INTERACTING SOLITONS AND THEIR REPRESENTATION AS A LINEAR SUPERPOSITION

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Untold want, by land and life ne'er granted

Now, voyager, sail thou forth - to seek, and find!

Walt Whitman

DEDICATION

To Dr. Marguerite Balhetchet

for her invaluable help and understanding

and

To my Mother

for her great patience and constant encouragement.

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DECLARATION

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ABSTRACT

The reasons for creating a nonlinear field theory of fundamental particles are advanced.The foundations of electrodynamics are discussed with especial regard to the difficulties which arise with elementary charges.The nonlinear field theory of Born and Infeld is reviewed.

The methods of solving soliton equations are presented.The multisoliton solutions of many nonlinear partial differential equations are summarised.

The attempts to treat solitons as particles in interaction are reviewed including the Bowtell-Stuart analysis of soliton interaction in terms of the singularities of the complex multisoliton solution.

The concepts of nonlinear and linear superposition of solitons are presented.

The author derives,by an original technique the multisoliton solution of the sG (solitons,antisolitons and breathers) using the theorem of permutability.The multisoliton solution is shown explicitly to decompose into a collection of solitons,antisolitons and breathers in the asymptotic limits of time.

The author proposes a new linear superposition principle for the multisoliton solutions of many equations.With this new principle the solitons are identified throughout the multisoliton interaction and the soliton interaction is analysed.The soliton positions (taken to be the projections on the real axis of the singularities of the complex multisoliton solution) are found to be related to the roots of a polynomial of degree N.In addition to providing a means of understanding soliton interaction,the new linear superposition principle of the author leads to remarkable connections between the multisoliton solutions of many equations.It also allows the author to find close global approximations to the multisoliton solutions.

PART ONE

Part 1. Synopsis

Chapter 1.

This is the introduction to the thesis.The need for a nonlinear field theory of elementary particles is discussed.

Chapter 2.

We review the foundations of electrodynamics and examine the fundamental difficulties therein.We review various attempts to place electrodynamics on a new footing and hence remove the previously unsolved difficulties.We have made the chapter as comprehensive as possible even to the extent of examining attempts at creating a field free electrodynamics(ie.the Wheeler-Feynman action-at-a-distance theory).

Chapter 3.

We review the methods developed to solve soliton equations,Bäcklund transformations, and list the N parameter solutions of many of the soliton equations(with which directly or indirectly our original contribution is concerned).

Chapter 4.

In this chapter we attempt to cover as many aspects of research done in the field of solitons in which solitons may be regarded as particles.This includes research done on how sG solitons interact,the Bowtell-Stuart technique,singularities of nonlinear partial differential equations in general, including rational solutions and connections with solvable many-body problems.We also review work done on another linear superposition principle for the KdV equation.We examine some research done on the particle like nature of sG solitons in bounded regions of space.In addition we review the controversial subject of solitons under perturbation, including the so called "non-Newtonian behaviour".This section of the chapter includes a great many references so as to give a picture of the depth of the controversy and of perturbed soliton equations in general.Finally we review a number of attempts at regarding "solitons" as elementary particles.

CHAPTER 1 : INTRODUCTION

Our present model of the universe is one in which the world can described in terms of "particles" and "fields";this division even exists in The General Theory of Relativity, where the fields are replaced by the concept of curved space-time.The field concept has of couse undergone change, due to the development of quantum mechanics,in that the energy of a configuration of field can only change by discrete amounts, and in so doing a "particle of field" is emitted(i.e a photon).However,ultimately,the concept of a universe divided into two essentially different constituents is an invention.

Generally speaking the field concept is more properly defined in Physics in that in the absence of "matter" the field is a mathematical function obeying certain partial differential equations.Such an abstract field can have physical significance when we are able to define such entities as energy,linear momentum and angular momentum.On the other hand the definition of "particles" seems to be on much shakier ground.Ultimately we seem forced to assume that they are mathematical points(provided we are not taking quantum mechanics into account-here we are left with the concept of the particle as a "process").

There is no doubt that the particle concept has in the macroscopic domain been extremely useful.However, when one attempts to understand the meaning of the phrase " a truly elementary particle" one is faced with great difficulty.Truly elementary particles cannot be described in terms of physical principles as they are the entities from which physical priciples are constructed.Thus we can only define truly elementary particles mathematically.Unfortunately no such definition currently exists in theoretical physics.This means that at best all of the established structure of contemporary physics is phenomenological.

The concept of a truly elementary particle as a point runs into severe difficulties classically.In chapter 2 we explore the difficulties of the point charge concept(i.e principally the infinite field energy problem),but we emphasize here that the same difficulties arise in gravitational theory also.Quantum mechanics has added to the difficulties by supplying us with a host of new indefinables,such as virtual particles.These virtual particles are supposed to be the arbiters of the interaction between other particles!

Einstein [1] was one of the main champions of the "particle free" physics and believed that a pure field physics could only be achieved with the introduction of nonlinear field equations.Unfortunately Einstein died before solitons were discovered.It is our contention that given the strides forward that have been made in the solution of nonlinear partial differential equations in recent years we should once again consider the problem of creating a particle free physics.

One of the extraordinary conceptual consequences of a pure field physics is the notion that the world may be an indivisible whole.This notion arises from a property only enjoyed by nonlinear differential equations-that if one finds two solutions of a nonlinear equation then the sum is not a solution,thus there do not exist independent units from which we can build the general solution.

The mechanistic idea of the universe as a piece of clockwork in which there were definable "separate" parts "interacting" with each other has had its day.We contend that the "apparatus determined" behaviour of, say, an electron interference experiment, clearly indicates the new wholeness aspect of the universe.The idea that Einstein's nonlinear field approach may enable us to not only achieve a more satisfactory theory of classical physics,but also a deeper understanding of Quantum theory has already started to be explored [2].

We argue that the currently known soliton equations can be used as test cases for the pure field view of physics.We will see in the course of this thesis that the multisoliton solutions although being exact solutions of the field equations themselves,do appear to *mimic* the appearance of a collection of "separate" particles interacting via certain force laws.We might choose to say that these "separate" solitons "generate" fields which act on other solitons.This idea introduces some important questions.Are the observable interaction fields of physics(eg electromagnetic field) just inferences from the behaviour of accelerating particles?Could there be another more fundamental underlying field with soliton-like solutions which so *choreographs* the motion of the solitons that we are led to believe that the solitons themselves generate interaction fields?In chapter 7 of this thesis we will see that the sine-Gordon equation certainly appears to choreograph the motion of solitons so that we might believe that they generate interaction fields of their own(we are referring to the retarded action of solitons).

The ultimate objective of the present author is to make advances on the road to a particle free physics and in particular to explore the possibility of creating a nonlinear theory of electrodynamics in which electrons are soliton-like excitations of the nonlinear field.It is particularly important in this respect to employ dimensionless quantities(such as the charge to mass ratio of the electron).This is an ambitious aim and many giants of theoretical physics have attempted in various ways to make progress on this subject(chapter 2 and references therein).

Of course one of the most troubling aspects of the pure field nonlinear approach is the existence of the many particles and various fields of force which exist in our universe.Even if one achieved a successful nonlinear theory of electrodynamics,how could one extend the equations to include apparently unrelated forces?One would have to admit that the proposed nonlinear field equations of electrodynamics were but an approximation to a more general set of equations which encompassed other force fields of matter.

The more specific objective of the thesis is to further explore to what extent various multisoliton solutions of soliton equations could be looked at as a collection of interacting solitons and to discover whether there were any unifying principles involved.

The study of the multisoliton solutions of soliton equations as a collection of interacting solitons has been unjustly neglected.Essentially the only researchers to explore this subject in any depth are Bowtell and Stuart(1977,83) [3,4] who investigated the motion of singularities of the complexified two soliton solutions of the sG and KdV equations.This is suprising as Kruskal had suggested the above as a worthy topic of investigation as long ago as 1974 [5].

If one was going to be able to look upon the multisoliton solutions of soliton equations as particles in interaction it seemed expedient to the author to discover whether the multisoliton solution could be written as a *linear superposition* of functions which could be clearly identified as solitons for all time.In this respect the discovery of a paper by Matsuda [6] on how the two parameter solutions of the sG in the centre of velocity frame could be written as an exact linear superposition of accelerating solitons and at the same time the discovery of the Bowtell-Stuart papers was most propitious.Beginning the PhD with these two discoveries immediately determined to some extent the direction in which the thesis had to go.

Clearly one had to determine how one could extend the results of Matsuda to any number of solitons and in any frame of reference.Matsuda's approach did not lend itself to any generalization and seemed totally specific to the centre of velocity frame.At the same time the technique of Bowtell and Stuart for determining the motion of sG solitons via the singularities of the complex Hamiltonian density met with great difficulties in non centre of velocity frames of reference primarily because the Hamiltonian density developed a multiplicity of singularities thus making it more dificult to identify those which we should associate with solitons.

Fortunately the present author discovered that there existed a simple algebraic way of obtaining the Matsuda results(Matsuda's technique involved much integration of a fairly complicated nature).This simple algebraic technique was immediately generalizable to any frame of reference and any number of solitons.It was found that the N parameter solution of the sG(by N parameter we mean a "soliton" solution which is a mixture of any number of solitons,antisolitons and breathers) Φ_N could be written,

$$\Phi_N = \sum_{i=1}^{N} 4\tan^{-1} f_i$$

where f_i were the roots of a certain N^{th} degree polynomial in f.In non centre of velocity frames and for three soliton solutions of the sG and higher, the linear superposition functions f_i were not separable in x(the position variable) and t(the time variable).This meant the elucidation of how the solitons and their associated complex singularities moved was going to be more difficult.One of the achievements of the thesis was the discovery of how the motion of the multisoliton complex singularities was related to the real functions involved in the linear superposition.

Bowtell and Stuart also observed [3] that the two and three soliton solutions of the sG could be built up in a similar way to the formula for the tangent of two or three angles added together and they surmised that the N soliton solution might be similarly constructed.The author took up this conjecture and rigorously proved it to be true and moreover showed how it could also be made to encompass antisolitons and breathers. Lamb [7] and Barnard [8] had developed an iterative technique for building higher parameter solutions of the sG,but they had not used it to determine the N parameter solution.The author developed an original technique for doing this and also demonstrated that it was indeed a multisoliton solution.

Another achievement of the thesis was the extension of the linear superposition principle originally proposed for the sG to many other soliton equations(KdV,MKdV,Boussinesq, Nonlinear Schrödinger,KP and the KdV hierarchy).In so doing it was discovered that the form of the polynomial determining the functions f_i involved in the various linear superpositions for the different equations had the *same* form.Thus a new connection between all these equations had been found.In addition this fact heightened the importance of the linear superposition for soliton equations.In actual fact another linear superposition principle exists for the KdV [9] which is totally different to that proposed by the author.In the course of the thesis we advance a number of reasons why the author's linear superposition is more general and significant.

With the connection between the motion of the singularities of the multisoliton complex solution and the real linear superposition principle we have been able to determine how the sG solitons move for up to five parameters and in so doing have discovered many interesting features of multisoliton interaction.

As a consequence of attempting to find good approximate formulae for how solitons move as a function of time for non centre of velocity cases of the sG, we have discovered close global approximations to multisoliton solutions of various soliton equations.In a very real sense these approximate multisoliton formulae *are* multisolitons in their own right.If we could find partial differential equations for which these new multisoliton solutions were exact solutions we would have made an important discovery.At present there are no soliton equations known whose multisoliton solutions are always close to the multisoliton solutions of other soliton equations.

The thesis is divided into two parts.In part one(chapters 2-4) we review the background into which our own original contribution(part 2) should be set.

CHAPTER 2 : ELECTRODYNAMICS

"What appears certain to me houever is that in the foundation of any consistent field theory ,the particle concept must not appear in addition to the field concept.The whole theory must be based solely on partial differential equations and their singularity free solution."

(A.Einstein ,Journal of the Franklin Institute,vol 221,1936)

§ 0. Introduction

Classical electrodynamics is a very interesting example of a physical theory.It accords with experience exceptionally well in certain areas ,yet it contains difficulties which have never satisfactorily been resolved.In this chapter we will explore some of these difficulties.To some extent the problems of electrodynamics represent an inadequacy in the whole of theoretical physics.

There is at present no satisfactory definition of a truly elementary particle.The theory of special relativity asserts that no signal can exceed the speed of light.Thus an extended body must necessarily deform.In physics the process of deformation of bodies is analysed in terms of an interaction between more primitive parts.Some authors [1] claim that an *elementary* extended body could not be deformable precisely because of the previous remark.The contentious conclusion is then reached that "within the framework of classical theory elementary particles must be treated as points"[1].This conclusion is disputable because,as we shall see later in this thesis,solitons are extended bodies which can deform and are elementary,yet they exist within *classical* field theory.

W.Pauli [2] expressed deep seated convictions about the problems facing electrodynamics,to quote:

"...there is no explanation for the fact that only multiples of a certain charge occur.The existence of an elementary charge has until now in no way been made plausible.It is still an open problem in theoretical physics.The electron itself is a stranger in the Maxwell-Lorentz theory as well as in the present-day quantum theory.

The field particle description presents a conceptual problem:although a field can be described mathematically without the need for any test charges,it cannot be *measured* without them.On the other hand,the test charge itself gives rise to a field.However it is impossible to measure an external field with a test charge and,at the same time,to determine the field due to this charge.A certain duality exists."

In part Born and Infeld's [8] motivation for their theory was precisely to replace the *dualistic view* of fields and particles with a *unitarian* view,in which there is only one physical entity,the electromagnetic field.The particles of matter are considered as singularities of the field.

To some extent an opposite view to Born and Infeld(BI) appears in the action-at-a-distance theory of Wheeler and Feynman [9].In this theory there are *only* charged particles. BI were unaware of the rich mathematics which would come with the discovery of solitons.There is renewed hope that one day a new unitarian field theory(without singularities) of electrodynamics will develop,involving soliton-like objects in an essential way.

In §1 we will review the Maxwell-Lorentz(ML) theory of electrodynamics.In §2 we discuss some of the fundamental difficulties prevalent in ML theory,with particular regard to problems arising from ascribing a point-like nature to the electron.In §3 we will present Dirac's important modification to ML theory [6],in which the infinities associated with radiative retardation were subtracted off in a relativistically invariant manner.Unfortunately the Dirac modified theory(DML) still has problems such as *runaway solutions* and *preacceleration*.We discuss these also in §3.

In §4 we review alternative theories to DML and we also discuss Wheeler-Feynman(WF) theory.In the WF theory a physical interpretation is given to the Dirac procedure,though it involves the properties of the entire universe [7].This is the so called *absorber* theory.

In §5 we concentrate on attempts to solve the relativistic two-body problem.A problem such as this is easily solved in Newtonian theory yet is beset with difficulties in the relativistic case.This problem is important to this thesis as later we shall see that to some extent we are faced with the *inverse* problem with regard to soliton interaction.

§6 is a discussion of attempts to solve the problems of electrodynamics within the context of a nonlinear theory in which Maxwell's equations are no longer true.We will discuss in the main the BI theory, although another theory by Dirac [41] has some interesting properties also.The chapter ends with a short summary.

§ 1. Foundations of classical electrodynamics

Maxwell's equations for the electric field intensity **E** and the magnetic field **B**, where ρ is the charge density, **j** the current density vector, c the speed of light in vacuo are,

$$\nabla . \mathbf{E} = 4\pi\rho \quad , \quad \nabla \wedge \mathbf{E} = -\frac{1}{c}\mathbf{B}_t \tag{2.1}$$

$$\nabla . \mathbf{B} = 0 \quad , \quad \nabla \wedge \mathbf{B} = \frac{1}{c}\mathbf{E}_t + \frac{4\pi}{c}\mathbf{j} \tag{2.2}$$

The force on a charge e moving with velocity **v** in an electromagnetic field is given by the Lorentz force equation,

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \wedge \mathbf{B} \tag{2.3}$$

We may interpret this as a *definition* of the electric and magnetic fields. All of the above equations can be derived from an action principle in four dimensional form in accordance with the fact that Maxwell's equations are Lorentz invariant. The action function S for the whole system of charges and the electromagnetic field is given by [1].

$$S = S_f + S_m + S_{mf} \tag{2.4}$$

where,

$$S_m = -\sum mc\int_a^b ds \tag{2.5}$$

$$S_{mf} = -\sum \frac{e}{c}\int A_k dx^k \tag{2.6}$$

$$S_f = -\frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega \ , \ d\Omega = cdtdxdydz \tag{2.7}$$

S_m is the action associated with free material particles(summation is over the particles). The integral $\int_a^b$ is along the world line of the particle between two particular events. ds, the infinitesimal interval is defined by,

$$ds = cdt(1-v^2/c^2)^{1/2} \tag{2.8}$$

Along with the requirements of Lorentz invariance S_m takes the particular form (2.5) so as to become the Newtonian action $S=\frac{1}{2}\int_a^b mv^2dt$ in the appropriate limit. The Lagrangian associated with (2.5) is given by,

$$L_m = -mc^2(1-v^2/c^2)^{1/2} \tag{2.9}$$

The quantity S_{mf} is the part of the action which depends on the interaction between the charged particles and the field.The four vector A_i is called the four-potential.In the integral A_i is evaluated at points on the world line of the particle.A^0 is the electrostatic potential and the space components of A_i (i=1,2,3) form the vector potential **A**.We write $A^i \equiv (\phi, \mathbf{A})$.We find (2.6) can be written,

$$S_{mf} = -\Sigma\int_{t_1}^{t_2} L_{mf}dt \quad , \quad L_{mf} = e\phi - \frac{e}{c}\,\mathbf{A}.\mathbf{v} \tag{2.10}$$

S_f is the action associated with the electromagnetic field itself and is the total action when no charges are present.The electromagnetic field tensor F^{ik} is defined by,†

$$F_{ik} = A_{k,i} - A_{i,k} \tag{2.11}$$

in terms of electric and magnetic fields,

$$F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.11a}$$

where E_x,B_x are the Cartesian x components of **E** and **B** etc.

In deriving Maxwell's equations from such an action S we firstly assume the fields to be given and consider variations of the trajectory which minimize the action.Secondly we assume the motion of the charges to be given and vary only the fields(via the potentials A_k) to find the minimum action. Strictly speaking both of these procedures are invalid,since varying the trajectory of a charge alters the field itself,and also,varying the potential alters the motion of the charges.Generally in deriving Maxwell's equations from an action principle such as (2.4) we imagine the given fields to

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† $_{,i} \equiv \partial/\partial x^i \equiv \partial_i \quad , \quad ^{,i} \equiv \partial/\partial x_i \equiv \partial^i$

be very large compared with the field of an individual particle.This assumption is justified provided we are not speaking of distances of the order of the classical electron radius(e^2/mc^2).Similarly internal contradictions begin to develop if we allow the fields to be of the order of the field of the electron at the electron radius(m^2c^4/e^3).In practice in most cases these are not serious restrictions,as the classical electron radius is very small($\approx$ 10^{-15}m) and the classical electron field is very large.

Now consider (2.4-7).When we vary the trajectory with a given field,S_f remains fixed in the variation and can be ignored.There is a variation of A_k as well as dx^k as in S_{mf} as A_k is evaluated on the varied trajectory.The Euler-Lagrange equations become in this case,

$$ms\frac{du^i}{ds} = \frac{e}{c} F^{ik}u_k \tag{2.12}$$

The time component of the above equation is

$$\frac{dE_{ke}}{dt} = e\mathbf{E}.\mathbf{v} \tag{2.13}$$

which expresses the fact that the time rate of change of the kinetic energy of the particle is the work done on the particle by the field per unit time.The space components of (2.12) give Lorentz's law(2.3) with

$$\mathbf{E} = -\frac{1}{c}\mathbf{A}_t - \nabla\phi \tag{2.14}$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} \tag{2.15}$$

Equations (2.14-5) give the two Maxwell's equations not involving ρ or $\mathbf{j}$.

Now if we vary the potentials A_k keeping the particle trajectory fixed(and therefore also the four current j^i=(ρ,$\mathbf{j}$)), we find [1] the equations necessary to minimize the action S are,

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i \tag{2.16}$$

The time component(i=0) of the above gives the first of Maxwell's equations(2.1).The space components give the remaining equation (second of (2.2)).

Energy-momentum tensor

Consider the action integral for some system having the form,

$$S = \int \mathcal{L}(q,q_{,i})dVdt = \frac{1}{c}\int \mathcal{L}d\Omega \tag{2.17}$$

where $d\Omega$ is an element of four dimensional volume and the Lagrangian $\mathcal{L}$ is taken to be a function of only the field variables q and $q_{,i}$.The Euler-Lagrange equations which stationarize the integral are

$$\partial_i\left(\frac{\partial\mathcal{L}}{\partial q_{,i}}\right) = \frac{\partial\mathcal{L}}{\partial q} \tag{2.18}$$

It can be shown that a tensor $T_i^{\ k}$ (energy-momentum tensor)

$$T_i^{\ k} = q_{,i}\frac{\partial\mathcal{L}}{\partial q_{,k}} - \delta_i^k\mathcal{L} \tag{2.19}$$

satisfies,

$$\partial_k T_i^{\ k} = 0 \tag{2.20}$$

A vanishing four divergence (above) leads to $\int T^{ik}dS_k$ over a hyperplane containing all of three-dimensional space,to be conserved.The four momentum associated with T^{ik} is given by,

$$P^i = \frac{1}{c}\int T^{ik}dS_k$$

If we carry out the integration over the hyperplane x^0=constant,the above becomes,

$$P^i = \frac{1}{c}\int T^{i0}dV \tag{2.21}$$

The energy density is given by T^{00} and the momentum density is the vector with components $\frac{1}{c}(T^{10},T^{20},T^{30})$.

For the pure electromagnetic field,

$$\mathcal{L} = -\frac{1}{16\pi}F_{kl}F^{kl} \tag{2.22}$$

The energy-momentum tensor associated with this Lagrangian is(after symmetrization-see[1]),

$$T^{ik} = \frac{1}{4\pi}\left(-F^{il}F^k_{\ l} + \frac{1}{4}g^{ik}F_{lm}F^{lm}\right) \tag{2.23}$$

g^{ik} is the metric tensor with signature (1,-1,-1,-1).

Gauge transformations and field invariants

We see from (2.14-5) that for a given $(\phi,\mathbf{A})$,**E** and **B** are unique.However we can change the A_k in such a way that **E** and **B** are unchanged.Such changes to A_k are known as *gauge transformations*.If we change A_k to A'_k via the transformation

$$A'_k = A_k - \partial_k f \quad \text{or} \quad \mathbf{A}' = \mathbf{A} + \nabla f \quad , \quad \phi' = \phi - \frac{1}{c} f_t \tag{2.24}$$

where f is an arbitrary scalar function of coordinates and time,the action S_{mf} (2.6) contains a new term which is the integral of a perfect differential.Thus it does not vary when the trajectory is varied,consequently the field equations are unchanged.The gauge invariance (2.24) means that we may choose a set of potentials $(\phi,\mathbf{A})$ obeying

$$\frac{1}{c}\phi_t + \nabla.\mathbf{A} = 0 \ , \ \partial^\mu A_\mu = 0 \tag{2.25}$$

This is known as the *Lorentz Gauge*.Actually we can even make changes to ϕ and **A** in (2.25) which leave the equation unchanged.We can add ∇f to **A** and subtract $\frac{1}{c}f_t$ from ϕ,provided f satisfies $\nabla^2 f - c^{-2} f_{tt} = 0$.The Lorentz gauge is invariant to Lorentz transformation.Other gauges can be chosen such as the *Coulomb gauge* ,defined by $\nabla.\mathbf{A} = 0$,which is not Lorentz invariant(see also [39] for another interesting gauge).

From the components of the electromagnetic field tensor F^{ik} we can form two invariants(under Lorentz transformation).These are the quantities,

$$F_{ik}F^{ik} \quad , \quad \varepsilon^{iklm}F_{ik}F_{lm} \tag{2.26}$$

ε^{iklm} is known as the *completely antisymmetric unit tensor* of fourth rank,and is defined by

$$\varepsilon^{iklm} = \begin{cases} 1 & \text{iklm even permutation of } 0,1,2,3 \\ -1 & \text{iklm odd permutation of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

From F^{ik} and ε^{iklm} we can form a new tensor $\tilde{F}^{lm}$ known as the *dual* of F^{ik}.$\tilde{F}^{lm} = \frac{1}{2}\varepsilon^{iklm}F_{ik}$.The invariant quantities (2.26) may be expressed in terms of $\tilde{F}^{lm}$ and the electric and magnetic fields in the following way.

$$\frac{1}{2}F_{lm}F^{lm} = B^2 - E^2 \quad , \quad \frac{1}{4}\tilde{F}_{lm}F^{lm} = E.B \tag{2.27}$$

For convenience we note(see 2.11a)

$$F^{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad , \quad \tilde{F}_{ij} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & -E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \tag{2.28}$$

Retarded potentials,fields of moving charges and radiation

If we choose Lorentz gauge (2.25),Maxwell's equations can be written,

$$\frac{\partial^2 A^i}{\partial x_k \partial x^k} = \frac{4\pi}{c} j^i \tag{2.29}$$

or,

$$\Box \mathbf{A} = -\frac{4\pi}{c}\mathbf{j} \quad , \quad \Box\phi = -4\pi\rho \tag{2.30}$$

$$\Box \equiv \partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2-c^{-2}\partial^2/\partial t^2$$

The above equations have solutions,

$$\phi_{ret} = \int \frac{\rho_{t-R/c}}{R}\,dV + (\phi_{ret})_o \quad , \quad \phi_{adv} = \int \frac{\rho_{t+R/c}}{R}\,dV + (\phi_{adv})_o \tag{2.31}$$

$$\mathbf{A}_{ret} = \int \frac{\mathbf{j}_{t-R/c}}{R}\,dV + (\mathbf{A}_{ret})_o \quad , \quad \mathbf{A}_{adv} = \int \frac{\mathbf{j}_{t+R/c}}{R}\,dV + (\mathbf{A}_{adv})_o$$

where R is the distance from the point where the field is evaluated to where the charge or current was(ret) ,or will be(adv), located at time t-R/c(ret) or t+R/c(adv).The times $t\pm R/c$ are known as the advanced or retarded times.Equation (2.31) states that in order to calculate the potentials at the present time ,we must use the charge densities and currents at the retarded or advanced time.Normally the advanced solutions are ruled out because of *causality*.Charges are supposed to *cause* disturbances in the fields which propagate away in the forward time direction.

It can be shown that the potentials for a charge moving along an arbitrary path $r=r_o(t)$ are given by,

$$\phi = \frac{e}{(R - \mathbf{v}.R/c)} \quad , \quad A = \frac{e\mathbf{v}}{c(R - \mathbf{v}.R/c)} \tag{2.32}$$

these are known as the Liénard-Wiechert potentials.R is the position of the field point relative to the charge point, $R = r - r_o(t)$ at the retarded time t',given by $t' = t - R(t')/c$.All quantities in (2.32) are evaluated at this time.v is the velocity of the charge.

If a charge is accelerated,but observed in a reference frame in which its velocity is small compared with light, then it can be shown that the total instantaneous power radiated,W,is given by,

$$W = 2e^2\dot{v}^2/3c^3 \tag{2.33}$$

§ 2. **Problems in Electrodynamics**

The energy of a static point charge is given by,

$$U = (8\pi)^{-1}\int E^2 dV \tag{2.34}$$

where E is the Coulomb field having magnitude

$$E = e/R^2 \tag{2.35}$$

When (2.34) is integrated with (2.35),over all space up to a radius a,we find the total energy U is given by

$$U = e^2/2a \equiv m_e c^2 \quad (m_e = e^2/2ac^2) \tag{2.36}$$

m_e is the electromagnetic mass of the charged particle.Clearly when a→0 U→∞ ,so the *self energy of a point charge is infinite.* When the momentum **P** of the electromagnetic field of a charge moving with small velocity **v** is calculated it is found to be given by [3],

$$\mathbf{P} = (4\mathbf{v}/3).m_e/(1-v^2/c^2)^{1/2} \tag{2.37}$$

In this formula the coefficient of **v** is *wrong* if all the mass of the particle is taken to be electromagnetic(m_e=1, according to special relativity).The discrepancy can only be removed by introducing *non-electromagnetic* forces which hold the charge together.This is consistent with the infinite self energy problem,as without some non-electromagnetic force holding a charge together,a point charge would explode.

When a charge is accelerated it emits radiation,this gives rise to a braking force known as *radiation reaction*.The first attempts to incorporate radiation reaction force in the equations of motion of charged particles in electromagnetic fields was made by Abraham and Lorentz [4].They attempted to construct a theory in which the mechanical momentum of a charged particle in an electromagnetic field was of purely electromagnetic origin.They assumed that a charged particle could be represented by a sharply localized charge density $\rho(x)$ in the particle's rest frame.Assuming there was no flow of momentum out of,or into a volume surrounding the charge,they wrote the conservation of momentum

$$\int(\rho\mathbf{E} + c^{-1}\mathbf{j}\wedge\mathbf{B})dV = 0 \qquad (2.38)$$

in the form of Newton's second law

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_{ext} \qquad (2.39)$$

where,

$$\mathbf{F}_{ext} = \int(\rho\mathbf{E}_{ext} + c^{-1}\mathbf{j}\wedge\mathbf{B}_{ext})dV \qquad (2.40)$$

$$\frac{d\mathbf{p}}{dt} = -\int(\rho\mathbf{E}_s + c^{-1}\mathbf{j}\wedge\mathbf{B}_s)dV \qquad (2.41)$$

The total fields **E** and **B** are the sum of the external fields $\mathbf{E}_{ext}$ and $\mathbf{B}_{ext}$ and the self fields $\mathbf{E}_s$ and $\mathbf{B}_s$ of the charged particle.To evaluate (2.41) they assumed:

A1. The particle is instantaneously at rest

A2. The charge distribution is rigid and spherically symmetric.

The first assumption meant $\mathbf{j} \simeq 0$.They expanded the retarded self fields in powers of the retardation time $\Delta t \simeq a/c$ where a is the dimension of the particle,and found that [2,4]

$$\frac{dp}{dt} = \sum_{n=0}^{\infty} K^{(n)} \tag{2.42}$$

with ,

$$K^{(0)} = -4m_e\dot{v}/3 \tag{2.43}$$

$$K^{(1)} = 2e^2\ddot{v}/3c^3 \tag{2.44}$$

$$n \geq 2 \qquad K^{(n)} \simeq a^{n-1} \tag{2.45}$$

where m_e is defined in (2.36).

If we let $a \to 0$ so that we can neglect higher terms in the expansion we get,

$$\frac{dp}{dt} = -4m_e\dot{v}/3 + 2e^2\ddot{v}/3c^3 \tag{2.46}$$

There were severe difficulties with the Abraham-Lorentz model of a charged particle:

1.The model is nonrelativistic(A2).

2.The electromagnetic mass m_e enters (2.46) with the wrong coefficient.

3.To ignore $K^{(n)}$ $n \geq 2$ we had to say $a \to 0$,but this makes m_e infinite.

4.For a localized charge density not to explode there would have to be powerful non-electromagnetic forces holding it together.

Poincaré,by accomodating 4 above into the theory in a relativistic manner was able to eliminate 2 and cure 1.He proposed that the total stress-energy momentum tensor should be given by

$$S_{ij} = T_{ij} + P_{ij} \tag{2.47}$$

where T_{ij} was given by (2.23) and P_{ij} was a non-electromagnetic stress-energy tensor.The four momentum is a true Lorentz invariant and is given by

$$P^i = c^{-1}\int S^{i0}dV \tag{2.48}$$

in accordance with (2.21).

The Poincaré model showed that it was erroneous to imagine that within the Maxwell-Lorentz theory the self energy could be purely electromagnetic in origin.Only the total self energy or mass m given by,

$$m = c^{-2}\int(T^{oo} + P^{oo})dV \qquad (2.49)$$

has any physical meaning.Of course the origin of the so called Poincaré stresses was completely unknown.

§ 3. Dirac's modification

The Lagrangian for a single particle in a free external field characterized by field tensor $F^{pq}_{in}(x)$,which starts the motion,and F^{pq} the self field of the particle,can be written [5](2.5-10).

$$\mathscr{L} = \mathscr{L}_{free} + \frac{1}{16\pi c}(F^{pq}+F^{pq}_{in})(F_{pq}+F^{in}_{pq}) + \frac{e}{c}\int ds\ \dot{z}_p \rho(x-z)(A^p+A^p_{in}) \qquad (2.50)$$

where $\mathscr{L}_{free} = -\mathscr{L}_m$,given in (2.9) and A^p,A^p_{in} correspond to F^{pq} and F^{pq}_{in} respectively.$\rho(x-z)$ is an invariant charge density.z is the four vector position of the particle on its world line.The dot over z_p refers to differentiation with respect to the interval(proper time).If A^p and A^p_{in} are varied independently,the equation of motion for the particle is

$$mc\ddot{z}_p = \frac{e}{c}\int ds\ \dot{z}^q\ \rho(x-z)[\ F^{ret}_{pq} + F^{in}_{pq}\ ] \qquad (2.51)$$

and the "equation of motion" for the self field is

$$F^{pq}_{ret,q} = -\frac{e}{c}\int ds \rho(x-z)\dot{z}^p \qquad (2.52)$$

The term F^{in}_{pq} in (2.51) represents the external force on the particle.The first term,involving the retarded field F^{ret}_{pq} represents the self force and is *infinite* for a point particle.In the previous section we saw that according to the Abraham-Lorentz model the self force contained an infinite part relating to the infinite self energy of a point particle and a finite part relating to radiative reaction.

Dirac [6] discovered a way of separating in a unique and

Lorentz invariant way the finite part of the self force connected with the radiation field,from the infinite part connected with the infinite self energy.This was accomplished by writing,

$$F^{ret}_{pq} = \frac{1}{2}F^{rad}_{pq} + \frac{1}{2}(F^{ret}_{pq} + F^{adv}_{pq}) \quad (2.53)$$

where the radiation field is defined by

$$F^{rad}_{pq} = F^{ret}_{pq} - F^{adv}_{pq} \quad (2.54)$$

and is calculated from,

$$A^{rad}_{p} = A^{ret}_{p} - A^{adv}_{p} \quad (2.55)$$

We then have(see [5] chap 4 §4)

$$A^{rad}_{p} = c^{-1}\int(D^{ret} - D^{adv})j_{p}(x')dx' = c^{-1}\int D(x-x')j_{p}(x')dx' \quad (2.56)$$

where,

$$D(x-x') = (2\pi)^{-1}\delta[(x-x')^{2}]\mathrm{signum}(x^{0} - x^{0\prime})$$

δ is the Dirac δ function.

It can be shown [5] that the self-force corresponding to $\frac{1}{2}F^{rad}_{pq}$ in (2.53) is finite and gives the following formula for the reaction force

$$K^{rad}_{p\ self} = -(2e^{2}/3c^{3})(\dddot{z}_{p} + \dot{z}_{p}\ddot{z}^{2}) \quad (2.57)$$

where the term dependent on $\dddot{z}_{p}$ corresponds to the reaction force discovered by Abraham and Lorentz(2.46).The term dependent on $\dot{z}_{p}\ddot{z}^{2}$ corresponds to the reaction force due to radiated energy,in accordance with Larmor's formula (2.33).

The infinite part of the self force is due to $\frac{1}{2}(F^{ret}_{pq} + F^{adv}_{pq})$ in (2.53).If we now evaluate the contribution this makes to the F^{ret}_{pq} term in (2.51),we find that the infinite part of the self-force $K^{\infty}_{p\ self}$ is given by

$$K^{\infty}_{p\ self} = \frac{e}{2c}\int ds\ \dot{z}^{q}\ \delta(x-z)(F^{ret}_{pq} + F^{adv}_{pq}) \quad (2.58)$$

Dirac [6] was able to show that K^{∞}_{pself} had the form

$$K^{\infty}_{p\ self} = -\delta m\ c\ \ddot{z}_{p} \quad (2.59)$$

In other words (2.58) reduced to a force equation for a single particle with inertial mass δm(infinite).Equations (2.57) and (2.59) taken together and substituted into (2.51) gave the force equation for a particle acted on by an external force characterized by the field tensor F^{pq}_{in}.

$$(m+\delta m)c\ddot{z}_p = \frac{e}{c} F^{in}_{pq}(z)\dot{z}^q + \frac{2}{3} e^2(\dddot{z}_p + \dot{z}_p \ddot{z}^2) \qquad (2.60)$$

We then put

$$m+\delta m = m_{exp} \qquad (2.61)$$

where m_{exp} is the experimentally measured mass of the elementary charge.This clever procedure enabled useful equations(i.e 2.60),involving no infinities to be obtained.In effect since δm is infinite we are taking the *bare mass* m to be negatively infinite.This essentially phenomenological procedure is known as *mass renormalization*.Such mass renormalizations are prevalent in Quantum Field Theory.It can be seen that such a procedure does not offer insight into the structure of a truly elementary charged body.

<u>Solutions of Lorentz-Dirac equation (2.60) in absence of external fields.</u>

In the absence of an external field (2.60) becomes

$$\ddot{z}_p = \tau(\dddot{z}_p + \dot{z}_p \ddot{z}^2) \qquad (2.62)$$

$$\tau = 2e^2/3mc^2$$

Clearly the "physical solution" $\ddot{z}_p=0$ satisfies (2.62). Unfortunately the equation has other "unphysical solutions".If we choose a frame of reference such that at s=0,$\dot{z}(0)$=(1,0,0,0), $\ddot{z}(0)$=(0,C,0,0),C being an arbitrary constant(acceleration in the rest frame at s=0,along x axis),we find the solution of (2.62) is [5,p196-7].

$$\dot{z}_0 = \cosh[\tau(e^{s/\tau}-1)] \;,\; \dot{z}_1 = \sinh[\tau(e^{s/\tau}-1)], \quad \dot{z}_2,\dot{z}_3 = 0 \qquad (2.64)$$

This solution is unphysical since the velocity ($v=\dot{z}_1 c/\dot{z}_0$) eventually equals light speed.This is an example of a *runaway* solution.Even in the low velocity limit, when the Larmor term

$(\dot{z}_p \ddot{z}^2)$ can be ignored (2.62) has runaway solutions [1,p207].

Eliezer [10] examined many situations in which the Lorentz-Dirac equation gave unphysical solutions and even claimed that in some situations there were *only* unphysical solutions.Plass [11] however refuted this and showed that a physical solution could *always* be found,though it is unsatisfactory that we should have to *choose* a solution.Even in these so called physical solutions a phenomenon known as *preacceleration* occurred.This had first been observed by Dirac [6] in the case of a charged particle which is disturbed by a momentary pulse of radiation.Here we give the treatment of this problem based on that of Plass [11].

We assume that the particle's velocity is small compared with light,and is moving along a single axis of the coordinate system.In this case (2.62) becomes with an external force f(t)

$$\ddot{z} = \tau \dddot{z} + f(t) \tag{2.65}$$

This has the exact solution

$$\ddot{z} = \tau^{-1}\int_0^\infty e^{-t'/\tau} f(t+t')dt' \tag{2.66}$$

If we let f(t) be a pulse

$$f(t) = k\delta(t-t_o) \tag{2.67}$$

we find from (2.66)

$$\ddot{z} = \begin{cases} (k/\tau)\exp[-(t_o-t)/\tau] , & t<t_o \\ 0 , & t>t_o \end{cases} \tag{2.68}$$

Clearly the particle has started to accelerate *before* the pulse arrives.After the pulse arrival the acceleration is zero,but the particle has increased its velocity.Plass rules out as unphysical other solutions to (2.65) in which once again the speed of the particle approaches that of light as t→∞ [6,10].

§ 4. Other theories of electrodynamics.Wheeler-Feynman theory.

Stabler [12] sought to remove the infinities associated with electron self energies and the *microacausality* inherent in the Lorentz-Dirac equation,by proposing a theory in which the action of a charge on itself is removed.This was accomplished by considering each discrete charged particle p to be a source of a retarded electromagnetic field $F^{(p)}_{ij}$.An energy-momentum stress tensor is then constructed having the form

$$(T_{ij})^{em} = (4\pi)^{-1}\sum_{p,q}{}' (F^{(p)}_{il}F^{(q)}_{jl} - \frac{1}{4} F^{(p)}_{lm}F^{(q)}_{lm}\delta_{ij}) \qquad (2.69)$$

where $\Sigma'_{p,q}$ indicates that p=q is excluded from the summation.The retarded electromagnetic field tensor $F^{(p)}_{ij}$ satisfies Maxwell's equations (2.16) and also $F_{ik,l} + F_{kl,i} + F_{li,k} = 0$,(which can also be deduced from (2.11).

In Stabler's theory the source of radiated energy is not the electromagnetic self energies of the particles(action of electron on itself),but is due to the net changes in the potential and kinetic energies of the particles in interaction.Particles only radiate in the field of another particle.According to Stabler's theory,single isolated,and accelerated particles would not radiate.The term *isolated* though not defined in detail appears to mean isolated electrically.Thus according to Stabler,a charged particle falling in a gravitational field would *not* radiate.

Cornish [13] presents a more general theory than Stabler's.He assumes that Maxwell's equations hold for point charges,but the equations of motion of the charges are regarded as being dependent on the particular energy-momentum tensor chosen.This method leads to a certain class of theories of which that of Lorentz and Dirac is just one.Stabler's theory also becomes one of the possible theories,which Cornish refers to as the *interaction theory*.Cornish shows that only when the number of charges is small do the Lorentz-Dirac and interaction theories differ significantly.

It is interesting to observe that the various arguments used to derive Larmor's theorem([4],chapter 14) merely assume we have an accelerating charge.*How* it is caused to undergo acceleration is not stipulated.Thus certainly as a consequence of conventional electrodynamics a charge should radiate as a consequence of acceleration in a gravitational field.

Wheeler and Feynman [7] in their paper entitled "Interaction with the Absorber as the Mechanism for Radiation" adopt the following postulates.

1.An accelerated charge in otherwise charge-free space does not radiate energy.

2.The fields which act on a given particle arise only from other particles.

3.These fields are represented by one-half the retarded plus one half the advanced Liénard-Wiechert solutions of Maxwell's equations.

The Wheeler-Feynman(WF) theory of electrodynamics [7,9] was based on a mathematical formulation of electrodynamics in which no direct use is made of the notion of field.This action-at-a-distance concept owed its origin to the papers of Schwartzchild [14],Tetrode [15] and Fokker [16].

In WF theory the action of a system of charged particles in interaction is given by,

$$S = -\sum_a m_a c\int ds^{(a)} + \frac{1}{c}\sum_{a<b} e_a e_b \int\int \delta[(x^{(a)}_m - x^{(b)}_m)(x^m_{(a)} - x^m_{(b)})]dx^{(a)}_n dx^n_{(b)} \tag{2.70}$$

where the differential interval for the particle a,$ds^{(a)}$ is also written,

$$ds^{(a)} = \sqrt{dx^{(a)}_i dx^i_{(a)}} \tag{2.71}$$

and $\delta(x)$ is the Dirac δ function ($\delta(x)=0, x\neq 0. \int_{-\infty}^{\infty}\delta(x)dx=1$).

Let the world line of a typical particle a be altered from $x^m_{(a)}(s^{(a)})$ to $x^m_{(a)}(s^{(a)})+\delta x^m_{(a)}(s^{(a)})$.We *define*(vector potential of particle b at point x),

$$A^{(b)}_n(x) = e_b\int \delta[(x_m - x^{(b)}_m)(x^m - x^m_{(b)})]dx^{(b)}_n(s^{(b)}) \quad (2.72)$$

Then the variation in S as we vary the world line of a typical particle is,

$$\delta S=-\sum_a m_a c\int \frac{dx^{(a)}_i}{ds^{(a)}}d(\delta x^i_{(a)}) + \sum_{b\neq a} \frac{e_a}{c} \int A^{(b)}_n(x^{(a)})d(\delta x^n_{(a)})+\delta A^{(b)}_n(x^{(a)})dx^n_{(a)}$$

Set $u^{(a)}_i \equiv dx^{(a)}_i/ds^{(a)}$ and integrate by parts all terms under the integral sign except the last[1,p60].Now noting that the integrated terms vanish at the limits of the variation of the world line,since $\delta x^i_{(a)}$ is zero,we obtain.

$$\delta S = \int \left[\sum_a m_a c \frac{du^{(a)}_i}{ds} + \sum_{b\neq a} \frac{e_a}{c} \left(\partial A^{(b)}_n/\partial x^i - \partial A^{(b)}_i/\partial x^n\right)u^n_{(a)}\right]\delta x^i_{(a)}ds$$

In view of the arbitrariness of $\delta x^i_{(a)}$ it follows that the integrand must be zero.Thus,

$$m_a c \frac{du^{(a)}_i}{ds} = e_a/c \sum_{b\neq a} F^{(b)}_{ni}(x^{(a)})u^n_{(a)} \quad (2.73)$$

with,

$$F^{(b)}_{ni}(x) \equiv \partial A^{(b)}_i/\partial x^n - \partial A^{(b)}_n/\partial x^i \quad (2.74)$$

Equations (2.73-4) are identical to Lorentz's equations(2.12) except that self actions are explicitly excluded.It can be shown from (2.72) and (2.74) that Maxwell's equations are obtained [9].The vector potential defined in (2.72) can be written,

$$A^{(b)}_n = [(A^{(b)}_n)_{ret} + (A^{(b)}_n)_{adv}]/2 \quad (2.75)$$

where $(A^{(b)}_n)_{()}$ is the retarded or advanced Liénard-Wiechert potential of particle b,depending on the subscript.From these equations can be derived all the familiar properties of electromagnetism in areas where we do not consider the self action of charges.

WF account for radiative retardation by invoking a supposed property of the *entire* universe:*perfect absorbtion*.This property means that when a charge is excited,all electromagnetic disturbances arising from it should tend to zero sufficiently rapidly at great distances from it.They express this using,$F_{ret}^{(k)}$ and $F_{adv}^{(k)}$,which are the retarded and advanced fields due to the k^{th} particle.They assume that an absorber surrounds the charge such that *outside the absorber*,

$$\sum_k (F_{ret}^{(k)} + F_{adv}^{(k)}) = 0 \tag{2.76}$$

WF then go on to show that as a consequence of this,

$$\sum_k (F_{ret}^{(k)} - F_{adv}^{(k)}) = 0 \quad , \quad everywhere \tag{2.77}$$

From this the entire field acting on the a^{th} charge is given, according to the action-at-a-distance theory outlined previously,by

$$\sum_{k \neq a} (F_{ret}^{(k)} + F_{adv}^{(k)})/2 \tag{2.78}$$

which can be written,

$$\sum_{k \neq a} F_{ret}^{(k)} + (F_{ret}^{(a)} - F_{adv}^{(a)})/2 - \sum_{all\ k} (F_{ret}^{(k)} - F_{adv}^{(k)})/2 \tag{2.79}$$

On account of (2.77) the third term vanishes.The second term was suggested by Dirac(2.54) and provides the radiative damping force.In the Abraham-Lorentz theory the radiative reaction force arose as a consequence of a charge acting on itself.

In the WF theory the radiation reaction force becomes a property of the entire universe.There are still problems in the WF theory connected with its time symmetry.In our experience,radiation is an irreversible phenomenon.This aspect is discussed further in [7].Naturally the WF theory is sensitive to the *structure* of the whole universe,further implications of this are explored in [17].

In concluding this section on linear electrodynamics,we point out that due to the difficulties of runaway solutions and preacceleration in the Lorentz-Dirac theory(with which the WF theory agrees),other modifications in electrodynamics have been proposed,aimed at removing the difficulties [18,19].A survey of different ways of deriving the Lorentz-Dirac equation which also discusses its limited applicability is given in [20].In this paper the author shows that the Lorentz-Dirac only applies when the acceleration which the charge undergoes is very small.This is due to the *limited* applicability of the Liénard-Wiechert formulae.

Another paper [21] compares the Lorentz-Dirac and WF theories with the little known formulation of electrodynamics due to Synge [22].The author shows that in most experimental situations the Synge theory(which considers only retarded interactions between particles and no self interactions) agrees closely with the other two.The agreement occurs because in the Synge theory *two or more* particles moving with similar velocities and accelerations *do* radiate when subjected to an accelerating field.It turns out that the Synge theory predicts the same radiation as the Lorentz-Dirac theory.At present experimental tests of the Lorentz-Dirac theory are inconclusive [23].

§ 5. Relativistic two-body problem

There is a very important problem which has never been solved either in electrodynamics or in relativistic field-particle interaction in general.This is the *two-body problem*,which is how to determine the trajectories of two particles interacting via relativistic fields (non-instantaneous).

The artificial case of two charges approaching each other along a line,such that one charge responds to the *retarded* L-W field and the other to the *advanced* L-W field,has been solved [24].

Further recent work on the relativistic two body problem has been carried out in [25].These authors consider the case of two particles of equal mass and charge, interacting in the centre of velocity frame,via half advanced-half retarded fields(as in WF theory).The results indicate a non-zero minimum distance of closest approach(even for initial speeds approaching that of light).A graph of acceleration versus time(centre-of-velocity) is *double-peaked* with a local *minimum* at t=0,when the particles are closest.For non-relativistic speeds the graph has a *single maximum* at t=0(as expected).The double peak is interpreted in the following way.As the particles approach,sometime before the point of minimum separation,their mutual advanced fields are a maximum(as if they were at the point of minimum separation).Thus the acceleration peaks *early*.Time symmetry gives a similar picture for the retarded interaction,where the acceleration peaks *late*.

As we shall see later in the thesis such behaviour is not reproduced with interacting sine-Gordon solitons.In this frame they *appear* to be interacting simultaneously.For an interesting discussion of instantaneous action-at-a-distance in relativistic mechanics see [28,29].

Calculations on a similar problem with particles interacting via retarded L-W potentials and no radiation reaction have also been carried out [26,27].The results indicate:

1.Charges do not have a minimum distance of closest approach as a function of initial velocity.

2.The maximum acceleration occurs *after* the charges have collided,and are on their way back.

Related to this latter point [27] is the fact that in this case,the speeds of the particles long after the collision were substantially *increased* on the speeds of the particles long before the collision.Thus, energy was not conserved.

Further references on the two-body problem may be found in [30].

§ 6 Nonlinear electrodynamics.Born-Infeld and others.

A theory of electrodynamics in which the primary physical entity is field and in which we have the possibility of particle-like configurations of this field,was proposed by Born and Infeld(BI).Some other early theories of electrodynamics are discussed in Pauli's book on relativity [31].

The basic idea of the BI theory was to change Maxwell's field equations in such a way that the total energy of a singularity in the field becomes finite.It is interesting that BI appeared to think that arbitrarily introduced charges would need to exist in their theory.Their primary motivation being to remove the infinite self energy of the point charge.However a much more interesting interpretation of the BI theory is that there is no need to introduce charges as the nonlinear field equations in "free space" support charge-like solutions.

In the BI theory the field tensor F_{mn} is still defined as in Maxwell's theory

$$F_{mn} = A_{n,m} - A_{m,n} \tag{2.80}$$

However the Lagrangian for the Maxwell field (2.22 with different units),

$$\mathcal{L}_M = -\frac{1}{4} F_{mn} F^{mn} \tag{2.81}$$

is replaced by,

$$\mathscr{L}_{BI} = b^2\left\{1 - \left[1+ (2b^2)^{-1}F_{mn}F^{mn} - (4b^2)^{-2}(\tilde{F}_{mn}F^{mn})^2\right]^{1/2}\right\} \qquad (2.82)$$

Note that $\mathscr{L}_{BI}$ preserves the relativistic invariance,by being built out of field invariants(2.27).Of course such a Lagrangian is not unique and Born originally proposed,

$$\mathscr{L} = b^2\left\{1 - [1+(2b^2)^{-1}F_{mn}F^{mn}]^{1/2}\right\}$$

b is some absolute unit of field strength(e.g the field at the surface of a classical electron).Clearly if $F_{mn}/b \ll 1$ $\mathscr{L}_{BI} \rightarrow \mathscr{L}_M$. The BI field equations are,firstly

$$\nabla \wedge \mathbf{E} = -c^{-1}\mathbf{B}_t \qquad (2.83)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (2.84)$$

which follow from (2.80) with the definition of the field tensor(2.11a).The other pair of field equations are found from the Euler-Lagrange equations derived from stationarizing $S=\int\mathscr{L}_{BI}d\Omega$, where the variation is carried out on the four potentials.These are ,

$$\partial_m\left(\frac{\partial\mathscr{L}}{\partial A_{l,m}}\right) = 0 \qquad (2.85)$$

which for $\mathscr{L}=\mathscr{L}_M$ give $\partial_m F^{ml} = 0$ (2.16,with $j^l = 0$).For the BI Larangian $\mathscr{L}_{BI}$ we have,†

$$\partial_m G^{ml} = 0 \qquad (2.86a)$$

where,

$$G^{ml} = \frac{F^{ml} - (2b^2)^{-1}(\tilde{F}_{ij}F^{ij})\tilde{F}^{ml}}{\left[1 + (2b^2)^{-1}F_{ij}F^{ij} - (4b^2)^{-2}(\tilde{F}_{ij}F^{ij})^2\right]^{1/2}} \qquad (2.86b)$$

We will find it useful to define scalar fields χ and ψ,

$$\chi(\mathbf{E},\mathbf{B}) \equiv \left[1 + (2b^2)^{-1}F_{ij}F^{ij} - (4b^2)^{-2}(\tilde{F}_{ij}F^{ij})^2\right]^{-1/2}$$

$$= \left[1 + \frac{B^2 - E^2}{b^2} - \frac{(\mathbf{E}.\mathbf{B})^2}{b^4}\right]^{-1/2}$$

$$\psi(\mathbf{E},\mathbf{B}) \equiv (2b^2)^{-1}\tilde{F}_{ij}F^{ij} = \frac{2(\mathbf{E}.\mathbf{B})}{b^2}$$

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† $$\frac{\partial F_{kl}F^{kl}}{\partial A_{l,k}} = 4F^{kl} \quad , \quad \frac{\partial \tilde{F}_{kl}F^{kl}}{\partial A_{l,k}} = 8\tilde{F}^{kl}$$

(2.86) then becomes in three dimensional form,

$$\nabla .\chi(E - \psi B) = 0 \tag{2.87}$$

$$\nabla \wedge \chi(B + \psi E) = c^{-1}\frac{\partial}{\partial t}\chi(E - \psi B) \tag{2.88}$$

These equations should be contrasted with Maxwell's equations derived from $\mathscr{L}_M$ in free space.

$$\nabla .E = 0 \quad , \quad \nabla \wedge B = c^{-1}E_t$$

BI discovered a time independent solution to the BI equations (2.83-4) and (2.87-8),in the case of radial symmetry.They assumed B = 0,from which it follows,

$$\nabla \wedge E = 0 \tag{2.89a}$$

$$E \neq E(t) \tag{2.89b}$$

$$\nabla .\chi E = 0 \tag{2.89c}$$

(2.89b) follows from (2.88) as $\psi(E,0)= 0$ and $\chi E = f(E)$,from the the definition of χ.(2.89c) follows from (2.87).For radial E (2.89a) is also satisfied(see B.Hague-Introduction to Vector Analysis).(2.89c) can be written

$$r^{-2}\frac{d}{dr}\left\{\frac{r^2 E}{(1 - E^2/b^2)^{1/2}}\right\} = 0 \tag{2.90}$$

which reduces to,

$$\frac{dE}{dr} + \frac{2E}{r}(1 - E^2/b^2) = 0 \tag{2.91}$$

This has solution

$$E = \pm b\,[1+(r/r_0)^4\,]^{-1/2} \tag{2.92}$$

where r_0 is a constant of integration.Since we require E to become Coulomb-like when r is large we find it necessary for r_0 to be given by,

$$r_0 = (e/b)^{1/2} \tag{2.93}$$

and hence we may write,

$$E = \pm e\, r_0^{-2}\,[1+(r/r_0)^4\,]^{-1/2} \tag{2.94}$$

This is the famous Born-Infeld static solution.As desired the solution is such that the electric field is finite for all r.

There appears to have arisen some controversy about solution (2.94) recently.S.Deser [32] erroneously claims that the *only* solution of the BI field equations(2.83-4,2.87-8) in the static case is the zero solution **B**=**E**=0.This is manifestly untrue for we have by straightforward argument a non-zero solution in the static case.Murphy [33] taking up from Deser in a paper romantically entitled "Requiem for the Born-Infeld electron", thought he had found the exact source of Born and Infeld's error.In the original BI paper BI indulged in the curious procedure of involving vector fields **D** and **H**,which normally enter Maxwell's equations in material media.A key step in BI's argument was $\nabla.\mathbf{D}=0$ which BI wrote for radial symmetry as $d(r^2D)/dr = 0$,hence they concluded $D = e/r^2$.This seemed to be necessary for BI to obtain the static solution (2.94).Murphy pointed out correctly that BI should have written $r^{-2}d(r^2D)/dr=0$.He then points out that $\mathbf{D}=e\mathbf{r}/r^3$ is *not* a solution of $\nabla.\mathbf{D}=0$, but instead is a solution of $\nabla.\mathbf{D}=4\pi e\delta(\mathbf{r})$, where $e\delta(\mathbf{r})$ is an external charge density.This mistake on BI's part led Murphy to believe (2.94) was incorrect.We have seen here however,that (2.94) is correct without involving **D**.

We now show that the energy associated with the static solution (2.94) is finite.We saw in equation (2.19) how to find the energy-momentum stress tensor associated with a given Lagrangian.In this case the "coordinates" are the vector potentials.Thus,

$$T_m{}^k = A_{l,m}\frac{\partial\mathcal{L}_{BI}}{\partial A_{l,k}} - \delta^k_m\,\mathcal{L}_{BI} \qquad (2.95)$$

where $\mathcal{L}_{BI}$ is defined in (2.82).Hence

$$T_m{}^k = -A_{l,m}G^{kl} - \delta^k_m\,\mathcal{L}_{BI} \qquad (2.96)$$

G^{kl} is defined in (2.86b).Multipying by g^{im} and rearranging on indices,we obtain,

$$T^{ik} = -\frac{\partial A^{l}}{\partial x_{i}} G^{k}{}_{l} - g^{ik}\mathscr{L}_{BI} \tag{2.97}$$

The above expression is not symmetric as an energy-momentum stress tensor should be.To symmetrize [1,p81] we add to T^{ik},

$$\frac{\partial A^{i}}{\partial x_{l}} G^{k}{}_{l} = \frac{\partial}{\partial x_{l}}(A^{i}G^{kl})$$

where we have used (2.86a), ∴

$$T^{ik} = \left(\frac{\partial A^{i}}{\partial x_{l}} - \frac{\partial A^{l}}{\partial x_{i}}\right) G^{k}{}_{l} - g^{ik}\mathscr{L}_{BI}$$

$$T^{ik} = -F^{il}G^{k}{}_{l} - g^{ik}\mathscr{L}_{BI} \tag{2.98}$$

The above expression is the symmetric expression we were seeking.The energy density is given by T^{oo}.Noting the definitions of χ and ψ and employing (2.86b) we find

$$T^{oo} = -\chi(F^{ol}F^{o}{}_{l} - \psi F^{ol}\tilde{F}^{o}{}_{l}) - \mathscr{L}_{BI}$$

bringing out g^{io} in the bracketed expression gives,

$$T^{oo} = \chi(\psi F^{ol}\tilde{F}_{ol} - F^{ol}F_{ol}) - \mathscr{L}_{BI} \tag{2.99}$$

Thus for the static solution,

$$T^{oo} = E^{2}(1 - E^{2}/b^{2})^{-1/2} - b^{2}[1 - (1-E^{2}/b^{2})^{1/2}] \tag{2.100}$$

This is the expression obtained by BI.When T^{oo} is integrated over all space we find,

$$T^{oo} = 0.098e^{2}/r_{o} = m_{e}c^{2}$$

where m_{e} is the mass of the electron.Thus the "radius" of the electron r_{o} becomes ,

$$r_{o} = 1.81\text{x}10^{-16}\text{metres}$$

Also the field constant b has an enormous magnitude thus justifying the use of Maxwell's equations except when distances of the order r_{o} are encountered.

Curiously BI also showed that the motion of just such an elementary charge in an external field satisfied an equation which was a generalization of the Lorentz force equation(2.3).Dirac also much later [34] gave a reformulation of the BI electrodynamics in which the action for a BI charge was described in a very similar way to conventional electrodynamics,but using the BI Lagrangian.

In retrospect,and in the light of recent soliton research,it is apparent that both Born and Infeld and Dirac did not fully appreciate the full revolutionary conceptual nature of the theory.The BI equations are nonlinear and therfore offer the hope of multicharge solutions.A two-charge solution would be one for which the total field as time $\rightarrow\pm\infty$,looked like a linear superposition of two single charge fields(2.94).In the nonrelativistic limit,at large separations we would expect the accelerations of the BI charges to be consistent with the mutual BI,Lorentz-like force field(i.e the acceleration of the charge would be proportional to the field at that point).

The BI equations should be looked on as describing a continuous underlying field.If we examine the energy density of the field,we should find certain small regions where the energy density is very great.These regions of high energy density would be the "charges".The particle-like nature of the BI charges would be a derived concept.Instead we would have a nonlinear BI field evolving in time from the initial state in such a way as to *mimic* the effect of two or more charges interacting with one another.Such concepts as "force fields" would be illusory.

With the recent developments in the theory of solitons and solitary waves the interpretation of the BI theory given above is gaining ground [35,36].The main dificulty is in solving the BI field equations,except in very special cases.There now appear to be connections between the so called "BI scalar field" and a nonlinear Liouville equation [37].

(Unfortunately these authors believe that the classical

Liouville equation($u_{tt}-u_{xx}=e^{u}$) has N-soliton solutions [38].However this has recently been shown to be incorrect [39].)

Other choices of Lagrangian than that of Born and Infeld may still lead to the Maxwell-Lorentz theory as an approximation [40].It may be that some other Lagrangian will lead to more tractable equations.

A different nonlinear theory of electrodynamics has been proposed by Dirac [41].He criticizes the adoption of the Lorentz gauge(2.25) in the Maxwell-Lorentz theory,as it leads to difficulties in the transition from a Lagrangian form of electrodynamics to a Hamiltonian form.The latter being essential in Dirac's view for making the transition to a quantized form of the theory.Dirac's attitude to the existence of gauge transformations is explained in his own words:
"they indicate that there are more variables present in the mathematics than are physically necessary."

Dirac's idea was to use the "superflous" variables in the theory *without* charges to describe the charges themselves.Gauge transformations in the new theory become *forbidden*.He considers destroying the gauge transformations by imposing the condition,

$$A_m A^m = k^2$$

where k is a universal constant.The new Lagrangian which takes this into account is then,

$$\mathscr{L} = -\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}\lambda(A_m A^m - k^2)$$

Dirac then obtained a theory in which the structure of elementary charges was not important,as in his view,"such detailed description is not needed when quantum phenomena are not being considered...".

More recently other workers [42] have discovered solutions of the Dirac equations which are static and spherically symmetric and may be regarded as charged particles.They also found that these solutions(one for each sign of charge) were connected with different vacua,and that a conserved topological current could be defined whose associated conserved charge was proportional to the charged particle number.

§ 7. Summary

We have presented in this chapter an outline of the Maxwell-Lorentz theory of electrodynamics.Ultimately we have seen that the theory becomes beset by difficulties in association with the description of an elementary charge.

We have reviewed various attempts to rid electrodynamics of these difficulties,including the interesting,but extaordinary theory of Wheeler and Feynman.In this theory the fields of interaction are *removed* and replaced by an action-at-a-distance theory which is sensitive to the properties of the entire universe.It is interesting to note that all these theories(§3,4) do not shed any light on the structure or meaning of an elementary charge.

Two theories which were at least capable of defining an elementary charge were those of Born and Infeld,and Dirac.Both these theories involved nonlinear partial differential equations.A fact which we should find,not entirely unexpected,since an multicharge solution can not be regarded as a true linear superposition of many single charge solutions.This is simply because charges interact with each other.Nonlinear equations have the potential for explaining this basic(but hitherto unexplained) property of the world.

In a nonlinear theory "particles" would become mere mental constructs.The essential reality being the underlying nonlinear field.The world would become a unity,incapable of division into seperate parts(except as an approximation,when "particles" were far apart).

Of course such a radical shift in the conceptual foundations of theoretical physics would also extend into the quantum domain,and perhaps supplant it with a new set of more fundamental,less phenomenological concepts.It is interesting to not that Dirac felt the search for a complete classical electrodynamics to be very important,since he says: "...the troubles of present quantum electrodynamics should be ascribed in my opinion,not to a fault in the general principles of quantization,but to our working from a wrong classical theory."[41,the first paper].

The author's view in this thesis is that,in a nonlinear theory, terms such as "particle","field of interaction" are only of secondary importance.However,they are central to both conventional classical theory and quantum theory.A nonlinear theory of "particles" may indeed cause a radical shift in the *whole* of theoretical physics,both classical *and* quantum.

CHAPTER 3 : SOLITON EQUATIONS

§ 0. Introduction

In the last chapter we saw that a nonlinear partial differential equation (nlpde) offered us the prospect of a continuous underlying field obeying the nlpde,forming a more fundamental description of particles in interaction.That prospect started to become much more of a reality when Zabusky and Kruskal [1] observed numerically the interaction of pulse-like entities having a $sech^2$ form.They had been investigating the properties of the Korteweg-de Vries(KdV) equation,

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (3.1)$$

Zabusky and Kruskal had found that an initial cosine profile decomposed into a series of pulses($sech^2$) travelling to the right,which appeared,then disappeared,and finally reappeared as the motion continued.The pulses were such that the larger ones moved with greater speed.The pulses were given the name "solitons" to indicate their solitary wave/particle properties.

When the interaction of solitons was investigated further,the following picture emerged.Let us suppose that long before the interaction,the larger soliton was to the left of the smaller.It was found that,long after the collision,the faster soliton would be found to the *right* of the smaller.Thus it appeared that the larger soliton had passed through the smaller soliton.In addition to this it was found that,the exact initial speeds possessed by the solitons were recovered.The only trace that there had been an interaction at all,was in the phase shifts of the solitons.That is to say they occupied different positions from those they would have occupied had they travelled at constant speed throughout the collision.

Looking at the interaction in more detail Zabusky and Kruskal found that when the leftmost(initially) soliton was much larger(and faster) than the other soliton,it appeared to swallow up the smaller one,re-emitting it later.However,when the solitons were of comparable size and speed,the interaction took a different form.As they approached one another the larger soliton started to decrease in amplitude and speed,while at the same time the smaller soliton increased both its amplitude and speed.In this case the solitons appeared to have exchanged their identities.

Lax [2] provided a more rigorous analysis of the evolution of the two soliton profile and discovered a third mode of interaction.In this intermediate case,the larger soliton emitted a separate pulse which was absorbed by the smaller soliton.As a consequence of this,the amplitude of the larger soliton diminished and the amplitude of the smaller soliton grew larger.A criticism of these modes of interaction was provided by Bowtell and Stuart [3].Further support for their analysis of KdV soliton interaction will be found in part two of this thesis.

We shall adopt the following definition of a soliton [4].A soliton is a solitary wave which preserves its shape and speed in collisions with another wave,or at worst exchanges its shape and speed with that of another soliton.

The first known observation of a soliton was by John Scott Russell [5].The KdV got its name from two early Dutch investigators of shallow water waves,Korteweg and de Vries [6]. Further references to Scott Russell and the early history of the KdV can be found in an appendix in reference [4].

It is not the primary concern of this thesis to examine physical systems which support the soliton equations.There is ample literature on the subject [4] and especially [7].

We now list a number of nlpde's which are at some stage considered or referred to in this thesis.

Korteweg-de Vries(KdV)

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3.1}$$

Replacing u with -u removes the sign from (3.1),we shall also refer to this as the KdV.(3.1) can be derived from the equation,

$$w_t - 3w_x^2 + w_{xxx} = 0$$

By differentiating the above with respect to x and setting $u=w_x$ we obtain (3.1).We will refer to the equation in w as the derivative KdV.

Modified Korteweg-de Vries(MKdV)

$$u_t + 6u^2u_x + u_{xxx} = 0 \tag{3.2}$$

This can also be derived from the equation,

$$w_t + 2w_x^3 + w_{xxx} = 0$$

by differentiating the above with respect to x and setting $u=w_x$ we obtain (3.2).The equation in w,we refer to as the derivative MKdV.

Kadomtsev-Petviashivili(KP)

This is also called the two-dimensional KdV.

$$(u_t - 6uu_x + u_{xxx})_x + u_{yy} = 0 \tag{3.3}$$

Boussinesq

$$(u_x - 6uu_x + u_{xxx})_x - u_{tt} = 0 \tag{3.4}$$

sine-Gordon

$$u_{xx} - u_{tt} = \sin u \tag{3.5}$$

The sinh-Gordon(not a soliton equation) is the same as the above with sin replaced by sinh,and can be simply obtained from (3.5) on replacing u with iu.

Nonlinear or cubic Schrödinger(NLS)

$$u_{xx} + iu_t + \varkappa|u|^2u = 0 \qquad (3.6)$$

Liouville

$$u_{xx} - u_{tt} = e^u \qquad (3.7)$$

There are also equations refered to as modified KP and modified Boussinesq,which are obtained by changing the power of u in (3.3-4) to 2,and changing its sign.The Liouville is the only equation in the list above which at present is not yet known to possess N-soliton solutions†.

We now give a short summary of the contents of this chapter.In §1 we will review the inverse scattering method as developed for the KdV by Gardner,Greene,Kruskal and Miura(GGKM).We will also discuss the linear superposition principle(lsp) developed for the KdV by GGKM.

In § 2 we will briefly review Lax's operator-theoretic generalization of the GGKM method for solving the KdV equation.

In § 3 we will discuss the very general technique for solving nlpde's developed by Ablowitz,Kaup,Newell and Segur(AKNS) which was inspired by Zakharov and Shabat's inverse scattering method for the NLS.

Bäcklund transformations and nonlinear superposition principles(nlsp) form the topic of § 4.Later in the thesis we shall see how the nlsp for the sG(also known by the grandiose title of "the theorem of permutability"),will enable us to *derive* the multisoliton solutions of the sG.

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†Andreev claims to have found N soliton solutions [8],but this is found to be false [9].Andreev's N soliton solution turns out to be a 1 soliton solution!

In §5 we list the N soliton solutions of the equations which concern us in this thesis.Originally this thesis was to be exclusively connected with the sG equation,however discoveries made by the author were found to be applicable to a large number of soliton equations,so as a result,the horizons of the thesis have widened.

The chapter ends(§6) with a discussion of certain topics which have arisen in the chapter and how they relate to the main topic of the next chapter.

§ 1. The Inverse Scattering Method

The essential ideas of the inverse scattering method(ISM) for solving the KdV equation were given by GGKM [10].It is their treatment of the problem which we outline here [11].

Consider the time independent Schrödinger equation,

$$\varphi_{xx} + (\lambda-u)\varphi = 0 \tag{3.8}$$

where φ would be known in quantum mechanics,as the wave function,and here,we shall refer to it as the eigenfunction.The potential u(x,t) is taken to be a solution of the KdV equation (3.1) and time t is just a parameter-having nothing to do with the time which enters the time dependent Schrödinger equation. $\lambda=\lambda(t)$ is the eigenvalue.If u in (3.8) is substituted into the KdV we find,

$$\lambda_t\varphi^2 + (\varphi Q_x-\varphi_x Q)_x = 0 \tag{3.9a}$$

where,

$$Q \equiv \varphi_t + \varphi_{xxx} - 3(u+\lambda)\varphi_x \tag{3.9b}$$

If we integrate (3.9a) with respect to x over the interval $(-\infty,+\infty)$,we find that,provided φ tends to zero sufficiently rapidly as $|x|\to\infty$,

$$\lambda_t = 0 \tag{3.10}$$

Substituting (3.10) into (3.9a) we obtain,

$$Q_{xx} - \frac{\varphi_{xx}}{\varphi} Q = 0$$

Noting (3.8),we see that Q is a solution of (3.8).Thus Q must be a sum of linearly independent solutions of (3.8),with coefficients possibly dependent on t.Thus,

$$Q = \varphi_t + \varphi_{xxx} - 3(u+\lambda)\varphi_x = F(t)\varphi + D(t)\varphi \int^x \varphi^{-2}dx \qquad (3.11)$$

Since φ vanishes as $|x| \to \infty$,we must have D(t)=0,to prevent Q from becoming unbounded.

Equation (3.11) determines the time evolution of φ.If the potential u(x,t) is given at some fixed,initial time t,(3.8) may have a finite number of bound states with discrete eigenvalues $\lambda_n = -k_n^2$,n=1,2,..,N and,a continuum state $\lambda=k^2>0$.The eigenfunction φ_n associated with the bound state eigenvalue,$\lambda_n=-k_n^2$ may be written,

$$\varphi_n(x,t) = c_n(t)\exp(-k_n x) \quad ,\text{as } x\to -\infty \qquad (3.12)$$

where we have chosen to normalize φ_n,so that,

$$\int_{-\infty}^{\infty} \varphi_n^2 dx = 1 \qquad (3.13)$$

The wave function associated with the unbound state is related to the situation of a steady plane wave being partly transmitted and reflected by the potential u(x,t) for some fixed time t.For large |x| a solution of (3.8) is a linear combination of plane waves exp(±ikx).The boundary conditions of a potential which may partly reflect and transmit,are written,

$$\varphi \sim e^{-ikx} + R(k,t)e^{ikx} \quad ,x\to \infty \qquad (3.14a)$$

and

$$\varphi \sim T(k,t)e^{-ikx} \quad ,x\to-\infty \qquad (3.14b)$$

In accordance with the quantum mechanical interpretation of $\varphi\varphi^*$ as a probability density,we have,

$$|R|^2 + |T|^2 = 1 \qquad (3.14c)$$

Essentially the problem to be solved,amounts to finding

$u(x,t)$ for a given $u(x,0)$ satisfying (3.8).This can be broken up into three steps.

1.The direct problem.

Solve $\varphi_{xx} + [\lambda - u(x,0)]\varphi = 0$ for given $u(x,0)$ to find any distinct eigenvalues λ_n and determine the *scattering data* $R(k,0),T(k,0)$ as $|x|\to\infty$.

2.Evolution of the scattering data in time.

From (3.11) together with (3.14) determine $R(k,t),T(k,t)$

3.The inverse problem.

Find $u(x,t)$ from $R(k,t),T(k,t)$ for arbitrary t.

In our case,since $u(x,t)$ is a solution of the KdV subject to the initial condition at t=0 ,$u(x,0)=u_o(x)$,the above procedure enables the KdV to be solved.

Now imagine we have completed step 1.We are in possession of a set of eigenfunctions $\varphi_n(x,0)$,associated with the eigenvalues $\lambda=-k_n^2$.We also have determined $R(k,0),T(k,0)$,by solving the Schrödinger equation (3.8) at t=0,with the asymptotic forms of φ (for many examples of this see §25 [12]).

For the discrete eigenfunctions φ_n,the time dependence is determined from (3.11) with D=0.Hence,

$$(\varphi_n)_t + (\varphi_n)_{xxx} - 3(u+\lambda_n)(\varphi_n)_x = F_n(t)\varphi_n \tag{3.15}$$

Multiplying by φ_n,and then integrating by parts over $(-\infty,+\infty)$,and also using (3.8),we find,

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}\varphi_n^2dx = F_n(t)\int_{-\infty}^{\infty}\varphi_n^2dx$$

However,the wave functions are normalized (3.13),thus $F_n(t)=0$. Clearly

$$(\varphi_n)_t + (\varphi_n)_{xxx} - 3(u+\lambda_n)(\varphi_n)_x = 0 \tag{3.16a}$$

$$\lambda_n = -k_n^2 \tag{3.16b}$$

Now assuming $u\to0$ as $x\to-\infty$ and noting (3.12),we find,

$$\frac{dc_n}{dt} = 4k_n^3 c_n$$

which has solution

$$c_n(t) = c_n(0)e^{4k_n^3 t}$$

Similarly the time dependence of the reflection and transmission coefficients follow from $\lambda=k^2$,putting (3.14) with $u\to 0$ when $|x|\to\infty$ into (3.11) with F and D equal to zero.We find,

$$T(k,t) = T(k,0), R(k,t) = R(k,0)e^{8ik^3 t}$$

Summarizing:

$$k_n(t) = k_n(0) \qquad (3.17a)$$

$$c_n(t) = c_n(0)e^{4k_n^3 t} \qquad (b)$$

$$T(k,t)= T(k,0) \qquad (c)$$

$$R(k,t)= R(k,0)e^{8ik^3 t} \qquad (d)$$

Thus we have completed step 2.Equations (3.17) specify the time evolution of the scattering data.We now turn to step 3.Fortunately this is a problem already solved in 1955 by Gel'fand and Levitan [13] and also by Marchenko (see [14],bibliography for a list of the Marchenko papers).

The potential u(x,t) is given by

$$u(x,t) = -2\frac{d}{dx} K(x,x,t) \qquad (3.18)$$

where K(x,x,t) is a solution of the Gel'fand-Levitan-Marchenko integral equation,

$$K(x,y,t) + B(x+y,t) + \int_x^\infty B(y+y',t)K(x,y',t)dy' = 0 \ ,y>x \qquad (3.19a)$$

with,

$$B(x,t) \equiv \sum_{n=1}^{N} c_n^2(t)e^{-k_n(t)x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ikx}dk \qquad (3.19b)$$

Thus we have seen how to solve a nlpde (KdV) in terms of solving two *linear* pde's.Namely,the time independent Schrödinger equation,with a given potential u(x,0),and the linear integral equation defined in (3.19).In general for arbitrary B(x,t) it is not very easy to solve (3.19a).

The following additional facts emerge from the previous analysis.

1.The number of solitons *moving to the right* which emerge from u(x,0) as time evolves are in one to one relationship with the number of bound states associated with (3.8) with potential u(x,0).Furthermore,as $t\to\infty$ the solitons are arranged in order of speed(amplitude),with the fastest,on the extreme right.

2.Non-soliton or oscillatory disturbances emerging from u(x,0) move to the *left*,and are present when $R(k,0)\neq 0$.

3.When u(x,0)>0 there is no bound state(potential hill),and no solitons emerge.In this case we only have oscillatory disturbances as in 2.

4.If u(x,0) admits bound states and R(k,0)=0 (zero reflection) then u(x,t) can be determined explicitly.In this case the asymptotic($|t|\to\infty$) case consists of a collection of solitons,well separated and each,moving at constant speed.

The N-soliton case is obtained by solving (3.19) with R(k,0)=0.It is found that [15],

$$K(x,x,t) = \frac{d}{dx}\Delta \tag{3.20a}$$

$$\Delta = \det(I+C) \tag{3.20b}$$

where I is the N×N unit matrix,and C is the NxN matrix defined below,

$$C = \left[\frac{c_m(t)c_n(t)}{k_m + k_n}.\exp[-(k_m+k_n)x]\right] \tag{3.20c}$$

and the N-soliton solution of the KdV is,

$$u(x,t) = -2\frac{d}{dx}K(x,x,t) \tag{3.21a}$$

thus,

$$u(x,t) = -2\frac{d^2}{dx^2}\ln\det(I+C) \tag{3.21b}$$

The above solution was first obtained using the inverse scattering technique by Wadati and Toda [16].Hirota had previously obtained the same solution using his own direct method [17].

In the 1-soliton case,

$$\det(I+C) = 1 + (2k_1)^{-1}c_1^2(t)e^{-2k_1x} = 1+(2k_1)^{-1}c_1^2(0)e^{-2k_1(x-4k_1^2t)}$$

Substituting this into (3.21b) gives,

$$u(x,t) = -2k_1^2\operatorname{sech}^2\xi_1 \qquad (3.22a)$$

$$\xi_1 = -2k_1(x-4k_1^2t-\delta) \qquad (b)$$

$$\delta = (2k_1)^{-1}\ln(c_1^2/2k_1) \qquad (c)$$

(3.22) is the well known single soliton solution of the KdV. When (3.19) are solved with $R(k,0)=0$,the initial step is to assume that,

$$K(x,y,t) = -\sum_{m=1}^{N} c_m(t)\psi_m(x,t)\exp(-k_my) \qquad (3.23)$$

so that the problem becomes to determine the unknown functions ψ_m.In fact ψ_m turn out to be the normalized eigenfunctions of the associated time independent Schrödinger equation(3.8)[11],

$$\mathcal{L}_m\psi_m \equiv \frac{d^2}{dx^2}\psi_m - [u(x,t)+k_m^2]\psi_m = 0 \qquad (3.24)$$

When (3.23) is substituted into (3.19),the ψ_m are found by solving (3.25) below,

$$(I+C)\psi = E \qquad (3.25)$$

where I,C are the matrices defined before(3.20c) and

$$\psi=(\psi_1,\psi_2,..,\psi_N)^T,\ E\equiv(c_1e^{-k_1x},c_2e^{-k_2x},..,c_Ne^{-k_Nx})^T \qquad (3.26)$$

Since $\det(I+C)\neq 0$ [11],I+C is non-singular and has an inverse,we may write,

$$\psi = (I+C)^{-1}E = \Delta^{-1}QE \qquad (3.27)$$

where Q is the classical adjoint matrix of I+C.Since I+C is symmetric Q is identical to the matrix of cofactors of I+C.

GGKM [11] were able to deduce a lsp for u.They showed that if

$\psi_m(x,t)$ are the N eigenfunctions in (3.24) and (3.27) then,

$$u(x,t) = -4\sum_{m=1}^{N} k_m \psi_m^2(x,t) \qquad (3.28)$$

In this lsp GGKM decided to *define* the quantity $-4k_m\psi_m^2(x,t)$ to be a soliton.This was very natural,especially since as $|t|\to\infty$,each such term does tend to a single soliton(3.22).We shall see in part two of this thesis that there is an alternative *equally natural* lsp which we can define for the multisoliton solution of the KdV.Like the GGKM soliton,our own soliton definition is also a soliton of form (3.22) in the $|t|\to\infty$ limit.There are also a number of other attractive features possessed by our own soliton definition,which are not possessed by the GGKM soliton.

§ 2. Lax's method

Lax [2] showed that the KdV was one of an infinite number of pde's that govern the variation of the potential in Schrödinger's equation (3.8) in such a way as to leave the eigenvalues fixed with respect to variations of the time parameter(in the potential).

Let us represent (3.8) in operator form,

$$L\varphi = \lambda\varphi \ , \ L \equiv D^2-u \ , \ D \equiv d/dx \qquad (3.29)$$

Now suppose,

$$\varphi_t = B\varphi \qquad (3.30)$$

where B is a linear differential operator.It is then found that the derivatives of λ with respect to the time parameter,λ_t are controlled by the equation,

$$(-u_t + [L,B])\varphi = \lambda_t\varphi \qquad (3.31a)$$

where,

$$[L,B] \equiv LB-BL \qquad (3.31b)$$

Thus, provided we choose B so that,

$$-u_t + [L,B] = 0 \tag{3.32}$$

then $\lambda_t = 0$. The simplest operator which can lead to constant eigenvalues is,

$$B_1 = aD \tag{3.33a}$$

we then find,

$$[L,B_1]\varphi = 2a_x D^2\varphi + a_{xx} D\varphi + au_x\varphi$$

Choosing a to be constant gives,

$$[L,B_1] = au_x \tag{3.33b}$$

Substituting this into (3.31a) we find,

$$(u_t - au_x)\varphi = -\lambda_t\varphi \tag{3.33c}$$

Thus, provided $u_t = au_x$, $\lambda_t = 0$. This result is not particularly exciting as it says that any potential of form $u(x+at)$ will leave λ fixed in time. When we pick an operator B_2 quadratic in D it so turns out that we once again obtain (3.33c). Lax discovered that only operators B_r, r odd, produced evolution equations for u different to (3.33c). Thus consider B_3 below,

$$B_3 = aD^3 + fD + g$$

where a is again constant, while f and g are functions of u and its spatial derivatives. We find,

$$[L,B_3]\varphi = (2f_x + 3au_x)D^2\varphi + (f_{xx} + 2g_x + 3au_{xx})D\varphi + (g_{xx} + au_{xxx} + fu_x)\varphi$$

Requiring the coefficients of D^2 and D to vanish leads eventually (choosing a=-4) to,

$$[L,B_3] = -(u_{xxx} - 6uu_x) + c_1 u_x$$

The pde satisfied by u which keeps $\lambda_t = 0$ is then (from 3.32),

$$-u_t - u_{xxx} + 6uu_x + c_1 u_x = 0$$

The last term may be elimated by change of variables $(dx \to dx + c_1(t)dt)$, so that eventually we find,

$$u_t - 6uu_x + u_{xxx} = 0$$

This is the KdV equation. Using this technique an infinite number of higher order equations characterized by B_{2n+1}, $n \in \mathbb{N}$, can be constructed which leave the eigenvalues of the Schrödinger equation invariant in time(see §5).

§ 3. The AKNS method

Zakharov and Shabat [18] extended the Lax method and developed an inverse scattering technique which enabled the NLS (3.6) to be solved.The Zakharov-Shabat method was further generalized by Ablowitz,Kaup,Newell,and Segur(AKNS) [19].They found that many nlpde's could be solved by the following method(sometimes referred to as the *two component* method).

Consider the equations,

$$v_{1x} + i\zeta v_1 = q(x,t)v_2 \tag{3.34}$$

$$v_{2x} - i\zeta v_2 = r(x,t)v_1$$

Choosing the time dependence of the eigenfunctions v_1,v_2 to be given by,

$$v_{1t} = A(x,t,\zeta)v_1 + B(x,t,\zeta)v_2 \tag{3.35}$$

$$v_{2t} = C(x,t,\zeta)v_1 - A(x,t,\zeta)v_2$$

We can obtain the conditions for the eigenvalues ζ to be time invariant by cross differentiation of systems (3.34-5).We find $\zeta_t = 0$ if,

$$A_x = qC - rB \tag{3.36}$$

$$B_x + 2i\zeta B = q_t - 2Aq$$

$$C_x - 2i\zeta C = r_t + 2Ar$$

Finite expansions of A,B,C in terms of the eigenvalue parameter $2i\zeta$ and the potentials q and r,determine equations solvable under the scheme.

CLASS 1

$$A = -4i\zeta^3 - 2iqr\zeta + rq_x - qr_x \tag{3.37}$$

$$B = 4q\zeta^2 + 2iq_x\zeta + 2q^2r - q_{xx}$$

$$C = 4r\zeta^2 - 2ir_x\zeta + 2qr^2 - r_{xx}$$

Substituting (3.37) into (3.36),we obtain,

$$q_t - 6rqq_x + q_{xxx} = 0 \tag{3.38}$$

$$r_t - 6rqr_x + r_{xxx} = 0$$

Setting r=1 gives the KdV equation (3.1) and $r=\pm q$ gives the MKdV (3.2).

CLASS 2

$$A = \frac{i\cos\phi}{4}\zeta^{-1}, B = -\frac{i}{2}q_t\zeta^{-1}, C = +\frac{i}{2}r_t\zeta^{-1} \tag{3.39}$$

(3.39) substituted into (3.36) give,

$$q_{xt} = q\cos\phi \quad , \quad r_{xt} = r\cos\phi \quad , \quad (\cos\phi)_x = 2(qr)_x \tag{3.40}$$

Setting $r=-q=\phi_x/2$ then gives the sG equation in the form

$$\phi_{xt} = \sin\phi$$

There are still more classes of equations solvable by this method [19].The solutions of all these pde's for q and r are given by the following.If the initial potentials,q(x,0),r(x,0) are sufficiently smooth and vanish rapidly as $|x| \to \infty$,then q(x,t) and r(x,t) are given by(for all time),

$$q(x,t) = -2K_1(x,x) \tag{3.41}$$

$$r(x,t) = -2\bar{K}_2(x,x)$$

where K and $\bar{K}$ are the solutions of the Gel'fand-Levitan-Marchenko equations,

$$K(x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\bar{F}(x+y) - \int_x^\infty \bar{K}(x,s)\bar{F}(s+y)ds = 0 \tag{3.42}$$

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\bar{F}(x+y) + \int_x^\infty \bar{K}(x,s)\bar{F}(s+y)ds = 0$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\xi,t)}{a(\xi)}e^{i\xi x}d\xi \quad - \quad i\sum_k c_k \exp(i\zeta_k x) \tag{3.43}$$

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{b}(\xi,t)}{\bar{a}(\xi)}e^{-i\xi x}d\xi \quad - \quad i\sum_k \bar{c}_k \exp(-i\bar{\zeta}_k x)$$

and,

$$K(x,y) = \begin{pmatrix} K_1(x,y) \\ K_2(x,y) \end{pmatrix} \quad , \quad \bar{K}(x,y) = \begin{pmatrix} \bar{K}_1(x,y) \\ \bar{K}_2(x,y) \end{pmatrix} \tag{3.44}$$

The $\zeta_k(\bar{\zeta}_k)$ are the eigenvalues of (3.34) which lie in the upper(lower) half plane;$a,b,c_k,\bar{a},\bar{b},\bar{c}_k$,are the scattering data and have time dependences,

$$a(\zeta) = a_o(\zeta) \quad , \quad \bar{a}(\zeta) = \bar{a}_o(\zeta) \tag{3.45}$$

$$b(\xi,t) = b_o(\xi)\exp[-2A_o(\xi)t]$$

$$\bar{b}(\xi,t) = \bar{b}_o(\xi)\exp[+2A_o(\xi)t]$$

$$c_k = c_{ko}\exp[-2A_o(\zeta_k)t]$$

$$\bar{c}_k = \bar{c}_{ko}\exp[+2A_o(\bar{\zeta}_k)t]$$

$$A(\zeta) = \lim_{|x|\to\infty} A(x,\zeta;\zeta)$$

The eigenvalues and constants above,are all determined by solving (3.34) at the initial time(using $q(x,0),r(x,0)$).The N soliton solutions arise as in the GGKM scheme,when the reflection coefficients $b_o(\xi)$ and $\bar{b}_o(\xi)$ are zero.For a complete analysis in the case of the sG,we refer to [20].The discrete eigenvalues ζ are either purely imaginary,or occur in complex conjugate pairs $\zeta,-\zeta^*$.The purely imaginary eigenvalues $\zeta=i\eta$,correspond to travelling wave solutions,while the paired complex eigenvalues correspond to soliton states which oscillate in time(breathers).

§ 4. **Bäcklund Transformations(BT's)**

A Bäcklund transformation for a second order pde for dependent variables $\varphi(\xi,\eta)$ is defined as a pair of equations,

$$\varphi'_{\xi} = P(\varphi',\varphi,\varphi_{\xi},\varphi_{\eta},\xi,\eta) \qquad (3.46)$$
$$\varphi'_{\eta} = Q(\varphi',\varphi,\varphi_{\xi},\varphi_{\eta},\xi,\eta)$$

The consistency condition $\varphi'_{\xi,\eta} = \varphi'_{\eta,\xi}$ provides a new equation for φ'.Sometimes the equation satisfied by φ' is the same as that satisfied by φ.In this case we refer to the transformation as an auto-Bäcklund transformation(aBT).To clarify the meaning of (3.46),consider the sG equation in the form,

$$\varphi_{\xi\eta} = \sin\varphi \qquad (3.47a)$$

where,

$$\xi = (x-t)/2\ ,\ \eta = (x+t)/2 \qquad (3.47b)$$

The equations,

$$\varphi'_{\xi} = \varphi_{\xi} + 2a\sin[(\varphi'+\varphi)/2] \qquad (3.48a)$$

$$\varphi'_{\eta} = -\varphi_{\eta} + 2a^{-1}\sin[(\varphi'+\varphi)/2] \qquad (3.48b)$$

are of the form (3.46),together they comprise an aBT for the sG equation.This is clear by cross differentiating (3.48) with respect to η and ξ,and then applying the consistency requirement.

Bäcklund transformations have a long history($\approx$ 100 years),and a detailed list of references to early work on the subject is given in [21].BT's generally relate the solutions of one equation with the solutions of another.A well known example of this is the BT which connects the Liouville equation $u_{xy}=e^{u}$ with the wave equation $u_{xy}=0$ [21,22].Since the latter has a general solution,one can also be found for the Liouville equation.The Liouville also possesses aBT's [27].The general method for determining BT's (when they exist) between pde's is known as Clairin's method [21,22].

Lamb pioneered the use of BT's,as a method of finding solutions to the sG equation [23].A first solution in the case of the sG can be obtained from (3.48),by choosing $\varphi=0$(an obvious solution of the sG).The resulting equations are,

$$\varphi'_\xi = 2a \sin[\varphi'/2] \qquad (3.49a)$$

$$\varphi'_\eta = 2a^{-1}\sin[\varphi'/2] \qquad (3.49b)$$

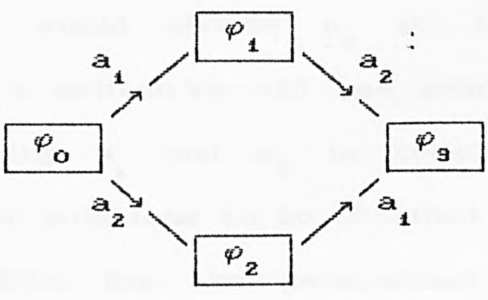
These equations can be integrated to give a 1-parameter solution(soliton or antisoliton) solution of the sG.Noting (3.47b) we have,

$$\varphi' = 4\tan^{-1}\exp[a\eta + \xi/a] = 4\tan^{-1}\exp[\pm\gamma(x-ut)] \qquad (3.50a)$$

where,

$$\gamma = (1-u^2)^{-1/2} \ , \ a = \pm[(1-u)/(1+u)]^{1/2} \ ,|u| < 1 \qquad (3.50b)$$

The positive sign in (3.50a) corresponds to the soliton(kink) and the negative sign to the antisoliton(antikink).We could in principle now use the solution (3.50) as φ in (3.48) and determine a two parameter solution φ'.Such a method involves complicated integrals,and anyway there exists a much simpler way of determining higher parameter solutions,known as the *theorem of permutability*(for the sG).This method involves only *algebraic* calculations.It was this method that Lamb [23] used to obtain the few known solutions of (3.47),incuding those found by Perring and Skyrme [25],prior to the discovery of the inverse scattering method [20,24].In discussing this method,it is useful to use a diagrammatic representation.



From a solution φ_0 of the sG we apply the BT(3.48) with parameter a_1,to generate φ_2.We do the same with parameter a_2 to generate φ_2.Then we apply (3.48) again to φ_1 with parameter a_2,similarly we apply (3.48) to φ_2 with parameter a_1.Both these final transformations may produce the same solution φ_3,having parameters a_1 and a_2.Considering (3.48a),we then have the following equations.

$$(\varphi_1 - \varphi_0)_\xi = 2a_1\sin[(\varphi_1+\varphi_0)/2] \quad (3.51a)$$

$$(\varphi_3 - \varphi_1)_\xi = 2a_2\sin[(\varphi_3+\varphi_1)/2] \quad (b)$$

$$(\varphi_2 - \varphi_0)_\xi = 2a_2\sin[(\varphi_2+\varphi_0)/2] \quad (c)$$

$$(\varphi_3 - \varphi_2)_\xi = 2a_1\sin[(\varphi_3+\varphi_2)/2] \quad (d)$$

The sum of the first pair of equations and the sum of the last pair have the same left hand side.Equating the right hand sides gives us an algebraic relation between $\varphi_3,\varphi_1,\varphi_2,\varphi_0$,which we write,

$$\tan[(\varphi_3-\varphi_0)/4] = k_{12}\tan[(\varphi_1-\varphi_2)/4] \quad (3.52a)$$

$$k_{12} = (a_1+a_2)(a_1-a_2)^{-1} \quad (b)$$

This is the *theorem of permutability*.Starting with the vacuum solution φ_0,higher parameter solutions can be built up iteratively.We may also begin at an N-2 parameter starting point,and find the N parameter solution as pointed out by Barnard [26].In addition Barnard stated implicitly the following.To build a strictly N *soliton* solution using (3.52), we must use 1 parameter soliton and antisoliton solutions, *alternately*.So that for instance,to build from (3.52) the three soliton solution,we would choose φ_0 to be a soliton,φ_1 an antisoliton and φ_2 a soliton.We will see exactly why this is so in chapter 5.Choosing a_1 and a_2 in complex conjugate pairs enables the breather solutions to be obtained [23].

Further work on BT's for the generalized sG(φ_{xt}=F(φ),including the sinh-Gordon and Liouville equations),in which exact conditions necessary for aBT's to exist are found,is given in [27].

It was long not after Lamb's pioneering work on aBT's for the sG,that Wahlquist and Estabrook discovered that the KdV also possessed aBT's [28].Writing the KdV in conservation form,

$$u_t+(-3u^2+u_{xx})_x = 0 \tag{3.53}$$

they introduced the function w such that $u=w_x$.(3.53) becomes, (derivative KdV-see introduction)

$$w_t = 3u^2-u_{xx} = 3w_x^2-w_{xxx} \tag{3.54}$$

If w and u are solutions of (3.53-4),then different solutions w',u' are defined by the BT

$$w'_x = -w_x-k^2/2 +(w'-w)^2/2 \tag{3.55a}$$

$$w'_t = -w_t-k^2u'+2u^2+u(w'-w)^2+2u_x(w'-w) \tag{3.55b}$$

The single soliton solutions of (3.53-4) are,

$$w = k\tanh\xi \tag{3.56a}$$

$$u = w_x = -(k^2/2)\text{sech}^2\xi \tag{3.56b}$$

$$\xi = -k(x-k^2t-x_o)/2 \tag{3.56c}$$

Also (3.53-4) have singular solutions,

$$\tilde{w} = k\coth\xi \tag{3.57a}$$

$$\tilde{u} = -(k^2/2)\text{cosech}^2\xi \tag{3.57b}$$

By succesively applying the BT's (3.55),in a similar way to the sG,Wahlquist and Estabrook were able to produce a nlsp for the derivative KdV,analogous to the sG(3.52).

$$w_3 = w_o + \frac{k_2^2 - k_1^2}{2(w_2-w_1)} \tag{3.58}$$

where w_o is the starting solution of (3.54) and w_2 and w_1 are solutions generated by the BT's with parameters k_2 and k_1.The N soliton solution of (3.54) could be generated by starting with the vacuum solution w_o=0 and using 1 parameter solutions w or $\tilde{w}$.Wahlquist and Estabrook noted that only a certain choice of $w_i,\tilde{w}_i$ will produce the N soliton solution by iteration.This was as follows.In the sequence of N 1-parameter solutions leading to the N soliton solution,the regular solutions(3.56a) and the singular solutions(3.57a) must be used *alternately*.For instance to build the 3 soliton solution from (3.58),w_0 must be a regular solution,w_1 a singular solution,and w_2 a regular solution.

Wahlquist and Estabrook note that "it is hard to resist the particle-antiparticle analogy suggested by this structure".We shall see when we discuss the N soliton solutions of various equations in §5,how the sequence of 1-parameter solutions needed to build the N soliton solution of the KdV,is exactly analogous to the sequence required to build up the N soliton solution of the sG.

It is now known that many soliton equations possess BT's and associated nlsp's.Wadati [29] discovered the aBT for the MKdV (in derivative form),

$$w_t + 2w_x^3 + w_{xxx} = 0 \qquad (3.59)$$

which when $u=w_x$ becomes the usual form of the MKdV(3.2).The BT's are,

$$w'_x = -w_x + 2k\sin(w'-w) \qquad (3.60)$$

$$w'_t = -w_t - 8k^2u - 4ku_x\cos(w'-w) - 4(2k^3+ku^2)\sin(w'-w)$$

From these(one only needs the first of the equations above) we obtain the nlsp,

$$\tan[(w_3-w_0)/2] = \left(\frac{k_1+k_2}{k_1-k_2}\right)\tan[(w_1-w_2)/2] \qquad (3.61)$$

This has identical form(apart from factor 2) to the sG equivalent(3.52).The reason the nlsp's for the MKdV and sG are so similar becomes evident when the connection between the BT's and the inverse scattering method(AKNS) are explored [30].

§ 5. N-soliton solutions

The KdV belongs to a hierarchy of equations as we saw in §2.This hierarchy is defined by [31],

$$u_{t_{2n+1}} = -2^{2n+1}(B_{n+1})_x \quad ,n=-1,0,1,\ldots \tag{3.62a}$$

$$B_{n+1} = LB_n \tag{3.62b}$$

$$L = -\frac{1}{4}\partial^2/\partial x^2 + u - \frac{1}{2}\int_{\infty}^{x} u_x dx \tag{3.62c}$$

$$B_0 = -1 \tag{3.62d}$$

where in the above u,u_x,u_{xx}, and all higher derivatives vanish when $|x|=\infty$.The above hierarchy has N soliton solutions,

$$u(x,t_{2n+1}) = -2(\ln\tau)_{xx} \tag{3.63a}$$

where,

$$\tau = \sum_{\alpha_j=0,1} \exp\left[\sum_{j=1}^{N} \alpha_j\xi_j + \sum_{1\le i<j\le N} A_{ij}\alpha_i\alpha_j \right] \tag{3.63b}$$

$$\xi_j(x,t_{2n+1}) = \bar{\xi}_j + (-1)^n k_j^{2n+1} t_{2n+1} \tag{3.63c}$$

$$\bar{\xi}_j = k_j x + \delta_j \tag{3.63d}$$

$$A_{ij} = \ln u_{ij}^2 \tag{3.63e}$$

$$u_{ij}^2 = (k_i-k_j)^2(k_i+k_j)^{-2} \tag{3.63f}$$

where $k_i,\delta_i \in \mathbb{R}$ and $k_i<k_j$,$i<j$.For example.

n=1 $$u_{t_3} = -(u_{xx}-3u^2)_x \quad ,\text{KdV} \tag{3.64a}$$

n=2 $$u_{t_5} = (u_{xxxx}-5u_x^2-10uu_{xx}+10u^3)_x \tag{3.64b}$$

If N=3 in (3.63b)

$$\tau=1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}+e^{\xi_1+\xi_2+A_{12}}+e^{\xi_1+\xi_3+A_{13}}+e^{\xi_2+\xi_3+A_{23}}+e^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}}$$

Also note that (3.63) may be written in the form [14]($1\le i,j\le N$)

$$\tau=\det M \tag{3.66a}$$

$$M_{ij} = \delta_{ij} + 2(k_i+k_j)^{1/2}(k_i+k_j)^{-1}\exp\frac{1}{2}(\xi_i+\xi_j) \tag{3.66b}$$

Solutions.Single soliton.

$$u(x,t) = -2k_1^2\operatorname{sech}^2\xi_1 \tag{3.67}$$

Two solitons.

$$u(x,t) = -2(k_2^2-k_1^2)\left[\frac{k_2^2\cosh^2\xi_1+k_1^2\sinh^2\xi_2}{(k_2\cosh\xi_1\cosh\xi_2-k_1\sinh\xi_1\sinh\xi_2)^2}\right] \tag{3.68}$$

Observe what happens if we allow the exponential in the single soliton solution,to be *negative*,and take the modulus of τ.

$$u(x,t) = -2[\ln|1 - e^{\xi_1}|]_{xx} = -2k_1^2\mathrm{cosech}^2\xi_1 \qquad (3.69)$$

This is the singular solution (3.57b).We can look at the inclusion of negative exponential terms in τ as analogous to the way in which antisolitons enter the multisoliton solution of the sG.In this way we see how it is not so unexpected,that we should have to use singular solutions in building up the N soliton solution of the KdV.Instead we see that the manner of building up multisoliton solutions of the KdV and sG are closely related.In fact we can even introduce complex conjugate amplitudes k_i in (3.63) and produce *singular breathers* [32].

The KdV also has the property,that as $|t|\to\infty$,[33]

$$u(x,t)=-2\sum_{n=1}^{N} k_n^2\mathrm{sech}^2(\xi_n+\delta_n^{\pm}) \qquad (3.70a)$$

where,

$$\delta_n^{\pm} = \ln B_n^{\pm}\ ,\ B_n^{+} = \prod_{j=1}^{n-1} u_{in}^2\ ,\ B_n^{-} = \prod_{j=n+1}^{N} u_{in}^2 \qquad (3.70b)$$

with u_{ij} given in (3.63f).We also have,

$$\sum_{n=1}^{N}\delta_n = 0,\ \delta_n = \delta_n^{+}-\delta_n^{-} \qquad (3.70c)$$

The superscripts on δ refer to the limits $t\to\pm\infty$.

<u>MKdV equation</u>

The N parameter solutions of the derivative MKdV($u=w_x$ satisfies equation (3.2)),

$$w_t + 2w_x^3 + w_{xxx} = 0 \qquad (3.59)$$

are given by [22,33,34].

$$w = -2\tan^{-1}\left(\frac{\mathrm{Im}\ \det(I-iM)}{\mathrm{Re}\ \det(I-iM)}\right) \qquad (3.71a)$$

where I is the NxN unit matrix and M is the NxN matrix with components,

$$m_{jl} = 2k_l\varepsilon_l(k_j+k_l)^{-1}\exp[(k_j+k_l)x-8k_l^3t+\eta_l] \qquad (3.71b)$$

where,

$\eta_j,k_j\in\mathbb{R}$ for solitons and antisolitons.$\varepsilon_l=1$ for solitons and breathers, $\varepsilon_l=-1$ for antisolitons. $\eta_j, k_j \in \mathbb{C}$ for breathers .Each breather requiring a pair of $(\eta_j,k_j),(\eta_{j+1},k_{j+1})$ such that $\eta_{j+1}=\eta_j^*$, $k_{j+1}=k_j^*$.

Solutions.

single soliton. N=1, $\varepsilon_1=1$,

$$m_{11} = e^{\xi_1}$$

$$\xi_1 = 2k_1(x-4k_1^2t)+\eta_1 \tag{3.72a}$$

Using (3.71a) we find,

$$w = 2\tan^{-1}\exp\xi_1 \ , \ u = 2k_1\mathrm{sech}\xi_1 \tag{3.72b}$$

Similarly,for the antisoliton($\varepsilon_1=-1$),

$$w = -2\tan^{-1}\exp\xi_1 \ , \ u = -2k_1\mathrm{sech}\xi_1 \tag{3.73}$$

Two solitons. N=2 . $\varepsilon_1=\varepsilon_2=1$.

$$\det(I-iM) = 1-u_{12}^2 e^{\xi_1+\xi_2} -i(e^{\xi_1} + e^{\xi_2}) \tag{3.74a}$$

$$\tan(w/2) = \left(\frac{e^{\xi_1} + e^{\xi_2}}{1 - u_{12}^2 e^{\xi_1+\xi_2}} \right) \tag{3.74b}$$

u_{12}^2 is defined as for the KdV(3.63f).

Soliton-antisoliton. $\varepsilon_1=1, \varepsilon_2=-1$.We find,

$$\tan(w/2) = \left(\frac{e^{\xi_1} - e^{\xi_2}}{1 + u_{12}^2 e^{\xi_1+\xi_2}} \right) \tag{3.74c}$$

Breather. $\varepsilon_1=\varepsilon_2=1$, $k_1=re^{+i\mu}$, $k_2=k_1^*$. $\beta\equiv r\cos\mu$, $\alpha\equiv r\sin\mu$.

$$u_{12} = (k_1-k_2)(k_1+k_2)^{-1} = i\tan\mu \tag{3.75}$$

det(I-iM) is determined by (3.74a) with,

$$\xi_1 = \Gamma+i\Omega+\eta_1 \ , \ \xi_2 = \Gamma-i\Omega+\eta_1^* \tag{3.76a}$$

$$\Gamma = 2\beta[x-4(\beta^2-3\alpha^2)t] \ , \ \Omega = 2\alpha[x-4(3\beta^2-\alpha^2)t] \tag{3.76b}$$

To obtain the standard breather solution we carry out the phase change below,

$$\eta_1 \rightarrow \eta_1 + \ln(\beta/\alpha) \tag{3.77}$$

we then find,

$$\tan(w/2) = (\beta/\alpha)\sin\Omega\,\mathrm{sech}\Gamma \tag{3.78}$$

This is the breather solution.

The asymptotic solution of (3.71) in the no breather case was investigated by Hirota [33] and Wadati [34].They found that as $t \to \pm\infty$,

$$u = 2\sum_{n=1}^{N} \varepsilon_n k_n \operatorname{sech}(\xi_n + \delta_n^{\pm}) \tag{3.79}$$

with $\delta_n^{\pm}$ defined as with the KdV.

sG equation

The N parameter solutions of the sG [22] are seen below.The solution for combinations of solitons and antisolitons was first obtained by Hirota using his direct method [35].

$$\phi = -4\tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\frac{\det(I-iM)}{\det(I-iM)}\right) \tag{3.80a}$$

where I is the NxN unit matrix and M is the NxN matrix with components,

$$m_{kl} = 2a_l\varepsilon_l(a_k+a_l)^{-1}\exp\frac{1}{2}\left[x\left(\frac{a_k+a_l}{2}+\frac{1}{a_l}\right)-t\left(\frac{a_k+a_l}{2}-\frac{1}{a_l}\right)+\eta_l\right] \tag{b}$$

where for solitons and antisolitons(u_l is the soliton/antisoliton asymptotic speed)

$$a_l = (1+u_l)^{1/2}(1-u_l)^{-1/2}, a_l \in \mathbb{R}^+, |u_l|<1 \tag{c}$$

for solitons and breathers $\varepsilon_l=1$,for antisolitons $\varepsilon_l=-1$.

Associated with each breather is a pair of complex parameters, a_l, a_{l+1} and a pair of complex phases η_l, η_{l+1}.

$$a_{l+1} = a_l^* = ae^{-i\mu}, \eta_{l+1} = \eta_l^*, a = (1+v_l)^{1/2}(1-v_l)^{-1/2}, a \in \mathbb{R}^+, |v_l|<1 \tag{d}$$

v_l are the breather speeds. Also we note that for breathers,

$$(a_l - a_{l+1})/(a_l + a_{l+1}) = i\tan\mu \tag{e}$$

Solutions.

Soliton.N=1,ε_1=1.We find

$$m_{11} = e^{X_1} \quad , X_1 = \gamma_1(x-u_1t)+\eta_1 \ , \ \gamma_1 = (1-u_1^2)^{-1/2} \tag{3.81}$$

Thus,

$$\phi = 4\tan^{-1}e^{X_1} \tag{3.82a}$$

Similarly for the antisoliton case(ε_1=-1).

$$\phi = 4\tan^{-1}\text{-}e^{X_1} \quad (3.82b)$$

Two solitons.N=2 $.\varepsilon_1=\varepsilon_2=1.$

$$\phi_{ss} = 4\tan^{-1}\left[\frac{e^{X_1} + e^{X_2}}{1 - u_{12}^2 e^{X_1+X_2}}\right] \quad (3.83a)$$

$$u_{12} = (a_1-a_2)/(a_1+a_2) = (1-u_1u_2-(1-u_1^2)^{1/2}(1-u_2^2)^{1/2})/(u_1-u_2) \quad (b)$$

Soliton-antisoliton.N=2 $.\varepsilon_1=1,\varepsilon_2=-1$.We find,

$$\phi_{sa} = 4\tan^{-1}\left[\frac{e^{X_1} - e^{X_2}}{1 + u_{12}^2 e^{X_1+X_2}}\right] \quad (3.84)$$

Breather.N=2 $.\varepsilon_1=\varepsilon_2=1$.From (3.80d) $a_2=a_1^*=a(\cos\mu-i\sin\mu)$.We find,

$$X_1=\Gamma+i\Omega \quad , \quad X_2=\Gamma-i\Omega \quad (3.85a)$$

$$\Gamma=\gamma_v\cos\mu.(x-vt)+\eta_R \ , \ \Omega=\gamma_v\sin\mu.(t-vx)+\eta_I \quad (b)$$

where in the above $\eta=\eta_R+i\eta_I$.Now,

$$\det(I-iM) = 1-(i\tan\mu)^2e^{2\Gamma}-ie^{\Gamma}(e^{i\Omega}+e^{-i\Omega})$$

We change phase as with the MKdV,$\eta \longrightarrow \eta-\ln\tan\mu$.The above becomes,

$$\det(I-iM) = 1+e^{2\Gamma}-2e^{\Gamma}(\sin\Omega)i/\tan\mu$$

Thus,from (3.80) we obtain the standard breather solution,

$$\tan(\phi_b/4) = \sin\Omega.\text{sech}\Gamma/\tan\mu \quad (3.86)$$

Note that if we go into the rest frame of the breather,by Lorentz transformation,

$$x \longrightarrow \gamma_v(x'+vt') \quad , \quad t \longrightarrow \gamma_v(t'+vx') \quad (3.87)$$

(3.86) becomes,after setting $\tan\mu=u$.

$$\tan(\phi_b/4) = \sin\sigma ut'.\text{sech}\sigma x'/u \quad (3.88a)$$

$$\sigma = (1+u^2)^{-1/2} \quad (b)$$

Since the "phase velocity" of a sG breather is equal to the translational velocity,we see that the sG breather is simpler than the MKdV breather(3.78).The asymptotic nature of the sG is similar in stucture to the KdV(although because the sG is bidirectional the asymptotic argument is simpler).The sG phase shifts have an identical form [35].

KP equation

The N soliton solution of the KP equation was first obtained by Satsuma [36] and also by Manakov.et al.[37].

The KP equation,

$$(u_t - 6uu_x + u_{xxx})_x + u_{yy} = 0 \qquad (3.3)$$

has solutions identical in structure to the KdV.

$$u(x,t) = -2(\ln \tau)_{xx} \qquad (3.89a)$$

where,

$$\tau = \sum_{\alpha_j=0,1} \exp\left(\sum_{j=1}^{N} \alpha_j \xi_j + \sum_{1\leq i<j\leq N} A_{ij}\alpha_i\alpha_j \right) \qquad (b)$$

$$\xi_j(x,y,t) = (p_j-q_j)x + (p_j^2-q_j^2)y + (p_j^3-q_j^3)t + \delta_j \qquad (c)$$

$$A_{ij} = \ln u_{ij}^2 \qquad (d)$$

$$u_{ij}^2 = (p_i-p_j)(q_i-q_j)(p_i-q_j)^{-1}(q_i-p_j)^{-1} \qquad (e)$$

where $p_j \neq q_j$.We note that the KdV is recovered from the above by setting $p_i=-q_i$ [38].τ may also be written $\tau=\det M$,where M is the NxN matrix with components,

$$m_{ij} = \delta_{ij} + \sqrt{\frac{(p_i-q_i)(p_j-q_j)}{(p_i-q_j)(p_j-q_i)}}\exp\frac{1}{2}(\xi_i+\xi_j) \qquad (3.89d)$$

In special circumstances the KP has *resonant soliton* solutions [37,39],in which a two space dimensional solution encloses a *finite* volume.These special solutions are called ZM solitons,after their discovers,Zakharov and Manakov.The KP equation considered here is the first(like the KdV) in a whole hierarchy of equations,which by certain transformations (e.g $p_i=-q_i$) may include the Boussinesq and others [38,40].

Boussinesq equation

The Boussinesq equation has solutions of the same functional form as the KdV,but with the great advantage of having bidirectional solutions and equal amplitude solitons.This means that two soliton collisions of equal amplitude are likely to be easily analysed(as we shall see in part two of the thesis).The sG is the only other soliton equation possessing these attributes that we consider in this thesis.

The N soliton solution of the Boussinesq equation,

$$(u_x - 6uu_x + u_{xxx})_x - u_{tt} = 0 \tag{3.4}$$

has the ubiquitous Hirota form [41],

$$u(x,t) = -2(\ln \tau)_{xx} \tag{3.90a}$$

$$\tau = \sum_{\alpha_j=0,1} \exp\left(\sum_{j=1}^{N} \alpha_j \xi_j + \sum_{1\le i<j\le N} A_{ij}\alpha_i\alpha_j \right) \tag{b}$$

$$\xi_j = k_j(x-\varepsilon_j v_j t) + \eta_j \tag{c}$$

$$v_j = (1+k_j^2)^{1/2} \tag{d}$$

$$A_{ij} = \ln u_{ij}^2 \tag{e}$$

$$u_{ij}^2 = \frac{(\varepsilon_i v_i - \varepsilon_j v_j)^2 + 3(k_i - k_j)^2}{(\varepsilon_i v_i - \varepsilon_j v_j)^2 + 3(k_i + k_j)^2} \tag{f}$$

$$\varepsilon_i = \pm 1 \tag{e}$$

The N soliton solution has the usual asymptotic properties.

Solutions.

Single soliton.

$$u(x,t) = -2k_1^2 \mathrm{sech}^2 \xi_1 \tag{3.91}$$

Two soliton.

A particularly interesting two soliton solution is one in which solitons having equal amplitudes,collide with equal and opposite speeds.This case is characterized by choosing $k_1=k=k_2$, $\varepsilon_1=1,\varepsilon_2=-1$.Then we have from (3.90),

$$\xi_1 = k(x-vt)+\eta_1 \;,\xi_1 = k(x+vt)+\eta_2 \;,u_{12}^2 = v^2/(v^2+3k^2) \tag{3.92}$$

After adjusting phases $\eta_i \to -\ln u_{12}$,we find,

$$u = -2[\ln(1 + 2u_{12}^{-1}\cosh vte^{kx} + e^{2kx})]_{xx} \tag{3.93}$$

In part two of this thesis we shall see how we can express the above as a linear superposition of accelerating solitons.

Nonlinear Schrödinger equation

The N soliton solution is given in [18,42].Bäcklund transformations and a nlsp can be found in [43,44].The N envelope soliton solution of the NLS,$(|u_N|^2)$

$$u_{xx} + iu_t + \varkappa|u|^2u = 0 \tag{3.6}$$

is given by,

$$|u_N|^2 = (2\varkappa)^{1/2}[\ln \det(I + CC^*)]_{xx} \tag{3.94}$$

where I is the NxN unit matrix and C is the NxN matrix with components,

$$c_{mn} = \frac{(c_m c_n^*)^{1/2}}{k_m - k_n^*} \exp i(k_m - k_n^*)x \quad , \quad k_m \text{ constant} \tag{3.95a}$$

$$c_m(t) = \exp(4ik_m^2 t + \gamma_m) \quad , \quad \gamma_m \text{ constant} \tag{b}$$

Hirota [42] gives the explicit N soliton solution to the NLS,

$$i\psi_t + \beta\psi_{xx} + \delta|\psi|^2\psi = 0 \tag{3.96}$$

We only quote the single soliton solution,which is like a real and imaginary breather added together.

The one-soliton solution is,

$$\psi = (P_1 + P_1^*)\exp \eta_1/(1+\exp[\eta_1+\eta_1^*]) \tag{3.97}$$

where,

$$\eta_1 = P_1 x - Q_1 t - \eta_1^{(o)} \equiv \varepsilon + i\delta \tag{3.98}$$

$$P_1 \equiv p_1 + iq_1 \; , Q_1 = -i\beta P_1^2 \equiv \sigma_1 + i\rho_1$$

$$\sigma_1 = 2p_1 q_1 \beta \quad , \quad \rho_1 = (q_1^2 - p_1^2)\beta$$

so,

$$\eta_1 = (p_1 x - \sigma_1 t + \varepsilon) + i(q_1 x - \rho_1 t + \delta)$$

Defining,

$$\Gamma = (\eta_1 + \eta_1^*)/2 \quad , \quad \Omega = (\eta_1 - \eta_1^*)/2 \tag{3.99}$$

(3.97) becomes,

$$\psi = p_1(\cos\Omega \operatorname{sech}\Gamma + i\sin\Omega \operatorname{sech}\Gamma) \tag{3.100}$$

or as it is usually written,

$$\psi = p_1 \exp i(q_1 x - \beta(q_1^2 - p_1^2)t + \delta).\text{sech}(p_1 x - 2\beta p_1 q_1 t + \varepsilon) \qquad (3.101)$$

This is the familiar envelope soliton.In (3.100) we see the similarity with the breather solutions of the sG or MKdV equations.

§ 6. Summary

In this chapter we have discussed the main soliton equations and have looked at the primary methods of solution.During the course of our exposition of the GGKM inverse scattering method for the KdV,we discussed how the N soliton solution could be expanded as a sum of N terms.Each term involved a squared eigenfunction of the associated Schrödinger equation.It seemed natural for GGKM to interpret each of these terms as representing a soliton for all time.

Others [45,46] have taken up this suggestion,and attempted to analyse,in detail,how solitons of the KdV interact.We will discuss these papers in more detail in the next chapter,which is specifically concerned with particle-like approaches in soliton theory.

We examined the Bäcklund method for finding solutions to soliton equations and looked at a the remarkable nlsp connected with them.We noted the peculiar use of alternating solutions (regular/singular or soliton/antisoliton) to build up N *soliton* solutions.We saw in an original argument that the alternating pattern for the KdV was a disguised form of the alternating pattern for the sG equation.

Examining N soliton solutions in §5,we saw how a large class of them are very similar.This is fortunate,since when we examine how a lsp may be used to study how the solitons interact for all time,our ideas will automatically apply to many equations.

Soliton equations are especially interesting because of the collisional properties of solitons.This was evident in Zabusky and Kruskal's 1965 paper [1].Yet,despite the amazing strides forward in our understanding of soliton equations,our understanding of how solitons interact is sketchy.In this thesis we aim to rectify the situation,and build on what progress has been made(chapter 4) to fully understand the detailed interaction of solitons.

CHAPTER 4 : SOLITONS AS PARTICLES AND THEIR INTERACTION.

§ 0. Introduction

This chapter is perhaps the most varied of the thesis.There are a great many aspects of soliton research which hint of the possibility of regarding solitons as "elementary particles".The work of Perring and Skyrme reviewed in §1 was a direct attempt to construct a theory of elementary particles based on solitons (of the sG).We concentrate on their method of determining the intersoliton potentials.

Over a period of time there have been a number of diverse attempts to understand how sG solitons(in particular) interact.We review these attempts in §1.The most successful approach to soliton interaction(two parameters) was devised by Bowtell and Stuart(BS).Their method consisted of allowing the space variable to become complex, thus allowing the complex Hamiltonian density to develop poles.Remarkably the dynamics of the singularities of the two soliton solutions could easily be determined(in the centre-of-velocity case).We review the BS findings in §2.

The singularities of the complexified sG can be simply mapped into the *real* singularities of the shG equation. Pogrebkov explored the motion of these real singularities and rediscovered some of the BS work.He also numerically analysed the motion of the real singularities of the "soliton-breather" solutions,and noted an interesting feature of their interaction.Namely,the incoming "free soliton" replaces the trapped breather soliton.The equivalent sG case and many others have been analysed by the author and the findings are presented in part two of the thesis.

In §3 we review how multisoliton solutions may be constructed by nonlinear superposition of asymptotic solitons(BS).We discover,how,via certain invariants we may switch the sG soliton interaction on or off.We also review Matsuda's remarkable,but generally unknown,discovery of an explicit linear superposition of accelerating solitons for the sG two parameter solutions.

Linear superposition as a means of understanding soliton interaction was suggested by GGKM for the KdV equation.Yoneyama and independently Caenepeel and Malfliet took up this suggestion by GGKM,and attempted to analyse two soliton KdV interaction in detail.The concept of linear superposition of solitons also arises with respect to the cnoidal solutions of soliton equations.Finally,linear superposition of accelerating solitary waves appears with regard to the two-solitary wave solutions of the ϕ^4 equation (Moshir).We review all these aspects in §3.

In §4 we review work done on the motion of singularities(real or complex) of solutions of nlpde's.Kruskal first suggested that allowing the spatial variable to become complex might be a profitable way of understanding soliton interaction(KdV).

In addition to soliton solutions many soliton equations have *rational* solutions possessing singularities for complex spatial variable.The motion of these singularities has been shown(KdV) to be related to a solvable many body problem,of particles interacting via a r^{-2} potential.This exposes the amazing richness of soliton equations.The motion of real singularities of the apparently non-soliton Liouville equation also has interesting dynamics.

Calogero and Degasperis discovered that certain coupled systems of nlpde's were solvable by the inverse scattering method.These new equations had interesting soliton dynamics.They found *single* solitons,moving as if under the influence of an external potential.The N soliton solutions of their equations("boomeron") developed poles in the complex plane.Analysing the two soliton solution of their equation which was Galilean invariant),they were able to obtain the dynamics of the solitons explicitly and discovered that they moved as if under the influence of a mutual $\mathrm{cosech}^2 r$ potential ($r\equiv$ relative separation of solitons).

The solution of nlpde's in bounded regions of space is a relatively new topic.We review some work done on the solutions of the sG in bounded regions in §5.We find solutions which can be regarded as particles bouncing off barriers or inside potential wells.This subject is particularly interesting as we might look for possible quantum-like behaviour of classical solitons.

In §5 we also review some of the research carried out on perturbed soliton equations.There has been considerable controversy connected with the subject of sG soliton dynamics under the influence of perturbations.We have included a very large number of references on soliton perturbation mainly for completeness but also to illustrate the difficulties of understanding soliton dynamics under perturbation.The behaviour of the sG soliton when exposed to weak applied fields can be quite suprising.Over short time scales the soliton moves in a manner such that its position is not proportional to t^2.This is the so-called "non-Newtonian" behaviour.This latter term should really read "non-Newtonian point-like" behaviour,as over short time scales the bulk movement of highly deformable bodies *is* "non-Newtonian" in general.

Finally in §5 we note that Gorshkov and Ostrovsky were able to deduce via an asymptotic analysis(which they pioneered) that the solitons of the KdV and MKdV *repel* each other.This has also been confirmed by a very recent numerical analysis.This repulsion is in agreement with BS and also our own work on the subject,and disagrees with the workers using the GGKM lsp to analyse soliton interaction.

In §6 we give a short review of some interesting papers on higher dimensional solitary waves.We also review briefly some interesting work on the quantum-like properties of classical solitons.We end the chapter with some conclusions.

§ 1. Attempts at deducing the force laws obeyed by sG solitons in interaction.

The possibility of regarding sG solitons as particles in interaction in the context of a field theory of matter was first put forward in 1958 by Skyrme [1-4].Perring and Skyrme [5] re-discovered the two parameter solutions of the sG†.

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi \qquad (4.1)$$

2-soliton solution

$$\tan(\varphi/4) = u\sinh\gamma x/\cosh\gamma ut \;,\gamma=(1-u^2)^{-1/2} \qquad (4.2)$$

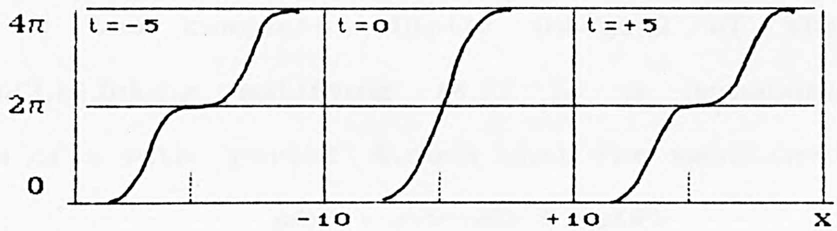
Soliton-antisoliton

$$\tan(\varphi/4) = \sinh\gamma ut/u\cosh\gamma x \qquad (4.3)$$

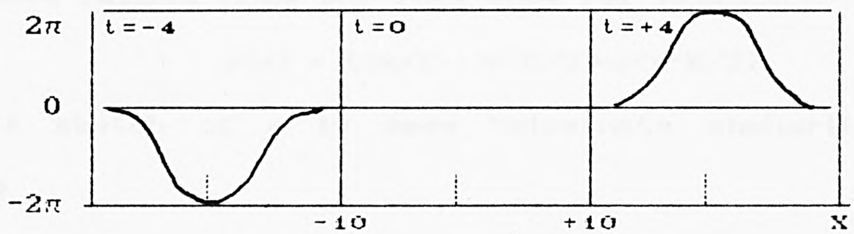
Breather

$$\tan(\varphi/4) = \sin\sigma ut/u\cosh\sigma x \;,\sigma=(1+u^2)^{-1/2} \qquad (4.4)$$

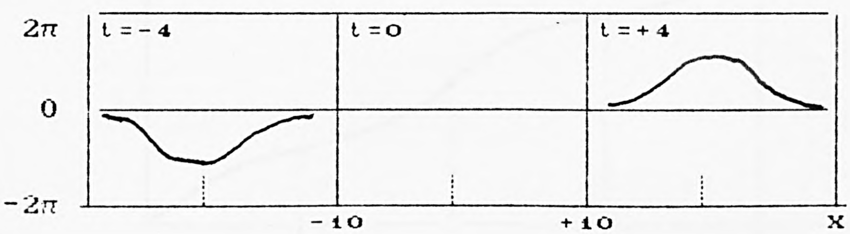
The appearance of these solutions is seen below.



two solitons



soliton-antisoliton



breather

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† Actually,these were discovered earlier by A.Seeger,H.Donth and A.Kochendörfer.Z.PHYS.134 (1953)pp173-193.

Perring and Skyrme defined the position of the soliton to be given by,

$$\cos\varphi = -1 \quad ,(\varphi = (2n+1)\pi,\ n\in \mathbb{Z}) \tag{4.5}$$

Equation (4.1) can be deduced from the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\,[\varphi_t^2 + \varphi_x^2 + 2(1-\cos\varphi)] \tag{4.6}$$

The sG admits static periodic solutions [6].For a "periodic array of solitons" separated by distance R,we have,

$$x = kF([\varphi-\pi]/2,k) \quad , \quad 0 \le x \le R/2 \tag{4.7a}$$

$$R = 2kK(k) \tag{4.7b}$$

where $F(\alpha,k)$ is the incomplete elliptic integral of the first kind,defined by,

$$F(\alpha,k) = \int_0^{\alpha}(1-k^2\sin^2\theta)^{-1/2}d\theta = u \ ,\alpha \equiv \mathrm{am}(u)$$

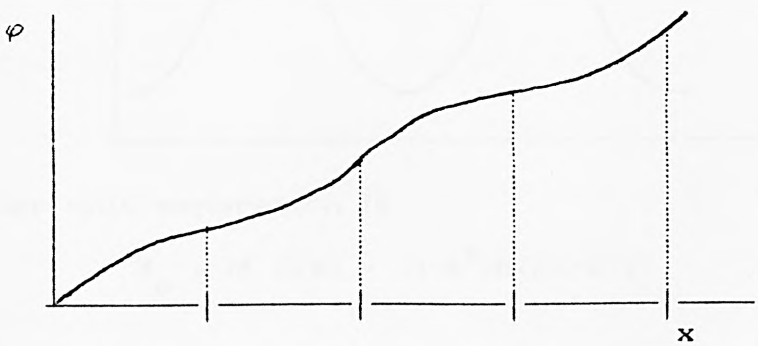
and K is the complete elliptic integral of the first kind $K(k)=F(\pi/2,k),0<k<1$.φ satifying (4.7) is a monotonic increasing function of x with "period" R,such that for $x\in[nR,(n+1)R],n\in \mathbb{N}^+$

$$\varphi(x) = \varphi(x-nR) + n\varphi(R)$$

The above follows from the fact that for $R/2\le x\le R$,

$$\varphi(x) = (4x/R-1)\varphi(R/2)-\varphi(x-R/2)$$

A sketch of φ is seen below(note similarity to soliton ladder).



The energy per unit separation R,or energy per soliton(one soliton per interval) is from (4.6)

$$\mathcal{E}_0 = \int_0^R[\varphi_x^2/2 + 1 - \cos\varphi]dx = 8[\ E(k)-(1-k^2)K(k)/2\ ]/k \tag{4.8}$$

where $E(k)$ is the complete elliptic integral of the second kind defined by,

$$E(k) = E(k,\pi/2) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}d\theta$$

Now from (4.7b) $R\gg1$ corresponds to $k\rightarrow1$ and $R\ll1$ corresponds to $k\rightarrow0$.Thus we find,

$R\gg1$ $$\mathcal{E}_0 \cong 8 + 32e^{-R} \tag{4.9a}$$

$R\ll1$ $$\mathcal{E}_0 \cong 8 + (2\pi^2/R - 8) \tag{4.9b}$$

Since the soliton rest mass is 8,if (4.9) represents two body forces,the intersoliton potential for large R should be of the order $16e^{-R}$.These results are essentially those of Perring and Skyrme,but we have used a treatment of them given by Hsu [6].

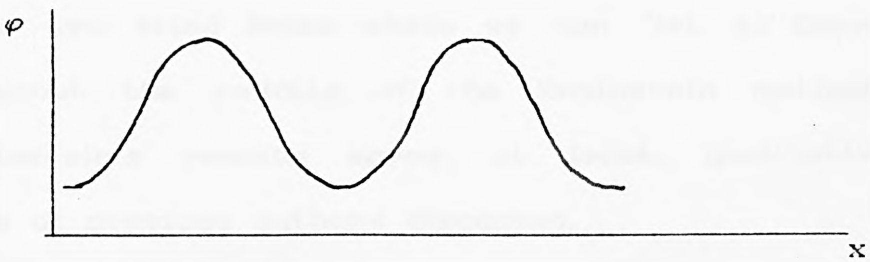
In a similar way we can obtain a static solution for a periodic array of solitons and antisolitons with separation R [6].In this case $\varphi(x)$ is defined by,

$$x = K(k) - F(\xi,k) \quad ,0\le x\le R/2 \ ,R>\pi \tag{4.10a}$$

$$R = 2K(k) \tag{4.10b}$$

$$\xi = \sin^{-1}(k^{-1}\cos(\varphi/2)) \tag{4.10c}$$

A sketch of φ is seen below.Note similarity to soliton antisoliton ladder.



The energy per unit separation is

$$\mathcal{E}_0 = 8[\ E(k) - (1-k^2)K(k)/2\ ]$$

and we find,

$R \gg \pi$ $$\mathcal{E}_0 \cong 8 - 32e^{-R} \tag{4.11a}$$

$R \gtrsim \pi$ $$\mathcal{E}_0 \cong 8 - (8-2R) \tag{4.11b}$$

When $R\rightarrow\pi$ the graph of φ versus x tends to the constant solution $\varphi = \pi$,and when $R < \pi, \varphi = \pi$ is the only solution.

Rajaraman [7] obtained (4.9) and (4.11) using a classical approximation to the quantized sG.

Rubinstein [8] attempted to obtain the intersoliton forces directly,by examining the two soliton and soliton-antisoliton solutions of the sG when the solitons were very far apart.He imagined a solution of the sG which was a small departure from (a) a sum of two fixed solitons and (b) a sum of a fixed soliton and a fixed antisoliton.He found that in case (a) after, a short time interval,the solitons had moved apart indicating repulsion.While in case (b) after the short interval the components had come closer together,indicating attraction.He calculated the "force" between two kinks as proportional to, $N_1N_2e^{-2q}$,where N_i=1(-1) for a soliton (antisoliton),and 2q is the relative separation of the kinks,which he assumes is large compared with their widths.

BS [9] criticized Rubinstein's approach.Firstly, Rubinstein's "force" was actually an impulse.Secondly two soliton solutions are strictly time dependent,hence we cannot imagine we have two fixed kinks which we can "let go".Despite these doubts about the validity of the Rubinstein method,we find that Rubinstein's results agree, at least, qualitatively with the results of previous authors discussed.

In Hsu's paper [6] a number of methods were developed for extracting information about multisoliton potentials.These methods were based on earlier work by Troost [10] and Vinciarelli [11].Vinciarelli had obtained an effective soliton-antisoliton potential,V_{sa}=$-2m\,\mathrm{sech}^2(mR/2)$,where m is the free soliton mass(8) and R the soliton-antisoliton separation. Hsu found a formula for V_{sa} which was more sharply decreasing than Vinciarelli's.Hsu used a technique which related the potential to the time delay/advance incurred by the interaction (with respect to no interaction).The technique is fairly complicated and does not give an explicit formula for the intersoliton potential.

Before we discuss the BS technique for analysing intersoliton interaction,we mention a paper by Ringwood [12](published 3 years after the BS paper).It is important because Ringwood obtained an exact formula for the intersoliton potential,as a function of time,by a completely different technique to any others'.

From the Lagrangian density for the sG,

$$\mathscr{L} = [\varphi_t^2 - \varphi_x^2 + 2(\cos\varphi - 1)]/2 \tag{4.12}$$

we may obtain the energy-momentum tensor $T^{\mu}{}_{\nu}$.

$$T^{\mu}{}_{\nu} = \varphi_{,\nu} \frac{\partial \mathscr{L}}{\partial \varphi_{,\mu}} - \delta^{\mu}{}_{\nu} \mathscr{L} \tag{4.13}$$

(where as usual $\varphi_{,\nu} \equiv \partial\varphi/\partial x^{\nu}$).The momentum in the spatial interval [a,b],$P|_a^b$ is given by,

$$P|_a^b = \int_a^b dxT^{01} \tag{4.14}$$

and the force in the spatial interval is,

$$\partial_0 \int_a^b dxT^{01} = -T^{11}|_a^b \tag{4.15}$$

For the soliton-soliton solution (4.2),Ringwood obtained the force($-T^{11}|_{-\infty}^{0}$) in the centre-of-velocity frame F_{ss},given by,

$$F_{ss} = -8u^2\gamma^2\text{sech}^2\gamma ut \tag{4.16}$$

and for the soliton-antisoliton solution he obtained,

$$F_{sa} = -8u^2\gamma^2[\sinh^2\gamma ut - u^2\cosh 2\gamma ut]/[\sinh^2\gamma ut + u^2]^2 \tag{4.17}$$

As we shall see (4.16) agrees with the BS results,however (4.17) does not.The BS technique gives an infinite force at t=0,while Ringwood's formula above is finite.It seems likely that Ringwood's technique may break down because of this singularity.

§ 2. The Bowtell-Stuart technique

The curious fact about the BS method [9] for analysing sG soliton interaction is that it has apparently gone unnoticed for so long.Hsu(1980) makes no reference to BS.Ringwood cites a number of researchers,including the Rajaraman paper which is in the *same* volume of Physical Review as that of BS!.In fact the *only* paper the author has found anywhere in the literature which cites the BS paper is one by Matsuda [13],which we will come to in the next section.Strangely, Matsuda's paper appears to have gone unnoticed also.

BS in an attempt to analyse the motion of solitons throughout the interaction introduced the idea of allowing the spatial variable to become complex.Kruskal [14] had earlier suggested that this might be a useful way at getting at KdV soliton interaction,though he did not pursue the subject.

The N soliton solution of the sG may be written(Hirota[15]),

$$\varphi = 4\tan^{-1}(g/f) \tag{4.18}$$

where,

$$f = \sum_{\mu_i=0,1}^{(e)} \exp\theta_{ij} \quad , \quad g = \sum_{\mu_i=0,1}^{(o)} \exp\theta_{ij} \tag{4.19}$$

$$\theta_{ij} = \sum_{i<j}^{N} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\gamma_i\xi_i + \alpha_i) \tag{4.20a}$$

$$\xi_i = x - u_i t \quad , \quad \gamma_i = (1-u_i^2)^{-1/2} \tag{4.20b}$$

Superscripts (e) and (o) on the summation signs denote summation over all possible combinations μ_i=0,1 under the condition $\sum_{i=1}^{N}\mu_i$ even, $\sum_{i=1}^{N}\mu_i$ odd ,respectively.BS showed that,

$$A_{ij}^2 = \ln u_{ij}^2 \tag{4.21}$$

where u_{ij} is the common speed of the i^{th} and j^{th} solitons in the c-of-v frame,and is given by,

$$u_{ij} = [1-u_iu_j-(1-u_i^2)^{1/2}(1-u_j^2)^{1/2}]/(u_i-u_j) \tag{4.22}$$

Substituting (4.18) into the sG Hamiltonian density (4.6) gives,

$$\mathcal{H} = 8(f^2+g^2)^{-2}[(fg_x-gf_x)^2+(fg_t-gf_t)^2+g^2f^2] \tag{4.23}$$

For real spatial variables x, $\mathcal{H}$ is always finite,however if $x\longrightarrow x+iy$,$\mathcal{H}$ may develop poles when,

$$f = \pm ig \tag{4.24}$$

Consider the single soliton solution(z=x+iy),

$$\varphi_s = 4\tan^{-1}\exp\gamma(z-ut) \tag{4.25}$$

$\mathcal{H}$ has poles at the branch points of φ_s when

$$z = ut + i\gamma(2n+1)\pi/2 \quad ,n\in \mathbb{Z} \tag{4.26}$$

Thus,for a single soliton we have a sequence of regularly spaced poles parallel to the imaginary axis with a single projection on the x axis,which moves at speed ut.

In the two soliton case the poles are determined by (4.2),

$$\sinh z = \pm i\cosh\gamma ut/u \tag{4.27}$$

solving for real and imaginary parts gives,

$$X = \pm \frac{1}{\gamma}\ln\left[\frac{\cosh\gamma ut+(\cosh^2\gamma ut-u^2)^{1/2}}{u}\right] \tag{4.48a}$$

$$y = (2n+1)\pi/2\gamma \quad , n\in \mathbb{Z} \tag{4.28b}$$

Thus, associated with the two soliton solution are two lines of poles lying parallel to the imaginary axis and positioned symmetrically about it,also there are only two real projections given by (4.28a).As time evolves the lines of poles, initially moving at speed u toward each other,slow down,stop, and then accelerate away from one another.

If we choose a pair of poles with fixed imaginary part,we find they move along a line parallel to the x axis.In this way we have a particle representation of interacting solitons.We identify the centre of mass of the real soliton with the position of the associated pole of the Hamiltonian density(or field branch point).

From the positions of the poles we can calculate exactly the velocity,acceleration and forces experienced by the pole particles.The force between the representative pole particles is found to be,

$$F = 8\gamma^2 \mathrm{sech}^2 \gamma X \tag{4.29}$$

where $|2X|$ is the distance between the pole particles.

In time dependent form(using (4.27)) the above can be written $F=8\gamma^2\mathrm{sech}^2\gamma ut$.This is the result found by Ringwood(4.16) 3 years later.Associated with (4.29) is the potential,

$$V(X) = 8\gamma(1-|\tanh\gamma X|) \tag{4.30}$$

When the solitons are far apart (4.29) gives,

$$F \cong 32\gamma^2 \exp(-\gamma|2X|) \tag{4.31}$$

Thus the results of §1 agree with the BS findings,in the appropriate limit.

The soliton-antisoliton solution(4.3) which BS took in the equivalent form($\varphi \longrightarrow -\varphi+2\pi$),is,

$$\varphi_{sa} = 4\tan^{-1}(u\cosh\gamma z/\sinh\gamma ut) \tag{4.32}$$

Poles of $\mathcal{H}$ occur when,

$$\cosh\gamma z = \pm i\sinh\gamma ut/u \tag{4.33a}$$

Solving we find,

$$X = \pm\frac{1}{\gamma}\ln\left[\frac{\sinh\gamma ut+(\sinh\gamma ut^2+u^2)^{1/2}}{u}\right] \tag{4.33b}$$

$$y = (2n+1)\pi/2\gamma \ , \ n\in \mathbb{Z} \tag{4.33c}$$

As with the two soliton case,we find pairs of pole particles moving along straight lines parallel to the x axis. We can calculate easily the velocity, acceleration and forces which the pole particles are subjected to. The force is found to be,

$$F = 8\gamma^2 \operatorname{cosech}^2 \gamma X \tag{4.34}$$

and this corresponds to a potential,

$$V(X) = 8\gamma(1-|\coth\gamma X|) \tag{4.35}$$

At t=0, X=0, so F is infinite. At this point the pole particles pass through each other in opposite directions, at the speed of light. BS also employed the same technique with the breather solution(4.4),

$$\varphi_b = 4\tan^{-1}(\sin\sigma ut/u\cosh\sigma z) \;,\; \sigma=(1+u^2)^{-1/2}, u\in \mathbb{R} \tag{4.36}$$

The pole positions turn out to be given by,

$$X = \pm \frac{1}{\sigma} \ln\left[\frac{\sin(\sigma ut)+(\sin^2\sigma ut+u^2)^{1/2}}{u}\right] \tag{4.37a}$$

$$y = (2n+1)\pi/2\sigma \;,\; n\in \mathbb{Z} \tag{4.37b}$$

The attractive force between breather poles has the same form as the soliton-antisoliton case,

$$F = 8\sigma^2 \operatorname{cosech}^2 \sigma X \tag{4.38}$$

$$V(X) = 8\sigma(1-|\coth\sigma X|) \tag{4.39}$$

The motion of the bound breather poles is periodic. They oscillate about x=0 with period τ and amplitude A given by,

$$\tau = 2\pi/\sigma u \tag{4.40}$$

$$A = \frac{1}{\sigma} \ln\left[\frac{1+(1+u^2)^{1/2}}{u}\right] \tag{4.41}$$

Pogrebkov [16] examined the motion of the real singularities of the shG, which is obtained from the sG by mapping $\varphi\rightarrow i\varphi$. This changes the inverse tangent into an inverse tanh. Otherwise all solutions are identical in form to those of the sG. The singular soliton solution of the shG is $(X\equiv\gamma(x-ut))$,

$$\varphi = 4\tanh^{-1}e^X = \ln\left(\frac{1+\exp X}{1-\exp X}\right)^2 = \ln[\tanh^2(X/2)]$$

In the above φ develops real singularities when $e^X = 1$,this is of course,identical to the condition that the pole projections of sG must satisfy.Actually the idea of mapping the sG to the shG can be quite useful,as in the laboratory frame,it is easily shown that the two parameter shG possesses *only two* real singularities.However analysis of the two parameter complex sG shows it can have an infinite number of poles(see §5 chapter 6).This confirms the author's contention that the extra poles are of no significance as only two have real projections.

We note that the two parameter solutions' associated poles(sG) can be described by a relativistic two body Hamiltonian,

$$\mathscr{H} = 8(1-\dot{x}_1^2)^{-1/2} + 8(1-\dot{x}_2^2)^{-1/2} + V \qquad (4.42)$$

where 8 is the rest mass of the solitons(or antisolitons) and $\dot{x}_1^2=\dot{x}_2^2$ are the squared velocities obtained by differentiating (4.28a,4.33b,4.37a).For the two soliton solution $V(X) = 16\gamma(1-|\tanh\gamma X|)$,while for the soliton-antisoliton solution $V(X) = 16\gamma(1-|\coth\gamma X|)$ and for the breather $V(X) = 16\sigma(1-|\coth\sigma X|)$.We note that for two soliton cases or soliton/antisoliton cases,as $t\longrightarrow\pm\infty,\mathscr{H}\longrightarrow16\gamma$,while in the breather case when $t=\tau,\dot{x}_1=\dot{x}_2=0$,and $\mathscr{H} = 16\sigma <$ rest energies of constituents.This indicates the bound state nature of the breather.

A generalisation of equation (4.42) is useful in higher parameter cases as it enables us to calculate numerically a global potential as a function of time(chap 7). Another point of interest here is the "gravitational" nature of the force between "neutral"(equal numbers of solitons and antisolitons) collections of solitons.Since the attractive force between opposites is greater than the repulsive force between like(4.29,4.34),we may profitably imagine that there are *two* potentials acting between solitons:gravitation-like,and electrostatic like.The "electric potential" generated by a soliton(+) or antisoliton(-) would be,

$$V_e = \pm 8\gamma\left(\frac{|\coth\gamma X| - |\tanh\gamma X|}{2} \right) \quad (4.43a)$$

and the "gravitational" would be given by,

$$V_g = 8\gamma\left(1 - \frac{|\tanh\gamma X| + |\coth\gamma X|}{2}\right) \quad (4.43b)$$

When X is large,

$$V_e \longrightarrow \pm\, 16\gamma\exp(-2\gamma X) \quad , \quad V_g \longrightarrow 16\gamma\exp(-8\gamma X) \quad (4.44)$$

For a large collection of N solitons and antisolitons in equal numbers,the total potential acting on a lone soliton or antisoliton would be $\sum 16\gamma_i \exp(-8\gamma_i X)$ due to the "gravitational" potential alone,where here X would represent the distance from the lone soliton to the "centre of mass" of the large collection.The "electric potential" would rapidly tend to zero.

§ 3. Linear and nonlinear superposition

Nonlinear superposition principles(nlsp) have been established for a number of important soliton equations.These were reviewed in the last chapter.We saw that in particular the sG and MKdV shared the same nlsp(apart from a factor of 2).

BS [9] gave an interesting presentation of the two and three soliton solutions of the sG.They found that by introducing the pairwise c-of-v speeds u_{ij}(4.22),the actual solutions of the sG,for two or three solitons,could be obtained from a *linear* superposition of single solitons.As we have seen,the single soliton solutions of the sG may be expressed,

$$i=1,2,3 \qquad \tan(\varphi_i/4)=\exp X_i, X_i=\gamma_i(x-u_i t), \gamma_i=(1-u_i^2)^{-1/2} \quad (4.45)$$

A linear superposition of these solutions is given by,

$$\varphi_{123} = \sum_{i=1}^{3} \varphi_i \tag{4.46}$$

Thus,

$$\tan(\varphi_{123}/4) = \frac{\sum_{i=1}^{3} \tan(\varphi_i/4) - \prod_{i=1}^{3} \tan(\varphi_i/4)}{1 - \sum_{i \neq j} \tan(\varphi_i/4)\tan(\varphi_j/4)} \tag{4.47}$$

To obtain the three soliton solution,BS observed that we merely "switch on" the interaction via invariants u_{ij}^2 .

$$\tan(\varphi_{123}/4) = \frac{\sum_{i=1}^{3} \tan(\varphi_i/4) - u_{12}^2 u_{13}^2 u_{23}^2 \prod_{i=1}^{3} \tan(\varphi_i/4)}{1 - \sum_{i \neq j} u_{ij}^2 \tan(\varphi_i/4)\tan(\varphi_j/4)} \tag{4.48}$$

We note,in addition to this,that the switching operation can be done in pairwise fashion in the following way.To switch off the interactions between solitons 1 and 2 with soliton 3,we let $u_{13}=1, u_{23}=1$,(4.48) can then be written,

$$\tan(\varphi_{123}/4) = \frac{\tan(\varphi_{12}/4) + \tan(\varphi_3/4)}{1 - \tan(\varphi_{12}/4)\tan(\varphi_3/4)} \tag{4.49a}$$

$$\tan(\varphi_{12}/4) = \frac{\tan(\varphi_1/4) + \tan(\varphi_2/4)}{1 - u_{12}^2 \tan(\varphi_1/4)\tan(\varphi_2/4)} \tag{4.49b}$$

φ_{12} is just the known two soliton solution of the sG.Thus from (4.49a),

$$\varphi_{123} = \varphi_{12} + \varphi_3 \tag{4.49c}$$

This demonstrates very clearly how the presence of $u_{ij} \neq 1$ generates the interaction.

Matsuda [13],as an intended approximation to the two soliton solutions of the sG,investigated whether the two soliton solution could be expressed as a linear superposition of accelerating kinks,

$$\varphi_{ss} = 4\tan^{-1}\exp\gamma[x+\zeta(t)] + 4\tan^{-1}\exp\gamma[x-\zeta(t)] \tag{4.50}$$

Using a rather obscure variational approach,involving the evaluation of some complicated integrals,Matsuda discovered that if,

$$\zeta(t) = \frac{1}{\gamma}\ln\left[\frac{\cosh\gamma ut+(\cosh^2\gamma ut-u^2)^{1/2}}{u}\right] \tag{4.51}$$

φ_{ss} is the *exact* two soliton solution,given by(after $\varphi\rightarrow\varphi+2\pi$),

$$\tan(\varphi_{ss}/4) = -\cosh\gamma ut/u\sinh\gamma x \tag{4.52}$$

He was also able to obtain exact results for the soliton antisoliton solution φ_{sa} and the breather solution φ_b.

$$\varphi_{sa} = 4\tan^{-1}\exp\gamma[x+\eta(t)] + 4\tan^{-1}\exp-\gamma[x-\eta(t)] \tag{4.53a}$$

$$\eta(t) = \frac{1}{\gamma}\ln\left[\frac{\sinh\gamma ut+(\sinh\gamma ut^2+u^2)^{1/2}}{u}\right] \tag{4.53b}$$

$$\tan(\varphi_{sa}/4) = -\cosh\gamma x/u\sinh\gamma ut \tag{4.53c}$$

$$\varphi_b = 4\tan^{-1}\exp\sigma[x+\xi(t)] + 4\tan^{-1}\exp-\sigma[x-\xi(t)] \tag{4.54a}$$

$$\xi(t) = \frac{1}{\sigma}\ln\left[\frac{\sin(\sigma ut)+(\sin^2\sigma ut+u^2)^{1/2}}{u}\right] \tag{4.54b}$$

$$\tan(\varphi_b/4) = -\cosh\sigma x/u\sin\sigma ut \tag{4.54c}$$

Comparing the above expressions with the moving pole formulae of BS we see that Matsuda's accelerating kinks move in an identical manner.Thus a connection is established between the BS poles and linear superposition of accelerating kinks.Unfortunately Matsuda's technique fails for general Lorentz frames and for higher parameter solutions(>2).The important feature of the lsp above is that each accelerating kink carries a branch point with a *single* projection on the real axis.

Linear superposition appears in a number of places in the literature.We have already seen it formulated for the KdV by GGKM [17] in the last chapter.Yoneyama [18] and Caenepeel and Malfliet [19] analysed in detail,soliton interaction via the GGKM lsp.We shall review their findings shortly.

Many soliton equations have solutions which have not been regarded as "soliton-like" in their nature.We refer to the cnoidal solutions.These were first found for the KdV by Kortweg and de Vries [20].Cnoidal solutions are periodic functions involving Jacobian elliptic functions.We have already met with static cnoidal waves for the sG in §1.We referred to these solutions as a "periodic array of solitons".In fact,cnoidal wave solutions of soliton equations can generally be thought of as a linear superposition of an infinite number of solitons. This was shown generally,by Zaitsev [21] and for the KdV by Korpel and Banerjee [22].Cnoidal waves have regularly repeating poles in the complex plane.The interaction of cnoidal waves with solitons(a subject as yet unexplored) might be profitably explored by the BS technique.

The "φ^4" equation,

$$\varphi_{xx} - \varphi_{tt} = \varphi - \varphi^3 \qquad (4.55)$$

has a kink solution,

$$\varphi = \varphi_c = \pm \tanh[\gamma(x-vt)/\sqrt{2}] \ , \ \gamma=(1-v^2)^{-1/2} \qquad (4.56)$$

Equation (4.55) is known not to support multisoliton solutions (as energy is lost in a multisolitary wave collision).However from our point of view it is interesting to note that the two kink solution φ_{12} can be written as a linear superposition of accelerating kinks plus some radiation,

$$\varphi_{12} = \varphi_c[f(t)(x-a(t)] + \varphi_c[f(t)(x+a(t)]-1+\varepsilon(x,t) \qquad (4.57)$$

The radiation component is the oscillatory term $\varepsilon(x,t)$.The above result was discovered in a numerical analysis by Moshir [23].Matsuda [13] had tried an ansatz like (4.57) for the φ^4 equation,but without the success he achieved with the sG.

Yoneyama and Caenepeel and Malfliet(CM) analysed the two soliton interaction of the KdV via the GGKM lsp(chapter 3),

$$u(x,t) = -\sum_{m=1}^{2} 4k_m^2\varphi_m^2(x,t) \qquad (4.58)$$

Defining each soliton for all time(after GGKM) by,

$$u_i \equiv -4k_i^2\varphi_i^2 \tag{4.59}$$

they found(we follow CM's treatment),

$$u_1 = -4k_1^2c_1^2(0)e^{-2k_1\xi_1}[1 + u_{12}c_2^2(0)e^{-2k_2\xi_2}/(2k_2)]^2/D^2 \tag{4.60a}$$

$$u_2 = -4k_2^2c_2^2(0)e^{-2k_2\xi_2}[1 + u_{12}c_1^2(0)e^{-2k_1\xi_1}/(2k_1)]^2/D^2 \tag{4.60b}$$

where,

$$u_{12} = (k_1-k_2)/(k_1+k_2) \ , \ k_1 > k_2 \tag{4.60c}$$

$$D=1 + \frac{c_1^2(0)}{2k_1}e^{-2k_1\xi_1} + \frac{c_2^2(0)}{2k_2}e^{-2k_2\xi_2} + u_{12}^2\frac{c_1^2(0)c_2^2(0)}{2k_1 \ 2k_2}e^{-2(k_1\xi_1+k_2\xi_2)} \tag{4.60d}$$

$$\xi_i = x-4k_i^2t \tag{4.60e}$$

CM study the particular example of the two soliton solution of the KdV which has the form at t=0,

$$u(x,0) = -6\text{sech}^2x \tag{4.61a}$$

The eigenvalues and eigenfunctions of the associated scattering problem (GGKM [17]) are ,

$$k_1=2 \qquad \varphi_1=(\sqrt{3}/2)\text{sech}^2x \qquad c_1(0)=2\sqrt{3} \tag{4.61b}$$

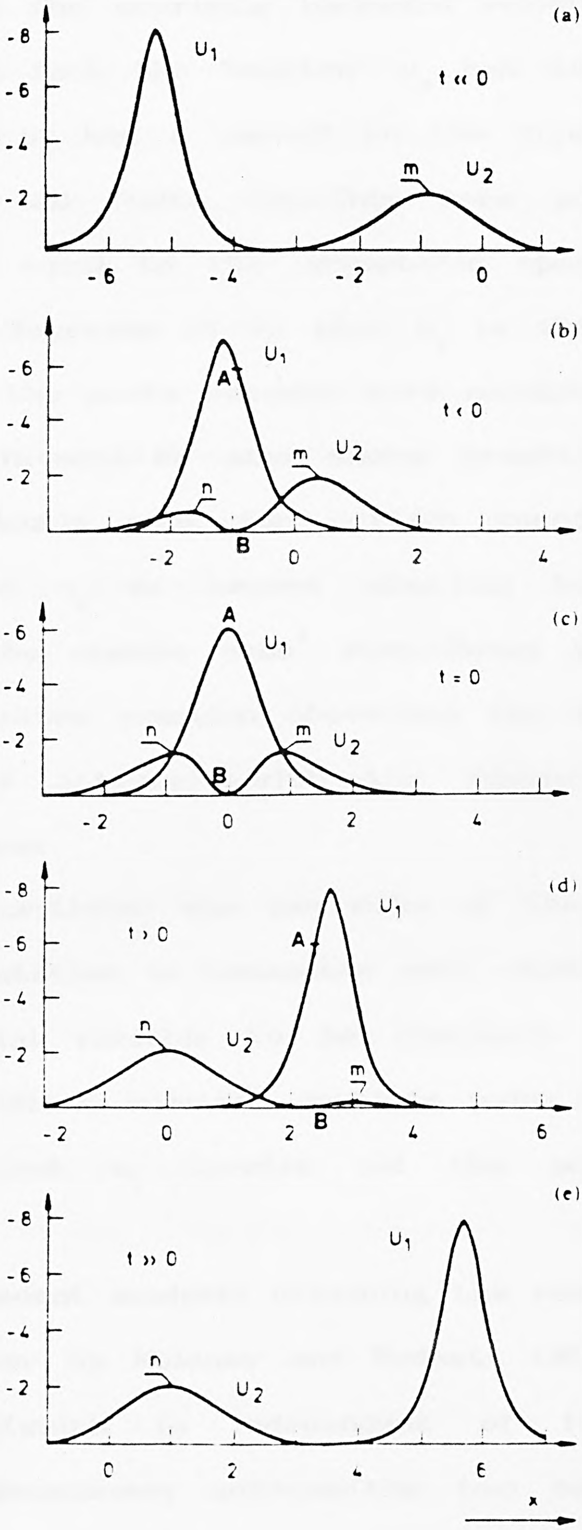
$$k_2=1 \qquad \varphi_2=(\sqrt{3}/2)\text{sech}x.\tanh x \qquad c_2(0)=\sqrt{6} \tag{4.61c}$$

CM obtain for $u_i(x,t)$,the following,

$$u_1(x,t) = \frac{-24\cosh^2\xi_2}{(2\cosh 2\xi_1\cosh\xi_2-\sinh 2\xi_1\sinh\xi_2)^2} \tag{4.62a}$$

$$u_2(x,t) = \frac{-6\sinh^2 2\xi_1}{(2\cosh 2\xi_1\cosh\xi_2-\sinh 2\xi_1\sinh\xi_2)^2} \tag{4.62b}$$

The time evolution and shapes of these functions,is shown in figures (a)-(e) taken from the CM paper.



The positions of the solitons are taken to be at the "centre-of-mass" defined below,

$$x_a^{(i)} = \int_{-\infty}^{\infty} x u_i \, dx \Big/ \int_{-\infty}^{\infty} u_i \, dx \qquad (4.63)$$

CM show that at t=0 the speeds of the solitons are given by,

$$\dot{x}_a^{(1)} \cong 19.2 \quad , \quad \dot{x}_a^{(2)} \cong -2.4 \qquad (4.64)$$

They note the suprising backward velocity of the smaller soliton.In actual fact the "soliton" u_2 has two maxima for all finite time.Also u_2 has a zero(B on the figures) between the double peak,for all finite time.This zero point moves at a constant speed equal to the asymptotic speed of the faster incoming soliton.Yoneyama shows that u_2 is always a double peak (though one of the peaks becomes very small,tending to zero,for large time magnitudes).He also shows graphically that u_1 can also have a double peak for certain speed ratios.As $t\rightarrow\pm\infty$ however u_1 and u_2 do become identical to constant speed solitons with the classic $sech^2$ shape.These authors using the definition of soliton position above,find the force between KdV solitons to be attractive,with the faster soliton passing through the slower.

A major criticism one can make of the above method of soliton representation is connected with what happens when we allow the spatial variable to be complex.It easily seen that the KdV two soliton solution develops poles when D=0 [24].This implies that each u_i carries *all* the poles.This is most unsatisfactory.

A more recent analysis obtaining the results above can be found in a paper by Moloney and Hodnett [25].It is clear from their approach(which is independent of inverse scattering theory) that Boussinesq solitons(the two soliton solution of the Boussinesq is identical in structure to the KdV two soliton solution) must interact in a very similar way to KdV solitons(i.e attractively).We shall see later in our own contribution to this thesis however ,that the Boussinesq equation has a two soliton solution which can be *exactly* decomposed into accelerating solitons of exact $sech^2$ form.Moreover these solitons are found to *repel* one another(each of these solitons carries a *single* pole).

Detailed analysis of the motion of the poles of the KdV two soliton solution can be found in a paper by BS [24].They found that the *faithful* poles interacted repulsively.Gorshkov et al [70] also came to the conclusion that the KdV(&MKdV) solitons interacted repulsively.Indeed a very recent numerical analysis by LeVeque [80] confirms Gorshkov and Ostrovsky's work.Both these papers were considering the interaction of KdV solitons,in the two soliton solution,provided the speed ratios were small.Actually Gorshkov and Ostrovsky's work is of great generality,we shall comment on it again in §5.The authors own original work on the subject also comes to the conclusion that KdV solitons interact repulsively and in the same manner as BS faithful poles.

§ 4. Singularities associated with nonlinear partial differential equations.

Kruskal [14] pioneered the polar representation of solitary waves.He noted that if one allowed the space variable(x) to become complex(z=x+iy),then the single soliton solutions of the KdV equation,

$$u_t + uu_x + u_{xxx} = 0 \qquad (4.65a)$$

$$u = 3c\,\text{sech}^2[\sqrt{c}(x-ct)/2] \qquad (4.65b)$$

possessed double poles at the positions,

$$z - ct = (2n+1)\pi i/\sqrt{c} \quad ,n\in \mathbb{Z} \qquad (4.66)$$

He noted that it would be easier to deal with the derivative KdV,

$$v_t + v_x^2 + v_{xxx} = 0 \qquad (4.67)$$

which has soliton solutions,

$$v(x,t) = 3\sqrt{c}\tanh[\sqrt{c}(x-ct)/2] \qquad (4.68)$$

The principal part of tanh may be written as a Laurent expansion around each pole given by,

$$\tanh z = \sum_{s,odd} (z-s\pi i/2)^{-1} \qquad (4.69)$$

so,

$$v(z,t) = 6 \sum_{s,odd} (z-ct-s\pi i/\sqrt{c})^{-1} \qquad (4.70)$$

Kruskal proposed that a soliton be thought of as a "parade of poles".He suggested that the two soliton solution of (4.67) be looked at,in the form,

$$v(z,t) = \sum_{s,odd} \frac{6}{z - \xi_1(t)} + \sum_{s,odd} \frac{6}{z - \xi_2(t)} \qquad (4.71a)$$

where when $t \to -\infty$,

$$\xi_1(t) \to c_1 t + s\pi i/\sqrt{c_1} \;,\quad \xi_2(t) \to c_2 t + r\pi i/\sqrt{c_2} \qquad (4.71b)$$

Kruskal noted that the trajectories of the poles in the complex plane coincided with the zeros of the Hirota function τ,as the solutions of the KdV (u) can be written,

$$u = 12(\ln \tau)_{xx} \qquad (4.72)$$

Thickstun [26] studied the pole motion in this way,but only for rational speed ratios(of solitons).A much more comprehensive analysis of the poles of u was given by BS [24].It turns out that there are a large collection of poles,with different modes of behaviour,associated with the two soliton solution.Their number and behaviour is intimately connected with the soliton speed ratio ρ.However it is possible [24] to identify two distinct solution functions $z_1(\rho,t)$ and $z_2(\rho,t)$ for all $\rho \in (1,\infty)$ and for which the imaginary parts of $z_i(\rho,t)$ are independent of time.These are the so called "faithful poles".

One of the many extraordinary features of soliton equations, is that the motion of poles of the *rational* solutions(KdV) can be related to the motion of n ,one-dimensional particles interacting via certain potentials [27-9].

The rational solutions of the KdV, $u_t-3uu_x+u_{xxx}/2 = 0$,have the form [30],

$$u(x,t) = 2\sum_{j=1}^{N}[x-x_j(t)]^{-2} \tag{4.73a}$$

Substituting this into the KdV requires the following($\dot{} \equiv d/dt$),

$$\dot{x}_j = 6\sum_{k\neq j}(x_j-x_k)^{-2} \tag{4.73b}$$

$$\sum_{k\neq j}(x_j-x_k)^{-3} = 0 \quad , 1 \le j \le N \tag{4.73c}$$

(4.73a) may be written in Hirota form,

$$u(x,t) = -2[\ln P_N(x,t)]_{xx} \tag{4.74a}$$

$$P_N(x,t) = \prod_{i=1}^{N}[x-x_i(t)] \tag{4.74b}$$

If N=2 there are no x_j satisfying (4.73c).When N=3,the roots are proportional to the cube roots of unity,and we have [31],

$$P_3 = x^3 + t \tag{4.75a}$$

$$u(x,t) = 6x(x^3-2t)/(x^3+t)^2 \tag{4.75b}$$

Equations (4.73b) are related to the integrable† many body problem of N particles on a line occupying positions $(x_1,..,x_N)$ and speeds $(y_1,..,y_N)$ and defined by the Hamiltonian [33],

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N}y_i^2 + \sum_{j<k}(x_j-x_k)^{-2} \tag{4.76}$$

The polynomials $P_N(x,t)$ in (4.74),satisfying the KdV,may be obtained by a limiting process [34] from the Hirota N soliton solutions.

We note that u in (4.73a) has been written as a linear superposition.This suggests that a connection may exist between the author's lsp for the KdV(chapter 6) with this limiting process, and the rational solutions(and in particular the $x_j(t)$).

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† Integrable means ∃ N independent constants of the motion $I_j(q,p)$ whose Poisson brackets $[I_i,I_k]=0$ ∀ i,k.Since one of the constants is the Hamiltonian itself,*all* the I_j are constant in time[32]. (q,p)= generalized position,momentum coordinates.

The Inverse Scattering Method has been considerably extended by Wadati [30] to matrix nlpde's,and even further still by Calogero and Degasperis [30].They discovered certain coupled nlpde's which had solutions in which,single solitons could move as if in an *external potential* (i.e single solitons moving with changing speed).The external potential commonly had the effect of causing incoming solitons to bounce back,and return to their original location.The term "boomeron" was coined to describe these solitons,the equations being known as boomeron equations.These equations also have N soliton solutions.The interaction of the solitons in the two soliton case was analysed by Calogero and Degasperis [35].

We will not consider their analysis in detail as it involves large sets of quite complicated equations.The important point as far as we are concerned,is that by considering the zeros in the complex plane of a function D(x,t) which was involved in the two soliton solution,the two soliton solution developed poles.There were two poles per periodic strip in the complex plane and the poles moved parallel to the real axis (cf BS poles).The boomeron equations are Galilean invariant,so that the two soliton solution could be analysed in the c-of-v frame.

It was found that the solitons moved as if repelling each other via a $g^2\operatorname{cosech}^2(pr)$ potential,where g and p are constants and r is the relative separation of the solitons.The boomeron equations have other solutions with solitons interacting via other potentials,indicating the rich structure present.

We have already seen how the singular solutions of the shG are in 1-1 correspondence with the soliton solutions of the sG.The Liouville equation,

$$\sigma_{xx} - \sigma_{tt} = \exp(\sigma) \qquad (4.77a)$$

belongs to the same family of generalized sG equations and possesses aBT's [36].Unfortunately it does not possess the zero solution and so a simple nlsp leading to a two parameter solution is not obtainable.The general solution of (4.77a) is [37-8](there is a BT from (4.77a) to the linear wave equation),

$$\sigma(x,t) = \ln\left[\frac{8A'(x+t)B'(x-t)}{[A(x+t)+B(x-t)]^2}\right] \tag{4.77b}$$

where A,B are thrice continuously differentiable functions of their arguments,and $A',B'>0$ ($'\equiv$ differentiation with respect to argument).If we choose [37],

$$A = 8\exp\left[(1-v)^{1/2}(1+v)^{-1/2}(x + t - x_o)/8\right] \tag{4.78a}$$

$$B = -8\exp\left[-(1-v)^{1/2}(1+v)^{-1/2}(x - t - x_o)/8\right] \tag{4.78a}$$

we find,

$$\exp\sigma_1 = (32)^{-1}\mathrm{sech}^2[\gamma(x-vt-x_o)/8] \ ,\gamma=(1-v^2)^{-1/2} \tag{4.79}$$

(4.79) is the so called solitary wave solution of (4.77a).σ_1 possesses a real singularity when $x=vt+x_o$,and more generally it will possess singularities whenever $A(x+t)+B(x-t)=0$.

It is shown in [38] that if the initial profile of σ is such as to possess N singularities,then this number is invariant in time.The singularities are found to move like classical particles.A Hamiltonian action-at-a-distance formulation is given for the motion of the singularities in [39].Investigations of nonlinear relativistic pde's which describe singularities interacting via lightlike fields are described in [39,40].

Clearly the investigation of the motion of singularities of nlpde's is a fascinating topic.However it is much more satisfactory to investigate the motion of singularities in the complex plane of nlpde's having multisoliton solutions.The solitons in the real plane have all the virtues of being non-singular,while the complex singularities give us the means of describing the interaction of solitons in terms of point particles.We should view the fact that many solitons have associated singularities in the complex plane, as neither trivial nor coincidental.

§ 5. Solitons in bounded regions of space and under perturbation.

We concentrate in this section on the sG equation,though we will give a very wide list of references on perturbed soliton equations.There has been much controversy regarding the behaviour of sG solitons perturbed by external fields.In fact, as the references show,the controversy is not merely restricted to the sG equation.First we look at the fascinating topic of solitons in bounded regions of space.Fascinating,because quantal objects reveal their true nature most markedly in bounded regions of space(quantized energy levels etc).The behaviour of solitons of the sG(being neither particle nor wave might reveal some surprising behaviour in bounded regions.

The solution of the sG in bounded regions of space has been investigated by DeLeonardis,Trullinger and Wallis(DTW) [41].

We first examine the solutions of the sG in the form,

$$\varphi_{tt} - c_o^2\varphi_{xx} + w_o^2\sin\varphi = 0 \tag{4.80}$$

When dimensionless variables $x \longrightarrow w_o x/c_o$,$t \longrightarrow w_o t$ are introduced (4.80) becomes the standard sG (4.1).

Firstly DTW solve this equation in the region $x \le 0$,subject to the "free" boundary condition,

$$\varphi_x(0,t) = 0 \tag{4.81}$$

To achieve this the parameters c_o,w_o in (4.80) are regarded as step functions and (4.80) is re-expressed,

$$\varphi_{tt} - \Big([c_o^2 - c_o^2\theta(x)]\varphi_x\Big)_x + [w_o^2 - w_o^2\theta(x)]\sin\varphi = 0 \tag{4.82}$$

where $\theta(x)$ is the step function, defined by $\theta(x)=0 \quad x\leq 0, 1, \quad x>0$. (4.82) can be rewritten as,

$$\varphi_{tt} - c_0^2[1-\theta(x)]\varphi_{xx} + c_0^2\delta(x)\varphi_x + w_0^2[1-\theta(x)]\sin\varphi = 0 \qquad (4.83)$$

where $\delta(x)$ is the Dirac δ function.

Now assume

$$\varphi = \varphi_{sa} + \psi \qquad (4.84)$$

where φ_{sa} is the familiar soliton-antisoliton solution of the sG (4.3)

$$\varphi_{sa} = 4\tan^{-1}(\sinh\gamma ut/u\cosh\gamma x) \qquad (4.85)$$

Substituting (4.84) into (4.83) we find ψ satisfies

$$\psi_{tt}-c_0^2\psi_{xx}+c_0^2\delta(x)[(\varphi_{sa})_x+\psi_x]+w_0^2[\sin(\varphi_{sa}+\psi)-\sin\varphi_{sa}] = 0 \qquad (4.86)$$

But from (4.85) $(\varphi_{sa})_x=0$ at $t=0$,$\therefore$ (4.86) has solution

$$\psi = 0 \qquad (4.87)$$

Now we know that (4.85) represents an antisoliton travelling from the right and passing through a soliton travelling from the left(at t=0).Thus a sG soliton impinging on a free boundary bounces off as an antisoliton.This kind of pulse inversion on reflection is familiar to us in the reflection of a pulse(on a rope say) at a fixed boundary.DTW examine also the solution of (4.82) with the "fixed" boundary condition

$$\varphi(0,t) = 0 \qquad (4.88)$$

The soliton-soliton solution of the sG

$$\varphi_{ss} = 4\tan^{-1}(u\sinh\gamma x/\cosh\gamma ut)$$

has just this property.Thus a soliton reflecting from a fixed boundary,reflects as a soliton.Allowing the space variable to be complex(BS) gives us the following picture.A soliton impinging on a free boundary specified by (4.81) is *attracted* towards the boundary.A soliton impinging on a fixed boundary,is *repelled* from the boundary.It slows down and stops *before* it hits the boundary and then accelerates back to where it came from.Using the Matsuda decomposition of the two parameter solutions,we can see that the boundary has an effect on the incoming soliton for all finite time.Far from being free the soliton is accelerating or decelerating,depending on the boundary condition.

Zakharov and Shabat [42] had observed a similar effect with the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0$$

(this is not the usual NLS which has a positive coefficient of $|\psi|^2$).

DTW go on to examine the solutions of the sG under two box type boundary conditions,

$$\varphi(0,t) = 0 \ , \ \varphi(L,t) = 0 \qquad (4.89a)$$

$$\varphi(0,t) = 0 \ , \ \varphi(L,t) = 2\pi \qquad (4.89b)$$

For boundary condition (4.89a) there is one independent solution,

$$\varphi(x,t) = 4\tan^{-1}\{A\,\mathrm{cn}[\beta(x-x_{ol});k_f].\mathrm{cn}[\Omega t;k_g]\} \qquad (4.90)$$

where,

$$\left.\begin{aligned} &\mathrm{cn}(u,k) = \cos\psi \ , \ \mathrm{sn}(u,k) = \sin\psi \ , \mathrm{sc}(u,k) = \tan\psi, \\ &\qquad \mathrm{dn}(u,k) = (1-k^2\sin^2\psi)^{1/2} \\ &\quad \psi = \mathrm{am}\ u \quad , \int_0^{\psi}(1-k^2\sin^2\theta)^{1/2}d\theta = u \\ &\beta l = 2lK(k_f) \quad , l=1,2,3,\ldots \quad , x_{ol} = L/2l \end{aligned}\right\} \qquad (4.91)$$

$K(k_f)$ is the quarter period of the cn function.cn is zero when its argument is an integer multiple of the quarter period.If A is taken to be independent $A\in \mathbb{R}^+$ then it is found that,

$$k_f^2 = \frac{A^2}{1+A^2}\left[1 + \frac{1}{\beta^2(1+A^2)}\right], k_g^2 = \frac{A^2}{1+A^2}\left[1 - \frac{1}{\Omega^2(1+A^2)}\right] \qquad (4.92a)$$

$$\Omega^2 = \beta^2 + (1-A^2)/(1+A^2) \qquad (4.92b)$$

(4.91) and (4.92a) can be used to determine β as a function of A.If the box size,$L\rightarrow\infty$ then it can be shown that (4.90) is the breather solution.The behaviour of the branch points for the sequence of solutions above,would be interesting.

Corresponding to boundary conditions (4.89b) is the solution,

$$\varphi(x,t) = 4\tan^{-1}[A.sc(\beta x;k_f)dn(\Omega t;k_g)] \quad (4.93)$$

where A must satisfy the condition $A_{th}<A<1$ with A_{th} specified by $L=(1-A^2_{th})K(k_f)$,where $k_f=(1-A^4_{th})^{1/2}$.For $A\neq A_{th}$,

$$k_f^2= 1 - A^2 + A^2/[\beta^2(1-A^2)],k_g^2= 1 - 1/A^2 + 1/[\Omega^2(1-A^2)] \quad (4.94a)$$

$$\beta L = K(k_f) \ , \ \Omega = A\beta \quad (4.94b)$$

(4.93) is possibly the more interesting solution,as it represents to some extent a moving single soliton.If the box size $L\longrightarrow \infty$ (4.93) can be shown to approach the two soliton solution φ_{ss}.Graphs of the spatial variation of φ satisfying (4.93) for a fixed length of box and different amplitudes A,for different times(φ is always a single kink),clearly suggest the picture of a particle bouncing around in a box.φ satisfying (4.93) is monotonic and DTW(as with Perring and Skyrme §1) *define* the position of the soliton to be $x_c(t)$,given by,

$$\varphi(x_c(t),t) = \pi \quad (4.95)$$

Again,the behaviour of the singularities of φ would be interesting.The motion of the soliton in the box(with position defined in (4.95)) has interesting dynamics and DTW were able to construct a model of a relativistic particle in a potential well whose motion closely agreed with the motion of the soliton.The force on the analogous particle was sharply repulsive near the edges of the box.The particle stopped and turned back before reaching the wall,achieving maximum kinetic energy in the centre.

Solitons under perturbation

In many physical situations soliton equations describe phenomena in rather idealized conditions.Thus it is natural to investigate the behaviour of solitons satisfying equations which differ from the pure soliton equation ,only by the presence of small perturbing terms.There has been much research on this topic [43-68].Many of the papers are connected with a controversy which arose concerning the behaviour of sG solitons satisfying the equation,

$$u_{tt} - u_{xx} + \sin u = \chi - \Gamma u_t \qquad (4.96)$$

where $\chi=\chi(t)\ll 1$ and Γ is a constant.χ is described as an applied field,and Γu_t as a damping term [43,50,51].

Fogel,Trullinger, Bishop,and Krumhansl(FTBK) [43] and Reinisch and Fernandez(RF) [50,51] used a method of linearized perturbations.

We present the treatment given by RF.Assume $\Gamma=0$ for the moment.The soliton or antisoliton solutions of (4.96) with $\chi=0$ are,

$$u^{(0)} = 4\tan^{-1}\exp\pm\gamma(x-vt) \qquad (4.97)$$

Consider the soliton in its rest frame,so that the argument of $4\tan^{-1}\exp(.)$ is $\pm x$.First consider a solution to (4.96) with $\chi,\Gamma=0$,

$$u = u^{(0)} + \psi \qquad (4.98a)$$

$$\psi = f(x)\exp(-i\omega t) \qquad (4.98b)$$

After substituting (4.98) into (4.96) and linearizing we find that f satisfies,

$$-f_{xx} + (1 - 2\mathrm{sech}^2 x)f = \omega^2 f \qquad (4.99)$$

This equation allows the "bound state" solution,

$$\omega_b = 0 \quad , \quad f_b(x) = (1/\sqrt{2})\mathrm{sech}x \qquad (4.100a)$$

and the continuum,

$$\omega_k^2 = 1+k^2 \ , \ f_k(x) = (1/\sqrt{2\pi})e^{ikx}(k+i\tanh x)/\omega_k \qquad (4.100b)$$

(4.100a) is regarded as being connected with the translational motion of the soliton,since,$\delta f_b(x) + u^{(0)}$ $\delta\ll 1$,corresponds to a translation of the soliton $u^{(0)}$ by an amount proportional to δ.

Since f_b and f_k form a complete set of functions which span the space of functions $\psi(x,t)$,we may express any function ψ

$$\psi(x,t) = \psi_b(t)f_b(x) + \int_{-\infty}^{\infty} dk\psi(k,t)f_k(x) \qquad (4.101)$$

We now turn to the perturbed sG with $\chi=\chi(t),\Gamma=0$.(4.101) may be substituted into (4.96) and equations for $\psi_b(t)$ and $\psi(k,t)$ determined,

$$(\psi_b)_{tt} = \pi\chi(t)/\sqrt{2} \qquad (4.102a)$$

$$(\psi(k,t))_{tt} + \omega_k^2\psi(k,t) = \chi(t)\int_{-\infty}^{\infty} dz' f_k^*(z') \qquad (4.102b)$$

where * refers to complex conjugation.Solving the above equations with $u(x,0)=u^{(0)}$ gives,

$$\psi_b(t)=(\pi/\sqrt{2})F(t) \ , \ \psi_k(t)=F_k(t)\int_{-\infty}^{\infty} dz' f_k^*(z') \qquad (4.103a)$$

$$F(t)=\int_0^t dt' \int_0^{t'} dt''\chi(t'') \quad , \quad F_k(t)=g(t)-g(0)\cos\omega_k t-\omega_k^{-1}\dot{g}(0)\sin\omega_k t \qquad (4.103b)$$

and $g(t)$ is a solution of

$$\ddot{g} + \omega_k^2 g = \chi(t) \qquad (4.103c)$$

RF define the position of the soliton to be the coefficient of $f_b(x)$ in (4.101) and the velocity $v(t)$ is given by,

$$v(t)=\pm(1/2\sqrt{2})\dot{\psi}_b(t) \qquad (4.104)$$

where the +(-) sign corresponds to a soliton(antisoliton).Using (4.103) RF obtain the solution of (4.96)

$$\psi_\chi(x,t)=(\pi/\sqrt{2})f_b(x)F(t) + \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dkF_k(t)f_k^*(x') \qquad (4.105)$$

they then show that,

$$\psi_\chi(x,t) = (\pi/\sqrt{2})f_b(x)F(t)\left[1 - \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty} dx' f_b(x')\right] + F(t) \qquad (4.106)$$

to the first order in $\varepsilon=\omega_k t \ll 1$(short time scales).However since $\int_{-\infty}^{\infty} dx' f_b(x')=\pi/\sqrt{2}$,the coefficient of $f_b(x)$ *vanishes*.Thus when the continous(phonon) spectrum is ignored (4.102a) leads to a Newtonian law of motion $\psi_b(t)=(\pi/\sqrt{2})F(t)$.This was the result obtained by FTBK.However over short time scales this motion is *cancelled* by the continous spectrum.

Thus according to RF,for small time scales there is no classical particle-like behaviour for sG solitons.RF also carried out numerical analysis which supported their conclusion.A number of papers have been written attempting to clear up the controversy [53,62,65],and have been answered by RF [54,63,66].Many other different approaches to the perturbed sG equation have been devised and they generally agree with the conclusions of FTBK,however they are over longer time scales.

A perturbation theory based on an inverse scattering theory method was devised by Kivshar and Kosevich [57,61];their findings agree with those of RF.

Finding a correct definition of the soliton position is central to the argument.Bergman et al [56] produce a much more physically based perturbation theory for the sG,in which the position of the soliton is more clearly defined.Noting that $u_x^{(0)}$ is a peaked function,

$$u_x^{(0)} = \pm 2\gamma \operatorname{sech}\gamma(x-ut) \tag{4.107}$$

they define the position of the soliton to be Q,

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} x u_x \, dx \tag{4.108}$$

and velocity $\dot{Q}$,

$$\dot{Q} = \frac{1}{2\pi}\int_{-\infty}^{\infty} x u_{xt} \, dx \tag{4.109}$$

The momentum can be obtained from the Lagrangian density (4.12) for the sG,and is given by,

$$P = \int_{-\infty}^{\infty} u_t u_x \, dx \tag{4.110}$$

Assuming $u_t \longrightarrow 0$ as $x \longrightarrow \pm\infty$

$$\dot{P} = \int_{-\infty}^{\infty} u_{tt} u_x \, dx \tag{4.111}$$

Bergman et al,then find that with $\Gamma=0$ for the perturbed sG(4.96),

$$\dot{P} = \int_{-\infty}^{\infty} \chi(t) u_x \, dx \tag{4.112}$$

thus using (4.107),

$$\dot{P} = 2\pi\chi(t) \qquad (4.113)$$

which is the Newtonian law.However according to Bergman et al, if $\chi(t)$ is switched on at time t=0 there is a transient time inversely proportional to Γ in which the soliton form is distorted.This is their explanation of the so called "non-Newtonian" behaviour discovered by RF.

Rice [60] developed a view in which the soliton could be regarded as a deformable particle whose translational motion is coupled to internal degrees of freedom.Thus it was possible that an external field applied to a soliton could,by exciting internal kinetic energy lead to a translational velocity which was not simply that of a Newtonian particle.We give the final remarks on this subject to Kaup [62].He makes the following points:

1.The soliton or kink is *not* rigid and is *not* a point particle.

2.An extended particle will respond with a time delay to an externally applied force.In Kaup's words,"What they(RF) observed were the combined transient effects of a soliton reshaping itself and experiencing a time delay."

3.RF's analysis was limited to short time scales.

4.The concept of the "soliton" arises by considering long time scales,as in multisoliton interaction.

5.The RF definition of soliton position(point where $u_x(x,t)$ is a minimum) differs from that used in other perturbation theories.

We note that,controversy also surrounds KdV solitons under perturbation [47,62].

The general concensus emerging on the subject of perturbed solitons is the following.Over short time intervals the soliton may move in an unexpected manner,while undergoing shape deformation.After longer time intervals the soliton is in general found to be accelerating(with some shape deformation) in accordance with Newtonian behaviour of particles.With any extended object there can be difficulties in defining the position.It certainly would be instructive to examine the motion of the poles associated with a perturbed Hamiltonian density as,we have seen how they provide a useful and unambiguous representation of the soliton position.

In closing this section on perturbation theory we mention the papers of Gorshkov and Ostrovsky [69,70].They investigate a small parameter asymptotic scheme for a system of solitary waves with close velocities.They found that a Lagrangian may be introduced analogous to that for classical particles in interaction via a pairwise potential.As we have already mentioned they found the interaction between KdV and MKdV solitons to be repulsive.The Gorshkov and Ostrovsky method is very general in character.LeVeque [80] investigated numerically exactly the same situation(in the two soliton case) as that envisaged by Gorshkov and Ostrovsky and also found repulsive interaction.He also produced an approximation to the two soliton KdV solution u(x,t) in the following form,

$$u(x,t)\cong A_1(t)\mathrm{sech}^2[\alpha_1(t)(x-\varphi_1(t))]+A_2(t)\mathrm{sech}^2[\alpha_2(t)(x-\varphi_2(t))]$$

finding exact formualae for A_i and φ_i.He showed that if the difference in the soliton speeds was $O(\varepsilon)$,then the above representation was valid to $O(\varepsilon^2)$.LeVeque's approximation is restricted to solitons travelling with nearly equal speeds.We shall see in part two of the thesis another approximation produced by the author which appears to provide a good approximation(of form above) for markedly different soliton speeds.

§ 6. "Solitons" as elementary particles

Perring and Skyrme were the first to specifically use the sG solitons as model particles in interaction [5].Enz [71] expressed in an independent examination,the attractive reasons which lie behind the idea of regarding elementary particles as solitons(or solitary waves).These were:

1.Finite energy and field.

2.Stability.

3.Discrete mass and charge.

The author has already expressed a number of other attractive reasons for expecting a soliton-like description of elementary particles.

One of the reasons why soliton equations in spatial dimensions higher than 1,are likely to be complicated was provided by Derrick [72].He considered relativistic fields Φ having a Lagrangian $\mathscr{L}$ of the form,

$$\mathscr{L} = \frac{1}{2}\,\partial_{\mu}\Phi\partial^{\mu}\Phi - U(\Phi) \tag{4.114}$$

Derrick showed that static field solutions $\varphi_c(\underline{x})$ having finite energy

$$E_c(\varphi) = \int[\tfrac{1}{2}(\nabla\varphi)^2 + U(\varphi)]d\underline{x} \tag{4.115}$$

were only possible in *one* spatial dimension.

Despite this restriction,higher dimensional "solitons" of relatively simple form have been found [73-4].

Enz [75-6] has discovered four dimensional "solitons" based on an action principle built from two coupled scalar fields $\theta(x,y,z,t)$ and $\phi(x,y,z,t)$.For a single field θ,the least action principle is introduced for W,

$$\delta W = 0 \tag{4.116a}$$

$$W = \int\{\ k\sin^2\theta + A[(\nabla\theta - c^{-2}\theta_t^2)]\ \}dxdydzdt \tag{4.116b}$$

This is essentially the sG Lagrangian(choosing 1 spatial dimension),and leads to the Euler-Lagrange equation,

$$\Box\,\theta = (k/2A)\sin 2\theta \qquad (4.117)$$

This is the higher dimensional sG and does not support finite energy solitons in 2 space dimensions or higher.However,by analogy to this Enz produced a new action priciple for two scalar fields θ and ϕ,

$$\delta W = 0\ ,\ W = \int (k\sin^2\theta + AD_o + ED_o^2)d^4x \qquad (4.118a)$$

$$D_o = [(\nabla\theta)^2 - c^{-2}\theta_t^2] + \sin^2\theta.[(\nabla\phi)^2 - c^{-2}\phi_t^2] \qquad (4.118b)$$

where K,A,E are constants.Enz showed that the Euler-Lagrange equations associated with (4.118) have a relatively simple solution with cylindrical symmetry.The field configuration is infinitely extended along the vertical axis,so that the solitons are of "string-like" type.Interesting toroidal "solitons" of equations derived from a generalization of the Enz Lagrangian density above,

$$\mathscr{L} = (\partial_\mu\Theta)^2 + \sin^2\Theta.(\partial_\mu\Phi)^2 + K^2\sin^2\Theta \qquad (4.119)$$

have been found by Williams [77](Enz uses K=0).Williams finds reasons to think that N soliton solutions may be possible.

In the conclusion to one of his papers,Enz [75],notes the interesting fact that many of the parameters associated with the field structures,are discrete.This includes parameters representing mass and charge.However the theory is *entirely classical in nature.*

Quantum-like features even enter into the sG.Klein [78] derives an equation from the sG which is shown to possess envelope solitons.These envelope solitons have the property that the energy of the soliton is a constant times the carrier frequency.This is reminiscent of de Broglie waves.

In [79],a classical field model is proposed involving a scalar field interacting with the electromagnetic field which is shown to possess *discrete* particle-like solutions.All the particles have the same charge.

At present,the only drawback to current higher dimensional soliton research is that as yet,multisoliton solutions have not been found.This means that we cannot explore the possibly very interesting soliton-soliton interactions.In the author's view it is *precisely* these intersoliton interactions which are physically most interesting.

§ 7. Conclusions

In this chapter we have explored various aspects of the particle-like nature of solitons.The soliton is an extended body,which in the case of the sG equation,is also relativistic.This means that it occupies a unique position in mathematical physics.

Classical physics describes a world consisting of points possessing mass and charge,and "giving rise to" fields of force such as gravity or electric force.In order to compare the soliton with the familiar world of classical physics we must find a point-like way of describing solitons.Bowtell and Stuart achieved this for the sG by associating singularities in the complex plane with solitons,and they were able to obtain explicitly the orbits of the solitons in the c-of-v frame.Hence they could determine the dynamics of the solitons.

Many other attempts have been made to get at how sG solitons interact.These were only partly successful,but they generally agreed with BS results in the appropriate limit.Matsuda made a remarkable discovery,that the two parameter solutions of the sG in the c-of-v frame,could be written as a linear superposition of accelerating kinks.The kinks moved exactly like the BS poles.This fact established the mutual importance of the results.

Many soliton equations have rational solutions and these turn out to be related to solvable many body problems.Since the rational solutions can be approached by a limiting process from soliton solutions,there may be a connection with soliton interaction and these solvable many body problems.

To some extent the KdV has provided an embarassment of riches as the N soliton solution can be represented using two different lsp's.When various researchers investigated the two soliton solution of the KdV using the GGKM lsp,they discovered that the solitons took on a quite different shape from their asymptotic counterparts.They also found the interaction to be attractive.In an analysis of the pole structure of the two soliton solution of the KdV,BS discovered that the poles repelled each other.

Our own original work on this subject in part two of the thesis,also supports the mutual repulsion of KdV solitons.Incidentally we also find the shapes of the accelerating solitons to be quite different from their asymptotic counterparts(except in the asymptotic limit).
Two independent pieces of research(Gorshkov et al,and LeVeque) also support repulsive KdV soliton interaction.

The topics covered in this chapter form a good case for thinking of solitons in some way to be like particles.The link between soliton interaction and the nlpde supporting the multisolitons however,is far from understood.We have seen signs that solitons may behave in some respects like quantum objects.This is not so surprising,since solitons cannot be described purely as point particles,neither can they be thought of as linear waves.They certainly hold out the prospect of a deeper understanding of the world.

PART TWO

Part 2. Synopsis

Chapter 5.

In this chapter we prove the three parameter solution of the sG directly by employing the theorem of permutability for the sG.The direct method becomes hopelessly complicated for more than three parameters,however we discover that the N parameter solution has a simple structure if certain constants k_{ij} are allowed to be ±1.Using this result we prove rigorously that the N parameter solution can only have a certain form when $k_{ij} \neq \pm 1$.

We transform the N parameter formula to another, by phase shift and then show that the new formula is a multisoliton solution of the sG.

Chapter 6

In this chapter we prove that the multisoliton solutions of the sG,MKdV,KdV and relatives can be written as a linear superposition of soliton like forms.We also demonstrate that a similar kind of linear superposition applies to the Nonlinear Schrödinger equation.All the linear superpositions found involve the roots f_i of an N^{th} degree polynomial having a common form.

We explore connections with the roots f_i of the linear superposition polynomial(lsp) and the technique used to find the N soliton solutions in Inverse Scattering Theory.

We investigate the properties of the f_i and derive the exact formulae for f_i up to the 4^{th} degree lsp.

We discover the relationship between the singularities of the complexified N soliton sG solution and the roots f_i of the real lsp.

Under certain circumstances the lsp for the sG takes on a simpler structure and we examine this.

Because of the symmetry enjoyed by the sG the solitons and antisolitons(and breathers) can be written in different ways.We show that one representation is to be preferred over others in that it produces lsp's with real roots.We also show that although the lsp derived in this chapter is natural it is not in fact unique.

Chapter 7

In this chapter we have analysed in detail the behaviour of the roots of the two parameter lsp.In a numerical analysis, we have produced graphs showing the appearance of the roots f_i of the lsp's for the sG,MKdV,and KdV .Although as we have seen the lsp's have the same form, we find that, in mixture cases and breathers,the f_i can develop points of inflexion.In the case of the two parameter sG solution,we find that points of inflexion do not arise.

We also examine both analytically and numerically the anomalous behaviour of the KdV two soliton solution as a sum of interacting separate parts.

We develop a mathematical technique for approximating f_i and discover that the approximations $\tilde{f}_i$ are very good in certain circumstances.We also propose a way of improving the approximation still further.As a result of these investigations we discover new soliton solutions which are good global approximations to known multisoliton solutions.

We present extensive numerical analysis concerning the motion of solitons and antisolitons for a number of multisoliton solutions of the sG(up to five parameters).We also demonstrate the retarded interaction of sG solitons and investigate soliton dynamics.

We end the thesis with some concluding remarks ,which include topics for further research.

CHAPTER 5 : NONLINEAR SUPERPOSITION OF SOLITONS

§ 0. Introduction

In this chapter we will derive from first principles the N parameter solution Φ_N of the sG equation,where N = ns+na+2nb and ns is the number of solitons,na the number of antisolitons and nb the number of breathers.The method we are using is original.The sG has a theorem of permutability (chapter 3,§4) which relates the N parameter solution to two N-1 parameter solutions and a N-2 parameter solution.This tempts one to try to prove Φ_N by induction.Unfortunately the algebra becomes impossibly complex for N≥4. We demonstrate this for N=3 in §1. $\Phi_N = \Phi_N(x,t;k_{ij})$,where $k_{ij} = (a_i+a_j)/(a_i-a_j)$ and a_i,a_j are the constants entering into the Bäcklund transformations.

If we set k_{ij} = signum(j-i) then $\tan(\Phi_N/4)$ has a very simple form,namely that of the tan of a sum of N angles,each angle being taken with alternating sign (if N=1,the + sign is taken).The functional dependence of Φ_N on x and t is thus determined.We establish this result in §1.

The constants k_{ij} enter the formula for $\tan(\Phi_N/4)$ only as multiplying coefficients,by virtue of theorem of permutability.The symmetry of Φ_N ,induced by the theorem of permutability means that to establish how the constants k_{ij} are included in the formula for Φ_N we need only consider a *single* k_{ij} This is carried out in §3.

From $\Phi_N(x,t;k_{ij})$ we prove an alternative formula for $\Phi_N(x,t;u^2_{ij})$,where $u^2_{ij} = k^{-2}_{ij}$. This Φ_N has the form suspected by Bowtell and Stuart [1].

In §4 we prove that Φ_N becomes asymptotically $(t \to \pm \infty)$ just a collection of solitons,antisolitons and breathers with the well known phase shift property.

In our concluding remarks we note that essentially identical arguments to those in preceeding sections can be used to prove the N parameter solution of the MKdV.This is because, as we saw in chapter 3, the sG and MKdV share the same theorem of permutabilty(apart from a factor of 2).It seems likely that the KdV might also be amenable to a similar treatment provided we work with the more fundamental Hirota τ function for which u = $(\ln \tau)_{xx}$ where u satisfies the KdV equation [2].

NOTATION

$t(1,.,N) \equiv \tan(\Phi_N/4)$

$t(1,.,N/i) \equiv \tan(\Phi_{N-1}/4)$,where Φ_{N-1} is a N-1 parameter solution not involving parameter i.

$t(1,..,N/i,r) \equiv \tan(\Phi_{N-2}/4)$,where Φ_{N-2} is a N-2 parameter solution not involving parameters i and r.

$t(i) \equiv \tan(\Phi_1(i)/4)$,where $\Phi_1(i)$ is a 1 parameter solution.

We express the theorem of permutability in the following form,

$$t(1,.,N) = \frac{t(1,..,N/i,r) + k_{ir}\left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]}{1 - k_{ir}t(1,..N/i,r)\left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]} \tag{5.1}$$

This is a rearrangement of the theorem of permutability as expressed by Barnard [3].

We also note the equation below which follows from the definition of k_{ij} .

$$k_{12}k_{13} + k_{21}k_{23} + k_{31}k_{32} = 1 \tag{5.2}$$

Parameters 1,2,3 represent any triple of parameters.

§ 1. The Three Parameter Solution of the sG

Consider (5.1) with N=3 and i=2 and r=3,

$$t(1,2,3) = \frac{t(1) + k_{23}\left(\dfrac{t(1,2) - t(1,3)}{1 + t(1,2)t(1,3)}\right)}{1 - k_{23}t(1)\left(\dfrac{t(1,2) - t(1,3)}{1 + t(1,2)t(1,3)}\right)} \tag{5.3}$$

where,

$$t(1,2) = N_{12}/D_{12} \quad \text{and} \quad t(1,3) = N_{13}/D_{13} \;,$$

and,

$$N_{12} = k_{12}[t(1)-t(2)] \qquad N_{13} = k_{13}[t(1)-t(3)]$$

$$D_{12} = 1+t(1)t(2) \qquad D_{13} = 1+t(1)t(3)$$

Denoting the numerator and denominator of (5.3) by N_{123} and D_{123} respectively,we obtain,

$$\begin{aligned} N_{123} = &(1+k_{12}k_{23}-k_{13}k_{23})t(1) - k_{23}k_{12}t(2) + k_{23}k_{13}t(3) \\ &+ t^2(1)t(2)(1-k_{12}k_{13}-k_{23}k_{13}) + t^2(1)t(3)(1-k_{12}k_{13}+k_{23}k_{12}) \\ &+ t(1)t(2)t(3)(k_{12}k_{13}-k_{23}k_{12}+k_{23}k_{13}) + k_{12}k_{13}t^3(1) \\ &+ t^3(1)t(2)t(3) \end{aligned}$$

Employing (5.2) we find,

$$N_{123} = [1+t^2(1)][k_{21}k_{31}t(1)+k_{12}k_{32}t(2)+k_{23}k_{13}t(3)+t(1)t(2)t(3)]$$

Similarly we find,

$$D_{123} = [1+t^2(1)][1+k_{13}k_{23}t(1)t(2)+k_{12}k_{32}t(1)t(3)+k_{21}k_{31}t(2)t(3)]$$

Thus the 3 parameter solution of the sG is

$$t(1,2,3) = \frac{[k_{12}k_{13}t(1)+k_{21}k_{23}t(2)+k_{31}k_{32}t(3)+t(1)t(2)t(3)]}{[1+k_{13}k_{23}t(1)t(2)+k_{12}k_{32}t(1)t(3)+k_{21}k_{31}t(2)t(3)]} \tag{5.4}$$

Now let k_{ij} = signum(j-i),(5.4) becomes,

$$t(1,2,3) = \frac{[t(1) - t(2) + t(3) + t(1)t(2)t(3)]}{[1 + t(1)t(2) - t(1)t(3) + t(2)t(3)]} \tag{5.5}$$

Clearly $\Phi_{123} = \Phi_1 - \Phi_2 + \Phi_3$.

Because of the symmetry $\Phi \to -\Phi$,enjoyed by the sG.We may choose an *alternating signature* for Φ_i or t(i),

$$t(i) \rightarrow (-1)^{i+1} t(i)$$

If t(i) are chosen to have alternating signature then in effect we are building a three soliton solution from two solitons and an antisoliton (i.e. t(2) is the antisoliton).The alternating structure of t(1,..,N) imposed by the theorem of permutability is of central importance as will be seen in the first lemma of the next section.

§ 2. The N-Parameter Solution when k_{ij} = signum(j-i)

LEMMA 5.1

If in a N-parameter solution of the sG ,t(1,..,N) we let,

$$k_{ij} \rightarrow \text{signum}(j-i) \ \forall \ i,j \qquad (5.6)$$

then,

$$t(1,..,N) = \tan\left(\sum_{m=1}^{N} (-1)^{m+1} \Phi_m / 4 \right) \qquad (5.7)$$

Proof: By induction on N

N = 1 $\qquad t(1) = \tan(\Phi_1/4)$,by definition

N = 2 $\qquad t(1,2) = k_{12}\left(\dfrac{t(1)-t(2)}{1 + t(1)t(2)}\right)$,follows directly

from (5.1).Applying (5.6) and noting definitions of t(i) we see immediately that (5.7) is true.

We now assume the lemma to be true for N-2 and N-1 parameters.

t(1,..,N/i,r) $=$ (5.8)

$$\tan\left(\sum_{m=1}^{i-1} (-1)^{m+1}\Phi_m/4 + \sum_{m=i+1}^{r-1} (-1)^{m}\Phi_m/4 + \sum_{m=r+1}^{N} (-1)^{m-1}\Phi_m/4 \right)$$

t(1,..,N/r) $=$ (5.9)

$$\tan\left(\sum_{m=1}^{i-1} (-1)^{m+1}\Phi_m/4 + (-1)^{i+1}\Phi_i/4 + \sum_{m=i+1}^{r-1} (-1)^{m+1}\Phi_m/4 + \sum_{m=r+1}^{N} (-1)^{m}\Phi_m/4 \right)$$

t(1,..,N/i) = (5.10)

$$\tan\left[\sum_{m=1}^{i-1}(-1)^{m+1}\Phi_m/4 + \sum_{m=i+1}^{r-1}(-1)^{m}\Phi_m/4 + (-1)^{r}\Phi_r/4 + \sum_{m=r+1}^{N}(-1)^{m}\Phi_m/4\right]$$

where the changes in the exponents of -1 are a result of the mising parameters altering the sequence of signs.

(5.1) gives with (5.6) (i < r) ,

$$t(1,..,N) = \frac{t(1,..,N/i,r) + \left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]}{1 - t(1,..N/i,r)\left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]} \quad (5.11)$$

Because of the definition of t() ,the quantity in large brackets can be written as tan(A) where noting (5.9-10) ,

$$A = \left[\sum_{m=1}^{i-1}(-1)^{m+1}\Phi_m/4 + (-1)^{i+1}\Phi_i/4 + \sum_{m=i+1}^{r-1}(-1)^{m+1}\Phi_m/4 + \sum_{m=r+1}^{N}(-1)^{m}\Phi_m/4\right]$$

$$-\left[\sum_{m=1}^{i-1}(-1)^{m+1}\Phi_m/4 + \sum_{m=i+1}^{r-1}(-1)^{m}\Phi_m/4 + (-1)^{r}\Phi_r/4 + \sum_{m=r+1}^{N}(-1)^{m}\Phi_m/4\right]$$

$$\therefore A = (-1)^{i+1}\Phi_i/4 + \sum_{m=i+1}^{r-1}(-1)^{m+1}\Phi_m/4 - \sum_{m=i+1}^{r-1}(-1)^{m}\Phi_m/4 - (-1)^{r}\Phi_r/4$$

$$\therefore A = (-1)^{i+1}\Phi_i/4 + (-1)^{r+1}\Phi_r/4 + 2\sum_{m=i+1}^{r-1}(-1)^{m+1}\Phi_m/4 \quad (5.12)$$

We note that for *sequential* i and r ,

$$A = A_s = (-1)^{i+1}(\Phi_i - \Phi_r)/4 \quad (5.13)$$

A considerable simplification.

(5.8) together with (5.12) when substituted into (5.11) gives, t(1,..,N) =

$$\tan\left[\sum_{m=1}^{i-1}(-1)^{m+1}\Phi_m/4+\sum_{m=i+1}^{r-1}(-1)^{m}\Phi_m/4+\sum_{m=r+1}^{N}(-1)^{m-1}\Phi_m/4+A\right]$$

with A given by (5.12).

Inspection shows that the above simply reduces to ,

$$t(1,..,N) = \tan\left[\sum_{m=1}^{N}(-1)^{m+1}\Phi_m/4\right]$$

This is the assertion of the lemma ■

We will need the formula for the tangent of N angles.This is given below.

$$\tan\sum_{l=1}^{N}\alpha(l) = \qquad (5.14)$$

$$\frac{\sum_{l=1}^{N}\alpha(l)-\sum_{l_j=1}^{N}\prod_{j=1}^{3}\alpha(l_j)+..(-1)^r\sum_{l_j=1}^{N}\prod_{j=1}^{2r+1}\alpha(l_j)+..(-1)^{n+1}\sum_{l_j=1}^{N}\prod_{j=1}^{n}\alpha(l_j)}{1-\sum_{l_j=1}^{N}\prod_{j=1}^{2}\alpha(l_j)+..(-1)^r\sum_{l_j=1}^{N}\prod_{j=1}^{2r}\alpha(l_j)+..(-1)^{m+1}\sum_{l_j=1}^{N}\prod_{j=1}^{m}\alpha(l_j)}$$

where n is the nearest odd integer $\leq$ N and m is the nearest even integer $\leq$ N.Thus we see all combinations of odd tuples appear on the numerator of (3.13) ,while all combinations of even tuples appear on the denominator.

We will also need the following definitions.

An *even(odd)* permutation of the N tuple 1,2,..,N which we denote by ε is one for which there are an even(odd) number of ordered pairs (i,j) taken from ε such that i>j.This is equivalent to saying that an even(odd) permutation requires an even(odd) number of pairwise rearrangements to recover the ordered N tuple ε.The sign of a permutation σ is defined by :

sgn σ is 1 if σ is even and -1 if σ is odd.

A *transposition* is a permutation τ defined by the following.

$\tau(i) = j$, $\tau(j) = i$, $\tau(k) = k$, $k \neq i,j$.

If $i<j$ then $\tau = 1,2,..(i-1),j,(i+1),..,(j-1),i,(j+1),..N$.

There are $2(j-i+1) + 1$ pairs (k,l) $k>l$,

(j,i) , (j,x) , (x,i) ,where $x = i+1,...,j-1$ .Thus τ is odd.

LEMMA 5.2

The coefficient of the m tuple in the tan of N parts taken with alternate signs is identical to the sign(sgn) of the associated N tuple,where the *associated N tuple* of the m tuple $j_1,..,j_m$ is $j_1,..,j_m,k_1,..,k_N$ where $k_i < k_j$ $i < j$.

Proof:

The coefficient of the m tuple in the tan of N parts taken with alternate signs is , see (5.13) ,$(-1)^{[m/2]}(-1)^e$ = $(-1)^\rho$,where e is the number of evens in the m tuple and ρ is the number of extra evens in the m tuple in excess of the number the m tuple would have if it was the identity m tuple 1,2,3,..,m, i.e $e=[m/2]+\rho$.([.] means nearest integer to).

To increase the number of evens in the m tuple by ρ would require ρ transpositions of the associated N tuple(i.e swapping ρ odds in the m tuple with evens in the remaining tuple).Thus the sign of just such a transposed associated N tuple is $(-1)^\rho$.This is precisely the coefficient of the m tuple in the tan of N parts taken with alternate sign ■

We give an example:Consider the three tuples taken over 7 parameters.The coefficient of t(1)t(2)t(3) is 1,but 123 is an identity 3 tuple of the associated 7 tuple 1234567.Now consider t(1)t(2)t(4) ; this has a coefficient in t(1,..,7) of -1.Now 124 |3567 has a sgn of -1 as it is an odd permutation.

COROLLARY 5.1

If an alternating signature for the alternating N parameter solution of the sG be chosen then the tan of the new N-parameter solution so formed is just the tan of a sum of N parts(1-parameter) taken with positive sign.

Proof:

The alternating signature is $\Phi_m \rightarrow (-1)^{m+1}\Phi_m$ .From lemma 5.1 the tan of the alternating N parameter solution is given by

$$t(1,..,N) = \tan\left[\sum_{m=1}^{N}(-1)^{m+1}\Phi_m/4\right]$$

Clearly under the signature change above ,

$$t(1,..,N) = \tan\left[\sum_{m=1}^{N}\Phi_m/4\right]$$ ∎

§ 3. The N-Parameter Solution of the sG

In the following lemma we will set all k_{ij} to 1 or -1 with one exception denoted k_{ir}.We will choose i and r to be sequential and avail ourselves of the considerable simplification this brings.By permuting the original input parameters(a_i) we can of course obtain any desired k_{ij} in place of k_{ir}.Examining the form of the N parameter solution so obtained will enable us to decide on what multipliers k_{ij} the coefficient of the m tuple depend (lemma 5.3).

In theorem 5.1 we shall deduce that there is only one possible function of the k_{ij} which is the multiplier of the m tuple $t(j_1)....t(j_m)$ in the N parameter solution of the sG.

In theorem 5.2 we will determine an alternative form of the N parameter solution (i.e that proposed by BS for solitons).

LEMMA 5.3

The coefficient of the m tuple $t(j_1)t(j_2)...t(j_m)$ in the N parameter solution of the sG equation depends on the set of $k_{j_i r_i}$,

$$\left\{ k_{j_i r_i} : 1 \le i \le m \ , \ j_i \ne j_l \ , \ j_i, r_i \in \{1,..,N\} \ r_i \ne j_i \right\}$$

e.g (5.14)

$N = 7 \quad m = 4 \quad j_1 = 2 \quad j_2 = 4 \quad j_3 = 6 \quad j_4 = 7$

The coefficient of the 4 tuple $t(2)t(4)t(6)t(7)$ depends on

$$\left\{ k_{21}, k_{23}, k_{25}, k_{41}, k_{43}, k_{45}\ k_{61}, k_{63}, k_{65}, k_{71}, k_{73}, k_{75} \right\}$$

We note the coefficient of the N tuple is 1

Proof:

Let k_{mn} = signum (m−n) except when m=i n=r=i+1.

Then the theorem of permutability gives(5.1)

$$t(1,..,N) = \frac{t(1,..,N/i,r) + k_{ir}\left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]}{1 - k_{ir}t(1,..N/i,r)\left[\dfrac{t(1,..,N/r) - t(1,..,N/i)}{1 + t(1,..,N/r)t(1,..,N/i)}\right]}$$

Using lemma 5.1 for sequential parameters i,r (5.13) the above becomes,

$$t(1,..,N) = \frac{\tan \sum_{l=1}' \Phi_l/4 + k_{ir}\tan\left[(-1)^{i+1}(\Phi_i - \Phi_r)/4\right]}{1 - k_{ir}\tan\left[(-1)^{i+1}(\Phi_i - \Phi_r)/4\right] \tan \sum_{l=1}' \Phi_l/4}$$

The prime over the summation sign indicates that i and r are missing.Noting that $k_{ir} = -k_{ri}$ and also denoting , $\tan\ (-1)^{i+1}\Phi_i/4$ by $t(i)$ and similarly for the expression with r as the parameter,we write the above equation.

$$t(1,..,N) = \frac{\tan \sum_{l=1}' \Phi_l/4 + k_{ir}\left[\frac{t(i) - t(r)}{1 + t(i)t(r)}\right]}{1 - k_{ir}\left[\frac{t(i) - t(r)}{1 + t(i)t(r)}\right] \tan \sum_{l=1}' \Phi_l/4}$$

This may be written ,

$$t(1,..,N) = \frac{\tan \sum_{l=1}' \Phi_l/4 \left[1 + t(i)t(r)\right] + k_{ir} t(i) + k_{ri} t(r)}{1 + t(i)t(r) - \left[k_{ir} t(i) + k_{ri} t(r)\right] \tan \sum_{l=1}' \Phi_l/4}$$

Now we substitute the formula for the tan of N parts (5.14) into the above equation .We have supressed the signs as they would arise in $\tan \Sigma' \Phi_l/4$ as we are only interested in how the k_{ij} attach themselves to particular multiplicative tuples of t(i).We obtain ,

$$t(1,..,N) = \frac{N_1 + N_2 + \ldots + N_{[(n+1)/2]}}{1 + D_1 + D_2 + \ldots + D_{[n/2]}}$$

where ,

$N_1 = \Sigma' t(l) + k_{ir} t(i) + k_{ri} t(r)$

$-N_2 =$

$\Sigma' \prod_{j=1}^{3} t(l_j) + k_{ir} t(i) \Sigma' \prod_{j=1}^{2} t(l_j) + k_{ri} t(r) \Sigma' \prod_{j=1}^{2} t(l_j) - t(i)t(r)\Sigma' t(l)$

$N_3 =$

$\Sigma' \prod_{j=1}^{5} t(l_j) + k_{ir} t(i) \Sigma' \prod_{j=1}^{4} t(l_j) + k_{ri} t(r) \Sigma' \prod_{j=1}^{4} t(l_j) - t(i)t(r)\Sigma' \prod_{j=1}^{3} t(l_j)$

⋮

$(-1)^{r+1} N_r =$

$\Sigma' \prod_{j=1}^{2r-1} t(l_j) + k_{ir} t(i) \Sigma' \prod_{j=1}^{2r-2} t(l_j) + k_{ri} t(r) \Sigma' \prod_{j=1}^{2r-2} t(l_j) - t(i)t(r)\Sigma' \prod_{j=1}^{2r-3} t(l_j)$

$-D_1 =$

$\Sigma' \prod_{j=1}^{2} t(l_j) + k_{ir} t(i)\, \Sigma' t(l) + k_{ri} t(r)\, \Sigma' t(l) - t(i)t(r)$

$D_2 =$

$\Sigma_{j=1}^{,4}\prod t(l_j)+ k_{ir}t(i)\Sigma_{j=1}^{,3}\prod t(l_j)+ k_{ri}t(r)\Sigma_{j=1}^{,3}\prod t(l_j)-t(i)t(r)\Sigma_{j=1}^{,2}\prod t(l_j)$

⋮

$(-1)^r D_r =$

$\Sigma_{j=1}^{,2r}\prod t(l_j)+ k_{ir}t(i)\Sigma_{j=1}^{,2r-1}\prod t(l_j)+ k_{ri}t(r)\Sigma_{j=1}^{,2r-1}\prod t(l_j)-t(i)t(r)\Sigma_{j=1}^{,2r-2}\prod t(l_j)$

Examine N_1,and in particular the term $k_{ir}t(i)$.Let us so permute the basic input parameters (a_i) that the consecutive parameters i and r achieve all possible k_{ir}.As we examine $k_{ir}t(i)$ for varying r we will see that the coefficient of t(i) can be any k_{ir} chosen from the set ,

$$\left\{ k_{ir} \ : \ r \in \{1,..,N\} \ r\neq i \right\}$$

Next examine the term $k_{ir}t(i) \ \Sigma_{j=1}^{2}\prod t(l_j)$ in N_2 we write this in the more explicit form $k_{ir}t(i)t(j)t(l)$ with $j,l \neq i$ or r .Now varying r,we see that the coefficient of t(i)t(j)t(l) can be any k_{ir} chosen from the set ,

$$\left\{ k_{ir} \ : \ r \neq i,j,l \right\}$$

But of course we may permute the parameters i,j,l so that we obtain the other possible terms $k_{jr}t(i)t(j)t(l)$ or $k_{lr}t(i)t(j)t(l)$.Clearly the coefficient of t(i)t(j)t(l) can be chosen from the set ,

$$\left\{ k_{ir} \ , \ k_{jr} \ , \ k_{lr} \ : \ r \neq i,j,l \ , \ r \in \{1,...,N\} \right\}$$

It is clear that this kind of reasoning can be applied for any tuple.Thus we find the coefficient of the m tuple $t(j_1)t(j_2)...t(j_m)$ in the N parameter solution of the sG must depend on ,

$$\left\{ k_{j_1 r_1} , k_{j_2 r_2} ,...., \ k_{j_m r_m} \ : \ r_i \in \{1,...,N\}\setminus\{j_i\} \right\}$$

This is the assertion of the lemma ■

We comment on a point which arises in relation to the above in the exact analysis of the three parameter case seen in §2.There we saw for example that the coefficients of t(1) t(2) and t(3) were the following :

t(1) : $1 + k_{12}k_{23} - k_{13}k_{23}$

t(2) : $k_{12}k_{32}$

t(3) : $k_{13}k_{23}$

It would seem that the coefficient of t(1) could depend on a combination of k_{ij} not considered in the above lemma.This is untrue,because had we allowed the "missing parameters" i and r in(5.2) to be other than 2 and 3 respectively(but selected from {1,2,3}) we would have explicitly got the coefficient of t(1) to be $k_{21}k_{31}$ .Since we are free to choose i and r arbitrarily , we can conclude that quantities like $1+k_{12}k_{23}-k_{13}k_{23}$ must be equal to $k_{21}k_{31}$ *without proving* it directly from the definition of k_{ij} .This illustrates the power of the approach used in the previous lemma.

We will need the following lemma in the theorem that follows.

LEMMA 5.4

The $\mu+1$ polynomials $(x+1)^{\mu-r}(x-1)^r$,r = 0,1..,μ. are linearly independent.

Proof:

Let

$y = \beta_1(x+1)^{\mu} + .. + \beta_r(x+1)^{\mu-r}(x-1)^r + .. + \beta_{\mu+1}(x-1)^{\mu} = 0$

Set x = 1 all terms vanish except the first.Thus $\beta_1 = 0$.

Similarly x = -1 gives $\beta_{\mu+1} = 0$.

Differentiating y with respect to x we find

$$y' = \sum_{r=1}^{\mu+1} \beta_r \, [\, (\mu-r+1)(x+1)^{\mu-r}(x-1)^{r-1} + (r-1)(x+1)^{\mu-r+1}(x-1)^{r-2} \,]$$

If $r = 2$ the second term in the square brackets is $(x+1)^{\mu-1}$.

If $r = \mu$ the first term in the square brackets is $(x-1)^{\mu-1}$.

All other terms involve at least one double product $(x+1)(x-1)$.Thus setting $x = -1$ in y' gives $\beta_2 = 0$ and $x = 1$ gives $\beta_\mu = 0$ (given that β_1 and $\beta_{\mu+1}$ are both zero).It is clear that at each successive differentiation two terms not involving the double product $(x+1)(x-1)$ as a factor will be released.Thus setting $x = 1$ and $x = -1$ causes a pair of coefficients β_i to vanish.Thus we find that the polynomials $(x+1)^{\mu-r}(x-1)^r$ $r=0,1,..,\mu$ are linearly independent ■

COROLLARY 5.2

The solution of the following is the trivial solution.

$$\sigma \beta = 0 \tag{5.15}$$

where,

$$\left.\begin{aligned} \sigma_{ij} &= \sigma_{i-1}(-1,-1,..,-1,1,1,..,1;j\text{-}1 \text{ negatives}) \quad i\neq 1 \\ \sigma_{1j} &= \sigma_0 \equiv 1 \;\forall\; j \end{aligned}\right\} \tag{5.16}$$

and $\sigma_k(x_1,x_2,..,x_m)$ are the elementary symmetric functions of their arguments i.e. $\sigma_1(x,y,z) = x+y+z$,$\sigma_2(x,y,z) = xy+xz+yz$.

$\beta = (\beta_1,\beta_2,..,\beta_{\mu+1})^T$.

Proof:

Since,

$$(x+1)^{\mu-r}(x-1)^r = \sum_{i=1}^{\mu+1} \sigma_{ij} \; x^{\mu-(i-1)}$$

with σ_{ij} defined above we may write

$$\beta_1(x+1)^\mu + .. + \beta_r(x+1)^{\mu-r}(x-1)^r + .. + \beta_{\mu+1}(x-1)^\mu = 0$$

Equating coefficients of x to zero we obtain $\sigma \beta = 0$ with σ and

β defined above,but in lemma 5.4 we saw that $\beta_i = 0 \ \forall$ i.This is precisely the trivial solution.Hence the corollary is proved ■

We now turn to the main result of this section.

THEOREM 5.1

The coefficient of the m tuple $t(j_1)....t(j_m)$,denoted $C(j_1,..,j_m)$ in the N parameter solution of the sG equation is given by ,

$$C(j_1,..,j_m) = \prod_{i=1}^{m} \prod_{l=1}^{N-m} k_{j_i r_l} \tag{5.17}$$

where it is to be understood that if $r_l \neq j_i \ \forall$ l,i (i.e m = N) then C( ) is one.

e.g. N = 5

$C(1,2,3) = k_{14} k_{15} k_{24} k_{25} k_{34} k_{35}$ $C(1) = k_{12} k_{13} k_{14} k_{15}$

Proof :

In lemma 5.2 we saw that $C(j_1,..,j_m)$ under mapping (5.6) was the sign of the associated N tuple.In lemma 5.3 we discovered the k_{ij} that $C(j_1,...,j_m)$ depended on.Since we can permute the j_i in the m tuple without affecting $C(j_1,...,j_m)$,it must be a symmetric function of the k_{ij} on which it depends.This means we can take $C(j_1,...,j_m)$ to be given by,

$$C(j_1,...,j_m) = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + ... \alpha_\mu \sigma_\mu \tag{5.18}$$

where σ_i are the elementary symmetric functions on the set (5.14) given by

$$\sigma_1 = \sum k_{j_i r_i}$$

$$\sigma_2 = \sum \prod^{2} k_{j_i r_i} \tag{5.19}$$

$$\vdots$$

$$\sigma_\mu = \prod^{m} k_{j_i r_i}$$

where $\Sigma \prod^{r}$ denotes the sum of all combinations of r tuple products taken from the set (5.14).

e.g. N = 3 m = 3 $j_1 = 1$ $j_2 = 2$ $j_3 = 3$

$$\sigma_1 = k_{14} + k_{15} + k_{24} + k_{25} + k_{34} + k_{35}$$

$$\sigma_6 = k_{14}k_{15}k_{24}k_{25}k_{34}k_{35}$$

Powers of σ_i have been excluded in (5.16) since we saw in lemma 5.3 that when k_{ij} = signum(j-i) with one exception k_{ir},only the first power in k_{ir} ocurred.If there are N parameters then the number of k_{ij} in the set (5.14) is $\mu = m(N-m)$.When k_{ij} is given by mapping (5.6) each pair (i,j) in the associated N tuple for which i<j is associated with $k_{ij} = -1$ and changes the sign of the associated N tuple once.Hence when the mapping (5.6) is introduced into (5.18) we obtain the following μ+1 equations in the μ unknowns α_i,where the right hand sides are a consequence of lemma 5.2 .

(5.20)

$$\alpha_1\sigma_1(1,1,..,1) + \alpha_2\sigma_2(1,1,..,1) + ..+ \alpha_\mu\sigma_\mu(1,1,..,1) = 1$$

$$\alpha_1\sigma_1(-1,1,..,1) + \alpha_2\sigma_2(-1,1,..,1) +..+ \alpha_\mu\sigma_\mu(-1,1,..,1) = -1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\alpha_1\sigma_1(-1,..,-1) + \alpha_2\sigma_2(-1,..,-1) +..+ \alpha_\mu\sigma_\mu(-1,..,-1) = (-1)^\mu$$

We write (5.20) in the following way,

$$\sigma^T\alpha = \iota \qquad (5.21)$$

where σ^T is a μ+1 × μ+1 matrix defined by ,

$$\left.\begin{aligned}(\sigma^T)_{ij} &= \sigma_{j-1}(-1,-1,..,-1,1,1,..,1;i-1 \text{ negatives}) \quad j\neq 1\\ (\sigma^T)_{i1} &= \sigma_0 \equiv 1 \ \forall\ i\end{aligned}\right\} \qquad (5.22)$$

and α is the column vector $(0,\alpha_1,\alpha_2,...,\alpha_\mu)^T$and ι is the column vector $(0,1,-1,1,...,(-1)^\mu)$.

The solution space of the associated homogeneous equations of (5.19) namely,

$$\sigma^T\alpha = 0 \qquad (5.23)$$

is trivial(α = 0) because of corollary 5.2.As a consequence of that corollary $\det(\sigma) \neq 0$ and consequently $\det(\sigma^T) \neq 0$.

The general solution of (5.21) is the sum of the solution of the associated homogeneous system (5.23) and a particular solution of (5.21).The particular solution is easily provided:

$$\alpha_i = 0 \;\; i < \mu \;,\; \alpha_\mu = 1 \tag{5.24}$$

This then is the general solution to (5.21) also.

Substituting (5.24) into (5.18) we see the theorem is proved ■

We give the four parameter solution of the sG as an example,

$$t(1,2,3,4) = \frac{A + B}{1 + C + D}$$

where,

$$A = k_{12}k_{13}k_{14}t(1) + k_{21}k_{23}k_{24}t(2) + k_{31}k_{32}k_{34}t(3) + k_{41}k_{42}k_{43}t(4)$$

$$\begin{aligned} B = \; & k_{14}k_{24}k_{34}t(1)t(2)t(3) + k_{13}k_{23}k_{43}t(1)t(2)t(4) \\ & + k_{12}k_{32}k_{42}t(1)t(3)t(4) + k_{21}k_{31}k_{41}t(2)t(3)t(4) \end{aligned}$$

$$\begin{aligned} C = \; & k_{13}k_{14}k_{23}k_{24}t(1)t(2) + k_{12}k_{14}k_{32}k_{34}t(1)t(3) \\ & + k_{12}k_{13}k_{42}k_{43}t(1)t(4) + k_{21}k_{24}k_{31}k_{34}t(2)t(3) \\ & + k_{21}k_{23}k_{41}k_{43}t(2)t(4) + k_{31}k_{32}k_{41}k_{42}t(3)t(4) \end{aligned}$$

$$D = t(1)t(2)t(3)t(4)$$

We now determine the N parameter solution of the sG in another form.

THEOREM 5.2

The N parameter solution of the sG is given by,

$$\tan \Psi_N/4 = \frac{\sum_{l=0}^{[(N-1)/2]} (-1)^l \sum \prod_{p=1}^{2l+1} t(p) \prod_{q=p+1}^{2l+1} u^2_{pq}}{1 + \sum_{m=1}^{[N/2]} (-1)^m \sum \prod_{p=1}^{2m} t(p) \prod_{q=p+1}^{2m} u^2_{pq}} \tag{5.25}$$

where the unscripted summation sign indicates taking all combinations of r=2l+1 or 2m integers from 1,..,N in place of 1,..,r.If there are ns solitons,na antisolitons and nb breathers (N=ns+na+2nb) then

(5.26)

$1 \le p \le ns$	$t(p) = \exp X_p$	(a)
$ns+1 \le p \le ns+na$	$t(p) = -\exp X_p$	(b)
$ns+na+1 \le p \le N-1$	$t(p) = i \exp(\Gamma_{p\,p+1} - i\Omega_{p\,p+1})$	(c)
$(p = ns+na+1, ns+na+3, \ldots, N-1)$		
"	$t(p+1) = -i \exp(\Gamma_{p\,p+1} + i\Omega_{p\,p+1})$	(d)

and where ,

$$X_p = \gamma_p(x - u_p t) + \alpha_p \tag{e}$$

$$\Gamma_{p\,p+1} = \gamma_p \cos \mu\ (x - v_p t) + \alpha_p \tag{f}$$

$$\Omega_{p\,p+1} = \gamma_p \sin \mu\ (t - v_p x) + \beta_p \tag{g}$$

and ,

$1 \le p \le ns+na$	$\gamma_p = (1 - u_p^2)^{-1/2}$	(h)
$ns+na \le p$	$\gamma_p = (1 - v_p^2)^{-1/2}$	(i)
$\sin \mu = 2\pi/\tau$	$\cos \mu = \tau^{-1}(\tau^2 - 4\pi^2)^{+1/2}$	(j)

$|u|,|v| < 1$ and $\tau > 4\pi$ α and β are real phases.

τ is the breather rest frame period. v is the breather asymptotic speed($t=-\infty$). u is the soliton or antisoliton asymptotic speed.

The quantities u^2_{pq} are defined below. (5.27)

$1 \le p \le ns+na$ $\qquad a_p = \gamma_p^{-1}(1 - u_p)^{-1}$ (a)

$ns+na+1 \le p \le N-1 \qquad a_p = \exp -i\mu \;, a_{p+1} = a_p^*$ (b)

$*$ ≡ complex conjugation.

$$u_{pq}^2 = (a_p - a_q)^2 (a_p + a_q)^{-2} \qquad (c)$$

Note $u_{pq}^2 = k_{pq}^{-2}$

---

e.g. $N = 4$

$$\tan \Phi_N/4 = \frac{A - B}{1 - C + D}$$

where,

$$A = t(1) + t(2) + t(3) + t(4)$$

$$B = u_{12}^2 u_{13}^2 u_{23}^2 t(1)t(2)t(3) + u_{12}^2 u_{14}^2 u_{24}^2 t(1)t(2)t(4) + u_{13}^2 u_{14}^2 u_{34}^2 t(1)t(3)t(4) + u_{23}^2 u_{24}^2 u_{34}^2 t(2)t(3)t(4)$$

$$C = u_{12}^2 t(1)t(2) + u_{13}^2 t(1)t(3) + u_{14}^2 t(1)t(4) + u_{23}^2 t(2)t(3) + u_{24}^2 t(2)t(4) + u_{34}^2 t(3)t(4)$$

$$D = u_{12}^2 u_{13}^2 u_{14}^2 u_{23}^2 u_{24}^2 u_{34}^2 t(1)t(2)t(3)t(4)$$

Proof:

When the u_{pq}^2 are replaced by 1 for all p,q then corollary 5.1 applies.Thus to prove the theorem in general we have only to deduce (5.25) from the application of theorem 5.1.This can be done via a phase shift as will be seen below.

In theorem 5.1 we saw in particular that the coefficients of the quantities t(j) were (i.e 1 tuples),

$$C(j) = \prod_{r \neq j} k_{jr} \qquad r \in \{1,..,N\} \setminus \{j\}$$

We map C(j)t(j) to $\bar{t}(j)$ by phase shift so that,

$$\bar{t}(j) = \prod_{r \neq j} k_{jr} t(j) \qquad (5.28)$$

After this transformation the m tuple $D(j_1, j_2, .., j_m)$becomes

$$D(j_1,..,j_m) = \prod_{r_i \neq j_i} k_{j_i r_i} t(j_i) = \frac{\prod_{r_i \neq j_i} k_{j_i r_i} \bar{t}(j_i)}{\prod_{j_i} \prod_{r_i \neq j_i} k_{j_i r}} \quad (5.29)$$

e.g. N = 4   2 tuple

$$k_{21} k_{24} k_{31} k_{34} t(2)t(3) = \frac{k_{21} k_{24} k_{31} k_{34} \bar{t}(2)\bar{t}(3)}{k_{21} k_{23} k_{24} k_{31} k_{32} k_{34}}$$

$$= -k_{23}^{-2} \bar{t}(2)\bar{t}(3) = -u_{23}^{2} \bar{t}(2)\bar{t}(3)$$

Without loss of generality we choose $j_i \in \{1,..,m\}$.Thus we write (5.29) ,

$$D(1,..,m) = \frac{\prod k_{1r_1} \prod k_{2r_2} \ldots \prod k_{mr_m} \bar{t}(1)\ldots\bar{t}(m)}{\prod_{r\neq 1} k_{1r} \prod_{r\neq 2} k_{2r} \ldots \prod_{r\neq m} k_{mr}}$$

where $r_i \notin \{1,..,m\}$ Now for $l \in \{1,..,m\}$,

$$\prod k_{lr_1} = k_{lm+1} k_{lm+2} \ldots k_{lN}$$

$$\prod k_{lr} = k_{l2} k_{l3} \ldots k_{lm} k_{lm+1} \ldots k_{lN}$$

Thus the coefficient of $\bar{t}(1)\ldots\bar{t}(m)$,$\bar{C}(1,..,m)$ is given by ,

$$\bar{C}(1,..,m) = \left(\prod_{r\neq 1}^{m} k_{1r} \prod_{r\neq 2}^{m} k_{2r} \ldots \prod_{r\neq m}^{m} k_{mr} \right)^{-1}$$

Setting out these quantities in a table ,

$$\prod_{r\neq 1}^{m} k_{1r} = k_{12} k_{13} k_{14} k_{15} \ldots k_{1m}$$

$$\prod_{r\neq 2}^{m} k_{2r} = k_{21} k_{23} k_{24} k_{25} \ldots k_{2m}$$

$$\vdots$$

$$\prod_{r\neq m}^{m} k_{mr} = k_{m1} k_{m2} k_{m3} k_{m4} \ldots k_{mm-1}$$

Clearly ,

$$\bar{C}(1,..,m) = (-1)^{m(m-1)/2} \prod_{j>i} k_{ij}^{-2} = (-1)^{[m/2]} \prod_{j>i} u_{ij}^{2}$$

This proves the theorem ■

So far we have been referring to "the N parameter solution of the sG".In the next theorem we establish that it is indeed a "N soliton " solution.We show that in the asymptotic time limits (5.25) does become a sum of ns solitons,na antisolitons and nb breathers.The only trace remaining in the future of a previous interaction being the so called phase shift.We also find that the sum of the phase shifts is zero.

§ 4. The N-Parameter Multisoliton Solution

THEOREM 5.3

The N parameter solution of the sG defined in (5.25) is a ns soliton,na antisoliton and nb breather solution.

Proof:

Let us suppose that all the speeds u (solitons/antisolitons) and v (breathers) have been arranged so that if $u_i < u_j$ or $u_i < v_j$ then $i > j$.

Let us move in a frame of reference moving at speed w_r along the positive x axis (i.e w = u or v depending on p).Thus $x_{old} \to \gamma\ (x + w_r t)$, $t_{old} \to \gamma\ (t + w_r x)$,where $\gamma = (1-w_r^2)^{-1/2}$

Substituting the above into (5.26e,f) we obtain ,

$$X_p \to \gamma\gamma_p[\, x(1 - u_p w_r) + (w_r - u_p)t] \tag{5.30a}$$

$$\Gamma_{pp+1} \to \gamma\gamma_p \sin\mu\ [\, x(1 - v_p w_r) + (w_r - v_p)t] \tag{5.30b}$$

As a consequence(5.26a-d), (5.31)

$$\lim_{t \to -\infty} t(p) = \begin{cases} 0 & r < p \\ \infty & r > p \\ \text{finite} & r = p \end{cases} \qquad \lim_{t \to +\infty} t(p) = \begin{cases} \infty & r < p \\ 0 & r > p \\ \text{finite} & r = p \end{cases}$$

where if p is an antisoliton parameter the limit is -∞ instead of ∞ above.

Let L represent the number of parameters for which $p < r$ and let M represent the number of parameters for which $p > r$.

Now consider (5.25) as $t \to -\infty$,where r is a soliton or antisoliton parameter (i.e. rest frame of soliton or antisoliton).There will be certain tuples which consist of all the terms t(p) which tend to an infinite limit plus a term t(p = r) having a finite limit.Clearly we can factor such terms out on the numerator(odd tuples) and denominator(even tuples) of the N parameter solution (5.25).When the numerator and denominator are factored in this way all that is left in the factorized part is unity plus many terms that tend to zero in the asymptotic limit.This is precisely because we have factored out the tuples containing *all* the t(j) which tend to infinity.

There will be two possible cases in the $t \to -\infty$ limit according to whether L is an odd or even integer.Thus ,

L odd CASE 1 (5.32)

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to \frac{(-1)^{\frac{L-1}{2}} \prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq}}{(-1)^{\frac{L+1}{2}} \prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq}} = - \left(\prod_{p=1}^{r-1} u^2_{pr}\, t(r) \right)^{-1}$$

e.g N = 8 r = 4 123<u>4</u>5678

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to \frac{-\, t(1)u^2_{12}u^2_{13}t(2)u^2_{23}t(3)}{t(1)u^2_{12}u^2_{13}u^2_{14}t(2)u^2_{23}u^2_{24}t(3)u^2_{34}t(4)}$$

L even CASE 2 (5.33)

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to \frac{(-1)^{L/2} \prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq}}{(-1)^{L/2} \prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq}} = \prod_{p=1}^{r-1} u^2_{pr}\, t(r)$$

e.g.N = 8 r = 5 1234<u>5</u>678

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to$$

$$\frac{t(1)u^2_{12}u^2_{13}u^2_{14}u^2_{15}\; t(2)u^2_{23}u^2_{24}u^2_{25}t(3)u^2_{34}u^2_{35}t(4)u^2_{45}t(5)}{t(1)u^2_{12}u^2_{13}u^2_{14}t(2)u^2_{23}u^2_{24}t(3)u^2_{34}t(4)}$$

When $t \to \infty$ we have the following ,

M odd CASE 3 (5.34)

$$\lim_{t \to \infty} \tan \Psi_N/4 \to \frac{(-1)^{\frac{M-1}{2}} \prod_{p=r+1}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq}}{(-1)^{\frac{M+1}{2}} \prod_{p=r}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq}} = -\left(\prod_{q=r+1}^{N} u^2_{rq}\, t(r) \right)^{-1}$$

e.g.N = 8 r = 5 1234<u>5</u>678

$$\lim_{t \to \infty} \tan \Psi_N/4 \to \frac{-\ t(6)u^2_{67}u^2_{68}t(7)u^2_{78}t(8)}{t(5)u^2_{56}u^2_{57}u^2_{58}t(6)u^2_{67}u^2_{68}t(7)u^2_{78}t(8)}$$

M even CASE 4 (5.35)

$$\lim_{t \to \infty} \tan \Psi_N/4 \to \frac{(-1)^{M/2} \prod_{p=r}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq}}{(-1)^{M/2} \prod_{p=r+1}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq}} = \prod_{q=r+1}^{N} u^2_{rq}\, t(r)$$

e.g.N = 8 r = 4 123<u>4</u>5678

$$\lim_{t \to \infty} \tan \Psi_N/4 \to$$

$$\frac{t(4)u^2_{45}u^2_{46}u^2_{47}u^2_{48}\ t(5)u^2_{56}u^2_{57}u^2_{58}t(6)u^2_{67}u^2_{68}t(7)u^2_{78}t(8)}{t(5)u^2_{56}u^2_{57}u^2_{58}t(6)u^2_{67}u^2_{68}t(7)u^2_{78}t(8)}$$

Cases 1 and 3 both can be brought to the same form as cases 1 and 2 by mapping $\Psi_N \to \Psi_N + 2\pi$,this cause $\tan \Psi_N/4$ to become its negative reciprocal.Clearly depending on whether the totel number of parameters is even or odd we can have four asymptotic cases (1 & 3) ,(1 & 4) ,(2 & 3) ,and (2 & 4).For all these cases we find only the rest frame one parameter term left in the asymptotic limits.Hence the theorem is proved for solitons and antisolitons.

We define the phases :

$$\lambda^-_r = \ln \prod_{p=1}^{r-1} u^2_{pr}\, t(r) \qquad , \quad \lambda^+_r = \ln \prod_{q=r+1}^{N} u^2_{rq}\, t(r) \qquad (5.36)$$

We now consider breather rest frames.The analysis is slightly more involved as dominating tuples come in a number of varieties.We may have tuples containing no breather rest frame terms,tuples containing one breather rest frame term and finally tuples containing both breather rest frame terms.As before, we take these dominating terms out as factors in the numerator and denominator of the N parameter solution (5.25).

L odd

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to$$

$$\frac{(-1)^{\frac{L+1}{2}}\left[-\prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq} + \prod_{p=1}^{r+1} t(p) \prod_{q=p+1}^{r+1} u^2_{pq}\right]}{(-1)^{\frac{L+1}{2}}\left[\prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq} + \prod_{p=1}^{r+1\backslash r} t(p) \prod_{q=p+1}^{r+1\backslash r} u^2_{pq}\right]}$$

Removing factors of $\prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq}$ from numerator and denominator in the above we find,

$$\lim_{t \to -\infty} \tan\Psi_N/4 =$$

$$-\frac{\left[1 - \prod_{p=1}^{r-1} u^2_{pr} \prod_{p=1}^{r-1} u^2_{pr+1} u^2_{rr+1} t(r)t(r+1)\right]}{\left[\prod_{p=1}^{r-1} u^2_{pr} t(r) + \prod_{p=1}^{r-1} u^2_{pr+1} t(r+1)\right]}$$

$$\lim_{t \to -\infty} \tan\Psi_N/4 =$$

$$-\frac{\left[1 - u^2_{rr+1} \exp(\lambda^-_r + \lambda^-_{r+1})t(r)t(r+1)\right]}{\exp(\lambda^-_r)t(r) + \exp(\lambda^-_{r+1})t(r+1)} \tag{5.37}$$

e.g.N = 8 r = 4 123<u>45</u>678

$$\lim_{t \to -\infty} \tan\Psi_N/4 =$$

$$\frac{-t(1)u^2_{12}u^2_{13}t(2)u^2_{23}t(3)\left[1 - u^2_{45}u^2_{14}u^2_{24}u^2_{34}t(4)u^2_{15}u^2_{25}u^2_{35}t(5)\right]}{t(1)u^2_{12}u^2_{13}t(2)u^2_{23}t(3)\left[u^2_{14}u^2_{24}u^2_{34}t(4) + u^2_{15}u^2_{25}u^2_{35}t(5)\right]}$$

L even

$$\lim_{t \to -\infty} \tan \Psi_N/4 \to$$

$$\frac{(-1)^{L/2}\left[\prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq} + \prod_{p=1}^{r+1\setminus r} t(p) \prod_{q=p+1}^{r+1\setminus r} u^2_{pq} \right]}{(-1)^{L/2}\left[\prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq} - \prod_{p=1}^{r+1} t(p) \prod_{q=p+1}^{r+1} u^2_{pq} \right]}$$

$$\lim_{t \to -\infty} \tan\Psi_N/4 =$$

$$\frac{\exp(\lambda^-_r)t(r) + \exp(\lambda^-_{r+1})t(r+1)}{\left[1 - u^2_{r\,r+1} \exp(\lambda^-_r + \lambda^-_{r+1})t(r)t(r+1) \right]} \qquad (5.38)$$

M odd $\quad t \to +\infty$

$$\lim_{t \to +\infty} \tan \Psi_N/4 \to$$

$$\frac{(-1)^{\frac{L-1}{2}}\left[\prod_{p=r+2}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq} - \prod_{p=r}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq} \right]}{(-1)^{\frac{L+1}{2}}\left[\prod_{p=r}^{N\setminus r+1} t(p) \prod_{q=p+1}^{N\setminus r+1} u^2_{pq} + \prod_{p=r+1}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq} \right]}$$

Extracting the factor $\prod_{p=r+2}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq}$ from the numerator and denominator we find ,

$$\lim_{t \to +\infty} \tan \Psi_N/4 \to$$

$$-\frac{\left[1 - u^2_{r\,r+1} \exp(\lambda^+_r + \lambda^+_{r+1})t(r)t(r+1) \right]}{\exp(\lambda^+_r)t(r) + \exp(\lambda^+_{r+1})t(r+1)} \qquad (5.39)$$

In the case M even we find by similar arguments that $\lim_{t \to +\infty} \tan \Psi_N/4$ equals an expression which is the negative reciprocal of (5.39).Thus we have shown the theorem to be true also for breathers.■

We note that it is a consequence of the above theorem that in the interaction of solitons ,antisolitons and breathers that like exchange with like, as if we begin in a soliton/antisoliton/breather rest frame in the remote past we once again find ourselves in a soliton/antisoliton/breather rest frame in the remote future.This also leads to the conservation of the numbers of solitons,antisolitons and breathers.

We comment on some properties of the phase shifts $\lambda_r^{+/-}$.

Suppose r represents a soliton or antisoliton parameter.Consider :

$$\exp(\lambda_r^-) = \prod_{p=1}^{r-1} u_{pr}^2 t(r) = \prod^{r-1} u_{kr}^2 t(r) \prod^{r-1} u_{lr}^2 t(r) u_{l+1r}^2 \qquad (5.40)$$

where the first product on the right hand side has k ranging over the set of soliton or antisoliton parameters less than r,while the second product sign has l ranging over only the soliton in breather parameters (first parameter of the pair) less than r.

Now from the definition of u_{lr}^2 (5.27c) ,

$$(u_{lr}^2)^* = [(a_l - a_r)^2 (a_l + a_r)^{-2}]^* = (a_l^* - a_r)^2 (a_l^* + a_r)^{-2}$$

where we have used the fact that a_r is real and a_l is complex.

Thus noting (5.27b) the above may be written ,

$$(u_{lr}^2)^* = u_{l+1r}^2$$

This means soliton and antisoliton phases are *real* as both sides of the above equation are multiplying each other in a soliton or antisoliton phase (5.40).

Now suppose r represents a soliton in a breather.Then it follows,

$$[\exp(\lambda_r^-)]^* = \left(\prod_{p=1}^{r-1} u_{pr}^2 t(r)\right)^* = \left(\prod^{r-1} u_{kr}^2 t(r) \prod^{r-1} u_{lr}^2 t(r) u_{l+1r}^2\right)^*$$

$$= \prod^{r-1} u_{kr+1}^2 t(r) \prod^{r-1} u_{l+1r+1}^2 t(r) \prod^{r-1} u_{lr+1}^2 t(r)$$

This last step is a consequence of the fact that k ranges over soliton or antisoliton parameters so that a_k is real ,while l ranges over all soliton in breather parameters($l \neq r$) so a_l is complex.We have also employed (5.27b-c).

We can write this last result,

$$\left[\prod_{p=1}^{r-1} u_{pr}^2 t(r) \right]^* = \prod_{p=1}^{r-1} u_{pr+1}^2 t(r)$$

Thus if r is a breather parameter (soliton) ,

$$(\lambda_r^-)^* = \lambda_{r+1}^-$$

Similar results obtain for λ_r^+.

COROLLARY 5.3

The sum of the phase shifts in the N parameter solution of the sG is zero.

Proof:

The phase shifts are defined by ,

$$\lambda_r^- = \ln \prod_{p=1}^{r-1} u_{pr}^2 t(r) \quad , \lambda_r^+ = \ln \prod_{q=r+1}^{N} u_{rq}^2 t(r)$$

Thus,

$$\sum_{r=1}^{N} (\lambda_r^+ - \lambda_r^-) = \sum_{r=1}^{N} \ln \left\{ \frac{\prod_{q=r+1}^{N} u_{rq}^2 t(r)}{\prod_{p=1}^{r-1} u_{pr}^2 t(r)} \right\} \tag{5.41}$$

The quantities in the numerator and denominator of (5.41) can be set out into two groups ,

Numerator	Denominator
$u_{12}u_{13}\ldots u_{1N}$	1
$u_{23}u_{24}\ldots u_{2N}$	u_{12}
$u_{34}u_{35}\ldots u_{3N}$	$u_{13}u_{23}$
$\vdots$	$u_{14}u_{24}u_{34}$
u_{N-1N}	$\vdots$
1	$u_{1N}u_{2N}\ldots u_{N-1N}$

These tables are clearly rearrangements of one another.The argument of the ln in (5.41) is therefore 1.So the result is proved ■

§ 5. **Concluding remarks**

We have seen in this chapter how the "N soliton" solution of the sG can be built via a nonlinear superposition of asymptotic components(solitons,antisolitons,breathers).Such a nonlinear superposition though formally important and interesting, does not however reveal in any obvious manner how solitons,antisolitons and breathers interact with one another.Neither does it tell us how the identity of solitons etc changes during the interaction.

In the next chapter we will see how the "N soliton" solution of the sG(MKdV,KdV and others) can be written as a *linear superposition of accelerating solitons*.This enables us not only to follow the motion of the solitons throughout the interaction but also to identify individually the solitons for all time and thus determine how they change their shape during the interaction.

Linear superposition plays an implicit role even in the arguments connected with nonlinear superposition used in this chapter.Firstly we saw in §2 how when the parameters k_{ij} = signum(j-i) the N parameter solution Φ_N became a linear superposition of single parameter terms.Further to this we also studied Φ_N when one of the k_{ij} (k_{ir}) was allowed not to be ±1.In this case we saw(proof of lemma 5.3) ,

$$\tan \Phi_N/4 = \frac{\tan \sum_{l=1}' \Phi_l/4 \quad + \quad k_{ir} \tan \left[(-1)^{i+1}(\Phi_i - \Phi_r)/4\right]}{1 \quad - \quad k_{ir} \tan \left[(-1)^{i+1}(\Phi_i - \Phi_r)/4\right] \tan \sum_{l=1}' \Phi_l/4}$$

where the prime on the summation sign meant that i and r were excluded.Now let ,

$$\tan \Phi_{ir}/4 = k_{ir} \tan \left[(-1)^{i+1}(\Phi_i - \Phi_r)/4 \right]$$

Φ_{ir} is the two parameter solution of the sG.With this the equation above becomes simply ,

$$\tan \Phi_N/4 = \tan \left[\sum_{l=1}' \Phi_l/4 + \Phi_{ir}/4 \right]$$

or,

$$\Phi_N = \sum_{l=1}' \Phi_l + \Phi_{ir}$$

That is to say the Φ_N being considered here is just a *linear superposition* of N-2 non interacting solitons and one interacting pair Φ_{ir}.

Although we have addressed all the arguments in this chapter specifically to the sG,essentially identical arguments can be used for the MKdV.Though the MKdV being a unidirectional Galilean invariant equation ,means that we cannot by Galilean transformation enter the rest frames of all the solitons.So that the MKdV equivalent of theorem 5.3 would be different.

A theorem of permutability for the Hirota τ function associated with the KdV which satisfies [2] ,

$$\tau\tau_{xt} - \tau_x \tau_t + \tau\tau_{xxxx} - 4\tau_x \tau_{xxx} + 3\tau_{xx}^2 = 0$$

is implicit in an article by Wahlquist [3].It is very likely that with it we could formulate similar arguments to derive the N-soliton solution as those advanced in this chapter.

We note the 1 and 2 soliton formulae for τ satisfying the above equation (u satisfying the KdV is given by $u = 2(\ln\tau)_{xx}$).

$$\tau_1 = 1 + \exp \theta_1 \quad ,\tau_{12} = 1 + \exp \theta_1 + \exp \theta_2 + u_{12}^2 \exp(\theta_1 + \theta_2)$$

τ_1 is the one soliton solution,τ_{12} the two soliton solution.

$$\theta_i = k_i x - k_i^3 t + \delta_i \quad ,u_{12}^2 = (k_1 - k_2)^2/(k_1 + k_2)^2 \quad ,k_1 > k_2$$

We saw in the chapter 3 how an alternating structure is present in the theorem of permutability satisfied by u.

Clearly further work on this topic would be valuable.

CHAPTER 6 : LINEAR SUPERPOSITION OF SOLITONS

§ 0. Introduction

In this chapter we introduce the linear superposition polynomials(lsp's) for the N parameter solutions of a large number of soliton equations.By N parameter we mean in general a mixture of solitons(sG,MKdV,NLS,KdV and relatives), antisolitons and breathers(sG,MKdV).The lsp's enable us to identify the solitons for all time and hence calculate how they interact with one another and also how they change their shape.

To the theoretical physicist interested in the particle-like nature of solitons,the detailed processes of their interaction are of great interest.Given the stability of solitons in collisions, and their characteristic asymptotic shapes,it is natural to suppose that the solitons are not lost in the multisoliton profile, and that perhaps their shapes are not altered very much.This is indeed the case as is explained in this chapter.The remarkable finding of §1 is that the basic form of the lsp is the same for all the soliton equations considered.

In §2 we find that the roots of the lsp turn out to be the eigenvalues of certain matrices which arise in inverse scattering theory.

In §3 we explore general properties of the roots of the lsp without solving the polynomial itself.As exact formulae for the roots of polynomials of degree higher than 4 do not in general exist, the exploration of the properties of the roots from general considerations is very important.We give the exact formulae for the roots of polynomials of degree less than or equal 4 in §4,though only the roots of the quadratic are revealing.

In §5 we explore the links between the motion of singularities of the N parameter complexified sG solution and the lsp,coming to the important conclusion that *points where the roots of the real lsp are equal to one(for solitons in breathers or otherwise) and minus one(for antisolitons in breathers or otherwise) coincide with the positions of the projections of the branch points on the real axis.*

In §6 we discuss how, with a special choice of phase(perfect phase), the soliton interaction is centred on the origin of the x,t plane, and we are able thus to determine the location of some of the solitons at time t=0 without solving the lsp.We also discuss how if we select the asymptotic speeds in a certain way,the behaviour of the roots of the lsp, and hence the solitons is completely time symmetric.

In §7 we discuss the reasons for choosing a particular antisoliton representation(sG,MKdV),thus discovering that whether the roots of the lsp are real, is sensitive to allowed transformations of the N parameter solution.We also discuss the *non-uniqueness* of the lsp(sG,MKdV).

The chapter ends with some concluding remarks.

§ 1. Multi-soliton solutions of the sG ,MKdV ,NLS ,KdV(& relatives) as linear superpositions of accelerating solitons.

In this section, we prove that the sG,MKdV,NLS,KdV and other equations having multisoliton solutions of KdV form(KP,Boussinesq,higher KdV etc), have multisoliton solutions which can be written as a linear superposition of characteristic functions.Formally the linear superposition(LS) is similar to a LS of non-interacting solitons,but where the argument which is a linear function of x and t(and y possibly) is replaced by a nonlinear function, $g(x,t) = \ln f(x,t)$.The functions $f_i(x,t)$ are the roots of the polynomial which we will define below.It is remarkable that the form of the polynomial is identical for *all* the equations under consideration.
We establish these facts in the theorems that follow.

THEOREM 6.1(sG)

The N-parameter solution of the sG equation is given by ,

$$\Phi_N = 4 \sum_{m=1}^{N} \tan^{-1} f_m \qquad (6.1)$$

where f_m are the roots of ,

$$H(f) = 0 \qquad (6.2)$$

and ,

$$H(f) = f^N + \sum_{l=1}^{N}\left\{(-1)^l \sum \prod_{p=1}^{l} t(p) \prod_{q=p+1}^{l} u^2_{pq}\ f^{N-l}\right\} \qquad (6.3)$$

where the unscripted summation sign indicates taking all combinations of l integers taken from 1,..,N.If there are ns solitons,na antisolitons and nb breathers (N=ns+na+2nb) then

(6.4)

$1 \le p \le ns$ $\qquad t(p) = \exp X_p$ (a)

$ns+1 \le p \le ns+na$ $\qquad t(p) = -\exp X_p$ (b)

$ns+na+1 \le p \le N-1$ $\qquad t(p) = -i \exp(\Gamma_{pp+1} + i\Omega_{pp+1})$ (c)

$(p = ns+na+1, ns+na+3, .., N-1)$

" $\qquad t(p+1) = +i \exp(\Gamma_{pp+1} - i\Omega_{pp+1})$ (d)

and where ,

$$X_p = \gamma_p(x - u_p t) + \alpha_p \quad (e)$$

$$\Gamma_{p\ p+1} = \gamma_p \cos \mu_p(x - v_p t) + \alpha_p \quad (f)$$

$$\Omega_{p\ p+1} = \gamma_p \sin \mu_p(t - v_p x) + \beta_p \quad (g)$$

and ,

$1 \le p \le ns+na$ $\qquad \gamma_p = (1 - u_p^2)^{-1/2}$ (h)

$ns+na \le p$ $\qquad \gamma_p = (1 - v_p^2)^{-1/2}$ (i)

$$\sin \mu_p = 2\pi/\tau_p \qquad \cos \mu_p = \tau_p^{-1}(\tau_p^2 - 4\pi^2)^{+1/2} \quad (j)$$

$|u|,|v| < 1$ and $\tau_p > 4\pi$ $\qquad \alpha$ and β are real phases.

τ_p is the breather rest frame period. v is the breather asymptotic speed($t=-\infty$). u is the soliton or antisoliton asymptotic speed.

The quantities u_{pq}^2 are defined below. (6.5)

$1 \le p \le ns+na$ $\qquad a_p = \gamma_p^{-1}(1 - u_p)^{-1}$ (a)

$ns+na+1 \le p \le N-1$ $\qquad a_p = b_p \exp +i\mu, a_{p+1} = a_p^*$ (b)

$$b_p = (1-v_p)^{1/2}(1+v_p)^{-1/2}$$

$*$ ≡ complex conjugation

$$u_{pq}^2 = (a_p - a_q)^2 (a_p + a_q)^{-2} \quad (c)$$

---

e.g. N = 3 f_i are the roots of the following polynomial ,

$$\begin{aligned} &f^3 \\ &- [t(1) + t(2) + t(3)]\, f^2 \\ &+ [u_{12}^2 t(1)t(2) + u_{13}^2 t(1)t(3) + u_{13}^2 t(2)t(3)]\, f \\ &- u_{12}^2 u_{13}^2 u_{23}^2 t(1)t(2)t(3) = 0 \end{aligned}$$

Proof:

We saw in corollary 5.1 and theorem 5.2 that $\tan\Phi_N/4$ has the form of a tan of N parts in t(p) with the addition of invariants u_{ij} .Now the theorem asserts that ,

$$\tan \Phi_N/4 = \tan \sum_{m=1}^{N} f_m$$

so that noting the formula for the tan of N parts given in §3 of the last chapter we see the above is the N parameter solution provided , (6.6)

$$\sigma_1(f_m) = f_1 + f_2 + \dots + f_N = t(1) + t(2) + \dots + t(N)$$

$$\vdots$$

$$\sigma_l(f_m) = \sum \prod_{p=1}^{l} f_p = \sum \prod_{p=1}^{l} t(p) \prod_{q=p+1}^{l} u_{pq}^{2}$$

$$\vdots$$

$$\sigma_N(f_m) = \prod_{p=1}^{N} f_p = \prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^{2}$$

where the σ_l are the elementary symmetric functions on the N variables f_m.

However the f_m are the N roots of (6.2-3) thus ,

$$H(f) = \prod_{m=1}^{N} (f-f_m) = f^N + \dots + (-1)^l \sigma_l(f_m) f^{N-l} + \dots + (-1)^N \prod_{p=1}^{N} f_p$$

Clearly substituting (6.6) into the above we obtain (6.3) ■

THEOREM 6.1(MKdV)

The N-parameter solution of the MKdV equation

$$w_t + 2\, w_x^3 + w_{xxx} = 0 \tag{6.7a}$$

is given by ,

$$w_N = 2 \sum_{m=1}^{N} \tan^{-1} f_m \tag{6.7b}$$

where f_m are the roots of (6.2) with H(f) defined in (6.3),and where ,

(6.8)

$1 \le p \le ns$ $\quad t(p) = \exp X_p$ (a)

$ns+1 \le p \le ns+na$ $\quad t(p) = -\exp X_p$ (b)

$ns+na+1 \le p \le N-1$ $\quad t(p) = -i \exp(\Gamma_{pp+1} + i\Omega_{pp+1})$ (c)

$(p = ns+na+1, ns+na+3, .., N-1)$

" $\quad t(p+1) = +i \exp(\Gamma_{pp+1} - i\Omega_{pp+1})$ (d)

and where ,

$$X_p = u_p^{1/2} (x - u_p t) + \alpha_p \quad \text{(e)}$$

$$\Gamma_{p\,p+1} = 2\theta_p [x - v_p t] + \alpha_p \quad \text{(f)}$$

$$\Omega_{p\,p+1} = 2\eta_p [x - w_p t] + \beta_p \quad \text{(g)}$$

and ,

$1 \le p \le ns+na$ $\quad u_p = 4 k_p^2$ (h)

$ns+na+1 \le p \le N-1$ $\quad v_p = 4 (\theta_p^2 - 3 \eta_p^2)$ (i)

$(p = ns+na+1, ns+na+3, .., N-1)$

" $\quad w_p = 4 (3 \theta_p^2 - \eta_p^2)$ (j)

where $k_i > k_j$ $i > j$ and $w_p > 3v_p$ α, β real phases, and period $\tau_p = \pi/\eta_p$. u_p are asymptotic soliton or antisoliton speeds, k_p are soliton or antisoliton amplitudes. v_p and w_p are breather velocity and breather phase velocity respectively. θ_p and η_p are the real and imaginary parts of the breather amplitude respectively.

The quantities u_{pq}^2 are defined below. (6.9)

$1 \le p \le ns+na$ $\quad a_p = k_p$ (a)

$ns+na+1 \le p \le N-1$ $\quad a_p = \theta_p + i\eta_p \;, \quad a_{p+1} = a_p^*$ (b)

$(p = ns+na+1, ns+na+3, .., N-1)$

$$u_{pq}^2 = (a_p - a_q)^2 (a_p + a_q)^{-2} \quad \text{(c)}$$

Proof:

As for the sG. The proof that this N parameter solution is a multisoliton solution has not been demonstrated in this thesis. However the solution (6.7) coincides with the multisoliton solution (no breathers) of Hirota [1]

The LS for the more common form of the MKdV ,

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{6.10}$$

where $u = -w_x$ is given below, by differentiating with respect to x equation (6.7b).

$$u_N = -2 \sum_{m=1}^{N} \frac{(f_m)_x}{1 + f_m^2} \tag{6.11}$$

or in terms of the argument g_m defined by ,

$$g_m = \ln f_m \tag{6.12}$$

$$u_N = - \sum_{m=1}^{N} (g_m)_x \text{ sech } g_m \tag{6.13}$$

THEOREM 6.2 (KdV,& RELATIVES)

The N-soliton solutions of the KdV equation ,

$$u_t - 6uu_x + u_{xxx} = 0 \tag{6.14}$$

the KP equation ,

$$(u_t - 6uu_x + u_{xxx})_x + u_{yy} = 0 \tag{6.15}$$

the Boussinesq equation ,

$$(u_x - 6uu_x + u_{xxx})_x + u_{tt} = 0 \tag{6.16}$$

are given by ,

$$u_N = -2 \sum_{m=1}^{N} (\ln [1 + f_m])_{xx} \tag{6.17}$$

f_m are the roots of (6.2) with H(f) defined by (6.3) If we once again express f_m in terms of g_m ($f_m = \exp g_m$),(6.17) becomes ,

$$u_N = - \sum_{m=1}^{N} \{ \frac{1}{2} A_m \text{ sech}^2(g_m/2) \} \tag{6.18}$$

with ,

$$A_m = g_{mx}^2 + (1 + f_m) g_{mxx} \tag{6.19}$$

where the quantities t(p) and u_{pq}^2 are given by the following equations.

$$t(p) = \exp X_p \tag{6.20 a}$$

For the KdV ,

$$X_p = u_p^{1/2} (x - u_p t) + \alpha_p \tag{b}$$

$$u_p = 4 k_p^2 \tag{c}$$

$$u_{pq}^2 = (k_p - k_q)^2 (k_p + k_q)^{-2} \tag{d}$$

For the KP ,

$$X_p = (r_p - s_p)x + (r_p^2 - s_p^2)y + (r_p^3 - s_p^3)t + \alpha_p \tag{e}$$

$$u_{pq}^2 = (r_p - r_q)^2 (r_p - s_q)^{-2} (s_p - s_q)^2 (s_p - r_q)^{-2} \tag{f}$$

For the Boussinesq ,

$$X_p = k_p (x - \varepsilon_p u_p t) + \alpha_p \tag{g}$$

$$u_p = (1 + k_p^2)^{1/2} , \varepsilon_p = \pm 1 \tag{h}$$

$$u_{pq}^2 = [\nu_{pq}^2 + 3(k_p - k_q)^2][\nu_{pq}^2 + 3(k_p + k_q)^2]^{-1} \tag{i}$$

where

$$\nu_{pq} = \varepsilon_p u_p - \varepsilon_q u_q \tag{j}$$

α_p are real phases and for the KdV $k_p \neq k_q$.For the KP $r_p \neq s_p$.

Proof:

As we saw in chapter 3 of this thesis all the above equations have multisoliton solutions which may be expressed in the following way ,

$$u_N = - 2(\ln \tau)_{xx} \tag{6.21}$$

where ,

$$\tau = \sum_{\mu_j=0,1} \exp \left[\sum_{j=1}^{N} \mu_j X_j + \sum_{1\leq i<j\leq N} \ln(u_{ij}^2) \mu_i \mu_j \right] \tag{6.22}$$

with X_j defined above.Now τ above can be written ,

$$\tau = 1 + \sum_{l=1}^{N} \sum \prod_{p=1}^{l} t(p) \prod_{q=p+1}^{l} u_{pq}^{2} \qquad (6.23)$$

where,as before the unscripted summation sign indicates summing over all combinations of l integers taken from N,and t(p) is defined in (6.20a).Setting $u_{pq}^{2} = 1$ in (6.23) gives ,

$$\tau = 1 + \sum_{l=1}^{N} \sigma_{l}(t(N)) \qquad (6.24)$$

where $\sigma_l(t(N))$ are the elementary symmetric functions on the N variables t(N).Of course (6.24) can be written in a yet more familiar form ,

$$\tau = \prod_{l=1}^{N} (1 + t(l))$$

It is clear that if when $u_{pq}^{2} \neq 1$ we let ,

$$\tau = \prod_{l=1}^{N} (1 + f_{l})$$

then the following equations must hold ,

$$\sigma_1(f_m) = f_1 + f_2 + \ldots + f_N = t(1) + t(2) + \ldots + t(N)$$

$$\vdots$$

$$\sigma_l(f_m) = \sum \prod_{p=1}^{l} f_p = \sum \prod_{p=1}^{l} t(p) \prod_{q=p+1}^{l} u_{pq}^{2}$$

$$\vdots$$

$$\sigma_N(f_m) = \prod_{p=1}^{N} f_p = \prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^{2}$$

These are familiar to us already from theorem 6.1(sG).Hence as then f_p must be the roots of (6.2) with H(f) defined in (6.3).■

e.g N = 3

$$\tau = 1 + f_1 + f_2 + f_3 + f_1 f_2 + f_1 f_3 + f_2 f_3 + f_1 f_2 f_3$$

$$= 1 + e^{X_1} + e^{X_2} + e^{X_3} + u_{12}^{2} e^{X_1+X_2} + u_{13}^{2} e^{X_1+X_3} + u_{23}^{2} e^{X_1+X_3} + u_{12}^{2} u_{13}^{2} u_{23}^{2} e^{X_1+X_2+X_3}$$

Naturally the linear superposition principle above also applies to the whole hierarchy of KdV equations which we saw defined in chapter 3.

CONJECTURE 6.0(NLS)

The N envelope soliton solutions of the Nonlinear Schrödinger equation ,

$$u_{xx} + iu_t + \varkappa|u|^2u = 0 \tag{6.25}$$

are given by ,

$$|u_N|^2 = \sqrt{2\varkappa} \sum_{m=1}^{N} (\ln [1 + f_m f_m^*])_{xx} \tag{6.26}$$

where f_m are the roots of (6.2-3) and $t(p)$ and u_{pq}^2 are defined by, (6.27)

$$t(p) = \frac{1}{i} \exp X_p \qquad (a)$$

$$X_p = -2\eta_p(x - 4\xi_p t) + \alpha_p \qquad (b)$$

$$u_{pq}^2 = w_{pq} w_{pq}^* \qquad w_{pq} = (k_p - k_q)/(k_p - k_q^*) \qquad (c)$$

where the amplitude k_p is given by ,

$$k_p = \xi_p + i\eta_p \qquad (d)$$

Proof:

We do not provide a proof of this.However, it is likely that a proof would be similar to that of the previous theorem but using the Hirota expression for the argument of the log in (6.26) [2].

§ 2. Roots of linear superposition polynomials(lsp's) as eigenvalues of matrices and links with inverse scattering theory.

We saw in chapter 3 how multisoliton solutions of the KdV type equations could be represented in the form ,

$$u_N = -2(\ln \det [I + C])_{xx} \qquad (6.28)$$

where I is the N x N unit matrix and C is the N x N matrix whose components c_{mn} may be written (KdV) ,

$$c_{mn} = 2k_m^{1/2} k_n^{1/2} (k_m + k_n)^{-1} \exp \frac{1}{2}(X_m + X_n) \qquad (6.29)$$

This may be expressed in terms of u_{mn}^2 defined in (6.20) ,

$$c_{mn} = (1 - u_{mn}^2)^{1/2} \exp \frac{1}{2}(X_m + X_n) \qquad (6.30)$$

In the above u_{mn}^2 can be any of the expressions defined in (6.20).The N envelope soliton solution of the NLS can also be written(chapter 3) in a form analogous to (6.28) ,

$$|u_N|^2 = \sqrt{2\varkappa}\,(\ln \det [I + CC^*])_{xx} \qquad (6.31)$$

where this time C is the matrix having components ,

$$c_{mn} = \frac{1}{i}(k_m - k_m^*)^{1/2} (k_n - k_n^*)^{1/2} (k_m - k_n^*)^{-1} \exp \frac{1}{2}(Y_m + Y_n^*) \qquad (6.32)$$

where ,

$$Y_m = X_m + 2i[\xi_m x + 2(\xi_m^2 - \eta_m^2)t] + \beta_m \qquad (6.33)$$

where X_m is given by (6.27b).β_m is a complex phase.

This can also be expressed in a similar form to (6.30),with u_{pq}^2 defined by(6.27c).It is clear from the form of (6.25) and the proof of the theorem(6.2 KdV) that if C is a diagonalizable matrix its eigenvalues are f_m where f_m are the roots of (6.2).A rigorous proof of this has been given by Bryan and Stuart(paper I in the appendix to this thesis.Also the eigenvalues f_m are shown to be distinct.This means that the eigenvectors ϕ_m associated with eigenvalues f_m and satisfying ,

$$C\ \phi_m = f_m \phi_m \tag{6.34}$$

are linearly independent.Thus the matrix T whose columns are the N eigenvectors ϕ_m diagonalizes C by a similarity transformation ,i.e

$$D = T^{-1} C\ T \tag{6.35}$$

where D is the diagonal matrix with elements f_m.

We give an example of some of the above remarks in the case of the two envelope soliton solution of the NLS.

The eigenvalues of C (6.32) are given by ,

$$\det \begin{pmatrix} \frac{1}{i} \exp \frac{1}{2}(Y_1+Y_1^*) - f & \frac{(k_1-k_1^*)^{1/2}(k_2-k_2^*)^{1/2}}{i(k_1-k_2^*)} \exp \frac{1}{2}(Y_1+Y_2^*) \\ \frac{(k_2-k_2^*)^{1/2}(k_1-k_1^*)^{1/2}}{i(k_2-k_1^*)} \exp \frac{1}{2}(Y_2+Y_1^*) & \frac{1}{i} \exp \frac{1}{2}(Y_2+Y_2^*) - f \end{pmatrix} = 0$$

(6.36)

which gives after some algebra the following equation for f ,

$$f^2 - \frac{1}{i}\left[\exp \frac{1}{2}(Y_1+Y_1^*) + \exp \frac{1}{2}(Y_2+Y_2^*)\right] f - u_{12}^2 \exp \frac{1}{2}(Y_1+Y_1^*+Y_2+Y_2^*) = 0$$

with u_{12} defined in (6.27c).Noting (6.33) this becomes an example of (6.2-3) with (6.27).According to (6.26) ,

$$|u_2|^2 = \sqrt{2\varkappa}\ (\ln [1 + f_1 f_1^*])_{xx} + \sqrt{2\varkappa}\ (\ln [1 + f_2 f_2^*])_{xx}$$

with f_1,f_2 the two roots of the above quadratic(both pure imaginary).

We note that if we let $f = h/i$ in (6.36) and define a matrix $B = iC$ then the eigenvalues of B are h_m and since B is Hermitian h_m are real.Note they are also positive and distinct.

We saw in chapter 3 that the wavefunctions ψ_i of the associated scattering problem for the KdV were given as the solutions of the equation below ,

$$(I+C)\psi = E \tag{6.37}$$

where in this case $\psi = (\psi_1,\psi_2,...,\psi_N)^T$ and

$$E = (\sqrt{2k_1}\exp X_1,...,\sqrt{2k_N}\exp X_N)^T$$

with C defined in (6.29).

Let us define a new set of "wave functions"

$$\chi \equiv T^{-1}\psi \tag{6.38}$$

where T is the matrix defined previously(after equation 6.34).

This allows us to write (6.37) in the form ,

$$(I+D)\chi = T^{-1}E \tag{6.39}$$

It is important to note that χ is not a superposition of ψ_i with constant coefficients(6.38) as the elements of T are functions of x and t

§ 3. General properties of the roots of the LSP.

In this section we attempt to discover as much information as we can about the functions f_m which are the roots of the lsp (6.2-3) without solving (6.2-3).As (6.3) is a polynomial we can only solve it exactly for N $\leq$ 4.For simplicity we do not address the following theorems and corollaries to the KP.

COROLLARY 6.1

The N roots of (6.2-3),f_m are never zero for $-\infty < x,t < \infty$.

Proof:

We saw in (6.6) that ,

$$\prod_{p=1}^{N} f_p = \prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^2$$

It is clear that the above product never vanishes in a bounded region of the x,t plane as for all the equations $\Pi t(p)$ has the form of an exponential in x and t.Thus no individual f_m can vanish ■

COROLLARY 6.2

If the N roots f_m are real then $\forall$ m f_m is either always positive or always negative.

Proof:

Since f_m are continuous functions of x and t which never acheive the value zero they must retain their sign ■

COROLLARY 6.3

For fixed time and $x \to \pm \infty$ (6.2-3) has N roots f_m whose moduli tend to zero in one limit and infinity in the other (modulus is used in the complex sense).

Proof:

It is clear from the various definitions of t(p) given for the NLPDE's in the earlier theorems that as $x \to -\infty$ $(+\infty)$ $|t(p)|$ tends to zero $\forall$ p .This means that in the appropriate limit $\sigma_l(f_p) \to 0$ $\forall$ l.Thus $H(f) \to f^N$ in this limit so that clearly $H(f) = 0$ has N zero roots in this limit.

Now consider the limit for which $|t(p)| \to \infty$ $\forall$ p. Let $\exists$ a f_p such that $|f_p|$ is finite.Clearly ,

$$|H(f_p)| = |\ f_p^N - [t(1)+..t(N)]f_p^{N-1}+...+(-1)^N \prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^2\ |$$

$$= |\ \prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^2\ |.|\ A\ |$$

where A just represents dividing H(f) term by term by the factor $\prod_{p=1}^{N} t(p) \prod_{q=p+1}^{N} u_{pq}^2$. Clearly the moduli of all the terms in A except for one which has unit modulus,tend to zero in the limit.Thus $|H(f_p)|$ becomes infinite.But $H(f) = 0.\to\leftarrow$ .Thus $|f_p|$ can not be finite.■

THEOREM 6.3

If the number of parameters N for the sG and MKdV equals the number of soliton parameters (antisoliton),then all the roots of (6.2-3) are real,positive(negative) and distinct.We include the KdV,Boussinesq in this.If we are talking about the NLS then the roots of (6.2-3) are purely imaginary.

Proof:

In paper I in the appendix it is proved by Bryan and Stuart that the roots f_m of (6.2-3) are identical to the eigenvalues of the matrix defined in (6.30).For the NLS similar arguments to those in I can also be applied to the matrix ; B = iC with C defined in (6.32).B can be written as the product of matrices(square brackets)) ,

$$B_{mn} = [\exp \frac{1}{2} \delta_{mn} Y_m] [h_{mn}] [\exp \frac{1}{2} \delta_{mn} Y^*_m] \qquad (6.40)$$

where δ_{mn} is the Kronecker delta function and h_{mn} is defined by,

$$h_{mn} = (k_m - k^*_m)^{1/2}(k_n - k^*_n)^{1/2}(k_m - k^*_n)^{-1} \qquad (6.41)$$

In paper I it was shown by Bryan and Stuart that matrices of the same form as (6.30) have real,positive(negative for antisolitons) and distinct eigenvalues.Clearly this independent of the definitions of X_m for the various equations.Similarly since B is Hermitian it has real eigenvalues and similar arguments as those in I give that the eigenvalues are distinct.Thus the eigenvalues of C are purely imaginary .■

We note that for mixtures of solitons,antisolitons and breathers(sG,MKdV) the matrix C is complex and symmetric (non Hermitian),so that the eigenvalues are not necessarily real,although for two parameter cases it is easy to demonstrate their reality.As yet,a proof of the reality of the eigenvalues in mixtures has not yet been devised

We now move on to prove the equivalent of theorem 5.3 for the lsp (6.2-3),that is to say we show that $4\tan^{-1}f_m$ where f_m are the roots of (6.2) tend asymptotically to constant speed solitons ,antisolitons or parts of breathers.To prove this we shall need the following lemma.

LEMMA 6.1

If the polynomial ($z \in \mathbb{C}, t \in \mathbb{R}$)

$$H(z) = \sum_{m=0}^{N} a_m(t)\, z^{N-m} \tag{6.42}$$

is such that ,

$$\lim_{t\to\pm\infty} a_m(t)/a_r(t) = \begin{cases} 0 & m \neq r, r+1 \\ 1 & m = r \\ k & m = r+1 \quad k \text{ finite and non zero} \end{cases} \tag{6.43}$$

then (6.42) possesses in the limit $t \to \pm\infty$ a single finite non zero root ,

$$|z| = k \tag{6.44}$$

and N-r-1 zero roots and r roots of infinite modulus.

Also if ,

$$\lim_{t\to\pm\infty} a_m(t)/a_r(t) = \begin{cases} 0 & m \neq r, r+1, r+2 \\ 1 & m = r \\ k_1 & m = r+1, k_i \text{ finite non zero} \\ k_2 & m = r+2 \end{cases} \tag{6.45}$$

then (6.42) possesses in the limit $t \to \pm\infty$ two finite non zero roots satisfying ,

$$|z|^2 - k_1|z| + k_2 = 0 \tag{6.46}$$

and N-r-2 zero roots and r roots of infinite modulus.

Proof:

We will consider only $t \to -\infty$ as the proof is readily adapted for $t \to \infty$ .Consider first the case of (6.42) together with (6.43).

Define ,

$$g_1(z) = (-1)^r a_r(t)\, z^{N-r} + (-1)^{r+1} a_{r+1}(t)\, z^{N-r-1}$$

$$= (-1)^r a_r(t)\, z^{N-r-1}\, [z - a_{r+1}(t)/a_r(t)\,]$$

and

$f_1(z) =$

$$a_0 z^N - a_1 z^{N-1} + ..(-1)^{r-1} a_{r-1} z^{N-r+1} + (-1)^{r+2} a_{r+2} z^{N-r-2} + ..(-1)^N a_N$$

We have that ,

$|f_1(z)/g_1(z)| \leq$

$$\frac{|a_0||z|^N + .. + |a_{r-1}||z|^{N-r+1} + |a_{r+2}||z|^{N-r-2} + ... + |a_N|}{|a_r||z|^{N-r-1}|z - a_{r+1}/a_r|}$$

Consider a circle C_1 of radius R_1 such that $k < |z| < |R_1|$, then on this circle ,

$|f_1(z)/g_1(z)| \quad <$

$$\frac{\left[\ \left|\frac{a_0}{a_r}\right||R_1|^{r+1} + ..+\left|\frac{a_{r-1}}{a_r}\right||R_1|^2 + \left|\frac{a_{r+2}}{a_r}\right||R_1|^{-1} + ..+\left|\frac{a_N}{a_r}\right||R_1|^{r+1-N}\right]}{|R_1| - |a_{r+1}/a_r|}$$

Noting (6.43) we see that as $t \to -\infty$ the quantity on the rhs of the inequality becomes arbitrarily small.Thus on C_1 $|g_1(z)| > |f_1(z)|$.Since f_1 and g_1 are analytic functions we may apply Rouché's theorem from complex analysis.Thus the number of zeros of $H(z) = f_1(z)+g_1(z)$ inside C_1 equals the number of zeros of $g_1(z)$ inside C_1.Clearly $g_1(z)$ has N-r zeros inside C_1.In addition since R_1 can be arbitrarily large H(z) must possess r roots of infinite modulus.

Now consider a circle C_2 of radius R_2 such that $|z| < k$ then using the same argument as before $|f_1/g_1|$ can be made arbitrarily small in the limit as $t \to -\infty$ .Thus the zeros of H(z) inside C_2 are equal in number to the zeros of g_1 inside C_2.There are N-r-1 zeros of g_1 inside C_2.Thus H(z) also has N-r-1 zeros inside C_2.It is clear that since a zero has been lost between C_1 and C_2, H(z) possesses a zero when $|z| = k$. Since R_2 above can be arbitrarily small we see that H(z) has N-r-1 roots of zero modulus.

We now consider the case of (6.42) together with (6.45).Define

$$g_2(z)=(-1)^r a_r(t)z^{N-r}+(-1)^{r+1}a_{r+1}(t)z^{N-r-1}+(-1)^{r+2}a_{r+2}(t)z^{N-r-2}$$
$$=(-1)^r a_r(t)z^{N-r-2}\,[z^2 - (a_{r+1}/a_r)z + a_{r+2}/a_r]$$
$$=(-1)^r a_r(t)z^{N-r-2}\,(z-z_1)(z-z_2)$$

with z_m the roots of the above quadratic($|z_1|<|z_2|$).

Also define ,

$$f_2(z) = a_0z^N - a_1z^{N-1} +..(-1)^{r-1}a_{r-1}z^{N-r+1} +(-1)^{r+3}a_{r+3}z^{N-r-3} +..(-1)^N a_N$$

then it follows that ,

$$|f_2(z)/g_2(z)| \le \frac{|a_0||z|^N +..+ |a_{r-1}||z|^{N-r+1}+ |a_{r+3}||z|^{N-r-3} +...+|a_N|}{|a_r||z|^{N-r-2}|z - z_1||z - z_2|}$$

Consider a circle C_3 of radius R_3 such that $|z_2| < |z| < R_3$, then clearly the above may be written ,

$$|f_2(z)/g_2(z)| < \frac{\left[\left|\frac{a_0}{a_r}\right||R_3|^{r+2}+ ..+\left|\frac{a_{r-1}}{a_r}\right||R_3|^3 + \left|\frac{a_{r+3}}{a_r}\right||R_3|^{-1} +..+\left|\frac{a_N}{a_r}\right||R_3|^{r+2-N}\right]}{(\ |R_3| - |z_1|\ )(\ |R_3| - |z_2|\ )}$$

Now, noting (6.45) we see that as $t \to -\infty$ the rhs of the inequality above becomes arbitrarily small.Thus on C_3 $|g_2(z)| > |f_2(z)|$.Again,by Rouché's theorem,the zeros of $H(z)=g_2(z)+f_2(z)$ inside C_3 are equal in number to the zeros of $g_2(z)$ inside C_3.Thus the number of zeros of H(z) inside C_3 is N-r.H(z) has r zeros outside C_3.

Now consider a circle C_4 of radius R_4 such that $|z| < |z_1|$, then using a similar argument to the above we find

$|f_2(z)/g_2(z)|$ can be made arbitrarily small in the limit as $t \to -\infty$. Thus the zeros of $H(z)$ inside C_4 are equal in number to the zeros of $g_2(z)$ inside C_4. There are $N-r-2$ zeros of $g_2(z)$ inside C_4. $H(z)$ has $r+2$ zeros outside C_4. Clearly $H(z)$ has two zeros in the region $|z_1| \le |z| \le |z_2|$.

Carrying out a similar argument as before in a circular region C_5 of radius R_5 such that $|z_1| < |R_5| < |z_2|$ we find once again the zeros of $H(z)$ inside C_5 are equal in number to the zeros of $g_2(z)$ inside C_5, $N-r-1$. But there were $N-r$ zeros inside C_3, thus $H(z)$ has a root when $|z|=|z_2|$. Similarly there were $N-r-2$ zeros of $g_2(z)$ inside C_4, thus a root was lost between C_4 and C_5, so $H(z)$ has a root when $|z|=|z_1|$. Clearly since R_3 in the argument above is arbitrarily large, $H(z)$ has r roots of infinite modulus and since R_4 is arbitrarily small $H(z)$ has $N-r-2$ roots of zero modulus inside C_4. This proves the theorem.■

THEOREM 6.4(sG)

In the limit as $t \to \pm\infty$ and x fixed, the only non zero finite roots of (6.2-3) in a frame of reference moving at speed $w_r = u_r$, where u_r is a soliton or antisoliton asymptotic speed (see 6.4), are given by ,

$$f = \exp(\lambda_r^{\pm})t(r) \tag{6.47}$$

or in a frame of reference moving at speed $w_r=v_r$, where v_r is a breather asymptotic speed(6.4), the only non zero finite roots of (6.2-3) are given by , (6.48)

$$f^2 - [\exp(\lambda_r^{\pm})t(r)+\exp(\lambda_{r+1}^{\pm})t(r+1)]f + \exp(\lambda_r^{\pm}+\lambda_{r+1}^{\pm})t(r)t(r+1)=0$$

where the $\pm$ superscripts refer to the limits t takes and where $t(r)$ are defined below. (6.49)

$1\le r\le ns$ $\qquad t(r) = \exp x$ (a)

$ns+1 \le r \le ns+na \qquad t(r) = -\exp x$ (b)

$ns+na+1 \le r \le N-1 \qquad t(r) = -i \exp[\Gamma_{rr+1} + i\Omega_{rr+1}]$ (c)

$(r = ns+na+1, ns+na+3, \ldots, N-1)$

" $\qquad t(r+1) = t(r)^*$ (6.49d)

where ,

$$\Gamma_{rr+1} = (\cos\mu_r)x \qquad \Omega_{rr+1} = (\sin\mu_r)t \tag{e}$$

μ_r defined in (6.4j).

$$\lambda_r^- = \ln \prod_{p=1}^{r-1} u_{pr}^2 \qquad \lambda_r^+ = \ln \prod_{q=r+1}^{N} u_{pr}^2 \tag{f}$$

u_{pq}^2 defined in (6.5).

Proof:

We assume that all the speeds u and v have been arranged so that if $u_i < u_j$ or $u_i < v_j$ then i>j.To enter the asymptotic rest frames of the solitons,antisolitons or breathers we carry out the following Lorentz transformation.

$$x_{old} \to \gamma\,(x + w_r t)\ ,\ t_{old} \to \gamma\,(t + w_r x)\ ,\gamma = (1-w_r^2)^{-1/2}$$

Substituting the above into (6.4) gives ,

$$X_p \to \gamma\gamma_p[x(1 - u_p w_r) + (w_r - u_p)t] + \alpha_p$$

$$\Gamma_{pp+1} \to \gamma\gamma_p \cos\mu_p[x(1 - v_p w_r) + (w_r - v_p)t] + \alpha_p$$

$$\Omega_{pp+1} \to \gamma\gamma_p \sin\mu_p[t(1 - v_p w_r) + (w_r - v_p)x] + \beta_p$$

and when p=r we find ,

$$X_p = x\ ,\ \Gamma_{pp+1} = (\cos\mu_r)x\ ,\ \Omega_{pp+1} = (\sin\mu_r)t$$

However noting (6.4) we find in general ,

$$\lim_{t\to -\infty}|t(p)| = \begin{cases} 0 & r<p \\ \infty & r>p \\ k_1 & r=p \end{cases} \qquad \lim_{t\to \infty}|t(p)| = \begin{cases} \infty & r<p \\ 0 & r>p \\ k_2 & r=p \end{cases} \tag{6.50}$$

k_1,k_2 finite non zero.

Consider soliton or antisoliton rest frames.It is clear that as $t\to -\infty$ $(+\infty)$ particular products of t(p) entering the polynomial H(f) (6.3) will dominate.These dominating products will involve p< r or p≤r (p>r or p≥r).Any other product when divided by a dominating product will involve $t(p)$ $p>r$ $(p<r)$ on the numerator and the uncancelled $t(p)$ $p<r$ $(p>r)$ on the denominator.Clearly from (6.50) such ratios will tend to zero.Numerous examples of this were given in the proof of theorem 5.3 in the last chapter.Denoting the general coefficient of (6.3) by $a_l(x,t)$,

$$a_l(x,t) = (-1)^l \sum \prod_{p=1}^{l} t(p) \prod_{q=p+1}^{l} u^2_{pq}$$

we find ,

$$\lim_{t\to -\infty} \frac{a_l(x,t)}{a_{r-1}(x,t)} = \begin{cases} 0 & l\neq r-1,r \\ 1 & l=r-1 \\ k & l=r \end{cases}$$

where

$$k = \prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq} \Big/ \prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq} = \exp \lambda^-_r$$

Similar results apply for the $t\to +\infty$ limit where we find that ,

$$k = \prod_{p=r}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq} \Big/ \prod_{p=r+1}^{N} t(p) \prod_{q=p+1}^{N} u^2_{pq} = \exp \lambda^+_r$$

In the case of breather rest frames,noting (6.50) we find that as $t\to -\infty$ $(t\to+\infty)$,dominating products of $t(p)$ entering $H(f)$ will be those for which $p < r$ $(p > r)$ or $p \leq r$ and $p \leq r+1$ $p\neq r$ $(p \geq r$ $p\neq r+1$ and $p \geq r+1)$ and finally $p \leq r+1$ $(p \geq r+2)$.Thus we find ,

$$\lim_{t\to -\infty} \frac{a_l(x,t)}{a_{r-1}(x,t)} = \begin{cases} 0 & m \neq r-1,r,r+1 \\ 1 & m = r-1 \\ k_1 & m = r \\ k_2 & m = r+1 \end{cases}$$

where,

$$k_1 = \left[\prod_{p=1}^{r} t(p) \prod_{q=p+1}^{r} u^2_{pq} + \prod_{\substack{p=1 \\ p\neq r}}^{r+1} t(p) \prod_{\substack{q=p+1 \\ q\neq r}}^{r+1} u^2_{pq} \right] \Big/ \prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq}$$

$$= \exp(\lambda^-_r)t(r) + \exp(\lambda^-_{r+1})t(r+1)$$

and

$$k_2 = \prod_{p=1}^{r+1} t(p) \prod_{q=p+1}^{r+1} u^2_{pq} \Big/ \prod_{p=1}^{r-1} t(p) \prod_{q=p+1}^{r-1} u^2_{pq}$$

$$= u^2_{rr+1} \exp(\lambda^-_r + \lambda^-_{r+1})t(r)t(r+1)$$

Similarly for $t\to +\infty$ .Now applying lemma 6.1 and noting corollary 6.2 that the roots of H(f) keep their sign,we see that the theorem is proved.■

COROLLARY 6.4 (sG)

In the limit as $t\to -\infty$ there are as many positive(negative) real roots of (6.2) as there are solitons(antisolitons) in breathers or otherwise.

Proof:

It is clear that if r of the previous theorem belongs to a soliton(antisoliton) then f_r is positive(negative).To conclude the proof, we must examine the roots of the quadratic (6.48).We write $\lambda_r^- = \alpha + i\beta$.We have already seen that $u^2_{rr+1} = -w^2_{rr+1}$ where w^2_{rr+1} is real.Thus (6.48) becomes

$$f^2 - 2\exp(\Gamma_{rr+1}+\alpha)\sin(\Omega_{rr+1}+\beta)\ f\ - w^2_{rr+1}\exp2(\Gamma_{rr+1}+\alpha) = 0$$

this has real roots of opposite sign and we may define the positive(negative) root to be the soliton(antisoliton) component.This completes the proof of the corollary.■

We now obtain immediately.

COROLLARY 6.5(sG)

If the N roots of (6.2) are real for all time then there are as many positive roots of (6.2) as there are solitons(in breathers or otherwise) and there are as many negative roots as there are antisolitons(in breathers or otherwise).

Proof:

Corollary 6.4 gives us that the assertion is true when $t\to-\infty$.Corollary 6.2 thus gives us that the assertion remains true for all time.■

Corollary 6.5 together with theorem 6.4 assures us that there is no transmutation of soliton(antisoliton) to antisoliton(soliton) and means that it is sensible to *define* a soliton to be represented by $4\tan^{-1}f$ where $f > 0$ is a root of (6.2) and an antisoliton to be represented by $4\tan^{-1}f$ where $f < 0$.Although it is not entirely satisfactory we must end this section on the properties of the roots of (6.2) with two conjectures.

CONJECTURE 6.1(sG,MKdV)

The N roots of (6.2) are real for all time(i.e.including mixtures of solitons,antisolitons and breathers).

CONJECTURE 6.2

The positive(negative) roots of (6.2) are monotonic increasing(decreasing).

Were we able to prove the second conjecture we would be assured that if a root of (6.2) f was positive (negative) then f=1(-1) would define the position of a single point as a function of time.In §5 we shall see that such points are of importance as they coincide with the projections on the real axis of the positions of the branch points of the complexified multisoliton solution.Certainly conjectures 1&2 are easily verified for two parameter cases as we shall see in the next section and especially §1 chapter 7.

§ 4. Exact solutions of the lsp.

For the case of two parameters (6.2-3) becomes simply,

$$f^2 - [t(1)+t(2)]f + u_{12}^2 t(1)t(2) = 0 \tag{6.51}$$

If we choose t(1) and t(2) to be solitons(i.e $t(m) = \exp X_m$),we obtain on solving the quadratic,the roots f_m,given by,

(6.52)

$$f_1=\exp\left\{ X_+ + \zeta(X_-)\right\} \qquad (a)$$

$$f_2=\exp\left\{ X_+ - \zeta(X_-) + \ln u_{12}^2\right\} \qquad (b)$$

$$X_+ = (X_1+X_2)/2 \qquad X_- = (X_1-X_2)/2 \qquad (c)$$

$$\zeta(r) = \ln[\cosh r + (\cosh^2 r - u_{12}^2)^{1/2}] \qquad (d)$$

where X_m are any of the X_m defined in §1.

Choosing the soliton-antisoliton solution (sG,MKdV) (i.e. $t(1) = \exp X_1$,$t(2) = - \exp X_2$) we obtain on solving (6.51) the roots , (6.53)

$$f_1=\exp\left\{ X_+ + \eta(X_-)\right\} \qquad (a)$$

$$f_2=-\exp\left\{ X_+ - \eta(X_-) +\ln u_{12}^2\right\} \qquad (b)$$

$$\eta(r) = \ln[\sinh r + (\sinh^2 r + u_{12}^2)^{1/2}] \qquad (c)$$

Finally choosing the breather solution(sG,MKdV) (using 6.4,6.8)

we obtain the roots where $w_{12} = |u_{12}|$,

(6.54)

$$f_1 = \exp\left\{\Gamma_{12} + \xi(\Omega_{12})\right\} \qquad (a)$$

$$f_2 = -\exp\left\{ \Gamma_{12} - \xi(\Omega_{12}) + \ln w_{12}^2\right\} \qquad (b)$$

$$\xi(r) = \ln[\sin r + (\sin^2 r + w_{12}^2)^{1/2}] \qquad (c)$$

We establish the fact that all the roots are monotonic functions of x in the next chapter.We also discover that the roots may in some cases develop points of inflexion.

In the case of the sG and Boussinesq(solitons only) in the centre of velocity frame(6.52-4) take on a particularly simple forms.We express these having chosen the phases α_m = -ln w_{12}.We refer to (6.4) and (6.20).

Two solitons

$$f_1 = \exp \gamma\, [x + \gamma^{-1}\zeta(\gamma ut)] \qquad (6.55a)$$

$$f_2 = \exp \gamma\, [x - \gamma^{-1}\zeta(\gamma ut) + \gamma^{-1}\ln u_{12}^2] \qquad (6.55b)$$

where u is the common centre of velocity speed(for the sG u <1) and for the sG $\gamma = (1-u^2)^{-1/2}$ while for the Boussinesq γ is a positive constant of arbitrary magnitude.

soliton and antisoliton(sG)

$$f_1 = \exp \gamma\, [x + \gamma^{-1}\eta(\gamma ut)] \qquad (6.56a)$$

$$f_2 = -\exp \gamma\, [x - \gamma^{-1}\eta(\gamma ut) + \gamma^{-1}\ln u_{12}^2] \qquad (6.56b)$$

breather(sG)

$$f_1 = \exp \sigma\, [x + \sigma^{-1}\xi(2\pi t/\tau)] \qquad (6.57a)$$

$$f_2 = \exp \sigma\, [x - \sigma^{-1}\xi(2\pi t/\tau) + \sigma^{-1}\ln w_{12}^2] \qquad (6.57b)$$

$$\sigma = \cos^{-1}\mu \qquad (6.57c)$$

where μ and τ are defined in (6.4j).

Thus we see that for these special cases the two parameter solutions of the sG and Boussinesq equations can be written as a linear superposition of *fixed shape* solitons.The accelerating solitons have shape $4\tan^{-1}\exp\, g_m$ (g=lnf) for the sG and for the Boussinesq are given by $-(\gamma^2/2)\mathrm{sech}^2(g_m/2)$.Note that here the amplitude factor(6.19) is a constant.We shall see in the next chapter how in general all the solitons of the various equations considered in this chapter change their shape during interaction.

It is clear from the polynomial nature of (6.3) that exact formulae for the roots are going to be limited to N $\le$ 4 ,due to the unsolvability of the quintic or higher.In actual fact the explicit formulae for the roots of (6.3) for N=3 or 4 are not very useful.We include them below just for completeness.

<u>Three parameter exact solutions</u>

The cubic (6.3) when N=3 has exact solutions,

$$f_1 = \sum t(m)/3 - 2(P/3)^{1/2}\cos\alpha \qquad (6.58a)$$

$$f_2 = \sum t(m)/3 + (P/3)^{1/2}(\cos\alpha + \sqrt{3}\sin\alpha) \qquad (b)$$

$$f_3 = \sum t(m)/3 + (P/3)^{1/2}(\cos\alpha - \sqrt{3}\sin\alpha) \qquad (c)$$

where ,

$$P = (\sum t(p))^2/3 - \sum u_{pq}^2 t(p)t(q) \qquad (6.59a)$$

$$Q = -2(\sum t(p))^3/27 + (\sum t(p))\sum u_{pq}^2 t(p)t(q)/3 - \prod u_{pq}^2 t(p) \qquad (b)$$

$$\alpha = 3^{-1}\cos^{-1}[-(27Q^2/4P^3)^{1/2}] \qquad (c)$$

where $\sum$ and $\prod$ are sums and product over the parameters p,q such that q > p.

<u>Four parameter exact solutions</u>

(6.2) and (6.3) when N=4 become ,

$$f^4 - Af^3 + Bf^2 - Cf + D = 0 \qquad (6.60a)$$

where ,

$$A = t(1)+t(2)+t(3)+t(4) \qquad (b)$$

$$B = u_{12}^2 t(1)t(2) + u_{13}^2 t(1)t(3) + u_{14}^2 t(1)t(4) \qquad (c)$$
$$+u_{23}^2 t(2)t(3) + u_{24}^2 t(2)t(4) + u_{34}^2 t(3)t(4)$$

$$C = u_{12}^2 u_{13}^2 u_{23}^2 t(1)(2)t(3) + u_{12}^2 u_{14}^2 u_{24}^2 t(1)t(2)t(4) \qquad (d)$$
$$+u_{13}^2 u_{14}^2 u_{34}^2 t(1)(3)t(4) + u_{23}^2 u_{24}^2 u_{34}^2 t(2)t(3)t(4)$$

$$D = u_{12}^2 u_{13}^2 u_{14}^2 u_{23}^2 u_{24}^2 u_{34}^2 t(1)t(2)t(3)t(4) \qquad (e)$$

(6.60a) has roots ,

$$f_1 = \sum t(m)/4 + \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q} \qquad (6.61a)$$

$$f_2 = \sum t(m)/4 + \sqrt{q}\sqrt{r} - \sqrt{r}\sqrt{p} - \sqrt{p}\sqrt{q} \qquad (b)$$

$$f_3 = \sum t(m)/4 - \sqrt{q}\sqrt{r} - \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q} \qquad (c)$$

$$f_4 = \sum t(m)/4 - \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} - \sqrt{p}\sqrt{q} \qquad (d)$$

where ,

$$M = (12H^2 - I)/6G \qquad (6.62a)$$

$$p = M - 2(P/3)^{1/2}\cos\alpha \qquad (b)$$

$$q = M + (P/3)^{1/2}(\cos\alpha + \sqrt{3}\sin\alpha) \qquad (c)$$

$$r = M + (P/3)^{1/2}(\cos\alpha - \sqrt{3}\sin\alpha) \qquad (d)$$

$$P = 3M^2 - 3H \qquad (e)$$

$$Q = -\frac{2}{3}M^2 - 3HM - G/2 \qquad (f)$$

$$\alpha = \frac{1}{3}\cos^{-1}[-(27Q^2/4P^3)] \qquad (g)$$

$$H = \frac{A}{16} - \frac{B}{6} \qquad (h)$$

$$I = D - \frac{AC}{4} + \frac{B}{12} \qquad (i)$$

$$G = \frac{C}{4} - \frac{AB}{8} + \frac{A^3}{32} \qquad (j)$$

When all the roots are real $12H^2 - I > 0$,$H > 0$,$G > 0$.

§ 5. Singularities of the complexified N parameter solution.

In this section we are going to examine in detail how we can determine the motion of the singularities of the complexified N parameter solution of the sG.It is clear from the argument used that it could be adapted to prove similar results with the other equations considered in this chapter.When the spatial variable x is allowed to be complex($x \to z = x+iy$) the N parameter solutions of all the equations considered in this chapter develop singularities(see also chapter 4).

We have seen that the N parameter solutions of the sG and MKdV can be written as a sum of inverse tangents of f_m ,where f_m are the roots of (6.2-3).Inverse tangents have branch points when their arguments are equal to ±i.Hence the N parameter solutions of the sG and MKdV develop branch points when ,

$$f_m = \pm i \qquad (6.63a)$$

Examining the linear superposition principle for the KdV class of equations(6.17) we see that the N soliton solution develops poles when ,

$$f_m = -1 \qquad (6.63b)$$

Also we find for the NLS, N soliton solution develops poles when ,

$$f_m f_m^* = -1 \qquad (6.63c)$$

In order to define the complex lsp consistently we must consider the consequence of not allowing the time variable to be complex.The sG is Lorentz invariant.Under a Lorentz transformation old variables x',t' are transformed to x,t as follows.

$$x = \gamma(x'-vt') \quad , \quad t = \gamma(t'-vx')$$

If we allow x' to be complex(x'→ z'),it would seem that we are forced into allowing t also to be complex.We do not wish to do this.So we must define a more restricted complex Lorentz transformation ,

$$z = \gamma(z'-vt') \quad , \quad t = \gamma(t'-vx') \quad ,\text{where } x' = \mathcal{Re}(z')$$

Noting this, we can now define the complex lsp for the sG.This can be written ,

$$H(fe^{i\theta}) = (fe^{i\theta})^N + \sum_{l=1}^{N}\left\{(-1)^l\; B_l(fe^{i\theta})^{N-l}\right\} \qquad (6.64)$$

where ,

$$B_l = \sum \exp\left\{ \sum_{p=1}^{l} X_p + i(\sum_{p=1}^{l} Y_p + \sum_{q=p+1}^{l} s_{pq})\right\} \prod_{q=p+1}^{l} r_{pq} \qquad (6.65)$$

where as before the unscripted summation sign means taking all combinations of l integers from 1,..,N and where ,

$1 \le p \le ns \qquad \underline{X}_p = X_p \quad , \quad Y_p = \gamma_p y + \nu_p \qquad (6.66a)$

$ns+1 \le p \le ns+na \qquad \underline{X}_p = X_p + i\pi \quad , \quad Y_p = \gamma_p y + \nu_p \qquad (b)$

$ns+na+1 \le p \le N-1 \qquad \underline{X}_p = X_p + i3\pi/2 \; , \; Y_p = \gamma_p \sigma_p y + \nu_p \qquad (c)$

$(p = ns+na+1, ns+na+3, .. , N-1)$

$$\underline{X}_{p+1} = X_{p+1} + i\pi/2 \; , \; Y_{p+1} = \gamma_p \sigma_p y + \nu_p \qquad (d)$$

The quantities X_p are as defined in (6.4) with the exception that the minus signs outside the exponentials and i's have been taken inside the exponentials in the form of complex constants. Also $\sigma_p = \cos\mu_p$. We also define X_p when $ns+na+1 \le p \le N-1$ to be given by $X_p \equiv \Gamma_{pp+1} + i\Omega_{pp+1}$ with $X_{p+1} = X^*_p . \nu_p$ are imaginary phases.Since u_{pq} is in general complex we have let $u^2_{pq} = r_{pq} \exp i(s_{pq})$.

The real lsp is recovered when ,

$$\theta = 0 \; , y = 0 \; , \nu_p = 0 \;\; \forall \; p \qquad (6.67)$$

We now define and prove the main result of this section.

THEOREM 6.5(sG)

The only branch points of the complexified N parameter solution of the sG which exist for all time ,move along a line in the complex plane parallel to the real axis.The line is defined(modulo 2π) by ,

$$Y_p = \pi/2 \quad \forall p \qquad (6.68a)$$

In addition ,the projections of the branch points on the real axis move in an identical manner to the points which satisfy,

for solitons in breathers or otherwise,

$$H(1) = 0 \qquad (6.68b)$$

and for antisolitons in breathers or otherwise,

$$H(-1) = 0 \qquad (6.68c)$$

Proof:

As we have seen the complexified N parameter solution of the sG develops branch points when $fe^{i\theta} = \pm i$ in (6.64).H(±i,z) is clearly an analytic function of z.For the moment we concentrate on the positive sign above.We define U(x,y,t) and V(x,y,t) to be the real and imaginary parts of H(i,z).The Cauchy-Riemann equations are ,

$$U_x = V_y \quad , \quad U_y = -V_x \tag{6.69}$$

Setting H(i,z) to zero(6.2) we find ,

$$U(x,y,t) = 0 \quad , \quad V(x,y,t) = 0 \tag{6.70}$$

(6.70) defines x and y as functions of time, and therefore the orbits in the complex plane of those points which satisfy $fe^{i\theta} = i$.Differentiating (6.70) with respect to t and noting (6.69) we find (dot indicates total derivative with respect to t) ,

$$\dot{y} = (U_x^2 + V_x^2)^{-1}(V_t U_x - V_x U_t) \equiv H_1(x,y,t) \tag{6.71}$$

Now if we can find an equation ,

$$\Theta(y) = 0 \quad \Rightarrow \quad U \text{ or } V = 0 \tag{6.72}$$

then $\dot{y}$ will also vanish(if both U_x and V_x vanish then $\dot{y}$ will be undefined).Differentiating (6.71) with respect to t again gives ,

$$\ddot{y} = H_{1x}\dot{x} + H_{1y}\dot{y} + H_{1t} \equiv H_2(x,y,t) \tag{6.73}$$

Clearly if (6.72) holds then H_{1x} and H_{1t} both vanish as they have the same y dependence as $\dot{y}$,thus $\ddot{y}$ vanishes also.Clearly any total derivative of y with respect to t vanishes when (6.72) holds.

We now examine the behaviour of (6.64) when $t \to -\infty$ to see if we can establish a condition of the form of (6.72).The asymptotic behaviour of the real lsp has been examined in theorem 6.4.It is clear from the argument used then that we may carry over to the complex case the results then obtained.This is because in the complex case we would be interested in $\lim_{t\to-\infty}|H(fe^{i\theta})|$ and this tends to terms with dominating moduli(see proof of theorem 6.4).

For consider two complex functions a(t) and b(t) which are such that $\lim_{t\to-\infty}|b(t)/a(t)| = 0$ then since ,

$$|a|(1-|b/a|) \le |a+b| \le |a|(1+|b/a|)$$

we must have $\lim_{t\to-\infty}|a+b| = |a|$ .Thus we may assert the following.

In the limit as $t\to-\infty$,

$1 \le p \le ns+na \qquad |f_p e^{i\theta_p}| = |\exp(X_p+iY_p+\lambda_p^-)| \qquad$ (6.74a)

$ns+na+1 \le p \le N-1 \qquad |f_p e^{i\theta_p}| = |\exp(X_{+p}+\xi(X_{-p})+\lambda_p^-+iY_p| \qquad$ (b)

(p=ns+na+1,ns+na+3,..N-1)

$|f_{p+1} e^{i\theta_{p+1}}| = |\exp(X_{+p}-\xi(X_{-p})+\lambda_{p+1}^-+iY_p| \qquad$ (c)

where $\xi()$ is defined in (6.54) and we have noted (6.66) and defined ,$X_{\pm p} = (X_p \pm X_{p+1})/2$.λ_p^- are defined in (6.49f).

Now the condition for branch points (6.63a) means $|f_p e^{i\theta_p}| = 1$ and thus we must have ,

$$Y_p = \pi/2(\text{mod } 2\pi) \ \forall\ p$$

This is an equation of the form (6.72).It now remains to be shown that this condition implies that either U or V is zero. We prove this below.

The general term in the expansion of H(i,z) is given by

$$H_l(i,z)=(-1)^l\sum \exp\left\{\sum_{p=1}^{l} X_p + i[\sum_{p=1}^{l} Y_p + \sum_{q=p+1}^{l} s_{pq} + (N-1)\frac{\pi}{2}]\right\}\prod_{q=p+1}^{l} r_{pq}$$

Substituting $Y_p = \pi/2$ into the above and rearranging ,

$$H_l(i,z) =$$

$$(-1)^l\sum \exp\left\{\sum_{p=1}^{l} X_p + i\sum_{q=p+1}^{l} s_{pq}\right\}\prod_{q=p+1}^{l} r_{pq} \ \exp\ i[\sum_{p=1}^{l}(\pi/2) + (N-1)\pi/2]$$

Thus we find.

$$H_l(i,z) = H_l(1)\exp(iN\pi/2)$$

where $H_l(1)$ is the general coefficient of the real polynomial with f = 1(6.2-3).Noting the definitions of U and V we see from the above that U or V equal zero.Solutions of H(i,z) = 0 are identical to ,

$$H(1) = 0$$

By very similar arguments we also conclude that solutions of H(-i,z) are identical to ,

$$H(-1) = 0$$

This proves the theorem.■

We now discuss a peculiar property of the lsp's (6.3).For simplicity, consider the two soliton case(though we have not proved an equivalent of the previous theorem for the KdV or other equations it is likely that an equivalent will be able to be proved).

The two soliton lsp for the sG (6.51) can be written,

$$f^2 - [\, e^{X_1} + e^{X_2} \,]f + u_{12}^2 e^{X_1+X_2} = 0 \qquad (6.75a)$$

where,

$$X_m = \gamma_m(x-ut)+\alpha_m \qquad (6.75b)$$

We have solved the quadratic (6.75a) and found the roots f_m to be real,positive and monotonic increasing(see §1 chapter 7).Therefore setting each f_m to 1 defines a single real point as a function of time.These points as we have seen above are the projections on the real axis of the branch points.Now consider substituting f=1 into (6.75a).We obtain.

$$1 - [\, e^{X_1} + e^{X_2} \,] + u_{12}^2 e^{X_1+X_2} = 0 \qquad (6.76)$$

Now define ,

$$a_m(t) \equiv \exp(-\gamma_m u_m t + \alpha_m)$$

and choose u_m so that ,

$$\gamma_m = p_m/q \qquad p_m > q \qquad p_2 < p_1 \qquad p_m,q \in \mathbb{N}^+$$

Substituting into (6.76) we find ,

$$1 - [\, a_1 e^{p_1 x/q} + a_2 e^{p_2 x/q} \,] + a_1 a_2 u_{12}^2 e^{(p_1+p_2)x/q} = 0 \qquad (6.77)$$

Defining $\phi \equiv e^{x/q}$ we can write the above as a polynomial of degree p_1+p_2 in ϕ.

$$1 - [\, a_1 \phi^{p_1} + a_2 \phi^{p_2} \,] + a_1 a_2 u_{12}^2 \phi^{(p_1+p_2)} = 0 \qquad (6.78)$$

(6.77) has p_1+p_2 roots most of which are complex(including negative real roots for ϕ).We know of course that (6.77) possesses only two real roots as it factorizes into $(1-f_1)(1-f_2)$ and f_m achieves the value 1 only once.

(6.78) is similar in form to a polynomial defining the rational poles(rational speed ratios) of the two soliton solution of the KdV [3,4].As we have seen in theorem (6.5) the positions of the projections of the branch points of the complexified N parameter solution of the sG on the real axis are defined by (6.68b),of which (6.77) is a special case.Clearly these projections cannot be complex.So the extra complex roots of (6.77) are of no significance.This argument really illustrates the usefullness of the *real* linear superposition.

In the case of the KdV although we have not proved a KdV version of theorem 6.5, as the underlying lsp has a similar form to the sG it is pretty clear that the above arguments would also apply to the KdV.

Finally in this section we consider the consequences of (6.68a).Solving (6.68a) we find that it can only be true if all the complex phases ν_p are functions of a *single* complex phase $\nu=\nu_1$(say) and we find ,

$$1 < p \le ns+na \qquad \nu_p = (4m_p+1)\pi/2 + [\nu-(4m_1+1)\pi/2]\gamma_p/\gamma_1 \qquad (6.79a)$$

$$ns+na+1 \le p \le N-1 \qquad \nu_p = (4m_p+1)\pi/2 + [\nu-(4m_1+1)\pi/2]\sigma_p\gamma_p/\gamma_1 \qquad (b)$$

$(p=ns+na+1, ns+na+3, \ldots, N-1)$

$$\nu_{p+1} = \nu_p \qquad\qquad m_p \in \mathbb{Z}$$

§ 6. Time symmetric lsp's and perfect phase(sG).

Although in general the lsp (6.2-3) is difficult to solve,under a special choice of phase it possesses some simpler properties.Also if the asymptotic speeds of the solitons and antisolitons are selected in a special way the lsp for the sG develops some useful features.We explore these topics in this section.

In the previous chapter we began by discussing the N parameter solution of the sG in terms of the constants $k_{ij}=u_{ij}^{-1}$. In actual fact the associated lsp with the constants k_{ij} has some desirable properties.We will describe such lsp's as having *perfect phase*.To illustrate these properties we examine the three parameter polynomial with perfect phase.

$$f^3 - [k_{12}k_{13}t(1) + k_{21}k_{23}t(2) + k_{31}k_{32}t(3)]f^2 \qquad (6.80)$$
$$- [k_{13}k_{23}t(1)t(2) + k_{12}k_{32}t(1)t(3) + k_{21}k_{31}t(2)t(3)]f$$
$$+ t(1)t(2)t(3) = 0$$

Now choose the phases α_m(6.4) to be zero and let x=0 and t=0.t(m) then become equal to unity and (6.80) represents a two soliton-one antisoliton interaction.(6.80) becomes ,

$$f^3-[k_{12}k_{13}+ k_{21}k_{23}+ k_{31}k_{32}]f^2-[k_{13}k_{23}+ k_{12}k_{32}+ k_{21}k_{31}]f +1 = 0$$

Now the coefficients of f^2 and f are both minus one(5.2),so that we find $f_1 = f_3 = 1$ and $f_2 = -1$(the subscripts are nominal).Thus from §5 we see that we can immediately see that the solitons and antisoliton are coincident at the origin(taking their position to be the projection on the real axis of the associated branch point).Choosing the three soliton case with perfect phase(t(2) → -t(2)) and carrying out similar steps to those before we find that (6.80) becomes ,

$$f^3-[k_{12}k_{13}- k_{21}k_{23}+ k_{31}k_{32}]f^2-[-k_{13}k_{23}+ k_{12}k_{32}- k_{21}k_{31}]f -1 = 0$$

This can be written ,

$$(f - 1)(f^2 - (1+2k_{12}k_{23})f + 1) = 0$$

Thus since one root is unity a single soliton is located at the origin at time t=0.

If the asymptotic speeds for these three parameter cases are chosen in the following way we find ourselves in the centre of velocity frame.

$$u_1 = -u_3 = u \quad u_2 = 0 \qquad (6.81)$$

Substituting (6.81) into (6.80) and noting the definitions of t(p) (6.4) with $\alpha_m = 0$.Also from the definition of k_{ij} we find $k_{12} = k_{23}$ .Thus (6.80) becomes ,

$$f^3 - [e^{\gamma x}\rho(t) - k_{12}^2 e^x]f^2 + [k_{12}^2 e^{2\gamma x} - e^{\gamma x+x}\rho(t)]f + e^{x+2\gamma x} = 0 \qquad (6.82)$$

where ,

$$\rho(t) = 2k_{12}k_{13}\cosh\gamma ut$$

Putting x=0 into (6.82) we find ,

$$f^3 - [\rho(t) - k_{12}^2]f^2 + [k_{12}^2 - \rho(t)]f + 1 = 0$$

which can be written ,

$$(f + 1)(f^2 - [1 + \rho(t)-k_{12}^2]f + 1) = 0$$

Thus in this case we see that an antisoliton is located at x=0 for all time.Similar results apply for the three soliton. case where we find (6.80) becomes ,

$$f^3 - [e^{\gamma x}\rho(t) + k_{12}^2 e^x]f^2 + [k_{12}^2 e^{2\gamma x} + e^{\gamma x+x}\rho(t)]f - e^{x+2\gamma x} = 0 \qquad (6.83)$$

Putting x=0 into the above and factorizing we find ,

$$(f - 1)(f^2 + [1 - \rho(t)-k_{12}^2]f + 1) = 0$$

So a soliton is located at x=0 for all time.We note that when t=0 in (6.83) it is easily seen that $f=e^x$ is not a solution.So,although the soliton does not move throughout the interaction,its characteristic function changes with time.It does this in such a way that when $t \to \pm\infty$ $f \to e^x$ but f(0,t)=1 for all time.This indicates that points such that f=1(-1 for antisolitons) are invariant,changes to f pivoting about this point.In the next chapter we will see how this happens numerically.

In general when we choose a centre of velocity frame such that,

$$u_r = -u_{N-r+1},\ r < (N+1)/2, N \text{ odd} \quad r \le N/2, N \text{ even} \qquad (6.84a)$$

where $u_{(N+1)/2} = 0$ N odd.We find ,

$$k_{mn} = k_{pq} \quad ,\text{where} \quad p=N-n+1 \quad \text{and} \quad q=N-m+1 \qquad (6.84b)$$

(6.81) is an example of this.

With this special choice of speeds numerical calculations suggest that the N^{th} degree lsp(6.2-3) becomes (when x=0 and t=0)

$$(f+1)^{[N/2]}(f-1)^{[(N+1)/2]} = 0$$

where [m] indicates the nearest integer $\le$ m .This indicates that all the solitons and antisolitons are coincident at time t=0.

We have examined numerically the N=3,5 cases in the next chapter.It is also clear that the speed pairing property of (6.84a) together with (6.84b) induces H(f) to be a symmetric function of time .Consider the triple product associated with parameters 124 in a five parameter polynomial.It is given by, noting (6.84a).

$$k_{13}k_{15}k_{23}k_{25}k_{43}k_{45}e^{\gamma_1(x-u_1t)}\ e^{\gamma_2(x-u_2t)}\ e^{\gamma_2(x+u_2t)}$$

this pairs with the triple product associated with 245 ,

$$k_{21}k_{23}k_{41}k_{43}k_{51}k_{53}e^{\gamma_2(x-u_2t)}\ e^{\gamma_2(x+u_2t)}\ e^{\gamma_1(x+u_1t)}$$

which becomes applying (6.84b) ,

$$k_{45}k_{43}k_{25}k_{23}k_{15}k_{13}e^{\gamma_2(x-u_2t)}\ e^{\gamma_2(x+u_2t)}\ e^{\gamma_1(x+u_1t)}$$

This clearly when added to the 124 term produces a time dependent term involving $\cosh\gamma_1u_1t$.

§ 7. Antisoliton representation and non-uniqueness of the lsp(sG,MKdV).

The sG possesses the symmetry $\varphi' \to \varphi + 2n\pi, n \in \mathbb{Z}$. This means that we can represent in particular an antisoliton in two ways

$$\varphi = 4\tan^{-1} - e^{x} \qquad \varphi \in (-2\pi, 0) \tag{6.85a}$$

or

$$\varphi' = 4\tan^{-1} e^{-x} \qquad \varphi' \in (0, 2\pi) \tag{b}$$

We chose (6.85a) as our representation of an antisoliton-this was no accident. Consider the following. The soliton-antisoliton solutions of the sG may be written in the following two ways ,

$$\tan \varphi/4 = u_{12}^{-1} \left(\frac{\exp X_1 - \exp X_2}{1 + \exp(X_1 + X_2)} \right) \tag{6.86a}$$

$$\tan\varphi'/4 = u_{12} \left(\frac{\exp X_1 + \exp - X_2}{1 + \exp(X_1 - X_2)} \right) \tag{6.86b}$$

We have already seen in this chapter how φ may be written in terms of the roots of a quadratic(f_1, f_2). Thus ,

$$\varphi = 4\tan^{-1} f_1 + 4\tan^{-1} f_2 \tag{6.87}$$

f_m are the real roots(one of which is negative) of ,

$$f^2 - u_{12}^{-1}(e^{x_1} - e^{x_2})f - e^{x_1 + x_2} = 0 \tag{6.88}$$

It would seem natural to represent φ' given by (6.86a) also in the form (6.87) but where f_m are the roots of ,

$$f^2 - u_{12}(e^{x_1} + e^{-x_2})f + e^{x_1 - x_2} = 0 \tag{6.89}$$

(6.89) has complex roots(for certain regions of the x-t plane).

Thus the accelerating soliton picture characterized by associating a single f_m with an individual soliton or antisoliton would break down here. It is clear from this that the particular antisoliton representation chosen is very important.

We can of course obtain a real functioned lsp for φ' given by (6.86b) by noticing that $\varphi' = \varphi + 2\pi$.

Thus we may write

$$\varphi' = 4\tan^{-1}f_1 + 4\tan^{-1}f_2 + 2\pi$$

where f_m are the roots of (6.88).Adding the 2π to $4\tan^{-1}f_2$ we obtain ,

$$\varphi' = 4\tan^{-1}f_1 + 4\tan^{-1}-f_2^{-1} \qquad (6.90)$$

Since f_2 is negative (6.90) gives us a real lsp for φ' in terms of the appropriate accelerating solitons.In general we would add $2.na.\pi$ to an N parameter solution of the sG having na antisolitons in the form (6.85b) to obtain the lsp involving arctans with real arguments.

We end this section by noting that the lsp (6.2-3) is not unique.It is quite possible to find other polynomials having roots h_m which are such that $\sum 4\tan^{-1}h_m = \Phi_N$.It must be said though that they are somewhat artificial and their interpretation at present remains unclear.We illustrate this for two parameter cases only.

Consider the quadratic ,

$$f^2 + B(x,t)f + C(x,t) = 0 \qquad (6.91)$$

Define ,

$$\varphi_f = 4\tan^{-1}f_1 + 4\tan^{-1}f_2 \qquad (6.92)$$

where f_m are the roots of (6.91).

Clearly ,

$$\tan \varphi_f/4 = \frac{-B}{1-C} \qquad (6.93)$$

Now consider the "rival" quadratic ,

$$h^2 + R(x,t)B(x,t)h + 1 - R(x,t)(1 - C(x,t)) = 0 \qquad (6.94)$$

Define ,

$$\varphi_h = 4\tan^{-1}h_1 + 4\tan^{-1}h_2 \qquad (6.95)$$

where h_m are the roots of (6.94).We conclude from (6.93-5) ,

$$\tan \varphi_h/4 = \frac{-RB}{1-(1-R(1-C))} = \frac{-B}{1-C} = \tan \varphi_f/4$$

Thus ,

$$\varphi_h = \varphi_f + 8k\pi \quad ,k \in \mathbb{Z}$$

So we see that the alternative quadratic produces essentially the same two parameter solution as the original.It is perfectly possible to choose R so that the roots of (6.94) are real yet the arguments of the arctans are very different from those produced by (6.91).R can also be chosen so that asymptotically the $4\tan^{-1}h_m$ become free solitons or antisolitons.

The behaviour of the branch points of the complexified sG N parameter solution is not sensitive to the individual lsp chosen(as we can look for the branch points of $\Phi_N=4\tan^{-1}A(z,t)$ by setting $A(z,t)=\pm i$ and $A(z,t)$ *is* unique).However the natural lsp (6.2-3) produces a soliton motion which coincides with the motion of the branch points(§4).This confirms its importance.

§ 8. Concluding Remarks

The results of this chapter are sufficiently attractive and common to a large number of soliton equations that we might hope that they may lead to a deeper understanding of the relationship between soliton equations and soliton interaction.

The fact that there are special points on the real accelerating solitons which move like the singularities of the complex solitons establishes the importance of the real linear superposition.There are still many questions to be answered.Numerical calculations described in the next chapter support the conjectures 6.1-2,yet they have resisted all attempts at proving them.

In addition we have not been able as yet to determine partial differential equations which are satisfied by the roots of the lsp.This is obviously a worthy topic of further investigation as it may lead to a new way of solving soliton equations.

Except in the case of two parameter solutions in the centre of velocity frame(sG,Boussinesq) the positions and shapes of the accelerating solitons can only be determined numerically.We shall see in the next chapter how we can determine good approximate formulae describing the shape change of solitons and their motion.

In the review chapters of this thesis we discussed a linear superposition principle (different from the author's) for the KdV.Apart from the obvious drawbacks possessed by this it presents a problem in that each proposed soliton carries *all* the poles of the complexified two soliton KdV solution.Thus at best we must conclude that though it may have mathematical merit,it is physically unimportant.

CHAPTER 7 : SOLITON INTERACTION AND THE BEHAVIOUR OF THE ROOTS OF THE LSP

§ 0. Introduction

In this chapter we study analytically (§1) and numerically(§2) the behaviour of the functions f_i which are the roots of the lsp defined in the last chapter(6.2-3).In so doing we illustrate many of the properties possessed by the f_i.

We shall see in §1 that f_i for two parameter cases are monotonic though it is possible for f_i to become stationary and also develop points of inflexion.We also see that numerical calculations support the conjectures (6.1-2) of the last chapter.We have solved the lsp (6.2-3) numerically for the sG and MKdV equations for up to three parameters.At no stage were roots lost. This indicates the conjectured reality of the roots(for cases involving mixtures of solitons and antisolitons or breathers).Also we always find the roots to be monotonic functions.

In addition to the above we also present time evolution graphs of two parameter cases and their decomposition into separate parts.We discover both analytically (§1) and numerically (§2) that as the two soliton solution of the KdV evolves,the two parts into which it can be decomposed develop some suprising properties.For $t \gtrsim 0$ the originally faster soliton develops(emits) another hump which moves away rapidly.Similarly we find the originally slower soliton develops a negative hump which again moves off rapidly.These extra humps eventually move off together (becoming infinitely distant from the solitons left behind) and their sum becomes zero.Such a fragmentation phenomenon does not manifest itself with either the sG or MKdV equations.We have explored the analytical reasons for this in §1.

In §3 we consider a method of approximating f_i as functions of the form exp $[\alpha_i(t)x + \beta_i(t)]$.By setting x=0 in the lsp we can obtain in certain cases explicit formulae for α_i and β_i.In fact this approximation proves to have zero error in two parameter centre of velocity frames for the sG and Boussinesq equations as shown in the previous chapter.Of course this form of approximation assumes the f_i to be perfectly exponential in appearance(as a fuction of x).Clearly it is in error in cases where f_i have points of inflexion.Nevertheless we find it to provide a good approximation when $f_i < 1$.

In §4 we study the motion of points such that $f_i=1$ or $f_i=-1$.We saw in the previous chapter(for the sG) that such points coincide with the positions of the projections on the real axis of the singularities of the complexified N parameter solution.As we saw in chapter 4 ,Bowtell and Stuart were able to obtain explicit formulae for the positions of such points as functions of time in the two parameter cases of the sG.Such exact results are not forthcoming in higher parameter cases.Though, using the approximations of §3 we can obtain surprisingly good approximate formulae for the positions of the solitons or antisolitons, especially in certain special cases(time symmetric,perfect phase-see §6 ,chapter 6).

We have carried out extensive numerical analysis of higher parameter solutions of the sG(up to 5 parameters).We have discovered that the breather in interaction with solitons or antisolitons, can be "broken-up" only to reform later,made out of different constituent parts.Thus although breathers cannot be destroyed ultimately,they can lose their identity for a period during the interaction with a soliton or antisoliton.

In §5 we consider approximate formulae for intersoliton forces and indications of retarded interaction amongst solitons.

The chapter ends with some concluding remarks.

Note that we have referred on the graphs to the MKdV and the derivative MKdV.These are respectively equations (6.7a) and (6.10).We also note that figures 7.1-43 were produced on the author's own home computer(Amstrad PC1512-hence the slightly rough appearance),figs 7.44-49 were produced using the City University computing facilities.

§ 1. Analytical properties of the roots of the two parameter lsp for the sG,MKdV and KdV equations.The analytical behaviour of the two parameter solutions as a sum of separate parts.The behaviour of the roots of the three parameter,two soliton-antisoliton solution at x=0,t=0.

In two parameter cases the lsp has roots of a relatively simple form as we saw in §4 chapter 6.We are going to discuss below such things as the monotonicity of f_i(roots of the lsp) and also stationary points and points of inflexion.Then when we examine the graphs of f_i computed numerically we will be able to compare our analytical findings with the features present on the graphs.Even in three parameter cases with perfect phase we will be able to deduce exactly,*without* solving the lsp,the properties of the f_i at position x=0 at time t=0.

<u>Two solitons</u>

The two soliton lsp with perfect phase is given by,

$$f^2 - \left(\frac{e^{X_1} + e^{X_2}}{u_{12}} \right) f + e^{X_1+X_2} = 0 \qquad (7.1a)$$

where for the sG,(u_1>u_2)

$$\left.\begin{aligned} X_i &= \gamma_i(x-u_i t) \\ u_{12} &= \frac{\gamma_2(1-u_2)-\gamma_1(1-u_1)}{\gamma_2(1-u_2)+\gamma_1(1-u_1)} \end{aligned}\right\} \qquad \text{(b)}$$

and for the MKdV and KdV, $(k_1 > k_2)$

$$\left.\begin{aligned} X_i &= 2k_i(x-4k_i^2 t) \\ u_{12} &= \frac{k_1-k_2}{k_1+k_2} \end{aligned}\right\} \qquad \text{(c)}$$

Solving (7.1) we find,

$$f_{1,2} = \exp g_{1,2} \qquad \text{(7.2a)}$$

$$g_{1,2} = X_+ \pm \ln\left[\frac{\cosh X_- + (\cosh^2 X_- - u_{12}^2)^{1/2}}{u_{12}}\right] \qquad \text{(b)}$$

where $X_{+,-} = (X_1 \pm X_2)/2$.

Differentiating (7.2b) with respect to x we find,

$$(g_{1,2})_x = k_+ \pm k_-\delta(x,t) \qquad \text{(7.3a)}$$

$$\delta(x,t) = \frac{\sinh X_-}{(\cosh^2 X_- - u_{12}^2)^{1/2}} \qquad \text{(b)}$$

where the constants $k_{+,-} = (X_{+,-})_x$. For the MKdV and KdV $k_{+,-} = k_1 \pm k_2$. We can see that $|\delta(x,t)| < 1 \; \forall \; x,t$, since $u_{12} < 1$, so that $|\sinh X_-| = (\cosh^2 X_- - 1)^{1/2} < (\cosh^2 X_- - u_{12}^2)^{1/2}$. Thus noting the definitions of $k_{+,-}$, $X_{+,-}$ and f_i we see easily that,

$$g_{ix}, f_{ix} > 0 \quad \forall \; x,t \qquad \text{(7.3c)}$$

f_i are also bounded below by zero. Examining (7.2) for fixed t we may also establish the following,

$$\lim_{x \to -\infty} g_{1,2} = X_{2,1} \mp \ln u_{12} \;, \lim_{x \to +\infty} g_{1,2} = X_{1,2} \mp \ln u_{12} \qquad \text{(7.3d)}$$

Also examining (7.2) for fixed X_i we find,

X_1 = constant

$$\lim_{t \to -\infty} g_1 = X_1 - \ln u_{12} \;, \; \lim_{t \to -\infty} g_2 = -\infty \qquad \text{(7.3e)}$$

$$\lim_{t \to +\infty} g_1 = +\infty \qquad , \; \lim_{t \to +\infty} g_2 = X_1 + \ln u_{12}$$

X_2 = constant

$$\lim_{t\to-\infty} g_1 = +\infty \quad , \lim_{t\to-\infty} g_2 = X_2 + \ln u_{12} \qquad (7.3f)$$

$$\lim_{t\to+\infty} g_1 = X_2 - \ln u_{12} \quad , \lim_{t\to+\infty} g_2 = -\infty$$

Differentiating (7.3a) with respect to x once and then twice, we find,

$$(g_{1,2})_{xx} = \pm k_-^2(1-u_{12}^2)\frac{\cosh X_-}{(\cosh^2 X_- - u_{12}^2)^{3/2}} \qquad (7.4a)$$

$$(g_{1,2})_{xxx} = \mp k_-^3(1-u_{12}^2)\frac{\sinh X_-.(2\cosh^2 X_- + u_{12}^2)}{(\cosh^2 X_- - u_{12}^2)^{5/2}} \qquad (7.4b)$$

It is clear from the above that $(g_{1,2})_{xx} \neq 0 \ \forall \ -\infty < x,t < \infty$.

Thus g_i do not possess points of inflexion.

We may write the derivative of f_i with respect to x ,

$$f_{ix} = g_{ix} f_i \qquad (7.5a)$$

$$f_{ixx} = (g_{ixx} + g_{ix}^2) f_i \qquad (7.5b)$$

From (7.4a) we see that $g_{1xx} > 0$ always. Thus from the above we see that f_1 cannot possess a point of inflexion. However since $g_{2xx} < 0$, f_2 has the possibility of points of inflexion.

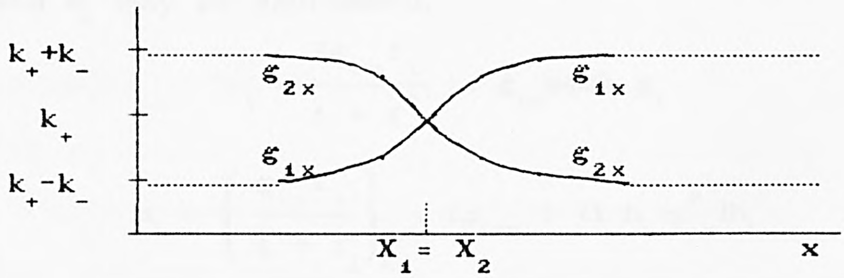
We note the following which is deduced from (7.3).

$$k_+ \mp k_- < (g_{1,2})_x < k_+ \pm k_- \ , \ \forall \ x,t \qquad (7.6a)$$

$$\lim_{x\to-\infty} (g_{1,2})_x = k_+ \mp k_- \ , \lim_{x\to+\infty} (g_{1,2})_x = k_+ \pm k_- \qquad (7.6b)$$

From the above analysis we may sketch g_i, g_{ix} and g_{ixx}.

fig a



From (7.4b) it is clear that the stationary points of g_{ixx} occur when $X_1 = X_2$. The sign of the gradient of g_{ixx} changes about this point. We see also from (7.4a) that g_{ixx} are symmetric functions about the point $X_1=X_2$. It is also clear that as $|x|\longrightarrow \pm\infty$, $g_{ixx}\longrightarrow 0$. Thus we have the following picture,

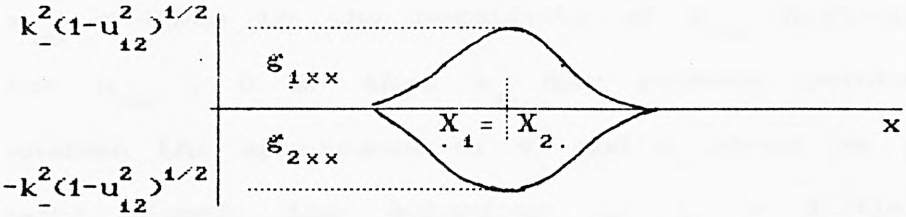


fig b

For the MKdV and KdV, if we allow $k_2\longrightarrow 0, k_-, k_+\longrightarrow k_1$ hence $u_{12}\longrightarrow 1$, so the maximum and minimum of g_{ixx} grow infinitely large. Thus provided u_{12} is close enough to 1, $g_{2xx}+g_{2x}^2$ will possess two zeros. Hence f_2 will from (7.5b) possess two points of inflexion. Finally g_i have the following appearance as a function of x for fixed t,

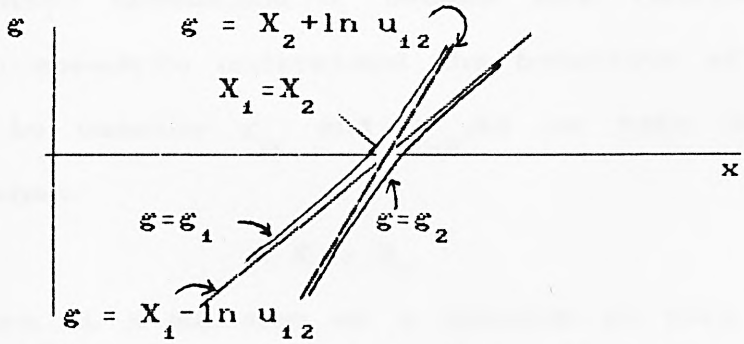


fig c

In the derivative MKdV equation the solitons are characterised by terms of the form $v_i=(2\tan^{-1}f_i)_x$ and in the case of the KdV solitons are characterised by terms of the form $u_i=w_{xx}$, where w is given by, $w=\ln(1+f_i)$. Thus in terms of g_i and f_i, v_i and u_i may be expressed,

$$v_i = \frac{2g_{ix}f_i}{1+f_i^2} = g_{ix}\operatorname{sech} g_i \tag{7.7a}$$

$$u_i = \left(\frac{g_{ix}f_i}{1+f_i}\right)_x = [g_{ixx} + (1-h_i)g_{ix}^2]h_i \tag{7.7b}$$

$$h_i = \frac{f_i}{1+f_i} \tag{7.7c}$$

As we have seen f_i are positive functions so that $0 < h_i < 1$, and as $x \rightarrow \infty$ $h_i \rightarrow 1, x \rightarrow -\infty$ $h_i \rightarrow 0$.From (7.3a),(7.4a) we see that w_1 does not have any points of inflexion as $u_1 = w_{1xx} \neq 0$.Due to the negativity of g_{2xx} however it is possible for $u_{2xx} = 0$ so that w_2 may possess points of inflexion.To analyse the appearance of v_i and u_i above we need to consider more closely the behaviour of $y_i = f_i/(1+f_i^2)$ and h_i.From established properties of f_i we can see that as $|x| \rightarrow \infty$,$y_i \rightarrow 0$. Differentiating y_i with respect to x,we find,

$$y_{ix} = g_{ix} f_i (1+f_i^2)^{-2}(1-f_i^2) \qquad (7.8)$$

Noting (7.3c) we see that y_i has stationary points only when $f_i = 1$.Using (7.3e-f) we can establish the appearance and motion of y_i and h_i as $|t| \rightarrow \infty$.y_i become sech functions moving at the soliton speeds,and h_i become kink functions moving at the soliton speeds.To understand the behaviour of v_i and u_i we need also to examine g_{ix} and g_{ixx}.As we have seen g_{ix}(fig a) intersect when.

$$X_1 = X_2 \qquad (7.9a)$$

Also g_{ixx} are at a maximum or a minimum at this point (fig b). The point with x coordinate x_{int} moves at constant speed given by(for MKdV and KdV),

$$x_{int} = v_{int} t \qquad (7.9b)$$

where,

$$v_{int} = 4\left(\frac{k_1^3 - k_2^3}{k_1 - k_2}\right) \qquad (7.9c)$$

It is easily shown that,

$$v_{int} > 4k_1^2 \qquad (7.9d)$$

Thus the intersection point moves at a speed greater than the greatest soliton speed.We now can sketch on the same graph the appearance of g_{ix} and y_i for $t \ll 0$ and $t \gg 0$.

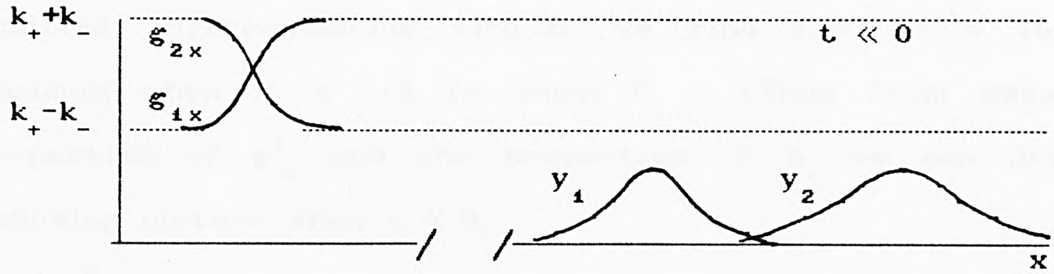


fig d

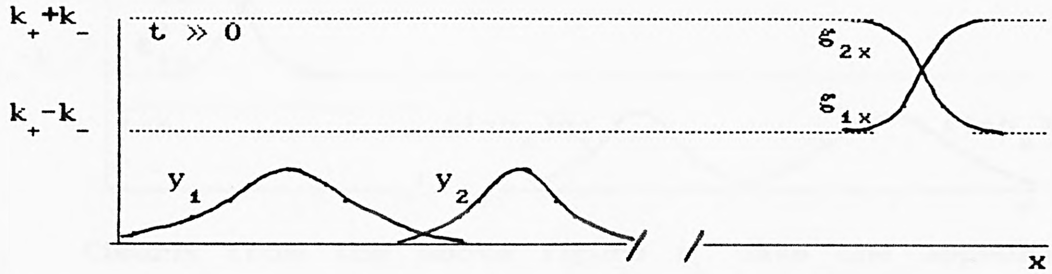


fig e

Thus $v_i = 2g_{ix}y_i$ can be formed by multiplying the respective graphs above and we find(note as t becomes more positive the whole graph translates along the x axis-so the origins of the x axes on the two graphs above do not coincide),

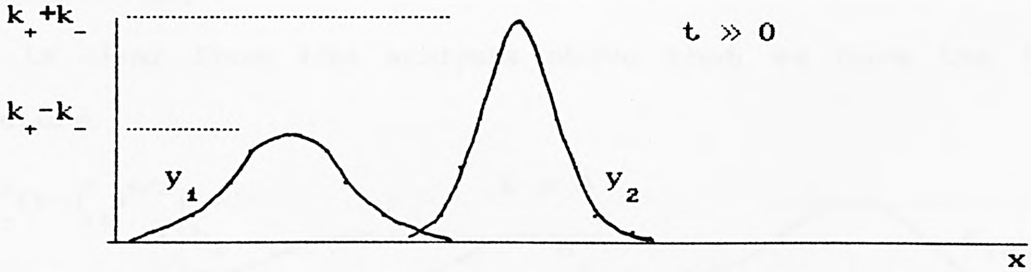
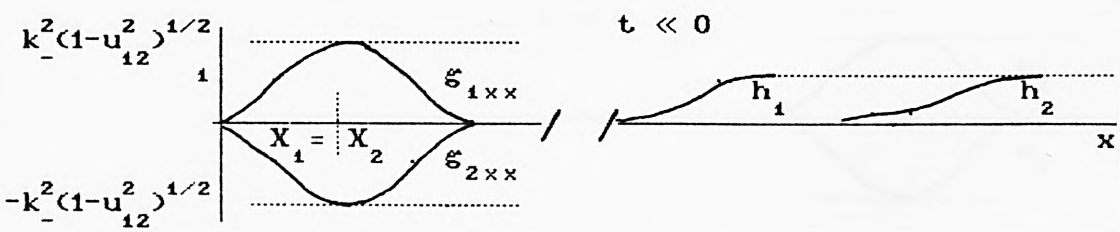


fig f

Similarly for $t \ll 0$.Examining u_i in (7.7b) we need to examine the two terms $r_i = g_{ixx}h_i$ and $s_i = g^2_{ix}(1-h_i)h_i$ separately.First we examine $t \ll 0$.From the above analysis we thus have the following.

fig g



Thus $r_i \cong 0 \ \forall \ x$ when $t \ll 0$.We now examine the terms g^2_{ix} and $(1-h_i)h_i$ which on multiplying produces s_i.Differentiating h_i (7.7c) we find $h_{ix} = f_{ix}/(1+f_i)^2$.Thus h_{ix} is always positive. Differentiating $(1-h_i)h_i$ we find for all t it is a maximum when $h_i = 1/2$ i.e when $f_i = 1$.Thus from established properties of g_{ix}^2 and the properties of h_i we can draw the following picture when $t \ll 0$.

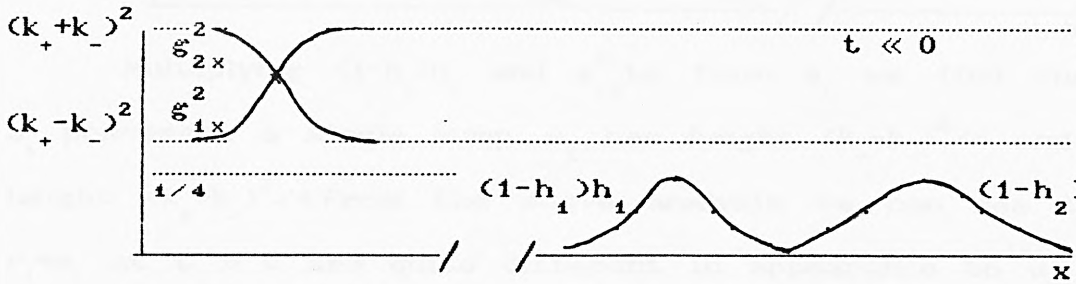
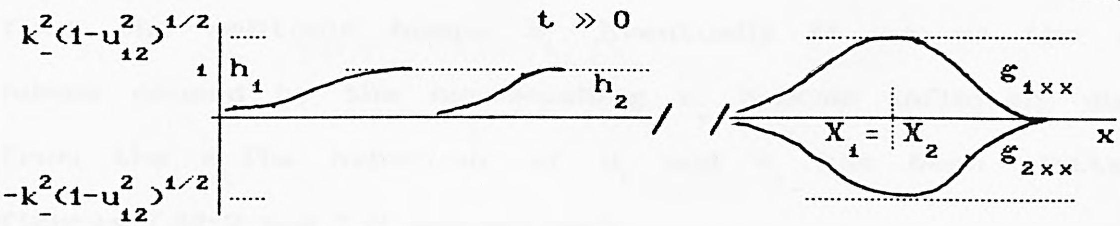


fig h

Clearly from the above figure s_i have the appearance of single humps of heights, $(k_+ + k_-)^2/4$ and $(k_+ - k_-)^2/4$.Thus we may conclude that $u_i = r_i + s_i \cong s_i$ when $t \ll 0$.Thus each u_i has a single hump in this limit.We now examine the markedly different situation which obtains when $t \gg 0$.As before we examine r_i and s_i separately.

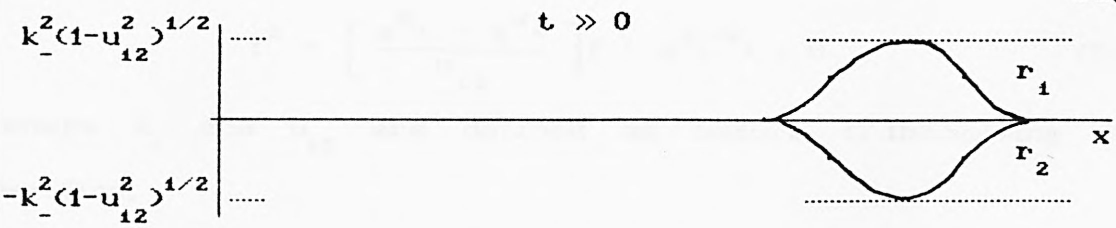
It is clear from the analysis above that we have the following picture.

fig i



Multipying h_i and g_{ixx} to form r_i we obtain the following graph for r_i.

fig j



Now we plot $(1-h_i)h_i$ and g_{ix}^2 for $t \gg 0$.We find,

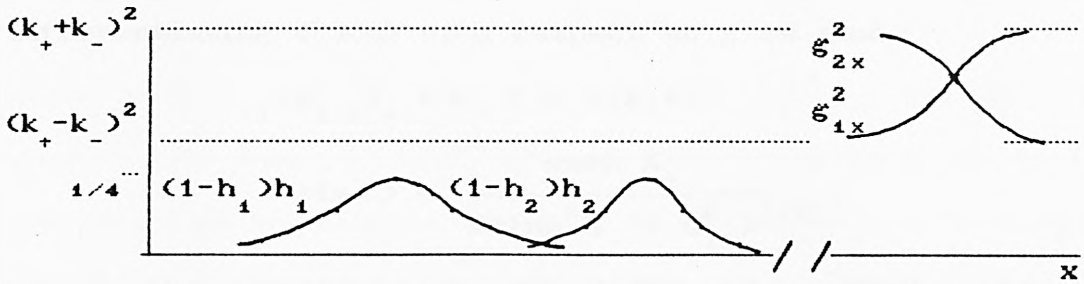


fig k

Multiplying $(1-h_i)h_i$ and g^2_{ix} to form s_i we find that each s_i possesses a single hump .s_1 has height $(k_+-k_-)^2/4$ and s_2 has height $(k_++k_-)^2/4$.From the above analysis we can see that$u_i = r_i+s_i$ at $t \gg 0$ are quite different in appearance to u_i at $t \ll 0$,this is because when $t \gg 0$ $r_i \not\equiv 0$,instead r_i has the appearance given in fig j.r_i is a very asymmetric function of time only departing from zero at times greater than or equal to the time when g_{ixx} and h_i overlap appreciably(maximum overlap occuring at time $t = 0$).We can therefore see that u_1 develops a second hump in the vicinity of the region where g_{1xx} differs from zero.Similarly we find that u_2 develops a negative hump in the same region.As t becomes larger the extra humps move away from the solitonic humps s_i .Eventually $(t \longrightarrow \infty)$ the extra humps caused by the nonvanishing r_i become infinitely distant from the s_i.The behaviour of u_i and v_i has been plotted in figures 7.32-3 and 7.11 respectively.

<u>Soliton-antisoliton</u>

The soliton-antisoliton lsp with perfect phase is given by,

$$f^2 - \left[\frac{e^{X_1} - e^{X_2}}{u_{12}} \right] f - e^{X_1+X_2} = 0 \qquad (7.10a)$$

where X_i and u_{12} are defined as before (7.1b).Solving (7.10a) we find,

$$f_{1,2} = \pm \exp g_{1,2} \qquad (7.10b)$$

$$g_{1,2} = X_+ \pm \ln \left[\frac{\sinh X_- + (\sinh^2 X_- + u^2_{12})^{1/2}}{u_{12}} \right] \qquad (c)$$

where as before $X_{+,-} = (X_1 \pm X_2)/2$.

Differentiating (7.10c) with respect to x we find,

$$(g_{1,2})_x = k_+ \pm k_- \varepsilon(x,t) \tag{7.11a}$$

$$\varepsilon(x,t) = \frac{\cosh X_-}{(\sinh^2 X_- + u_{12}^2)^{1/2}} \tag{b}$$

where the constants $k_{+,-} = (X_{+,-})_x$. For the MKdV and KdV $k_{+,-} = k_1 \pm k_2$. We can see that $0 < \varepsilon(x,t) < 1 \ \forall \ x,t$, since $u_{12} < 1$, so that $\cosh X_- = (\sinh^2 X_- + 1)^{1/2} > (\sinh^2 X_- + u_{12}^2)^{1/2}$. Examining (7.10c) for fixed t we may also establish the following,

$$\lim_{x \to -\infty} g_{1,2} = X_{1,2} \mp \ln u_{12} \ , \lim_{x \to +\infty} g_{1,2} = X_{1,2} \mp \ln u_{12} \tag{7.12}$$

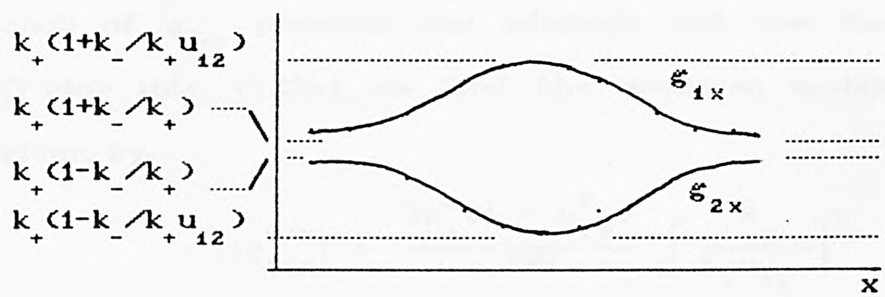
Differentiating (7.11a) with respect to x once and then twice, we find,

$$(g_{1,2})_{xx} = \mp k_-^2 (1-u_{12}^2) \frac{\sinh X_-}{(\sinh^2 X_- + u_{12}^2)^{3/2}} \tag{7.13a}$$

$$(g_{1,2})_{xxx} = \pm k_-^3 (1-u_{12}^2) \frac{\cosh X_- . (2\sinh^2 X_- - u_{12}^2)}{(\sinh^2 X_- + u_{12}^2)^{5/2}} \tag{b}$$

g_{ix} have stationary points when $g_{ixx} = 0$. From (7.13a) we see that these occur when $X_1 = X_2$. Each g_{ix} has a single stationary point. Clearly from (7.13b) g_{1x} has a maximum at $X_1 = X_2$ and g_{2x} a minimum. It is clear from (7.11) that as $|x| \to \infty$ $\varepsilon(x,t)$ for fixed t, tends to zero. Thus from these results we draw the following picture for g_{ix}.

fig 1



We note that for the MKdV $k_+(1-k_-/k_+u_{12}) = 0$. For the sG we have,

$k_-/k_+ = (\gamma_1-\gamma_2)/(\gamma_1+\gamma_2)$, $u_{12} = (\gamma_1-\gamma_2+\gamma_1 u_1-\gamma_2 u_2)/(\gamma_1+\gamma_2-\gamma_1 u_1-\gamma_2 u_2)$

Thus $k_-/(k_+ u_{12}) < 1$ so the minimum of g_{2x} is not zero.Also note that when $X_1 = X_2$, $g_{ixx} = 0$ (7.13a).Thus from (7.5) we can conclude that since for the sG equation $g_{ix} \neq 0 \; \forall \; x,t$,f_i do not possess stationary points.Also it is clear that for the MKdV that f_1(the soliton function) does not possess stationary points.However f_2 does possess one stationary point when $X_1 = X_2$.

It also clear from the above analysis (using (7.5)) that for the sG ,f_i do not possess points of inflexion at the point $X_1 = X_2$ .In the case of the MKdV,f_1 does not possess a point of inflexion at $X_1 = X_2$,but f_2 does.

We can show quite generally that f_1 possesses no point of inflexion $\forall$ x,t for the sG and the MKdV,by the following argument.

Examine (7.13b).g_{ixx} has stationary points when $g_{ixxx} = 0$.These occur when,

$$\sinh X_- = \pm u_{12}/\sqrt{2} \tag{7.14a}$$

It is clear from (7.13a) that $|g_{ixx}| \to 0$ as $|x| \to \infty$. Thus at the pair of points determined by (7.14a) g_{ixx} is at a maximum or a minimum(i.e. (7.14a) does not determine points of inflexion).Since the point where $X_1 = X_2$ determines a zero of g_{ixx} and g_{ixxx} is not zero at this point,we can conclude that each of g_{ixx} possess one minimum and one maximum.Substituting (7.14a) into (7.13a) we find the maximum modulus of g_{ixx} to be given by,

$$|g^{max}_{ixx}| = \frac{2k_+^2(1 - u_{12}^2)}{3\sqrt{3}} \cdot \left(\frac{k_-}{k_+ u_{12}} \right)^2 \tag{7.14b}$$

We have already seen the minimum of g_{1x}^2 is given by,

$$(g_{1x}^2)_{min} = k_+^2(1 + k_-/k_+)^2 \tag{7.14c}$$

For both the sG and MKdV $(g^2_{1x})_{min} > 1$.But $k_-/k_+u_{12} < 1$ and $u_{12} < 1$.Therefore $(g^2_{1x})_{min} > |g^{max}_{1xx}|$.Thus the coefficient of f_1 in the expression for f_{1xx} given by (7.5b) is never zero.

It is much more difficult to ascertain whether f_{2xx} can be zero at points other than when $X_1 = X_2$.Numerical evidence is so far inconclusive as to whether further points of inflexion exist. However the numerical evidence (figs 7.3 ,7.14) accords with the analysis above concerning stationary points of inflexion of the antisoliton function of the MKdV.

We now turn our attention to the breather lsp.

<u>Breather</u>

The breather lsp is given by,

$$f^2 - \left(\frac{e^{\Gamma+i\Omega} - e^{\Gamma-i\Omega}}{u_{12}} \right) f - e^{2\Gamma} = 0 \tag{7.15a}$$

where for the sG,

$$\left.\begin{aligned} \Gamma &= \gamma_v \cos\mu(x-vt)+\alpha \quad , \quad \Gamma_x = \gamma_v \cos\mu \\ \Omega &= \gamma_v \sin\mu(t-vx)+\beta \quad , \quad \Omega_x = -v\gamma_v \sin\mu \\ u_{12} &= i\tan\mu \quad , \quad \tan\mu < 1 \end{aligned}\right\} \tag{7.15b}$$

and for the MKdV,

$$\left.\begin{aligned} \Gamma &= 2\theta(x-vt)+\alpha \quad , \quad \Gamma_x = 2\theta \\ \Omega &= 2\eta(x-wt)+\beta \quad , \quad \Omega_x = 2\eta \\ u_{12} &= i\eta/\theta \equiv i\tan\mu \end{aligned}\right\} \tag{7.15c}$$

Solving (7.15a) we find,

$$f_{1,2} = \pm\exp g_{1,2} \tag{7.16a}$$

$$g_{1,2} = \Gamma \pm \ln\left[\frac{\sin\Omega + (\sin^2\Omega + \tan^2\mu)^{1/2}}{\tan\mu}\right] \tag{7.16b}$$

Differentiating (7.16b) with respect to x we find,

$$(g_{1,2})_x = k_+ \pm k_-\lambda(x,t) \tag{7.17a}$$

$$\lambda(x,t) = \frac{\cos\Omega}{(\sin^2\Omega + \tan^2\mu)^{1/2}} \tag{7.17b}$$

where the constants k_+,k_- are given by,

$$k_+ = \Gamma_x \quad , \quad k_- = \Omega_x \tag{7.17c}$$

Clearly $|\lambda(x,t)| < (\tan\mu)^{-1}$.Thus we find from (7.17a),

$$k_+(1 - |k_-/k_+\tan\mu|) \le g_{ix} \le k_+(1 + |k_-/k_+\tan\mu|) \qquad (7.18a)$$

where for the sG,

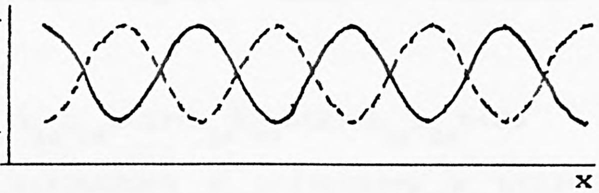
$$|k_-/k_+\tan\mu| = v < 1 \qquad (7.18b)$$

and for the MKdV,

$$|k_-/k_+\tan\mu| = 1 \qquad (7.18c)$$

(7.18b-c) follow from (7.17c) and (7.15b-c).Clearly g_{ix} have the following appearance.

fig m



We note that the minimum of g_{ix} in the case of the MKdV is zero.Differentiating (7.17) with respect to x we obtain,

$$(g_{1,2})_{xx} = \mp k_-^2(1 + \tan^2\mu)\frac{\sin\Omega}{(\sin^2\Omega + \tan^2\mu)^{3/2}} \qquad (7.19)$$

Once again noting (7.5) and the nonvanishing g_{ix} for the sG we see that f_i do not possess stationary points.Although g_{ixx} vanishes when $\Omega = 0$ we do not have points of inflexion at these points also.In the case of the MKdV the situation is different.When $\sin\Omega = 0$ one of the f_i develops a stationary point and we find that f_1 has stationary points when $\Omega=(2n+1)\pi$ where $n \in \mathbb{Z}$ ($\cos\Omega=-1$),and f_2 has stationary points when $\Omega=2n\pi$ ($\cos\Omega=1$).It is clear from (7.19) that f_i also have points of inflexion when f_i is stationary.The features discussed above can clearly be seen in figs 7.5(sG breather) and 7.23(MKdV breather).

We now discuss how despite the considerable complexity of the roots of the three parameter polynomial(§4 chapter 6) we can determine the magnitudes of $f_i(0,0)$,$f_{ix}(0,0)$,$f_{ixx}(0,0)$ in the perfect phase case *without* solving the three parameter polynomial.Such an analysis will enable us to explain certain notable features of the three parameter graphs of f_i produced numerically.We only consider here for brevity the two soliton antisoliton case,though it is apparent from the technique explained below that any three parameter case can be handled with equal ease.

The roots of the two soliton-antisoliton lsp with perfect phase obey the following(§6 chapter 6).Note f_2 is the antisoliton function.

$$f_1+f_2+f_3 = k_{12}k_{13}t(1)+k_{21}k_{23}t(2)+k_{31}k_{32}t(3) \quad (7.20a)$$

$$f_1f_2+f_1f_3+f_2f_3=-[k_{13}k_{23}t(1)t(2)+k_{12}k_{32}t(1)t(3)+k_{21}k_{31}t(2)t(3)] \quad (b)$$

$$f_1f_2f_3 = -t(1)t(2)t(3) \quad (c)$$

where for the sG,

$$t(i) = \exp\gamma_i(x-u_it) \quad , \quad t_x(i) = \gamma_it(i) \quad (7.21a)$$

and for the MKdV,

$$t(i) = \exp2k_i(x-4k_i^2t) \quad , \quad t_x(i) = 2k_it(i) \quad (7.21b)$$

Henceforth we will use the symbol γ_i to represent $2k_i$ in the MKdV case.Now differentiate (7.20) with respect to x. We obtain,

$$f_{1x}+f_{2x}+f_{3x}=\gamma_1k_{12}k_{13}t(1)+\gamma_2k_{21}k_{23}t(2)+\gamma_3k_{31}k_{32}t(3) \quad (7.22a)$$

$$f_1(f_{2x}+f_{3x})+f_2(f_{1x}+f_{3x})+f_3(f_{1x}+f_{2x})= \quad (b)$$
$$-[k_{13}k_{23}(\gamma_1+\gamma_2)t(1)t(2)+k_{12}k_{32}(\gamma_1+\gamma_3)t(1)t(3)+k_{21}k_{31}(\gamma_2+\gamma_3)t(2)t(3)]$$

$$f_{1x}f_2f_3+f_{2x}f_1f_3+f_{3x}f_1f_2 = -(\gamma_1+\gamma_2+\gamma_3)t(1)t(2)t(3) \quad (c)$$

Differentiating (7.22) with respect to x,

$$f_{1xx}+f_{2xx}+f_{3xx}=\gamma_1^2k_{12}k_{13}t(1)+\gamma_2^2k_{21}k_{23}t(2)+\gamma_3^2k_{31}k_{32}t(3) \quad (7.23a)$$

$$f_1(f_{2xx}+f_{3xx})+f_2(f_{1xx}+f_{3xx})+f_3(f_{1xx}+f_{2xx})+ \quad (b)$$
$$2(f_{1x}f_{2x}+f_{1x}f_{3x}+f_{2x}f_{3x}) = -[k_{13}k_{23}(\gamma_1+\gamma_2)^2t(1)t(2)+$$
$$k_{12}k_{32}(\gamma_1+\gamma_3)^2t(1)t(3)+k_{21}k_{31}(\gamma_2+\gamma_3)^2t(2)t(3)]$$

$$f_{1xx}f_2f_3+f_{2xx}f_1f_3+f_{3xx}f_1f_2+ \quad (c)$$

$$f_{1x}(f_{2x}f_3+f_2f_{3x})+f_{2x}(f_{1x}f_3+f_1f_{3x})+f_{3x}(f_{1x}f_2+f_1f_{2x}) =$$

$$-(\gamma_1+\gamma_2+\gamma_3)^2t(1)t(2)t(3)$$

Differentiating (7.23) with respect to x, (7.24)

$$f_{1xxx}+f_{2xxx}+f_{3xxx}=\gamma_1^3k_{12}k_{13}t(1)+\gamma_2^3k_{21}k_{23}t(2)+\gamma_3^3k_{31}k_{32}t(3) \quad (a)$$

$$f_{1x}(f_{2xx}+f_{3xx})+f_1(f_{2xxx}+f_{3xxx})+ \quad (b)$$

$$f_{2x}(f_{1xx}+f_{3xx})+f_2(f_{1xxx}+f_{3xxx})+$$

$$f_{3x}(f_{1xx}+f_{2xx})+f_3(f_{1xxx}+f_{2xxx})+$$

$$2[f_{1xx}f_{2x}+f_{1x}f_{2xx}+f_{1xx}f_{3x}+f_{1x}f_{3xx}+f_{2xx}f_{3x}+f_{2x}f_{3xx}] =$$

$$-[k_{13}k_{23}(\gamma_1+\gamma_2)^3t(1)t(2)+k_{12}k_{32}(\gamma_1+\gamma_3)^3t(1)t(3)$$

$$+k_{21}k_{31}(\gamma_2+\gamma_3)^3t(2)t(3)]$$

$$f_{1xxx}f_2f_3+f_{1xx}(f_{2x}f_3+f_2f_{3x})+ \quad (c)$$

$$f_{2xxx}f_1f_3+f_{2xx}(f_{1x}f_3+f_1f_{3x})+$$

$$f_{3xxx}f_1f_2+f_{3xx}(f_{1x}f_2+f_1f_{2x})+$$

$$f_{1xx}(f_{2x}f_3+f_2f_{3x})+f_{1x}(f_{2xx}f_3+2f_{2x}f_{3x}+f_2f_{3xx})+$$

$$f_{2xx}(f_{1x}f_3+f_1f_{3x})+f_{2x}(f_{1xx}f_3+2f_{1x}f_{3x}+f_1f_{3xx})+$$

$$f_{3xx}(f_{1x}f_2+f_1f_{2x})+f_{3x}(f_{1xx}f_2+2f_{1x}f_{2x}+f_1f_{2xx}) =$$

$$-(\gamma_1+\gamma_2+\gamma_3)^3t(1)t(2)t(3)$$

Now examine the point x=0 t=0.Substituting into (7.20) we find,

$$f_1+f_2+f_3=1 \quad , \quad f_1f_2+f_1f_3+f_2f_3=-1 \quad , \quad f_1f_2f_3=-1 \quad (7.25)$$

where we have employed the equation

$$k_{12}k_{13}+k_{21}k_{23}+k_{31}k_{32}=1 \quad (7.26)$$

and $k_{ij}=-k_{ji}$.Thus from (7.25),

$$f_1=f_3=1 \quad , \quad f_2=-1 \quad (7.27)$$

Henceforth we shall concentrate on the MKdV and only later examine the centre-of-velocity solution of the sG equation.Substituting (7.27) into (7.22b) and noting from (7.21b) that at (0,0) t(i)=1.We obtain ($f_{()}(0,0) \equiv f_{()}$)

$$f_{2x}=-\frac{1}{2}[k_{13}k_{23}(\gamma_1+\gamma_2)+k_{12}k_{32}(\gamma_1+\gamma_3)+k_{21}k_{31}(\gamma_2+\gamma_3)] \quad (7.28a)$$

From the definition of k_{ij} for the MKdV $(k_{ij}=(k_i+k_j)/(k_i-k_j))$ and the definition of γ_i given earlier ,it can be seen that the right hand side of (7.28a) is identically zero.Hence,

$$f_{2x} = 0 \qquad (7.28b)$$

Substituting the above into (7.22a) we find,

$$f_{1x} + f_{3x} = \gamma_1 k_{12} k_{13} + \gamma_2 k_{21} k_{23} + \gamma_3 k_{31} k_{32} \qquad (7.28c)$$

(7.23) become at (0,0) using (7.27) and (7.28b) we find,

$$f_{1xx}+f_{2xx}+f_{3xx} = \gamma_1^2 k_{12} k_{13} + \gamma_2^2 k_{21} k_{23} + \gamma_3^2 k_{31} k_{32} \qquad (7.29a)$$

$$2f_{2xx}+2f_{1x}f_{3x} = -[k_{13}k_{23}(\gamma_1+\gamma_2)^2+k_{12}k_{32}(\gamma_1+\gamma_3)^2+k_{21}k_{31}(\gamma_2+\gamma_3)^2] \quad (b)$$

$$f_{2xx} - f_{1xx} - f_{3xx} - 2f_{1x}f_{3x} = -(\gamma_1+\gamma_2+\gamma_3)^2 \qquad (c)$$

Adding (7.29a) and (7.29c) we find,

$$2f_{2xx}-2f_{1x}f_{3x} = \gamma_1^2 k_{12} k_{13} + \gamma_2^2 k_{21} k_{23} + \gamma_3^2 k_{31} k_{32} - (\gamma_1+\gamma_2+\gamma_3)^2 \qquad (d)$$

Now adding (7.29b) and (7.29d) we find,

$$4f_{2xx} = \gamma_1^2 k_{12} k_{13} + \gamma_2^2 k_{21} k_{23} + \gamma_3^2 k_{31} k_{32} - (\gamma_1+\gamma_2+\gamma_3)^2 \qquad (7.30)$$
$$-[k_{13}k_{23}(\gamma_1+\gamma_2)^2+k_{12}k_{32}(\gamma_1+\gamma_3)^2+k_{21}k_{31}(\gamma_2+\gamma_3)^2]$$

The last term above can be shown to be identically zero using the definition of k_{ij} and γ_i.Also after tedious calculation the first two terms above can be shown to be identically zero.Thus we find,

$$f_{2xx} = 0 \qquad (7.31)$$

Substituting this into (7.29b) or (7.29d) and we find,

$$f_{1x}f_{3x} = 0 \qquad (7.32a)$$

Now consider (7.22c) at (0,0),we find on substituting (7.27) and (7.28b),

$$f_{1x}+f_{3x} = \gamma_1+\gamma_2+\gamma_3 \qquad (7.32b)$$

Clearly from (7.32) choosing f_{3x} to be the zero root,

$$f_{3x} = 0 \;,\; f_{1x} = \gamma_1+\gamma_2+\gamma_3 = 2(k_1+k_2+k_3) \qquad (7.33)$$

We now investigate f_{1xx} and f_{3xx}.Examine (7.24b).Substituting (7.27),(7.28b),(7.31) and (7.33) into

(7.24b) at (0,0),gives,

$$3f_{1x}f_{3xx}+2f_{2xxx}=-[k_{13}k_{23}(\gamma_1+\gamma_2)^3+k_{12}k_{32}(\gamma_1+\gamma_3)^3+k_{21}k_{31}(\gamma_2+\gamma_3)^3] \tag{7.34a}$$

Also examining (7.24c) we find,

$$f_{2xxx}-f_{1xxx}-f_{3xxx}-3f_{3xx}f_{1x} = -(\gamma_1+\gamma_2+\gamma_3)^3 \tag{7.34b}$$

Now substituting (7.24a) at (0,0) into (7.34b)

$$2f_{2xxx}-3f_{3xx}f_{1x} = -(\gamma_1+\gamma_2+\gamma_3)^3+\gamma_1^3k_{12}k_{13}+\gamma_2^3k_{21}k_{23}+\gamma_3^3k_{31}k_{32} \tag{7.34c}$$

We saw earlier that for the MKdV(after equation (7.30)),

$$\gamma_1^2k_{12}k_{13} + \gamma_2^2k_{21}k_{23} + \gamma_3^2k_{31}k_{32}-(\gamma_1+\gamma_2+\gamma_3)^2 = 0$$

Multiplying this by $(\gamma_1+\gamma_2+\gamma_3)$ we obtain,

$$-(\gamma_1+\gamma_2+\gamma_3)^3+\gamma_1^3k_{12}k_{13}+\gamma_2^3k_{21}k_{23}+\gamma_3^3k_{31}k_{32} = \tag{7.35a}$$
$$\gamma_3^2(\gamma_1+\gamma_2)k_{13}k_{23}+\gamma_2^2(\gamma_1+\gamma_3)k_{12}k_{32}+\gamma_1^2(\gamma_2+\gamma_3)k_{21}k_{31}$$

Now it can be shown by taking $k_{13}k_{12}k_{23}$ out as a factor and cancelling that,

$$k_{13}k_{23}(\gamma_1+\gamma_2)^3+k_{12}k_{32}(\gamma_1+\gamma_3)^3+k_{21}k_{31}(\gamma_2+\gamma_3)^3= \tag{7.35b}$$
$$\gamma_1\gamma_2(\gamma_1+\gamma_2)k_{13}k_{23}+\gamma_1\gamma_3(\gamma_1+\gamma_3)k_{12}k_{32}+\gamma_2\gamma_3(\gamma_2+\gamma_3)k_{12}k_{13}$$

Similarly by taking $k_{13}k_{12}k_{23}$ out as a factor and rearranging the right hand side of (7.35b) can be shown to be identical to the right hand side of (7.35a).Thus we find,

$$-(\gamma_1+\gamma_2+\gamma_3)^3+\gamma_1^3k_{12}k_{13}+\gamma_2^3k_{21}k_{23}+\gamma_3^3k_{31}k_{32} = \tag{7.35c}$$
$$k_{13}k_{23}(\gamma_1+\gamma_2)^3+k_{12}k_{32}(\gamma_1+\gamma_3)^3+k_{21}k_{31}(\gamma_2+\gamma_3)^3$$

Subtracting (7.34a) from (7.34c) and substituting (7.35c) we obtain,

$$6f_{3xx}f_{1x} = 0 \tag{7.36a}$$

But we saw earlier in (7.33) that $f_{1x} \neq 0$ thus,

$$f_{3xx} = 0 \tag{7.36b}$$

Substituting this ,(7.31) and (7.33) into (7.29c) gives,

$$f_{1xx} = (\gamma_1+\gamma_2+\gamma_3)^2 \tag{7.36c}$$

We summarize our findings,which apply only to the MKdV equation:In the two soliton-antisoliton lsp with perfect phase the roots are found to have the following properties at $x=0,t=0$.The antisoliton function f_2 and one of the soliton functions f_3 have a stationary point of inflexion.The remaining soliton function f_1 possesses neither a stationary point nor a point of inflexion.In addition,$f_2=-1$ and $f_1=f_3=1$.

We now consider the perfect phase lsp for the centre-of-velocity frame of the two soliton-antisoliton solution of the sG at x=0 ,t=0.

As we saw in previous chapters,

$$k_{ij}=(a_i+a_j)/(a_i-a_j) \;,a_i=\gamma_i^{-1}(1+u_j) \;,\gamma_i^{-1}=(1-u_i^2)^{1/2} \qquad (7.37)$$

Choosing the speeds of the solitons to be u,-u and the antisoliton speed to be zero,we find,

$$k_{12}=k_{23}=(\gamma_1(1-u_1)+1)/(-\gamma_1(1-u_1)+1) \;,\; k_{13}=u_1^{-1} \qquad (7.38a)$$

(7.26) becomes,

$$2k_{12}k_{13} - k_{12}^2 = 1 \qquad (7.38b)$$

Substituting the above equations into the equation for f_{2x} (7.28a) and the equation for f_{2xx} (7.30) we find,

$$f_{2x} = 0 \quad , \quad f_{2xx} = 0 \qquad (7.39)$$

Similar calculations lead to the conclusion that f_{1x},f_{3x} are not zero and do not possess points of inflexions at (0,0).The above calculations confirm the numerical results seen in figs 7.37-8.

We close this section by examining the speeds of points such that $f_i(x,t) = k$, $\dot{x}_i$,where k is a constant.As we have already seen in this thesis for the sG equation, the points where $f_i=1$(for solitons in breathers or otherwise) or $f_i=-1$(for antisolitons in breathers or otherwise) represent the positions of the solitons and antisolitons.Indeed there is every reason to believe the same is true for the MKdV and the KdV.With this in mind we now examine $\dot{x}_i$ at (0,0).Differentiating the equation

$f_i=k$ we obtain(noting $f_i = \pm\exp g_i$),

$$\dot{x}_i = -g_{it}/g_{ix} \tag{7.40}$$

We saw that in all two soliton cases that $g_{ix}\neq0$ $\forall$ x,t.Thus $\dot{x}_i$ are all finite.In the soliton-antisoliton case we found that in the case of the sG $g_{ix}\neq0$ $\forall$ x,t,so again,$\dot{x}_i$ is finite.However in the MKdV case although the soliton moves at finite speed as $g_{1x}\neq0$ the antisoliton has *infinite* speed at (0,0).It is easy to see from the analysis in that case that $g_{2t}(0,0)\neq0$ but $g_{2x}(0,0)=0$.Clearly in interaction solitons and antisolitons obeying the MKdV equation do not obey a law similar to Newton's Third Law.

With the breather case of the sG we again found $g_{ix}\neq0$ $\forall$ x,t,so once again there was no infinite speed behaviour.It is easily shown that in the breather MKdV case $g_{it}\neq0$.However we saw that $g_{ix}=0$ periodically.Thus solitons and antisolitons within MKdV breathers periodically move at infinite speed(not at the same time).

Turning now to the two soliton-antisoliton case,it can be shown by calculations of a very similar nature to those before(i.e by differentiating (7.20) with respect to t several times),that at (0,0) $g_{it}\neq0$.Thus in the two soliton-antisoliton interaction of the MKdV one soliton and the antisoliton move with infinite speed at x=0 t=0,but the remaining soliton moves with finite speed.This as we have proved occurs when they are all coincident at (0,0).

Finally we saw that in the centre-of-velocity frame of the two soliton antisoliton solution of the sG equation,the antisoliton function $g_{2x}(0,0)=0$.In this particular case $\dot{x}_2$ is not necessarily infinite as it is easily shown from the time equivalent of equation (7.22b) that $f_{2t}(0,0)=0$.So in this case the speed is undefined by the above analysis.Clearly the limit of $\dot{x}_2(0,t)$ would have to be carefully examined as $t \rightarrow 0$.

Numerical studies(§3) support the finding above,though in this thesis we have not studied the motion of MKdV,KdV solitons directly.

§ 2. Numerical studies of the time evolution of the roots of the lsp.Two parameter solutions of the sG,MKdV and KdV as an interaction between separate parts.

We have computed the exact roots of the two parameter lsp for the sG,MKdV and KdV(same as soliton case of MKdV) given by formulae (6.32-4),except that we have chosen perfect phase.This ensures that the interactions are centred on x=0 at time t=0.The results can be seen in the figures at the end of §2.We note the somewhat unexpected differences between the sG and the MKdV.In the case of the sG all the functions f_i(two parameters only) are very similar in appearance to simple exponential functions.In the case of the MKdV the situation is quite different(except with pure soliton cases).We note in fig 7.14 that in a soliton-antisoliton case the antisoliton root of the lsp develops a point of inflexion.From figure 7.20 we see that the point of inflexion develops in a small neighbourhood of time t=0.

As we have seen the differences between the two equations manifest themselves in the definition of k_{ij}.This causes a considerable difference in the exact f_i and the approximate f_i.

One might have expected a noticeable difference in the f_i for the two equations in the breather case on account of the more complicated nature of the MKdV breather.As we can see in fig 7.23 both the f_i for the MKdV breather can develop points of inflexion(as opposed to neither in the case of the sG see fig 7.5),though only one root possessess a point of inflexion at any one time(see fig 7.24 where $\tan^{-1}f_i$ compressess the points of inflexion of f_i into a finite strip.Actually despite these marked differences in the behaviour of the breather f_i, the actual two parameter solution for the two equations(figs 7.6 and 7.24) are not so dissimilar.We note from fig 7.13,17,35 that the points where $f_i = \pm 1$ are very close to the maxima or minima of the solitons or antisolitons plotted.

The roots of the three parameter lsp for various situations are plotted in figs 7.36-43.These graphs were obtained by numerically solving the relevant cubic instead of using the exact formulae (6.58-9).The problem one has in using the exact formulae is computational in origin.These formulae involve small quantities being multiplied by large quantities.The trouble is a computer which recognizes numbers less than 10^{-38} to be zero will also reckon $10^{15}\times10^{-38}$ to be zero!

Examining figs 7.36-43 we see that three soliton f_i for the sG and the MKdV are quite similar to each other and to pure exponentials.We also note that there is a gradual change in gradient of the f_i over the time interval chosen.Although we have chosen a centre of velocity frame for the sG the results in any other frame are broadly similar(i.e points of inflexion do not appear as a result of Lorentz transformation).Comparing the two soliton -one antisoliton case for the two equations we immediately see marked differences.

In the sG's case the antisoliton(which happens not to be moving throughout the interaction) develops a stationary point of inflexion at time t=0.Fig 7.38 indicates that the antisoliton f_i only develops the point inflexion *exactly* at t=0.At the same time the two soliton f_i retain their pure exponential-like appearance but we find that they intersect each other at x=0,t=0.This latter fact indicates that the solitons pass through each other(and the antisoliton).

Fig 7.1

Time evolution of the roots (f_i) of the two soliton lsp for the sG equation. Initial soliton speeds .6,.2

The solitons are located at $f_i = 1$.

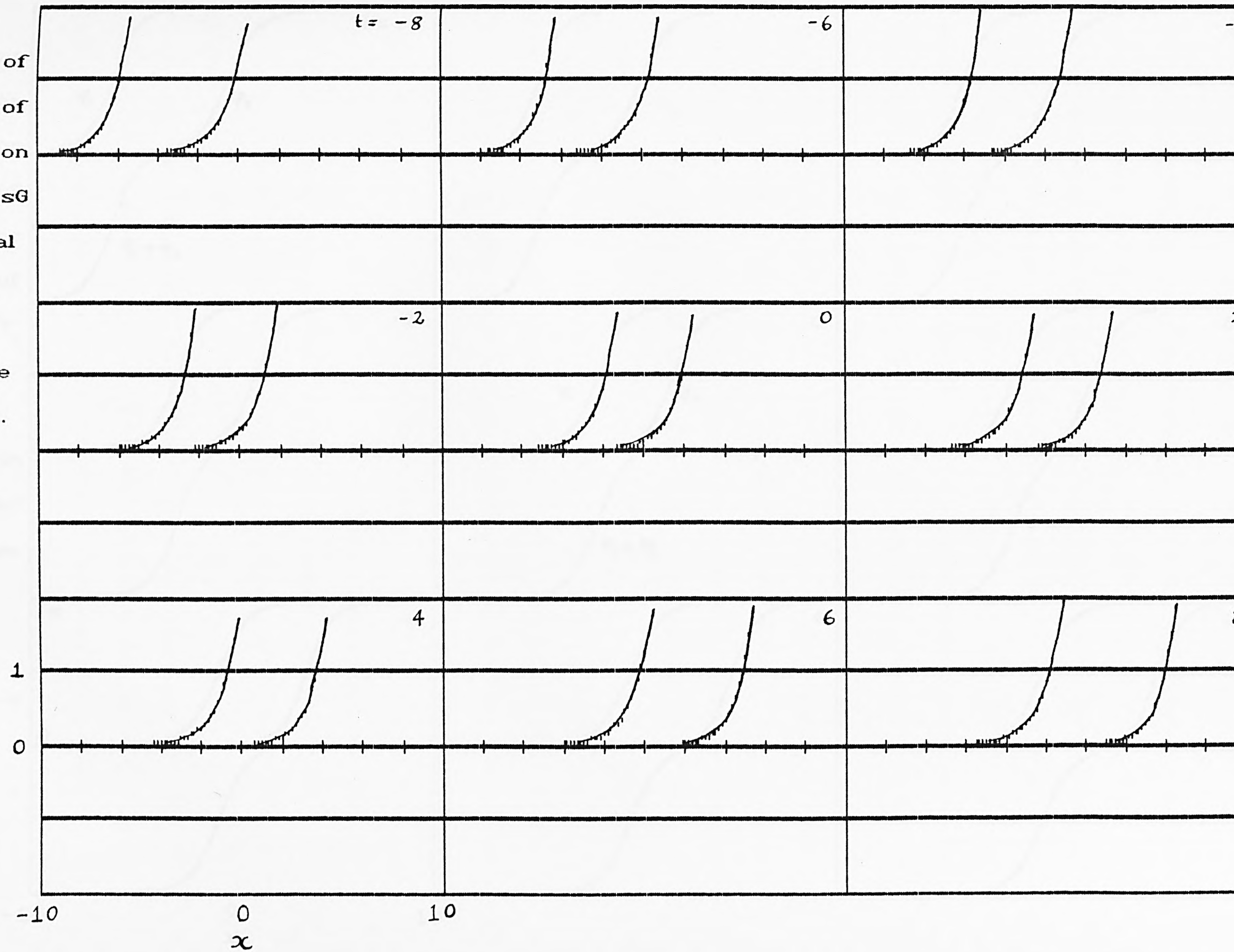
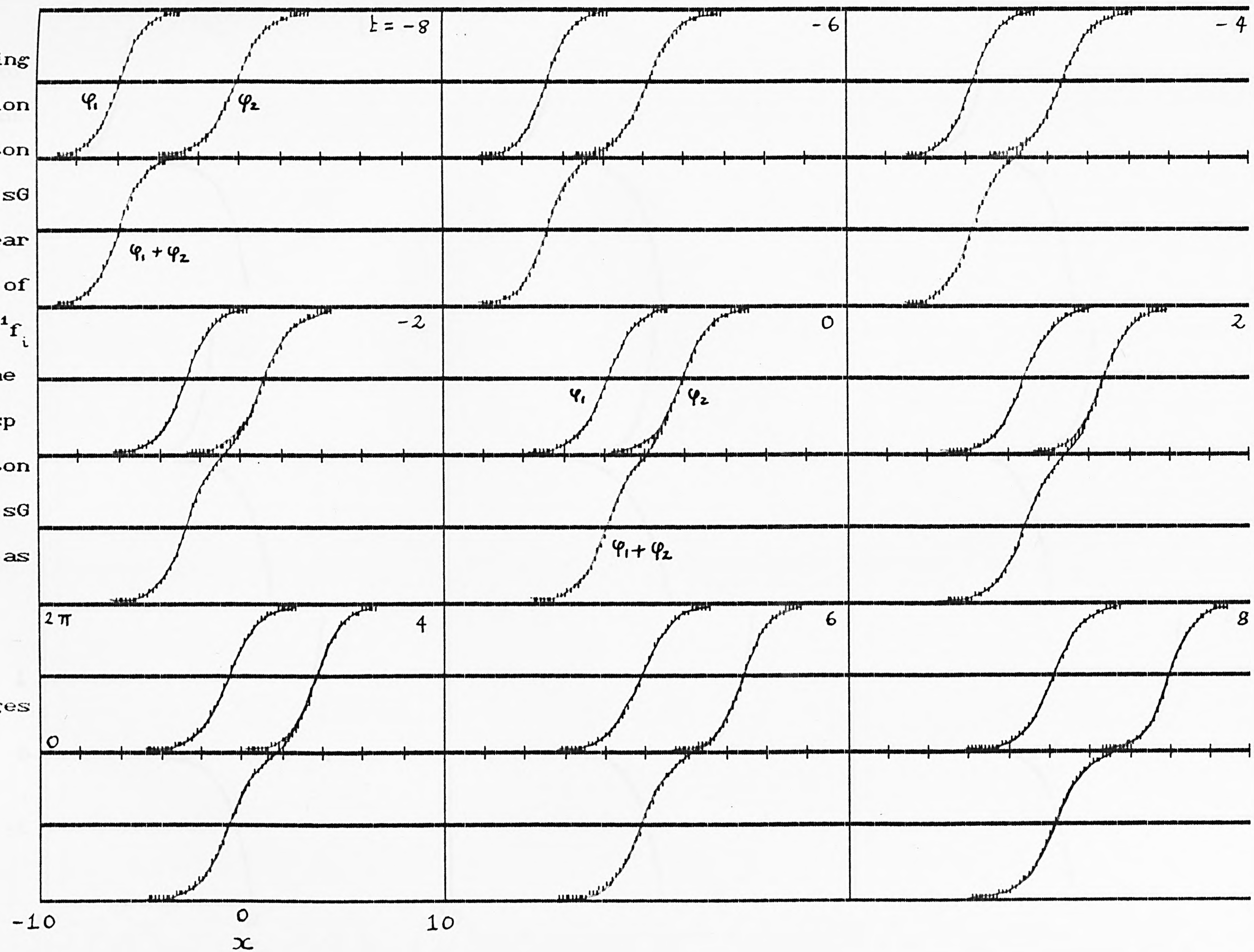


Fig 7.2

A graph showing the time evolution of the two soliton solution of the sG as a linear superposition of functions $\varphi_i = 4\tan^{-1} f_i$ where f_i are the roots of the lsp for the 2 soliton solution of the sG (soliton speeds as in fig7.1).

φ_i range over $(0, 2\pi)$. $\varphi_1 + \varphi_2$ ranges over interval $(0, 4\pi)$.



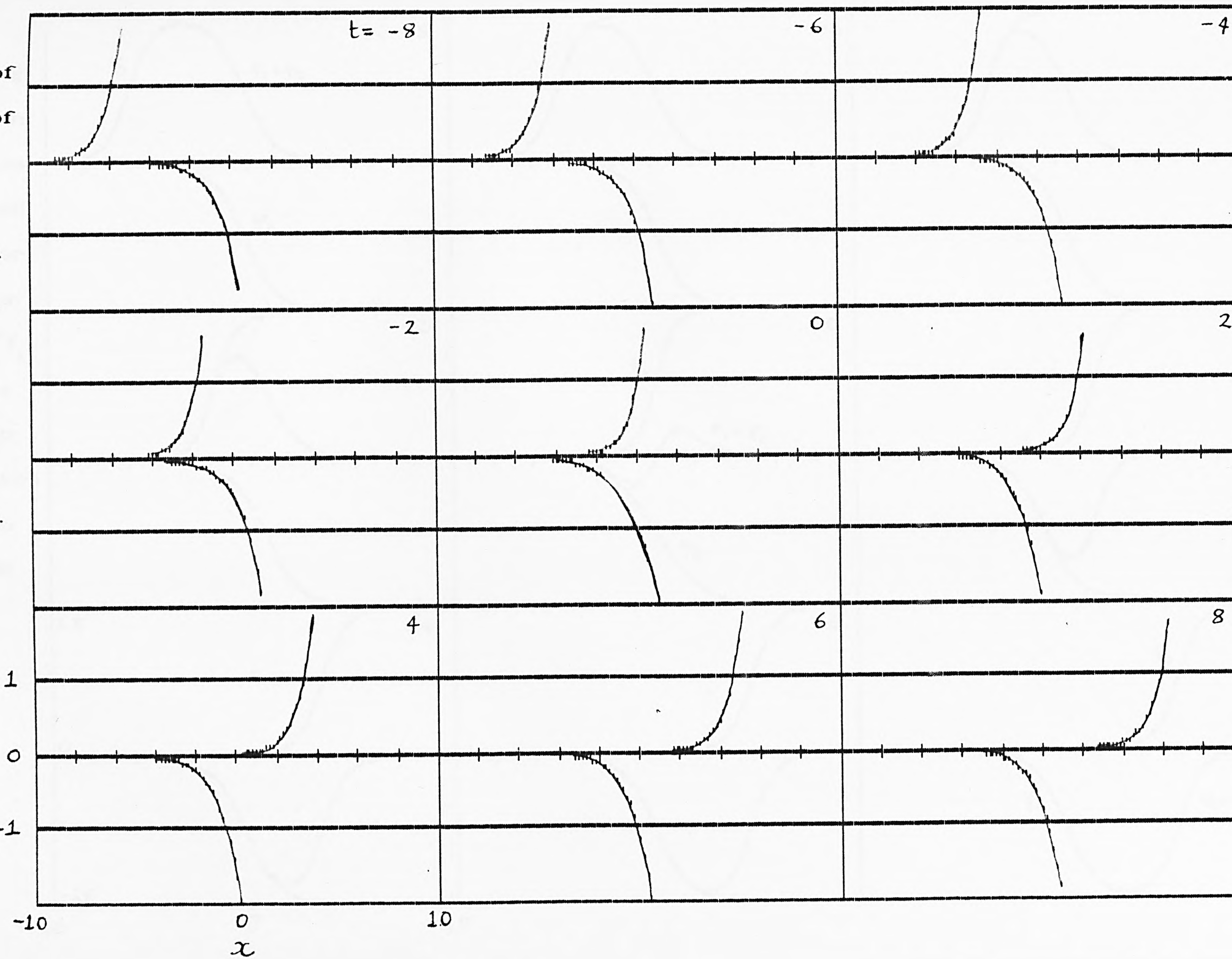


Fig 7.3

Time evolution of the roots (f_i) of the soliton, anti-soliton lsp for the sG equation. Initial speeds, soliton .6, antisoliton .2 . The solitons and antisolitons are located at $|f_i|=1$.

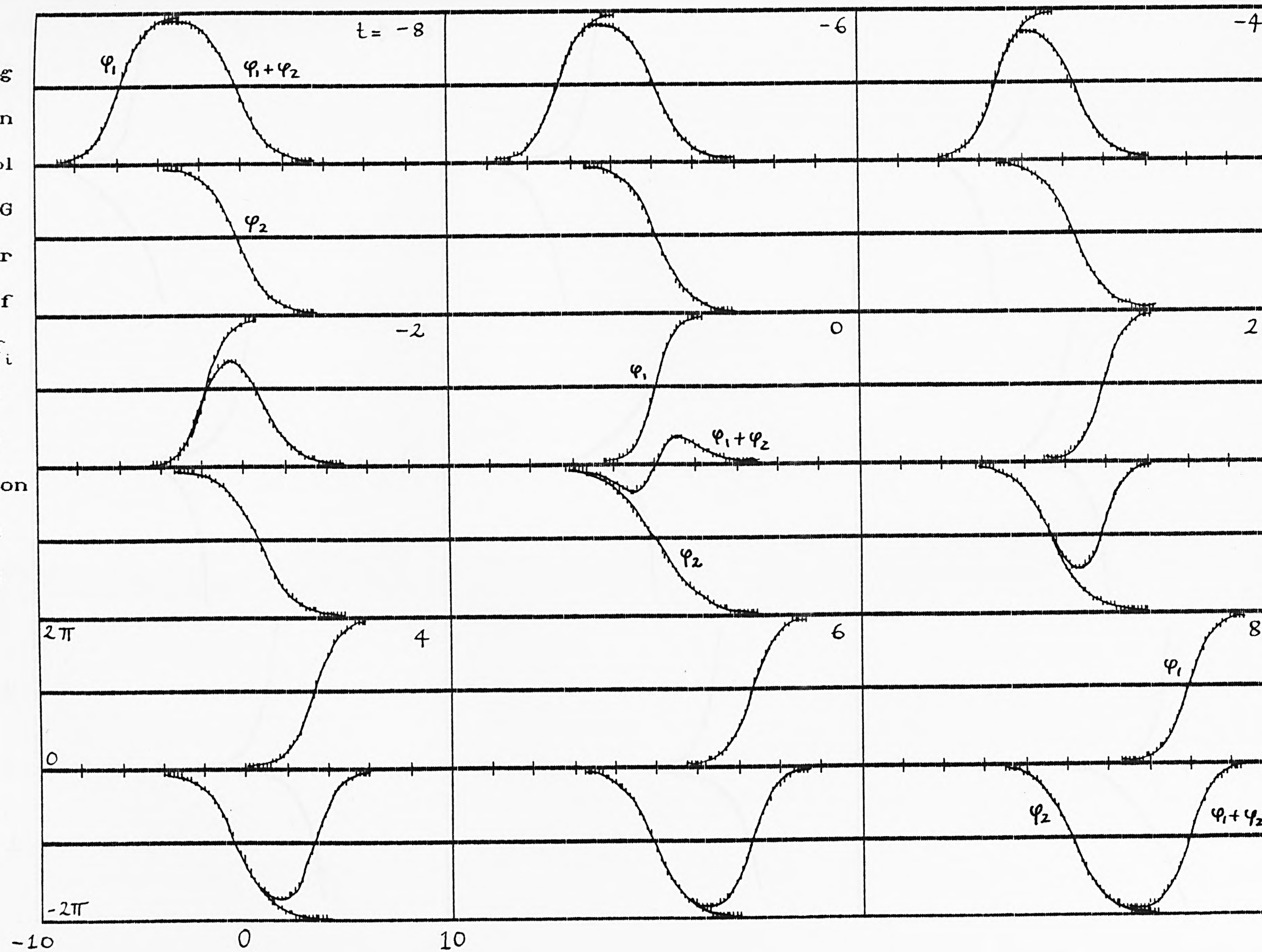


Fig 7.4

A graph showing the time evolution of the sol/antisol solution of the sG as a linear superposition of functions $\varphi_i = 4\tan^{-1} f_i$ where f_i are the roots of the lsp for the soliton /antisol. solution of the sG (speeds as in fig7.3).

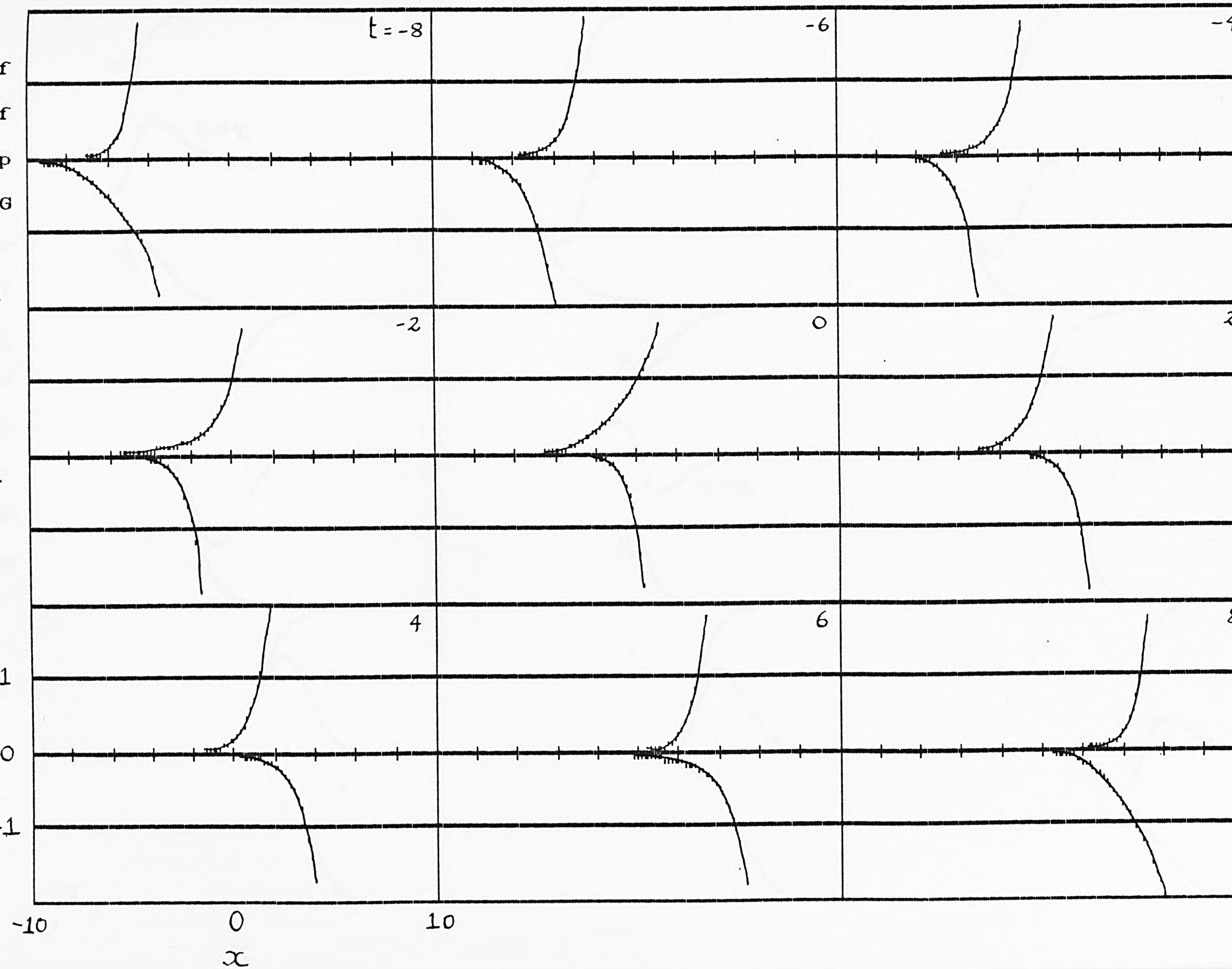


Fig 7.5

Time evolution of the roots (f_i) of the breather lsp for the sG equation. Breather speed .6 ,period 2. .

The solitons and antisolitons are located at $|f_i|=1$.

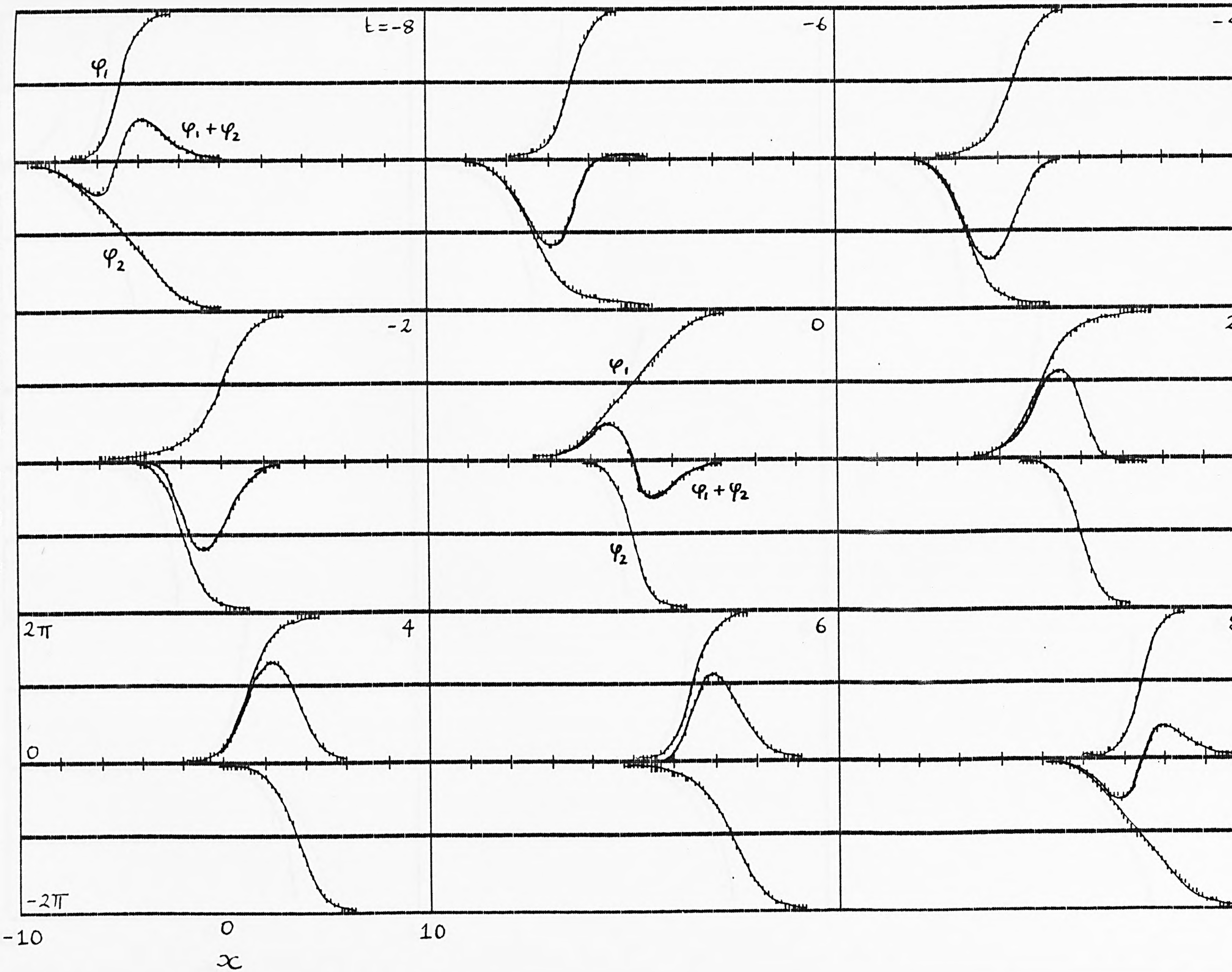


Fig 7.6

A graph showing the time evolution of the breather solution of the sG as a linear superposition of functions $\varphi_i = 4\tan^{-1} f_i$ where f_i are the roots of the lsp for the breather solution of the sG (period & speed as in fig 7.5)

Fig 7.7

Time evolution of the roots (f_i) of the breather lsp for the sG equation, compared with approximation (shown dotted). Breather speed .6 period 2. .

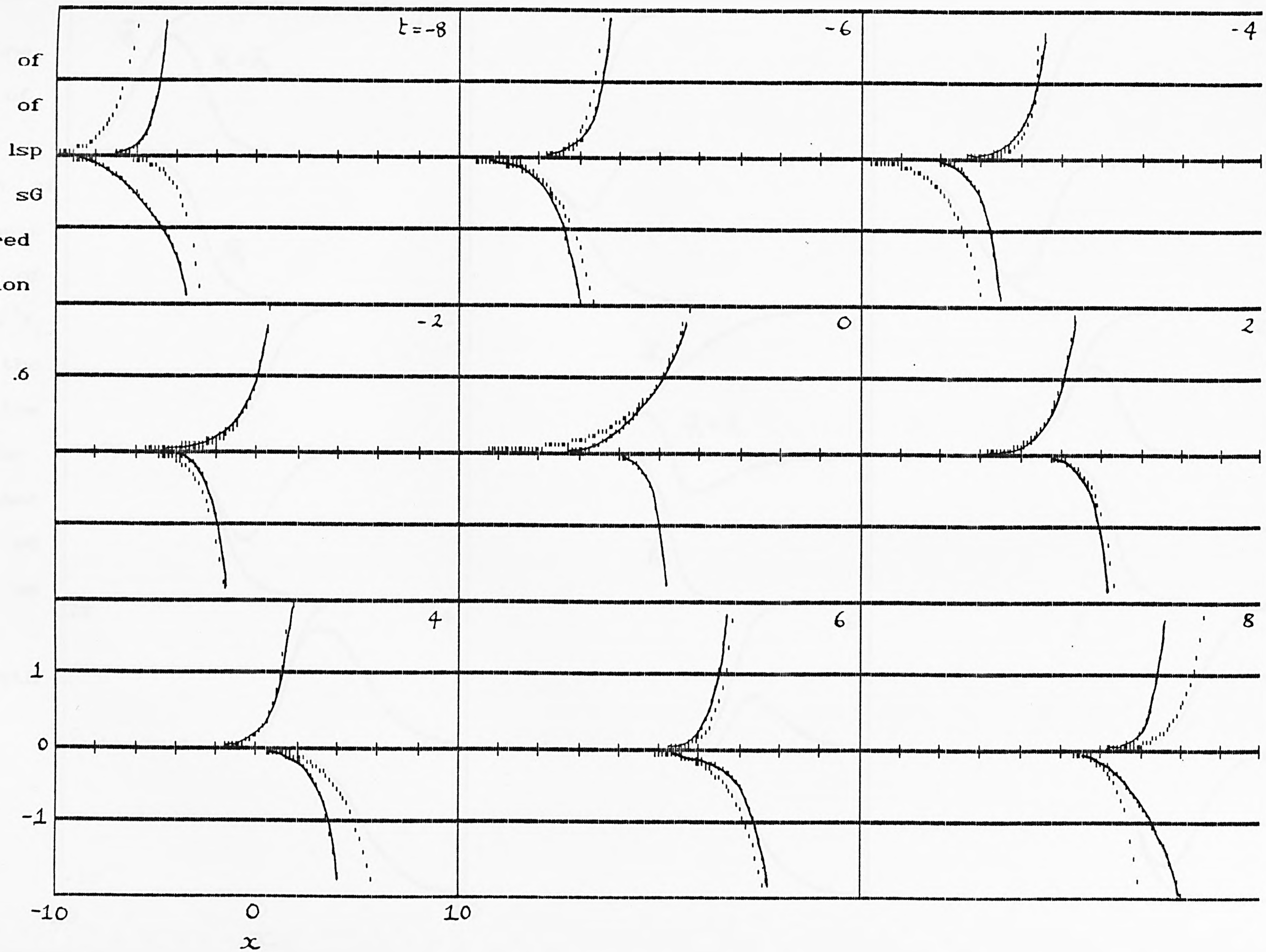
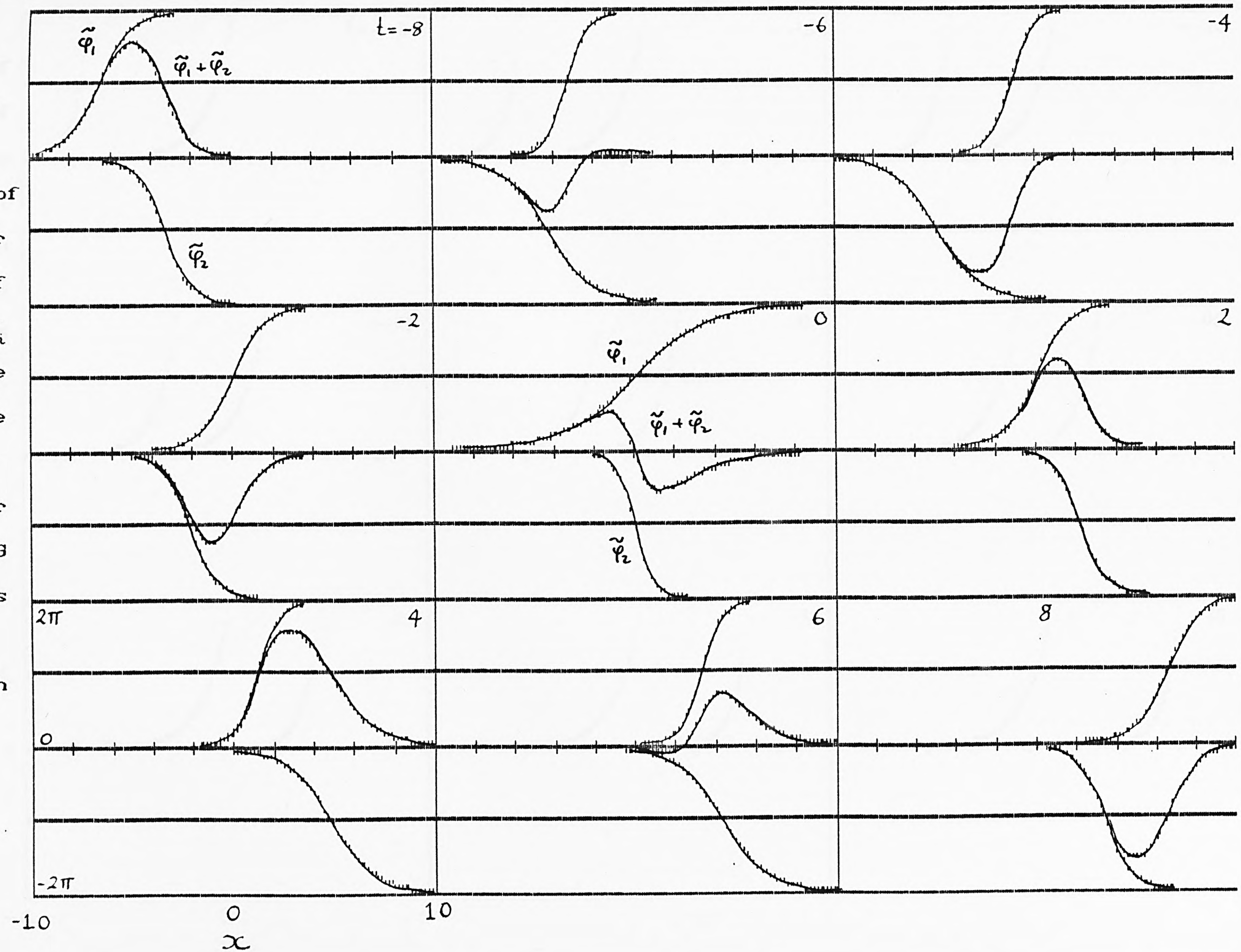


Fig 7.8

A graph showing the time evolution of the approximate breather solution of the sG as a linear superposition of functions $\tilde{\varphi}_i = 4\tan^{-1} f_i$ where $\tilde{f}_i$ are the roots of the approximate lsp for the breather solution of the sG (period & speed as in fig 7.7) Compare this with fig 7.6.



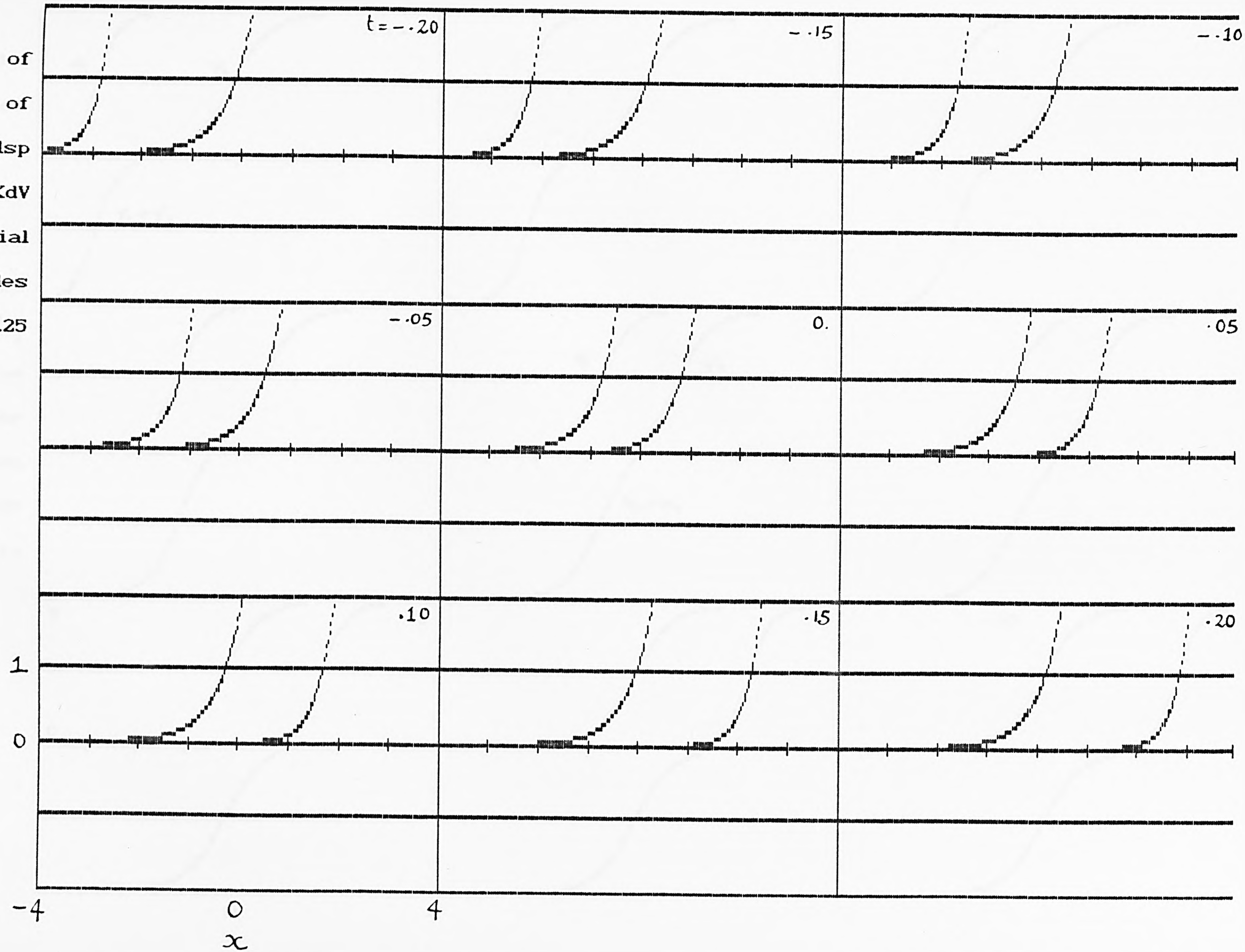


Fig 7.9

Time evolution of the roots (f_i) of the two soliton lsp for the MKdV equation. Initial soliton amplitudes 1.75,1.(speeds 12.25 ,4.)

The solitons are located at f_i=1.

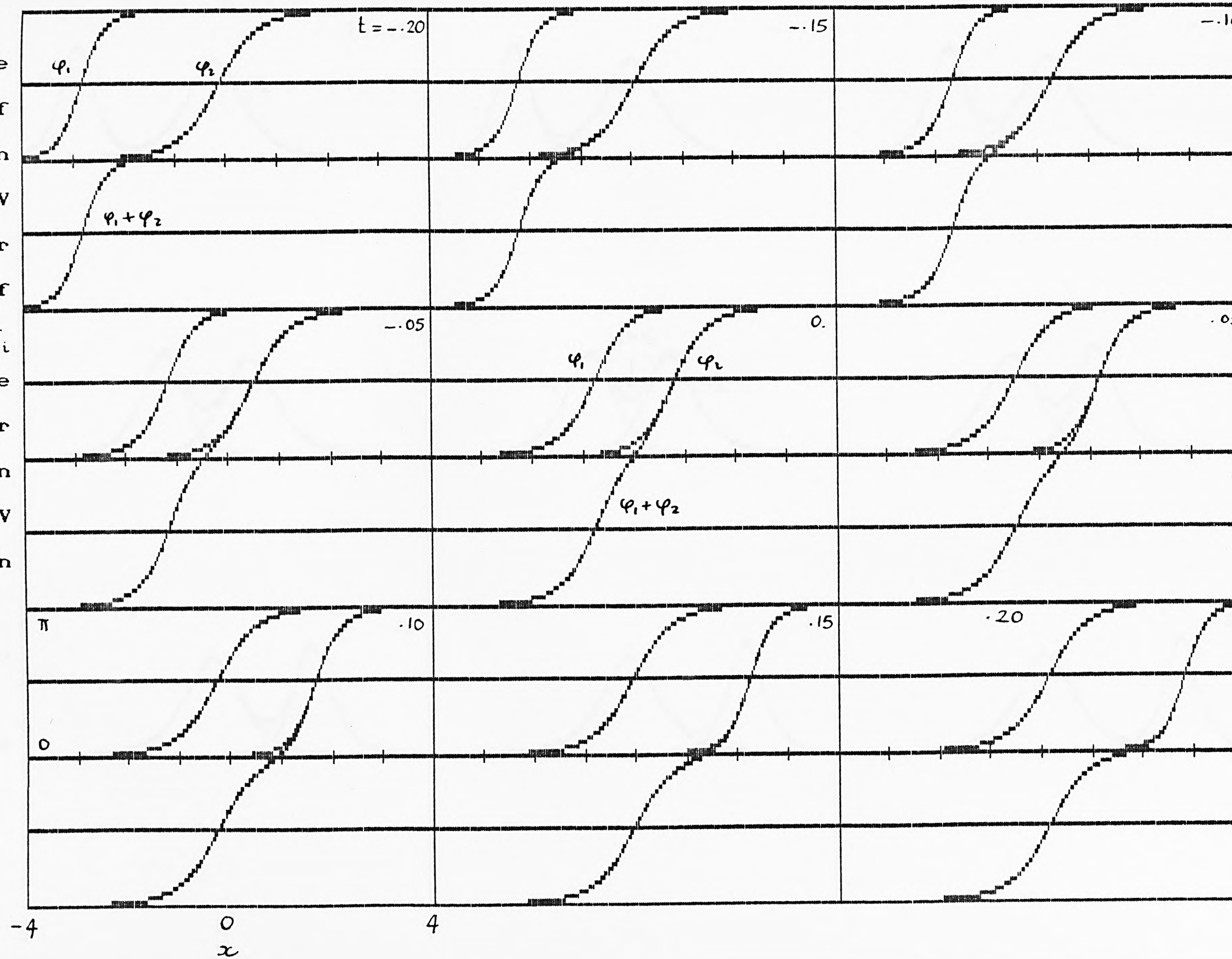


Fig 7.10

A graph showing the time evolution of the two soliton solution of the MKdV as a linear superposition of functions $\varphi_i = 2\tan^{-1} f_i$ where f_i are the roots of the lsp for the two soliton solution of the MKdV (amplitudes as in fig 7.9)

Fig 7.11

A graph showing the time evolution of the two soliton solution of the derivative MKdV as a linear superposition of functions $v_i = g_{ix} \operatorname{sech} g_i$ where $g_i = e^{f_i}$ and f_i are the roots of the lsp for the two soliton solution of the MKdV (amplitudes as in fig 7.9) Y axis scale is nominal.

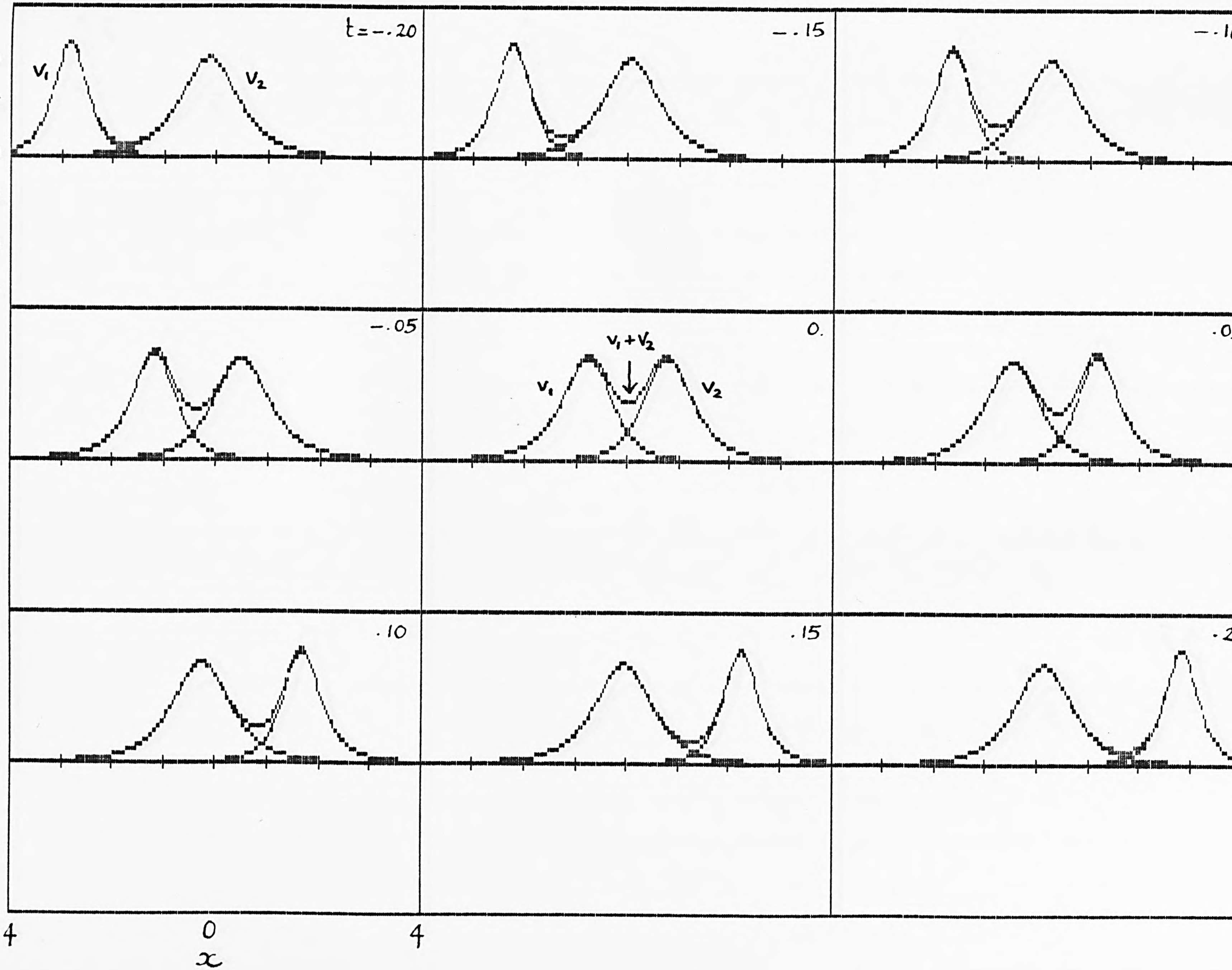


Fig 7.12

As for fig 7.11 but comparing the approximate roots with the exact roots. Solid lines are the exact roots.

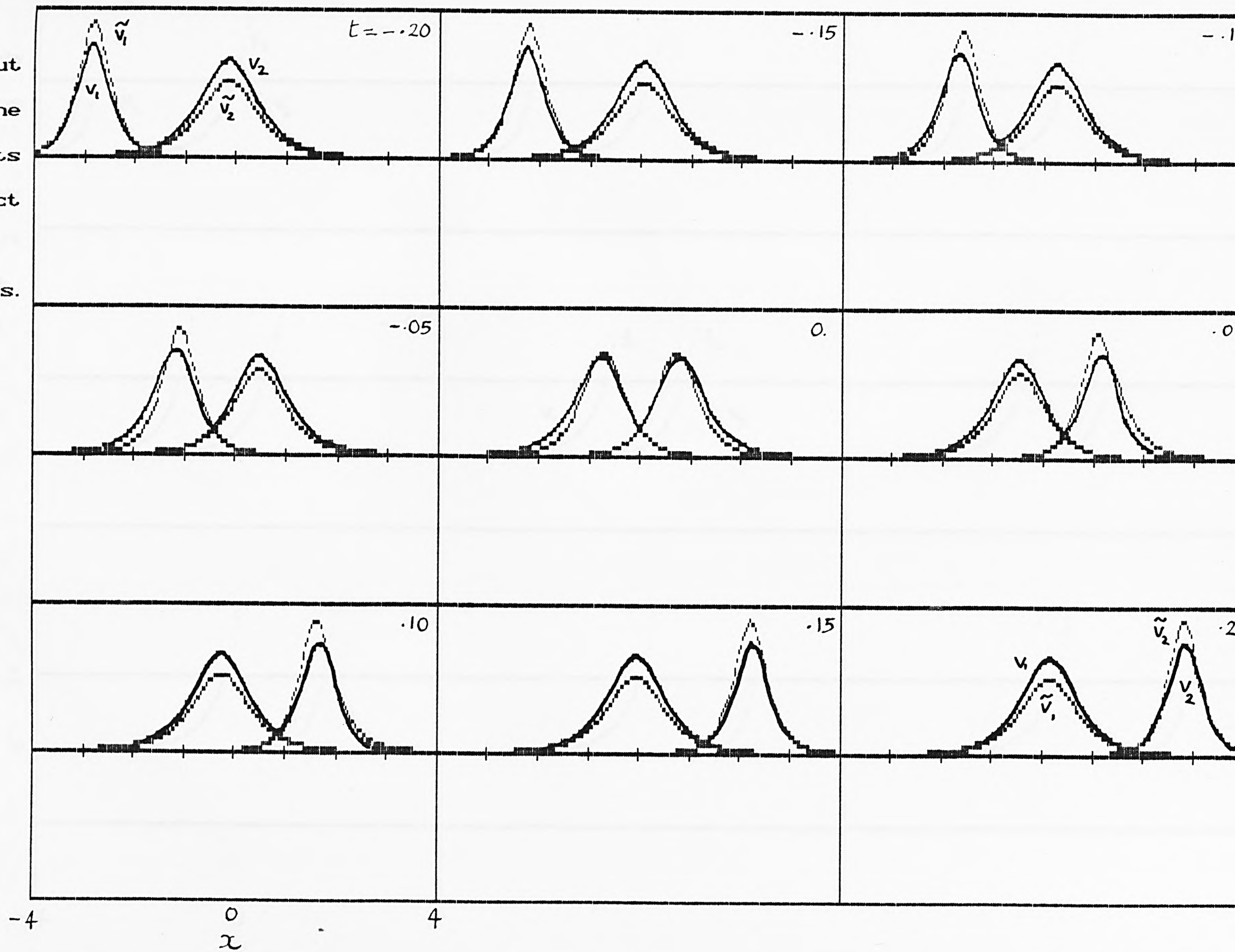


Fig 7.13

As for fig 7.11 only with the functions f_i plotted in addition. Notice how the points where $f_i=1$ appear to coincide with the maxima of $v_i = g_{ix} \operatorname{sech} g_i$. The scale on the y axis refers only to f_i.

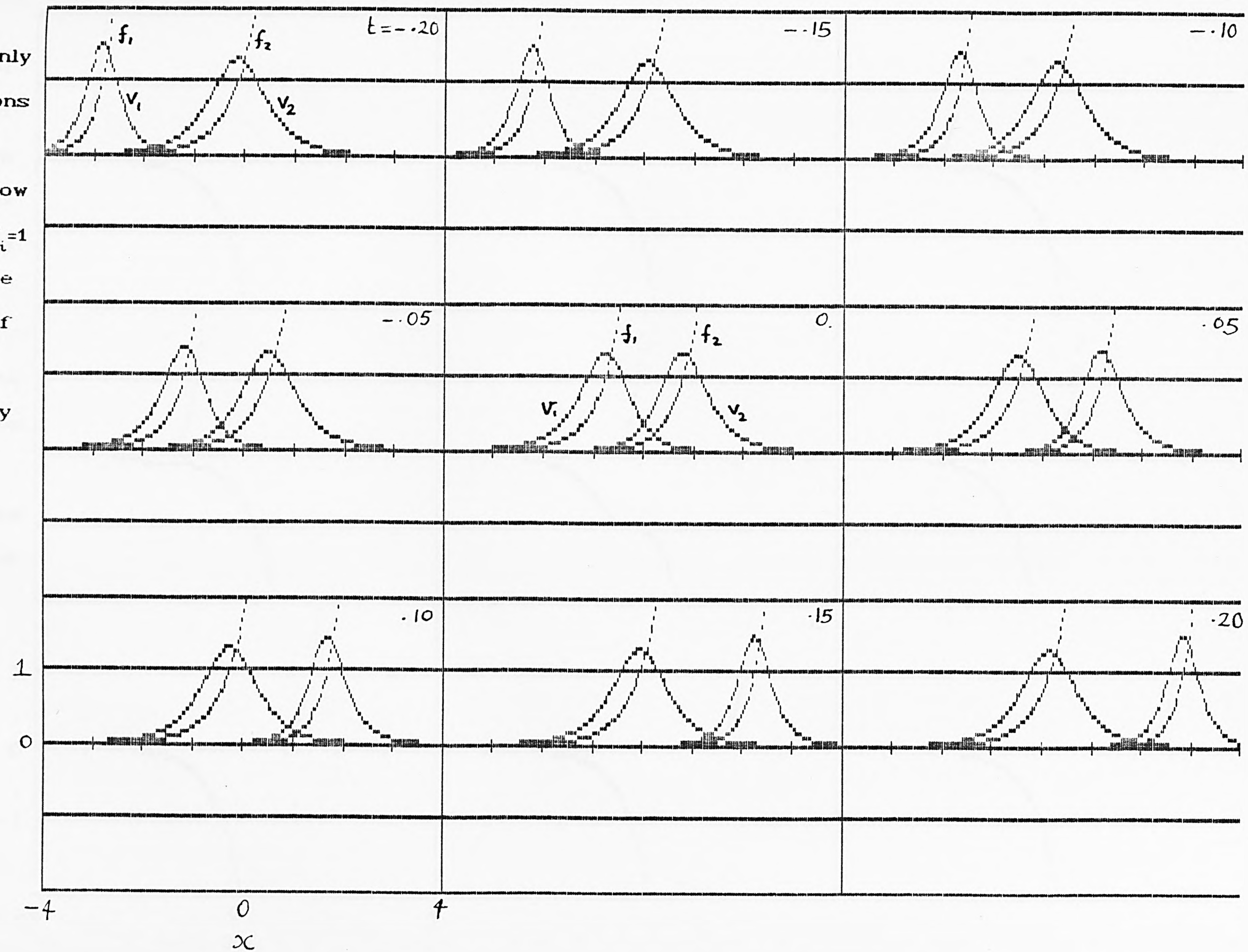
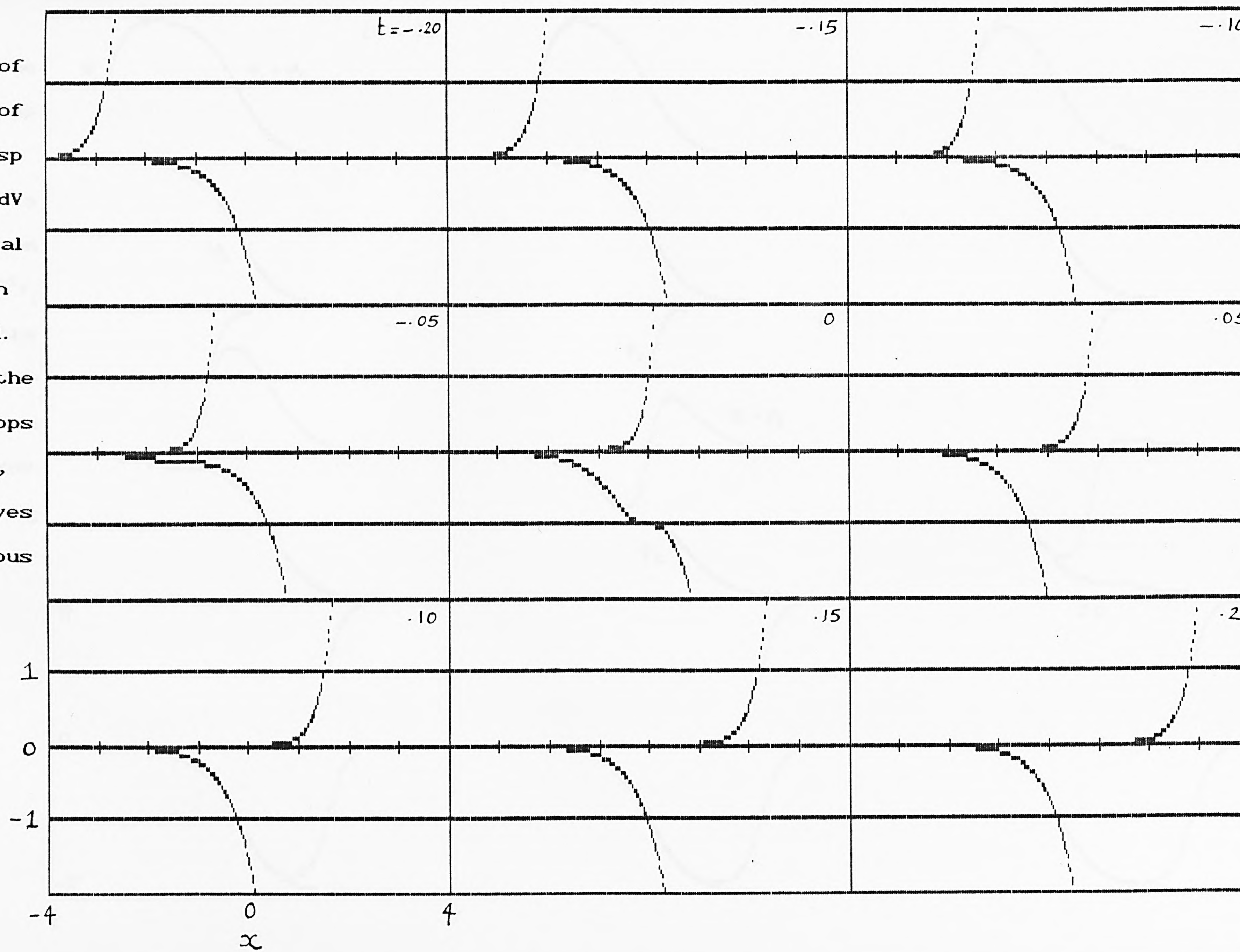


Fig 7.14

Time evolution of the roots (f_i) of the sol/antisol lsp for the MKdV equation. Initial amplitudes ,soliton 1.75,antisoliton 1. Note how the antisoliton develops a point of inflexion, and also how it moves backwards at various times.

The solitons and antisolitons are located at $|f_i|=1$



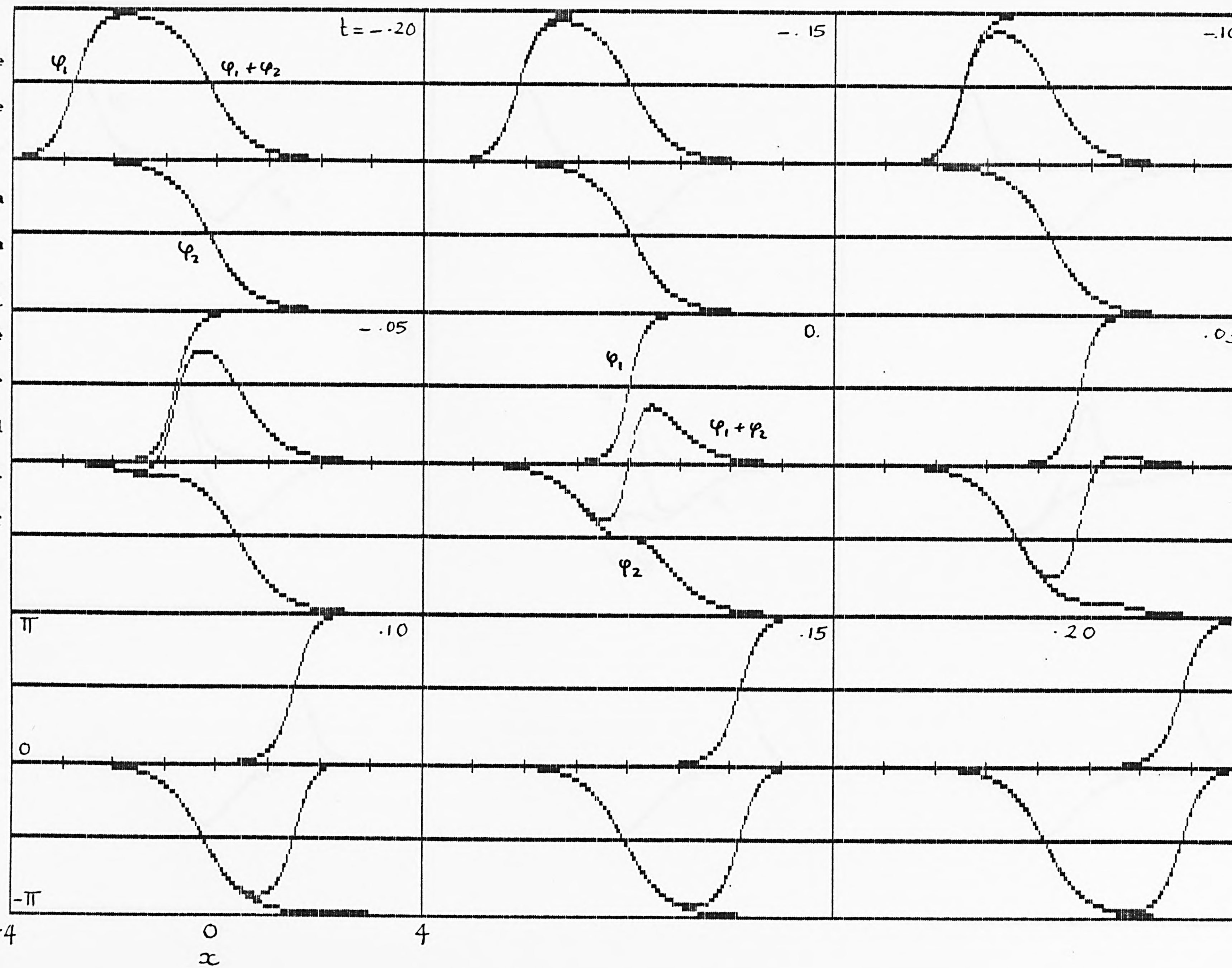


Fig 7.15

A graph showing the time evolution of the sol/antisol solution of the MKdV as a linear superposition of functions $\varphi_i = 2\tan^{-1} f_i$ where f_i are the roots of the lsp for the soliton /antisol solution of the MKdV. (amplitudes as in fig 7.14).

Fig 7.16

As for fig 7.14 except that we have plotted the soliton/antisoliton solution of the derivative MKdV as a linear superposition of parts $V_i = g_{ix} \operatorname{sech} g_i$. The dotted lines are the parts and the solid line is that part of the sum which is not indistinguishable from the parts.

The derivative MKdV is the equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0$$

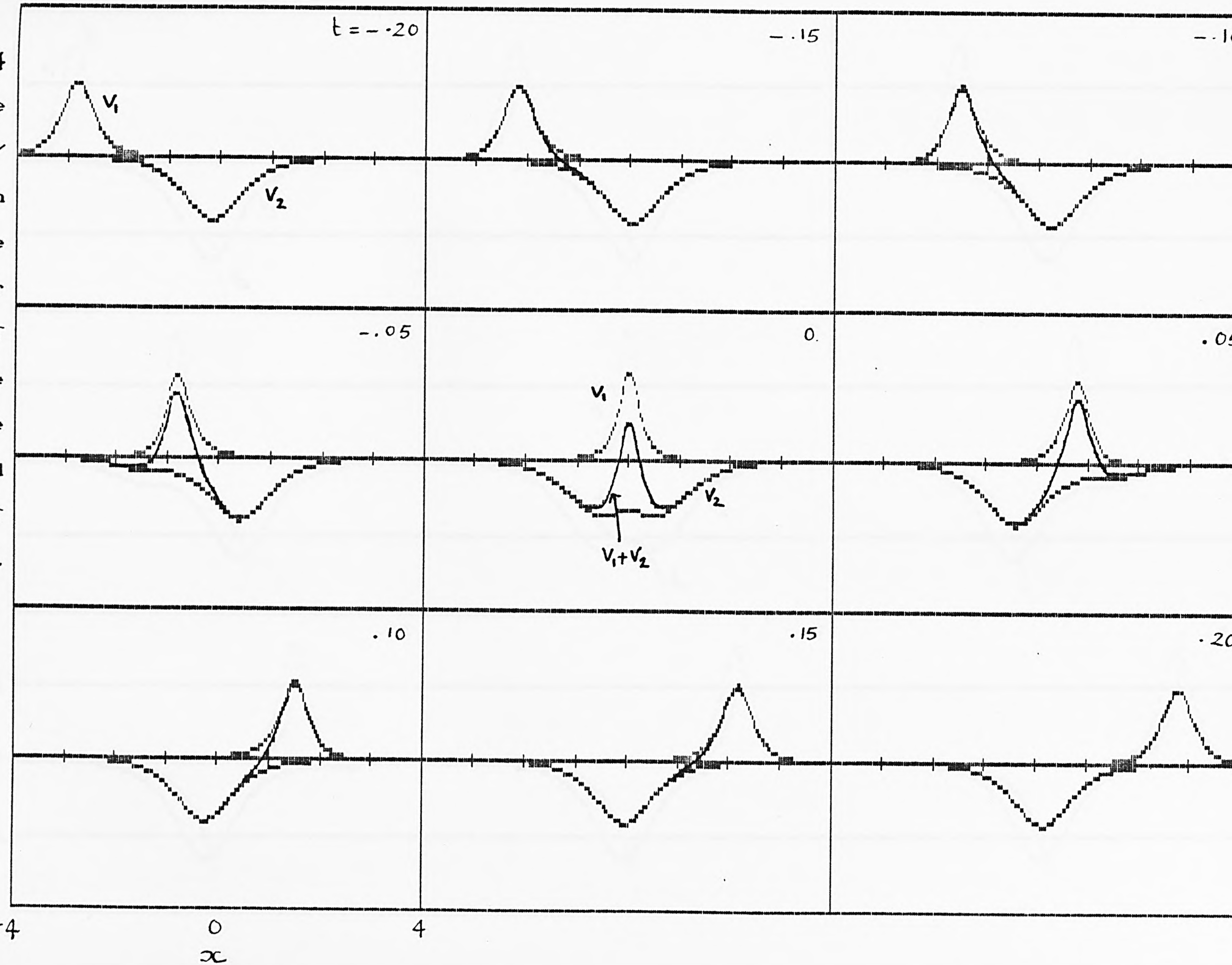
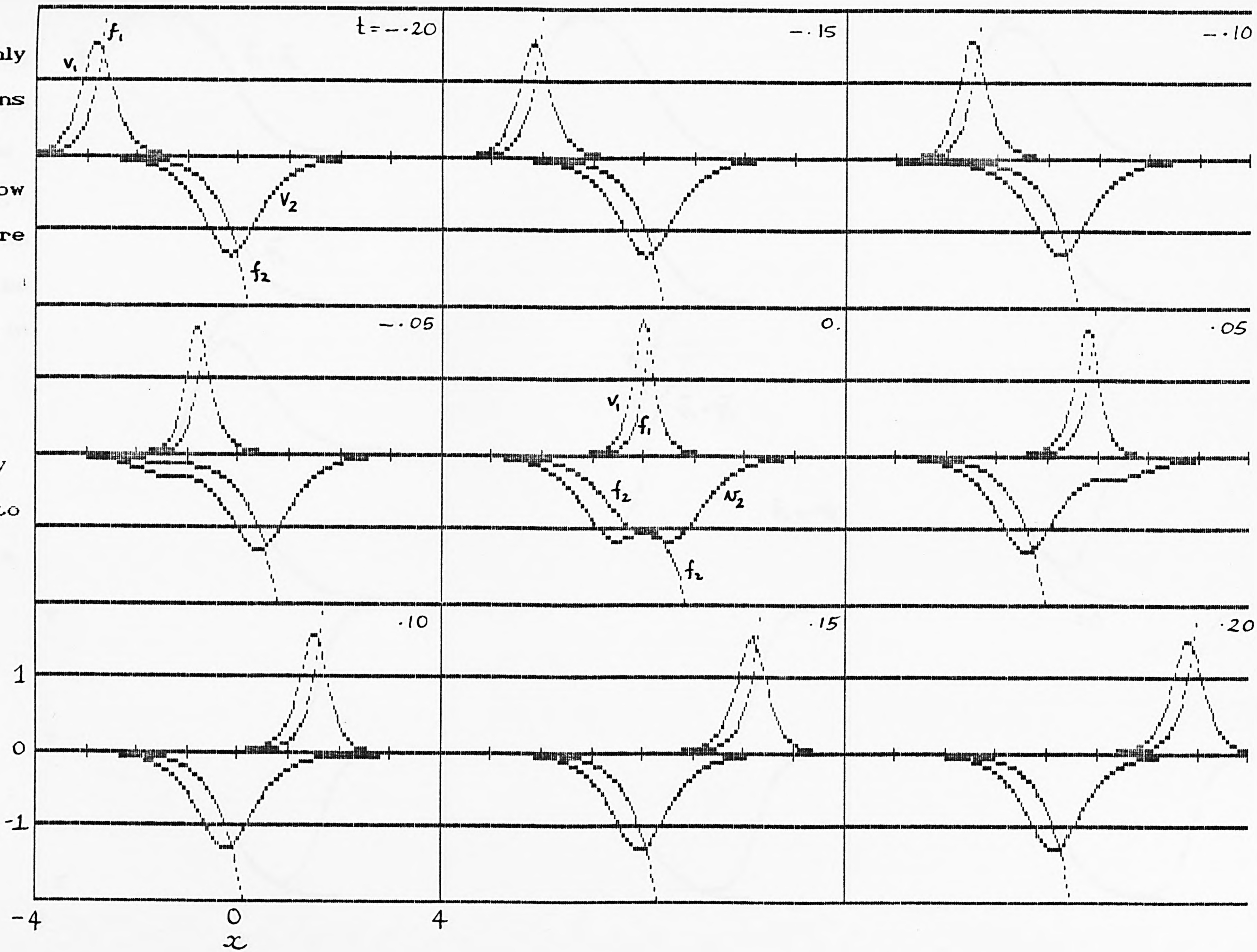


Fig 7.17

As for fig 7.16 only with the functions f_i plotted in addition. Notice how the points where $f_i = \pm 1$ appear to coincide with the maxima or minima of $v_i = g_{ix} \operatorname{sech} g_i$. The scale on the y axis refers only to f_i.



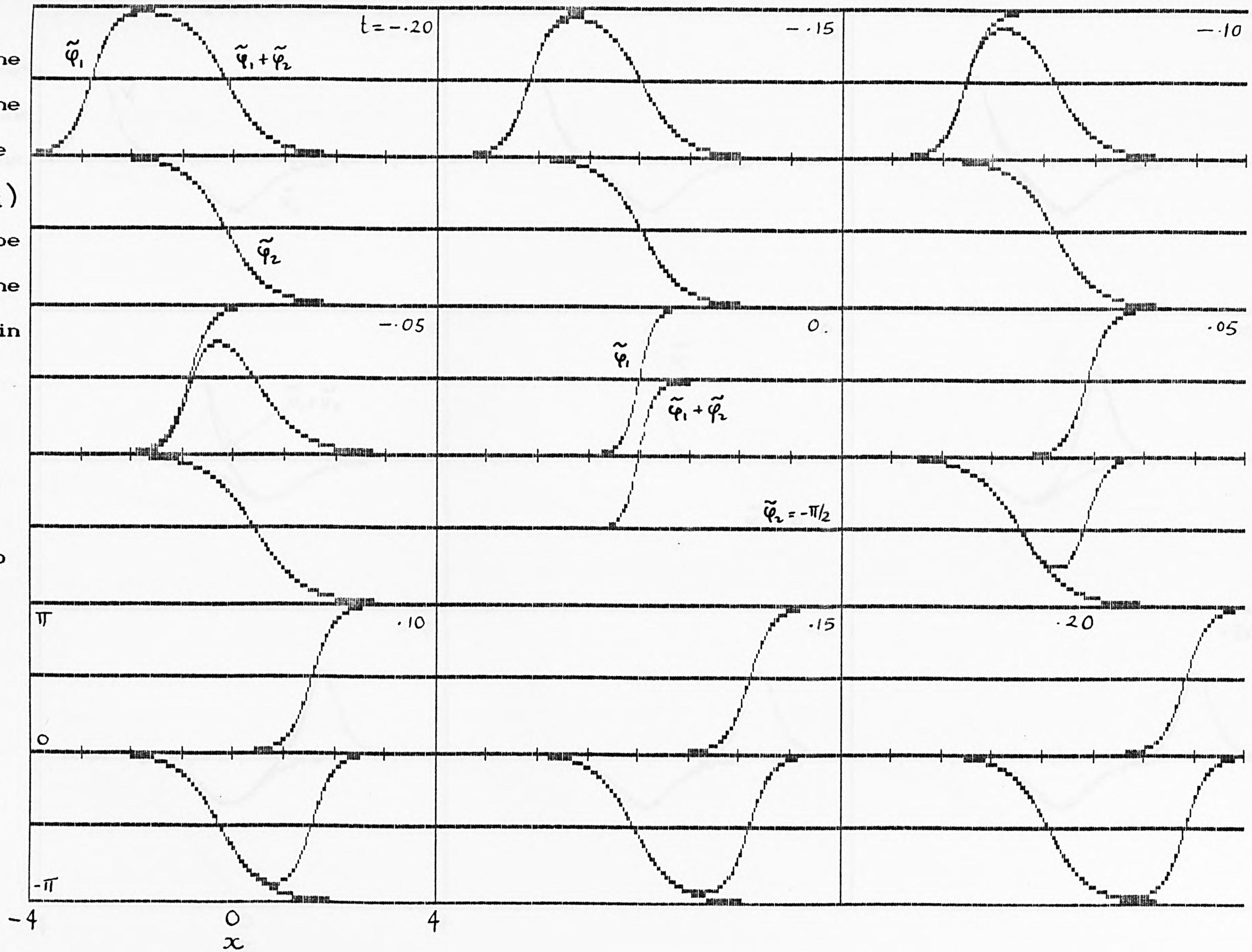


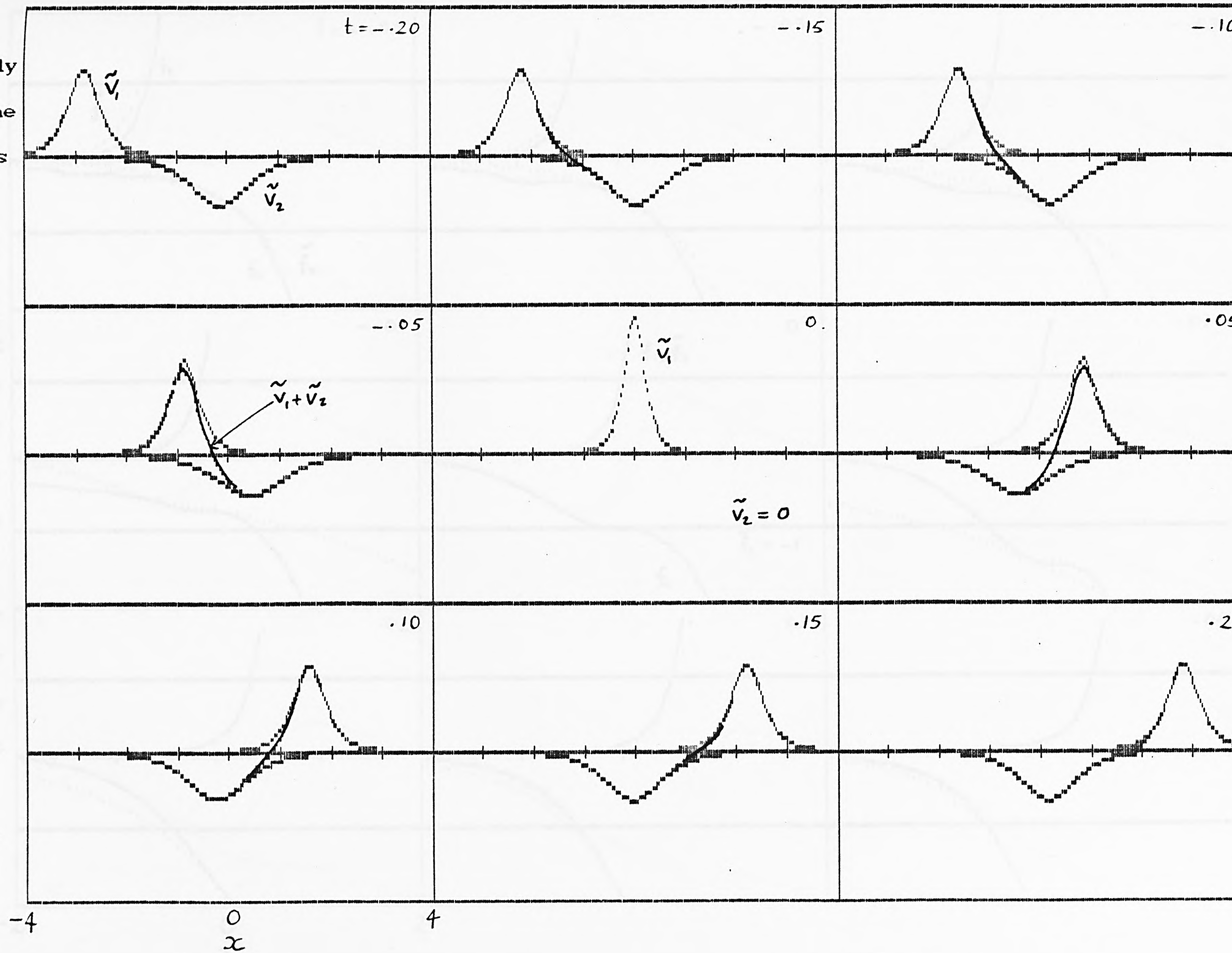
Fig 7.18

A graph showing the time evolution of the approximation to the functions $2\tan^{-1}f_i$ $(\tilde{\varphi}_i)$ This should be compared with the exact functions in fig 7.15.

The approximation departs little from the exact solution except when close to time t=0.

Fig 7.19

As for fig 7.16 only we have plotted the approximate functions



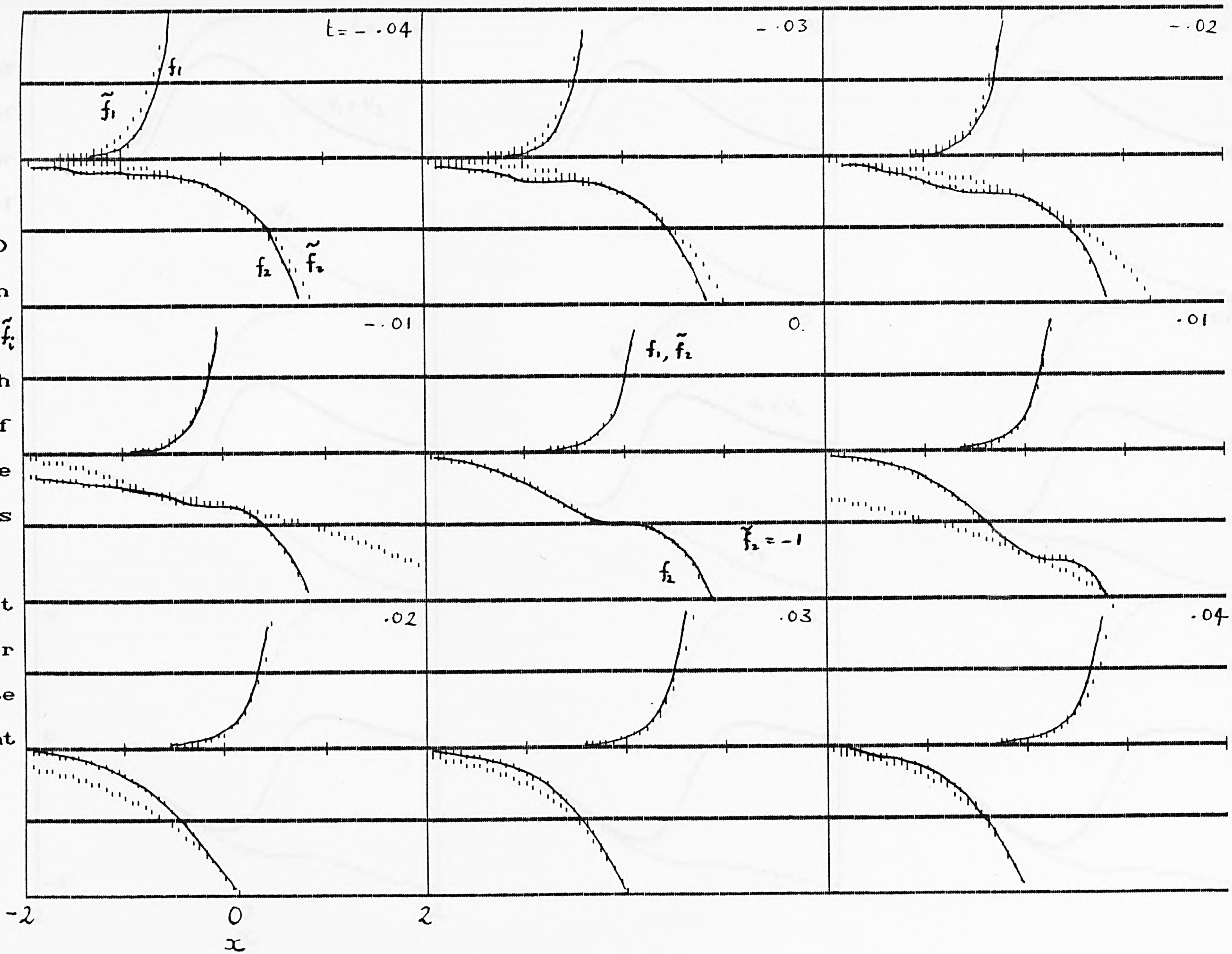


Fig 7.20 Time evolution of the roots (f_i) of the sol/antisol lsp for the MKdV equation(solid lines) as compared with an approximation(dotted), $\tilde{f}_i$. We have chosen a much smaller range of times as the approximation is indistinguishable from the exact functions for larger times. The approximate function equals -1 at time t=0. Amplitudes as in fig 7.14.

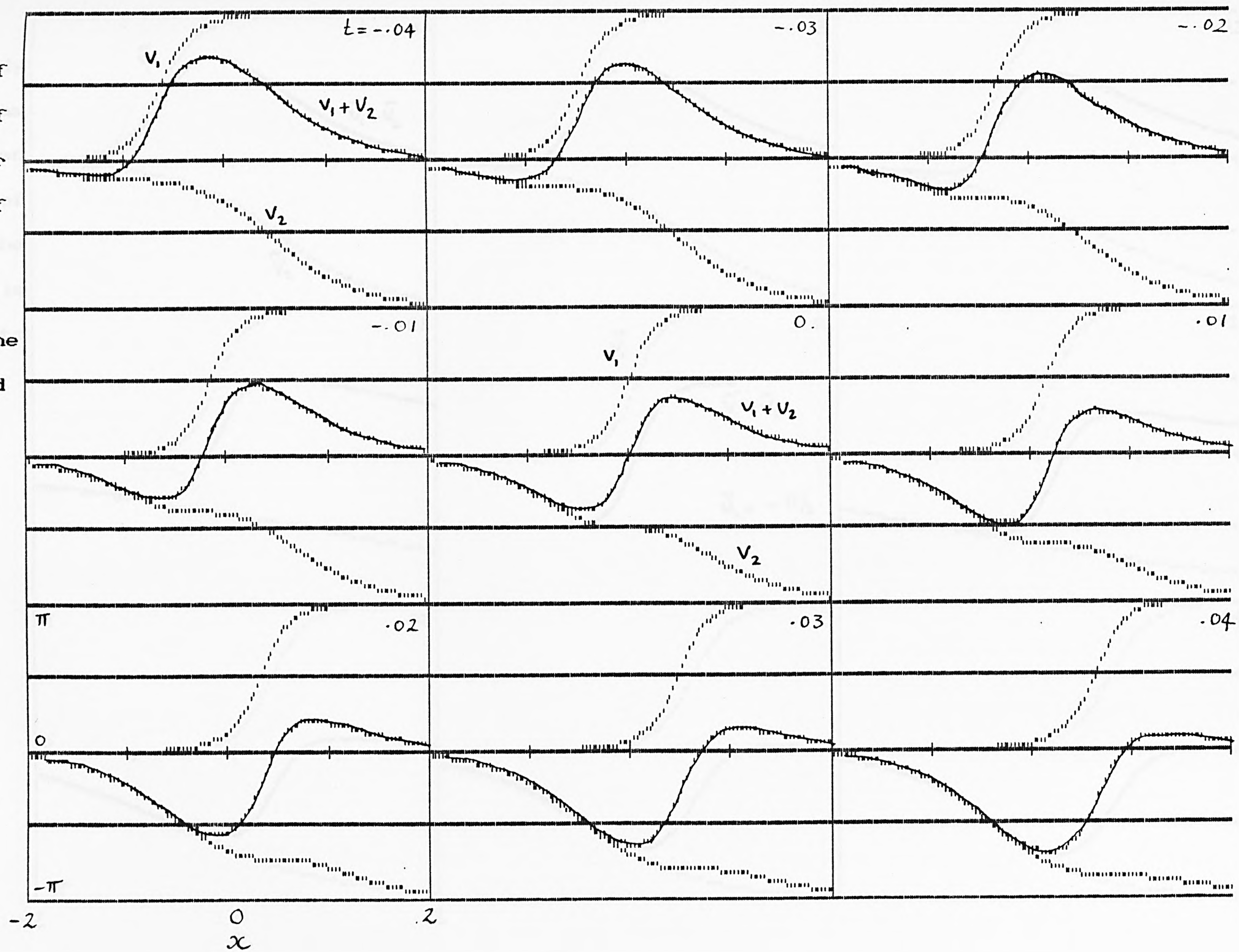
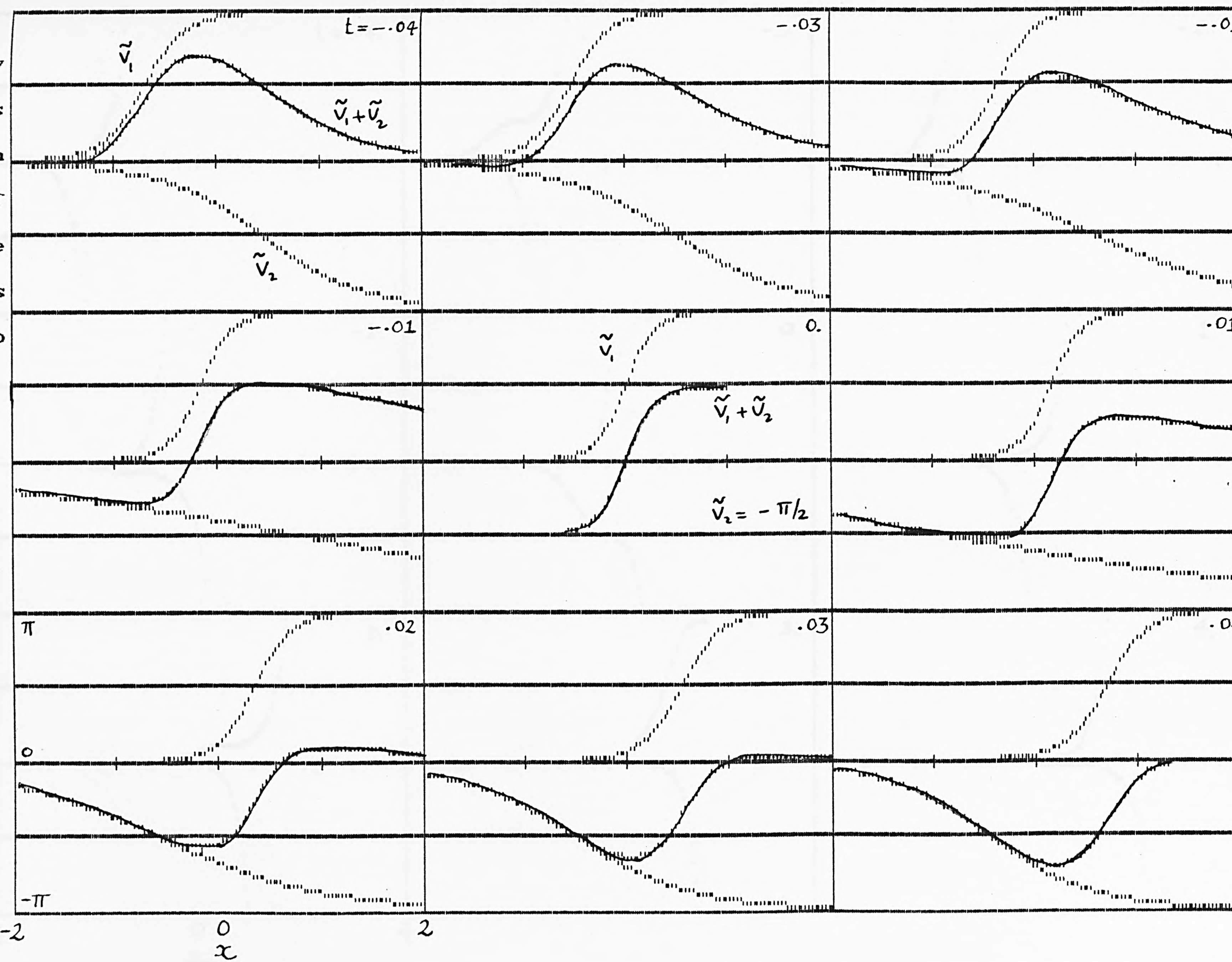


Fig 7.21

Time evolution of exact sol/asoliton of MKdV as a linear superposition of parts $V_i = 2\tan^{-1} f_i$. Amplitudes as before. The solid curve is the sum of the dotted parts.

Fig 7.22

As for fig 7.21 only the approximation has been plotted.Even over this sequence of times the approximation is still quite close to the exact solution.



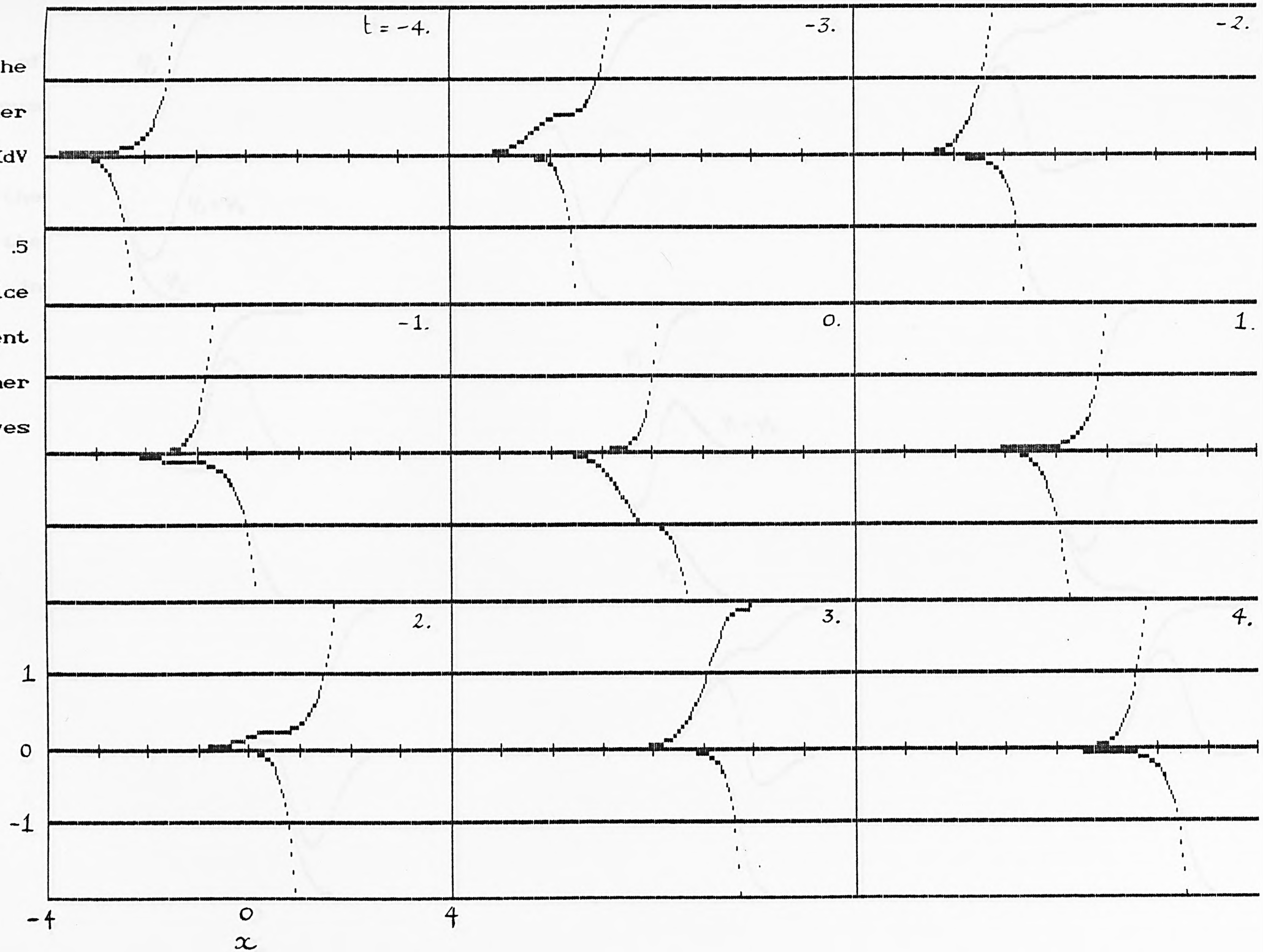


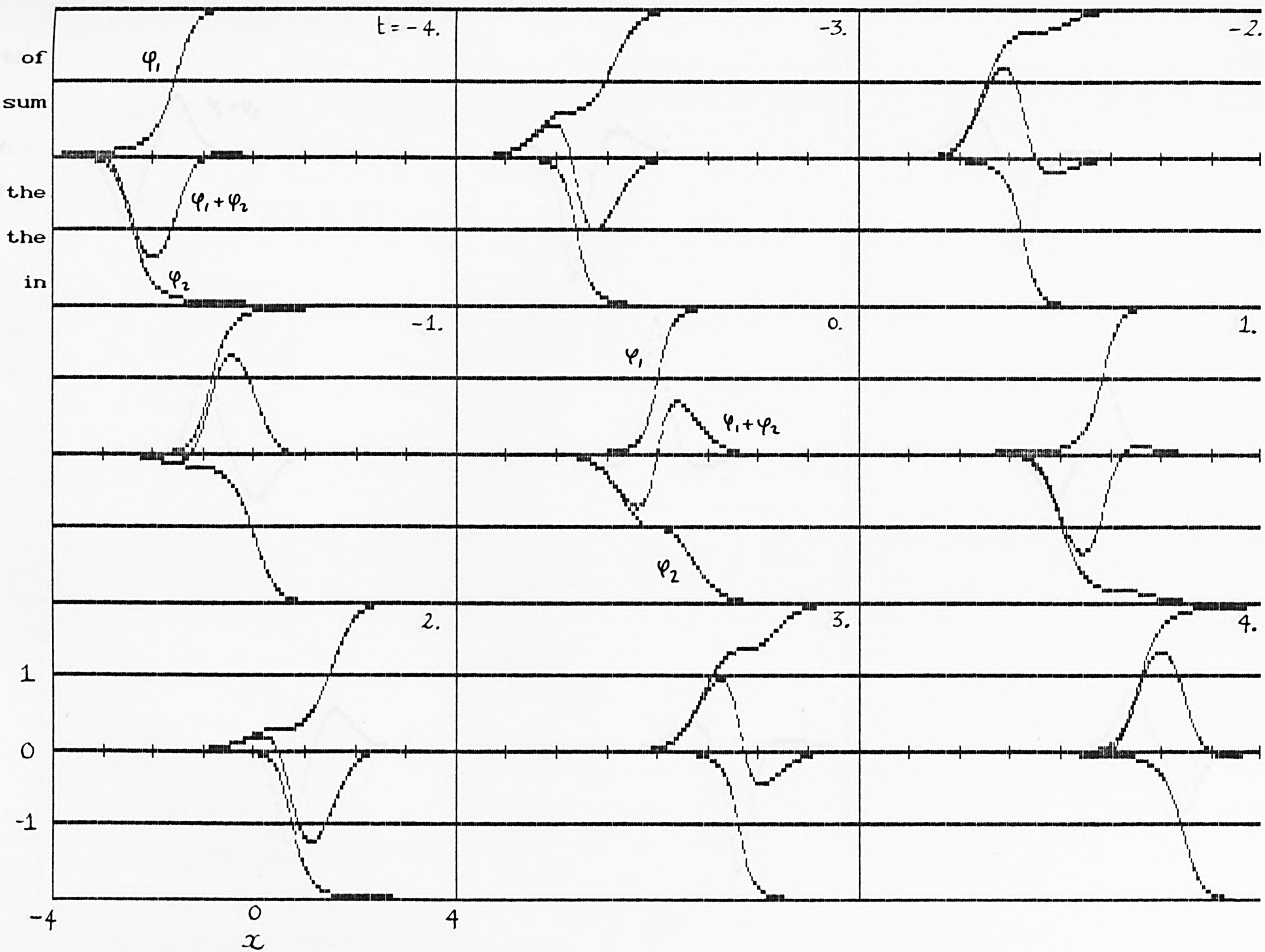
Fig 7.23

Time evolution of the roots of the breather lsp for the MKdV equation. Breather moving at speed .5 with period 4. Notice how as each component overtakes the other the latter moves backwards.

Also note the periodic presence of stationary points of inflexion(§2).

Fig 7.24

Time evolution of $2\tan^{-1} f_i$ and their sum (the breather solution of the MKdV). f_i are the functions plotted in fig 7.23



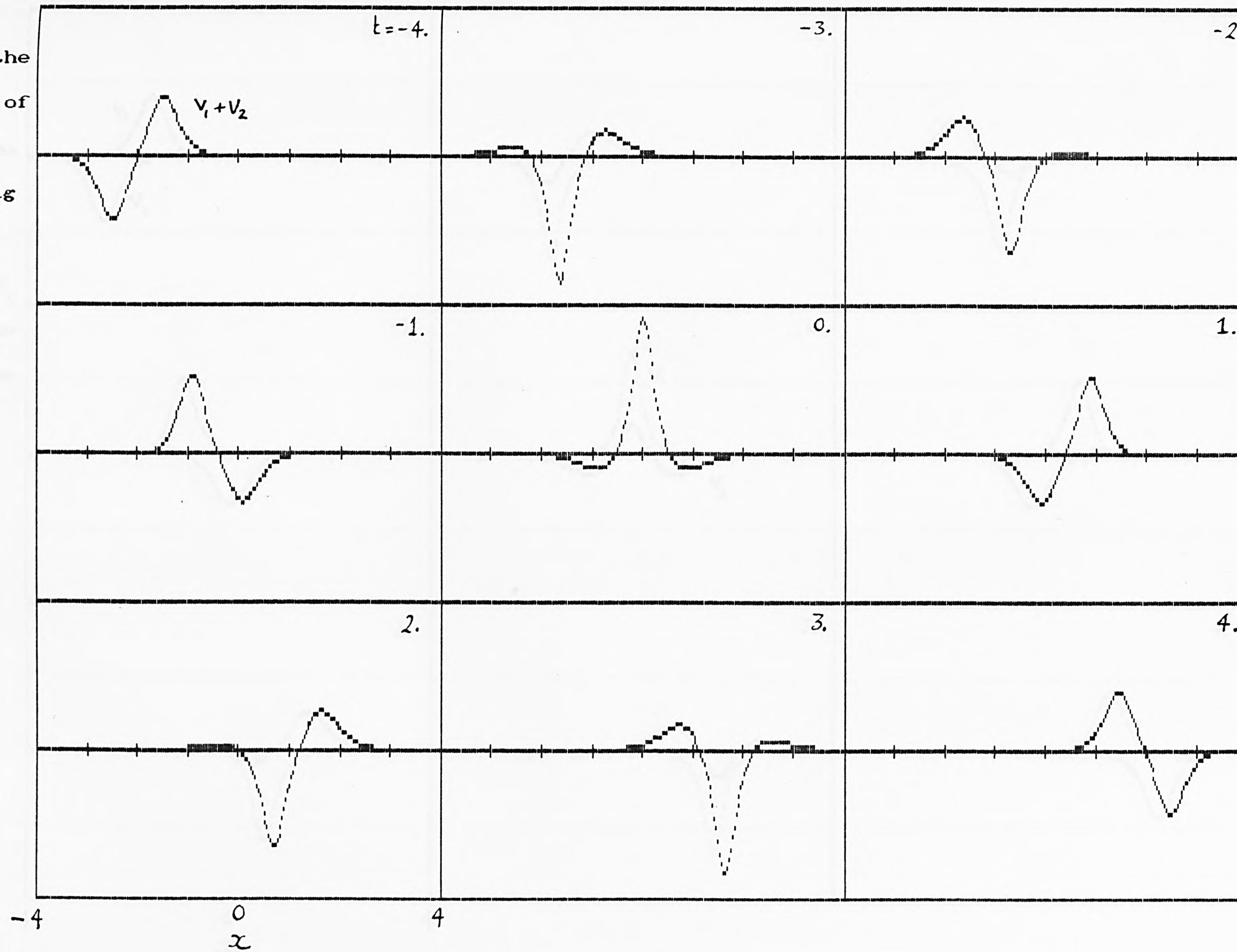


Fig 7.25

Time evolution of the breather solution of the derivative MKdV. Parameters as in fig 7.23-4.

The derivative MKdV is the equation,

$$v_t+6v^2v_x+v_{xxx}=0$$

Fig 7.26

The breather solution of fig 7.24 decomposed into a sum of terms of form, $V_i = g_{ix} \operatorname{sech} g_i$ where $g_i = \ln f_i$. f_i are the roots of the breather lsp for the MKdV

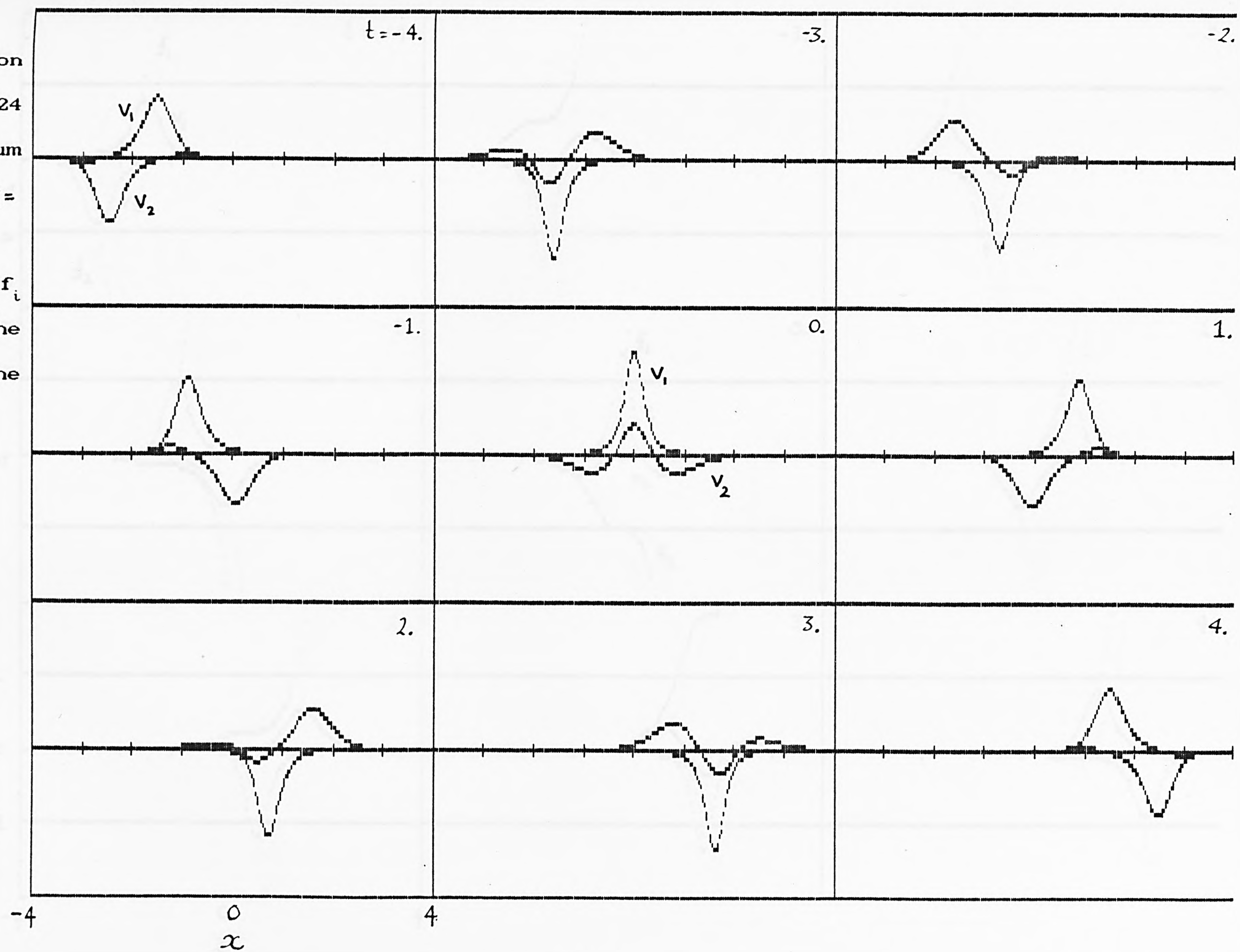


Fig 7.27

The components of the derivative MKdV breather and f_i (fig 7.23)are superimposed Note that the points where $f_i = \pm 1$ are always located near (but not neccessarily coincident with) maxima or minima of the breather components.

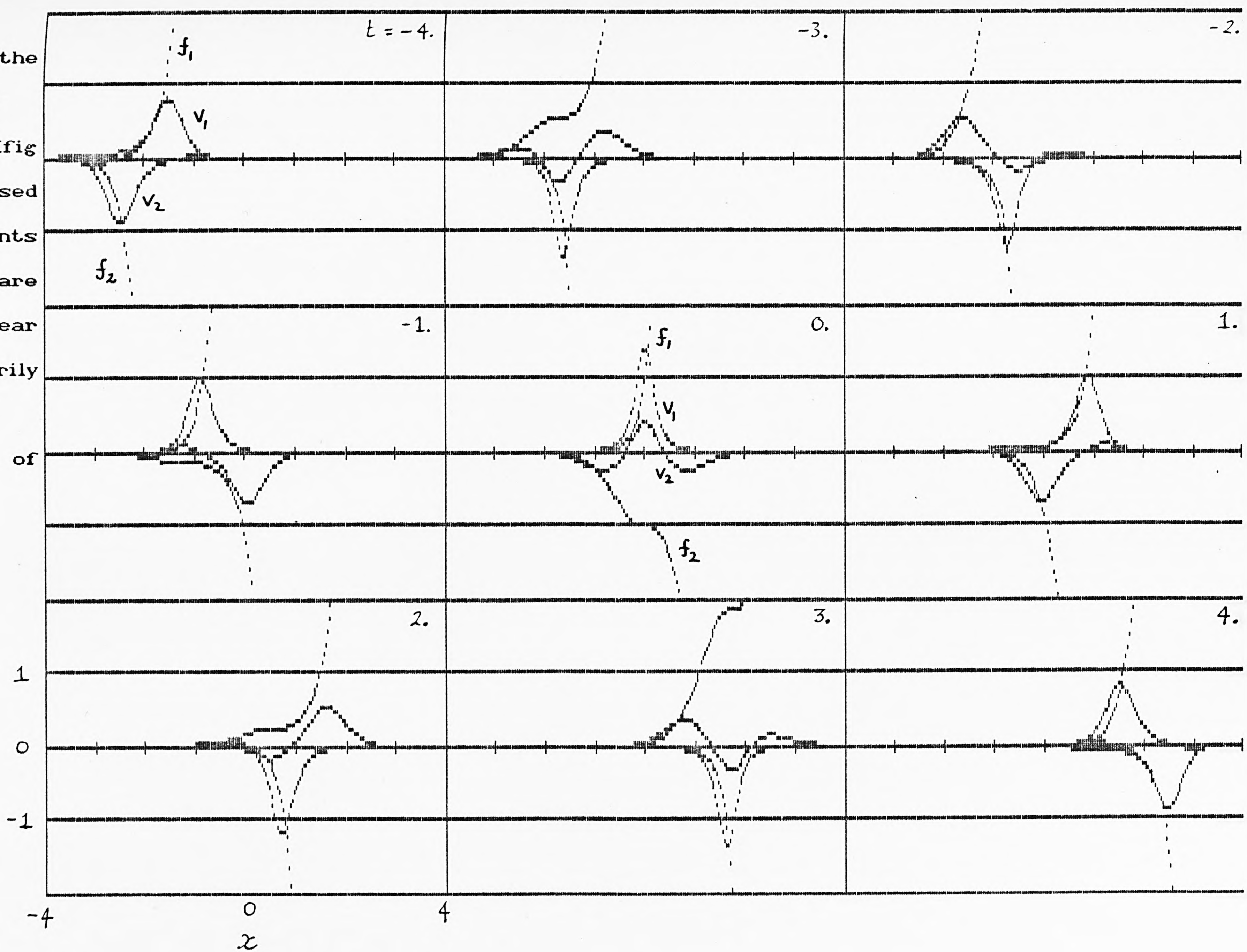


Fig 7.28

The roots of the MKdV breather lsp are compared with an approximation(shown dotted).Clearly the approximation can become very innacurate at certain times.

The parameters are the same as fig 7.23.

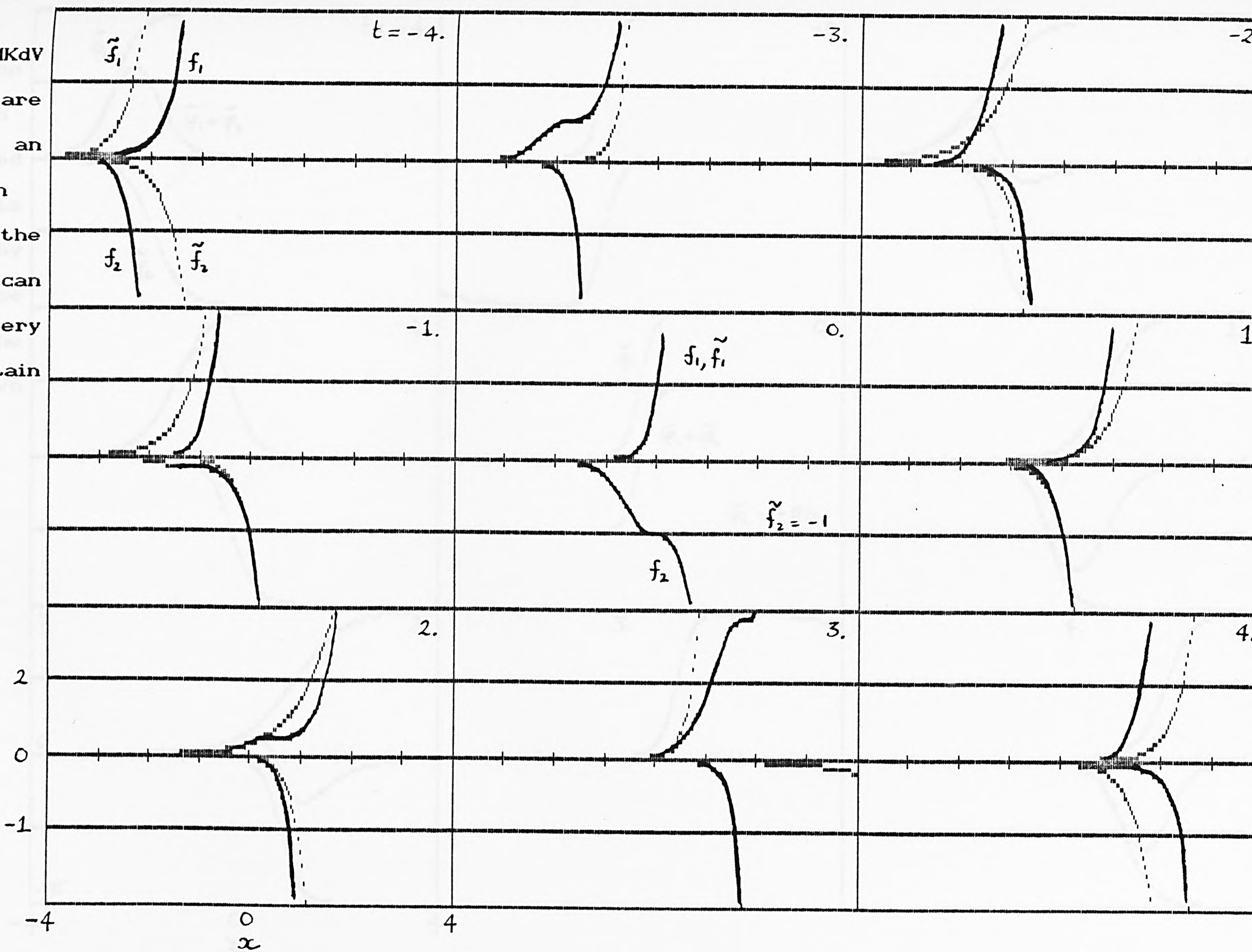


Fig 7.29

The approximation functions plotted in Fig 7.28 are plotted as inverse tangents multiplied by two. These should be compared with the exact functions drawn in fig 7.24.

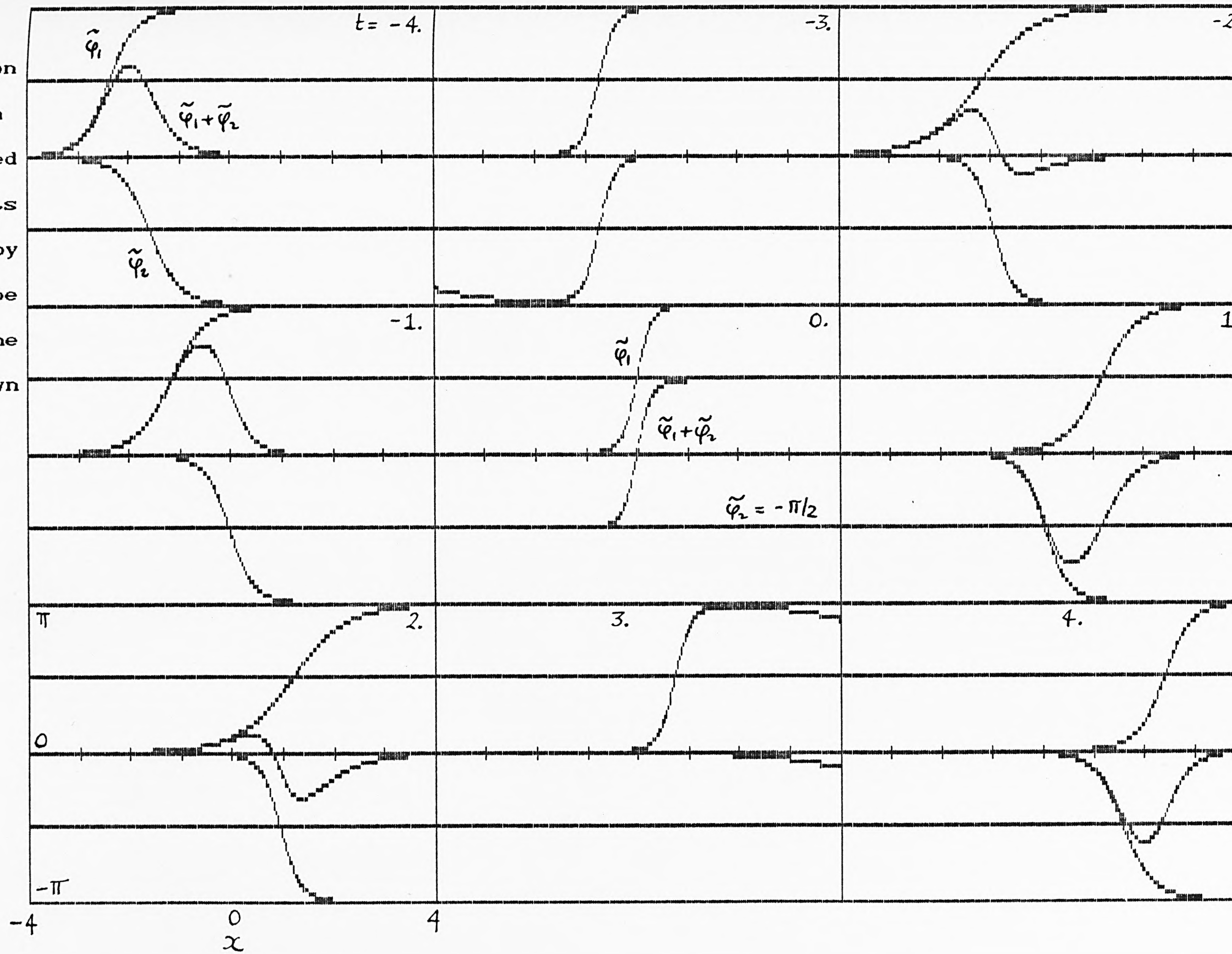


Fig 7.30

Using the approximate solutions drawn in Fig 7.28 .The approximate breather solution of the derivative MKdV has been drawn for various times.Note the scale has been increased over that used in fig7.26.This was necessary as on the same vertical scale as in fig 7.26 the curves in this figure could not be contained in the regions defined.

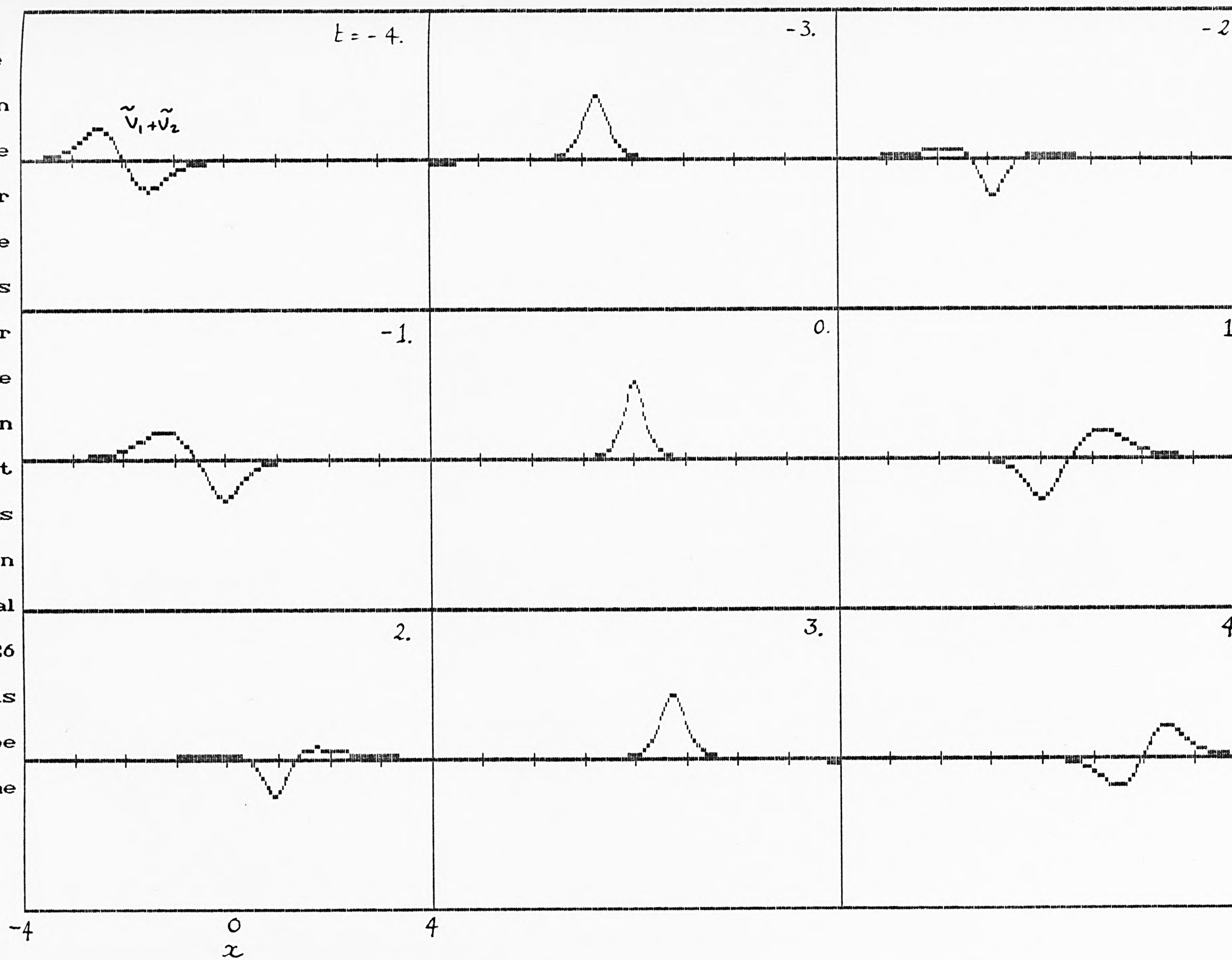
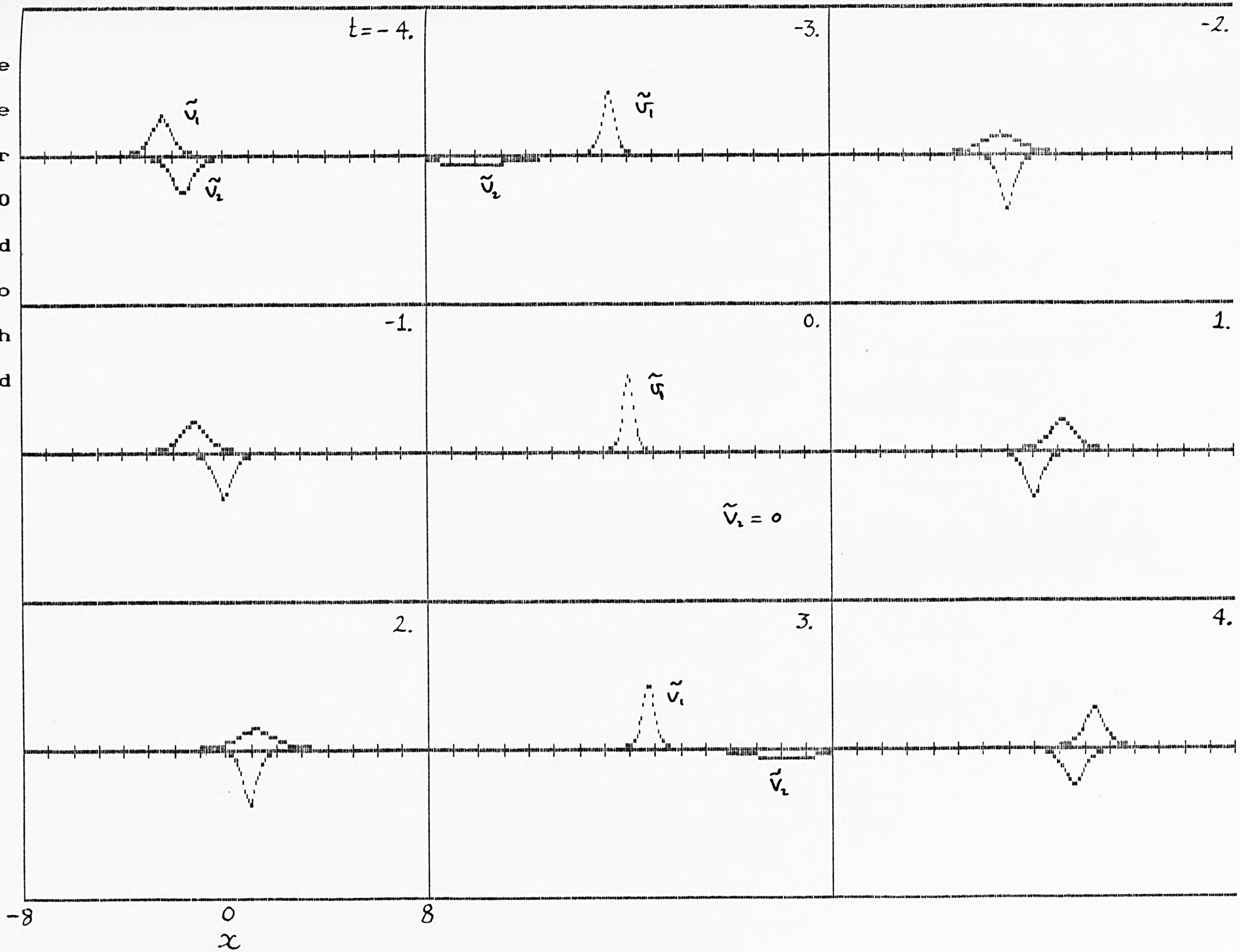


Fig 7.31

Over a larger range of x values the approximate breather solution of fig 7.30 is drawn decomposed into a sum of two parts.This graph should be compared with fig 7.26.



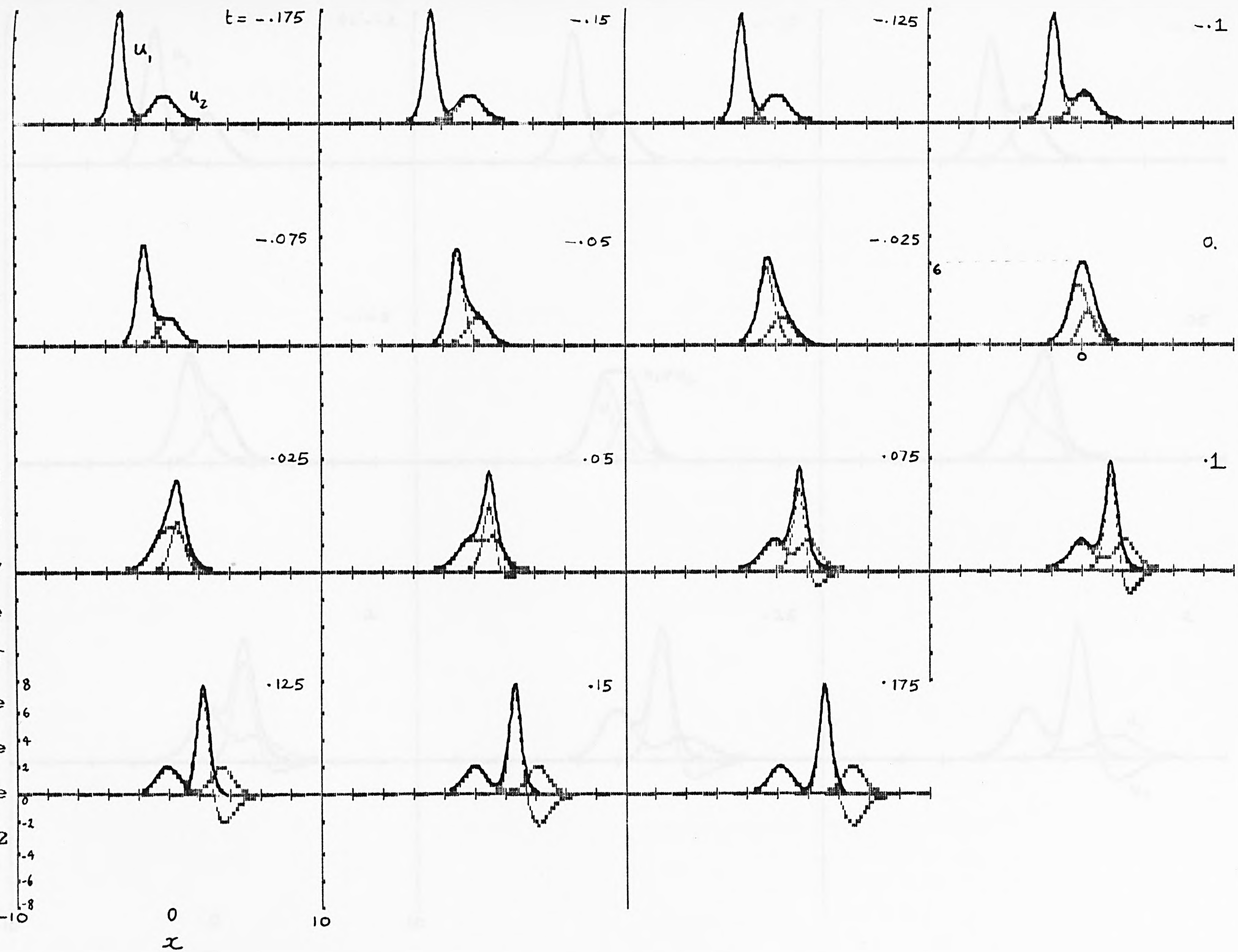


Fig 7.32

Two soliton solution of the KdV as a linear superposition of accelerating solitons(dotted).The soliton amplitudes are 2.,1..Note the peculiar double peak which appears on the smaller soliton.This rightmost peak is almost completely cancelled by the negative portion of the taller peak.The small positive remainder defines the tail of the actual 2 soliton solution.

Fig 7.33a

The two soliton solution of the KdV with amplitudes 1.68,1. is decomposed into two accelerating solitons.Amplitudes have been chosen to fall within the Lax pulse emission mode.

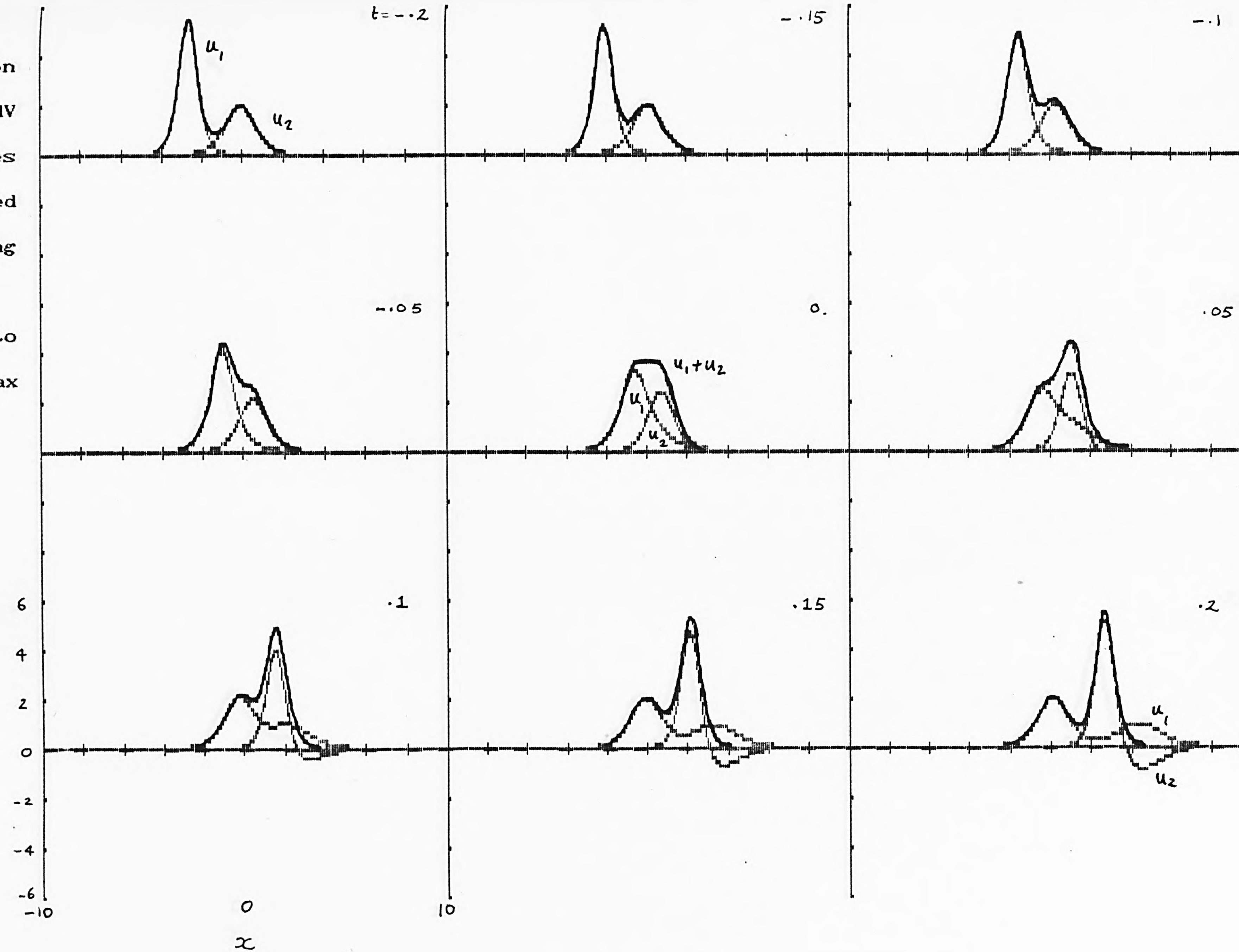


Fig 7.33b

The two soliton solution of the KdV with amplitudes 1.5,1. is decomposed into two accelerating solitons.

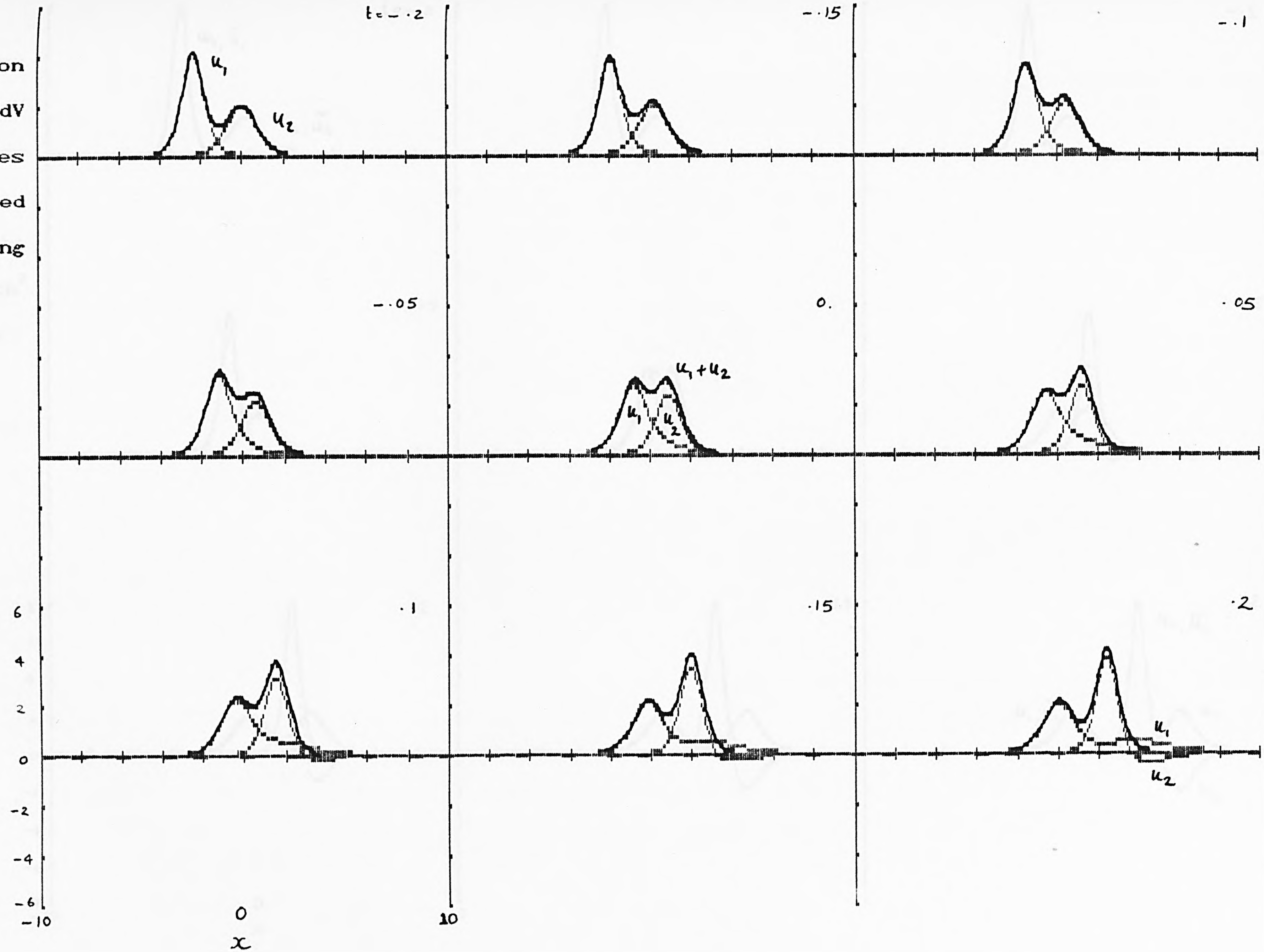


Fig 7.34a

Exact accelerating solitons for the KdV two soliton solution with amplitudes 2.,1. compared with the exact sech^2 approximate solitons (solid).

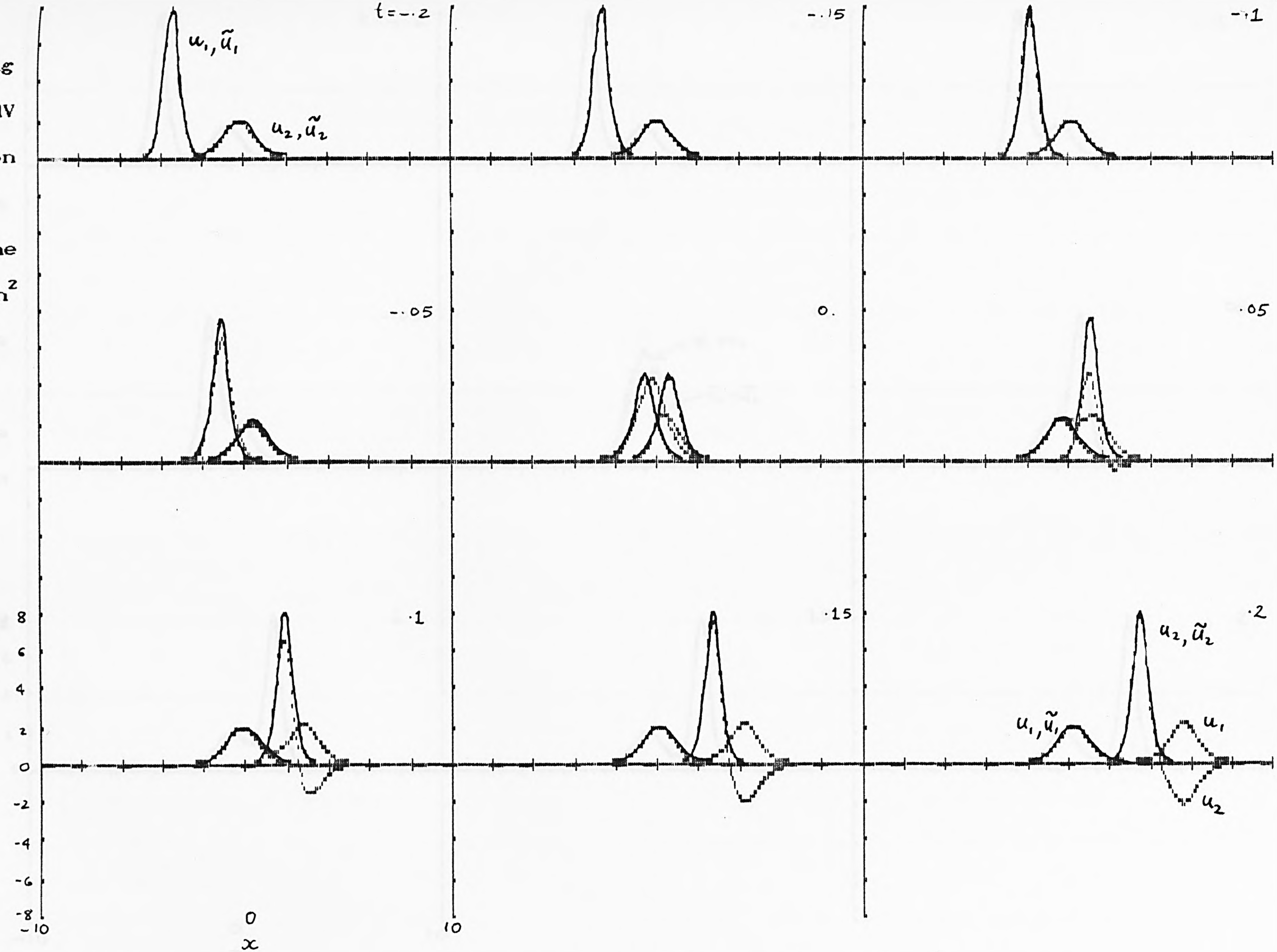
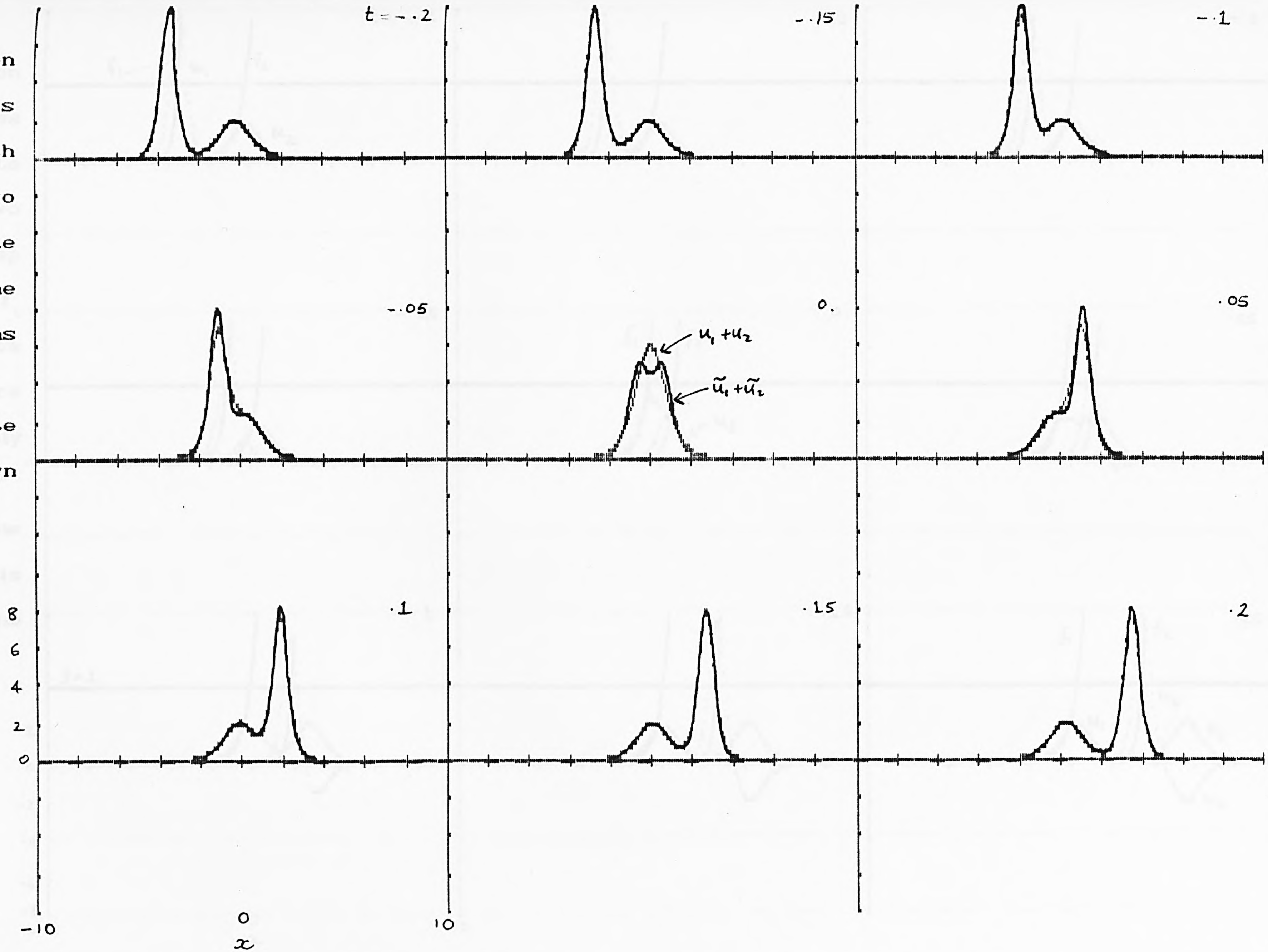


Fig 7.34b

Two soliton solution of the KdV, amplitudes 2.,1. compared with approximate two soliton solution(i.e sum of the approximate solitons shown in fig 7.34a). The approximate solution is drawn solid.



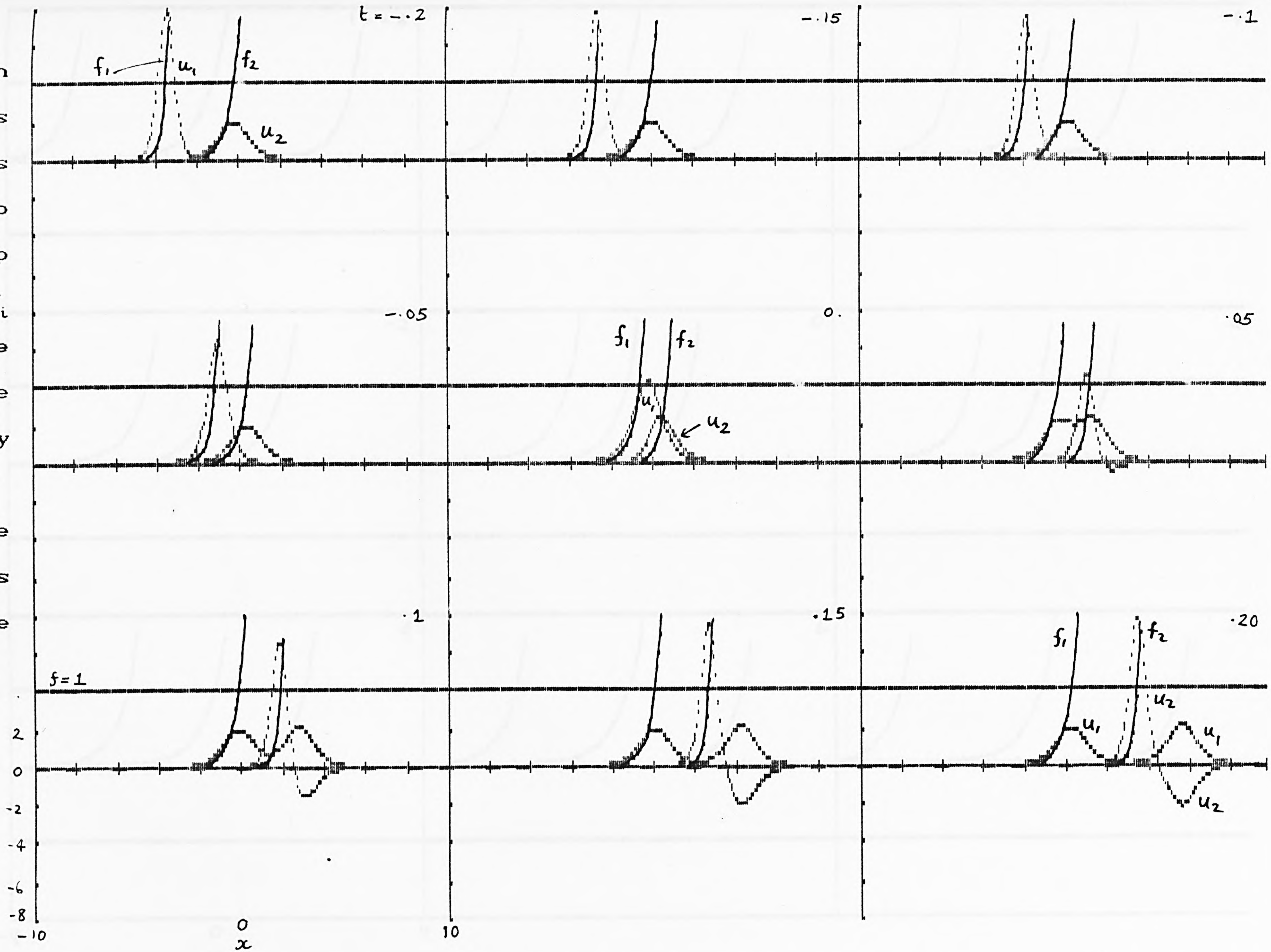


Fig 7.35

Two soliton solution of the KdV,amplitudes 2.,1. with the roots f_i of the KdV two soliton lsp superimposed.The f_i are drawn solid.Note how the points where f_i=1 coincide closely with the "principal" maxima of the solitons.Their is some departure in the region close to t=0.

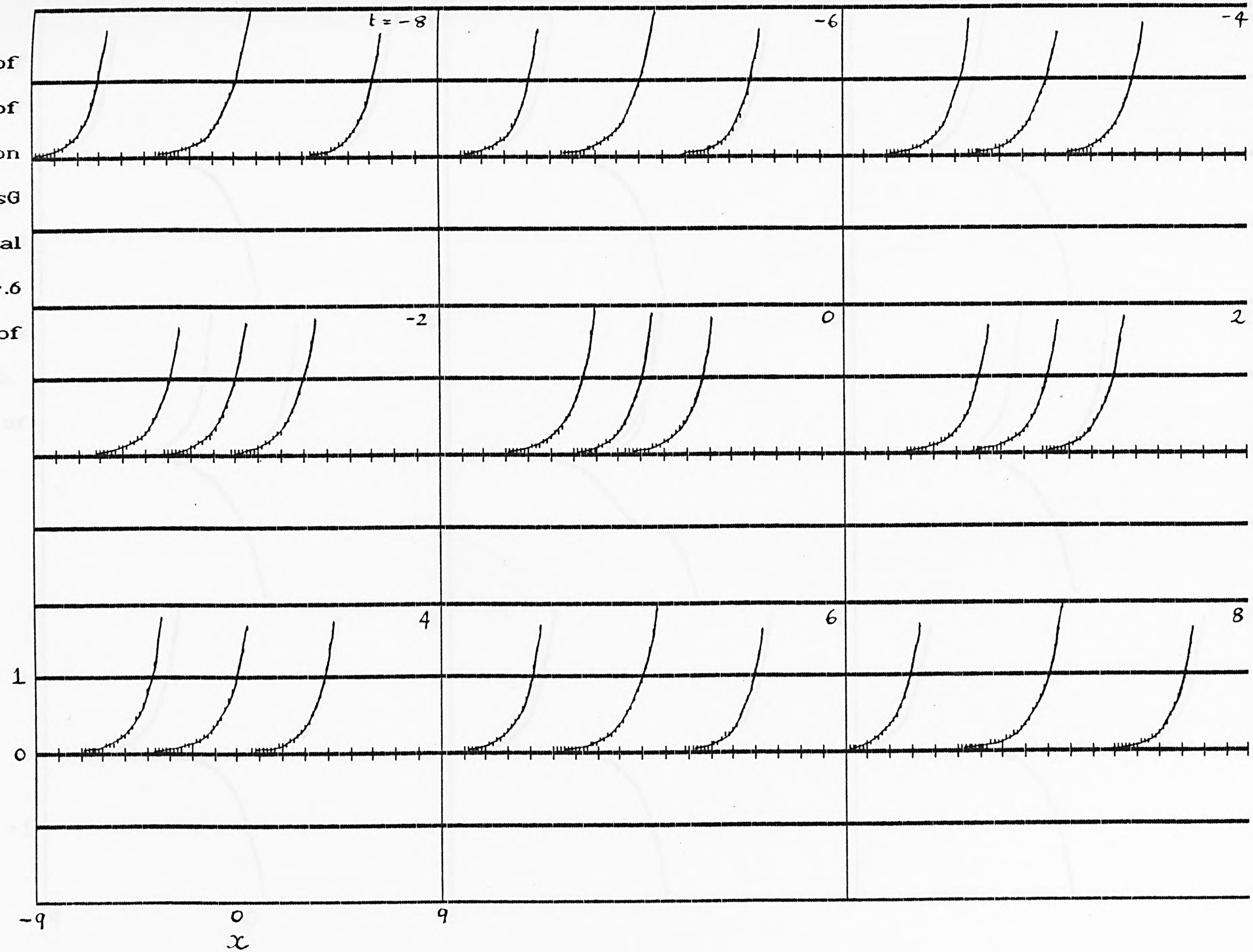


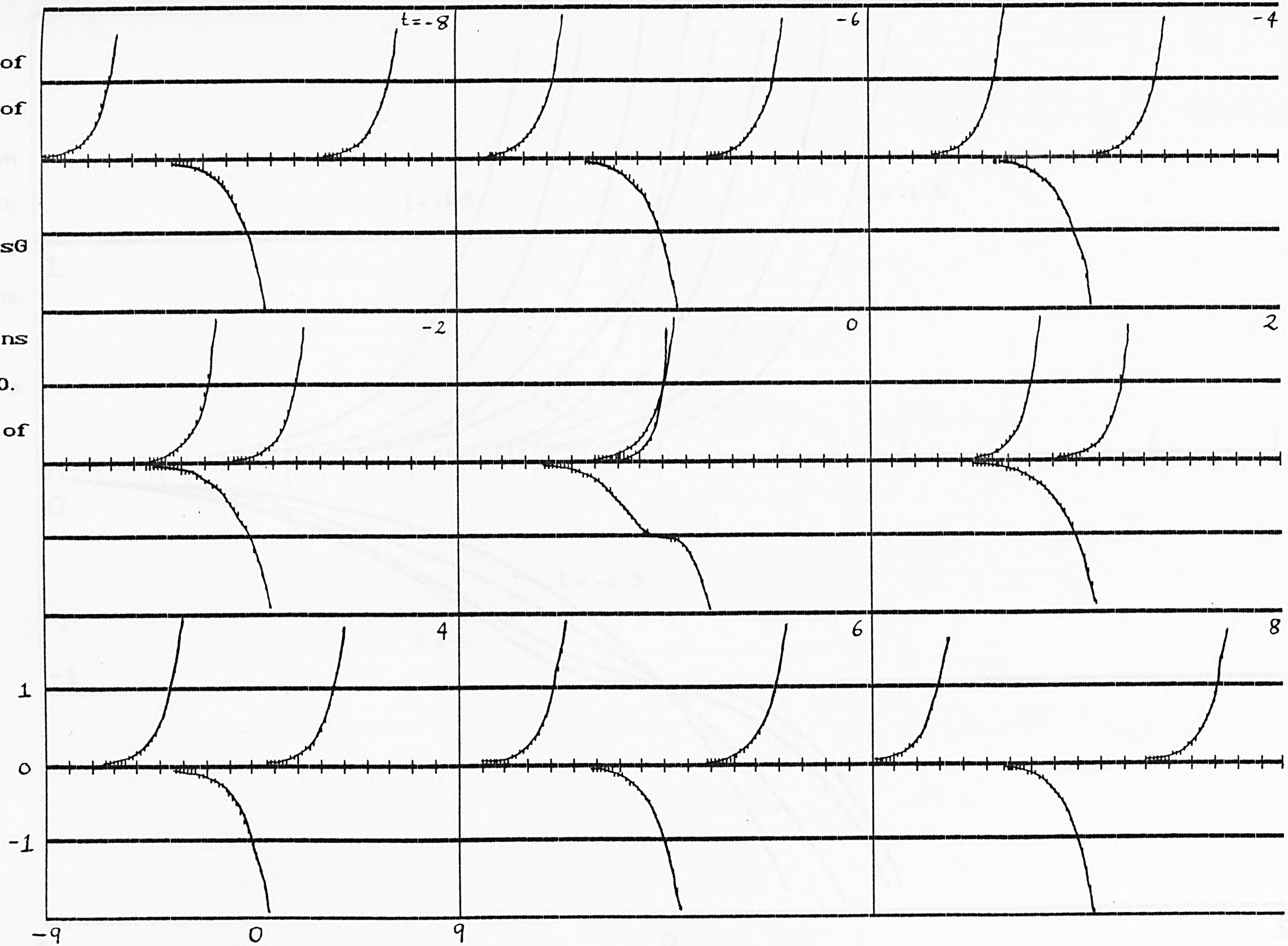
Fig 7.36

Time evolution of the roots (f_i) of the three soliton lsp for the sG equation. Initial speeds, .6,0.,-.6

This is a centre of velocity frame.

Fig 7.37

Time evolution of the roots(f_i) of the two soliton, one antisoliton lsp for the sG equation.Initial speeds ,solitons ±.6,antisoliton 0. This is a centre of velocity frame.



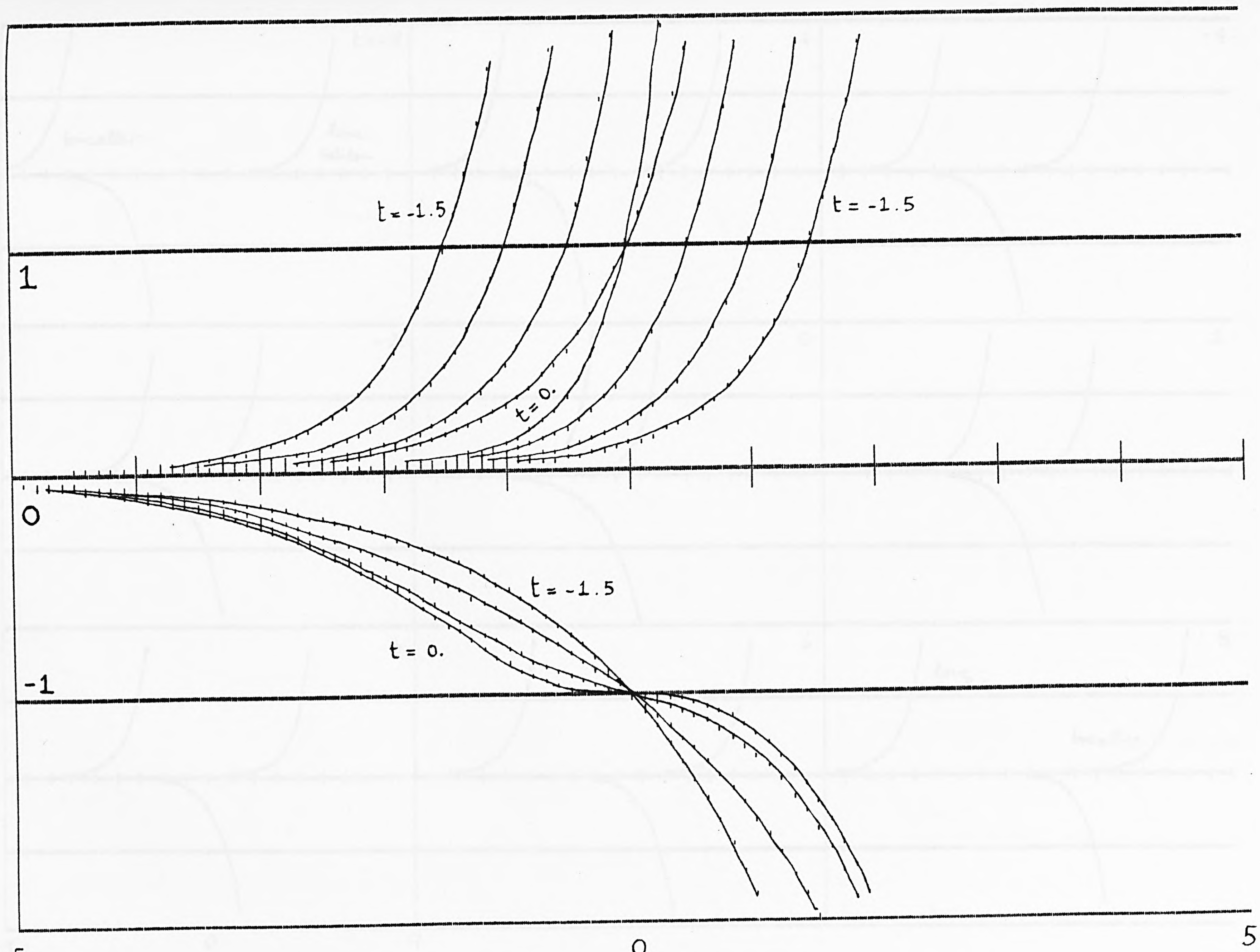


Fig 7.38

A closer look at the f_i in the strong interaction region of fig 7.37..The point of inflexion in the antisoliton function appears only at time t=0.

Fig 7.39

Time evolution of the roots of the breather-soliton lsp for the sG equation.Breather initial speed .5, period 4.,Lone soliton initial speed -.5.This is a centre of velocity case.

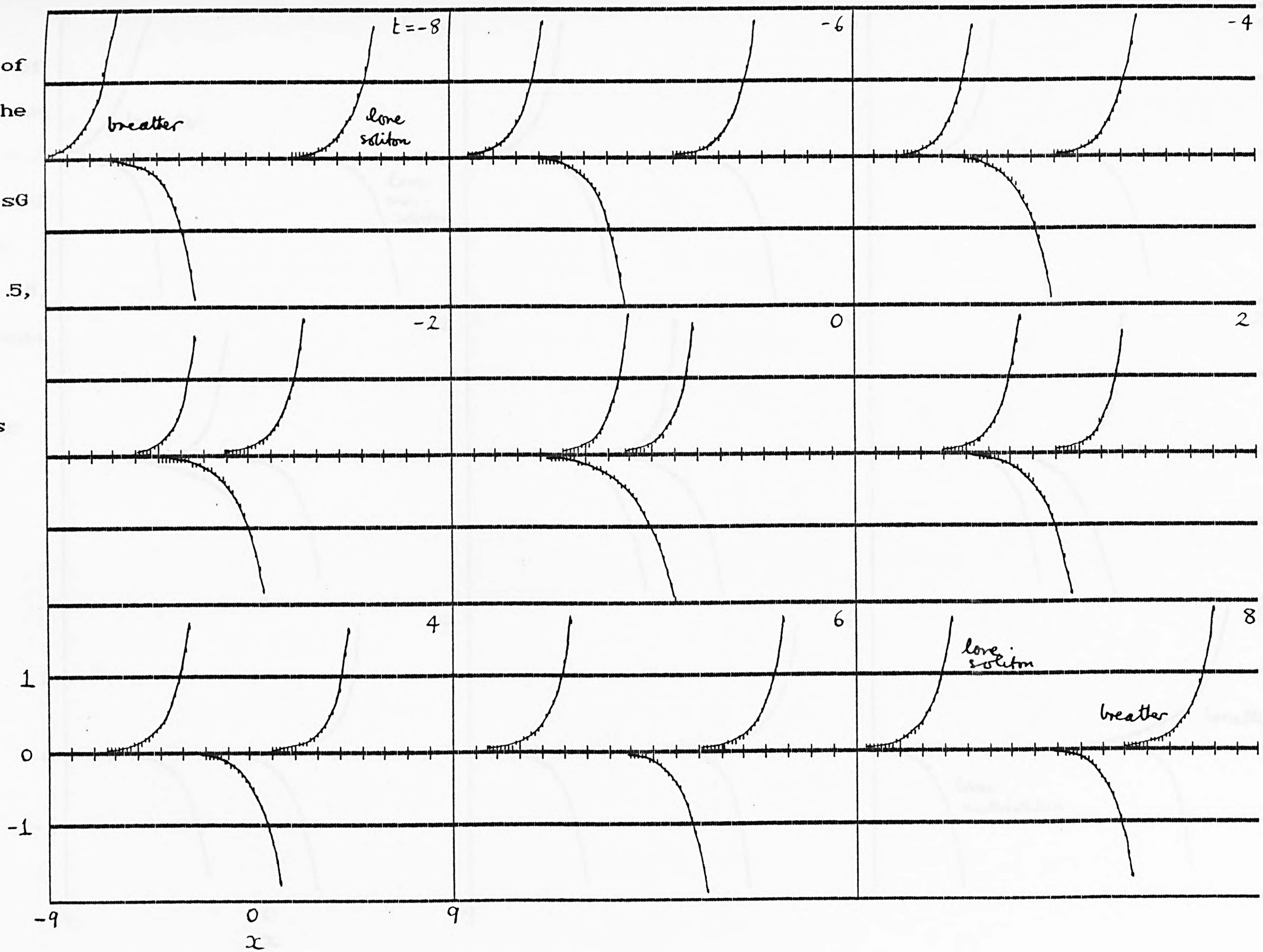
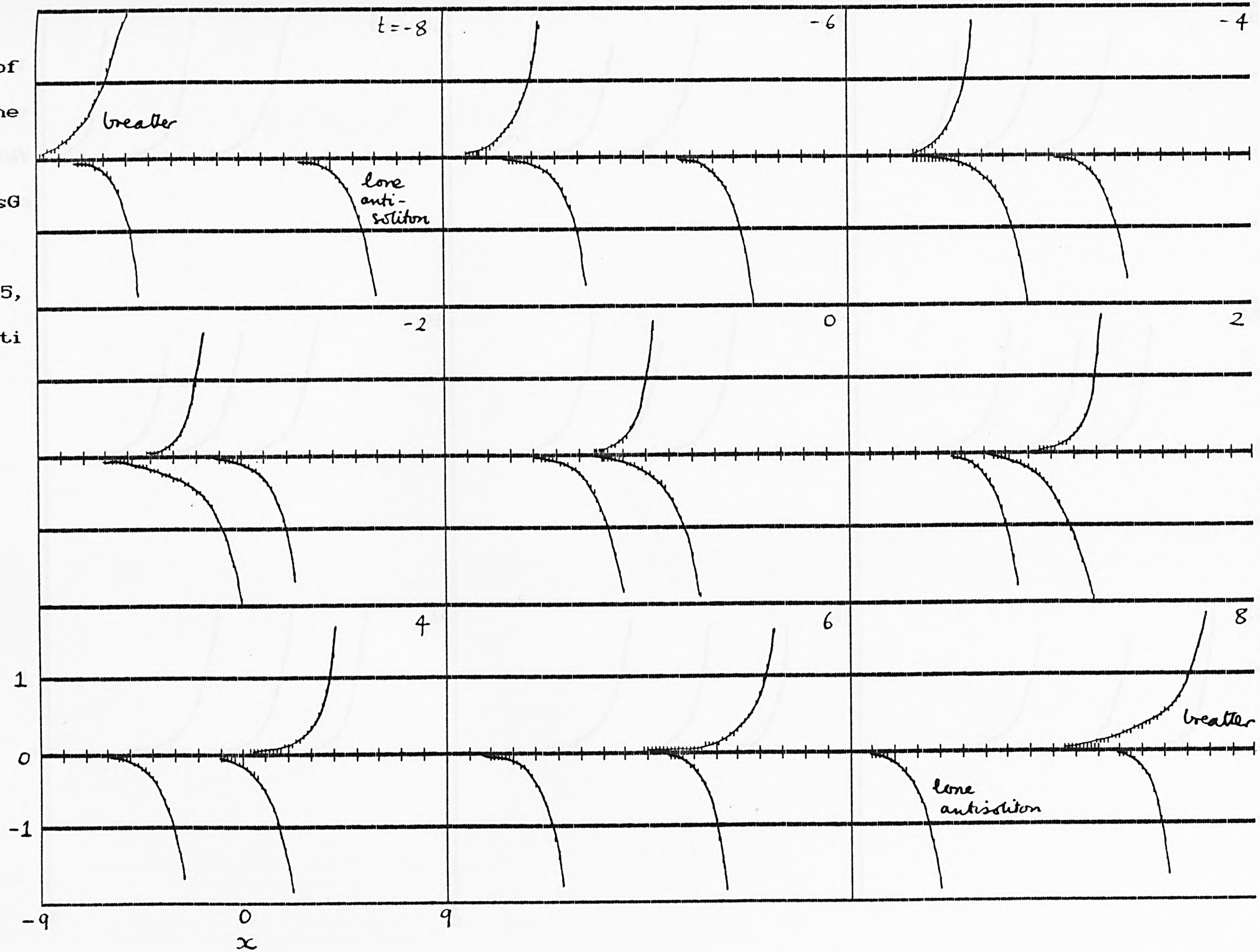


Fig 7.40

Time evolution of the roots of the breather-antisol lsp for the sG equation.Breather initial speed .5, period 4..lone anti soliton initial speed -.5.This is a centre of velocity case.



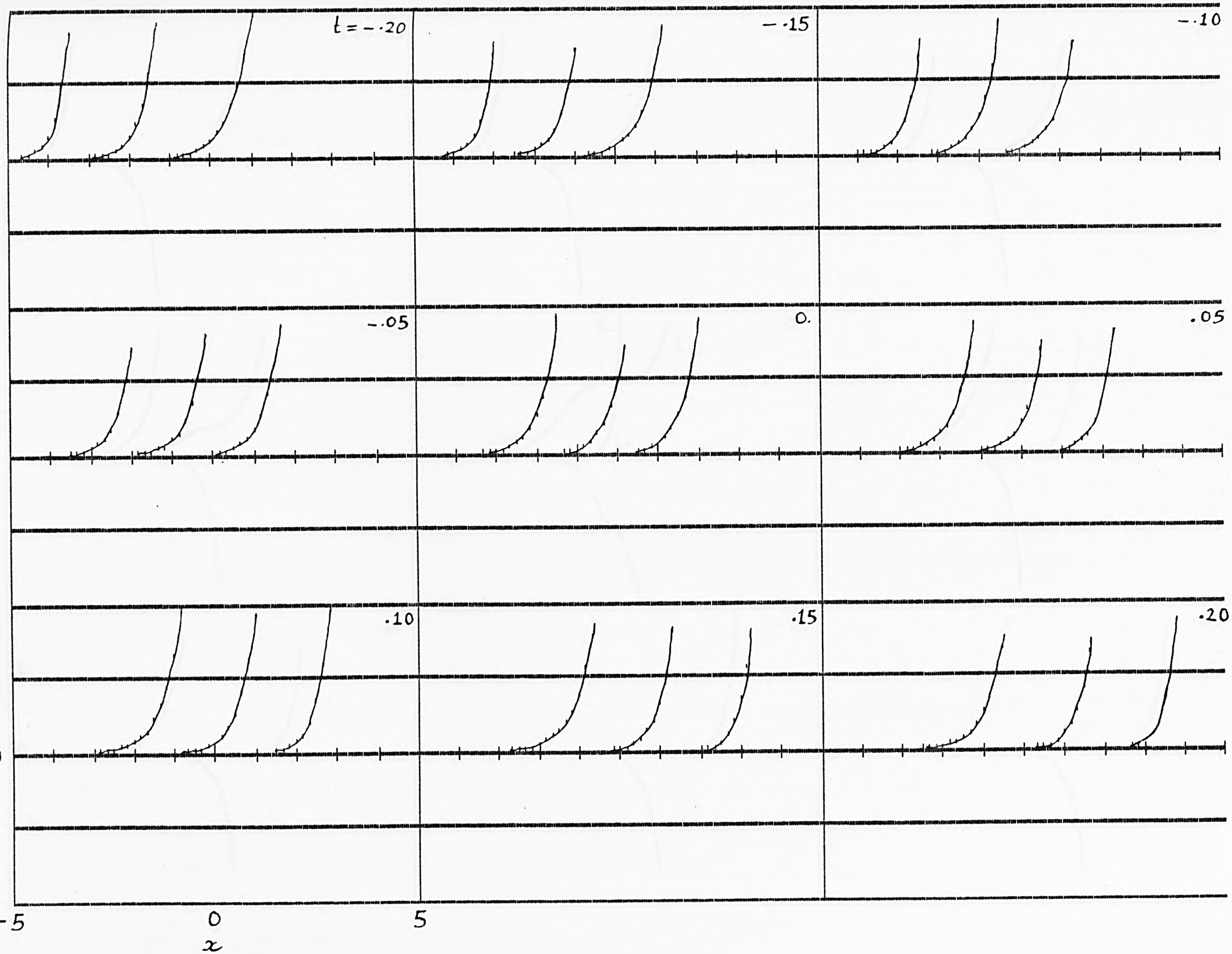


Fig 7.41

Time evolution of the roots of the three soliton MKdV lsp. Initial soliton amplitudes 1.75, 1.5, 1.

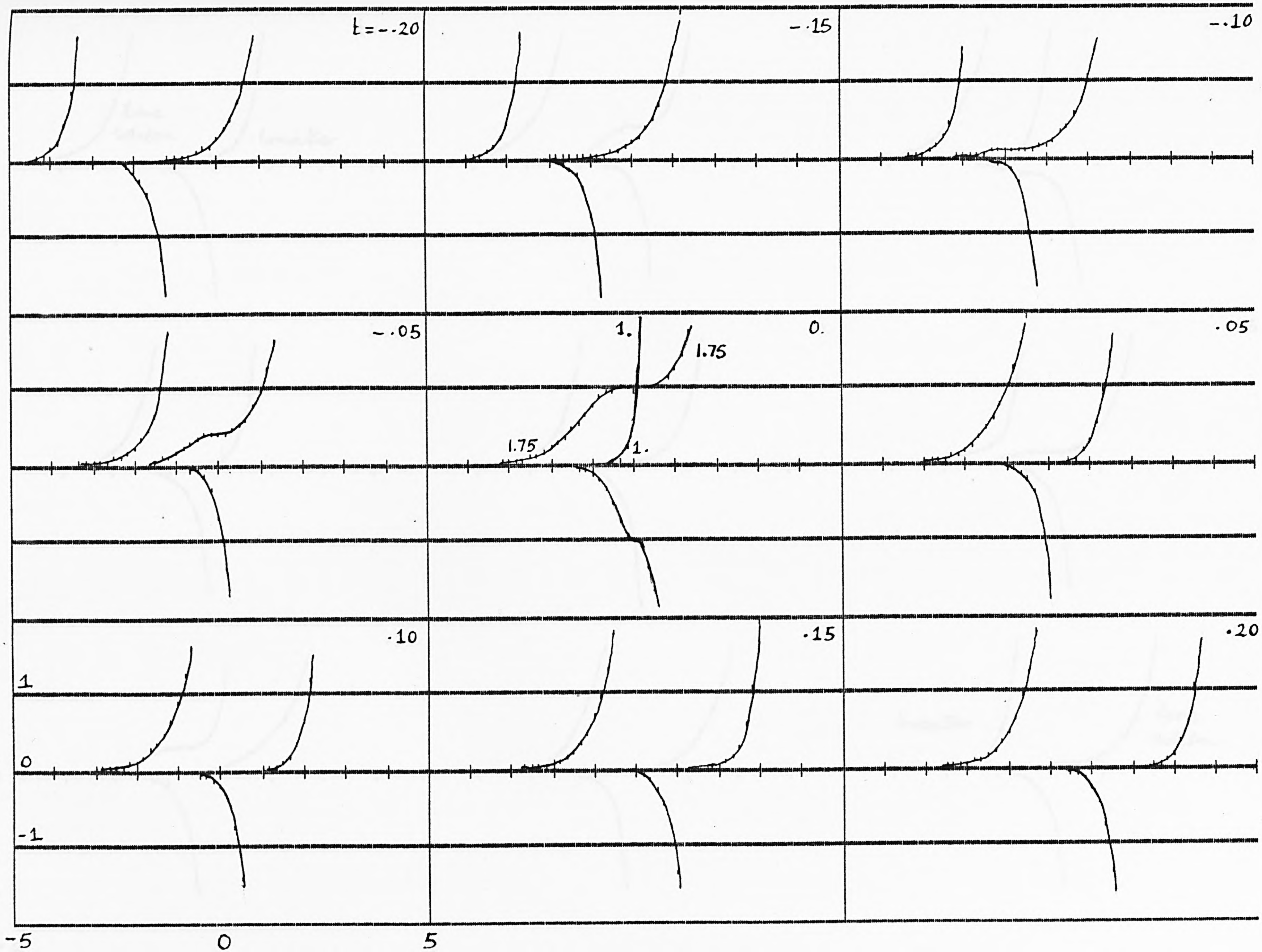
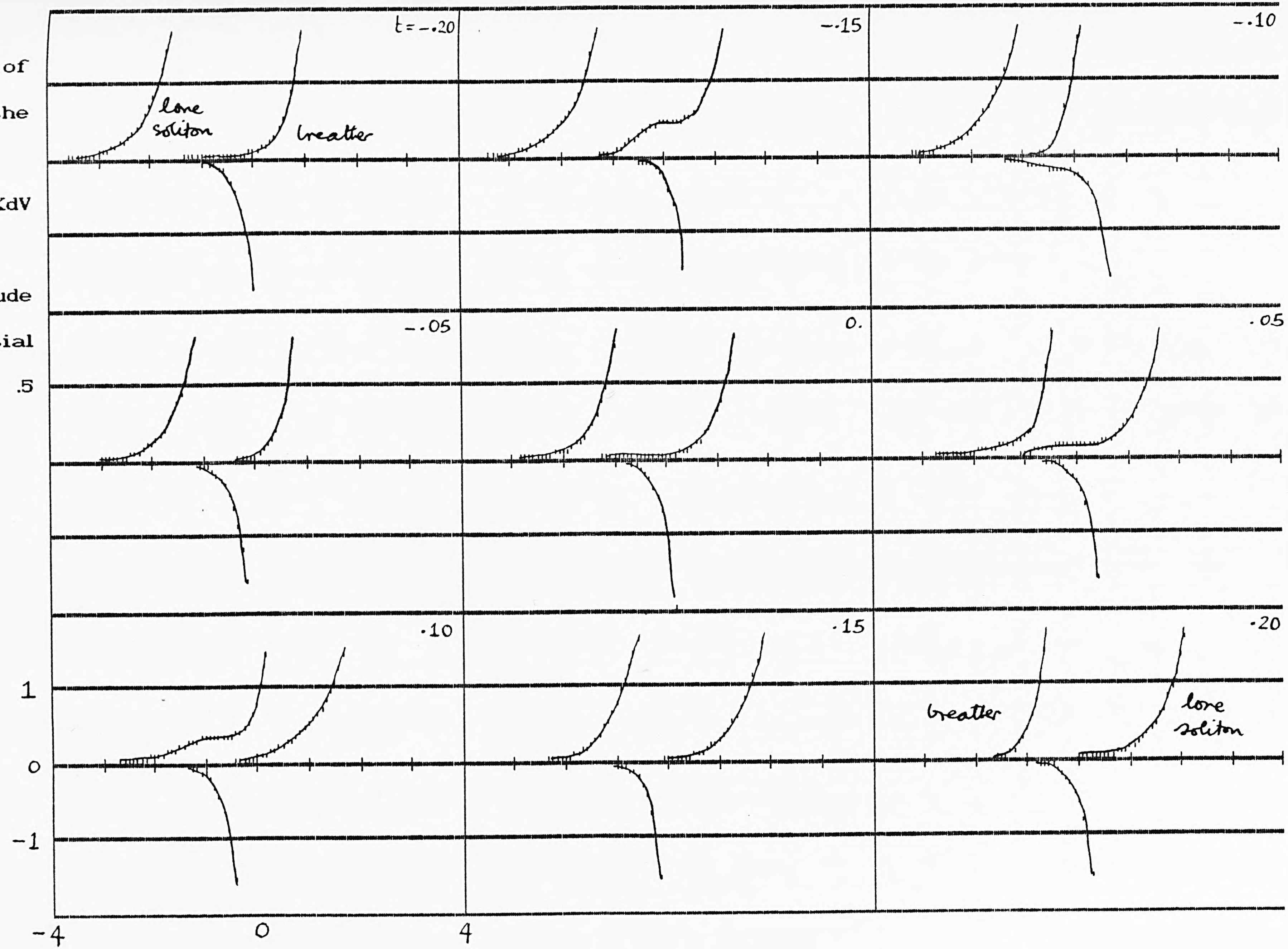


Fig 7.42

Time evolution of the roots of the two soliton one antisoliton lsp for the MKdV equation. Initial amplitudes, 1.75 ,1.5 ,1. . Note how the rightmost soliton develops a point of inflexion .The picture here is quite different from the equivalent sG case(even in a general frame-where we find that the soliton functions never possess points of inflexion.)

Fig 7.43

Time evolution of the roots of the breather-soliton lsp for the MKdV equation.Initial soliton amplitude 1. .Initial breather speed .5 ,period 4.



In the case of the MKdV (fig 7.42)we find the behaviour of the soliton f_i is quite different,as was also discovered in §1.One of the soliton functions(associated with the slower soliton) develops a stationary point of inflexion some time before collision.As with the sG the antisoliton function also develops a stationary point of inflexion and we find the solitons and antisolitons to be coincident at t=0.We can also deduce from fig 7.42(by looking at the slopes of the functions) that there is an exchange of speeds .The initially slower soliton picks up speed as a result of the collision, while the initially faster soliton slows down.

Comparing figs 7.39 with 7.43 we see that in both cases of the breather in interaction with a lone soliton,the antisoliton component of the breather moves away from the original soliton component of the breather,and forms a breather with the originally lone soliton.

§ 3. Approximations to the roots of the lsp

In this section we discuss a method of approximating the exact roots of the lsp's for the sG,MKdV and KdV by simpler functions.These approximations enable us to determine formulae for the shapes of solitons and their motion.The latter is especially useful as in most cases the motion of the solitons can only be deduced numerically.The motion of the approximate solitons agrees very closely with the actual motion of the solitons in cases where breathers are not present.For $|f_i|<1$ the agreement between the approximate f_i ,denoted $\tilde{f}_i$ and f_i is so good that we have not plotted the $\tilde{f}_i$ for the two parameter non breather cases except in the case of the soliton antisoliton solution of the MKdV(fig 7.20) in a small neighbourhood of $t=0$.

Note however even in this case the position of the approximate antisoliton does not depart greatly from the actual antisoliton position.

$\tilde{f}_i$ when breathers are present are not such good approximations to f_i,although we observe from figs 7.7 and 7.28 that the agreement is better with the sG than with the MKdV.We also note that in the case of the MKdV there are times when $\tilde{f}_i$ become very inaccurate.

The approximation method's basic assumption is the following ,

$$\tilde{f}_i = \pm \exp[\alpha_i(t)x].b_i(t) \tag{7.41}$$

Consider the two parameter lsp with perfect phase,

$$f^2 - k_{12}[t(1) + t(2)]f + t(1)t(2) = 0 \tag{7.42}$$

where $t(1)$,$t(2)$ and k_{12} are defined in chapter 6.

Assuming the roots to be given by (7.41) we have ,

$$\tilde{f}_1 + \tilde{f}_2 = k_{12}[t(1) + t(2)] \tag{7.43a}$$

$$\tilde{f}_1\tilde{f}_2 = t(1)t(2) \tag{7.43b}$$

Differentiating (7.43a) with respect to x we find ,

$$\alpha_1\tilde{f}_1 + \alpha_2\tilde{f}_2 = k_{12}[t_x(1) + t_x(2)] \tag{7.43c}$$

Now setting $x=0$ in (7.43) we find ($t(i;0) \equiv t(i;x=0)$) ,

$$b_1 + b_2 = k_{12}[t(1;0) + t(2;0)] \tag{7.44a}$$

$$b_1 b_2 = t(1;0)t(2;0) \tag{7.44b}$$

$$\alpha_1 b_1 + \alpha_2 b_2 = k_{12}[t_x(1;0) + t_x(2;0)] \tag{7.44c}$$

$\alpha_1+\alpha_2$ can be deduced from (7.43b) and (7.44b).(7.44a-b) define a quadratic which we can solve for b_i.Thus from the remaining equations α_i can be determined.

The technique is clearly readily generalised to higher parameter cases also.The procedure for three parameters is given below.The three parameter perfect phase polynomial is given by (6.80).The approximation method gives,

$$\tilde{f}_1 + \tilde{f}_2 + \tilde{f}_3 = k_{12}k_{13}t(1) + k_{21}k_{23}t(2) + k_{31}k_{32}t(3) \qquad (7.45a)$$

$$\tilde{f}_1\tilde{f}_2+\tilde{f}_1\tilde{f}_3+\tilde{f}_2\tilde{f}_3=-\{k_{13}k_{23}t(1)t(2)+k_{12}k_{32}t(1)t(3)+k_{21}k_{31}t(2)t(3)\} \quad (b)$$

$$\tilde{f}_1\tilde{f}_2\tilde{f}_3 = -t(1)t(2)t(3) \qquad (c)$$

Differentiating (7.45a-b) with respect to x we find ,

$$\alpha_1\tilde{f}_1 + \alpha_2\tilde{f}_2 + \alpha_3\tilde{f}_3 = k_{12}k_{13}t_x(1) + k_{21}k_{23}t_x(2) + k_{31}k_{32}t_x(3) \qquad (d)$$

$$(\alpha_1+\alpha_2)\tilde{f}_1\tilde{f}_2 + (\alpha_1+\alpha_3)\tilde{f}_1\tilde{f}_3 + (\alpha_2+\alpha_3)\tilde{f}_2\tilde{f}_3= \qquad (e)$$

$$-\{ k_{13}k_{23}[t_x(1)t(2) + t(1)t_x(2)]+k_{12}k_{32}[t_x(1)t(3)+t(1)t_x(3)]$$

$$+ k_{21}k_{31}[t_x(2)t(3)+t(2)t_x(3)] \}$$

Setting x=0 in (7.45) we find ,

$$b_1 + b_2 + b_3 = k_{12}k_{13}t(1;0) + k_{21}k_{23}t(2;0) + k_{31}k_{32}t(3;0) \qquad (7.46a)$$

$$b_1b_2+b_1b_3+b_2b_3 = -\{k_{13}k_{23}t(1;0)t(2;0) + k_{12}k_{32}t(1;0)t(3;0) \qquad (b)$$

$$+k_{21}k_{31}t(2;0)t(3;0) \}$$

$$b_1b_2b_3 = -t(1;0)t(2;0)t(3;0) \qquad (c)$$

$$\alpha_1b_1 + \alpha_2b_2 + \alpha_3b_3 = k_{12}k_{13}t_x(1;0)+k_{21}k_{23}t_x(2;0)+k_{31}k_{32}t_x(3;0) \quad (d)$$

$$(\alpha_1+\alpha_2)b_1b_2 + (\alpha_1+\alpha_3)b_1b_3 + (\alpha_2+\alpha_3)b_2b_3 = \qquad (e)$$

$$-\{ k_{13}k_{23}[t_x(1;0)t(2;0)+t(1;0)t_x(2;0)]$$

$$+ k_{12}k_{32}[t_x(1;0)t(3;0)+t(1;0)t_x(3;0)]$$

$$+ k_{21}k_{31}[t_x(2;0)t(3;0)+t(2;0)t_x(3;0)] \}$$

(7.46a-c) provide us with a cubic which we can solve for b_i and (7.46c) together with (7.45c) provide an equation for $\alpha_1+\alpha_2+\alpha_3$,this together with (7.46d-e) provides three simultaneous equations for α_i.Thus we can completely determine the approximating functions $\tilde{f}_i$.

We now consider the exact formulae for the approximating functions for the sG,MKdV and KdV.

Two solitons

We find on solving (7.4) ,

$$b_1(t) = \exp [y_+ + \zeta(y_-) + \ln k_{12}] \qquad (7.47a)$$

$$b_2(t) = \exp [y_+ - \zeta(y_-) - \ln k_{12}] \qquad (b)$$

$$\alpha_1(t) = \lambda \left\{ 1 + \nu \frac{\sinh y_-}{(\cosh^2 y_- - u_{12}^2)^{1/2}} \right\} \qquad (c)$$

$$\alpha_2(t) = \lambda \left\{ 1 - \nu \frac{\sinh y_-}{(\cosh^2 y_- - u_{12}^2)^{1/2}} \right\} \qquad (d)$$

where $\zeta(r)$ is defined in (6.52d) and $y_\pm = X_\pm(x=0)$ is defined in (6.52c),$k_{12} = u_{12}^{-1}$,is defined in (6.5,9c).We find that for the sG (γ and k are defined in (6.4) and (6.8) respectively),

$$\lambda = (\gamma_1 + \gamma_2)/2 \quad , \quad \nu = (\gamma_1 - \gamma_2)/(\gamma_1 + \gamma_2) \qquad (7.48a)$$

while for the MKdV and KdV ,

$$\lambda = (k_1 + k_2) \quad , \quad \nu = (k_1 - k_2)/(k_1 + k_2) = u_{12} \qquad (b)$$

Soliton-antisoliton

We find on solving (7.44) ,

$$b_1(t) = \exp [y_+ - \eta(y_-) - \ln k_{12}] \qquad (7.49a)$$

$$b_2(t) =-\exp [y_+ + \eta(y_-) + \ln k_{12}] \qquad (b)$$

$$\alpha_1(t) = \lambda \left\{ 1 + \nu \frac{\cosh y_-}{(\sinh^2 y_- + u_{12}^2)^{1/2}} \right\} \qquad (c)$$

$$\alpha_2(t) = \lambda \left\{ 1 - \nu \frac{\cosh y_-}{(\sinh^2 y_- + u_{12}^2)^{1/2}} \right\} \qquad (d)$$

where $\eta(r)$ is defined in (6.53c).

Breather

$$b_1(t) = \exp\,[y_+ + \varepsilon\xi(y_-) - \varepsilon \ln w \quad] \tag{7.50a}$$

$$b_2(t) = -\exp\,[y_+ - \varepsilon\xi(y_-) + \varepsilon \ln w \quad] \tag{b}$$

$$\alpha_1(t) = \lambda \left\{ 1 - \nu\varepsilon \frac{\cos y_-}{(\sin^2 y_- + w^2)^{1/2}} \right\} \tag{c}$$

$$\alpha_2(t) = \lambda \left\{ 1 + \nu\varepsilon \frac{\cos y_-}{(\sin^2 y_- + w^2)^{1/2}} \right\} \tag{d}$$

where $\xi(r)$ is defined in (6.54c) and for the sG,

$$y_+ = -\gamma \cos\mu\, vt \quad , \quad y_- = \gamma \sin\mu\, t \tag{7.51a}$$

$$\lambda = \gamma \cos\mu \quad , \quad \nu = vw \tag{b}$$

$$\varepsilon = 1 \quad , \quad w = \tan\mu \tag{c}$$

with cos μ ,sin μ and γ defined in terms of the speed v and the period τ of the breather in (6.4).

For the MKdV we have ,

$$y_+ = -2\theta vt \quad , \quad y_- = 2\eta wt \tag{7.52a}$$

$$\lambda = 2\theta \quad , \quad \nu = \eta/\theta \tag{b}$$

$$\varepsilon = -1 \quad , \quad w = \eta/\theta = \nu \tag{c}$$

where θ,η are defined in terms of the speed v and the period of the breather in (6.8).

Now examining (7.49-50) we can immediately see the inaccuracy of the approximation in the case of the MKdV at certain times.Let t=0 in (7.9),we find ,

$$\alpha_1(0) = (k_1 + k_2) \quad , \quad \alpha_2(0) = 0 \tag{7.53a}$$

$$b_1(0) = 1 \quad , \quad b_2(0) = 1 \tag{b}$$

$$\tilde{f}_1(x,0) = e^{(k_1+k_2)x} \quad , \quad \tilde{f}_2(x,0) = -1 \tag{c}$$

While in the case of the breather the constancy of either of the $\tilde{f}_i$ at t=0 is a periodic phenomenon.We find ,

$t = n\pi/\eta w \quad n \in \mathbb{Z}$

$$\alpha_1(0) = 4\theta \quad , \quad \alpha_2(0) = 0 \qquad (7.54a)$$

$$b_1(0) = 1 \quad , \quad b_2(0) = 1 \qquad (b)$$

$$\tilde{f}_1(x,0) = e^{4\theta x} \quad , \quad \tilde{f}_2(x,0) = -1 \qquad (c)$$

and at times $t = \pi(n + 1/4)/\eta w$ we find similar results to the above with the subscripts 1 and 2 interchanged.None of these difficulties arise with the sG which for all times has $\tilde{f}_i$ which are exponential functions of x.

Clearly the problem with the approximation for the mixture cases of the MKdV lies in the fact that f_i possess points of inflexion.To obtain a more exact approximation in these cases one would have to consider $\tilde{f}_i$ having the form ,

$$\tilde{f}_i = \pm\, b_i(t)\exp\,[\alpha_i(t)x + \delta_i(t)x^3] \qquad (7.55)$$

Actually when one applies this one finds that $\delta_2 = -\,\delta_1$ and the equations for b_i and α_i are identical to those without the term in x^3.By differentiating (7.43a) three times we obtain ,

$$(\alpha_1^3 + 6\delta_1)b_1 + (\alpha_2^3 + 6\delta_2)b_2 = k_{12}[t_{xxx}(1;0) + t_{xxx}(2;0)] \quad (7.56)$$

Thus δ_i can be determined.

Of course an approximation of the form (7.55) no longer makes it so easy to obtain the approximate motion of points such that $f_i=\pm1$.However it is certainly possible that the equation ,

$$\delta_i x^3 + \alpha_i x - \ln b_i = 0$$

only has one real root.We have not investigated this point in this thesis.

We now move on to discuss how for the sG in the centre of velocity frame for three solitons or two solitons and one antisoliton with one component fixed for all time we can solve equations (7.46) exactly.

As observed in the previous chapter in the centre of velocity frames of reference(when they exist) the behaviour of the f_i is time symmetric.This coupled with the fact that one soliton or antisoliton does not move for all time leads us to consider $\tilde{f}_i$ with the following properties,

$$\alpha_2(t) = \alpha_1(t) = \alpha(t), b_2(t) = 1/b_1(t) = b(t), b_3(t) = 1 \qquad (7.57)$$

<u>Three solitons</u>

The soliton speeds are u,-u,0.$k_{12} = k_{23}$ .Denote $k_{13} \equiv k$. (7.46) becomes noting (6.4) where in perfect phase t(2)<0.

$$b + \frac{1}{b} + 1 = 2k_{12}k\cosh\gamma ut + k_{12}^2 \qquad (7.58a)$$

$$\alpha(b + \frac{1}{b}) + \alpha_3 = 2k_{12}k\,\gamma\cosh\gamma ut + k_{12}^2 \qquad (b)$$

Also (7.45c) becomes ,

$$2\alpha + \alpha_3 = 1 + 2\gamma \qquad (c)$$

Solving (7.58a) for b we obtain (the negative root interchanges b_1 with b_2),

$$b(t) = c(t) + (c^2(t) - 1)^{1/2} \qquad (7.59a)$$

$$c(t) = (2k_{12}k\cosh\gamma ut + k_{12}^2 - 1)/2 \qquad (b)$$

Also we find ,

$$\alpha(t) = \gamma - \frac{(\gamma - 1)(k_{12}^2 - 1)}{2k_{12}k\cosh\gamma ut + k_{12}^2 - 3} \qquad (c)$$

$$\alpha_3(t) = 1 + 2\frac{(\gamma - 1)(k_{12}^2 - 1)}{2k_{12}k\cosh\gamma ut + k_{12}^2 - 3} \qquad (d)$$

Analysing the two soliton antisoliton case($b_3 = -1$) we find ,

$$b(t) = d(t) + (d^2(t) - 1)^{1/2} \qquad (7.60a)$$

$$d(t) = (2k_{12}k\cosh\gamma ut - k_{12}^2 + 1)/2 \qquad (b)$$

$$\alpha(t) = \gamma + \frac{(\gamma - 1)(k_{12}^2 - 1)}{2k_{12}k\cosh\gamma ut + k_{12}^2 + 3} \qquad (c)$$

$$\alpha_3(t) = 1 - 2\frac{(\gamma - 1)(k_{12}^2 - 1)}{2k_{12}k\cosh\gamma ut + k_{12}^2 + 3} \qquad (d)$$

Clearly $c(t)$,$d(t)$ are at a minimum when $t=0$.Now using (5.2) which in this case becomes ,

$$2k_{12}k - k_{12}^2 = 1 \qquad (7.61)$$

we then find ,

$$c(0) = k_{12}^2 > 1 \ , \ d(0) = 1$$

Thus $b(t)$ is real for all t.

It is found that for all t $\tilde{f}_i(t)$ are very close to $f_i(t)$ in the pure soliton case(7.59).However in the two soliton antisoliton case, $\tilde{f}_3(t)$ for the antisoliton(i.e $-\exp\ \alpha_3(t)x$) is found to depart significantly from $f_3(t)$ in a region close to t= 0,by virtue of the fact that f_3 has a point of inflexion at x=0 and t=0(see fig 7.38).Despite this $\tilde{f}_3$ provides a reasonable approximation(note $\tilde{f}_3(0,0) = -1 = f_3(0,0)$).The soliton functions $\tilde{f}_{1,2}$ are very close to $f_{1,2}$ for all t.

Using $\tilde{f}_i$ we can find approximate multisoliton solutions of the derivative MKdV ,

$$u_t + 6u^2u_x + u_{xxx} = 0 \qquad (7.62)$$

and the KdV ,

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (7.63)$$

The approximate multisoliton solutions of (7.62) are the derivatives with respect to x of a sum of terms of form $2\tan^{-1}\tilde{f}_i$.Thus the approximate multisoliton solution of (7.62),$\tilde{u}_N$ is given by ,

$$\tilde{u}_N = \sum (2\tan^{-1}\tilde{f}_i)_x = \sum \varepsilon_i\alpha_i(t)\operatorname{sech}[\alpha_i(t)x + \ln|b_i(t)|] \qquad (7.64)$$

where $\varepsilon_i = 1$ for solitons and -1 for antisolitons.

The approximate multisoliton solution of (7.63) is given by(noting (6.58-9)),

$$\tilde{u}_N = -\frac{1}{2}\sum \alpha_i^2(t)\operatorname{sech}^2\frac{1}{2}[\alpha_i(t)x + \ln|b_i(t)|] \qquad (7.65)$$

$\tilde{u}_2$ for (7.62-3) are seen in the figures in §2.

§ **4. The motion of sG solitons,antisolitons,breathers in interaction.**

We have analysed numerically a number of higher parameter solutions of the sG.As we saw in the previous chapter in a multiparameter solution of the sG we can determine the motion of the solitons or solitons within breathers by setting f=1 in the sG lsp (6.2-4),similarly to determine the motion of antisolitons in breathers or otherwise we set f=-1 into the sG lsp.Computationally speaking this has an immediate advantage in that only under unusual circumstances do the roots of either of the resulting transcendental equations coincide(i.e solitons close to solitons,or antisolitons close to antisolitons).

The computer program created by the author to solve these trancendental equations was designed to be as general as possible,as such the number of solitons ,antisolitons and breathers in interaction that it would handle, was only limited by two factors ;memory required to store all the double precision(17 decimal places) numbers involved and the execution time.

For more than four parameters,especially involving breathers the execution time was quite great,depending of course on the range and number of positions of the particles.Since during the course of the interaction there were times when the particles came very close to one another or where the speeds changed very rapidly it was necessary to determine a variable number of positions of the particles per unit interval on the x axis.

This was accomplished by running the program several times with some regions of the x,t plane being examined much more closely than others.

Later, after all the data making up a very detailed picture of the particle interaction had been stored in various files,another program read all the files involved and created one very large data file,which could be subsequently used to produce the graphs seen in this section.It was by this method that data could be collected without exceeding the execution time for a single run of the program.

The accuracy of the program was checked against the exact two parameter solutions of Bowtell and Stuart(chapter 4).The agreement was exact up to 17 decimal places.

One of the problems which had to be overcome in writing the computer program was connected with the highest exponent(powers of e) that the computer would recognise,88($\simeq 10^{38}$).The lsp for the sG(and others) involves many exponential functions of x and t being multiplied together,e.g consider a 3 parameter lsp at time t=-20 x=20,the minimum power involved would be $\simeq$ 40,the maximum $\simeq$ 120 and in a four parameter case with t=-65, x= +20 one can have powers ranging from 85 to 340(see fig 7.46h).

These enormous ranges of exponents could of course not be handled by the computer directly,and in fact the lsp had to be rescaled by the multiplication by a suitably large number.Of couse this had to be done by subtracting a suitable exponent from each term in the lsp before taking powers, and then a comparison made of the *log* of the lsp with the exponent subracted.We give a simple example:Suppose we want to find the zero of the function,$H(x,t)= e^{x-t}-1$ at time t=-50.We start by putting in values of x beginning at x=50.$H(50,-50)=e^{100}-1$.To ascertain the sign of H we imagine multiplying the equation by e^{-20}.Thus we look at the sign of $\ln(e^{80})-(-20)$.This the computer can handle.

The computer program was made "user friendly" so that after it had been designed the program could be run with the minimum of fuss.One could choose the frame of reference at will though likely frames of frames of reference could be opted for automatically.

We must point out that the data on the graphs relating to breathers was later discarded in favour of simply breather speed and period(asymptotic).On the graphs are seen a pair of speeds for breathers u_1,u_2 say.The breather asymptotic speed v can determined from this pair of speeds by the following formula ,

$$v = (1 + u_1 u_2 - \sqrt{1 - u_1^2}\ \sqrt{1 - u_2^2}\ )/(u_1 + u_2)$$

The breather asymptotic period τ could be determined from,

$$\tau = 2\pi/\sigma u\gamma$$

where ,

$$\sigma = (1 + u^2)^{-1/2},\ \gamma = (1 - v^2)^{-1/2}$$

$$u = (1 - u_1 u_2 - \sqrt{1 - u_1^2}\ \sqrt{1 - u_2^2}\ )/(u_1 - u_2)$$

The force plotted in the graphs is the relativistic formula for force for a particle of rest mass 8(soliton/antisoliton) moving with speed v(t) and acceleration a(t) and is given by ,

$$F(t) = 8a(t)/(1 - v(t)^2)^{3/2}$$

The potential for a system of solitons,antisolitons and breathers is defined by the sum of the following terms,

$8\gamma_i$ for each soliton and antisoliton,

$16(1 + v_b^2\ )^{-1/2}$ for each breather

$-8(1 - v_i(t)^2)^{-1/2}$ for each soliton or antisoliton in breathers or otherwise, where v_i are the soliton or antisoliton speeds in breathers or otherwise as functions of time,and $\gamma_i = (1 - v_i(-\infty)^2)^{-1/2}.v_b$ is the breathers' aymptotic speed.

A plot of the potential as a function of time gives us a global indication of the attractiveness or repulsiveness of the forces involved.

Most of the graphs (figs 7.44a-7.48e) are self explanatory,the solid lines always refer to solitons whether in breathers or otherwise,the dotted lines refer to antisolitons in breathers or otherwise.

It is clear from fig 7.46a that the antisoliton component of the breather espouses another soliton leaving the original partner to become the lone soliton.There is then a range of times over which the soliton-breather system cannot be thought of as consisting of soliton and breather separately.One can see from fig 7.46g that overall the system is weakly repulsive(note small magnitude of positive potential).However there is something very puzzling about the soliton-breather interaction,this is shown in fig 7.46h.At a time close to -45 the soliton and antisoliton components become coincident,however the interaction between them at this time is *not* attractive.

The behaviour of the lone soliton at this time appears to be almost asymptotic yet clearly the behaviour of the breather is not asymptotic as normally the antisoliton component of the breather would pass through the soliton component(achieving light speed at the moment of coincidence).We note from fig 7.46j that the global potential starts to become more attractive as we approach time -50 but then grows more repulsive again appearing to reach a maximum at approximately

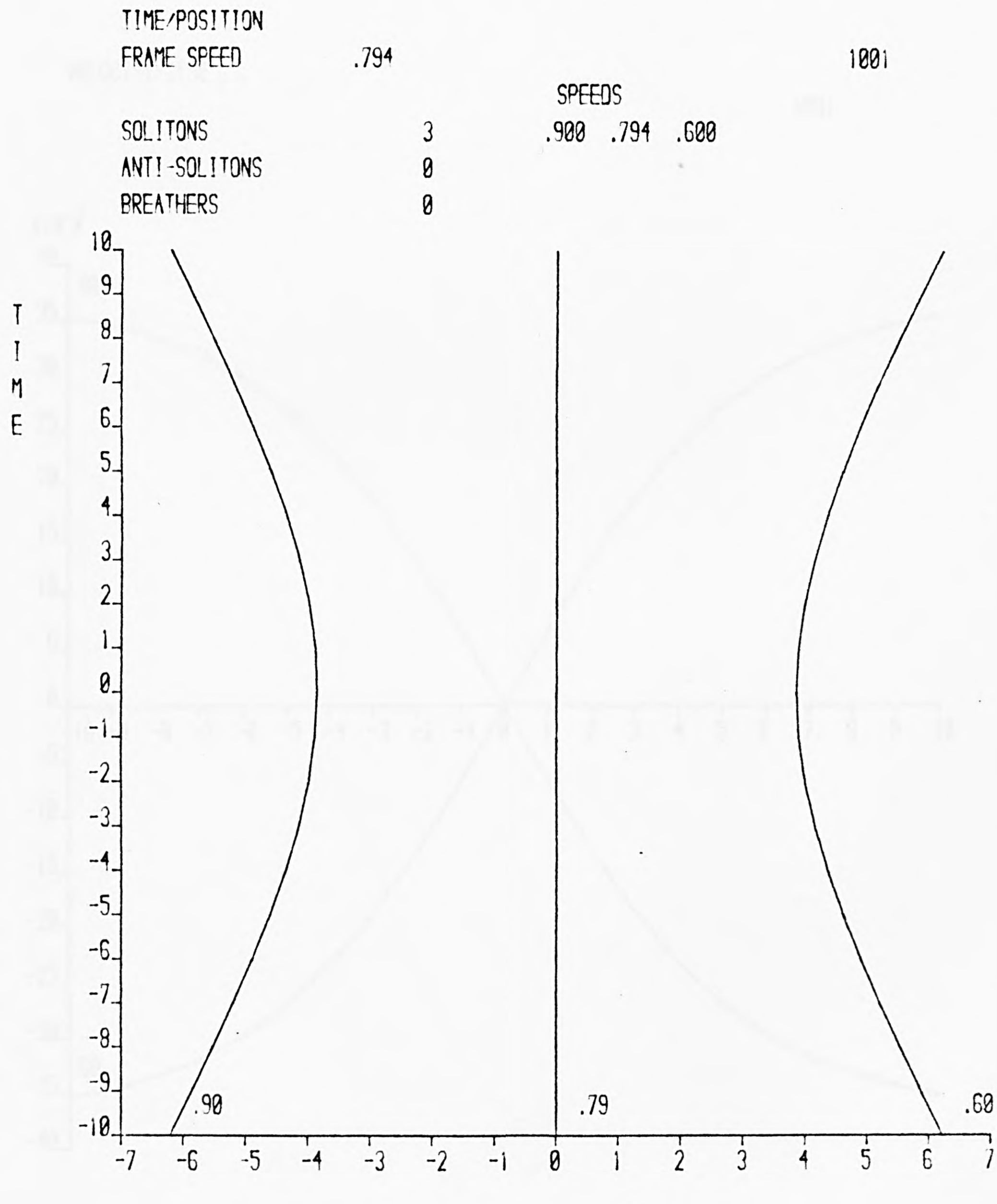

TIME/POSITION
FRAME SPEED .794
1001
SPEEDS
SOLITONS 3 .900 .794 .600
ANTI-SOLITONS 0
BREATHERS 0
TIME
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
.90
.79
.60
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

Fig 7.44a

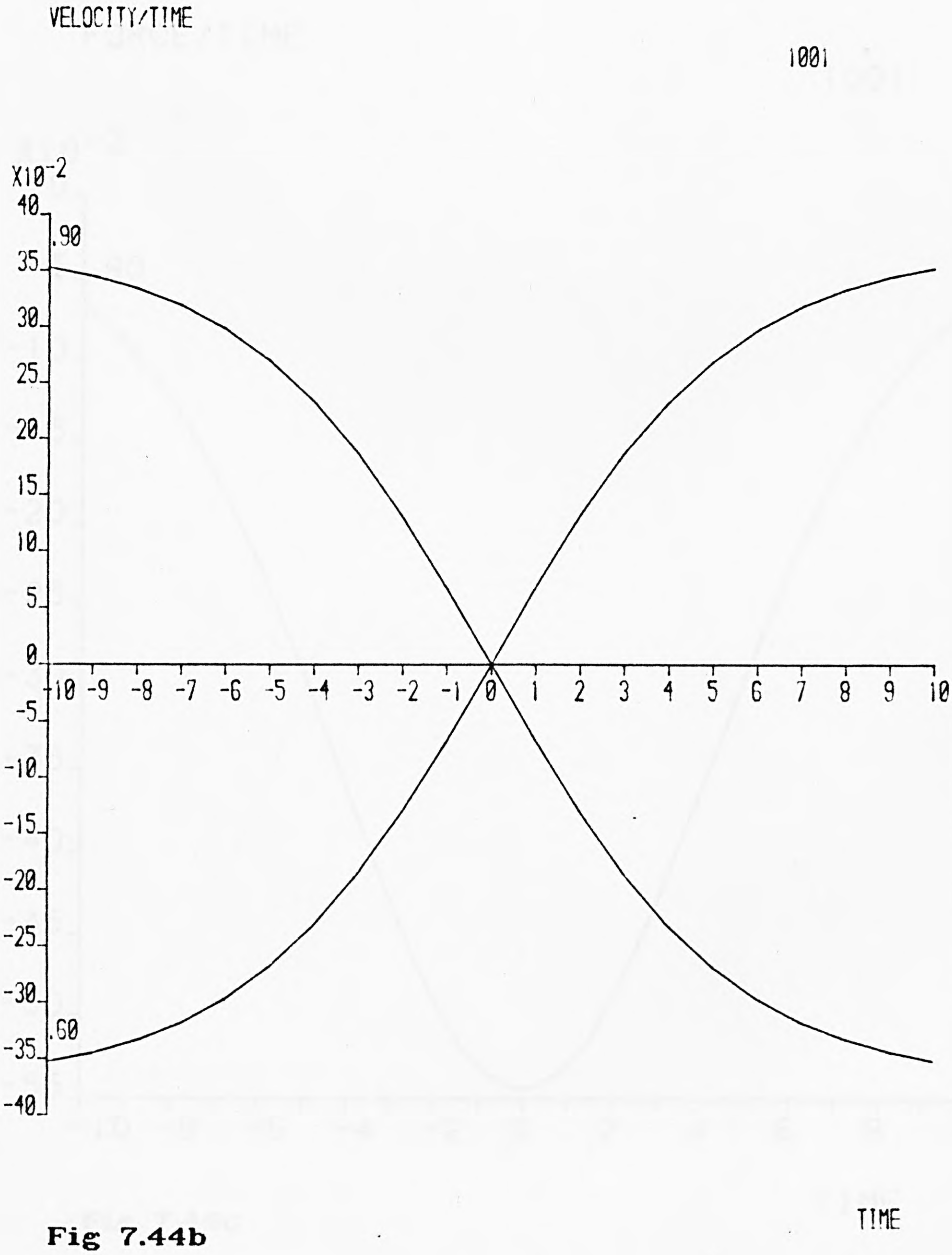

VELOCITY/TIME
1001
X10^-2
40
.90
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
.60
-35
-40
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
TIME

Fig 7.44b

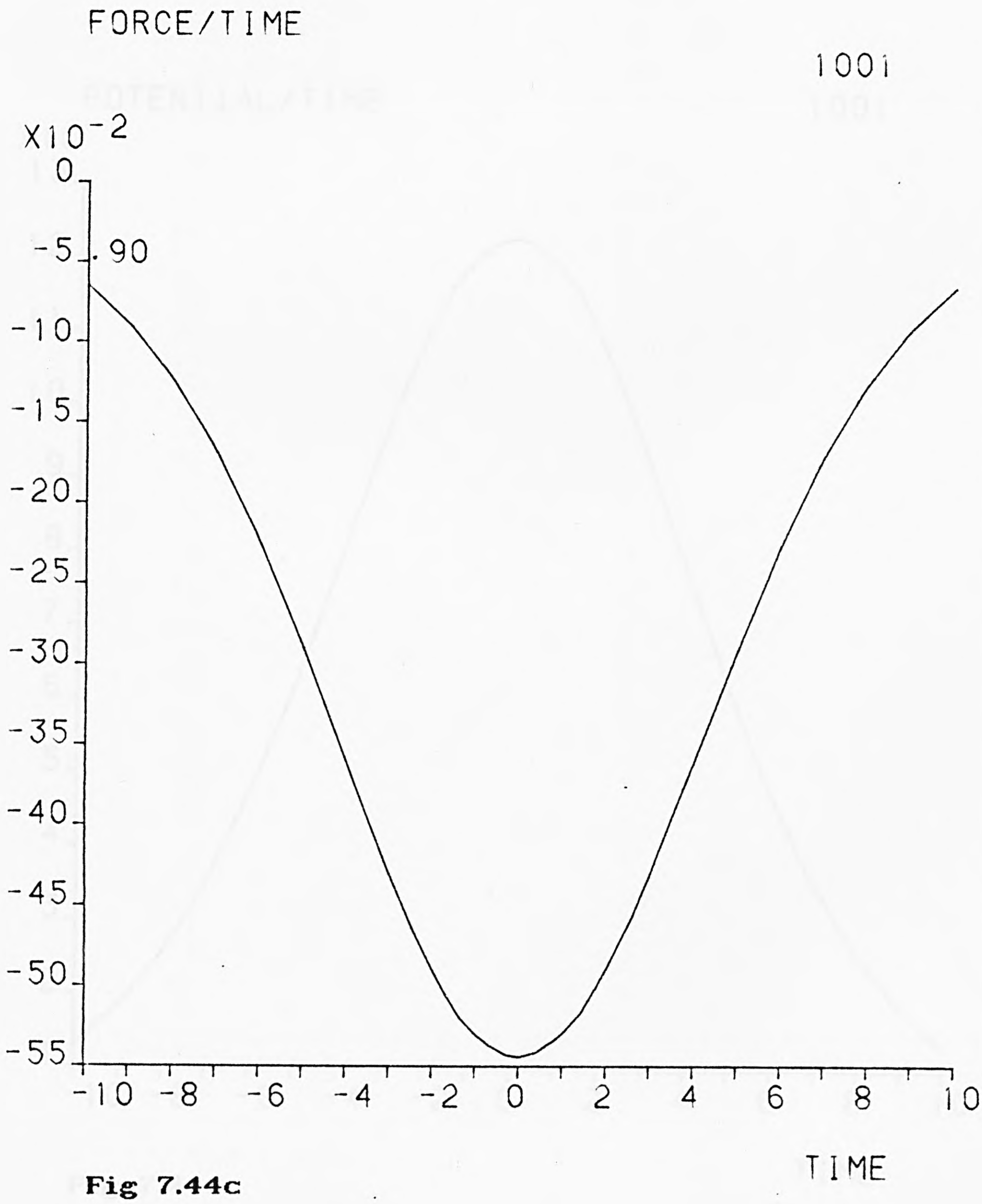

FORCE/TIME
1001
X10^-2
10
-5
.90
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-10
-8
-6
-4
-2
0
2
4
6
8
10
TIME

Fig 7.44c

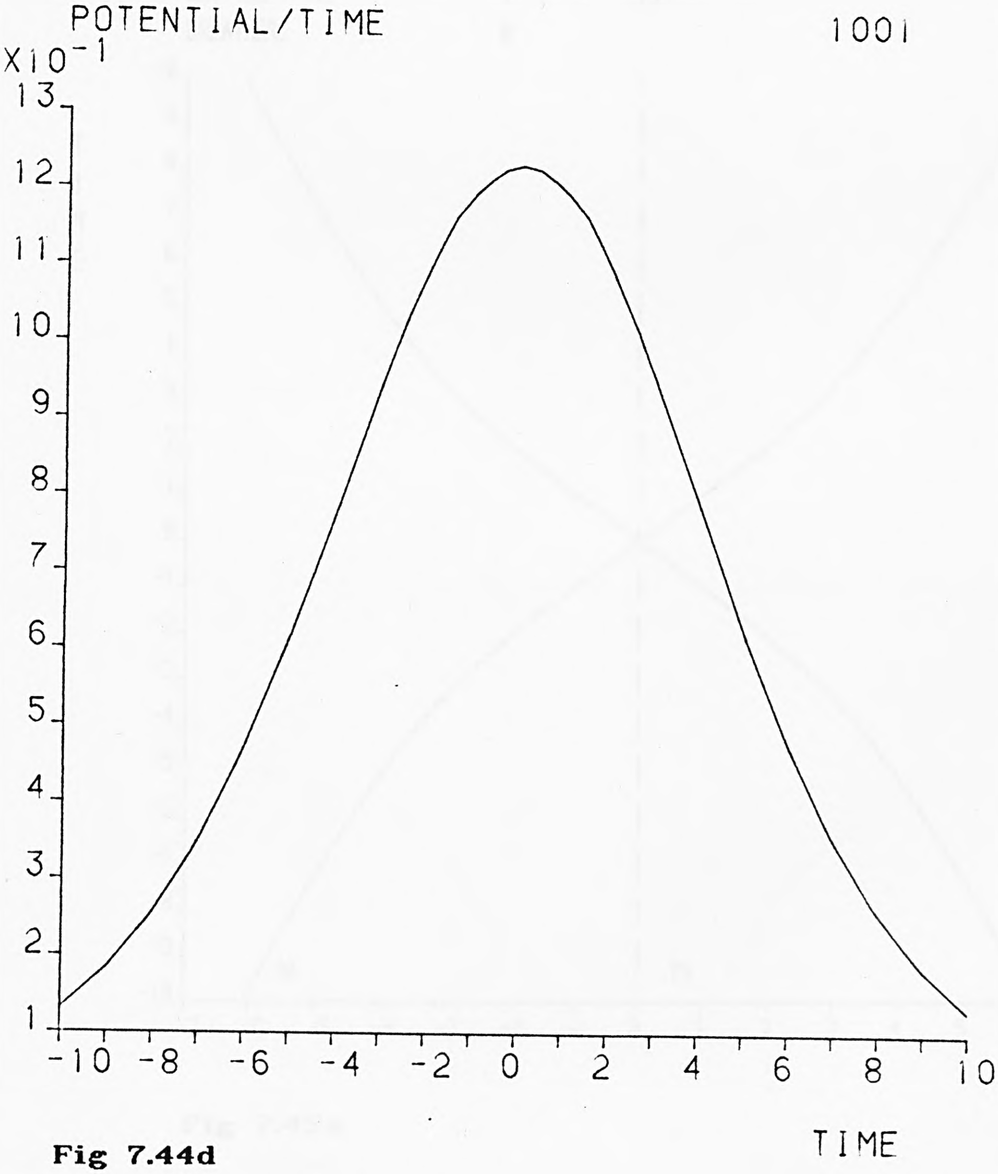

POTENTIAL/TIME
1001
X10^-1
13
12
11
10
9
8
7
6
5
4
3
2
1
-10
-8
-6
-4
-2
0
2
4
6
8
10
TIME

Fig 7.44d

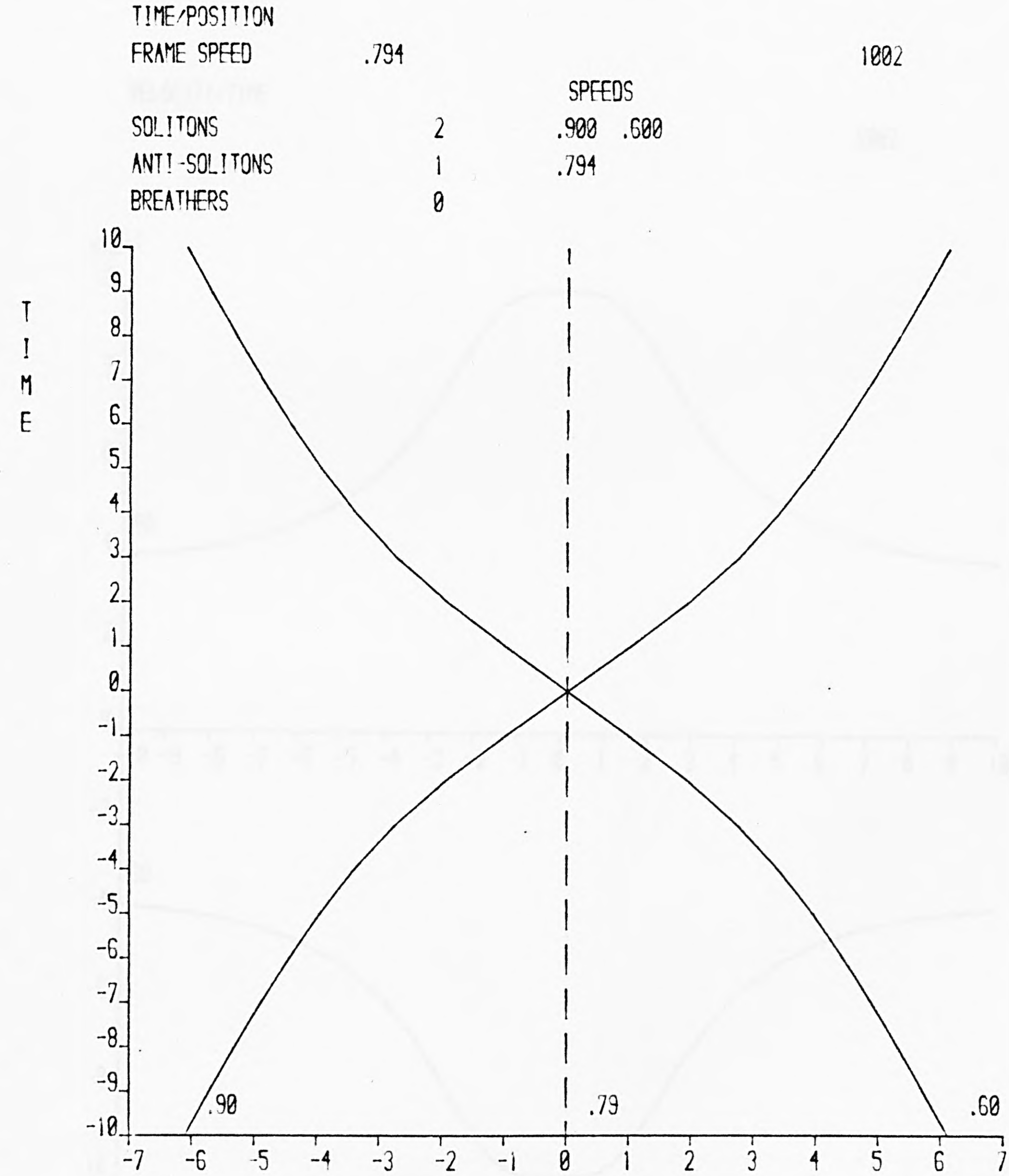

TIME/POSITION
FRAME SPEED .794 1002
SPEEDS
SOLITONS 2 .900 .600
ANTI-SOLITONS 1 .794
BREATHERS 0
TIME
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
.90
.79
.60
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

Fig 7.45a

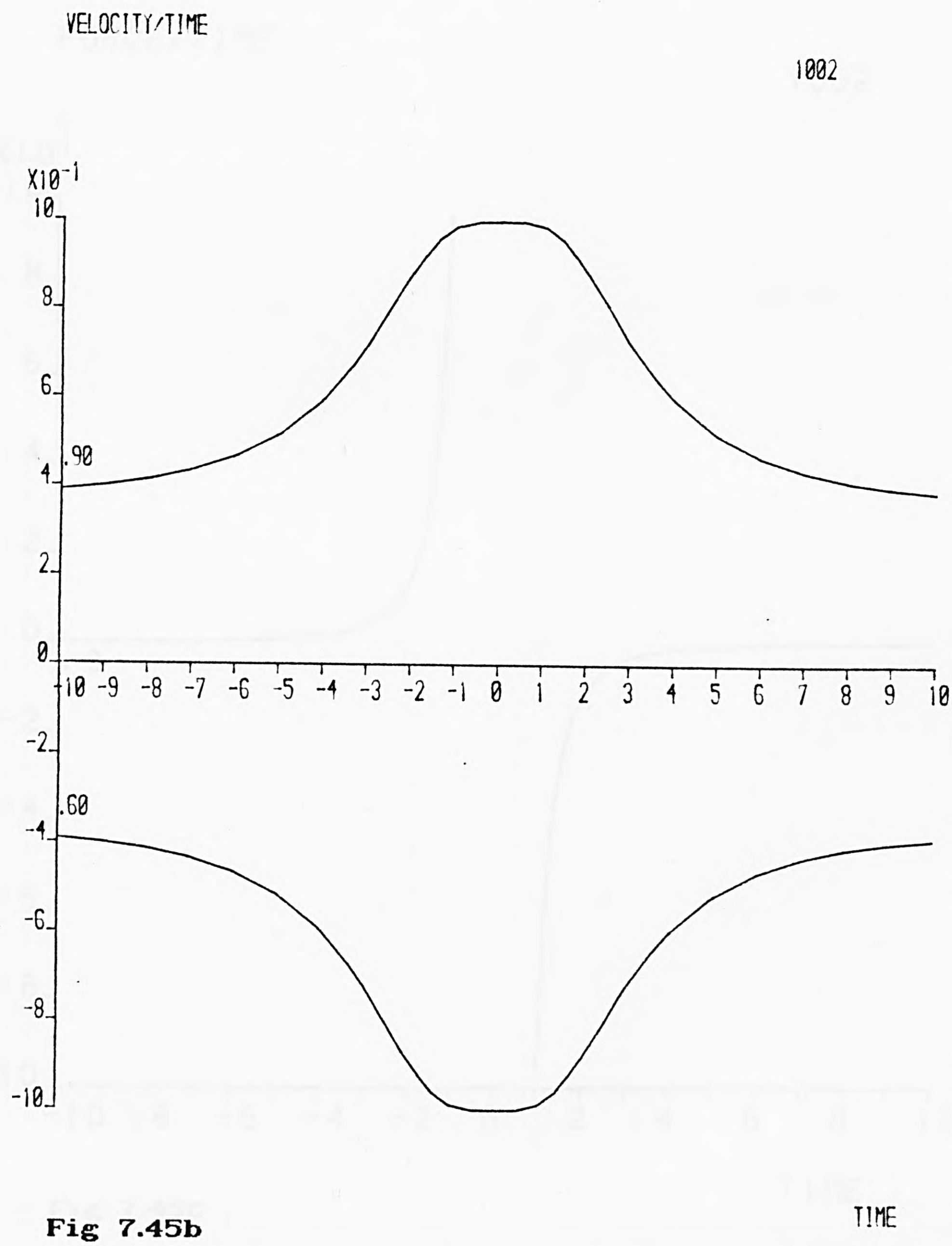

VELOCITY/TIME
1002
X10^-1
10
8
6
4
.90
2
0
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
-2
.60
-4
-6
-8
-10
TIME

Fig 7.45b

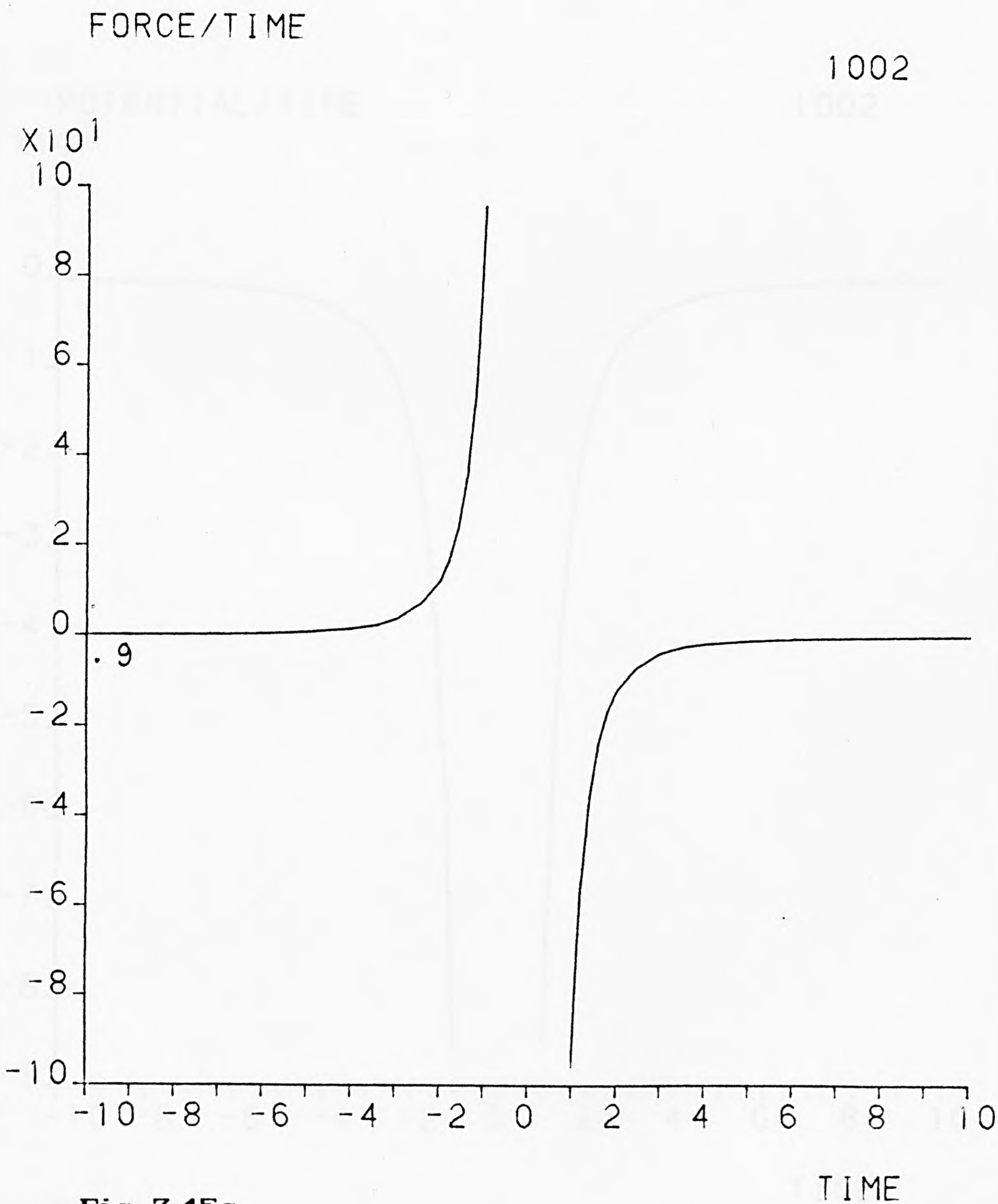


Fig 7.45c

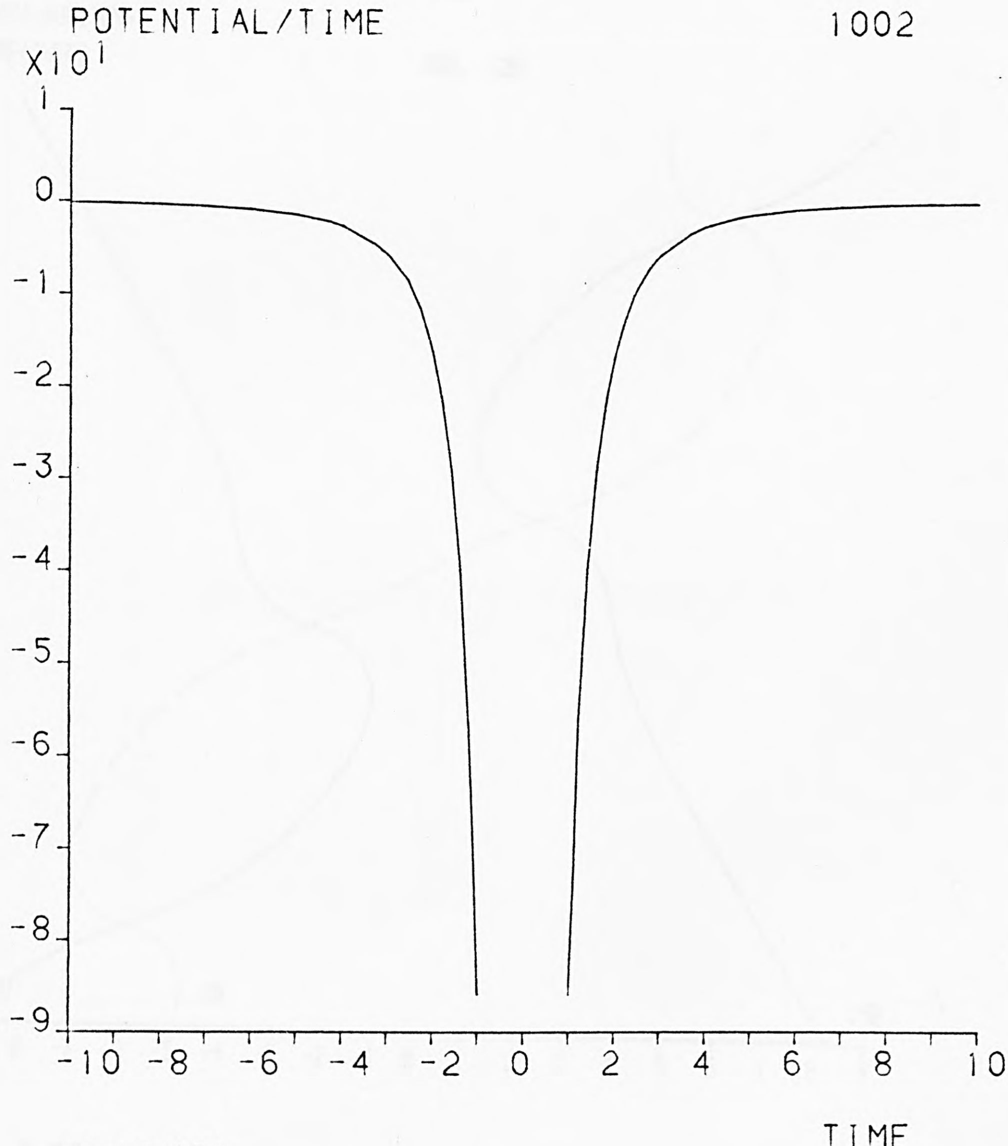
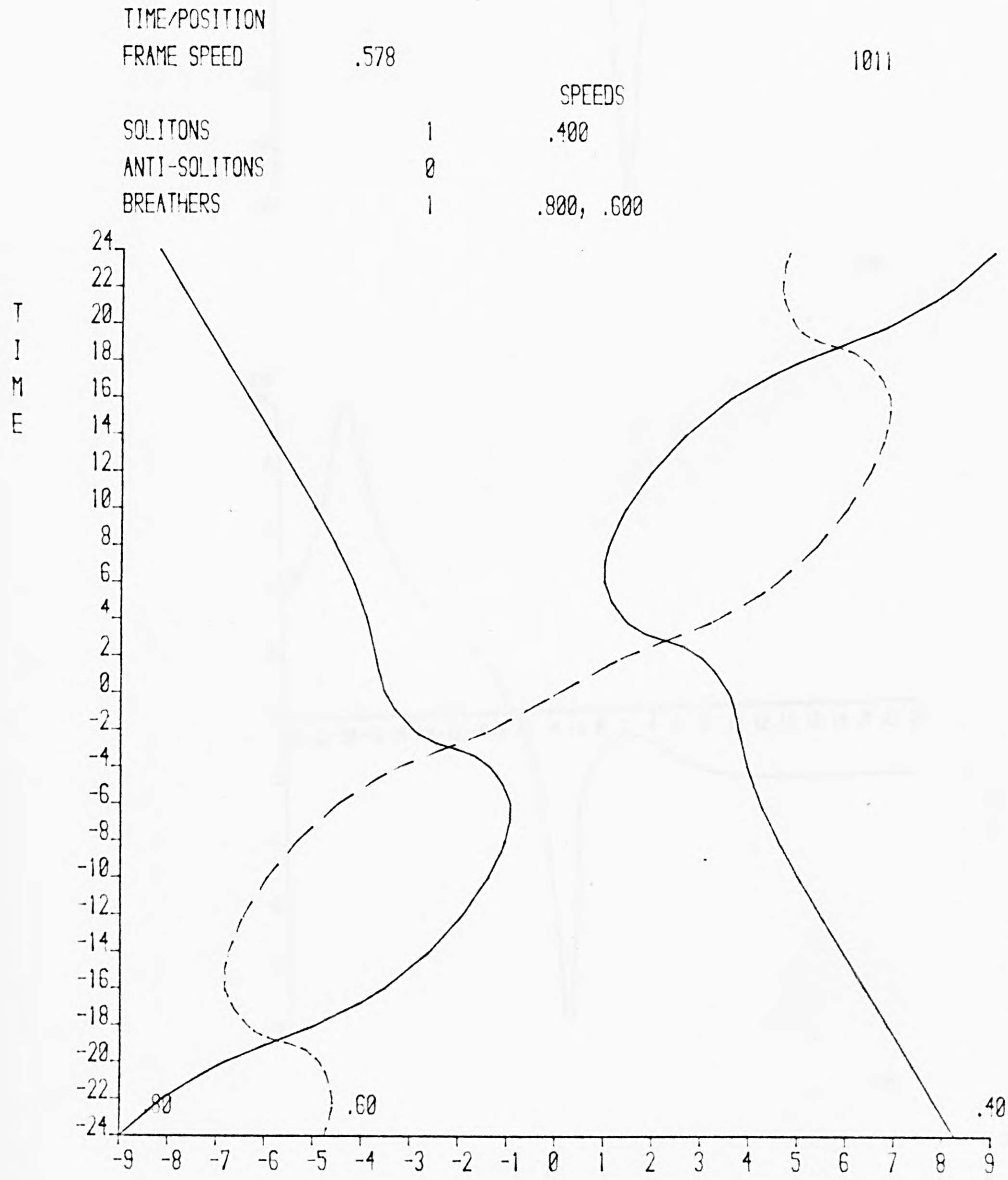
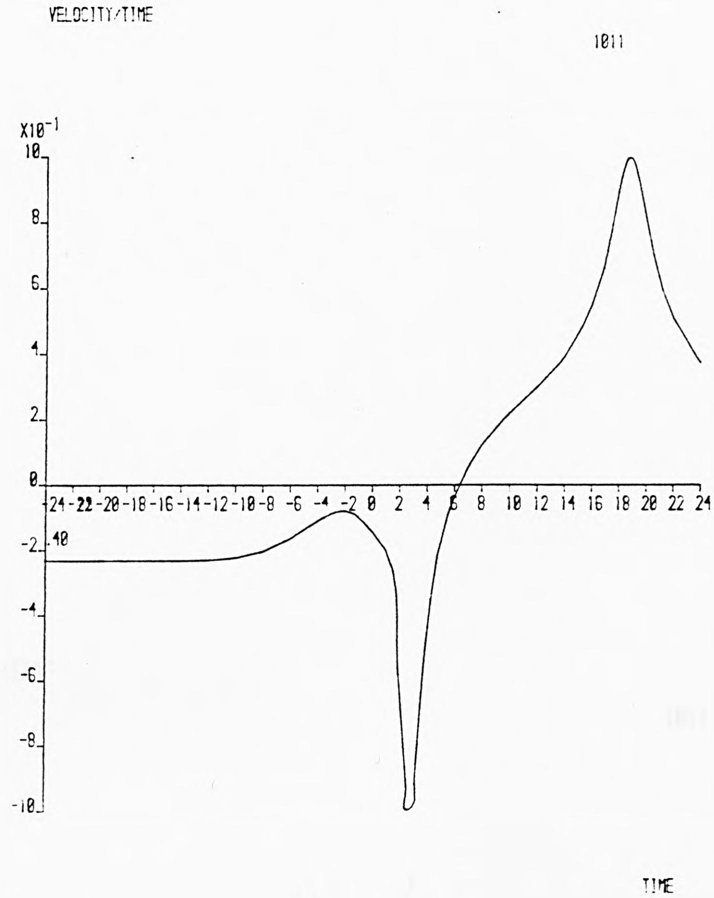


Fig 7.45d



TIME/POSITION
FRAME SPEED .578
1011
SPEEDS
SOLITONS 1 .400
ANTI-SOLITONS 0
BREATHERS 1 .800, .600
TIME
.80
.60
.40

Fig 7.46a


VELOCITY/TIME
1011
X10^-1
10
8
6
4
2
0
-2
-4
-6
-8
-10
-24 -22 -20 -18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22 24
.40
TIME

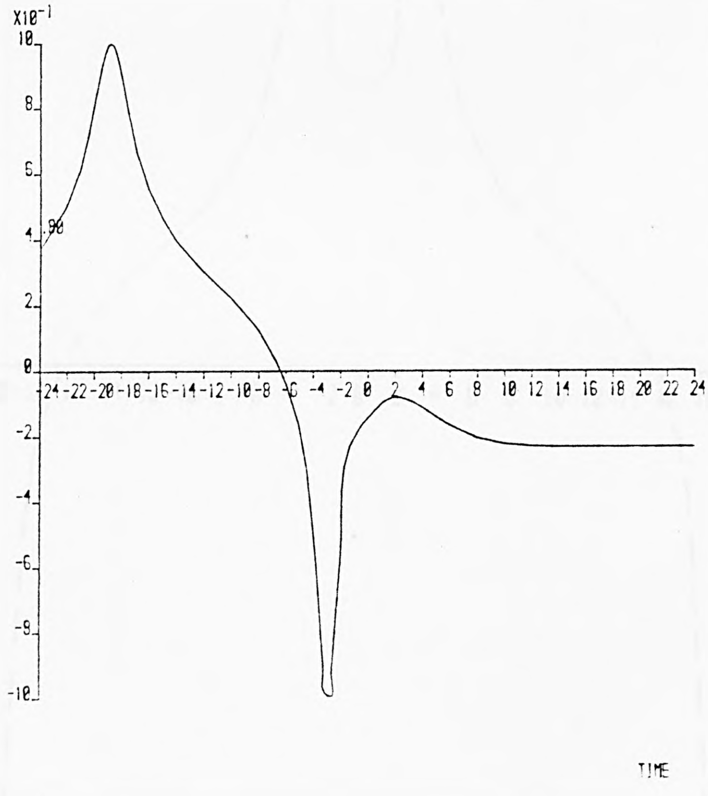

X10^-1
10
8
6
4
2
0
-2
-4
-6
-8
-10
-24 -22 -20 -18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22 24
.90
TIME

Fig 7.46b

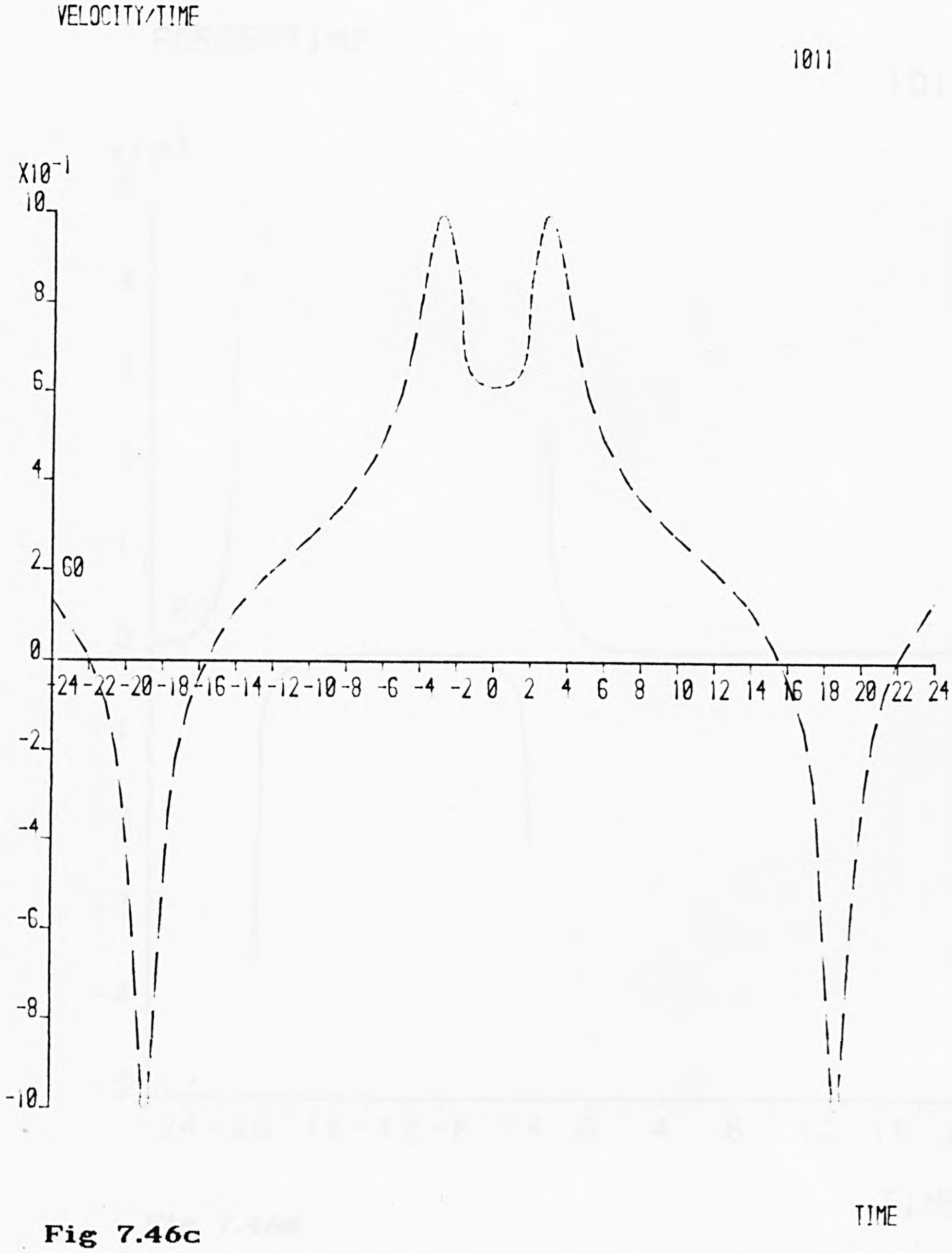

VELOCITY/TIME
1011
X10^-1
10
8
6
4
2
0
-2
-4
-6
-8
-10
GO
-24 -22 -20 -18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22 24
TIME

Fig 7.46c

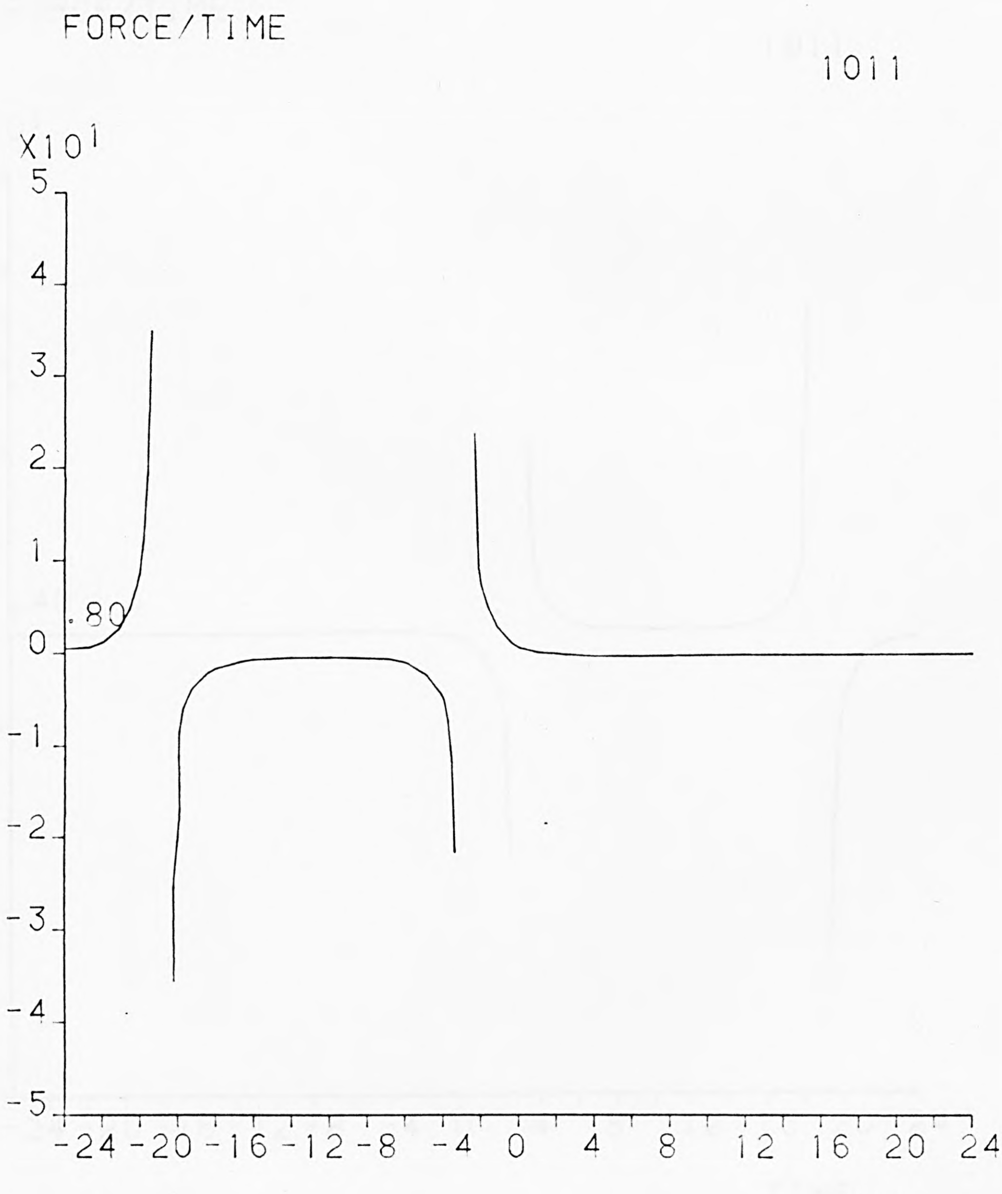

FORCE/TIME
1011
X10^1
5
4
3
2
1
.80
0
-1
-2
-3
-4
-5
-24 -20 -16 -12 -8 -4 0 4 8 12 16 20 24
TIME

Fig 7.46d

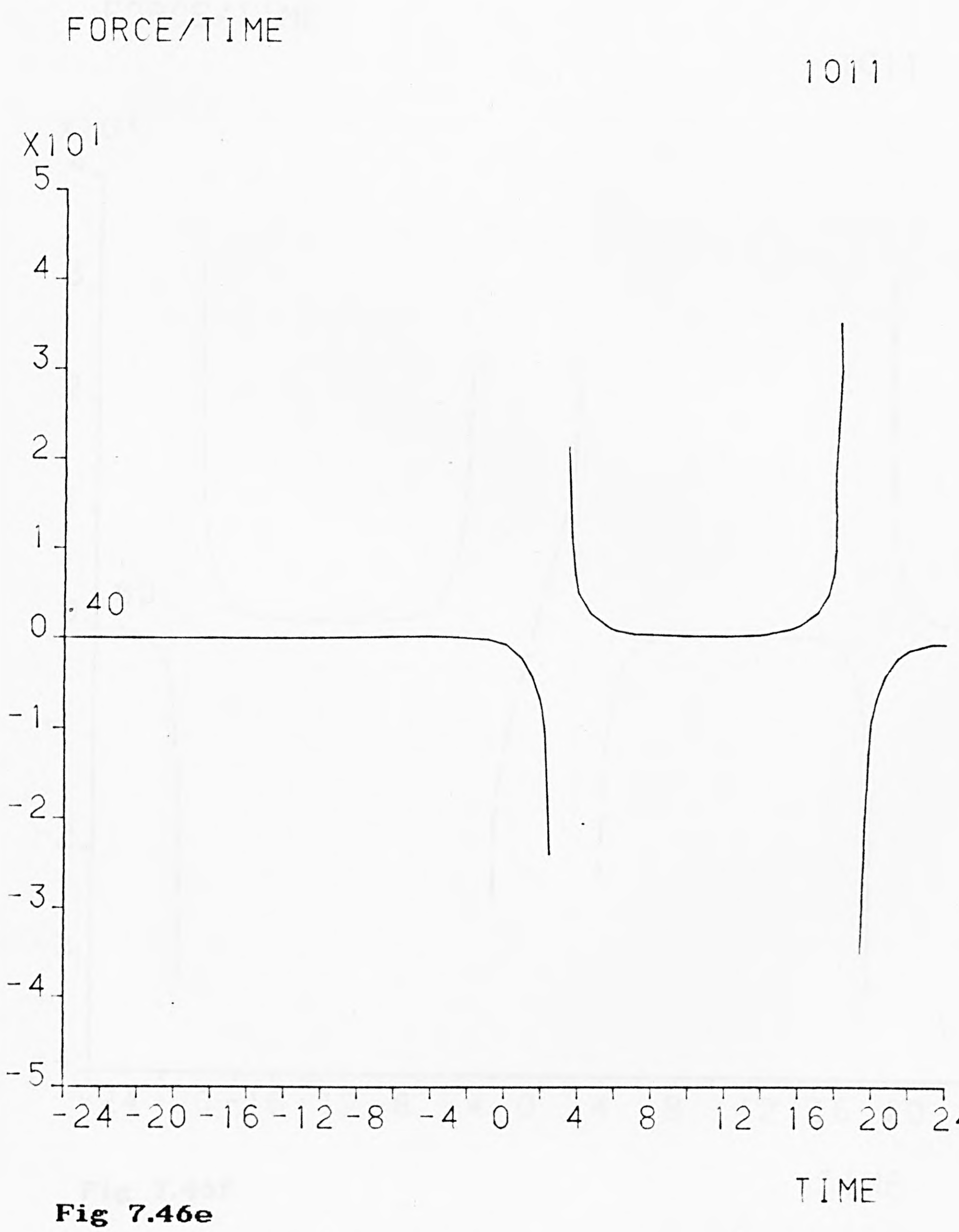

FORCE/TIME
1011
X10^1
5
4
3
2
1
.40
0
-1
-2
-3
-4
-5
-24 -20 -16 -12 -8 -4 0 4 8 12 16 20 24
TIME

Fig 7.46e

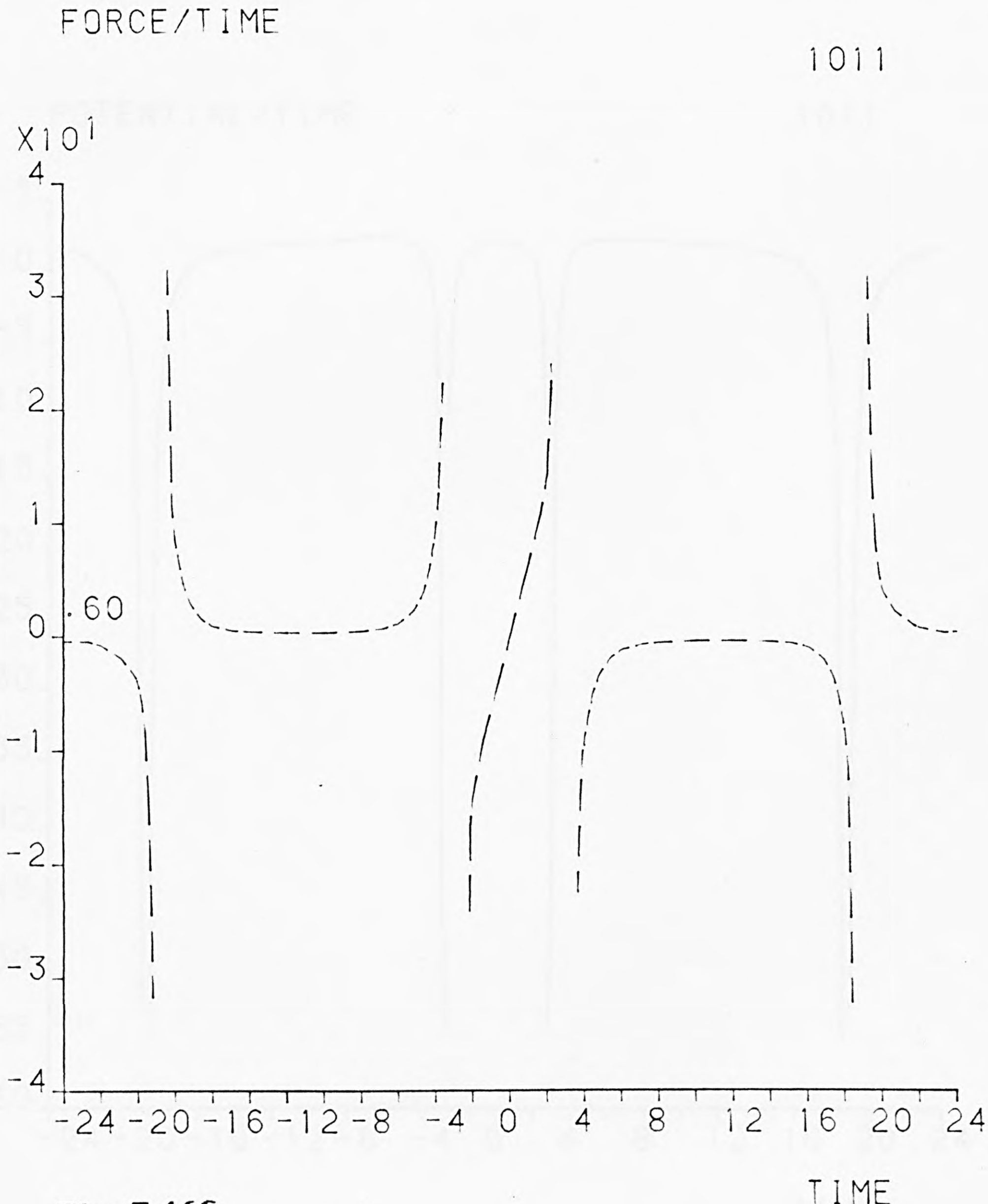

FORCE/TIME
1011
X10^1
4
3
2
1
.60
0
-1
-2
-3
-4
-24 -20 -16 -12 -8 -4 0 4 8 12 16 20 24
TIME

Fig 7.46f

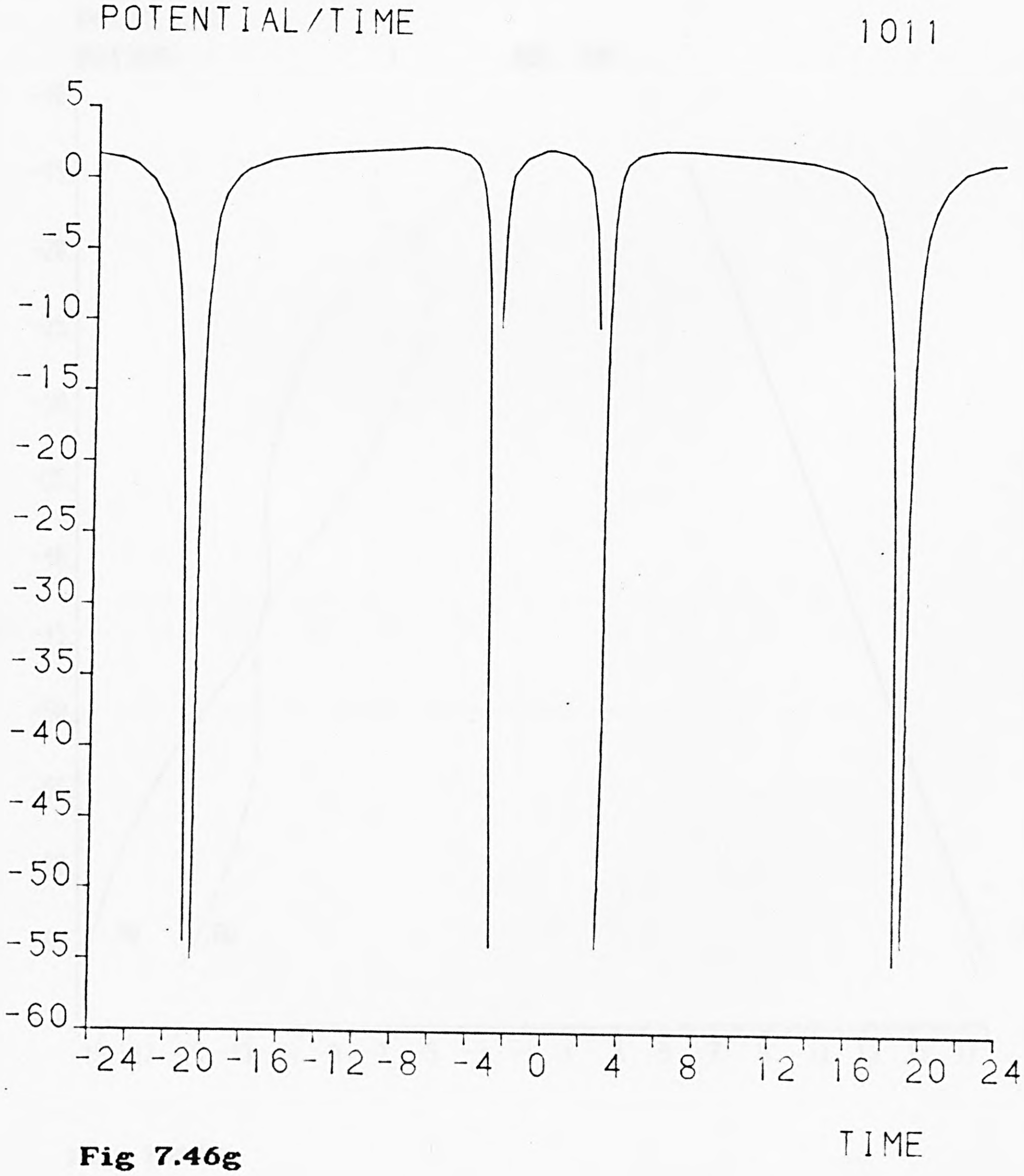
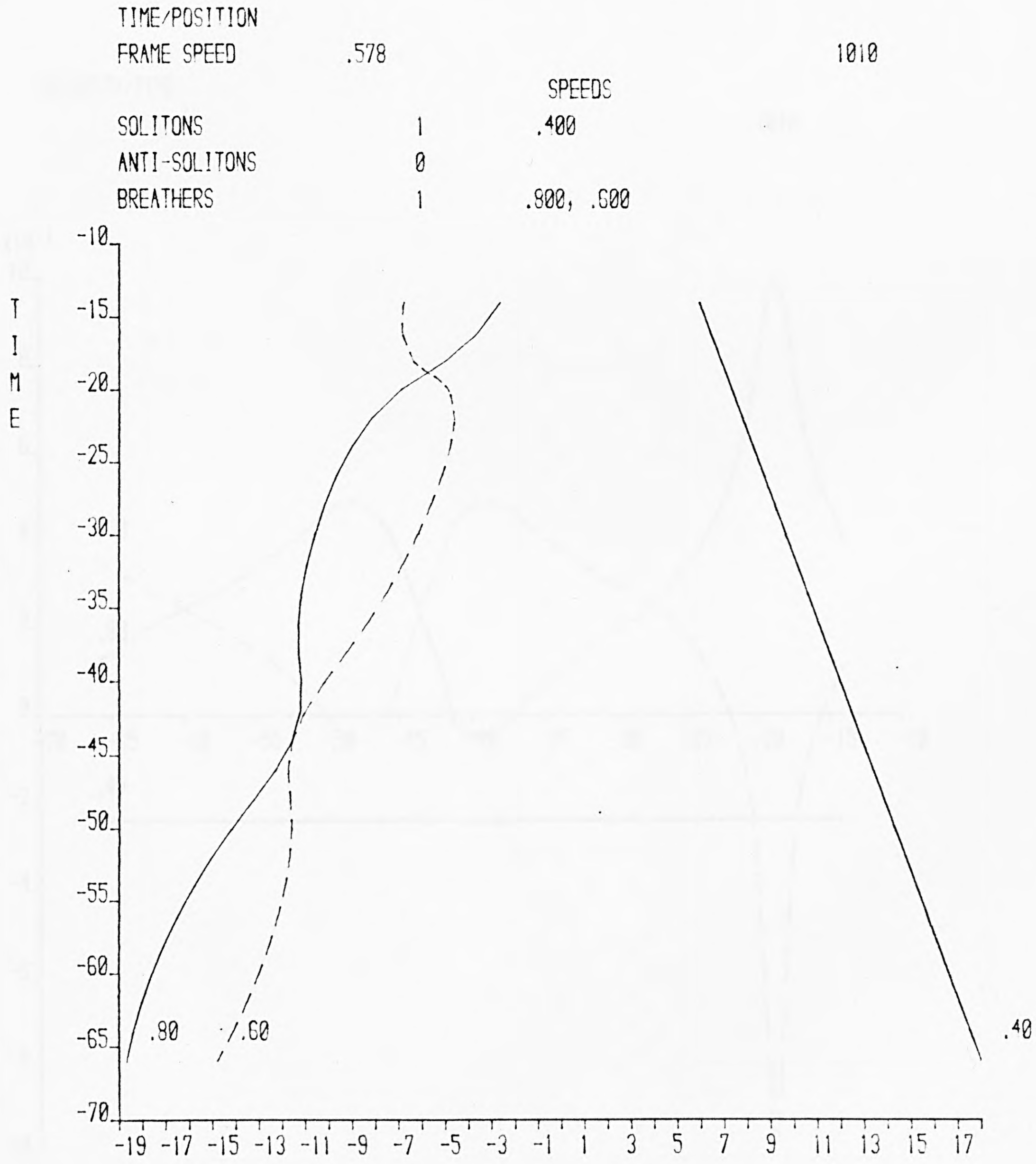
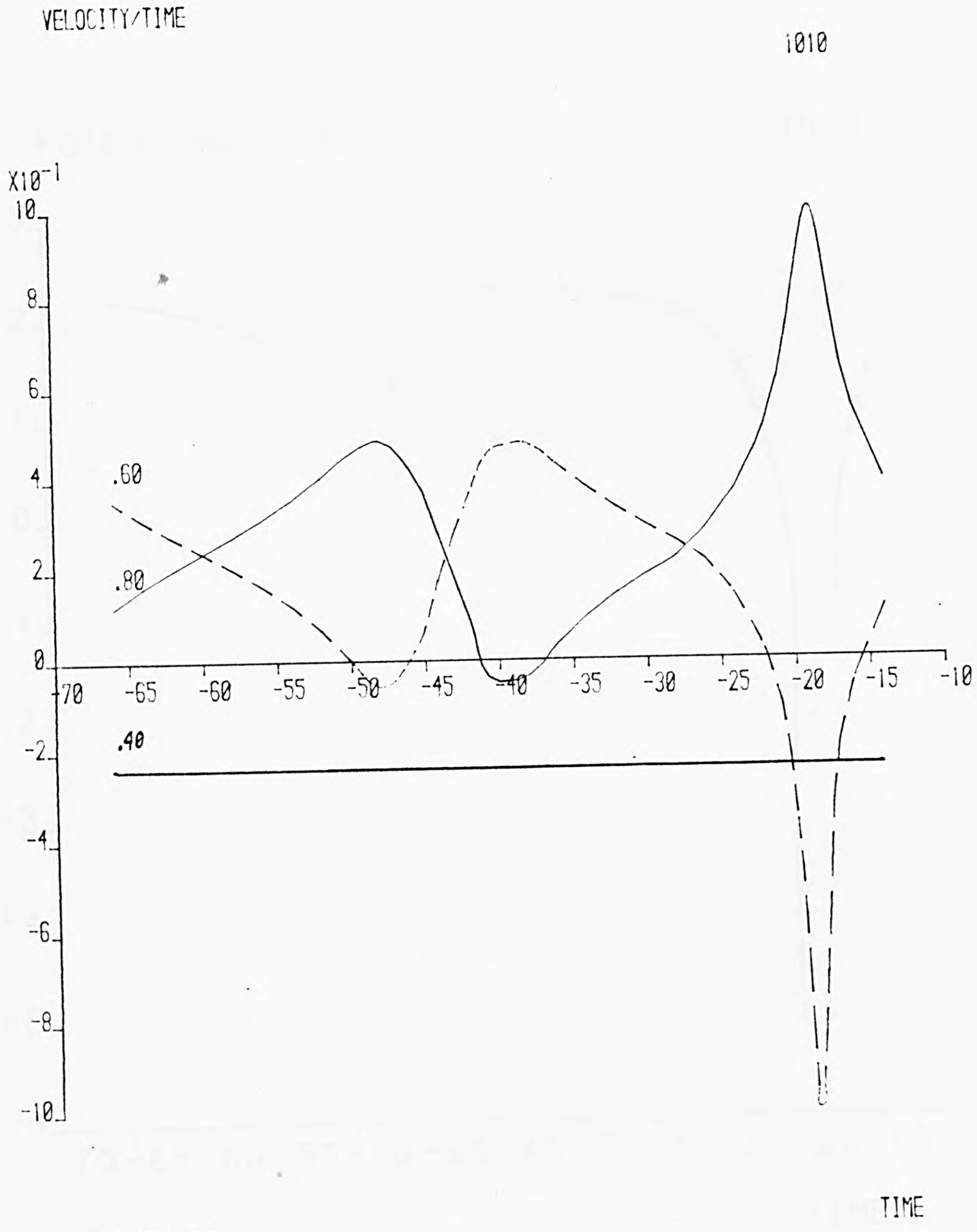

POTENTIAL/TIME
1011
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
-24 -20 -16 -12 -8 -4 0 4 8 12 16 20 24
TIME

Fig 7.46g



TIME/POSITION
FRAME SPEED .578 1010
SPEEDS
SOLITONS 1 .400
ANTI-SOLITONS 0
BREATHERS 1 .800, .600
TIME
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
-65
-70
.80
.60
.40
-19 -17 -15 -13 -11 -9 -7 -5 -3 -1 1 3 5 7 9 11 13 15 17

Fig 7.46h



VELOCITY/TIME
1010
X10^-1
10
8
6
4
2
0
-2
-4
-6
-8
-10
.60
.80
.40
-70
-65
-60
-55
-50
-45
-40
-35
-30
-25
-20
-15
-10
TIME

Fig 7.46i

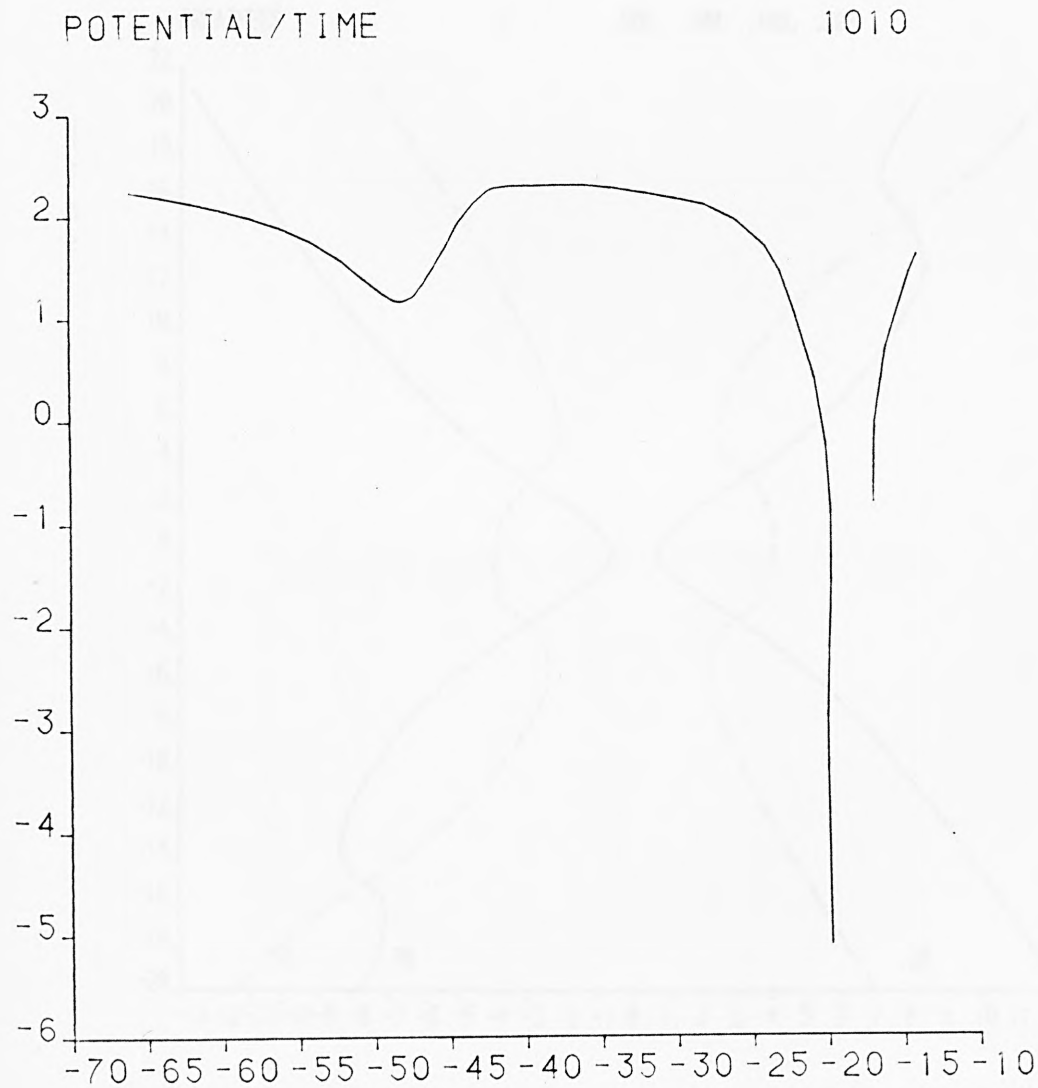

POTENTIAL/TIME
1010
3
2
1
0
-1
-2
-3
-4
-5
-6
-70 -65 -60 -55 -50 -45 -40 -35 -30 -25 -20 -15 -10
TIME

Fig 7.46j

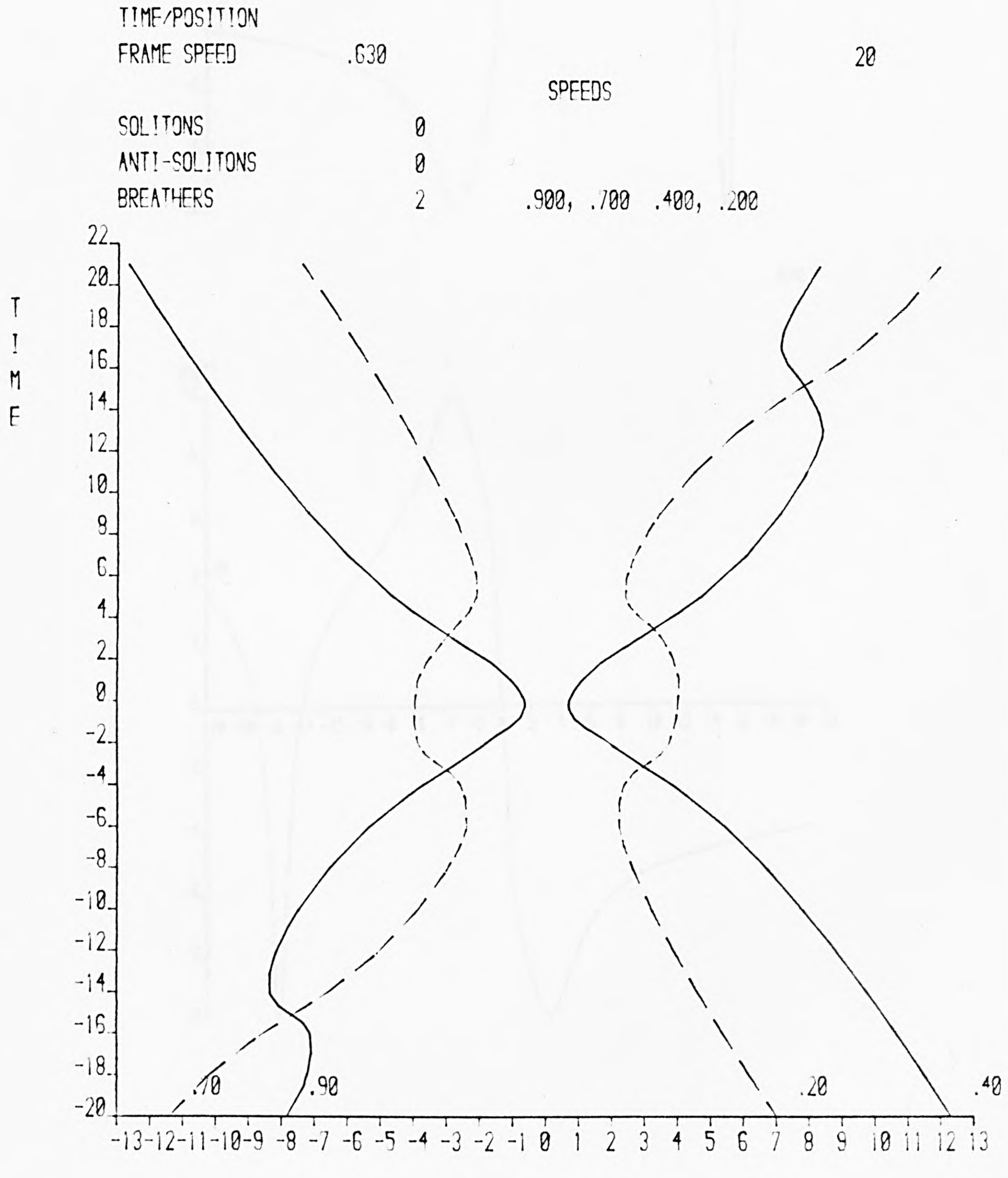
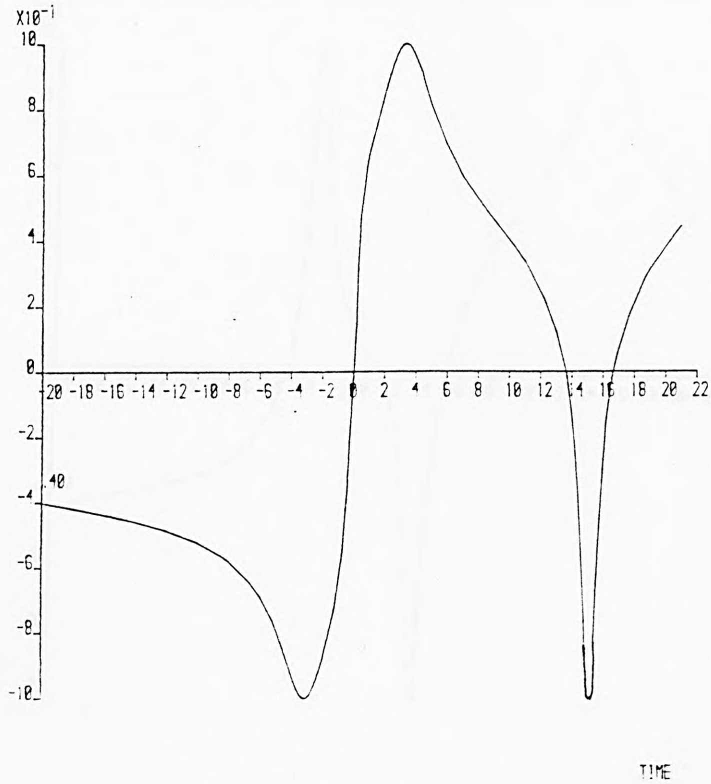

TIME/POSITION
FRAME SPEED .630
20
SPEEDS
SOLITONS 0
ANTI-SOLITONS 0
BREATHERS 2 .900, .700 .400, .200
TIME
22
20
18
16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
.70
.90
.20
.40
-13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13

Fig 7.47a

VELOCITY/TIME

20



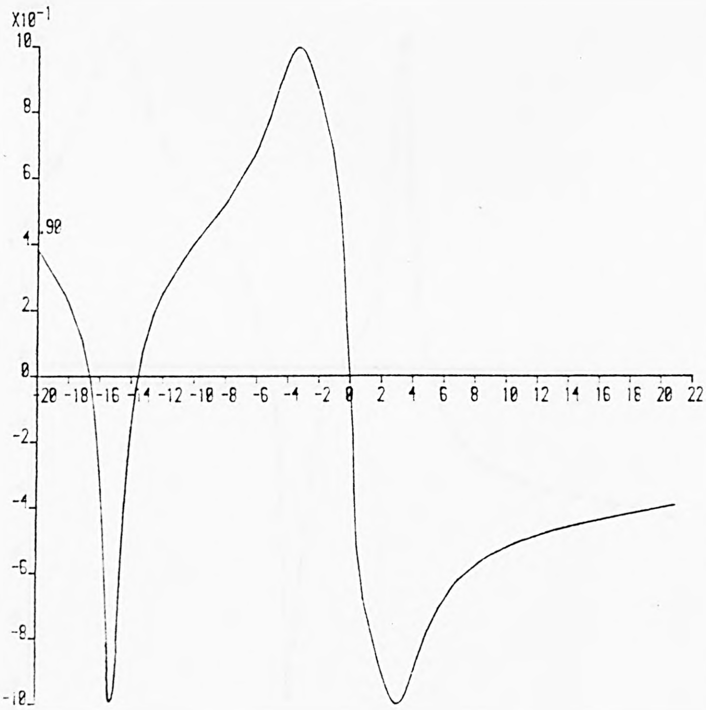
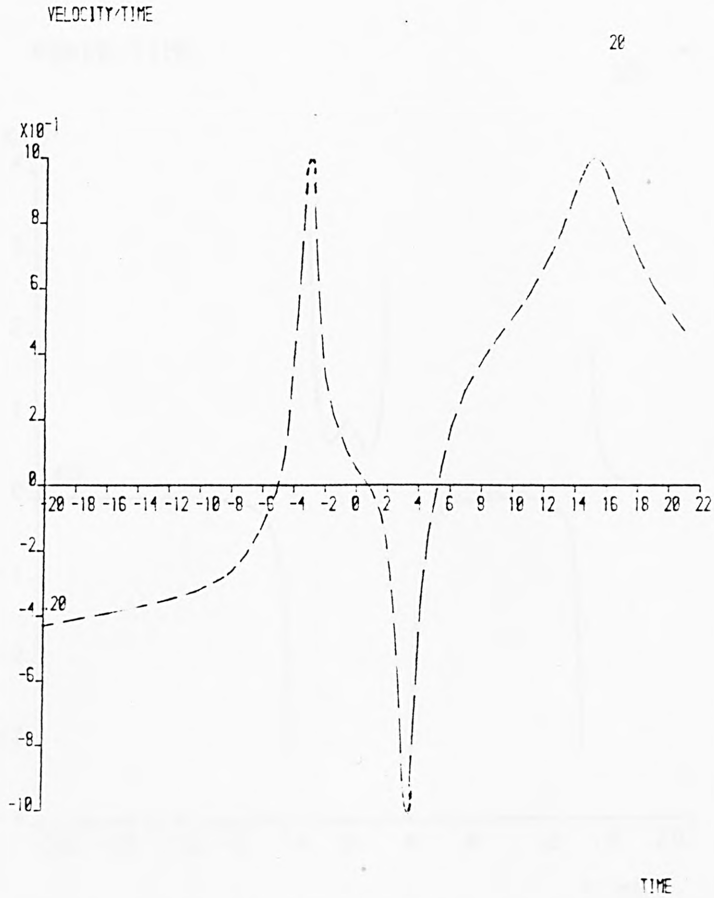


Fig 7.47b


VELOCITY/TIME
20
X10⁻¹
10
8
6
4
2
0
-2
-4
-6
-8
-10
.20
-20 -18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22
TIME

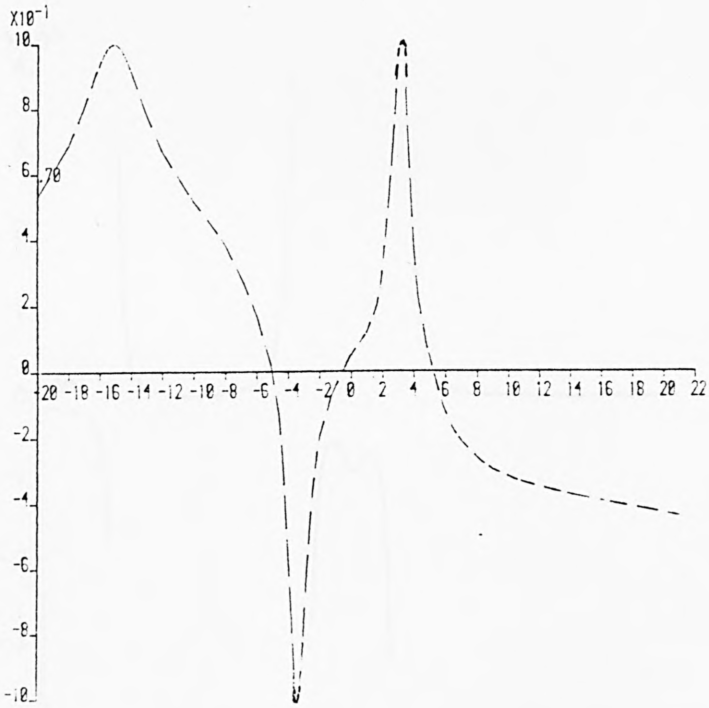

X10⁻¹
10
8
6
4
2
0
-2
-4
-6
-8
-10
.70
-20 -18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22

Fig 7.47c

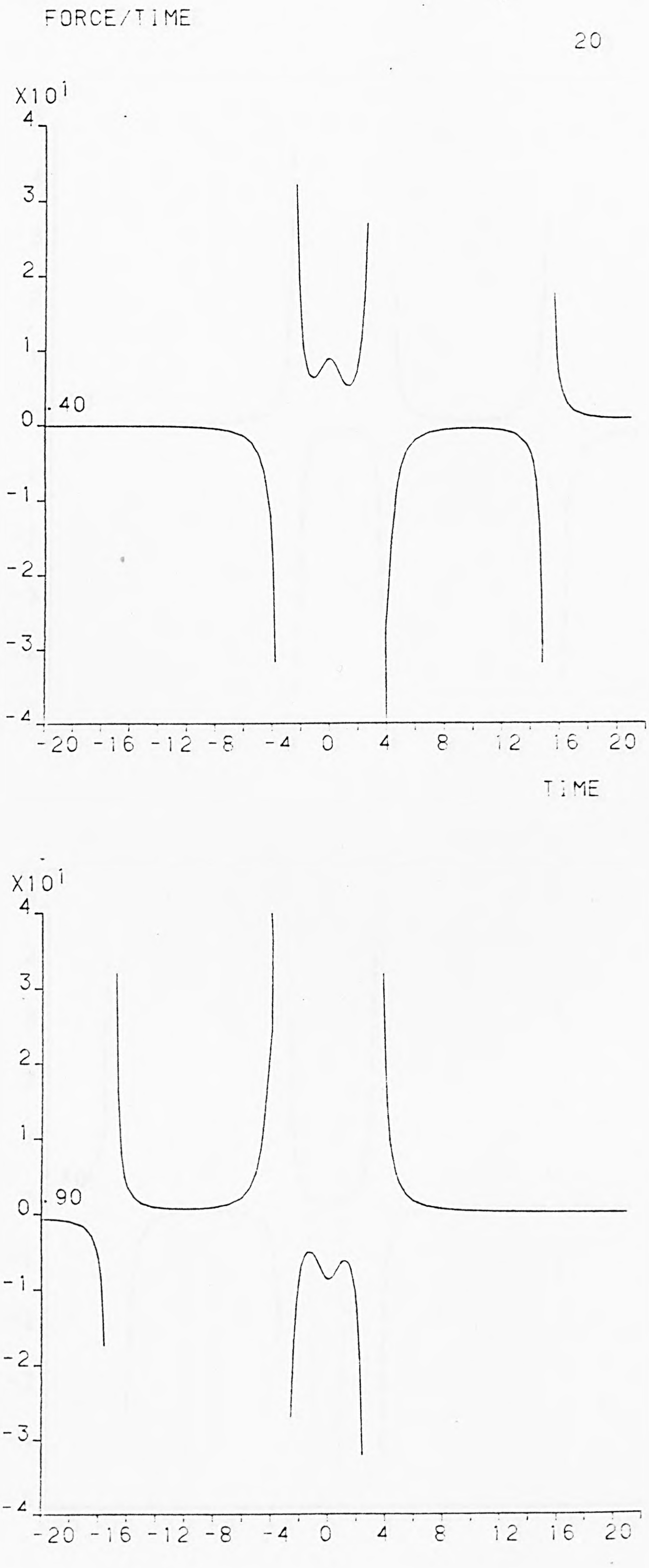

FORCE/TIME
20
X10¹
4
3
2
1
0
-1
-2
-3
-4
.40
-20 -16 -12 -8 -4 0 4 8 12 16 20
TIME
X10¹
.90
-20 -16 -12 -8 -4 0 4 8 12 16 20

Fig 7.47d

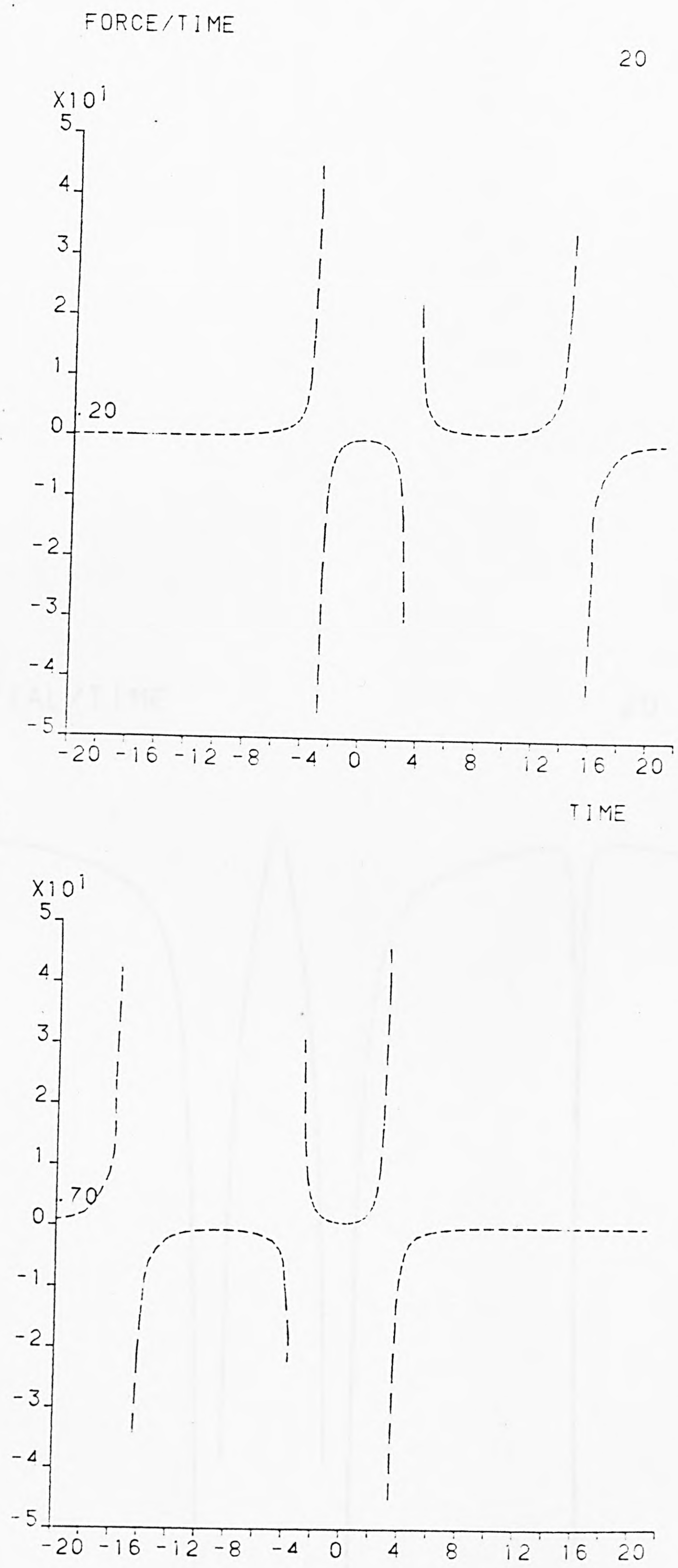

FORCE/TIME
20
X10¹
.20
-20 -16 -12 -8 -4 0 4 8 12 16 20
TIME
X10¹
.70
-20 -16 -12 -8 -4 0 4 8 12 16 20

Fig 7.47e

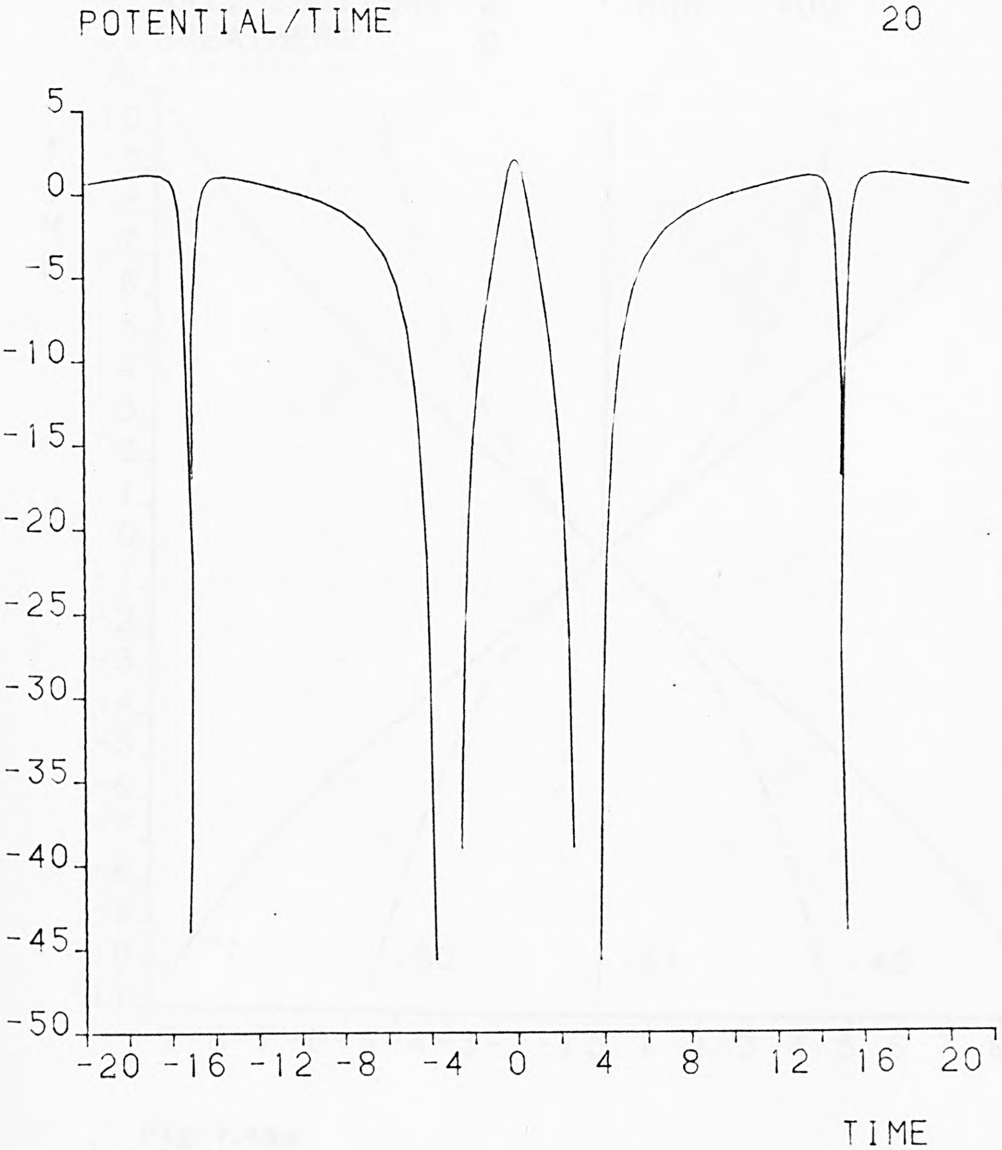

POTENTIAL/TIME
20
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-20 -16 -12 -8 -4 0 4 8 12 16 20
TIME

Fig 7.47f

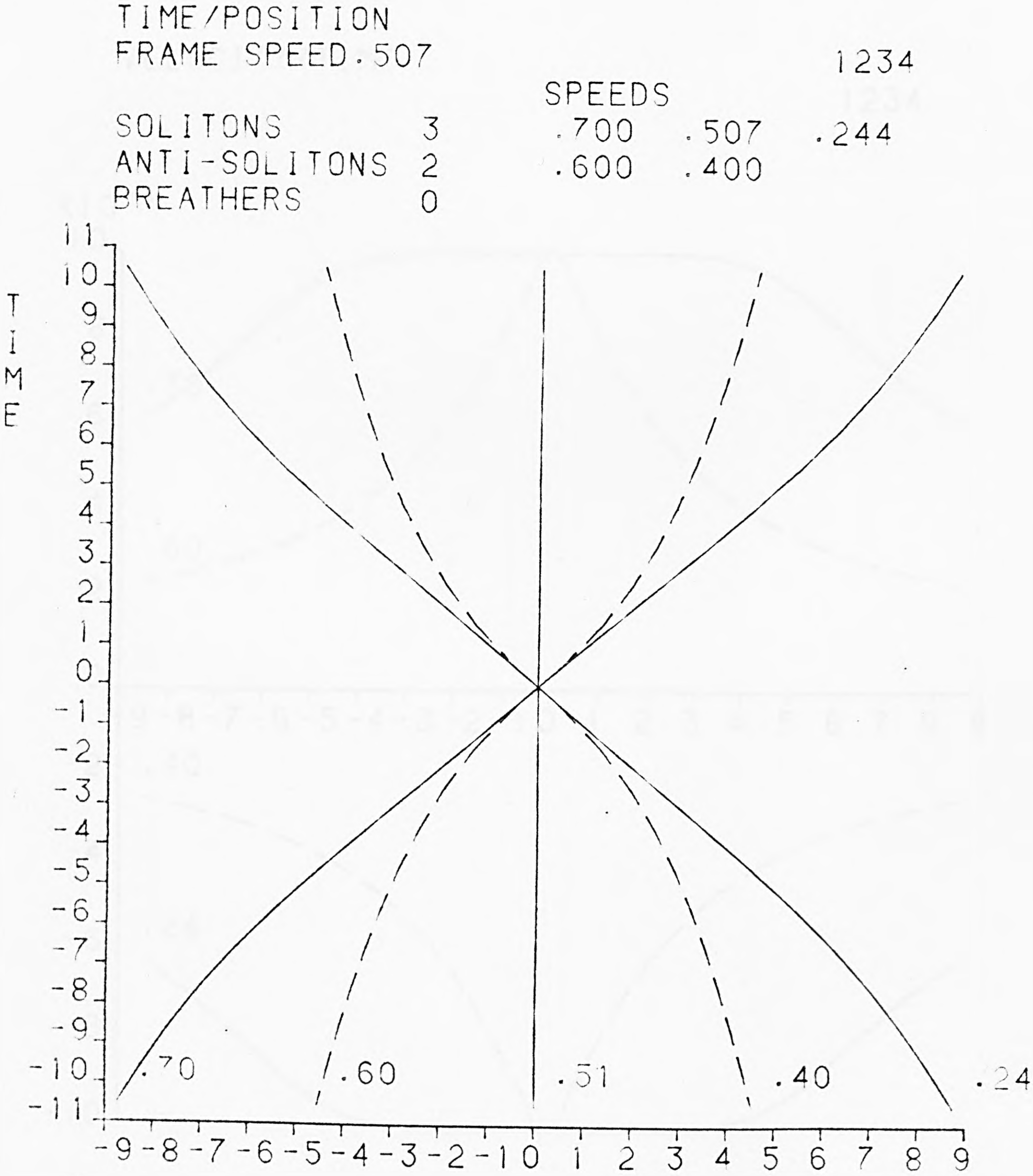

TIME/POSITION
FRAME SPEED.507
1234
SPEEDS
SOLITONS 3 .700 .507 .244
ANTI-SOLITONS 2 .600 .400
BREATHERS 0
TIME
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
.70
.60
.51
.40
.24
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

Fig 7.48a

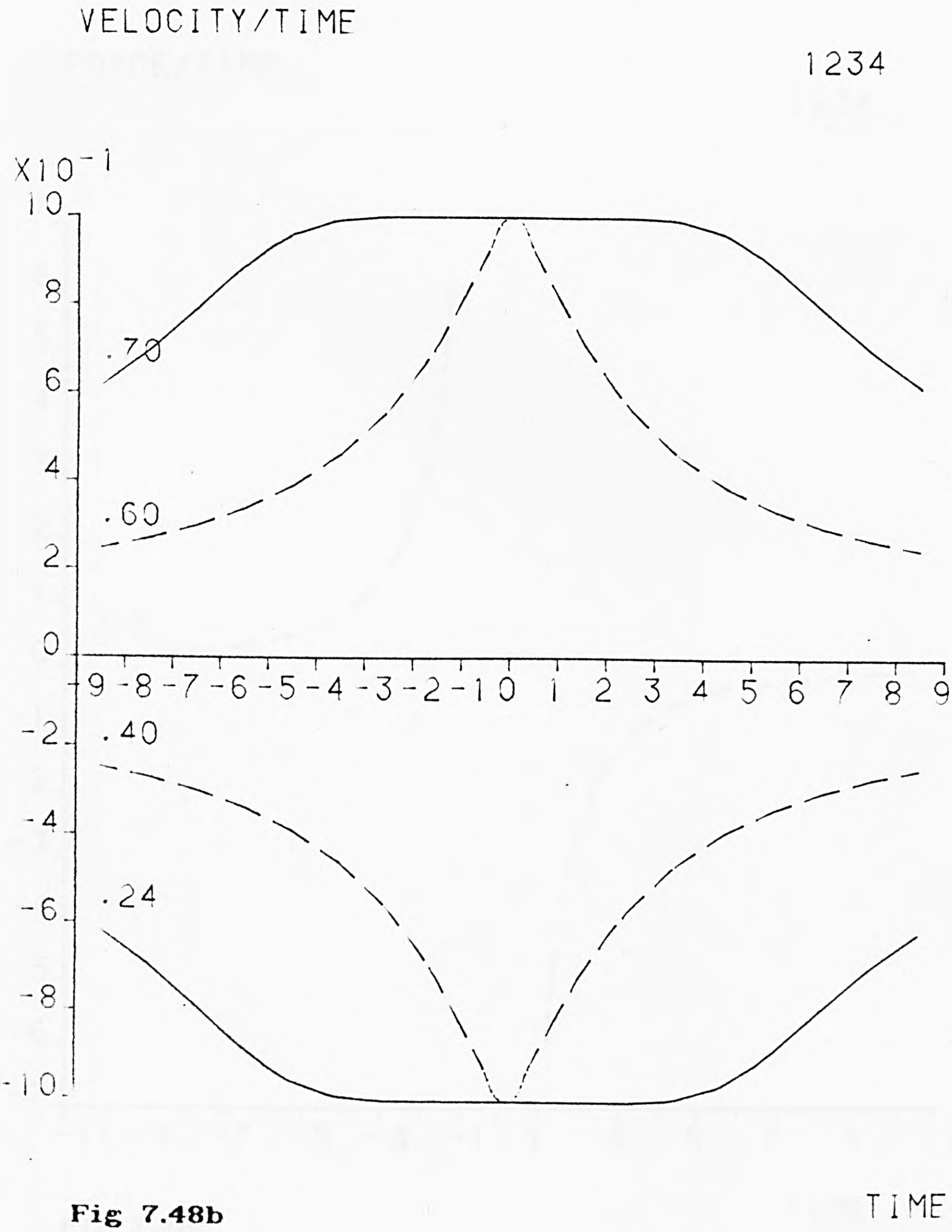

VELOCITY/TIME
1234
X10^{-1}
10
8
6
4
2
0
-2
-4
-6
-8
-10
.70
.60
.40
.24
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
TIME

Fig 7.48b

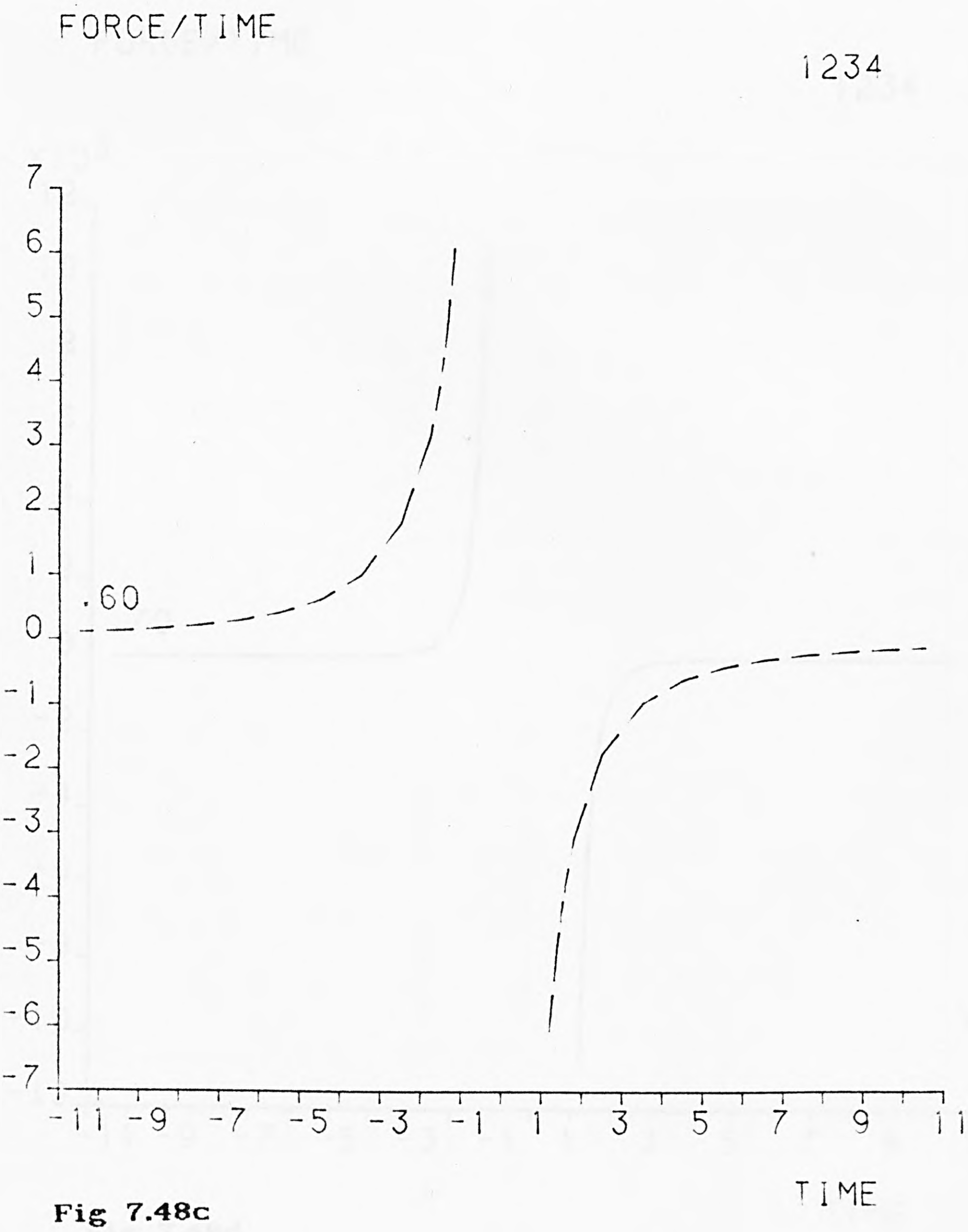


Fig 7.48c

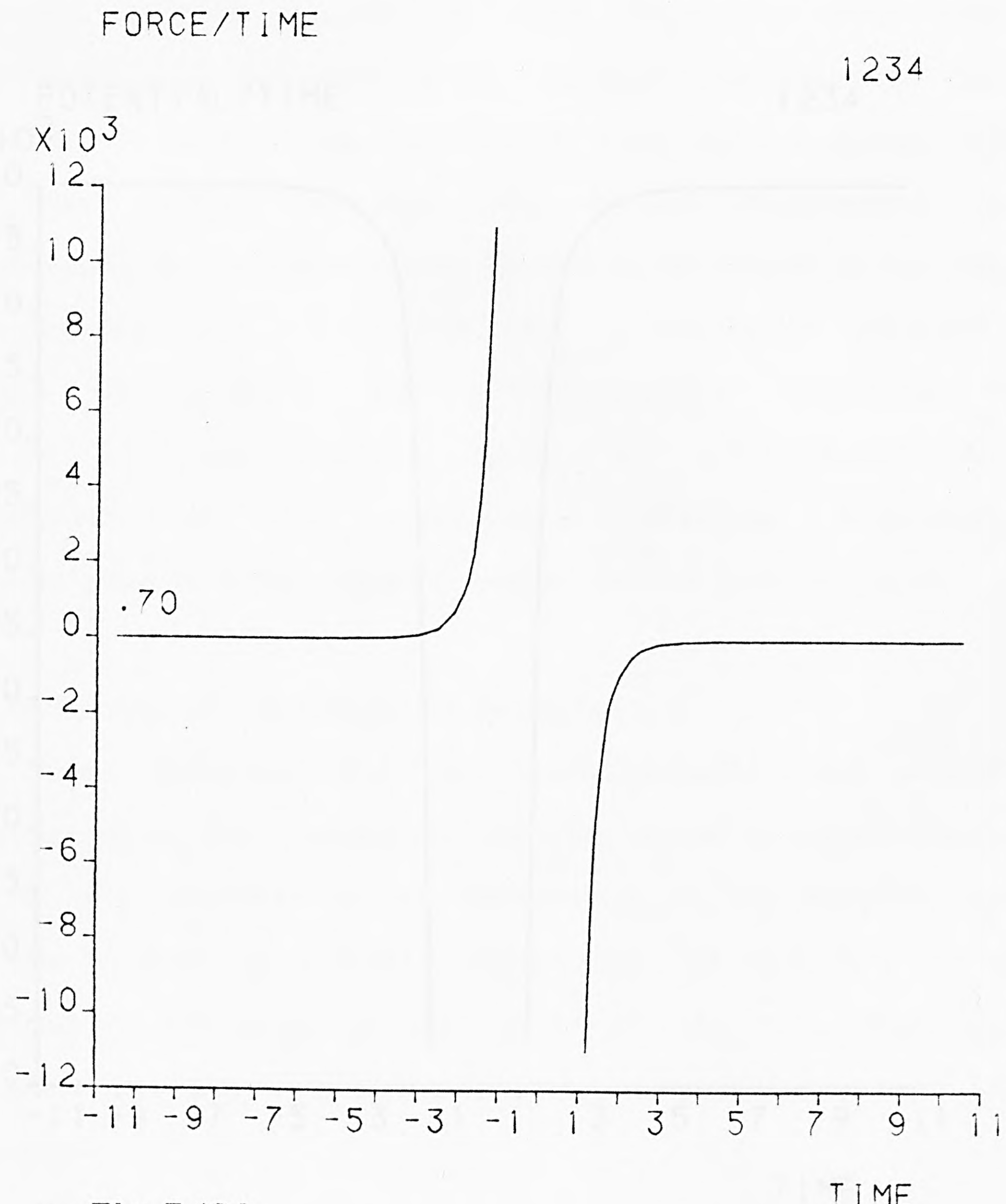
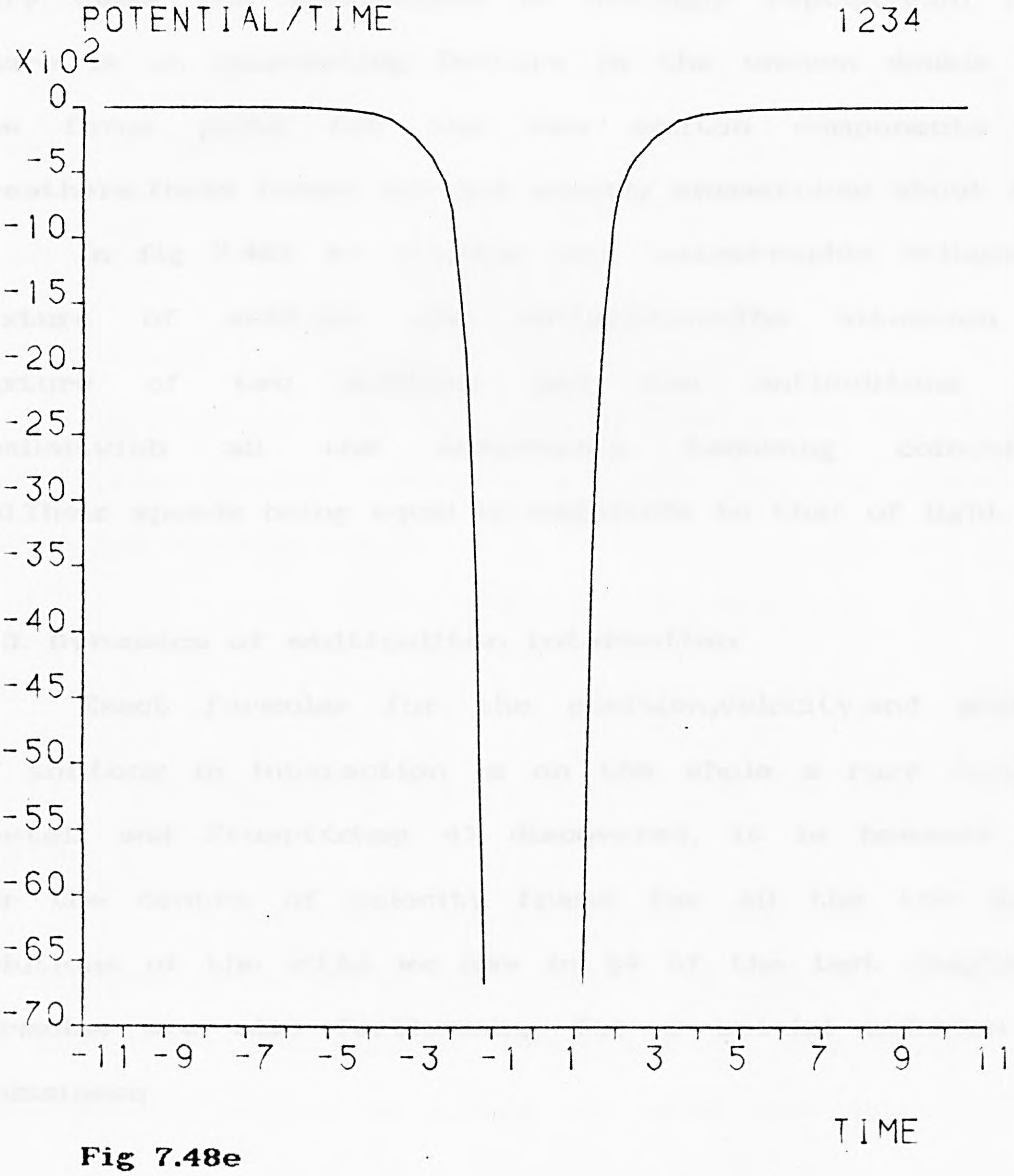

FORCE/TIME
1234
X10^3
12
10
8
6
4
2
.70
0
-2
-4
-6
-8
-10
-12
-11 -9 -7 -5 -3 -1 1 3 5 7 9 11
TIME

Fig 7.48d



POTENTIAL/TIME
1234
X10^2
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
-65
-70
-11 -9 -7 -5 -3 -1 1 3 5 7 9 11
TIME

Fig 7.48e

time -45 seconds.Also,we note from fig 7.46i that over this entire region there is no discernable change of speed of the lone soliton.Even if one supposes this curious behaviour to be induced by the lone soliton the interaction is extremely strange, as at time -20 approximately, the soliton and antisoliton components of the breather *do* pass through one another,but here the lone soliton is much closer.The graph(fig 7.46h) appears to suggest a many body interaction and not a sum of two body interactions.

From fig 7.47f we can conclude that when breathers are far apart they are weakly repulsive then gradually the potential becomes attractive, until eventually when they are very close the interaction is strongly repulsive.On fig 7.47d there is an interesting feature in the uneven double hump on the force plots for the two soliton components of the breathers.These humps are not exactly symmetrical about time 0.

In fig 7.48a we witness the "catastrophic collapse" of a mixture of solitons and antisolitons.The situation for a mixture of two solitons and two antisolitons is also similar,with all the components becoming coincident at t=0.Their speeds being equal in magnitude to that of light.

§ 5. Dynamics of multisoliton interaction

Exact formulae for the position,velocity,and acceleration of solitons in interaction is on the whole a rare occurence.As Bowtell and Stuart(chap 4) discovered, it is however possible for the centre of velocity frame for all the two parameter solutions of the sG.As we saw in §4 of the last chapter, exact formulae are also forthcoming for a special solution of the Boussinesq.

These formulae are identical in form to the sG, except in the value taken by certain constants,and in the formula for the force as a function of separation(due to the Boussinesq being non-relativistic).The acceleration of two Boussinesq solitons of equal amplitude moving in opposite directions can be shown(see (6.55)) to be ,

$$\ddot{x} = \gamma^{-1}\coth(\gamma r/2)\mathrm{cosech}^2\gamma r/2 \qquad (7.66)$$

where γ = k the amplitude of the solitons and r is their mutual separation.

Although in general the two parameter lsp with f_i =1 can only be solved numerically there is a large class of solutions which we can obtain exactly.The two soliton lsp with f_i=1 can be written(with perfect phase), (7.67a)

$$1-u_{12}^{-1}\left[\exp a_1(x-b_1t)+\exp a_2(x-b_2t)\right]+\exp[(a_1+a_2)x-(a_1b_1+a_2b_2)t]=0$$

where for the MKdV,

$$a_i = 2k_i \ , \ b_i = 4k_i^2 \qquad (b)$$

and for the sG,

$$a_i = \gamma_i \ , \ b_i = u_i \qquad (c)$$

(see (6.4,8,20)).

Choosing $a_1 = 2a_2$ and substituting $y=\exp a_2x$ into (7.67a) and multipying the resulting equation by $\exp a_2t(2b_1+b_2)$ we obtain the folowing cubic ,

$$y^3 + a_1y^2 + a_2y + a_3 = 0 \qquad (7.68)$$

$$a_1 = -u_{12}^{-1}e^{a_2b_2t} \ , \ a_2 = -u_{12}^{-1}e^{2a_2b_1t} \ , \ a_3 = e^{a_2(2b_1+b_2)t}$$

It can be shown that the condition for real roots $Q^3+R^2<0$, where,

$$Q = (3a_2-a_1^2)/9 \ , \ R = (9a_1a_2-27a_3-2a_1^3)/54 \qquad (7.69)$$

is satisfied.Hence since the product of the roots is negative we must have two positive roots and one negative,but y is an exponential function of x,thus we need only consider the two positive roots(note the discussion at the end of §5 chapter 6).

We give an example in the case of the MKdV,KdV two soliton case where the amplitudes are $k_1=2$,$k_2=1$,we find $u_{12}=1/3$.

Substituting these values into (7.67b) we find ,

$$Q = -e^{48t}(1+e^{48t})^3 \ , \ R = e^{48t}(1+e^{48t})^2$$

and we find the roots of (7.67a) become (using the standard formula for roots of a cubic,noting which are positive) after taking the log of y_i.

$$x_1 = 4t + \frac{1}{2} \ln [1+2(1+e^{48t})^{1/2}\cos\alpha] \qquad (7.70)$$

$$x_2 = 4t + \frac{1}{2} \ln [1+ (1+e^{48t})^{1/2}(\sqrt{3}\sin\alpha - \cos\alpha)]$$

$$\alpha = \frac{1}{3} \cos^{-1} (1+e^{48t})^{1/2}$$

We find that,

$$t\to-\infty \quad x_1\to \ 4t + \ \ln\sqrt{3} \quad , x_2\to 16t - \tfrac{1}{2}\ln\sqrt{3} \qquad (7.71)$$

$$t\to+\infty \quad x_1\to 16t + \tfrac{1}{2}\ln\sqrt{3} \quad , x_2\to \ 4t - \ \ln\sqrt{3}$$

This is the expected asymptotic behaviour.

Unfortunately (7.70) are sufficiently complicated to render further differentiation uninstructive and the explicit dynamics(i.e force as a function of separation) is very difficult to obtain.

To obtain a better picture of the dynamics of interacting solitons it might seem useful to employ the approximations developed in §3,but even here we are faced with great complication.If we set $|b_i(t)| = \exp \beta_i(t)$ (see §3) we may write the position,speed and acceleration of the approximate solitons(antisolitons) by

$$x_i(t) = -\beta_i(t)/\alpha_i(t) \tag{7.72a}$$

$$\dot{x}_i(t) = -(\dot{\beta}_i\alpha_i - \beta_i\dot{\alpha}_i)/\alpha_i^2 \tag{b}$$

$$\ddot{x}_i(t) = -\left[(\ddot{\beta}_i\alpha_i - \beta_i\ddot{\alpha}_i) - 2(\dot{\beta}_i\alpha_i - \beta_i\dot{\alpha}_i)\dot{\alpha}_i\right] \Big/ \alpha_i^2 \tag{c}$$

As can be seen from the above,the expressions become very complicated both for the two parameter approximate solutions(7.47-52) and the special three parameter sG solutions(7.58-60).This complication is due to the time dependent "shape factor" α_i.In the centre of velocity two parameter sG and Boussinesq cases the shape factor is a constant.

If we look at the asymptotic region $t\to-\infty$ the formulae do simplify.Examining the approximate solitons(whose motion becomes arbitrarily close to the motion of the real solitons in the asymptotic limits) we obtain the following results.

Note that from (7.72b) we must examine carefully the limiting behaviour of both terms in the numerator.In fact we find that for two parameter cases as $t\to-\infty$ the term in $\dot{\alpha}$ in the numerator of (7.72b) tends to zero,while the term in $\dot{\beta}$ does not.In the following λ,ν,y_+,y_- are defined as before in §3.

<u>Two solitons</u>

$$x_1 \to -\lambda^{-1}(1-\nu)^{-1}[y_+ + \ln(2k_{12}\cosh y_-)] \tag{7.73a}$$

$$x_2 \to -\lambda^{-1}(1+\nu)^{-1}[y_+ - \ln(2k_{12}\cosh y_-)] \tag{b}$$

$$\dot{x}_1 \to -\lambda^{-1}(1-\nu)^{-1}[\dot{y}_+ + \dot{y}_-\tanh y_-] \tag{c}$$

$$\dot{x}_2 \to -\lambda^{-1}(1+\nu)^{-1}[\dot{y}_+ - \dot{y}_-\tanh y_-] \tag{d}$$

$$\ddot{x}_1 \to -\lambda^{-1}(1-\nu)^{-1}\,\dot{y}_-^2\,\mathrm{sech}^2 y_- \tag{e}$$

$$\ddot{x}_2 \to +\lambda^{-1}(1+\nu)^{-1}\,\dot{y}_-^2\,\mathrm{sech}^2 y_- \tag{f}$$

Soliton-antisoliton

$$x_1 \to -\lambda^{-1}(1+\nu)^{-1}[y_+ - \ln(2k_{12}\sinh y_-)] \quad (7.74a)$$

$$x_2 \to -\lambda^{-1}(1-\nu)^{-1}[y_+ + \ln(2k_{12}\sinh y_-)] \quad (b)$$

$$\dot{x}_1 \to -\lambda^{-1}(1+\nu)^{-1}[\dot{y}_+ - \dot{y}_- \coth y_-] \quad (c)$$

$$\dot{x}_2 \to -\lambda^{-1}(1-\nu)^{-1}[\dot{y}_+ + \dot{y}_- \coth y_-] \quad (d)$$

$$\ddot{x}_1 \to -\lambda^{-1}(1+\nu)^{-1}\,\dot{y}_-^2\,\mathrm{cosech}^2 y_- \quad (e)$$

$$\ddot{x}_2 \to +\lambda^{-1}(1-\nu)^{-1}\,\dot{y}_-^2\,\mathrm{cosech}^2 y_- \quad (f)$$

In the three parameter centre of velocity cases(sG) we find we can no longer ignore $\dot{\alpha}$ in the expression for soliton speed.The following results are obtained(see (7.57-9))

Three sG solitons in centre of velocity frame

$$x_1 \to -\gamma^{-1}\chi(t) \quad (7.75a)$$

$$\dot{x}_1 \to -u\,\tanh\gamma ut \quad (b)$$

$$\ddot{x}_1 \to -\gamma u^2\mathrm{sech}^2\gamma ut - \tfrac{1}{2}k_{12}^{-1}k^{-1}(\gamma-1)(k_{12}^2-1)u^2\mathrm{sech}\gamma ut.\chi(t) \quad (c)$$

$$\chi(t)= \ln(4k_{12}k\,\cosh\gamma ut) \quad (d)$$

$$x_2 = -x_1 \quad (e)$$

The results for the two soliton-one antisoliton centre of velocity case for the sG in the asymptotic region $t\to-\infty$ are given by,

$$x_1 \to -\gamma^{-1}\chi(t) \quad (7.76a)$$

$$\dot{x}_1 \to -u\,\tanh\gamma ut \quad (b)$$

$$\ddot{x}_1 \to -\gamma u^2\mathrm{sech}^2\gamma ut + \tfrac{1}{2}k_{12}^{-1}k^{-1}(\gamma-1)(k_{12}^2-1)u^2\mathrm{sech}\gamma ut.\chi(t) \quad (c)$$

$$\chi(t)= \ln(4k_{12}k\,\cosh\gamma ut) \quad (d)$$

$$x_2 = -x_1 \quad (e)$$

The smaller term in (7.75-6c) is related to the force between the two solitons which are furthest apart as

$$\exp-\alpha(x_2-x_1)\to (4k_{12}k)^{-1}\mathrm{sech}^2\gamma ut$$

and the fact that the acceleration is a sum of two terms

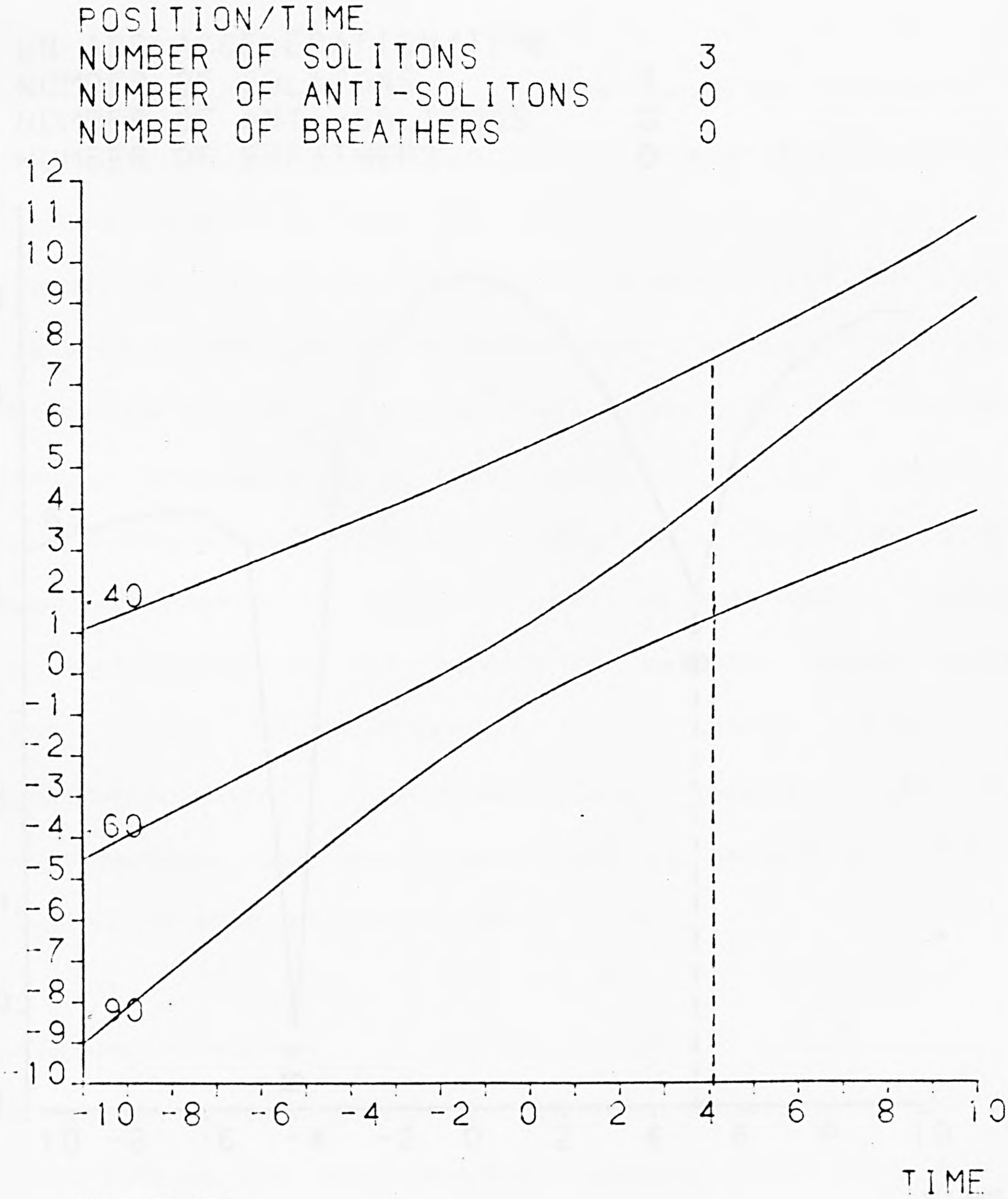
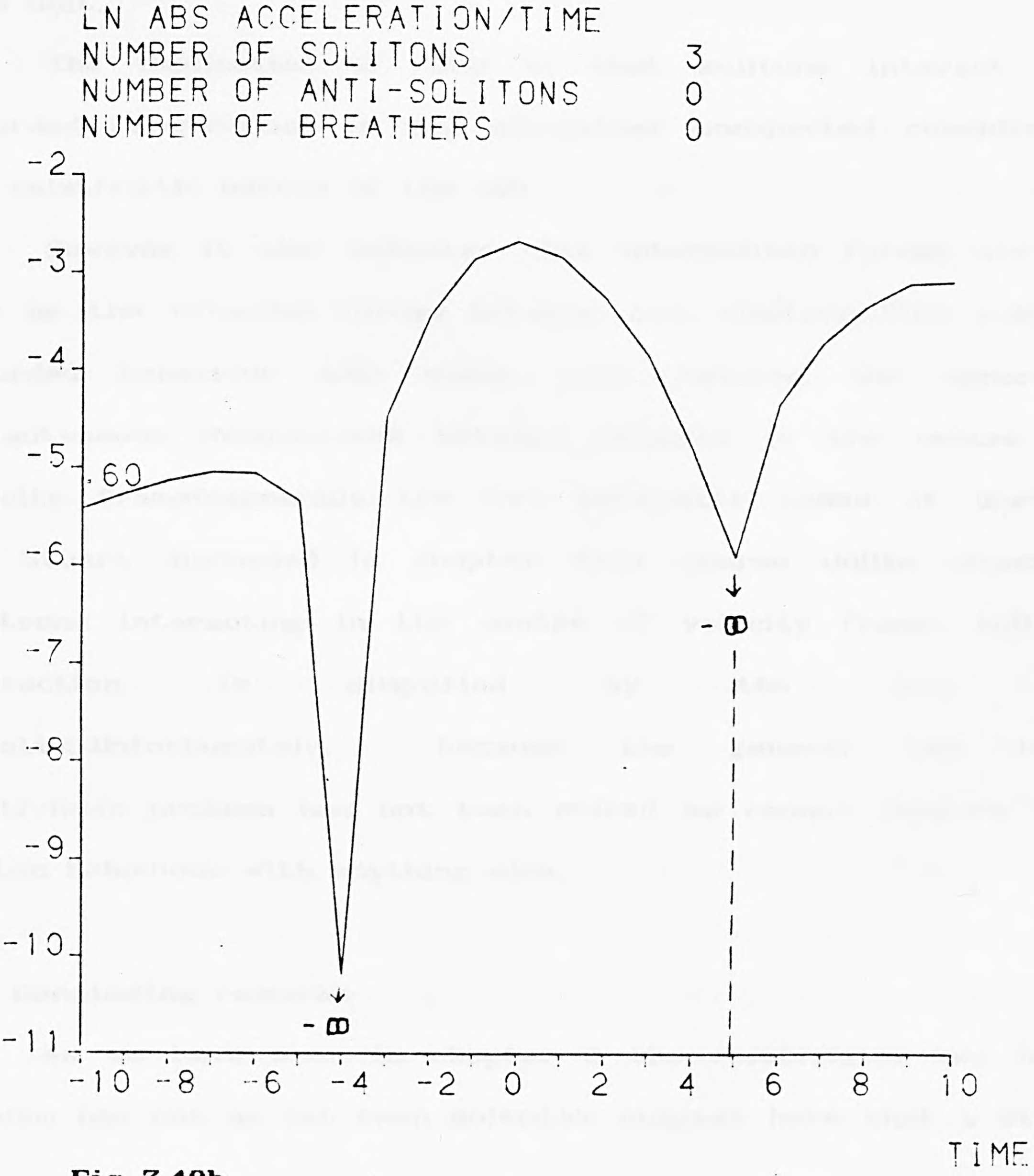

POSITION/TIME
NUMBER OF SOLITONS 3
NUMBER OF ANTI-SOLITONS 0
NUMBER OF BREATHERS 0
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
.40
.60
.90
-10 -8 -6 -4 -2 0 2 4 6 8 10
TIME

Fig 7.49a



LN ABS ACCELERATION/TIME
NUMBER OF SOLITONS 3
NUMBER OF ANTI-SOLITONS 0
NUMBER OF BREATHERS 0
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
.60
-∞
-∞
-10 -8 -6 -4 -2 0 2 4 6 8 10
TIME

Fig 7.49b

suggests the two body nature of the intersoliton forces.

Finally we discuss the implications of figs 7.49,which are graphs relating to the three soliton interaction for the sG.The speeds of the solitons are written close to the curves relating to them.The dotted line on fig 7.49a indicates the time at which the middle soliton is equidistant from its partners, we denote this time t_o.If the forces acting between solitons were instantaneous then one would expect the acceleration of the middle soliton to be zero at time t_o.Figure 7.49b however clearly shows that the acceleration of the middle soliton becomes zero at a time $t_o+\Delta$,where Δ is of the order one time unit.

The implication of this is that solitons interact via retarded forces(which is not altogether unexpected considering the relativistic nature of the sG).

However it also indicates that intersoliton forces are as real as the retarded forces between say, electrons.This soliton retarded behaviour also makes more puzzling the apparent instantaneous interactions between solitons in the centre of velocity frame(especially the two parameter cases of Bowtell and Stuart discussed in chapter 4).Of course unlike classical electrons interacting in the centre of velocity frame, soliton interaction is simplified by the lack of radiation.Unfortunately, because the general two body relativistic problem has not been solved we cannot compare the soliton behaviour with anything else.

§ 6. Concluding remarks

As we have seen in chapter 2 the relativistic two body problem has not as yet been solved.We suggest here that a study of the interaction of relativistic solitons may be of use in our attempts to understand the problem.The relativistic two body problem begins with an assumed potential acting between the particles, and the problem becomes one of determining the positions of the particles as a function of time,given that the interaction between the particles is retarded by the finite speed of propagation of signals between the particles.We are fortunate to possess in the two soliton solutions of the sG a knowledge of the exact positions of the solitons as functions of time.Our problem becomes the inverse of the standard one,in that we would like to know the potential that acts between solitons which is such that by retarded interaction it *produces* the known soliton positions.

Another feature of this chapter is the development of the approximate soliton solutions of various equations.As we have seen the approximate solitons are remarkably similar in most situations to the exact solitons.The interesting feature here is that we may regard the approximate solitons in *their own right*.Divorcing ourselves from the question as to whether the approximate solitons provide a good approximation to the real solitons,there is no doubt that looked at in isolation they *are* solitons.

If partial differential equations could be found which had the approximate two soliton solution of another pde as an *exact* solution we would have discovered not only a new soliton equation but one which in some sense was close to known soliton equations.This would be a very interesting development in soliton theory, especially as the sG equation is often thought of as being a *unique* equation, possessing no close relatives.Such topics will no doubt merit future study.

CONCLUSION

In this thesis we have strengthened the case for thinking of solitons as particle-like objects in interaction.

With the linear superposition principle we have discovered a way of identifying the solitons,during the interaction,and we have found the motion of these interacting solitons to be correlated with the motion of the singularities of the complex multisoliton solution.We have found that each complex interacting soliton carries a singularity with a *single* real projection. Thus,we have established a field/ particle duality for the multisoliton solutions of some soliton equations.

As we have seen;there is a pressing need to cure the problems associated with point particles in field theory.We have amply illustrated this,with our discussion of the foundations of electrodynamics.These problems provided us with keen motivation for the study of solitons and their interaction.

The progress of physics has often proceeded by the challenging of absolutes.This was most notable in the creation of the Special Theory of Relativity,where the notions of absolute space and time were challenged.We have argued in this thesis for an alternative to the *absolute point particle*.We have provided many good reasons for believing that the soliton (or solitary wave) is the best model we have for a *truly elementary particle*.

The study of physics has taught us that it is useful to imagine that the world is comprised of point particles interacting via mediating fields.We have suggested that it would be more fruitful to describe the world in terms of a multisoliton (solitary wave) solution of a hitherto unknown nonlinear partial differential equation.In this way we would be able to provide a mathematical mechanism for the phenomenon of interacting particles.

A radical shift in our understanding of "particles" would have far reaching consequences not only in classical theory,but also in quantum theory.It is our view that the supposed perplexities of the wave particle duality paradox are due in no small measure to our inability to define *mathematically* the meaning of the term "particle".This is why the discovery of solitons was so important.

The Born-Infeld nonlinear field theory was a most important attempt to provide a "particle" free field theory.The fact that the Born-Infeld theory is regarded as not being quantizable does not in our view diminish its importance.This is because without knowing the solutions of the Born-Infeld field equations ,we cannot be sure that quantization is even necessary.

We take the view that a nonlinear partial differential equation with multisoliton solutions is a theoretical laboratory.Our attempts to analyse the motion of the solitons mathematically can be likened to the experiments physicists perform in order to understand the world.Each new equation is like a new universe.It is our goal, eventually, to find an equation in which solitons behave similar to known particles.

<u>Specific suggestions for future research.</u>

We suggest that further attempts be made to prove the main conjectures made in chapter 6.Namely:

1.The roots of the lsp(6.2-3) are real for mixtures of solitons ,antisolitons and breathers.

2.The roots of the lsp are monotonic.

We also suggest that it be proved

3.That with perfect phase,and non-alternating signature the roots of the lsp are all ±1 at x=0,t=0.

The ubiquity of the lsp for many soliton equations suggests strongly that the equations may be able to be solved directly.It is therefore extremely important that ,partial differential equations which have the lsp roots as solutions, be sought.

As we have seen in the thesis the multisoliton solutions of many soliton equations can be written in terms of the determinant of a matrix.We found the roots of the lsp to be the eigenvalues of that matrix.We therefore must ask why is it that all these matrices are diagonalizable?

We saw that the interacting solitons of the KdV involved quantities,

$$w = \ln(1+f_i) = \int df_i/(1+f_i)$$

and that the interacting solitons of the sG and MKdV involved

$$v = \tan^{-1}f_i = \int df_i/(1+f_i^2)$$

This suggest that it might be interesting to study equations which have one soliton solutions,

$$u_n = \int df/(1+f^n) \quad , \text{ where } f= \exp[\alpha x+\beta t]$$

We also mentioned in the thesis that when certain limiting processes are applied to the multisoliton solutions,we can obtain the *rational* solutions.We suggest that a study of the behaviour of the roots of the lsp under these limiting processes might be worthy of our attention.

We saw in the latter part of chapter 7 that in a general frame of reference the solitons (in a three soliton collision) did not appear to be interacting instantaneously.This is obviously a topic for further investigation.We could analyse the interaction of an antisoliton with two solitons,such that at t=0 say, the antisoliton was equidistant from the solitons.

If retardation is confirmed as definitely present,then we are faced with a theoretical puzzle.How do the solitons and antisolitons *appear* to act on each other instantaneously in the centre of velocity frames?

A further interesting topic to study would be the investigation of whether there was a connection between nonlinear superposition principles and Bäcklund transformations.We give an example of a new nonlinear superposition.

The solitary wave,

$$\phi = 1/(1+\ln[\cosh X_1]) \;,\; X_1 = \gamma(x-ut) \;,\; \gamma=(1-u^2)^{-1/2}$$

satisfies the Liouville equation,

$$\psi_{xx} - \psi_{tt} = e^2 e^{-2\psi}$$

where $\phi = 1/\psi$.

It is readily confirmed that the above solitary wave is a true higher dimensional solitary wave solution of the equation,

$$\psi_{yy} + \psi_{xx} - \psi_{tt} = e^2 e^{-2\psi}$$

with $\phi = 1/\psi$.

A linear superposition of the solitary waves ϕ is

$$\phi_2 = 1/(1+\ln[\cosh X_1]) + 1/(1+\ln[\cosh X_2])$$

$$= \frac{2 + \ln[\cosh X_1] + \ln[\cosh X_2]}{1 + \ln[\cosh X_1] + \ln[\cosh X_2] + \ln[\cosh X_1]\ln[\cosh X_2]}$$

We can create a nonlinear superposition by replacing

$$\ln[\cosh X_1] + \ln[\cosh X_2] \text{ with } \Big(\ln[\cosh X_1] + \ln[\cosh X_2]\Big)\Big/u_{12}$$

where u_{12} is defined the same way as with the sG.We can then write this nonlinear superposition in the linear form using functions f_i,

$$\phi_2 = 1/(1+f_1) + 1/(1+f_2)$$

where f_i are the roots of the polynomial,

$$f^2 - \left(\frac{\ln[\cosh X_1] + \ln[\cosh X_2]}{u_{12}}\right)f + \ln[\cosh X_1]\ln[\cosh X_2] = 0$$

The above polynomial has real roots as is easily seen by considering the following quadratic,

$$f^2 - k(t_1+t_2)f + t_1t_2 = 0$$

where $k > 1$ and $t_i \in \mathbb{R}$.Clearly $(t_1-t_2)^2 \geq 0$ .Adding $4t_1t_2$ to both sides of this inequality,we obtain $(t_1+t_2)^2 \geq 4t_1t_2$.Thus $k(t_1+t_2)^2 \geq 4t_1t_2$.

We can invent an infinite number of multisoliton formulas using the roots f_i of the lsp of chapter 6.A simple candidate for instance is,

$$\varphi_N = \sum_{i=1}^{N} (1+f_i)^{-1}$$

The obvious question is can we find the pde's which have these multisoliton formulas as solutions? Given the ubiquity of the lsp of chapter 6, it is natural to expect it to provide us with new multisoliton solutions.

Finally we suggest that it would be interesting to see to what extent some of the ideas expressed in this thesis may be carried over to the quantised sG equation.

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CHAPTER 5

[1] Chapter 4 ref 9

[2] Chapter 3 ref 31 and also article by H.D.Wahlquist in the book ref 21 chapter 3.

[3] Chapter 3 ref 26

CHAPTER 6

[1] Chapter 3 ref 33

[2] Chapter 3 ref 42

[3] Chapter 4 ref 26

[4] Chapter 3 ref 3

APPENDIX

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A Linear Superposition Formula for the Sine-Gordon Multisoliton Solutions
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