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**Citation:** Dey, S., Fring, A. & Mathanaranjan, T. (2014). Non-Hermitian systems of Euclidean Lie algebraic type with real energy spectra. Annals of Physics, 346, pp. 28-41. doi: 10.1016/j.aop.2014.04.002

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# Non-Hermitian systems of Euclidean Lie algebraic type with real energy spectra

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Abstract: We study several classes of non-Hermitian Hamiltonian systems, which can be expressed in terms of bilinear combinations of Euclidean Lie algebraic generators. The classes are distinguished by different versions of antilinear (PT)-symmetries exhibiting various types of qualitative behaviour. On the basis of explicitly computed non-perturbative Dyson maps we construct metric operators, isospectral Hermitian counterparts for which we solve the corresponding time-independent Schrödinger equation for specific choices of the coupling constants. In these cases general analytical expressions for the solutions are obtained in the form of Mathieu functions, which we analyze numerically to obtain the corresponding energy spectra. We identify regions in the parameter space for which the corresponding spectra are entirely real and also domains where the PT symmetry is spontaneously broken and sometimes also regained at exceptional points. In some cases it is shown explicitly how the threshold region from real to complex spectra is characterized by the breakdown of the Dyson maps or the metric operator. We establish the explicit relationship to models currently under investigation in the context of beam dynamics in optical lattices.

## 1. Introduction

Quasi-exactly solvable models [1] of Lie algebraic type are believed to be almost all related to $sl_2(\mathbb{C})$ with their compact and non-compact real forms $su(2)$ and $su(1,1)$, respectively [2]. The nature of those models dictates that essentially all the wavefunctions related to solutions for the time-independent Schrödinger equation of these type of models may be expressed in terms of hypergeometric functions. Non-Hermitian variants of these models expressed generically in terms of $su(2)$ or $su(1,1)$ generators have been investigated systematically in [3, 4] and large classes of models were found to possess real or partially spectra despite their non-Hermitian nature. Under certain constraints on the coupling constants

the models could be mapped to Hermitian isospectral counterparts. Positive Hermitian metric operators were shown to exist, such that a consistent quantum mechanical description of these models is possible when following the general techniques developed over the last years [5, 6, 7] in the context of $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics.

It is, however, also well known that there exists an interesting subclass of solvable models related to Mathieu functions which are known to possess solutions, which are not expressible in terms of hypergeometric functions. In a more generic setting these type of models are known to be related to specific representations of the Euclidean algebra rather than to its subalgebra $sl_2(\mathbb{C})$. This feature makes models based on them interesting objects of investigation from a mathematical point of view. In a more applied setting it is also well known that the Mathieu equation arises in optics as a reduction from the Helmholtz equation. This analogue setting of complex quantum mechanics is currently under intense investigation. Concrete versions of complex potentials leading to real Mathieu potentials have recently been studied from a theoretical as well as experimental point of view in [8, 9, 10, 11, 12, 13]. Further applications are found for instance in the investigation of complex crystals [14].

It was recently shown that for $E_2$ [15] and $E_3$ [16] some simple non-Hermitian versions also possess real spectra. Here we will follow the line of thought of [3] and investigate systematically the analogues of quasi-exactly solvable models of Lie algebraic type, that is those models which can be written as bilinear combinations in terms of the Euclidean algebra generators.

Our manuscript is organized as follows: At the beginning of section 2 we discuss five different types of $\mathcal{PT}$-symmetries for the $E_2$-algebra and present the computation of the adjoint action on their generators. In the following five subsection we derive Dyson maps and isospectral counterparts for generic non-Hermitian Hamiltonians invariant under these different types of symmetries. For the last symmetry we present a more detailed analysis of the time-independent Schrödinger equation. We derive some explicit analytical solutions, which we analyze numerically to compute the corresponding energy spectra leading to three qualitatively different scenarios: entirely real energies, spectra with spontaneously broken $\mathcal{PT}$-symmetry at exceptional points characterized by two or three disconnected regions in the parameter space. We propose a measurable quantity that can be used as a criterium to identify the spontaneously broken $\mathcal{PT}$-symmetric regime. In section 3 we discuss the $\mathcal{PT}$-symmetries for the $E_3$-algebra, present the computation of the adjoint action on its generators and indicate how to obtain simple examples of explicit isospectral pairs of an $E_3$-invariant non-Hermitian and Hermitian Hamiltonian.

## 2. $\mathcal{PT}$-symmetric $\mathbf{E_2}$-invariant non-Hermitian Hamiltonians

We take here the commutation relations obeyed by the three generators $u$,$v$ and $J$ as the defining relations of the Euclidean-algebra $E_2$

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad \text{and} \qquad [u, v] = 0. \tag{2.1}$$

Obviously there are many representations for this algebra, as for instance one used in the context of quantizing strings on tori [17] acting on square integrable wavefunctions $L^2(\mathbf{S}^1, d\theta)$ with

$$J := -i\partial_\theta, \qquad u := \sin\theta, \qquad \text{and} \qquad v := \cos\theta, \tag{2.2}$$

or a two-dimensional one in terms of generators of the Heisenberg canonical commutators $q_j$, $p_j$ satisfying $[q_j, p_k] = i\delta_{jk}$ for $j, k = 1, 2$

$$J := q_1 p_2 - p_1 q_2, \qquad u := p_2, \qquad \text{and} \qquad v := p_1. \tag{2.3}$$

For our purposes it is important to note that the $E_2$-algebra is left invariant with regard to an antilinear symmetry [18]. As previously noted [19, 20, 21] in dimensions larger than one there are in general various types of antilinear symmetries, which by a slight abuse of language we all refer to as $\mathcal{PT}$-symmetries. For instance, it is easy to see that the algebra (2.1) is left invariant under the following antilinear maps

$$\begin{array}{lllll}
\mathcal{PT}_1: & J \to -J, & u \to -u, & v \to -v, & i \to -i, \\
\mathcal{PT}_2: & J \to -J, & u \to u, & v \to v, & i \to -i, \\
\mathcal{PT}_3: & J \to J, & u \to v, & v \to u, & i \to -i, \\
\mathcal{PT}_4: & J \to J, & u \to -u, & v \to v, & i \to -i, \\
\mathcal{PT}_5: & J \to J, & u \to u, & v \to -v, & i \to -i.
\end{array} \tag{2.4}$$

Each of these symmetries may be utilized to describe different types of physical scenarios. For instance, $\mathcal{PT}_1$ was considered in [15] with $\mathcal{P}_1 : \theta \to \theta + \pi$ corresponding to a reflection of the particle to the opposite side of the circle for the representation (2.2). For the same representation we can identify the remaining symmetries as $\mathcal{P}_2 : \theta \to \theta + 2\pi$, $\mathcal{P}_3 : \theta \to \pi/2 - \theta$, $\mathcal{P}_4 : \theta \to \pi - \theta$ and $\mathcal{P}_5 : \theta \to -\theta$. Of course other representations allow for different interpretations. For instance, in the two dimensional representation (2.3) the symmetry $\mathcal{PT}_3$ can be used when describing systems with two particle species as it may be viewed as a particle exchange, or an annihilation of a particle of one species accompanied by the creation a particle of another species, together with a simultaneous reflection $\mathcal{PT}_3 : p_1 \leftrightarrow p_2$, $q_1 \leftrightarrow -q_2$.

$\mathcal{PT}_i$-invariant Hamiltonians $H$ in term of bilinear combinations of $E_2$-generators are then easily written down. Crucially, this very general symmetry allows for non-Hermitian Hamiltonians to be considered since it is antilinear [18]. Following the general techniques developed over the last years [5, 6, 7] in the context of $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics we attempt to map these non-Hermitian Hamiltonians $H \neq H^\dagger$ to isospectral Hermitian counterparts $h = h^\dagger$ by means of a similarity transformation $h = \eta H \eta^{-1}$. When $\eta$, often referred to as the Dyson map, is Hermitian the latter equation is equivalent to $H^\dagger = \eta^2 H \eta^{-2}$, which is another equation one might utilize to determine $\eta$. Taking here the Dyson map to be of the general form

$$\eta = e^{\lambda J + \rho u + \tau v}, \qquad \text{for } \lambda, \tau, \rho \in \mathbb{R}, \tag{2.5}$$

we can easily compute the adjoint action of this operator on the $E_2$-generators. We find

$$\eta J \eta^{-1} = J + i(\rho v - \tau u)\frac{\sinh\lambda}{\lambda} + (\rho u + \tau v)\frac{1-\cosh\lambda}{\lambda}, \tag{2.6}$$

$$\eta u \eta^{-1} = u\cosh\lambda - iv\sinh\lambda, \tag{2.7}$$

$$\eta v \eta^{-1} = v\cosh\lambda + iu\sinh\lambda. \tag{2.8}$$

Once $\eta$ is identified the metric operators needed for a consistent quantum mechanical formulation can in general be taken to be $\rho = \eta^\dagger\eta$. Let us now construct isospectral counterparts, if they exist, for non-Hermitian Hamiltonians symmetric with regard to the various different types of $\mathcal{PT}$-symmetries. It should be noted that exact computations of this type remain a rare exception and even for some of the simplest potentials the answer is only known perturbatively, as for instance even for the simple prototype non-Hermitian potential $V = i\varepsilon x^3$ [22, 23, 24].

### 2.1 $\mathcal{PT}_1$-invariant Hamiltonians of $E_2$-Lie algebraic type

The most general $\mathcal{PT}_1$-invariant Hamiltonian expressed in terms of bilinear combinations of the $E_2$-generators is

$$H_{\mathcal{PT}_1} = \mu_1 J^2 + i\mu_2 J + i\mu_3 u + i\mu_4 v + \mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv, \tag{2.9}$$

with $\mu_i \in \mathbb{R}$ for $i = 1, \ldots, 9$. Clearly the Hamiltonian $H_{\mathcal{PT}_1}$ is non-Hermitian with regard to the standard inner product when considering it for a Hermitian representation with $J^\dagger = J$, $v^\dagger = v$ and $u^\dagger = u$, unless $\mu_2 = 0$, $\mu_5 = -2\mu_4$, $\mu_6 = 2\mu_3$. The specific case $H_{BK} = J^2 + igv$ when $\mu_i = 0$ for $i \neq 1, 4$ was studied in [15], where partially real spectra were found but no isospectral counterparts were constructed. Using the relations (2.6)-(2.8), we compute the adjoint action of $\eta$ on $H$ and subsequently demand the result to be Hermitian. This requirement will constrain our 12 free parameters $\mu_i, \lambda, \tau, \rho$. A priori it is unclear whether solutions to the resulting set of equations exist. For $H_{\mathcal{PT}_1}$ we find the manifestly Hermitian isospectral counterpart

$$h_{\mathcal{PT}_1} = \mu_1 J^2 + \mu_3\{v, J\} - \mu_4\{u, J\} - \frac{2\mu_3\mu_4}{\mu_1}uv + \frac{\mu_4^2 - \mu_3^2}{\mu_1}u^2 + \mu_8(u^2 + v^2). \tag{2.10}$$

As usual, we denote by $\{A, B\} := AB + BA$ the anti-commutator. Without loss of generality we may set $\mu_8 = 0$ since $C = u^2 + v^2$ is a Casimir operator for the $E_2$-algebra and can therefore always be added to $H$ having simply the effect of shifting the ground state energy. The remaining constants $\mu_i$ have been constrained to

$$\tau = \frac{\lambda\mu_3}{\mu_1},\ \rho = -\frac{\lambda\mu_4}{\mu_1},\ \mu_2 = 0,\ \mu_5 = -2\mu_4,\ \mu_6 = 2\mu_3,\ \mu_7 = \mu_8 + \frac{\mu_4^2 - \mu_3^2}{\mu_1},\ \mu_9 = -\frac{2\mu_3\mu_4}{\mu_1}, \tag{2.11}$$

by the requirement that $h_{\mathcal{PT}_1}$ ought to be Hermitian, whereas $\lambda, \mu_1, \mu_3, \mu_4$ are chosen to be free. We observe that we have been led to the constraints (2.11), of which a subset stated that $H_{\mathcal{PT}_1}$ is already Hermitian before the transformation. We also note that the

constraints (2.11) do not allow a reduction to the Hamiltonian $H_{BK}$, dealt with in [15], as for instance $\mu_5 = 0$ implies $\mu_4 = 0$.

Having guaranteed that $H_{\mathcal{PT}_1}$ possess real energies under certain constraints we may now also compute the corresponding solutions to the time-independent Schrödinger equation $h_{\mathcal{PT}_1}\phi = E\phi$ or equivalently to $H_{\mathcal{PT}_1}\psi = E\psi$ with $\psi = \eta^{-1}\phi$. We find

$$\phi(\theta) = e^{-\frac{i\mu_4\cos\theta}{\mu_1} - i\frac{\sin\theta}{\mu_1}\mu_3}\left[c_1\exp\left(-i\theta\sqrt{\frac{E}{\mu_1}+\frac{\mu_3^2}{\mu_1^2}}\right) + \frac{i}{2\sqrt{\frac{E}{\mu_1}+\frac{\mu_3^2}{\mu_1^2}}}c_2\exp\left(i\theta\sqrt{\frac{E}{\mu_1}+\frac{\mu_3^2}{\mu_1^2}}\right)\right], \tag{2.12}$$

with normalization constants $c_1$, $c_2$. Imposing either bosonic or fermionic boundary conditions, i.e. $\psi(\theta+2\pi) = \pm\psi(\theta)$, we obtain the discrete real energy spectra

$$\text{bosonic:}\;\; E_n = \mu_1\left(n^2 - \frac{\mu_3^2}{\mu_1^2}\right), \qquad \text{fermionic:}\;\; E_n = \mu_1\left(n^2+n+\frac{1}{4} - \frac{\mu_3^2}{\mu_1^2}\right), \quad n\in\mathbb{Z}. \tag{2.13}$$

As expected, the wavefunctions are eigenstates of the $\mathcal{PT}$-operator, selecting different behaviours for the two linearly independent parts of $\phi(\theta)$, acting as $\mathcal{PT}_1\phi_n(c_1) = (-1)^n\phi_n(c_1)$ and $\mathcal{PT}_1\phi_n(c_2) = (-1)^{n+1}\phi_n(c_2)$.

### 2.2 $\mathcal{PT}_2$-invariant Hamiltonians of $E_2$-Lie algebraic type

Similarly as in the previous subsection we use the adjoint action of $\eta$ as specified in (2.5) to map the general $\mathcal{PT}_2$-symmetric and for $\mu_2 \neq 0$, $\mu_5 \neq 2\mu_4$, $\mu_6 = -2\mu_3$ non-Hermitian Hamiltonian

$$H_{\mathcal{PT}_2} = \mu_1 J^2 + i\mu_2 J + \mu_3 u + \mu_4 v + i\mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv, \tag{2.14}$$

to the Hermitian isospectral counterpart

$$\begin{aligned} h_{\mathcal{PT}_2} = {} & \mu_1 J^2 + \mu_3\tanh\frac{\lambda}{2}\{u,J\} + \mu_4\tanh\frac{\lambda}{2}\{u,J\} + \frac{2\mu_3\mu_4}{\mu_1}\tanh^2\frac{\lambda}{2}uv \\ & + \frac{\mu_3^2}{\mu_1}\frac{\cosh\lambda}{\cosh^2\frac{\lambda}{2}}u^2 + \left(\frac{\mu_3^2}{\mu_1}+\frac{\mu_4^2}{\mu_1}\tanh^2\frac{\lambda}{2}\right)v^2 + \mu_8(u^2+v^2). \end{aligned} \tag{2.15}$$

In this case the coupling constants are constraint to

$$\rho = \tau\frac{\mu_3}{\mu_4} = \frac{\mu_3\lambda\coth\lambda}{\mu_1}, \;\; \mu_2 = 0, \;\; \mu_5 = 2\mu_4, \; \mu_6 = -2\mu_3, \;\; \mu_7 = \mu_8 + \frac{\mu_3^2-\mu_4^2}{\mu_1}, \; \mu_9 = \frac{2\mu_3\mu_4}{\mu_1}, \tag{2.16}$$

We note that once again we have only the four free parameters $\lambda, \mu_1, \mu_3, \mu_4$ left at our disposal, as $\mu_8$ may be set to zero for the above mentioned reason. As in the previous case these conditions imply also that the original Hamiltonian $H_{\mathcal{PT}_2}$ is already Hermitian when these type of constraints are imposed.

### 2.3 $\mathcal{PT}_3$-invariant Hamiltonians of $E_2$-Lie algebraic type

As the general $\mathcal{PT}_3$-invariant Hamiltonian of Lie algebraic type we consider

$$H_{\mathcal{PT}_3} = \mu_1 J^2 + \mu_2 J + \mu_3(u+v) + i\mu_4(u-v) + \mu_5(u+v)J + i\mu_6(u-v)J + i\mu_7(v^2-u^2) \\ +\mu_8(v^2+u^2) + \mu_9 uv. \tag{2.17}$$

For Hermitian representations of the $E_2$-generators this Hamiltonian is non-Hermitian unless $\mu_6 = \mu_7 = 0$ and $\mu_5 = 2\mu_4$. As isospectral Hermitian counterpart we find in this case

$$h_{\mathcal{PT}_3} = \mu_1 J^2 + \mu_2 J + \frac{1}{2}\left(\mu_5 + \mu_6 \tanh\frac{\lambda}{2}\right)\{u+v, J\} \tag{2.18}$$
$$+\left\{\frac{1}{2\mu_1}\left[\mu_5^2 + \mu_6^2 \tanh^2\frac{\lambda}{2} + \mu_6\mu_5\frac{4+4\cosh\lambda-2\cosh(2\lambda)}{\sinh(2\lambda)}\right] + \frac{2\mu_7}{\sinh(2\lambda)}\right\} uv$$
$$+\left[\mu_3 - \frac{\mu_6}{2} + \left(\mu_4 - \frac{\mu_5}{2}\right)\tanh\frac{\lambda}{2}\right](u+v) + \left[\mu_8 + \frac{\mu_5\mu_6\sinh\lambda + \mu_6^2\cosh\lambda}{2\mu_1(1+\cosh\lambda)}\right](u^2+v^2)$$

with only four constraining equations

$$\rho = \tau = \frac{\lambda\,(\mu_5 + \mu_6\coth\lambda)}{2\mu_1}, \quad \coth\lambda = \frac{\mu_2\mu_5 + \mu_1\,(\mu_6 - 2\mu_3)}{\mu_1\,(2\mu_4 - \mu_5) - \mu_2\mu_6}, \tag{2.19}$$

$$\mu_9 = \frac{\mu_5^2 + \mu_6^2 + 2\mu_6\mu_5\coth(2\lambda)}{2\mu_1} + 2\mu_7\coth(2\lambda). \tag{2.20}$$

Thus, in this case we have eight free parameters left. We also note that unlike as for the $\mathcal{PT}_1$ and $\mathcal{PT}_2$ symmetric cases we are not led to constraints which render the original Hamiltonian $H_{\mathcal{PT}_3}$ Hermitian. For $\mu_1 = 1$, $\mu_7 = 2q$ and all other coupling constants vanishing the Schrödinger equation with representation (2.2) converts into the standard Mathieu differential equation, see e.g. [25],

$$-\phi''(\theta) + 2iq\cos(2\theta)\phi(\theta) = E\phi(\theta). \tag{2.21}$$

with purely complex coupling constant. Unfortunately for this choice of the coupling constants the Dyson map is no longer well-defined, because of the last equation in (2.19), such that it remains an open problem to find the corresponding isospectral counterpart for this scenario.

### 2.4 $\mathcal{PT}_4$-invariant Hamiltonians of $E_2$-Lie algebraic type

The general $\mathcal{PT}_4$-invariant Hamiltonian we consider is

$$H_{\mathcal{PT}_4} = \mu_1 J^2 + \mu_2 J + i\mu_3 u + \mu_4 v + i\mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv. \tag{2.22}$$

This Hamiltonian is non-Hermitian unless $\mu_5 = \mu_9 = 0$ and $\mu_6 = 2\mu_3$. Constraining now the parameters as

$$\rho = 0, \quad \tau = \frac{\lambda\,(\mu_5\coth\lambda + \mu_6)}{2\mu_1}, \quad \coth(2\lambda) = \frac{4\mu_1(\mu_8 - \mu_7) - \mu_5^2 - \mu_6^2}{2\mu_5\mu_6}, \tag{2.23}$$

$$\mu_3 = \frac{\mu_1\mu_5 + \mu_2\mu_6 - 2\mu_1\mu_4}{2\mu_1}\tanh\lambda + \frac{\mu_2\mu_5}{2\mu_1} + \frac{\mu_6}{2}, \qquad \mu_9 = 0, \tag{2.24}$$

we map this to the isospectral counterpart

$$\begin{aligned} h_{\mathcal{PT}_4} &= \mu_1 J^2 + \mu_2 J + \frac{1}{2}\left(\mu_6 + \mu_5 \tanh\frac{\lambda}{2}\right)\{v, J\} \\ &+ \left[\frac{\mu_2 \tanh\left(\frac{\lambda}{2}\right)(\mu_5 + \mu_6 \tanh\lambda)}{2\mu_1} + \left(\mu_4 - \frac{\mu_5}{2}\right)\operatorname{sech}\lambda\right] v \\ &+ \left[\frac{\mu_5^2\left(\tanh^2\frac{\lambda}{2} - \cosh(2\lambda)\right) - 2\mu_6^2 \sinh^2\lambda + 2\mu_5\mu_6\left(\tanh\frac{\lambda}{2} - \sinh(2\lambda)\right)}{8\mu_1}\right. \\ &\left. + \frac{\mu_8 - \mu_7}{2}\cosh(2\lambda)\right]\left(v^2 - u^2\right) + \frac{\mu_5^2\cosh\lambda + \mu_5\mu_6\sinh\lambda}{4\mu_1(1+\cosh\lambda)} + \frac{1}{2}\left(\mu_7 + \mu_8\right). \end{aligned} \tag{2.25}$$

Thus, in this case we have seven free parameters left to our disposal. Also in this case we obtained a genuine non-Hermitian/Hermitian isospectral pair of Hamiltonians.

### 2.5 $\mathcal{PT}_5$-invariant Hamiltonians of $E_2$-Lie algebraic type

As general $\mathcal{PT}_5$-invariant Hamiltonian we consider

$$H_{\mathcal{PT}5} = \mu_1 J^2 + \mu_2 J + \mu_3 u + i\mu_4 v + \mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv. \tag{2.26}$$

This Hamiltonian is non-Hermitian unless $\mu_6 = \mu_9 = 0$ and $\mu_5 = -2\mu_4$. In the same manner as in the previous subsections we construct the isospectral counterpart

$$\begin{aligned} h_{\mathcal{PT}_5} &= \mu_1 J^2 + \mu_2 J + \frac{1}{2}\left(\mu_5 - \mu_6 \tanh\frac{\lambda}{2}\right)\{u, J\} \\ &+ \left[\frac{2\mu_5^2\sinh^2\lambda + \mu_6^2(\operatorname{sech}^2\frac{\lambda}{2} + \cosh(2\lambda) - 1) + 2(\tanh\frac{\lambda}{2} - \sinh(2\lambda))\mu_5\mu_6}{8\mu_1}\right. \\ &\left. + \frac{\mu_8 - \mu_7}{2}\cosh(2\lambda)\right]\left(v^2 - u^2\right) + \left[\operatorname{csch}\lambda\left(\mu_4 + \frac{1}{2}\mu_5\right) + \frac{\mu_2}{2\mu_1}(\mu_5 - \coth\lambda\mu_6)\right] u \\ &+ \frac{\mu_6^2\cosh\lambda - \mu_5\mu_6\sinh\lambda}{4\mu_1(1+\cosh\lambda)} + \frac{1}{2}\left(\mu_7 + \mu_8\right), \end{aligned} \tag{2.27}$$

where the constants are constraint to

$$\tau = 0, \quad \rho = \frac{\lambda\left(\mu_5 - \mu_6\coth\lambda\right)}{2\mu_1}, \quad \coth(2\lambda) = \frac{\mu_5^2 + \mu_6^2 - 4\mu_1\mu_7 + 4\mu_1\mu_8}{2\mu_5\mu_6}, \tag{2.28}$$

$$\mu_3 = \frac{(2\mu_1\mu_4 + \mu_1\mu_5 - \mu_2\mu_6)\coth(\lambda)}{2\mu_1} + \frac{\mu_2\mu_5}{2\mu_1} - \frac{\mu_6}{2}, \quad \mu_9 = 0. \tag{2.29}$$

Thus, in this case we have also seven free parameters left to our disposal.

Having obtained the Hermitian counterpart, let us construct in this case some solutions to the time-independent Schrödinger equation. The discussion of the entire parameter space is a formidable task, but as we shall see it will be sufficient to focus on some special parameter choices in order to extract different types of qualitative behaviour. We will also make contact to some special cases previously treated in the literature, notably in the area of complex optical lattices.

### 2.5.1 Maps to a three parameter real Mathieu equation

First we specify our parameters further such that only three are left free

$$\mu_1 = 1, \quad \mu_2 = 0, \quad \mu_5 = -2\mu_4, \quad \mu_6 = -2\mu_3, \quad \mu_8 = \mu_9 = 0, \tag{2.30}$$

$$\tau = 0, \quad \rho = \lambda\left(\mu_3 \coth\lambda - \mu_4\right), \quad \coth(2\lambda) = \frac{\mu_3^2 + \mu_4^2 - \mu_7}{2\mu_3\mu_4}. \tag{2.31}$$

The corresponding isospectral pair of Hamiltonians simplifies in this case to

$$H^{(3)}_{\mathcal{PT}_5} = J^2 - i\mu_3\{v, J\} - \mu_4\{u, J\} + \mu_7 u^2, \tag{2.32}$$

$$h^{(3)}_{\mathcal{PT}_5} = J^2 + \alpha\{u, J\} + \beta u^2 + \gamma, \tag{2.33}$$

where $\alpha$, $\beta$, $\gamma$ are functions of $\mu_3$, $\mu_4$, $\mu_7$

$$\alpha = \mu_3 \tanh\frac{\lambda}{2} - \mu_4, \tag{2.34}$$

$$\beta = \frac{2\mu_3}{1+\cosh\lambda}(\mu_3\cosh\lambda - \mu_4\sinh\lambda) + \mu_7 - 2\gamma, \tag{2.35}$$

$$\gamma = (\mu_3\cosh\lambda - \mu_4\sinh\lambda)^2 - \mu_7\sinh^2\lambda. \tag{2.36}$$

For the representation (2.2) the standard Mathieu differential equation (2.21) with real coupling constant is easily converted into the time-independent Schrödinger equation

$$h^{(3)}_{\mathcal{PT}_5}\psi(\theta) = E\psi(\theta) \tag{2.37}$$

with the transformations $\phi(\theta) \to e^{-i\alpha\cos\theta}\psi(\theta)$, $q \to (\alpha^2-\beta)/4$ and $E \to E+(\alpha^2-\beta)/2-\gamma$. Therefore (2.37) is solved by

$$\psi(\theta) = e^{i\alpha\cos\theta}\left[c_1 C\left(E + \frac{\alpha^2-\beta}{2} - \gamma, \frac{\alpha^2-\beta}{4}, \theta\right) + c_2 S\left(E + \frac{\alpha^2-\beta}{2} - \gamma, \frac{\alpha^2-\beta}{4}, \theta\right)\right] \tag{2.38}$$

where $C$ and $S$ denote the even and odd Mathieu function, respectively. A discrete energy spectrum is extracted in the usual way by imposing periodic boundaries $\psi(\theta + 2\pi) = e^{i\pi s}\psi(\theta)$ as quantization condition. While in general anyonic conditions are possible in dimensions lower than 4, we present here only the bosonic and fermionic case, that is $s = 0$ and $s = 1$, respectively. As the Mathieu function is known to possess infinitely many periodic solutions, the boundary condition as such is not sufficient to obtain a unique solution. However, the latter is achieved by demanding in addition the continuity of the energy levels at $q = 0$. The inclusion of all values for $s$ will naturally lead to band structures.

We commence our numerical analysis by taking $\mu_7 = 0$. In this case the map $\eta$ is well-defined, except when $\mu_3 = \mu_4$ for which $\lambda \to \infty$ by (2.31). Thus we expect an entirely real energy spectrum. In figure 1 we present the results of our numerical solutions for the computation of the lowest seven energy levels, demonstrating that this is indeed the case for the even and odd solutions for bosonic as well as fermionic boundary conditions.

For nonzero values of $\mu_7$ we can enter the ill-defined region for the Dyson map as for the last constraint in (2.31) we may encounter values on the right hand side between $-1$

and 1. Viewing the energies as functions of $\mu_{3/4}$ we expect therefore to find four exceptional points at $\mu_{3/4} = \pm\mu_{4/3} \pm \sqrt{\mu_7}$. As an example we fix $\mu_{3/4} = 1$ and $\mu_7 = 4$, such that $\eta(\mu_{4/3})$ is only well defined for $|\mu_{4/3}| < 1$ or $|\mu_{4/3}| > 3$. Indeed our numerical solutions for this choice presented in figure 2 confirm this prediction. We observe that the energies acquire a complex part when $1 < \mu_{3/4} < 3$ and $-3 < \mu_{3/4} < -1$ and is real otherwise. We present here only the spectrum for bosonic boundary condition with an even wavefunction since the qualitative behaviour for the other cases and levels are very similar as already noted in the previous example.

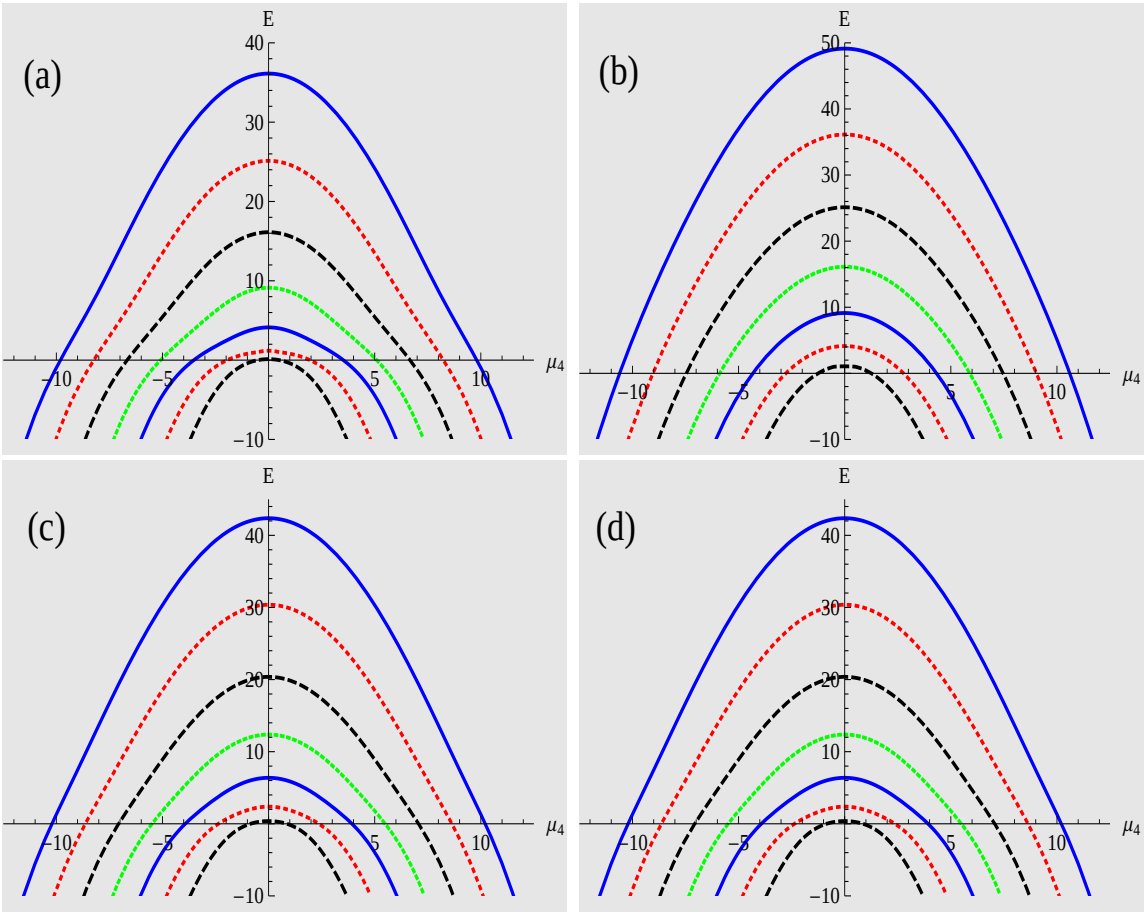


**Figure 1:** Entirely real energy spectrum for the non-Hermitian Hamiltonian $H^{(3)}_{\mathcal{PT}_5}$ as a function of $\mu_4$ with $\mu_3 = 1/2$ and $\mu_7 = 0$. The values for even (odd) eigenfunctions with bosonic and fermionic boundary conditions are displayed in the panels a and c (b and d), respectively.

We clearly observe the typical behaviour of spontaneously broken $\mathcal{PT}$-symmetry in form of two of the real energies merging into complex conjugate pairs at exceptional points. We further note that there are three disconnected regions $|\mu_{3/4}| < 1$ or $|\mu_{3/4}| > 3$ in which all the energies are real.

Alternatively we may also view the energy spectra as functions of $\mu_7$, in which case we expect just two exceptional points at $(\mu_3 \pm \mu_4)$ [2]. Our numerical solutions for this choice are presented in figure 3, which clearly confirms these values and the predicted qualitative

behaviour.

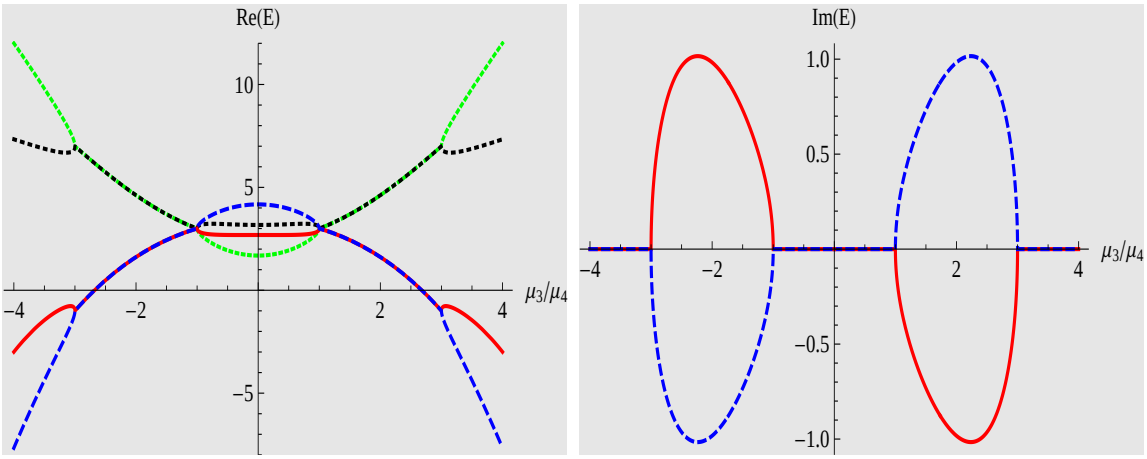


**Figure 2:** Spontaneously broken energy spectra for $H^{(3)}_{\mathcal{PT}_5}$ as a function of $\mu_3$ with fixed values $\mu_4 = 1$ and $\mu_7 = 4$ with even (green, short dashed) and odd (black, dotted) eigenfunctions for bosonic boundary conditions and as a function of $\mu_4$ with fixed values $\mu_3 = 1$ and $\mu_7 = 4$ with even (red, solid) and odd (blue, dashed) eigenfunctions for bosonic boundary conditions. The exceptional points are located at $(\mu_{3/4} = \pm 1, E = 3)$, $(\mu_3 = \pm 3, E = 7)$ and $(\mu_4 = \pm 3, E = -1)$.

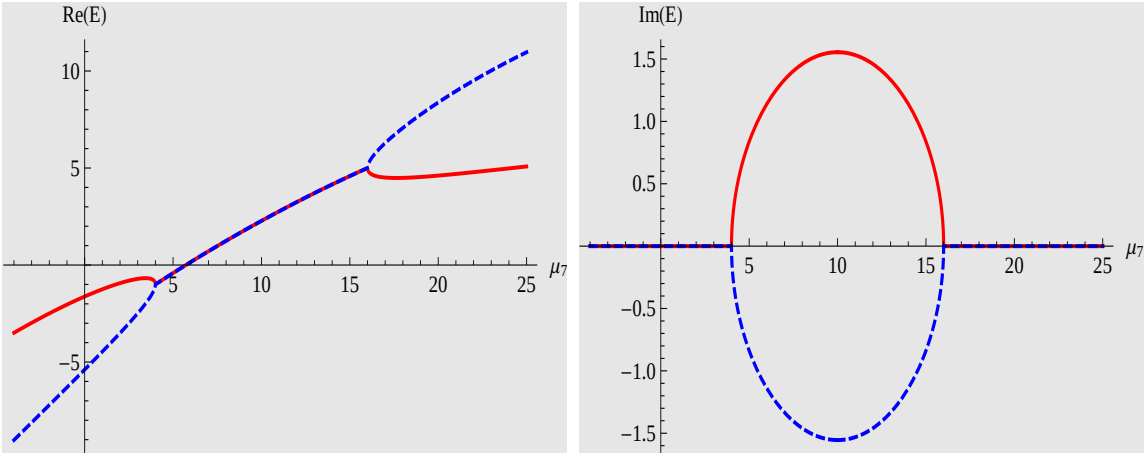


**Figure 3:** Spontaneously broken energy spectra for $H^{(3)}_{\mathcal{PT}_5}$ as a function of $\mu_7$ with fixed values $\mu_3 = 1$ and $\mu_4 = 3$ with even (red, solid) and odd (blue, dashed) eigenfunctions. The exceptional points are located at $(\mu_7 = 4, E = -1)$ and $(\mu_7 = 16, E = 5)$.

We conclude this subsection by considering the behaviour of some intensities, as in principle these quantities are experimentally accessible. In figure 4 we display the intensity $I(\theta) = |\psi(\theta)|^2$ for an odd and even wavefunction merging at the exceptional points whose energy spectrum as displayed in figure 2. In the spontaneously broken $\mathcal{PT}$-regime we clearly observe the loss/gain symmetry around the line $I_{\max}(\theta)/2$, which is absent in the unbroken $\mathcal{PT}$-regime. Searching for a measurable quantity that can be used to identify the

symmetry breaking we observe that

$$I(\theta) := |\psi_{\text{even}}(\theta)|^2 + |\psi_{\text{odd}}(\theta)|^2 - |\psi_{\text{even}}(0)|^2 \begin{cases} = 0 & \text{for broken } \mathcal{PT}\text{-symmetry} \\ \neq 0 & \text{for unbroken } \mathcal{PT}\text{-symmetry} \end{cases} . \quad (2.39)$$

We note that the change from one regime to the other is very abrupt and sharp. This effect is very strongly displayed in figure 5, where we scan over a larger range for the coupling constant $\mu_3$ entering and leaving the broken $\mathcal{PT}$-regime. We depict $I(\theta)$ as defined in (2.39) and clearly observe an oscillatory behaviour in the unbroken $\mathcal{PT}$-regime ($\mu_3 < 1$ and $\mu_3 > 3$) and complete annihilation in the region where the symmetry is spontaneously broken ($1 < \mu_3 < 3$). This qualitative behaviour is somewhat reminiscent of the symmetric gain/loss behaviour observed in complex optical potentials [10].

Based on our observation we propose (2.39) as a measurable quantity that can be used as a criterium to distinguish between unbroken $\mathcal{PT}$-symmetric and spontaneously broken $\mathcal{PT}$-symmetric regimes. At this point this behaviour remains an observation for which we have no rigorous explanation.

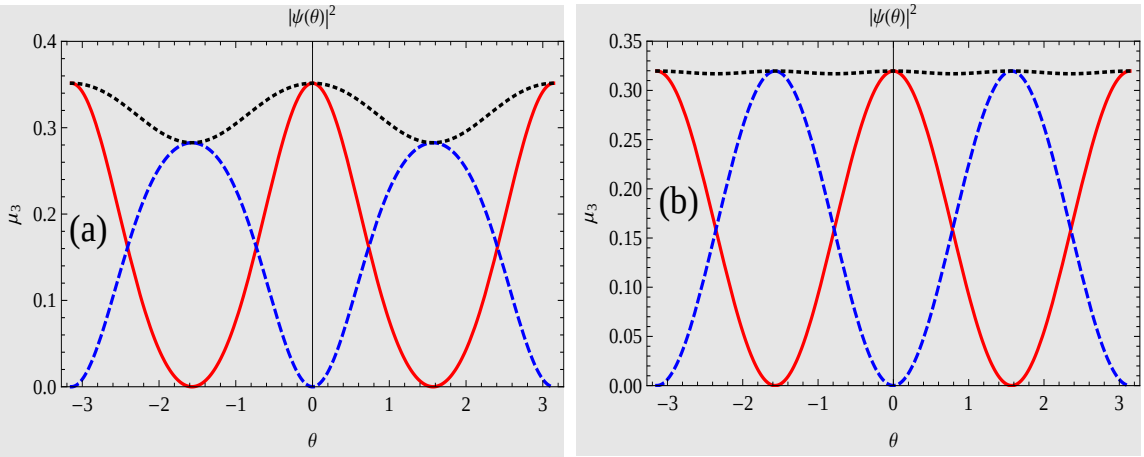


**Figure 4:** Intensities for a merging an even (red, solid) and odd (blue, dashed) wavefunction together with their sum (black, dotted) in the unbroken with $\mu_3 = 0.8$, $\mu_4 = 1$, $\mu_7 = 4$ and broken $\mathcal{PT}$-regime with $\mu_3 = 1.2$, $\mu_4 = 1$, $\mu_7 = 4$, panel (a) and (b), respectively.

### 2.5.2 Sinusoidal optical lattices

For different choices we can also make contact with a simpler example currently of great interest, since it can be realized experimentally in form of optical lattices. Making the simple choice

$$\mu_1 = 1, \quad \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0 \quad \tau = \rho, \quad \coth(2\lambda) = \frac{\mu_7 - \mu_8}{\mu_9}, \quad (2.40)$$

we obtain the isospectral Hermitian counterpart

$$h^{(ol)}_{\mathcal{PT}_{4/5}} = J^2 + \frac{1}{2}\sqrt{(\mu_7 - \mu_8)^2 - \mu_9^2}(v^2 - u^2) + \frac{1}{2}(\mu_7 + \mu_8). \quad (2.41)$$

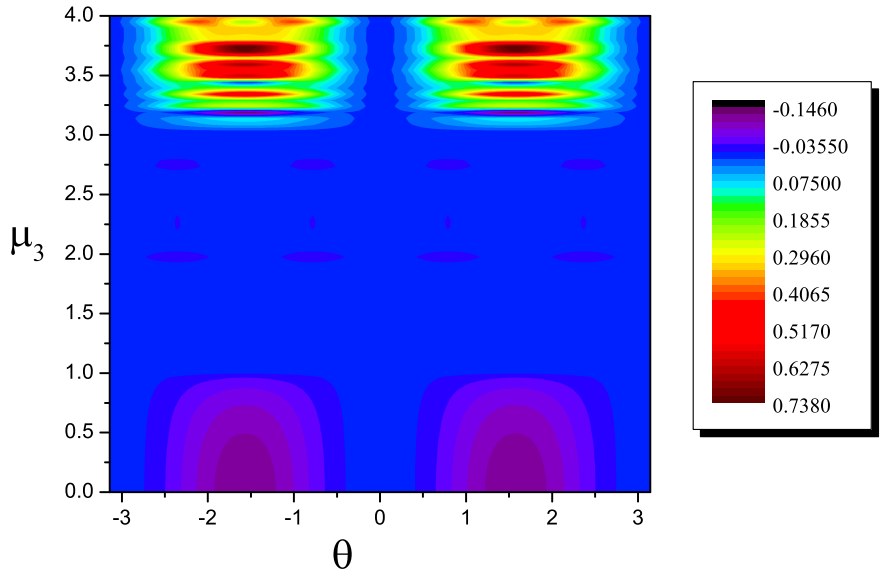


**Figure 5:** Intensity sum $I(\theta) = |\psi_{\text{even}}(\theta)|^2 + |\psi_{\text{odd}}(\theta)|^2 - |\psi_{\text{even}}(0)|^2$ as a function of $\mu_3$ with fixed values $\mu_4 = 1$ and $\mu_7 = 4$.

Taking the representation (2.2) in (2.41), the further special choices $\mu_7 = 0$, $\mu_8 = -4$, $\mu_9 = -8V_0$ or $\mu_7 = -\mu_8 = A/2$, $\mu_9 = -2AV_0$ reduce the potential to the sinusoidal optical lattice potential dealt with in [11] or [12], respectively. In both cases the requirement for the validity of the Dyson map $|(\mu_7 - \mu_8)/\mu_9| < 1$, implied by the last equation in (2.40), boils down to $|V_0| < 1/2$ confirming the finding in [11] and [12] that only in this regime the corresponding potential leads to a real energy spectrum.

### 2.5.3 Complex Mathieu equation

We conclude by discussing the parameter choice

$$\mu_1 = 1, \quad \mu_2 = 0, \quad \mu_3 = -\frac{\mu_6}{2}, \quad \mu_5 = -\mu_4, \quad \mu_7 = \frac{\mu_4^2}{2}, \quad \mu_8 = -\frac{\mu_6^2}{4}, \quad \mu_9 = -\frac{\mu_4\mu_6}{2}. \tag{2.42}$$

In that case the reported similarity transformation is invalid. However, similarly as in the previous case we may solve the corresponding Schrödinger equation exactly by mapping it to the Mathieu equation, which is however complex in this case. We then find the solution

$$\psi(\theta) = e^{-i\mu_4/2\cos\theta + \mu_6/2\sin\theta}\left[c_1 C\left(4E, i\mu_4, \theta/2\right) + c_2 S\left(4E, i\mu_4, \theta/2\right)\right]. \tag{2.43}$$

As in the previous case we impose bosonic or fermionic boundary conditions to determine the spectrum. Our results are depicted in figure 6.

We clearly observe the usual merger of two energy levels at the exceptional points where they split into complex conjugate pairs. Since the real part of the energy is monotonically increasing we note that the spectrum is entirely real for $|\mu_4| \leq 1.46876$. It remains an

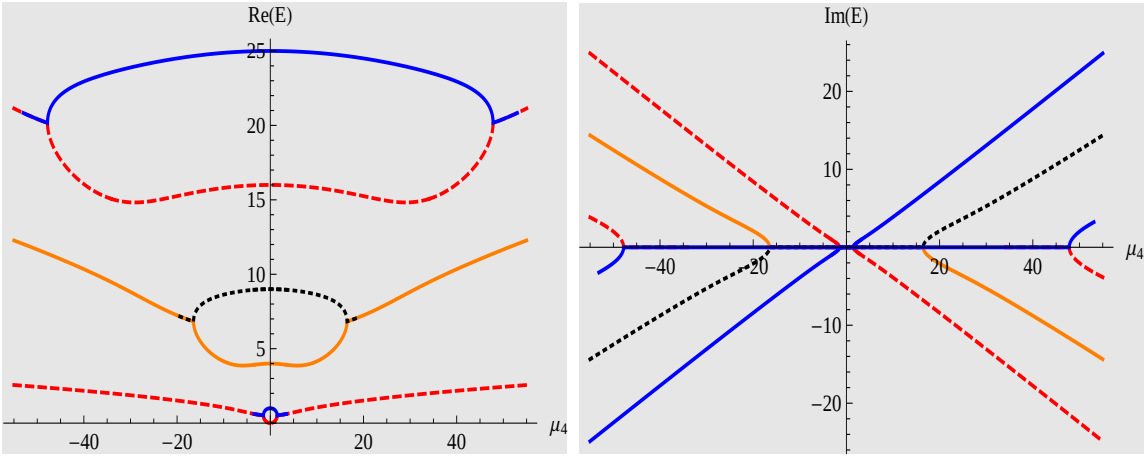


**Figure 6:** Spontaneously broken energy spectra for the parameter choice (2.37) as a function of $\mu_4$ with even eigenfunctions for bosonic boundary conditions. The exceptional points are located at $(\mu_4 = \pm 1.4687, E = 0.5205)$, $(\mu_4 = \pm 16.47116, E = 6.8323)$ and $(\mu_4 = \pm 47.80596, E = 20.1677)$.

open challenge to explain the origin of this value for instance by finding an exact similarity transformation. As we expect, this behaviour is similar to the one reported in [15].

## 3. $\mathcal{PT}$-symmetric $\mathbf{E}_3$-invariant systems

The $E_3$-algebra is the rank 3 extension of the $E_2$-algebra, spanned by six generators $J_i$, $P_i$ for $i = 1, 2, 3$ satisfying the algebra

$$[J_j, J_k] = i\varepsilon_{jkl} J_l, \qquad [J_j, P_k] = i\varepsilon_{jkl} P_l, \qquad \text{and} \qquad [P_j, P_k] = 0. \tag{3.1}$$

Evidently every subset $\{J_j, P_k, P_l\}$ with $j \neq k \neq l$ constitutes an $E_2$-subalgebra. It is convenient to introduce the following combinations of the generators

$$J_z = 2J_1, \quad J_\pm = J_2 \pm iJ_3, \quad P_z = P_1, \qquad \text{and} \qquad P_\pm = \pm P_2 + iP_3, \tag{3.2}$$

such that we obtain the commutation relations

$$[J_z, J_\pm] = \pm 2J_\pm, \quad [J_+, J_-] = J_z, \quad [J_z, P_\pm] = \pm 2P_\pm, \quad [J_\pm, P_z] = -P_\pm, \quad [J_\pm, P_\mp] = -2P_z, \tag{3.3}$$

with all remaining ones vanishing. In [26] the following representation was provided for this algebra

$$\begin{aligned} J_z &:= x\partial_x - y\partial_y, & J_+ &:= x\partial_y, & J_- &:= y\partial_x, \\ P_z &:= -xy\partial_z, & P_+ &:= x^2\partial_z, & P_- &:= y^2\partial_z. \end{aligned} \tag{3.4}$$

Similarly as $E_2$, also $E_3$ is left invariant with respect to various types of $\mathcal{PT}$-symmetries

$$\begin{aligned} \mathcal{PT}_1: &\quad J_k \to -J_k, & P_k &\to -P_k, & i &\to -i; \\ \mathcal{PT}_2: &\quad J_k \to -J_k, & P_k &\to P_k, & i &\to -i; \\ \mathcal{PT}_3: &\quad J_k \to J_k, & P_1 &\to P_1, & P_2 &\leftrightarrow P_3, & i &\to -i; \\ \mathcal{PT}_4: &\quad J_1 \to -J_1, & J_{2/3} &\to J_{2/3}, & P_{1/3} &\leftrightarrow -P_{1/3}, & P_2 &\leftrightarrow P_2, \quad i \to -i; \end{aligned} \tag{3.5}$$

for $k = 1, 2, 3$.

Once again we wish to find the Dyson map to map non-Hermitian Hamiltonians expressed in terms of bilinear combinations of these generators to Hermitian ones. For the $E_3$-algebra we take it to be of the general form

$$\eta = e^{\lambda_z J_z + \lambda_+ J_+ + \lambda_- J_- + \kappa_z P_z + \kappa_+ P_+ + \kappa_- P_-}, \qquad \text{for } \lambda_z, \lambda_\pm, \kappa_z, \kappa_\pm \in \mathbb{R}. \tag{3.6}$$

For the adjoint action of this operator on the $E_3$-generators we compute

$$\eta P_\ell \eta^{-1} = \mu_{\ell z} P_z + \mu_{\ell +} P_+ + \mu_{\ell -} P_- \qquad \text{for } \ell = z, \pm \tag{3.7}$$

with constant coefficients

$$\begin{aligned}
\mu_{zz} &= 1 + 2c(\omega)\lambda_+\lambda_-, \quad \mu_{\pm\pm} = 1 + (2\lambda_z^2 + \lambda_+\lambda_-)c(\omega) \pm 2s(\omega)\lambda_z, \\
\mu_{\pm\mp} &= c(\omega)\lambda_\mp^2, \quad \mu_{\pm z} = \mp 2c(\omega)\lambda_z\lambda_\mp - 2s(\omega)\lambda_\mp, \quad \mu_{z\pm} = \mp c(\omega)\lambda_z\lambda_\pm - s(\omega)\lambda_\pm,
\end{aligned}$$

and

$$\eta J_\ell \eta^{-1} = \nu_{\ell z} J_z + \nu_{\ell +} J_+ + \nu_{\ell -} J_- + \rho_{\ell z} P_z + \rho_{\ell +} P_+ + \rho_{\ell -} P_- \qquad \text{for } \ell = z, \pm \tag{3.8}$$

with constant coefficients

$$\begin{aligned}
\nu_{zz} &= 1 + 2c(\omega)\lambda_+\lambda_-, \quad \nu_{\pm\pm} = 1 + \tilde{\omega}^2 c(\omega) \pm 2s(\omega)\lambda_z, \quad \nu_{\pm\mp} = -c(\omega)\lambda_\mp^2, \\
\nu_{\pm z} &= \mp s(\omega)\lambda_\mp - c(\omega)\lambda_z\lambda_\mp, \quad \nu_{z\pm} = -2c(\omega)\lambda_z\lambda_\pm \mp 2s(\omega)\lambda_\pm,
\end{aligned}$$

$$\begin{aligned}
\rho_{zz} &= 4\left[(\lambda_-\kappa_+ - \lambda_+\kappa_-)\,c(\omega) - \frac{\lambda_+\lambda_-}{\omega^2}\mu(c(\omega) - s(\omega))\right] \\
\rho_{z\pm} &= c(\omega)(\pm\lambda_\pm\kappa_z - 2\lambda_z\kappa_\pm) \mp 2s(\omega)(\kappa_\pm + \lambda_\pm\kappa_z) \pm \frac{2c(\omega)}{\omega^2}\lambda_\pm\nu + \frac{s(\omega)}{\omega^2}\lambda_\pm\,(\mu \mp 2\nu) \\
&\quad -\frac{\cosh(2\omega)}{\omega^2}\mu\lambda_\pm \\
\rho_{\pm z} &= c(\omega)(\lambda_\mp\kappa_z \pm 2\lambda_z\kappa_\mp) + 2s(\omega)(\kappa_\mp - \lambda_\mp\kappa_z) + \frac{2c(\omega)}{\omega^2}\lambda_\mp\nu \pm \frac{s(\omega)}{\omega^2}\lambda_\mp\,(\mu \mp 2\nu) \\
&\quad \mp\frac{\cosh(2\omega)}{\omega^2}\mu\lambda_\mp \\
\rho_{\pm\pm} &= \pm c(\omega)\tilde{\mu} + s(\omega)\kappa_z \pm \mu\frac{\tilde{\omega}^2}{\omega^2}[s(\omega) - c(\omega)] + \frac{\cosh(2\omega) - s(\omega)}{\omega^2}\lambda_z\mu \\
\rho_{\pm\mp} &= -2c(\omega)\lambda_\mp\kappa_\mp \pm \frac{\mu\lambda_\mp^2}{\omega^2}[s(\omega) - c(\omega)]
\end{aligned}$$

where we abbreviated $\omega := \sqrt{\lambda_z^2 + \lambda_+\lambda_-}$, $\tilde{\omega} := \sqrt{2\lambda_z^2 + \lambda_+\lambda_-}$, $\mu := \kappa_z\lambda_z + \kappa_+\lambda_- - \kappa_-\lambda_+$, $\tilde{\mu} := 2\kappa_z\lambda_z + \kappa_+\lambda_- - \kappa_-\lambda_+$, $\nu := \kappa_+\lambda_z\lambda_- - \kappa_z\lambda_+\lambda_- - \kappa_-\lambda_z\lambda_+$, $c(\omega) := (\cosh(2\omega) - 1)/(2\omega^2)$ and $s(\omega) := \sinh(2\omega)/(2\omega)$.

The construction of isospectral counterparts, if they exist, for non-Hermitian Hamiltonians symmetric with regard to various different types of $\mathcal{PT}$-symmetries is far more involved in this for this algebra. The most generic cases are very complicated in this

case as they involve 25 free parameters. One may therefore restrict the discussion to simpler examples, such as for instance the complements of $E_2$ in $E_3$ constitutes well-defined subclasses

For instance, we may consider a $\mathcal{PT}_1$-invariant Hamiltonians of $E_3/E_2$-Lie algebraic type. Selecting $\{J_z, P_\pm\}$ as the generators of the $E_2$-subalgebra the most general Hamiltonian of this type is

$$\tilde{H}_{\mathcal{PT}_1} = \mu_1 J_+^2 + \mu_2 J_-^2 + \mu_3 P_z^2 + \mu_4 P_z J_+ + \mu_5 P_z J_- + \mu_6 J_+ J_- + i\mu_7 J_+ + i\mu_8 J_- + i\mu_9 P_z. \quad (3.9)$$

All the necessary tools have been provided here to find the corresponding counterparts etc. We leave this discussion for future investigations [27].

## 4. Conclusion

We presented five different types of $\mathcal{PT}$-symmetries (2.4) for the Euclidean algebra $E_2$ (2.1). Considering the most general invariant non-Hermitian Hamiltonians in terms of bilinear combinations of the generators of this algebra, we have systematically constructed isospectral counterparts from Dyson maps $\eta$ of the general form (2.5) by exploiting its adjoint action on the Lie algebraic generators. In this process some of the coupling constants involved had to be constrained. We noted that the different versions of the symmetries also lead to qualitatively quite different isospectral counterparts. For the symmetries $\mathcal{PT}_1$ and $\mathcal{PT}_2$ the required constraints rendered the original Hamiltonians $H_{\mathcal{PT}_{1/2}}$ Hermitian, such that the adjoint action of $\eta$ maps Hermitian Hamiltonians to Hermitian ones. It should be noted that the maps are non-trivial, albeit the distinguishing features of the obtained Hamiltonians $h_{\mathcal{PT}_{1/2}}$ remain unclear. More interesting are the transformation properties of the non-Hermitian Hamiltonians invariant under the symmetries $\mathcal{PT}_3$, $\mathcal{PT}_4$ and $\mathcal{PT}_5$, as they lead to genuine non-Hermitian/Hermitian isospectral pairs constructed from an explicit non-perturbative Dyson map.

For the representation (2.2) we analyzed the $\mathcal{PT}_5$-system in further detail by solving the corresponding time-dependent Schrödinger equation. For some parameter choices we found simple transformations of the real Mathieu equation as solutions. In a subset of cases the corresponding energy spectra were identified to be entirely real, see figure 1. For other choices we observed spontaneously broken $\mathcal{PT}$-symmetry with region in the parameter space where the whole spectrum remained real. It is possible to consider the spectra as functions of coupling constants in such a way that its monotonic variation leads to an initial break down of the $\mathcal{PT}$-symmetry at some exceptional points which is subsequently regained, see figure 2. This numerically observed behaviour is completely understood from the explicit formulae for the Dyson maps, which break down at the exceptional points. In section 2.5.2. we have made contact to some simple systems of optical lattices and it should be highly interesting to investigate further whether the more involved systems with richer structure we considered here may also be realized experimentally. We have verified the typical gain/loss symmetry for one of those models.

Clearly we have not exhausted the discussion for the entire parameter space for the $\mathcal{PT}_5$-system and also left the analysis of time-dependent Schrödinger equation $\mathcal{PT}_3$ and

$\mathcal{PT}_4$ for further investigation. An additional open problem is the analysis of alternative representations such as (2.3) and many more not mentioned here. Also still an intriguing open challenge is the computation of the explicit Dyson map for systems of the type dealt with in section 2.5.3. We established that they certainly require a different type of Ansatz for the Dyson map $\eta$ as the one in (2.5).

The completion of the above mentioned programme is far from being finished for the Euclidean algebra $E_3$. For that case we have provided the far more complicated adjoint action on the generators and left the further analysis, which can be carried out along the same lines as for $E_2$, for future investigations [27].

**Acknowledgments:** SD is supported by a City University Research Fellowship. TM is funded by an Erasmus Mundus scholarship and thanks City University for kind hospitality.

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