2
 YB=0.01667*001+0.01667*002+4.0*001*SS1-4.0*002*SS2
 ZB=-0.01061-1.27*SS1+1.27*SS2
 WB=0.01061+1.27*SS1-1.27*SS2
 AAA=(-0.01061+AA*G)*BB-0.000042688
 CCA=(-0.0067547+CA*G)*BB
 BBA=(-0.0067547+BA*G)*BB
 RHSA=AP-G*BB*(YYA*YY+ZZA*ZZ+WWA*WW+YA*Y+ZA*Z+WA*W)
1+0.516771*Y+0.0043064*YY
1+(0.329*R*YY)/BX-YA*UM*G*BB
 AAC=(-0.0067547+AC*G)*BB
 CCC=(-0.0086778+CC*G)*BB-0.000072036
 BBC=(-0.0035368+BC*G)*BB
 RHSC=CP-G*BB*(YYC*YY+ZZC*ZZ+WWC*WW+YC*Y+ZC*Z+WC*W)
1+0.872051*Z+0.0072671*ZZ
1+((0.04973*ZZ+0.06667*YY)*R)/BX-YC*UM*G*BB
 AAB=(-0.0067547+AB*G)*BB
 CCB=(-0.0035368+CB*G)*BB
 BBB=(-0.0086778+BB1*G)*BB-0.000072036
 RHSB=BP-G*BB*(YYB*YY+ZZB*ZZ+WWB*WW+YB*Y+ZB*Z+WB*W)
1+0.872051*W+0.0072671*WW
1+((0.04973*WW+0.06667*YY)*R)/BX-YB*UM*G*BB
 V1=CCC*BBB-CCB*BBC
 V2=CCA*BBB-CCB*BBA
 V3=CCA*BBC-CCC*BBA
 AN=RHSA*V1-RHSC*V2+RHSB*V3
 AD=AAA*V1-AAC*V2+AAB*V3
 A=AN/AD
 V4=AAC*BBB-AAB*BBC
 V5=AAA*BBB-AAB*BBA
 V6=AAA*BBC-AAC*BBA
 CN=RHSA*V4-RHSC*V5+RHSB*V6
 CD=CCA*V4-CCC*V5+CCB*V6
 C=CN/CD
 V7=AAC*CCB-CCC*AAB
 V8=AAA*CCB-CCA*AAB
 V9=AAA*CCC-CCA*AAC
 BN=RHSA*V7-RHSC*V8+RHSB*V9
 BD=BBA*V7-BBC*V8+BBB*V9
 B=BN/BD
 Y=A/3600.0+YY/60.0+Y
 YY=A/30.0+YY
 Z=C/3600.0+ZZ/60.0+Z
 ZZ=C/30.0+ZZ
 W=B/3600.0+WW/60.0+W
 WW=B/30.0+WW
 DY=48.0*Y
 DZ=81.0*Z
 DW=81.0*W
 WRITE(6,2) DY,DZ,DW
 Q=Q+1.0
 U=20.0
 DO14 M=1,20
 OOQ1=SIN(3.14159*(4.0*U-1.0)/160.0)
 OOQ2=SIN(3.14159*(4.0*U-3.0)/160.0)
 QQO1=1.0-COS(3.14159*(4.0*U-1.0)/160.0)
 QQO2=1.0-COS(3.14159*(4.0*U-3.0)/160.0)
 BP=-0.0053052*QQO1-0.0053052*QQO2
 BB1=0.000003777*OOQ1*OOQ1+0.00001565*OOQ2*OOQ2+0.006359*QQO1*QQO1
1-0.02864*QQO2*QQO2-0.0003272*OOQ1*QQO1+0.0003272*OOQ2*QQO2
 WWB=0.0006256*OOQ1*OOQ1+0.00134*OOQ2*OOQ2+1.27*QQO1*QQO1
1-1.27*QQO2*QQO2-0.03998*OOQ1*QQO1+0.04248*OOQ2*QQO2
 WB=0.11781*OOQ1*OOQ1+0.11781*OOQ2*OOQ2-6.0*OOQ1*QQO1+6.0*OOQ2*QQO2
```

Cont.

```
 AAA=-0.01061*BB-0.000042688
 CCA=-0.0067547*BB
 BBA=-0.0067547*BB
 RHSA=0.516771*Y+0.0043064*YY
1+(0.329*R*YY)/BX
 AAC=-0.0067547*BB
 CCC=-0.0086778*BB-0.000072036
 BBC=-0.0035368*BB
 RHSC=0.872051*Z+0.0072671*ZZ
1+((0.04973*ZZ+0.06667*YY)*R)/BX
 AAB=-0.0067547*BB
 CCB=-0.0035368*BB
 BBB=(-0.0086778+BB1*G)*BB-0.000072036
 RHSB=BP-G*BB*(WWB*WW+WB*W)+0.872051*W+0.0072671*WW
1+((0.04973*WW+0.06667*YY)*R)/BX
 V1=CCC*BBB-CCB*BBC
 V2=CCA*BBB-CCB*BBA
 V3=CCA*BBC-CCC*BBA
 AN=RHSA*V1-RHSC*V2+RHSB*V3
 AD=AAA*V1-AAC*V2+AAB*V3
 A=AN/AD
 V4=AAC*BBB-AAB*BBC
 V5=AAA*BBB-AAB*BBA
 V6=AAA*BBC-AAC*BBA
 CN=RHSA*V4-RHSC*V5+RHSB*V6
 CD=CCA*V4-CCC*V5+CCB*V6
 C=CN/CD
 V7=AAC*CCB-CCC*AAB
 V8=AAA*CCB-CCA*AAB
 V9=AAA*CCC-CCA*AAC
 BN=RHSA*V7-RHSC*V8+RHSB*V9
 BD=BBA*V7-BBC*V8+BBB*V9
 B=BN/BD
 Y=A/3600.0+YY/60.0+Y
 YY=A/30.0+YY
 Z=C/3600.0+ZZ/60.0+Z
 ZZ=C/30.0+ZZ
 W=B/3600.0+WW/60.0+W
 WW=B/30.0+WW
 DY=48.0+Y
 DZ=81.0+Z
 DW=81.0+W
 WRITE(6,2) DY,DZ,DW
 U=U-1.0
 BX=BX+0.25
 STOP
 END
```

PROGRAMME USED IN CHAPTER 7.

```
γ —  G=0.5
β —  BX=1.0
ξ —  SYZ=0.4
     DO11 I=1,5
     BBP=3.14159/BX
     BB=BBP*BBP
     Y=0.0
     YY=0.0
     Z=0.0
     ZZ=0.0
     P=1.0
     DO12 K=1,60
     001=COS(3.14159*(4.0*P-3.0)/240.0)
     002=COS(3.14159*(4.0*P-1.0)/240.0)
     QQ1=SIN(3.14159*(4.0*P-3.0)/240.0)
     QQ2=SIN(3.14159*(4.0*P-1.0)/240.0)
     AP=-0.0053052*QQ1-0.0053052*QQ2
     AA=0.0000016818*001*001+0.000006968*002*002+0.006359*QQ1*QQ1
    1-0.02864*QQ2*QQ2+0.0002181*001*QQ1-0.0002181*002*QQ2
     YYA=1.273*QQ1*QQ1-1.273*QQ2*QQ2+0.00027773*001*001
    1+0.0005948*002*002+0.026655*001*QQ1-0.02832*002*QQ2
     YA=0.05236*001*001+0.05236*002*002+4.0*001*QQ1-4.0*002*QQ2
     CA=0.006359*QQ1-0.02864*QQ2+0.0001768*001+0.0003787*002
     ZZA=1.273*QQ1-1.273*QQ2+0.016667*001+0.016667*002
     AC=0.0063588*QQ1-0.02864*QQ2+0.0000413*001-0.000597*002
     YYC=1.273*QQ1-1.273*QQ2+0.009988*001-0.044987*002
     YC=4.0*001-4.0*002
     CC=-0.022281
     AAA=(-0.01061+AA*G)*BB-0.000042688
     CCA=CA*G*BB
     RHSA=AP-G*BB*(YYA*YY+ZZA*ZZ+YA*Y)+0.516775*Y+0.004306*YY
     AAC=AC*G*BB
     CCC=CC*G*BB-0.000042688*SYZ
     RHSC=-G*BB*(YYC*YY+YC*Y)+0.516775*Z*SYZ+0.004306*ZZ*SYZ
     ACD=CCC*AAA-AAC*CCA
     AN=RHSA*CCC-RHSC*CCA
     CN=RHSC*AAA-RHSA*AAC
     A=AN/ACD
     C=CN/ACD
     Y=A/3600.0+YY/60.0+Y
     YY=A/30.0+YY
     Z=C/3600.0+ZZ/60.0+Z
     ZZ=C/30.0+ZZ
     DY=48.0*Y
     DZ=Z/Y
     SD=48.0*(Y+Z)
     WRITE(6,2)DY,DZ,SD
2    FORMAT (3(3X,F15.7))
12   P=P+1.0
11   BX=BX+0.5
     STOP
     END
```

5.4b KINETIC ENERGY (T_2) OF A MOVING LOAD ON THE MAIN SPAN

During the sixty stages whilst the load is on the main span its position will be given by:

$$x_M = \left(\frac{n-1}{60}\right) L + ut$$

(x_M is here measured from the beginning of the main span.)

The instantaneous deflection of the main span, at the position of the moving load can now be expressed as follows:

$$\begin{aligned} y_M = & (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \sin\left(\Pi\left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right) \\ & + (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)\left(1 - \frac{n-1}{60} - \frac{ut}{L}\right) \\ & + (Bt^2 + \dot{W}_o t + W_o + b \sin qt)\left(\frac{n-1}{60} + \frac{ut}{L}\right) \end{aligned}$$

$$\begin{aligned} \dot{y}_M = & (At^2 + \dot{Y}_o t + Y_o + a \sin qt) \frac{\Pi u}{L} \cos\left(\Pi\left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right) \\ & + (2At + \dot{Y}_o + qa \cos qt) \sin\left(\Pi\left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right) \\ & + (Ct^2 + \dot{Z}_o t + Z_o + c \sin qt)\left(-\frac{u}{L}\right) \\ & + (2Ct + \dot{Z}_o + qc \cos qt)\left(1 - \frac{n-1}{60} - \frac{ut}{L}\right) \\ & + (Bt^2 + \dot{W}_o t + W_o + b \sin qt)\left(\frac{u}{L}\right) \\ & + (2Bt + \dot{W}_o + qb \cos qt)\left(\frac{n-1}{60} + \frac{ut}{L}\right) \end{aligned}$$

Let,

$$\cos\left(\Pi\left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right) = \cos\frac{\Pi x_M}{L} = \theta\theta$$

$$\sin\left(\Pi\left(\frac{n-1}{60}\right) + \frac{\Pi ut}{L}\right) = \sin\frac{\Pi x_M}{L} = QQ$$

$$1 - \frac{n-1}{60} - \frac{ut}{L} = 1 - \frac{x_M}{L} = RR$$

$$\frac{n-1}{60} + \frac{ut}{L} = \frac{x_M}{L} = SS$$

The average values of the above functions, in each half stage (following a similar approximation procedure to that explained in para. 5.4a) can be expressed as follows:

$$\theta\theta 1 = \cos\Pi\left(\frac{4n - 3}{240}\right) \quad ; \quad \theta\theta 2 = \cos\Pi\left(\frac{4n - 1}{240}\right)$$

$$QQ1 = \sin\Pi\left(\frac{4n - 3}{240}\right) \quad ; \quad QQ2 = \sin\Pi\left(\frac{4n - 1}{240}\right)$$

$$RR1 = 1 - \frac{4n - 3}{240} \quad ; \quad RR2 = 1 - \frac{4n - 1}{240}$$

$$SS1 = \frac{4n - 3}{240} \quad ; \quad SS2 = \frac{4n - 1}{240}$$

$$\text{Now } T_2 = \tfrac{1}{2}M\dot{y}_M{}^2$$

δT_2 will now involve all three variations. The expressions for δT_{2a}, δT_{2b}, δT_{2c}, followed by their integrals, are shown on the following six pages.

$\Pi/2q$ to 0 $\quad$ Π/q to $\Pi/2q$

$$\frac{\delta T_{2a}}{M.a} = (\Theta\Theta 1^2 \; ; \; \Theta\Theta 2^2) \; \frac{\Pi^2 u^2}{L^2} \; (At^2 + \dot{Y}_o t + Y_o) \sin qt$$

$$+ \; (\Theta\Theta 1.QQ1 \; ; \; \Theta\Theta 2.QQ2) \; \frac{\Pi u}{L} \; (At^2 + \dot{Y}_o t + Y_o) q \cos qt$$

$$+ \; (\Theta\Theta 1.QQ1 \; ; \; \Theta\Theta 2.QQ2) \frac{\Pi u}{L} (2At + \dot{Y}_o) \sin qt$$

$$+ \; (QQ1^2 \; ; \; QQ2^2 \;) \; (2At + \dot{Y}_o) \; q \cos qt$$

$$- \; (\Theta\Theta 1 \; ; \; \Theta\Theta 2 \;) \; \frac{\Pi u^2}{L^2} \; (Ct^2 + \dot{Z}_o t + Z_o) \sin qt$$

$$- \; (QQ1 \; ; \; QQ2 \;) \; \frac{u}{L} \; (Ct^2 + \dot{Z}_o t + Z_o) q \cos qt$$

$$+ \; (\Theta\Theta 1.RR1 \; ; \; \Theta\Theta 2.RR2) \; \frac{\Pi u}{L} \; (2Ct + \dot{Z}_o) \sin qt$$

$$+ \; (QQ1.RR1 \; ; \; QQ2.RR2) \; (2Ct + \dot{Z}_o) \; q \cos qt$$

$$+ \; (\Theta\Theta 1 \; ; \; \Theta\Theta 2 \;) \; \frac{\Pi u^2}{L^2} \; (Bt^2 + \dot{W}_o t + W_o) \sin qt$$

$$+ \; (QQ1 \; ; \; QQ2 \;) \; \frac{u}{L} \; (Bt^2 + \dot{W}_o t + W_o) q \cos qt$$

$$+ \; (\Theta\Theta 1.SS1 \; ; \; \Theta\Theta 2.SS2) \; \frac{\Pi u}{L} \; (2Bt + \dot{W}_o) \sin qt$$

$$+ \; (QQ1.SS1 \; ; \; QQ2.SS2) \; (2Bt + \dot{W}_o) \; q \cos qt$$

$\Pi/2q$ to 0 $\quad$ Π/q to $\Pi/2q$

$$\frac{1}{M.a}\int_{o}^{L/60u} \delta T_{2a}\,dt \quad =$$

$$A\,\frac{L}{u}\left(\begin{array}{l} +\ .000001682(\theta\theta 1)^2 + .000006968(\theta\theta 2)^2 + .006359(QQ1)^2 \\ -\ .02864(QQ2)^2 \quad + .0002181(\theta\theta 1)(QQ1) - .0002181(\theta\theta 2)(QQ2) \end{array}\right)$$

$$\dot{Y}_o\left(\begin{array}{l} +\ .0002777(\theta\theta 1)^2 \quad + .0005948(\theta\theta 2)^2 + {}^{*}1.27(QQ1)^2 - {}^{*}1.27(QQ2)^2 \\ +\ .02665(\theta\theta 1)(QQ1) - .02832(\theta\dot{\theta} 2)(QQ2) \end{array}\right)$$

$$Y_o\,\frac{u}{L}\left(.05236(\theta\theta 1)^2 \quad + .05236(\theta\theta 2)^2 \quad + {}^{*}4(\theta\theta 1)(QQ1) - {}^{*}4(\theta\theta 2)(QQ2)\right)$$

$$C\,\frac{L}{u}\left(\begin{array}{l} -\ .0000005353(\theta\theta 1) - .000002218(\theta\theta 2) - .00001315(QQ1) \\ +\ .0001900(QQ2) \quad + .0001768(\theta\theta 1)(RR1) + .0003787(\theta\theta 2)(RR2) \\ +\ .006359(QQ1)(RR1) - .02864(QQ2)(RR2) \end{array}\right)$$

$$\dot{Z}_o\left(\begin{array}{l} -\ .00008840(\theta\theta 1) \quad - .0001893(\theta\theta 2) - .003179(QQ1) \\ +\ .01432(QQ2) \quad + .01667(\theta\theta 1)(RR1) + .01667(\theta\theta 2)(RR2) \\ +\ {}^{*}1.27(QQ1)(RR1) \quad - {}^{*}1.27(QQ2)(RR2) \end{array}\right)$$

$$Z_o\,\frac{u}{L}\left(-\ .01667(\theta\theta 1) \quad - .01667(\theta\theta 2) - {}^{*}1.27(QQ1) + {}^{*}1.27(QQ2)\right)$$

$$B\,\frac{L}{u}\left(\begin{array}{l} +\ .0000005353(\theta\theta 1) + .000002218(\theta\theta 2) \quad + .00001315(QQ1) \\ -\ .0001900(QQ2) \quad + .0001769(\theta\theta 1)(SS1) + .0003787(\theta\theta 2)(SS2) \\ +\ .006359(QQ1)(SS1) - .02864(QQ2)(SS2) \end{array}\right)$$

$$\dot{W}_o\left(\begin{array}{l} +\ .00008840(\theta\theta 1) \quad + .0001893(\theta\theta 2) \quad + .003179(QQ1) \\ -\ .01432(QQ2) \quad + .01667(\theta\theta 1)(SS1) + .01667(\theta\theta 2)(SS2) \\ +\ {}^{*}1.27(QQ1)(SS1) \quad - {}^{*}1.27(QQ2)(SS2) \end{array}\right)$$

$$W_o\,\frac{u}{L}\left(+\ .01667(\theta\theta 1) \quad + .01667(\theta\theta 2) \quad + {}^{*}1.27(QQ1) - {}^{*}1.27(QQ2)\right)$$

* Improvement factor 4/Π introduced as in para 5.4a.

A factor 1.05 has been applied to terms involving $\int qt \cos qt\, dt$.

} These remarks apply to other similar pages also.

	Π/2q to 0		Π/q to Π/2q		
$\frac{\delta T_{2c}}{M.c} =$	$-(\theta\theta 1$	;	$\theta\theta 2)$	$\frac{\Pi u^2}{L^2}$	$(At^2 + \dot{Y}_o t + Y_o)\sin qt$
	$+(\theta\theta 1.RR1$	:	$\theta\theta 2.RR2)$	$\frac{\Pi u}{L}$	$(At^2 + \dot{Y}_o t + Y_o)q \cos qt$
	$-(QQ1$	;	$QQ2\ )$	$\frac{u}{L}$	$(2At + \dot{Y}_o)\sin qt$
	$+(QQ1.RR1$	;	$QQ2.RR2)$		$(2At + \dot{Y}_o)q \cos qt$
	$+(\ 1$	;	$1\ )$	$\frac{u^2}{L^2}$	$(Ct^2 + \dot{Z}_o t + Z_o)\sin qt$
	$-(RR1$	;	$RR2\ )$	$\frac{u}{L}$	$(Ct^2 + \dot{Z}_o t + Z_o)q \cos qt$
	$-(RR1$	;	$RR2\ )$	$\frac{u}{L}$	$(2Ct + \dot{Z}_o)\sin qt$
	$+(RR1^2$	;	$RR2^2\ )$		$(2Ct + \dot{Z}_o)q \cos qt$
	$-(\ 1$	;	$1\ )$	$\frac{u^2}{L^2}$	$(Bt^2 + \dot{W}_o t + W_o)\sin qt$
	$+(RR1$	;	$RR2\ )$	$\frac{u}{L}$	$(Bt^2 + \dot{W}_o t + W_o)q \cos qt$
	$-(SS1$	;	$SS2\ )$	$\frac{u}{L}$	$(2Bt + \dot{W}_o)\sin qt$
	$+(RR1.SS1$	;	$RR2;SS2)$		$(2Bt + \dot{W}_o)q \cos qt$

Π/2q to 0   Π/q to Π/2q

$$\frac{1}{M.c}\int_{o}^{L/60u} \delta T_{2c}dt =$$

$$A\frac{L}{u}\left(\begin{array}{lll} -.0000005353(\theta\theta 1) & -.000002218(\theta\theta 2) & +.00004131(\theta\theta 1)(RR1) \\ -.0005969(\theta\theta 2)(RR2) & -.00005628(QQ1) & -.0001205(QQ2 \\ +.006359(QQ1)(RR1) & -.02864(QQ2)(RR2) & \end{array}\right)$$

$$\dot{Y}_o\left(\begin{array}{lll} -.00008840(\theta\theta 1) & -.0001893(\theta\theta 2) & +.009988(\theta\theta 1)(RR1) \\ -.04499(\theta\theta 2)(RR2) & -.0053052(QQ1) & -.0053052(QQ2) \\ +\overset{*}{1.27}(QQ1)(RR1) & -\overset{*}{1.27}(QQ2)(RR2) & \end{array}\right)$$

$$Y_o\frac{u}{L}\left(-.01667(\theta\theta 1) \quad -.01667(\theta\theta 2) \quad +4(\theta\theta 1)(RR1) - 4(\theta\theta 2)(RR2)\right)$$

$$C\frac{L}{u}\left(\begin{array}{ll} +.0000008764 & -.00006944(RR1) + .00006944(RR2) \\ +.006359(RR1)^2 & -.02864(RR2)^2 \end{array}\right)$$

$$\dot{Z}_o\left(\begin{array}{ll} +.00008841 & -.008485(RR1) \quad +.009015(RR2) \\ +\overset{*}{1.27}(RR1)^2 & -\overset{*}{1.27}(RR2)^2 \end{array}\right)$$

$$Z_o\frac{u}{L}\left(+.010610 \quad - \overset{*}{1.27}(RR1) \quad + \overset{*}{1.27}(RR2)\right)$$

$$B\frac{L}{u}\left(\begin{array}{l} -.0000008764 + .00001315(RR1) - .0001900(RR2) \\ -.00005628(SS1) - .0001205(SS2) \\ +.006359(RR1)(SS1) - .02864(RR2)(SS2) \end{array}\right)$$

$$\dot{W}_o\left(\begin{array}{l} -.00008841 + .003179(RR1) - .01432(RR2) - .005305(SS1) \\ -.005305(SS2) + \overset{*}{1.27}(RR1)(SS1) - \overset{*}{1.27}(RR2)(SS2) \end{array}\right)$$

$$W_o\frac{u}{L}\left(-.01061 + \overset{*}{1.27}(RR1) - \overset{*}{1.27}(RR2)\right)$$

$$
\begin{array}{llllll}
 & & \Pi/2q \text{ to } 0 & \Pi/q \text{ to } \Pi/2q & & \\
\dfrac{\delta T_{2b}}{M.b} & = & +(\theta\theta 1 & ;\ \theta\theta 2 &)\ \dfrac{\Pi u^2}{L^2} & (At^2 + \dot{Y}_o t + Y_o)\sin qt \\
 & & +(\theta\theta 1.SS1 & ;\ \theta\theta 2.SS2 &)\ \dfrac{\Pi u}{L} & (At^2 + \dot{Y}_o t + Y_o)q \cos qt \\
 & & +(QQ1 & ;\ QQ2 &)\ \dfrac{u}{L} & (2At + \dot{Y}_o)\sin qt \\
 & & +(QQ1.SS1 & ;\ QQ2.SS2 &) & (2At + \dot{Y}_o)q \cos qt \\
 & & -(\ 1 & ;\ 1 &)\ \dfrac{u^2}{L^2} & (Ct^2 + \dot{Z}_o t + Z_o)\sin qt \\
 & & -(SS1 & ;\ SS2 &)\ \dfrac{u}{L} & (Ct^2 + \dot{Z}_o t + Z_o)q \cos qt \\
 & & +(RR1 & ;\ RR2 &)\ \dfrac{u}{L} & (2Ct + \dot{Z}_o)\sin qt \\
 & & +(RR1.SS1 & ;\ RR2.SS2 &) & (2Ct + \dot{Z}_o)q \cos qt \\
 & & +(\ 1 & ;\ 1 &)\ \dfrac{u^2}{L^2} & (Bt^2 + \dot{W}_o t + W_o)\sin qt \\
 & & +(SS1 & ;\ SS2 &)\ \dfrac{u}{L} & (Bt^2 + \dot{W}_o t + W_o)q \cos qt \\
 & & +(SS1 & ;\ SS2 &)\ \dfrac{u}{L} & (2Bt + \dot{W}_o)\sin qt \\
 & & +(SS1^2 & ;\ SS2^2 &) & (2Bt + \dot{W}_o)q \cos qt \\
 & & \Pi/2q \text{ to } 0 & \Pi/q \text{ to } \Pi/2q & &
\end{array}
$$

$$\frac{1}{M.b}\int_{o}^{L/60u} \delta T_{2b}dt \;=$$

$$A\,\frac{L}{u}\left(\begin{array}{l} +\;.0000005353(\theta\theta 1)\;+\;.000002218(\theta\theta 2)\;+.00004131(\theta\theta 1)(SS1) \\ -\;.0005969(\theta\theta 2)(SS2)\;+\;.0005628(QQ1)\;+\;0001205(QQ2) \\ +\;.006359(QQ1)(SS1)\;-\;.02864(QQ2)(SS2) \end{array}\right)$$

$$\dot{Y}_o\left(\begin{array}{l} +\;.00008841(\theta\theta 1)\;+\;.0001893(\theta\theta 2)\;+\;.009988(\theta\theta 1)(SS1) \\ -\;.04499(\theta\theta 2)(SS2)\;+\;.005305(QQ1)\;+\;.005305(QQ2) \\ +\;\overset{*}{1.27}(QQ1)(SS1)\;-\;\overset{*}{1.27}(QQ2)(SS2) \end{array}\right)$$

$$Y_o\,\frac{u}{L}\left(+\;.01667(\theta\theta 1)\;+.01667(\theta\theta 2)\;+\;\overset{*}{4}(\theta\theta 1)(SS1)\;-\;\overset{*}{4}(\theta\theta 2)(SS2)\right)$$

$$C\,\frac{L}{u}\left(\begin{array}{l} -\;.0000008764\;-\;.00001315(SS1)\;+\;0001900(SS2) \\ +\;.00005628(RR1)\;+\;.0001205(RR2) \\ +\;.006359(RR1)(SS1)\;-\;.02864(RR2)(SS2) \end{array}\right)$$

$$\dot{Z}_o\left(\begin{array}{l} -\;.00008841\;-\;.003179(SS1)\;+\;.01432(SS2) \\ +\;.005305(RR1)\;+\;.005305(RR2)\;+\;\overset{*}{1.27}(RR1)(SS1) \\ -\;\overset{*}{1.27}(RR2)(SS2) \end{array}\right)$$

$$Z_o\,\frac{u}{L}\left(-\;\overset{*}{1.27}(SS1)\;-.010610\;+\;\overset{*}{1.27}(SS2)\right)$$

$$B\,\frac{L}{u}\left(\begin{array}{l} +\;.0000008764\;+\;.00006944(SS1)\;-\;.00006944(SS2) \\ +\;.006359(SS1)^2\;-\;.02864(SS2)^2 \end{array}\right)$$

$$\dot{W}_o\left(\begin{array}{l} +\;.00008841\;+\;.008485(SS1)\;-\;.009015(SS2) \\ +\;\overset{*}{1.27}(SS1)^2\;-\;\overset{*}{1.27}(SS2)^2 \end{array}\right)$$

$$W_o\,\frac{u}{L}\left(+\;.010610\;+\;\overset{*}{1.27}(SS1)\;-\;\overset{*}{1.27}(SS2)\right)$$

5.4c KINETIC ENERGY (T_2) OF A MOVING LOAD ON THE RIGHT HAND CANTILEVER

The load position coordinate x_M is here measured from the right hand end of the cantilever whilst the load still moves from left to right as in paras. 5.4a and 5.4b.

$$x_M = \frac{nL}{60} - ut$$

where n has the successive values 20, 19, 18 ....2, 1, for the twenty stages during which the load traverses this cantilever.

The instantaneous deflection of the right hand cantilever at the position of the moving load, can now be expressed as follows:

$$w_M = (Bt^2 + \dot{W}_o t + W_o + b \sin qt)\left(1 - \cos \frac{3\Pi x_M}{2L}\right)$$

$$\text{i.e. } w_M = (Bt^2 + \dot{W}_o t + W_o + b \sin qt)\left(1 - \cos\left(\frac{n\Pi}{40} - \frac{3\Pi ut}{2L}\right)\right)$$

$$\therefore \dot{w}_M = -\frac{3\Pi u}{2L}(Bt^2 + \dot{W}_o t + W_o + b \sin qt)\sin\left(\frac{n\Pi}{40} - \frac{3\Pi ut}{2L}\right)$$

$$+ (2Bt + \dot{W}_o + qb \cos qt)\left(1 - \cos\left(\frac{n\Pi}{40} - \frac{3\Pi ut}{2L}\right)\right)$$

$$\text{Let, } \sin\left(\frac{n\Pi}{40} - \frac{3\Pi ut}{2L}\right) = \sin \frac{3\Pi x_M}{2L} = \theta\theta Q$$

$$\text{and } 1 - \cos\left(\frac{n\Pi}{40} - \frac{3\Pi ut}{2L}\right) = 1 - \cos \frac{3\Pi x_M}{2L} = QQ\theta$$

$$\text{Now } T_2 = \tfrac{1}{2}M\dot{w}_M^2$$

$$\begin{array}{ll} \Pi/2q \text{ to } 0 & \Pi/q \text{ to } \Pi/2q \end{array}$$

$$\therefore \frac{\delta T_{2b}}{M.b} = (\theta\theta Q1^2 \quad ; \quad \theta\theta Q2^2 \quad) \frac{9\Pi^2u^2}{4L^2} (Bt^2 + \dot{W}_o t + W_o)\sin qt$$

$$+ (QQ01^2 \quad ; \quad QQ02^2 \quad)(2Bt + \dot{W}_o)q \cos qt$$

$$- \frac{3\Pi u}{2L} (\theta\theta Q1.QQ\theta 1;\ \theta\theta Q2.QQ\theta 2)(Bt^2 + \dot{W}_o t + W_o)q \cos qt)$$

$$- \frac{3\Pi u}{2L} (\theta\theta Q1.QQ\theta 1;\ \theta\theta Q2.QQ\theta 2)(2Bt + \dot{W}_o)\sin qt$$

$$\begin{array}{ll} \Pi/2q \text{ to } 0 & \Pi/q \text{ to } \Pi/2q \end{array}$$

$$\therefore \frac{1}{M.b} \int_o^{L/60u} \delta T_{2b} dt =$$

$$B \frac{L}{u} \left(\begin{array}{l} .000003777(\theta\theta Q1)^2 + .00001565(\theta\theta Q2)^2 \\ +.006359(QQ\theta 1)^2 \quad - .02864(QQ\theta 2)^2 \\ -.0003272(\theta\theta Q1)(QQ\theta 1) + .0003272(\theta\theta Q2)(QQ\theta 2) \end{array} \right)$$

$$+\dot{W}_o \left(\begin{array}{l} .0006256(\theta\theta Q1)^2 \quad + \quad .001340(\theta\theta Q2)^2 \\ + \overset{*}{1}.27(QQ\theta 1)^2 - \overset{*}{1}.27(QQ\theta 2)^2 \\ - 0.03998(\theta\theta Q1)(QQ\theta 1) \ +.04248(\theta\theta Q2)(QQ\theta 2) \end{array} \right)$$

$$+W_o \frac{u}{L} \left(\begin{array}{l} .11781(\theta\theta Q1)^2 + .11781(\theta\theta Q2)^2 \\ - \overset{*}{6}(\theta\theta Q1)(QQ\theta 1) \ + \overset{*}{6}(\theta\theta Q2)(QQ\theta 2) \end{array} \right)$$

REFERENCES

1 Mang, Herbert A., Cedolin, L., "Cooling Tower Analysis by a Finite Element Technique Based on a Modified Hamilton's Principle", Meccanica, Vol. 13, No.4, December 1974, pp. 208-224.

2 Cavin, R.K., Dusto, A.R., "Hamilton's Principle: Finite Element Methods and Flexible Body Dynamics", AIAA J., Vol. 15, No. 12, Dec. 1977, pp. 1684-1690.

3 Smith, Donald R., Smith, C.V., "When is Hamilton's Principle an Extremum Principle?", AIAA J., Vo. 12, No. 11, Nov. 1974, pp. 1573-1576.

4 Taya, M., Mura T., "Dynamic Plastic Behaviour of Structures under Impact Loading Investigated by the Extended Hamilton's Principle", Int. J. Solids Struct. Vol. 10, No. 2, Feb. 1974, pp. 197-209.

5 Mucichescu, D., "On the Choice of a Reference State in the Application of Hamilton's Principle in Elastodynamics", Rev. Roum. Sci. Tech., Ser. Mec. Appl., Vol. 17, No.2, 1972, pp. 251-264.

6 Vance, John M., Sitchin, A., "Numerical Solution of Dynamical Systems by Direct Application of Hamilton's Principle", Int. J. Numer. Methods Eng., Vol. 4, No. 2, Mar-April, 1972, pp. 207-216.

7 Vance, John M., Sitchin, A., "Derivation of First-order Difference Equations for Dynamical Systems by Direct Application of Hamilton's Principle", ASME Pap. 70-APM-PP, 1970.

8. E.C. Ting, J. Genin and J.H. Ginsberg, A General Algorithm for Moving Mass problems. Journal of Sound and Vibration, Vol. 33, 1974.

9. J.H. Ginsberg, J. Genin and E.C. Ting, Parametric Study of the Interaction of Bridges and Moving Vehicles, Applied Scientific Research, Vol. 32, 1976.

10. F.V. Filho, Finite Element Analysis of Structures under Moving Load, Shock and Vibration Digest, Vol. 10, No. 8, 1978.

11. T.E. Blejwas, C.C. Feng and R.S. Ayre, Dynamic Interation of Moving Vehicles and Structures, Journal of Sound and Vibration, Vol. 67, 1979.

12. N. Nagaraju, K.S. Jagadish and K.T.S. R. Iyengar, Dynamic Behaviour of Cantilever Bridges under Moving Loads, International Association for Bridge and Structural Engineering, Vol. 33, 1973.

13. J.L. Pahwa, The Free Vibration of Double Cantilever Bridges, Dissertation for M.E. Indian Institute of Sciences, Bangalore.